Journal of Number Theory
Revised Manuscript and Responses to Referee Feedback

Exact formulas for partial sums of the Möbius function expressed by partial sums of weighted Liouville functions

Revision 4 – v2022-03-01

Summary of changes in this revised manuscript:

I have been fortunate enough to be in touch with Jeff Lagarias over the past few months on Zoom. He has taken literally hours of his time to provide feedback and discussion of the last submitted revision of the article emphasizing changes to the section and appendix-wise organization, stylistic modifications aimed at making the prior notation less cumbersome, and on enforcing general practices with respect to good mathematical writing style. Key changes in this revision include the following:

- The title has changed to be more descriptive;
- The abstract has been modified to reflect the newer changes in the article;
- The introduction has been shortened and reorganized thoughtfully to motivate the key takeaways and interpretations of the article, and to present statements of the main new results proved within;
- The results on the limiting distributions of the two primary unsigned sequences we study in the article are re-tooled and stated as conjectures (see Section 5). As proofs of these results are outside of the scope of this article, and would substantially lengthen the already long and technical contents, we aim to only provide enough motivation to support arriving at the conjectured results in the new separate section;
- There are several new discrete plots and numerical experiments featured in the figures given in Section 5 and Section 6. These figures are carefully explained in the relevant subsections of the article. The figures have been chosen carefully to showcase interesting properties of the auxiliary functions studied in the article and to point out apparent features of the new functions in comparison with the classical summatory functions, M(x) and L(x);
- The main sections of the article (non-appendix sections) have been shifted to reflect a more comprehensible exposition by proof method and by topic according to the auxiliary function to which the results in each section pertain;
- Several sections of the article that can be logically separated from the main body sections have been moved to separate appendices;
- Subsections of the article that do not provide extra information on which we build and extend in our new results and conjectures have been removed to shorten the article. For example, the former subsection from the introduction providing preliminaries, or a discussion of known results on M(x), is now omitted from the manuscript.

Please note that most of the referee responses from the last returned revision of the article now refers to a subsection on prior known results on the Mertens function that has now been removed from the manuscript. The motivation for removing this section on the “preliminary” results on M(x) is that we do not utilize nor require these existing results in the proofs and key takeaways that we wish to emphasize within the article.

Thank you very much to the editor and referee for allowing me the time to update the article and make these modifications to improve the overall quality of the manuscript.

Responses to the referee from the first revised manuscript:

Referee Point:
It takes a long time to see exactly what theorems the author is proving, and when you see ‘em, it’s unclear why anyone would do this.
Author Response:
I have received feedback from the JNT editor and from other sources in making changes to the article. A summary of the changes made in this revision of the manuscript are given in the bulleted list appearing on the previous pages of this referee response document. The changes in this revision follow hours of detailed online discussion with Jeff Lagarias. The modifications and updated structure and presentation within this revision have been made and copyedited carefully to improve the presentation of the article. Please proceed by re-reading the article from the start sections. I think you will find that the thoughtful discussion of how to best present and motivate and outline key takeaways of these new results was a productive effort.

Referee Point:
Sound’s paper is in Crelle, not Annals.
Author Response:
Thank you for catching this missed citation from my old BibTeX file. Some of the results I originally had attributed to Sound’s article with the reference are actually contained in the bibliography item by Humphries. Please note that this comment refers to a section of the article that has been removed in the latest revision of the article.

Referee Point:
Sound’s result, quoted before on page 4, just before \S 1.1.3 should have “14” as the exponent of \log\log x, and not 5/2 + \epsilon.
Author Response:
Thank you for catching this typographical error. Please note that this comment refers to a section of the article that has been removed in the latest revision of the article.

Referee Point:
Above this, Walfisz’s result should have, as the exponent of \log\log x “-3/5” instead of “-1/5”.
Author Response:

Thank you for pointing out this typographical error. Please note that this comment refers to a section of the article that has been removed in the latest revision of the article.

Referee Point:
“te Riele” and “Odlyzko” have interesting variations on their spelling throughout the article.
Author Response:
Thank you for pointing out that I needed to check the spellings more carefully. Please note that this comment refers to a section of (and corresponding bibliography references in) the article that has been removed in the latest revision of the article.