# Lower bounds on the summatory function of the Möbius function along infinite subsequences

Maxie Dion Schmidt
Georgia Institute of Technology
School of Mathematics

Last Revised: Tuesday 14th July, 2020 @ 16:32:27 – Compiled with LaTeX2e

**Abstract**

The Mertens function, $M(x) = \sum_{n \leq x} \mu(n)$, is classically defined as the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture states that $|M(x)| < C \cdot \sqrt{x}$ for some absolute $C > 0$ for all $x \geq 1$. This classical conjecture has a well-known disproof due to Odlyzko and té Riele by computation of non-trivial zeta function zeros in conjunction with integral formulas expressing $M(x)$. We prove the unboundedness of $|M(x)|/\sqrt{x}$ using new methods by showing that

$$\limsup_{x \rightarrow \infty} \frac{|M(x)|\sqrt{\log\log x} \cdot (\log\log\log x)^2}{\sqrt{x} \cdot (\log x)^{\frac{1}{4}}} \geq 0.106408.$$

There is a distinct stylistic flavor and new element of combinatorial analysis to our proof combined with the standard methods from analytic, additive and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

**Keywords and Phrases:** *Möbius function; Mertens function; summatory function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting functions; Dirichlet generating function; asymptotic lower bounds; Mertens conjecture.*
**Math Subject Classifications (MSC 2010):** *11N37; 11A25; 11N60; and 11N64.*

# Glossary of special notation and conventions

| Symbol | Definition |
|---|---|
| $\approx$ | We write that $f(x) \approx g(x)$ if $\lvert f(x) - g(x)\rvert = O(1)$ as $x \rightarrow \infty$. |
| $\mathbb{E}[f(x)], \overset{\mathbb{E}}{\sim}$ | We adapt the expectation notation $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \overset{\mathbb{E}}{\sim} h(x)$, to denote that $f$ has an *average order* growth rate of $h(x)$. This means that $\frac{1}{x}\sum_{n \leq x} f(n) \sim h(x)$, or equivalently that $$\lim_{x\rightarrow\infty} \frac{\frac{1}{x}\sum_{n\leq x} f(n)}{h(x)} = 1.$$ |
| $B$ | The absolute constant $B \approx 0.2614972$ from the statement of Mertens theorem. |
| $C_k(n)$ | The sequence is defined recursively for $n \geq 1$ as follows where we assume that $1 \leq k \leq \Omega(n)$: $$C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d\mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$ |
| $[q^n]F(q)$ | The coefficient of $q^n$ in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n\geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n\geq 0} f_n q^n$. |
| $\varepsilon(n)$ | The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic $f$ we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (see below). |
| $f * g$ | The Dirichlet convolution of $f$ and $g$, $(f * g)(n) := \sum\limits_{d\mid n} f(d) g(n/d)$, where the sum is taken over the divisors $d$ of $n$ for $n \geq 1$. |
| $f^{-1}(n)$ | The Dirichlet inverse of $f$ with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ for $n \geq 2$ with $f^{-1}(1) = 1/f(1)$. The Dirichlet inverse of $f$ exists if and only if $f(1) \neq 0$. This inverse function, denoted by $f^{-1}$ when it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$. |
| $\gg, \ll, \asymp$ | For functions $A, B$ in $x$, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A \ll B$ and $B \gg A$, we write $A \asymp B$. |
| $g^{-1}(n), G^{-1}(x)$ | The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$. |
| $H_n$ | The *first-order harmonic numbers*, $H_n := \sum_{k=1}^{n} \frac{1}{k}$, satisfy the limiting asymptotic relation $$\lim_{n\rightarrow\infty} \left[H_n - \log(n)\right] = \gamma,$$ where $\gamma \approx 0.5772157$ denotes Euler's gamma constant. |

| **Symbol** | **Definition** |
|---|---|
| $[n=k]_\delta$,$[\texttt{cond}]_\delta$ | The symbol $[n=k]_\delta$ is a synonym for $\delta_{n,k}$ which is one if and only if $n=k$, and is zero otherwise. For boolean-valued conditions, `cond`, $[\texttt{cond}]_\delta$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention*. |
| $\lambda(n)$ | The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. That is, $\lambda(n) \in \{\pm 1\}$ for all integers $n \geq 1$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$. |
| $\mu(n)$ | The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever $n$ is squarefree. |
| $M(x)$ | The Mertens function is the summatory function of $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum\limits_{n \leq x} \mu(n)$. |
| $\nu_p(n)$ | The valuation function that extracts the maximal exponent of $p$ in the prime factorization of $n$, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^\alpha \vert\vert n$ (or when $p^\alpha$ exactly divides $n$) for $p$ prime, $\alpha \geq 1$ and $n \geq 2$. |
| $\omega(n)$,$\Omega(n)$ | We define the strongly additive function $\omega(n) := \sum_{p \mid n} 1$ and the completely additive function $\Omega(n) := \sum_{p^\alpha \vert\vert n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$. |
| $\pi_k(x), \widehat{\pi}_k(x)$ | The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leq n \leq x$ for $x > 1$ with exactly $k$ distinct prime factors: $\pi_k(x) := \#\{n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$ for $x \geq 2$. |
| $P(s)$ | For complex $s$ with $\operatorname{Re}(s) > 1$, we define the *prime zeta function* to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$. For $\operatorname{Re}(s) > 1$, the prime zeta function is related to $\zeta(s)$ according to the formula $P(s) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log[\zeta(ks)]$. |
| $Q(x)$ | For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. More precisely, this function is summed and identified with its limiting asymptotic formula as $x \to \infty$ in the following form: $Q(x) := \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O(\sqrt{x})$. |
| $\sim$ | We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x \to \infty} \frac{A(x)}{B(x)} = 1$. |
| $\zeta(s)$ | The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\operatorname{Re}(s) > 1$, and by analytic continuation on the rest of the complex plane with the exception of a simple pole at $s = 1$ of residue one. |

# 1 Introduction

## 1.1 Definitions

We define the *Möbius function* to be the signed indicator function of the squarefree integers in the form of [14, A008683]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \text{ and } n \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

There are many variants and special properties of the Möbius function and its generalizations [13, *cf.* §2]. One crucial role of the classical $\mu(n)$ is that the function forms an inversion relation for the divisor sums formed by arithmetic functions convolved with one through *Möbius inversion*:

$$g(n) = (f * 1)(n) \iff f(n) = (g * \mu)(n), \forall n \geq 1.$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined on the positive integers as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as follows [14, A002321]:

$$\{M(x)\}_{x \geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \ldots\}.$$

Clearly, a positive integer $n \geq 1$ is *squarefree*, or contains no (prime power) divisors which are squares, if and only if $\mu^2(n) = 1$. A related summatory function which counts the number of *squarefree* integers $n \leq x$ satisfies [2, §18.6] [14, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers characterized by the sign of the Möbius function are in fact equal as $x \rightarrow \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{x} \overset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{x} \xrightarrow{x \rightarrow \infty} \frac{3}{\pi^2}.$$

## 1.2 Properties

A conventional approach to evaluating the limiting asymptotic behavior of $M(x)$ for large $x \rightarrow \infty$ results by considering an inverse Mellin transformation of the reciprocal of the Riemann zeta function. In particular, since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \cdot \int_1^\infty \frac{M(x)}{x^{s+1}} dx, \operatorname{Re}(s) > 1,$$

we obtain that

$$M(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

The previous two representations lead us to the exact expression of $M(x)$ for any real $x > 0$ given by the next theorem due to Titchmarsh.

**Theorem 1.1** (Analytic Formula for $M(x)$)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each $k$ such that for any real $x > 0$*

$$M(x) = \lim_{k \rightarrow \infty} \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |\operatorname{Im}(\rho)| < T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp\left(-C \cdot \log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan more recently proved new updated estimates bounding $M(x)$ from above for large $x$ in the following forms [15]:

$$\begin{aligned} M(x) &\ll \sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right), \ \forall \epsilon > 0. \end{aligned}$$

## 1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C \cdot \sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens for $C = 1$ and all $x < 10000$. Since its beginnings in 1897, the Mertens conjecture has been disproven by computation of non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele [10]. Since the truth of the conjecture would have implied the RH, more recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound along infinite subsequences, e.g., considering the asymptotics of the function in the limit supremum and limit infimum senses.

A precise statement of this problem is to produce an unconditional proof of whether $\limsup_{x\rightarrow\infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x\rightarrow\infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there are infinite subsequences of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound towards either $\pm\infty$ along the subsequence. We cite that it is only known by computation that [12, *cf.* §4.1] [14, *cf.* A051400; A051401]

$$\limsup_{x\rightarrow\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x\rightarrow\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leq -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [10, 5, 6, 3]. Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies [9]

$$\limsup_{x\rightarrow\infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log\log\log x)^{5/4}} = O(1).$$

## 2 An overview of the core components to the proof

We offer an initial step-by-step summary overview of the core components to our proof outlined in the next points. We hope that this sketch of the logical components to this argument makes the article easier to parse.

**(1)** We prove a matrix inversion formula relating the summatory functions of an arithmetic function $f$ and its Dirichlet inverse $f^{-1}$ (for $f(1) \neq 0$). See Theorem 3.1 in Section 4.

**(2)** This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the prime counting function, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is stated in (1). The link relating our new formula in (1) to canonical additive functions and their distributions lends a recent distinguishing element to the success of the methods in our proof.

**(3)** We tighten bounds from a less classical result proved in [8, §7] providing uniform asymptotic formulas for the summatory functions, $\widehat{\pi}_k(x)$, large $x \gg e$ and $1 \leq k \leq \log\log x$ (see Theorem 3.7). We use this result to bound sums of the form $\sum_{n\leq x} \lambda(n) f(n)$ from below for particular positive arithmetic functions $f$ when $x$ is large.

**(4)** We then turn to estimating the limiting asymptotics of the quasi-periodic function, $|g^{-1}(n)|$, by proving several formulas bounding its average order as $x \to \infty$ in Section 6. We eventually use these estimates to prove a substantially unique new lower bound formulas for the summatory function $G^{-1}(x) := \sum_{n\leq x} \lambda(n)|g^{-1}(n)|$ along certain asymptotically large infinite subsequences (see Theorem 8.6).

**(5)** In Section **??**, we transform the new formula for $M(x)$ in (1) into an integral representation with small bounded error term. To prove that this error term in correct as $x \to \infty$, we require proofs of Erdös-Kac like theorems providing semi-distributions on the values of key component functions and their sums.

**(6)** When we return to step **(2)** with our new lower bounds at hand, we are led to a new unconditional proof of the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. In fact, we recover a quick and rigorous proof of Theorem 3.9 given at the conclusion of Section 8.2.

# 3 A concrete new approach to bounding $M(x)$ from below

## 3.1 Summatory functions of Dirichlet convolutions of arithmetic functions

**Theorem 3.1** (Summatory functions of Dirichlet convolutions)**.** *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of $f$ and $h$, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse of $f$. We have the following exact expressions for the summatory function of $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned}
\pi_{f*h}(x) &:= \sum_{n \leq x} \sum_{d \mid n} f(d) h(n/d) \\
&= \sum_{d \leq x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].
\end{aligned}$$

*Moreover, for all $x \geq 1$*

$$\begin{aligned}
H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\
&= \sum_{n=1}^{x} f^{-1}(n) \pi_{f*h}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).
\end{aligned}$$

**Corollary 3.2** (Convolutions arising from Möbius inversion)**.** *Suppose that $g$ is an arithmetic function such that $g(1) \neq 0$. Define the summatory function of the convolution of $g$ with $\mu$ by $\widetilde{G}(x) := \sum_{n \leq x} (g * \mu)(n)$. The Mertens function is expressed by the sum*

$$M(x) = \sum_{k=1}^{x} \left( \sum_{j = \left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geq 1.$$

**Corollary 3.3** (A motivating special case)**.** *We have exactly that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

## 3.2 An exact expression for $M(x)$ in terms of strongly additive functions

Fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute exactly that (see Table T.1 starting on page 47 of the appendix section)

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these positive terms is given by $\operatorname{sgn}(g^{-1}(n)) = \frac{g^{-1}(n)}{|g^{-1}(n)|} = \lambda(n)$ for all $n \geq 1$ (see Proposition 4.1).

There is not an easy, nor subtle direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences. The distribution of distinct sets of prime exponents is still fairly regular so that $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of $g^{-1}(n)$ over $n \geq 2$:

**Heuristic 3.4** (Symmetry in $g^{-1}(n)$ in the prime factorizations of $n$)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for $= \omega(n_i) \geq 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, $g^{-1}$ has the same values on the squarefree integers with exactly one, two, three, and so on prime factors.

**Conjecture 3.5.** *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

***(A)*** $g^{-1}(1) = 1$*;*

***(B)*** *For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*

***(C)*** *For all squarefree integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!;$$

***(D)*** *If $n \geq 2$ and $\Omega(n) = k$, then*

$$2 \leq |g^{-1}(n)| \leq \sum_{m=0}^{k} \binom{k}{m} \cdot m!.$$

We illustrate parts (B)–(D) of the conjecture clearly using the computation of initial values of this inverse sequence in Table T.1. A proof of (C) in fact follows from Lemma 6.3 stated on page 21. The realization that the beautiful and remarkably simple combinatorial form of property (C) in Conjecture 3.5 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through sums of auxiliary sequences of arithmetic functions (see Section 6).

We prove that (see Proposition 7.2)

$$M(x) \approx G^{-1}(x) + \left(1 - \frac{2}{\log 2}\right) G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)} + o(\sqrt{x}).$$

This formula implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by bounding appropriate estimates of the summatory function $G^{-1}(x)$.

### 3.3 Uniform asymptotics from enumerative bivariate DGFs from Mongomery and Vaughan

**Theorem 3.6** (Montgomery and Vaughan)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

*For $R < 2$ we have that uniformly for all $1 \leq k \leq R \log\log x$*

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right],$$

*where*

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, 0 \leq |z| \leq R.$$

The proof of the next result is combinatorially motivated in so much as it interprets lower bounds on a key infinite product factor of $\mathcal{G}(z)$ defined in Theorem 3.6 as corresponding to an ordinary generating function of certain homogeneous symmetric polynomials involving the primes. This interpretation allows us to recover the following uniform lower bounds on $\widehat{\pi}_k(x)$ as $x \to \infty$:

**Theorem 3.7.** *For all sufficiently large $x$ we have uniformly for $1 \leq k \leq \log\log x$ that*

$$\widehat{\pi}_k(x) \gg \frac{x^{3/4}}{(\log x)^{1/2}} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

### 3.3.1 Applications of the new uniform lower bound estimates

Our inspiration for the new bounds found in the last sections of this article allows us to approximate finite partial sums of certain bounded non-negative arithmetic functions weighted by the Liouville lambda function $\lambda(n)$.

**Lemma 3.8.** ***(TODO)*** *Suppose that $f(n)$ is an arithmetic function defined such that $f(n) > 0$ for all $n > u_0$ where $f(n) \gg \widehat{\tau}_\ell(n) > 0$ whenever $n > u_0$ as $n \to \infty$. Assume also that the bounding function $\widehat{\tau}_\ell(t)$ is a continuously differentiable function of $t$ for all large enough $t \gg u_0$. We define the $\lambda$-sign-scaled summatory function of $f$ as follows:*

$$F_\lambda(x) := \sum_{u_0 < n \leq x} \lambda(n) f(n).$$

*Let the summatory weight function be defined as*

$$A_\Omega(t) := \sum_{k=1}^{\lfloor \log\log t \rfloor} (-1)^k \widehat{\pi}_k(t).$$

*Suppose that $|A_\Omega(t)| \gg |A_\Omega^{(\ell)}(t)|$ as $t \to \infty$, the function $|A_\Omega^{(\ell)}(t)|$ is monotone increasing for $t > u_0$ large, and that $\left|\widehat{\tau}_\ell\left(\frac{\log\log x}{2}\right) - \widehat{\tau}_\ell\left(\frac{\log\log x}{2} - \frac{1}{2}\right)\right| = O\left(\frac{\widehat{\tau}_\ell(x)}{\log\log x}\right)$ as $x \to \infty$. Then we have that for sufficiently large $x > e$*

$$|F_\lambda(x)| \gg \left| \left|A_\Omega^{(\ell)}(x) \widehat{\tau}_\ell(x)\right| - \int_{\frac{\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log x}{2}} \left|A_\Omega^{(\ell)}\left(e^{e^{2t}}\right) \widehat{\tau}_\ell'\left(e^{e^{2t}}\right)\right| e^{e^{2t}} dt \right|. \tag{2}$$

### 3.3.2 Remarks

We emphasize the relevant recency of the method demonstrated by Montgomery and Vaughan in constructing a proof of Theorem 3.6. To the best of our knowledge, this textbook reference is one of the first clear-cut applications documenting something of a hybrid DGF-and-OGF approach to enumerating sequences of arithmetic functions and their summatory functions. The hybrid method is motivated by the fact that it does not require a direct appeal to traditional highly oscillatory DGF-only inversions and integral formulas involving the Riemmann zeta function. This newer interpretaion of certain bivariate DGFs offers a window into the best of both generating function series worlds: It combines the additivity implicit to the coefficients indexed by a formal power series variable formed by multiplication of these structures, while coordinating the distinct DGF-best property of the multiplicativity of distinct prime powers invoked by taking powers of an Euler product.

## 3.4 Cracking the classical unboundedness barrier

In Section 8, we are able to state what forms a bridge between the results we carefully prove up to that point the article. What we obtain at the conclusion of the section is the next summary theorem that unconditionally resolves the classical question of the unboundedness of the scaled function Mertens function $q(x) := |M(x)|/\sqrt{x}$ in the limit supremum sense.

**Theorem 3.9** (Unboundedness of the the Mertens function, $q(x)$)**.** *We have that*

$$\limsup_{x \to \infty} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

In establishing the rigorous proof of Theorem 3.9 based on our new methods, we not only show unboundedness of $q(x)$, but also set a minimal rate (along a large infinite subsequence) at which this form of the scaled Mertens function grows without bound.

# 4 Preliminary proofs of new results

## 4.1 Establishing the summatory function properties and inversion identities

We will offer a proof of Theorem 3.1 suggested by an intuitive construction through matrix methods. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95].

*Proof of Theorem 3.1.* Let $h, g$ be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of $h$ and $g$, respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of $g$ with $h$. We have that the following formulas hold for all $x \geq 1$:

$$\begin{aligned} \pi_{g*h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right] H(i). \end{aligned} \tag{3}$$

The first formula above is well known. The second formula is justified directly using summation by parts as [–A–]

$$\begin{aligned} \pi_{g*h}(x) &= \sum_{d=1}^{x} h(d) G\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i \leq x} \left(\sum_{j \leq i} h(j)\right) \times \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right]. \end{aligned}$$

We next form the invertible matrix of coefficients associated with this linear system defining $H(j)$ for all $1 \leq j \leq x$ in (3) by defining

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right), 1 \leq j \leq x.$$

Since $g_{x,x} = G(1) = g(1)$ and $g_{x,j} = 0$ for all $j > x$, the matrix we must invert in this problem is lower triangular with a non-zero constant on its diagonals, and is hence invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressed by applying an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

Here, $U$ is a square matrix with sufficiently large finite dimensions whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$\left[(I - U^T)^{-1}\right]_{i,j} = [j \leq i]_{\delta}.$$

Observe that

$$\left\lfloor \frac{x}{j} \right\rfloor - \left\lfloor \frac{x-1}{j} \right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

The previous property implies that

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

[–A–] For any arithmetic functions, $u_n, v_n$, with $U_j := u_1 + u_2 + \cdots + u_j$ for $j \geq 1$, we have that [11, §2.10(ii)]

$$\sum_{j=1}^{n-1} u_j \cdot v_j = U_{n-1} v_n + \sum_{j=1}^{n-1} U_j \left(v_j - v_{j+1}\right), n \geq 2.$$

We use the last property in (4) to shift the matrix $\hat{G}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of $g$ in the following form:

$$\left[(I-U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Our target matrix in the inversion problem is defined by

$$(g_{x,j}) = (I-U^T)\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators as follows:

$$\begin{aligned}
(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k)\right)(I-U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1}\right\rfloor} g^{-1}(k)\right).
\end{aligned}$$

Hence, the summatory function $H(x)$ is given exactly for any $x \geq 1$ by a vector product with the inverse matrix from the previous equation in the next form.

$$H(x) = \sum_{k=1}^{x} g_{x,k}^{-1} \cdot \pi_{g*h}(k) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right)\cdot \pi_{g*h}(k)$$

We can prove an inversion formula providing the coefficients of $G^{-1}(i)$ for $1 \leq i \leq x$ given by the last equation by adapting our argument to prove (3) above. This leads to the identity that

$$H(x) = \sum_{k=1}^{x} g^{-1}(x)\pi_{g*h}\left(\left\lfloor \frac{x}{k}\right\rfloor\right). \qquad \square$$

### 4.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of $n$. Then we can easily prove using DGFs that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega+1) * \mu. \tag{5}$$

When combined with Corollary 3.2 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 3.3.

**Proposition 4.1** (The signedness property of $g^{-1}(n)$). *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0, \pm 1\}$ *denote the sign of the arithmetic function $h$ at integers $n \geq 1$. For the Dirichlet invertible function, $g(n) := \omega(n)+1$, we have that* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ *for all $n \geq 1$.*

*Proof.* The function $D_f(s) := \sum_{n\geq 1} f(n)n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ for $\sigma_f$ the abcissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = 1/\zeta(s)$ and $D_\omega(s) = P(s)\zeta(s)$ for $\operatorname{Re}(s) > 1$. Then by (5) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any arithmetic function $f$ such that $f(1) \neq 0$, we have for all $\operatorname{Re}(s) > 1$ that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{6}$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. This observation implies that $\operatorname{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make our claims rigorous.

By the recurrence relation that defines the Dirichlet inverse function of any arithmetic function $h$ such that $h(1) = 1$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d \mid n \\ d > 1}} h(d) h^{-1}(n/d), & n \geq 2. \end{cases} \tag{7}$$

For $n \geq 2$, the summands in (7) can be simply indexed over the primes $p \mid n$ given our definition of $h$ from above. This observation yields that we can inductively unfold these sums into nested divisor sums provided the depth of the expanded divisor sums does not exceed the capacity to index summations over the primes dividing $n$. Namely, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p \mid n} h^{-1}\left(\frac{n}{p}\right), & \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1 \mid n} \sum_{p_2 \mid \frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right), & \text{if } \Omega(n) \geq 2 \\ &= -\sum_{p_1 \mid n} \sum_{p_2 \mid \frac{n}{p_1}} \sum_{p_3 \mid \frac{n}{p_1 p_2}} h^{-1}\left(\frac{n}{p_1 p_2 p_3}\right), & \text{if } \Omega(n) \geq 3. \end{aligned}$$

Then by induction with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1 \mid n} \sum_{p_2 \mid \frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)} \mid \frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1, n \geq 2. \tag{8}$$

In fact, by a combinatorial argument we recover exactly that

$$h^{-1}(n) = \lambda(n) \frac{(\alpha_1 + \cdots + \alpha_{\omega(n)})!}{\alpha_1! \alpha_2! \cdots \alpha_{\omega(n)}!} = \lambda(n) \binom{\Omega(n)}{\alpha_1, \alpha_2, \ldots, \alpha_{\omega(n)}}. \tag{9}$$

These expansions imply that the following property holds for all $n \geq 1$:

$$\operatorname{sgn}(h^{-1}(n)) = \lambda(n).$$

Since $\lambda$ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$ for all $d \mid n$ and $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever $n$ is squarefree, so that we obtain

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d \mid n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1. \qquad \square$$

### 4.3 Statements of known limiting asymptotics

**Theorem 4.2** (Mertens theorem). *For all $x \geq 2$ we have that*

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log\log x + B + o(1), \text{ as } x \to \infty,$$

*where* $B \approx 0.2614972128476427837554$ *is an absolute constant* [–B–].

**Corollary 4.3** (Product form of Mertens theorem)**.** *We have that for all sufficiently large* $x \gg 2$

$$\prod_{p \leq x}\left(1-\frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\left(1+o(1)\right), \text{ as } x \rightarrow \infty,$$

*where the notation for the absolute constant* $0 < B < 1$ *coincides with the definition of Mertens constant from Theorem 4.2. Hence, for any real* $z \geq 0$ *we obtain that*

$$\prod_{p \leq x}\left(1-\frac{1}{p}\right)^z \sim \frac{e^{-\gamma z}}{(\log x)^z}, \text{ as } x \rightarrow \infty.$$

Proofs of Theorem 4.2 and Corollary 4.3 are given in [2, §22.7; §22.8]. We have a related analog of Corollary 4.3 that is justified using the Euler product representation for the Riemann zeta function:

$$\prod_{p \leq x}\left(1+\frac{1}{p}\right) = \prod_{p \leq x}\frac{(1-p^{-2})}{(1-p^{-1})} = \zeta(2)e^{\gamma}(\log x)(1+o(1)), \text{ as } x \rightarrow \infty.$$

**Facts 4.4** (Exponential integrals and the incomplete gamma function)**.** Two variants of the *exponential integral function* are defined by the integral next representations [11, §8.19].

$$\begin{aligned}
\mathrm{Ei}(x) &:= \int_{-x}^{\infty}\frac{e^{-t}}{t}dt, x \in \mathbb{R} \\
E_1(z) &:= \int_{1}^{\infty}\frac{e^{-tz}}{t}dt, \mathrm{Re}(z) \geq 0
\end{aligned}$$

These functions are related by $\mathrm{Ei}(-kz) = -E_1(kz)$ for real $k, z > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $\mathrm{Ei}(\pm x)$ for all real $x > 0$:

$$\begin{aligned}
\gamma + \log x - x \leq \mathrm{Ei}(-x) &\leq \gamma + \log x - x + \frac{x^2}{4}, \qquad \text{(10a)} \\
1 + \gamma + \log x - \frac{3}{4}x \leq \mathrm{Ei}(x) \quad &\leq 1 + \gamma + \log x - \frac{3}{4}x + \frac{11}{36}x^2.
\end{aligned}$$

The (upper) *incomplete gamma function* is defined by [11, §8.4]

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1}e^{-t}dt, \mathrm{Re}(s) > 0.$$

The following properties of $\Gamma(s, x)$ hold:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1}\frac{x^k}{k!}, s \in \mathbb{Z}^+, x > 0, \qquad \text{(10b)}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, s > 0, \text{ as } x \rightarrow \infty. \qquad \text{(10c)}$$

[–B–]Precisely, we have that the *Mertens constant* is defined by [14, A077761]

$$B = \gamma + \sum_{m \geq 2}\frac{\mu(m)}{m}\log\left[\zeta(m)\right].$$

# 5 Components to the asymptotic analysis of lower bounds for sums of arithmetic functions weighted by $\lambda(n)$

## 5.1 A discussion of the results proved by Montgomery and Vaughan

**Remark 5.1** (Intuition and constructions in Theorem 3.6)**.** For $|z| < 2$ and $\operatorname{Re}(s) > 1$, let

$$F(s, z) := \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \tag{11}$$

and define the DGF coefficients, $a_z(n)$ for $n \geq 1$, by the product

$$\zeta(s)^z \cdot F(s, z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s}, \operatorname{Re}(s) > 1.$$

Suppose that $A_z(x) := \sum_{n \leq x} a_z(n)$ for $x \geq 1$. Then we obtain the next generating function like identity in $z$ enumerating the $\widehat{\pi}_k(x)$ for $1 \leq k \leq \log\log x$ [–A–]

$$A_z(x) = \sum_{n \leq x} z^{\Omega(n)} = \sum_{k \geq 0} \widehat{\pi}_k(x) z^k \tag{12}$$

Thus for $r < 2$, by Cauchy's integral formula we have

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz.$$

Selecting $r := \frac{k-1}{\log\log x}$ for $1 \leq k < 2\log\log x$ leads to the uniform asymptotic formulas for $\widehat{\pi}_k(x)$ given in Theorem 3.6. Montgomery and Vaughan then consider individual analysis of the main and error terms for $A_z(x)$ to prove that

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

We will require estimates of $A_{-z}(x)$ from below to form summatory functions that weight the terms of $\lambda(n)$ in our new formulas derived in the next sections.

## 5.2 New uniform asymptotics based on refinements of Theorem 3.6

**Proposition 5.2.** *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \geq 2.$$

*When $s := 1$, we have the asymptotic formula from Mertens theorem (see Theorem 4.2). For all integers $s \geq 2$ there is are absolutely defined quasi-polynomial bounding functions $\gamma_0(s, x)$ and $\gamma_1(s, x)$ in $s, x$ such that*

$$\gamma_0(s, x) + o(1) \leq P_s(x) \leq \gamma_1(s, x) + o(1), \text{ as } x \to \infty.$$

*It suffices to define the bounds in the previous equation by the functions*

$$\gamma_0(s, x) = s \log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}s(s-1)^2 \log^2(2)$$
$$\gamma_1(s, x) = s \log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) + \frac{1}{4}s(s-1)^2 \log^2(x).$$

[–A–] In fact, for any additive arithmetic function $a(n)$, characterized by the property that $a(n) = \sum_{p^\alpha || n} a(p^\alpha)$ for all $n \geq 2$, we have that [4, *cf.* §1.7]

$$\prod_p \left(1 - \sum_{m \geq 1} \frac{z^{a(p^m)}}{p^{ms}}\right)^{-1} = \sum_{n \geq 1} \frac{z^{a(n)}}{n^s}, \operatorname{Re}(s) > 1.$$

*Proof.* Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$, and where our target function smooth function is $f(t) = t^{-s}$ so that $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= \mathrm{Ei}(-(s-1)\log x) - \mathrm{Ei}(-(s-1)\log 2) + o(1), \text{ as } x \to \infty. \end{aligned}$$

Now using the inequalities in Facts 4.4, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned} \frac{P_s(x)}{s} &\geq \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}(s-1)^2 \log^2(2) + o(1) \\ \frac{P_s(x)}{s} &\leq \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) + \frac{1}{4}(s-1)^2 \log^2(x) + o(1). \end{aligned}$$ □

We will first prove the stated form of the lower bound on $\mathcal{G}(-z)$ for $z := \frac{k-1}{\log\log x}$. Then we will discuss the technical adaptations to Montgomery and Vaughan's proof of Theorem 3.6 in Remark 5.3 to justify the new asymptotic lower bounds on $\widehat{\pi}_k(x)$ that hold uniformly for all $1 \leq k \leq \log\log x$.

*Proof of Theorem 3.7.* For $0 \leq z < 2$ and integers $x \geq 2$, the right-hand-side of the following product is finite.

$$\widehat{P}(z, x) := \prod_{p \leq x} \left(1 - \frac{z}{p}\right)^{-1}.$$

For fixed, finite $x \geq 2$ let

$$\mathbb{P}_x := \{n \geq 1 : \text{all prime divisors } p | n \text{ satisfy } p \leq x\}.$$

Then we can see that

$$\prod_{p \leq x} \left(1 - \frac{z}{p^s}\right)^{-1} = \sum_{n \in \mathbb{P}_x} \frac{z^{\Omega(n)}}{n^s}, x \geq 2. \tag{13}$$

By extending the argument in the proof given in [8, §7.4], we have that the formulas

$$A_{-z}(x) := \sum_{n \leq x} \lambda(n) z^{\Omega(n)} = \sum_{k \geq 0} \widehat{\pi}_k(x)(-z)^k,$$

If we let $a_n(z, x)$ be defined by the DGF

$$\widehat{P}(z, x) := \sum_{n \geq 1} \frac{a_n(z, x)}{n^s},$$

then we show that

$$\sum_{n \leq x} a_n(-z, x) = \sum_{n \leq x} \lambda(n) z^{\Omega(n)} = \sum_{k=0}^{\log_2(x)} \widehat{\pi}_k(x)(-z)^k + \sum_{k > \log_2(x)} e_k(x)(-z)^k.$$

This assertion if correct since the products of all non-negative integral powers of the primes $p \leq x$ generate the integers $\{1 \leq n \leq x\}$ as a subset. Thus we capture all of the relevant terms needed to express $(-1)^k \cdot \widehat{\pi}_k(x)$ via the Cauchy integral formula representation over $A_{-z}(x)$ by replacing the corresponding infinite product terms with $\widehat{P}(-z, x)$ in the definition of $\mathcal{G}(-z)$.

Now we must argue that

$$\mathcal{G}(-z) \gg \prod_{p \leq x} \left(1 + \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{-z}, 0 \leq z < 1, x \geq 2.$$

For $0 \leq z < 1$ and $x \geq 2$, we see that

$$\begin{aligned}
\mathcal{G}(-z) &= \exp\left(-\sum_p \left[\log\left(1+\frac{z}{p}\right) + \log\left(1-\frac{1}{p}\right)\right]\right) \\
&\gg \exp\left(-z \times \sum_{p>x}\left[\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right] - \sum_{p\leq x}\left[\log\left(1+\frac{z}{p}\right)+\log\left(1-\frac{1}{p}\right)\right]\right) \\
&= \widehat{P}(-z,x) \times \exp\left(-z(B+o(1))\right) \gg_z \widehat{P}(-z,x), \text{ as } x \rightarrow \infty.
\end{aligned}$$

Next, we have for all integers $0 \leq k \leq m < \infty$, and any sequence $\{f(n)\}_{n\geq 1}$ with sufficiently bounded partial power sums, that [7, §2]

$$[z^k] \prod_{1\leq i\leq m} (1-f(i)z)^{-1} = [z^k] \exp\left(\sum_{j\geq 1}\left(\sum_{i=1}^m f(i)^j\right)\frac{z^j}{j}\right), |z|<1. \tag{14}$$

In our case we have that $f(i)$ denotes the reciprocal of the $i^{th}$ prime in the generating function expansion of (14). It follows from Proposition 5.2 that for any real $0 \leq z < 1$ we obtain

$$\begin{aligned}
\log\left[\prod_{p\leq x}\left(1+\frac{z}{p}\right)^{-1}\right] &\geq -(B+\log\log x)z + \sum_{j\geq 0}\left[\log\left(\frac{\log x}{\log 2}\right) - (2j+1)\log\left(\frac{x}{2}\right) - (2j+1)^2\frac{\log^2 2}{4}\right] z^{2j+2} \\
&\qquad + \sum_{j\geq 0}\left[\log\left(\frac{\log x}{\log 2}\right) - (2j+2)\log\left(\frac{x}{2}\right) + (2j+2)^2\frac{\log^2 x}{4}\right] z^{2j+3} \\
&= -(B+\log\log x)z + z^2 \times \sum_{j\geq 0}\left[\log\left(\frac{\log x}{\log 2}\right) - (j+1)\log\left(\frac{x}{2}\right)\right](-z)^j \\
&\qquad - \frac{z^2}{4} \times \sum_{j\geq 0}\left[\log^2 2 + \log^2 x\right](j+1)^2 z^j \\
&= -(B+\log\log x)z + z^2\left[\log\left(\frac{\log x}{\log 2}\right)\frac{1}{1+z} - \log\left(\frac{x}{2}\right)\frac{1}{(1+z)^2}\right] \\
&\qquad + \left(\log^2 2 + \log^2 x\right)\frac{z^2(1+z)}{4\cdot(1-z)^3} \\
&=: \widehat{\mathcal{B}}(x;z).
\end{aligned} \tag{15}$$

We adjust the uniform bound parameter $R$ so that

$$z \equiv z(k,x) = \frac{k-1}{\log\log x} \in [0,1),$$

whenever $1 \leq k \leq \log\log x$ in the notation of Theorem 3.6. This implies that $(1+z)^{-1} \in \left(\frac{1}{2}, 1\right]$. Then we have from (15) that

$$\begin{aligned}
\widehat{\mathcal{B}}(x;z) &\gg \left(\frac{\log x}{\log 2}\right)^{\frac{z^2}{2}} \cdot \left(\frac{2}{x}\right)^{\frac{1}{4}} \cdot \exp\left(\frac{z^2(1+z)}{4\cdot(1-z)^3}\cdot \log^2 x\right) \\
&\gg \frac{(\log x)^{1/2}}{x^{1/4}}.
\end{aligned}$$

In summary, we have arrived at a proof that as $x \rightarrow \infty$

$$\frac{e^{\gamma z}}{(\log x)^{-z}} \times \exp\left(\widehat{\mathcal{B}}(u,x;z)\right) \gg \frac{(\log x)^{1/2}}{x^{1/4}}. \tag{16}$$

Finally, to finish our proof of the new form of the lower bound on $\mathcal{G}(-z)$, we need to bound the reciprocal factor of $\Gamma(1-z)$. Since $z \equiv z(k,x) = \frac{k-1}{\log\log x}$ and $k \in [1, \log\log x]$, or again with $z \in [0,1)$, we obtain for minimal $k$ and all large enough $x \gg 1$ that $\Gamma(1-z) = \Gamma(1) = 1$, and for $k$ towards the upper range of its interval that

$$\Gamma(1-z) \approx \Gamma\left(\frac{1}{\log\log x}\right) = \frac{1}{\log\log x}\Gamma\left(1+\frac{1}{\log\log x}\right) \approx \frac{1}{\log\log x}. \qquad \square$$

**Remark 5.3** (Technical adjustments in the proof of Theorem 3.7)**.** We now discuss the differences between our construction and that in the technical proof of Theorem 3.6 in the reference when we bound $\mathcal{G}(-z)$ from below as in Theorem 3.7. The reference proves that for real $0 \leq z < 2$

$$A_{-z}(x) = -\frac{zF(1,-z)}{\Gamma(1-z)} \cdot x(\log x)^{-(z+1)} + O\left(x(\log x)^{-\operatorname{Re}(z)-2}\right). \tag{17}$$

Recall that for $r < 2$ we have by Cauchy's integral formula that

$$(-1)^k \widehat{\pi}_k(x) = \frac{1}{2\pi\imath}\int_{|z|=r} \frac{A_{-z}(x)}{z^{k+1}} dz. \tag{18}$$

We first claim that uniformly for large $x$ and $1 \leq k \leq \log\log x$ we have

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{1-k}{\log\log x}\right) \times \frac{x(\log\log x)^{k-1}}{(\log x)(k-1)!}\left[1+O\left(\frac{k}{(\log\log x)^3}\right)\right]. \tag{19}$$

Then since we have proved in Theorem 3.6 above that

$$\mathcal{G}\left(\frac{1-k}{\log\log x}\right) \gg \frac{2^{1/4}(\log x)^{1/2}}{\sqrt{\log 2}\cdot x^{1/4}} \cdot \frac{(k-1)}{\log\log x},$$

the result in (19) implies our stated uniform asymptotic bound. Namely, we obtain that

$$\widehat{\pi}_k(x) \gg \frac{2^{1/4}}{\sqrt{\log 2}} \cdot \frac{x^{3/4}}{(\log x)^{1/2}} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1+O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

We have to provide analogs to the two separate bounds corresponding to the error and main terms of our estimate according to (17) and (18). The error term estimate is simpler, so we tackle it first in the next argument. The second part of our proof establishing the main term in (19) requires us to duplicate and adjust significant parts of the fine-tuned reasoning given in the reference.

*Error Term Bound.* To prove that the error term bound holds, we estimate that

$$\begin{aligned}\left|\frac{1}{2\pi\imath}\int_{|z|=r} \frac{x}{(\log x)^2}\frac{(\log x)^{-\operatorname{Re}(z)}}{z^{k+1}}\right| &\ll x(\log x)^{-(r+2)} r^{-(k+1)} \ll \frac{x}{(\log x)^2}\frac{(\log\log x)^{k+1}}{e^{k-1}(k-1)^{k+1}} \\ &\ll \frac{x}{(\log x)^2}\frac{(\log\log x)^{k+1}}{e^{2(k-1)}(k-1)!(k-1)} \ll \frac{x}{(\log x)^2}\frac{(\log\log x)^{k+1}}{(k-1)!} \\ &\ll \frac{x}{\log x}\frac{(\log\log x)^{k-4}}{(k-1)!}.\end{aligned} \tag{20}$$

We can calculate that for $0 \leq z < 1$

$$\begin{aligned}\prod_p \left(1+\frac{z}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{-z} &= \exp\left(-\sum_p \left[\log\left(1+\frac{z}{p}\right) + z\log\left(1-\frac{1}{p}\right)\right]\right) \\ &\sim \exp\left(-o(z) \times \sum_p \frac{1}{p^2}\right)\end{aligned}$$

$$\gg \exp\left(-o(z)\frac{\pi^2}{6}\right) \gg_z 1.$$

In other words, we have that $\mathcal{G}\left(\frac{1-k}{\log\log x}\right) \gg 1$. So the error term in (20) is majorized by taking $O\left(\frac{k}{(\log\log x)^3}\right)$ as our upper bound.
*Main Term Bounds.* Notice that the main term estimate corresponfing to (17) and (18) is given by $\frac{x}{\log x} I$, where

$$I := \frac{(-1)^{k-1}}{2\pi\imath}\int_{|z|=r} G(-z)(\log x)^{-z} z^{-k} dz.$$

In particular, we can write $I = I_1 + I_2$ where we define

$$\begin{aligned}
I_1 &:= \frac{(-1)^{k-1}G(-r)}{2\pi\imath}\int_{|z|=r}(\log x)^{-z}z^{-k}dz \\
&= \frac{G(-r)(\log\log x)^{k-1}}{(k-1)!} \\
I_2 &:= \frac{(-1)^{k-1}}{2\pi\imath}\int_{|z|=r}(G(-z)-G(-r))(\log x)^{-z}z^{-k}dz \\
&= \frac{(-1)^{k-1}}{2\pi\imath}\int_{|z|=r}(G(-z)-G(-r)+G'(-r)(z+r))(\log x)^{-z}z^{-k}dz.
\end{aligned}$$

We have by a power series expansion of $G''(-w)$ about $-z$ and integrating the resulting series termwise with respect to $w$ that

$$\left|G(-z)-G(-r)+G'(-r)(z+r)\right| = \left|\int_{-r}^{z}(z+w)G''(-w)dw\right| \ll G''(-r)\times|z+r|^2 \ll |z+r|^2.$$

Now we parameterize the curve in the contour for $I_2$ by writing $z = re^{2\pi\imath t}$ for $t\in[-1/2,1/2]$. This leads us to the bounds

$$\begin{aligned}
|I_2| &= r^{3-k}\times\int_{-1/2}^{1/2}|e^{2\pi\imath t}+1|^2\cdot(\log x)^{re^{2\pi\imath t}}\cdot e^{2\pi\imath t}dt \\
&\ll r^{3-k}\times\int_{-1/2}^{1/2}\sin^2(\pi t)\cdot e^{(1-k)\cos(2\pi t)}dt.
\end{aligned}$$

Whenever $|x|\leq 1$, we know that $|\sin x|\leq|x|$. We can construct bounds on $-\cos(2\pi t)$ for $t\in[-1/2,1/2]$ by writing $\cos(2x) = 1-2\sin^2 x$ for $|x|<1/2$. Then by the alternating Taylor series expansions of the sine function

$$\begin{aligned}
1-2\sin^2(2\pi t) \geq 1-2\left(1-\frac{\pi t}{3}\right)^2 &\geq -1-\frac{2\pi^2t^2}{9} \qquad \implies \\
-\cos(2\pi t) \leq 1+\frac{2\pi^2t^2}{9} &\leq \left(4+\frac{2\pi^2}{9}\right)t^2 \leq 1+3t^2.
\end{aligned}$$

So it follows that

$$\begin{aligned}
|I_2| &\ll r^{3-k}e^{k-1}\times\left|\int_0^\infty t^2e^{3(k-1)t^2}dt\right| \\
&\ll \frac{r^{3-k}e^{k-1}}{(k-1)^{3/2}} = \frac{(\log\log x)^{k-3}e^{k-1}}{(k-1)^{k-3/2}} \\
&\ll \frac{k\cdot(\log\log x)^{k-3}}{(k-1)!}.
\end{aligned}$$

Thus the contribution from the term $|I_2|$ can then be absorbed into the error term bound in (19).

### 5.3 The distribution of exceptional values of $\Omega(n)$

The next theorems reproduced from [8, §7.4] characterize the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \log\log x$. The tendency of this canonical completely additive function to not deviate substantially from its average order is an extraordinary property that allows us to prove asymptotic relations on summatory functions that are weighted by its parity without having to account for significant local oscillations when we average over a large interval.

**Theorem 5.4** (Upper bounds on exceptional values of $\Omega(n)$ for large $n$)**.** *Let*

$$\begin{aligned} A(x,r) &:= \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\}, \\ B(x,r) &:= \#\left\{n \leq x : \Omega(n) \geq r \cdot \log\log x\right\}. \end{aligned}$$

*If $0 < r \leq 1$ and $x \geq 2$, then*

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x \rightarrow \infty.$$

*If $1 \leq r \leq R < 2$ and $x \geq 2$, then*

$$B(x,r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \rightarrow \infty.$$

Theorem 5.5 is an analog to the celebrated Erdös-Kac theorem typically stated for the normally distributed values of the scaled-shifted $\omega(n)$ function over $n \leq x$ as $x \rightarrow \infty$.

**Theorem 5.5** (Exact bounds on exceptional values of $\Omega(n)$ for large $n$)**.** *We have that as $x \rightarrow \infty$*

$$\#\left\{3 \leq n \leq x : \Omega(n) - \log\log n \leq 0\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

**Remark 5.6.** The key interpretation we need to take away from the statements of Theorem 5.4 and Theorem 5.5 is the result proved in the next corollary. The role of the parameter $R$ involved in stating the previous theorem is a critical bound as the scalar factor in the upper bound on $k \leq R\log\log x$ in Theorem 3.6 up to which our uniform bounds given by Theorem 3.7 hold. In contrast, for $n \geq 2$ we can actually have contributions from values distributed throughout the range $1 \leq \Omega(n) \leq \log_2(n)$ infinitely often. It is then crucial that we can show that the main term in the asymptotic formulas we obtain for these summatory functions is captured by summing only over the truncated range of $k \in [1, \log\log x]$ where the uniform bounds hold.

**Corollary 5.7.** *Using the notation for $A(x,r)$ and $B(x,r)$ from Theorem 5.4, we have that for $x \geq 2$ and $\delta > 0$,*

$$o(1) \leq \frac{B(x, 1+\delta)}{A(x,1)} \ll 2, \text{ as } \delta \rightarrow 0^+, x \rightarrow \infty.$$

*Proof.* The lower bound stated above is clear. To show that the asymptotic upper bound is correct, we compute using Theorem 5.4 and Theorem 5.5 that

$$\frac{B(x,1+\delta)}{A(x,1)} \ll \frac{x \cdot (\log x)^{\delta - \delta\log(1+\delta)}}{O(1) + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right)} \sim o_\delta(1),$$

as $x \rightarrow \infty$. Notice that since $\mathbb{E}[\Omega(n)] = \log\log n + B$, with $0 < B < 1$ the absolute constant from Mertens theorem, when we denote the range of $k > \log\log x$ as holding in the form of $k > (1+\delta)\log\log x$ for $\delta > 0$ at large $x$, we can assume that $\delta \rightarrow 0^+$ as $x \rightarrow \infty$. In particular, this holds since $k > \log\log x$ implies that

$$\lfloor \log\log x \rfloor + 1 \geq (1+\delta)\log\log x \quad \Longrightarrow \quad \delta \leq \frac{1 + \{\log\log x\}}{\log\log x} = o(1), \text{ as } x \rightarrow \infty.$$

The key consequence is that $B(x, 1+\delta)$ is at most a bounded constant multiple of $A(x,1)$ for all large $x$. $\square$

# 6 Average case analysis of bounds on the Dirichlet inverse functions, $g^{-1}(n)$

The pages of tabular data given as Table T.1 in the appendix section (refer to page 47) are intended to provide clear insight into why we arrived at the approximations to $g^{-1}(n)$ proved in this section. The table provides illustrative numerical data by examining the approximate behavior at hand for the cases of $1 \leq n \leq 500$ with *Mathematica*.

## 6.1 Definitions and basic properties of component function sequences

We define the following auxiliary coefficient sequence for integers $n \geq 1, k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{21}$$

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (21) inductively. By the same argument, we see that at fixed $n$, the function $C_k(n)$ is seen to be non-zero only for positive integers $k \leq \Omega(n)$ whenever $n \geq 2$. A sequence of relevant signed semi-diagonals of the functions $C_k(n)$ begins as [14, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n\geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \ -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

**Example 6.1** (Special cases of the functions $C_k(n)$ for small $k$)**.** We cite the following special cases which are verified by explicit computation using (21) [14, A066922] [–A–]:

$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= d(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} - \gcd\left(\Omega(n), \omega(n)\right). \end{aligned}$$

The connection between the functions $C_k(n)$ and the inverse sequence $g^{-1}(n)$ is clarified precisely in Section 6.3. Before we can prove explicit bounds on $|g^{-1}(n)|$ through its relation to these functions, we will require a perspective on the lower asymptotic order of $C_k(n)$ for fixed $k$ when $n$ is large.

## 6.2 Uniform asymptotics of $C_k(n)$ for large all $n$ and fixed $k$

The next theorem formally proves a minimal growth rate of the class of functions $C_k(n)$ as functions of fixed $k$ and $n \to \infty$. In the statement of the result that follows, we view $k$ as a fixed variable which is necessarily bounded in $n$, but is still taken as an independent parameter of $n$.

**Theorem 6.2** (Asymptotics of the functions $C_k(n)$)**.** *For $k := 0$, we have by definition that $C_0(n) = \delta_{n,1}$. For all sufficiently large $n > 1$ and any fixed $1 \leq k \leq \Omega(n)$ taken independently of $n$, we obtain that the asymptotic main term for the expected order of $C_k(n)$ is bounded uniformly from below as*

$$\mathbb{E}[C_k(n)] \gg (\log\log n)^{2k-1}, \text{ as } n \to \infty.$$

[–A–] For all $n, k \geq 2$, we have the following recurrence relation satisfied by $C_k(n)$ between successive values of $k$:

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=0}^{\nu_p(n)-1} C_{k-1}\left(dp^i\right), n \geq 1.$$

*Proof.* We prove our bounds by induction on $k$. We can see by Example 6.1 that $C_1(n)$ satsfies the formula we must establish when $k := 1$ since $\mathbb{E}[\omega(n)] = \log\log n$. Suppose that $k \geq 2$ and let our inductive assumption provide that for all $1 \leq m < k$ and $n \geq 2$

$$\mathbb{E}[C_m(n)] \gg (\log\log n)^{2m-1}.$$

For all large $x > e$, we cite that the summatory function of $\omega(n)$ satisfies [2, §22.10]

$$\sum_{n \leq x} \omega(n) = x \log\log x + Bx + O\left(\frac{x}{\log x}\right).$$

Now using the recursive formula we used to define the sequences of $C_k(n)$ in (21), we have that as $n \to \infty$

$$\begin{aligned} \mathbb{E}[C_k(n)] &= \mathbb{E}\left[\sum_{d|n} \omega(n/d) C_{k-1}(d)\right] \\ &= \frac{1}{n} \times \sum_{d \leq n} C_{k-1}(d) \times \sum_{r=1}^{\left\lfloor \frac{n}{d} \right\rfloor} \omega(r) \\ &\sim \sum_{d \leq n} C_{k-1}(d) \left[\frac{\log\log(n/d) \left[d \leq \frac{n}{e}\right]_\delta}{d} + \frac{B}{d} + o(1)\right] \\ &\sim \sum_{d \leq \frac{n}{e}} \left[\sum_{m<d} \frac{\mathbb{E}[C_{k-1}(m)]}{m} \log\log\left(\frac{n}{m}\right) + B \cdot \mathbb{E}[C_{k-1}(d)] + B \cdot \sum_{m<d} \frac{\mathbb{E}[C_{k-1}(m)]}{m}\right] \qquad (22) \\ &\gg \sum_{d \leq \frac{n}{e}} \frac{\mathbb{E}[C_{k-1}(m)]}{m} \\ &\gg (\log n)(\log\log n)^{2k-3}. \end{aligned}$$

In transitioning from the previous step, we have used that $(\log n) \gg (\log\log n)^2$ as $n \to \infty$. We have also used that for large $n$ and fixed $m$, by an asymptotic approximation to the incomplete gamma function we have that

$$\int_e^n \frac{(\log\log t)^m}{t} dt \sim (\log n)(\log\log n)^m, \text{ as } n \to \infty.$$

Hence, the claim follows by mathematical induction for large $n \to \infty$ whenever $1 \leq k \leq \Omega(n)$. □

### 6.3 Relating the auxiliary functions $C_k(n)$ to formulas approximating $g^{-1}(n)$

**Lemma 6.3** (An exact formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

*Proof.* We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d) + 1) g^{-1}(n/d) \quad \Longrightarrow \quad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \qquad (23)$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (23) in the form of $(g^{-1} * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m \cdot (C_m(-) * g^{-1})(n)$, or so that

$$F_m(n) = -\begin{cases} \sum\limits_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum\limits_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & m \geq 2, \\ (\omega * g^{-1})(n), & m = 1. \end{cases}$$

By repeatedly expanding the right-hand-side of the previous equation, we find that for $m := \Omega(n)$

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \tag{24}$$

The formula then follows from (24) by Möbius inversion applied to each side of the last equation. ☐

**Corollary 6.4.** *For all squarefree integers $n \geq 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \tag{25}$$

*Proof.* Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geq 2$ and that $n$ is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where $p_i$ is prime for all $1 \leq i \leq \omega(n)$. Since all divisors of any squarefree $n$ are necessarily also squarefree, we can transform the exact divisor sum guaranteed for all $n$ in Lemma 6.3 into a sum that partitions the divisors according to the number of distinct prime factors:

$$\begin{aligned}
g^{-1}(n) &= \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i}(-1)^i \cdot C_{\Omega(d)}(d) \\
&= \lambda(n) \times \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d) \\
&= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).
\end{aligned}$$

The signed contributions in the first of the previous equations is justified by noting that $\lambda(n) = (-1)^{\omega(n)}$ whenever $n$ is squarefree, and that for $d \geq 1$ squarefree we have the correspondence $\omega(d) = k \implies \Omega(d) = k$ for $1 \leq k \leq \log_2(d)$. ☐

Since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ using the notation defined in the the proof of Proposition 4.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for squarefree $n \geq 1$. A proof of part (C) of Conjecture 3.5 follows as an immediate consequence.

**Lemma 6.5.** *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{26}$$

*Proof.* By applying Lemma 6.3, Proposition 4.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$. Lemma 6.3 implies

$$\begin{aligned}
|g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\
&= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\
&= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d).
\end{aligned}$$

In the last equation, we see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. ☐

Combined with the signedness property of $g^{-1}(n)$ guaranteed by Proposition 4.1, Lemma 6.5 shows that the summatory function is expressed as

$$G^{-1}(x)=\sum_{d\leq x}\lambda(d)C_{\Omega(d)}(d)M\left(\left\lfloor\frac{x}{d}\right\rfloor\right).$$

Since $\lambda(d)C_{\Omega(d)}(d)=(g^{-1}*1)^{-1}(d)=(\chi_{\mathbb{P}}+\varepsilon)(d)$ where $\chi_{\mathbb{P}}$ denotes the characteristic function of the primes, we also clearly recover by inversion that

$$M(x)=G^{-1}(x)+\sum_{p\leq x}G^{-1}\left(\left\lfloor\frac{x}{p}\right\rfloor\right), x\geq 1.$$

**Corollary 6.6.** *We have that*

$$(\log n)(\log\log n)\ll\mathbb{E}|g^{-1}(n)|\leq\mathbb{E}\left[\sum_{d|n}C_{\Omega(d)}(d)\right].$$

*Proof.* To prove the lower bound, recall from the introduction that the summatory function of the squarefree integers is given by

$$Q(x):=\sum_{n\leq x}\mu^2(n)=\frac{6x}{\pi^2}+O(\sqrt{x}).$$

Then since $C_{\Omega(d)}(d)\geq 1$ for all $d\geq 1$, and since $\mathbb{E}[C_k(d)]$ is minimized when $k:=1$ according to Theorem 6.2, we obtain by summing over (26) that

$$\begin{aligned}
\frac{1}{x}\times\sum_{n\leq x}|g^{-1}(n)| &= \frac{1}{x}\times\sum_{d\leq x}C_{\Omega(d)}(d)Q\left(\left\lfloor\frac{x}{d}\right\rfloor\right)\\
&\sim\sum_{d\leq x}C_{\Omega(d)}(d)\left[\frac{6}{d\cdot\pi^2}+O\left(\frac{1}{\sqrt{dx}}\right)\right]\\
&=\frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(x)}(x)]+\sum_{d<x}\frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right]+O\left(\frac{1}{\sqrt{x}}\times\int_0^x t^{-1/2}dt\right)\\
&\gg\left[\sum_{e\leq d\leq x}\frac{\log\log d}{d}\right]+O(1)\\
&\sim\times\int_e^x\frac{\log\log t}{t}dt+O(1)\\
&\gg(\log x)(\log\log x),\ \text{as } x\to\infty.
\end{aligned}$$

To prove the upper bound, notice that by Lemma 6.3 and Corollary 6.4,

$$|g^{-1}(n)|\leq\sum_{d|n}C_{\Omega(d)}(d), n\geq 1.$$

Now since both of the above quantities are positive for all $n\geq 1$, we clearly obtain the upper bound stated above when we average over $n\leq x$ for all large $x$. □

### 6.3.1 A connection to the distribution of the primes

**Remark 6.7.** The combinatorial complexity of $g^{-1}(n)$ is deeply tied to the distribution of the primes $p\leq n$ as $n\to\infty$. While the magnitudes and dispersion of the primes $p\leq x$ certainly restricts the repeating of these distinct sequence values we can see in the contributions to $G^{-1}(x)$, the following statement is still clear about the relation of the weight functions $|g^{-1}(n)|$ to the distribution of the primes: The value of $|g^{-1}(n)|$ is entirely dependent on the pattern of the *exponents* (viewed as multisets) of the distinct prime factors of $n \geq 2$. The relation of the repitition of the distinct values of $|g^{-1}(n)|$ in forming bounds on $G^{-1}(x)$ makes another clear tie to $M(x)$ through Proposition 7.2 in the next section.

**Example 6.8** (Combinatorial significance to the distribution of $g^{-1}(n)$)**.** We have a natural extremal behavior with respect to distinct values of $\Omega(n)$ corresponding to squarefree integers, and prime powers. Namely, if for $k \geq 1$ we define the infinite sets $M_k$ and $m_k$ to correspond to the maximal (minimal) positive integers such that

$$M_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\},$$

$$m_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\},$$

then any element of $M_k$ is squarefree and any element of $m_k$ is a prime power. In particular, we have that for any $N_k \in M_k$ and $n_k \in m_k$

$$N_k = \sum_{j=0}^{k} \binom{k}{j} \cdot j!, \qquad \text{and} \qquad n_k = 2 \cdot (-1)^k.$$

The formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 4.1 implies that we can express an exact formula for $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of $n$. Namely, for $n \geq 2$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z \cdot \nu_p(n)) = [z^k] \prod_{p^{\alpha}||n} (1 + \alpha z), 0 \leq k \leq \omega(n).$$

Then we have essentially shown using (9) and (26) that we can expand

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, n \geq 2.$$

The combinatorial formula for $h^{-1}(n) = \lambda(n) \cdot (\Omega(n))! \times \prod_{p^{\alpha}||n} (\alpha!)^{-1}$ we derived in the proof of the key signedness proposition in Section 4 suggests further patterns and more regularity in the contributions of the distinct weighted terms for $G^{-1}(x)$ when we sum over all of the distinct prime exponent patterns that factorize $n \leq x$.

# 7 New formulas, bounds and error terms for $M(x)$

## 7.1 Proving new formulas that identify main and error terms

**Lemma 7.1.** *Let the error term function be defined as follows for all sufficiently large $x \geq 2$:*

$$E_M(x) := \sum_{k=1}^{x/2-1} G^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].$$

*Then there is an infinite sequence $\mathbb{X}_E \subseteq \mathbb{Z}^+$ such that for all $x \in \mathbb{X}_E$, $E_M(x) = o(\sqrt{x})$ as $x \to \infty$.*

We will require something more in the form of an Erdös-Kac like theorem providing a limiting semi-distribution form on the values of $|g^{-1}(n)|$ to prove the lemma. These theorems are established in the next subsection. The next result is our stated as our central new formula for $M(x)$.

**Proposition 7.2.** *For all sufficiently large $x$, we have that (TODO)*

$$M(x) \approx G^{-1}(x) + \left(1 - \frac{2}{\log 2}\right) G^{-1}\left(\frac{x}{2}\right) + x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt + E_M(x). \tag{27}$$

*Proof.* We know by applying Corollary 3.3 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k) \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right) \\ &= G^{-1}(x) + \sum_{k=1}^{x/2} g^{-1}(k) \pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right), \end{aligned} \tag{28}$$

where the upper bound on the sum is truncated by the fact that $\pi(1) = 0$. We seek to approximate the function $\pi(j) \sim \frac{j}{\log j}$ by the main term in its limiting asymptotic approximation. To make this substitution useful in our formulas, we need not only that $\pi(j) - \frac{j}{\log j}$ is small, but also that the summatory functions over the difference of these terms weighted by $g^{-1}$ are small enough in order. In short, we need to make sure that the non-negligible error terms in our approximation are not too unruly as to make our main term results imprecise as $x \to \infty$. Now observe that

$$\begin{aligned} D_g^{-1}(x) &:= \sum_{k=1}^{x/2} g^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \frac{x}{k \cdot \log\left(\frac{x}{k}\right)}\right] \\ &= \left(1 - \frac{2}{\log 2}\right) G^{-1}\left(\frac{x}{2}\right) + \sum_{k=1}^{x/2-1} G^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right) + \frac{x}{k \cdot \log\left(\frac{x}{k}\right)} - \frac{x}{(k+1) \cdot \log\left(\frac{x}{k+1}\right)}\right]. \end{aligned} \tag{29}$$

By applying summation by parts, (29) implies that we have

$$\begin{aligned} \sum_{k=1}^{x/2-1} G^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)\right] &= \sum_{k=1}^{x/2-1} G^{-1}(k) \left[\frac{x}{k \cdot \log\left(\frac{x}{k}\right)} - \frac{x}{(k+1) \cdot \log\left(\frac{x}{k+1}\right)}\right] + E_M(x) \\ &\sim \sum_{k=1}^{x/2-1} G^{-1}(k) \left[\frac{x}{k \cdot \log(x/k)} - \frac{x}{(k+1) \cdot \log(x/k)}\right] + E_M(x) \qquad (30a) \\ &\approx \sum_{k=1}^{x/2-1} G^{-1}(k) \frac{x}{k^2 \cdot \log(x/k)} + E_M(x). \qquad (30b) \end{aligned}$$

Indeed, step (30b) is justified by writing

$$\frac{x}{(k+1)\log\left(\frac{x}{k+1}\right)} = \frac{x}{k+1}\cdot\frac{1}{\left[\log\left(\frac{x}{k}\right)+\log\left(1-\frac{1}{k+1}\right)\right]} = \frac{x}{(k+1)\log\left(\frac{x}{k}\right)}\cdot\frac{1}{1+\frac{\log\left(1-\frac{1}{k+1}\right)}{\log x\left[1-\frac{\log k}{\log x}\right]}}$$
$$\sim \frac{x}{(k+1)\log\left(\frac{x}{k}\right)},\text{ as } x\rightarrow\infty.$$

The correctness of the transition from step (30a) to (30b) is verified by seeing that for $\mathrm{Re}(s)>1$, we have that

$$\left|\sum_{k\geq 1}\frac{G^{-1}(k)}{k^{s+1}}\right| = \left|\int_1^{\infty}\frac{G^{-1}(x)}{x^{s+1}}dx\right| = \left|\frac{1}{s\cdot(P(s)+1)\zeta(s)}\right| < \infty.$$

When $s:=\frac{3}{2}$, we obtain that

$$0\leq\left|\sum_{k\geq 1}\frac{G^{-1}(k)}{k^2(k+1)}\right| \leq \left|\sum_{k\geq 1}\frac{G^{-1}(k)}{k^{\frac{5}{2}}}\right| < \infty.$$

The difference of the terms in forming the approximation in this step is bounded above and below by absolute constants as

$$\left|\sum_{k=1}^{\frac{x}{2}} G^{-1}(k)\left[\frac{1}{k^2}-\frac{1}{k(k+1)}\right]\right| \leq \left|\sum_{k=1}^{\frac{x}{2}}\frac{G^{-1}(k)}{k^2(k+1)}\right| = O(1).$$

For $x$ large enough the summand factor $\frac{x}{k^2\cdot\log(x/k)}$ is monotonic as $k$ ranges over $k\in[1,x/2]$ in ascending order. Because this summand factor is a smooth function of $k$ (and $x$) where $G^{-1}(x)$ is a summatory function with jumps only in steps of the positive integers, we can finally approximate $M(x)$ for any finite $x\geq 2$ as follows:

$$M(x)\approx G^{-1}(x)+\left(1-\frac{2}{\log 2}\right)G^{-1}\left(\frac{x}{2}\right)+x\cdot\int_1^{x/2}\frac{G^{-1}(t)}{t^2\cdot\log(x/t)}dt+E_M(x).$$

We will later only use unsigned lower bound approximations to this function in the next theorems so that the signedness of the summatory function term in the integral formula above does not require more restrictive attention in constructing limiting cases as $x\rightarrow\infty$. □

### 7.2 Exact probabilistic bounds on the distributions of component sequences

We have remarked already in the introduction that the relation of the component functions, $g^{-1}(n)$ and $C_k(n)$, to the canonical additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties of these functions witnessed in Table T.1. In particular, each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdös-Kac theorem that shows that a shifted and scaled variant of each of the sets of these function values can be expressed through a limiting normal distribution as $n\rightarrow\infty$. This extremely regular tendency of these functions towards their average order is inherited by the component function sequences we are summing in the approximation of $M(x)$ stated by Proposition 7.2. In the remainder of this section we establish more technical analytic proofs of related properties of our key sequences, again in the spirit of Montgomery and Vaughan's reference.

**Proposition 7.3.** *For $|z|<2$, let the summatory function be defined as*

$$\widehat{A}_z(x) := \sum_{n\leq x}(-1)^{\omega(n)}C_{\Omega(n)}(n)z^{\Omega(n)}.$$

*Let the function $F(s,z)$ is defined for $\mathrm{Re}(s)>1$ in terms of the exponential of the prime zeta function by*

$$F(s,z) := \exp\left(z\cdot P(s)\right)\times\prod_p\left(1-\frac{1}{p^s}\right)^z.$$

*Then we have that for large* $x$

$$\widehat{A}_z(x) = \frac{x \cdot F(2,z)}{\Gamma(z)} (\log x)^{z-1} + O_{R,z}\left(x \cdot (\log x)^{\mathrm{Re}(z)-2}\right),$$

*Proof.* **(TODO)** We know from the proof of Proposition 4.1 that for $n \geq 2$

$$C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}.$$

Then we can generate the denominator terms by the Dirichlet series

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! \cdot p^{rs}}\right)^{-1} = \exp\left(z \cdot P(s)\right).$$

So by computing a Laplace transform on the right-hand-side of the above with respect to the variable $z$, we obtain

$$\sum_{n \geq 1} C_{\Omega(n)}(n) \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp\left(tz \cdot P(s)\right) dt = \frac{1}{1 - P(s)z}.$$

It follows that

$$\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times F(s,z),$$

where

$$F(s,z) := \frac{1}{1 - P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z, \mathrm{Re}(s) > 1, |z| \leq R < 2.$$

Note that we are unable to sum this result in exactly the same format as in the reference [8, §7.4; Thm. 7.18] by effectively setting $s := 1$, especially given the weighting action of the Laplace transform on the coefficients of this product in $z$. However, since for any $|z| \leq R < 2$ we have that

$$\left|\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^2}\right| < +\infty,$$

we will adapt the details to the traditional case where this method arises in the reference application so that we can sum over our modified function depending on $\Omega(n)$. In fact, we notice that since $|z|^{\Omega(n)} \leq n$, we have the exact DGF

$$\mathcal{H}(s) := \sum_{n \geq 1} \frac{\lambda(n) C_{\Omega(n)}(n)}{n^s},$$

which is absolutely convergent for $\mathrm{Re}(s) > 1$. The DGF $\mathcal{H}(s)$ is thus an analytic function of $s$ whenever $\mathrm{Re}(s) > 1$, and so we can differentiate it any integer $m \geq 0$ number of times to still obtain an absolutely convergent series of the form

$$\left|\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) (\log n)^m z^{\Omega(n)}}{n^s}\right| < +\infty, \mathrm{Re}(s) > 2.$$

Let the function $d_z(n)$ have DGF $\zeta(s)^z$ for $\mathrm{Re}(s) > 1$, with corresponding summatory function $D_z(x) := \sum_{n \leq x} d_z(n)$. Adopting the notation from the reference, we set $b_z(n) := (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$, let the convolution $a_z(n) := \sum_{d|n} b_z(d) d_z(n/d)$, and define the summatory function $A_z(x) := \sum_{n \leq x} a_z(n)$. Then we have that

$$A_z(x) = \sum_{m \leq x/2} b_z(m) D_z(x/m) + \sum_{x/2 < m \leq x} b_z(m)$$

$$= \frac{x}{\Gamma(z)} \times \sum_{m \leq x/2} \frac{b_z(m)}{m^2} \log\left(\frac{x}{m}\right)^{z-1} + O\left(x \sum_{m \leq x} \frac{|b_z(m)|}{m^2} \times \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2}\right).$$

The error term in the previous equation satisfies

$$x \sum_{m \leq x} \frac{|b_z(m)|}{m^2} \times \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2} \ll x(\log x)^{\mathrm{Re}(z)-2} \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m^2} + x(\log x)^{-(R+2)} \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m^2} (\log m)^{2R}$$
$$\ll x(\log x)^{\mathrm{Re}(z)-2}.$$

In the main term estimate for $A_z(x)$, when $m \leq \sqrt{x}$ we have

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\mathrm{Re}(z)-2}\right).$$

The main term sum over the interval $m \leq x/2$ corresponds to bounding

$$\sum_{m \leq x/2} b_z(m) D_z(x/m) = x(\log x)^{z-1} \sum_{m \leq x/2} \frac{b_z(m)}{m^2}$$
$$+ O\left(x(\log x)^{\mathrm{Re}(z)-2} \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m^2} + x(\log x)^{R-1} \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m^2}\right)$$
$$= x(\log x)^{z-1} F(2, z) + O\left(x(\log x)^{\mathrm{Re}(z)-2} \sum_{m \geq 1} \frac{b_z(m)(\log m)^{2R+1}}{m^2}\right). \qquad \square$$

**Remark 7.4** (A standard simplifying assumption)**.** Let the constant $\widehat{c} \approx 1.5147$ be defined explicitly as the product of primes

$$\widehat{c} := \frac{1}{6} \times \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right)^{-1}.$$

This constant is related to expressions of the asymptotic densities of the sets

$$N_k(x) := \{n \leq x : \Omega(n) - \omega(n) = k\},$$

for integers $k \geq 0$ in the form of [8, §2.4]

$$N_k(x) = d_k x + O\left(\left(\frac{3}{4}\right)^k \sqrt{x} (\log x)^{4/3}\right),$$

where for each natural number $k \geq 0$, $d_k > 0$ is an absolute constant that satisfies

$$d_k = \frac{\widehat{c}}{2^k} + O\left(5^{-k}\right).$$

The limiting distribution of $\Omega(n) - \omega(n)$ is utilized in the proof of Theorem 7.5.

For $m \leq \omega_{\max}$ and $k \leq \Omega_{\max}$, as $n \to \infty$ we expect

$$\mathbb{P}\left(\omega(n) = m | \Omega(n) = k\right) \approx \frac{\omega_{\max} + 1 - k}{\omega_{\max}},$$

so that the conditional distribution of $\omega(n), \Omega(n)$ is not uniform over its bounded range. However, we do as is standard fare in proofs of the more traditional Erdös-Kac theorems require the simplifying assumption that as $n \to \infty$, we expect independently that $\omega(n), \Omega(n)$ are approximately equally likely to assume any values in some bounded $[1, M]$. This means we can treat the difference $\Omega(n) - \omega(n)$ as being approximately randomly distributed over some bounded range of its possible values. The form of this assumption we draw upon is that for large $x$,

$$\#\{n \leq x : \Omega(n) - \omega(n) = k\} \sim d_k.$$

For a more rigorous treatment of this underlying principle see [**?**, **?**].

**Theorem 7.5.** *We have uniformly for $1 \leq k < P(2) \cdot \log\log x$ that as $x \rightarrow \infty$*

$$\widehat{C}_k(x) := \sum_{\substack{n \leq x \\ \Omega(n)=k}} \lambda(n)(-1)^{\omega(n)} C_k(n) \asymp \frac{x}{\log x} \cdot \frac{(-1)^k \cdot P(2)^{k-1} \cdot (\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^3}\right)\right].$$

*Proof.* The proof is a similar adaptation of the method of Montgomery and Vaughan we cited in Remark 5.3 to prove our variant of Theorem 3.7. We begin by bounding a contour integral over the error term for fixed large $x$ for $r := \frac{k-1}{\log\log x}$ with $r \leq P(2) \approx 0.452247$:

$$\begin{aligned}\left|\int_{|z|=r} \frac{x \cdot (\log x)^{-(\operatorname{Re}(z)+2)}}{z^{k+1}} dz\right| &\ll x(\log x)^{-(r+2)} r^{-(k+1)} \ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{k+1}} \cdot \frac{1}{e^{k-1}} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{3/2}} \cdot \frac{1}{e^{2k}(k-1)!} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \ll \frac{x}{\log x} \cdot \frac{k \cdot (\log\log x)^{k-5}}{(k-1)!}.\end{aligned}$$

By induction we can compute the coefficients $[z^k]\widehat{A}_z(x)$ with respect to $x$ for fixed $k \leq P(2) \cdot \log\log x$ using the Cauchy integral formula for integers $m \geq 0$:

$$\frac{1}{m!} \times \frac{\partial^{(m)}}{\partial z^{(m)}} \left[\frac{(\log x)^{-z}}{1 + P(2)z}\right] = \sum_{j=0}^{m} \frac{(-P(2))^{m-j} (\log\log x)^j}{j!} = e^{-\frac{\log\log x}{P(2)}} \frac{(-P(2))^m}{m!} \times \Gamma\left(m+1, -\frac{\log\log x}{P(2)}\right).$$

Now by parameterizing the countour around $|z| = r := \frac{k-1}{\log\log x}$ we find the the main term of our approximation corresponds to

$$\begin{aligned}-\int_{|z|=r} \frac{x \cdot \exp(-P(2)z)(\log x)^{-z}}{(\log x) z^k (1 + P(2)z)} dz &\sim \frac{x}{\log x} \cdot \frac{(-1)^k P(2)^{k-1} (\log\log x)^{k-1}}{\Gamma\left(1 - \frac{k}{\log\log x}\right)(k-1)!} \\ &\asymp \frac{x}{\log x} \cdot \frac{(-1)^k P(2)^{k-1} (\log\log x)^{k-1}}{(k-1)!}.\end{aligned}$$ □

The signs of the functions estimated in the next theorem are again dictated by the differences of the prime omega functions as $(-1)^{\Omega(n)-\omega(n)}$. It happens, as we have summarized above, that this distribution is fairly regular. This signedness property, in place of the more natural $\lambda(n)$ weights as appear in Proposition 4.1, is necessary to simplify the DGF expansion we used to obtain the asymptotics for the summatory functions $\widehat{A}_z(x)$ in Proposition 7.3. In particular, an exact DGF expression for $\lambda(n)C_{\Omega(n)}(n)$ is complicated by the need to estimate the asymptotics of the coefficients of the product

$$\sum_{n \geq 1} \frac{\lambda(n) C_{\Omega(n)}(n) z^{\Omega(n)}}{(\Omega(n))! \cdot n^s} = \prod_p \left(2 - \exp\left(-z \cdot p^{-s}\right)\right)^{-1}, \operatorname{Re}(s) > 1, |z| < \log 2.$$

**Lemma 7.6.** *We have that as $x \rightarrow \infty$*

$$\left|\mathbb{E}\left[\sum_{n \leq x} \lambda(n)(-1)^{\omega(n)} C_{\Omega(n)}(n)\right]\right| \asymp \frac{1}{\sqrt{2\pi P(2)}} \cdot \frac{(\log x)^{P(2)}}{\sqrt{\log\log x}}.$$

*Proof.* We observe that

$$\sum_{n \leq x} \lambda(n)(-1)^{\omega(n)} C_{\Omega(n)}(n) = \sum_{k=1}^{\log_2(x)} \sum_{\substack{n \leq x \\ \Omega(n)=k}} \lambda(n)(-1)^{\omega(n)} C_{\Omega(n)}(n) = \sum_{k=1}^{\log_2(x)} \widehat{C}_k(x).$$

We claim that

$$\sum_{k=1}^{\log_2(x)} \widehat{C}_k(x) \asymp \sum_{k=1}^{P(2)\log\log x} \widehat{C}_k(x). \tag{31}$$

To prove (31), it suffices to show that

$$\left| \frac{\sum\limits_{P(2)\log\log x < k \leq \log_2(x)} \widehat{C}_k(x)}{\sum\limits_{k=1}^{P(2)\log\log x} \widehat{C}_k(x)} \right| = o(1), \text{ as } x \to \infty. \tag{32}$$

We first compute the absolute value of the following summatory function by applying Theorem 7.5 for large $x \to \infty$:

$$\begin{aligned} \left|\sum_{k=1}^{P(2)\log\log x} \widehat{C}_k(x)\right| & = \left|\frac{x}{\log x} \times \sum_{k=1}^{P(2)\log\log x} \frac{(-1)^k P(2)^{k-1} (\log\log x)^{k-1}}{(k-1)!}\right| \left[1 + O\left(\frac{k}{(\log\log x)^3}\right)\right] \\ & = \left|\frac{x}{\log x} \times \sum_{k=1}^{P(2)\log\log x} \frac{(-1)^k P(2)^{k-1} (\log\log x)^{k-1}}{(k-1)!}\right| \left[1 + O\left(\frac{1}{(\log\log x)^2}\right)\right] \\ & = \left|\frac{x \cdot \Gamma(P(2)\log\log x, -P(2)(\log\log x))(\log x)^{P(2)-1}}{\Gamma(P(2)\log\log x)}\right| \left[1 + O\left(\frac{1}{(\log\log x)^2}\right)\right] \\ & = \frac{x}{\log x} \cdot \frac{(P(2)\log\log x)^{P(2)\log\log x - 1}}{\Gamma(P(2)\log\log x)} \left[1 + O\left(\frac{1}{(\log\log x)^2}\right)\right] \\ & = \frac{x}{\sqrt{2\pi P(2)}} \cdot \frac{(\log x)^{P(2)}}{\sqrt{\log\log x}} \left[1 + O\left(\frac{1}{(\log\log x)^2}\right)\right]. \end{aligned} \tag{33}$$

To show that (32) holds, observe that whenever $\Omega(n) = k$, we have that $C_{\Omega(n)}(n) \leq k!$. Then we compute using Theorem 5.4 to show that for large $x \to \infty$ we have for $0 < \varepsilon < 1$

$$\begin{aligned} & \sum_{P(2)\log\log x < k \leq (\log\log x)^{\varepsilon \frac{\log\log x}{\log\log\log x}}} \widehat{C}_k(x) \leq \sum_{P(2)\log\log x < k \leq (\log\log x)^{\varepsilon \frac{\log\log x}{\log\log\log x}}} \sum_{\substack{n \leq x \\ \Omega(n) = k}} C_{\Omega(n)}(n) \\ & \ll \sum_{k=P(2)\log\log x + 1}^{\log\log x - 1} \frac{\widehat{\pi}_k(x)}{x} \cdot k! + \sum_{k=\log\log x}^{(\log\log x)^{\varepsilon \frac{\log\log x}{\log\log\log x}}} \frac{\widehat{\pi}_k(x)}{x} \cdot k! \\ & \ll \sum_{k=P(2)\log\log x + 1}^{\log\log x} (\log x)^{P(2) - 1 - P(2)\log P(2)} (\log\log x)^k \sqrt{k} \\ & \quad + \sum_{k=\log\log x}^{(\log\log x)^{\varepsilon \frac{\log\log x}{\log\log\log x}}} (\log x)^{\frac{k}{\log\log x} - 1 - \frac{k}{\log\log x}(\log k - \log\log\log x)} \cdot \left(\frac{k}{e}\right)^k \sqrt{2\pi k} \\ & \ll \sum_{k=P(2)\log\log x + 1}^{\log\log x} (\log x)^{P(2) - 1 - P(2)\log P(2)} (\log\log x)^k \sqrt{k} + \sum_{k=\log\log x}^{\varepsilon \frac{\log\log x}{\log\log\log x}} (\log x)^{k \frac{\log\log\log x}{\log\log x} - 1} \sqrt{k} \\ & \ll (\log x)^{P(2) - 1 - P(2)\log P(2)} \int_{P(2)\log\log x}^{\log\log x} (\log\log x)^t \sqrt{t} dt \frac{1}{(\log x)} \times \int_{\log\log x}^{\varepsilon \frac{\log\log x}{\log\log\log x}} (\log x)^t \sqrt{t} dt \\ & \ll (\log x)^{P(2) - 1 - P(2)\log P(2)} \sqrt{\log\log x} (\log\log x)^{\log\log x} + \frac{1}{(\log x)} \sqrt{\varepsilon \frac{\log\log x}{\log\log\log x}} (\log\log x)^{\varepsilon \frac{\log\log x}{\log\log\log x}} = o(x), \end{aligned}$$

where $\lim_{x\to\infty} (\log x)^{\frac{1}{\log\log x}} = e$ and $P(2) - 1 - P(2)\log P(2) \approx -0.188883$. By (33) this form of the ratio in (32) clearly tends to zero. If we have a contribution from the terms $\widehat{\pi}_k(x)$ as $\varepsilon \to 1$, e.g., if $x$ is a power of two, then $C_{\Omega(x)}(x) = 1$ by the formula in (9), so this contribution is negligible. The formula for the expectation claimed in the statement of this lemma above then follows from (33) by scaling by $\frac{1}{x}$ and dropping the asymptotically lesser error terms in the bound. □

**Corollary 7.7** (Summatory functions of unsigned sequences). *We have that for large $x \geq 2$ and $1 \leq k \leq \log\log x$*

$$\sum_{n\leq x} C_{\Omega(n)}(n) = \frac{3x}{2\widehat{c}\sqrt{2\pi}P(2)(1+P(2))} \cdot \frac{(\log x)^{1+P(2)}}{\sqrt{\log\log x}} \left[1 + O\left(\frac{1}{\log\log x}\right)\right]$$

$$\sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \sim \frac{3}{2\widehat{c}} \cdot \frac{x \cdot P(2)^{k-1}}{(\log x)^2} \left[\frac{(\log\log x)^{k-1}}{(k-1)!} - \frac{(\log\log x)^{k-2}}{(k-2)!}\right].$$

*Proof.* We handle transforming our previous results for the first sum. Let

$$\begin{aligned}
\widehat{C}_*(x) &:= \left|\sum_{n\leq x} \lambda(n)(-1)^{\omega(n)} C_{\Omega(n)}(n)\right| \\
&= \frac{x}{P(2)\sqrt{x}} \cdot \frac{(\log x)^{\log P(2)}}{\sqrt{\log\log x}} \left[1 + O\left(\frac{1}{(\log\log x)^2}\right)\right],
\end{aligned}$$

where the second equation follows from the proof of Lemma 7.6. Let the summatory function

$$\begin{aligned}
L_*(t) := \sum_{n\leq x} \lambda(n)(-1)^{\omega(n)} &= \sum_{k=0}^{\log_2(t)} (-1)^k \cdot \#\{n \leq t : \Omega(n) - \omega(n) = k\} \\
&\sim \sum_{k=0}^{\log_2(t)} \widehat{c}t \cdot \frac{(-1)^k}{2^k} = \frac{2\widehat{c}t}{3} + o(1), \text{ as } t \to \infty.
\end{aligned}$$

Then integrating by parts in the Abel summation formula, we obtain that

$$\begin{aligned}
\widehat{C}_*(x) &= \int_1^x L_*'(t) C_{\Omega(t)}(t)dt \qquad \Longrightarrow \\
C_{\Omega(x)}(x) &\sim \frac{\widehat{C}_*'(x)}{L_*'(x)} \\
&\sim \frac{3}{4\widehat{c}\sqrt{2\pi}P(2)} \cdot \frac{(\log x)^{P(2)-1}}{(\log\log x)^{3/2}} \left[2(\log x)(\log\log x) + 2\log P(2)(\log\log x) - 1\right] \\
&\sim \frac{3(\log x)^{P(2)}}{2\widehat{c}\sqrt{2\pi}P(2)\sqrt{\log\log x}} =: \widehat{C}_{**}(x).
\end{aligned}$$

So by Abel summation (again integrating by parts), we obtain (TODO)

$$\begin{aligned}
\sum_{n\leq x} C_{\Omega(n)}(n) &\asymp \int \widehat{C}_{**}(x)dx = (x-1)\int \frac{\widehat{C}_{**}(x)}{x}dx \\
&= \frac{3(x-1)}{2\widehat{c}\sqrt{2}P(2)\sqrt{1+P(2)}} \operatorname{erfi}\left(\sqrt{1+P(2)} \cdot \sqrt{\log\log x}\right) \\
&= \frac{3x}{2\widehat{c}\sqrt{2\pi}P(2)(1+P(2))} \cdot \frac{(\log x)^{1+P(2)}}{\sqrt{\log\log x}} \left[1 + O\left(\frac{1}{\log\log x}\right)\right].
\end{aligned}$$

In the previous equation, we have used a known asymptotic expansion of the function $\operatorname{erfi}(z)$ about infinity in the form of

$$\operatorname{erfi}(z) = \frac{e^{z^2}}{\sqrt{\pi}}\left(z^{-1} + \frac{1}{2}z^{-3} + \frac{3}{4}z^{-5} + \cdots\right), \text{ as } |z| \to \infty.$$

This proves the first formula.

A similar method is used to adapt the asymptotic formulas for the sums $\widehat{C}_k(x)$ we proved in Theorem 7.5. Namely, since

$$(x-1)\int \frac{d^2}{dx^2}\left[\frac{x}{\log x} \cdot \frac{P(2)^{k-1}(\log\log x)^{k-1}}{(k-1)!}\right] \cdot x dx = \frac{x \cdot P(1)^{k-1}}{(\log x)^2}\left[\frac{(\log\log x)^{k-2}}{(k-2)!} - \frac{(\log\log x)^{k-1}}{(k-1)!}\right],$$

we obtain the stated result using a similar procedure to what we just proved above. □

**Corollary 7.8** (Expectation formulas)**.** *We have that as $n \to \infty$*

$$\mathbb{E}|g^{-1}(n)| = \frac{9}{\pi^2\sqrt{2\pi}\widehat{c}P(2)(1+\log P(2))(2+P(2))} \cdot \frac{(\log n)^{2+P(2)}}{\sqrt{\log\log n}},$$

*where* $2 + \log P(2) \approx 1.20647$.

*Proof.* Because of the first formula in Corollary 7.7, we have that as $n \to \infty$

$$\mathbb{E}[C_{\Omega(n)}(n)] = \frac{3}{2\widehat{c}\sqrt{2\pi}P(2)(1+P(2))} \cdot \frac{(\log n)^{1+P(2)}}{\sqrt{\log\log n}}.$$

This implies that

$$\begin{aligned}
\int_2^x \frac{\mathbb{E}[C_{\Omega(t)}(t)]}{t} dt &= \frac{3}{2\sqrt{2}\widehat{c}P(2)(1+P(2))} \cdot \frac{\operatorname{erfi}\left(\sqrt{2+P(2)} \cdot \sqrt{\log\log x}\right)}{\sqrt{2+P(2)}} \\
&= \frac{3}{2\sqrt{2\pi}\widehat{c}P(2)(1+P(2))(2+P(2))} \cdot \frac{(\log x)^{2+P(2)}}{\sqrt{\log\log x}}\left[1 + O\left(\frac{1}{\log\log x}\right)\right].
\end{aligned}$$

Therefore, by the formula we derived in the proof of Corollary 6.6 we have

$$\begin{aligned}
\mathbb{E}|g^{-1}(n)| &= \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d<n}\frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O(1) \\
&= \frac{9}{\pi^2\sqrt{2\pi}\widehat{c}P(2)(1+P(2))(2+P(2))} \cdot \frac{(\log n)^{2+P(2)}}{\sqrt{\log\log n}}.
\end{aligned}$$

□

**Theorem 7.9.** *Let the mean and variance analogs be denoted by*

$$\mu_x(C) := P(2)\log\log x, \qquad \text{and} \qquad \sigma_x(C) := \sqrt{\mu_x(C)}$$

*Set $Y > 0$ and suppose that $z \in [-Y, Y]$. Then we have uniformly for all $-Y \leq z \leq Y$ as $x \to \infty$ that*

$$\frac{1}{x}\cdot\#\left\{2 \leq n \leq x : \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)} \leq z\right\} = \frac{3}{2\widehat{c}\cdot(\log x)^{2-P(2)}(\log\log x)}\cdot\Phi(z) + O\left(\frac{1}{(\log x)^{2-P(2)}(\log\log x)^{3/2}}\right).$$

*In the previous equation, the denominator exponent is approximately* $2 - P(2) \approx 1.54775$.

*Proof.* For large $x$ and $n \leq x$, define the following auxiliary variables:

$$\alpha_n := \frac{C_{\Omega(n)}(n) - \mu_n(C)}{\sigma_n(C)}, \quad \beta_{n,x} := \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)}.$$

Let the corresponding densities (whose limiting distributions we must verify) be defined by the functions

$$\Phi_1(x,z) := \frac{1}{x} \cdot \#\{n \leq x : \alpha_n \leq z\},$$

and

$$\Phi_2(x,z) := \frac{1}{x} \cdot \#\{n \leq x : \beta_{n,x} \leq z\}.$$

We first argue that it suffices to consider the distribution of $\Phi_2(x,z)$ as $x \to \infty$ in place of $\Phi_1(x,z)$ to obtain our desired result statement. In particular, the difference of the two auxiliary variables is neglibible as $x \to \infty$ for $n, x$ taken over the ranges that contribute the non-trivial weight to the main term of each density function. We have for $\sqrt{x} \leq n \leq x$ and $C_{\Omega(n)}(n) \leq 2 \cdot \mu_x(C)$ that

$$|\alpha_n - \beta_{n,x}| \ll \frac{1}{\sigma_x(C)} \xrightarrow{x \to \infty} 0.$$

So we naturally prefer to estimate the easier forms of the distribution function $\Phi_2(x,z)$ when $x$ is large, and for any fixed $z \in \mathbb{R}$. (TODO) ... $\square$

**Corollary 7.10.** *Let $Y > 0$ and $z \in [-Y, Y]$. Then uniformly for all $-Y \leq z \leq Y$ as $x \to \infty$ we have that*

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq z\right\} = \frac{3}{2\widehat{c} \cdot (\log x)^{2-P(2)}(\log\log x)} \cdot \Phi\left(\frac{\frac{\pi^2}{6}z - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{1}{(\log x)^{2-P(2)}(\log\log x)^{3/2}}\right).$$

*Proof.* **(TODO)** We compute using the argument sketched in the proof of Corollary 6.6 from Section 6.3 that

$$|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2} C_{\Omega(n)}(n).$$

Then the result follows from Theorem 7.9. $\square$

### 7.3 Proof of the error term bound

*Proof of Lemma 7.1 (Part I).* First of all, we notice that

$$\frac{x}{k} - \frac{x}{k+1} = \frac{x}{k(k+1)} \sim \frac{x}{k^2},$$

so that $\frac{x}{k^2} \geq 1 \implies k \leq \sqrt{x}$ **(TODO)**. Thus we can re-write the error term sum as

$$E_M(x) \sim \sum_{k=1}^{\sqrt{x}} G^{-1}(k)\left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)\right] \tag{34}$$

$$= \sum_{k=1}^{\sqrt{x}} g^{-1}(k)\pi\left(\frac{x}{k}\right). \tag{35}$$

Since $\pi\left(\frac{x}{k}\right) \leq \pi(\sqrt{x})$ for $1 \leq k \leq \sqrt{x}$, we have that

$$E_M(x) = O\left(\frac{\sqrt{x}}{\log x} G^{-1}(\sqrt{x})\right). \tag{36}$$

So to show that $E_M(x) = o(\sqrt{x})$ infinitely often along an infinite subsequence of positive integers, it suffices to show that $G^{-1}(\sqrt{x}) = o(\log x)$ for infinitely many positive integers. We will require a few more results before we can prove this current conjectured property. $\square$

# 8 Lower bounds for $M(x)$ along infinite subsequences

## 8.1 Establishing initial lower bounds on the summatory functions $G^{-1}(x)$

Let the summatory function $G_E^{-1}(x)$ be defined for $x \geq 1$ by

$$G_E^{-1}(x) := \sum_{n \leq (\log x)^5 (\log\log x)} \lambda(n) \times \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log\log d}. \tag{37}$$

The subscript of $E$ is a formality of notation that does not correspond to an actual parameter or any implicit dependence on $E$ in the function defined above.

**Theorem 8.1.** *For almost all sufficiently large integers $x \to \infty$, we have that*

$$|G^{-1}(x)| \gg |G_E^{-1}(x)|.$$

*Proof.* First, consider the following upper bound on $|G_E^{-1}(x)|$:

$$\begin{aligned}
|G_E^{-1}(x)| &= \left| \sum_{e \leq n \leq (\log x)^5 (\log\log x)} \lambda(n) \times \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log\log d} \right| \\
&\ll \sum_{e < d \leq (\log x)^5 (\log\log x)} \frac{(\log d)^{\frac{1}{4}}}{\log\log d} \cdot \left\lfloor \frac{(\log x)^5 (\log\log x)^{16}}{d} \right\rfloor \\
&\ll (\log x)^5 (\log\log x) \times \int_e^{(\log x)^5 (\log\log x)} \frac{(\log t)^{\frac{1}{4}}}{t \cdot \log\log t} dt \\
&= (\log x)^5 (\log\log x) \times \mathrm{Ei}\left(\frac{5}{4} \log\log\left((\log x)^5 (\log\log x)\right)\right) \\
&\ll \frac{25}{64} \cdot (\log x)^5 (\log\log x) (\log\log\log x)^2.
\end{aligned} \tag{38}$$

We compute that for almost every sufficiently large $x \to \infty$:

$$\frac{|G^{-1}(x)|}{x} = \frac{1}{x} \times \left| \sum_{\substack{d \leq x \\ \lambda(d)=+1}} |g^{-1}(d)| - \sum_{\substack{d \leq x \\ \lambda(d)=-1}} |g^{-1}(d)| \right| \gg \left| \mathbb{E}|g^{-1}(x)| - \frac{2}{x} \times \sum_{\substack{d \leq x \\ \lambda(d)=-1}} |g^{-1}(d)| \right|.$$

Let the summation in the previous equation be defined by

$$S_-(x) := \sum_{\substack{d \leq x \\ \lambda(d)=-1}} |g^{-1}(d)|.$$

We will find upper and lower bounds on this sum that show $\mathbb{E}|g^{-1}(x)| \gg \frac{S_-(x)}{x}$.
For the positive summands of $S_-(x)$ to be at their largest, we require that for $d \geq 2$

$$|g^{-1}(d)| = \sum_{j=0}^{\omega(d)} \binom{\omega(d)}{j} \cdot j!.$$

Then we have that

$$S_-(x) \ll \sum_{1 \leq k \leq \log_2(x)} \widehat{\pi}_k(x) \times \sum_{j=0}^{k} \binom{k}{j} \cdot j!. \tag{39}$$

We can bound the summatory function terms by

$$\widehat{\pi}_k(x) \leq \frac{\widehat{\pi}_k(x) \cdot \pi_k(x)}{\#\left\{n \leq x : \Omega(n) = \omega(n) \wedge \Omega(n) = k\right\}}.$$

By an argument with conditional probabilities of set densities, we find

$$\begin{aligned}\#\left\{n \leq x : \Omega(n) = \omega(n) \wedge \Omega(n) = k\right\} &\geq \frac{1}{x} \cdot \#\left\{n \leq x : n \text{ squarefree} \wedge \mu(n) = (-1)^k\right\} \times \widehat{\pi}_k(x) \\ &= \frac{3}{\pi^2}\widehat{\pi}_k(x), \text{ as } x \rightarrow \infty.\end{aligned}$$

So from (39), we obtain that

$$S_-(x) \ll \sum_{1 \leq k \leq \log_2(x)} \frac{\pi^2}{3}\pi_k(x) \times \sum_{j=0}^{k} \binom{k}{j} \cdot j!. \tag{40}$$

We weight by the known asymptotic formula for the summatory functions $\pi_k(x) \sim \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}(1 + o(1))$ as $x \rightarrow \infty$ to find that

$$\begin{aligned}S_-(x) &\ll \frac{\pi^2}{3} \times \sum_{1 \leq k \leq \log_2(x)} \pi_k(x) \times \sum_{j=0}^{k} \binom{k}{j} \cdot j! \\ &\ll \frac{\pi^2}{3} \times \frac{x}{(\log x)(\log\log x)} \times \sum_{k \geq 1} k \cdot (\log\log x)^k \sum_{j=0}^{k} \frac{1}{j!} \\ &\ll \frac{\pi^2}{3} \times \frac{ex}{(\log x)(\log\log x)} \times \sum_{k \geq 1} k \cdot (\log\log x)^k \\ &\ll \frac{\pi^2}{3} \times \frac{ex}{(\log x)(\log\log x)^2}.\end{aligned}$$

Thus, over these choices bounding the $g^{-1}(d)$, we obtain that $\frac{S_-(x)}{x} = o(1)$ as $x \rightarrow \infty$.
On the other hand, we can choose the summands to satisfy $|g^{-1}(d)| \geq 2$. We define the following densities for large $x \geq 2$ [16, *cf.* §1]:

$$\begin{aligned}\mathcal{L}_+(x) &:= \frac{1}{x} \cdot \#\{n \leq x : \lambda(n) = +1\} \overset{\mathbb{E}}{\sim} \frac{1}{2} \\ \mathcal{L}_-(x) &:= \frac{1}{x} \cdot \#\{n \leq x : \lambda(n) = -1\} \overset{\mathbb{E}}{\sim} \frac{1}{2}.\end{aligned}$$

Now we see that

$$S_-(x) \gg 2x \cdot \min\left(\mathcal{L}_-(x), 1 - \mathcal{L}_-(x)\right).$$

This implies that $\frac{S_-(x)}{x} = O(1)$. In either of these extreme bounds on $S_-(x)$, we have by Corollary 6.6 that

$$\frac{|G^{-1}(x)|}{x} \gg \frac{6}{\pi^2}(\log x)(\log\log x).$$

Then naturally from (38) we have proved that as $x \rightarrow \infty$, $|G^{-1}(x)| \gg |G_E^{-1}(x)|$. □

**Remark 8.2.** Note that the only cases of $x \geq 1$ we need to be wary of in the *almost everywhere* clause to applying the statement of Theorem 8.1 happen when $G^{-1}(x) = 0$. This singularity in the distribution of $G^{-1}(x)$ can only occur when

$$G^{-1}(x) = \frac{g^{-1}(x)}{2} + \frac{1}{2\pi\imath}\int_{c-\imath\infty}^{c+\imath\infty} \frac{x^s}{s \cdot (P(s)+1)\zeta(s)} ds = 0, \text{ for } c > 1.$$

It suffices to assume that $G^{-1}(x) \neq 0$ on an asymptotically dense subset of the integers for the bounds we need to prove Corllary 3.9 in the last subsection. In particular, we only require that

$$\lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leq x : G^{-1}(n) \neq 0\} \geq \frac{1}{2}.$$

**Corollary 8.3.** *We have that for almost every sufficiently large $x$, that as $x \to \infty$*

$$\left|G_E^{-1}(x)\right| \gg \frac{\widehat{C}_0}{2\sqrt{2\pi}} \times \frac{(\log x)^{\frac{5}{4}}}{(\log\log x)^{\frac{1}{4}}\sqrt{\log\log\log x}} \times \left|\sum_{e<d\leq \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d^{1/4}\cdot \log\log d}\right|.$$

*Proof.* Using the definition in (37), we obtain on average that [–A–]

$$\begin{aligned}\left|G_E^{-1}(x)\right| &= \left|\sum_{n\leq (\log x)^5(\log\log x)} \lambda(n) \times \sum_{\substack{d\mid n\\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log\log d}\right| \\ &= \left|\sum_{e<d\leq (\log x)^5(\log\log x)} \frac{(\log d)^{\frac{1}{4}}}{\log\log d} \times \sum_{n=1}^{\left\lfloor\frac{(\log x)^5(\log\log x)}{d}\right\rfloor} \lambda(dn)\right|.\end{aligned}$$

We see that by complete additivity of $\Omega(n)$ (complete multiplicativity of $\lambda(n)$) that

$$\sum_{n=1}^{\left\lfloor\frac{x}{d}\right\rfloor} \lambda(dn) = \sum_{n=1}^{\left\lfloor\frac{x}{d}\right\rfloor} \lambda(d)\times\lambda(n) = \lambda(d) \times \sum_{n\leq\left\lfloor\frac{x}{d}\right\rfloor} \lambda(n).$$

From Theorem 3.7 and Lemma 8.4 (see below), we can establish that

$$\left|\sum_{k\leq\log\log x} (-1)^k \cdot \widehat{\pi}_k(x)\right| \gg \frac{\widehat{C}_0}{\sqrt{2\pi}} \cdot \frac{x^{\frac{1}{4}}}{(\log x)^{\frac{1}{2}}\sqrt{\log\log x}} =: \widehat{L}_0(x), \text{ as } x\to\infty. \tag{41}$$

The sign of the sum obtained by taking the right-hand-side of (41) without the absolute value operation is given by $(-1)^{1+\lfloor\log\log x\rfloor}$. The precise formula for the limiting lower bound stated above for $\widehat{L}_0(x)$ is computed by symbolic summation in *Mathematica* using the new bounds on $\widehat{\pi}_k(x)$ guaranteed by the theorem, and then by applying subsequent standard asymptotic estimates to the resulting formulas for large $x\to\infty$ in the form of (10c) and Stirling's formula. It follows that

$$|G_E^{-1}(x)| \gg \left|\sum_{e<d\leq(\log x)^5(\log\log x)} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\left\lfloor\log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor} \cdot \widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right|. \tag{42}$$

**Outline for the remainder of the proof.** We sketch the following steps remaining to prove our claimed lower bound on $|G_E^{-1}(x)|$:

**(A)** We identify an initial subinterval $\mathcal{R}_x$ where we can expect constant sign term contributions resulting from the inputs to the function $\widehat{L}_0$ involving both $(d, x)$ for $x$ large and $d$ on this smaller subinterval.

[–A–]For any arithmetic functions $f, h$, we have that [1, *cf.* §3.10; §3.12]

$$\sum_{n\leq x} h(n)\times\sum_{d\mid n} f(d) = \sum_{d\leq x} f(d)\times\sum_{n=1}^{\left\lfloor\frac{x}{d}\right\rfloor} h(dn).$$

**(B)** We factor out easily bounded terms from the expansion of the monotone $\widehat{L}_0$ on this interval.

**(C)** We determine additional asymptotic formulas we will refer to in later sections for the resulting lower bounds on $|G_E^{-1}(x)|$ that are formed by restricting the range of $d$ in (42) to $\mathcal{R}_x$.

**(D)** We argue that the sums of oscillatory terms on the upper end of the deleted interval for $d \in \left(e, (\log x)^5(\log\log x)\right] \setminus \mathcal{R}_x$ cannot generate trivial bounds by cancellation with the new lower bounds.

**Part A.** We will simplify (42) by proving that there are ranges of consecutive integers over which we obtain essentially constant sign contributions from the function $\widehat{L}_0((\log x)^5(\log\log x)/d)$ as $x \to \infty$. In particular, consider that

$$\log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right) = \log\log\left((\log x)^5(\log\log x)\right) + \log\left(1 - \frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)}\right), \text{ as } x \to \infty.$$

If we take $d \in (e, \log x] =: \mathcal{R}_x$, we have that

$$\frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)} = o(1) \to 0, \text{ as } x \to \infty.$$

For $d$ within $\mathcal{R}_x$, we expect that for almost every $x$ there are at most a handful of negligible cases of comparitively small order $d \leq d_{0,x}$ such that

$$\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor \sim \left\lfloor \log\log\left((\log x)^5(\log\log x)\right) + o(1)\right\rfloor,$$

changes in parity transitioning from $d \mapsto d+1$. An argument making this assertion precise brings leads us to two primary cases that rely on the small-order distribution of the fractional parts $f_x := \left\{\log\log\left((\log x)^5(\log\log x)\right)\right\}$ within $[0,1)$ for large $x \to \infty$ and any $\log d \in \mathcal{R}_x$:

**(1)** If the fractional part $f_x = 0$, then

$$\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor = \left\lfloor \log\log\left((\log x)^5(\log\log x)\right)\right\rfloor + \left\lfloor - \frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)}\right\rfloor.$$

This implies that provided that

$$-1 \leq -\frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)} < 0,$$

we obtain a constant multplier as $\operatorname{sgn}\left(\widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right)$ whenever $d \in \mathcal{R}_x$. Since $d$ is positive and maximized at $\log x$, this condition clearly happens for any sufficiently large $x$.

**(2)** If the fractional part $f_x \in (0,1)$, then

$$\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor = \left\lfloor \log\log\left((\log x)^5(\log\log x)\right)\right\rfloor + \left\lfloor \left\{\log\log\left((\log x)^5(\log\log x)\right)\right\} - \frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)}\right\rfloor.$$

Define shorthand notation for the function $\mathcal{B}(x) := (\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)$. We require that

$$-1 \leq f_x - \frac{\log d}{\mathcal{B}(x)} < 0 \iff (1+f_x)\cdot\mathcal{B}(x) \geq \log d > 0.$$

This property is similarly clearly attained for $d \in \mathcal{R}_x$ since $(1+f_x)\cdot\mathcal{B}(x) \geq \mathcal{B}(x)$ as $x \to \infty$ .

**Part B.** Provided that the sign term involving both $d$ and $x$ from (42) does not change for $d \in \mathcal{R}_x$, we can remove any oscillations in the sums due to sign changes in the monotonically decreasing function $\widehat{L}_0(d,x) := \widehat{L}_0\left((\log x)^5(\log\log x)/d\right)$. The function $\widehat{L}_0(d,x)$ is monotone decreasing in the variable $d$ for fixed $x$ as we sum along the subinterval $\mathcal{R}_x$ in ascending order. We can see that this function is decreasing in $d$ by computing its partial derivative and evaluating the asymptotic main terms as having a leading negative sign for all large $x$. Thus we should select $d := \log x$ in (42) to obtain a global lower bound on $|G_E^{-1}(x)|$ if we truncate the sum to range only over the subset of original indices $d \in \mathcal{R}_x$.

**Part C.** Let the magnitudes of the signed remainder term sums be defined for all sufficiently large $x$ by

$$R_E(x) := \left| \sum_{\log x < d \leq \frac{(\log x)^5(\log\log x)}{e^2}} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor} \cdot \widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right) \right|.$$

Set the function $T_E(x)$ to correspond to the easily factored dependence of the less simply integrable factors in $\widehat{L}_0(d,x)$ when we set $d := \log x$ on $\mathcal{R}_x$. This function is defined for all large enough $x$ as

$$T_E(x) \gg \frac{1}{\log\left[(\log x)^4(\log\log x)\right]^{\frac{1}{2}} \sqrt{\log\log\left[(\log x)^4(\log\log x)\right]}} \gg \frac{1}{2(\log\log x)^{\frac{1}{2}}\sqrt{\log\log\log x}}. \tag{43}$$

Then in limiting cases the lower bounding function satisfies

$$\begin{aligned} S_{E,1}(x) &:= \left| \sum_{e < d \leq (\log x)^5(\log\log x)} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor} \widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right) \right| \\ &\gg \frac{\widehat{C}_0}{\sqrt{2\pi}} \times (\log x)^{\frac{5}{4}}(\log\log x)^{\frac{1}{4}} T_E(x) \times \left| \sum_{e < d \leq \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d^{1/4} \cdot \log\log d} \right| \\ &\gg \frac{\widehat{C}_0}{2\sqrt{2\pi}} \times \frac{(\log x)^{\frac{5}{4}}}{(\log\log x)^{\frac{1}{4}} \cdot \sqrt{\log\log\log x}} \times \left| \sum_{e < d \leq \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d^{1/4} \cdot \log\log d} \right|. \end{aligned} \tag{44}$$

The formulas in (42) and (44) imply the following lower bound by the triangle inequality as $x \to \infty$:

$$|G_E^{-1}(x)| \gg \left| S_{E,1}(x) - R_E(x) \right| \gg S_{E,1}(x), \text{ as } x \to \infty. \tag{45}$$

We have claimed that we can in fact drop the sum terms over upper range of $d \notin \mathcal{R}_x$ and still obtain the asymptotic lower bound on $|G_E^{-1}(x)|$ stated in (45). To justify this step in the proof, we will provide limiting lower bounds on $R_E(x)$ that show that the contribution from the deleted interval in absolute value exceeds the magnitude of the corresponding sums over $d \in \mathcal{R}_x$ defined by $S_{E,1}(x)$ when $x$ is large.

**Part D.** We want to arrange the signed weight coefficients $\varepsilon_{x,d} \mapsto \{\pm 1\}$ so that the function

$$M_{\pm}(x) := \left| \sum_{\log x < d \leq \frac{(\log x)^5(\log\log x)}{e^2}} \frac{\varepsilon_{x,d} \cdot \lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times \widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right) \right|,$$

is minimal. We need to prove that this minimal sum exceeds the bound for $S_E(x)$ given in (44) in asymptotic order. That is, we prove that

$$S_{E,1}^{(\ell)}(x) := \frac{\widehat{C}_0}{2\sqrt{2\pi}} \times \frac{(\log x)^{\frac{5}{4}}}{(\log\log x)^{\frac{1}{4}} \cdot \sqrt{\log\log\log x}} \times \left| \sum_{e < d \leq \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d^{1/4} \cdot \log\log d} \right| = o\left(M_{\pm}(x)\right), \text{ as } x \to \infty.$$

Notice that by considering the sum term in the previous definition as being unsigned, we have that

$$\begin{aligned} S_{E,1}^{(\ell)}(x) &\ll \frac{(\log x)^{\frac{5}{4}}}{(\log\log x)^{\frac{1}{4}} \cdot \sqrt{\log\log\log x}} \times \int_e^{\log x} \frac{(\log t)^{1/4}}{t \cdot (\log\log t)} dt \\ &\ll \frac{(\log x)^2}{(\log\log x)^{\frac{1}{4}} \cdot \sqrt{\log\log\log x}} \times \operatorname{Ei}\left(\frac{5}{4}\log\log\log x\right) \\ &\ll \frac{(\log x)^2 (\log\log\log x)^{\frac{3}{2}}}{(\log\log x)^{\frac{1}{4}}}. \end{aligned} \tag{46}$$

We need to show that $M_{\pm}(x)$ always exceeds this bound. Since the function $L_0(x,d)$ is decreasing in $d$ for $d \in \left(\log x, \frac{(\log x)^5 (\log\log x)}{e^2}\right] =: \overline{\mathcal{R}}_x$, we obtain that

$$\widehat{L}_0\left(\frac{(\log x)^5 (\log\log x)}{d}\right) \gg \frac{(\log x)^{5/4} (\log\log x)^{1/4}}{d}, d \in \overline{\mathcal{R}}_x.$$

So we need to find a global lower bound on the sum

$$S_{\pm}(x) := (\log x)^{5/4} (\log\log x)^{1/4} \times \left| \sum_{\log x < d \leq \frac{(\log x)^5 (\log\log x)}{e^2}} \frac{\varepsilon_{x,d} \cdot \lambda(d) (\log d)^{\frac{1}{4}}}{d^{1/4} \cdot \log\log d} \right|,$$

that holds for any choice of the signed weights $\varepsilon_{x,d}$. Notice that for any $d > \log x$ and $\delta \geq 1$, by an expansion of convergent geometric and binomial series, the next difference of terms satisfies

$$\begin{aligned} &\left| \frac{(\log d)^{\frac{1}{4}}}{d^{\frac{1}{4}} \cdot (\log\log d)} - \frac{\log(d+\delta)^{\frac{1}{4}}}{(d+\delta)^{\frac{1}{4}} \cdot \log\log(d+\delta)} \right| \\ &\quad \sim \frac{\log(d+\delta)^{\frac{1}{4}}}{(d+\delta)^{\frac{1}{4}} \cdot \log\log(d+\delta)} \times \left| \frac{\left(1 - \frac{\delta}{\log(d+\delta)}\right)^{\frac{1}{4}}}{\left(1 - \frac{\delta}{(d+\delta)}\right)^{\frac{1}{4}} \left(1 - \frac{\delta}{(d+\delta)\log(d+\delta)\log\log(d+\delta)}\right)} - 1 \right| \\ &\quad \gg \frac{\delta}{(d+\delta)^{\frac{1}{4}} \log(d+\delta)^{\frac{3}{4}} \log\log(d+\delta)}. \end{aligned}$$

Let the number of sign changes of the terms in our sum on the interval $\overline{\mathcal{R}}_x$ be defined by

$$N_x := \#\left\{d \in \overline{\mathcal{R}}_x : \varepsilon_{x,d+1}\lambda(d+1) = -\varepsilon_{x,d}\lambda(d)\right\}.$$

Define the maximum number of consecutively signed terms on this interval to be $\delta_{\max} \geq 1$. Then by the difference property we noted above, we have that

$$\begin{aligned} \frac{S_{\pm}(x)}{(\log x)^{5/4} (\log\log x)^{1/4}} &\gg \sum_{k=0}^{\frac{N_x}{2}} \frac{\delta_{\max}}{(\log x + (2k+1)\delta_{\max})^{\frac{1}{4}} \log(\log x + (2k+1)\delta_{\max})^{\frac{3}{4}} \log\log(\log x + (2k+1)\delta_{\max})} \\ &\gg \int_{\log x + \delta_{\max}}^{\frac{(\log x)^5 (\log\log x) - e^2 \delta_{\max}}{2e^2 N_x}} \frac{2\delta_{\max}^2 \cdot dt}{t^{1/4} \cdot (\log t)^{3/4} (\log\log t)} \\ &\gg \delta_{\max} \cdot (\log x + \delta_{\max})^{\frac{3}{4}} \times \operatorname{Ei}\left(\frac{1}{4}\log\log t\right) \Bigg|_{t=\log x + \delta_{\max}}^{t=\frac{(\log x)^5 (\log\log x) - e^2 \delta_{\max}}{2e^2 N_x}} \\ &\gg \left| (\log x + \delta_{\max})^{\frac{3}{4}} \times \left(\log\left(\frac{1}{4}\right) + \log\log\log x\right) \right| \end{aligned}$$

$$\gg (\log x)^{\frac{3}{4}}.$$

Therefore, we have that

$$|R_E(x)| \gg S_{\pm}(x) \gg (\log x)^2 (\log\log x)^{1/4}.$$

Clearly, the right-hand-side lower bound in the previous equation is asymptotically larger than the maximum order bound on $S_{E,1}^{(\ell)}(x)$ we proved above in (46). □

### 8.1.1 A few more necessary results

We now use the superscript and subscript notation of $(\ell)$ not to denote a formal parameter to the functions we define below, but instead to denote that these functions form *lower bound* (rather than exact) approximations to other forms of the functions without the scripted $(\ell)$.

**Lemma 8.4.** *Suppose that $\widehat{\pi}_k^{(\ell)}(x) = o\left(\widehat{\pi}_k(x)\right)$ where $\widehat{\pi}_k^{(\ell)}(x) \geq 1$ for all integers $1 \leq k \leq \log\log x$ as $x \to \infty$. Let the weighted summatory functions be defined as*

$$A_{\Omega}^{(\ell)}(x) := \sum_{k \leq \log\log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x)$$
$$A_{\Omega}(x) := \sum_{k \leq \log\log x} (-1)^k \widehat{\pi}_k(x).$$

*Futhermore, suppose that $|A_{\Omega}(x)| \not\to 0$ as $x \to \infty$ and that*

$$\liminf_{k \to \infty} \frac{\widehat{\pi}_k^{(\ell)}(x)}{\widehat{\pi}_k(x)} \geq x^{-\rho_0}$$
$$\limsup_{k \to \infty} \frac{\widehat{\pi}_k^{(\ell)}(x)}{\widehat{\pi}_k(x)} \leq x^{-\rho_1},$$

*as $x \to \infty$ for some $\rho_0, \rho_1 > 0$. Then for all sufficiently large $x$, we have that*

$$|A_{\Omega}(x)| \gg |A_{\Omega}^{(\ell)}(x)|.$$

*Proof.* By the second conditions above, we find that

$$\left|A_{\Omega}(x) - A_{\Omega}^{(\ell)}(x)\right| \leq |A_{\Omega}(x)| \left(1 - \inf_{1 \leq k \leq \log\log x} \frac{\widehat{\pi}_k^{(\ell)}(x)}{\widehat{\pi}_k(x)}\right) = |A_{\Omega}(x)|(1 + o(1))$$
$$\left|A_{\Omega}(x) - A_{\Omega}^{(\ell)}(x)\right| \geq |A_{\Omega}(x)| \left(1 - \sup_{1 \leq k \leq \log\log x} \frac{\widehat{\pi}_k^{(\ell)}(x)}{\widehat{\pi}_k(x)}\right) = |A_{\Omega}(x)|(1 + o(1)).$$

Similarly, we can see that

$$|A_{\Omega}(x)|(1 + o(1)) \leq \left|A_{\Omega}(x) + A_{\Omega}^{(\ell)}(x)\right| \leq |A_{\Omega}(x)|(1 + o(1)).$$

This implies that

$$|A_{\Omega}(x)|(1 + o(1)) \ll \left||A_{\Omega}(x)| \pm |A_{\Omega}^{(\ell)}(x)|\right| \ll |A_{\Omega}(x)|(1 + o(1)), \text{ as } x \to \infty.$$

Because we have that $|A_{\Omega}(x)| \not\to 0$, the previous equation shows that $|A_{\Omega}^{(\ell)}(x)|$ is bounded above and below by a constant times $|A_{\Omega}(x)|$. In other words, $|A_{\Omega}(x)| \gg |A_{\Omega}^{(\ell)}(x)|$ whenever $x$ is sufficiently large. □

*Proof of Lemma 3.8.* We can form an accurate $C^1(\mathbb{R})$ approximation by the smoothness of $\widehat{\pi}_k^{(\ell)}(x)$ that allows us to apply the Abel summation formula using the summatory function $A_\Omega(t)$ for $t$ on any bounded connected subinterval of $[1,\infty)$. Namely, we obtain

$$\begin{aligned} |F_\lambda(x)| &\gg \left| A_\Omega(x) f(x) - \int_{u_0}^x A_\Omega(t) f'(t) dt \right| \\ &\gg \left| |A_\Omega(x) f(x)| - \int_{u_0}^x |A_\Omega(t) f'(t)| dt \right| \\ &\gg \left| |A_\Omega^{(\ell)}(x) \widehat{\tau}_\ell(x)| - \int_{u_0}^x |A_\Omega(t) f'(t)| dt \right|. \end{aligned} \tag{47}$$

The stated lower bound formula for $|F_\lambda(x)|$ in (47) above is valid whenever

$$0 \leq \left| \frac{\sum\limits_{\log\log t < k \leq \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_\Omega(t)} \right| \ll 2, \text{ as } t \to \infty,$$

Indeed, by Corollary 5.7, we have that the assertion above holds as $t \to \infty$. This property remarkably holds even when we should technically index over all $k \in [1, \log_2(x)]$ to obtain an exact formula for this summatory weight function given by $L(x) := \sum_{n \leq x} \lambda(n)$.

Let the function

$$\widehat{I}_\ell(x) := \int_{\frac{\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log x}{2}} \left| A_\Omega^{(\ell)}\left(e^{e^{2t}}\right) \widehat{\tau}'_\ell\left(e^{e^{2t}}\right) \right| e^{e^{2t}} dt.$$

We argue that two key properties of this function hold as $x \to \infty$:

**(1)** $\int_{u_0}^x |A_\Omega(t) f'(t)| dt \gg \widehat{I}_\ell(x)$; and

**(2)** $\widehat{I}_\ell(x) = O\left(A_\Omega^{(\ell)}(\log\log x) \widehat{\tau}_\ell(\log\log x)\right)$.

To prove property **(1)**, observe that by hypothesis since $|A_\Omega(x)| \gg |A_\Omega^{(\ell)}(x)|$ as $x \to \infty$, we have that

$$\begin{aligned} \int_{u_0}^x |A_\Omega(t) f'(t)| dt &\gg \int_{u_0}^x |A_\Omega(t) \widehat{\tau}'_\ell(t)| dt \\ &\gg \left| \sum_{k=u_0}^{\log\log x} (-1)^k \left| A_\Omega\left(e^{e^k}\right) \widehat{\tau}'_\ell\left(e^{e^k}\right) \right| \cdot \left(e^{e^k} - e^{e^{k-1}}\right) \right| \\ &\gg \left| \sum_{k=u_0}^{\frac{\log\log x}{2}} \left[ \left| A_\Omega\left(e^{e^{2k}}\right) \widehat{\tau}'_\ell\left(e^{e^{2k}}\right) \right| \cdot e^{e^{2k}} - \left| A_\Omega\left(e^{e^{2k-1}}\right) \widehat{\tau}'_\ell\left(e^{e^{2k-1}}\right) \right| \cdot e^{e^{2k-1}} \right] \right| \\ &\gg \int_{\frac{\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log x}{2}} \left| A_\Omega\left(e^{e^{2t}}\right) \widehat{\tau}'_\ell\left(e^{e^{2t}}\right) \right| e^{e^{2t}} dt \\ &\gg \int_{\frac{\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log x}{2}} \left| A_\Omega^{(\ell)}\left(e^{e^{2t}}\right) \widehat{\tau}'_\ell\left(e^{e^{2t}}\right) \right| e^{e^{2t}} dt. \end{aligned}$$

To prove property **(2)**, we see by the mean value theorem, the monotonicity of $|A_\Omega^{(\ell)}(x)|$ as $x \to \infty$, and the hypothesis $\left| \widehat{\tau}_\ell\left(\frac{\log\log x}{2}\right) - \widehat{\tau}_\ell\left(\frac{\log\log x}{2} - \frac{1}{2}\right) \right| = O\left(\frac{\widehat{\tau}_\ell(x)}{\log\log x}\right)$ as $x \to \infty$, that for some $c \in \left[\frac{\log\log x}{2} - \frac{1}{2}, \frac{\log\log x}{2}\right]$ we have

$$\widehat{I}_\ell(x) = \left| A_\Omega^{(\ell)}\left(e^{e^{2c}}\right) \right| e^{e^{2c}} \times \left| \widehat{\tau}_\ell\left(\frac{\log\log x}{2}\right) - \widehat{\tau}_\ell\left(\frac{\log\log x}{2} - \frac{1}{2}\right) \right|$$

$$= O\left(\log\log x \cdot A_\Omega^{(\ell)}(\log\log x) \times \left|\widehat{\tau}_\ell\left(\frac{\log\log x}{2}\right) - \widehat{\tau}_\ell\left(\frac{\log\log x}{2} - \frac{1}{2}\right)\right|\right)$$
$$= O\left(A_\Omega^{(\ell)}(\log\log x)\widehat{\tau}_\ell(\log\log x)\right).$$

Combined with the last equation in (47), properties **(1)** and **(2)** imply the stated result. □

**Corollary 8.5** (Conditions on our central bounding functions)**.** *Let the smooth bounding functions be defined for large $t \gg e$ as*

$$\widehat{\tau}_\ell(t) := \frac{(\log t)^{\frac{1}{4}}}{t^{\frac{1}{4}} \cdot (\log\log t)},$$
$$A_\Omega^{(\ell)}(t) := \frac{\widehat{C}_0}{\sqrt{2\pi}} \cdot \frac{t^{\frac{1}{4}}}{(\log t)^{\frac{1}{2}}\sqrt{\log\log t}}.$$

*Then we have that as $x \to \infty$*

$$|G_E^{-1}(x)| \gg \frac{\widehat{C}_0}{2\sqrt{2\pi}} \cdot \frac{(\log x)^{5/4}}{(\log\log x)^{1/4}\sqrt{\log\log\log x}} \times \left|A_\Omega^{(\ell)}(\log x)\widehat{\tau}_\ell(\log x) - \int_{\frac{\log\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log\log x}{2}} A_\Omega^{(\ell)}\left(e^{e^{2t}}\right)\widehat{\tau}_\ell\left(e^{e^{2t}}\right)e^{e^{2t}}dt\right|.$$

*Proof.* By Corollary 8.3, we have that

$$|G_E^{-1}(x)| \gg \frac{\widehat{C}_0}{2\sqrt{2\pi}} \cdot \frac{(\log x)^{5/4}}{(\log\log x)^{1/4}\sqrt{\log\log\log x}} \times \left|\sum_{e<d\leq\log x} \frac{\lambda(d)(\log d)^{1/4}}{d^{1/4}\cdot\log\log d}\right|, \text{ as } x \to \infty. \tag{48}$$

The crux of the remainder of the proof boils down to checking hypotheses in Lemma 8.4 and Lemma 3.8. We first apply Lemma 8.4 with the lower bound function resulting from Theorem 3.7 as follows:

$$\widehat{\pi}_k^{(\ell)}(x) := \frac{\widehat{C}_0 x^{\frac{1}{4}}}{(\log x)^{\frac{3}{2}}} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

This shows that the necessary hypotheses on the function $A_\Omega^{(\ell)}(t)$ required by Lemma 3.8 are satisfied according to the sums for the function approximated by (41) for large $t$.

We next select this non-negative arithmetic function $f(d) := \frac{(\log d)^{1/4}}{d^{1/4}\cdot\log\log d}$ in applying Lemma 3.8. In particular, we can take the function $\widehat{\tau}_\ell(t) := \frac{(\log t)^{1/4}}{t^{1/4}\cdot\log\log t}$, which is non-negative and monotone for all $t > e$. Furthermore, we compute that for large $x$ we have

$$\left|\widehat{\tau}_\ell\left(\frac{x}{2}\right) - \widehat{\tau}_\ell\left(\frac{x}{2} - \frac{1}{2}\right)\right| = \widehat{\tau}_\ell\left(\frac{x}{2}\right) \times \left|1 - \frac{\left(1 + \frac{1}{\log x}\cdot\log\left(1 - \frac{1}{x}\right)\right)^{1/4}}{\left(1 - \frac{1}{x}\right)^{1/4} \times \log\left(1 + \frac{1}{\log x}\cdot\log\left(1 - \frac{1}{x}\right)\right)}\right|$$
$$= \widehat{\tau}_\ell\left(\frac{x}{2}\right) \times \left|\frac{1}{4x} + \frac{3}{4x(\log x)} - \frac{1}{4x^2(\log x)^2} + \frac{3}{16x^2(\log x)} + O\left(\frac{1}{x^3(\log x)^2}\right)\right|$$
$$= O\left(\frac{\widehat{\tau}_\ell(x)}{x}\right).$$

This argument proves that all of the requirements in Lemma 3.8 on our choice of $\widehat{\tau}_\ell(t)$ are also satisfied. So the stated result follows from (48) and Lemma 3.8. □

### 8.1.2 The proof of a central lower bound on the magnitude of $G_E^{-1}(x)$

The next central theorem is the last barrier required to prove Theorem 3.9 in the next subsection. Combined with Theorem 8.1 proved in the last section, the new lower bounds we establish below provide us with a sufficient mechanism to bound the formula from Proposition 7.2.

**Theorem 8.6** (Asymptotics and bounds for the summatory function $G^{-1}(x)$)**.** *Let $C_{\ell,1} > 0$ be the absolute constant defined by*

$$\widehat{C}_{\ell,1} = \frac{\widehat{C}_0^2}{32\pi} = \frac{(\log 2) \cdot \exp\left(-\frac{15}{16}(\log 2)^2\right)}{8\sqrt{2\pi}} \approx 0.00792203.$$

*We obtain the following limiting estimate for the bounding function $G_E^{-1}(x)$ as $x \to \infty$:*

$$\left|G_E^{-1}(x)\right| \gg \frac{\left(8 - e^{1/4}\right)\widehat{C}_{\ell,1} \cdot (\log x)^{5/4}}{\sqrt{\log\log x} \cdot (\log\log\log x)^2}.$$

*Proof.* We can form a lower summatory function indicating the signed contributions over the distinct parity of $\Omega(n)$ for all $n \leq x$ as follows by applying (10b) and Stirling's approximation as already noted in the proof of Corollary 8.3:

$$\left|A_\Omega^{(\ell)}(t)\right| = \left|\sum_{k \leq \log\log t} (-1)^k \widehat{\pi}_k(t)\right| \gg \frac{\widehat{C}_0}{\sqrt{2\pi}} \times \frac{t^{\frac{1}{4}}}{(\log t)^{\frac{1}{2}}\sqrt{\log\log t}}, \text{ as } t \to \infty. \tag{49}$$

We select the functions $\widehat{\tau}_0(t) := \frac{(\log t)^{1/4}}{t^{1/4} \cdot \log\log t}$ and $-\widehat{\tau}_0'(t) \gg \frac{(\log t)^{1/4}}{4t^{5/4} \cdot \log\log t}$ in the form of the next equation using the notation in Corollary 8.5.

$$-\widehat{\tau}_0'(t) = -\frac{d}{dt}\left[\frac{(\log t)^{\frac{1}{4}}}{t^{\frac{1}{4}} \cdot \log\log t}\right] \gg \frac{(\log t)^{1/4}}{4t^{\frac{5}{4}} \cdot \log\log t} \tag{50}$$

Moreover, we have that we can select the initial form of the lower bound to be defined as follows:

$$\begin{aligned} G_E^{-1}(x) \gg \frac{\widehat{C}_0}{2\sqrt{2\pi}} \cdot \frac{(\log x)^{5/4}}{(\log\log x)^{1/4}\sqrt{\log\log\log x}} \times \\ \times \left| A_\Omega^{(\ell)}(\log x)\widehat{\tau}_0(\log x) - \int_{\frac{\log\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log\log x}{2}} \left| A_\Omega^{(\ell)}\left(e^{e^{2t}}\right)\widehat{\tau}_0'\left(e^{e^{2t}}\right)\right| e^{e^{2t}} dt \right|. \end{aligned} \tag{51}$$

We express the integrand function as the following function of $t$:

$$\widehat{I}_\ell(t) := \left|A_\Omega^{(\ell)}\left(e^{e^{2t}}\right)\widehat{\tau}_0'\left(e^{e^{2t}}\right)\right| e^{e^{2t}} \gg \frac{\widehat{C}_0}{16\sqrt{\pi}} \cdot \frac{e^{-t/2}}{t^{3/2}}. \tag{52}$$

We find from the mean value theorem applied to the monotone function from (52) that

$$\begin{aligned} \frac{\widehat{C}_0}{2\sqrt{2\pi}} \cdot \frac{(\log x)^{5/4}}{(\log\log x)^{1/4}\sqrt{\log\log\log x}} \times \int_{\frac{\log\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log\log x}{2}} \widehat{I}_\ell(t)dt &\ll \frac{1}{2}\widehat{I}_\ell\left(\frac{\log\log\log x}{2} - \frac{1}{2}\right) \\ &= \frac{e^{1/4} \cdot \widehat{C}_{\ell,1} \cdot (\log x)^{5/4}}{\sqrt{\log\log x} \cdot (\log\log\log x)^2}. \end{aligned} \tag{53}$$

Consider the following expansion for the leading term in the Abel summation formula from (51) for comparison with (53):

$$\frac{\widehat{C}_0}{2\sqrt{2\pi}} \cdot \frac{(\log x)^{5/4}}{(\log\log x)^{1/4}\sqrt{\log\log\log x}} \times \left|A_\Omega^{(\ell)}(\log x)\widehat{\tau}_0(\log x)\right| \gg \frac{8\widehat{C}_{\ell,1} \cdot (\log x)^{5/4}}{\sqrt{\log\log x} \cdot (\log\log\log x)^2} \tag{54}$$

Hence, we conclude that we can take $\left|G_E^{-1}(x)\right|$ bounded below by the difference of terms in (54) and (53). $\square$

## 8.2 Proof of the unboundedness of the scaled Mertens function

We finally address the main conclusion of our arguments given so far with the following proof:

*Proof of Theorem 3.9.* We split the interval of integration from Proposition 7.2 over $t \in [1, x/2]$ into two disjoint subintervals: one that is easily bounded from $1 \leq t \leq \sqrt{x}$ and the other that will conveniently give us our slow-growing tendency towards infinity along the subsequence when evaluated using Theorem 8.6. Given a fixed large infinitely tending $x$, we have some (at least one) point $x_0 \in \left[\sqrt{x}, \frac{x}{2}\right]$ defined such that $|G^{-1}(t)|$ is minimal and non-vanishing as

$$\left|G^{-1}(x_0)\right| := \min_{\substack{\sqrt{x} \leq t \leq \frac{x}{2} \\ G^{-1}(t) \neq 0}} |G^{-1}(t)|.$$

We can then apply Proposition 7.2 to bound the function as follows:

$$\begin{aligned}
\frac{|M(x)|}{\sqrt{x}} &= \frac{1}{\sqrt{x}} \left| G^{-1}(x) + \frac{2}{(\log 2)} G^{-1}\left(\frac{x}{2}\right) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt + o(\sqrt{x}) \right| \\
&\gg \left| \left| \frac{G^{-1}(x)}{\sqrt{x}} + \frac{2}{(\log 2)} G^{-1}\left(\frac{x}{2}\right) \right| - \sqrt{x} \int_1^{x/2} \frac{|G^{-1}(t)|}{t^2 \cdot \log(x/t)} dt + o(1) \right| \qquad (55) \\
&\gg \sqrt{x} \times \left| \int_{\sqrt{x}}^{x/2} \frac{|G^{-1}(t)|}{t^2 \cdot \log(x/t)} dt \right| \\
&\gg \left( \min_{\substack{\sqrt{x} \leq t \leq \frac{x}{2} \\ G^{-1}(t) \neq 0}} |G^{-1}(t)| \right) \times \left| \int_{\sqrt{x}}^{\frac{x}{2}} \frac{2\sqrt{x}}{t^2 \cdot \log(x_0)} dt \right| \\
&\gg \frac{2\left|G^{-1}(x_0)\right|}{\log(x_0)}. \qquad (56)
\end{aligned}$$

In the second to last step, we observe that $G^{-1}(x) = 0$ for $x$ on a set of asymptotic density *at least* bounded below by $\frac{1}{2}$, so that our claim is accurate as the integral bound does not trivially vanish at large $x$.

To complete the logic to the formula we arrived at in (56), first observe that the difference of terms we have in (55) corresponds to the first term having a bound from below of the form (see the proof of Theorem 8.1)

$$\begin{aligned}
\left| \frac{G^{-1}(x)}{\sqrt{x}} + \frac{2}{(\log 2)} G^{-1}\left(\frac{x}{2}\right) \right| &\gg \frac{6\sqrt{x}}{\pi^2} \left| (\log x)(\log\log x) - \frac{2}{\sqrt{2}(\log 2)} (\log x - \log 2)(\log\log x + o(1)) \right| \\
&\gg \frac{12\sqrt{x}}{\pi^2\sqrt{2}} (\log\log x), \text{ for a.e. } x, \text{ as } x \to \infty.
\end{aligned}$$

Secondly, for the sake of argument, suppose that there is a smooth approximation for $|G^{-1}(t)|$ so that by the mean value theorem for some $c_0 \in [1, \sqrt{x}]$ and $c_1 \in \left[\sqrt{x}, \frac{x}{2}\right]$ we have

$$\begin{aligned}
&\sqrt{x} \left| \int_1^{x/2} \frac{|G^{-1}(t)|}{t^2 \cdot \log(x/t)} dt \right| \\
&\gg \left| \frac{\sqrt{x} \cdot |G^{-1}(c_0)|}{c_0} \times \left| \int_1^{\sqrt{x}} \frac{dt}{t \log(x/t)} \right| + \sqrt{x} \cdot |G^{-1}(c_1)| \times \left| \int_{\sqrt{x}}^{x/2} \frac{dt}{t^2 \log(x)} \right| \right| \\
&\gg \left| \left( \min_{\substack{1 \leq c \leq \sqrt{x} \\ G^{-1}(c) \neq 0}} |G^{-1}(c)| \right) \times \log\log x + \left( \min_{\substack{\sqrt{x} \leq c^* \leq \frac{x}{2} \\ G^{-1}(c^*) \neq 0}} |G^{-1}(c^*)| \right) \times \left( \frac{1}{\log x} + o\left(\frac{1}{\log x}\right) \right) \right|.
\end{aligned}$$

Since $G^{-1}(x)$ changes stepwise only at $x \in \mathbb{Z}^+$, what we in fact precisely arrive at is a close variant of this mean value theorem type observation.

By Theorem 8.1, the result in (56) implies that

$$\frac{|M(x)|}{\sqrt{x}} \gg \frac{2\left|G_E^{-1}(x_0)\right|}{\log(x_0)}. \tag{57}$$

Define the infinite increasing subsequence, $\{x_{0,y}\}_{y \geq Y_0}$, by $x_{0,y} := e^{2e^{e^{2y+1}}}$ for the sequence indices $y$ starting at some sufficiently large finite integer $Y_0 \gg 1$. We can verify that for sufficiently large $y \to \infty$, this infinitely tending subsequence is well defined as $\hat{x}_{0,y+1} > \hat{x}_{0,y}$ whenever $y \geq Y_0$. When we assume that $x \mapsto x_{0,y}$ is taken along this subsequence, we can transform the bound in the last equation into a statement about a lower bound for $|M(x)|/\sqrt{x}$ by applying Theorem 8.6 to (57) in the following form:

$$\frac{|M(x_{0,y})|}{\sqrt{x_{0,y}}} \gg \frac{2\left(8-e^{1/4}\right) \cdot \widehat{C}_{\ell,1} \cdot (\log\sqrt{x_{0,y}})^{\frac{1}{4}}}{(\log\log\sqrt{x_{0,y}})^{\frac{1}{2}}(\log\log\log\sqrt{x_{0,y}})^2}, \text{ as } y \to \infty. \tag{58}$$

We evaluate the following limit to conclude unboundedness where $\sqrt{x_{0,y}} \to +\infty$ as $y \to +\infty$:

$$\lim_{x\to\infty}\left[\frac{(\log x)^{\frac{1}{4}}}{(\log\log x)^{\frac{1}{2}}(\log\log\log x)^2}\right] = +\infty.$$

There is a small, but nonetheless insightful point to explain about a technicality in stating (58). Namely, we are not asserting that $|M(x)|/\sqrt{x}$ grows unbounded along the precise subsequence of $x \mapsto x_{0,y}$ itself as $y \to \infty$. Rather, we are asserting that the unboundedness of this function can be witnessed along some subsequence whose points are taken within a large interval window of $\hat{x}_{0,y} \in \left[\sqrt{x_{0,y}}, \frac{x_{0,y}}{2}\right]$ as $y \to \infty$. We choose to state the lower bound given on the right-hand-side of (58) using the monotonicity of the lower bound on $|G_E^{-1}(x)|$ we proved in Theorem 8.6 with $\hat{x}_{0,y} \geq \sqrt{x_{0,y}}$ for all $y \geq Y_0$. □

## References


[1] T. M. Apostol. *Introduction to Analytic Number Theory.* Springer–Verlag, 1976.

[2] G. H. Hardy and E. M. Wright, editors. *An Introduction to the Theory of Numbers.* Oxford University Press, 2008 (Sixth Edition).

[3] G. Hurst. Computations of the Mertens function and improved bounds on the Mertens conjecture. *https://arxiv.org/pdf/1610.08551/*, 2017.

[4] H. Iwaniec and E. Kowalski. *Analytic Number Theory*, volume 53. AMS Colloquium Publications, 2004.

[5] T. Kotnik and H. té Riele. The Mertens conjecture revisited. *Algorithmic Number Theory*, $7^{th}$ International Symposium, 2006.

[6] T. Kotnik and J. van de Lune. On the order of the Mertens function. *Exp. Math.*, 2004.

[7] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford: The Clarendon Press, 1995.

[8] H. L. Montgomery and R. C. Vaughan. *Multiplicative Number Theory: I. Classical Theory.* Cambridge, 2006.

[9] N. Ng. The distribution of the summatory function of the Móbius function. *Proc. London Math. Soc.*, 89(3):361–389, 2004.

[10] A. M. Odlyzko and H. J. J. té Riele. Disproof of the Mertens conjecture. *J. REINE ANGEW. MATH*, 1985.

[11] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. *NIST Handbook of Mathematical Functions.* Cambridge University Press, 2010.

[12] P. Ribenboim. *The new book of prime number records.* Springer, 1996.

[13] J. Sándor and B. Crstici. *Handbook of Number Theory II.* Kluwer Academic Publishers, 2004.

[14] N. J. A. Sloane. The Online Encyclopedia of Integer Sequences, 2020.

[15] K. Soundararajan. Partial sums of the Möbius function. *Annals of Mathematics*, 2009.

[16] T. Tao and J. Teräväinen. Value patterns of multiplicative functions and related sequences. *Forum of Mathematics, Sigma*, 7, 2019.

[17] E. C. Titchmarsh. *The theory of the Riemann zeta function.* Clarendon Press, 1951.

## T.1 Table: The Dirichlet inverse function $g^{-1}(n)$ and the distribution of its summatory function

| $n$ | **Primes** | **Sqfree** | **PPower** | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $1^1$ | Y | N | 1 | 0 | 1.0000000 | 1.000000 | 0.000000 | 1 | 1 | 0 |
| 2 | $2^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.500000 | 0.500000 | $-1$ | 1 | $-2$ |
| 3 | $3^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.333333 | 0.666667 | $-3$ | 1 | $-4$ |
| 4 | $2^2$ | N | Y | 2 | 0 | 1.5000000 | 0.500000 | 0.500000 | $-1$ | 3 | $-4$ |
| 5 | $5^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.400000 | 0.600000 | $-3$ | 3 | $-6$ |
| 6 | $2^1 3^1$ | Y | N | 5 | 0 | 1.0000000 | 0.500000 | 0.500000 | 2 | 8 | $-6$ |
| 7 | $7^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.428571 | 0.571429 | 0 | 8 | $-8$ |
| 8 | $2^3$ | N | Y | $-2$ | 0 | 2.0000000 | 0.375000 | 0.625000 | $-2$ | 8 | $-10$ |
| 9 | $3^2$ | N | Y | 2 | 0 | 1.5000000 | 0.444444 | 0.555556 | 0 | 10 | $-10$ |
| 10 | $2^1 5^1$ | Y | N | 5 | 0 | 1.0000000 | 0.500000 | 0.500000 | 5 | 15 | $-10$ |
| 11 | $11^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.454545 | 0.545455 | 3 | 15 | $-12$ |
| 12 | $2^2 3^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.416667 | 0.583333 | $-4$ | 15 | $-19$ |
| 13 | $13^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.384615 | 0.615385 | $-6$ | 15 | $-21$ |
| 14 | $2^1 7^1$ | Y | N | 5 | 0 | 1.0000000 | 0.428571 | 0.571429 | $-1$ | 20 | $-21$ |
| 15 | $3^1 5^1$ | Y | N | 5 | 0 | 1.0000000 | 0.466667 | 0.533333 | 4 | 25 | $-21$ |
| 16 | $2^4$ | N | Y | 2 | 0 | 2.5000000 | 0.500000 | 0.500000 | 6 | 27 | $-21$ |
| 17 | $17^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.470588 | 0.529412 | 4 | 27 | $-23$ |
| 18 | $2^1 3^2$ | N | N | $-7$ | 2 | 1.2857143 | 0.444444 | 0.555556 | $-3$ | 27 | $-30$ |
| 19 | $19^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.421053 | 0.578947 | $-5$ | 27 | $-32$ |
| 20 | $2^2 5^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.400000 | 0.600000 | $-12$ | 27 | $-39$ |
| 21 | $3^1 7^1$ | Y | N | 5 | 0 | 1.0000000 | 0.428571 | 0.571429 | $-7$ | 32 | $-39$ |
| 22 | $2^1 11^1$ | Y | N | 5 | 0 | 1.0000000 | 0.454545 | 0.545455 | $-2$ | 37 | $-39$ |
| 23 | $23^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.434783 | 0.565217 | $-4$ | 37 | $-41$ |
| 24 | $2^3 3^1$ | N | N | 9 | 4 | 1.5555556 | 0.458333 | 0.541667 | 5 | 46 | $-41$ |
| 25 | $5^2$ | N | Y | 2 | 0 | 1.5000000 | 0.480000 | 0.520000 | 7 | 48 | $-41$ |
| 26 | $2^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.500000 | 0.500000 | 12 | 53 | $-41$ |
| 27 | $3^3$ | N | Y | $-2$ | 0 | 2.0000000 | 0.481481 | 0.518519 | 10 | 53 | $-43$ |
| 28 | $2^2 7^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.464286 | 0.535714 | 3 | 53 | $-50$ |
| 29 | $29^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.448276 | 0.551724 | 1 | 53 | $-52$ |
| 30 | $2^1 3^1 5^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.433333 | 0.566667 | $-15$ | 53 | $-68$ |
| 31 | $31^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.419355 | 0.580645 | $-17$ | 53 | $-70$ |
| 32 | $2^5$ | N | Y | $-2$ | 0 | 3.0000000 | 0.406250 | 0.593750 | $-19$ | 53 | $-72$ |
| 33 | $3^1 11^1$ | Y | N | 5 | 0 | 1.0000000 | 0.424242 | 0.575758 | $-14$ | 58 | $-72$ |
| 34 | $2^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.441176 | 0.558824 | $-9$ | 63 | $-72$ |
| 35 | $5^1 7^1$ | Y | N | 5 | 0 | 1.0000000 | 0.457143 | 0.542857 | $-4$ | 68 | $-72$ |
| 36 | $2^2 3^2$ | N | N | 14 | 9 | 1.3571429 | 0.472222 | 0.527778 | 10 | 82 | $-72$ |
| 37 | $37^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.459459 | 0.540541 | 8 | 82 | $-74$ |
| 38 | $2^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473684 | 0.526316 | 13 | 87 | $-74$ |
| 39 | $3^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487179 | 0.512821 | 18 | 92 | $-74$ |
| 40 | $2^3 5^1$ | N | N | 9 | 4 | 1.5555556 | 0.500000 | 0.500000 | 27 | 101 | $-74$ |
| 41 | $41^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.487805 | 0.512195 | 25 | 101 | $-76$ |
| 42 | $2^1 3^1 7^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.476190 | 0.523810 | 9 | 101 | $-92$ |
| 43 | $43^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.465116 | 0.534884 | 7 | 101 | $-94$ |
| 44 | $2^2 11^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.454545 | 0.545455 | 0 | 101 | $-101$ |
| 45 | $3^2 5^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.444444 | 0.555556 | $-7$ | 101 | $-108$ |
| 46 | $2^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.456522 | 0.543478 | $-2$ | 106 | $-108$ |
| 47 | $47^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.446809 | 0.553191 | $-4$ | 106 | $-110$ |
| 48 | $2^4 3^1$ | N | N | $-11$ | 6 | 1.8181818 | 0.437500 | 0.562500 | $-15$ | 106 | $-121$ |

**Table T.1:** ***Computations with*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for*** $\mathbf{1 \leq n \leq 500}$.

- ► *The column labeled* `Primes` *provides the prime factorization of each $n$ so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled* `Sqfree` *and* `PPower`*, respectively, list inclusion of $n$ in the sets of squarefree integers and the prime powers.*
- ► *The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.*
- ► *The last several columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{n} \cdot \#\{n \leq x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geq 1$.*

| $n$ | **Primes** | **Sqfree** | **PPower** | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G^{-1}_+(n)$ | $G^{-1}_-(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 49 | $7^2$ | N | Y | 2 | 0 | 1.5000000 | 0.448980 | 0.551020 | −13 | 108 | −121 |
| 50 | $2^1 5^2$ | N | N | −7 | 2 | 1.2857143 | 0.440000 | 0.560000 | −20 | 108 | −128 |
| 51 | $3^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.450980 | 0.549020 | −15 | 113 | −128 |
| 52 | $2^2 13^1$ | N | N | −7 | 2 | 1.2857143 | 0.442308 | 0.557692 | −22 | 113 | −135 |
| 53 | $53^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.433962 | 0.566038 | −24 | 113 | −137 |
| 54 | $2^1 3^3$ | N | N | 9 | 4 | 1.5555556 | 0.444444 | 0.555556 | −15 | 122 | −137 |
| 55 | $5^1 11^1$ | Y | N | 5 | 0 | 1.0000000 | 0.454545 | 0.545455 | −10 | 127 | −137 |
| 56 | $2^3 7^1$ | N | N | 9 | 4 | 1.5555556 | 0.464286 | 0.535714 | −1 | 136 | −137 |
| 57 | $3^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473684 | 0.526316 | 4 | 141 | −137 |
| 58 | $2^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482759 | 0.517241 | 9 | 146 | −137 |
| 59 | $59^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.474576 | 0.525424 | 7 | 146 | −139 |
| 60 | $2^2 3^1 5^1$ | N | N | 30 | 14 | 1.1666667 | 0.483333 | 0.516667 | 37 | 176 | −139 |
| 61 | $61^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.475410 | 0.524590 | 35 | 176 | −141 |
| 62 | $2^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 40 | 181 | −141 |
| 63 | $3^2 7^1$ | N | N | −7 | 2 | 1.2857143 | 0.476190 | 0.523810 | 33 | 181 | −148 |
| 64 | $2^6$ | N | Y | 2 | 0 | 3.5000000 | 0.484375 | 0.515625 | 35 | 183 | −148 |
| 65 | $5^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492308 | 0.507692 | 40 | 188 | −148 |
| 66 | $2^1 3^1 11^1$ | Y | N | −16 | 0 | 1.0000000 | 0.484848 | 0.515152 | 24 | 188 | −164 |
| 67 | $67^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.477612 | 0.522388 | 22 | 188 | −166 |
| 68 | $2^2 17^1$ | N | N | −7 | 2 | 1.2857143 | 0.470588 | 0.529412 | 15 | 188 | −173 |
| 69 | $3^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478261 | 0.521739 | 20 | 193 | −173 |
| 70 | $2^1 5^1 7^1$ | Y | N | −16 | 0 | 1.0000000 | 0.471429 | 0.528571 | 4 | 193 | −189 |
| 71 | $71^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.464789 | 0.535211 | 2 | 193 | −191 |
| 72 | $2^3 3^2$ | N | N | −23 | 18 | 1.4782609 | 0.458333 | 0.541667 | −21 | 193 | −214 |
| 73 | $73^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.452055 | 0.547945 | −23 | 193 | −216 |
| 74 | $2^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.459459 | 0.540541 | −18 | 198 | −216 |
| 75 | $3^1 5^2$ | N | N | −7 | 2 | 1.2857143 | 0.453333 | 0.546667 | −25 | 198 | −223 |
| 76 | $2^2 19^1$ | N | N | −7 | 2 | 1.2857143 | 0.447368 | 0.552632 | −32 | 198 | −230 |
| 77 | $7^1 11^1$ | Y | N | 5 | 0 | 1.0000000 | 0.454545 | 0.545455 | −27 | 203 | −230 |
| 78 | $2^1 3^1 13^1$ | Y | N | −16 | 0 | 1.0000000 | 0.448718 | 0.551282 | −43 | 203 | −246 |
| 79 | $79^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.443038 | 0.556962 | −45 | 203 | −248 |
| 80 | $2^4 5^1$ | N | N | −11 | 6 | 1.8181818 | 0.437500 | 0.562500 | −56 | 203 | −259 |
| 81 | $3^4$ | N | Y | 2 | 0 | 2.5000000 | 0.444444 | 0.555556 | −54 | 205 | −259 |
| 82 | $2^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.451220 | 0.548780 | −49 | 210 | −259 |
| 83 | $83^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.445783 | 0.554217 | −51 | 210 | −261 |
| 84 | $2^2 3^1 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.452381 | 0.547619 | −21 | 240 | −261 |
| 85 | $5^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.458824 | 0.541176 | −16 | 245 | −261 |
| 86 | $2^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.465116 | 0.534884 | −11 | 250 | −261 |
| 87 | $3^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.471264 | 0.528736 | −6 | 255 | −261 |
| 88 | $2^3 11^1$ | N | N | 9 | 4 | 1.5555556 | 0.477273 | 0.522727 | 3 | 264 | −261 |
| 89 | $89^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.471910 | 0.528090 | 1 | 264 | −263 |
| 90 | $2^1 3^2 5^1$ | N | N | 30 | 14 | 1.1666667 | 0.477778 | 0.522222 | 31 | 294 | −263 |
| 91 | $7^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483516 | 0.516484 | 36 | 299 | −263 |
| 92 | $2^2 23^1$ | N | N | −7 | 2 | 1.2857143 | 0.478261 | 0.521739 | 29 | 299 | −270 |
| 93 | $3^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 34 | 304 | −270 |
| 94 | $2^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.489362 | 0.510638 | 39 | 309 | −270 |
| 95 | $5^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.494737 | 0.505263 | 44 | 314 | −270 |
| 96 | $2^5 3^1$ | N | N | 13 | 8 | 2.0769231 | 0.500000 | 0.500000 | 57 | 327 | −270 |
| 97 | $97^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.494845 | 0.505155 | 55 | 327 | −272 |
| 98 | $2^1 7^2$ | N | N | −7 | 2 | 1.2857143 | 0.489796 | 0.510204 | 48 | 327 | −279 |
| 99 | $3^2 11^1$ | N | N | −7 | 2 | 1.2857143 | 0.484848 | 0.515152 | 41 | 327 | −286 |
| 100 | $2^2 5^2$ | N | N | 14 | 9 | 1.3571429 | 0.490000 | 0.510000 | 55 | 341 | −286 |
| 101 | $101^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.485149 | 0.514851 | 53 | 341 | −288 |
| 102 | $2^1 3^1 17^1$ | Y | N | −16 | 0 | 1.0000000 | 0.480392 | 0.519608 | 37 | 341 | −304 |
| 103 | $103^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.475728 | 0.524272 | 35 | 341 | −306 |
| 104 | $2^3 13^1$ | N | N | 9 | 4 | 1.5555556 | 0.480769 | 0.519231 | 44 | 350 | −306 |
| 105 | $3^1 5^1 7^1$ | Y | N | −16 | 0 | 1.0000000 | 0.476190 | 0.523810 | 28 | 350 | −322 |
| 106 | $2^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.481132 | 0.518868 | 33 | 355 | −322 |
| 107 | $107^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.476636 | 0.523364 | 31 | 355 | −324 |
| 108 | $2^2 3^3$ | N | N | −23 | 18 | 1.4782609 | 0.472222 | 0.527778 | 8 | 355 | −347 |
| 109 | $109^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.467890 | 0.532110 | 6 | 355 | −349 |
| 110 | $2^1 5^1 11^1$ | Y | N | −16 | 0 | 1.0000000 | 0.463636 | 0.536364 | −10 | 355 | −365 |
| 111 | $3^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.468468 | 0.531532 | −5 | 360 | −365 |
| 112 | $2^4 7^1$ | N | N | −11 | 6 | 1.8181818 | 0.464286 | 0.535714 | −16 | 360 | −376 |
| 113 | $113^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.460177 | 0.539823 | −18 | 360 | −378 |
| 114 | $2^1 3^1 19^1$ | Y | N | −16 | 0 | 1.0000000 | 0.456140 | 0.543860 | −34 | 360 | −394 |
| 115 | $5^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.460870 | 0.539130 | −29 | 365 | −394 |
| 116 | $2^2 29^1$ | N | N | −7 | 2 | 1.2857143 | 0.456897 | 0.543103 | −36 | 365 | −401 |
| 117 | $3^2 13^1$ | N | N | −7 | 2 | 1.2857143 | 0.452991 | 0.547009 | −43 | 365 | −408 |
| 118 | $2^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.457627 | 0.542373 | −38 | 370 | −408 |
| 119 | $7^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.462185 | 0.537815 | −33 | 375 | −408 |
| 120 | $2^3 3^1 5^1$ | N | N | −48 | 32 | 1.3333333 | 0.458333 | 0.541667 | −81 | 375 | −456 |
| 121 | $11^2$ | N | Y | 2 | 0 | 1.5000000 | 0.462810 | 0.537190 | −79 | 377 | −456 |
| 122 | $2^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467213 | 0.532787 | −74 | 382 | −456 |
| 123 | $3^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.471545 | 0.528455 | −69 | 387 | −456 |
| 124 | $2^2 31^1$ | N | N | −7 | 2 | 1.2857143 | 0.467742 | 0.532258 | −76 | 387 | −463 |

| $n$ | **Primes** | **Sqfree** | **PPower** | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 125 | $5^3$ | N | Y | $-2$ | 0 | 2.0000000 | 0.464000 | 0.536000 | $-78$ | 387 | $-465$ |
| 126 | $2^1 3^2 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.468254 | 0.531746 | $-48$ | 417 | $-465$ |
| 127 | $127^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.464567 | 0.535433 | $-50$ | 417 | $-467$ |
| 128 | $2^7$ | N | Y | $-2$ | 0 | 4.0000000 | 0.460938 | 0.539062 | $-52$ | 417 | $-469$ |
| 129 | $3^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.465116 | 0.534884 | $-47$ | 422 | $-469$ |
| 130 | $2^1 5^1 13^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.461538 | 0.538462 | $-63$ | 422 | $-485$ |
| 131 | $131^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.458015 | 0.541985 | $-65$ | 422 | $-487$ |
| 132 | $2^2 3^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.462121 | 0.537879 | $-35$ | 452 | $-487$ |
| 133 | $7^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.466165 | 0.533835 | $-30$ | 457 | $-487$ |
| 134 | $2^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.470149 | 0.529851 | $-25$ | 462 | $-487$ |
| 135 | $3^3 5^1$ | N | N | 9 | 4 | 1.5555556 | 0.474074 | 0.525926 | $-16$ | 471 | $-487$ |
| 136 | $2^3 17^1$ | N | N | 9 | 4 | 1.5555556 | 0.477941 | 0.522059 | $-7$ | 480 | $-487$ |
| 137 | $137^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.474453 | 0.525547 | $-9$ | 480 | $-489$ |
| 138 | $2^1 3^1 23^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.471014 | 0.528986 | $-25$ | 480 | $-505$ |
| 139 | $139^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.467626 | 0.532374 | $-27$ | 480 | $-507$ |
| 140 | $2^2 5^1 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.471429 | 0.528571 | 3 | 510 | $-507$ |
| 141 | $3^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475177 | 0.524823 | 8 | 515 | $-507$ |
| 142 | $2^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478873 | 0.521127 | 13 | 520 | $-507$ |
| 143 | $11^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482517 | 0.517483 | 18 | 525 | $-507$ |
| 144 | $2^4 3^2$ | N | N | 34 | 29 | 1.6176471 | 0.486111 | 0.513889 | 52 | 559 | $-507$ |
| 145 | $5^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.489655 | 0.510345 | 57 | 564 | $-507$ |
| 146 | $2^1 73^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493151 | 0.506849 | 62 | 569 | $-507$ |
| 147 | $3^1 7^2$ | N | N | $-7$ | 2 | 1.2857143 | 0.489796 | 0.510204 | 55 | 569 | $-514$ |
| 148 | $2^2 37^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.486486 | 0.513514 | 48 | 569 | $-521$ |
| 149 | $149^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.483221 | 0.516779 | 46 | 569 | $-523$ |
| 150 | $2^1 3^1 5^2$ | N | N | 30 | 14 | 1.1666667 | 0.486667 | 0.513333 | 76 | 599 | $-523$ |
| 151 | $151^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.483444 | 0.516556 | 74 | 599 | $-525$ |
| 152 | $2^3 19^1$ | N | N | 9 | 4 | 1.5555556 | 0.486842 | 0.513158 | 83 | 608 | $-525$ |
| 153 | $3^2 17^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.483660 | 0.516340 | 76 | 608 | $-532$ |
| 154 | $2^1 7^1 11^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.480519 | 0.519481 | 60 | 608 | $-548$ |
| 155 | $5^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 65 | 613 | $-548$ |
| 156 | $2^2 3^1 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.487179 | 0.512821 | 95 | 643 | $-548$ |
| 157 | $157^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.484076 | 0.515924 | 93 | 643 | $-550$ |
| 158 | $2^1 79^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487342 | 0.512658 | 98 | 648 | $-550$ |
| 159 | $3^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490566 | 0.509434 | 103 | 653 | $-550$ |
| 160 | $2^5 5^1$ | N | N | 13 | 8 | 2.0769231 | 0.493750 | 0.506250 | 116 | 666 | $-550$ |
| 161 | $7^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.496894 | 0.503106 | 121 | 671 | $-550$ |
| 162 | $2^1 3^4$ | N | N | $-11$ | 6 | 1.8181818 | 0.493827 | 0.506173 | 110 | 671 | $-561$ |
| 163 | $163^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.490798 | 0.509202 | 108 | 671 | $-563$ |
| 164 | $2^2 41^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.487805 | 0.512195 | 101 | 671 | $-570$ |
| 165 | $3^1 5^1 11^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.484848 | 0.515152 | 85 | 671 | $-586$ |
| 166 | $2^1 83^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487952 | 0.512048 | 90 | 676 | $-586$ |
| 167 | $167^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.485030 | 0.514970 | 88 | 676 | $-588$ |
| 168 | $2^3 3^1 7^1$ | N | N | $-48$ | 32 | 1.3333333 | 0.482143 | 0.517857 | 40 | 676 | $-636$ |
| 169 | $13^2$ | N | Y | 2 | 0 | 1.5000000 | 0.485207 | 0.514793 | 42 | 678 | $-636$ |
| 170 | $2^1 5^1 17^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.482353 | 0.517647 | 26 | 678 | $-652$ |
| 171 | $3^2 19^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.479532 | 0.520468 | 19 | 678 | $-659$ |
| 172 | $2^2 43^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.476744 | 0.523256 | 12 | 678 | $-666$ |
| 173 | $173^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.473988 | 0.526012 | 10 | 678 | $-668$ |
| 174 | $2^1 3^1 29^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.471264 | 0.528736 | $-6$ | 678 | $-684$ |
| 175 | $5^2 7^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.468571 | 0.531429 | $-13$ | 678 | $-691$ |
| 176 | $2^4 11^1$ | N | N | $-11$ | 6 | 1.8181818 | 0.465909 | 0.534091 | $-24$ | 678 | $-702$ |
| 177 | $3^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.468927 | 0.531073 | $-19$ | 683 | $-702$ |
| 178 | $2^1 89^1$ | Y | N | 5 | 0 | 1.0000000 | 0.471910 | 0.528090 | $-14$ | 688 | $-702$ |
| 179 | $179^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.469274 | 0.530726 | $-16$ | 688 | $-704$ |
| 180 | $2^2 3^2 5^1$ | N | N | $-74$ | 58 | 1.2162162 | 0.466667 | 0.533333 | $-90$ | 688 | $-778$ |
| 181 | $181^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.464088 | 0.535912 | $-92$ | 688 | $-780$ |
| 182 | $2^1 7^1 13^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.461538 | 0.538462 | $-108$ | 688 | $-796$ |
| 183 | $3^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.464481 | 0.535519 | $-103$ | 693 | $-796$ |
| 184 | $2^3 23^1$ | N | N | 9 | 4 | 1.5555556 | 0.467391 | 0.532609 | $-94$ | 702 | $-796$ |
| 185 | $5^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.470270 | 0.529730 | $-89$ | 707 | $-796$ |
| 186 | $2^1 3^1 31^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.467742 | 0.532258 | $-105$ | 707 | $-812$ |
| 187 | $11^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.470588 | 0.529412 | $-100$ | 712 | $-812$ |
| 188 | $2^2 47^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.468085 | 0.531915 | $-107$ | 712 | $-819$ |
| 189 | $3^3 7^1$ | N | N | 9 | 4 | 1.5555556 | 0.470899 | 0.529101 | $-98$ | 721 | $-819$ |
| 190 | $2^1 5^1 19^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.468421 | 0.531579 | $-114$ | 721 | $-835$ |
| 191 | $191^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.465969 | 0.534031 | $-116$ | 721 | $-837$ |
| 192 | $2^6 3^1$ | N | N | $-15$ | 10 | 2.3333333 | 0.463542 | 0.536458 | $-131$ | 721 | $-852$ |
| 193 | $193^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.461140 | 0.538860 | $-133$ | 721 | $-854$ |
| 194 | $2^1 97^1$ | Y | N | 5 | 0 | 1.0000000 | 0.463918 | 0.536082 | $-128$ | 726 | $-854$ |
| 195 | $3^1 5^1 13^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.461538 | 0.538462 | $-144$ | 726 | $-870$ |
| 196 | $2^2 7^2$ | N | N | 14 | 9 | 1.3571429 | 0.464286 | 0.535714 | $-130$ | 740 | $-870$ |
| 197 | $197^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.461929 | 0.538071 | $-132$ | 740 | $-872$ |
| 198 | $2^1 3^2 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.464646 | 0.535354 | $-102$ | 770 | $-872$ |
| 199 | $199^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.462312 | 0.537688 | $-104$ | 770 | $-874$ |
| 200 | $2^3 5^2$ | N | N | $-23$ | 18 | 1.4782609 | 0.460000 | 0.540000 | $-127$ | 770 | $-897$ |

| $n$ | **Primes** | **Sqfree** | **PPower** | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G^{-1}_+(n)$ | $G^{-1}_-(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 201 | $3^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.462687 | 0.537313 | −122 | 775 | −897 |
| 202 | $2^1 101^1$ | Y | N | 5 | 0 | 1.0000000 | 0.465347 | 0.534653 | −117 | 780 | −897 |
| 203 | $7^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467980 | 0.532020 | −112 | 785 | −897 |
| 204 | $2^2 3^1 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.470588 | 0.529412 | −82 | 815 | −897 |
| 205 | $5^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473171 | 0.526829 | −77 | 820 | −897 |
| 206 | $2^1 103^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475728 | 0.524272 | −72 | 825 | −897 |
| 207 | $3^2 23^1$ | N | N | −7 | 2 | 1.2857143 | 0.473430 | 0.526570 | −79 | 825 | −904 |
| 208 | $2^4 13^1$ | N | N | −11 | 6 | 1.8181818 | 0.471154 | 0.528846 | −90 | 825 | −915 |
| 209 | $11^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473684 | 0.526316 | −85 | 830 | −915 |
| 210 | $2^1 3^1 5^1 7^1$ | Y | N | 65 | 0 | 1.0000000 | 0.476190 | 0.523810 | −20 | 895 | −915 |
| 211 | $211^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.473934 | 0.526066 | −22 | 895 | −917 |
| 212 | $2^2 53^1$ | N | N | −7 | 2 | 1.2857143 | 0.471698 | 0.528302 | −29 | 895 | −924 |
| 213 | $3^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.474178 | 0.525822 | −24 | 900 | −924 |
| 214 | $2^1 107^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476636 | 0.523364 | −19 | 905 | −924 |
| 215 | $5^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.479070 | 0.520930 | −14 | 910 | −924 |
| 216 | $2^3 3^3$ | N | N | 46 | 41 | 1.5000000 | 0.481481 | 0.518519 | 32 | 956 | −924 |
| 217 | $7^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 37 | 961 | −924 |
| 218 | $2^1 109^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486239 | 0.513761 | 42 | 966 | −924 |
| 219 | $3^1 73^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488584 | 0.511416 | 47 | 971 | −924 |
| 220 | $2^2 5^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.490909 | 0.509091 | 77 | 1001 | −924 |
| 221 | $13^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493213 | 0.506787 | 82 | 1006 | −924 |
| 222 | $2^1 3^1 37^1$ | Y | N | −16 | 0 | 1.0000000 | 0.490991 | 0.509009 | 66 | 1006 | −940 |
| 223 | $223^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.488789 | 0.511211 | 64 | 1006 | −942 |
| 224 | $2^5 7^1$ | N | N | 13 | 8 | 2.0769231 | 0.491071 | 0.508929 | 77 | 1019 | −942 |
| 225 | $3^2 5^2$ | N | N | 14 | 9 | 1.3571429 | 0.493333 | 0.506667 | 91 | 1033 | −942 |
| 226 | $2^1 113^1$ | Y | N | 5 | 0 | 1.0000000 | 0.495575 | 0.504425 | 96 | 1038 | −942 |
| 227 | $227^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.493392 | 0.506608 | 94 | 1038 | −944 |
| 228 | $2^2 3^1 19^1$ | N | N | 30 | 14 | 1.1666667 | 0.495614 | 0.504386 | 124 | 1068 | −944 |
| 229 | $229^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.493450 | 0.506550 | 122 | 1068 | −946 |
| 230 | $2^1 5^1 23^1$ | Y | N | −16 | 0 | 1.0000000 | 0.491304 | 0.508696 | 106 | 1068 | −962 |
| 231 | $3^1 7^1 11^1$ | Y | N | −16 | 0 | 1.0000000 | 0.489177 | 0.510823 | 90 | 1068 | −978 |
| 232 | $2^3 29^1$ | N | N | 9 | 4 | 1.5555556 | 0.491379 | 0.508621 | 99 | 1077 | −978 |
| 233 | $233^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.489270 | 0.510730 | 97 | 1077 | −980 |
| 234 | $2^1 3^2 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.491453 | 0.508547 | 127 | 1107 | −980 |
| 235 | $5^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493617 | 0.506383 | 132 | 1112 | −980 |
| 236 | $2^2 59^1$ | N | N | −7 | 2 | 1.2857143 | 0.491525 | 0.508475 | 125 | 1112 | −987 |
| 237 | $3^1 79^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493671 | 0.506329 | 130 | 1117 | −987 |
| 238 | $2^1 7^1 17^1$ | Y | N | −16 | 0 | 1.0000000 | 0.491597 | 0.508403 | 114 | 1117 | −1003 |
| 239 | $239^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.489540 | 0.510460 | 112 | 1117 | −1005 |
| 240 | $2^4 3^1 5^1$ | N | N | 70 | 54 | 1.5000000 | 0.491667 | 0.508333 | 182 | 1187 | −1005 |
| 241 | $241^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.489627 | 0.510373 | 180 | 1187 | −1007 |
| 242 | $2^1 11^2$ | N | N | −7 | 2 | 1.2857143 | 0.487603 | 0.512397 | 173 | 1187 | −1014 |
| 243 | $3^5$ | N | Y | −2 | 0 | 3.0000000 | 0.485597 | 0.514403 | 171 | 1187 | −1016 |
| 244 | $2^2 61^1$ | N | N | −7 | 2 | 1.2857143 | 0.483607 | 0.516393 | 164 | 1187 | −1023 |
| 245 | $5^1 7^2$ | N | N | −7 | 2 | 1.2857143 | 0.481633 | 0.518367 | 157 | 1187 | −1030 |
| 246 | $2^1 3^1 41^1$ | Y | N | −16 | 0 | 1.0000000 | 0.479675 | 0.520325 | 141 | 1187 | −1046 |
| 247 | $13^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.481781 | 0.518219 | 146 | 1192 | −1046 |
| 248 | $2^3 31^1$ | N | N | 9 | 4 | 1.5555556 | 0.483871 | 0.516129 | 155 | 1201 | −1046 |
| 249 | $3^1 83^1$ | Y | N | 5 | 0 | 1.0000000 | 0.485944 | 0.514056 | 160 | 1206 | −1046 |
| 250 | $2^1 5^3$ | N | N | 9 | 4 | 1.5555556 | 0.488000 | 0.512000 | 169 | 1215 | −1046 |
| 251 | $251^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.486056 | 0.513944 | 167 | 1215 | −1048 |
| 252 | $2^2 3^2 7^1$ | N | N | −74 | 58 | 1.2162162 | 0.484127 | 0.515873 | 93 | 1215 | −1122 |
| 253 | $11^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486166 | 0.513834 | 98 | 1220 | −1122 |
| 254 | $2^1 127^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488189 | 0.511811 | 103 | 1225 | −1122 |
| 255 | $3^1 5^1 17^1$ | Y | N | −16 | 0 | 1.0000000 | 0.486275 | 0.513725 | 87 | 1225 | −1138 |
| 256 | $2^8$ | N | Y | 2 | 0 | 4.5000000 | 0.488281 | 0.511719 | 89 | 1227 | −1138 |
| 257 | $257^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.486381 | 0.513619 | 87 | 1227 | −1140 |
| 258 | $2^1 3^1 43^1$ | Y | N | −16 | 0 | 1.0000000 | 0.484496 | 0.515504 | 71 | 1227 | −1156 |
| 259 | $7^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486486 | 0.513514 | 76 | 1232 | −1156 |
| 260 | $2^2 5^1 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.488462 | 0.511538 | 106 | 1262 | −1156 |
| 261 | $3^2 29^1$ | N | N | −7 | 2 | 1.2857143 | 0.486590 | 0.513410 | 99 | 1262 | −1163 |
| 262 | $2^1 131^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488550 | 0.511450 | 104 | 1267 | −1163 |
| 263 | $263^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.486692 | 0.513308 | 102 | 1267 | −1165 |
| 264 | $2^3 3^1 11^1$ | N | N | −48 | 32 | 1.3333333 | 0.484848 | 0.515152 | 54 | 1267 | −1213 |
| 265 | $5^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486792 | 0.513208 | 59 | 1272 | −1213 |
| 266 | $2^1 7^1 19^1$ | Y | N | −16 | 0 | 1.0000000 | 0.484962 | 0.515038 | 43 | 1272 | −1229 |
| 267 | $3^1 89^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486891 | 0.513109 | 48 | 1277 | −1229 |
| 268 | $2^2 67^1$ | N | N | −7 | 2 | 1.2857143 | 0.485075 | 0.514925 | 41 | 1277 | −1236 |
| 269 | $269^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.483271 | 0.516729 | 39 | 1277 | −1238 |
| 270 | $2^1 3^3 5^1$ | N | N | −48 | 32 | 1.3333333 | 0.481481 | 0.518519 | −9 | 1277 | −1286 |
| 271 | $271^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.479705 | 0.520295 | −11 | 1277 | −1288 |
| 272 | $2^4 17^1$ | N | N | −11 | 6 | 1.8181818 | 0.477941 | 0.522059 | −22 | 1277 | −1299 |
| 273 | $3^1 7^1 13^1$ | Y | N | −16 | 0 | 1.0000000 | 0.476190 | 0.523810 | −38 | 1277 | −1315 |
| 274 | $2^1 137^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478102 | 0.521898 | −33 | 1282 | −1315 |
| 275 | $5^2 11^1$ | N | N | −7 | 2 | 1.2857143 | 0.476364 | 0.523636 | −40 | 1282 | −1322 |
| 276 | $2^2 3^1 23^1$ | N | N | 30 | 14 | 1.1666667 | 0.478261 | 0.521739 | −10 | 1312 | −1322 |
| 277 | $277^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.476534 | 0.523466 | −12 | 1312 | −1324 |

| $n$ | **Primes** | **Sqfree** | **PPower** | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 278 | $2^1 139^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478417 | 0.521583 | −7 | 1317 | −1324 |
| 279 | $3^2 31^1$ | N | N | −7 | 2 | 1.2857143 | 0.476703 | 0.523297 | −14 | 1317 | −1331 |
| 280 | $2^3 5^1 7^1$ | N | N | −48 | 32 | 1.3333333 | 0.475000 | 0.525000 | −62 | 1317 | −1379 |
| 281 | $281^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.473310 | 0.526690 | −64 | 1317 | −1381 |
| 282 | $2^1 3^1 47^1$ | Y | N | −16 | 0 | 1.0000000 | 0.471631 | 0.528369 | −80 | 1317 | −1397 |
| 283 | $283^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.469965 | 0.530035 | −82 | 1317 | −1399 |
| 284 | $2^2 71^1$ | N | N | −7 | 2 | 1.2857143 | 0.468310 | 0.531690 | −89 | 1317 | −1406 |
| 285 | $3^1 5^1 19^1$ | Y | N | −16 | 0 | 1.0000000 | 0.466667 | 0.533333 | −105 | 1317 | −1422 |
| 286 | $2^1 11^1 13^1$ | Y | N | −16 | 0 | 1.0000000 | 0.465035 | 0.534965 | −121 | 1317 | −1438 |
| 287 | $7^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.466899 | 0.533101 | −116 | 1322 | −1438 |
| 288 | $2^5 3^2$ | N | N | −47 | 42 | 1.7659574 | 0.465278 | 0.534722 | −163 | 1322 | −1485 |
| 289 | $17^2$ | N | Y | 2 | 0 | 1.5000000 | 0.467128 | 0.532872 | −161 | 1324 | −1485 |
| 290 | $2^1 5^1 29^1$ | Y | N | −16 | 0 | 1.0000000 | 0.465517 | 0.534483 | −177 | 1324 | −1501 |
| 291 | $3^1 97^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467354 | 0.532646 | −172 | 1329 | −1501 |
| 292 | $2^2 73^1$ | N | N | −7 | 2 | 1.2857143 | 0.465753 | 0.534247 | −179 | 1329 | −1508 |
| 293 | $293^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.464164 | 0.535836 | −181 | 1329 | −1510 |
| 294 | $2^1 3^1 7^2$ | N | N | 30 | 14 | 1.1666667 | 0.465986 | 0.534014 | −151 | 1359 | −1510 |
| 295 | $5^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467797 | 0.532203 | −146 | 1364 | −1510 |
| 296 | $2^3 37^1$ | N | N | 9 | 4 | 1.5555556 | 0.469595 | 0.530405 | −137 | 1373 | −1510 |
| 297 | $3^3 11^1$ | N | N | 9 | 4 | 1.5555556 | 0.471380 | 0.528620 | −128 | 1382 | −1510 |
| 298 | $2^1 149^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473154 | 0.526846 | −123 | 1387 | −1510 |
| 299 | $13^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.474916 | 0.525084 | −118 | 1392 | −1510 |
| 300 | $2^2 3^1 5^2$ | N | N | −74 | 58 | 1.2162162 | 0.473333 | 0.526667 | −192 | 1392 | −1584 |
| 301 | $7^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475083 | 0.524917 | −187 | 1397 | −1584 |
| 302 | $2^1 151^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476821 | 0.523179 | −182 | 1402 | −1584 |
| 303 | $3^1 101^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478548 | 0.521452 | −177 | 1407 | −1584 |
| 304 | $2^4 19^1$ | N | N | −11 | 6 | 1.8181818 | 0.476974 | 0.523026 | −188 | 1407 | −1595 |
| 305 | $5^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478689 | 0.521311 | −183 | 1412 | −1595 |
| 306 | $2^1 3^2 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.480392 | 0.519608 | −153 | 1442 | −1595 |
| 307 | $307^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.478827 | 0.521173 | −155 | 1442 | −1597 |
| 308 | $2^2 7^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.480519 | 0.519481 | −125 | 1472 | −1597 |
| 309 | $3^1 103^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482201 | 0.517799 | −120 | 1477 | −1597 |
| 310 | $2^1 5^1 31^1$ | Y | N | −16 | 0 | 1.0000000 | 0.480645 | 0.519355 | −136 | 1477 | −1613 |
| 311 | $311^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.479100 | 0.520900 | −138 | 1477 | −1615 |
| 312 | $2^3 3^1 13^1$ | N | N | −48 | 32 | 1.3333333 | 0.477564 | 0.522436 | −186 | 1477 | −1663 |
| 313 | $313^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.476038 | 0.523962 | −188 | 1477 | −1665 |
| 314 | $2^1 157^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477707 | 0.522293 | −183 | 1482 | −1665 |
| 315 | $3^2 5^1 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.479365 | 0.520635 | −153 | 1512 | −1665 |
| 316 | $2^2 79^1$ | N | N | −7 | 2 | 1.2857143 | 0.477848 | 0.522152 | −160 | 1512 | −1672 |
| 317 | $317^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.476341 | 0.523659 | −162 | 1512 | −1674 |
| 318 | $2^1 3^1 53^1$ | Y | N | −16 | 0 | 1.0000000 | 0.474843 | 0.525157 | −178 | 1512 | −1690 |
| 319 | $11^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476489 | 0.523511 | −173 | 1517 | −1690 |
| 320 | $2^6 5^1$ | N | N | −15 | 10 | 2.3333333 | 0.475000 | 0.525000 | −188 | 1517 | −1705 |
| 321 | $3^1 107^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476636 | 0.523364 | −183 | 1522 | −1705 |
| 322 | $2^1 7^1 23^1$ | Y | N | −16 | 0 | 1.0000000 | 0.475155 | 0.524845 | −199 | 1522 | −1721 |
| 323 | $17^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476780 | 0.523220 | −194 | 1527 | −1721 |
| 324 | $2^2 3^4$ | N | N | 34 | 29 | 1.6176471 | 0.478395 | 0.521605 | −160 | 1561 | −1721 |
| 325 | $5^2 13^1$ | N | N | −7 | 2 | 1.2857143 | 0.476923 | 0.523077 | −167 | 1561 | −1728 |
| 326 | $2^1 163^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478528 | 0.521472 | −162 | 1566 | −1728 |
| 327 | $3^1 109^1$ | Y | N | 5 | 0 | 1.0000000 | 0.480122 | 0.519878 | −157 | 1571 | −1728 |
| 328 | $2^3 41^1$ | N | N | 9 | 4 | 1.5555556 | 0.481707 | 0.518293 | −148 | 1580 | −1728 |
| 329 | $7^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483283 | 0.516717 | −143 | 1585 | −1728 |
| 330 | $2^1 3^1 5^1 11^1$ | Y | N | 65 | 0 | 1.0000000 | 0.484848 | 0.515152 | −78 | 1650 | −1728 |
| 331 | $331^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.483384 | 0.516616 | −80 | 1650 | −1730 |
| 332 | $2^2 83^1$ | N | N | −7 | 2 | 1.2857143 | 0.481928 | 0.518072 | −87 | 1650 | −1737 |
| 333 | $3^2 37^1$ | N | N | −7 | 2 | 1.2857143 | 0.480480 | 0.519520 | −94 | 1650 | −1744 |
| 334 | $2^1 167^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482036 | 0.517964 | −89 | 1655 | −1744 |
| 335 | $5^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483582 | 0.516418 | −84 | 1660 | −1744 |
| 336 | $2^4 3^1 7^1$ | N | N | 70 | 54 | 1.5000000 | 0.485119 | 0.514881 | −14 | 1730 | −1744 |
| 337 | $337^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.483680 | 0.516320 | −16 | 1730 | −1746 |
| 338 | $2^1 13^2$ | N | N | −7 | 2 | 1.2857143 | 0.482249 | 0.517751 | −23 | 1730 | −1753 |
| 339 | $3^1 113^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483776 | 0.516224 | −18 | 1735 | −1753 |
| 340 | $2^2 5^1 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.485294 | 0.514706 | 12 | 1765 | −1753 |
| 341 | $11^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486804 | 0.513196 | 17 | 1770 | −1753 |
| 342 | $2^1 3^2 19^1$ | N | N | 30 | 14 | 1.1666667 | 0.488304 | 0.511696 | 47 | 1800 | −1753 |
| 343 | $7^3$ | N | Y | −2 | 0 | 2.0000000 | 0.486880 | 0.513120 | 45 | 1800 | −1755 |
| 344 | $2^3 43^1$ | N | N | 9 | 4 | 1.5555556 | 0.488372 | 0.511628 | 54 | 1809 | −1755 |
| 345 | $3^1 5^1 23^1$ | Y | N | −16 | 0 | 1.0000000 | 0.486957 | 0.513043 | 38 | 1809 | −1771 |
| 346 | $2^1 173^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488439 | 0.511561 | 43 | 1814 | −1771 |
| 347 | $347^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.487032 | 0.512968 | 41 | 1814 | −1773 |
| 348 | $2^2 3^1 29^1$ | N | N | 30 | 14 | 1.1666667 | 0.488506 | 0.511494 | 71 | 1844 | −1773 |
| 349 | $349^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.487106 | 0.512894 | 69 | 1844 | −1775 |
| 350 | $2^1 5^2 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.488571 | 0.511429 | 99 | 1874 | −1775 |

| $n$ | **Primes** | **Sqfree** | **PPower** | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 351 | $3^3 13^1$ | N | N | 9 | 4 | 1.5555556 | 0.490028 | 0.509972 | 108 | 1883 | −1775 |
| 352 | $2^5 11^1$ | N | N | 13 | 8 | 2.0769231 | 0.491477 | 0.508523 | 121 | 1896 | −1775 |
| 353 | $353^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.490085 | 0.509915 | 119 | 1896 | −1777 |
| 354 | $2^1 3^1 59^1$ | Y | N | −16 | 0 | 1.0000000 | 0.488701 | 0.511299 | 103 | 1896 | −1793 |
| 355 | $5^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490141 | 0.509859 | 108 | 1901 | −1793 |
| 356 | $2^2 89^1$ | N | N | −7 | 2 | 1.2857143 | 0.488764 | 0.511236 | 101 | 1901 | −1800 |
| 357 | $3^1 7^1 17^1$ | Y | N | −16 | 0 | 1.0000000 | 0.487395 | 0.512605 | 85 | 1901 | −1816 |
| 358 | $2^1 179^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488827 | 0.511173 | 90 | 1906 | −1816 |
| 359 | $359^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.487465 | 0.512535 | 88 | 1906 | −1818 |
| 360 | $2^3 3^2 5^1$ | N | N | 145 | 129 | 1.3034483 | 0.488889 | 0.511111 | 233 | 2051 | −1818 |
| 361 | $19^2$ | N | Y | 2 | 0 | 1.5000000 | 0.490305 | 0.509695 | 235 | 2053 | −1818 |
| 362 | $2^1 181^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491713 | 0.508287 | 240 | 2058 | −1818 |
| 363 | $3^1 11^2$ | N | N | −7 | 2 | 1.2857143 | 0.490358 | 0.509642 | 233 | 2058 | −1825 |
| 364 | $2^2 7^1 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.491758 | 0.508242 | 263 | 2088 | −1825 |
| 365 | $5^1 73^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493151 | 0.506849 | 268 | 2093 | −1825 |
| 366 | $2^1 3^1 61^1$ | Y | N | −16 | 0 | 1.0000000 | 0.491803 | 0.508197 | 252 | 2093 | −1841 |
| 367 | $367^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.490463 | 0.509537 | 250 | 2093 | −1843 |
| 368 | $2^4 23^1$ | N | N | −11 | 6 | 1.8181818 | 0.489130 | 0.510870 | 239 | 2093 | −1854 |
| 369 | $3^2 41^1$ | N | N | −7 | 2 | 1.2857143 | 0.487805 | 0.512195 | 232 | 2093 | −1861 |
| 370 | $2^1 5^1 37^1$ | Y | N | −16 | 0 | 1.0000000 | 0.486486 | 0.513514 | 216 | 2093 | −1877 |
| 371 | $7^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487871 | 0.512129 | 221 | 2098 | −1877 |
| 372 | $2^2 3^1 31^1$ | N | N | 30 | 14 | 1.1666667 | 0.489247 | 0.510753 | 251 | 2128 | −1877 |
| 373 | $373^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.487936 | 0.512064 | 249 | 2128 | −1879 |
| 374 | $2^1 11^1 17^1$ | Y | N | −16 | 0 | 1.0000000 | 0.486631 | 0.513369 | 233 | 2128 | −1895 |
| 375 | $3^1 5^3$ | N | N | 9 | 4 | 1.5555556 | 0.488000 | 0.512000 | 242 | 2137 | −1895 |
| 376 | $2^3 47^1$ | N | N | 9 | 4 | 1.5555556 | 0.489362 | 0.510638 | 251 | 2146 | −1895 |
| 377 | $13^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490716 | 0.509284 | 256 | 2151 | −1895 |
| 378 | $2^1 3^3 7^1$ | N | N | −48 | 32 | 1.3333333 | 0.489418 | 0.510582 | 208 | 2151 | −1943 |
| 379 | $379^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.488127 | 0.511873 | 206 | 2151 | −1945 |
| 380 | $2^2 5^1 19^1$ | N | N | 30 | 14 | 1.1666667 | 0.489474 | 0.510526 | 236 | 2181 | −1945 |
| 381 | $3^1 127^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490814 | 0.509186 | 241 | 2186 | −1945 |
| 382 | $2^1 191^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492147 | 0.507853 | 246 | 2191 | −1945 |
| 383 | $383^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.490862 | 0.509138 | 244 | 2191 | −1947 |
| 384 | $2^7 3^1$ | N | N | 17 | 12 | 2.5882353 | 0.492188 | 0.507812 | 261 | 2208 | −1947 |
| 385 | $5^1 7^1 11^1$ | Y | N | −16 | 0 | 1.0000000 | 0.490909 | 0.509091 | 245 | 2208 | −1963 |
| 386 | $2^1 193^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492228 | 0.507772 | 250 | 2213 | −1963 |
| 387 | $3^2 43^1$ | N | N | −7 | 2 | 1.2857143 | 0.490956 | 0.509044 | 243 | 2213 | −1970 |
| 388 | $2^2 97^1$ | N | N | −7 | 2 | 1.2857143 | 0.489691 | 0.510309 | 236 | 2213 | −1977 |
| 389 | $389^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.488432 | 0.511568 | 234 | 2213 | −1979 |
| 390 | $2^1 3^1 5^1 13^1$ | Y | N | 65 | 0 | 1.0000000 | 0.489744 | 0.510256 | 299 | 2278 | −1979 |
| 391 | $17^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491049 | 0.508951 | 304 | 2283 | −1979 |
| 392 | $2^3 7^2$ | N | N | −23 | 18 | 1.4782609 | 0.489796 | 0.510204 | 281 | 2283 | −2002 |
| 393 | $3^1 131^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491094 | 0.508906 | 286 | 2288 | −2002 |
| 394 | $2^1 197^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492386 | 0.507614 | 291 | 2293 | −2002 |
| 395 | $5^1 79^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493671 | 0.506329 | 296 | 2298 | −2002 |
| 396 | $2^2 3^2 11^1$ | N | N | −74 | 58 | 1.2162162 | 0.492424 | 0.507576 | 222 | 2298 | −2076 |
| 397 | $397^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.491184 | 0.508816 | 220 | 2298 | −2078 |
| 398 | $2^1 199^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492462 | 0.507538 | 225 | 2303 | −2078 |
| 399 | $3^1 7^1 19^1$ | Y | N | −16 | 0 | 1.0000000 | 0.491228 | 0.508772 | 209 | 2303 | −2094 |
| 400 | $2^4 5^2$ | N | N | 34 | 29 | 1.6176471 | 0.492500 | 0.507500 | 243 | 2337 | −2094 |
| 401 | $401^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.491272 | 0.508728 | 241 | 2337 | −2096 |
| 402 | $2^1 3^1 67^1$ | Y | N | −16 | 0 | 1.0000000 | 0.490050 | 0.509950 | 225 | 2337 | −2112 |
| 403 | $13^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491315 | 0.508685 | 230 | 2342 | −2112 |
| 404 | $2^2 101^1$ | N | N | −7 | 2 | 1.2857143 | 0.490099 | 0.509901 | 223 | 2342 | −2119 |
| 405 | $3^4 5^1$ | N | N | −11 | 6 | 1.8181818 | 0.488889 | 0.511111 | 212 | 2342 | −2130 |
| 406 | $2^1 7^1 29^1$ | Y | N | −16 | 0 | 1.0000000 | 0.487685 | 0.512315 | 196 | 2342 | −2146 |
| 407 | $11^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488943 | 0.511057 | 201 | 2347 | −2146 |
| 408 | $2^3 3^1 17^1$ | N | N | −48 | 32 | 1.3333333 | 0.487745 | 0.512255 | 153 | 2347 | −2194 |
| 409 | $409^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.486553 | 0.513447 | 151 | 2347 | −2196 |
| 410 | $2^1 5^1 41^1$ | Y | N | −16 | 0 | 1.0000000 | 0.485366 | 0.514634 | 135 | 2347 | −2212 |
| 411 | $3^1 137^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486618 | 0.513382 | 140 | 2352 | −2212 |
| 412 | $2^2 103^1$ | N | N | −7 | 2 | 1.2857143 | 0.485437 | 0.514563 | 133 | 2352 | −2219 |
| 413 | $7^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486683 | 0.513317 | 138 | 2357 | −2219 |
| 414 | $2^1 3^2 23^1$ | N | N | 30 | 14 | 1.1666667 | 0.487923 | 0.512077 | 168 | 2387 | −2219 |
| 415 | $5^1 83^1$ | Y | N | 5 | 0 | 1.0000000 | 0.489157 | 0.510843 | 173 | 2392 | −2219 |
| 416 | $2^5 13^1$ | N | N | 13 | 8 | 2.0769231 | 0.490385 | 0.509615 | 186 | 2405 | −2219 |
| 417 | $3^1 139^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491607 | 0.508393 | 191 | 2410 | −2219 |
| 418 | $2^1 11^1 19^1$ | Y | N | −16 | 0 | 1.0000000 | 0.490431 | 0.509569 | 175 | 2410 | −2235 |
| 419 | $419^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.489260 | 0.510740 | 173 | 2410 | −2237 |
| 420 | $2^2 3^1 5^1 7^1$ | N | N | −155 | 90 | 1.1032258 | 0.488095 | 0.511905 | 18 | 2410 | −2392 |
| 421 | $421^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.486936 | 0.513064 | 16 | 2410 | −2394 |
| 422 | $2^1 211^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488152 | 0.511848 | 21 | 2415 | −2394 |
| 423 | $3^2 47^1$ | N | N | −7 | 2 | 1.2857143 | 0.486998 | 0.513002 | 14 | 2415 | −2401 |
| 424 | $2^3 53^1$ | N | N | 9 | 4 | 1.5555556 | 0.488208 | 0.511792 | 23 | 2424 | −2401 |
| 425 | $5^2 17^1$ | N | N | −7 | 2 | 1.2857143 | 0.487059 | 0.512941 | 16 | 2424 | −2408 |

| $n$ | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 426 | $2^1 3^1 71^1$ | Y | N | −16 | 0 | 1.0000000 | 0.485915 | 0.514085 | 0 | 2424 | −2424 |
| 427 | $7^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487119 | 0.512881 | 5 | 2429 | −2424 |
| 428 | $2^2 107^1$ | N | N | −7 | 2 | 1.2857143 | 0.485981 | 0.514019 | −2 | 2429 | −2431 |
| 429 | $3^1 11^1 13^1$ | Y | N | −16 | 0 | 1.0000000 | 0.484848 | 0.515152 | −18 | 2429 | −2447 |
| 430 | $2^1 5^1 43^1$ | Y | N | −16 | 0 | 1.0000000 | 0.483721 | 0.516279 | −34 | 2429 | −2463 |
| 431 | $431^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.482599 | 0.517401 | −36 | 2429 | −2465 |
| 432 | $2^4 3^3$ | N | N | −80 | 75 | 1.5625000 | 0.481481 | 0.518519 | −116 | 2429 | −2545 |
| 433 | $433^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.480370 | 0.519630 | −118 | 2429 | −2547 |
| 434 | $2^1 7^1 31^1$ | Y | N | −16 | 0 | 1.0000000 | 0.479263 | 0.520737 | −134 | 2429 | −2563 |
| 435 | $3^1 5^1 29^1$ | Y | N | −16 | 0 | 1.0000000 | 0.478161 | 0.521839 | −150 | 2429 | −2579 |
| 436 | $2^2 109^1$ | N | N | −7 | 2 | 1.2857143 | 0.477064 | 0.522936 | −157 | 2429 | −2586 |
| 437 | $19^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478261 | 0.521739 | −152 | 2434 | −2586 |
| 438 | $2^1 3^1 73^1$ | Y | N | −16 | 0 | 1.0000000 | 0.477169 | 0.522831 | −168 | 2434 | −2602 |
| 439 | $439^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.476082 | 0.523918 | −170 | 2434 | −2604 |
| 440 | $2^3 5^1 11^1$ | N | N | −48 | 32 | 1.3333333 | 0.475000 | 0.525000 | −218 | 2434 | −2652 |
| 441 | $3^2 7^2$ | N | N | 14 | 9 | 1.3571429 | 0.476190 | 0.523810 | −204 | 2448 | −2652 |
| 442 | $2^1 13^1 17^1$ | Y | N | −16 | 0 | 1.0000000 | 0.475113 | 0.524887 | −220 | 2448 | −2668 |
| 443 | $443^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.474041 | 0.525959 | −222 | 2448 | −2670 |
| 444 | $2^2 3^1 37^1$ | N | N | 30 | 14 | 1.1666667 | 0.475225 | 0.524775 | −192 | 2478 | −2670 |
| 445 | $5^1 89^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476404 | 0.523596 | −187 | 2483 | −2670 |
| 446 | $2^1 223^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477578 | 0.522422 | −182 | 2488 | −2670 |
| 447 | $3^1 149^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478747 | 0.521253 | −177 | 2493 | −2670 |
| 448 | $2^6 7^1$ | N | N | −15 | 10 | 2.3333333 | 0.477679 | 0.522321 | −192 | 2493 | −2685 |
| 449 | $449^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.476615 | 0.523385 | −194 | 2493 | −2687 |
| 450 | $2^1 3^2 5^2$ | N | N | −74 | 58 | 1.2162162 | 0.475556 | 0.524444 | −268 | 2493 | −2761 |
| 451 | $11^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476718 | 0.523282 | −263 | 2498 | −2761 |
| 452 | $2^2 113^1$ | N | N | −7 | 2 | 1.2857143 | 0.475664 | 0.524336 | −270 | 2498 | −2768 |
| 453 | $3^1 151^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476821 | 0.523179 | −265 | 2503 | −2768 |
| 454 | $2^1 227^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477974 | 0.522026 | −260 | 2508 | −2768 |
| 455 | $5^1 7^1 13^1$ | Y | N | −16 | 0 | 1.0000000 | 0.476923 | 0.523077 | −276 | 2508 | −2784 |
| 456 | $2^3 3^1 19^1$ | N | N | −48 | 32 | 1.3333333 | 0.475877 | 0.524123 | −324 | 2508 | −2832 |
| 457 | $457^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.474836 | 0.525164 | −326 | 2508 | −2834 |
| 458 | $2^1 229^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475983 | 0.524017 | −321 | 2513 | −2834 |
| 459 | $3^3 17^1$ | N | N | 9 | 4 | 1.5555556 | 0.477124 | 0.522876 | −312 | 2522 | −2834 |
| 460 | $2^2 5^1 23^1$ | N | N | 30 | 14 | 1.1666667 | 0.478261 | 0.521739 | −282 | 2552 | −2834 |
| 461 | $461^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.477223 | 0.522777 | −284 | 2552 | −2836 |
| 462 | $2^1 3^1 7^1 11^1$ | Y | N | 65 | 0 | 1.0000000 | 0.478355 | 0.521645 | −219 | 2617 | −2836 |
| 463 | $463^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.477322 | 0.522678 | −221 | 2617 | −2838 |
| 464 | $2^4 29^1$ | N | N | −11 | 6 | 1.8181818 | 0.476293 | 0.523707 | −232 | 2617 | −2849 |
| 465 | $3^1 5^1 31^1$ | Y | N | −16 | 0 | 1.0000000 | 0.475269 | 0.524731 | −248 | 2617 | −2865 |
| 466 | $2^1 233^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476395 | 0.523605 | −243 | 2622 | −2865 |
| 467 | $467^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.475375 | 0.524625 | −245 | 2622 | −2867 |
| 468 | $2^2 3^2 13^1$ | N | N | −74 | 58 | 1.2162162 | 0.474359 | 0.525641 | −319 | 2622 | −2941 |
| 469 | $7^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475480 | 0.524520 | −314 | 2627 | −2941 |
| 470 | $2^1 5^1 47^1$ | Y | N | −16 | 0 | 1.0000000 | 0.474468 | 0.525532 | −330 | 2627 | −2957 |
| 471 | $3^1 157^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475584 | 0.524416 | −325 | 2632 | −2957 |
| 472 | $2^3 59^1$ | N | N | 9 | 4 | 1.5555556 | 0.476695 | 0.523305 | −316 | 2641 | −2957 |
| 473 | $11^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477801 | 0.522199 | −311 | 2646 | −2957 |
| 474 | $2^1 3^1 79^1$ | Y | N | −16 | 0 | 1.0000000 | 0.476793 | 0.523207 | −327 | 2646 | −2973 |
| 475 | $5^2 19^1$ | N | N | −7 | 2 | 1.2857143 | 0.475789 | 0.524211 | −334 | 2646 | −2980 |
| 476 | $2^2 7^1 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.476891 | 0.523109 | −304 | 2676 | −2980 |
| 477 | $3^2 53^1$ | N | N | −7 | 2 | 1.2857143 | 0.475891 | 0.524109 | −311 | 2676 | −2987 |
| 478 | $2^1 239^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476987 | 0.523013 | −306 | 2681 | −2987 |
| 479 | $479^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.475992 | 0.524008 | −308 | 2681 | −2989 |
| 480 | $2^5 3^1 5^1$ | N | N | −96 | 80 | 1.6666667 | 0.475000 | 0.525000 | −404 | 2681 | −3085 |
| 481 | $13^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476091 | 0.523909 | −399 | 2686 | −3085 |
| 482 | $2^1 241^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477178 | 0.522822 | −394 | 2691 | −3085 |
| 483 | $3^1 7^1 23^1$ | Y | N | −16 | 0 | 1.0000000 | 0.476190 | 0.523810 | −410 | 2691 | −3101 |
| 484 | $2^2 11^2$ | N | N | 14 | 9 | 1.3571429 | 0.477273 | 0.522727 | −396 | 2705 | −3101 |
| 485 | $5^1 97^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478351 | 0.521649 | −391 | 2710 | −3101 |
| 486 | $2^1 3^5$ | N | N | 13 | 8 | 2.0769231 | 0.479424 | 0.520576 | −378 | 2723 | −3101 |
| 487 | $487^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.478439 | 0.521561 | −380 | 2723 | −3103 |
| 488 | $2^3 61^1$ | N | N | 9 | 4 | 1.5555556 | 0.479508 | 0.520492 | −371 | 2732 | −3103 |
| 489 | $3^1 163^1$ | Y | N | 5 | 0 | 1.0000000 | 0.480573 | 0.519427 | −366 | 2737 | −3103 |
| 490 | $2^1 5^1 7^2$ | N | N | 30 | 14 | 1.1666667 | 0.481633 | 0.518367 | −336 | 2767 | −3103 |
| 491 | $491^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.480652 | 0.519348 | −338 | 2767 | −3105 |
| 492 | $2^2 3^1 41^1$ | N | N | 30 | 14 | 1.1666667 | 0.481707 | 0.518293 | −308 | 2797 | −3105 |
| 493 | $17^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482759 | 0.517241 | −303 | 2802 | −3105 |
| 494 | $2^1 13^1 19^1$ | Y | N | −16 | 0 | 1.0000000 | 0.481781 | 0.518219 | −319 | 2802 | −3121 |
| 495 | $3^2 5^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.482828 | 0.517172 | −289 | 2832 | −3121 |
| 496 | $2^4 31^1$ | N | N | −11 | 6 | 1.8181818 | 0.481855 | 0.518145 | −300 | 2832 | −3132 |
| 497 | $7^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482897 | 0.517103 | −295 | 2837 | −3132 |
| 498 | $2^1 3^1 83^1$ | Y | N | −16 | 0 | 1.0000000 | 0.481928 | 0.518072 | −311 | 2837 | −3148 |
| 499 | $499^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.480962 | 0.519038 | −313 | 2837 | −3150 |
| 500 | $2^2 5^3$ | N | N | −23 | 18 | 1.4782609 | 0.480000 | 0.520000 | −336 | 2837 | −3173 |