

We have that

$$\# \{3 \leq n \leq x : \sqrt{2}(n) \leq \sqrt{2}n\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{2}x}\right)$$

→ note the bounds is flipped in your comment

Using the reversed inequality. From for the function $B(x, r)$, $1 \leq r < 2$ and $x \geq 2$, it should be that

$$B(x, 1) \ll x$$

$$\Rightarrow \frac{1}{x} |N_w(x)| \ll 1$$

Really, it still gives the correct result:

$$\left| \frac{N_w(x)}{D_w(x)} \right|^{-1} = O\left(\frac{1}{\sqrt{2}x}\right) = o(1)$$

So, I guess I need to write at the end of the proof that

$$1 + o(1) \ll \left| \frac{D_w(x)}{A_w(x)} \right| \ll 1 + o(1),$$

but the idea is the same: You capture the dominant asymptotic main term by only summing over $\Pi_k(x)$ in the range $1 \leq k \leq \ell x$ where we get the nice uniform bounds. That is what I use in Section 4 that is important here.