Hence, integration by parts and Proposition A.2 (from the appendix) yield the main term

$$\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n) \sim \left|\int \widehat{C}_{k,**}(x)dx\right| \tag{19}$$

$$\sim \frac{4\sqrt{2\pi}\cdot x(\log\log x)^{k-1/2}}{(2k-1)(k-1)!} + \frac{2\sqrt{2\pi}\cdot x\Gamma\left(k-\frac{1}{2},\log\log x\right)}{(k-1)!} - \frac{2\sqrt{2\pi}\cdot x\Gamma\left(k-\frac{3}{2},\log\log x\right)}{(k-1)!}$$

$$\sim \frac{4\sqrt{2\pi}\cdot x(\log\log x)^{k-1/2}}{(2k-1)(k-1)!}.$$

□

4.2 Average orders of the unsigned sequences

Lemma 4.4. *As $x\to\infty$, we have that*

$$\left|\sum_{n\leq x}(-1)^{\omega(n)}\right| \ll \frac{x}{\sqrt{\log\log x}}.$$

Plausible

Proof. By the Erdős-Kac theorem for $\omega(n)$ [9, §1.7] [13, *cf.* §7], for all $k\geq 1$ we have that as $x\to\infty$

This is false.

$$\frac{1}{x}\times\#\{n\leq x : k<\omega(n)\leq k+1\} = \Phi\left(\frac{k+1-\log\log x}{\sqrt{\log\log x}}\right) - \Phi\left(\frac{k-\log\log x}{\sqrt{\log\log x}}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right)$$

As $z\to+\infty$, the CDF for the standard normal distribution satisfies [19, §7]

$$\Phi(z) = \frac{1}{2}\left(1+\operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right) = \frac{3}{2} - \frac{e^{-z^2}}{\sqrt{2\pi}}\left(\frac{1}{z}+O\left(\frac{1}{z^3}\right)\right).$$

An argument based on the last asymptotic expansion that shows

$$\lim_{x\to\infty}\left|\frac{\sum\limits_{k\geq 1}(-1)^k\pi_k(x)}{\sum\limits_{1\leq k\leq\log\log x}(-1)^k\pi_k(x)}\right| \leq A_0+o(1),\ \text{ for some } A_0\in(0,+\infty),$$

What if this sum is zero?

is an absolute constant. In particular, we see that

$$\frac{1}{x}\times\left|\sum_{k>\log\log x}(-1)^k\pi_k(x)\right| \ll \sum_{1\leq k\leq\log\log x}\left|\Phi\left(\frac{k+1}{\sqrt{\log\log x}}\right)-\Phi\left(\frac{k}{\sqrt{\log\log x}}\right)\right|$$

$$\ll \sqrt{\log\log x}\times\sum_{1\leq k\leq\log\log x}\left|\frac{e^{-\frac{(k+1)^2}{\log\log x}}}{k+1} - \frac{e^{-\frac{k^2}{\log\log x}}}{k}\right|$$

$$\ll \frac{(\log\log x)^{3/2}}{\log x}.$$

Hence, using Lemma A.3 from the appendix, we have that as $x\to\infty$

$$\left|\sum_{n\leq x}(-1)^{\omega(n)}\right| \leq A_0\times\left|\sum_{1\leq k\leq\log\log x}\frac{(-1)^k x}{\log x}\cdot\frac{(\log\log x)^{k-1}}{(k-1)!}\right| + O(E_\omega(x))$$

$$= \frac{A_0 x}{2\sqrt{2\pi}\log\log x} + O\left(\frac{x}{(\log\log x)^{3/2}} + E_\omega(x)\right),$$

Proof. By the Erdős-Kac theorem for $\omega(n)$ [9, §1.7] [13, *cf.* §7], for all $k \ge 1$ we have that as $x \to \infty$

$$\frac{1}{x} \times \#\{n \le x : k < \omega(n) \le k+1\} = \Phi\left(\frac{k+1-\log\log x}{\sqrt{\log\log x}}\right) - \Phi\left(\frac{k-\log\log x}{\sqrt{\log\log x}}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right).$$

As $x \to \infty$, the CDF for the standard normal distribution satisfies [10, §7]

You cant use a CLT like this.

I can derive a contradiction assuming it is true.

$$\frac{1}{x} \times \#\{n \le x : k < \omega(n) \le k+1\} = \Phi\left(\frac{k+1-\log\log x}{\sqrt{\log\log x}}\right) - \Phi\left(\frac{k-\log\log x}{\sqrt{\log\log x}}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right).$$

As $x \to \infty$, the CDF for the standard normal distribution satisfies [10, §7]

Let $k = 0$. Then

$$\frac{1}{x}\#\{n \le x : \omega(n) = 1\} \simeq \frac{1}{\log x}$$

$$= O\left(\frac{1}{\sqrt{\log\log x}}\right)$$

$\boxed{\text{Contradiction}}$

CLT's only give bounds for $\mathbb{P}(Y_n \le \mu + \sigma t)$

$$\simeq \Phi(t) \quad n \to \infty$$

As a consequence, if $t_1 < t_2$ are fixed, and $n \longrightarrow \infty$

$$\mathbb{P}(\mu + \sigma t_1 \le Y_n \le \mu + \sigma t_2)$$

$$\longrightarrow \Phi(t_2) - \Phi(t_1).$$

* The 'gap' has to be governed by σ.

* The uniformity in t_1 & t_2 is typically quite limited.