Exact formulas for partial sums of the Möbius function expressed by partial sums of weighted Liouville functions

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High-level overview and takeaways of the talk

- Study new expressions for partial sums of a signed classical function
- Identify new unsigned sequences through which we can express these partial sums, or summatory functions
- Try to keep things in perspctive at a high level
    1. Write the Mertens function via partial sums depending on $\lambda(n)$-signed terms; and then **motivate why we should care** by arguing that the unsigned magnitudes of these summands are "nicer"
    2. Conjecture limiting CLT type results for the distributions certain unsigned sequences we will precisely identify in thew coming slides.
    3. State formulas for smaller-order moments of the logarithm of the unsigned functions

Definitions and notation

- The function $\omega(n)$ (and $\Omega(n)$) counts the number of distinct prime factors of any n without (and with, respectively) multiplicity.
- Recall that the **Möbius function** is defined as

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \text{ (i.e., if this } n \geq 2 \text{ is squarefree);} \\ 0, & \text{otherwise.} \end{cases}$$

- The summatory function given by the **Mertens function** is defined as follows:

$$M(x) := \sum_{n \leq x} \mu(n), \text{ for } x \geq 1.$$

- Related functions include the **Liouville lambda function**, $\lambda(n) := (-1)^{\Omega(n)}$ for $n \geq 1$, and its partial sums $L(x) := \sum_{n \leq x} \lambda(n)$ for $x \geq 1$.

Definitions of auxiliary unsigned functions

- We fix the notation for the Dirichlet inverse function (inverse taken with respect to the operation of Dirichlet convolution, e.g., $(f * h)(n) = \sum_{d|n} f(d) h\left(\frac{n}{d}\right)$) as follows:

$$g(n) := (\omega + \mathbb{1})^{-1}(n), \text{ for } n \geq 1.$$

- We define the partial sums for $x \geq 1$ as

$$G(x) := \sum_{n \leq x} g(n) = \sum_{n \leq x} \lambda(n) |g(n)|.$$

- Where did the definition of $g(n)$ come from? Its partial sums are related to the classical prime counting function by

$$\chi_{\mathbb{P}}(n) + \delta_{n,1} = (\omega + \mathbb{1}) * \mu(n), n \geq 1,$$

by Möbius inversion since

$$\omega(n) = \sum_{p|n} 1 = \sum_{d|n} \chi_{\mathbb{P}}(d).$$

New explicit formulas for $M(x)$

Theorem

For all $x \geq 1$

(1a) $$M(x) = G(x) + \sum_{1 \leq k \leq x} |g(k)| \pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) \lambda(k),$$

(1b) $$M(x) = G(x) + \sum_{1 \leq k \leq \frac{x}{2}} \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right) G(k),$$

(1c) $$M(x) = G(x) + \sum_{p \leq x} G\left(\left\lfloor \frac{x}{p} \right\rfloor\right).$$

Remarks on the significance of the new formulas for $M(x)$ in terms of $G(x)$

- The summands are sign-weighted by $\lambda(n)$ with unsigned magnitudes that have "nicer" properties.
- For comparision, we have the less predictably signed expansion:

$$M(x) = \sum_{d \leq \sqrt{x}} \mu(d) L\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), \text{ for } x \geq 1. \tag{2}$$

- Why are the unsigned summands in the previous theorem so much "nicer" than classical expansions of $M(x)$ like in (2)?
- We conjecture that there is limiting CLT that "characterizes" the spread of the unsigned values of $|g(n)|$ from $2 \leq n \leq x$ as $x \to \infty$.

Properties of the unsigned sequences

- For all $n \geq 1$, $\operatorname{sgn}(g(n)) = \lambda(n)$
- An exact expression is given by:

$$\lambda(n)g(n) = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_\Omega(d), n \geq 1,$$

where

$$C_\Omega(n) = \begin{cases} 1, & \text{if } n = 1; \\ (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}, & \text{if } n \geq 2. \end{cases}$$

- For all squarefree integers $n \geq 1$, we have that

$$|g(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \times m!$$

Properties of the unsigned sequences (cont'd)

- Recall the Erdős-Kac theorem:

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : \frac{\omega(n) - \log\log x}{\sqrt{\log\log x}} \leq z\right\} = \Phi(z) + o(1).$$

- In analog, we conjecture that there are absolute constants $B_0, D_0 > 0$ such that for $z \in \mathbb{R}$ as $x \to \infty$

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : \frac{\log C_\Omega(n) - B_0 \cdot (\log\log x)(\log\log\log x)}{D_0 \cdot (\log\log x)(\log\log\log x)} \leq z\right\} = \Phi(z) + o(1).$$

- If the previous conjecture holds, then for any $y > 0$

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : -y \leq |g(n)| - \frac{1}{n} \times \sum_{k \leq n} |g(k)| \leq y\right\}$$
$$= \Phi\left(\frac{\log\left(\frac{\pi^2 y}{6}\right) - B_0 \cdot (\log\log x)(\log\log\log x)}{D_0 \cdot (\log\log x)(\log\log\log x)}\right) + o(1).$$

Average order asymptotics

- There is an absolute constant $B_0 > 0$ so that

$$\frac{1}{n} \times \sum_{k \leq n} \log C_{\Omega}(k) = B_0 \cdot (\log\log n)(\log\log\log n)(1 + o(1)).$$

- The average order of $\log|g(n)|$ is given by

$$\frac{1}{n} \times \sum_{k \leq n} \log|g(k)| = \left(\frac{B_0}{2} \cdot (\log\log n)(\log\log\log n) - \frac{1}{2}\log\left(\frac{\pi^2}{6}\right)\right)(1 + o(1)).$$

- The variance of $\log C_{\Omega}(n)$ (and $\log|g(n)|$ up to a constant factor) is

$$\sigma_{\Omega}(x) = D_0\sqrt{x}(\log\log x)(\log\log\log x)(1 + o(1)),$$

for $D_0 > 0$ an absolute constant.

Conclusions – Taking a step back – What we've done

- We defined $g(n) := (\omega + \mathbb{1})^{-1}(n)$ as the shifted Dirichlet inverse of the strongly additive function, $\omega(n)$.
- We precisely connected $C_\Omega(n)$ to $g(n)$ and used it to prove formulas for the average orders of the unsigned sequences.
- We have conjectured a limiting CLT for the distribution of $\log C_\Omega(n)$ (and so $|g(n)|$) for $n \leq x$ as $x \to \infty$.
- We connected the Mertens function $M(x)$ with the partial sums $G(x) := \sum_{n \leq x} \lambda(n)|g(n)|$ via exact formulas for all $x \geq 1$.

Conclusions

The End

Questions?

Comments?

Feedback?

Thank you for attending!

n	n	$g(n)$	$\frac{\sum_{d\mid n} C_{\Omega}(d)}{\lvert g(n)\rvert}$	$G(x)$
2	2^1	-2	1.000	-1
3	3^1	-2	1.000	-3
4	2^2	2	1.500	-1
5	5^1	-2	1.000	-3
6	$2^1 3^1$	5	1.000	2
7	7^1	-2	1.000	0
8	2^3	-2	2.000	-2
9	3^2	2	1.500	0
10	$2^1 5^1$	5	1.000	5
11	11^1	-2	1.000	3
12	$2^2 3^1$	-7	1.286	-4
13	13^1	-2	1.000	-6
14	$2^1 7^1$	5	1.000	-1
15	$3^1 5^1$	5	1.000	4
16	2^4	2	2.500	6
17	17^1	-2	1.000	4
18	$2^1 3^2$	-7	1.286	-3
19	19^1	-2	1.000	-5
20	$2^2 5^1$	-7	1.286	-12
21	$3^1 7^1$	5	1.000	-7
22	$2^1 11^1$	5	1.000	-2
23	23^1	-2	1.000	-4
24	$2^3 3^1$	9	1.556	5
25	5^2	2	1.500	7
26	$2^1 13^1$	5	1.000	12
27	3^3	-2	2.000	10

n	n	$g(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}}{\lvert g(n)\rvert}$	$G(x)$
28	$2^2 7^1$	-7	1.286	3
29	29^1	-2	1.000	1
30	$2^1 3^1 5^1$	-16	1.000	-15
31	31^1	-2	1.000	-17
32	2^5	-2	3.000	-19
33	$3^1 11^1$	5	1.000	-14
34	$2^1 17^1$	5	1.000	-9
35	$5^1 7^1$	5	1.000	-4
36	$2^2 3^2$	14	1.357	10
37	37^1	-2	1.000	8
38	$2^1 19^1$	5	1.000	13
39	$3^1 13^1$	5	1.000	18
40	$2^3 5^1$	9	1.556	27
41	41^1	-2	1.000	25
42	$2^1 3^1 7^1$	-16	1.000	9
43	43^1	-2	1.000	7
44	$2^2 11^1$	-7	1.286	0
45	$3^2 5^1$	-7	1.286	-7
46	$2^1 23^1$	5	1.000	-2
47	47^1	-2	1.000	-4
48	$2^4 3^1$	-11	1.818	-15
49	7^2	2	1.500	-13
50	$2^1 5^2$	-7	1.286	-20
51	$3^1 17^1$	5	1.000	-15
52	$2^2 13^1$	-7	1.286	-22
53	53^1	-2	1.000	-24