An approach to the $\gcd(\binom{2n}{n}, 105) = 1$ problem

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1 A basic heuristic

First, let me give a heuristic as to why we expect there to be infinitely many integers $n \geq 1$ with $\gcd(\binom{2n}{n}, 105) = 1$: a well-known consequence of Kummer's Theorem of p-divisibilty of binomial coefficients is that $\binom{2n}{n}$ is coprime to 105 if and only if when adding n to itself base-3, 5 and 7, there are no carries. In other words, the digits of n in base-3 are all 0 or 1; the digits in base-5 are $0, 1$, or 2; and the digits base-7 are $0, 1, 2$, or 3.

Now, if we choose an integer $n \leq x$ at random, we can ask what the probability is that all its digits in those bases are the possibilities listed. We can't calculate this probability exactly – if we did, then we could already solve the problem – but we can give a plausible heuristic as to its value, by assuming the bases act independently. For x large, the probability that all the digits are 0 or 1 in base-3 is about $(2/3)^{\log(x)/\log(3)}$, since there are at most about $\log(x)/\log(3)$ digits base-3 among all $n \leq x$. For base-5 the probability estimate is $(3/5)^{\log(x)/\log(5)}$, and for base-7 it is $(4/7)^{\log(x)/\log(7)}$. So, the number of integers $n \leq x$ that hit all three requirements, in all three bases, assuming indepedence, is

$$x(2/3)^{\log(x)/\log(3)}(3/5)^{\log(x)/\log(5)}(4/7)^{\log(x)/\log(7)}$$

Taking logs, this is

$$\log(x)\,(1 + \log(2/3)/\log(3) + \log(3/5)/\log(5) + \log(4/7)/\log(7)) \approx 0.02595 \log x.$$

So, there should be about $x^{0.02595}$ integers $n \leq x$ such that $\binom{2n}{n}$ is coprime to 105.

2 An initial working hypothesis

We claim that the following holds:

Working Hypothesis. There exist integers $1 \le B_1 < B_2$ such that for every set of four real numbers $\alpha, \beta, \gamma, \delta$ satisfying

$$0 < \beta,\ \delta < 3^{-B_1};\ 1 \le \alpha < 5;\ 1 \le \gamma < 7,$$

we have that there exist distinct integers

$$1 \le x_1 < x_2 < \cdots < x_t \le B_2,$$

such that if we write the number

$$\alpha\left(\frac{1}{3^{x_1}} + \frac{1}{3^{x_2}} + \cdots + \frac{1}{3^{x_t}}\right) + \beta$$

in its base-5 expansion,

$$\frac{d_1}{5} + \frac{d_2}{5^2} + \cdots + \frac{d_u}{5^u} + \varepsilon_1,\ u = \lceil x_t \log(3)/\log 5\rceil,$$

then

$$d_1, d_2, ..., d_u \in \{0, 1, 2\},\ \text{and } 0 < \varepsilon_1 < 3^{-B_1}5^{-u};$$

and if we write

$$\gamma\left(\frac{1}{3^{x_1}} + \frac{1}{3^{x_2}} + \cdots + \frac{1}{3^{x_t}}\right) + \delta$$

in its base-7 expansion,

$$\frac{e_1}{7} + \frac{e_2}{7^2} + \cdots + \frac{e_v}{7^v} + \varepsilon_2,\ v = \lceil x_t \log(3)/\log 7\rceil,$$

then

$$e_1, e_2, ..., e_v \in \{0, 1, 2, 3\},\ \text{and } 0 < \varepsilon_2 < 3^{-B_1}7^{-v}.$$

This appears to be an infinite problem, since $\alpha, \beta, \gamma, \delta$ are allowed to be real numbers; however, it only suffices to verify the above for all such variables looking at the top few base-3 digits – that is, we may assume that $\alpha, \beta, \gamma, \delta$ have the form

$$\frac{f_1}{3} + \frac{f_2}{3^2} + \cdots + \frac{f_{B_2+m}}{3^{B_2+m}}.$$

3 Expanding to a larger value of B_2

Here, we show that if the working hypothesis in the previous section holds for a pair of integers $B_1 < B_2$, then we must have that it holds for the same B_1 and where we can choose B_2 as large as we wish. To establish this, we will show that we may simple enlarge the value of B_2 – and thus, we can iteratively enlarge it as much as desired.

So, suppose the claim in the previous section holds for the pair $B_1 < B_2$, let $\alpha, \beta, \gamma, \delta$ be any constants satisfying the constraints of that hypothesis; and let $x_1, ..., x_t$, u, v, and ε_1 and ε_2 be as indicated.

Now let

$$\alpha' \ := \ \alpha 3^{-x_t} 5^u \ \in \ [1,\ 5),\ \beta' \ := \ \varepsilon_1 5^u \ \in \ [0,\ 3^{-B_1}).$$

and let

$$\gamma' \ := \ \alpha 3^{-x_t} 7^v \ \in \ [1,\ 7),\ \delta' \ := \ \varepsilon_2 7^v \ \in \ [0,\ 3^{-B_1}).$$

Then we apply the hypothesis to these new choices for $\alpha, \beta, \gamma, \delta$, and find that there exist $1 \leq y_1 < y_2 < \cdots < y_z \leq B_2$ as the new sequence of x_i's.

Now we combine our two lists of numbers together as follows: we leave $x_1, ..., x_t$ as it was before; but now set

$$x_{t+i} \ := \ x_t + y_i.$$

Then, we have that

$$\alpha \left(\frac{1}{3^{x_1}} + \frac{1}{3^{x_2}} + \cdots + \frac{1}{3^{x_{t+z}}} \right) + \beta$$

can be written as

$$\alpha \left(\frac{1}{3^{x_1}} + \cdots + \frac{1}{3^{x_t}} \right) + \beta + \alpha' 5^{-u} \left(\frac{1}{3^{y_1}} + \cdots + \frac{1}{3^{y_z}} \right).$$

This may, in turn, be written as

$$\frac{d_1}{5} + \cdots + \frac{d_u}{5^u} + \frac{d_{u+1}}{5^{u+1}} + \cdots + \frac{d_{u'}}{5^{u'}} + \varepsilon_1 - 5^{-u}\beta' + 5^{-u}\varepsilon_1',$$

where $\varepsilon_1' < 3^{-B_1} 5^{-u'+u}$. Since $\varepsilon_1 = 5^{-u}\beta'$, this simplifies to

$$\frac{d_1}{5} + \cdots + \frac{d_{u'}}{5^{u'}} + \varepsilon_1'',$$

where $\varepsilon'' < 3^{-B_1} 5^{-u'}$.

And, again, we have that all the $d_i \in \{0, 1, 2\}$.

The analogous result hold for the base-7 expansion. So, we conclude that if we find a viable pair $B_1 < B_2$, we can expand B_2 indefinitely.

4 The working hypothesis implies gcd$(\binom{2n}{n}, 105) = 1$ infinitely often

We begin with the well-known fact that $\gcd(\binom{2n}{n}, 105) = 1$ if and only if the base-3 digits of n are in $\{0, 1\}$, the base-5 digits are in $\{0, 1, 2\}$, and the base-7 digits are in $\{0, 1, 2, 3\}$.

We will use the claim from the previous section to algorithmically construct integers n with the desired property. Rather than list out the exact steps of this algorithm, we will here only present how the first two steps work – the other steps just repeat the second step (so, it will be obvious how the algorithm works).

We begin by letting letting $h \geq 1$ be any integer such that

$$|3^h - 5^k| \ < \ 3^{-B_1}, \text{ and } |3^h - 7^\ell| \ < \ 3^{-B_1},$$

where B_1 is as in the previous section (we assume we have a pair $B_1 < B_2$ that satisfies the hypothesis, and choose that value for our B_1 here) and where

$$k \ = \ \lfloor h \log(3)/\log 5 \rfloor, \text{ and } \ell \ = \ \lfloor h \log(3)/\log 7 \rfloor.$$

This is just saying that the upper few digits of 3^h in base 5 and base 7 begin with 1, followed by a few 0's.

Next, we set

$$\alpha \ := \ 3^h 5^{-k}, \text{ and } \gamma \ := \ 3^h 7^{-\ell}.$$

and set

$$\beta \ := \ \delta \ := \ 0,$$

Applying the "working hypothesis" from the previous section to $\alpha, \beta, \gamma, \delta$ we find that there exist $B_1 < B_2$, with B_2 arbitrarily large, and $x_1, x_2, ..., x_t$ satisfying the conclusion of that claim. This implies that

$$3^{h-x_1} + \cdots + 3^{h-x_t} \ = \ 5^k(\alpha(3^{-x_1} + \cdots + 3^{-x_t}) + \beta) \ = \ d_1 5^{k-1} + d_2 5^{k-2} + \cdots + d_u 5^{k-u} + 5^k \varepsilon_1,$$

where $d_1, d_2, ..., d_u \in \{0,1,2\}$, and $5^k\varepsilon_1 < 3^{-B_1}5^{k-u}$ – so, the lower digits in base-5 our of sum of powers of 3 (below the coefficients of 5^{k-u}) will be 0 for a while, then it could be something different.

Similarly, we have

$$3^{h-x_1} + \cdots + 3^{h-x_t} \ = \ e_1 7^{\ell-1} + \cdots + e_v 7^{k-v} + 7^{\ell}\varepsilon_2,$$

where $e_1, ..., e_v \in \{0,1,2,3\}$, and $7^{\ell}\varepsilon_2 < 3^{-B_1}7^{\ell-v}$.

So, the upper few digits of $3^{h-x_1} + \cdots + 3^{h-x_t}$ look good bases $3, 5$, and 7. If B_2 is large enough, this will cover all the base-5 and 7 digits below 5^0 and 7^0, which gives us an n whose base-3, base-5 and base-7 digits are such that there are no carries when adding the number to itself in these bases.