Partition function transformations:

$\longrightarrow$ idea: perform an invertible transformation on a variably signed sequence to "decode it" to a form where we can predict the signs...

definitions:

① $P_1(N) := [q^N] \prod_{m \geq 1} (1+q^m)$, $N \geq 0 \mapsto \{1,1,1,2,2,3,4,5,6,8,10,12,15,\ldots$

$P_2(N) := [q^N] \prod_{m \geq 1} (1+q^m)^{-1}$, $N \geq 0 \mapsto \{1,-1,0,-1,1,-1,1,-1,2,-2,2,3,\ldots$

② Let f be an arithmetic function such that $f(1) \neq 0$. Then its Dirichlet inverse f^{-1} exists, $(f * f^{-1})(N) = \delta_{N,1}$ and can be computed recursively as

$$f^{-1}(N) = \begin{cases} \frac{1}{f(1)}, & N = 1 \\ -\sum_{\substack{d \mid N \\ d < N}} \frac{f^{-1}(d) \cdot f\left(\frac{N}{d}\right)}{f(1)}, & N \geq 2 \end{cases}$$

heuristic: if f is "nice" and well behaved, then f^{-1} is signed, oscillatory, and hard to work wi

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encode the transform:

$$S_{f^{-1}}(N) := \sum_{J=1}^{N} f^{-1}(J) \cdot P_1(N-J)$$

decode the transform:

$$f^{-1}(N) = \sum_{J=1}^{N} S_{f^{-1}}(J) \cdot P_2(N-J)$$

Properties of the partition functions:

- $\forall N \geq 0, \quad P_1(N) \geq 1$
- $\forall N \geq 0, \quad |P_2(N)| \geq 1$
- P_2 alternates in sign: $\mathrm{sgn}(P_2(N)) = (-1)^{N}$

Key observation:

for any (presumably bounded) arithmetic f s.t. $f(1) \neq 0$, the transform, or encoding, of its inverse $f^{-1}(N)$ satisfies:

→ for all large enough $N \geq N_f$, $\mathrm{sgn}(S_{f^{-1}}(N))$ is constant, either always $+1$, or always -1!

→ That is, the encoding takes a crazy, signed, unpredictable f^{-1}, and somehow scales it so that its sign is predictable for all sufficiently large $N \geq N >> N_0$

→ Examples are given in the attached notebook file.

Questions: how do we prove this? Are there any reasonable restrictions to place, or bounds to satisfy, on f so that this property is in fact always true?

(Enjoy, Maxie) → I have plenty of applications to cite if we can give a proof of this "magic" theorem.