ASYMPTOTIC BOUNDS FOR THE MERTENS FUNCTION: A PROOF THAT THE MERTENS FUNCTION IS UNBOUNDED WITH GENERALIZATIONS

MAXIE D. SCHMIDT

SCHOOL OF MATHEMATICS
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GA 30332

MAXIEDS@GMAIL.COM
MSCHMIDT34@GATECH.EDU

Abstract. The Mertens function is defined as the average order of the Möbius function, or as the summatory function $M(x) = \sum_{n \leq x} \mu(n)$, for all $x \geq 1$. There are many open problems are related to determining optimal asymptotic bounds for this function. The famous statement of Mertens' conjecture which says that $|M(x)| < \sqrt{x}$ has been disproved, though is it known that the Riemann Hypothesis is equivalent to showing that $|M(x)| \ll \sqrt{x} \exp\left(B \frac{\log x}{\log \log x}\right)$ for some constant B. Another unresolved problem related to this function is whether $\limsup_{x \to \infty} |M(x)|/\sqrt{x} = \infty$. In this article, we employ the recent construction of new formulas for the generalized sum-of-divisors functions proved by the author to obtain new results which exactly sum the classical Mertens function for all finite x. We state and prove analogous results for the generalized Mertens function which we define to be $M_\alpha^*(x) = \sum_{n \leq x} n^\alpha \mu(n)$ for any fixed $\alpha \in \mathbb{C}$.

1. Introduction

1.1. Mertens summatory functions. The Mertens summatory function, or *Mertens function*, is defined as

$$M(x) = \sum_{n \leq x} \mu(n), \ x \geq 1,$$

where $\mu(n)$ denotes the Möbius function which is in some sense a signed indicator function for the squarefree integers. A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

We define the notion of a *generalized Mertens summatory function* for fixed $\alpha \in \mathbb{C}$ as

$$M_\alpha^*(x) = \sum_{n \leq x} n^\alpha \mu(n), \ x \geq 1,$$

where the special case of $M_0^*(x)$ corresponds to the definition of the classical Mertens function $M(x)$ defined above. The plots shown in Figure 1.1 illustrate the chaotic behavior of the growth of these functions for x in small intervals when $\alpha \in \{-1, 0, 1, 2\}$.

1.2. Open problems. There are many open problems related to bounding $M(x)$ for large x. For example, the Riemann Hypothesis is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. For $\mathrm{Re}(\alpha) < 1$, we know the limiting absolute behavior of these functions as $x \to \infty$ as the Dirichlet generating function

$$\frac{1}{\zeta(\alpha)} = \lim_{x \longrightarrow \infty} M_\alpha^*(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{-\alpha}},$$

which is definitively bounded for all large x. It is still unresolved whether

$$\limsup_{x \to \infty} |M(x)|/\sqrt{x} = \infty,$$

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although computational evidence suggests that this is a likely conjecture [5, 4]. We make a newly well-founded attempt to prove that this conjecture is true in Theorem 2.1. There is a rich history to the original statement of the *Mertens conjecture* which states that

$$|M(x)| < c \cdot x^{1/2}, \quad \text{some constant } c > 0,$$

which was first verified by Mertens for $c = 1$ and $x < 10000$, although since its beginnings in 1897 has since been disproved by computation. We cite that prior to this point it is known that [10, *cf.* §4.1]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad \text{(now 1.826054)},$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad \text{(now } -1.837625\text{)},$$

although based on work by Odlyzko and te Riele it seems probable that each of these limits should be $\pm\infty$, respectively [8, 6, 5, 4]. While it is known that $M(x) = \Omega_{\pm}(\sqrt{x})$ and $M(x)/\sqrt{x} = \Omega_{\pm}(1)$, we appear to offer the first complete proof that the function $M(x)/\sqrt{x}$ is in fact *unbounded* in the next sections of this article.

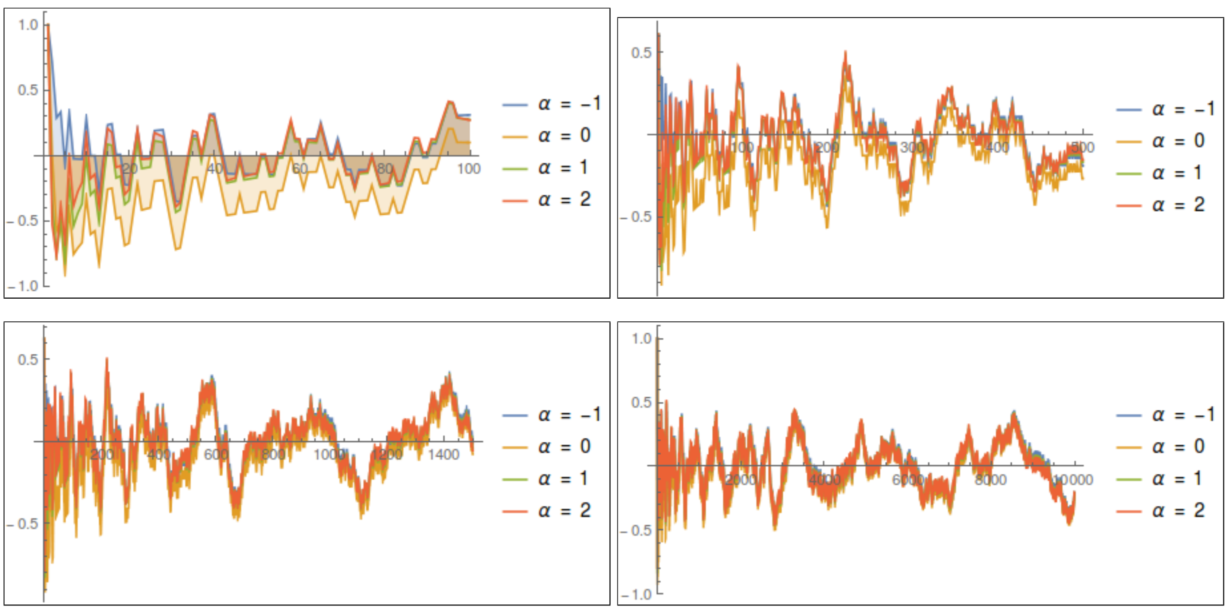

Figure 1.1. *Comparison of the Mertens Summatory Functions $M_\alpha(x)/x^{\frac{1}{2}+\alpha}$ for Small x and α*

1.3. **Exact formulas for the generalized sum-of-divisors functions.** The author has recently proved (2017) several new exact formulas for the *generalized sum-of-divisors functions*, $\sigma_\alpha(x)$, defined for any $x \geq 1$ as

$$\sigma_\alpha(x) = \sum_{d|x} d^\alpha, \ \ \alpha \in \mathbb{C}.$$

In particular, if we let $H_n^{(r)} = \sum_{k=1}^n k^{-r}$ denote the sequence of *r-order harmonic numbers*, where [9, §2.4(iii)]

$$H_n^{(-t)} = \frac{B_{t+1}(n+1) - B_{t+1}}{(t+1)} = \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + \sum_{k=1}^{t-1} \binom{t}{k} \frac{B_{k+1} n^{t-k}}{(k+1)}, \tag{1}$$

is a *Bernoulli polynomial* for any $n \geq 0$ when $t \in \mathbb{Z}^+$, then we can restate the next theorem from [12]. Within this article we assume that an index of summation p denotes that the sum is taken over only prime values of p. We also use the notation that the valuation function

$$\nu_p(x) = m \quad \text{if and only if} \quad p^m \| x,$$

to denote the exact exponent of the prime p dividing x.

Theorem 1.1 (Schmidt, 2017)**.** *For any fixed $\alpha \in \mathbb{C}$ and all $x \geq 1$, we have that*

$$\begin{aligned}\sigma_\alpha(x) = H_x^{(1-\alpha)} + \sum_{d|n} \tau_x^{(\alpha+1)}(d) + \sum_{2\leq p\leq x} \sum_{k=1}^{\nu_p(x)+1} p^{\alpha k} H^{(1-\alpha)}_{\left\lfloor \frac{x}{p^k}\right\rfloor} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p}\right) \\ + \sum_{3\leq p\leq x} \sum_{k=1}^{\nu_p(x)+1} 2^{\alpha-1} p^{\alpha k} (-1)^{\lfloor x/p^{k-1}\rfloor} H^{(1-\alpha)}_{\left\lfloor \frac{x}{2p^k}\right\rfloor} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p}\right),\end{aligned}$$

where the divisor sum over the function $\tau_x^{(\alpha)}(d)$ is defined precisely by Lemma 2.3.

Remark 1.2 (Restatement of the Theorem)**.** For $x \geq 1$ and fixed $\alpha \in \mathbb{C}$, we define the sums

$$\begin{aligned}S_1^{(\alpha+1)}(x) &= \sum_{2\leq p\leq x} \sum_{k=1}^{\nu_p(x)+1} p^{\alpha k} H^{(1-\alpha)}_{\left\lfloor \frac{x}{p^k}\right\rfloor} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p}\right) \\ S_2^{(\alpha+1)}(x) &= \sum_{3\leq p\leq x} \sum_{k=1}^{\nu_p(x)+1} 2^{\alpha-1} p^{\alpha k} (-1)^{\lfloor x/p^{k-1}\rfloor} H^{(1-\alpha)}_{\left\lfloor \frac{x}{2p^k}\right\rfloor} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p}\right).\end{aligned}$$

Then we prefer to work with the next form of Theorem 1.1 stated in terms of our new shorthand sum functions as follows:

$$\left| \sum_{d|x} \tau_x^{(\alpha+1)}(d) \right| = \left| H_x^{(1-\alpha)} - \sigma_\alpha(x) + S_1^{(\alpha+1)}(x) + S_2^{(\alpha+1)}(x) \right|. \tag{2}$$

The statement of the theorem given in (2) is important and significant since it implies deep connections between the sum-of-divisors functions, the generalized Mertens summatory functions, and the partial sums of the Riemann zeta function for real $\alpha < 0$, each related to one another in a convolved formula taken over sums of successive powers of the primes $p \leq x$. Thus we immediately see new relations from the restatement of the key results in [12] above. Moreover, from the previous result, we then obtain our main new results in the article given in the results in the next section as consequences of this restatement in terms of the Mertens functions.

2. New results and proofs of key lemmas

2.1. **Statement of the main theorem.**

Theorem 2.1 (The Limit Supremum of $M(x)$ and Its Values at Large Prime Powers)**.** *Let $x = q^r$ denote a large odd prime power for some $r \geq 1$. Then we have that*

$$\limsup_{\substack{x\to\infty \\ x=q^r}} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

Proof (Sketch). The complete proof of the theorem is given at conclusion of this section. For now, we will elaborate on the key steps in proving the theorem. We begin by noting that

$$\left| \sum_{d|x} \tau_x^{(\alpha+1)}(d) \right| \leq \left| T_v^{(\alpha)}(x) \right| + \left(2 \cdot \sup_{1\leq i\leq x} |M_\alpha^*(i)| + x^\alpha \right) \times D_v^{(\alpha)}(x) - M_\alpha^*(1) d_x^{(\alpha)}(1),$$

where the upper bound is obtained by summation by parts. We then need to show that infinitely and predictably often at least (and not necessarily for all large x) that we can bound the ratio of the next sums by $x \log\log x$. We consider the cases of large x when $x := q^r$ is a large prime power for some $r \geq 1$ and employ the resulting expansions to complete our proof. The next step in the proof is to show that (2) is approximately

$$\begin{aligned}\left| \sum_{d|x} \tau_x^{(\alpha+1)}(d) \right| &= \left| H_x^{(1-\alpha)} - \sigma_\alpha(x) + S_1^{(\alpha+1)}(x) + S_2^{(\alpha+1)}(x) \right| \\ &\geq \widetilde{C} \cdot x \log\log(x-1),\end{aligned}$$

where for sufficiently large x we can take $\widetilde{C} = \frac{3}{2}$. We then prove by a key (and not at all obvious) construction in Proposition 2.7 that the functions $|T_v^{(0)}(x)| \leq x \log\log\log x$ and $D_v^{(0)}(x) \leq O(\sqrt{x})$ when x is a sufficiently large prime. Then for all large primes x we have that

$$\frac{1}{\sqrt{x}}\left(\sup_{1\leq i\leq x}|M(i)|\right)\geq \log\log(x-1).$$

Thus as the lower bound stated in the previous equations increases with x and tends to infinity infinitely often, i.e., whenever we input x as one of our large primes, we see that the right-hand-side supremum must tend to infinity infinitely often as well. This is the basic sketch of the argument we will employ when we give the full proof of Theorem 2.1 in the next subsections. For now, we need to develop more machinery and state several lemmas to establish this claim. □

2.2. **Key asymptotic bounds and formulas.**

Definition 2.2 (Prime Power Indicator Functions)**.** For $x \geq 1$, let $\chi_{\mathrm{pp}}(x)$ denote the indicator function for prime powers, i.e., the function defined precisely as

$$\chi_{\mathrm{pp}}(x) = \begin{cases} 1, & \text{if } x = p^k \text{ for some prime } p\geq 2 \text{ and } k\geq 1; \\ 0, & \text{otherwise,} \end{cases}$$

and define the composite indicator function for the prime powers $p^k, 2p^k$ as follows where $\chi_{\mathrm{pp}}(x)=0$ if $x\in\mathbb{Q}\setminus\mathbb{Z}$[1]:

$$\begin{aligned}\widetilde{\chi}_{\mathrm{pp}}(x) &= [\chi_{\mathrm{pp}}(x)=0]_{\delta}\left[\chi_{\mathrm{pp}}\left(\frac{x}{2}\right)=0\right]_{\delta} \\ &= (1-\chi_{\mathrm{pp}}(x))\left(1-\chi_{\mathrm{pp}}\left(\frac{x}{2}\right)\right).\end{aligned}$$

Lemma 2.3 (Exact Formulas for the Divisor Sums $\sum_{d|x}\tau_x^{(\alpha)}(d)$)**.** *Let $v(x,d)$ be a boolean-valued function whose negation is given by $\neg v(x,d)$. For $\alpha\in\mathbb{N}$, $m\geq 1$, and $x\geq 12$, let*

$$d_{x,v}^{(\alpha)}(m) = \sum_{k=1}^{x}\sum_{\substack{d|k \\ v(x,d)}}\sum_{r|(d,x)} r^{\alpha+1}\left(\frac{k}{d}\right)^{\alpha}\widetilde{\chi}_{\mathrm{pp}}(d)\left[m=\frac{d}{r}\right]_{\delta},$$

and let

$$T_v^{(\alpha)}(x) = \sum_{k=1}^{x}\sum_{\substack{d|k \\ \neg v(x,d)}}\sum_{r|(d,x)} r\cdot\mu(d/r)\widetilde{\chi}_{\mathrm{pp}}(d)\cdot k^{\alpha}.$$

Then for any condition v on the divisors in the above sums, we can expand the divisor sums in Theorem 1.1 exactly in the following forms:

$$\sum_{d|x}\tau_x^{(\alpha+1)}(d) = T_v^{(\alpha)}(x) + \sum_{m=1}^{x}\mu(m)m^{\alpha}\cdot d_{x,v}^{(\alpha)}(m).$$

Proof. We start with the following formula for computing the divisor sum over $\tau_x^{(\alpha)}(d)$ from [12, §2]:

$$\begin{aligned}\sum_{d|x}\tau_x^{(\alpha+1)}(d) &= [q^x]\left(\sum_{k=1}^{x}\sum_{d|k}\sum_{r|d}\frac{r\cdot\widetilde{\chi}_{\mathrm{pp}}(d)\cdot\mu(d/r)}{(1-q^r)}k^{\alpha}\right) \\ &= \sum_{k=1}^{x}\sum_{r|x}\sum_{d|k} r\cdot\widetilde{\chi}_{\mathrm{pp}}(d)\cdot\mu(d/r)\cdot[r|d]_{\delta}\cdot k^{\alpha} \\ &= \sum_{k=1}^{x}\sum_{d|k}\sum_{r|(d,x)} r\cdot\widetilde{\chi}_{\mathrm{pp}}(d)\cdot\mu(d/r)\cdot k^{\alpha} \qquad (3) \\ &= T_v^{(\alpha)}(x) + \sum_{m=1}^{x}\mu(m)m^{\alpha}\cdot d_{x,v}^{(\alpha)}(m). \qquad (4)\end{aligned}$$

[1] Notation: We use *Iverson's convention* $[\mathtt{cond}=\mathtt{True}]_{\delta}\equiv\delta_{\mathtt{cond},\mathtt{True}}$ according to whether the condition cond is true or false where $\delta_{n,k}$ denotes Kronecker's delta function.

We can also expand the right-hand-side of (3) as

$$\sum_{d|x} \tau_x^{(\alpha+1)}(d) = \sum_{d=1}^{x} \left(\sum_{r|(d,x)} r\mu(d/r) \right) \widetilde{\chi}_{\mathrm{PP}}(d) H^{(-\alpha)}_{\left\lfloor \frac{x}{d} \right\rfloor},$$

which for x a target prime simplifies substatially to

$$\begin{aligned} \sum_{d|x} \tau_x^{(\alpha+1)}(d) &= \sum_{d=1}^{x} \mu(d) \widetilde{\chi}_{\mathrm{PP}}(d) H^{(-\alpha)}_{\left\lfloor \frac{x}{d} \right\rfloor} \\ &= \sum_{d=2}^{x} \mu(d) H^{(-\alpha)}_{\left\lfloor \frac{x}{d} \right\rfloor} + \sum_{2 \leq p \leq x} H^{(-\alpha)}_{\left\lfloor \frac{x}{p} \right\rfloor} - \sum_{3 \leq p \leq x/2} H^{(-\alpha)}_{\left\lfloor \frac{x}{2p} \right\rfloor}. \end{aligned} \tag{5}$$

We notice that the left-hand-side divisor sum in the previous equations is given by the large-order case of $T^{(\alpha)}_{d \leq x}(x)$, or say $T^{(\alpha)}_{d \leq \infty}(x)$ to distinguish between our cases. The delicate balance that the role of the parameter v plays in bounding this divisor sum motivates the result in Proposition 2.7. □

Remark 2.4. A few remarks about these special divisor sums before we continue:

▸ *A brief explanation of notation.* Notice that the boolean-valued condition functions $v(x, d)$ in the definitions of $T_v^{(\alpha)}(x)$ and $d_{x,v}^{(\alpha)}(m)$ (and then in the implied definition of the function $D_v^{(\alpha)}(x)$ given in the next lemma) we are effectively "splitting the difference" in which divisors we choose to include in each sum variation. We add the over-head notation to indicate this distinction so that we can divide the treatment of the complete bounds we require to complete the proof in Section 2.3 into two distinct cases – one for each resulting function case. In other words, the functions $v(x, d)$ allow us to attribute the corresponding divisor forms in each sum variant we employ in the article according to whether they lie in some interval defined by x and their other properties such as squarefree-ness. The absolute condition function v which we define in Proposition 2.7 below allows us to split the divisors d in the two functions $T_v^{(\alpha)}(x)$ and $D_v^{(\alpha)}(x)$ so that we can bound the magnitude of each sum as appropriate functions of x that ensure that the next formulas for the supremums of the Mertens function $|M(n)|$ taken over each interval $n \leq x$ given in Lemma 2.5 do not lead to trivial bounds on the function, i.e., so that the lower bounds for these sups are both positive and tending to infinity along a sequence of large odd primes x.

▸ *Connection to Ramanujan's Sum.* We have a deep connection between the divisor sums in Lemma 2.3 and Ramanujan's sum $c_q(n)$ given by

$$\begin{aligned} \sum_{d|x} \tau_x^{(1)}(d) &= \sum_{k=1}^{x} \sum_{d|k} \widetilde{\chi}_{\mathrm{PP}}(d) \cdot c_d(x) \\ &= \sum_{k=1}^{x} \sum_{d|k} \widetilde{\chi}_{\mathrm{PP}}(d) \cdot \mu\left(\frac{d}{(d,x)} \right) \frac{\varphi(d)}{\varphi\left(\frac{d}{(d,x)} \right)}, \end{aligned}$$

where $\varphi(x)$ denotes Euler's totient function. These identities follow by expanding out Ramanujan's sum in the form of [9, §27.10] [7, §A.7] [3, *cf.* §5.6]

$$c_q(n) = \sum_{d|(q,n)} d \cdot \mu(q/d),$$

and then applying the formula in (3) from the proof of the lemma above.

Lemma 2.5 (A Lower Bound for the Magnitude of $M(x)$)**.** *For fixed integers $\alpha \geq 0$ and $x \geq 1$, let*

$$D_v^{(\alpha)}(x) = \sum_{m=1}^{x} \left| d_{x,v}^{(\alpha)}(m) \right| = \sum_{k=1}^{x} \sum_{\substack{d|k \\ v(x,d)}} \sum_{r|(d,x)} r^{\alpha+1} \left(\frac{k}{d} \right)^{\alpha} \widetilde{\chi}_{\mathrm{PP}}(d).$$

For all sufficiently large $x \geq 14$ and any $v \geq 0$, we have the following bound on the supremum of $|M(i)|$ taken over all $i \leq x$:

$$\frac{\left| \sum_{d|x} \tau_x^{(1)}(d) \right| - \left| T_v^{(0)}(x) \right| + d_{x,v}^{(0)}(1)}{2 \cdot D_v^{(0)}(x)} - \frac{x^{\alpha}}{2} \leq \sup_{1 \leq i \leq x} |M(i)|.$$

Proof. We first observe the equivalences of the sums for (2) given in Lemma 2.3. For fixed x, we then proceed from here by summation by parts to obtain that

$$\begin{aligned}
\left|\sum_{d|x} \tau_x^{(\alpha+1)}(d)\right| &\leq \left|T_v^{(\alpha)}(x)\right| + \left|\sum_{m=1}^{x} \mu(m) m^{\alpha} \cdot d_{x,v}^{(\alpha)}(m)\right| \\
&\leq \left|T_v^{(\alpha)}(x)\right| + \left|M_\alpha^*(x)\right| d_{x,v}^{(\alpha)}(x) + \sum_{m=1}^{x-1} \left|M_\alpha^*(x)\right| \left|d_{x,v}^{(\alpha)}(m+1) - d_{x,v}^{(\alpha)}(m)\right| \\
&\leq \left|T_v^{(\alpha)}(x)\right| + 2\sum_{m=1}^{x} \left|M_\alpha^*(x)\right| d_{x,v}^{(\alpha)}(m) + \sum_{m=1}^{x} m^{\alpha} |\mu(m)| \cdot |d_{x,v}^{(\alpha)}(m)| - M_\alpha^*(1) d_{x,v}^{(\alpha)}(1) \\
&\leq \left|T_v^{(\alpha)}(x)\right| + \left(2 \sup_{1 \leq i \leq x} |M_\alpha^*(i)| + x^{\alpha}\right) \cdot D_v^{(\alpha)}(x) - d_{x,v}^{(\alpha)}(1)
\end{aligned}$$

We then consider the special case when $\alpha := 0$ to obtain the correct bound for the Mertens function $M(x)$ stated above. □

Remark 2.6 (Appropriate Condition Functions v)**.** There is a delicate balancing act we must now play to ensure that we can bound the supremum of the $|M(i)|$ for $i \leq x$ in the lemma below by an increasingly large function of x tending to infinity. Since (3) implies that

$$\sum_{d|x} \tau_x^{(\alpha+1)}(d) = T_{d \leq \infty}^{(\alpha)}(x),$$

we can always subtract off our factor of $T_v^{(\alpha)}(x)$ and obtain a correct bound in the form of Lemma 2.5. In particular, we need to select a function $v \equiv v(x,d)$ such that the following two properties hold whenever $x = q^r$ is a power of a large odd prime for some $r \geq 1$:

- **(P1)** $T_v^{(0)}(x) \leq x \log\log\log x$, and
- **(P2)** $D_v^{(0)}(x) = O\left(x^{1/2-\delta} \log\log x\right)$ for some $\delta > 0$, *or alternately*, $D_v^{(0)}(x) = O\left(\sqrt{x}\right)$.

We will require the next Proposition 2.7 in the complete proof of Theorem 2.1 given in the next subsection.

Proposition 2.7 (The Absolute Condition Function v)**.** *Let the condition function v defined by*

$$v(x,d) \equiv d \in S_{D,x} \wedge d \text{ is squarefree} \vee d = p^k, 2p^k,$$

where we construct the non-empty set $S_{D,x}$ according to the proof below such that $|S_{D,x}| \leq x^{2/3}$. Then for all large odd primes x we have that

i. $D_v^{(0)}(x) \leq O\left(\sqrt{x}\right)$, *and*
ii. $T_v^{(0)}(x) \leq \frac{x}{2} \log\log x$.

Proof (By Heuristic). We first sketch our proof by presenting a constructive heuristic that allows us to verify that it is indeed possible to obtain the bounds roughly of the form in (i) and (ii) above when x is taken to be a sufficiently large odd prime.

▸ Since the factor of $\widetilde{\chi}_{\rm pp}(d)$ sets all divisors of the form $d = p^k, 2p^k$ to zero for p a prime with $k \geq 1$, we reserve the definition of the appropriately-sized set $S_{D,x}$ consisting of squarefree integers such that $d \neq p^k, 2p^k$ in the following expansion to absorb any remaining factors after we bound the function $T_v^{(0)}(x)$ below:

$$D_v^{(0)}(x) = \sum_{k=1}^{x} \sum_{\substack{d|k \\ v(x,d)}} \sum_{r|(d,x)} r \widetilde{\chi}_{\rm pp}(d) = \sum_{1 \leq k \leq x} \sum_{\substack{d|k \\ d \in S_{D,x}}} 1 \leq \sum_{d \in S_{D,x}} \frac{x}{d}.$$

The idea here is that since

$$x \int_1^x \frac{dt}{t^{3/2}} = O(\sqrt{x}),$$

we can select any convenient subset of approximately $x^{2/3}$ of the divisors in the range $1 \leq d \leq x$ as our set $S_{D,x}$ and obtain a correct bound of the form in (i).

▸ The procedure for obtaining the desired bound in (ii) is more subtle and involved. We first cite a result of Erdös which provides an asymptotic formula for the number of squarefree integers $n \leq x$ with exactly k prime divisors where $y = \log\log x$ [2, p. 10]:

$$\pi_k''(x) = \frac{6}{\pi^2}(1+o(1))\frac{x}{\log x} \cdot \frac{y^{k-1}}{(k-1)!}. \tag{6}$$

Next, we recall that

$$\left|T_v^{(0)}(x)\right| = \left|\sum_{k=1}^{x} \sum_{\substack{d \mid k \\ d \notin S_{D,x}}} \mu(d)\widetilde{\chi}_{\mathrm{pp}}(d)\right| \leq \left|\sum_{\substack{d=1 \\ d \in S_{T,x}}}^{x} \frac{6}{\pi^2}\frac{x \cdot \mu(d)}{d}\right|. \tag{7}$$

The idea here is to use the signed-ness of $\mu(d)$ to induce enough cancellation in the reciprocal divisor sums that we get out stated bound in (ii). With this in mind, let the counts $B_{2k}(x)$ and $B_{2k+1}(x)$ denote the respective numbers of squarefree integers in $S_{T,x} := [1, x] \setminus S_{D,x}$ whose number of distinct prime factors are even (odd), i.e., according to the resulting sign of the Möbius function at these divisors d. Then we have the following estimates from Erdös' bound in (6):

$$\begin{aligned}
B_{2k}(x) &\leq \sum_{k=0}^{\infty} \pi_{2k+2}''(x) = \sum_{k=0}^{\infty} \frac{6}{\pi^2}(1+o(1))\frac{x}{\log x} \cdot \frac{y^{2k+1}}{(2k+1)!} \\
&= \frac{6}{\pi^2}\frac{(1+o(1))}{2}\frac{x}{\log x}\left(e^y - e^{-y}\right) \\
&= \frac{6}{\pi^2}\frac{(1+o(1))}{2}x\left(1 - \frac{1}{(\log x)^2}\right) \\
B_{2k+1}(x) &\leq \sum_{k=0}^{\infty} \pi_{2k+3}''(x) = \sum_{k=0}^{\infty} \frac{6}{\pi^2}(1+o(1))\frac{x}{\log x} \cdot \frac{y^{2k+2}}{(2k+2)!} \\
&= \frac{6}{\pi^2}\frac{(1+o(1))}{2}\frac{x}{\log x}\left(e^y + e^{-y} - 2\right) \\
&= \frac{6}{\pi^2}\frac{(1+o(1))}{2}x\left(1 + \frac{1}{(\log x)^2} - \frac{2}{\log x}\right) \\
&\sim \frac{3x}{\pi^2}\left(\frac{(\log x)^2 + 1}{(\log x)^2}\right).
\end{aligned}$$

In order to overestimate the bound on our sum in (7) we suppose that we can place all of the divisors in $S_{T,x}$ with an even number of prime factors at the beginning of the interval, and then place all of the divisors with an odd number of prime divisors in the last portion of the interval (of course in reality, this is just a heuristic and these divisors are actually interleaved in less predictable alignments). More precisely, we bound the sum

$$\begin{aligned}
\left|\sum_{d=1}^{B_{2k}(x)} \frac{6}{\pi^2}\frac{x}{d} - \sum_{d=x-1-B_{2k+1}(x)}^{x} \frac{6}{\pi^2}\frac{x}{d}\right| &= \left|\frac{6x}{\pi^2}\log\left\{\frac{3}{\pi^2}\left(1 - \frac{1}{(\log x)^2}\right)\left(1 - \frac{3}{((x-1)\pi^2 - 3)(\log x)^2}\right)\right\}\right| \\
&\leq \frac{x}{2}\log\log x, \quad \text{for } x \gg 1.
\end{aligned}$$

In practice, we can choose the constant on the right-hand-side of (ii) and in the previous equation to be slightly less than 1/2. Thus by just a hair, we have proved our required bounds. □

Proposition 2.8 (Asymptotic Bound for the Tau Function Divisor Sum). *Let $x = q^r$ denote a power of the large prime q for some $r \geq 1$. Then when the x tending to infinity of these forms is sufficiently large, we obtain*

$$\left|\sum_{d \mid x} \tau_x^{(\alpha+1)}(d)\right| \geq \widetilde{C} \cdot x \log\log(x-1),$$

for some absolute constant which we may effectively take as $\widetilde{C} = \frac{3}{2}$ for x sufficiently large.

Proof. By the statement of Theorem 1.1 rephrased in (2), we see that

$$\left|\sum_{d \mid x} \tau_x^{(1)}(d)\right| = \left|x - \sigma_1(x) + S_1^{(1)}(x) + S_2^{(1)}(x)\right|$$

$$\begin{aligned}
&= \left| S_1^{(1)}(q^r) + S_2^{(1)}(q^r) - \left(1 + q + \cdots + q^{r-1}\right) \right| \\
&\geq \left| S_1^{(1)}(q^r) + S_2^{(1)}(q^r) \right| - \frac{x-1}{x^{\frac{1}{r}} - 1}.
\end{aligned}$$

We next use the result of *Mertens' theorem* which implies that [7, §6.3] [1, §4.9] [3, §22.8] [9, §27.11]

$$\sum_{p \leq x} \frac{1}{p} = \log\log(x) + A + O\left(\frac{1}{\log x}\right),$$

where A is a limiting constant. In particular, when x is large we can expand the sum for $S_1^{(1)}(x)$ as

$$\begin{aligned}
S_1^{(1)}(x) &= \sum_{\substack{2 \leq p < q^r \\ p \neq q}} p \cdot \left\lfloor \frac{q^r}{p} \right\rfloor \left(\left\lfloor \frac{q^r}{p} \right\rfloor - \left\lfloor \frac{q^r}{p} - \frac{1}{p} \right\rfloor - \frac{1}{p} \right) + \sum_{k=1}^{r+1} q^k \left\lfloor q^{r-k} \right\rfloor \left(\left\lfloor q^{r-k} \right\rfloor - \left\lfloor q^{r-k} - \frac{1}{q} \right\rfloor - \frac{1}{q} \right) \\
&= \sum_{\substack{2 \leq p < q^r \\ p \neq q}} -\frac{p}{p} \left[\frac{q^r}{p} - \left\{ \frac{q^r}{p} \right\} \right] + \sum_{k=1}^{r} q^r \left(1 - \frac{1}{q}\right) \\
&= C_1 \pi(q^r - 1) - q^r \left(\log\log(q^r - 1) + A + O\left(\frac{1}{\log(q^r - 1)}\right)\right) + r \cdot q^{r-1}(q-1) \\
&\sim \frac{C_1(x-1)}{\log(x-1)} - x\left(\log\log(x-1) + A\right) + r \cdot x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1\right),
\end{aligned}$$

and similarly, the sum $S_2^{(1)}(x)$ is expanded as

$$\begin{aligned}
S_2^{(1)}(x) &= \sum_{\substack{2 \leq p < q^r \\ p \neq q}} -p \left\lfloor \frac{q^r}{2p} \right\rfloor \cdot \frac{1}{p} - \sum_{k=1}^{r} q^k \left\lfloor \frac{q^{r-k}}{2} \right\rfloor \frac{(q-1)}{q} - 2C_3 \left\lfloor \frac{q^r}{4} \right\rfloor \cdot \frac{1}{2} \\
&= \sum_{\substack{2 \leq p < q^r \\ p \neq q}} -\left(\frac{q^r}{2p} - C_4 \right) - \sum_{k=1}^{r} q^k \left(\frac{q^{r-k}}{2} - C_5 \right) \frac{(q-1)}{q} - C_3 \left(\frac{q^r}{4} - C_6 \right) \\
&\sim -\frac{1}{2} x \left(\log\log(x-1) + A\right) - \frac{C_2 C_4 (x-1)}{\log(x-1)} - \frac{r}{2} x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1\right) + C_5(x-1) - \frac{C_3}{4} x + C_3 C_6.
\end{aligned}$$

Hence when we add these two sums cancellation of symmetric terms results in

$$\begin{aligned}
S_1^{(1)}(x) + S_2^{(1)}(x) \sim \frac{(C_1 - C_2 C_4)(x-1)}{\log(x-1)} &- \frac{3}{2} x \left(\log\log(x-1) + A\right) \\
&+ \frac{r}{2} x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1\right) - \frac{C_3}{4} x + C_3 C_6,
\end{aligned}$$

which proves our result. In the previous equation, we have that each constant $0 \leq C_i < 1$ since the fractional parts corresponding to the floor function terms in the respective bounds for $S_i^{(\alpha+1)}(x)$ are in this range. $\square$

2.3. **The complete proof of Theorem 2.1.** We are now at the point where we can assemble the complete results necessary to prove Theorem 2.1. The key idea here is that while the value of $|M(x)|$ is oscillating with x, we can bound the value of $\sup_{1 \leq i \leq x} |M(i)|$ below by something increasingly large and tending to infinity infinitely often, i.e., since there are an infinitude of large primes $q \to \infty$. Then using the lower bound in Lemma 2.5, and combining the bounds in Proposition 2.8 and Proposition 2.7, we see that when $x = q$ is large we have

$$\frac{1}{\sqrt{x}} \left(\sup_{1 \leq i \leq x} |M(i)| \right) \geq \widetilde{C} \cdot \log\log(x-1) - \log\log\log x. \tag{8}$$

Next, for $x = q$ a large odd prime q let

$$x_{0,q} = \operatorname*{argmax}_{1 \leq i \leq q} |M(i)|.$$

Then we see from (8) that

$$\frac{|M(x_{0,q})|}{\sqrt{x_{0,q}}} \geq \frac{|M(x_{0,q})|}{\sqrt{q}} \geq \log\log(q-1).$$

Moreover, since the lower bound in (8) and in the previous equation is increasing with q, i.e., as $q \to \infty$, we see that the non-decreasing sequence of $x_{0,q}$ must gradually increase with larger and larger q. Thus we see that for any $L > 0$, there are infinitely many $x \in \mathbb{N}$ such that $|M(x)|/\sqrt{x} > L$. Hence the result is proved.

3. Generalizations to Higher-Order (Weighted) Mertens Functions

We remark that Theorem 2.1 can be effectively generalized to a conjecture of the more general form

$$\limsup_{\substack{x\to\infty \\ x=q^r}} \frac{|M_\alpha^*(x)|}{(\sqrt{x})^{2\alpha+1}} = +\infty.$$

The only caveat here is that we need to know more precise forms of Mertens' theorem for general sums of the form $\sum_{p\leq x} p^{\alpha k}$ depending on the parameter $\alpha \geq 0$. This generalization is a simple enough jump to make when we consider $\alpha \in \mathbb{Z}^+$. In particular, by the integral form of the summation by parts formula in [13, §2.1], where we take $\pi(x) = x/\log x + O(\sqrt{x})$, we see that

$$\sum_{p\leq x} p^\alpha = \pi(x)x^\alpha - \alpha\,\mathrm{Ei}((\alpha+2)\log x) + O_\alpha\left(x^{\alpha+1/2}\right),$$

where $\mathrm{Ei}(w)$ denotes the *exponential integral function* [9, §6]. The exponential integral satisfies the following asymptotic expansion for any natural number $N \geq 1$:

$$\mathrm{Ei}(w) = \frac{e^{-w}}{w} \times \sum_{n=0}^{N-1} \frac{(-1)^n n!}{w^n} + O\left(N! w^{-N}\right).$$

Then we see that

$$\sum_{p\leq x} p^\alpha = \frac{x^{\alpha+1}}{\log x} - \frac{\alpha}{x^{\alpha+2}\log(x^{\alpha+2})} + O\left(x^{\alpha+1/2}\right),\ \alpha \geq 1.$$

The next key component in formulating the generalized result is the expansion given in (5). Finally, for x prime, the functions $D_v^{(\alpha)}(x)$ and $T_v^{(\alpha)}(x)$ can be expanded as in in (5) in the forms of (TODO: Check)

$$D_v^{(\alpha)}(x) = \sum_{\substack{d=1 \\ v(x,d)}}^{x} \frac{\widetilde{\chi}_{\mathrm{PP}}(d)}{d^\alpha} H_{\lfloor x/d \rfloor}^{(-\alpha)}$$

$$T_v^{(\alpha)}(x) = \sum_{\substack{d=1 \\ \neg v(x,d)}}^{x} \widetilde{\chi}_{\mathrm{PP}}(d)\mu(d) H_{\lfloor x/d \rfloor}^{(-\alpha)},$$

where the generalized harmonic numbers in the previous two equations correspond to the polynomials in (1) whose coefficients are rational multiples of the Bernoulli numbers. These formulas suggest a nice generalized analog to Proposition 2.7 for higher-order $\alpha \geq 1$. For now, we will leave the remainder of the generalizations of our main theorem as an exercise for future research on the generalized Mertens summatory functions $M_\alpha^*(x)$ defined in the introduction.

4. Short Appendix: The proof of Theorem 1.1

Given that the core elements of the proof of the main result in Theorem 2.1 first proved in this article follow from a careful asymptotic treatment of the first results in Theorem 1.1, and its restatement given in (2), we provide this appendix on the proof of these first results given by the author in 2017 [12]. The next results present a concise statement of the key results in the proof of that theorem given in the original reference.

5. Conclusions

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School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332
E-mail address: maxieds@gmail.com, mschmidt34@gatech.edu