5 Components to the asymptotic analysis of lower bounds for sums of arithmetic functions weighted by $\lambda(n)$

5.1 A discussion of the results proved by Montgomery and Vaughan

Remark 5.1 (Intuition and constructions in Theorem 3.6)**.** For $|z| < 2$ and $\operatorname{Re}(s) > 1$, let

$$F(s, z) := \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \tag{11}$$

and define the DGF coefficients, $a_z(n)$ for $n \geq 1$, by the product

$$\zeta(s)^z \cdot F(s, z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s}, \operatorname{Re}(s) > 1.$$

Suppose that $A_z(x) := \sum_{n \leq x} a_z(n)$ for $x \geq 1$. Then we obtain the next generating function like identity in z enumerating the $\widehat{\pi}_k(x)$ for $1 \leq k \leq \log\log x$ [–A–]

$$A_z(x) = \sum_{n \leq x} z^{\Omega(n)} = \sum_{k \geq 0} \widehat{\pi}_k(x) z^k \tag{12}$$

Thus for $r < 2$, by Cauchy's integral formula we have

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz.$$

Selecting $r := \frac{k-1}{\log\log x}$ for $1 \leq k < 2\log\log x$ leads to the uniform asymptotic formulas for $\widehat{\pi}_k(x)$ given in Theorem 3.6. Montgomery and Vaughan then consider individual analysis of the main and error terms for $A_z(x)$ to prove that

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

We will require estimates of $A_{-z}(x)$ from below to form summatory functions that weight the terms of $\lambda(n)$ in our new formulas derived in the next sections.

5.2 New uniform asymptotics based on refinements of Theorem 3.6

Proposition 5.2. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \geq 2.$$

When $s := 1$, we have the asymptotic formula from Mertens theorem (see Theorem 4.2). For all integers $s \geq 2$ there is an absolutely defined bounding function $\gamma_0(s, x)$ such that

$$\gamma_0(s, x) + o(1) \leq P_s(x), \text{ as } x \to \infty.$$

It suffices to define the bound in the previous equation as as the quasi-polynomial function in s and x given by

$$\gamma_0(s, x) = s \log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4} s(s-1)^2 \log^2(2).$$

[–A–]In fact, for any additive arithmetic function $a(n)$, characterized by the property that $a(n) = \sum_{p^\alpha || n} a(p^\alpha)$ for all $n \geq 2$, we have that [4, *cf.* §1.7]

$$\prod_p \left(1 - \sum_{m \geq 1} \frac{z^{a(p^m)}}{p^{ms}}\right)^{-1} = \sum_{n \geq 1} \frac{z^{a(n)}}{n^s}, \operatorname{Re}(s) > 1.$$

For $s > 1$, $x \gtrsim c_s$

$$\gamma_0(s, x) < 0 < P_s$$

So your bound is very confusing.

Corollary 5.7. *Using the notation for $A(x,r)$ and $B(x,r)$ from Theorem 5.4, we have that for $x \geq 2$ and $\delta > 0$,*

$$o(1) \leq \frac{B(x,1+\delta)}{A(x,1)} \ll 2, \text{ as } \delta \to 0^+, x \to \infty.$$

Proof. The lower bound stated above is clear. To show that the asymptotic upper bound is correct, we compute using Theorem 5.4 and Theorem 5.5 that

$$\frac{B(x,1+\delta)}{A(x,1)} \ll \frac{x \cdot (\log x)^{\delta - \delta \log(1+\delta)}}{O(1) + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right)} \sim \frac{(\log x)^{\delta-\delta\log(1+\delta)}}{\frac{1}{2} + o(1)} \xrightarrow{\delta \to 0^+} 2,$$

as $x \to \infty$. Notice that since $\mathbb{E}[\Omega(n)] = \log\log n + B$, with $0 < B < 1$ the absolute constant from Mertens theorem, when we denote the range of $k > \log\log x$ as holding in the form of $k > (1+\delta)\log\log x$ for $\delta > 0$ at large x, we can assume that $\delta \to 0^+$ as $x \to \infty$. In particular, this holds since $k > \log\log x$ implies that

$$\lfloor \log\log x \rfloor + 1 \geq (1+\delta)\log\log x \qquad \Longrightarrow \qquad \delta \leq \frac{1 + \{\log\log x\}}{\log\log x} = o(1), \text{ as } x \to \infty. \qquad \square$$

$A(x,1)/x/2 \longrightarrow 1$

$B(x, 1+\delta) \leq x$

So your upper bound is trivial.

And a better bound is true,

7 Lower bounds for $M(x)$ along infinite subsequences

7.1 A formula for $M(x)$

Lemma 7.1. *As $x \to \infty$, we have that*

$$\frac{1}{x} \times \sum_{k=1}^{\log x} \widehat{\pi}_k(x) \cdot k! \ll \frac{1}{x} \times \sum_{k=1}^{\log\log x} \widehat{\pi}_k(x) \cdot k! = o(\sqrt{x}).$$

Proof. First, observe that

$$\begin{aligned} &\ll \frac{1}{x} \times \sum_{k=1}^{\log\log x} \widehat{\pi}_k(x) \cdot k! \ll \frac{1}{\log x} \times \sum_{k=1}^{\log\log x} k(\log\log x)^{k-1} \\ &\ll \frac{1}{(\log x)(\log\log x)} \times (\log\log x)^{\log\log x} = o(\sqrt{x}), \text{ as } x \to \infty. \end{aligned}$$

Now to bound the tail end of the sum over $k \in (\log\log x, \log x]$, we use Theorem 5.4 to show that we have for large $x \to \infty$

$$\begin{aligned} \frac{1}{x} \times \sum_{k=\log\log x+1}^{\log x} \widehat{\pi}_k(x) \cdot k! &\ll \sum_{k=\log\log x}^{\log x} (\log x)^{\frac{k}{\log\log x} - 1 - \frac{k}{\log\log x}(\log k - \log\log\log x)} \cdot \left(\frac{k}{e}\right)^k \sqrt{2\pi k} \\ &\ll \sum_{k=\log\log x}^{\log x} (\log x)^{\frac{\log\log\log x}{\log\log x} - 1} e^{-k\log k} k^{k+\frac{1}{2}} \\ &= \sum_{k=\log\log x}^{\log x} (\log x)^{\frac{\log\log\log x}{\log\log x} - 1} \sqrt{k} \ll \frac{\log\log x}{\log x} \times \int_{\log\log x}^{\log x} \sqrt{t} dt \\ &\ll (\log x)^{\frac{1}{2}} (\log\log x) = o(\sqrt{x}), \end{aligned}$$

where $\lim_{x\to\infty} (\log x)^{\frac{1}{\log\log x}} = e$. □

Lemma 7.2. *As $x \to \infty$, we have that*

$$\sum_{k=1}^{x/2} g^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \frac{x}{k \cdot \log(x/k)}\right] = o(\sqrt{x}).$$

Proof. We will show two separate parts of this result:

$\boxed{\text{A+B} \not\Rightarrow \text{Lemma}}$

(A) $\sum_{k=1}^{x/2} g^{-1}(k) \frac{x}{k \cdot \log(x/k)} = o(\sqrt{x})$, as $x \to \infty$; and

(B) $\sum_{\substack{k=1 \\ \lambda(k)=-1}}^{x/2} g^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \frac{x}{k \cdot \log(x/k)}\right] = o(\sqrt{x})$, as $x \to \infty$.

Then since the are absolute constants $B, C > 0$ so that whenever $x \geq 2$ we have

$$\frac{B \cdot x}{\log x} \leq \pi(x) - \frac{x}{\log x} \leq \frac{C \cdot x}{\log x},$$

we can see that

$$\sum_{k=1}^{x/2} g^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \frac{x}{k \cdot \log(x/k)}\right] = O\left(\left|\sum_{k=1}^{x/2} g^{-1}(k) \frac{x}{k \cdot \log(x/k)}\right| + \left|\sum_{\substack{k=1 \\ \lambda(k)=-1}}^{x/2} g^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \frac{x}{k \cdot \log(x/k)}\right]\right|\right).$$

So in this case, **(A)** and **(B)** imply our result.
Proof of (A): By Lemma 6.3 and Theorem 3.6 we have that

$$\sum_{k=1}^{x/2} g^{-1}(k) \frac{x}{k \cdot \log(x/k)} \ll \sum_{k=1}^{x/2} g^{-1}(k) \cdot \frac{x}{k}$$
$$\left|\sum_{k=1}^{x/2} g^{-1}(k) \cdot \frac{x}{k}\right| \sim x \times \left|\mathbb{E}\left[\sum_{d \mid \frac{x}{2}} g^{-1}(d)\right]\right| \ll x \times \mathbb{E}\left[C_{\Omega(x)}(x)\right] = \sum_{k=1}^{x} C_{\Omega(k)}(k)$$
$$\ll \frac{1}{x} \times \sum_{k=1}^{\log\log x} \widehat{\pi}_k(x) \cdot k! \ll \frac{1}{\log x} \times \sum_{k=1}^{\log\log x} k(\log\log x)^{k-1} \tag{27}$$
$$\ll \frac{1}{(\log x)(\log\log x)} \times (\log\log x)^{\log\log x} = o(\sqrt{x}), \text{ as } x \to \infty.$$

It suffices to only take the sum over $k \leq \log\log x$ in (27) by applying Lemma 7.1.
Proof of (B): We know by the properties established at the end of Section 6.3 that if $\Omega(n) = k$, then for $n \geq 2$

$$|g^{-1}(n)| \leq \sum_{j=0}^{k} \binom{k}{j} \cdot j!.$$

It follows that for large $n \to \infty$,

$$|g^{-1}(n)| \ll \sum_{k=1}^{\log\log n} \frac{\widehat{\pi}_k(n)}{n} \times \sum_{j=0}^{k} \binom{k}{j} \cdot j!$$
$$\ll \frac{1}{(\log n)} \times \sum_{k=1}^{\log\log n} k \cdot (\log\log n)^{k-1}$$
$$\ll \frac{1}{(\log n)} (\log\log n)^{\log\log n}.$$

(B) only sums over $\lambda(x) = -1$

To arrive at the last bound, we have used a crude form of the bound provided by Theorem 3.6 and applied Lemma 7.1.
Let the summatory function

$$\widetilde{L}(x) := \sum_{n \leq x} \lambda(n).$$

I dont Follow

By again applying a crude form of the bound given by Theorem 3.6, followed by an asymptotic approximation to the incomplete gamma function given by (10c), we see from Corollary 5.7 that

$$|\widetilde{L}(x)| \ll \left|\sum_{k=1}^{\log\log x} (-1)^k \widehat{\pi}_k(x)\right| \ll \frac{x}{\sqrt{\log\log x}},$$

where for this approximation $\operatorname{sgn}\left(\widetilde{L}(x)\right) = (-1)^{\lfloor \log\log x \rfloor}$. We can also compute that

$$\frac{d}{dt}|\widetilde{L}(t)| \ll \frac{1}{\sqrt{\log\log t}}.$$

??

We want to sum over the intervals where $\widetilde{L}(x) < 0$ to capture the contribution from the negative magnitude terms $g^{-1}(k)$ such that $\lambda(k) = -1$. Then by Abel summation and integration by parts

$$\left|\sum_{\substack{k=1 \\ \lambda(k)=-1}}^{x/2} g^{-1}(k) \frac{x}{k \cdot \log(x/k)} \left[\frac{\pi\left(\frac{x}{k}\right)}{x/k \cdot \log^{-1}(x/k)} - 1\right]\right|$$

(B) Is harder than your claim! The only cancelation now comes from

$$\pi(x/k) - \frac{x/k}{\log x/k}$$

$$\ll \sum_{\substack{k=1 \\ \lambda(k)=-1}}^{x/2} |g^{-1}(k)| \frac{x}{k \cdot \log(x/k)} \ll \sum_{\substack{k=1 \\ \lambda(k)=-1}}^{x/2} |g^{-1}(k)| \frac{x}{k}$$

$$\ll \int_e^{\frac{\log\log x}{2}} \left.\frac{x \cdot |\widetilde{L}'(t)|(\log\log t)^{\log\log t}}{t \cdot (\log t)}\right|_{t=e^{e^{2k}}} \times e^{e^k} dk$$

$$\ll x \times \left(\max_{1 \leq c \leq \frac{\log\log x}{2}} (2c)^{2c-\frac{1}{2}} \cdot \exp\left(e^c - e^{2c} - 2c\right)\right) \times \frac{\log\log x}{2}$$

$$\ll \frac{(\log\log x)^{\log\log x + \frac{1}{2}}}{\log x} e^{\sqrt{\log x}} = o(\sqrt{x}).$$ □

Why the last proof of (B) makes sense: Notice that as we have argued above, we are basically trying to prove the bound

$$\sum_{\substack{k=1 \\ \lambda(k)=-1}}^{x/2} \frac{x}{k} g^{-1}(k) = \sum_{k=1}^{x/2} \sum_{\substack{r|k \\ \lambda(r)=-1}} g^{-1}(r) = o(\sqrt{x}).$$

We expect the sequence $g^{-1}(k)$ to have a random sign so that we might as well (in explaining this approximation) assume the the sign of $\lambda(k)$ is uniformly distibuted. Now we expect that on average

$$\begin{aligned}
\mathbb{E}|g^{-1}(x)| &= \frac{1}{x} \times \sum_{n \leq x} C_{\Omega(n)}(n) Q\left(\frac{x}{n}\right) \\
&\sim \frac{6}{\pi^2} \times \sum_{n \leq x} \frac{C_{\Omega(n)}(n)}{n} \\
&\ll \int_e^x \frac{(\log\log t)^{\log\log t}}{t} dt \ll (\log x)(\log\log x)^{\log\log x} = o(x^\rho), \text{ for any } \rho > 0.
\end{aligned}$$

We assume that there is a slight sign bias in the distribution of $\lambda(k)$ so that when $\pi(k) \equiv 1 \pmod 2$, we expect there to be more negatively signed terms. At any rate, the sign bias we recover is inherent to the summatory function approximation to $\widetilde{L}(x)$ noted above. So because we are summing over $k \in \left[e^{e^{2m}}, e^{e^{2m+1}}\right)$ for $m \in \mathbb{Z}^+$, we obtain that for any small $\varepsilon > 0$

$$\begin{aligned}
\sum_{k=1}^{x/2} \frac{x}{k} |g^{-1}(k)| \left[\lambda(k) = -1\right]_\delta &\ll \sum_{k=1}^{x/2} \frac{x}{k^{1-\varepsilon}} \left[\lambda(k) = -1\right]_\delta \ll \int_e^{\frac{\log\log x}{2}} \frac{x}{e^{(1-\varepsilon)e^{2k}}} \cdot e^{e^k} dk \\
&\ll x \cdot (\log\log x) \exp\left(\sqrt{\log x} - (\log x)^{1-\varepsilon}\right) \ll \lim_{\varepsilon \to 0} x^{2\varepsilon} (\log\log x) e^{\sqrt{\log x}}.
\end{aligned}$$

We have somewhat oversimplified the bound in the last lines, though this rough back-of-the-envelope argument should make it clearer why we can expect to get this type of cancellation out of our formulas.

Proposition 7.3. *For all sufficiently large x, we have that*

$$M(x) \approx G^{-1}(x) + \frac{2}{\log 2} G^{-1}\left(\frac{x}{2}\right) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt + o(\sqrt{x}). \tag{28}$$

Proof. We know by applying Corollary 3.3 that

$$\begin{aligned}
M(x) &= \sum_{k=1}^{x} g^{-1}(k) \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right) \\
&= G^{-1}(x) + \sum_{k=1}^{x/2} g^{-1}(k) \pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right),
\end{aligned} \tag{29}$$

Exact formula for $g^{-1}(n)$

says
$$|g^{-1}(n)| \geq C_{\Omega(n)}(n)$$

and $$\mathbb{E}\, C_{\Omega(n)}(n) \gg (\log\log n)$$

So $$\sum_{\substack{\lambda(n)=-1 \\ n \leq x}} g^{-1}(k) \frac{x/k}{\log x/k} \ll \sqrt{x}$$

Seems very unlikely.

Maxie — My frustration stems from: I point out serious errors. You say 'oh I can fix it' and the error is not addressed.

I have looked at 3 versions of the argument that all have mistakes in logic and what look like same conceptual error.

I cant be doing this.

I still have serious reservations about this approach — as I have explained, the expected result seems to be well beyond what one would expect as a 1st result on a very famous problem.

Your versions continue to make same conceptual errors. I feel like my attention to each new version is a game of tracing the appearence of same errors further and further down the proof tree.

That suggests that you really need some new ideas. And that I should stop reading these manuscripts.

$\boxed{\text{I will not read another manuscript until Labor Day.}}$

What you <u>should</u> do is stop working on this until you have some new ideas.

Rules for Next version

I stop reading if I see

- old error/confusion not addressed
- "monotonicity principle"
- Elementary mistake in analytical reasoning.
- Faulty application of summation by parts. Their use in oscillatory sums is highly suspect.

I will not read next version for 3 months, i.e. Dec. 1.