Starting with

$$\sum_{k=1}^{n} \frac{(-1)^k t^{k-1}}{(k-1)!} = -e^{-t} \frac{\Gamma(n,-t)}{(n-1)!},$$

we assume that $n = at + \xi = at + \mathcal{O}(1)$ with n and t being large and $0 < a < 1.763226846$. We apply the third formula in Proposition A.2 with $a = n$, $z = t$ and $1/\lambda = a + \xi/t$, to deduce

$$\sum_{k=1}^{n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{1}{(n-1)!} \frac{t^{n-1}}{1+a} \left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right).$$

Using

$$(n-1)! = \Gamma(at+\xi) = (at)^{n-1/2} e^{-at} \sqrt{2\pi} \left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right),$$

we obtain

$$\sum_{k=1}^{n} \frac{(-1)^k t^{k-1}}{(k-1)!} = \frac{(-1)^n}{a^{n-1}\sqrt{2\pi a t}} \frac{e^{at}}{1+a} \left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right).$$

Taking $t = \log\log x$ and $n = \lfloor a \log\log x \rfloor$, we find

$$\begin{aligned}\frac{x}{\log x} \left| \sum_{k=1}^{\lfloor a \log\log x \rfloor} \frac{(-1)^k (\log\log x)^{k-1}}{(k-1)!} \right| &= \frac{\sqrt{a} x}{(1+a)\sqrt{2\pi}} \frac{(\log x)^{a-1} a^{-n}}{\sqrt{\log\log x}} \left(1 + \mathcal{O}\left(\frac{1}{\log\log x}\right)\right) \\ &= \frac{\sqrt{a} x}{a^{\{a \log\log x\}}(1+a)\sqrt{2\pi}} \frac{(\log x)^{a-1-a\log a}}{\sqrt{\log\log x}} \left(1 + \mathcal{O}\left(\frac{1}{\log\log x}\right)\right)\end{aligned}$$

with $\{w\} = w - \lfloor w \rfloor$.