ASYMPTOTIC BOUNDS FOR GENERALIZED MERTENS SUMMATORY FUNCTIONS

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Abstract. The Mertens function is defined as the average order of the Möbius function, or as the summatory function $M(x) = \sum_{n \leq x} \mu(n)$, for all $x \geq 1$. There are many open problems are related to determining optimal asymptotic bounds for this function. The famous statement of Merten's conjecture which says that $|M(x)| < \sqrt{x}$ has been disproved, though is it known that the Riemann Hypothesis is equivalent to showing that $|M(x)| \ll \sqrt{x} \exp\left(B \frac{\log x}{\log \log x}\right)$ for some constant B. Another unresolved problem related to this function is whether $\limsup_{x \to \infty} |M(x)|/\sqrt{x} = \infty$. In this article, we employ the recent construction of new formulas for the generalized sum-of-divisors functions proved by Schmidt to obtain new results which exactly sum the classical Mertens function for all finite x. We state and prove analogous results for the generalized Mertens function which we define to be $M_\alpha^*(x) = \sum_{n \leq x} n^\alpha \mu(n)$ for any fixed $\alpha \in \mathbb{C}$.

1. Introduction

1.1. Mertens summatory functions. The Mertens summatory function, or *Mertens function*, is defined as

$$M(x) = \sum_{n \leq x} \mu(n),\ x \geq 1,$$

where $\mu(n)$ denotes the Möbius function which is in some sense a signed indicator function for the squarefree integers. A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

We define the notion of a *generalized Mertens summatory function* for fixed $\alpha \in \mathbb{C}$ as

$$M_\alpha^*(x) = \sum_{n \leq x} n^\alpha \mu(n),\ x \geq 1,$$

where the special case of $M_0^*(x)$ corresponds to the definition of the classical Mertens function $M(x)$ defined above. The plots shown in Figure 1 illustrate the chaotic behavior of the growth of these functions for x in small intervals when $\alpha \in \{-1, 0, 1, 2\}$. In particular, there are many open problems related to bounding $M(x)$ for large x. The Riemann Hypothesis is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. It is still unresolved whether

$$\limsup_{x \to \infty} |M(x)|/\sqrt{x} = \infty,$$

although computational evidence suggests that this is a likely conjecture [5, 4]. We make a well-founded attempt to prove that this conjecture is true in Theorem 2.3.

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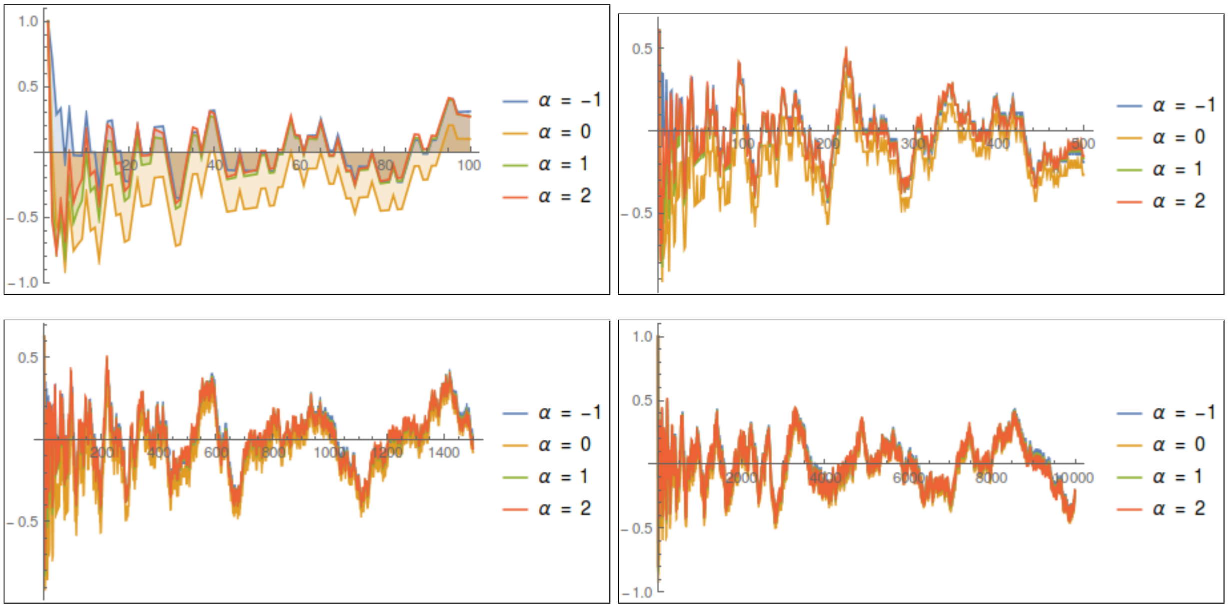

FIGURE 1. Comparison of the Mertens Summatory Functions $M_\alpha(x)/x^{\frac{1}{2}+\alpha}$ for Small x and α

1.2. **Exact formulas for the generalized sum-of-divisors functions.** Schmidt has recently proved (2017) several new exact formulas for the *generalized sum-of-divisors functions*, $\sigma_\alpha(x)$, defined for any $x \geq 1$ as

$$\sigma_\alpha(x) = \sum_{d|n} d^\alpha, \ \alpha \in \mathbb{C}.$$

In particular, if we let $H_n^{(r)} = \sum_{k=1}^n k^{-r}$ denote the sequence of r*-order harmonic numbers*, where [7, §2.4(iii)]

$$H_n^{(-t)} = \frac{B_{t+1}(n+1) - B_{t+1}}{(t+1)} = \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + \sum_{k=1}^{t-1} \binom{t}{k} \frac{B_{k+1} n^{t-k}}{(k+1)},$$

is a *Bernoulli polynomial* for any $n \geq 0$ when $t \in \mathbb{N}$, then we can restate the next theorem from [9]. Within this article we assume that an index of summation p denotes that the sum is taken over only prime values of p. We also use the notation that the function

$$\varepsilon_p(x) = \sum_{k \geq 1} \left\lfloor \frac{x}{p^k} \right\rfloor = m \quad \text{if and only if} \quad p^m \| x,$$

to denote the exact exponent of the prime p dividing x.

Theorem 1.1 (Schmidt, 2017)**.** *For any fixed $\alpha \in \mathbb{C}$ and all $x \geq 1$, we have that*

$$\sigma_\alpha(x) = H_x^{(1-\alpha)} + \sum_{d|n} \tau_x^{(\alpha)}(d) + \sum_{2 \leq p \leq x} \sum_{k=1}^{\varepsilon_p(x)+1} p^{\alpha k} H^{(1-\alpha)}_{\left\lfloor \frac{x}{p^k} \right\rfloor} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x - p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right)$$
$$+ \sum_{3 \leq p \leq x} \sum_{k=1}^{\varepsilon_p(x)+1} 2^{\alpha-1} p^{\alpha k} (-1)^{\lfloor x/p^{k-1} \rfloor} H^{(1-\alpha)}_{\left\lfloor \frac{x}{2p^k} \right\rfloor} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x - p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right),$$

where the divisor sum over the function $\tau_x^{(\alpha)}(d)$ is defined precisely by Lemma 2.1.

Remark 1.2 (Restatement of the Theorem)**.** For $x \geq 1$ and fixed $\alpha \in \mathbb{C}$, we define the sums

$$S_1^{(\alpha)}(x) = \sum_{2 \leq p \leq x} \sum_{k=1}^{\varepsilon_p(x)+1} p^{\alpha k} H_{\left\lfloor \frac{x}{p^k} \right\rfloor}^{(1-\alpha)} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right)$$

$$S_2^{(\alpha)}(x) = \sum_{3 \leq p \leq x} \sum_{k=1}^{\varepsilon_p(x)+1} 2^{\alpha-1} p^{\alpha k} (-1)^{\lfloor x/p^{k-1} \rfloor} H_{\left\lfloor \frac{x}{2p^k} \right\rfloor}^{(1-\alpha)} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right).$$

Then we prefer to work with the next form of Theorem 1.1 stated in terms of our new shorthand sum functions and Lemma 2.1 as follows:

$$\left| \sum_{d|x} \tau_x^{(\alpha)}(d) \right| = \left| H_x^{(1-\alpha)} - \sigma_\alpha(x) + S_1^{(\alpha)}(x) + S_2^{(\alpha)}(x) \right|. \tag{1}$$

The statement of the theorem given in (1) is important and significant since it implies deep connections between the sum-of-divisors functions, the generalized Mertens summatory functions, and the partial sums of the Riemann zeta function for real $\alpha < 0$, each related to one another in a convolved formula taken over sums of successive powers of the primes $p \leq x$. Thus we immediately see new relations from the restatement of the key results in [9] above. Moreover, from the previous result, we then obtain our main new results in the article given in the results in the next section as consequences of this restatement in terms of the Mertens functions.

2. New Results and Proof of the Key Lemma

Lemma 2.1 (Key Relation to the Mertens Summatory Functions)**.** *For sufficiently large $x \geq 1$ and any $\alpha \in \mathbb{N}$, we have the bound*

$$\left| \sum_{d|x} \tau_x^{(\alpha+1)}(d) \right| \leq \frac{\pi^2}{3} 2^\alpha x^{\alpha+1} \left| M_{\alpha-1}^*(x) \right| \left(1 - \frac{1}{\log(x)} \right).$$

In particular, we have a lower asymptotic bound for the Mertens function $M(x)$ given by

$$\left| \sum_{d|x} \tau_x^{(1)}(d) \right| \leq \frac{\pi^2}{3} x |M(x)| \left(1 - \frac{1}{\log(x)} \right).$$

Proof. Let the sets $S_{i,x}$ be defined as in [9, §1], i.e., such that $S_{i,x}$ consists of the integers s in the range $[12, x]$ such that

- **(1)** The set index i divides s: $i|s$;
- **(2)** Either $\varepsilon_2(s) \geq 2$ or there are at least two odd primes dividing s;
- **(3)** The quotient s/i is *squarefree*: $\mu(s/i) \neq 0$; and
- **(4)** If $i = 2^k$ is a power of two, then $s/i > 2$.

We define the auxiliary union set, S_x, to denote

$$\begin{aligned} S_x &= \bigcup_{i=12}^{x} S_{i,i} \\ &= \{12, 15, 20, 21, 24, 28, 30, 33, 35, 36, 39, 40, 42, 44, 45, 48, 51, 52, 55, \ldots\} \cap \{n \in \mathbb{N} : 12 \leq n \leq x\}. \end{aligned}$$

For $x \geq 1$, let $\chi_{\mathrm{pp}}(x)$ denote the indicator function for prime powers, i.e., the function defined precisely as

$$\chi_{\mathrm{pp}}(x) = \begin{cases} 1, & \text{if } x = p^k \text{ for some prime } p \geq 2 \text{ and } k \geq 1; \\ 0, & \text{otherwise,} \end{cases}$$

and define the composite indicator function for the prime powers $p^k, 2p^k$ as follows where $\chi_{\mathrm{pp}}(x) = 0$ if $x \in \mathbb{Q} \setminus \mathbb{Z}$:

$$\widetilde{\chi}_{\mathrm{pp}}(x) = \chi_{\mathrm{pp}}(x) + \chi_{\mathrm{pp}}\left(\frac{x}{2}\right).$$

Next, we start with the following formula for computing the divisor sum over $\tau_x^{(\alpha)}(d)$ from [9, §2]:

$$\begin{aligned}
\sum_{d|x} \tau_x^{(\alpha+1)}(d) &= [q^x] \left(\sum_{k=1}^{x} \sum_{d|k} \sum_{r|d} \frac{r \cdot \widetilde{\chi}_{\mathrm{pp}}(d) \cdot \mu(d/r)}{(1-q^r)} k^\alpha \right) \\
&= \sum_{k=1}^{x} \sum_{r|x} \sum_{d|k} r \cdot \widetilde{\chi}_{\mathrm{pp}}(d) \cdot \mu(d/r) \cdot [r|d]_\delta \\
&= \sum_{s \in S_x} \sum_{d|s} s^\alpha \cdot \mu(s/d) \cdot d\, [d|x]_\delta \\
&= \sum_{s \in S_x} \sum_{d|\gcd(s,x)} s^\alpha \cdot \mu(s/d) \cdot d. \qquad \text{(i)}
\end{aligned}$$

By (i) above we see that we have factors of Ramanujan's sum which leads to the identity that [7, §27.10] [6, §A.7] [3, *cf.* §5.6]

$$\sum_{d|x} \tau_x^{(\alpha+1)}(d) = \sum_{s \in S_x} \mu\left(\frac{s}{\gcd(s,x)}\right) \frac{\varphi(s) \cdot s^\alpha}{\varphi\left(\frac{s}{\gcd(s,x)}\right)},$$

where $\varphi(x)$ denotes Euler's totient function. However, since this last sum (while interesting on its own) does not provide the necessary form of our desired bound, we move forward with a different expansion.

We next proceed to bound the second to last sum in (i) by

$$\begin{aligned}
\sum_{s \in S_x} \sum_{d|\gcd(s,x)} s^\alpha \cdot \mu(s/d) \cdot d &\leq \left| \sum_{m=1}^{x} \mu(m) m^\alpha \left(\sum_{s \in S_x} \sum_{d|(s,x)} d^{\alpha+1} \left[m = \frac{s}{d}\right]_\delta \right) \right| \\
&\leq \left| \sum_{m=1}^{x} \mu(m) m^\alpha \cdot |S_x \cap \{n \geq m\}| \right| \cdot \left(\sup_{d|x} \sigma_{\alpha+1}(d) \right) \\
&\leq |M_\alpha^*(x)| \left| \sum_{m=1}^{x} |S_x \cap \{n \geq m\}| \right| \cdot \frac{1}{\sum_{m=1}^{x} [\mu(m) \neq 0]_\delta} \cdot \left(\sup_{d|x} \sigma_{\alpha+1}(d) \right) \\
&\leq (2x)^{\alpha+1} |M_\alpha^*(x)| (x - \pi(x)) \frac{1}{\frac{6x}{\pi^2}} \\
&= \frac{\pi^2}{3} 2^\alpha x^{\alpha+1} |M_\alpha^*(x)| \left(1 - \frac{1}{\log(x)}\right),
\end{aligned}$$

for sufficiently large x. In the previous equations, we have used the facts that $\sigma_{\alpha+1}(x) \leq (2x)^{\alpha+1}$ for all x, the asymptotic bound for the function $Q(x)$ indicating the number of squarefree integers $n \leq x$ from the introduction, a modified form of Chebyshev's sum inequality, and finally that

$$|S_x| \leq x - \pi(x),$$

which follows from the fact that by construction the set S_x contains no primes. Thus we have proved our desired bound. □

Remark 2.2. In the proof of the next theorem, we use the result of *Mertens' theorem* which implies that [6, §6.3] [1, §4.9] [3, §22.8] [7, §27.11]

$$\sum_{p \leq x} \frac{1}{p} = \log\log(x) + A + O\left(\frac{1}{\log x}\right),$$

where A is a constant. Moreover, if we assume the Riemann Hypothesis we have that the *prime number theorem* is equivalent to

$$\pi(x) = \frac{x}{\log x} + O(\sqrt{x}).$$

The next theorem attempts a proof that the limit supremum of $|M(x)|/\sqrt{x}$ is infinite! (Assuming the conclusion of Theorem 2.3 is true,) Prior to this point it is known that [8, *cf.* §4.1]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.06,$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009,$$

although based on work by Odlyzyko and te Riele (1985) it seems probable that each of these limits should be $\pm\infty$, respectively. Thus the above lemma, which is central to the proof of the following theorem, is extremely significant, as is the work in [9] on which it is based.

Theorem 2.3 (The Limit Supremum of $M(x)$ and Its Values at Large Prime Powers)**.** *Let's first assume that the Riemann Hypothesis is true. Let $x = q^r$ be a large odd prime power for some $r \geq 2$. Then we have the following bound on $M(x)$ where $\gamma_E \approx 0.577216$ is the Euler-Mascheroni gamma constant, $C_0 > 0$ is some bounded real-valued constant, and $\varepsilon > 0$ is small tending to 0:*

$$|M_0^*(x)| \geq \frac{3\log(x)}{\pi^2 x(\log(x)-1)} \left[\frac{C_0 \cdot x(x-1)}{\log(x-1)} + o\left(x^{1+\varepsilon}\right)\right]$$

In particular, we have the consequence that

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

Proof. Using the statement of Theorem 1.1 given in (1), we see that

$$\begin{aligned}
\left|\sum_{d|x} \tau_x^{(1)}(d)\right| &= \left|x - \sigma_1(x) + S_1^{(1)}(x) + S_2^{(1)}(x)\right| \\
&= \left|S_1^{(1)}(q^r) + S_2^{(1)}(q^r) - \left(1 + q + \cdots + q^{r-1}\right)\right| \\
&\geq C_0 \cdot \left|\sum_{p<q^r} p \cdot \frac{q^r}{p}\left(1 - \frac{1}{p}\right) + \sum_{k=1}^{r} q^r(q-1) + \frac{1}{q}\right| - \frac{q^r-1}{q-1} \\
&= C_0 \left|x\left(\pi(x-1) - \log\log(x-1) - A\right) + r(q-1)q^r\right| - \frac{q^r-1}{q-1} \\
&= C_0 \left|x\left(\frac{(x-1)}{\log(x-1)} + O(\sqrt{x-1}) - \left[\log\log(x-1) + A + O\left(\frac{1}{\log(x-1)}\right)\right]\right) + rx(x^{\frac{1}{r}}-1)\right| \\
&\quad - \frac{x-1}{x^{\frac{1}{r}}-1},
\end{aligned}$$

by the consequence of Mertens' theorem stated in (2.2) above. Thus we can see from Lemma 2.1 that

$$\begin{aligned}
&\limsup_{x\to\infty} \left\{\frac{|M(x)|}{\sqrt{x}}\right\} \\
&\quad \geq \lim_{x\to\infty} \left\{\frac{3C_0\log(x)}{\pi^2\sqrt{x}(\log(x)-1)} \left|\left(\frac{(x-1)}{\log(x-1)} + O(\sqrt{x}) - \log\log(x-1) - A\right) + r(x^{\frac{1}{r}}-1)\right| \right. \\
&\quad\quad \left. - \frac{3\log(x)}{\pi^2 x^{3/2}\left(\log(x)-1\right)}\left(\frac{x-1}{x^{\frac{1}{r}}-1}\right)\right\} \\
&\quad \longrightarrow +\infty
\end{aligned}$$

when $x = q^r$ is sufficiently large, i.e., as $x \to \infty$. Hence our claimed consequence holds. $\square$

3. Conclusions

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