New characterizations of the summatory function of the Möbius function

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Abstract

The Mertens function, $M(x) := \sum_{n \leq x} \mu(n)$, is defined as the summatory function of the Möbius function for $x \geq 1$. The inverse function sequence $\{g^{-1}(n)\}_{n \geq 1}$ taken with respect to Dirichlet convolution is defined in terms of the strongly additive function $\omega(n)$ that counts the number of distinct prime factors of any integer $n \geq 2$. For large x and $n \leq x$, we associate a natural combinatorial significance to the magnitude of the distinct values of the function $g^{-1}(n)$ that depends directly on the exponent patterns in the prime factorizations of the integers in $\{2, 3, \ldots, x\}$ viewed as multisets.

We prove an Erdös-Kac theorem analog for the distribution of the unsigned sequence $|g^{-1}(n)|$ with a limiting central limit theorem type tendency towards normal as $x \to \infty$. For all $x \geq 1$, discrete convolutions of $G^{-1}(x) := \sum_{n \leq x} \lambda(n)|g^{-1}(n)|$ with the prime counting function $\pi(x)$ determine exact formulas and asymptotic bounds for $M(x)$. In this way, we prove another concrete link between the distributions of both $L(x) := \sum_{n \leq x} \lambda(n)$ and the Mertens function and connect these classical summatory functions with sums weighted by an explicit normal tending probability distribution at large x. The proofs of these resulting combinatorially motivated new characterizations of $M(x)$ in the article are rigorous and unconditional.

Keywords and Phrases: *Möbius function; Mertens function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function or DGF; Erdös-Kac theorem; strongly additive function.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; 11N64; and 11-04.*

Glossary of notation and conventions

Symbol	Definition
$\approx$	We write that $f(x) \approx g(x)$ if $\lvert f(x) - g(x)\rvert = O(1)$ as $x \rightarrow \infty$.
$\mathbb{E}[f(x)], \overset{\mathbb{E}}{\sim}$	We use the expectation notation of $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \overset{\mathbb{E}}{\sim} h(x)$, to denote that f has an *average order* of $h(x)$. This means that $\frac{1}{x}\sum_{n\leq x} f(n) \sim h(x)$, or equivalently that $$\lim_{x\rightarrow\infty} \frac{\frac{1}{x}\sum\limits_{n\leq x} f(n)}{h(x)} = 1.$$
B	The absolute constant $B \approx 0.2614972$ from the statement of Mertens theorem.
$\chi_{\mathbb{P}}(n)$	The characteristic (or indicator) function of the primes equals one if and only if $n \in \mathbb{Z}^+$ is prime, and is zero-valued otherwise.
$C_k(n)$	The sequence is defined recursively for $n \geq 1$ as follows: $$C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d\mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$ It represents the multiple, k-fold convolution of the function $\omega(n)$ with itself.
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function (OGF) of some sequence, $\{f_n\}_{n\geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n\geq 0} f_n q^n$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (see definition below).
$f * g$	The Dirichlet convolution of f and g, $(f * g)(n) := \sum\limits_{d\mid n} f(d) g(n/d)$, where the sum is taken over the divisors of any $n \geq 1$.
$f^{-1}(n)$	The Dirichlet inverse of f exists if and only if $f(1) \neq 0$. The Dirichlet inverse of any f such that $f(1) \neq 0$ with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ for $n \geq 2$ with $f^{-1}(1) = 1/f(1)$. This inverse function, denoted by f^{-1} when it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$.
γ	The Euler gamma constant defined by $\gamma := \lim\limits_{n\rightarrow\infty} \left(\sum\limits_{k=1}^{n} \frac{1}{k} - \log n\right) \approx 0.5772157$.
$\gg, \ll, \asymp$	For functions A, B, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A \ll B$ and $B \ll A$, we write $A \asymp B$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$.

Symbol	Definition
$[n=k]_{\delta}$,$[\texttt{cond}]_{\delta}$	The symbol $[n=k]_{\delta}$ is a synonym for $\delta_{n,k}$ which is one if and only if $n=k$, and is zero otherwise. For boolean-valued conditions, `cond`, the symbol $[\texttt{cond}]_{\delta}$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention*.
$\lambda_*(n)$	For positive integers $n \geq 2$, we define the next variant of the Liouville lambda function, $\lambda(n)$, as follows: $\lambda_*(n) := (-1)^{\omega(n)}$. We have the initial condition that $\lambda_*(1) = 1$.
$\lambda(n), L(x)$	The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. Its summatory function is defined by $L(x) := \sum_{n \leq x} \lambda(n)$.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree.
$\mu_x(C), \sigma_x(C)$	We define these analogs to the mean and variance of the function $C_{\Omega(n)}(n)$ in the context of our new Erdös-Kac like theorems as $\mu_x(C) := \log\log x + \log(2) - \log\left(\hat{C}_*(C)\right)$ and $\sigma_x(C) := \sqrt{\log\log x}$ where $\hat{C}_*(C) > 0$ is an absolute constant.
$M(x)$	The Mertens function is the summatory function of $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum\limits_{n \leq x} \mu(n)$.
$\Phi(z)$	For $x \in \mathbb{R}$, we define the function giving the normal distribution CDF by $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{z} e^{-t^2/2} dt$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^{\alpha} \vert\vert n$ (or when p^{α} exactly divides n) for p prime, $\alpha \geq 1$ and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p \vert n} 1$ and the completely additive function $\Omega(n) := \sum_{p^{\alpha} \vert\vert n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leq n \leq x$ for $x \geq 2$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$ for $x \geq 2$.
$P(s)$	For complex s with $\mathrm{Re}(s) > 1$, we define the prime zeta function to be the Dirichlet generating function $P(s) = \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s}$.
$Q(x)$	For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. More precisely, this function is summed and identified with its limiting asymptotic formula as $x \rightarrow \infty$ in the following form: $Q(x) := \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O(\sqrt{x})$.
$\sim$	We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} = 1$.
$\zeta(s)$	The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation on the rest of the complex plane with the exception of a simple pole at $s = 1$ of residue one.

1 Introduction

1.1 Preliminaries

1.1.1 Definitions

We define the *Möbius function* to be the signed indicator function of the squarefree integers in the form of [20, A008683]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \text{ and } n \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined on the positive integers as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as follows [20, A002321]:

$$\{M(x)\}_{x \geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \ldots\}.$$

The Mertens function satisfies that $\sum_{n \leq x} M\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = 1$, and is related to the summatory function $L(x) := \sum_{n \leq x} \lambda(n)$ via the relation [6, 10]

$$L(x) = \sum_{d \leq \sqrt{x}} M\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), x \geq 1.$$

A positive integer $n \geq 1$ is *squarefree*, or contains no divisors (other than one) which are squares, if and only if $\mu^2(n) = 1$. A related summatory function which counts the number of *squarefree* integers $n \leq x$ satisfies [5, §18.6] [20, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers characterized by the sign of the Möbius function are in fact equal as $x \to \infty$:

$$\begin{aligned} \mu_+(x) &:= \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{x} \xrightarrow{x \to \infty} \frac{3}{\pi^2} \\ \mu_-(x) &:= \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{x} \xrightarrow{x \to \infty} \frac{3}{\pi^2}. \end{aligned}$$

1.1.2 Properties

A conventional approach to evaluating the limiting asymptotic behavior of $M(x)$ for large $x \to \infty$ considers an inverse Mellin transformation of the reciprocal of the Riemann zeta function. In particular, since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \cdot \int_1^\infty \frac{M(x)}{x^{s+1}} dx, \operatorname{Re}(s) > 1,$$

we obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

The previous two representations lead us to the exact expression of $M(x)$ for any real $x > 0$ given by the next theorem.

Theorem 1.1 (Analytic Formula for $M(x)$, Titchmarsh)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence* $\{T_k\}_{k\geq 1}$ *satisfying* $k \leq T_k \leq k+1$ *for each* k *such that for any real* $x > 0$

$$M(x) = \lim_{k\to\infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |\operatorname{Im}(\rho)|<T_k}} \frac{x^\rho}{\rho\cdot\zeta'(\rho)} - 2 + \sum_{n\geq 1} \frac{(-1)^{n-1}}{n\cdot(2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2}\left[x\in\mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x\cdot\exp\left(-C\cdot\log^{\frac{3}{5}}(x)(\log\log x)^{-\frac{3}{5}}\right).$$

Under the assumption of the RH, Soundararajan proved new updated estimates bounding $M(x)$ from above for large x in the following forms [21]:

$$\begin{aligned} M(x) &\ll \sqrt{x}\cdot\exp\left((\log x)^{\frac{1}{2}}(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x}\cdot\exp\left((\log x)^{\frac{1}{2}}(\log\log x)^{\frac{5}{2}+\epsilon}\right)\right),\ \forall\epsilon>0. \end{aligned}$$

1.1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$ for any $0 < \epsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C\cdot\sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens himself for $C = 1$ and all $x < 10000$ without the benefit of modern computation. Since its beginnings in 1897, the Mertens conjecture has been disproven by computational methods with non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele [13]. More recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound along infinite subsequences, e.g., considering the asymptotics of the function in the limit supremum and limit infimum senses.

We cite that it is known by computation that [16, *cf.* §4.1] [20, *cf.* A051400; A051401]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leq -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [13, 8, 9, 7]. Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies [12]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}\cdot(\log\log\log x)^{\frac{5}{4}}} = O(1).$$

1.2 A concrete new approach to characterizing $M(x)$

The main interpretation to take away from the article is that we have rigorously motivated an equivalent alternate characterization of $M(x)$ by constructing combinatorially relevant sequences related to the distribution of the primes and to standard strongly additive functions that have not yet been studied in the literature surrounding the Mertens function. The prime-related combinatorics at hand here are discussed in more detail by the remarks given in Section 3.3. This new perspective offers equivalent exact characterizations of $M(x)$ for all $x \geq 1$ through the formulas involving discrete convolutions of $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ with the prime counting function $\pi(x)$ given in Section 5. The new sequence $g^{-1}(n)$ defined precisely below and $G^{-1}(x)$ are crucially tied to standard, canonical examples of strongly and completely additive functions, e.g., $\omega(n)$ and $\Omega(n)$, respectively. The definitions of the core subsequences we define and the parameterized bivariate DGF based proof method that is given in the spirit of Montgomery and Vaughan's work allow us to reconcile the property of strong additivity in a novel way with signed sums of multiplicative functions and their classical importance.

The proofs of key properties of these new sequences bundles with it a scaled normal tending probability distribution for the unsigned magnitude of $|g^{-1}(n)|$ that is similar in many ways to the Erdös-Kac theorems for $\omega(n)$ and $\Omega(n)$. Since we prove that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$, it follows that we have a new probabilistic perspective from which to express distributional features of the summatory functions $G^{-1}(x)$ as $x \to \infty$ in terms of the properties of $|g^{-1}(n)|$ and $L(x) := \sum_{n \leq x} \lambda(n)$. Formalizing the properties of the distribution of $L(x)$ is typically viewed as a problem on par with, or equally as difficult in order to understanding the distribution of $M(x)$ well as $x \to \infty$. The results in this article concretely connect the distributions of $L(x)$, a well defined scaled normally tending probability distribution, and $M(x)$ as $x \to \infty$.

1.2.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 1.2 (Summatory functions of Dirichlet convolutions)**.** *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of f and h, respectively, and that $F^{-1}(x) := \sum_{n \leq x} f^{-1}(n)$ denotes the summatory function of the Dirichlet inverse of f for any $x \geq 1$. We have the following exact expressions for the summatory function of the convolution $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned}
\pi_{f*h}(x) &:= \sum_{n \leq x} \sum_{d|n} f(d) h(n/d) \\
&= \sum_{d \leq x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].
\end{aligned}$$

Moreover, for all $x \geq 1$

$$\begin{aligned}
H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\
&= \sum_{k=1}^{x} f^{-1}(k) \cdot \pi_{f*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right).
\end{aligned}$$

Corollary 1.3 (Convolutions arising from Möbius inversion)**.** *Suppose that h is an arithmetic function such that $h(1) \neq 0$. Define the summatory function of the convolution of h with μ by $\widetilde{H}(x) := \sum_{n \leq x} (h * \mu)(n)$. Then the Mertens function is expressed by the sum*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} h^{-1}(j) \right) \widetilde{H}(k), \forall x \geq 1.$$

Corollary 1.4 (A motivating special case)**.** *We have that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

1.2.2 An exact expression for $M(x)$ via strongly additive functions

Fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute exactly that (see Table T.1 starting on page 34)

$$\{g^{-1}(n)\}_{n\geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

There is not a simple meaningful direct recursion between the distinct values of $g^{-1}(n)$. The distribution of distinct sets of prime exponents is still clearly quite regular since $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of this inverse function we notice below over $n \geq 2$:

Heuristic 1.5 (Symmetry in $g^{-1}(n)$ in the prime factorizations of $n \leq x$)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for $\omega(n_i) \geq 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly one, two, three, and so on prime factors.

Conjecture 1.6 (Characteristic properties of the inverse sequence)**.** *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) *For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*

(B) *For all squarefree integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!;$$

(C) *If $n \geq 2$ and $\Omega(n) = k$, then*

$$2 \leq |g^{-1}(n)| \leq \sum_{j=0}^{k} \binom{k}{j} \cdot j!.$$

We illustrate the conjecture clearly using the computation of initial values of this inverse sequence in Table T.1. The signedness property in (A) is proved precisely in Proposition 2.1. A proof of (B) in fact follows from Lemma 3.1 stated on page 15. The realization that the beautiful and remarkably simple combinatorial form of property (B) in Conjecture 1.6 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through the sums of auxiliary subsequences $C_k(n)$ (see Section 3). That is, we observe a familiar formula for $g^{-1}(n)$ on an asymptotically dense infinite subset of integers and the seek to extrapolate by proving there are regular tendencies of this sequence viewed more generally over any $n \geq 2$.

An exact expression for $g^{-1}(n)$ is given by

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d), n \geq 1,$$

where the sequence $\lambda(n) C_{\Omega(n)}(n)$ has DGF $(P(s) + 1)^{-1}$ for $\operatorname{Re}(s) > 1$ (see Proposition 2.1). In Corollary 4.5, we prove that for an absolute constant $\hat{C}_*(C) > 0$, the approximate average order of the unsigned sequence satisfies

$$\mathbb{E}|g^{-1}(n)| \sim \hat{C}_*(C) \cdot (\log n)^2 \sqrt{\log\log n}, \text{ as } n \to \infty.$$

In Section 4, we prove the next variant of an Erdös-Kac theorem like analog for a component sequence $C_{\Omega(n)}(n)$. This leads us to conclude the following uniform statement for any fixed $Y > 0$, $\mu_x(C) := \log\log x + \log 2 - \log(\hat{C}_*(C))$, $\sigma_x(C) := \sqrt{\log\log x}$, and for all $-Y \leq y \leq Y$ (see Corollary 4.7):

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : |g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq y\right\} = \Phi\left(\frac{\frac{\pi^2}{6} y - \mu_x(C)}{(\log x)(\log\log x)}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

The regularity and quasi-periodicity we have alluded to in the remarks above are actually then quantifiable in so much as the distrubution of $|g^{-1}(n)|$ for $n \leq x$ tends to its average order with a non-central normal tendency depending on x as $x \to \infty$. That is, if $x \geq 2$ is sufficiently large and if we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds:

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq 0\right) = \frac{1}{2} + o(1) \tag{A}$$

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{\alpha\mathbb{E}|g^{-1}(x)|\sqrt{\log\log x}}{\log x}\right) = \Phi\left(\frac{\alpha\pi^2\hat{G}_*}{6}\right) + o(1), \alpha \in \mathbb{R}. \tag{B}$$

In the previous statement of (B), $\hat{G}_* > 0$ is an absolute constant. It follows from the last property that as $n \to \infty$,

$$|g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|,$$

on an infinite set of the integers with asymptotic density one.

Remark 1.7 (Uniform asymptotics from certain bivariate counting DGFs)**.** We emphasize the recency of the method demonstrated by Montgomery and Vaughan in constructing their original proof of Theorem 2.5 (stated below). To the best of our knowledge, this textbook reference is one of the first clear cut applications documenting something of a hybrid DGF-and-OGF type approach to enumerating sequences of arithmetic functions and their summatory functions. This interpretation of certain bivariate DGFs offers a window into the best of both generating function series worlds. It combines the additivity implicit to the coefficients indexed by a formal power series variable formed by multiplication of these structures, while coordinating the distinct DGF-best property of the multiplicativity with respect to distinct prime powers invoked by taking powers of a reciprocal Euler type product. A proof constructed from this type of bivariate power series DGF is given in Section 4.

1.2.3 Formulas illustrating the new characterizations of $M(x)$

Let $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ for integers $x \geq 1$. We prove that (see Proposition 5.2)

$$\begin{aligned} M(x) &= G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right] \\ &= G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1. \end{aligned} \tag{2}$$

This formula implies that we can establish new asymptotic bounds on $M(x)$ along large infinite subsequences by bounding the summatory function $G^{-1}(x)$. The take on the regularity of $|g^{-1}(n)|$ is then imperative to our arguments that formally bound the growth of $M(x)$ by its new identification with $G^{-1}(x)$. A more combinatorial approach to summing $G^{-1}(x)$ for large x based on the distribution of the primes is outlined in our remarks in Section 3.3.

The results in Theorem 5.1 prove that for almost every sufficiently large x[A]

$$G^{-1}(x) = O\left(\max_{1 \leq t \leq x} |L(t)| \times \mathbb{E}|g^{-1}(x)|\right).$$

If the RH is true, then we have the following result for any $\varepsilon > 0$ and almost every integer $x \geq 1$:

$$G^{-1}(x) = O\left(\sqrt{x} \cdot (\log x)^2 \sqrt{\log\log x} \times \exp\left(\sqrt{\log x} \cdot (\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right).$$

[A] By *almost every* large integer x, we mean that the result holds for all large x taken within an infinite subset of $\mathbb{Z}^+$ with asymptotic density one.

In Corollary 5.4, we prove that as $x \to \infty$

$$M(x) = O\left(G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) + \frac{x}{\log x} \times \max_{1 \leq k \leq \sqrt{x}} |G^{-1}(k)| + (\log x)^2 (\log\log x)^{\frac{3}{2}}\right).$$

Moving forward, a discussion of the properties of the summatory functions $G^{-1}(x)$ motivates more study in the future to extend the full range of possibilities for viewing this new structure behind $M(x)$.

2 Initial proofs of new results

2.1 Establishing the summatory function properties and inversion identities

We will offer a proof of Theorem 1.2 suggested by an intuitive construction through matrix based methods. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95]. It is not difficult to prove the related identity

$$\sum_{n\leq x} h(n)(f*g)(n) = \sum_{n\leq x} f(n) \times \sum_{k\leq \left\lfloor \frac{x}{n}\right\rfloor} g(k)h(kn).$$

Proof of Theorem 1.2. Let h, g be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n\leq x} h(n)$ and $G(x) = \sum_{n\leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h. We have that the following formulas hold for all $x \geq 1$:

$$\begin{aligned} \pi_{g*h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d}\right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i}\right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1}\right\rfloor\right)\right] H(i). \end{aligned} \tag{3}$$

The first formula above is well known. The second formula is justified directly using summation by parts as [14, §2.10(ii)]

$$\begin{aligned} \pi_{g*h}(x) &= \sum_{d=1}^{x} h(d) G\left(\left\lfloor \frac{x}{d}\right\rfloor\right) \\ &= \sum_{i\leq x} \left(\sum_{j\leq i} h(j)\right) \times \left[G\left(\left\lfloor \frac{x}{i}\right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1}\right\rfloor\right)\right]. \end{aligned}$$

We next form the invertible matrix of coefficients associated with this linear system defining $H(j)$ for all $1 \leq j \leq x$ in (3) by setting

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j}\right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1}\right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j}\right\rfloor\right), 1 \leq j \leq x.$$

Since $g_{x,x} = G(1) = g(1)$ and $g_{x,j} = 0$ for all $j > x$, the matrix we must work with in this problem is lower triangular with a non-zero constant on its diagonals, and is hence invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressed by applying an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

Here, U is a square matrix with sufficiently large finite dimensions whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$\left[(I - U^T)^{-1}\right]_{i,j} = [j \leq i]_\delta.$$

Observe that

$$\left\lfloor \frac{x}{j}\right\rfloor - \left\lfloor \frac{x-1}{j}\right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

The previous property implies that

$$G\left(\left\lfloor \frac{x}{j}\right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j}\right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

We use the last property in (4) to shift the matrix $\hat{G}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of g in the following form:

$$\left[(I-U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

In particular, our target matrix in the inversion problem is defined by

$$(g_{x,j}) = (I-U^T)\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators as

$$\begin{aligned}(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k)\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1}\right\rfloor} g^{-1}(k)\right).\end{aligned}$$

Hence, the summatory function $H(x)$ is given exactly for any $x \geq 1$ by a vector product with the inverse matrix from the previous equation as

$$H(x) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right)\cdot \pi_{g*h}(k).$$

We can prove an inversion formula providing the coefficients of the summatory function $G^{-1}(i)$ for $1 \leq i \leq x$ given by the last equation by adapting our argument to prove (3) above. This leads to the following equivalent identity:

$$H(x) = \sum_{k=1}^{x} g^{-1}(x)\cdot \pi_{g*h}\left(\left\lfloor \frac{x}{k}\right\rfloor\right). \qquad \square$$

2.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, let $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n. Then we can easily prove using DGFs (or other elementary methods) that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu. \tag{5}$$

When combined with Corollary 1.3 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 1.4.

Proposition 2.1 (The signedness property of $g^{-1}(n)$)**.** *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0, \pm 1\}$ *denote the sign of the arithmetic function h at integers $n \geq 1$. For the Dirichlet invertible function $g(n) := \omega(n)+1$, we have that* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n) \neq 0$ *for all $n \geq 1$.*

Proof. The function $D_f(s) := \sum_{n\geq 1} f(n)n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ for σ_f the abcissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = \zeta(s)^{-1}$ and $D_\omega(s) = P(s)\zeta(s)$ for $\mathrm{Re}(s) > 1$. Then by (5) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any arithmetic function f such that $f(1) \neq 0$, we have for all $\mathrm{Re}(s) > 1$ that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{6}$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\mathrm{sgn}(h^{-1}) = \lambda$. This observation implies that $\mathrm{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make our claims based on this intuition rigorous.

By the recurrence relation that defines the Dirichlet inverse function of any arithmetic function h such that $h(1) = 1$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d|n \\ d>1}} h(d)h^{-1}(n/d), & n \geq 2. \end{cases} \tag{7}$$

For $n \geq 2$, the summands in (7) can be simply indexed over the primes $p|n$ given our definition of h from above. This observation yields that we can inductively unfold these sums into nested divisor sums provided the depth of the expanded divisor sums does not exceed the capacity to index summations over the primes dividing n. Namely, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p|n} h^{-1}\left(\frac{n}{p}\right), && \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1p_2}\right), && \text{if } \Omega(n) \geq 2 \\ &= -\sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1p_2}} h^{-1}\left(\frac{n}{p_1p_2p_3}\right), && \text{if } \Omega(n) \geq 3. \end{aligned}$$

Then by induction with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1p_2\cdots p_{\Omega(n)-1}}} 1, n \geq 2. \tag{8}$$

In fact, by a combinatorial argument related to multinomial coefficient expansions of the sums in (8), we recover exactly that

$$h^{-1}(n) = \lambda(n)(\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}, n \geq 2. \tag{9}$$

The last two expansions imply that the following property holds for all $n \geq 1$:

$$\mathrm{sgn}(h^{-1}(n)) = \lambda(n).$$

In particular, since λ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$ for all divisors $d|n$ when $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever n is squarefree, so that we obtain the following result:

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right)|h^{-1}(n)|, n \geq 1. \qquad \square$$

2.3 Statements of known limiting asymptotic formulas

Facts 2.2 (The incomplete gamma function)**.** The (upper) *incomplete gamma function* is defined by [14, §8.4]

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt, \operatorname{Re}(s) > 0.$$

The following properties of $\Gamma(s,x)$ hold:

$$\Gamma(s,x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, x > 0, \tag{10a}$$

$$\Gamma(s,x) \sim x^{s-1} \cdot e^{-x}, s > 0, \text{ as } x \to \infty. \tag{10b}$$

2.4 Results on the distribution of exceptional values of $\omega(n)$ and $\Omega(n)$

The next theorems reproduced from [11, §7.4] characterize the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \log\log x$. Since $\mathbb{E}[\Omega(n)] = \log\log n + B$ for $B \in (0,1)$ an absolute constant, these results imply a very regular, normal tendency of this arithmetic function towards its average order.

Theorem 2.3 (Upper bounds on exceptional values of $\Omega(n)$ for large n)**.** *Let*

$$\begin{aligned} A(x,r) &:= \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\}, \\ B(x,r) &:= \#\left\{n \leq x : \Omega(n) \geq r \cdot \log\log x\right\}. \end{aligned}$$

If $0 < r \leq 1$ *and* $x \geq 2$*, then*

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

If $1 \leq r \leq R < 2$ *and* $x \geq 2$*, then*

$$B(x,r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

Theorem 2.4 is a special case analog to the celebrated Erdös-Kac theorem typically stated for the normally distributed values of the scaled-shifted function $\omega(n)$ over $n \leq x$ as $x \to \infty$ [11, *cf.* Thm. 7.21].

Theorem 2.4 (Exact limiting bounds on exceptional values of $\Omega(n)$ for large n)**.** *We have that as* $x \to \infty$

$$\#\left\{3 \leq n \leq x : \Omega(n) - \log\log n \leq 0\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Theorem 2.5 (Montgomery and Vaughan)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $R < 2$ *we have that uniformly for all* $1 \leq k \leq R \cdot \log\log x$

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right],$$

where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1}\left(1 - \frac{1}{p}\right)^z, 0 \leq |z| < R.$$

Remark 2.6. We can extend the work in [11] with $\Omega(n)$ to see that for $0 < R < 2$

$$\pi_k(x) = \widehat{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{k!}\left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right], \text{ uniformly for } 1 \leq k \leq R\log\log x.$$

The analogous function to express these bounds for $\omega(n)$ is defined by $\widehat{\mathcal{G}}(z) := \widehat{F}(1,z)/\Gamma(1+z)$ where we take

$$\widehat{F}(s,z) := \prod_p \left(1+\frac{z}{p^s-1}\right)\left(1-\frac{1}{p^s}\right)^z, \mathrm{Re}(s) > \frac{1}{2}; |z| \leq R < 2.$$

Let the functions

$$\begin{aligned} C(x,r) &:= \#\{n \leq x : \omega(n) \leq r \log\log x\} \\ D(x,r) &:= \#\{n \leq x : \omega(n) \geq r \log\log x\}. \end{aligned}$$

Then we have the next uniform upper bounds given by

$$\begin{aligned} C(x,r) &\ll x(\log x)^{r-1-r\log r}, \text{ uniformly for } 0 < r \leq 1, \\ D(x,r) &\ll x(\log x)^{r-1-r\log r}, \text{ uniformly for } 1 \leq r \leq R < 2. \end{aligned}$$

Corollary 2.7. *Suppose that for $x > e$ we define the functions*

$$\begin{aligned} \mathcal{N}_\omega(x) &:= \left|\sum_{k>\log\log x} (-1)^k \pi_k(x)\right| \\ \mathcal{D}_\omega(x) &:= \left|\sum_{k\leq\log\log x} (-1)^k \pi_k(x)\right| \\ \mathcal{A}_\omega(x) &:= \left|\sum_{k\geq 1} (-1)^k \pi_k(x)\right|. \end{aligned}$$

Then as $x \to \infty$, we have that $\mathcal{N}_\omega(x)/\mathcal{D}_\omega(x) = o(1)$ and $\mathcal{A}_\omega(x) \asymp \mathcal{D}_\omega(x)$.

Proof. First, we sum the function $\mathcal{D}_\omega(x)$ exactly, and then apply the limiting asymptotics for the incomplete gamma function from (10b) and Stirling's formula, to obtain that

$$\begin{aligned} \mathcal{D}_\omega(x) &= \left|\sum_{k\leq\log\log x} \frac{(-1)^k \cdot x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\right| \\ &= \left|\frac{x}{(\log x)^2} \cdot \frac{\Gamma(\log\log x, -\log\log x)}{\Gamma(\log\log x)}\right| \\ &\asymp \frac{x}{\log x} \cdot \frac{(\log\log x)^{\log\log x - 1}}{\Gamma(\log\log x)} \\ &\asymp \frac{x}{\sqrt{\log\log x}}, \text{ as } x \to \infty. \end{aligned}$$

Next, for any $\delta_{x,k} > 0$ when we define $r \log\log x \leq k := \log\log x + \delta_{x,k}$, we obtain the bounds that $1 \leq r \leq \frac{\log x}{\log\log x}$ for all large $x > e$. Expanding logarithms leads to

$$\frac{D(x,r)}{x} \ll (\log x)^{r-1-r\log r} \ll \frac{x^{1+\log\log\log x}}{(\log x)^{\log x}}.$$

Then we see that

$$\frac{\mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)} \ll \frac{x^{1+\log\log\log x}\sqrt{\log\log x}}{(\log x)^{1+\log x}} = o(1), \text{ as } x \to \infty.$$

Now we have the following bounds for large x:

$$1 + o(1) = \frac{\mathcal{D}_\omega(x) - \mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)} \ll \frac{\mathcal{A}_\omega(x)}{\mathcal{D}_\omega(x)} \ll \frac{\mathcal{D}_\omega(x) + \mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)} = 1 + o(1).$$

The last equation implies that $\mathcal{A}_\omega(x) \asymp \mathcal{D}_\omega(x)$ as $x \to \infty$. Hence, we can accurately approximate asymptotic order of the sums $\mathcal{A}_\omega(x)$ for large x by only considering the truncated sums $\mathcal{D}_\omega(x)$ where we have the uniform bounds for $1 \leq k \leq \log\log x$. $\square$

3 Auxiliary sequences to express the Dirichlet inverse function $g^{-1}(n)$

The pages of tabular data given as Table T.1 in the appendix section (refer to page 34) are intended to provide clear insight into why we eventually arrived at the approximations to $g^{-1}(n)$ initially proved in this section. The table provides illustrative numerical data by examining the approximate behavior at hand for the cases of $1 \leq n \leq 500$ with *Mathematica* [19]. In Section 4, we will use these relations between $g^{-1}(n)$ and $C_{\Omega(n)}(n)$ to prove an Erdös-Kac like analog for the distribution of the unsigned function $|g^{-1}(n)|$.

3.1 Definitions and properties of triangular component function sequences

We define the following auxiliary coefficient sequence for integers $n \geq 1$ and $k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{11}$$

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (11) inductively. By the same argument, we see that at fixed n, the function $C_k(n)$ is seen to be non-zero only for positive integers $k \leq \Omega(n)$ whenever $n \geq 2$. A sequence of relevant signed semi-diagonals of the functions $C_k(n)$ begins as follows [20, A008480]:

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \ -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

We can see that $C_{\Omega(n)}(n) \leq (\Omega(n))!$ for all $n \geq 1$. In fact, $h^{-1}(n) \equiv \lambda(n) C_{\Omega(n)}(n)$ is the same function given by the formula in (9) from Proposition 2.1.

3.2 Relating the function $C_{\Omega(n)}(n)$ to exact formulas for $g^{-1}(n)$

Lemma 3.1 (An exact initial formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) g^{-1}(n/d) \quad \implies \quad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \tag{12}$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (12) in the form of $(g^{-1} * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m \cdot (C_m(-) * g^{-1})(n)$, or so that

$$F_m(n) = -\begin{cases} \sum\limits_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum\limits_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & 2 \leq m \leq \Omega(n), \\ (\omega * g^{-1})(n), & m = 1. \end{cases}$$

By repeatedly expanding the right-hand-side of the previous equation, we find that for $m := \Omega(n)$ (i.e., with the expansions taken to a maximal depth in the previous equation)

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \tag{13}$$

The formula then follows from (13) by Möbius inversion applied to each side of the last equation. $\square$

Corollary 3.2. *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{14}$$

Proof. By applying Lemma 3.1, Proposition 2.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$, Lemma 3.1 implies

$$\begin{aligned}
|g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\
&= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\
&= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d).
\end{aligned}$$

In the last equation, we see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. □

Since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ using the notation defined in the the proof of Proposition 2.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for squarefree $n \geq 1$. A proof of part (B) of Conjecture 1.6 follows as an immediate consequence.

Remark 3.3. Combined with the signedness property of $g^{-1}(n)$ guaranteed by Proposition 2.1, Corollary 3.2 shows that its summatory function is expressed as

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_{\Omega(d)}(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Additionally, equation (5) implies that

$$\lambda(d) C_{\Omega(d)}(d) = (g^{-1} * 1)(d) = (\chi_{\mathbb{P}} + \varepsilon)^{-1}(d),$$

where $\chi_{\mathbb{P}}$ denotes the characteristic function of the primes. We clearly recover by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1.$$

It can infact be seen that

$$\mu(n) = g^{-1}(n) + \sum_{p|n} g^{-1}\left(\frac{n}{p}\right), n \geq 1.$$

This connection between the summatory function of $g^{-1}(n)$ and the primes is also conveyed by the form of the identity we prove for $M(x)$ in Proposition 5.2 involving the prime counting function, $\pi(x)$.

3.3 A connection to the distribution of the primes

The combinatorial complexity of $g^{-1}(n)$ is deeply tied to the distribution of the primes $p \leq n$ as $n \to \infty$. The magnitudes and dispersion of the primes $p \leq x$ certainly restricts the repeating of these distinct sequence values. Nonetheless, we can see that the following is still clear about the relation of the weight functions $|g^{-1}(n)|$ to the distribution of the primes: The value of $|g^{-1}(n)|$ is entirely dependent on the pattern of the *exponents* (viewed as multisets) of the distinct prime factors of $n \geq 2$, rather than on the prime factor weights themselves (*cf.* Heuristic 1.5).

Example 3.4 (Combinatorial significance to the distribution of $g^{-1}(n)$)**.** We have a natural extremal behavior with respect to distinct values of $\Omega(n)$ corresponding to squarefree integers and prime powers. Namely, if for integers $k \geq 1$ we define the infinite sets M_k and m_k to correspond to the maximal (minimal) sets of positive integers such that

$$M_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

$$m_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

then any element of M_k is squarefree and any element of m_k is a prime power. In particular, for any fixed $k \geq 1$ we have that for any $N_k \in M_k$ and $n_k \in m_k$

$$(-1)^k \cdot g^{-1}(N_k) = \sum_{j=0}^{k} \binom{k}{j} \cdot j!, \qquad \text{and} \qquad (-1)^k \cdot g^{-1}(n_k) = 2.$$

Remark 3.5. The formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 2.1 implies that we can express an exact formula for $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n. Namely, for $n \geq 2$ and $0 \leq k \leq \omega(n)$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z \cdot \nu_p(n)) = [z^k] \prod_{p^\alpha || n} (1 + \alpha z).$$

Then we have essentially shown using (9) and (14) that we can expand formulas for these inverse functions in the following form:

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, n \geq 2.$$

The combinatorial formula[A] for $h^{-1}(n) = \lambda(n) \cdot (\Omega(n))! \times \prod_{p^\alpha || n} (\alpha!)^{-1}$ we derived in the proof of the key signedness proposition from Section 2 suggests additional patterns and more regularity in the contributions of the distinct weighted terms for $G^{-1}(x)$.

[A] This seqence is also considered using a different motivation based on the DGFs $(1 \pm P(s))^{-1}$ in [4, §2].

4 The distributions of the unsigned sequences $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$

We have already suggested in the introduction that the relation of the component functions, $g^{-1}(n)$ and $C_{\Omega(n)}(n)$, to the canonical additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties of these functions cited in Table T.1. Each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdös-Kac theorem that shows that the density of a shifted and scaled variant of each of the sets of these function values for $n \leq x$ can be expressed through a limiting normal distribution as $x \to \infty$ [3, 2, 15]. In the remainder of this section we establish more analytical proofs of related properties of these key sequences used to express $G^{-1}(x)$, again in the spirit of Montgomery and Vaughan's modern reference manual (*cf.* Remark 1.7).

Proposition 4.1. *Let the function $\widehat{F}(s,z)$ is defined for $\operatorname{Re}(s) \geq 2$ and $|z| < |P(s)|^{-1}$ in terms of the prime zeta function by*

$$\widehat{F}(s,z) := \frac{1}{1-P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

For $|z| < P(2)^{-1} \approx 2.21118$, the summatory function of the coefficients of $\widehat{F}(s,z)$ expanded as a DGF are defined as follows:

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

Moreover, we have that for all sufficiently large x

$$\widehat{A}_z(x) = \frac{x}{\Gamma(z)} \cdot \widehat{F}(2,z) \cdot (\log x)^{z-1} + O_z\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right), |z| < P(2)^{-1}.$$

Proof. We can see by adapting the notation from the proof of Proposition 2.1 that

$$C_{\Omega(n)}(n) = \begin{cases} (\Omega(n))! \times \prod\limits_{p^\alpha || n} \frac{1}{\alpha!}, & n \geq 2; \\ 1, & n = 1. \end{cases}$$

We can then generate scaled forms of these terms through a product identity of the following form:

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! \cdot p^{rs}}\right)^{-1} = \exp\left(z \cdot P(s)\right), \operatorname{Re}(s) \geq 2, z \in \mathbb{C}.$$

This product based expansion is similar in construction to the parameterized bivariate DGF used in [11, §7.4]. By computing a Laplace transform on the right-hand-side of the above equation, we obtain

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n) \cdot (-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp\left(tz \cdot P(s)\right) dt = \frac{1}{1-P(s)z}, \operatorname{Re}(s) > 1, |z| < |P(s)|^{-1}.$$

It follows that

$$\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times \widehat{F}(s,z), \operatorname{Re}(s) > 1, |z| < |P(s)|^{-1}.$$

Since $\widehat{F}(s,z)$ is an analytic function of s for all $\operatorname{Re}(s) > 1$ whenever the parameter $|z| < |P(s)|^{-1}$, if the sequence $\{b_z(n)\}_{n \geq 1}$ indexes the coefficients in the DGF expansion of $\widehat{F}(s,z)$ whenever $\operatorname{Re}(s) > 1$, then

$$\left| \sum_{n \geq 1} \frac{b_z(n) (\log n)^{2R+1}}{n^2} \right| < +\infty,$$

is uniformly bounded for $|z| \leq R < +\infty$. This fact follows by repeated termwise differentiation with respect to s.

Let the function $d_z(n)$ be generated as the coefficients of the DGF

$$\zeta(s)^z = \sum_{n\geq 1} \frac{d_z(n)}{n^s}, \mathrm{Re}(s) > 1,$$

with corresponding summatory function defined by $D_z(x) := \sum_{n\leq x} d_z(n)$. The theorem proved in the reference [11, Thm. 7.17; §7.4] shows that for any $z \in \mathbb{C}$ such that $0 < |z| < 2$ and all integers $x \geq 2$

$$D_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\left(x \cdot (\log x)^{\mathrm{Re}(z)-2}\right).$$

We set $b_z(n) := (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$, let the convolution $a_z(n) := \sum_{d|n} b_z(d) d_z(n/d)$, and define its summatory function by $A_z(x) := \sum_{n\leq x} a_z(n)$. Then we have that

$$\begin{aligned} A_z(x) &= \sum_{m\leq x/2} b_z(m) D_z(x/m) + \sum_{x/2<m\leq x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \times \sum_{m\leq x/2} \frac{b_z(m)}{m^2} \times m \cdot \log\left(\frac{x}{m}\right)^{z-1} + O\left(\sum_{m\leq x} \frac{x \cdot |b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2}\right). \end{aligned} \tag{15}$$

We can sum the coefficients for integers $u > e$ taken sufficiently large as follows:

$$\sum_{m\leq u} \frac{b_z(m)}{m} = \left(\widehat{F}(2,z) + O(u^{-2})\right) u - \int_1^u \left(\widehat{F}(2,z) + O(t^{-2})\right) dt = \widehat{F}(2,z) + O(u^{-1}).$$

Suppose that $|z| \leq R < P(2)^{-1}$. Then the error term in (15) satisfies

$$\begin{aligned} \sum_{m\leq x} \frac{x \cdot |b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2} &\ll x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m\leq\sqrt{x}} \frac{|b_z(m)|}{m} \\ &\quad + x(\log x)^{-(R+2)} \times \sum_{m>\sqrt{x}} \frac{|b_z(m)|}{m} (\log m)^{2R} \\ &= O_z\left(x \cdot (\log x)^{\mathrm{Re}(z)-2}\right), |z| \leq R. \end{aligned}$$

In the main term estimate for $A_z(x)$ from (15), when $m \leq \sqrt{x}$ we have

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\mathrm{Re}(z)-2}\right).$$

The total sum over the interval $m \leq x/2$ then corresponds to bounding the following sum components when we take $|z| \leq R$:

$$\begin{aligned} \sum_{m\leq x/2} b_z(m) D_z(x/m) &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \times \sum_{m\leq x/2} \frac{b_z(m)}{m} \\ &\quad + O_z\left(x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m\leq\sqrt{x}} \frac{|b_z(m)|}{m} + x(\log x)^{R-1} \times \sum_{m>\sqrt{x}} \frac{|b_z(m)|}{m}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \widehat{F}(2,z) + O_z\left(x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m\geq 1} \frac{b_z(m)(\log m)^{2R+1}}{m^2}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \widehat{F}(2,z) + O_{z,R}\left(x(\log x)^{\mathrm{Re}(z)-2}\right). \end{aligned}$$

□

Theorem 4.2. *As $x \to \infty$, we have that uniformly for $1 \leq k < \log\log x$*

$$\widehat{C}_k(x) := \sum_{\substack{n \leq x \\ \Omega(n)=k}} (-1)^{\omega(n)} C_k(n) \asymp \frac{x}{\log x} \cdot \frac{(\log\log x + \log\zeta(2))^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

Proof. We begin by bounding a contour integral over the error term for fixed large x when $r := \frac{k-1}{\log\log x}$ with $r < 2$ and $k \geq 2$:

$$\begin{aligned}\left|\int_{|v|=r} \frac{x \cdot (\log x)^{-(\mathrm{Re}(v)+2)}}{v^{k+1}} dv\right| &\ll x(\log x)^{-(r+2)} r^{-(k+1)} \ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{k+1}} \cdot \frac{1}{e^{k-1}} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{3/2}} \cdot \frac{1}{e^{2k}(k-1)!} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \ll \frac{x}{\log x} \cdot \frac{k \cdot (\log\log x)^{k-5}}{(k-1)!}.\end{aligned}$$

When $k = 1$ we have directly by the Cauchy integral formula that

$$\begin{aligned}\left|\int_{|v|=r} \frac{x \cdot (\log x)^{-(\mathrm{Re}(v)+2)}}{v^2} dv\right| &= \left|\frac{1}{1!} \times \frac{d}{dv}\left[x \cdot (\log x)^{-(\mathrm{Re}(v)+2)}\right]\right| \\ &\ll \left|\frac{d}{dr}\left[\frac{x}{(\log x)^2} \cdot \exp(-r\log\log x)\right]\right| \\ &\ll \frac{x}{(\log x)(\log\log x)^2}.\end{aligned}$$

We must now find an asymptotically accurate main term approximation to the coefficients of the following contour integral for $r \in [0, z_{\max}]$ where $z_{\max} < P(2)^{-1}$ according to Proposition 4.1:

$$\widetilde{A}_r(x) := \int_{|v|=r} \frac{x \cdot (\log x)^{-v} \zeta(2)^v}{(\log x)\Gamma(1+v) \cdot v^k (1+P(2)v)} dv. \tag{16}$$

We can show that provided a restriction to $1 \leq r < 1$, we can approximate the contour integral in (16) where the resulting main term is accurate up to a bounded constant factor. This procedure removes the gamma function term in the denominator of the integrand by essentially applying a mean value theorem type analog for smoothly parameterized contours. The logic used to justify this simplification is discussed next.

We observe that for $r := 1$, the function $|\Gamma(1 + re^{2\pi\imath t})|$ has a singularity (pole) when $t := \frac{1}{2}$. We restrict the range of $|v| = r$ so that $0 \leq r < 1$ when we parameterize $v = re^{2\pi\imath t}$ by a real-line integral over $t \in [0, 1]$. We can numerically evaluate the finite extremal values of this function as

$$\min_{\substack{0 \leq r < 1 \\ 0 \leq t \leq 1}} |\Gamma(1 + re^{2\pi\imath t})| = |\Gamma(1 + re^{2\pi\imath t})|\Big|_{(r,t)\approx(1,0.740592)} \approx 0.520089$$

$$\max_{\substack{0 \leq r < 1 \\ 0 \leq t \leq 1}} |\Gamma(1 + re^{2\pi\imath t})| = |\Gamma(1 + re^{2\pi\imath t})|\Big|_{(r,t)\approx(1,0.999887)} \approx 1.$$

This shows that

$$\widetilde{A}_r(x) \asymp \int_{|v|=r} \frac{x \cdot (\log x)^{-v} \zeta(2)^v}{(\log x) \cdot v^k (1+P(2)v)} dv, \tag{17}$$

where as $x \to \infty$

$$\frac{\widetilde{A}_r(x)}{\int_{|v|=r} \frac{x(\log x)^{-v}\zeta(2)^v}{(\log x)\cdot v^k(1+P(2)v)} dv} \in [1, 1.92275].$$

By induction we can compute the remaining coefficients $[z^k]\Gamma(1+z)\times \widehat{A}_z(x)$ with respect to x for fixed $k \leq \log\log x$ using the Cauchy integral formula. Namely, it is not difficult to see that for any integer $m \geq 0$, we have the m^{th} partial derivative of the integrand with respect to z has the following limiting expansion by applying (10b):

$$\begin{aligned} \frac{1}{m!} \times \frac{\partial^{(m)}}{\partial v^{(m)}} \left[\frac{(\log x)^{-v}\zeta(2)^v}{1+P(2)v}\right]\Bigg|_{v=0} &= \sum_{j=0}^{m} \frac{(-1)^m P(2)^j (\log\log x + \log\zeta(2))^{m-j}}{(m-j)!} \\ &= \frac{(-P(2))^m (\log x)^{\frac{1}{P(2)}} \zeta(2)^{\frac{1}{P(2)}}}{m!} \times \Gamma\left(m+1, \frac{\log\log x + \log\zeta(2)}{P(2)}\right) \\ &\sim \frac{(-1)^m(\log\log x + \log\zeta(2))^m}{m!}. \end{aligned}$$

Now by parameterizing the countour around $|z| = r := \frac{k-1}{\log\log x} < 1$ we deduce that the the main term of our approximation corresponds to

$$\int_{|v|=r} \frac{x\cdot(\log x)^{-v}\zeta(2)^{-v}}{(\log x)v^k(1+P(2)v)} dv \asymp \frac{x}{\log x}\cdot \frac{(-1)^k(\log\log x + \log\zeta(2))^{k-1}}{(k-1)!}. \qquad \square$$

Corollary 4.3. *We have that for large $x \geq 2$ uniformly for $1 \leq k \leq \log\log x$*

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp 2\sqrt{2\pi}\cdot x \times \frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!}.$$

Proof. We have an integral formula involving the non-sign-weighted sequence that results by applying ordinary Abel summation (and integrating by parts) in the form of the next equations.

$$\begin{aligned} \sum_{n\leq x} \lambda_*(n)h(n) &= \left(\sum_{n\leq x}\lambda_*(n)\right)h(x) - \int_1^x \left(\sum_{n\leq t}\lambda_*(n)\right)h'(t)dt \qquad (18) \\ & \left\{\begin{array}{ll} u_t = L_*(t) & v_t' = h'(t)dt \\ u_t' = L_*'(t)dt & v_t = h(t) \end{array}\right\} \\ &\asymp \int_1^x \frac{d}{dt}\left[\sum_{n\leq t}\lambda_*(n)\right]h(t)dt \qquad (19) \end{aligned}$$

Let the signed left-hand-side summatory function for our function corresponding to (18) be defined for large $x > e$ and integers $k \geq 1$ by

$$\begin{aligned} \widehat{C}_{k,*}(x) &:= \sum_{\substack{n\leq x \\ \Omega(n)=k}} (-1)^{\omega(n)} C_{\Omega(n)}(n) \\ &\asymp \frac{x}{\log x}\cdot\frac{(\log\log x + \log\zeta(2))^{k-1}}{(k-1)!}\left[1+O\left(\frac{1}{\log\log x}\right)\right] \\ &\asymp \frac{x}{\log x}\cdot\frac{(\log\log x)^{k-1}}{(k-1)!}\left[1+O\left(\frac{1}{\log\log x}\right)\right]. \end{aligned}$$

The second equation above follows from the proof of Theorem 4.2. We adopt the notation that $\lambda_*(n) = (-1)^{\omega(n)}$ for $n \geq 1$ and define its summatory function by $L_*(x) := \sum_{n\leq x}\lambda_*(n)$ for $x \geq 1$.

We transform our previous results for the partial sums over the signed sequences $\lambda_*(n)\cdot C_{\Omega(n)}(n)$ such that $\Omega(n) = k$. The argument is based on approximating $L_*(t)$ for large t using the following uniform asymptotics for $\pi_k(x)$ when $1 \leq k \leq \log\log x$:

$$\pi_k(x) \asymp \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}(1+o_k(1)), \text{ as } x\to\infty.$$

We have by an asymptotic approximation to the incomplete gamma function and Corollary 2.7 that

$$L_*(t) := \left|\sum_{n\leq t}(-1)^{\omega(n)}\right| \asymp \left|\sum_{k=1}^{\log\log t}(-1)^k\pi_k(x)\right| \sim \frac{t}{\sqrt{2\pi}\sqrt{\log\log t}}, \text{ as } t\to\infty.$$

The main term for the reciprocal of the derivative of the main term approximation to this summatory function is asymptotic to

$$\frac{1}{L_*'(t)} \asymp \sqrt{2\pi}\cdot(\log\log t)^{\frac{1}{2}}.$$

After applying the formula from (18), we thus deduce that the unsigned summatory function variant satisfies

$$\begin{aligned}\widehat{C}_{k,*}(x) &= \int_1^x L_*'(t)C_{\Omega(t)}(t)dt \qquad \implies C_{\Omega(x)}(x) \asymp \frac{\widehat{C}_{k,*}'(x)}{L_*'(x)} \qquad \implies \\ C_{\Omega(x)}(x) &\asymp \sqrt{2\pi}\cdot\frac{(\log\log x)^{\frac{1}{2}}}{\log x}\cdot\left[\frac{(\log\log x)^{k-1}}{(k-1)!}\left(1-\frac{1}{\log x}\right)+\frac{(\log\log x)^{k-2}}{(\log x)(k-2)!}\right] \\ &\asymp \sqrt{2\pi}\cdot\frac{(\log\log x)^{k-\frac{1}{2}}}{(\log x)(k-1)!} =: \widehat{C}_{k,**}(x), \text{ as } x\to\infty.\end{aligned}$$

The ordinary Abel summation formula, and integration by parts, implies that we obtain the main term

$$\begin{aligned}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n) &\asymp \int \widehat{C}_{k,**}(x)dx \\ &\asymp 2\sqrt{2\pi}\cdot x\times\frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!}.\end{aligned}$$

□

Lemma 4.4. *We have that as $n\to\infty$*

$$\mathbb{E}\left[C_{\Omega(n)}(n)\right] \asymp 2\sqrt{2\pi}\cdot(\log n)\sqrt{\log\log n}.$$

Proof. We first compute the following summatory function by applying Corollary 4.3:

$$\sum_{k=1}^{\log\log x}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp 2\sqrt{2\pi}\cdot x\cdot(\log x)\sqrt{\log\log x}. \tag{20}$$

We claim that

$$\sum_{n\leq x}C_{\Omega(n)}(n) = \sum_{k=1}^{\log_2(x)}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp \sum_{k=1}^{\log\log x}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n). \tag{21}$$

If equation (21) holds, then (20) clearly implies our result. To prove (21) it suffices to show that

$$\frac{\sum\limits_{\log\log x<k\leq\log_2(x)}\sum\limits_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n)}{\sum\limits_{k=1}^{\log\log x}\sum\limits_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n)} = o(1), \text{ as } x\to\infty. \tag{22}$$

We define the following component sums for large x and $0<\varepsilon<1$ so that $(\log\log x)^{\frac{\varepsilon\log\log x}{\log\log\log x}} = o(\log x)$:

$$S_{2,\varepsilon}(x) := \sum_{\log\log x<k\leq\frac{1}{\log 2}\cdot(\log\log x)^{\frac{\varepsilon\log\log x}{\log\log\log x}}}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n).$$

Then

$$\sum_{k=\log\log x}^{\log_2(x)} \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \gg S_{2,\varepsilon}(x),$$

with equality as $\varepsilon \to 1$ so that the upper bound of summation tends to $\log_2 x = \frac{\log x}{\log 2}$. Observe that whenever $\Omega(n) = k$, we have that $C_{\Omega(n)}(n) \leq k!$, with equality at the upper bound precisely when $\mu^2(n) = 1$, or equivalently when n is squarefree. We can then bound the sums defined above using Theorem 2.3 with $r := \frac{k}{\log\log x}$ for large $x \to \infty$ by

$$\begin{aligned}
S_{2,\varepsilon}(x) &= \sum_{\log\log x \leq k \leq \frac{1}{\log 2}\cdot(\log\log x)^{\frac{\varepsilon\log\log x}{\log\log\log x}}} \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \\
&\ll \sum_{k=\log\log x}^{\frac{1}{\log 2}\cdot(\log\log x)^{\frac{\varepsilon\log\log x}{\log\log\log x}}} \frac{\widehat{\pi}_k(x)}{x} \cdot k! \\
&\ll \sum_{k=\log\log x}^{\frac{1}{\log 2}\cdot(\log\log x)^{\frac{\varepsilon\log\log x}{\log\log\log x}}} (\log x)^{\frac{k}{\log\log x}-1-\frac{k}{\log\log x}(\log k - \log\log\log x)} \cdot \left(\frac{k}{e}\right)^k \sqrt{2\pi k} \\
&\ll \sum_{k=\log\log x}^{\frac{1}{\log 2}\cdot(\log\log x)^{\frac{\varepsilon\log\log x}{\log\log\log x}}} (\log x)^{\frac{k}{\log\log x}-1} e^{-k}\sqrt{k} \\
&\ll \frac{1}{(\log x)} \times \int_{\log\log x}^{\frac{1}{\log 2}\cdot(\log\log x)^{\frac{\varepsilon\log\log x}{\log\log\log x}}} (\log x)^{\frac{t}{\log\log x}} e^{-t}\sqrt{t}\cdot dt \\
&\ll \sqrt{\log x}.
\end{aligned}$$

Thus by (20) the ratio in (22) clearly tends to zero. ☐

Corollary 4.5. *We have that as $n \to \infty$, the average order of the unsigned inverse sequence satisfies*

$$\mathbb{E}|g^{-1}(n)| \asymp (\log n)^2 \sqrt{\log\log n}.$$

Proof. We use the formula from Lemma 4.4 to find $\mathbb{E}[C_{\Omega(n)}(n)]$ up to a small bounded multiplicative constant factor as $n \to \infty$. This implies that for large t

$$\begin{aligned}
\int \frac{\mathbb{E}[C_{\Omega(t)}(t)]}{t} dt &\asymp \sqrt{2\pi}\cdot(\log t)^2\sqrt{\log\log t} - \frac{\pi}{2}\operatorname{erfi}\left(\sqrt{2\log\log t}\right) \\
&\asymp \sqrt{2\pi}\cdot(\log t)^2\sqrt{\log\log t}.
\end{aligned}$$

Recall from the introduction that the summatory function of the squarefree integers is approximated by

$$Q(x) := \sum_{n\leq x} \mu^2(n) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Therefore summing over the formula from (14) we find that

$$\begin{aligned}
\mathbb{E}|g^{-1}(n)| &= \frac{1}{n} \times \sum_{d\leq n} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{n}{d} \right\rfloor\right) \\
&\sim \sum_{d\leq n} C_{\Omega(d)}(d) \left[\frac{6}{d\cdot\pi^2} + O\left(\frac{1}{\sqrt{dn}}\right)\right]
\end{aligned}$$

$$\asymp \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d<n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right]$$
$$\asymp \frac{6\sqrt{2}}{\pi^{\frac{3}{2}}}(\log n)^2\sqrt{\log\log n}. \qquad \square$$

Theorem 4.6. *Let the absolute constant*

$$\hat{C}_*(C) := \lim_{n\to\infty} \frac{\mathbb{E}[C_{\Omega(n)}(n)]}{(\log n)\sqrt{2\sqrt{2\pi}\cdot\log\log n}}.$$

Let the mean and variance parameter analogs be denoted by

$$\mu_x(C) := \log\log x - \log\left(\frac{1}{2}\hat{C}_*(C)\right), \qquad \text{and} \qquad \sigma_x(C) := \sqrt{\log\log x}.$$

Let $Y > 0$ be fixed. Then we have uniformly for all $-Y \le z \le Y$ that

$$\frac{1}{x}\cdot\#\left\{2 \le n \le x : \frac{C_{\Omega(n)}(n) - \mu_x(C)}{(\log x)\sqrt{\log\log x}\cdot\sigma_x(C)} \le z\right\} = \Phi(z) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x\to\infty.$$

Proof. Fix any $Y > 0$ and set $z \in [-Y, Y]$. For large x and $n \le x$, define the following auxiliary variables:

$$\alpha_n := \frac{C_{\Omega(n)}(n) - \mu_n(C)}{\sigma_n(C)}, \quad \text{and} \quad \beta_{n,x} := \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)}.$$

Let the corresponding densities be defined by the functions

$$\Phi_1(x,z) := \frac{1}{x}\cdot\#\{n \le x : \alpha_n \le z\},$$

and

$$\Phi_2(x,z) := \frac{1}{x}\cdot\#\{n \le x : \beta_{n,x} \le z\}.$$

We first argue that it suffices to consider the distribution of $\Phi_2(x,z)$ as $x\to\infty$ in place of $\Phi_1(x,z)$ to obtain our desired result. The difference of the two auxiliary variables is neglibible as $x\to\infty$ for $(n,x)\in[1,\infty)^2$ taken over the ranges that contribute the non-trivial weight to the main term of each density function. In particular, we have for $\sqrt{x} \le n \le x$ and $C_{\Omega(n)}(n) \le 2\cdot\mu_x(C)$ that the following is true:

$$|\alpha_n - \beta_{n,x}| \ll \frac{1}{\sigma_x(C)} \xrightarrow{x\to\infty} 0.$$

Thus we can replace α_n by $\beta_{n,x}$ and estimate the limiting densities corresponding to these alternate terms. The rest of our argument follows the method in the proof of the related theorem in [11, Thm. 7.21; §7.4] closely. Readers familiar with the reference will see many parallels to those constructions.

We use the formula proved in Corollary 4.3 to estimate the densities claimed within the ranges bounded by z as $x\to\infty$. Let $k\ge 1$ be a natural number and set $k := t + \mu_x(C)$. We define the small parameter $\delta_{t,x} := \frac{t}{\mu_x(C)}$. When $|t| \le \frac{1}{2}\mu_x(C)$, we have by Stirling's formula that

$$\frac{1}{x}\times\sum_{\substack{n\le x\\ \Omega(n)=k}} C_{\Omega(n)}(n)\hat{C}_*(C)\times\frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!}$$
$$\sim \frac{(\log x)}{(1+\frac{1}{2k})}\cdot e^t(1+o(1))^{k+\frac{1}{2}}\times(1+\delta_{t,x})^{-\mu_x(C)(1+\delta_{t,x})-\frac{1}{2}}.$$

We have the uniform estimate that $\log(1+\delta_{t,x}) = \delta_{t,x} - \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)$ whenever $|\delta_{t,x}| \leq \frac{1}{2}$. Then we can expand the factor involving $\delta_{t,x}$ in the previous equation as follows:

$$\begin{aligned}(1+\delta_{t,x})^{-\mu_x(C)(1+\delta_{t,x})-\frac{1}{2}} &= \exp\left(\left(\frac{1}{2}+\mu_x(C)(1+\delta_{t,x})\right) \times \left(-\delta_{t,x} + \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)\right)\right) \\ &= \exp\left(-t - \frac{t+t^2}{2\mu_x(C)} + \frac{t^2}{4\mu_x(C)^2} + O\left(\frac{|t|^3}{\mu_x(C)^2}\right)\right).\end{aligned}$$

For both $|t| \leq \mu_x(C)^{1/2}$ and $\mu_x(C)^{1/2} < |t| \leq \mu_x(C)^{2/3}$, we see that

$$\frac{t}{\mu_x(C)} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Similarly, for $|t| \leq 1$ and $|t| > 1$, we see that both

$$\frac{t^2}{\mu_x(C)^2} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Let the corresponding error terms in (x,t) be denoted by

$$\widetilde{E}(x,t) := O\left(\frac{1}{\sigma_x(C)} + \frac{|t|^3}{\mu_x(C)^2}\right).$$

Combining these estimates with the previous computations, we can deduce that uniformly for $|t| \leq \mu_x(C)^{2/3}$

$$\frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!} \sim \frac{(\log x)\sqrt{\log\log x}}{\sqrt{2\pi}\cdot\sigma_x(C)} \cdot \exp\left(-\frac{t^2}{2\sigma_x(C)^2}\right) \times \left[1+\widetilde{E}(x,t)\right].$$

It follows that for $1 \leq k \leq \log\log x$

$$f(k,x) := \frac{1}{x} \times \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \sim \frac{(\log x)\sqrt{\log\log x}}{\sqrt{2\pi}\cdot\sigma_x(C)} \cdot \exp\left(-\frac{(k-\mu_x(C))^2}{2\sigma_x(C)^2}\right) \times \left[1+\widetilde{E}(x,|k-\mu_x(C)|)\right].$$

By the same argument utilized in the proof of Lemma 4.4, we see that the contributions of these summatory functions for $k \leq \mu_x(C) - \mu_x(C)^{2/3}$ is negligible. We also require that $k \leq \log\log x$ for all large x as we have worked out in Theorem 4.2. We then sum over a corresponding range of

$$\mu_x(C) - \mu_x(C)^{2/3} \leq k \leq \mu_x(C) + z\cdot\sigma_x(C),$$

to approximate the stated normalized densities.

Since our target probability density function approximating the PDF of the normal distribution is given by

$$f(k,x) \to \frac{1}{\sqrt{2\pi}\cdot\sigma_x(C)} \times e^{\frac{-(t-\mu_x(C))^2}{2\sigma_x^2}},$$

we perform the change of variable $t \mapsto (\log x)\sqrt{\log\log x}$ to obtain the normalized form of our theorem stated above. Then finally as $x\to\infty$, the three terms that result (one main term and two error terms, respectively) can be considered to each correspond to a Riemann sum for an associated integral whose limiting formula corresponds to a main term given by the standard normal CDF at z. $\square$

Corollary 4.7. *Let $Y > 0$. Suppose that $\mu_x(C)$ and $\sigma_x(C)$ are defined as in Theorem 4.6 for large $x > e$. Uniformly for $Y > 0$ and all $-Y \leq z \leq Y$ we have that*

$$\frac{1}{x}\cdot\#\left\{2\leq n\leq x : |g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq z\right\} = \Phi\left(\frac{\frac{\pi^2}{6}z - \mu_x(C)}{(\log x)(\log\log x)}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x\to\infty.$$

Proof. We claim that

$$|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2}C_{\Omega(n)}(n).$$

Since $|g^{-1}(n)| = \sum_{d|n} C_{\Omega(d)}(d)$ for all squarefree $n \geq 1$, we obtain that

$$\frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| = \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(x)}(x)] + \sum_{d<x} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O(1).$$

Let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) := f(x) - f(x-1)$. Then from the proof of Corollary 4.5, we see that for large n

$$\begin{aligned} |g^{-1}(n)| &= \Delta_n(n \cdot \mathbb{E}|g^{-1}(n)|) \sim \Delta_n\left(\sum_{d \leq n} \frac{6}{\pi^2} \cdot C_{\Omega(d)}(d) \cdot \frac{n}{d}\right) \\ &= \frac{6}{\pi^2}\left[C_{\Omega(n)}(n) + \sum_{d<n} C_{\Omega(d)}(d)\frac{n}{d} - \sum_{d<n} C_{\Omega(d)}(d)\frac{(n-1)}{d}\right] \\ &\sim \frac{6}{\pi^2}C_{\Omega(n)}(n) + \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n-1)|, \text{ as } n \to \infty. \end{aligned}$$

Since $\mathbb{E}|g^{-1}(n-1)| \sim \mathbb{E}|g^{-1}(n)|$ for all sufficiently large n, by Corollary 4.5, the result finally follows by a normalization of Theorem 4.6. □

Lemma 4.8. *Suppose that $\mu_x(C)$ and $\sigma_x(C)$ are defined as in Theorem 4.6 for large $x > e$. Let the absolute constant*

$$\hat{G}_* := \lim_{n\to\infty} \frac{\mathbb{E}|g^{-1}(n)|}{(\log n)^2\sqrt{\log\log n}}.$$

If x is sufficiently large and we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds as $x \to \infty$:

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq 0\right) = \frac{1}{2} + o(1) \tag{A}$$

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{\alpha \cdot \mathbb{E}|g^{-1}(x)| \cdot \sqrt{\log\log x}}{\log x}\right) = \Phi\left(\frac{\alpha\pi^2\hat{G}_*}{6}\right) + o(1), \alpha \in \mathbb{R}. \tag{B}$$

Proof. Each of these results is a consequence of Corollary 4.7. Let the densities $\gamma_z(x)$ be defined for $z \in \mathbb{R}$ and sufficiently large $x > e$ as follows:

$$\gamma_z(x) := \frac{1}{x} \cdot \#\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq z\}.$$

To prove (A), observe that by Corollary 4.7 for $z := 0$ we have that

$$\gamma_0(x) = \Phi\left(-\frac{\mu_x(C)}{(\log x)(\log\log x)}\right) + o(1), \text{ as } x \to \infty.$$

We have the reflection identity for the normal distribution CDF $\Phi(z) = 1 - \Phi(-z)$, where

$$\Phi(z) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right).$$

Since we have a MacLaurin series for the error function for $|z| < 1$ given by

$$\Phi(-z) = 1 - \Phi(z) = \frac{1}{2} - \frac{z}{\sqrt{2\pi}}\left(1 - \frac{z^2}{12} + O(|z|^5)\right),$$

the result in (A) clearly holds as $x \to \infty$. The result in (B) follows easily by rearrangement and applying Corollary 4.5. □

Remark 4.9. It follows from the lemma that as $n \to \infty$,

$$|g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|,$$

on a set of asymptotic density one. That is, we can see that

$$\lim_{x\to\infty} \frac{1}{x} \cdot \#\left\{n \leq x : |g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|\right\} = 1.$$

5 Proofs of new formulas and limiting relations for $M(x)$

5.1 Establishing initial asymptotic bounds on the summatory function $G^{-1}(x)$

The most recent best known upper bound on $L(x)$ (assuming the RH) is established by Humphries based on Soundararajan's result bounding $M(x)$ stated in the following form [6]:

$$L(x) = O\left(\sqrt{x} \cdot \exp\left((\log x)^{\frac{1}{2}} (\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right), \text{ for any } \varepsilon > 0; \text{ as } x \rightarrow \infty. \tag{23}$$

Theorem 5.1. *Let $L(x) := \sum_{n \leq x} \lambda(n)$ for $x \geq 1$. We have that for almost every sufficiently large x, the summatory function of $g^{-1}(n)$ is bounded by*

$$G^{-1}(x) = O\left(\max_{1 \leq t \leq x} |L(t)| \cdot \mathbb{E}|g^{-1}(x)|\right).$$

If the RH is true, then for any $\varepsilon > 0$ and almost every large integer $x \geq 1$

$$G^{-1}(x) = O\left(\sqrt{x} \cdot (\log x)^2 \sqrt{\log\log x} \times \exp\left(\sqrt{\log x} \cdot (\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right).$$

Proof. We write the next formulas for $G^{-1}(x)$ by Abel summation for almost every large $x \geq 1$ by applying the mean value theorem:

$$\begin{aligned} G^{-1}(x) &= \sum_{n \leq x} \lambda(n) |g^{-1}(n)| \\ &= L(x)|g^{-1}(x)| - \int L(x) \frac{d}{dx} \left|g^{-1}(x)\right| dx \\ &= O\left(\max_{1 \leq t \leq x} |L(t)| \cdot \mathbb{E}|g^{-1}(x)|\right). \end{aligned} \tag{24}$$

The proof of this result appeals to the material we used to establish the more probabilistic interpretations of the distribution of $|g^{-1}(n)|$ as $n \rightarrow \infty$ from Section 4.

We need to bound the sums of the maximal values of $|g^{-1}(n)|$ over $n \leq x$ as $x \rightarrow \infty$. We know by a result of Robin that [17]

$$\omega(n) \ll \frac{\log n}{\log\log n}, \text{ as } n \rightarrow \infty.$$

Recall that the values of $|g^{-1}(n)|$ are maximized when n is squarefree with

$$|g^{-1}(n)| \leq \sum_{j=0}^{\omega(n)} \binom{\omega(n)}{j} j! \ll \Gamma(\omega(n) - 1) \ll \left(\frac{\log n}{\log\log n}\right)^{\frac{\log n}{\log\log n} + \frac{1}{2}}.$$

Now because we have deduced that the set of n on which $|g^{-1}(n)|$ is substantially larger than its average order is asymptotically thin, we can see that the contributions at these maximal values such that the values are of order greater than $\mathbb{E}|g^{-1}(x)|$ are bounded by

$$\begin{aligned} \left| \int_{x - o\left(\frac{1}{x}\right)}^{x} L'(t) |g^{-1}(t)| dt \right| &\ll \int_{x - o\left(\frac{1}{x}\right)}^{x} \left(\frac{\log t}{\log\log t}\right)^{\frac{\log t}{\log\log t} + \frac{1}{2}} dt \\ &\ll x \left(\frac{\log x}{\log\log x}\right)^{\frac{\log x}{\log\log x} + \frac{1}{2}} \times \int_{1 - o\left(\frac{1}{x^2}\right)}^{1} dv \\ &\ll o\left(\frac{(\log x)^{\frac{3}{2}}}{\sqrt{\log\log x}}\right), \text{ as } x \rightarrow \infty. \end{aligned}$$

The formula from (18) then implies that for almost every large x and some finite, bounded $\delta > 0$

$$\begin{aligned} G^{-1}(x) &= O\left(\int L'(x)|g^{-1}(x)|dx\right) \\ &= O\left(\mathbb{E}|g^{-1}(x)| \times \int L'(x)dx\right). \end{aligned}$$

Thus we obtain the conslusion of the second result by applying Humpries' result in (23) in tandem with Corollary 4.5. □

5.2 Bounding $M(x)$ by asymptotics for $G^{-1}(x)$

Proposition 5.2. *For all sufficiently large x, we have that the Mertens function satisfies*

$$M(x) = G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} G^{-1}(k)\left[\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor\frac{x}{k+1}\right\rfloor\right)\right]. \tag{25}$$

Proof. We know by applying Corollary 1.4 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)\left[\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) + 1\right] \\ &= G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} g^{-1}(k)\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) \\ &= G^{-1}(x) + G^{-1}\left(\left\lfloor\frac{x}{2}\right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left[\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor\frac{x}{k+1}\right\rfloor\right)\right]. \end{aligned}$$

The upper bound on the sum is truncated in the second equation above due to the fact that $\pi(1) = 0$. The third formula follows from summation by parts. □

Lemma 5.3. *For sufficiently large x, integers $k \in \left[\sqrt{x}, \frac{x}{2}\right]$ and $m \geq 0$, we have that*

$$\frac{x}{k \cdot \log^m\left(\frac{x}{k}\right)} - \frac{x}{(k+1)\cdot \log^m\left(\frac{x}{k+1}\right)} \gg \frac{x}{(\log x)^m \cdot k(k+1)}, \tag{A}$$

and

$$\sum_{k=\sqrt{x}}^{\frac{x}{2}} \frac{x}{k(k+1)} = \sum_{k=\sqrt{x}}^{\frac{x}{2}} \frac{x}{k^2} + O(1). \tag{B}$$

Proof. The proof of (A) is obvious since for $k_0 \in \left[\sqrt{x}, \frac{x}{2}\right]$ we have that

$$\log(2)(1+o(1)) \leq \log\left(\frac{x}{k_0}\right) \leq \log(x).$$

To prove (B), notice that

$$\frac{x}{k(k+1)} - \frac{x}{k^2} = -\frac{x}{k^2(k+1)}.$$

Then we see that

$$\left|\int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^2(t+1)}dt\right| \leq \left|\int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^3}dt\right| \asymp 1.$$ □

Corollary 5.4. *We have that as $x \to \infty$*

$$M(x) = O\left(G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) + \frac{x}{\log x} \times \max_{1 \leq k \leq \sqrt{x}} |G^{-1}(k)| + (\log x)^2 (\log\log x)^{\frac{3}{2}}\right).$$

Proof. We need to first bound the prime counting function differences in the formula given by Proposition 5.2. We will require the following known bounds on the prime counting function due to Rosser and Schoenfeld for large $x > 59$ [18, Thm. 1]:

$$\frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right). \tag{26}$$

The result in (26) together with Lemma 5.3 implies that for $\sqrt{x} \leq k \leq \frac{x}{2}$

$$\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) = O\left(\frac{x}{k^2 \cdot \log\left(\frac{x}{k}\right)}\right).$$

We will rewrite the intermediate formula from the proof of Proposition 5.2 as a sum of two components with summands taken over disjoint intervals. For large $x > e$, let

$$\begin{aligned} S_1(x) &:= \sum_{1 \leq k \leq \sqrt{x}} g^{-1}(k) \pi\left(\frac{x}{k}\right) \\ S_2(x) &:= \sum_{\sqrt{x} < k \leq \frac{x}{2}} g^{-1}(k) \pi\left(\frac{x}{k}\right). \end{aligned}$$

We assert by the asymptotic formulas for the prime counting function that

$$S_1(x) = O\left(\frac{x}{\log x} \times \sum_{k \leq \sqrt{x}} \frac{|G^{-1}(k)|}{k^2}\right)$$

To bound the second sum, we perform summation by parts as in the proof of the proposition, apply the bound above for the difference of the summand functions to obtain that

$$\begin{aligned} S_2(x) &= O\left(G^{-1}\left(\frac{x}{2}\right) + \int_{\sqrt{x}}^{\frac{x}{2}} \frac{G^{-1}(t)}{t^2 \log\left(\frac{x}{t}\right)} dt\right) \\ &= O\left(G^{-1}\left(\frac{x}{2}\right) + (\log\log x) \times \max_{\sqrt{x} < k < \frac{x}{2}} \frac{|G^{-1}(k)|}{k}\right). \end{aligned}$$

The rightmost term in the previous bound is known to satisfy $\frac{|G^{-1}(k)|}{k} \ll \mathbb{E}|g^{-1}(k)|$ as $k \to \infty$. The conclusion follows since the average order of $|g^{-1}(n)|$ is non-decreasing with sufficiently large n. $\square$

6 Conclusions

We have identified a key sequence, $\{g^{-1}(n)\}_{n\geq 1}$, which is the Dirichlet inverse of the shifted additive function, $g := \omega + 1$. In general, we find that the Dirichlet inverse of any arithmetic function f such that $f(1) \neq 0$ is expressed at each $n \geq 2$ as a signed sum of m-fold convolutions of f with itself for $1 \leq m \leq \Omega(n)$. As we discussed in the remarks in Section 3.3, it happens that there is a natural combinatorial interpretation to the distribution of distinct values of $|g^{-1}(n)|$ for $n \leq x$ involving the primes $p \leq x$ at large x. In particular, the magnitude of $|g^{-1}(n)|$ depends only on the pattern of the exponents of the prime factorization of n in so much as $|g^{-1}(n_1)| = |g^{-1}(n_2)|$ whenever $\omega(n_1) = \omega(n_2)$, $\Omega(n_1) = \Omega(n_2)$, and where the is a one-to-one correspondence $\nu_{p_1}(n_1) = \nu_{p_2}(n_2)$ between the distinct primes $p_1|n_1$ and $p_2|n_2$. The signedness of $g^{-1}(n)$ is given by $\lambda(n)$ for all $n \geq 1$. This leads to a familiar dependence of the summatory functions $G^{-1}(x)$ on the distribution of the summatory function $L(x)$.

Section 5 provides equivalent characterizations of the limiting properties of $M(x)$ by exact formulas and asymptotic relations involving $G^{-1}(x)$ and $L(x)$. We emphasize that our new work on the Mertens function proved within this article is significant in providing a new window through which we can view bounding $M(x)$, rather than in proving explicit new best known bounds on the classical function at this point. The computational data generated in Table T.1 suggests numerically, especially when compared to the initial values of $M(x)$, that the distribution of $|G^{-1}(x)|$ may be easier to work with that those of $|M(x)|$ or $|L(x)|$. The remarks given in Section 3.3 about the direct combinatorial relation of the distinct (and repitition of) values of $|g^{-1}(n)|$ for $n \leq x$ are also suggestive towards bounding main terms for $G^{-1}(x)$ along infinite subsequences.

One topic that we do not touch on in the article is to consider the correlation between $\lambda(n)$ and the unsigned sequence of $|g^{-1}(n)|$ whose limiting distribution is proved in Corollary 4.7. Much in the same way that variants of the Erdös-Kac theorem are proved by defining the random variables related to $\omega(n)$, we suggest an analysis of the summatory function $G^{-1}(x)$ by scaling the explicitly distributed $|g^{-1}(n)|$ for $n \leq x$ as $x \to \infty$ by its signed weight of $\lambda(n)$ using an initial heuristic along these lines. Another experiment illustrated in the online computational reference [19] suggests that for many, if not most sufficiently large x, we may consider replacing the summatory function with terms weighted by $\lambda(n)$

$$G^{-1}(x) := \sum_{n\leq x} \lambda(n)|g^{-1}(n)|, x \geq 1,$$

by alternate sums that average these sequences differently while still preserving the original asymptotic order of $|G^{-1}(x)|$. For example, each of the following three summatory functions offers a unique interpretation of an average of sorts that "mixes" the values of $\lambda(n)$ with the unsigned sequence $|g^{-1}(n)|$ over $1 \leq n \leq x$:

$$\begin{aligned}
G_{*}^{-1}(x) &:= \sum_{n\leq x} \frac{1}{2\gamma - 1 + \log n} \times \sum_{d|n} \lambda\left(\frac{n}{d}\right) |g^{-1}(d)| \\
G_{**}^{-1}(x) &:= \sum_{n\leq x} \frac{1}{2\gamma - 1 + \log n} \times \sum_{d|n} \lambda\left(\frac{n}{d}\right) g^{-1}(d) \\
G_{***}^{-1}(x) &:= \sum_{n\leq x} \frac{1}{2\gamma - 1 + \log n} \times \sum_{d|n} g^{-1}(d).
\end{aligned}$$

Then based on preliminary numerical results, a large proportion of the $y \leq x$ for large x satisfy

$$\left|\frac{G_{*}^{-1}(y)}{G^{-1}(y)}\right|^{-1}, \left|\frac{G_{**}^{-1}(y)}{G^{-1}(y)}\right|, \left|\frac{G_{***}^{-1}(y)}{G^{-1}(y)}\right| \in (0, 3].$$

Variants of this type of summatory function identity exchange are similarly suggested for future work on these topics.

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