Asymptotic Bounds for the Mertens Function

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Last Revised: May 7, 2018

Abstract.

Keywords: Mertens function; Möbius function; Lioville's function; sum-of-divisors function; Ramanujan's sum.
2010 MSC: 11A25; 11N37; 11N56.
Pages: 1–20.

Article Contents

Chapter 1

New asymptotics of the Mertens functions

1.1 Introduction

1.1.1 Mertens summatory functions

The Mertens summatory function, or *Mertens function*, is defined as

$$M(x) = \sum_{n \leq x} \mu(n),\ \ x \geq 1,$$

where $\mu(n)$ denotes the Möbius function which is in some sense a signed indicator function for the squarefree integers. A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

We define the notion of a *generalized Mertens summatory function* for fixed $\alpha \in \mathbb{C}$ as

$$M_{\alpha}^{*}(x) = \sum_{n \leq x} n^{\alpha} \mu(n),\ \ x \geq 1,$$

where the special case of $M_0^*(x)$ corresponds to the definition of the classical Mertens function $M(x)$ defined above. The plots shown in Figure 1.1.1 illustrate the chaotic behavior of the growth of these functions for x in small intervals when $\alpha \in \{-1, 0, 1, 2\}$.

1.1.2 Open problems

There are many open problems related to bounding $M(x)$ for large x. For example, the Riemann Hypothesis is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. For $\operatorname{Re}(\alpha) < 1$, we know the limiting absolute behavior of these functions as $x \to \infty$ as the Dirichlet generating function

$$\frac{1}{\zeta(\alpha)} = \lim_{x \longrightarrow \infty} M_{\alpha}^{*}(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{-\alpha}},$$

which is definitively bounded for all large x. It is still unresolved whether

$$\limsup_{x \to \infty} |M(x)|/\sqrt{x} = \infty,$$

although computational evidence suggests that this is a likely conjecture [9, 6]. We prove that this conjecture is true in Theorem **??**. There is a rich history to the original statement of the *Mertens conjecture* which states that

$$|M(x)| < c \cdot x^{1/2},\ \ \text{some constant } c > 0,$$

which was first verified by Mertens for $c = 1$ and $x < 10000$, although since its beginnings in 1897 has since been disproved by computation. We cite that prior to this point it is known that [16, *cf.* §4.1]

$$\limsup_{x \to \infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad \text{(now 1.826054)},$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } -1.837625),$$

although based on work by Odlyzko and te Riele it seems probable that each of these limits should be $\pm\infty$, respectively [14, 10, 9, 6]. While it is known that $M(x) = \Omega_{\pm}(\sqrt{x})$ and $M(x)/\sqrt{x} = \Omega_{\pm}(1)$, we appear to offer the first complete proof that the function $M(x)/\sqrt{x}$ is in fact *unbounded* in the next sections of this article.

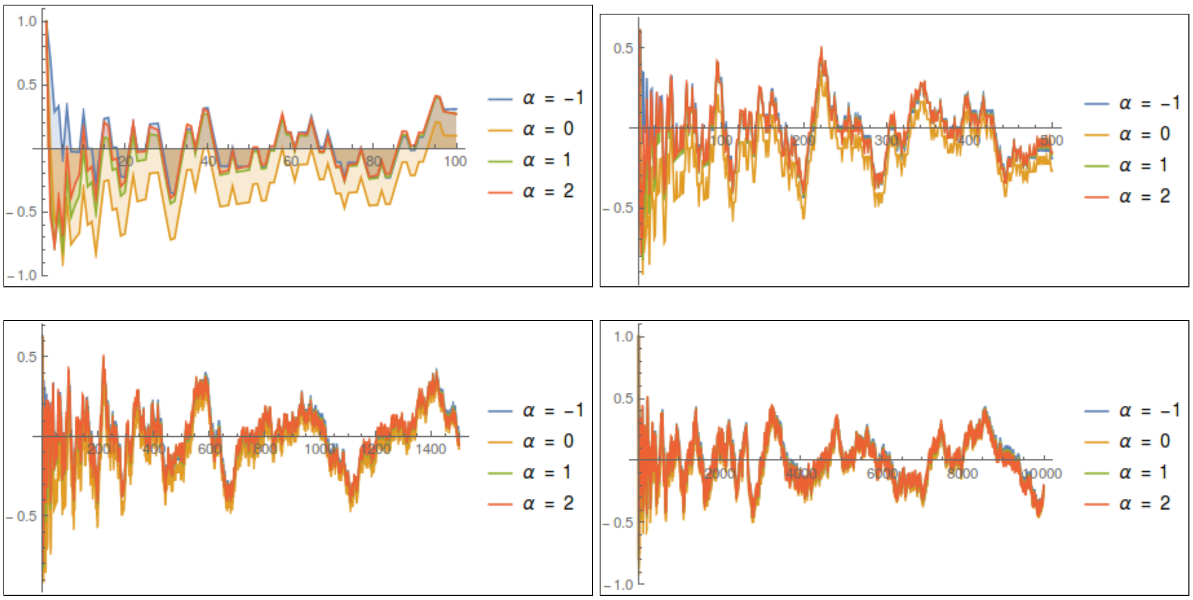

Figure 1.1.1: *Comparison of the Mertens Summatory Functions $M_\alpha(x)/x^{\frac{1}{2}+\alpha}$ for Small x and α*

1.1.3 Exact formulas for the generalized sum-of-divisors functions

The author has recently proved (2017) several new exact formulas for the *generalized sum-of-divisors functions*, $\sigma_\alpha(x)$, defined for any $x \geq 1$ as

$$\sigma_\alpha(x) = \sum_{d|x} d^\alpha, \ \ \alpha \in \mathbb{C}.$$

In particular, if we let $H_n^{(r)} = \sum_{k=1}^n k^{-r}$ denote the sequence of r*-order harmonic numbers*, where [15, §2.4(iii)]

$$H_n^{(-t)} = \frac{B_{t+1}(n+1) - B_{t+1}}{(t+1)} = \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + \sum_{k=1}^{t-1} \binom{t}{k} \frac{B_{k+1} n^{t-k}}{(k+1)}, \tag{1.1}$$

is a *Bernoulli polynomial* for any $n \geq 0$ when $t \in \mathbb{Z}^+$, then we can restate the next theorem from [18]. Within this article we adopt the convention that an index of summation p denotes that the sum is taken over only prime values of p. We also use the notation that the valuation function

$$\nu_p(x) = m \quad \text{if and only if} \quad p^m \| x,$$

to denote the exact exponent of the prime p dividing x.

Theorem 1.1 (Schmidt, 2017). For any fixed $\alpha \in \mathbb{C}$ and all $x \geq 1$, we have that

$$\begin{aligned}\sigma_\alpha(x) = H_x^{(1-\alpha)} + \sum_{d|n} \tau_x^{(\alpha+1)}(d) + \sum_{2\leq p\leq x} \sum_{k=1}^{\nu_p(x)+1} p^{\alpha k} H_{\left\lfloor \frac{x}{p^k}\right\rfloor}^{(1-\alpha)} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p}\right) \\ + \sum_{3\leq p\leq x} \sum_{k=1}^{\nu_p(x)+1} 2^{\alpha-1} p^{\alpha k} (-1)^{\lfloor x/p^{k-1}\rfloor} H_{\left\lfloor \frac{x}{2p^k}\right\rfloor}^{(1-\alpha)} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p}\right),\end{aligned}$$

where the divisor sum over the function $\tau_x^{(\alpha)}(d)$ is defined precisely by Lemma **??**.

Remark 1.2 (Restatement of the Theorem). For $x \geq 1$ and fixed $\alpha \in \mathbb{C}$, we define the sums

$$\begin{aligned} S_1^{(\alpha+1)}(x) &= \sum_{2 \leq p \leq x} \sum_{k=1}^{\nu_p(x)+1} p^{\alpha k} H_{\left\lfloor \frac{x}{p^k} \right\rfloor}^{(1-\alpha)} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right) \\ S_2^{(\alpha+1)}(x) &= \sum_{3 \leq p \leq x} \sum_{k=1}^{\nu_p(x)+1} 2^{\alpha-1} p^{\alpha k} (-1)^{\lfloor x/p^{k-1} \rfloor} H_{\left\lfloor \frac{x}{2p^k} \right\rfloor}^{(1-\alpha)} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right). \end{aligned}$$

Then we prefer to work with the next form of Theorem 1.1 stated in terms of our new shorthand sum functions as follows:

$$\left| \sum_{d|x} \tau_x^{(\alpha+1)}(d) \right| = \left| H_x^{(1-\alpha)} - \sigma_\alpha(x) + S_1^{(\alpha+1)}(x) + S_2^{(\alpha+1)}(x) \right|. \tag{1.2}$$

The short appendix given in Chapter 2 starting on page 15 sketches the details of the proof of this result from [18].

1.2 New results and proofs of key lemmas

1.2.1 Connections to Ramanujan sums

We adopt the new convention that the divisor sum terms in Theorem 1.1 are denoted in shorthand by (see also the appendix in Chapter 2 at the end of the article)

$$\begin{aligned} \tau_\alpha(x) &= \sum_{d|x} \tau_x^{(\alpha+1)}(d) \\ &= [q^x] \left(\sum_{k=1}^{x} \sum_{\substack{d|k \\ d \neq p^k, 2p^k}} \sum_{r|d} \frac{r \cdot \mu(d/r)}{(1-q^r)} k^\alpha \right) \\ &= \sum_{k=1}^{x} \sum_{\substack{d|k \\ d \neq p^k, 2p^k}} \sum_{r|(d,x)} r \cdot \mu(d/r) \cdot k^\alpha. \end{aligned} \tag{1.3}$$

We can also expand the right-hand-side of (1.3) as

$$\tau_\alpha(x) = \sum_{\substack{d=1 \\ d \neq p^k, 2p^k}}^{x} \left(\sum_{r|(d,x)} r\mu(d/r) \right) H_{\left\lfloor \frac{x}{d} \right\rfloor}^{(-\alpha)} - x. \tag{1.4}$$

Remark 1.3 (Connection to Ramanujan's Sum). We have a deep connection between the divisor sums in (1.1) and (1.4) and Ramanujan's sum $c_q(n)$ given by

$$\begin{aligned} \tau_0(x) &= \sum_{k=1}^{x} \sum_{\substack{d|k \\ d \neq p^k, 2p^k}} c_d(x) - x \\ &= \sum_{k=1}^{x} \sum_{\substack{d|k \\ d \neq p^k, 2p^k}} \mu\left(\frac{d}{(d,x)} \right) \frac{\varphi(d)}{\varphi\left(\frac{d}{(d,x)} \right)} - x, \end{aligned}$$

where $\varphi(x)$ denotes Euler's totient function. These identities follow by expanding out Ramanujan's sum in the form of [15, §27.10] [12, §A.7] [4, *cf.* §5.6]

$$c_q(n) = \sum_{d|(q,n)} d \cdot \mu(q/d),$$

and then applying the formula in (1.3) from the proof of the lemma above. Ramanujan's sum also satisfies the convenient bound that $|c_q(n)| \leq (n,q)$ for all $n, q \geq 1$, which can be used to obtain asymptotic estimates in the form of upper bounds for these sums when x is not prime or a prime power.

1.2.2 New connections to the Mertens function

Definition 1.4 (Component Divisor Sums Defining the Mertens Function). For positive natural numbers $x \geq 1$, define the following component sums which form a combined expansion of the initial summation form given in (1.4) when $\alpha := 0$:

$$T_1(x) := \sum_{d=1}^{x} \mu(x)\chi_{\mathrm{pp}}(d) \left\lfloor \frac{x}{d} \right\rfloor$$

$$T_2(x) := x \cdot \sum_{d|x} \left(\sum_{\substack{r|d \\ r<d}} \frac{\mu(r)}{r} \right) \chi_{\mathrm{pp}}(d)$$

$$T_3(x) := \sum_{\substack{d \leq x \\ 1<(d,x)<d}} \left\lfloor \frac{x}{d} \right\rfloor \chi_{\mathrm{pp}}(d) \left(\sum_{\substack{r|(d,x) \\ r>1}} r\mu(d/r) \right).$$

In particular, by construction the previous definitions allow us to expand the function $\tau_0(x)$ as

$$\begin{aligned} \tau_0(x) &= T_1(x) + T_2(x) + T_3(x) - x \\ &= \sum_{d=1}^{x} \left\lfloor \frac{x}{d} \right\rfloor \chi_{\mathrm{pp}}(d) \left(\sum_{r|(d,x)} r\mu(d/r) \right) - x. \end{aligned} \tag{1.5}$$

We are now in a position to state and prove a new exact formula for $M(x)$ expanded by the component functions in the previous definition for all $x \geq 1$. Our new estimates proved later in this subsection are derived by bounding the next formula from below.

Proposition 1.5 (A New Exact Formula for $M(x)$). *For all integers $x \geq 1$, the Mertens function satsifies the following formula:*

$$M(x) = \sum_{k=1}^{x} \sum_{d|k} \left(\tau_0(d) - \tau_0(d-1) + T_2(d-1) - T_2(d) + T_3(d-1) - T_3(d)\right) \mu(k/d) - \pi(x) + \pi\left(\left\lfloor \frac{x}{2} \right\rfloor\right).$$

Proof. By considering differences of the floored fraction terms $\left\lfloor \frac{x}{d} \right\rfloor$ over x, we first obtain that

$$T_1(x) - T_1(x-1) = \sum_{k|x} \mu(k)\chi_{\mathrm{pp}}(k). \tag{1.6}$$

Then by Möbius inversion and considering that $1 - \chi_{PP}(d)$ is the indicator function of the natural numbers of the form $p^k, 2p^k$ for primes p and some $k \geq 1$, we see easily also that

$$M(x) = \sum_{k=1}^{x} \sum_{d|k} \left(T_1(d) - T_1(d-1)\right) \mu(k/d) - (\pi(x) + 1) + \pi\left(\left\lfloor \frac{x}{2} \right\rfloor\right). \tag{1.7}$$

This observation is the critical step in forming the expansion for $M(x)$ claimed in the statement above. Since we have constructed the functions $T_i(x)$ so that (1.5) holds, we can rearrange terms of these sums and again apply Möbius inversion to see that our formula is correct. Notice that we are implicitly using the known fact that the convolution $(\mu * 1)(n) = \varepsilon(n) = \delta_{n,1}$ to simplify parts of the initial sum formed in this step. □

The statement of the last proposition is particularly encouraging because it provides statements of new exact formulas for the Mertens function which can be attacked from the standpoint of other standard and classical techniques. The divisor sums from Definition 1.4 are strikingly similar to exponential and Dedkind sums which have been studied and bounded by many mathematicians. In particular, the Ramanujan sums implicit to the definitions of these sums are special cases of the well-studied exponential sums.

Remark 1.6 (Definitions of Summatory Functions From the Last Proof). Moving forward, we will now turn our attention to (1.6) and (1.7) given in the proof of Proposition 1.5. Notice that the right-hand-side of (1.6), redefined as the function

$$f(n) := \sum_{k|n} \mu(k)\chi_{\mathrm{pp}}(k),$$

is expanded explicitly in terms of the distinct prime divisor counting function, $\omega(n)$, as follows where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative Dirichlet identity:

$$\begin{aligned} f(n) &= \sum_{d|n} \mu(d) - \sum_{p^k|n} \mu(p^k) - \sum_{2p^k|n} \mu(2p^k) \\ &= \varepsilon(n) + \sum_{p|n} 1 - \sum_{p|\left\lfloor \frac{x}{2} \right\rfloor} \mu(2p) \\ &= \varepsilon(n) + \omega(n) - \omega\left(\lfloor n/2 \rfloor\right) [n \text{ even}]_{\delta}. \end{aligned}$$

It is easily seen then that he corresponding summatory function over this special additive function case is defined and expanded simply by cases as

$$F(x) := \sum_{n \leq x} f(n) = S_{\mathrm{odd}}(x) + \left\lfloor \frac{x}{2} \right\rfloor + 1,$$

where we define the odd-indexed variant of the well-known average order sums over $\omega(n)$ from the introduction according to

$$S_{\mathrm{odd}}(x) := \sum_{n \leq x} \omega(n) [n \text{ odd}]_{\delta}. \tag{1.8}$$

As we shall see soon, the properties of these two variant summatory functions will form an important tool in sharpening our estimates and understanding of the alternate exact expansions of the Mertens function offered through the next corollary.

Corollary 1.7 (Another New Exact Recurrence Expression for $M(x)$). *For all natural numbers $x \geq 1$, we have that*

$$M(x) = \sum_{i=1}^{x} \left\{ F\left(\left\lfloor \frac{x-1}{i} \right\rfloor\right) - F\left(\left\lceil \frac{x+1}{i+1} \right\rceil\right) + [i|x]_{\delta} f\left(\frac{x}{i}\right) \right\} M(i) - (\pi(x)+1) - \pi\left(\left\lfloor \frac{x}{2} \right\rfloor\right), \tag{1.9a}$$

where $\pi(x) \sim x/\log x$ denotes the prime counting function. Equivalently, if $f(1) = 1$ then for $m \geq 2$ we have that

$$\sum_{i=1}^{m} \left[F\left(\left\lfloor \frac{2m-2}{i} \right\rfloor\right) - F\left(\left\lceil \frac{2m}{i+1} \right\rceil\right) + [i|2m-1]_{\delta} f\left(\frac{2m-1}{i}\right) \right] M(i) = \pi(2m-1) + 1 - \pi(m-1). \tag{1.9b}$$

Proof. The construction of the extra divisor sum term in the identity is a convenience that allows us to use sums in the forms of Proposition 1.5 to construct our recursive identities, but that also ensure that the coefficient of the leading $M(x)$ function term is non-zero. Other than that configuration, the identity follows by a simple construction of a summation by parts argument. Let f and g be any arithmetic functions defined such that $g(1) := 1$ and let F and G denote their corresponding summatory functions. Then we can sum over the convolutions of these two functions for all $n \leq x$ in the following forms:

$$\begin{aligned} \sum_{n=1}^{x} \sum_{d|n} f(d) g(n/d) &= \sum_{d=1}^{x} f(d) G\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{k=1}^{x} F(k) \Big[G\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + G\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \\ &\quad + G\left(\left\lfloor \frac{x-1}{k} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{k+1} \right\rfloor\right) \Big] + \sum_{k|x} G(k) F(x/k) \end{aligned}$$

$$= \sum_{i=1}^{x} \left(\sum_{\lceil \frac{x+1}{i+1} \rceil \leq n \leq \lfloor \frac{x-1}{i} \rfloor} f(n) \right) G(i) + \sum_{k|x} G(k) f(x/k).$$

Recall from the proof of the first formula proposition given above that $\lfloor x/k \rfloor - \lfloor (x-1)/k \rfloor = 1$ precisely when $k|x$ which we have observed to justify the middle step in the previous equations. □

1.2.3 Interpreting the Recurrence Relation for $M(x)$ as an Invertible Matrix Equation

What the somewhat "twisted" recursive properties of the Mertens function from the last result implies is antoher new matrix determinant method for exactly expressing this special function. In particular, we know by inversion formulas for the matrix equation $(C_{i,j})_{2 \leq i,j \leq x} \vec{M} = \vec{X}_0$ that

$$\begin{aligned} C_{i,j}^{(-1)} &= -\sum_{m=1}^{i-j} C_{i,j+m}^{(-1)} \cdot C_{j+m,j} + [i=j]_{\delta} && \text{(1.10a)} \\ &= -\sum_{m=1}^{i-j} C_{j-1-m,j}^{(-1)} \cdot C_{i,j-1-m} + [i=j]_{\delta} \\ C_{i,j}^{(-1)} &= \sum_{m=1}^{i-j} (-1)^m \left(\sum_{j+1 \leq i_1 < i_2 < \cdots < i_m \leq i-1} C_{i_1,j} C_{i_2,i_1} \times \cdots \times C_{i,i_{m-1}} \right), && \text{(1.10b)} \end{aligned}$$

for any invertible matrix and its inverse. We shall use these properties to construct recurrence relations for our matrices of interest in the Mertens function matrix problem implicitly stated by Corollary 1.7.

Definition 1.8 (Mertens Function Matrix Equation). Following as in Remark 1.6 and in the second form of Corollary 1.7, we make the following definitions:

$$\begin{aligned} f(x) &:= \varepsilon(x) + \omega(x) \left[x \equiv 1 \bmod 2 \right]_{\delta} + \left[x \equiv 0 \bmod 2 \right]_{\delta} \\ F(x) &:= \begin{cases} S_{\mathrm{odd}}(x) + \left\lfloor \frac{x}{2} \right\rfloor + 1, & x \geq 1; \\ 0, & \text{otherwise,} \end{cases} \\ G_{x,j} &:= F\left(\left\lfloor \frac{2x-2}{j} \right\rfloor \right) - F\left(\left\lceil \frac{2x}{j+1} \right\rceil \right) + f\left(\frac{2x-1}{j} \right) [j | 2x-1]_{\delta}. \end{aligned}$$

We also define the constant terms on the matrix equation by

$$\widetilde{X}_0(k) := \pi(2k+1) + 1 - \pi(k) \sim \frac{k}{\log k}.$$

Table 1.2.1 provides symbolic and numerical computations of these functions. Likewise, Table 1.2.2 provides the corresponding listings of the inverse matrices defined immediately below in the expansion of the formulas in (1.11) (see also Remark 3.2).

We can shift the initial conditions on the matrix by taking the first row to correspond to $x := 2$, the second row to $x := 4$, the third to $x := 5$, and so on. This results in a matrix that is lower triangular with all ones on the diagonal (as opposed to the default route which would leave a 2 in the first diagonal position). Then using the definitions above we have that

$$M(x) = t_{x,1} + \sum_{k=2}^{x} t_{x,k} \widetilde{X}_0(k), \tag{1.11a}$$

where $(t_{i,j})_{1 \leq i,j \leq x}$ is the inverse matrix of $(G_{i,j})_{1 \leq i,j \leq x}$, i.e., so that

$$\begin{aligned} t_{x,j} &= -\sum_{m=1}^{x-j} t_{x,j+m} G_{j+1+m,j} + [x=j]_{\delta} && \text{(1.11b)} \\ &= -\sum_{m=1}^{x-j} t_{x-m,j} G_{x+1,x-m} + [x=j]_{\delta}. \end{aligned}$$

We note that this expression for $M(x)$ is <u>exact</u> up to the point where we begin to estimate the implicit matrix coefficients asymptotically in the analysis in the next subsections.

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
6	2	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
7	2	1	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
9	2	2	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0
10	3	1	1	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0
11	3	2	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0
13	3	2	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	1
14	4	2	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
15	4	2	1	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
16	4	3	1	1	1	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0
17	5	2	2	1	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0
17	6	3	1	1	1	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0
19	6	3	2	1	1	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0
21	7	3	1	2	0	1	1	0	0	1	0	0	0	0	0	1	0	0	0	0
22	7	3	2	1	1	1	0	1	0	0	1	0	0	0	0	0	1	0	0	0
24	7	4	2	1	1	1	0	1	0	0	0	1	0	0	0	0	0	1	0	0
25	8	3	2	2	1	0	1	0	1	0	0	1	0	0	0	0	0	0	1	0

Numerical values of the function $G(x,j)$ for $1 \leq j \leq x \leq 21$

0	f_1	0	0	0	0	0	0	0	0	0	0	0	0
f_2	0	0	f_1	0	0	0	0	0	0	0	0	0	0
f_3	f_2	0	0	0	f_1	0	0	0	0	0	0	0	0
f_4	f_3	f_2	0	0	0	0	f_1	0	0	0	0	0	0
f_4+f_5	f_3	0	f_2	0	0	0	0	0	f_1	0	0	0	0
f_5+f_6	f_4	f_3	0	f_2	0	0	0	0	0	0	f_1	0	0
f_6+f_7	f_4+f_5	0	f_3	0	f_2	0	0	0	0	0	0	0	f_1
$f_6+f_7+f_8$	f_5	f_4	f_3	0	0	f_2	0	0	0	0	0	0	0
$f_7+f_8+f_9$	f_5+f_6	f_4	0	f_3	0	0	f_2	0	0	0	0	0	0
$f_8+f_9+f_{10}$	f_6+f_7	f_5	f_4	0	f_3	0	0	f_2	0	0	0	0	0
$f_8+f_9+f_{10}+f_{11}$	f_6+f_7	f_5	f_4	0	f_3	0	0	0	f_2	0	0	0	0
$f_9+f_{10}+f_{11}+f_{12}$	f_7+f_8	f_6	f_5	f_4	0	f_3	0	0	0	f_2	0	0	0
$f_{10}+f_{11}+f_{12}+f_{13}$	$f_7+f_8+f_9$	f_6	f_5	f_4	0	0	f_3	0	0	0	f_2	0	0
$f_{10}+f_{11}+f_{12}+f_{13}+f_{14}$	f_8+f_9	f_6+f_7	f_5	0	f_4	0	f_3	0	0	0	0	f_2	0
$f_{11}+f_{12}+f_{13}+f_{14}+f_{15}$	$f_8+f_9+f_{10}$	f_7	f_6	f_5	f_4	0	0	f_3	0	0	0	0	f_2

Symbolic values of the function $G(x,j)$ for $2 \leq j \leq x \leq 15$.

Table 1.2.1: *Intuition for the first few rows of the matrix coefficients $G(x,j)$.*

1.2.4 Asymptotics of the matrix equations

Lemma 1.9 (Average Order of the Sum Over Odd-Indexed $\omega(n)$). *We have the following asymptotic relation on these sums:*

$$S_{\mathrm{odd}}(x) = \frac{x}{2}\log\log(x) + \frac{(2B_1-1)x}{4} + \left\{\frac{x}{4}\right\} - [x \equiv 2, 3 \bmod 4]_{\delta} + O\left(\frac{x}{\log x}\right).$$

Proof. The derivation of the key asymptotic formula in this case is not new and can be found for example in Hardy and Wright in the form of

$$\sum_{n \leq x} \omega(n) = x\log\log(x) + B_1 x + o(x),$$

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	−1	−1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	−1	0	−1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−2	1	−1	0	−1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	−1	1	−1	−1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	−1	0	0	−1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	−1	1	−1	0	0	−1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
2	−2	1	−1	0	0	−1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
−2	1	−1	1	0	−1	0	−1	0	0	0	1	0	0	0	0	0	0	0	0	0
−2	1	0	0	−1	0	0	0	−1	0	0	0	1	0	0	0	0	0	0	0	0
3	−3	3	−2	0	1	−1	0	−1	0	0	0	0	1	0	0	0	0	0	0	0
5	−1	−1	1	0	−1	0	0	0	−1	0	0	0	0	1	0	0	0	0	0	0
0	2	−2	1	0	−1	1	−1	0	0	−1	0	0	0	0	1	0	0	0	0	0
2	0	−1	1	0	0	0	−1	0	0	−1	0	0	0	0	0	1	0	0	0	0
6	−4	3	−2	0	1	−1	1	−1	0	0	−1	0	0	0	0	0	1	0	0	0
5	−3	1	−1	1	0	−1	0	0	0	0	0	−1	0	0	0	0	0	1	0	0
3	−1	−1	1	0	−1	1	−1	1	−1	0	0	−1	0	0	0	0	0	0	1	0
5	1	−5	3	1	−2	1	−1	1	−1	0	0	0	−1	0	0	0	0	0	0	1
−4	2	1	−1	−1	1	0	0	−1	1	−1	0	0	0	−1	0	0	0	0	0	0
−2	1	1	−1	−1	1	0	0	−1	1	−1	0	0	0	−1	0	0	0	0	0	0
8	−5	3	−3	1	2	−2	1	−1	0	1	−1	0	0	0	−1	0	0	0	0	0
3	0	−1	−1	0	0	0	1	0	−1	1	−1	0	0	0	0	−1	0	0	0	0
5	−2	0	−1	1	−1	0	0	1	−1	1	0	−1	0	0	0	−1	0	0	0	0
−4	4	−4	3	0	−2	1	−1	1	0	−1	1	−1	0	0	0	0	−1	0	0	0
−8	7	−5	3	−1	−1	2	−1	0	0	−1	0	1	−1	0	0	0	0	−1	0	0
−6	6	−5	3	−1	−1	2	−1	0	0	−1	0	1	−1	0	0	0	0	−1	0	0
−4	0	3	−3	0	1	−1	1	−1	1	0	−1	1	0	−1	0	0	0	0	−1	0
−6	−2	8	−7	−1	4	−2	2	−2	1	0	−1	0	1	−1	0	0	0	0	0	−1
−4	−4	8	−5	−1	3	−2	1	−1	0	1	0	−1	1	0	−1	0	0	0	0	−1
7	−4	0	0	2	−1	−1	0	1	−1	1	0	−1	0	1	−1	0	0	0	0	0
2	−1	0	−1	2	−2	0	0	1	−1	1	0	−1	0	1	0	−1	0	0	0	0
4	1	−4	2	1	−3	1	0	2	−2	1	0	0	−1	1	0	−1	0	0	0	0

Numerical values of the function $t_{x,j}$ for $1 \leq j \leq 21$ and $1 \leq x \leq 35$

1	0	0	0	0
$-G_{3,1}$	1	0	0	0
$G_{3,1}G_{4,2} - G_{4,1}$	$-G_{4,2}$	1	0	0
$-G_{5,1} + G_{3,1}G_{5,2} + G_{4,1}G_{5,3} - G_{3,1}G_{4,2}G_{5,3}$	$G_{4,2}G_{5,3} - G_{5,2}$	$-G_{5,3}$	1	0

Symbolic values of the function $t_{x,j}$ for $1 \leq j \leq x < 6$.

Table 1.2.2: *The effective "non-homogeneous-like" matrix entries corresponding to the power sum coefficients $t_{x,j}$.*

where $B_1 \approx 0.2614972128$ is the constant in Merten's theorem bounding finite sums over reciprocals of the primes $p \leq x$. Our result is formed by relating the odd-indexed sums $S_{\mathrm{odd}}(x)$ to the corresponding average order sums for $\omega(n)$, which we will denote here by $S(x)$. In particular, with the exception of tailoring a few corner cases on the end bounds of summation depending on whether x is even or odd, or whether it has an even divisor, the basic argument follows by noticing that

$$\omega(2n) = \begin{cases} \omega(n) + 1, & \text{if } n \text{ is odd;} \\ \omega(n), & \text{if } n \text{ is even.} \end{cases}$$

Then in the case where $x \equiv 1 \pmod 4$ (again ignoring bothersome corner case terms in the other three cases) we have that

$$\begin{aligned} S(x) &= S_{\mathrm{odd}}(x) + \sum_{n \leq \lfloor \frac{x}{2} \rfloor} \omega(2n) \\ &= S_{\mathrm{odd}}(x) + \sum_{n \leq \lfloor \frac{x}{4} \rfloor} (\omega(4n) + \omega(4n+2)) \\ &= S_{\mathrm{odd}}(x) + \sum_{n \leq \lfloor \frac{x}{4} \rfloor} (\omega(2n) + \omega(2n+1) + 1) \\ &= S_{\mathrm{odd}}(x) + S\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + \left\lfloor \frac{x}{4} \right\rfloor. \end{aligned}$$

By combining the original asymptotic results for $S(x)$ and $S(x/2)$, and making simplifications based on the fact that the ratio of two logarithms whose arguments differ only by a constant always tends to one, we obtain the precise statement given above. □

Proposition 1.10 (Asymptotics for $F(x)$). *For x sufficienty large and natural numbers $1 \leq i \leq x$, we have the following bounds:*

$$\begin{aligned} F\left(\left\lceil \frac{x-1}{i} \right\rceil\right) - F\left(\left\lceil \frac{x+1}{i+1} \right\rceil\right) &= \left\lceil \frac{x-1}{2i} \right\rceil - \left\lceil \frac{x+1}{2(i+1)} \right\rceil + \frac{1}{i+1} \log\left(\frac{i+1}{x+1}\right) - \frac{2B_1+1}{2} + o(x/i) \\ &= \frac{x-1}{2i(i+1)} - \frac{\log(x+1)}{i+1} - \frac{2B_1-1}{2} + o(x/i). \end{aligned}$$

Proof. We use the previous lemma together with the properties from Remark 1.6 providing exact special function expansions of both $f(n)$ and its summatory function $F(x)$. Notably, we obtain the formula that

$$F(x) = \frac{x}{2} \log\log(x) + \frac{(2B_1+1)x}{4} + \widetilde{C}_2(x) + O\left(\frac{x}{\log x}\right),$$

where the constant term (depending on x) corresponds to $\widetilde{C}_2(x) = \{x/4\} - \{x/2\} + 1$ and is always bounded in the range $0 < \widetilde{C}_2(x) < 2$ for any x. □

Notice that in practice, when $i \approx 1$ the coefficients $G_{x,i}$ are small constants so that we need not worry too much about the error term in Proposition 1.10 dominating our calculations. Since it is extremely difficult to account for the extra function f term present only under certain divisibility conditions on the x, i, we will typicially omit this term from our asymptotic analysis. The dominant term left out is as small as 1 and as large as order $\log(x)/\log\log x$ when x is primorial. The function $\omega(x)$ has average order of $\log\log x$, which is still smaller than the dominant term in the asymptotic formula in the proposition [4, §22.10–22.11].

1.2.5 Towards Better Bounds for $M(x)$, A Start

There is an *enormous amount* of cancellation going on in computing the inverse matrices from the initital matrix terms $G_{x,j}$. For my current suggested approach, consider for example the following expansions of the first column of the inverse matrices corresponding to the ordinary matrices $(u \cdot G_{i,j})_{1 \leq i,j \leq x}$, i.e., indexed with a formal variable to distinguish the number of weighted G terms:

$$\begin{aligned} t_{1,1} &= 1 && \longmapsto 1 \\ t_{2,1} &= -2u && \longmapsto -2 \\ t_{3,1} &= 2u^2 - 3u && \longmapsto -1 \\ t_{4,1} &= -2u^3 + 5u^2 - 4u && \longmapsto -1 \\ t_{5,1} &= 2u^4 - 7u^3 + 9u^2 - 5u && \longmapsto -1 \\ t_{6,1} &= -2u^5 + 7u^4 - 11u^3 + 12u^2 - 6u && \longmapsto 0 \\ t_{7,1} &= 2u^6 - 7u^5 + 13u^4 - 19u^3 + 17u^2 - 7u && \longmapsto -1 \\ t_{8,1} &= -2u^7 + 7u^6 - 15u^5 + 26u^4 - 30u^3 + 22u^2 - 9u && \longmapsto -1 \end{aligned}$$

$$
\begin{aligned}
t_{9,1} &= 2u^8 - 7u^7 + 15u^6 - 28u^5 + 39u^4 - 38u^3 + 27u^2 - 10u && \longmapsto 0 \\
t_{10,1} &= -2u^9 + 7u^8 - 15u^7 + 30u^6 - 46u^5 + 51u^4 - 48u^3 + 32u^2 - 11u && \longmapsto -2.
\end{aligned}
$$

We notice immediately that the corresponding coefficients of powers of u^k in these expansions are considerably more friendly and regular than the chaotic, oscillating and sign-changing versions of these coefficients exhibited in Table 1.2.2.

Coefficient asymptotics

We can next use the generic formula expanded in (1.10b) in combination with Proposition 1.10 to find asymptotic formulas for the (implied) coefficients of u^k in $t_{x,j}$ when $x \geq 1$ and $1 \leq j \leq x$. Notably, we can easily expand formulas for the first three cases of k and then attempt to extrapolate the behavior in our results from there. More precisely, we have[1]:

$$
\begin{aligned}
[u]t_{x,j} &= -G_{x,j} \approx C + \frac{\log(x+1)}{j+1} - \frac{x-1}{2j(j+1)} \\
[u^2]t_{x,j} &= \sum_{i=j+1}^{x-1} G_{i,j} \cdot G_{x,i} \\
&\approx \frac{\left[(2j-1)x - 4Cj^2 - (4C-2)j - 3\right]}{4j(j+1)\cdot x} (\log j - \log x) \log x \\
&\quad - \frac{1}{4j(j+1)^2 \cdot x} \Big[x^2 \left(-4c^2 j^3 + \left(-4c^2 - c - 2\right) j - 2c(4c-1)j^2 - 3c + 2\right) \\
&\qquad + x\left(4c^2 j^4 + 2\left(6c^2 - 3c + 1\right) j^2 + \left(4c^2 - c + 2\right) j + 3c(4c-1)j^3 + 2(c-2)\right) + 2cj^3 + 2(2c-1)j^2 \\
&\qquad + \log(x+1)\left(-4cj^3 - 4(2c-1)j^2 + x\left(2(2c+3)j^2 + 4(c-1)j + 4j^3 + 2\right) - 4cj + \left(-4j^2 - 2j + 2\right)x^2 - 4\right) \\
&\qquad + x^3(cj+c) + 2cj + 2 \Big] + \frac{C}{j+1} \log\left(\frac{x!}{j!}\right) \\
[u^3]t_{x,j} &= \sum_{i_1=j+1}^{x-1} \sum_{i_2=i_1+1}^{x-1} G_{i_1,j} \cdot G_{i_2,i_1} G_{x,i_2}.
\end{aligned}
$$

In general, it is apparent that we have a (not so immediately useful) recurrence relation relating these "*ghost*" (like ghost notes on a drum) *coefficients* of $t_{x,j}$ of the form

$$
[u^k]t_{x,j} = \sum_{i=m+1}^{x-1} G_{x,m} \cdot [u^{k-1}]t_{m,j},
$$

which can be used as above to obtain successive asymptotic approximations to the left-hand-side coefficients provided that a reasonable formula for $[u^{k-1}]t_{x,j}$ has already been computed beforehand.

Mertens function asymptotics

We can use (at least in these special cases) Abel's summation formula to add up the contribution of the k–weight terms for $k = 1, 2, 3$ in the formula for $M(x)$. For example, when $k = 1$ we can use the previous calculations to sum[2]

$$
\begin{aligned}
[u]M(x) &= C + \frac{1}{2}\log(x+1) - \frac{x-1}{4} + \sum_{k=2}^{x} \left(C + \frac{\log(x+1)}{k+1} - \frac{x-1}{2k(k+1)}\right) \frac{k}{\log k} \\
&= C_{1,5} - (C-1)\operatorname{li}(x^2) + \frac{2(C-1)x^2 + 4x - 2}{\log x} + \left(2\gamma + 2\log x - \frac{17}{4}\right) x + \frac{1}{2}\log(x+1) \\
&\quad - \log\log x + (1-\gamma)\operatorname{li}(x).
\end{aligned}
$$

[1] To simplify logarithms in certain terms we have used that $\log(i+1) \approx i$.

[2] With $f(t) := t/\log(t)$ and slightly fudging the indices for purposes of smooth integration to obtain that $A(x) := Cx\log(x)(\log x + \gamma - 1) - (x-1)(1 - 1/x)$. We also use that $\operatorname{li}(x^2) \equiv \operatorname{Ei}(2\log x)$ when $x \geq 2$.

Now since we are interested only in the limiting behavior of $M(x)/\sqrt{x}$ when x is large, we need not keep track of all of the smaller-order terms in completing our rudimentary asymptotic analysis using this method. More to the point, we are interested in the following terms: (TODO)

$$\frac{[u]M(x)}{\sqrt{x}} \sim \frac{(C-1)x^{3/2}}{2\log x} + \left(\frac{9}{4} + \frac{\gamma-1}{\log x}\right)\sqrt{x}. \tag{1.12}$$

Obviously, we expect to get a great deal of cancellation from the other coefficients of u^k for $k \geq 2$ when we add them into our formula below. *What we are looking to get infinitely often is a little off-cancellation of these gigantic terms so that we can recover something like a power of* $\log\log x$ *(or say like* $(\log\log\log x)^{5/4}$ *as in Gonek's unproven conjecture) remaining when we divide* $M(x)$ *through by* $\sqrt{x}$ *as sum:* $M(x)/\sqrt{x} = \sum_{k=1}^{x-1}[u]M(x)/\sqrt{x}$. Let's see what we get with the next coefficient contribution when $k = 2$[3]:

$$\begin{aligned}
\frac{[u^2]M(x)}{\sqrt{x}} &\sim -\frac{Cx^{3/2}}{8} + \frac{\sqrt{x}}{8}\left((4C+1)\log x + 8C^2 + C + 1 - \gamma\right) + \sqrt{x}f_2(x) - \frac{1}{\sqrt{x}}\int_2^x t\cdot f_2'(t)dt \\
&= -\frac{C^2x^{5/2}}{6} + \frac{x^{3/2}}{8}\left(2C\log\log x - (5C+4) - \frac{C(4C-1)}{\log x}\right) \\
&\quad - \frac{\sqrt{x}}{8}\Big((8C^2 - 3C - 17 - \gamma) + (4C+6)\log x - \frac{49C^2 - 12C - 4}{\log x} \\
&\quad + (24C^2 - 10C - 4)\log\log x + (8C+6)\log x\log\log x\Big).
\end{aligned}$$

An alternate route

The computations in the previous section are quickly becoming too tedious to manage on a case-by-case basis, so we turn to the next identity for sums of the coefficients $t_{x,j}$. In particular, we can start with the next equation from the relatively clean first formula in (1.12) to continue with the special case computations.

$$\begin{aligned}
M(x) &= -\sum_{i=1}^{x-1} G_{x,i}\cdot M(i) + \widetilde{X}_0(x) \\
[u^k]M(x) &= \sum_{i=1}^{x-1}\left\{\frac{1-x}{2}\left(\frac{1}{i} - \frac{1}{i+1}\right) + \frac{\log(x+1)}{i+1} + C\right\}[u^{k-1}]M(i) + \frac{x}{\log x}
\end{aligned}$$

Then following from (1.12) we can re-compute the contributions from the $k := 2, 3$ coefficient cases[4]:

$$\begin{aligned}
\frac{[u^2]M(x)}{\sqrt{x}} &\sim \frac{(2C-1)(C-1)x^{5/2}}{12\log x} + \frac{1}{16}\left(26C - 17 + \frac{4(3C-2)(\gamma-1)}{\log x}\right)x^{3/2} \\
&\quad + \frac{1}{4}\left(17\gamma - 35 + \frac{(2\gamma-1)(2\gamma-3)+C}{\log x} + 19\log x\right)\sqrt{x}
\end{aligned} \tag{1.13}$$

$$\frac{[u^3]M(x)}{\sqrt{x}} \sim \tag{1.14}$$

Why do I believe there are Bernoulli numbers involved? For large x, these should form a series roughly converging to some generating function (probably exponential and/or involving polynomial powers of $\log(x)$...

[3] Here we have selected $A(x) := x$ and differentiated the long expression from the previous subsection with respect to j, denoted by $f_2'(j)$ in the integral formula below.

[4] In respective order, we take $f(t) := [u^{k-1}]M(t)$ and $A_1(x) := 1 - \frac{1}{x}$, $A_2(x) := H_x - 1 \sim \log x + \gamma - 1$, and $A_3(x) := x$.

1.2.6 Necessary conditions of the limiting behavior of the matrix solutions

RH is equivalent to the limit ...
Questions on properties;
Reciprocal zeta function integral;

However, we expect in fact that $M(x)$ changes sign inifinitely often and is unbounded (though still oscillating) in both directions [19, A028442] [14].

1.3 Conclusions

1.3.1 Summary

We have touched on a "famous" open conjecture showing that along the sequence of sufficiently large primes $q \sim m \log m$, we have that $|M(q)|/\sqrt{q}$ grows increasingly unbounded as $q \rightarrow \infty$. We do not consider the local oscillations of the functions between the large odd primes q here, though our exact formulas bounding the Mertens function employed in the proof of Theorem **??** certainly suggest an approach to a more complicated analysis of those properties as well. As already mentioned in Remark 1.3, the deep connection of this class of divisor sums to Ramanujan's well-studied sum, $c_q(n)$, suggests new approaches for the general limiting cases of any subsequence of natural numbers approaching infinity. There is much work that still needs to be done to strategically utilize these new formulas to their fullest extent in finding new asymptotic relations for $M(x)$ and $M_\alpha^*(x)$ when $\alpha \geq 1$ is an integer. We also cannot neglect to acknowlege the deeper analytic connections between these summatory functions over the Möbius function, the Riemann zeta function, and its non-trivial zeros.

1.3.2 Unexpected results

One unexpected result from the start of the project is that the method we employ here to produce the lower bounds on the classical case of $M(x)$ nicely extend to the "weighted" Mertens function cases of $M_\alpha^*(x)$ defined in the introduction. We believe that we are the first to formally define these generalized analogs to the Mertens' function $M(x)$ and note that these definitions fall naturally out of the component divisor sums in the formulas proved by the author in [18]. Another unexpected bound for the summatory function for Liouville's function follows from the key re-characterization of the divisor sums in Lemma **??** according to the identity in (2.2) from Lemma 2.2 in the last appendix summarizing the preliminary theorem from [18]. The realization of this new identity for the Möbius function involving Liouville's function $\lambda(n)$ allows us to effectively reuse the bounds we obtain to prove Theorem **??** in Section 1.2 in the new context of the summatory functions $L_{-\alpha}(x)$ which we define and consider in Section **??**.

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Chapter 2

Short Appendix: The proof of Theorem 1.1

2.1 Motivation

Given that the core elements of the proof of the main result in Theorem **??** first proved in this article follow from a careful asymptotic treatment of the first results in Theorem 1.1, and its restatement given in (1.2), we provide this appendix on the proof of these first results given by the author in 2017 [18]. The next results present a concise statement of the key results in the proof of that theorem given in the original reference.

Definition 2.1 (Notation). For $n \geq 1$ and any fixed indeterminate q, we define the next rational functions related to the logarithmic derivatives of the cyclotomic polynomials, $\Phi_n(q)$.

$$\begin{aligned}\Pi_n(q) &:= \sum_{j=0}^{n-2} \frac{(n-1-j)q^j(1-q)}{(1-q^n)} = \frac{(n-1)+nq-q^n}{(1-q)} \\ \widetilde{\Phi}_n(q) &:= \frac{1}{q} \cdot \frac{d}{dw}\left[\log \Phi_n(w)\right]\Big|_{w \to \frac{1}{q}}.\end{aligned} \tag{2.1}$$

For fixed q and any $n \geq 1$, we define the component sums, $\widetilde{S}_{i,n}(q)$ for $i = 0, 1, 2$ as follows:

$$\begin{aligned}\widetilde{S}_{0,n}(q) &= \sum_{\substack{d \mid n \\ d > 1 \\ d \neq p^k, 2p^k}} \widetilde{\Phi}_d(q) \\ \widetilde{S}_{1,n}(q) &= \sum_{p \mid n} \Pi_{p^{\nu_p(n)}}(q) \\ \widetilde{S}_{2,n}(q) &= \sum_{2p \mid n} \Pi_{p^{\nu_p(n)}}(q).\end{aligned}$$

2.2 Statements and proof of key components

Lemma 2.2 (Key Characterizations of the Tau Divisor Sums). *For integers $n \geq 1$ and any indeterminate q, we have the following expansion of the functions in* (2.1)*:*

$$\widetilde{\Phi}_n(q) = \sum_{d \mid n} \frac{d \cdot \mu(n/d)}{(1-q^d)}.$$

In particular, we have that

$$\begin{aligned}\widetilde{S}_{0,n}(q) &= \sum_{d \mid n} \sum_{r \mid d} \frac{r \cdot \widetilde{\chi}_{\mathrm{pp}}(d) \cdot \mu(d/r)}{(1-q^r)} \\ &= \sum_{d \mid n} \sum_{r \mid d} \frac{r \cdot \widetilde{\chi}_{\mathrm{pp}}(d) \cdot |\mu(d/r)| \lambda(d/r)}{(1-q^r)}.\end{aligned}$$

Proof. The proof is essentially the same as the example given in the reference. Since we can refer to this illustrative example, we only need to sketch the details to the remainder of the proof. In particular, we notice that since we have the known identity for the cyclotomic polynomials given by

$$\Phi_n(x) = \prod_{d|n}(1-q^d)^{\mu(n/d)} = \prod_{d|n}(1-q^d)^{|\mu(n/d)|\lambda(n/d)}, \tag{2.2}$$

where $\lambda(n) = (-1)^{\Omega(n)}$ matches the value of $\mu(n)$ exactly when n is squarefree, we can take logarithmic derivatives to obtain that

$$\frac{1}{x}\cdot\frac{d}{dq}\left[\log\left(1-q^d\right)^{\pm 1}\right]\Bigg|_{q\to 1/q} = \mp\frac{d}{q^d\left(1-\frac{1}{q^d}\right)} = \pm\frac{d}{1-q^d},$$

which applied inductively leads us to our result. □

Theorem 2.3 (Exact Formulas for the Generalized Sum-of-Divisors Functions). For any fixed $\alpha\in\mathbb{C}$ and natural numbers $x\geq 1$, we have the following generating function formula:

$$\sigma_\alpha(x) = H_x^{(1-\alpha)} + [q^x]\left(\sum_{n=1}^{x}\widetilde{S}_{0,n}(q)n^\alpha + \widetilde{S}_{1,n}(q)n^\alpha + \widetilde{S}_{2,n}(q)n^\alpha\right).$$

Proof. We begin with a well-known divisor product formula involving the cyclotomic polynomials when $n\geq 1$ and q is fixed:

$$q^n - 1 = \prod_{d|n}\Phi_d(q).$$

Then by logarithmic differentiation we can see that

$$\begin{aligned}\frac{q^n}{1-q^n} &= -1 + \frac{1}{n(1-q)} + \sum_{\substack{d|n\\ d>1}}\widetilde{\Phi}_d(q) \\ &= -1 + \frac{1}{n(1-q)} + \widetilde{S}_{0,n} + \widetilde{S}_{1,n} + \widetilde{S}_{2,n}.\end{aligned} \tag{2.3}$$

The last equation is obtained from the first expansion above by noting the equivalence of the next two sums as

$$\Pi_n(1/q) = \widetilde{\Phi}_n(q) = \sum_{j=0}^{n-2}\frac{(n-1-j)q^j(1-q)}{1-q^n},$$

where it is known that

$$\sum_{\substack{d|n\\ d>1}}\widetilde{\Phi}_d(q) = \frac{nq^{n-1}}{q^n-1} - \frac{1}{q-1} = \frac{(n-1)q^{n-2}+(n-2)q^{n-3}+\cdots+2q+1}{q^{n-1}+q^{n-2}+\cdots+q+1}.$$

Here we are implicitly using the known expansions of the cyclotomic polynomials which condense the order n of the polynomials by exponentiation of the indeterminate q when n contains a factor of a prime power given by

$$\Phi_{2p}(q) = \Phi_p(-q), \Phi_{p^k}(q) = \Phi_p\left(q^{p^{k-1}}\right), \Phi_{p^k r}(q) = \Phi_{pr}\left(q^{p^{k-1}}\right), \Phi_{2^k}(q) = q^{2^{k-1}}+1, \tag{2.4}$$

for p and odd prime, $k\geq 1$, and where $p\nmid r$. Finally, we complete the proof by summing the right-hand-side of (2.3) over $n\leq x$ times the weight n^α to obtain the x^{th} partial sum of the Lambert series generating function for $\sigma_\alpha(x)$ [4, §17.10] [15, §27.7], which since each term in the summation contains a power of q^n is $(x+1)$-order accurate to the terms in the infinite series. □

Proposition 2.4 (Series Coefficients of the Component Sums). *For any fixed $\alpha\in\mathbb{C}$ and integers $x\geq 1$, we have the following components of the partial sums of the Lambert series generating functions in Theorem 2.3:*

$$[q^x]\sum_{n=1}^{x}\widetilde{S}_{0,n}(q)n^\alpha = \sum_{d|n}\tau_x^{(\alpha)}(d) \tag{i}$$

$$[q^x]\sum_{n=1}^{x}\widetilde{S}_{1,n}(q)n^{\alpha} = \sum_{p\le x}\sum_{k=1}^{\varepsilon_p(x)+1} p^{\alpha k-1}H^{(1-\alpha)}_{\left\lfloor\frac{x}{p^k}\right\rfloor}\left(p\left\lfloor\frac{x}{p^k}\right\rfloor - p\left\lfloor\frac{x}{p^k}-\frac{1}{p}\right\rfloor - 1\right) \tag{ii}$$

$$[q^x]\sum_{n=1}^{x}\widetilde{S}_{2,n}(q)n^{\alpha} = \sum_{3\le p\le x}\sum_{k=1}^{\varepsilon_p(x)+1} \frac{p^{\alpha k-1}}{2^{1-\alpha}}H^{(1-\alpha)}_{\left\lfloor\frac{x}{2p^k}\right\rfloor}(-1)^{\left\lfloor\frac{x}{p^{k-1}}\right\rfloor}\left(p\left\lfloor\frac{x}{p^k}\right\rfloor - p\left\lfloor\frac{x}{p^k}-\frac{1}{p}\right\rfloor - 1\right). \tag{iii}$$

Proof. The identity in (i) follows from Lemma 2.2. Since $\Phi_{2p}(q) = \Phi_p(-q)$ for any prime p, we are essentially in the same case with the two component sums in (ii) and (iii). We outline the proof of our expansion for the first sum, $\widetilde{S}_{1,n}(q)$, and note the small changes necessary along the way to adapt the proof to the second sum case. By the properties of the cyclotomic polynomials expanded in (2.4) , we may factor the denominators of $\Pi_{p^{\varepsilon_p(n)}}(q)$ into smaller irreducible factors of the same polynomial, $\Phi_p(q)$, with inputs varying as special prime-power powers of q. More precisely, we may expand

$$\widetilde{S}_{1,n}(q) = \sum_{p\le n}\sum_{k=1}^{\varepsilon_p(n)} \underbrace{\frac{\sum_{j=0}^{p-2}(p-1-j)q^{p^{k-1}j}}{\sum_{i=0}^{p-1} q^{p^{k-1}i}}}_{:=Q^{(n)}_{p,k}(q)} \cdot p^{k-1}.$$

In performing the sum $\sum_{n\le x} Q^{(n)}_{p,k}(q)p^{k-1}n^{\alpha-1}$, these terms of the $Q^{(n)}_{p,k}(q)$ occur again, or have a repeat coefficient, every p^k terms, so we form the coefficient sums for these terms as

$$\sum_{i=i}^{\left\lfloor\frac{x}{p^k}\right\rfloor}\left(ip^k\right)^{\alpha-1}\cdot p^{k-1} = p^{k\alpha-1}\cdot H^{(1-\alpha)}_{\left\lfloor\frac{x}{p^k}\right\rfloor}.$$

We can also compute the inner sums in the previous equations exactly for any fixed t as

$$\sum_{j=0}^{p-2}(p-1-j)t^j = \frac{(p-1)+pt-t^p}{(1-t)^2},$$

where the corresponding paired denominator sums in these terms are given by $1+t+t^2+\cdots+t^{p-1} = (1-t^p)/(1-t)$. We now assemble the full sum over $n\le x$ we are after in this proof as follows:

$$\sum_{n\le x}\widetilde{S}_{1,n}(q)\cdot n^{\alpha-1} = \sum_{p\le x}\sum_{k=1}^{\varepsilon_p(x)} p^{k\alpha-1}H^{(1-\alpha)}_{\left\lfloor\frac{x}{p^k}\right\rfloor}\frac{(p-1)-pq^{p^{k-1}}+q^{p^k}}{(1-q^{p^{k-1}})(1-q^{p^k})}.$$

The corresponding result for the second sums is obtained similarly with the exception of sign changes on the coefficients of the powers of q in the last expansion.

We compute the series coefficients of one of the three cases in the previous equation to show our method of obtaining the full formula. In particular, the right-most term in these expansions leads to the double sum

$$\begin{aligned} C_{3,x,p} &:= [q^x]\frac{q^{p^k}}{(1\mp q^{p^{k-1}})(1\mp q^{p^k})} \\ &= [q^x]\sum_{n,j\ge 0}(\pm 1)^{n+j}q^{p^{k-1}(n+p+jp)}. \end{aligned}$$

Thus we must have that $p^{k-1}|x$ in order to have a non-zero coefficient and for $n := x/p^{k-1}-jp-p$ with $0\le j\le x/p^k-1$ we can compute these coefficients explicitly as

$$C_{3,x,p} := (\pm 1)^{\lfloor x/p^{k-1}\rfloor}\times\sum_{j=0}^{\lfloor x/p^k-1\rfloor} 1 = (\pm 1)^{\lfloor x/p^{k-1}\rfloor}\left\lfloor\frac{x}{p^k}-1\right\rfloor + 1 = (\pm 1)^{\lfloor x/p^{k-1}\rfloor}\left\lfloor\frac{x}{p^k}\right\rfloor.$$

With minimal simplifications we have arrived at our claimed result in the proposition. ∎

Proposition 2.5 (Asymptotic Bound for the Tau Function Divisor Sum). *Let $x = q^r$ denote a power of the large prime q for some $r \geq 1$. Then when the x tending to infinity of these forms is sufficiently large, we obtain*

$$|\tau_0(x)| \geq \widetilde{C} \cdot x \log\log(x-1) + \frac{Ax}{2},$$

for some absolute constant which we may effectively take as $\widetilde{C} = \frac{1}{2}$ for x sufficiently large, and where $A \approx 0.2614972128$ is the constant from Mertens' theorem[1].

Proof. By the statement of Theorem 1.1 rephrased in (1.2), we see that

$$\begin{aligned} |\tau_0(x)| &= \left| x - \sigma_1(x) + S_1^{(1)}(x) + S_2^{(1)}(x) \right| \\ &= \left| S_1^{(1)}(q^r) + S_2^{(1)}(q^r) - \left(1 + q + \cdots + q^{r-1}\right) \right| \\ &\geq \left| S_1^{(1)}(q^r) + S_2^{(1)}(q^r) \right| - \frac{x-1}{x^{\frac{1}{r}} - 1}. \end{aligned}$$

We next use the result of *Mertens' theorem* which implies that [12, §6.3] [1, §4.9] [4, §22.8] [15, §27.11]

$$\sum_{p \leq x} \frac{1}{p} = \log\log(x) + A + O\left(\frac{1}{\log x}\right),$$

where $A \approx 0.2614972128$ is a limiting constant [20, *cf.* §1.3]. In particular, when x is large we can expand the sum for $S_1^{(1)}(x)$ as

$$\begin{aligned} S_1^{(1)}(x) &= \sum_{\substack{2 \leq p < q^r \\ p \neq q}} p \cdot \left\lfloor \frac{q^r}{p} \right\rfloor \left(\left\lfloor \frac{q^r}{p} \right\rfloor - \left\lfloor \frac{q^r}{p} - \frac{1}{p} \right\rfloor - \frac{1}{p} \right) + \sum_{k=1}^{r+1} q^k \left\lfloor q^{r-k} \right\rfloor \left(\left\lfloor q^{r-k} \right\rfloor - \left\lfloor q^{r-k} - \frac{1}{q} \right\rfloor - \frac{1}{q} \right) \\ &= \sum_{\substack{2 \leq p < q^r \\ p \neq q}} -\frac{p}{p} \left[\frac{q^r}{p} - \left\{ \frac{q^r}{p} \right\} \right] + \sum_{k=1}^{r} q^r \left(1 - \frac{1}{q}\right) \\ &= C_1 \pi(q^r - 1) - q^r \left(\log\log(q^r - 1) + A + O\left(\frac{1}{\log(q^r - 1)}\right) \right) + r \cdot q^{r-1}(q-1) \\ &\sim \frac{C_1(x-1)}{\log(x-1)} - x\left(\log\log(x-1) + A\right) + r \cdot x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1\right), \end{aligned}$$

and similarly, the sum $S_2^{(1)}(x)$ is expanded as

$$\begin{aligned} S_2^{(1)}(x) &= \sum_{\substack{2 \leq p < q^r \\ p \neq q}} p \left\lfloor \frac{q^r}{2p} \right\rfloor \cdot \frac{1}{p} + \sum_{k=1}^{r} q^k \left\lfloor \frac{q^{r-k}}{2} \right\rfloor \frac{(q-1)}{q} + 2C_3 \left\lfloor \frac{q^r}{4} \right\rfloor \cdot \frac{1}{2} \\ &= \sum_{\substack{2 \leq p < q^r \\ p \neq q}} \left(\frac{q^r}{2p} - C_4 \right) + \sum_{k=1}^{r} q^k \left(\frac{q^{r-k}}{2} - C_5 \right) \frac{(q-1)}{q} + C_3 \left(\frac{q^r}{4} - C_6 \right) \\ &\sim \frac{1}{2} x \left(\log\log(x-1) + A\right) + \frac{C_2 C_4 (x-1)}{\log(x-1)} + \frac{r}{2} x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1\right) - C_5(x-1) + \frac{C_3}{4} x - C_3 C_6. \end{aligned}$$

Hence when we add these two sums cancellation of symmetric terms results in

$$S_1^{(1)}(x) + S_2^{(1)}(x) \sim \frac{(C_2 C_4 - C_1)(x-1)}{\log(x-1)} + \frac{x}{2}\left(\log\log(x-1) + A\right),$$

which proves our result. In the previous equation, we have that each constant $0 \leq C_i < 1$ since the fractional parts corresponding to the floor function terms in the respective bounds for $S_i^{(\alpha+1)}(x)$ are in this range. ∎

[1] More precisely, as in Theorem 4 from [20, §1.3] we have

$$A = \gamma + \sum_{n \geq 2} \mu(n) \frac{\log(\zeta(n))}{n}.$$

Chapter 3

TODO / Woring Topics

Corollary 3.1 (Newer Upper Bounds on the Mertens Function and the Function $q(x)$). *We define the auxiliary function $q(x)$ as is standard in the references to be the scaled Mertens function $q(x) := M(x)/\sqrt{x}$ for all $x \in \mathbb{C}$ where we take $M(x)$ to be defined. Then we have the following inequality with sharpened outer sub-square-root error term included for all real $x > 685$ given by*

$$|q(x)| \leq C_{0,M}(x) + \frac{2 \times 0.58782}{\sqrt{x}} \int_{\log(3)}^{\log(x+1)} \left[\log\left\{\log(x+1) - \log(e^s+1)\right\} + \frac{2B_1+1}{4} + o\left(\frac{x}{e^s}\right)\right] \frac{e^s}{\log^{11/9}(e^s+1)}\, ds,$$

where we define the x-dependent constant term from the integral expression above to be

$$C_{0,M}(x) = \frac{1}{\sqrt{x}} \log\log\left(\frac{x+1}{2}\right) + \frac{2B_1+1}{4\sqrt{x}} + o\left(\sqrt{x}\right).$$

We similarly obtain the next estimate on the double exponential of the derivative of the Mertens function for all sufficiently large real x tending to infinity.

$$\exp\left(\exp\left\{\frac{d}{dx}\left[|M(x)| - \log\log\left(\frac{x+1}{2}\right)\right] \times \frac{\log^{11/9}(x+2)}{x} \cdot \frac{(x+1)}{2 \times 0.58782} - \frac{2B_1+1}{4} + o(x)\right\}\right) \longrightarrow 1$$

Proof. There are many known upper bounds for the Mertens function, integrals of it, and related functions and sums. One of my current favorites in so much as it is comparitively simple effective bound that is easy to integrate (both symbolically and numerically) provides the following bound for all $x > 685$ [**?**]:

$$|M(x)| < \frac{0.58782 \cdot x}{(\log x)^{11/9}}.$$

We can then apply this estimate with Proposition 1.10 to the right-hand-side Mertens terms in the inner sums of Corollary 1.9b to obtain the next integral.

$$|M(x)| \leq \left\{F\left(\left\lfloor\frac{x-1}{i}\right\rfloor\right) - F\left(\left\lceil\frac{x+1}{i+1}\right\rceil\right)\right\}\Bigg|_{i=1} + 2\int_2^x \left[F\left(\left\lfloor\frac{x-1}{t}\right\rfloor\right) - F\left(\left\lceil\frac{x+1}{t+1}\right\rceil\right) + o(x/t)\right] \frac{0.58782 \cdot t}{(\log t)^{11/9}}\, dt.$$

We then substitute $s := \log(t)$ in the previous equation to determine the first result which is a gamma function (exponential integral) in basic form. The second result we cite above is an immediate consequence of the inequality involving the first integral where we differentiate with respect to x, shift terms, and then finally exponentiate twice to find the limiting behavior for sufficiently large essentially $x > 685$ or so. $\square$

Remark 3.2 (Effective Summation Bounds on an Inverse Matrix of Mostly Zeros). Obtaining specific asymptotic approximations of the $t_{x,j}$ for fixed j at large x will vary according to the recursive-based definition of the triangle defined in the previous equation. However, what is clear is that $t_{x,x} = 1$, the second to last term in each row indexed by x is valued -1, and that $t_{x,x-r}$ is zero-valued for all $1 \leq r < \lfloor\sqrt{x}\rfloor$. Additionally, since

$$\frac{2(x+i-1)}{(i+1)^2} \leq \frac{2x-2}{i} - \frac{2x}{i+1} = \frac{2(x+i-1)}{i(i+1)} \leq \frac{2(x+i-1)}{i^2},$$

we can obtain approximate formulas (depending on x) for the columnwise distances between the $t_{x,x-r} \neq 0$ for $r \geq 0$ (these coefficients are apparently sparse by computation). For a fixed x, let i_m $(m \geq 0)$ denote the m^{th} $(x-r)$ term such that $t_{x,x-r} \neq 0$ in row x, i.e., so that $t_{x,i_m} \neq 0$ for all finite m where there is a sequence of increasing $r_m < r_{m+1}$ such that $i_m = x - r_m$. Then by solving a quadratic inequality obtained by bounding the previous equation, we can see approximately that $i_m = x - \left\lfloor \frac{\sqrt{x}}{2^m} \right\rfloor$ for $m \geq 0$.

(TODO)