Hence, integration by parts and Proposition A.2 (from the appendix) yield the main term

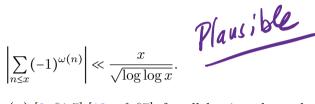
$$\sum_{\substack{n \leq x \\ \Omega(n) = k}} C_{\Omega(n)}(n) \sim \left| \int \widehat{C}_{k,**}(x) dx \right| \tag{19}$$

$$\sim \frac{4\sqrt{2\pi} \cdot x (\log \log x)^{k-1/2}}{(2k-1)(k-1)!} + \frac{2\sqrt{2\pi} \cdot x \Gamma\left(k - \frac{1}{2}, \log \log x\right)}{(k-1)!} - \frac{2\sqrt{2\pi} \cdot x \Gamma\left(k - \frac{3}{2}, \log \log x\right)}{(k-1)!}$$

$$\sim \frac{4\sqrt{2\pi} \cdot x (\log \log x)^{k-1/2}}{(2k-1)(k-1)!}.$$

## 4.2 Average orders of the unsigned sequences

**Lemma 4.4.** As  $x \to \infty$ , we have that



*Proof.* By the Erdős-Kac theorem for  $\omega(n)$  [9, §1.7] [13, cf. §7], for all  $k \ge 1$  we have that as  $x \to \infty$ 

$$\frac{1}{x} \times \# \left\{ n \le x : k < \omega(n) \le k + 1 \right\} = \Phi\left(\frac{k + 1 - \log\log x}{\sqrt{\log\log x}}\right) - \Phi\left(\frac{k - \log\log x}{\sqrt{\log\log x}}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right) \right\}$$

As  $z \to +\infty$ , the CDF for the standard normal distribution satisfies [19, §7]

$$\Phi(z) = \frac{1}{2} \left( 1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) \right) = \frac{3}{2} - \frac{e^{-z^2}}{\sqrt{2\pi}} \left(\frac{1}{z} + O\left(\frac{1}{z^3}\right)\right).$$

An argument based on the last asymptotic expansion that shows

$$\lim_{x \to \infty} \frac{\sum_{k \ge 1} (-1)^k \pi_k(x)}{\sum_{1 \le k \le \log \log x} (-1)^k \pi_k(x)} \le A_0 + o(1), \text{ for some } A_0 \in (0, +\infty),$$
ant. In particular, we see that

is an absolute constant. In particular, we see that

$$\frac{1}{x} \times \left| \sum_{k>\log\log x} (-1)^k \pi_k(x) \right| \ll \sum_{1 \le k \le \log x} \left| \Phi\left(\frac{k+1}{\sqrt{\log\log x}}\right) - \Phi\left(\frac{k}{\sqrt{\log\log x}}\right) \right| \\
\ll \sqrt{\log\log x} \times \sum_{1 \le k \le \log x} \left| \frac{e^{-\frac{(k+1)^2}{\log\log x}}}{k+1} - \frac{e^{-\frac{k^2}{\log\log x}}}{k} \right| \\
\ll \frac{(\log\log x)^{3/2}}{\log x}.$$

Hence, using Lemma A.3 from the appendix, we have that as  $x \to \infty$ 

$$\left| \sum_{n \le x} (-1)^{\omega(n)} \right| \le A_0 \times \left| \sum_{1 \le k \le \log \log x} \frac{(-1)^k x}{\log x} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \right| + O(E_{\omega}(x))$$

$$= \frac{A_0 x}{2\sqrt{2\pi \log \log x}} + O\left(\frac{x}{(\log \log x)^{3/2}} + E_{\omega}(x)\right),$$

Proof. By the Erdős-Kac theorem for  $\omega(n)$  [9, §1.7] [13, cf. §7], for all  $k \ge 1$  we have that as  $x \to \infty$   $\frac{1}{x} \times \# \{n \le x : k < \omega(n) \le k+1\} = \Phi\left(\frac{k+1-\log\log x}{\sqrt{\log\log x}}\right) - \Phi\left(\frac{k-\log\log x}{\sqrt{\log\log x}}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right).$ 

As a contract the CDE for the standard named distribution satisfies [10, 97]

## You cant use a CLT like this.

I can delive a controdiction assuming it in true.

$$\frac{1}{x} \times \# \left\{ n \le x : k < \omega(n) \le k+1 \right\} = \Phi\left(\frac{k+1-\log\log x}{\sqrt{\log\log x}}\right) - \Phi\left(\frac{k-\log\log x}{\sqrt{\log\log x}}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right).$$

As a contract the CDD for the standard mannel distribution satisfies [10, 97]

$$\frac{1}{x} \# \left\{ n \leq x : |\omega(n) \geq 1 \right\} \simeq \frac{1}{\log x}$$

$$=O\left(\frac{1}{\sqrt{\log\log x}}\right)$$

Contradiction

CLT's only give bounds

for  $P(Y_n \leq \mu + \sigma t)$   $\simeq \overline{\Phi}(t)$   $n \to \infty$ 

As a consequence, if  $t_1 < t_2$  are fixed, and  $n \longrightarrow \infty$ 

 $\mathbb{P}\left(\mu+\sigma t_1 \leq Y_1 \leq \mu+\sigma_0 t_2\right)$ 

 $\longrightarrow \overline{\Phi}(t_2) - \overline{\Phi}(t_1).$ 

\* The 'gap' has to be governed by O.

\* The uniformity in to it to in typically quite limited.