defs:

$$\hat{F}(s,z) := \prod_p \left[\left(1+\frac{z}{p^s-1}\right)\left(1-\frac{1}{p^s}\right)^z\right], \quad |z|<2, \ \mathrm{Re}(z)\geq 1$$
so that $\left|\frac{z}{p}\right|<1$

$$\hat{G}(z) := \hat{F}(1,z)/\Gamma(1+z)$$

claim that: $\hat{G}\left(\frac{k-1}{\log\log x}\right) \asymp 1$ (e.g., $\gg$ and $\ll 1$) as $x\to\infty$ whenever $1\leq k\leq \log\log x$.

recall Merten's (second) Theorem:

$$\sum_{p\leq x}\frac{1}{p} = \log\log x + B + O\left(\frac{1}{\log x}\right), \quad B\simeq 0.26\ldots,$$
an abs. constant

steps: ① $\prod_p\left(1+\frac{z}{p-1}\right)^{-1} \in e^{o(1)}[1-o(1), 1]$;

② $\prod_p\left(1-\frac{1}{p}\right)^z \in e^{o(1)}[1-o(1),1]$;

③ For $z:=\frac{k-1}{\log\log x}$, $1\leq k\leq \log\log x$,

$\Gamma(1+z)\asymp 1$. (easiest step first)

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③/: $\Gamma(1+z) = (1+z)^{-1}\Gamma(2+z)$

$\geq (1+z)^{-1}\Gamma\left(2-\frac{1}{\log\log x}\right) \gg 1$

(at the upper end, when $k\to \ell\ell x$)

$\Gamma(1+z)\leq \Gamma(1) = 1 \ll 1$ (at the lower end, as $k\to 1^+$).

①and②: (These are similar):

For①:

②

$$\log \prod_p \left(1+\frac{z}{p-1}\right)^{-1} \sim -z\cdot \sum_p \frac{1}{p-1} = -z\cdot\sum_p \frac{1}{p}(1+o(1))$$

$$= -z\cdot \lim_{x\to+\infty}\left(\log\log x + B + O\left(\frac{1}{\log x}\right)\right) \quad (*).$$

So for $|z| = r := \frac{k-1}{\log\log x}$, $1 \le k \le \log\log x$,

$$(*) = -Bz - (k-1) + o_z(1)$$

$$\Rightarrow \prod_p \left(1+\frac{z}{p-1}\right)^{-1} \in e^{o(1)-Bz}\left[1-o(1), 1\right] \asymp 1 \checkmark$$

Similarly, for②: Again, $|z| := \frac{k-1}{\log\log x}$:

$$\prod_p \left(1-\frac{1}{p}\right)^{z} \sim \exp(-(k-1))\cdot(1\pm o(1))$$

$$\in [1, e] \asymp 1.$$

Again, we write $A(x) \asymp B(x)$ if $A \ll B$, and $B \geq 0$ s.t. $B \ll A$ (same order up to an abs. limiting constant factor).

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①+②+③ $\Longrightarrow$ The claim: $\hat{G}\left(\frac{k-1}{\log\log x}\right) \asymp e^{o(1)} \asymp 1$

$\square$