Explorations of a Prime–Related Sequence and Closed–Forms

Overview

I have always been drawn to the forms and properties of recurrence relations. In the summer before my first semester at the University of Illinois I became consumed with the pursuit of forms related to the sequence of prime numbers. Inspired by the unusual result of Binet's formula for the seemingly simple Fibonacci sequence, I started racking my brain to find such a non–intuitive result for the fascinating sequence of primes.

I explored the divisibility properties between primes through well–known theorems on modular arithmetic and related transformations. This approach led me to investigate the integer divisibility of the function $\mathrm{J}_x(y)$ (1).

$$\mathrm{J}_x(y) = -\frac{1}{x}\left[2\ (x-1)^{\frac{(y-1)}{2}} - 2 + x\ y\right] \tag{1}$$

Consider the following pair of set definitions where $\mathbb{Z}$ denotes the set of integers.

$$\begin{aligned} \mathbb{K} \quad &\equiv \{y \in \mathbb{Z} \mid y = 4k+1\ (k \in \mathbb{Z}^+) \text{ and } y \mid \mathrm{J}_5(y)\} \\ \mathbb{P}_{4k+1} &\equiv \{y \in \mathbb{Z} \mid y \text{ is prime and } y = 4k+1\} \end{aligned}$$

Extensive empirical evidence suggests that I) All elements of $\mathbb{P}_{4k+1}$ are contained in $\mathbb{K}$ ($\mathbb{P}_{4k+1} \subseteq \mathbb{K}$); and II) the distribution of the elements $p \in \mathbb{K}$ such that $p \notin \mathbb{P}_{4k+1}$ over the positive integers is sparse. It follows from the given properties that finding an expression for the elements of $\mathbb{K}$ ordered over the positive integers is essentially equivalent to finding a countable, closed–form function that generates a huge subset of the sequence of prime numbers. The sequences $\mathbb{P}_{4k+1} = \{13,\ 17,\ 29,\ 37,\ 41,\ \ldots\}$ and $\mathbb{K} - \mathbb{P}_{4k+1} = \{341,\ 561,\ 1729,\ 2701,\ \ldots\}$ are discussed in many forms by well–known literature on prime and pseudo–prime numbers [Rib96]. It is well–known that the sequence of primes $\mathbb{P}_{4k+1} \subseteq \mathbb{P}$ contains infinitely–many elements [Apo76] and so by extension the ordered sequence of the elements of $\mathbb{K}$ is infinite as well. The elements $4k+1 \in \mathbb{K}$ can be generalized somewhat to more exclusive arithmetic progressions of the form $4ak+1$ for positive integer a by performing the substitution $k \mapsto ak$ in the results for the original sequence.

Factorial Expansions and Binomial Coefficients

Consider the triangular recurrence relation (2) and corresponding coefficient polynomial enumeration (3) [Sch09].

$$\begin{bmatrix} n \\ k \end{bmatrix}_\alpha = (\alpha n + 1 - 2\alpha)\begin{bmatrix} n-1 \\ k \end{bmatrix}_\alpha + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_\alpha \tag{2}$$

$$\sum_{n=0}^{\infty} \begin{bmatrix} x \\ x-n \end{bmatrix}_\alpha \frac{(x-n-1)!}{x!}\ z^n = e^{(1-\alpha)z}\left(\frac{\alpha z e^{\alpha z}}{e^{\alpha z}-1}\right)^x \tag{3}$$

The forms of (2) and (3) are similar in a number of respects to the triangle of unsigned Stirling numbers of the first kind and the sequence of Stirling convolution polynomials [GKP94; Knu92; Sch09]. The definition (2) provides the generalization of the expansions of integer–valued j–factorial functions demonstrated by the next identity (4).

$$\prod_{j=0}^{n} (s-1-\alpha j) = \sum_{k=1}^{n+2} \begin{bmatrix} n+2 \\ k \end{bmatrix}_\alpha (-1)^{n+k}\ s^{k-1} \tag{4}$$

Application of the factorial expansion identity (4) provides expressions for the two equivalent binomial coefficient expansions in (5) and (6) that characterize the non–trivial divisibility properties of (1).

$$(x-1)^{\frac{y-1}{2}} = \sum_m \left[\sum_{j=0}^{k-1} \begin{bmatrix} j+1 \\ m+1 \end{bmatrix}_2 \frac{(-1)^{j+m}}{2^j\ j!}[(-x)^j + (-x)^{2k-j}] + \begin{bmatrix} k+1 \\ m+1 \end{bmatrix}_2 \frac{(-1)^m x^k}{2^k\ k!}\right] y^m \tag{5}$$

$$(x-1)^{2k} = \sum_m \left[\sum_{j=0}^{k-1} \begin{bmatrix} j+1 \\ m+1 \end{bmatrix} \frac{(-1)^{j+m}}{j!}[(-x)^j + (-x)^{2k-j}] + \begin{bmatrix} k+1 \\ m+1 \end{bmatrix} \frac{(-1)^m x^k}{k!}\right] (2k)^m \tag{6}$$

The expansions in (5) and (6) may be expressed in terms of the Stirling polynomials, the generalized Bernoulli polynomials, and the Nőrlund polynomials as well through the special case identities derived for the expansion coefficients (2) [GKP94; Sch09] [Rom84, §2.2].

The form of (1) corresponding to the form of (5) satisfies the additional property that $y | \sum_m [y^{2m}]\,\mathrm{J}_5(y)$. The observation is the motivation for the next enumeration (7) [Sch09]. The notation $[z^k]F(z)$ refers to the coefficient of the term z^k in the Taylor series expansion of the function $F(z)$. The bracket notation for series coefficients is equivalently defined as $[z^k]\sum_{n=0}^{\infty} f_n z^n = f_k$.

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} [y^{2m}](x-1)^{\frac{y-1}{2}} y^{2m} z^k \\ &\qquad = \frac{\left(1-x^2z^2\right)\cosh\left(\frac{1}{2}y\log(1-xz)\right)}{(z-1)\sqrt{1-xz}\,(x^2z-1)} \end{aligned} \tag{7}$$

Generating Functions and Coefficient Identities

The parameters in the following closed–form generating function enumerations are defined symbolically in terms of the shorthands $y = 4k + 1$ and $s = 2k$.

$$(x-1)^{\frac{y-1}{2}} = [z^k]\frac{2^{1-k}x^{k+1}z^{k+1}e^{z-yz}(\coth(z)+1)^k}{x-(x-1)e^{2z}} + [z^k]\frac{2^{-k}x^{k+1}z^{k+1}e^{z-yz}(\coth(z)+1)^{k+1}}{(x-1)e^{2z}+1} \tag{8}$$

$$(x-1)^{2k} = [z^k]\frac{zx^{k+2}\left(\frac{ze^{xz}}{e^{xz}-1}\right)^k e^{(1-s)xz}}{(e^{xz}-1)\left((x-1)e^{xz}+1\right)} - [z^k]\frac{zx^k\left(\frac{ze^{\frac{z}{x}}}{e^{\frac{z}{x}}-1}\right)^k e^{-\frac{sz}{x}}}{(x-1)e^{\frac{z}{x}}-x} \tag{9}$$

Observe that the following general result for a parametrized series expansion involved in the forms of both (8) and (9) holds where Φ denotes the Hurwitz Lerch transcendent, Li denotes the polylogarithm function, and $\delta_{n,k}$ denotes Kronecker's delta function.

$$[z^k]\frac{1}{ae^{rz}+b} = \frac{r^k\Phi\left(-\frac{a}{b},-k,0\right)}{b\Gamma(k+1)} = \frac{r^k\mathrm{Li}_{-k}\left(-\frac{a}{b}\right)}{b\Gamma(k+1)} + \frac{\delta_{0,k}}{b}$$

The distinct terms in the closed–form identities given by (10) and (11) demonstrate in a succinct listing the variety of forms that result from the series expansions of (8) and (9). The proof of these results follows from performing discrete convolutions of the coefficients corresponding to the separated generating function products involving the classical enumerations for the Stirling polynomials, $\sigma_n(x)$, the Bernoulli numbers, B_n, the generalized Bernoulli polynomials, $B_n^{(a)}$, and the Nőrlund polynomials, $B_n^{(a)}(x)$, [GKP94; Sch09] [Rom84, §2.2] in addition to special cases of the noted exponential series.

The divisibility of the given characteristic generating series and series coefficient identities are the foundation of the topics I am currently considering in my primes research. It is through investigation of the divisibility of these forms that I aim to find an exact closed–form for the sequence $\mathbb{K}$ that closely approximates the distribution of the prime subset $\mathbb{P}_{4k+1}$.

$$(x-1)^{\frac{y-1}{2}} = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i}\frac{2^{-j}(1-y)^j x^k k\sigma_i(k)\,\Phi\left(\frac{x-1}{x}, i+j-k+1, 0\right)}{\Gamma(j+1)\Gamma(k-i-j)} + \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i}\frac{2^{j-k+1}x^k B_j^{(k)}\left[(2k-y+1)^{k-1-i-j}\right]\left(\Gamma(i+1)\delta_{0,i} + 2^i\mathrm{Li}_{-i}\left(\frac{x-1}{x}\right)\right)}{\Gamma(i+1)\Gamma(j+1)\Gamma(k-i-j)} \tag{10}$$

$$(x-1)^{2k} = \sum_{j=0}^{k-1}\frac{x^k B_{k-1-j}^{(k)}(k-s)\left(x^j\Gamma(j+1)\delta_{0,j} + \mathrm{Li}_{-j}\left(\frac{x-1}{x}\right)\right)}{\Gamma(j+1)\Gamma(k-j)} + \sum_{i=0}^{k}\sum_{j=0}^{k-i}\sum_{m=0}^{k-i-j}\frac{B_j x^{k+1}(1-s)^m k\sigma_i(k)\,\Phi(1-x, i+j-k+m, 0)}{\Gamma(j+1)\Gamma(m+1)\Gamma(k+1-i-j-m)} \tag{11}$$

Concluding Remarks and Applications

The study of prime numbers is fundamental to nearly all branches of pure mathematical number theory. The classical fundamental theorem of arithmetic that provides a unique prime factorization of each natural number also gives implicit importance to primes in the vast areas of integer and combinatorial sequences research. Prime number results are important to many other disciplines such as computer and physical sciences as well, with applications to public–key cryptography, integer factorization problems, pseudo–random number generation, hash tables, and quantum mechanics. Additionally, the distribution of the prime numbers over the positive integers is intimately related to the frequency of non–trivial complex zeros of the Zeta function, $\zeta(s)$, the topic of perhaps the most well–known and important outstanding mathematical conjecture of the Riemann hypothesis. Since prime numbers are involved in the applied theory of so many scientific fields, understanding the prime sequence in more detail, or at least a sizable subset of its elements exactly, will yield immediate applications to further theoretical developments in these fields of study as well – even far outside of the interests of pure mathematical research.

References

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