$$\sim \frac{4\sqrt{2\pi}\cdot x(\log\log x)^{k-1/2}}{(2k-1)(k-1)!} + \frac{2\sqrt{2\pi}\cdot x\Gamma\left(k-\frac{1}{2},\log\log x\right)}{(k-1)!} - \frac{2\sqrt{2\pi}\cdot x\Gamma\left(k-\frac{3}{2},\log\log x\right)}{(k-1)!}$$
$$\sim \frac{4\sqrt{2\pi}\cdot x(\log\log x)^{k-1/2}}{(2k-1)(k-1)!} \qquad \square$$

4.2 Average orders of the unsigned sequences

Lemma 4.4. *As $x \to \infty$, we have that*

$$\left|\sum_{n\leq x}(-1)^{\omega(n)}\right| \ll \frac{x}{\sqrt{\log\log x}}.$$

Proof. An adaptation of the proof of Lemma A.3 from the appendix shows that for any fixed $a \in (1, 1.76323]$ we have that

$$\frac{x}{\log x} \times \left|\sum_{k=1}^{a\log\log x} \frac{(-1)^k(\log\log x)^{k-1}}{(k-1)!}\right| = \frac{\sqrt{a}\cdot x}{2\sqrt{2\pi}} \cdot \frac{(\log x)^{a-1-a\log a}}{\sqrt{\log\log x}}\left(1 + O\left(\frac{1}{\log\log x}\right)\right). \tag{20}$$

Suppose that we take $a := 3/2$ so that $a - 1 - a\log a = \frac{1}{2}\left(1 - 3\log\left(\frac{3}{2}\right)\right) \approx -0.108198$. We can write the summatory function

$$L_{**}(x) := \left|\sum_{n\leq x}(-1)^{\omega(n)}\right| = \left|\sum_{k\leq\log x}(-1)^k\pi_k(x)\right|.$$

By the uniform asymptotics for $\pi_k(x)$ as $x \to \infty$ when $1 \leq k \leq R\log\log x$ for $1 \leq R < 2$ guaranteed by the results from Remark 2.5, we have by Lemma A.3 (from the appendix) and (20) above that for large x

$$L_{**}(x) \ll \frac{x}{2\sqrt{2\pi\log\log x}} + \frac{\sqrt{3}\cdot x}{4\sqrt{\pi}(\log x)^{0.108198}\sqrt{\log\log x}} + \#\left\{n \leq x : \omega(x) \geq \frac{3}{2}\log\log x\right\} + O\left(\frac{x}{(\log\log x)^{3/2}}\right).$$

Similarly, by applying the second set of results stated in Remark 2.5, we see that

$$\#\left\{n \leq x : \omega(x) \geq \frac{3}{2}\log\log x\right\} \ll \frac{x}{(\log x)^{0.108198}}.$$

The result follows by removing constant factors from the main term in the second to last inequality above. $\square$

Proposition 4.5. *We have that as $n \to \infty$*

$$\mathbb{E}\left[C_{\Omega(n)}(n)\right] = \frac{2\sqrt{2\pi}(\log n)}{\sqrt{\log\log n}}(1 + o(1)).$$

Proof. We first compute the following summatory function by applying Corollary 4.3 and Lemma A.4 from the appendix:

$$\sum_{k=1}^{2\log\log x}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n) = \frac{2\sqrt{2\pi}\cdot x\log x}{\sqrt{\log\log x}} + O\left(\frac{x\log x}{(\log\log x)^{3/2}}\right). \tag{21}$$

We claim that

$$\frac{1}{x} \times \sum_{n\leq x} C_{\Omega(n)}(n) = \frac{1}{x} \times \sum_{k\geq 1}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n)$$

$$\frac{x}{\log x} \times \left| \sum_{k=1}^{a \log\log x} \frac{(-1)^k (\log\log x)^{k-1}}{(k-1)!} \right| = \frac{\sqrt{a} \cdot x}{2\sqrt{2\pi}} \cdot \frac{(\log x)^{a-1-a\log a}}{\sqrt{\log\log x}} \left(1 + O\left(\frac{1}{\log\log x}\right)\right). \tag{20}$$

$$\sum_{k=0}^{a\lambda} \frac{(-\lambda)^k}{k!} = C_a \frac{e^{\lambda(a - a\log a)}}{\sqrt{\lambda}} \left(1 + O\left(\frac{1}{\log\log x}\right)\right)$$

by setting $\lambda = \log\log x$, so $\log x = e^{\lambda}$

A sufficient estimate is elementary

$\frac{\lambda^j}{j!}$ decreases geometrically for $j > a\lambda$, since $a > 1$

namely

$$\frac{\lambda^{j+1}/(j+1)!}{\lambda^j/j!} \leq \frac{\lambda}{j+1} \leq \frac{1}{a}$$

We can sum geometric eseries

So

$$\left|\sum_{j>\lambda a} \frac{(-\lambda)^j}{j!}\right| \leq \frac{\lambda^{\lambda a}}{\lambda a!} \frac{1}{1-1/a}$$

$$\lesssim_a \frac{\lambda^{\lambda a}}{(\lambda a)^{\lambda a} e^{-\lambda a} \sqrt{\lambda a}}$$

$$=_a \left(\frac{e}{a}\right)^{\lambda a} / \sqrt{\lambda}$$

For both $|t| \leq \mu_x(C)^{1/2}$ and $\mu_x(C)^{1/2} < |t| \leq \mu_x(C)^{2/3}$, we can see that

$$\frac{t}{\mu_x(C)} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Similarly, for both $|t| \leq 1$ and $|t| > 1$, we have that

$$\frac{t^2}{\mu_x(C)^2} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Let the corresponding error terms in x and t be denoted by

$$\widetilde{E}(x,t) := O\left(\frac{1}{\sigma_x(C)} + \frac{|t|^3}{\mu_x(C)^2}\right).$$

Combining these estimates with the previous computations, we deduce uniformly for $|t| \leq \mu_x(C)^{2/3}$ that

$$\frac{4\sqrt{2\pi}(\log\log x)^{k-\frac{1}{2}}}{(2k-1)(k-1)!} \sim \frac{\log x}{\sqrt{2\pi\log\log x}\cdot\sigma_x(C)} \times \exp\left(-\frac{t_x^2}{2\sigma_x(C)^2}\right) \times \left[1 + \widetilde{E}(x, t_x)\right].$$

It follows that uniformly for $1 \leq k \leq \log\log x$

$$\begin{aligned} f(k,x) &:= \frac{1}{x} \times \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \\ &\sim \frac{(\log x)}{\sqrt{2\pi\log\log x}\cdot\sigma_x(C)} \times \exp\left(-\frac{(k-\mu_x(C))^2\sqrt{\log\log x}}{2(\log x)\sigma_x(C)^2}\right) \times \left[1 + \widetilde{E}\left(x, \frac{|k-\mu_x(C)|\sqrt{\log\log x}}{(\log x)}\right)\right]. \end{aligned}$$

Since our target probability density function approximating the PDF (in t) of the normal distribution is given here by

$$\frac{f(k,x)\sqrt{\log\log x}}{(\log x)} \to \frac{1}{\sqrt{2\pi}\cdot\sigma_x(C)} \times \exp\left(-\frac{t^2}{2\sigma_x(C)^2}\right),$$

we perform the change of variable $t \mapsto \frac{t\sqrt{\log\log x}}{(\log x)}$ to obtain the form of our theorem stated above. By the same argument utilized in the proof of Proposition 4.5, we see that the contributions of these summatory functions for $k \leq \mu_x(C) - \mu_x(C)^{2/3}$ is negligible. We also require that $k \leq 2\log\log x$ for all large x as we required by Theorem 4.2. We then sum over a corresponding range of

$$\mu_x(C) - \mu_x(C)^{2/3} \leq k \leq \mu_x(C) + z\sigma_x(C),$$

to approximate the stated normalized densities. As $x \to \infty$ the three terms that result (one main term and two error terms, respectively) can be considered to each correspond to a Riemann sum for an associated integral whose limiting formula corresponds to a main term given by the standard normal CDF, $\Phi(z)$. □

Corollary 4.8. *Let $Y > 0$. Suppose that $\mu_x(C)$ and $\sigma_x(C)$ are defined as in Theorem 4.7 for large $x > e$. For $Y > 0$ and we have uniformly for all $-Y \leq y \leq Y$ that as $x \to \infty$*

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : |g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq y\right\} = \Phi\left\{\frac{6\sigma_x(C)}{\pi^2}\left(\frac{\pi^2 y}{6} + \sigma_x(C)\right) - \frac{6}{\pi^2}\log\left(4\sqrt{2\pi}\right)\right\} + o(1).$$

Proof. We claim that

$$|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2}C_{\Omega(n)}(n), \text{ as } n \to \infty.$$

This is a very strange CLT

$$= \Phi\left\{\frac{6\sigma_x(C)}{\pi^2}\left(\frac{\pi^2 y}{6}+\sigma_x(C)\right) - \underbrace{\frac{6}{\pi^2}\log\left(4\sqrt{2\pi}\right)}_{=\,\theta}\right\} + ($$

$$\sigma_x(c)y + \frac{6\,\sigma_x(c)^2}{\pi^2} - \theta$$

$$= \sigma_x(c)\left\{y + \frac{6\,\sigma_x(c)}{\pi^2} - \theta/\sigma_x(c)\right\}$$

This means $|g^{-1}(n)|$ becomes more concentrated around $\frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|$ as $n \to \infty$.

So this is a very odd conclusion. Very hard for me to beleive it is true.

Up to some smaller errors, you are saying

$$\sqrt{\vartheta_x(C)}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|\right) \longrightarrow N(0,1)$$

It does <u>not</u> concentrate around the mean?

The std. deviation decreases with n?

Nothing like any CLT I have seen before.

As in the proof of Corollary 4.6, we obtain that

$$\frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| = \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(x)}(x)] + \sum_{d<x} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O(1).$$

Let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) := f(x) - f(x-1)$. We see that for large n (*cf.* last lines in the proof of Corollary 4.6)

$$\begin{aligned}
|g^{-1}(n)| &= \Delta_n(n \cdot \mathbb{E}|g^{-1}(n)|) \sim \Delta_n\left(\sum_{d \leq n} \frac{6}{\pi^2} \cdot C_{\Omega(d)}(d) \cdot \frac{n}{d}\right) \\
&= \frac{6}{\pi^2}\left[C_{\Omega(n)}(n) + \sum_{d<n} C_{\Omega(d)}(d)\frac{n}{d} - \sum_{d<n} C_{\Omega(d)}(d)\frac{(n-1)}{d}\right] \\
&\sim \frac{6}{\pi^2}C_{\Omega(n)}(n) + \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n-1)|, \text{ as } n \to \infty.
\end{aligned}$$

Since $\mathbb{E}|g^{-1}(n-1)| \sim \mathbb{E}|g^{-1}(n)|$ for all sufficiently large n, the result finally follows by a normalization of Theorem 4.7. $\square$

4.4 Probabilistic interpretations

Assuming Cor 4.8, this is not right.

Lemma 4.9. *For all x sufficiently large, if we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds:*

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}(\log\log x)\right) = \frac{1}{2} + o(1) \tag{A}$$

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}(\alpha + \log\log x)\right) = \Phi(\alpha) + o(1), \alpha \in \mathbb{R}. \tag{B}$$

?

Proof. Each of these results is a consequence of Corollary 4.8. The result in (A) follows since $\Phi(0) = \frac{1}{2}$ by taking

$$z = \frac{\left(\alpha + \frac{6}{\pi^2}\log\left(4\sqrt{2\pi}\right)\right)}{\sigma_x(C)} - \frac{6}{\pi^2}\sigma_x(C),$$

in Corollary 4.8 for $\alpha = 0$ for $\sigma_x(C) := \log\log x$. Note that as $\alpha \to +\infty$, we get that the right-hand-side of (B) tends to one for large $x \to \infty$. $\square$

It follows from Lemma 4.9 and Corollary 4.6 that

$$\lim_{x\to\infty} \frac{1}{x} \times \#\left\{n \leq x : |g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|(1 + o(1))\right\} = 1.$$

That is, for almost every sufficiently large integer n we recover that

$$|g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|(1 + o(1)).$$

$$= \Phi\left\{ \underbrace{\frac{6\sigma_x(C)}{\pi^2}\left(\frac{\pi^2 y}{6}+\sigma_x(C)\right) - \frac{6}{\pi^2}\log\left(4\sqrt{2\pi}\right)}_{=0} \right\} + ($$

$$\frac{\pi^2 y}{6} + \sigma_x(C) = \frac{\log\left(4\sqrt{2\pi}\right)}{\sigma_x(C)}$$

$$y = \frac{6\log\left(4\sqrt{2\pi}\right)}{\pi^2 \sigma_x(C)} - \frac{6\sigma_x(C)}{\pi^2}$$

$$= \cancel{\frac{c}{\sqrt{\log\log x}}} - \frac{6\sqrt{\log\log x}}{\pi^2}$$

5 New formulas and limiting relations characterizing $M(x)$

5.1 Establishing initial asymptotic bounds on the summatory function $G^{-1}(x)$

Let $L(x) := \sum_{n \leq x} \lambda(n)$ for $x \geq 1$. A recent upper bound on $L(x)$ (assuming the RH) is proved by Humphries based on Soundararajan's result bounding $M(x)$. It is stated in the following form [6]:

$$L(x) = O\left(\sqrt{x} \times \exp\left((\log x)^{\frac{1}{2}} (\log\log x)^{\frac{5}{2}+\epsilon}\right)\right), \text{ for any } \epsilon > 0, \text{ as } x \rightarrow \infty. \tag{27}$$

Theorem 5.1. *We have that for almost every sufficiently large x, there exists $1 \leq t_0 \leq x$ such that*

$$G^{-1}(x) = O\left(L(t_0) \times \mathbb{E}|g^{-1}(x)|\right).$$

If the RH is true, then for any $\epsilon > 0$ and all large integers $x > e$ we have that

$$G^{-1}(x) = O\left(\frac{\sqrt{x}(\log x)^2}{\sqrt{\log\log x}} \times \exp\left(\sqrt{\log x}(\log\log x)^{\frac{5}{2}+\epsilon}\right)\right).$$

Proof. We write the next formulas for $G^{-1}(x)$ at almost every large $x > e$ by Abel summation and applying the mean value theorem:

$$\begin{aligned} G^{-1}(x) &= \sum_{n \leq x} \lambda(n)|g^{-1}(n)| \\ &= L(x)|g^{-1}(x)| - \int_1^x L(x) \frac{d}{dx} \left|g^{-1}(x)\right| dx \\ &= O\left(L(t_0) \times \mathbb{E}|g^{-1}(x)|\right), \text{ for some } t_0 \in [1, x]. \end{aligned} \tag{28}$$

The proof of this result appeals to the last few results we used to establish the probabilistic interpretations of the distribution of $|g^{-1}(n)|$ as $n \rightarrow \infty$ in Section 4.4.

We need to bound the sums of the maximal extreme values of $|g^{-1}(n)|$ over $n \leq x$ as $x \rightarrow \infty$ to prove the second claim. We know by a result of Robin that [22]

$$\omega(n) \ll \frac{\log n}{\log\log n}, \text{ as } n \rightarrow \infty.$$

Recall that the values of $|g^{-1}(n)|$ are locally maximized when n is squarefree with

$$|g^{-1}(n)| \leq \sum_{j=0}^{\omega(n)} \binom{\omega(n)}{j} \times j! \ll \Gamma(\omega(n)+1) \ll \left(\frac{\log n}{\log\log n}\right)^{\frac{\log n}{\log\log n}+\frac{1}{2}}.$$

Since we deduced that the set of $n \leq x$ on which $|g^{-1}(n)|$ is substantially larger than its average order is asymptotically thin at the end of the last section, we find the largest possible bounds asserting that

$$\begin{aligned} \left|\int_{x-o(1)}^x L'(t)|g^{-1}(t)|dt\right| &\ll \int_{x-o(1)}^x \left(\frac{\log t}{\log\log t}\right)^{\frac{\log t}{\log\log t}+\frac{1}{2}} dt = o\left(\left(\frac{\log x}{\log\log x}\right)^{\frac{\log x}{\log\log x}+\frac{1}{2}}\right) \\ &\ll o\left(\frac{x}{(\log x)^{m-1/2}(\log\log x)^r}\right), \text{ for any } m, r = o\left(\frac{\log\log\log x}{\log\log x}\right), \text{ as } x \rightarrow \infty. \end{aligned}$$

Indeed, we can see that the limit

$$\lim_{x\rightarrow\infty} \frac{1}{x}\left(\frac{\log x}{\log\log x}\right)^{\frac{\log x}{\log\log x}+\frac{1}{2}} (\log x)^{m-1/2}(\log\log x)^r \ll \lim_{x\rightarrow\infty} x^{-\frac{\log\log\log x}{\log\log x}} (\log x)^{m+r}$$

$$\int_1^x L(t)\,\frac{d}{dt}|g^{-1}(t)|\,dt$$

$$= x\,L(t_0)\,\mathbb{E}\,|g^{-1}(x)|$$

need the x for homogeneity reasons.

$$L(x)\,|g^{-1}(x)| = O\big(x\,L(t_0)\,\mathbb{E}|g^{-1}(x)|\big)$$

is false, since $|g^{-1}(x)|$ could be much larger than $\mathbb{E}\,|g^{-1}(x)|$

Proof. To prove (A), we first notice that for any $k \in [1, \sqrt{x}]$

$$\frac{\log\left(\frac{x}{k}\right)}{\log\left(\frac{x}{k+1}\right)} = \frac{1 - \frac{\log k}{\log x}}{1 - \frac{\log k}{\log x} + O\left(\frac{1}{k \log x}\right)} = 1 + O\left(\frac{1}{k \log x\left(1 - \frac{\log k}{\log x}\right)}\right) = 1 + o(1), \text{ as } x \to \infty.$$

Then for any $m \geq 0$ and k within these bounds, we see that

$$\begin{aligned}\frac{x}{k \cdot \log^m\left(\frac{x}{k}\right)} - \frac{x}{(k+1) \cdot \log^m\left(\frac{x}{k+1}\right)} &= \frac{x}{\log^m\left(\frac{x}{k+1}\right)}\left[\frac{(1+o(1))^m}{k} - \frac{1}{k+1}\right] \\ &\asymp \frac{x}{(\log x)^m}\left[\frac{1}{k(k+1)} + o\left(\frac{1}{k}\right)\right],\end{aligned}$$

where for any $k \in [1, \sqrt{x}]$ we have that $o\left(k^{-1}\right) = o(1)$ for all large $x \to \infty$.

To prove (B), notice that

$$\frac{x}{k(k+1)} - \frac{x}{k^2} = -\frac{x}{k^2(k+1)}.$$

Then we see that

$$\left|\int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^2(t+1)} dt\right| \leq \left|\int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^3} dt\right| = O(1). \qquad \square$$

Corollary 5.4. *We have that as $x \to \infty$*

$$M(x) = O\left(G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) + \frac{x}{\log x} \times \sum_{k \leq \sqrt{x}} \frac{G^{-1}(k)}{k^2} + (\log x)^2 \sqrt{\log\log x}\right).$$

Proof. We need to first bound the prime counting function differences in the formula given by Proposition 5.2. We will require the following known bounds on the prime counting function due to Rosser and Schoenfeld for all sufficiently large $x > 59$ [23, Thm. 1]:

$$\frac{x}{\log x}\left(1 + \frac{1}{2 \log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{3}{2 \log x}\right). \tag{30}$$

The bounds in (30) together with Lemma 5.3 implies that for $\sqrt{x} \leq k \leq \frac{x}{2}$

$$\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) = O\left(\frac{x}{k^2 \log\left(\frac{x}{k}\right)}\right). \tag{31}$$

??

I dont agree

We will rewrite the intermediate formula from the proof of Proposition 5.2 as a sum of two components with summands taken over disjoint intervals. For large $x > e$, let

$$\begin{aligned}S_1(x) &:= \sum_{1 \leq k \leq \sqrt{x}} g^{-1}(k) \pi\left(\frac{x}{k}\right) \\ S_2(x) &:= \sum_{\sqrt{x} < k \leq \frac{x}{2}} g^{-1}(k) \pi\left(\frac{x}{k}\right).\end{aligned}$$

We then assert by the asymptotic formulas for the prime counting function that

$$S_1(x) = O\left(\frac{x}{\log x} \times \sum_{k \leq \sqrt{x}} \frac{G^{-1}(k)}{k^2}\right).$$

The bounds in (30) together with Lemma 5.3 implies that for $\sqrt{x} \le k \le \frac{x}{2}$

$$\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) = O\left(\frac{x}{k^2 \log\left(\frac{x}{k}\right)}\right). \tag{31}$$

$$\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)$$

$$\le \frac{x/k}{\log x/k}\left(1 + \frac{3}{2\log x/k}\right)$$

$$- \frac{x/k+1}{\log x/k+1}\left(1 + \frac{1}{2\log x/k+1}\right)$$

$$\sim \frac{x/k}{(\log x/k)^2}$$