Determinants

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1 Main Proposition

Suppose

$$K(t) \;=\; (c_1\theta_1^{-t},\; c_2\theta_2^{-t},\; ...,\; c_r\theta_r^{-t}),$$

and suppose that

$$F \;\subseteq\; \{0, 1, ..., P-1\}^r$$

is the set of all vectors $(x_1, ..., x_r)$ such that if $(y_1, ..., y_r) = K(t)$, for some $t \in [0, 1)$, then exists $(x_1, ..., x_r) \in F$ such that

$$(y_1, ..., y_r) \;\in\; \frac{1}{P}(x_1, x_2, ..., x_r) + [0, 1/P)^r.$$

Note that if $c_1, ..., c_r \in (0, 1)$ and $\theta_1, ..., \theta_r > 0$, we will have that

$$|F| \;\gg\; P,$$

where the implied constant depends on $c_1, ..., c_r, \theta_1, ..., \theta_r$. Now, let

$$A_1,\; A_2,\; ...,\; A_r \;\subseteq\; \mathbb{F}_P,\; \text{with } |A_1|, ..., |A_r| \;\geq\; P^{1-\varepsilon}.$$

We claim that for all but at most $o(P)$ elements $(x_1, ..., x_r) \in F$, there exist

$$1 \leq n \leq P^{7r^3\varepsilon},\; \text{and } (\delta_1, ..., \delta_r) \in \{0, 1, ..., [P^{7r^3\varepsilon}]\}^r,$$

such that

$$n\cdot(x_1, ..., x_r) + (\delta_1, ..., \delta_r) \;\in\; (A_1{+}A_1{+}A_2)\times(A_2{+}A_2{+}A_2)\times\cdots\times(A_r{+}A_r{+}A_r). \tag{1}$$

1.1 Proof of the Main Proposition

1.1.1 Basic setup

Parseval tells us that

$$\sum_{0\leq s_1,...,s_r\leq P-1} |\widehat{1}_{A_1\times A_2\times\cdots\times A_r}(s_1, ..., s_r)|^2 \;=\; P^r|A_1|\cdots|A_r|.$$

Thus, if Q is the set of all places $(s_1, ..., s_r)$ where

$$|\widehat{1}_{A_1 \times A_2 \times \cdots \times A_r}(s_1, ..., s_r)| \ \geq \ P^{r(1-3\varepsilon)},$$

then

$$|Q| \ \leq \ P^{-2r(1-3\varepsilon)} P^r |A_1| \cdots |A_r| \ \leq \ P^{6r\varepsilon}$$

Let $Q' \subseteq Q$ be all those places $(s_1, ..., s_r) \in Q$, $|s_i| < P/2$, satisfying the stronger constraint that

$$|s_i| \ \leq \ P^{1-(7r^3-r)\varepsilon}, \ i = 1, 2, ..., r. \tag{2}$$

Let $N = |Q'|$, and note that

$$N \ \leq \ |Q| \ \leq \ P^{6r\varepsilon}.$$

Now we let E denote the set of all $(x_1, ..., x_r) \in F$, such that there exists $(s_1, ..., s_r) \in Q'$, $(s_1, ..., s_r) \neq (0, ..., 0)$, such that

$$\left\| \frac{(x_1, ..., x_r) \cdot (s_1, ..., s_r)}{P} \right\| \ = \ \left\| \frac{x_1 s_1 + \cdots + x_r s_r}{P} \right\| \ < \ \frac{1}{P^{(7r^3-2r)\varepsilon}}. \tag{3}$$

1.1.2 Theorem follows if we can show $|E| = o(P)$

We will show that $|E| = o(P)$. If this holds, then let us see how it implies the conclusion of the Proposition: let $L = [\log P]$,

$$U \ := \ \{0, 1, 2, ..., [P^{7r^3\varepsilon}/L]\}^r,$$

and define $g(\vec{\delta}) = g(\delta_1, ..., \delta_r)$ to be the following L-fold convolution

$$g(\vec{\delta}) \ := \ 1_{-U} * 1_{-U} * \cdots * 1_{-U}(\delta_1, ..., \delta_r).$$

Now, let

$$(x_1, ..., x_r) \ \in \ F \setminus E \tag{4}$$

be any of the $|F| - o(P)$ vectors such that (3) fails to hold, for every $(s_1, ..., s_r) \in Q'$. Let

$$M \ := \ [P^{7r^3\varepsilon}],$$

and let f be the indicator function for the set

$$\{(-nx_1, -nx_2, ..., -nx_r) \ : \ 1 \leq n \leq M\}.$$

Then, we have that if

$$1_{A_1 \times \cdots \times A_r} * 1_{A_1 \times \cdots \times A_r} * 1_{A_1 \times \cdots \times A_r} * g * f(\vec{0}) \ > \ 0, \tag{5}$$

then there exists $1 \leq n \leq M$ and $(\delta_1, ..., \delta_r)$, so that (1) holds.

Expressing the left-hand-side of (5) in terms of Fourier transforms, one sees that it equals:

$$\begin{aligned} P^{-r} \sum_{(s_1,...,s_r)\in\mathbb{F}_P^r} & \widehat{1}_{A_1\times\cdots\times A_r}(s_1,...,s_r)^3 \widehat{g}(s_1,...,s_r)\widehat{f}(s_1,...,s_r) \\ = \ & P^{-r} \sum_{\vec{s}\in\mathbb{F}_P^r} \widehat{1}_{A_1\times\cdots\times A_r}(\vec{s})^3 \widehat{1}_{-U}(\vec{s})^L \widehat{f}(\vec{s}). \end{aligned} \tag{6}$$

We split the terms in the second sum into the term with $(s_1,...,s_r) = (0,...,0)$, the terms $(s_1,...,s_r) \in Q$, and then the remaining terms.

The contribution of the term $(s_1,...,s_r) = (0,...,0)$ is

$$P^{-r} M |U|^L |A_1|^3 \cdots |A_r|^3. \tag{7}$$

Now suppose $(s_1,...,s_r) \in Q \setminus Q'$. Then, for some $i = 1,...,r$ we have that $P^{1-7r^3\varepsilon} < |s_i| < P/2$. Thus,

$$|\widehat{g}(\vec{s})| \ll \prod_{i=1}^{r} \min(|P^{7r^3\varepsilon}/L|^L, \|s_i/P\|^{-L}) < |P^{7r^3\varepsilon}/L|^{L(r-1)} P^{(7r^3-r)\varepsilon L} \le |U|^L (LP^{-r\varepsilon})^L.$$

It follows, then, that the contribution of all such $(s_1,...,s_r) \in Q \setminus Q'$ to the right-hand-side of (6) is bounded from above by

$$P^{-r} N |A_1|^3 \cdots |A_r|^3 |U|^L (LP^{-r\varepsilon})^L M,$$

which is much smaller than (7), on account of the $(LP^{-r\varepsilon})^L$ factor, even when using the crude upper bounds: $|A_i| \le P$, $i = 1,...,r$, and $M \le P^{7r^3\varepsilon}$, $N \le P^{6r\varepsilon}$.

Next, we consider the contribution of all terms with $(s_1,...,s_r) \in Q'$. Then, since $(x_1,...,x_r)$ satisfies (4), and in particular that it is not E, we have that

$$\begin{aligned} |\widehat{f}(s_1,...,s_r)| &= \left| \sum_{1\le n\le M} e^{2\pi i n (x_1,...,x_r)\cdot(s_1,...,s_r)/P} \right| \\ &\ll \frac{1}{\|(x_1,...,x_r)\cdot(s_1,...,s_r)/P\|} \\ &\le P^{(7r^3-2r)\varepsilon}. \end{aligned}$$

So, the contribution of the terms in (6) with $(s_1,...,s_r) \in Q'$, $(s_1,...,s_r) \ne (0,...,0)$, is, by Parseval,

$$\begin{aligned} &\ll P^{-r} P^{(7r^3-2r)\varepsilon} |U|^L \sum_{0\le s_1,...,s_r\le P-1} |\widehat{1}_{A_1\times\cdots\times A_r}(s_1,...,s_r)|^3 \\ &\le P^{-r+(7r^3-2r)\varepsilon} |U|^L |A_1|\cdots|A_r| \sum_{0\le s_1,...,s_r\le P-1} |\widehat{1}_{A_1\times\cdots\times A_r}(s_1,...,s_r)|^2 \\ &\le P^{(7r^3-2r)\varepsilon} |U|^L |A_1|^2 \cdots |A_r|^2 \\ &\ll P^{-(1+\varepsilon)r} M |U|^L |A_1|^3 \cdots |A_r|^3, \end{aligned}$$

which is smaller than the contribution of the term with $(s_1, ..., s_r) = (0, ..., 0)$ given in (7).

Finally, we consider the contribution of the remaining terms. For these terms we have

$$|\widehat{1}_{A_1\times\cdots\times A_r}(s_1, ..., s_r)| \ < \ P^{r(1-3\varepsilon)} \ \leq \ |A_1|\cdots|A_r|P^{-2r\varepsilon}.$$

Using this in those terms on the right-hand-side of (6), we find that, using Parseval again, they contribute at most

$$\begin{aligned} P^{-r-2r\varepsilon}M|U|^L|A_1|\cdots|A_r| & \sum_{0\leq s_1,...,s_r\leq P-1} |\widehat{1}_{A_1\times\cdots\times A_r}(s_1, ..., s_r)|^2 \\ \leq \ P^{-2r\varepsilon}M|U|^L|A_1|^2\cdots|A_r|^2 \ & \leq \ P^{-r-r\varepsilon}M|U|^L|A_1|^3\cdots|A_r|^3, \end{aligned}$$

which is also appreciably smaller than the contribution of the term with $(s_1, ..., s_r) = (0, ..., 0)$, as in (7).

Thus, there exists $1 \leq n \leq M$ and $0 \leq \delta_1, ..., \delta_r \leq P^{7r^3\varepsilon}$ so that

$$n(x_1, ..., x_r) + (\delta_1, ..., \delta_r) \ \in \ (3A_1) \times (3A_2) \times \cdots \times (3A_r).$$

And since this holds for $(1 - o(1))|F|$ vectors $(x_1, ..., x_r) \in F$, the proposition is proved.

1.1.3 Proving $|E| = o(P)$

We begin by noting, by the pigeonhole principle, that there exist $(s_1, ..., s_r) \in Q'$, such that (3) holds for at least $|E|/N$ vectors $(x_1, ..., x_r) \in E$. Call this new set of vectors $E' \subseteq E$; so, we have

$$|E'| \ \geq \ |E|/N.$$

(We suppose $N \geq 1$, since the set Q' can be taken to include 0.)

Let us suppose, for proof by contradiction, that

$$|E|/N \ > \ P^{1-7r\varepsilon}. \tag{8}$$

Note that if we establish a contradiction, then we would be forced to conclude that

$$|E| \ \leq \ NP^{1-7r\varepsilon} \ \leq \ P^{1-r\varepsilon},$$

which would imply $|E| = o(P)$, and which is just what we wanted to show.

For each $\vec{x} = (x_1, ..., x_r) \in E'$, let $t = t(\vec{x})$ be any value of t, so that if $\vec{y} = K(t)$, then

$$\vec{y} \ \in \ (x_1/P, ..., x_r/P) + [0, 1/P)^r. \tag{9}$$

Also, for any vector $\vec{v} \in [0, 1)^r$, let $\pi(v)$ denote the unique $\vec{x} \in \{0, ..., P-1\}^r$, so that

$$\vec{v} \ \in \ \frac{1}{P}\vec{x} + \left[0,\ \frac{1}{P}\right]^r.$$

Now, if we consider the set of all points in a cube

$$\vec{w} + \left[0, \frac{1}{P}\right]^r, \tag{10}$$

where $\vec{w}$ is some arbitrary r-dimensional vector, the function π will map that set to a set of size at most 2^r. Thus, if we let

$$T \ := \ \{t(\vec{x}) \ : \ \vec{x} \in E'\},$$

then we claim that in any interval of width $(P \log P)^{-1}$ there can be at most 2^r elements of T; and therefore any interval of width P^{-1} has at most $2^r \log P$ elements of T. The reason this holds is that if we restrict t to an interval I of width at most $(P \log P)^{-1}$, then the coordinates of $K(t)$ will vary by $o(1/P)$; and so, the set $\{K(t) \ : \ t \in I\}$ will be contained in one of the cubes (10).

By picking at most one element of T in each interval of width P^{-1}, we can pass to a subset

$$T' \ \subseteq \ T, \text{ where } |T'| > 2^{-r}|T|(\log P)^{-1} = 2^{-r}|E'|(\log P)^{-1} > P^{1-7r\varepsilon-o(1)},$$

such that every pair of elements of T' is at least $1/P$ apart.

Furthermore, we eliminate the elements of T' that are $\leq P^{-8r\varepsilon}$ in size. Call this new set $T'' \subseteq T'$. There can be at most $P^{1-8r\varepsilon+o(1)}$ elements in T' that are $\leq P^{-8r\varepsilon}$; and so,

$$|T''| \ \geq \ |T'| - P^{1-8r\varepsilon+o(1)} \ \geq \ P^{1-7r\varepsilon-o(1)}.$$

Now we index the elements or T'' as follows:

$$T'' \ := \ \{t_1, t_2, ..., t_n\},$$

where

$$t_1 \ < \ t_2 \ < \ \cdots \ < \ t_n.$$

Then, we extract disjoint subsets $T_1, ..., T_r \subseteq T''$ as follows: we let

$$T_i \ := \ \{t_j \ : \ (2i-2)n/2r < j < (2i-1)n/2r\},$$

which satisfies

$$|T_i| \ \gg \ n/r \ \gg \ |T''| \ > \ P^{1-7r\varepsilon-o(1)} \tag{11}$$

Let

$$d(T_i, T_j) \ := \ \min_{t \in T_i, u \in T_j} |t - u|.$$

Since the elements of T'' are spaced at least $1/P$ apart, we must have that

$$\min_{1 \leq i < j \leq r} d(T_i, T_j) \ \geq \ n/2rP \ > \ P^{-7r\varepsilon-o(1)}. \tag{12}$$

Define, also, the associated intervals

$$I_i \ := \ [t_{\lceil (2i-2)n/2r \rceil}, \ t_{\lfloor (2i-1)n/2r \rfloor}].$$

Note that if $t \in T_i$, then $t \in I_i$.

We now define $u_1, ..., u_r$ as follows: we let u_i be any element in the interval I_i such that $|h'(u)|$ is minimal, where

$$h(t) \; := \; (s_1, ..., s_r) \cdot K(t) \; = \; s_1 c_1 \theta_1^{-t} + \cdots + s_r c_r \theta_r^{-t}.$$

Note that

$$h'(t) \; := \; -s_1 c_1 \theta_1^{-t} \log \theta_1 - s_2 c_2 \theta_2^{-t} \log \theta_2 - \cdots - s_r c_r \theta_r^{-t} \log \theta_r.$$

Bundling together $h'(u_1), ..., h'(u_r)$, we get the following matrix equation

$$\begin{bmatrix} \theta_1^{-u_1} & \theta_2^{-u_1} & \cdots & \theta_r^{-u_1} \\ \theta_1^{-u_2} & \theta_2^{-u_2} & \cdots & \theta_r^{-u_2} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1^{-u_r} & \theta_2^{-u_r} & \cdots & \theta_r^{-u_r} \end{bmatrix} \begin{bmatrix} -s_1 c_1 \log \theta_1 \\ -s_2 c_2 \log \theta_2 \\ \vdots \\ -s_r c_r \log \theta_r \end{bmatrix} = \begin{bmatrix} h'(u_1) \\ h'(u_2) \\ \vdots \\ h'(u_r) \end{bmatrix}. \tag{13}$$

Now we need the following lemma:

Lemma 1 *Let*

$$0 \; < \; x_1 \; < \; x_2 \; < \; \cdots \; < \; x_r, \text{ and } 0 \; < \; y_1 \; < \; y_2 \; < \; \cdots \; < \; y_r$$

be two sets of increasing real numbers. Define the matrix

$$A \; := \; \begin{bmatrix} x_1^{y_1} & x_2^{y_1} & \cdots & x_r^{y_1} \\ x_1^{y_2} & x_2^{y_2} & \cdots & x_r^{y_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{y_r} & x_2^{y_r} & \cdots & x_r^{y_r} \end{bmatrix}.$$

Let

$$\sigma \; := \; \min_{\substack{(c_1,\ldots,c_r) \\ \|(c_1,\ldots,c_r)\|_2 = 1}} \|[c_1 \; \cdots \; c_r] \cdot A\|_2 \; = \; \min_{\substack{(c_1,\ldots,c_r) \\ \|(c_1,\ldots,c_r)\|_2 = 1}} \left\|[c_1 \; \cdots \; c_r] \cdot A^T\right\|_2.$$

Then,

$$\sigma \; \geq \; r^{-r+1/2} (x_r + 1)^{-(r-1)y_r} (x_r/x_1)^{-y_1} x_1^{y_1 + y_2 + \cdots + y_r} \sigma_0, \tag{14}$$

where

$$\sigma_0 \; := \; \min_{i=1,\ldots,r-1} \prod_{1 \leq j < \ell \leq r} ((x_{i+1}/x_i)^{y_\ell/(r-1)} - (x_{i+1}/x_i)^{y_j/(r-1)}).$$

Note that when $r = 1$ this gives $\sigma \geq x_1^{y_1}$ (and $\sigma_0 = 1$ since the minimum is empty), and it is easy to see that σ exactly equals $x_1^{y_1}$ in this case.

Applying this lemma to (13), using $x_i = \theta_i^{-1}$ and $y_i = u_i$, $i = 1, ..., r$, reordering the columns as necessary (because $\theta_1^{-1}, \theta_2^{-1}, ...$ may not be in increasing order), and shuffling the ordering of the coordinates of the column vector in left-hand-side of (13) accordingly (if you reorder the columns of the square matrix, you have to do the same for the column vector), we conclude that

$$\begin{aligned} \|(h'(u_1), ..., h'(u_r))\|_2 \quad &\geq \quad \sigma\|(s_1 \log \theta_1, s_2c_2 \log \theta_2, ..., s_rc_r \log \theta_r)\|_2 \\ &\geq \quad \sigma \cdot \min_i |c_i \log \theta_i| \cdot \|(s_1, ..., s_r)\|_2, \end{aligned}$$

where σ satisfies (14). Letting $i = 1, ..., r$ be any value where $|h'(u_i)|$ is maximal; that is,

$$|h'(u_i)| \;=\; \max_{j=1,...,r} |h'(u_j)|,$$

we will have that for every $t \in I_i$,

$$|h'(t)| \;\geq\; |h'(u_i)| \;\geq\; r^{-1/2}\|(h'(u_1), ..., h'(u_r))\|_2 \;\geq\; r^{-1/2}\sigma \cdot \min_j |c_j \log \theta_j| \cdot \|(s_1, ..., s_r)\|_2. \tag{15}$$

By the Cauchy-Schwarz inequality we also have the following upper bound that holds for any $t \in I_i$:

$$|h'(t)| \;\leq\; \max_j |c_j\theta_j^{-t} \log \theta_j| \cdot \|(s_1, ..., s_r)\|_2.$$

We wish to bound σ from below. First, note that for $\alpha > 1$ and $0 < u < t < 1$,

$$\begin{aligned} \alpha^t - \alpha^u \;=\; \alpha^u(\alpha^{t-u} - 1) \quad &= \quad \alpha^u(e^{(t-u)\log \alpha} - 1) \\ &> \quad \alpha^u(t - u) \log \alpha. \end{aligned}$$

Thus, since

$$P^{-8r\varepsilon} \;<\; u_1 \;<\; \cdots \;<\; u_r \;\leq\; 1,$$

and since (12) holds, one sees that for any $i, i' = 1, ..., r$, and $\theta_i/\theta_{i'} > 1$, $j < \ell$,

$$\begin{aligned} (\theta_i/\theta_{i'})^{u_\ell/(r-1)} - (\theta_i/\theta_{i'})^{u_j/(r-1)} \quad &> \quad (\theta_i/\theta_{i'})^{u_j/(r-1)} \frac{(u_\ell - u_j)(\log \theta_i/\theta_{i'})}{r-1} \\ &> \quad \kappa P^{-7r\varepsilon - o(1)} \log \kappa, \end{aligned}$$

where

$$\kappa \;:=\; \min_{\substack{i,i',j=1,...,r \\ \theta_i > \theta_{i'}}} (\theta_i/\theta_{i'})^{u_j/(r-1)}.$$

Thus, (14) implies

$$\sigma \;>\; P^{-7r^3\varepsilon/2 - o(1)}.$$

(The implied constants in the $o(1)$ depend on r, ε, the x_i's and y_i's; the term $o(1)$ tends to 0 as $P \to \infty$.) It follows from (17) that for every $t \in I_i$ that

$$|h'(t)| \;\geq\; P^{-7r^3\varepsilon/2 - o(1)}\|(s_1, ..., s_r)\|_2. \tag{16}$$

In particular, this means that $h'(t) \neq 0$ for all $t \in I_i$, so that $h(t)$ is either strictly increasing on the interval I_i, or strictly decreasing on the interval I_i.

Now, for $t \in T_i$ we have that

$$|h(t)| \ = \ |(s_1, ..., s_r) \cdot K(t)| \ \leq \ \|(s_1, ..., s_r)\|_2 \|K(t)\|_2 \ \ll \ \|(s_1, ..., s_r)\|_2. \quad (17)$$

Applying (3), (2), and (9), we also conclude that

$$\|h(t)\| \ \ll \ P^{-(7r^3 - 2r)\varepsilon}, \quad (18)$$

where $\| \cdot \|$ denotes the nearest integer function.

Now, combining (11) and (17), and applying the Pigeonhole Principle, we let T_i' be a maximal subset of T_i where the nearest integer to all the $h(t)$, $t \in T_i'$, is the same. Thus, there exists an integer z such that

$$\text{For every } t \in T_i', \ |h(t) - z| \ \leq \ 1/2; \text{ and } |T_i'| \ > \ P^{1 - 7r\varepsilon - o(1)} \|(s_1, ..., s_r)\|_2^{-1}. \quad (19)$$

From (18) we know that the $h(t)$ come within $P^{-(7r^3-2r)\varepsilon}$ of an integer; and so, we must have that the following upper bound also holds:

$$\text{For every } t \in T_i', \ |h(t) - z| \ \leq \ P^{-(7r^3-2r)\varepsilon}. \quad (20)$$

However, we will see that this cannot hold, by using the Mean Value Theorem and the bound (16): without loss of generality, assume $h(t)$ is *increasing* in I_i (we know it is either increasing or decreasing, and it doesn't matter which). Write the set T_i' in increasing order as

$$t_1' \ < \ t_2' \ < \ \cdots \ < \ t_{n'}'.$$

Since h in increasing across this set, we have that

$$h(t_1') \ < \ h(t_2') \ < \ \cdots \ < \ h(t_{n'}').$$

Now, from the Mean Value Theorem, (16), the fact that the t_j''s are spaced at least $1/P$ apart, and our bound on $|T_i'|$ in (19), we have that

$$\begin{aligned} |h(t_1') - h(t_{n'}')| \ \gg \ |t_1' - t_{n'}'| \min_{t \in [t_1', t_{n'}']} |h'(t)| \quad &\geq \quad (n'/P) P^{-7r^3\varepsilon/2 - o(1)} \|(s_1, ..., s_r)\|_2 \\ &\geq \quad P^{-(7r^3/2 + 7r)\varepsilon - o(1)}. \end{aligned}$$

This is impossible, since from (20) we deduce from the triangle inequality that

$$|h(t_1') - h(t_{n'}')| \ \ll \ P^{-(7r^3-2r)\varepsilon}.$$

We conclude that (8) is false, and so our theorem is proved.

2 Proof of Lemma 1

The claim clearly holds for $r = 1$. Assume we've proved it for all $1 \leq r \leq k$. Now we prove it for $r = k + 1$: So, we assume we have a matrix of that size; and assume, for proof by contradiction, that (14) fails to hold.

We let $(c_1, ..., c_{k+1})$ denote a vector of norm 1 such that

$$\|[c_1 \ \cdots \ c_{k+1}] \cdot A\| \ = \ \sigma.$$

Define

$$f(x) \ := \ \sum_{j=1}^{k+1} c_j x^{y_j},$$

and note that since $f(x_i)$ is the ith coordinate of $[c_1 \ \cdots \ c_{k+1}] \cdot A$, we must have

$$|f(x_i)| \ \leq \ \sigma, \ i = 1, ..., k+1, \tag{21}$$

all of which are rather small in magnitude, since we are assuming (14) fails to hold, making σ very small. We wish to show (for reasons explained below) that there exist $z_1, ..., z_k$, where

$$x_i \ < \ z_i \ < \ x_{i+1}, \ i = 1, 2, ..., k,$$

such that

$$|f(z_i)| \ > \ (z_i/x_i)^{y_1} |f(x_i)|, \text{ and } |f(z_i)| \ > \ |f(x_{i+1})|, \ i = 1, ..., k.$$

To see that such z_i exist, let $\delta > 0$ be such that

$$\delta \ = \ \frac{\log x_{i+1}}{\log x_i} \ - \ 1 \ > \ 0,$$

which means

$$x_i^{1+\delta} \ = \ x_{i+1}.$$

Then, consider the numbers

$$f(x_i), \ f(x_i^{1+\delta/k}), \ f(x_i^{1+2\delta/k}), \ ..., \ f(x_i^{1+\delta}) = f(x_{i+1}).$$

Written as a row vector we have

$$[f(x_i) \ f(x_i^{1+\delta/k}) \ \cdots \ f(x_{i+1})] \ = \ [c_1 \ c_2 \ \cdots \ c_{k+1}] \ \cdot \ V,$$

where

$$V \ := \ \begin{bmatrix} x_i^{y_1} & (x_i^{y_1})^{1+\delta/k} & \cdots & (x_i^{y_1})^{1+\delta} \\ x_i^{y_2} & (x_i^{y_2})^{1+\delta/k} & \cdots & (x_i^{y_2})^{1+\delta} \\ \vdots & \vdots & \ddots & \vdots \\ x_i^{y_{k+1}} & (x_i^{y_{k+1}})^{1+\delta/k} & \cdots & (x_i^{y_{k+1}})^{1+\delta} \end{bmatrix}.$$

The square matrix V here is a Vandermonde (well, after dividing out by certain factors down columns), so its determinant can be explicitly computed:

$$\det(V) \ = \ x_i^{y_1+y_2+\cdots+y_{k+1}} \prod_{1 \leq j < \ell \leq k+1} (x_i^{y_\ell \delta/k} - x_i^{y_j \delta/k}). \tag{22}$$

Letting $J = VV^T$, we then also have

$$\det(J) \;=\; \det(V)^2 \;=\; x_i^{2y_1+2y_2+\cdots+2y_{k+1}} \prod_{1\le j<\ell\le k+1} (x_i^{y_\ell \delta/k} - x_i^{y_j\delta/k})^2.$$

A crude upper bound on the largest eigenvalue of J can be found as follows: let μ be the maximum value of the entries of J. We note that

$$\mu \;\le\; (k+1) \max_{j,\ell=1,\ldots,k+1} |x_j^{y_\ell}|^2 \;\le\; (k+1)(x_{k+1}+1)^{2y_{k+1}}.$$

Then, for any vector $\vec{v} := (v_1, ..., v_{k+1})$ satisfying $\|\vec{v}\|_2 = 1$, we have that all the entries of $J\vec{v}$ can be bounded from above by

$$\mu\|\vec{v}\|_1 \;\le\; \mu(k+1)\|\vec{v}\|_\infty.$$

Thus,

$$\mu(k+1) \;\le\; (k+1)^2(x_{k+1}+1)^{2y_{k+1}}$$

is an upper bound for any eigenvalue for J.

Also, if $\alpha > 0$ is the smallest eigenvalue (in magnitude) of J, and $\beta > 0$ the largest eigvenvalue (in magnitude) of J, then since $\det(J)$ is the product of its eigenvalues,

$$\det(J) \;\le\; \alpha\beta^k.$$

So, recalling that $\|(c_1, ..., c_{k+1})\|_2 = 1$, we have

$$\begin{aligned}
\|(f(x_i), f(x_i^{1+\delta/k}), ..., f(x_i^{1+\delta}))\|_2^2 \;&=\; (c_1, ..., c_{k+1})VV^T(c_1, ..., c_{k+1})^T \\
&\ge\; \alpha \\
&\ge\; \det(J)\cdot\beta^{-k} \\
&>\; \det(J)\cdot((k+1)^2(x_{k+1}+1)^{2y_{k+1}})^{-k} \\
&=\; \det(V)^2((k+1)^2(x_{k+1}+1)^{2y_{k+1}})^{-k}.
\end{aligned}$$

Note that the first inequality here is by the Rayleigh Principle:

$$\min_{\substack{(c_1,\ldots,c_{k+1})\\ \|(c_1,\ldots,c_{k+1})\|=1}} (c_1, ..., c_{k+1})J(c_1, ..., c_{k+1})^T \;=\; \alpha.$$

Thus,

$$\max_{j=0,\ldots,k}|f(x_i^{1+j\delta/k})| \;\ge\; (k+1)^{-k-1/2}(x_{k+1}+1)^{-ky_{k+1}}|\det(V)|.$$

Now, since we are operating under the assumption that (14) fails to hold, expressing (21) in terms of $\det(V)$ (and using (22)), we find that

$$|f(x_i)| \;\le\; \sigma \;\le\; (k+1)^{-k-1/2}(x_{k+1}+1)^{-ky_{k+1}}(x_{k+1}/x_1)^{-y_1}|\det(V)|.$$

Thus

$$\max_{j=0,\ldots,k}|f(x_i^{1+j\delta/k})| \ge\; (x_{k+1}/x_1)^{y_1}|f(x_i)|, \text{ and } \;\ge\; (x_{k+1}/x_1)^{y_1}|f(x_{i+1})|.$$

We, therefore, have found the z_i we were looking for, since: First, for $j = 0, ..., k$ we have that

$$x_i \ \leq \ z_i \ := \ x_i^{1+j\delta/k} \ \leq \ x_{i+1},$$

where the j arising from the max above cannot be $j = 0$ or $j = k$, since the max is bigger than $|f(x_i)|$ and $|f(x_{i+1})|$. And, second, we also have

$$|f(z_i)| \ \geq \ (x_{k+1}/x_1)^{y_1} |f(x_i)| \ > \ (z_i/x_i)^{y_1} |f(x_i)|,$$

and

$$|f(z_i)| \ > \ |f(x_{i+1})|.$$

Now, if we let

$$g(x) \ := \ x^{-y_1} f(x),$$

then note that for this choice of j (chosen by the max above),

$$|g(z_i)| \ = \ z_i^{-y_1} |f(z_i)| \ > \ x_i^{-y_1} |f(x_i)| \ = \ |g(x_i)|,$$

and, likewise,

$$|g(z_i)| \ = \ z_i^{-y_1} |f(z_i)| \ \geq \ z_i^{-y_1} |f(x_{i+1})| \ > \ x_{i+1}^{-y_1} |f(x_{i+1})| \ = \ |g(x_{i+1})|.$$

Thus, by Rolle's Theorem, there exists a point $w_i \in (x_i, x_{x+1})$ where the derivative

$$g'(w_i) \ = \ 0, \ i = 1, 2, ..., k. \tag{23}$$

But, since

$$g(w) \ = \ w^{-y_1} f(w) \ = \ \sum_{j=1}^{k+1} c_j w^{y_j - y_1},$$

we find that

$$g'(w) \ = \ \sum_{j=2}^{k+1} c_j (y_j - y_1) w^{y_j - y_1};$$

so, we have that

$$\begin{bmatrix} w_1^{y_2-y_1} & w_1^{y_3-y_1} & \cdots & w_1^{y_{k+1}-y_1} \\ w_2^{y_2-y_1} & w_2^{y_3-y_1} & \cdots & w_2^{y_{k+1}-y_1} \\ \vdots & \vdots & \ddots & \vdots \\ w_k^{y_2-y_1} & w_k^{y_3-y_1} & \cdots & w_k^{y_{k+1}-y_1} \end{bmatrix} \begin{bmatrix} c_2(y_2 - y_1) \\ c_3(y_3 - y_1) \\ \vdots \\ c_{k+1}(y_{k+1} - y_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

However, the induction hypotheses for the case $r = k$ tells us that the square matrix on the left is non-singular (in fact, it gives a non-trivial lower bound, in magnitude, for its smallest singular value). So this is impossible.

We conclude that our assumption that (14) was wrong is incorrect, and so the induction is proved.