Theorem 4.8 (Generating functions of symmetric functions)**.** *We obtain lower bounds of the following form on the function $\mathcal{G}(z)$ from Theorem 4.7 for $A_0 > 0$ an absolute constant, for $C_0(z)$ a strictly linear function only in z, and where we must take $0 \leq z \leq 1$, or equivalently $1 \leq k \leq \log\log x$ for x large:*

$$\mathcal{G}(z) \geq A_0 \cdot (1-z)^3 \cdot C_0(z)^z.$$

It suffices to take the components to the bound in the previous equation as

$$\begin{aligned} A_0 &= \frac{2^{9/16} \exp\left(-\frac{55}{4}\log^2(2)\right)}{(3e\log 2)^3 \cdot \Gamma\left(\frac{5}{2}\right)} \approx 3.81296 \times 10^{-6} \\ C_0(z) &= \frac{4(1-z)}{3e\log 2}. \end{aligned}$$

In particular, with $0 \leq z \leq 1$ and $z \equiv z(k,x) = \frac{k-1}{\log\log x}$, by Theorem 4.7, we have that

$$\widehat{\pi}_k(x) \overset{\blacktriangle}{\gtrsim} \frac{A_0 \cdot x}{\log x \cdot (\log\log x)^4 \cdot (k-1)!} \cdot \left(\frac{4}{3e\log 2}\right)^k.$$

4.5 Rigorous proofs jusifying that so-called average order lower bounds are meaningful with respect to our problem

Theorem 4.9. *Let the summatory function $G_E^{-1}(x)$ be defined for $x \geq 1$ by*

$$G_E^{-1}(x) := \sum_{n \leq x} \lambda(n) \times \sum_{\substack{d|n \\ d > e^e}} \mathbb{E}[C_{\Omega(d)}(d)].$$

If for some respectively minimally and maximally defined absolute constants $B, C \in [0,1)$, we have that as $x \to \infty$

$$B + o(1) \leq \frac{1}{x} \cdot \#\left\{n \leq x : |G^{-1}(n)| - |G_E^{-1}(n)| \leq 0\right\} \leq C + o(1),$$

then for all sufficiently large x we have some $x_0 \in [(1-C)x, x]$ such that

$$|G^{-1}(x_0)| \geq \left|G_E^{-1}(x_0)\right|.$$

We prove Theorem 4.9, and rigorously justify that its hypothesis holds, in Section 8. This result combines to allow us to take lower bounds based on average order estimates of certain arithmetic functions we have defined to approximate $g^{-1}(n)$ and still recover an infinite subsequence along which we can witness the unboundedness in Corollary 4.12 stated below.

The following observation that is suggestive of the quasi-periodicity at play with the distinct values of $g^{-1}(n)$ distributed over $n \geq 2$:

Heuristic 4.10 (Symmetry in $g^{-1}(n)$ in the exponents in the prime factorization of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly two, three, and so on prime factors (see Table T.1 starting on page 46).

There does not appear to be an easy, nor subtle direct recursion between the distinct g^{-1} values, except through auxiliary function sequences. However, the distribution of distinct sets of prime exponents is fairly regular with $\omega(n)$ and $\Omega(n)$ playing a crucial role in the repitition of common values of $g^{-1}(n)$. The next remark makes clear what our intuiton ought suggest about the relation of the actual function values to the average case expectation of $g^{-1}(n)$ for $n \leq x$ when x is large.

Remark 4.11 (Essential components of the proof)**.** Given that we have chosen to work with a representation for $M(x)$ that depends critically on the distribution of the values of the additive functions, $\omega(n)$ and $\Omega(n)$, there is substantial intuition involved á priori that suggests our sums over these functions ought behave regularly on average. Notably, we have an Erdös-Kac like theorem for each of $\omega(n)$ and $\Omega(n)$, which when the bounding parameter is set to $z := 0$, we provably ought expect these sums involving the classically "nice" functions to tend towards their average case asymptotic nature infinitely often, and predictably near any large x [9, §1.7] (see Theorem 6.1). Thus the choice in stating (1) as it depends on the cannonical additive function examples we have cited is *absolutely essential* to the success of our proof making "magic" happen out of the average case scenario we easily bound from below.

4.6 Nearly cracking the classical unboundedness barrier

In Section 9, we provide the culmination of what we build up to in the proofs established in prior sections of the article. What we obtain at the conclusion of the section is the following important summary corollary that comes close (by a factor of $\log x$) to resolving the classical question of the unboundedness of the scaled function Mertens function $|M(x)|/\sqrt{x}$ in the limit supremum sense:

Corollary 4.12 (Lower Bounds for the Mertens function)**.** *Let* $u_0 := e^{e^{e^{e^{e^{e^{e}}}}}}$ *and define the infinite increasing subsequence,* $\{x_{0,n}\}_{n\geq 1}$*, by* $x_{0,n} := e^{e^{e^{e^{4n\cdot\left\lceil e^{e^{4n}}\right\rceil}}}}$. *We have that along the increasing subsequence* x_y*, for some* $x_y \in (x_{0,y-1}, x_{0,y+1})$*, for large all sufficiently large* $y \gg \max\left(\lceil x_{0,1}\rceil + 1, u_0 + 2\right)$ *the following bound holds:*

$$\frac{|M(x_y)|\log\sqrt{x_y}}{\sqrt{x_y}} \overset{\blacktriangle}{\gtrsim} 2C_{\ell,1}\cdot(\log\log\sqrt{x_y})\frac{(\log\log\log\sqrt{x_y})^{2\log 2+\frac{1}{3\log 2}-1}}{(\log\log\log\log\sqrt{x_y})^{\frac{5}{2}}}\cdot\frac{\log_*^5(\sqrt{x_y})^{2\log 2+\frac{1}{3\log 2}}}{\log_*^6(\sqrt{x_y})^{\frac{5}{2}}}, \text{ as } y\to\infty.$$

In the previous equation, we adopt the notation for the absolute constant $C_{\ell,1} > 0$ *defined more precisely by*

$$C_{\ell,1} := \frac{128\cdot 2^{1/8}}{6561\cdot e^6\pi\log^6(2)}\exp\left(-\frac{55}{2}\log^2(2)\right) \approx 2.76631\times 10^{-10}.$$

This is all to say that in establishing the rigorous proof of Corollary 4.12 based on our new methods, we not only show that

$$\limsup_{x\to\infty}\frac{|M(x)|\log x}{\sqrt{x}} = +\infty,$$

but also set a minimal rate (along a large infinite subsequence) at which this form of the scaled Mertens function grows without bound.