

New characterizations of the summatory function of the Möbius function

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Abstract

The Mertens function, $M(x) := \sum_{n \leq x} \mu(n)$, is defined as the summatory function of the classical Möbius function for $x \geq 1$. The inverse function sequence $\{g^{-1}(n)\}_{n \geq 1}$ taken with respect to Dirichlet convolution is defined in terms of the strongly additive function $\omega(n)$ that counts the number of distinct prime factors of any integer $n \geq 2$. For large x and $n \leq x$, we associate a natural combinatorial significance to the magnitude of the distinct values of the function $g^{-1}(n)$ that depends directly on the exponent patterns in the prime factorizations of the integers in $\{2, 3, \dots, x\}$ viewed as multisets.

We prove an Erdős-Kac theorem analog for the distribution of the unsigned sequence $|g^{-1}(n)|$ over $n \leq x$ with a limiting central limit theorem type tendency towards normal as $x \rightarrow \infty$. For all $x \geq 1$, discrete convolutions of $G^{-1}(x) := \sum_{n \leq x} \lambda(n) |g^{-1}(n)|$ with the prime counting function $\pi(x)$ determine exact formulas and new characterizations of asymptotic bounds for $M(x)$. In this way, we prove another concrete link of the distribution of $L(x) := \sum_{n \leq x} \lambda(n)$ with the Mertens function and connect these classical summatory functions with an explicit normal tending probability distribution at large x . The proofs of these resulting combinatorially motivated new characterizations of $M(x)$ are rigorous and unconditional.

Keywords and Phrases: *Möbius function; Mertens function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function; Erdős-Kac theorem; strongly additive function.*

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1 Introduction

1.1 Preliminaries

1.1.1 Definitions

We define the *Möbius function* to be the signed indicator function of the squarefree integers in the form of [25, A008683]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \wedge n \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined on the positive integers as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as follows [25, A002321]:

$$\{M(x)\}_{x \geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \dots\}.$$

The Mertens function satisfies that $\sum_{n \leq x} M\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = 1$, and is related to the summatory function $L(x) := \sum_{n \leq x} \lambda(n)$ via the relation [6, 12]

$$L(x) = \sum_{d \leq \sqrt{x}} M\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), x \geq 1.$$

A positive integer $n \geq 1$ is *squarefree*, or contains no divisors (other than one when $n \geq 2$) which are squares, if and only if $\mu^2(n) = 1$. The summatory function that counts the number of *squarefree* integers $n \leq x$ satisfies [5, §18.6] [25, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

1.1.2 Properties

A conventional approach to evaluating the limiting asymptotic behavior of $M(x)$ for large $x \rightarrow \infty$ considers an inverse Mellin transformation of the reciprocal of the Riemann zeta function. In particular, since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \cdot \int_1^\infty \frac{M(x)}{x^{s+1}} dx, \text{ for } \operatorname{Re}(s) > 1,$$

we obtain that

$$M(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{T-i\infty}^{T+i\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

The previous two representations lead us to the exact expression of $M(x)$ for any real $x > 0$ given by the next theorem.

Theorem 1.1 (Analytic Formula for $M(x)$, Titchmarsh). *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any real $x > 0$*

$$M(x) = \lim_{k \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ |\operatorname{Im}(\rho)| < T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} [x \in \mathbb{Z}^+]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp\left(-C \cdot \log^{\frac{3}{5}}(x)(\log \log x)^{-\frac{3}{5}}\right).$$

Under the assumption of the RH, Soundararajan improved estimates bounding $M(x)$ from above for large x in the following form [26]:

$$M(x) = O\left(\sqrt{x} \cdot \exp\left((\log x)^{\frac{1}{2}}(\log \log x)^{\frac{5}{2}+\epsilon}\right)\right), \quad \forall \epsilon > 0.$$

1.1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$ for any $0 < \epsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C \cdot \sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens himself for $C = 1$ and all $x < 10000$ without the benefit of modern computation. Since its beginnings in 1897, the Mertens conjecture has been disproved by computational methods with non-trivial simple zeta function zeros with comparatively small imaginary parts in the famous paper by Odlyzko and té Riele [18]. More recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $q(x) := M(x)/\sqrt{x}$ grows with or without bound along infinite subsequences, e.g., considering the asymptotics of $q(x)$ in the limit supremum and limit infimum senses.

It is verified by computation that [21, cf. §4.1] [25, cf. [A051400](#); [A051401](#)]

$$\limsup_{x \rightarrow \infty} \frac{M(x)}{\sqrt{x}} > 1.060 \quad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x \rightarrow \infty} \frac{M(x)}{\sqrt{x}} < -1.009 \quad (\text{now } \leq -1.837625).$$

Based on work by Odlyzko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [18, 10, 11, 7]. A famous conjecture due to Gonek asserts that in fact $M(x)$ satisfies [17]

$$\limsup_{x \rightarrow \infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log \log \log x)^{\frac{5}{4}}} = O(1).$$

1.2 A concrete new approach to characterizing $M(x)$

The main interpretation to take away from the article is our rigorous motivation of an equivalent characterization of $M(x)$ formed by constructing combinatorially relevant sequences related to the distribution of the primes through convolutions of strongly additive functions. These sequences and their summatory functions have not yet been studied in the literature surrounding the Mertens function. The prime-related combinatorics at hand are discussed by the remarks given in Section 3.3. This new perspective offers new exact characterizations of $M(x)$ for all $x \geq 1$ through the formulas involving discrete convolutions of $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ with the prime counting function $\pi(x)$ proved in Section 5.

The sequence $g^{-1}(n)$ defined precisely below and $G^{-1}(x)$ are crucially tied to canonical number theoretic examples of strongly and completely additive functions, e.g., to $\omega(n)$ and $\Omega(n)$, respectively. The definitions of the primary subsequences we define, and the corresponding parameterized bivariate DGF based proof methods that are given in the spirit of Montgomery and Vaughan's work, allow us to reconcile the property of strong additivity with signed sums of multiplicative functions. The proofs of characteristic properties of

these new sequences imply a scaled normal tending probability distribution for the unsigned magnitude of $|g^{-1}(n)|$ that is analogous to the Erdős-Kac theorems for $\omega(n)$ and $\Omega(n)$.

Since we prove that $\text{sgn}(g^{-1}(n)) = \lambda(n)$, it follows that we have a new probabilistic perspective from which to express distributional features of the summatory functions $G^{-1}(x)$ as $x \rightarrow \infty$ in terms of the properties of $|g^{-1}(n)|$ and $L(x) := \sum_{n \leq x} \lambda(n)$. Formalizing the properties of the distribution of $L(x)$ is typically viewed as a problem that is equally as difficult as understanding the distribution of $M(x)$ well for large x . The new results in this article then precisely connect the distributions of $L(x)$, a well defined scaled normally tending probability distribution, and $M(x)$ as $x \rightarrow \infty$.

1.2.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 1.2 (Summatory functions of Dirichlet convolutions). *Let $f, h : \mathbb{Z}^+ \rightarrow \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of f and h , respectively, and that $F^{-1}(x) := \sum_{n \leq x} f^{-1}(n)$ denotes the summatory function of the Dirichlet inverse of f for any $x \geq 1$. We have the following exact expressions for the summatory function of the convolution $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned} \pi_{f*h}(x) &:= \sum_{n \leq x} \sum_{d|n} f(d)h(n/d) \\ &= \sum_{d \leq x} f(d)H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{k=1}^x H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right]. \end{aligned}$$

Moreover, for all $x \geq 1$

$$\begin{aligned} H(x) &= \sum_{j=1}^x \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \right] \\ &= \sum_{k=1}^x f^{-1}(k) \cdot \pi_{f*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right). \end{aligned}$$

Corollary 1.3 (Applications of Möbius inversion). *Suppose that h is an arithmetic function such that $h(1) \neq 0$. Define the summatory function of the convolution of h with μ by $\tilde{H}(x) := \sum_{n \leq x} (h * \mu)(n)$. Then the Mertens function is expressed by the sum*

$$M(x) = \sum_{k=1}^x \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} h^{-1}(j) \right) \tilde{H}(k), \forall x \geq 1.$$

Corollary 1.4. *We have that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^x (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1 \right]. \quad (1)$$

1.2.2 An exact expression for $M(x)$ via strongly additive functions

Fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$ [25, [A341444](#)]. We can compute exactly that (see Table B on page 40)

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \dots\}.$$

There is not a simple direct recursion between the distinct values of $g^{-1}(n)$ that holds for all $n \geq 1$. The distribution of distinct sets of prime exponents is still clearly quite regular since $\omega(n)$ and $\Omega(n)$ play a crucial role in the repetition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of this inverse function over $n \geq 2$:

Heuristic 1.5 (Symmetry in $g^{-1}(n)$ from the prime factorizations of $n \leq x$). Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \dots q_r^{\beta_r}$ for $\omega(n_i) \geq 1$. If $\{\alpha_1, \dots, \alpha_r\} \equiv \{\beta_1, \dots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly one, two, three, and so on prime factors.

Conjecture 1.6 (Characteristic properties of the inverse sequence). *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) For all $n \geq 1$, $\text{sgn}(g^{-1}(n)) = \lambda(n)$;

(B) For all squarefree integers $n \geq 2$, we have that

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!;$$

(C) If $n \geq 2$ and $\Omega(n) = k$, then

$$2 \leq |g^{-1}(n)| \leq \sum_{j=0}^k \binom{k}{j} \cdot j!.$$

The signedness property in (A) is proved precisely in Proposition 2.1. A proof of (B) in fact follows from Lemma 3.1 stated on page 15. The realization that the beautiful and remarkably simple combinatorial form of property (B) in Conjecture 1.6 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through the sums of auxiliary subsequences $C_k(n)$ in Section 3. That is, we observe a familiar formula for $g^{-1}(n)$ on an asymptotically dense infinite subset of integers, e.g., that holds for all squarefree $n \geq 2$, and then seek to extrapolate by proving there are regular tendencies of this sequence viewed more generally over any $n \geq 2$. An exact expression for $g^{-1}(n)$ is given by

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d), n \geq 1,$$

where the sequence $\lambda(n)C_{\Omega(n)}(n)$ has DGF $(P(s) + 1)^{-1}$ for $\text{Re}(s) > 1$ (see Proposition 2.1). The function $C_{\Omega(n)}(n)$ has been previously considered in [4] with its exact formula given by (cf. [8])

$$C_{\Omega(n)}(n) = \begin{cases} 1, & \text{if } n = 1; \\ (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}, & \text{if } n \geq 2. \end{cases}$$

In Corollary 4.6, we prove that the average order of the unsigned sequence is bounded by by

$$\mathbb{E}|g^{-1}(n)| = \frac{3\sqrt{2}}{\pi^{3/2}} (\log n)^2 \sqrt{\log \log n} (1 + o(1)), \text{ as } n \rightarrow \infty.$$

In Section 4, we prove a variant of the Erdős-Kac theorem that characterizes the distribution of the sequence $C_{\Omega(n)}(n)$. This leads us to conclude the following statement for any fixed $Y > 0$, with $\mu_x(C) := \log \log x - \log(4\sqrt{2\pi})$ and $\sigma_x(C) := \sqrt{\log \log x}$, that holds uniformly for any $-Y \leq y \leq Y$ as $x \rightarrow \infty$ (see Corollary 4.8):

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : |g^{-1}(n)| - \frac{6}{\pi^2} \mathbb{E}|g^{-1}(n)| \leq z\right\} = \Phi\left\{\frac{6\sigma_x(C)}{\pi^2} \left(\frac{\pi^2 z}{6} + \sigma_x(C)\right) - \frac{6}{\pi^2} \log(4\sqrt{2\pi})\right\} + O\left(\frac{1}{\sqrt{\log \log x}}\right).$$

The regularity and quasi-periodicity we have alluded to in the remarks above are then quantifiable in so much as the distribution of $|g^{-1}(n)|$ for $n \leq x$ tends to its average order with a non-central normal tendency depending on x as $x \rightarrow \infty$. That is, if $x > e$ is sufficiently large and if we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds:

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}(\log \log x)\right) = \frac{1}{2} + o(1) \quad (\text{D})$$

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}(\alpha + \log \log x)\right) = \Phi(\alpha) + o(1), \alpha \in \mathbb{R}. \quad (\text{E})$$

It follows from the last property that as $n \rightarrow \infty$,

$$|g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|(1 + o(1)),$$

on an infinite set of the integers with asymptotic density one.

1.2.3 Formulas illustrating the new characterizations of $M(x)$

Let $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ for integers $x \geq 1$ [25, A341472]. We prove that (see Proposition 5.2)

$$\begin{aligned} M(x) &= G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right] \\ &= G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1. \end{aligned} \quad (2)$$

This formula implies that we can establish new asymptotic bounds on $M(x)$ along large infinite subsequences by sharply bounding the summatory function $G^{-1}(x)$. The take on the regularity of $|g^{-1}(n)|$ is as such imperative to our arguments that formally bound the growth of $M(x)$ by its new identification with $G^{-1}(x)$. A combinatorial approach to summing $G^{-1}(x)$ for large x based on the distribution of the primes is outlined in our remarks in Section 3.3.

Theorem 5.1 proves that for almost every sufficiently large $x^{\mathbf{A}}$ there exists some $1 \leq t_0 \leq x$ such that

$$G^{-1}(x) = O\left(L(t_0) \cdot \mathbb{E}|g^{-1}(x)|\right).$$

If the RH is true, then for any $\varepsilon > 0$ and sufficiently large $x > e$ we have that

$$G^{-1}(x) = O\left((\log x)^2 \sqrt{\log \log x} \sqrt{x} \times \exp\left(\sqrt{\log x} (\log \log x)^{\frac{5}{2}+\varepsilon}\right)\right).$$

In Corollary 5.4, we also prove that

$$M(x) = O\left(G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) + \frac{x}{\log x} \times \sum_{k \leq \sqrt{x}} \frac{G^{-1}(k)}{k^2} + (\log x)^2 (\log \log x)^{3/2}\right).$$

Moving forward, a discussion of the properties of the summatory functions $G^{-1}(x)$ motivates more study in the future to extend the full range of possibilities for viewing the new structure behind $M(x)$ we identify within this article.

1.3 Notation and conventions

The next listing provides a glossary of common notation, conventions and abbreviations used throughout the article.

^ABy *almost every* large integer x , we mean that the result holds for all large x taken within an infinite subset of \mathbb{Z}^+ with asymptotic density one.

| Symbol | Definition |
|--|---|
| \approx, \sim | We write that $f(x) \approx g(x)$ if $ f(x) - g(x) = O(1)$ as $x \rightarrow \infty$. Two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} = 1$. |
| $\mathbb{E}[f(x)]$ | We use the expectation notation of $\mathbb{E}[f(x)] = h(x)$ to denote that f has an <i>average order</i> of $h(x)$. This means that $\frac{1}{x} \sum_{n \leq x} f(n) \sim h(x)$. |
| $\chi_{\mathbb{P}}(n)$ | The characteristic (or indicator) function of the primes equals one if and only if $n \in \mathbb{Z}^+$ is prime, and is zero-valued otherwise. |
| $C_k(n), C_{\Omega(n)}(n)$ | The sequence is defined recursively for $n \geq 1$ as follows: $C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum_{d n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$ |
| | It represents the multiple, k -fold convolution of the function $\omega(n)$ with itself. |
| $[q^n]F(q)$, OGF | The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function (or OGF) of some sequence, $\{f_n\}_{n \geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n \geq 0} f_n q^n$. |
| $\varepsilon(n)$ | The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (see definition below). |
| $*, f * g$ | The Dirichlet convolution of f and g , $(f * g)(n) := \sum_{d n} f(d)g(n/d)$, where the sum is taken over the divisors of any $n \geq 1$. |
| $f^{-1}(n)$ | The Dirichlet inverse f^{-1} of f exists if and only if $f(1) \neq 0$. The Dirichlet inverse of any f such that $f(1) \neq 0$ with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d n \\ d > 1}} f(d) f^{-1}(n/d)$ for $n \geq 2$ with $f^{-1}(1) = 1/f(1)$. When it exists, this inverse function is unique and satisfies the characteristic relations that $f^{-1} * f = f * f^{-1} = \varepsilon$. |
| \gg, \ll, \asymp | For functions A, B , the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A \ll B$ and $B \ll A$, we write $A \asymp B$. |
| $g^{-1}(n), G^{-1}(x)$ | The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$. |
| $[n = k]_{\delta}, [\mathbf{cond}]_{\delta}$ | The symbol $[n = k]_{\delta}$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise. For boolean-valued conditions, \mathbf{cond} , the symbol $[\mathbf{cond}]_{\delta}$ evaluates to one precisely when \mathbf{cond} is true, and to zero otherwise. This notation is called <i>Iverson's convention</i> . |
| $\lambda(n), L(x), \lambda_*(n)$ | The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. Its summatory function is defined by $L(x) := \sum_{n \leq x} \lambda(n)$. For positive integers $n \geq 2$, we define $\lambda_*(n) := (-1)^{\omega(n)}$. We have the initial condition that $\lambda_*(1) = 1$. |

| Symbol | Definition |
|--------------------------------|---|
| $\mu(n), M(x)$ | The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers $n \geq 1$ where $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree. The Mertens function is the summatory function of $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum_{n \leq x} \mu(n)$. |
| $\Phi(z)$ | For $x \in \mathbb{R}$, we define the CDF of the standard normal distribution to be $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^z e^{-t^2/2} dt$. |
| $\nu_p(n)$ | The valuation function that extracts the maximal exponent of p in the prime factorization of n , e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^\alpha \parallel n$ (e.g., when p^α exactly divides n) for p prime, $\alpha \geq 1$ and $n \geq 2$. |
| $\omega(n), \Omega(n)$ | We define the strongly additive function $\omega(n) := \sum_{p n} 1$ and the completely additive function $\Omega(n) := \sum_{p^\alpha \parallel n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we set $\omega(1) = \Omega(1) = 0$. |
| $\pi_k(x), \widehat{\pi}_k(x)$ | For integers $k \geq 1$, the prime counting function variant $\pi_k(x)$ denotes the number of $2 \leq n \leq x$ with exactly k distinct prime factors: $\pi_k(x) := \#\{2 \leq n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{2 \leq n \leq x : \Omega(n) = k\}$ for $x \geq 2$. |
| $P(s)$ | For complex s with $\operatorname{Re}(s) > 1$, we define the prime zeta function to be the Dirichlet generating function (or DGF) $P(s) = \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s} = \sum_{k \geq 2} \frac{\mu(k)}{k} \log \zeta(ks)$. |
| $Q(x)$ | For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. That is, $Q(x) := \sum_{n \leq x} \mu^2(n)$. |
| $W(x)$ | For $x, y \in \mathbb{R}_{\geq 0}$, we write that $x = W(y)$ if and only if $xe^x = y$. |
| $\zeta(s)$ | The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\operatorname{Re}(s) > 1$, and by analytic continuation on the rest of the complex plane with the exception of a simple pole at $s = 1$ of residue one. |

2 Initial elementary proofs of new results

2.1 Establishing the summatory function properties and inversion identities

We will offer a proof of Theorem 1.2 suggested by an intuitive construction through matrix based methods. Related results on summations of Dirichlet convolutions and their inversion appear in [1, §2.14; §3.10; §3.12; cf. §4.9, p. 95]. It is not difficult to prove the related identity that

$$\sum_{n \leq x} h(n)(f * g)(n) = \sum_{n \leq x} f(n) \times \sum_{k \leq \lfloor \frac{x}{n} \rfloor} g(k)h(kn).$$

Proof of Theorem 1.2. Let h, g be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of h and g , respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h . We have that the following formulas hold for all $x \geq 1$:

$$\begin{aligned} \pi_{g*h}(x) &:= \sum_{n=1}^x \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^x g(d)H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^x \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] H(i). \end{aligned} \quad (3)$$

The first formula above is well known in the references. The second formula is justified directly using summation by parts as [19, §2.10(ii)]

$$\begin{aligned} \pi_{g*h}(x) &= \sum_{d=1}^x h(d)G\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i \leq x} \left(\sum_{j \leq i} h(j) \right) \times \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right]. \end{aligned}$$

We next form the invertible matrix of coefficients associated with this linear system defining $H(j)$ for all $1 \leq j \leq x$ in (3) by setting

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right), 1 \leq j \leq x.$$

Since $g_{x,x} = G(1) = g(1)$ and $g_{x,j} = 0$ for all $j > x$, the matrix we must work with in this problem is lower triangular with a non-zero constant on its diagonals, and is hence invertible. If we let $\hat{G} := (G_{x,j})$, then this matrix is expressed by applying an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

The square matrix U of sufficiently large finite dimensions $N \times N$ has $(i, j)^{th}$ entries for all $1 \leq i, j \leq N$ that are defined by $(U)_{i,j} = \delta_{i+1,j}$ and such that

$$[(I - U^T)^{-1}]_{i,j} = [j \leq i]_{\delta}.$$

Observe that

$$\left\lfloor \frac{x}{j} \right\rfloor - \left\lfloor \frac{x-1}{j} \right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

The previous property implies that

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

We use the last property in (4) to shift the matrix \hat{G} , and then invert the result to obtain a matrix involving the Dirichlet inverse of g in the following form:

$$[(I - U^T)\hat{G}]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

In particular, our target matrix in the inversion problem is defined by

$$(g_{x,j}) = (I - U^T) \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right) (I - U^T)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators as

$$\begin{aligned} (g_{x,j})^{-1} &= (I - U^T)^{-1} \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right) (I - U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k)\right) (I - U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} g^{-1}(k)\right). \end{aligned}$$

Hence, the summatory function $H(x)$ is given exactly for any $x \geq 1$ by a vector product with the inverse matrix from the previous equation by the formula

$$H(x) = \sum_{k=1}^x \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor+1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j)\right) \cdot \pi_{g*h}(k).$$

We can prove another inversion formula providing the coefficients of the summatory function $G^{-1}(i)$ for $1 \leq i \leq x$ from the last equation by adapting our argument to prove (3) above. This leads to the following equivalent identity expressing $H(x)$:

$$H(x) = \sum_{k=1}^x g^{-1}(x) \cdot \pi_{g*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right). \quad \square$$

2.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, let $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n . We can easily prove using DGFs (or other elementary methods) that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu. \quad (5)$$

When combined with Corollary 1.3 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 1.4.

Proposition 2.1 (The signedness property of $g^{-1}(n)$). *Let the operator $\text{sgn}(h(n)) = \frac{h(n)}{|h(n)| + [h(n)=0]_\delta} \in \{0, \pm 1\}$ denote the sign of the arithmetic function h at integers $n \geq 1$. For the Dirichlet invertible function $g(n) := \omega(n) + 1$, we have that $\text{sgn}(g^{-1}(n)) = \lambda(n)$ for all $n \geq 1$.*

Proof. The function $D_f(s) := \sum_{n \geq 1} f(n)n^{-s}$ defines the *Dirichlet generating function* (or DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\text{Re}(s) > \sigma_f$ with σ_f the abscissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = \zeta(s)^{-1}$ and $D_\omega(s) = P(s)\zeta(s)$ for $\text{Re}(s) > 1$. Then by (5) and the known property that whenever $f(1) \neq 0$, the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of the arithmetic function f , we have for all $\text{Re}(s) > 1$ that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \quad (6)$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\text{sgn}(h^{-1}) = \lambda$. This observation then implies that $\text{sgn}(h^{-1} * \mu) = \lambda$.

By the recurrence relation that defines the Dirichlet inverse function of any arithmetic function h such that $h(1) = 1$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ - \sum_{\substack{d|n \\ d > 1}} h(d)h^{-1}(n/d), & n \geq 2. \end{cases} \quad (7)$$

For $n \geq 2$, the summands in (7) can be simply indexed over the primes $p|n$ given our definition of h from above. We can inductively unfold these sums into nested divisor sums provided the depth of the expanded divisor sums does not exceed the capacity to index non-trivial summations over the primes dividing n . Namely, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= - \sum_{p|n} h^{-1}\left(\frac{n}{p}\right), & \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right), & \text{if } \Omega(n) \geq 2 \\ &= - \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1 p_2}} h^{-1}\left(\frac{n}{p_1 p_2 p_3}\right), & \text{if } \Omega(n) \geq 3. \end{aligned}$$

Then by induction with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1, n \geq 2. \quad (8)$$

Moreover, by a combinatorial argument related to multinomial coefficient expansions of the sums in (8), we recover exactly that

$$h^{-1}(n) = \lambda(n)(\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}, n \geq 2. \quad (9)$$

The last two expansions imply that the following property holds for all $n \geq 1$:

$$\text{sgn}(h^{-1}(n)) = \lambda(n).$$

Since λ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$ for all divisors $d|n$ when $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever n is squarefree, so that we obtain the following result:

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1. \quad \square$$

The conclusion of the proof of Proposition 2.1 in fact implies the stronger result that

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d),$$

where we adopt the notation that for $n \geq 2$, $C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}$, where the sequence is taken to be one at $n := 1$.

2.3 Results on the distribution of exceptional values of $\omega(n)$ and $\Omega(n)$

The next theorems reproduced from [13, §7.4] characterize the relative scarcity of the distributions of $\omega(n)$ and $\Omega(n)$ for $n \leq x$ such that $\omega(n), \Omega(n) > \log \log x$. Since $\mathbb{E}[\omega(n)], \mathbb{E}[\Omega(n)] = \log \log n + B$ for $B \in (0, 1)$ an absolute constant in each case, these results imply a regular, normal tendency of these additive arithmetic functions towards their respective average orders.

Theorem 2.2 (Upper bounds on exceptional values of $\Omega(n)$ for large n). *Let*

$$\begin{aligned} A(x, r) &:= \#\{n \leq x : \Omega(n) \leq r \cdot \log \log x\}, \\ B(x, r) &:= \#\{n \leq x : \Omega(n) \geq r \cdot \log \log x\}. \end{aligned}$$

If $0 < r \leq 1$ and $x \geq 2$, then

$$A(x, r) \ll x(\log x)^{r-1-r \log r}, \quad \text{as } x \rightarrow \infty.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x, r) \ll_R x \cdot (\log x)^{r-1-r \log r}, \quad \text{as } x \rightarrow \infty.$$

Theorem 2.3 is a special case analog to the celebrated Erdős-Kac theorem typically stated for the normally distributed values of the scaled-shifted function $\omega(n)$ over $n \leq x$ as $x \rightarrow \infty$ [13, cf. Thm. 7.21] [9, cf. §1.7].

Theorem 2.3 (Exact limiting bounds on exceptional values of $\Omega(n)$ for large n). *We have that as $x \rightarrow \infty$*

$$\#\{3 \leq n \leq x : \Omega(n) \leq \log \log n\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log \log x}}\right).$$

Theorem 2.4 (Montgomery and Vaughan). *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $0 < R < 2$ we have that uniformly for all $1 \leq k \leq R \cdot \log \log x$

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log \log x}\right) \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!} \left[1 + O_R\left(\frac{k}{(\log \log x)^2}\right)\right],$$

where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, \quad 0 \leq |z| < R.$$

Remark 2.5. We can extend the work in [13] on the distribution of $\Omega(n)$ to find analogous results bounding the distribution of $\omega(n)$. We have that for $0 < R < 2$

$$\pi_k(x) = \widehat{\mathcal{G}}\left(\frac{k-1}{\log \log x}\right) \frac{x}{\log x} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \left[1 + O_R\left(\frac{k}{(\log \log x)^2}\right)\right], \quad \text{unif. for } 1 \leq k \leq R \log \log x. \quad (10)$$

The analogous function to express these bounds for $\omega(n)$ is defined by $\widehat{\mathcal{G}}(z) := \widehat{F}(1, z)/\Gamma(1 + z)$ where we take

$$\widehat{F}(s, z) := \prod_p \left(1 + \frac{z}{p^s - 1}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \operatorname{Re}(s) > \frac{1}{2}; |z| \leq R < 2.$$

Let the functions

$$C(x, r) := \#\{n \leq x : \omega(n) \leq r \log \log x\}$$

$$D(x, r) := \#\{n \leq x : \omega(n) \geq r \log \log x\}.$$

Then we have the next uniform upper bounds given by

$$C(x, r) \ll x(\log x)^{r-1-r \log r}, \text{ uniformly for } 0 < r \leq 1,$$

$$D(x, r) \ll x(\log x)^{r-1-r \log r}, \text{ uniformly for } 1 \leq r \leq R < 2.$$

3 Auxiliary sequences expressing the Dirichlet inverse function $g^{-1}(n)$

The computational data given as Table B in the appendix section (refer to page 40) is intended to provide clear insight into why we eventually arrived at the approximations to $g^{-1}(n)$ proved in this section. The table provides illustrative numerical data by examining the approximate behavior at hand for the cases of $1 \leq n \leq 500$ with *Mathematica* [24]. In Section 4, we will use these relations between $g^{-1}(n)$ and $C_{\Omega(n)}(n)$ to prove an Erdős-Kac like analog that characterizes the distribution of the unsigned function $|g^{-1}(n)|$.

3.1 Definitions and properties of triangular component function sequences

We define the following sequence for integers $n \geq 1$ and $k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \quad (11)$$

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (11) inductively. By the same argument, we see that at fixed n , the function $C_k(n)$ is seen to be non-zero only for positive integers $k \leq \Omega(n)$ whenever $n \geq 2$. A sequence of relevant signed semi-diagonals of the functions $C_k(n)$ begins as follows [25, A008480]:

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, -3, 2, 2, -1, 4, 1, 2, \dots\}.$$

We can see that $C_{\Omega(n)}(n) \leq (\Omega(n))!$ for all $n \geq 1$. In fact, $h^{-1}(n) \equiv \lambda(n) C_{\Omega(n)}(n)$ is the same function given by the formula in (9) from Proposition 2.1.

3.2 Relating the function $C_{\Omega(n)}(n)$ to exact formulas for $g^{-1}(n)$

Lemma 3.1 (An initial exact formula for $g^{-1}(n)$). *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse as

$$g^{-1}(n) = - \sum_{\substack{d|n \\ d>1}} (\omega(d) + 1) g^{-1}(n/d) \implies (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \quad (12)$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (12) in the form of $(g^{-1} * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m \cdot (C_m(-) * g^{-1})(n)$, or so that

$$F_m(n) = - \begin{cases} \sum_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & 2 \leq m \leq \Omega(n), \\ (\omega * g^{-1})(n), & m = 1. \end{cases}$$

By repeatedly expanding the right-hand-side of the previous equation, we find that for $m := \Omega(n)$ (i.e., with the expansions taken to a maximal depth in the previous equation)

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \quad (13)$$

The formula then follows from (13) by Möbius inversion applied to each side of the last equation. \square

Corollary 3.2. *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \quad (14)$$

Proof. By applying Lemma 3.1, Proposition 2.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and since at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$, Lemma 3.1 implies

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \end{aligned}$$

We see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. \square

Since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ using the notation defined in the the proof of Proposition 2.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for squarefree $n \geq 1$. A proof of part (B) of Conjecture 1.6 follows as an immediate consequence.

Remark 3.3. Combined with the signedness property of $g^{-1}(n)$ guaranteed by Proposition 2.1, Corollary 3.2 shows that the summatory function of this sequence satisfies

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_{\Omega(d)}(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Additionally, equation (5) implies that

$$\lambda(d) C_{\Omega(d)}(d) = (g^{-1} * 1)(d) = (\chi_{\mathbb{P}} + \varepsilon)^{-1}(d),$$

where $\chi_{\mathbb{P}}$ denotes the characteristic function of the primes. We clearly recover by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1.$$

It can in fact be shown that

$$\mu(n) = g^{-1}(n) + \sum_{p|n} g^{-1}\left(\frac{n}{p}\right), n \geq 1.$$

3.3 Another connection to the distribution of the primes

The combinatorial properties of $g^{-1}(n)$ are deeply tied to the distribution of the primes $p \leq n$ as $n \rightarrow \infty$. The magnitudes and dispersion of the primes $p \leq n$ certainly restricts the repeating of these distinct sequence values. Nonetheless, we can see that the following is still clear about the relation of the weight functions $|g^{-1}(n)|$ to the distribution of the primes: The value of $|g^{-1}(n)|$ is entirely dependent on the pattern of the *exponents* (viewed as multisets) of the distinct prime factors of $n \geq 2$, rather than on the prime factor weights themselves (*cf.* Heuristic 1.5). This observation implies that $|g^{-1}(n)|$ has an inherently additive, rather than multiplicative, structure behind the distribution of its distinct values over $n \leq x$.

Example 3.4. We have a natural extremal behavior with respect to distinct values of $\Omega(n)$ corresponding to squarefree integers and prime powers. If for integers $k \geq 1$ we define the infinite sets M_k and m_k to correspond to the maximal (minimal) sets of positive integers such that

$$M_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

$$m_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

then any element of M_k is squarefree and any element of m_k is a prime power. Moreover, for any fixed $k \geq 1$ we have that for any $N_k \in M_k$ and $n_k \in m_k$

$$(-1)^k \cdot g^{-1}(N_k) = \sum_{j=0}^k \binom{k}{j} \cdot j!, \quad \text{and} \quad (-1)^k \cdot g^{-1}(n_k) = 2.$$

The formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 2.1 implies that we can express an exact formula for $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n . Namely, for $n \geq 2$ and $0 \leq k \leq \omega(n)$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z \cdot \nu_p(n)) = [z^k] \prod_{p^\alpha || n} (1 + \alpha z).$$

Then we can prove using (9) and (14) that we can expand exact formulas for the signed inverse sequence in the following form:

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, \quad n \geq 2.$$

The combinatorial formula^B for $h^{-1}(n) = \lambda(n) \cdot (\Omega(n))! \times \prod_{p^\alpha || n} (\alpha!)^{-1}$ we discovered in the proof of the key signedness proposition from Section 2 suggests additional patterns and more regularity in the contributions of the distinct weighted terms in the summands of $G^{-1}(x)$. A preliminary analysis suggests that bounds of this type may improve upon those we are able to prove for $G^{-1}(x)$ in Section 5.1.

^BThis sequence is also considered using a different motivation based on the DGFs $(1 \pm P(s))^{-1}$ in [4, §2].

4 The distributions of $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$

We have already suggested in the introduction that the relation of the component functions, $g^{-1}(n)$ and $C_{\Omega(n)}(n)$, to the canonical additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties of these functions cited in Table B. Each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdős-Kac theorem that provides a central limit type theorem for the distributions of these functions over $n \leq x$ as $x \rightarrow \infty$ [3, 2, 20] (cf. [8]). In the remainder of this section we establish more analytical proofs of related properties of these key sequences used to express $G^{-1}(x)$.

4.1 Analytic proofs and adaptations of DGF methods for summing additive functions

Theorem 4.1. *Let the function $\widehat{F}(s, z)$ be defined in terms of the prime zeta function, $P(s)$, for $\operatorname{Re}(s) \geq 2$ and $|z| < |P(s)|^{-1}$ by*

$$\widehat{F}(s, z) := \frac{1}{1 + P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

For $|z| < P(2)^{-1}$, the summatory function of the DGF coefficients of $\widehat{F}(s, z) \cdot \zeta(s)^z$ correspond to

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

We have that for all sufficiently large $x \geq 2$ and any $|z| < P(2)^{-1}$

$$\widehat{A}_z(x) = \frac{x}{\Gamma(z)} \cdot \widehat{F}(2, z) \cdot (\log x)^{z-1} + O_z \left(x \cdot (\log x)^{\operatorname{Re}(z)-2} \right).$$

Proof. We can see from the proof of Proposition 2.1 that

$$C_{\Omega(n)}(n) = \begin{cases} 1, & n = 1; \\ (\Omega(n))! \times \prod_{p^\alpha \parallel n} \frac{1}{\alpha!}, & n \geq 2. \end{cases}$$

We can then generate exponentially scaled forms of these terms through a product identity of the following form:

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! \cdot p^{rs}} \right)^{-1} = \exp(-z \cdot P(s)), \operatorname{Re}(s) \geq 2 \wedge \operatorname{Re}(P(s)z) > -1.$$

This product based expansion is similar in construction to the parameterized bivariate DGF used in the reference [13, §7.4]. By computing a Laplace transform on the right-hand-side of the above equation, we obtain

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n) \cdot (-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp(-tz \cdot P(s)) dt = \frac{1}{1 + P(s)z}, \operatorname{Re}(s) > 1 \wedge \operatorname{Re}(P(s)z) > -1.$$

It follows that

$$\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times \widehat{F}(s, z), \operatorname{Re}(s) > 1 \wedge |z| < |P(s)|^{-1}.$$

Since $\widehat{F}(s, z)$ is an analytic function of s for all $\operatorname{Re}(s) \geq 2$ whenever the parameter $|z| < |P(s)|^{-1}$, if the sequence $\{b_z(n)\}_{n \geq 1}$ indexes the coefficients in the DGF expansion of $\widehat{F}(s, z) \cdot \zeta(s)^z$, then

$$\left| \sum_{n \geq 1} \frac{b_z(n) (\log n)^{2R+1}}{n^s} \right| < +\infty, \operatorname{Re}(s) \geq 2$$

is uniformly bounded for $|z| \leq R < +\infty$. This fact follows by repeated termwise differentiation $\lceil 2R + 1 \rceil$ times with respect to s .

For fixed $0 < |z| < 2$, let the sequence $d_z(n)$ be generated as the coefficients of the DGF

$$\zeta(s)^z = \sum_{n \geq 1} \frac{d_z(n)}{n^s}, \operatorname{Re}(s) > 1,$$

with corresponding summatory function defined by $D_z(x) := \sum_{n \leq x} d_z(n)$. The theorem proved in the reference [13, Thm. 7.17; §7.4] shows that for any $0 < |z| < 2$ and all integers $x \geq 2$

$$D_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right).$$

We set $b_z(n) := (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$, define the convolution function $a_z(n) := \sum_{d|n} b_z(d) d_z(n/d)$, and take its summatory function to be $A_z(x) := \sum_{n \leq x} a_z(n)$. Then we have that

$$\begin{aligned} A_z(x) &= \sum_{m \leq x/2} b_z(m) D_z(x/m) + \sum_{x/2 < m \leq x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \times \sum_{m \leq x/2} \frac{b_z(m)}{m^2} \times m \log\left(\frac{x}{m}\right)^{z-1} + O\left(\sum_{m \leq x} \frac{x|b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\operatorname{Re}(z)-2}\right). \end{aligned} \quad (15)$$

We can sum the coefficients $b_z(m)/m$ for integers $m \leq u$ with $u > e$ taken sufficiently large as follows:

$$\sum_{m \leq u} \frac{b_z(m)}{m} = (\widehat{F}(2, z) + O(u^{-2}))u - \int_1^u (\widehat{F}(2, z) + O(t^{-2})) dt = \widehat{F}(2, z) + O(u^{-1}).$$

Suppose that $|z| \leq R < P(2)^{-1} \approx 2.21118$. The error term in (15) satisfies

$$\begin{aligned} \sum_{m \leq x} \frac{x \cdot |b_z(m)|}{m^2} \times m \log\left(\frac{2x}{m}\right)^{\operatorname{Re}(z)-2} &\ll x(\log x)^{\operatorname{Re}(z)-2} \times \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m} \\ &\quad + x(\log x)^{-(R+2)} \times \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m} (\log m)^{2R} \\ &= O_z\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right), |z| \leq R. \end{aligned}$$

In the main term estimate for $A_z(x)$ from (15), when $m \leq \sqrt{x}$ we have

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\operatorname{Re}(z)-2}\right).$$

The total sum over the interval $m \leq x/2$ corresponds to bounding the sum components when we take $|z| \leq R$ as follows:

$$\begin{aligned} \sum_{m \leq x/2} b_z(m) D_z(x/m) &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \times \sum_{m \leq x/2} \frac{b_z(m)}{m} \\ &\quad + O_z\left(x(\log x)^{\operatorname{Re}(z)-2} \times \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m} + x(\log x)^{R-1} \times \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \widehat{F}(2, z) + O_R\left(x(\log x)^{\operatorname{Re}(z)-2} \times \sum_{m \geq 1} \frac{b_z(m)(\log m)^{2R+1}}{m^2}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \widehat{F}(2, z) + O_R\left(x(\log x)^{\operatorname{Re}(z)-2}\right). \end{aligned} \quad \square$$

Theorem 4.2. For large $x > e$ and integers $k \geq 1$, let

$$\widehat{C}_{k,*}(x) := \sum_{\substack{n \leq x \\ \Omega(n)=k}} (-1)^{\omega(n)} C_k(n)$$

Let the function $\widehat{G}(z) := \widehat{F}(2, z)/\Gamma(z+1)$ for $|z| < P(2)^{-1}$ where the function $\widehat{F}(s, z)$ is defined as in Theorem 4.1 for $\operatorname{Re}(s) > 1$. As $x \rightarrow +\infty$, we have uniformly for any $1 \leq k \leq \log \log x$ that

$$\widehat{C}_{k,*}(x) = -\widehat{G}\left(\frac{k-1}{\log \log x}\right) \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!} \left[1 + O_k\left(\frac{k}{(\log \log x)^2}\right)\right].$$

Proof. When $k = 1$, we have that $\Omega(n) = \omega(n)$ for all $n \leq x$ such that $\Omega(n) = k$. The $n \leq x$ that satisfy this requirement are precisely the primes $p \leq x$. Thus we get that the bound is satisfied as

$$\sum_{p \leq x} (-1)^{\omega(p)} C_1(p) = -\sum_{p \leq x} 1 = -\frac{x}{\log x} \left[1 + O\left(\frac{1}{\log x}\right)\right].$$

Since $O((\log x)^{-1}) = O((\log \log x)^{-2})$, we obtain the required error term bound when $k := 1$.

For $2 \leq k \leq \log \log x$, we will apply the error estimate from Theorem 4.1 at $r := \frac{k-1}{\log \log x}$. At large x , the error term from this bound contributes that is bounded from above by

$$\begin{aligned} x(\log x)^{-(r+2)} r^{-(k+1)} &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log \log x)^{k+1}}{(k-1)^{k+1}} \cdot \frac{1}{e^{k-1}} \ll \frac{x}{(\log x)^2} \cdot \frac{(\log \log x)^{k+1}}{(k-1)^{3/2}} \cdot \frac{1}{e^{2k}(k-1)!} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \ll \frac{x}{\log x} \cdot \frac{k \cdot (\log \log x)^{k-5}}{(k-1)!}. \end{aligned}$$

We find an asymptotically accurate main term approximation to the coefficients of the following contour integral for $r \in [0, z_{\max}]$ where $z_{\max} < P(2)^{-1}$ to satisfy Theorem 4.1:

$$\widetilde{A}_r(x) := - \int_{|v|=r} \frac{x \cdot (\log x)^{-v}}{(\log x) \Gamma(1+v) \cdot v^k (1 - P(2)v)} dv. \quad (16)$$

The main term for the sums $\widehat{C}_{k,*}(x)$ is given by $-\frac{x}{\log x} \cdot I_k(r, x)$, where we take

$$\begin{aligned} I_k(r, x) &= \frac{(-1)^k}{2\pi i} \times \int_{|v|=r} \frac{\widehat{G}(v)(\log x)^{-v}}{v^k \cdot (1 - P(2)v)} dv \\ &=: I_{1,k}(r, x) + I_{2,k}(r, x). \end{aligned}$$

The first of the component integrals in the last equation is defined to be

$$I_{1,k}(r, x) := \frac{\widehat{G}(r)}{2\pi i} \times \int_{|v|=r} \frac{(\log x)^{-v}}{v^k \cdot (1 - P(2)v)} dv.$$

We can inductively compute the remaining coefficients $[z^k]I_{1,k}(r, x)$ with respect to x for fixed $k \leq \log \log x$ to apply the Cauchy integral formula. It is not difficult to see that for any integer $m \geq 0$, we have the m^{th} partial derivative of the scaled integrand with respect to z has the following limiting expansion by applying (A.3) at large x :

$$\begin{aligned} \frac{(-1)^m}{m!} \times \frac{\partial^{(m)}}{\partial v^{(m)}} \left[\frac{(\log x)^{-v}}{1 - P(2)v} \right] \Bigg|_{v=0} &= \sum_{j=0}^m \frac{(-1)^j P(2)^j (\log \log x)^{m-j}}{(m-j)!} \\ &= \frac{P(2)^m (\log x)^{-\frac{1}{P(2)}}}{m!} \times \Gamma\left(m+1, -\frac{\log \log x}{P(2)}\right) \end{aligned}$$

$$= \frac{(\log \log x)^m}{m!} + O\left(\frac{(\log \log x)^{m-2}}{m!}\right).$$

Note that we have restricted the asymptotic analysis of the limiting dominant terms in the above formula to cases of $m+1 < \log \log x$. We see by taking $v = r = \frac{k-1}{\log \log x}$ that (cf. (30) in Appendix A for the error term bounds)

$$I_{1,k}(r, x) = \frac{\widehat{G}(r)(\log \log x)^{k-1}}{(k-1)!} + O\left(\frac{(\log \log x)^{k-2}}{(\log x)(k-1)!}\right).$$

The second component integral, $I_{2,k}(r, x)$, corresponds to error terms in our approximation that we must bound. This function is defined by

$$I_{2,k}(r, x) := \frac{1}{2\pi i} \times \int_{|v|=r} (\widehat{G}(v) - \widehat{G}(r)) \frac{(\log x)^{-v}}{z^k \cdot (1 - P(2)v)} dv.$$

After integrating by parts, we write that

$$I_{2,k}(r, x) := \frac{1}{2\pi i} \times \int_{|v|=r} (\widehat{G}(v) - \widehat{G}(r) - \widehat{G}'(r)(v-r)) (\log x)^{-v} \left[\sum_{i \geq 0} v^{i-k} P(2)^i \right] dz.$$

Notice that

$$\widehat{G}(v) - \widehat{G}(r) - \widehat{G}'(r)(v-r) = \int_r^v (v-w) \widehat{G}''(w) dw \ll |v-r|^2.$$

We then define component integrands for $I_{2,k}(r, x)$ as follows for any integers $i \geq 0$:

$$T_{k,i}(r, x) := \frac{1}{2\pi i} \times \int_{|v|=r} (\widehat{G}(v) - \widehat{G}(r) - \widehat{G}'(r)(v-r)) (\log x)^{-v} v^{i-k} dz.$$

With the parameterization $z = re^{2\pi i \theta}$ for real $\theta \in [-1/2, 1/2]$, we get that

$$T_{k,i}(r, x) \ll r^{3-k+i} \int_{-1/2}^{1/2} (\sin \pi \theta)^2 e^{(k-i-1) \cos(2\pi \theta)} d\theta.$$

Since $|\sin x| \leq |x|$ for all $|x| < 1$ and $\cos(2\pi \theta) \leq 1 - 8\theta^2$ whenever $-1/2 \leq \theta \leq 1/2$, we obtain bounds of the next forms by setting $r := \frac{k-1}{\log \log x}$.

$$\begin{aligned} T_{k,i}(r, x) &\ll r^{3-k+i} e^{k-i-1} \times \int_0^\infty \theta^2 e^{-8(k-i-1)\theta^2} d\theta \\ &\ll \frac{r^{3+i-k} e^{k-i-1}}{(k-i-1)^{3/2}} \ll \frac{(\log \log x)^{k-3-i} e^{k-i-1}}{(k-1-i)^{3/2} (k-1)^{k-3-i}} \\ &\ll \frac{k \cdot (\log \log x)^{k-3-i}}{(k-1)!}. \end{aligned}$$

Then it follows that with $r := \frac{k-1}{\log \log x}$, the sums

$$\sum_{i \geq 0} |T_{k,i}(r, x)| P(2)^i \ll \frac{k \cdot (\log \log x)^{k-3}}{(k-1)!} (1 + o(1)).$$

Finally, we see that whenever $1 \leq k \leq \log \log x$, we have

$$\widehat{G}\left(\frac{k-1}{\log \log x}\right) = \frac{1}{\Gamma\left(1 + \frac{k-1}{\log \log x}\right)} \cdot \frac{\zeta(2)^{(1-k)/\log \log x}}{\left(1 + \frac{k-1}{\log \log x}\right)} \gg 1.$$

In fact, we can show that the the function on the left-hand-side of the last equation is asymptotic to $e^{o(1)}$ as $x \rightarrow \infty$. This implies the stated result of our theorem. \square

With the next lemma, we can accurately approximate asymptotic order of the sums $\mathcal{A}_\omega(x)$ (defined below) for large x by only considering the truncated sums $\mathcal{D}_\omega(x)$ (defined below) where we have the known uniform bounds on the summands for $1 \leq k \leq \log \log x$.

Lemma 4.3. *Suppose that for $x > e$ we define the following functions:*

$$\begin{aligned}\mathcal{N}_\omega(x) &:= \left| \sum_{k > \log \log x} (-1)^k \pi_k(x) \right| \\ \mathcal{D}_\omega(x) &:= \left| \sum_{k \leq \log \log x} (-1)^k \pi_k(x) \right| \\ \mathcal{A}_\omega(x) &:= \left| \sum_{k \geq 1} (-1)^k \pi_k(x) \right|.\end{aligned}$$

As $x \rightarrow \infty$, we have that $\mathcal{D}_\omega(x)/\mathcal{N}_\omega(x) = o(1)$ and $\mathcal{A}_\omega(x) \sim \mathcal{D}_\omega(x)$.

Proof. First, we sum the main term for the function $\mathcal{D}_\omega(x)$ by applying the limiting asymptotics for the incomplete gamma function derived in Lemma A.3 to obtain that

$$\begin{aligned}\mathcal{D}_\omega(x) &= \left| \sum_{1 \leq k \leq \log \log x} \frac{(-1)^k \cdot x}{\log x} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \right| + O(E_\omega(x)) \\ &= \frac{x}{2\sqrt{2\pi} \log \log x} + O\left(\frac{x}{(\log \log x)^{3/2}} + E_\omega(x)\right),\end{aligned}$$

The error term from the bound in the previous equation is defined according to (10) with $\widehat{\mathcal{G}}\left(\frac{k-1}{\log \log x}\right) \gg 1$ for all $1 \leq k \leq \log \log x$ as

$$\begin{aligned}E_\omega(x) &:= \sum_{k \leq \log \log x} \frac{x}{\log x} \cdot \frac{k(\log \log x)^{k-3}}{(k-1)!} \leq \frac{x}{\log x} \times \sum_{1 \leq k \leq \log \log x} \frac{(\log \log x)^{k-2}}{(k-1)!} \\ &\leq \frac{x}{(\log x)(\log \log x)} e^{\log \log x} \leq \frac{x}{\log \log x}.\end{aligned}$$

The right-hand-side expression in the previous equation follows by applying Lemma A.3.

Next, we utilize the notation for and bounds on the function $D(x, r)$ from Remark 2.5 to bound the function $\mathcal{N}_\omega(x)$ as follows:

$$\frac{1}{x} \times |\mathcal{N}_\omega(x)| \leq \sum_{k \geq \log \log x} \frac{\pi_k(x)}{x} = \frac{1}{x} \times \sum_{k \geq \log \log x} \# \{2 \leq n \leq x : \omega(n) = k\} \ll 1.$$

Then we see that

$$\left| \frac{\mathcal{D}_\omega(x)}{\mathcal{N}_\omega(x)} \right| = O\left(\frac{1}{\sqrt{\log \log x}}\right) = o(1), \text{ as } x \rightarrow \infty.$$

Equivalently, we have shown that $\mathcal{D}_\omega(x) = o(\mathcal{N}_\omega(x))$. The following results from the triangle inequality when x is large:

$$1 + o(1) = \left(\frac{\mathcal{D}_\omega(x) - \mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)} \right)^{-1} \ll \frac{\mathcal{D}_\omega(x)}{\mathcal{A}_\omega(x)} \ll \left(\frac{\mathcal{D}_\omega(x) + \mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)} \right)^{-1} = 1 + o(1).$$

The last equation implies that $\mathcal{A}_\omega(x) \sim \mathcal{D}_\omega(x)$ as $x \rightarrow \infty$. □

Corollary 4.4. *We have for large $x > e$ and $1 \leq k \leq \log \log x$ that*

$$\widehat{C}_k(x) := \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \sim \frac{4\sqrt{2\pi} \cdot x}{(2k-1)!} \cdot \frac{(\log \log x)^{k-1/2}}{(k-1)!}.$$

Proof. We have an integral formula involving the unsigned summand sequence that results by applying Abel summation in the form of the next equations.

$$\sum_{n \leq x} \lambda_*(n) h(n) = \left(\sum_{n \leq x} \lambda_*(n) \right) h(x) - \int_1^x \left(\sum_{n \leq t} \lambda_*(n) \right) h'(t) dt \quad (17a)$$

$$\sim \int_1^x \frac{d}{dt} \left[\sum_{n \leq t} \lambda_*(n) \right] h(t) dt \quad (17b)$$

Let the signed left-hand-side summatory function for our function in (17a) be defined precisely for large $x > e$ and any integers $1 \leq k \leq \log \log x$ by

$$\begin{aligned} \widehat{C}_{k,*}(x) &:= \sum_{\substack{n \leq x \\ \Omega(n)=k}} (-1)^{\omega(n)} C_{\Omega(n)}(n) \\ &\sim \frac{x}{\log x} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{1}{\log \log x}\right) \right]. \end{aligned}$$

The second equation above follows from the proof of Theorem 4.2 where we note that $\widehat{G}((k-1)/\log \log x) \sim e^{o(1)}$ as $x \rightarrow \infty$. We adopt the notation that $\lambda_*(n) = (-1)^{\omega(n)}$ for $n \geq 1$ and set $L_*(x) := |\sum_{n \leq x} \lambda_*(n)|$ for $x \geq 1$.

We can then transform our previous results for the partial sums over the signed sequences $\lambda_*(n) \cdot C_{\Omega(n)}(n)$ such that $\Omega(n) = k$ to approximate the same sum over the unsigned summands $C_{\Omega(n)}(n)$. The argument is based on approximating $L_*(t)$ for large t using the following uniform asymptotics for $\pi_k(x)$ that hold when $1 \leq k \leq \log \log x$ ^C:

$$\pi_k(x) = \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!} \left(1 + O\left(\frac{k}{(\log \log x)^2}\right) \right), \text{ as } x \rightarrow \infty.$$

We have by Lemma A.3 and Lemma 4.3 that

$$L_*(t) := \left| \sum_{n \leq t} (-1)^{\omega(n)} \right| \sim \left| \sum_{k=1}^{\log \log t} (-1)^k \pi_k(t) \right| = \frac{t}{2\sqrt{2\pi \log \log t}} + O\left(\frac{t}{(\log \log t)}\right), \text{ as } t \rightarrow \infty. \quad (18)$$

The error term in the previous equation corresponds to the asymptotics of the following sum as $x \rightarrow +\infty$ over the error term for $\pi_k(x)$ above resulting from an application of Stirling's formula, (29a) and (29c).

$$\frac{x}{\log x} \times \sum_{1 \leq k \leq \log \log x} \frac{(\log \log x)^{k-2}}{(k-1)!} = \frac{\Gamma(\log \log x, \log \log x)}{\Gamma(\log \log x + 1)} \sim \frac{x}{2(\log \log x)}.$$

The main term for the reciprocal of the derivative of the main term approximation of this summatory function is given by computation as

$$\frac{1}{L'_*(t)} \sim 2\sqrt{2\pi \log \log t}.$$

We apply the formula from (17a), to deduce that the unsigned summatory function variant satisfies

$$\widehat{C}_{k,*}(x) = \int_1^x L'_*(t) C_{\Omega(t)}(t) [\Omega(t) = k]_\delta dt \quad \implies$$

^CWe can in fact show that for any $1 \leq k \leq x$, the function $\widehat{G}(z)$ defined in Remark 2.5 satisfies

$$\widehat{G}\left(\frac{k-1}{\log \log x}\right) = e^{o(1)} \xrightarrow{x \rightarrow \infty} 1.$$

$$C_{\Omega(x)}(x) [\Omega(x) = k]_{\delta} \sim \frac{\widehat{C}'_{k,*}(x)}{L'_*(x)} \implies C_{\Omega(x)}(x) [\Omega(x) = k]_{\delta} \sim 2\sqrt{2\pi \log \log x} \cdot \widehat{C}'_{k,*}(x)(1 + o(1)) =: \widehat{C}_{k,**}(x).$$

We have that

$$\widehat{C}_{k,**}(x) \sim -2\sqrt{2\pi \log \log x} \left[\frac{(\log \log x)^{k-1}}{(\log x)(k-1)!} \left(1 - \frac{1}{\log x}\right) + \frac{(\log \log x)^{k-2}}{(\log x)^2(k-2)!} \right].$$

Hence, integration by parts and Proposition A.2 yield the main term

$$\begin{aligned} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) &\sim \left| \int \widehat{C}_{k,**}(x) dx \right| \\ &\sim \frac{4\sqrt{2\pi} \cdot x (\log \log x)^{k-1/2}}{(2k-1)(k-1)!} + \frac{2\sqrt{2\pi} \cdot x \Gamma\left(k - \frac{1}{2}, \log \log x\right)}{(k-1)!} - \frac{2\sqrt{2\pi} \cdot x \Gamma\left(k - \frac{3}{2}, \log \log x\right)}{(k-1)!} \\ &\sim \frac{4\sqrt{2\pi} \cdot x (\log \log x)^{k-1/2}}{(2k-1)(k-1)!}. \end{aligned} \quad (19) \quad \square$$

4.2 Average order of the unsigned sequences

Proposition 4.5. *We have that as $n \rightarrow \infty$*

$$\mathbb{E}[C_{\Omega(n)}(n)] = \frac{\sqrt{2\pi}(\log n)}{\sqrt{\log \log n}}(1 + o(1)).$$

Proof. We first compute the following summatory function by applying Corollary 4.4 and Lemma A.4:

$$\sum_{k=1}^{\log \log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \sim \frac{\sqrt{2\pi} \cdot x (\log x)}{\sqrt{\log \log x}}. \quad (20)$$

We claim that

$$\begin{aligned} \frac{1}{x} \times \sum_{n \leq x} C_{\Omega(n)}(n) &= \frac{1}{x} \times \sum_{k \geq 1} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \\ &= \frac{1}{x} \times \sum_{k=1}^{\log \log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n)(1 + o(1)), \text{ as } x \rightarrow \infty. \end{aligned} \quad (21)$$

To prove (21) it suffices to show that

$$\frac{1}{x} \times \sum_{k > \log \log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) = o\left(\frac{(\log x)}{\sqrt{\log \log x}}\right), \text{ as } x \rightarrow \infty. \quad (22)$$

We know from Theorem 4.1 that for all sufficiently large x

$$\sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)} = x \frac{\widehat{F}(2, z)}{\Gamma(z)} (\log x)^{z-1} + O_z\left(x (\log x)^{\operatorname{Re}(z)-2}\right).$$

By (18) we have that the summatory function

$$\left| \sum_{n \leq x} (-1)^{\omega(n)} \right| = \frac{x}{2\sqrt{2\pi \log \log x}} + O\left(\frac{x}{\log \log x}\right).$$

We can argue as in the proof of Corollary 4.4 by integration by parts with the Abel summation formula that whenever $1 < |z| < P(2)^{-1}$ and $x > e$ is sufficiently large

$$\begin{aligned} \sum_{n \leq x} C_{\Omega(n)}(n) z^{\Omega(n)} &\ll \frac{\widehat{F}(2, z)}{\Gamma(z)} \times \int_e^x \frac{\sqrt{\log \log t}}{t} \frac{\partial}{\partial t} [t(\log t)^{z-1}] dt \\ &\ll \frac{x \widehat{F}(2, z)}{\Gamma(z)} \left[\frac{(\log x)^{z-1} (z + \log x)}{z} \sqrt{\log \log x} - \frac{\sqrt{\pi}}{2\sqrt{z-1}} \operatorname{erfi} \left(\sqrt{(z-1) \log \log x} \right) \right. \\ &\quad \left. - \frac{\sqrt{\pi}}{2z^{3/2}} \operatorname{erfi} \left(\sqrt{z \log \log x} \right) \right] \\ &\ll \frac{x \widehat{F}(2, z)}{\Gamma(1+z)} (\log x)^z \sqrt{\log \log x}. \end{aligned}$$

An error term for the last formula follows from the asymptotic series for $\operatorname{erfi}(z)$ in (23) below. Namely, as $|z| \rightarrow \infty$, the *imaginary error function*, denoted by $\operatorname{erfi}(z)$, has the following asymptotic expansion [19, §7.12]:

$$\operatorname{erfi}(z) := \frac{2}{\sqrt{\pi} \cdot i} \times \int_0^{iz} e^{t^2} dt = \frac{e^{z^2}}{\sqrt{\pi}} \left(z^{-1} + \frac{z^{-3}}{2} + \frac{3z^{-5}}{4} + \frac{15z^{-7}}{8} + O(z^{-9}) \right). \quad (23)$$

For all large enough $x > e$, we define

$$\widehat{B}(x, r) := \sum_{\substack{n \leq x \\ \Omega(n) \geq r \log \log x}} C_{\Omega(n)}(n).$$

We argue as in the proof from the reference [13, Thm. 7.20; §7.4] that for $1 \leq r < P(2)^{-1}$

$$\begin{aligned} \sum_{\substack{n \leq x \\ \Omega(n) \geq r \log \log x}} C_{\Omega(n)}(n) r^{\Omega(n)} &\ll x (\log x)^{-r \log r} \times \sum_{n \leq x} C_{\Omega(n)}(n) r^{\Omega(n)} \\ &\sim \frac{x \widehat{F}(2, z)}{\Gamma(1+z)} \sqrt{\log \log x} (\log x)^{r-r \log r}. \end{aligned}$$

Since $\widehat{F}(2, r) = \frac{\zeta(2)^{-r}}{1+P(2)^r} \ll 1$ for $r \in [1, P(2)^{-1})$, and similarly we have that $\frac{1}{\Gamma(1+r)} \gg 1$ for r taken within this same range, we get that

$$\sum_{\substack{n \leq x \\ \Omega(n) \geq r \log \log x}} C_{\Omega(n)}(n) r^{\Omega(n)} \ll x \sqrt{\log \log x} \times (\log x)^{r-r \log r}, \text{ for all } 1 \leq r < P(2)^{-1}.$$

When $1 \leq r < P(2)^{-1}$ we have

$$x \sqrt{\log \log x} (\log x)^{r-r \log r} \gg \sum_{\substack{n \leq x \\ \Omega(n) \geq r \log \log x}} C_{\Omega(n)}(n) r^{\Omega(n)} \gg \sum_{\substack{n \leq x \\ \Omega(n) \geq r \log \log x}} C_{\Omega(n)}(n) r^{r \log \log x}.$$

This implies that for $r \in (1, P(2)^{-1})$ we have

$$\widehat{B}(x, r) \ll x (\log x)^{r-2r \log r} \sqrt{\log \log x} = o \left(\frac{x}{\sqrt{\log \log x}} \right) \quad (24)$$

We wish to evaluate the limiting asymptotics of the sum

$$S_2(x) := \frac{1}{x \sqrt{\log \log x}} \times \sum_{k \geq \log \log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \ll \widehat{B}(x, 1).$$

We have proved that $S_2(x) \sqrt{\log \log x} = o((\log x) \sqrt{\log \log x})$ as $x \rightarrow \infty$. \square

Corollary 4.6. *We have that as $n \rightarrow \infty$, the average order of the unsigned inverse sequence satisfies*

$$\mathbb{E}|g^{-1}(n)| = \frac{6}{\sqrt{2} \cdot \pi^{3/2}} \frac{(\log n)^2}{\sqrt{\log \log n}} (1 + o(1)).$$

Proof. We use the formula from Proposition 4.5 to find $\mathbb{E}[C_{\Omega(n)}(n)]$ as $n \rightarrow \infty$. This result implies that for all sufficiently large $t \rightarrow +\infty$

$$\int \frac{\mathbb{E}[C_{\Omega(t)}(t)]}{t} dt = \pi \operatorname{erfi}(\sqrt{2 \log \log t}) = \sqrt{\frac{\pi}{2}} \cdot \frac{(\log t)^2}{\sqrt{\log \log t}} (1 + o(1)).$$

Recall that the summatory function of the squarefree integers is approximated for large x by

$$Q(x) := \sum_{n \leq x} \mu^2(n) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Therefore summing over the formula from (14) and applying (23) we find that

$$\begin{aligned} \mathbb{E}|g^{-1}(n)| &= \frac{1}{n} \times \sum_{d \leq n} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{n}{d} \right\rfloor\right) \\ &\sim \sum_{d \leq n} C_{\Omega(d)}(d) \left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dn}}\right) \right] \\ &= \frac{6}{\pi^2} \left(\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d < n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d} \right) + O(1). \end{aligned}$$

□

4.3 Erdős-Kac theorem analogs for the distributions of the unsigned sequences

Theorem 4.7 (Central limit theorem for the distribution of $C_{\Omega(n)}(n)$). *Set the mean and variance parameter analogs be defined by*

$$\mu_x(C) := \log \log x - \log(4\sqrt{2\pi}), \quad \text{and} \quad \sigma_x(C) := \sqrt{\log \log x}.$$

Let $Y > 0$ be fixed. We have uniformly for all $-Y \leq z \leq Y$ that

$$\frac{1}{x} \cdot \# \left\{ 2 \leq n \leq x : \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)} \leq z \right\} = \Phi(z) + O\left(\frac{1}{\sqrt{\log \log x}}\right), \text{ as } x \rightarrow \infty.$$

Proof. Fix any $Y > 0$ and set $z \in [-Y, Y]$. For large x and $2 \leq n \leq x$, define the following auxiliary variables:

$$\alpha_n := \frac{C_{\Omega(n)}(n) - \mu_n(C)}{\sigma_n(C)}, \quad \text{and} \quad \beta_{n,x} := \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)}.$$

Let the corresponding densities be defined by the functions

$$\Phi_1(x, z) := \frac{1}{x} \cdot \#\{n \leq x : \alpha_n \leq z\},$$

and

$$\Phi_2(x, z) := \frac{1}{x} \cdot \#\{n \leq x : \beta_{n,x} \leq z\}.$$

We assert that it suffices to consider the distribution of $\Phi_2(x, z)$ as $x \rightarrow \infty$ in place of $\Phi_1(x, z)$ to obtain our desired result. The normalizing terms $\mu_n(C)$ and $\sigma_n(C)$ hardly change over $\sqrt{x} \leq n \leq x$. Namely, we see that for $n \in [\sqrt{x}, x]$

$$|\mu_n(C) - \mu_x(C)| \leq \frac{\log 2}{\log x} + O\left(\frac{1}{(\log x)^2}\right),$$

and

$$|\sigma_n(C) - \sigma_x(C)| \leq \frac{\log 2}{(\log x)\sqrt{\log \log x}} + O\left(\frac{1}{(\log \log x)(\log x)^2}\right).$$

In particular, for $\sqrt{x} \leq n \leq x$ and $C_{\Omega(n)}(n) \leq 2 \cdot \mu_x(C)$ we can show using (24) that the following is true:

$$|\alpha_n - \beta_{n,x}| \xrightarrow{x \rightarrow \infty} 0.$$

Thus we can replace α_n by $\beta_{n,x}$ and estimate the limiting densities corresponding to these alternate terms. The rest of our argument follows the method in the proof of the related theorem in [13, Thm. 7.21; §7.4] closely. Readers familiar with the reference will see many parallels to those constructions. The crux of the remainder of the proof emulates the methods from Montgomery and Vaughan. After a change of variable we obtain the limiting CLT statement in analog to their analytic proof of the Erdős-Kac theorem for the distributions of $\omega(n)$ and $\Omega(n)$.

We use the formula proved in Corollary 4.4 to estimate the densities claimed within the ranges bounded by z as $x \rightarrow \infty$. Let $k \geq 1$ be a natural number such that $k := t_x + \mu_x(C)$ where $t_x := \frac{t\sqrt{\log \log x}}{(\log x)}$. For fixed large x , we define the small parameter $\delta_{t,x} := \frac{t_x}{\mu_x(C)}$. When $|t| \leq \frac{1}{2}\mu_x(C)$, we have by Stirling's formula that

$$\begin{aligned} \frac{1}{x} \times \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) &\sim \frac{4\sqrt{2\pi}(\log \log x)^{k-\frac{1}{2}}}{(2k-1)(k-1)!} \\ &\sim \frac{(\log x)}{\sqrt{2\pi \log \log x} \cdot \sigma_x(C) \left(1 - \frac{1}{2k}\right)} \times e^{t_x} (1 + o(1))^{k-\frac{1}{2}} \times (1 + \delta_{t,x})^{-\mu_x(C)(1+\delta_{t,x})-\frac{1}{2}}. \end{aligned}$$

Notice that

$$\begin{aligned} \frac{1}{1 - \frac{1}{2k}} &\sim \sum_{m \geq 0} \frac{1}{(2\mu_x(C))^m (1 + \delta)^m} \sim 1 + \frac{1}{2\mu_x(C)} (1 + \delta + O(\delta^2)) \\ &= 1 + o_\delta(1), \text{ for } \delta \approx 0 \text{ as } x \rightarrow \infty. \end{aligned}$$

We have the uniform estimate that $\log(1 + \delta_{t,x}) = \delta_{t,x} - \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)$ whenever $|\delta_{t,x}| \leq \frac{1}{2}$. Then we can expand the factor involving $\delta_{t,x}$ from the previous equation as follows:

$$\begin{aligned} (1 + \delta_{t,x})^{-\mu_x(C)(1+\delta_{t,x})-\frac{1}{2}} &= \exp\left(\left(\frac{1}{2} + \mu_x(C)(1 + \delta_{t,x})\right) \times \left(-\delta_{t,x} + \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)\right)\right) \\ &= \exp\left(-t_x - \frac{t_x + t_x^2}{2\mu_x(C)} + \frac{t_x^2}{4\mu_x(C)^2} + O\left(\frac{|t_x|^3}{\mu_x(C)^2}\right)\right). \end{aligned}$$

For both $|t| \leq \mu_x(C)^{1/2}$ and $\mu_x(C)^{1/2} < |t| \leq \mu_x(C)^{2/3}$, we can see that

$$\frac{t}{\mu_x(C)} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Similarly, for both $|t| \leq 1$ and $|t| > 1$, we have that

$$\frac{t^2}{\mu_x(C)^2} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Let the corresponding error terms in (x, t) be denoted by

$$\tilde{E}(x, t) := O\left(\frac{1}{\sigma_x(C)} + \frac{|t|^3}{\mu_x(C)^2}\right).$$

Combining these estimates with the previous computations, we can deduce that uniformly for $|t| \leq \mu_x(C)^{2/3}$

$$\frac{4\sqrt{2\pi}(\log \log x)^{k-\frac{1}{2}}}{(2k-1)(k-1)!} \sim \frac{(\log x)\sqrt{\log \log x}}{\sqrt{2\pi} \cdot \sigma_x(C)} \times \exp\left(-\frac{t_x^2}{2\sigma_x(C)^2}\right) \times [1 + \tilde{E}(x, t_x)].$$

It follows that for $1 \leq k \leq \log \log x$

$$\begin{aligned} f(k, x) &= \frac{1}{x} \times \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \\ &\sim \frac{(\log x)}{\sqrt{2\pi \log \log x} \cdot \sigma_x(C)} \times \exp\left(-\frac{(k - \mu_x(C))^2 \sqrt{\log \log x}}{2(\log x) \sigma_x(C)^2}\right) \times \left[1 + \tilde{E}\left(x, \frac{|k - \mu_x(C)| \sqrt{\log \log x}}{(\log x)}\right)\right]. \end{aligned}$$

Since our target probability density function approximating the PDF (in t) of the normal distribution is given by

$$\frac{f(k, x)\sqrt{\log \log x}}{(\log x)} \rightarrow \frac{1}{\sqrt{2\pi} \cdot \sigma_x(C)} \times \exp\left(-\frac{t^2}{2\sigma_x(C)^2}\right),$$

we perform the change of variable $s \mapsto \frac{t\sqrt{\log \log x}}{(\log x)}$ to obtain the normalized form of our theorem stated above.

By the same argument utilized in the proof of Proposition 4.5, we see that the contributions of these summatory functions for $k \leq \mu_x(C) - \mu_x(C)^{2/3}$ is negligible. We also require that $k \leq \log \log x$ for all large x as we required by Theorem 4.2. We then sum over a corresponding range of

$$\mu_x(C) - \mu_x(C)^{2/3} \leq k \leq \mu_x(C) + z \cdot \sigma_x(C),$$

to approximate the stated normalized densities. As $x \rightarrow \infty$ the three terms that result (one main term and two error terms, respectively) can be considered to each correspond to a Riemann sum for an associated integral whose limiting formula corresponds to a main term given by the standard normal CDF at z . \square

Corollary 4.8. *Let $Y > 0$. Suppose that $\mu_x(C)$ and $\sigma_x(C)$ are defined as in Theorem 4.7 for large $x > e$. For $Y > 0$ and all $-Y \leq z \leq Y$ we have uniformly that as $x \rightarrow \infty$*

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : |g^{-1}(n)| - \frac{6}{\pi^2} \mathbb{E}|g^{-1}(n)| \leq z\right\} = \Phi\left\{\frac{6\sigma_x(C)}{\pi^2} \left(\frac{\pi^2 z}{6} + \sigma_x(C)\right) - \frac{6}{\pi^2} \log(4\sqrt{2\pi})\right\} + o(1).$$

Proof. We claim that

$$|g^{-1}(n)| - \frac{6}{\pi^2} \mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2} C_{\Omega(n)}(n), \text{ as } n \rightarrow \infty.$$

As in the proof of Corollary 4.6, we obtain that

$$\frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| = \frac{6}{\pi^2} \left[\mathbb{E}[C_{\Omega(x)}(x)] + \sum_{d < x} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d} \right] + O(1).$$

Let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) := f(x) - f(x-1)$. We see that for large n

$$\begin{aligned} |g^{-1}(n)| &= \Delta_n(n \cdot \mathbb{E}|g^{-1}(n)|) \sim \Delta_n\left(\sum_{d \leq n} \frac{6}{\pi^2} \cdot C_{\Omega(d)}(d) \cdot \frac{n}{d}\right) \\ &= \frac{6}{\pi^2} \left[C_{\Omega(n)}(n) + \sum_{d < n} C_{\Omega(d)}(d) \frac{n}{d} - \sum_{d < n} C_{\Omega(d)}(d) \frac{(n-1)}{d} \right] \\ &\sim \frac{6}{\pi^2} C_{\Omega(n)}(n) + \frac{6}{\pi^2} \mathbb{E}|g^{-1}(n-1)|, \text{ as } n \rightarrow \infty. \end{aligned}$$

Since $\mathbb{E}|g^{-1}(n-1)| \sim \mathbb{E}|g^{-1}(n)|$ for all sufficiently large n , the result finally follows by a normalization of Theorem 4.7. \square

Lemma 4.9. *For all x sufficiently large, if we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds:*

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}(\log \log x)\right) = \frac{1}{2} + o(1) \quad (\text{A})$$

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}(\alpha + \log \log x)\right) = \Phi(\alpha) + o(1), \alpha \in \mathbb{R}. \quad (\text{B})$$

Proof. Each of these results is a consequence of Corollary 4.8. The result in (A) follows since $\Phi(0) = \frac{1}{2}$ by taking

$$z = \frac{6}{\pi^2} \cdot \frac{(\log(4\sqrt{2\pi}) + \alpha)}{\sigma_x(C)} - \sigma_x(C),$$

in Corollary 4.8 for $\alpha = 0$ and with $\sigma_x(C) := \log \log x$. Note that as $\alpha \rightarrow +\infty$, we get that the right-hand-side of (B) tends to one for large $x \rightarrow +\infty$. \square

It follows from Lemma 4.9 and Corollary 4.6 that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \cdot \#\left\{n \leq x : |g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|(1 + o(1))\right\} = 1.$$

That is, for almost every sufficiently large integer n we recover that

$$|g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|(1 + o(1)).$$

5 Proofs of new formulas and limiting relations for $M(x)$

5.1 Establishing initial asymptotic bounds on the summatory function $G^{-1}(x)$

Let $L(x) := \sum_{n \leq x} \lambda(n)$ for $x \geq 1$. The most recent known upper bound on $L(x)$ (assuming the RH) is established by Humphries based on Soundararajan's result bounding $M(x)$. It is stated in the following form [6]:

$$L(x) = O\left(\sqrt{x} \cdot \exp\left((\log x)^{\frac{1}{2}}(\log \log x)^{\frac{5}{2}+\varepsilon}\right)\right), \text{ for any } \varepsilon > 0; \text{ as } x \rightarrow \infty. \quad (25)$$

Theorem 5.1. *We have that for almost every sufficiently large x , there exists $1 \leq t_0 \leq x$ such that*

$$G^{-1}(x) = O\left(L(t_0) \cdot \mathbb{E}|g^{-1}(x)|\right).$$

If the RH is true, then for any $\varepsilon > 0$ and all large integers $x > e$

$$G^{-1}(x) = O\left((\log x)^2 \sqrt{\log \log x} \sqrt{x} \times \exp\left(\sqrt{\log x}(\log \log x)^{\frac{5}{2}+\varepsilon}\right)\right).$$

Proof. We write the next formulas for $G^{-1}(x)$ at almost every large $x > e$ by Abel summation and applying the mean value theorem:

$$\begin{aligned} G^{-1}(x) &= \sum_{n \leq x} \lambda(n) |g^{-1}(n)| \\ &= L(x) |g^{-1}(x)| - \int L(x) \frac{d}{dx} |g^{-1}(x)| dx \\ &= O\left(|L(t_0)| \cdot \mathbb{E}|g^{-1}(x)|\right), \text{ for some } t_0 \in [1, x]. \end{aligned} \quad (26)$$

The proof of this result appeals to the last few results we used to establish the probabilistic interpretations of the distribution of $|g^{-1}(n)|$ as $n \rightarrow \infty$ in Section 4.

We need to bound the sums of the maximal extreme values of $|g^{-1}(n)|$ over $n \leq x$ as $x \rightarrow \infty$ to prove the second bound. We know by a result of Robin that [22]

$$\omega(n) \ll \frac{\log n}{\log \log n}, \text{ as } n \rightarrow \infty.$$

Recall that the values of $|g^{-1}(n)|$ are locally maximized when n is squarefree with

$$|g^{-1}(n)| \leq \sum_{j=0}^{\omega(n)} \binom{\omega(n)}{j} j! \ll \Gamma(\omega(n) + 1) \ll \left(\frac{\log n}{\log \log n}\right)^{\frac{\log n}{\log \log n} + \frac{1}{2}}.$$

Since we have deduced that the set of $n \leq x$ on which $|g^{-1}(n)|$ is substantially larger than its average order is asymptotically thin, we find the bounds

$$\begin{aligned} \left| \int_{x-o(1)}^x L'(t) |g^{-1}(t)| dt \right| &\ll \int_{x-o(1)}^x \left(\frac{\log t}{\log \log t}\right)^{\frac{\log t}{\log \log t} + \frac{1}{2}} dt = o\left(\left(\frac{\log x}{\log \log x}\right)^{\frac{\log x}{\log \log x} + \frac{1}{2}}\right) \\ &\ll o\left(\frac{x}{(\log x)^m (\log \log x)^r}\right), \text{ for any } m, r = o\left(\frac{(\log x)(\log \log \log x)}{\log \log x}\right), \text{ as } x \rightarrow \infty. \end{aligned}$$

Indeed, we can see that the limit

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{1}{x} \left(\frac{\log x}{\log \log x}\right)^{\frac{\log x}{\log \log x} + \frac{1}{2}} (\log x)^m (\log \log x)^r &\ll \lim_{x \rightarrow \infty} x^{-\frac{(\log x)(\log \log \log x)}{\log \log x}} (\log x)^{m+r} \\ &= \lim_{x \rightarrow \infty} \exp\left((m+r) \log x - (\log x)^2 \frac{\log \log \log x}{\log \log x}\right) = \lim_{t \rightarrow \infty} e^{-t} = 0. \end{aligned}$$

For large x , let $\mathcal{R}_x := \{t \leq x : |g^{-1}(t)| \gg \mathbb{E}|g^{-1}(t)|\}$ such that $|\mathcal{R}_x| = o(1)$. The formula from (17a) implies that for large x and any $m, r = o\left(\frac{(\log x)(\log \log \log x)}{\log \log x}\right)$

$$\begin{aligned} G^{-1}(x) &= O\left(\int L'(x)|g^{-1}(x)|dx\right) = O\left(\mathbb{E}|g^{-1}(x)| \times \int L'(x)dx + \int_{x-|\mathcal{R}_x|}^x |L'(t)| \cdot |g^{-1}(t)|dt\right) \\ &= O\left(\mathbb{E}|g^{-1}(x)| \cdot |L(x)| + o\left(\frac{x}{(\log x)^m (\log \log x)^r}\right)\right). \end{aligned}$$

If the RH is true, by applying Humphries' result in (25) in tandem with Corollary 4.6, then for any $\varepsilon > 0$, $m, r = o\left(\frac{(\log x)(\log \log \log x)}{\log \log x}\right)$ and almost every large integer $x \geq 1$ we have that

$$\begin{aligned} G^{-1}(x) &= O\left((\log x)^2 \sqrt{\log \log x} \cdot \sqrt{x} \times \exp\left(\sqrt{\log x} \cdot (\log \log x)^{\frac{5}{2}+\varepsilon}\right) + o\left(\frac{x}{(\log x)^m (\log \log x)^r}\right)\right), \\ &= O\left((\log x)^2 \sqrt{\log \log x} \cdot x^{\frac{1}{2} + \frac{(\log \log x)^{5/2+\varepsilon}}{\sqrt{\log x}}} + o\left(x^{1-\log \log \log x}\right)\right). \end{aligned}$$

To obtain the conclusion of the second result, we take limits as $x \rightarrow \infty$ to see that the dominant term is given by the leftmost term in the last equation. \square

5.2 Bounding $M(x)$ by asymptotics for $G^{-1}(x)$

Proposition 5.2. *For all sufficiently large x , we have that the Mertens function satisfies*

$$M(x) = G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} G^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right]. \quad (27)$$

Proof. We know by applying Corollary 1.4 that

$$\begin{aligned} M(x) &= \sum_{k=1}^x g^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1 \right] \\ &= G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} g^{-1}(k) \pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) \\ &= G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right]. \end{aligned}$$

The upper bound on the sum is truncated to $k \in [1, \frac{x}{2}]$ in the second equation above due to the fact that $\pi(1) = 0$. The third formula follows from summation by parts. \square

Lemma 5.3. *For sufficiently large x , integers $k \in [1, \sqrt{x}]$ and $m \geq 0$, we have that*

$$\frac{x}{k \cdot \log^m\left(\frac{x}{k}\right)} - \frac{x}{(k+1) \cdot \log^m\left(\frac{x}{k+1}\right)} \asymp \frac{x}{(\log x)^m \cdot k(k+1)}, \quad (\text{A})$$

and

$$\sum_{k=\sqrt{x}}^{\frac{x}{2}} \frac{x}{k(k+1)} = \sum_{k=\sqrt{x}}^{\frac{x}{2}} \frac{x}{k^2} + O(1). \quad (\text{B})$$

Proof. The proof of (A) is obvious since for $k_0 \in [1, \frac{x}{2}]$ we have that

$$\log(2)(1 + o(1)) \leq \log\left(\frac{x}{k_0}\right) \leq \log(x).$$

To prove (B), notice that

$$\frac{x}{k(k+1)} - \frac{x}{k^2} = -\frac{x}{k^2(k+1)}.$$

Then we see that

$$\left| \int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^2(t+1)} dt \right| \leq \left| \int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^3} dt \right| \asymp 1. \quad \square$$

Corollary 5.4. *We have that as $x \rightarrow \infty$*

$$M(x) = O\left(G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) + \frac{x}{\log x} \times \sum_{k \leq \sqrt{x}} \frac{G^{-1}(k)}{k^2} + (\log x)^2 (\log \log x)^{3/2}\right).$$

Proof. We need to first bound the prime counting function differences in the formula given by Proposition 5.2. We will require the following known bounds on the prime counting function due to Rosser and Schoenfeld for all large $x > 59$ [23, Thm. 1]:

$$\frac{x}{\log x} \left(1 + \frac{1}{2 \log x}\right) \leq \pi(x) \leq \frac{x}{\log x} \left(1 + \frac{3}{2 \log x}\right). \quad (28)$$

The result in (28) together with Lemma 5.3 implies that for $\sqrt{x} \leq k \leq \frac{x}{2}$

$$\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) = O\left(\frac{x}{k^2 \cdot \log\left(\frac{x}{k}\right)}\right).$$

We will rewrite the intermediate formula from the proof of Proposition 5.2 as a sum of two components with summands taken over disjoint intervals. For large $x > e$, let

$$\begin{aligned} S_1(x) &:= \sum_{1 \leq k \leq \sqrt{x}} g^{-1}(k) \pi\left(\frac{x}{k}\right) \\ S_2(x) &:= \sum_{\sqrt{x} < k \leq \frac{x}{2}} g^{-1}(k) \pi\left(\frac{x}{k}\right). \end{aligned}$$

We assert by the asymptotic formulas for the prime counting function that

$$S_1(x) = O\left(\frac{x}{\log x} \times \sum_{k \leq \sqrt{x}} \frac{G^{-1}(k)}{k^2}\right).$$

To bound the second sum, we perform summation by parts as in the proof of the proposition and apply the bound above for the difference of the summand functions to obtain that

$$\begin{aligned} S_2(x) &= O\left(G^{-1}\left(\frac{x}{2}\right) + \int_{\sqrt{x}}^{\frac{x}{2}} \frac{G^{-1}(t)}{t^2 \log\left(\frac{x}{t}\right)} dt\right) \\ &= O\left(G^{-1}\left(\frac{x}{2}\right) + \max_{\sqrt{x} < k < \frac{x}{2}} \frac{|G^{-1}(k)|}{k} \times \int_{\sqrt{x}}^{\frac{x}{2}} \frac{dt}{t \cdot \log\left(\frac{x}{t}\right)}\right) \\ &= O\left(G^{-1}\left(\frac{x}{2}\right) + (\log \log x) \times \max_{\sqrt{x} < k < \frac{x}{2}} \frac{|G^{-1}(k)|}{k}\right). \end{aligned}$$

The rightmost maximum term in the previous equation is known to satisfy $\frac{|G^{-1}(k)|}{k} \ll \mathbb{E}|g^{-1}(k)|$ as $k \rightarrow \infty$. The conclusion follows since the average order of $|g^{-1}(n)|$ is increasing for sufficiently large n . \square

6 Conclusions

We have identified a new sequence, $\{g^{-1}(n)\}_{n \geq 1}$, which is the Dirichlet inverse of the shifted additive function, $g := \omega + 1$. In general, we find that the Dirichlet inverse of any arithmetic function f such that $f(1) \neq 0$ is expressed at each $n \geq 2$ as a signed sum of m -fold convolutions of f with itself for $1 \leq m \leq \Omega(n)$. As we discussed in the remarks in Section 3.3, it happens that there is a natural combinatorial interpretation to the distribution of distinct values of $|g^{-1}(n)|$ for $n \leq x$ involving the distribution of the primes $p \leq x$ at large x . In particular, the magnitude of $|g^{-1}(n)|$ depends only on the pattern of the exponents of the prime factorization of n in so much as $|g^{-1}(n_1)| = |g^{-1}(n_2)|$ whenever $\omega(n_1) = \omega(n_2)$, $\Omega(n_1) = \Omega(n_2)$, and where there is a one-to-one correspondence $\nu_{p_1}(n_1) = \nu_{p_2}(n_2)$ between the distinct primes $p_1|n_1$ and $p_2|n_2$.

The signedness of $g^{-1}(n)$ is given by $\lambda(n)$ for all $n \geq 1$. This leads to a familiar dependence of the summatory functions $G^{-1}(x)$ on the distribution of the summatory function $L(x)$. Section 5 provides equivalent characterizations of the limiting properties of $M(x)$ by exact formulas and asymptotic relations involving $G^{-1}(x)$ and $L(x)$. We emphasize that our new work on the Mertens function proved within this article is significant in providing a new window through which we can view bounding $M(x)$. The computational data generated in Table B suggests numerically that the distribution of $G^{-1}(x)$ may be easier to work with than those of $M(x)$ or $L(x)$. The remarks given in Section 3.3 about the direct combinatorial relation of the distinct (and repetition of) values of $|g^{-1}(n)|$ for $n \leq x$ are suggestive towards bounding main terms for $G^{-1}(x)$ along infinite subsequences.

One topic that we do not touch on in the article is to consider what correlation (if any) exists between $\lambda(n)$ and the unsigned sequence of $|g^{-1}(n)|$ with the limiting distribution proved in Corollary 4.8. Much in the same way that variants of the Erdős-Kac theorem are proved by defining random variables related to $\omega(n)$, we suggest an analysis of the summatory function $G^{-1}(x)$ by scaling the explicitly distributed $|g^{-1}(n)|$ for $n \leq x$ as $x \rightarrow \infty$ by its signed weight of $\lambda(n)$ using an initial heuristic along these lines for future work.

Another experiment illustrated in the online computational reference [24] suggests that for many, if not most sufficiently large x , we may consider replacing the summatory function with other summands weighted by $\lambda(n)$. These alternate sums can be seen to average these sequences differently while still preserving the original asymptotic order of $|G^{-1}(x)|$ heuristically. For example, each of the following three summatory functions offer a unique interpretation of an average of sorts that “mixes” the values of $\lambda(n)$ with the unsigned sequence $|g^{-1}(n)|$ over $1 \leq n \leq x$:

$$\begin{aligned} G_*^{-1}(x) &:= \sum_{n \leq x} \frac{1}{2\gamma - 1 + \log n} \times \sum_{d|n} \lambda\left(\frac{n}{d}\right) |g^{-1}(d)| \\ G_{**}^{-1}(x) &:= \sum_{n \leq x} \frac{1}{2\gamma - 1 + \log n} \times \sum_{d|n} \lambda\left(\frac{n}{d}\right) g^{-1}(d) \\ G_{***}^{-1}(x) &:= \sum_{n \leq x} \frac{1}{2\gamma - 1 + \log n} \times \sum_{d|n} g^{-1}(d). \end{aligned}$$

Then based on preliminary numerical results, a large proportion of the $y \leq x$ for large x satisfy

$$\left| \frac{G_*^{-1}(y)}{G^{-1}(y)} \right|^{-1}, \left| \frac{G_{**}^{-1}(y)}{G^{-1}(y)} \right|, \left| \frac{G_{***}^{-1}(y)}{G^{-1}(y)} \right| \in (0, 3].$$

Variants of this type of summatory function identity exchange are similarly suggested for future work to extend the topics and new results proved in this article.

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A Appendix: Asymptotic formulas

We thank Gergő Nemes from the Alfréd Rényi Institute of Mathematics for his careful notes on the limiting asymptotics for the sums identified in this section. We have adapted his proofs to establish the next few lemmas.

Facts A.1 (The incomplete gamma function). The (upper) *incomplete gamma function* is defined by [19, §8.4]

$$\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt, a \in \mathbb{R}, |\arg z| < \pi.$$

In the previous equation, z is defined on the universal covering of $\mathbb{C} \setminus \{0\}$ by analytic continuation. For $a \in \mathbb{Z}^+$, $\Gamma(a, z)$ is an entire function of z . The following properties of $\Gamma(a, z)$ hold [19, §8.4; §8.11(i)]:

$$\Gamma(a, z) = (a-1)! \cdot e^{-z} \times \sum_{k=0}^{a-1} \frac{z^k}{k!}, \text{ for } a \in \mathbb{Z}^+, z \in \mathbb{C}, \quad (29a)$$

$$\Gamma(a, z) \sim z^{a-1} e^{-z}, \text{ for fixed } a \in \mathbb{C}, \text{ as } z \rightarrow +\infty. \quad (29b)$$

Moreover, for real $z > 0$, as $z \rightarrow +\infty$ we have that [14]

$$\Gamma(z, z) \sim \sqrt{\frac{\pi}{2}} z^{z-\frac{1}{2}} e^{-z} + O(z^{z-1} e^{-z}), \quad (29c)$$

If $z, a \rightarrow \infty$ with $z = \lambda a$ for some $\lambda > 1$ such that $(\lambda - 1)^{-1} = o(|a|^{1/2})$, then [14]

$$\Gamma(a, z) = z^a e^{-z} \times \sum_{n \geq 0} \frac{(-a)^n b_n(\lambda)}{(z-a)^{2n+1}}, \quad (29d)$$

where the sequence $b_n(\lambda)$ satisfies the characteristic relation that $b_0(\lambda) = 1$ and^D

$$b_n(\lambda) = \lambda(1-\lambda)b'_{n-1}(\lambda) + \lambda(2n-1)b_{n-1}(\lambda), n \geq 1.$$

Proposition A.2. Suppose that $a, z, \lambda > 0$ are such that $z = \lambda a$. If $\lambda > 1$, then as $z \rightarrow +\infty$

$$\Gamma(a, z) = \frac{z^{a-1} e^{-z}}{1 - \lambda^{-1}} + O_\lambda(z^{a-2} e^{-z}).$$

If $\lambda > 0.567142 > W(1)$ where $W(x)$ denotes the principal branch of the Lambert W -function, then as $z \rightarrow +\infty$

$$\Gamma(a, ze^{\pm \pi i}) = -e^{\pm \pi i a} \frac{z^{a-1} e^z}{1 + \lambda^{-1}} + O_\lambda(z^{a-2} e^z).$$

Note that we cannot write this expansion as $\Gamma(a, -z)$ directly unless $a \in \mathbb{Z}^+$ as the incomplete gamma function has a branch point at the origin with respect to its second variable. This function becomes a single-valued analytic function of its second input on the universal covering of $\mathbb{C} \setminus \{0\}$.

Proof. Using the notation from (29d) and [15], we have that

$$\Gamma(a, z) = \frac{z^{a-1} e^{-z}}{1 - \lambda^{-1}} + z^a e^{-z} R_1(a, \lambda).$$

^D An exact formula for $b_n(\lambda)$ is given in terms of the *second-order Eulerian number triangle* [25, A008517] as follows:

$$b_n(\lambda) = \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \lambda^{k+1}.$$

From the bounds in [15, §3.1], we get

$$|z^a e^{-z} R_1(a, \lambda)| \leq z^a e^{-z} \times \frac{a \cdot b_1(\lambda)}{(z-a)^3} = \frac{z^{a-2} e^{-z}}{(1-\lambda^{-1})^3}$$

The main and error terms in the previous equation can also be seen by applying the asymptotic series in (29d) directly.

The proof of the second equation above follows from the following asymptotics [14, Eq. (1.1)]

$$\Gamma(-a, z) \sim z^{-a} e^{-z} \times \sum_{n \geq 0} \frac{a^n b_n(-\lambda)}{(z+a)^{2n+1}},$$

by setting $a \mapsto ae^{\pm\pi i}$ and $z \mapsto ze^{\pm\pi i}$ with $\lambda = z/a > 0.567142 > W(1)$. The restriction on the range of λ over which the second formula holds is made to ensure that the last formula from the reference is valid at negative real a . \square

Lemma A.3. *For $x \rightarrow +\infty$, we have that*

$$S_1(x) := \frac{x}{\log x} \times \left| \sum_{1 \leq k \leq \log \log x} \frac{(-1)^k (\log \log x)^{k-1}}{(k-1)!} \right| = \frac{x}{2\sqrt{2\pi \log \log x}} + O\left(\frac{x}{(\log \log x)^{3/2}}\right).$$

Proof. We have that for $n \geq 1$ and any $t > 0$ by (29a) and [19, §8.4]

$$\sum_{1 \leq k \leq n} \frac{(-1)^k t^{k-1}}{(k-1)!} = -e^{-t} \times \frac{\Gamma(n, -t)}{(n-1)!}.$$

Suppose that $t = n + \xi$ with $\xi = O(1)$. By the second formula in Proposition A.2 with the parameters $(a, z, \lambda) \mapsto (n, t, 1 + \frac{\xi}{n})$, we deduce that as $n, t \rightarrow +\infty$.

$$\Gamma(n, -t) = (-1)^{n+1} \times \frac{t^n e^t}{t+n} + O\left(\frac{nt^n e^t}{(t+n)^3}\right) = (-1)^{n+1} \frac{t^n e^t}{2n} + O\left(\frac{t^n e^t}{nt}\right). \quad (30)$$

Accordingly, we see that

$$\sum_{1 \leq k \leq n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{t^n}{2n!} + O\left(\frac{t^{n-1}}{n!}\right).$$

By a variant of Stirling's formula in [19, cf. Eq. (5.11.8)]

$$n! = \Gamma(1+t-\xi) = \sqrt{2\pi} \cdot t^{t-\xi+1/2} e^{-t} (1 + O(t^{-1})) = \sqrt{2\pi} \cdot t^{n+1/2} e^{-t} (1 + O(t^{-1})).$$

Hence, as $n \rightarrow +\infty$ with $t := n + \xi$ and $\xi = O(1)$, we obtain

$$\sum_{k=1}^n \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{e^t}{2\sqrt{2\pi}t} + O\left(\frac{e^t}{t^{3/2}}\right).$$

The conclusion follows by taking $n = \lfloor \log \log x \rfloor$ and $t = \log \log x$ and applying the triangle inequality to obtain the desired result. \square

Lemma A.4. *For $x \rightarrow +\infty$, we have that*

$$S_3(x) := \sum_{1 \leq k \leq \log \log x} \frac{(\log \log x)^{k-1/2}}{(2k-1)(k-1)!} = \frac{(\log x)}{4\sqrt{\log \log x}} + O\left(\frac{\log x}{\log \log x}\right).$$

Proof. Notice that by (23) we can write

$$\begin{aligned} S_3(x) &= \sum_{k \geq 1} \frac{(\log \log x)^{k-1/2}}{(2k-1)(k-1)!} + O\left(\sum_{k \geq 0} \frac{(\log \log x)^{\log \log x + k + 1/2}}{(2k+2 \log \log x + 1)\Gamma(k + \log \log x + 1)}\right) \\ &= \frac{\sqrt{\pi}}{2} \operatorname{erfi}\left(\sqrt{\log \log x}\right) \left(1 + O\left(\frac{1}{\log \log x}\right)\right) \\ &= \frac{\log x}{2\sqrt{\log \log x}} + O\left(\frac{\log x}{(\log \log x)^{3/2}}\right). \end{aligned}$$

The second formula involving the imaginary error function in the previous equations is obtained by applying the symbolic summation procedures in *Mathematica*. The last result nearly implies the stated formula in this lemma with less tight error term up to a factor of $\frac{1}{2}$. We will give an exact proof of the result directly using the bounds for the incomplete gamma function established in the recent reference [16]. The reference takes into account the behavior of the incomplete gamma function $\Gamma(a, z)$ near the transition point $\lambda = z/a = 1$ as $a, z \rightarrow +\infty$.

For $n \geq 1$ and any $t > 0$, let

$$\tilde{S}_n(t) := \sum_{1 \leq k \leq n} \frac{t^{k-1}}{(2k-1)(k-1)!}.$$

By the formula in (29a) and a change of variable, we get that

$$\begin{aligned} \tilde{S}_n(t) &= \int_0^1 \left(\sum_{k=1}^n \frac{(s^2 t)^{k-1}}{(k-1)!} \right) ds \\ &= \frac{1}{(n-1)!} \times \int_0^1 e^{s^2 t} \Gamma(n, s^2 t) ds \\ &= \frac{1}{2t(n-1)!} \times \int_0^1 e^y \Gamma(n, y) dy. \end{aligned}$$

Integration by parts performed one time with

$$\left\{ \begin{array}{ll} u_x = \Gamma(n, y) & v'_x = e^y dy \\ u'_x = -y^{n-1} e^{-y} dy & v_x = e^y \end{array} \right\},$$

implies that

$$\begin{aligned} \tilde{S}_n(t) &= \frac{1}{2t(n-1)!} \times \left[\Gamma(n, t) e^t - \Gamma(n) + \int_0^t y^{n-1} dy \right] \\ &= \frac{\Gamma(n, t) e^t}{2t(n-1)!} - \frac{1}{2t} + \frac{t^{n-1}}{2n!}. \end{aligned} \tag{31}$$

Suppose that $t = n + \xi$ where $\xi = O(1)$. According to the reference [16, Eq. (2.4)], we have that as $t \rightarrow +\infty$

$$\Gamma(n, t) = t^n e^{-t} \sqrt{\frac{\pi}{2t}} \left(1 + O(t^{-1/2})\right).$$

By the variant of Stirling's formula expressed in [19, §5.11(i)], we again have that

$$(n-1)! = \sqrt{2\pi} \cdot t^{n-1/2} e^{-t} \left(1 + O(t^{-1})\right).$$

Whence, with $t = n + O(1)$ and as $n \rightarrow +\infty$, we obtain

$$\tilde{S}_n(t) = \frac{e^t}{4t} + O\left(\frac{e^t}{nt^{3/2}}\right).$$

The conclusion follows taking $n = \lfloor \log \log x \rfloor$, $t = \log \log x$ and multiplying $\tilde{S}_n(t)$ by $(\log \log x)^{1/2}$. \square

B Table: The Dirichlet inverse function $g^{-1}(n)$

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum_{d n} C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|--|---------------|---------------|-------------|--|---|--------------------|--------------------|-------------|---------------|---------------|
| 1 | 1 ¹ | Y | N | 1 | 0 | 1.000000 | 1.000000 | 0.000000 | 1 | 1 | 0 |
| 2 | 2 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.500000 | 0.500000 | -1 | 1 | -2 |
| 3 | 3 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.333333 | 0.666667 | -3 | 1 | -4 |
| 4 | 2 ² | N | Y | 2 | 0 | 1.500000 | 0.500000 | 0.500000 | -1 | 3 | -4 |
| 5 | 5 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.400000 | 0.600000 | -3 | 3 | -6 |
| 6 | 2 ¹ 3 ¹ | Y | N | 5 | 0 | 1.000000 | 0.500000 | 0.500000 | 2 | 8 | -6 |
| 7 | 7 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.428571 | 0.571429 | 0 | 8 | -8 |
| 8 | 2 ³ | N | Y | -2 | 0 | 2.000000 | 0.375000 | 0.625000 | -2 | 8 | -10 |
| 9 | 3 ² | N | Y | 2 | 0 | 1.500000 | 0.444444 | 0.555556 | 0 | 10 | -10 |
| 10 | 2 ¹ 5 ¹ | Y | N | 5 | 0 | 1.000000 | 0.500000 | 0.500000 | 5 | 15 | -10 |
| 11 | 11 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.454545 | 0.545455 | 3 | 15 | -12 |
| 12 | 2 ² 3 ¹ | N | N | -7 | 2 | 1.285714 | 0.416667 | 0.583333 | -4 | 15 | -19 |
| 13 | 13 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.384615 | 0.615385 | -6 | 15 | -21 |
| 14 | 2 ¹ 7 ¹ | Y | N | 5 | 0 | 1.000000 | 0.428571 | 0.571429 | -1 | 20 | -21 |
| 15 | 3 ¹ 5 ¹ | Y | N | 5 | 0 | 1.000000 | 0.466667 | 0.533333 | 4 | 25 | -21 |
| 16 | 2 ⁴ | N | Y | 2 | 0 | 2.500000 | 0.500000 | 0.500000 | 6 | 27 | -21 |
| 17 | 17 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.470588 | 0.529412 | 4 | 27 | -23 |
| 18 | 2 ¹ 3 ² | N | N | -7 | 2 | 1.285714 | 0.444444 | 0.555556 | -3 | 27 | -30 |
| 19 | 19 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.421053 | 0.578947 | -5 | 27 | -32 |
| 20 | 2 ² 5 ¹ | N | N | -7 | 2 | 1.285714 | 0.400000 | 0.600000 | -12 | 27 | -39 |
| 21 | 3 ¹ 7 ¹ | Y | N | 5 | 0 | 1.000000 | 0.428571 | 0.571429 | -7 | 32 | -39 |
| 22 | 2 ¹ 11 ¹ | Y | N | 5 | 0 | 1.000000 | 0.454545 | 0.545455 | -2 | 37 | -39 |
| 23 | 23 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.434783 | 0.565217 | -4 | 37 | -41 |
| 24 | 2 ³ 3 ¹ | N | N | 9 | 4 | 1.555556 | 0.458333 | 0.541667 | 5 | 46 | -41 |
| 25 | 5 ² | N | Y | 2 | 0 | 1.500000 | 0.480000 | 0.520000 | 7 | 48 | -41 |
| 26 | 2 ¹ 13 ¹ | Y | N | 5 | 0 | 1.000000 | 0.500000 | 0.500000 | 12 | 53 | -41 |
| 27 | 3 ³ | N | Y | -2 | 0 | 2.000000 | 0.481481 | 0.518519 | 10 | 53 | -43 |
| 28 | 2 ² 7 ¹ | N | N | -7 | 2 | 1.285714 | 0.464286 | 0.535714 | 3 | 53 | -50 |
| 29 | 29 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.448276 | 0.551724 | 1 | 53 | -52 |
| 30 | 2 ¹ 3 ¹ 5 ¹ | Y | N | -16 | 0 | 1.000000 | 0.433333 | 0.566667 | -15 | 53 | -68 |
| 31 | 31 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.419355 | 0.580645 | -17 | 53 | -70 |
| 32 | 2 ⁵ | N | Y | -2 | 0 | 3.000000 | 0.406250 | 0.593750 | -19 | 53 | -72 |
| 33 | 3 ¹ 11 ¹ | Y | N | 5 | 0 | 1.000000 | 0.424242 | 0.575758 | -14 | 58 | -72 |
| 34 | 2 ¹ 17 ¹ | Y | N | 5 | 0 | 1.000000 | 0.441176 | 0.558824 | -9 | 63 | -72 |
| 35 | 5 ¹ 7 ¹ | Y | N | 5 | 0 | 1.000000 | 0.457143 | 0.542857 | -4 | 68 | -72 |
| 36 | 2 ² 3 ² | N | N | 14 | 9 | 1.357142 | 0.472222 | 0.527778 | 10 | 82 | -72 |
| 37 | 37 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.459459 | 0.540541 | 8 | 82 | -74 |
| 38 | 2 ¹ 19 ¹ | Y | N | 5 | 0 | 1.000000 | 0.473684 | 0.526316 | 13 | 87 | -74 |
| 39 | 3 ¹ 13 ¹ | Y | N | 5 | 0 | 1.000000 | 0.487179 | 0.512821 | 18 | 92 | -74 |
| 40 | 2 ³ 5 ¹ | N | N | 9 | 4 | 1.555556 | 0.500000 | 0.500000 | 27 | 101 | -74 |
| 41 | 41 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.487805 | 0.512195 | 25 | 101 | -76 |
| 42 | 2 ¹ 3 ¹ 7 ¹ | Y | N | -16 | 0 | 1.000000 | 0.476190 | 0.523810 | 9 | 101 | -92 |
| 43 | 43 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.465116 | 0.534884 | 7 | 101 | -94 |
| 44 | 2 ² 11 ¹ | N | N | -7 | 2 | 1.285714 | 0.454545 | 0.545455 | 0 | 101 | -101 |
| 45 | 3 ² 5 ¹ | N | N | -7 | 2 | 1.285714 | 0.444444 | 0.555556 | -7 | 101 | -108 |
| 46 | 2 ¹ 23 ¹ | Y | N | 5 | 0 | 1.000000 | 0.456522 | 0.543478 | -2 | 106 | -108 |
| 47 | 47 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.446809 | 0.553191 | -4 | 106 | -110 |
| 48 | 2 ⁴ 3 ¹ | N | N | -11 | 6 | 1.818181 | 0.437500 | 0.562500 | -15 | 106 | -121 |

Table B: Computations with $g^{-1}(n) \equiv (\omega + 1)^{-1}(n)$ for $1 \leq n \leq 500$.

- The column labeled **Primes** provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled **Sqfree** and **PPower**, respectively, list inclusion of n in the sets of squarefree integers and the prime powers.
- The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.
- The last columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{n} \cdot \#\{n \leq x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geq 1$.

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\Sigma_{d n} C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|----------------|--------|--------|-------------|--|---|--------------------|--------------------|-------------|---------------|---------------|
| 49 | 7^2 | N | Y | 2 | 0 | 1.5000000 | 0.448980 | 0.551020 | -13 | 108 | -121 |
| 50 | $2^1 5^2$ | N | N | -7 | 2 | 1.2857143 | 0.440000 | 0.560000 | -20 | 108 | -128 |
| 51 | $3^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.450980 | 0.549020 | -15 | 113 | -128 |
| 52 | $2^2 13^1$ | N | N | -7 | 2 | 1.2857143 | 0.442308 | 0.557692 | -22 | 113 | -135 |
| 53 | 53^1 | Y | Y | -2 | 0 | 1.0000000 | 0.433962 | 0.566038 | -24 | 113 | -137 |
| 54 | $2^1 3^3$ | N | N | 9 | 4 | 1.5555556 | 0.444444 | 0.555556 | -15 | 122 | -137 |
| 55 | $5^1 11^1$ | Y | N | 5 | 0 | 1.0000000 | 0.454545 | 0.545455 | -10 | 127 | -137 |
| 56 | $2^3 7^1$ | N | N | 9 | 4 | 1.5555556 | 0.464286 | 0.535714 | -1 | 136 | -137 |
| 57 | $3^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473684 | 0.526316 | 4 | 141 | -137 |
| 58 | $2^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482759 | 0.517241 | 9 | 146 | -137 |
| 59 | 59^1 | Y | Y | -2 | 0 | 1.0000000 | 0.474576 | 0.525424 | 7 | 146 | -139 |
| 60 | $2^2 3^1 5^1$ | N | N | 30 | 14 | 1.1666667 | 0.483333 | 0.516667 | 37 | 176 | -139 |
| 61 | 61^1 | Y | Y | -2 | 0 | 1.0000000 | 0.475410 | 0.524590 | 35 | 176 | -141 |
| 62 | $2^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 40 | 181 | -141 |
| 63 | $3^2 7^1$ | N | N | -7 | 2 | 1.2857143 | 0.476190 | 0.523810 | 33 | 181 | -148 |
| 64 | 2^6 | N | Y | 2 | 0 | 3.5000000 | 0.484375 | 0.515625 | 35 | 183 | -148 |
| 65 | $5^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492308 | 0.507692 | 40 | 188 | -148 |
| 66 | $2^1 3^1 11^1$ | Y | N | -16 | 0 | 1.0000000 | 0.484848 | 0.515152 | 24 | 188 | -164 |
| 67 | 67^1 | Y | Y | -2 | 0 | 1.0000000 | 0.477612 | 0.522388 | 22 | 188 | -166 |
| 68 | $2^2 17^1$ | N | N | -7 | 2 | 1.2857143 | 0.470588 | 0.529412 | 15 | 188 | -173 |
| 69 | $3^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478261 | 0.521739 | 20 | 193 | -173 |
| 70 | $2^1 5^1 7^1$ | Y | N | -16 | 0 | 1.0000000 | 0.471429 | 0.528571 | 4 | 193 | -189 |
| 71 | 71^1 | Y | Y | -2 | 0 | 1.0000000 | 0.464789 | 0.535211 | 2 | 193 | -191 |
| 72 | $2^3 3^2$ | N | N | -23 | 18 | 1.4782609 | 0.458333 | 0.541667 | -21 | 193 | -214 |
| 73 | 73^1 | Y | Y | -2 | 0 | 1.0000000 | 0.452055 | 0.547945 | -23 | 193 | -216 |
| 74 | $2^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.459459 | 0.540541 | -18 | 198 | -216 |
| 75 | $3^1 5^2$ | N | N | -7 | 2 | 1.2857143 | 0.453333 | 0.546667 | -25 | 198 | -223 |
| 76 | $2^2 19^1$ | N | N | -7 | 2 | 1.2857143 | 0.447368 | 0.552632 | -32 | 198 | -230 |
| 77 | $7^1 11^1$ | Y | N | 5 | 0 | 1.0000000 | 0.454545 | 0.545455 | -27 | 203 | -230 |
| 78 | $2^1 3^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.448718 | 0.551282 | -43 | 203 | -246 |
| 79 | 79^1 | Y | Y | -2 | 0 | 1.0000000 | 0.443038 | 0.556962 | -45 | 203 | -248 |
| 80 | $2^4 5^1$ | N | N | -11 | 6 | 1.8181818 | 0.437500 | 0.562500 | -56 | 203 | -259 |
| 81 | 3^4 | N | Y | 2 | 0 | 2.5000000 | 0.444444 | 0.555556 | -54 | 205 | -259 |
| 82 | $2^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.451220 | 0.548780 | -49 | 210 | -259 |
| 83 | 83^1 | Y | Y | -2 | 0 | 1.0000000 | 0.445783 | 0.554217 | -51 | 210 | -261 |
| 84 | $2^2 3^1 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.452381 | 0.547619 | -21 | 240 | -261 |
| 85 | $5^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.458824 | 0.541176 | -16 | 245 | -261 |
| 86 | $2^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.465116 | 0.534884 | -11 | 250 | -261 |
| 87 | $3^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.471264 | 0.528736 | -6 | 255 | -261 |
| 88 | $2^3 11^1$ | N | N | 9 | 4 | 1.5555556 | 0.477273 | 0.522727 | 3 | 264 | -261 |
| 89 | 89^1 | Y | Y | -2 | 0 | 1.0000000 | 0.471910 | 0.528090 | 1 | 264 | -263 |
| 90 | $2^1 3^2 5^1$ | N | N | 30 | 14 | 1.1666667 | 0.477778 | 0.522222 | 31 | 294 | -263 |
| 91 | $7^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483516 | 0.516484 | 36 | 299 | -263 |
| 92 | $2^2 23^1$ | N | N | -7 | 2 | 1.2857143 | 0.478261 | 0.521739 | 29 | 299 | -270 |
| 93 | $3^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 34 | 304 | -270 |
| 94 | $2^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.489362 | 0.510638 | 39 | 309 | -270 |
| 95 | $5^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.494737 | 0.505263 | 44 | 314 | -270 |
| 96 | $2^5 3^1$ | N | N | 13 | 8 | 2.0769231 | 0.500000 | 0.500000 | 57 | 327 | -270 |
| 97 | 97^1 | Y | Y | -2 | 0 | 1.0000000 | 0.494845 | 0.505155 | 55 | 327 | -272 |
| 98 | $2^1 7^2$ | N | N | -7 | 2 | 1.2857143 | 0.489796 | 0.510204 | 48 | 327 | -279 |
| 99 | $3^2 11^1$ | N | N | -7 | 2 | 1.2857143 | 0.484848 | 0.515152 | 41 | 327 | -286 |
| 100 | $2^2 5^2$ | N | N | 14 | 9 | 1.3571429 | 0.490000 | 0.510000 | 55 | 341 | -286 |
| 101 | 101^1 | Y | Y | -2 | 0 | 1.0000000 | 0.485149 | 0.514851 | 53 | 341 | -288 |
| 102 | $2^1 3^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.480392 | 0.519608 | 37 | 341 | -304 |
| 103 | 103^1 | Y | Y | -2 | 0 | 1.0000000 | 0.475728 | 0.524272 | 35 | 341 | -306 |
| 104 | $2^3 13^1$ | N | N | 9 | 4 | 1.5555556 | 0.480769 | 0.519231 | 44 | 350 | -306 |
| 105 | $3^1 5^1 7^1$ | Y | N | -16 | 0 | 1.0000000 | 0.476190 | 0.523810 | 28 | 350 | -322 |
| 106 | $2^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.481132 | 0.518868 | 33 | 355 | -322 |
| 107 | 107^1 | Y | Y | -2 | 0 | 1.0000000 | 0.476636 | 0.523364 | 31 | 355 | -324 |
| 108 | $2^2 3^3$ | N | N | -23 | 18 | 1.4782609 | 0.472222 | 0.527778 | 8 | 355 | -347 |
| 109 | 109^1 | Y | Y | -2 | 0 | 1.0000000 | 0.467890 | 0.532110 | 6 | 355 | -349 |
| 110 | $2^1 5^1 11^1$ | Y | N | -16 | 0 | 1.0000000 | 0.463636 | 0.536364 | -10 | 355 | -365 |
| 111 | $3^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.468468 | 0.531532 | -5 | 360 | -365 |
| 112 | $2^4 7^1$ | N | N | -11 | 6 | 1.8181818 | 0.464286 | 0.535714 | -16 | 360 | -376 |
| 113 | 113^1 | Y | Y | -2 | 0 | 1.0000000 | 0.460177 | 0.539823 | -18 | 360 | -378 |
| 114 | $2^1 3^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.456140 | 0.543860 | -34 | 360 | -394 |
| 115 | $5^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.460870 | 0.539130 | -29 | 365 | -394 |
| 116 | $2^2 29^1$ | N | N | -7 | 2 | 1.2857143 | 0.456897 | 0.543103 | -36 | 365 | -401 |
| 117 | $3^2 13^1$ | N | N | -7 | 2 | 1.2857143 | 0.452991 | 0.547009 | -43 | 365 | -408 |
| 118 | $2^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.457627 | 0.542373 | -38 | 370 | -408 |
| 119 | $7^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.462185 | 0.537815 | -33 | 375 | -408 |
| 120 | $2^3 3^1 5^1$ | N | N | -48 | 32 | 1.3333333 | 0.458333 | 0.541667 | -81 | 375 | -456 |
| 121 | 11^2 | N | Y | 2 | 0 | 1.5000000 | 0.462810 | 0.537190 | -79 | 377 | -456 |
| 122 | $2^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467213 | 0.532787 | -74 | 382 | -456 |
| 123 | $3^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.471545 | 0.528455 | -69 | 387 | -456 |
| 124 | $2^2 31^1$ | N | N | -7 | 2 | 1.2857143 | 0.467742 | 0.532258 | -76 | 387 | -463 |

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\Sigma_{d n} C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|----------------|--------|--------|-------------|--|---|--------------------|--------------------|-------------|---------------|---------------|
| 125 | 5^3 | N | Y | -2 | 0 | 2.0000000 | 0.464000 | 0.536000 | -78 | 387 | -465 |
| 126 | $2^1 3^2 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.468254 | 0.531746 | -48 | 417 | -465 |
| 127 | 127^1 | Y | Y | -2 | 0 | 1.0000000 | 0.464567 | 0.535433 | -50 | 417 | -467 |
| 128 | 2^7 | N | Y | -2 | 0 | 4.0000000 | 0.460938 | 0.539062 | -52 | 417 | -469 |
| 129 | $3^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.465116 | 0.534884 | -47 | 422 | -469 |
| 130 | $2^1 5^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.461538 | 0.538462 | -63 | 422 | -485 |
| 131 | 131^1 | Y | Y | -2 | 0 | 1.0000000 | 0.458015 | 0.541985 | -65 | 422 | -487 |
| 132 | $2^2 3^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.462121 | 0.537879 | -35 | 452 | -487 |
| 133 | $7^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.466165 | 0.533835 | -30 | 457 | -487 |
| 134 | $2^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.470149 | 0.529851 | -25 | 462 | -487 |
| 135 | $3^3 5^1$ | N | N | 9 | 4 | 1.5555556 | 0.474074 | 0.525926 | -16 | 471 | -487 |
| 136 | $2^3 17^1$ | N | N | 9 | 4 | 1.5555556 | 0.477941 | 0.522059 | -7 | 480 | -487 |
| 137 | 137^1 | Y | Y | -2 | 0 | 1.0000000 | 0.474453 | 0.525547 | -9 | 480 | -489 |
| 138 | $2^1 3^1 23^1$ | Y | N | -16 | 0 | 1.0000000 | 0.471014 | 0.528986 | -25 | 480 | -505 |
| 139 | 139^1 | Y | Y | -2 | 0 | 1.0000000 | 0.467626 | 0.532374 | -27 | 480 | -507 |
| 140 | $2^2 5^1 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.471429 | 0.528571 | 3 | 510 | -507 |
| 141 | $3^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475177 | 0.524823 | 8 | 515 | -507 |
| 142 | $2^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478873 | 0.521127 | 13 | 520 | -507 |
| 143 | $11^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482517 | 0.517483 | 18 | 525 | -507 |
| 144 | $2^4 3^2$ | N | N | 34 | 29 | 1.6176471 | 0.486111 | 0.513889 | 52 | 559 | -507 |
| 145 | $5^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.489655 | 0.510345 | 57 | 564 | -507 |
| 146 | $2^1 73^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493151 | 0.506849 | 62 | 569 | -507 |
| 147 | $3^1 7^2$ | N | N | -7 | 2 | 1.2857143 | 0.489796 | 0.510204 | 55 | 569 | -514 |
| 148 | $2^2 37^1$ | N | N | -7 | 2 | 1.2857143 | 0.486486 | 0.513514 | 48 | 569 | -521 |
| 149 | 149^1 | Y | Y | -2 | 0 | 1.0000000 | 0.483221 | 0.516779 | 46 | 569 | -523 |
| 150 | $2^1 3^1 5^2$ | N | N | 30 | 14 | 1.1666667 | 0.486667 | 0.513333 | 76 | 599 | -523 |
| 151 | 151^1 | Y | Y | -2 | 0 | 1.0000000 | 0.483444 | 0.516556 | 74 | 599 | -525 |
| 152 | $2^3 19^1$ | N | N | 9 | 4 | 1.5555556 | 0.486842 | 0.513158 | 83 | 608 | -525 |
| 153 | $3^2 17^1$ | N | N | -7 | 2 | 1.2857143 | 0.483660 | 0.516340 | 76 | 608 | -532 |
| 154 | $2^1 7^1 11^1$ | Y | N | -16 | 0 | 1.0000000 | 0.480519 | 0.519481 | 60 | 608 | -548 |
| 155 | $5^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 65 | 613 | -548 |
| 156 | $2^2 3^1 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.487179 | 0.512821 | 95 | 643 | -548 |
| 157 | 157^1 | Y | Y | -2 | 0 | 1.0000000 | 0.484076 | 0.515924 | 93 | 643 | -550 |
| 158 | $2^1 79^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487342 | 0.512658 | 98 | 648 | -550 |
| 159 | $3^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490566 | 0.509434 | 103 | 653 | -550 |
| 160 | $2^5 5^1$ | N | N | 13 | 8 | 2.0769231 | 0.493750 | 0.506250 | 116 | 666 | -550 |
| 161 | $7^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.496894 | 0.503106 | 121 | 671 | -550 |
| 162 | $2^1 3^4$ | N | N | -11 | 6 | 1.8181818 | 0.493827 | 0.506173 | 110 | 671 | -561 |
| 163 | 163^1 | Y | Y | -2 | 0 | 1.0000000 | 0.490798 | 0.509202 | 108 | 671 | -563 |
| 164 | $2^2 41^1$ | N | N | -7 | 2 | 1.2857143 | 0.487805 | 0.512195 | 101 | 671 | -570 |
| 165 | $3^1 5^1 11^1$ | Y | N | -16 | 0 | 1.0000000 | 0.484848 | 0.515152 | 85 | 671 | -586 |
| 166 | $2^1 83^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487952 | 0.512048 | 90 | 676 | -586 |
| 167 | 167^1 | Y | Y | -2 | 0 | 1.0000000 | 0.485030 | 0.514970 | 88 | 676 | -588 |
| 168 | $2^3 3^1 7^1$ | N | N | -48 | 32 | 1.3333333 | 0.482143 | 0.517857 | 40 | 676 | -636 |
| 169 | 13^2 | N | Y | 2 | 0 | 1.5000000 | 0.485207 | 0.514793 | 42 | 678 | -636 |
| 170 | $2^1 5^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.482353 | 0.517647 | 26 | 678 | -652 |
| 171 | $3^2 19^1$ | N | N | -7 | 2 | 1.2857143 | 0.479532 | 0.520468 | 19 | 678 | -659 |
| 172 | $2^2 43^1$ | N | N | -7 | 2 | 1.2857143 | 0.476744 | 0.523256 | 12 | 678 | -666 |
| 173 | 173^1 | Y | Y | -2 | 0 | 1.0000000 | 0.473988 | 0.526012 | 10 | 678 | -668 |
| 174 | $2^1 3^1 29^1$ | Y | N | -16 | 0 | 1.0000000 | 0.471264 | 0.528736 | -6 | 678 | -684 |
| 175 | $5^2 7^1$ | N | N | -7 | 2 | 1.2857143 | 0.468571 | 0.531429 | -13 | 678 | -691 |
| 176 | $2^4 11^1$ | N | N | -11 | 6 | 1.8181818 | 0.465909 | 0.534091 | -24 | 678 | -702 |
| 177 | $3^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.468927 | 0.531073 | -19 | 683 | -702 |
| 178 | $2^1 89^1$ | Y | N | 5 | 0 | 1.0000000 | 0.471910 | 0.528090 | -14 | 688 | -702 |
| 179 | 179^1 | Y | Y | -2 | 0 | 1.0000000 | 0.469274 | 0.530726 | -16 | 688 | -704 |
| 180 | $2^2 3^2 5^1$ | N | N | -74 | 58 | 1.2162162 | 0.466667 | 0.533333 | -90 | 688 | -778 |
| 181 | 181^1 | Y | Y | -2 | 0 | 1.0000000 | 0.464088 | 0.535912 | -92 | 688 | -780 |
| 182 | $2^1 7^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.461538 | 0.538462 | -108 | 688 | -796 |
| 183 | $3^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.464481 | 0.535519 | -103 | 693 | -796 |
| 184 | $2^3 23^1$ | N | N | 9 | 4 | 1.5555556 | 0.467391 | 0.532609 | -94 | 702 | -796 |
| 185 | $5^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.470270 | 0.529730 | -89 | 707 | -796 |
| 186 | $2^1 3^1 31^1$ | Y | N | -16 | 0 | 1.0000000 | 0.467742 | 0.532258 | -105 | 707 | -812 |
| 187 | $11^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.470588 | 0.529412 | -100 | 712 | -812 |
| 188 | $2^2 47^1$ | N | N | -7 | 2 | 1.2857143 | 0.468085 | 0.531915 | -107 | 712 | -819 |
| 189 | $3^3 7^1$ | N | N | 9 | 4 | 1.5555556 | 0.470899 | 0.529101 | -98 | 721 | -819 |
| 190 | $2^1 5^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.468421 | 0.531579 | -114 | 721 | -835 |
| 191 | 191^1 | Y | Y | -2 | 0 | 1.0000000 | 0.465969 | 0.534031 | -116 | 721 | -837 |
| 192 | $2^6 3^1$ | N | N | -15 | 10 | 2.3333333 | 0.463542 | 0.536458 | -131 | 721 | -852 |
| 193 | 193^1 | Y | Y | -2 | 0 | 1.0000000 | 0.461140 | 0.538860 | -133 | 721 | -854 |
| 194 | $2^1 97^1$ | Y | N | 5 | 0 | 1.0000000 | 0.463918 | 0.536082 | -128 | 726 | -854 |
| 195 | $3^1 5^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.461538 | 0.538462 | -144 | 726 | -870 |
| 196 | $2^2 7^2$ | N | N | 14 | 9 | 1.3571429 | 0.464286 | 0.535714 | -130 | 740 | -870 |
| 197 | 197^1 | Y | Y | -2 | 0 | 1.0000000 | 0.461929 | 0.538071 | -132 | 740 | -872 |
| 198 | $2^1 3^2 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.464646 | 0.535354 | -102 | 770 | -872 |
| 199 | 199^1 | Y | Y | -2 | 0 | 1.0000000 | 0.462312 | 0.537688 | -104 | 770 | -874 |
| 200 | $2^3 5^2$ | N | N | -23 | 18 | 1.4782609 | 0.460000 | 0.540000 | -127 | 770 | -897 |

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\Sigma_{d n} C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|-------------------|--------|--------|-------------|--|---|--------------------|--------------------|-------------|---------------|---------------|
| 201 | $3^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.462687 | 0.537313 | -122 | 775 | -897 |
| 202 | $2^1 101^1$ | Y | N | 5 | 0 | 1.0000000 | 0.465347 | 0.534653 | -117 | 780 | -897 |
| 203 | $7^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467980 | 0.532020 | -112 | 785 | -897 |
| 204 | $2^2 3^1 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.470588 | 0.529412 | -82 | 815 | -897 |
| 205 | $5^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473171 | 0.526829 | -77 | 820 | -897 |
| 206 | $2^1 103^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475728 | 0.524272 | -72 | 825 | -897 |
| 207 | $3^2 23^1$ | N | N | -7 | 2 | 1.2857143 | 0.473430 | 0.526570 | -79 | 825 | -904 |
| 208 | $2^4 13^1$ | N | N | -11 | 6 | 1.8181818 | 0.471154 | 0.528846 | -90 | 825 | -915 |
| 209 | $11^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473684 | 0.526316 | -85 | 830 | -915 |
| 210 | $2^1 3^1 5^1 7^1$ | Y | N | 65 | 0 | 1.0000000 | 0.476190 | 0.523810 | -20 | 895 | -915 |
| 211 | 211^1 | Y | Y | -2 | 0 | 1.0000000 | 0.473934 | 0.526066 | -22 | 895 | -917 |
| 212 | $2^2 53^1$ | N | N | -7 | 2 | 1.2857143 | 0.471698 | 0.528302 | -29 | 895 | -924 |
| 213 | $3^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.474178 | 0.525822 | -24 | 900 | -924 |
| 214 | $2^1 107^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476636 | 0.523364 | -19 | 905 | -924 |
| 215 | $5^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.479070 | 0.520930 | -14 | 910 | -924 |
| 216 | $2^3 3^3$ | N | N | 46 | 41 | 1.5000000 | 0.481481 | 0.518519 | 32 | 956 | -924 |
| 217 | $7^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 37 | 961 | -924 |
| 218 | $2^1 109^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486239 | 0.513761 | 42 | 966 | -924 |
| 219 | $3^1 73^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488584 | 0.511416 | 47 | 971 | -924 |
| 220 | $2^2 5^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.490909 | 0.509091 | 77 | 1001 | -924 |
| 221 | $13^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493213 | 0.506787 | 82 | 1006 | -924 |
| 222 | $2^1 3^1 37^1$ | Y | N | -16 | 0 | 1.0000000 | 0.490991 | 0.509009 | 66 | 1006 | -940 |
| 223 | 223^1 | Y | Y | -2 | 0 | 1.0000000 | 0.488789 | 0.511211 | 64 | 1006 | -942 |
| 224 | $2^5 7^1$ | N | N | 13 | 8 | 2.0769231 | 0.491071 | 0.508929 | 77 | 1019 | -942 |
| 225 | $3^2 5^2$ | N | N | 14 | 9 | 1.3571429 | 0.493333 | 0.506667 | 91 | 1033 | -942 |
| 226 | $2^1 113^1$ | Y | N | 5 | 0 | 1.0000000 | 0.495575 | 0.504425 | 96 | 1038 | -942 |
| 227 | 227^1 | Y | Y | -2 | 0 | 1.0000000 | 0.493392 | 0.506608 | 94 | 1038 | -944 |
| 228 | $2^2 3^1 19^1$ | N | N | 30 | 14 | 1.1666667 | 0.495614 | 0.504386 | 124 | 1068 | -944 |
| 229 | 229^1 | Y | Y | -2 | 0 | 1.0000000 | 0.493450 | 0.506550 | 122 | 1068 | -946 |
| 230 | $2^1 5^1 23^1$ | Y | N | -16 | 0 | 1.0000000 | 0.491304 | 0.508696 | 106 | 1068 | -962 |
| 231 | $3^1 7^1 11^1$ | Y | N | -16 | 0 | 1.0000000 | 0.489177 | 0.510823 | 90 | 1068 | -978 |
| 232 | $2^3 29^1$ | N | N | 9 | 4 | 1.5555556 | 0.491379 | 0.508621 | 99 | 1077 | -978 |
| 233 | 233^1 | Y | Y | -2 | 0 | 1.0000000 | 0.489270 | 0.510730 | 97 | 1077 | -980 |
| 234 | $2^1 3^2 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.491453 | 0.508547 | 127 | 1107 | -980 |
| 235 | $5^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493617 | 0.506383 | 132 | 1112 | -980 |
| 236 | $2^2 59^1$ | N | N | -7 | 2 | 1.2857143 | 0.491525 | 0.508475 | 125 | 1112 | -987 |
| 237 | $3^1 79^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493671 | 0.506329 | 130 | 1117 | -987 |
| 238 | $2^1 7^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.491597 | 0.508403 | 114 | 1117 | -1003 |
| 239 | 239^1 | Y | Y | -2 | 0 | 1.0000000 | 0.489540 | 0.510460 | 112 | 1117 | -1005 |
| 240 | $2^4 3^1 5^1$ | N | N | 70 | 54 | 1.5000000 | 0.491667 | 0.508333 | 182 | 1187 | -1005 |
| 241 | 241^1 | Y | Y | -2 | 0 | 1.0000000 | 0.489627 | 0.510373 | 180 | 1187 | -1007 |
| 242 | $2^1 11^2$ | N | N | -7 | 2 | 1.2857143 | 0.487603 | 0.512397 | 173 | 1187 | -1014 |
| 243 | 3^5 | N | Y | -2 | 0 | 3.0000000 | 0.485597 | 0.514403 | 171 | 1187 | -1016 |
| 244 | $2^2 61^1$ | N | N | -7 | 2 | 1.2857143 | 0.483607 | 0.516393 | 164 | 1187 | -1023 |
| 245 | $5^1 7^2$ | N | N | -7 | 2 | 1.2857143 | 0.481633 | 0.518367 | 157 | 1187 | -1030 |
| 246 | $2^1 3^1 41^1$ | Y | N | -16 | 0 | 1.0000000 | 0.479675 | 0.520325 | 141 | 1187 | -1046 |
| 247 | $13^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.481781 | 0.518219 | 146 | 1192 | -1046 |
| 248 | $2^3 31^1$ | N | N | 9 | 4 | 1.5555556 | 0.483871 | 0.516129 | 155 | 1201 | -1046 |
| 249 | $3^1 83^1$ | Y | N | 5 | 0 | 1.0000000 | 0.485944 | 0.514056 | 160 | 1206 | -1046 |
| 250 | $2^1 5^3$ | N | N | 9 | 4 | 1.5555556 | 0.488000 | 0.512000 | 169 | 1215 | -1046 |
| 251 | 251^1 | Y | Y | -2 | 0 | 1.0000000 | 0.486056 | 0.513944 | 167 | 1215 | -1048 |
| 252 | $2^2 3^2 7^1$ | N | N | -74 | 58 | 1.2162162 | 0.484127 | 0.515873 | 93 | 1215 | -1122 |
| 253 | $11^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486166 | 0.513834 | 98 | 1220 | -1122 |
| 254 | $2^1 127^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488189 | 0.511811 | 103 | 1225 | -1122 |
| 255 | $3^1 5^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.486275 | 0.513725 | 87 | 1225 | -1138 |
| 256 | 2^8 | N | Y | 2 | 0 | 4.5000000 | 0.488281 | 0.511719 | 89 | 1227 | -1138 |
| 257 | 257^1 | Y | Y | -2 | 0 | 1.0000000 | 0.486381 | 0.513619 | 87 | 1227 | -1140 |
| 258 | $2^1 3^1 43^1$ | Y | N | -16 | 0 | 1.0000000 | 0.484496 | 0.515504 | 71 | 1227 | -1156 |
| 259 | $7^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486486 | 0.513514 | 76 | 1232 | -1156 |
| 260 | $2^2 5^1 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.488462 | 0.511538 | 106 | 1262 | -1156 |
| 261 | $3^2 29^1$ | N | N | -7 | 2 | 1.2857143 | 0.486590 | 0.513410 | 99 | 1262 | -1163 |
| 262 | $2^1 131^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488550 | 0.511450 | 104 | 1267 | -1163 |
| 263 | 263^1 | Y | Y | -2 | 0 | 1.0000000 | 0.486692 | 0.513308 | 102 | 1267 | -1165 |
| 264 | $2^3 3^1 11^1$ | N | N | -48 | 32 | 1.3333333 | 0.484848 | 0.515152 | 54 | 1267 | -1213 |
| 265 | $5^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486792 | 0.513208 | 59 | 1272 | -1213 |
| 266 | $2^1 7^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.484962 | 0.515038 | 43 | 1272 | -1229 |
| 267 | $3^1 89^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486891 | 0.513109 | 48 | 1277 | -1229 |
| 268 | $2^2 67^1$ | N | N | -7 | 2 | 1.2857143 | 0.485075 | 0.514925 | 41 | 1277 | -1236 |
| 269 | 269^1 | Y | Y | -2 | 0 | 1.0000000 | 0.483271 | 0.516729 | 39 | 1277 | -1238 |
| 270 | $2^1 3^3 5^1$ | N | N | -48 | 32 | 1.3333333 | 0.481481 | 0.518519 | -9 | 1277 | -1286 |
| 271 | 271^1 | Y | Y | -2 | 0 | 1.0000000 | 0.479705 | 0.520295 | -11 | 1277 | -1288 |
| 272 | $2^4 17^1$ | N | N | -11 | 6 | 1.8181818 | 0.477941 | 0.522059 | -22 | 1277 | -1299 |
| 273 | $3^1 7^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.476190 | 0.523810 | -38 | 1277 | -1315 |
| 274 | $2^1 137^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478102 | 0.521898 | -33 | 1282 | -1315 |
| 275 | $5^2 11^1$ | N | N | -7 | 2 | 1.2857143 | 0.476364 | 0.523636 | -40 | 1282 | -1322 |
| 276 | $2^2 3^1 23^1$ | N | N | 30 | 14 | 1.1666667 | 0.478261 | 0.521739 | -10 | 1312 | -1322 |
| 277 | 277^1 | Y | Y | -2 | 0 | 1.0000000 | 0.476534 | 0.523466 | -12 | 1312 | -1324 |

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\Sigma d n C_{\Omega(d)}^{(d)}}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|--------------------|--------|--------|-------------|--|--|--------------------|--------------------|-------------|---------------|---------------|
| 278 | $2^1 139^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478417 | 0.521583 | -7 | 1317 | -1324 |
| 279 | $3^2 31^1$ | N | N | -7 | 2 | 1.2857143 | 0.476703 | 0.523297 | -14 | 1317 | -1331 |
| 280 | $2^3 5^1 7^1$ | N | N | -48 | 32 | 1.3333333 | 0.475000 | 0.525000 | -62 | 1317 | -1379 |
| 281 | 281^1 | Y | Y | -2 | 0 | 1.0000000 | 0.473310 | 0.526690 | -64 | 1317 | -1381 |
| 282 | $2^1 3^1 47^1$ | Y | N | -16 | 0 | 1.0000000 | 0.471631 | 0.528369 | -80 | 1317 | -1397 |
| 283 | 283^1 | Y | Y | -2 | 0 | 1.0000000 | 0.469965 | 0.530035 | -82 | 1317 | -1399 |
| 284 | $2^2 71^1$ | N | N | -7 | 2 | 1.2857143 | 0.468310 | 0.531690 | -89 | 1317 | -1406 |
| 285 | $3^1 5^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.466667 | 0.533333 | -105 | 1317 | -1422 |
| 286 | $2^1 11^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.465035 | 0.534965 | -121 | 1317 | -1438 |
| 287 | $7^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.466899 | 0.533101 | -116 | 1322 | -1438 |
| 288 | $2^5 3^2$ | N | N | -47 | 42 | 1.7659574 | 0.465278 | 0.534722 | -163 | 1322 | -1485 |
| 289 | 17^2 | N | Y | 2 | 0 | 1.5000000 | 0.467128 | 0.532872 | -161 | 1324 | -1485 |
| 290 | $2^1 5^1 29^1$ | Y | N | -16 | 0 | 1.0000000 | 0.465517 | 0.534483 | -177 | 1324 | -1501 |
| 291 | $3^1 97^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467354 | 0.532646 | -172 | 1329 | -1501 |
| 292 | $2^2 73^1$ | N | N | -7 | 2 | 1.2857143 | 0.465753 | 0.534247 | -179 | 1329 | -1508 |
| 293 | 293^1 | Y | Y | -2 | 0 | 1.0000000 | 0.464164 | 0.535836 | -181 | 1329 | -1510 |
| 294 | $2^1 3^1 7^2$ | N | N | 30 | 14 | 1.1666667 | 0.465986 | 0.534014 | -151 | 1359 | -1510 |
| 295 | $5^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467797 | 0.532203 | -146 | 1364 | -1510 |
| 296 | $2^3 37^1$ | N | N | 9 | 4 | 1.5555556 | 0.469595 | 0.530405 | -137 | 1373 | -1510 |
| 297 | $3^3 11^1$ | N | N | 9 | 4 | 1.5555556 | 0.471380 | 0.528620 | -128 | 1382 | -1510 |
| 298 | $2^1 149^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473154 | 0.526846 | -123 | 1387 | -1510 |
| 299 | $13^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.474916 | 0.525084 | -118 | 1392 | -1510 |
| 300 | $2^2 3^1 5^2$ | N | N | -74 | 58 | 1.2162162 | 0.473333 | 0.526667 | -192 | 1392 | -1584 |
| 301 | $7^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475083 | 0.524917 | -187 | 1397 | -1584 |
| 302 | $2^1 151^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476821 | 0.523179 | -182 | 1402 | -1584 |
| 303 | $3^1 101^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478548 | 0.521452 | -177 | 1407 | -1584 |
| 304 | $2^4 19^1$ | N | N | -11 | 6 | 1.8181818 | 0.476974 | 0.523026 | -188 | 1407 | -1595 |
| 305 | $5^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478689 | 0.521311 | -183 | 1412 | -1595 |
| 306 | $2^1 3^2 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.480392 | 0.519608 | -153 | 1442 | -1595 |
| 307 | 307^1 | Y | Y | -2 | 0 | 1.0000000 | 0.478827 | 0.521173 | -155 | 1442 | -1597 |
| 308 | $2^2 7^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.480519 | 0.519481 | -125 | 1472 | -1597 |
| 309 | $3^1 103^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482201 | 0.517799 | -120 | 1477 | -1597 |
| 310 | $2^1 5^1 31^1$ | Y | N | -16 | 0 | 1.0000000 | 0.480645 | 0.519355 | -136 | 1477 | -1613 |
| 311 | 311^1 | Y | Y | -2 | 0 | 1.0000000 | 0.479100 | 0.520900 | -138 | 1477 | -1615 |
| 312 | $2^3 3^1 13^1$ | N | N | -48 | 32 | 1.3333333 | 0.477564 | 0.522436 | -186 | 1477 | -1663 |
| 313 | 313^1 | Y | Y | -2 | 0 | 1.0000000 | 0.476038 | 0.523962 | -188 | 1477 | -1665 |
| 314 | $2^1 157^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477707 | 0.522293 | -183 | 1482 | -1665 |
| 315 | $3^2 5^1 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.479365 | 0.520635 | -153 | 1512 | -1665 |
| 316 | $2^2 79^1$ | N | N | -7 | 2 | 1.2857143 | 0.477848 | 0.522152 | -160 | 1512 | -1672 |
| 317 | 317^1 | Y | Y | -2 | 0 | 1.0000000 | 0.476341 | 0.523659 | -162 | 1512 | -1674 |
| 318 | $2^1 3^1 53^1$ | Y | N | -16 | 0 | 1.0000000 | 0.474843 | 0.525157 | -178 | 1512 | -1690 |
| 319 | $11^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476489 | 0.523511 | -173 | 1517 | -1690 |
| 320 | $2^6 5^1$ | N | N | -15 | 10 | 2.3333333 | 0.475000 | 0.525000 | -188 | 1517 | -1705 |
| 321 | $3^1 107^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476636 | 0.523364 | -183 | 1522 | -1705 |
| 322 | $2^1 7^1 23^1$ | Y | N | -16 | 0 | 1.0000000 | 0.475155 | 0.524845 | -199 | 1522 | -1721 |
| 323 | $17^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476780 | 0.523220 | -194 | 1527 | -1721 |
| 324 | $2^2 3^4$ | N | N | 34 | 29 | 1.6176471 | 0.478395 | 0.521605 | -160 | 1561 | -1721 |
| 325 | $5^2 13^1$ | N | N | -7 | 2 | 1.2857143 | 0.476923 | 0.523077 | -167 | 1561 | -1728 |
| 326 | $2^1 163^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478528 | 0.521472 | -162 | 1566 | -1728 |
| 327 | $3^1 109^1$ | Y | N | 5 | 0 | 1.0000000 | 0.480122 | 0.519878 | -157 | 1571 | -1728 |
| 328 | $2^3 41^1$ | N | N | 9 | 4 | 1.5555556 | 0.481707 | 0.518293 | -148 | 1580 | -1728 |
| 329 | $7^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483283 | 0.516717 | -143 | 1585 | -1728 |
| 330 | $2^1 3^1 5^1 11^1$ | Y | N | 65 | 0 | 1.0000000 | 0.484848 | 0.515152 | -78 | 1650 | -1728 |
| 331 | 331^1 | Y | Y | -2 | 0 | 1.0000000 | 0.483384 | 0.516616 | -80 | 1650 | -1730 |
| 332 | $2^2 83^1$ | N | N | -7 | 2 | 1.2857143 | 0.481928 | 0.518072 | -87 | 1650 | -1737 |
| 333 | $3^2 37^1$ | N | N | -7 | 2 | 1.2857143 | 0.480480 | 0.519520 | -94 | 1650 | -1744 |
| 334 | $2^1 167^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482036 | 0.517964 | -89 | 1655 | -1744 |
| 335 | $5^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483582 | 0.516418 | -84 | 1660 | -1744 |
| 336 | $2^4 3^1 7^1$ | N | N | 70 | 54 | 1.5000000 | 0.485119 | 0.514881 | -14 | 1730 | -1744 |
| 337 | 337^1 | Y | Y | -2 | 0 | 1.0000000 | 0.483680 | 0.516320 | -16 | 1730 | -1746 |
| 338 | $2^1 13^2$ | N | N | -7 | 2 | 1.2857143 | 0.482249 | 0.517751 | -23 | 1730 | -1753 |
| 339 | $3^1 113^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483776 | 0.516224 | -18 | 1735 | -1753 |
| 340 | $2^2 5^1 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.485294 | 0.514706 | 12 | 1765 | -1753 |
| 341 | $11^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486804 | 0.513196 | 17 | 1770 | -1753 |
| 342 | $2^1 3^2 19^1$ | N | N | 30 | 14 | 1.1666667 | 0.488304 | 0.511696 | 47 | 1800 | -1753 |
| 343 | 7^3 | N | Y | -2 | 0 | 2.0000000 | 0.486880 | 0.513120 | 45 | 1800 | -1755 |
| 344 | $2^3 43^1$ | N | N | 9 | 4 | 1.5555556 | 0.488372 | 0.511628 | 54 | 1809 | -1755 |
| 345 | $3^1 5^1 23^1$ | Y | N | -16 | 0 | 1.0000000 | 0.486957 | 0.513043 | 38 | 1809 | -1771 |
| 346 | $2^1 173^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488439 | 0.511561 | 43 | 1814 | -1771 |
| 347 | 347^1 | Y | Y | -2 | 0 | 1.0000000 | 0.487032 | 0.512968 | 41 | 1814 | -1773 |
| 348 | $2^2 3^1 29^1$ | N | N | 30 | 14 | 1.1666667 | 0.488506 | 0.511494 | 71 | 1844 | -1773 |
| 349 | 349^1 | Y | Y | -2 | 0 | 1.0000000 | 0.487106 | 0.512894 | 69 | 1844 | -1775 |
| 350 | $2^1 5^2 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.488571 | 0.511429 | 99 | 1874 | -1775 |

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum d n C_{\Omega(d)}^{(d)}}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|--------------------|--------|--------|-------------|--|--|--------------------|--------------------|-------------|---------------|---------------|
| 351 | $3^3 13^1$ | N | N | 9 | 4 | 1.5555556 | 0.490028 | 0.509972 | 108 | 1883 | -1775 |
| 352 | $2^5 11^1$ | N | N | 13 | 8 | 2.0769231 | 0.491477 | 0.508523 | 121 | 1896 | -1775 |
| 353 | 353^1 | Y | Y | -2 | 0 | 1.0000000 | 0.490085 | 0.509915 | 119 | 1896 | -1777 |
| 354 | $2^1 3^1 59^1$ | Y | N | -16 | 0 | 1.0000000 | 0.488701 | 0.511299 | 103 | 1896 | -1793 |
| 355 | $5^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490141 | 0.509859 | 108 | 1901 | -1793 |
| 356 | $2^2 89^1$ | N | N | -7 | 2 | 1.2857143 | 0.488764 | 0.511236 | 101 | 1901 | -1800 |
| 357 | $3^1 7^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.487395 | 0.512605 | 85 | 1901 | -1816 |
| 358 | $2^1 179^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488827 | 0.511173 | 90 | 1906 | -1816 |
| 359 | 359^1 | Y | Y | -2 | 0 | 1.0000000 | 0.487465 | 0.512535 | 88 | 1906 | -1818 |
| 360 | $2^3 3^2 5^1$ | N | N | 145 | 129 | 1.3034483 | 0.488889 | 0.511111 | 233 | 2051 | -1818 |
| 361 | 19^2 | N | Y | 2 | 0 | 1.5000000 | 0.490305 | 0.509695 | 235 | 2053 | -1818 |
| 362 | $2^1 181^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491713 | 0.508287 | 240 | 2058 | -1818 |
| 363 | $3^1 11^2$ | N | N | -7 | 2 | 1.2857143 | 0.490358 | 0.509642 | 233 | 2058 | -1825 |
| 364 | $2^2 7^1 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.491758 | 0.508242 | 263 | 2088 | -1825 |
| 365 | $5^1 73^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493151 | 0.506849 | 268 | 2093 | -1825 |
| 366 | $2^1 3^1 61^1$ | Y | N | -16 | 0 | 1.0000000 | 0.491803 | 0.508197 | 252 | 2093 | -1841 |
| 367 | 367^1 | Y | Y | -2 | 0 | 1.0000000 | 0.490463 | 0.509537 | 250 | 2093 | -1843 |
| 368 | $2^4 23^1$ | N | N | -11 | 6 | 1.8181818 | 0.489130 | 0.510870 | 239 | 2093 | -1854 |
| 369 | $3^2 41^1$ | N | N | -7 | 2 | 1.2857143 | 0.487805 | 0.512195 | 232 | 2093 | -1861 |
| 370 | $2^1 5^1 37^1$ | Y | N | -16 | 0 | 1.0000000 | 0.486486 | 0.513514 | 216 | 2093 | -1877 |
| 371 | $7^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487871 | 0.512129 | 221 | 2098 | -1877 |
| 372 | $2^2 3^1 31^1$ | N | N | 30 | 14 | 1.1666667 | 0.489247 | 0.510753 | 251 | 2128 | -1877 |
| 373 | 373^1 | Y | Y | -2 | 0 | 1.0000000 | 0.487936 | 0.512064 | 249 | 2128 | -1879 |
| 374 | $2^1 11^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.486631 | 0.513369 | 233 | 2128 | -1895 |
| 375 | $3^1 5^3$ | N | N | 9 | 4 | 1.5555556 | 0.488000 | 0.512000 | 242 | 2137 | -1895 |
| 376 | $2^3 47^1$ | N | N | 9 | 4 | 1.5555556 | 0.489362 | 0.510638 | 251 | 2146 | -1895 |
| 377 | $13^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490716 | 0.509284 | 256 | 2151 | -1895 |
| 378 | $2^1 3^3 7^1$ | N | N | -48 | 32 | 1.3333333 | 0.489418 | 0.510582 | 208 | 2151 | -1943 |
| 379 | 379^1 | Y | Y | -2 | 0 | 1.0000000 | 0.488127 | 0.511873 | 206 | 2151 | -1945 |
| 380 | $2^2 5^1 19^1$ | N | N | 30 | 14 | 1.1666667 | 0.489474 | 0.510526 | 236 | 2181 | -1945 |
| 381 | $3^1 127^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490814 | 0.509186 | 241 | 2186 | -1945 |
| 382 | $2^1 191^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492147 | 0.507853 | 246 | 2191 | -1945 |
| 383 | 383^1 | Y | Y | -2 | 0 | 1.0000000 | 0.490862 | 0.509138 | 244 | 2191 | -1947 |
| 384 | $2^7 3^1$ | N | N | 17 | 12 | 2.5882353 | 0.492188 | 0.507812 | 261 | 2208 | -1947 |
| 385 | $5^1 7^1 11^1$ | Y | N | -16 | 0 | 1.0000000 | 0.490909 | 0.509091 | 245 | 2208 | -1963 |
| 386 | $2^1 193^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492228 | 0.507772 | 250 | 2213 | -1963 |
| 387 | $3^2 43^1$ | N | N | -7 | 2 | 1.2857143 | 0.490956 | 0.509044 | 243 | 2213 | -1970 |
| 388 | $2^2 97^1$ | N | N | -7 | 2 | 1.2857143 | 0.489691 | 0.510309 | 236 | 2213 | -1977 |
| 389 | 389^1 | Y | Y | -2 | 0 | 1.0000000 | 0.488432 | 0.511568 | 234 | 2213 | -1979 |
| 390 | $2^1 3^1 5^1 13^1$ | Y | N | 65 | 0 | 1.0000000 | 0.489744 | 0.510256 | 299 | 2278 | -1979 |
| 391 | $17^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491049 | 0.508951 | 304 | 2283 | -1979 |
| 392 | $2^3 7^2$ | N | N | -23 | 18 | 1.4782609 | 0.489796 | 0.510204 | 281 | 2283 | -2002 |
| 393 | $3^1 131^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491094 | 0.508906 | 286 | 2288 | -2002 |
| 394 | $2^1 197^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492386 | 0.507614 | 291 | 2293 | -2002 |
| 395 | $5^1 79^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493671 | 0.506329 | 296 | 2298 | -2002 |
| 396 | $2^2 3^2 11^1$ | N | N | -74 | 58 | 1.2162162 | 0.492424 | 0.507576 | 222 | 2298 | -2076 |
| 397 | 397^1 | Y | Y | -2 | 0 | 1.0000000 | 0.491184 | 0.508816 | 220 | 2298 | -2078 |
| 398 | $2^1 199^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492462 | 0.507538 | 225 | 2303 | -2078 |
| 399 | $3^1 7^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.491228 | 0.508772 | 209 | 2303 | -2094 |
| 400 | $2^4 5^2$ | N | N | 34 | 29 | 1.6176471 | 0.492500 | 0.507500 | 243 | 2337 | -2094 |
| 401 | 401^1 | Y | Y | -2 | 0 | 1.0000000 | 0.491272 | 0.508728 | 241 | 2337 | -2096 |
| 402 | $2^1 3^1 67^1$ | Y | N | -16 | 0 | 1.0000000 | 0.490050 | 0.509950 | 225 | 2337 | -2112 |
| 403 | $13^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491315 | 0.508685 | 230 | 2342 | -2112 |
| 404 | $2^2 101^1$ | N | N | -7 | 2 | 1.2857143 | 0.490099 | 0.509901 | 223 | 2342 | -2119 |
| 405 | $3^4 5^1$ | N | N | -11 | 6 | 1.8181818 | 0.488889 | 0.511111 | 212 | 2342 | -2130 |
| 406 | $2^1 7^1 29^1$ | Y | N | -16 | 0 | 1.0000000 | 0.487685 | 0.512315 | 196 | 2342 | -2146 |
| 407 | $11^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488943 | 0.511057 | 201 | 2347 | -2146 |
| 408 | $2^3 3^1 17^1$ | N | N | -48 | 32 | 1.3333333 | 0.487745 | 0.512255 | 153 | 2347 | -2194 |
| 409 | 409^1 | Y | Y | -2 | 0 | 1.0000000 | 0.486553 | 0.513447 | 151 | 2347 | -2196 |
| 410 | $2^1 5^1 41^1$ | Y | N | -16 | 0 | 1.0000000 | 0.485366 | 0.514634 | 135 | 2347 | -2212 |
| 411 | $3^1 137^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486618 | 0.513382 | 140 | 2352 | -2212 |
| 412 | $2^2 103^1$ | N | N | -7 | 2 | 1.2857143 | 0.485437 | 0.514563 | 133 | 2352 | -2219 |
| 413 | $7^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486683 | 0.513317 | 138 | 2357 | -2219 |
| 414 | $2^1 3^2 23^1$ | N | N | 30 | 14 | 1.1666667 | 0.487923 | 0.512077 | 168 | 2387 | -2219 |
| 415 | $5^1 83^1$ | Y | N | 5 | 0 | 1.0000000 | 0.489157 | 0.510843 | 173 | 2392 | -2219 |
| 416 | $2^5 13^1$ | N | N | 13 | 8 | 2.0769231 | 0.490385 | 0.509615 | 186 | 2405 | -2219 |
| 417 | $3^1 139^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491607 | 0.508393 | 191 | 2410 | -2219 |
| 418 | $2^1 11^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.490431 | 0.509569 | 175 | 2410 | -2235 |
| 419 | 419^1 | Y | Y | -2 | 0 | 1.0000000 | 0.489260 | 0.510740 | 173 | 2410 | -2237 |
| 420 | $2^2 3^1 5^1 7^1$ | N | N | -155 | 90 | 1.1032258 | 0.488095 | 0.511905 | 18 | 2410 | -2392 |
| 421 | 421^1 | Y | Y | -2 | 0 | 1.0000000 | 0.486936 | 0.513064 | 16 | 2410 | -2394 |
| 422 | $2^1 211^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488152 | 0.511848 | 21 | 2415 | -2394 |
| 423 | $3^2 47^1$ | N | N | -7 | 2 | 1.2857143 | 0.486998 | 0.513002 | 14 | 2415 | -2401 |
| 424 | $2^3 53^1$ | N | N | 9 | 4 | 1.5555556 | 0.488208 | 0.511792 | 23 | 2424 | -2401 |
| 425 | $5^2 17^1$ | N | N | -7 | 2 | 1.2857143 | 0.487059 | 0.512941 | 16 | 2424 | -2408 |

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum d n C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|--------------------|--------|--------|-------------|--|---|--------------------|--------------------|-------------|---------------|---------------|
| 426 | $2^1 3^1 71^1$ | Y | N | -16 | 0 | 1.0000000 | 0.485915 | 0.514085 | 0 | 2424 | -2424 |
| 427 | $7^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487119 | 0.512881 | 5 | 2429 | -2424 |
| 428 | $2^2 107^1$ | N | N | -7 | 2 | 1.2857143 | 0.485981 | 0.514019 | -2 | 2429 | -2431 |
| 429 | $3^1 11^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.484848 | 0.515152 | -18 | 2429 | -2447 |
| 430 | $2^1 5^1 43^1$ | Y | N | -16 | 0 | 1.0000000 | 0.483721 | 0.516279 | -34 | 2429 | -2463 |
| 431 | 431^1 | Y | Y | -2 | 0 | 1.0000000 | 0.482599 | 0.517401 | -36 | 2429 | -2465 |
| 432 | $2^4 3^3$ | N | N | -80 | 75 | 1.5625000 | 0.481481 | 0.518519 | -116 | 2429 | -2545 |
| 433 | 433^1 | Y | Y | -2 | 0 | 1.0000000 | 0.480370 | 0.519630 | -118 | 2429 | -2547 |
| 434 | $2^1 7^1 31^1$ | Y | N | -16 | 0 | 1.0000000 | 0.479263 | 0.520737 | -134 | 2429 | -2563 |
| 435 | $3^1 5^1 29^1$ | Y | N | -16 | 0 | 1.0000000 | 0.478161 | 0.521839 | -150 | 2429 | -2579 |
| 436 | $2^2 109^1$ | N | N | -7 | 2 | 1.2857143 | 0.477064 | 0.522936 | -157 | 2429 | -2586 |
| 437 | $19^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478261 | 0.521739 | -152 | 2434 | -2586 |
| 438 | $2^1 3^1 73^1$ | Y | N | -16 | 0 | 1.0000000 | 0.477169 | 0.522831 | -168 | 2434 | -2602 |
| 439 | 439^1 | Y | Y | -2 | 0 | 1.0000000 | 0.476082 | 0.523918 | -170 | 2434 | -2604 |
| 440 | $2^3 5^1 11^1$ | N | N | -48 | 32 | 1.3333333 | 0.475000 | 0.525000 | -218 | 2434 | -2652 |
| 441 | $3^2 7^2$ | N | N | 14 | 9 | 1.3571429 | 0.476190 | 0.523810 | -204 | 2448 | -2652 |
| 442 | $2^1 13^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.475113 | 0.524887 | -220 | 2448 | -2668 |
| 443 | 443^1 | Y | Y | -2 | 0 | 1.0000000 | 0.474041 | 0.525959 | -222 | 2448 | -2670 |
| 444 | $2^2 3^1 37^1$ | N | N | 30 | 14 | 1.1666667 | 0.475225 | 0.524775 | -192 | 2478 | -2670 |
| 445 | $5^1 89^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476404 | 0.523596 | -187 | 2483 | -2670 |
| 446 | $2^1 223^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477578 | 0.522422 | -182 | 2488 | -2670 |
| 447 | $3^1 149^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478747 | 0.521253 | -177 | 2493 | -2670 |
| 448 | $2^6 7^1$ | N | N | -15 | 10 | 2.3333333 | 0.477679 | 0.522321 | -192 | 2493 | -2685 |
| 449 | 449^1 | Y | Y | -2 | 0 | 1.0000000 | 0.476615 | 0.523385 | -194 | 2493 | -2687 |
| 450 | $2^1 3^2 5^2$ | N | N | -74 | 58 | 1.2162162 | 0.475556 | 0.524444 | -268 | 2493 | -2761 |
| 451 | $11^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476718 | 0.523282 | -263 | 2498 | -2761 |
| 452 | $2^2 113^1$ | N | N | -7 | 2 | 1.2857143 | 0.475664 | 0.524336 | -270 | 2498 | -2768 |
| 453 | $3^1 151^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476821 | 0.523179 | -265 | 2503 | -2768 |
| 454 | $2^1 227^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477974 | 0.522026 | -260 | 2508 | -2768 |
| 455 | $5^1 7^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.476923 | 0.523077 | -276 | 2508 | -2784 |
| 456 | $2^3 3^1 19^1$ | N | N | -48 | 32 | 1.3333333 | 0.475877 | 0.524123 | -324 | 2508 | -2832 |
| 457 | 457^1 | Y | Y | -2 | 0 | 1.0000000 | 0.474836 | 0.525164 | -326 | 2508 | -2834 |
| 458 | $2^1 229^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475983 | 0.524017 | -321 | 2513 | -2834 |
| 459 | $3^3 17^1$ | N | N | 9 | 4 | 1.5555556 | 0.477124 | 0.522876 | -312 | 2522 | -2834 |
| 460 | $2^2 5^1 23^1$ | N | N | 30 | 14 | 1.1666667 | 0.478261 | 0.521739 | -282 | 2552 | -2834 |
| 461 | 461^1 | Y | Y | -2 | 0 | 1.0000000 | 0.477223 | 0.522777 | -284 | 2552 | -2836 |
| 462 | $2^1 3^1 7^1 11^1$ | Y | N | 65 | 0 | 1.0000000 | 0.478355 | 0.521645 | -219 | 2617 | -2836 |
| 463 | 463^1 | Y | Y | -2 | 0 | 1.0000000 | 0.477322 | 0.522678 | -221 | 2617 | -2838 |
| 464 | $2^4 29^1$ | N | N | -11 | 6 | 1.8181818 | 0.476293 | 0.523707 | -232 | 2617 | -2849 |
| 465 | $3^1 5^1 31^1$ | Y | N | -16 | 0 | 1.0000000 | 0.475269 | 0.524731 | -248 | 2617 | -2865 |
| 466 | $2^1 233^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476395 | 0.523605 | -243 | 2622 | -2865 |
| 467 | 467^1 | Y | Y | -2 | 0 | 1.0000000 | 0.475375 | 0.524625 | -245 | 2622 | -2867 |
| 468 | $2^2 3^2 13^1$ | N | N | -74 | 58 | 1.2162162 | 0.474359 | 0.525641 | -319 | 2622 | -2941 |
| 469 | $7^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475480 | 0.524520 | -314 | 2627 | -2941 |
| 470 | $2^1 5^1 47^1$ | Y | N | -16 | 0 | 1.0000000 | 0.474468 | 0.525532 | -330 | 2627 | -2957 |
| 471 | $3^1 157^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475584 | 0.524416 | -325 | 2632 | -2957 |
| 472 | $2^3 59^1$ | N | N | 9 | 4 | 1.5555556 | 0.476695 | 0.523305 | -316 | 2641 | -2957 |
| 473 | $11^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477801 | 0.522199 | -311 | 2646 | -2957 |
| 474 | $2^1 3^1 79^1$ | Y | N | -16 | 0 | 1.0000000 | 0.476793 | 0.523207 | -327 | 2646 | -2973 |
| 475 | $5^2 19^1$ | N | N | -7 | 2 | 1.2857143 | 0.475789 | 0.524211 | -334 | 2646 | -2980 |
| 476 | $2^2 7^1 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.476891 | 0.523109 | -304 | 2676 | -2980 |
| 477 | $3^2 53^1$ | N | N | -7 | 2 | 1.2857143 | 0.475891 | 0.524109 | -311 | 2676 | -2987 |
| 478 | $2^1 239^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476987 | 0.523013 | -306 | 2681 | -2987 |
| 479 | 479^1 | Y | Y | -2 | 0 | 1.0000000 | 0.475992 | 0.524008 | -308 | 2681 | -2989 |
| 480 | $2^5 3^1 5^1$ | N | N | -96 | 80 | 1.6666667 | 0.475000 | 0.525000 | -404 | 2681 | -3085 |
| 481 | $13^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476091 | 0.523909 | -399 | 2686 | -3085 |
| 482 | $2^1 241^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477178 | 0.522822 | -394 | 2691 | -3085 |
| 483 | $3^1 7^1 23^1$ | Y | N | -16 | 0 | 1.0000000 | 0.476190 | 0.523810 | -410 | 2691 | -3101 |
| 484 | $2^2 11^2$ | N | N | 14 | 9 | 1.3571429 | 0.477273 | 0.522727 | -396 | 2705 | -3101 |
| 485 | $5^1 97^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478351 | 0.521649 | -391 | 2710 | -3101 |
| 486 | $2^1 3^5$ | N | N | 13 | 8 | 2.0769231 | 0.479424 | 0.520576 | -378 | 2723 | -3101 |
| 487 | 487^1 | Y | Y | -2 | 0 | 1.0000000 | 0.478439 | 0.521561 | -380 | 2723 | -3103 |
| 488 | $2^3 61^1$ | N | N | 9 | 4 | 1.5555556 | 0.479508 | 0.520492 | -371 | 2732 | -3103 |
| 489 | $3^1 163^1$ | Y | N | 5 | 0 | 1.0000000 | 0.480573 | 0.519427 | -366 | 2737 | -3103 |
| 490 | $2^1 5^1 7^2$ | N | N | 30 | 14 | 1.1666667 | 0.481633 | 0.518367 | -336 | 2767 | -3103 |
| 491 | 491^1 | Y | Y | -2 | 0 | 1.0000000 | 0.480652 | 0.519348 | -338 | 2767 | -3105 |
| 492 | $2^2 3^1 41^1$ | N | N | 30 | 14 | 1.1666667 | 0.481707 | 0.518293 | -308 | 2797 | -3105 |
| 493 | $17^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482759 | 0.517241 | -303 | 2802 | -3105 |
| 494 | $2^1 13^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.481781 | 0.518219 | -319 | 2802 | -3121 |
| 495 | $3^2 5^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.482828 | 0.517172 | -289 | 2832 | -3121 |
| 496 | $2^4 31^1$ | N | N | -11 | 6 | 1.8181818 | 0.481855 | 0.518145 | -300 | 2832 | -3132 |
| 497 | $7^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482897 | 0.517103 | -295 | 2837 | -3132 |
| 498 | $2^1 3^1 83^1$ | Y | N | -16 | 0 | 1.0000000 | 0.481928 | 0.518072 | -311 | 2837 | -3148 |
| 499 | 499^1 | Y | Y | -2 | 0 | 1.0000000 | 0.480962 | 0.519038 | -313 | 2837 | -3150 |
| 500 | $2^2 5^3$ | N | N | -23 | 18 | 1.4782609 | 0.480000 | 0.520000 | -336 | 2837 | -3173 |