limiting) parameter on the input $r \in (1, R)$ to the functions $B(x, r)$. The precise way in which the bound stated in this cited theorem depends on this bounded, indeterminate paramater R can be reviewed for reference in the proof algebra and relations cited in the reference [11, §7].

The role of the parameter R involved in stating the previous theorem is notably important as a scalar factor the upper bound on $k \leq R \log\log x$ in Theorem 4.7 up to which we obtain the valid uniform bounds in x on the asymptotics for $\widehat{\pi}_k(x)$. Namely, we have a discrepancy to work out in so much as we can only form summatory functions over the $\widehat{\pi}_k(x)$ for $1 \leq k \leq R \log\log x$ using the desirable, or "nice", asymptotic formulas guaranteed by Theorem 4.7, even though we can actually have contributions from values distributed throughout the range $1 \leq \Omega(n) \leq \log_2(n)$ to handle over the full range of $n \leq x$ (i.e., when k is required as an index into the full interval of $1 \leq k \leq \log_2(x) \leq x$).

It is then crucial that we can show that the dominant growth of the asymptotic formulas we obtain for these summatory functions is captured by summing only over k in the truncated range where the uniform formulas hold. In particular, we will require a proof that we can discard the terms in the summatory function asymptotic formulas as negligible (up to at most a constant) for large x when they happen to fall in the limiting exceptional range of $\Omega(n) > R \log\log x$ for $n \leq x$.

Corollary 6.4. *Using the notation for $A(x, r)$ and $B(x, r)$ from Theorem 6.1, we have that for $\delta > 0$,*

$$0 \leq \left|\frac{B(x, 1+\delta)}{A(x, 1)}\right| \ll 2, \text{ as } \delta \to 0^+, x \to \infty.$$

Proof. The lower bound stated above should be clear. To show that the asymptotic upper bound is correct, we compute using Theorem 6.1 and Theorem 6.2 that

$$\boxed{\delta - \delta\log(1+\delta) > 0}$$

?

$$\begin{aligned}
\left|\frac{B(x, 1+\delta)}{A(x, 1)}\right| &\ll \left|\frac{x \cdot (\log x)^{\delta - \delta\log(1+\delta)}}{\widehat{\pi}_1(x) + \widehat{\pi}_2(x) + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right)}\right| \\
&\sim \left|\frac{x \cdot (\log x)^{\delta - \delta\log(1+\delta)}}{\frac{x}{\log x} + \frac{x \cdot (\log\log x)}{\log x} + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right)}\right| \\
&= \left|\frac{(\log x)^{1+\delta - \delta\log(1+\delta)}}{1 + \log\log x + \frac{\log x}{2} + o(1)}\right| \\
&\xrightarrow{\delta \to 0^+} \left|\frac{(\log x)}{1 + \log\log x + \frac{\log x}{2} + o(1)}\right| \\
&\sim 2,
\end{aligned}$$

This looks too hard.

This argument is wrong but can be fixed.

as $x \to \infty$. ∎

We again emphasize that Corollary 6.4 implies that for sums involving $\widehat{\pi}_k(x)$ indexed by k, we can capture the dominant asymptotic behavior of these sums by taking k in the truncated range $1 \leq k \leq \log\log x$, e.g., $0 \leq z \leq 1$ in Theorem 4.7. This fact will be important when we prove Theorem 8.4 in Section 8 using a sign-weighted summatory function in Abel summation that depends on these functions (see Lemma 8.2).

6.2 The key new results utilizing Theorem 4.7

We will require a handle on partial sums of integer powers of the reciprocal primes as functions of the integral exponent and the upper summation index x. The next corollary is not a triviality as it comes in handy when we take to the task of proving Theorem 4.8 below. The next statement of Corollary 6.5 effectively generalizes Mertens theorem stated previously as Theorem 5.3 by providing a coarse rate in x below which the reciprocal prime sums tend to absolute constants given by the prime zeta function, $P(s)$.

Corollary to what??

Corollary 6.5. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \gg 2.$$

When $s := 1$, we have the known bound in Mertens theorem (see Theorem 5.3). For $s > 1$, we obtain that

$$P_s(x) \approx E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1).$$

For $s > 2$, $P_s(x) \leq \frac{\pi^2}{6}$

It follows that for $s \geq 2$ we have that

$$P_s(x) \leq \gamma_1(s, x) + o(1).$$

It suffices to take the bounding function in the previous equation as

$$\boxed{\gamma_1(s, x) = -s\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}s(s-1)\log(x/2) + \frac{11}{36}s(s-1)^2\log^2(2).}$$

Proof. Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$ and where our target function $f(t) = t^{-s}$ with $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned}
P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\
&= E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1), |x| \to \infty.
\end{aligned}$$

Now using the inequalities in Facts 5.5, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned}
\frac{P_s(x)}{s} &\geq -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1)\log(x/2) - \frac{11}{36}(s-1)^2\log^2(x) \\
\frac{P_s(x)}{s} &\leq -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1)\log(x/2) + \frac{11}{36}(s-1)^2\log^2(2).
\end{aligned}$$

This completes the proof of the bounds cited above in the statement of this lemma. □

Proof of Theorem 4.8. We have that for all integers $0 \leq k \leq m$

$$[z^k] \prod_{1 \leq i \leq m} (1 - f(i)z)^{-1} = [z^k]\exp\left(\sum_{j \geq 1}\left(\sum_{i=1}^m f(i)^j\right)\frac{z^j}{j}\right). \tag{9}$$

In our case we have that $f(i)$ denotes the i^{th} prime. Hence, summing over all $p \leq ux$ in place of $0 \leq k \leq m$ in the previous formula, and in tandem with Corollary 6.5, we obtain that the logarithm of the generating function series we are after when we sum over all $p \leq ux$ for some parameter u that we must next determine corresponds to

$$\begin{aligned}
\log\left[\prod_{p \leq ux}\left(1 - \frac{z}{p}\right)^{-1}\right] &\geq (B + \log\log(ux))z + \sum_{j \geq 2}\left[a(ux) + b(ux)(j-1) + c(ux)(j-1)^2\right]z^j \\
&= (B + \log\log(ux))z - a(ux)\left(1 + \frac{1}{z-1} + z\right) \\
&\qquad + b(ux)\left(1 + \frac{2}{z-1} + \frac{1}{(z-1)^2}\right) \\
&\qquad - c(ux)\left(1 + \frac{4}{z-1} + \frac{5}{(z-1)^2} + \frac{2}{(z-1)^3}\right) \\
&=: \widehat{\mathcal{B}}(u, x; z).
\end{aligned}$$