The 105 Problem

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1 The Hypothesis

The Working Hypothesis I told you about before is good; but there is a slightly weaker one that should be easier to satisfy, which we will now call, simply, "The Hypothesis". In this section I will describe it; in the next section I will describe how the Hypothesis may be iterated to solve the 105 problem; and then in Section 3 I will discuss a "generative process" for verifying it, that will also include the meet-in-the-middle trick.

First, though, we need a definition: given a set of vectors $V := \{(\alpha_i, \beta_i, \gamma_i, \delta_i)\}_{i=1}^m$, with

$$\alpha_i = \frac{d_{i,1}}{5} + \frac{d_{i,2}}{5^2} + \cdots + \frac{d_{i,k}}{5^k},\ d_{i,j} \in \{0,1,2,3,4\},\ k = k(i),$$

$$\beta_i = \frac{e_{i,1}}{5} + \frac{e_{i,2}}{5^2} + \cdots + \frac{e_{i,\ell}}{5^k},\ e_{i,j} \in \{0,1,2,3,4\},$$

$$\gamma_i = \frac{f_{i,1}}{7} + \frac{f_{i,2}}{7^2} + \cdots + \frac{f_{i,\ell}}{7^\ell},\ f_{i,j} \in \{0,1,2,3,4,5,6\},\ \ell = \ell(i),$$

and

$$\delta_i = \frac{g_{i,1}}{7} + \frac{g_{i,2}}{7^2} + \cdots + \frac{g_{i,\ell}}{7^\ell},\ g_{i,j} \in \{0,1,2,3,4,5,6\},$$

where $k = k(i)$ and $\ell = \ell(i)$ are certain positive integers that are functions of i, we say that V *covers* a subset $S \subseteq [0,1]^4$ if for every vector $(x,y,z,w) \in S$ we have that there exists i such that

$$0 < x - \alpha_i,\ y - \beta_i < 5^{-k(i)},\ \text{and } 0 < z - \gamma_i,\ w - \delta_i < 7^{-\ell(i)}.$$

1.1 Hypothesis with parameters (B, C)

Let S denote the set of all vectors in $(\alpha, \beta, \gamma, \delta) \in [0,1]^4$, such that

$$1/5 \ \le \ \alpha \ < 1, \ 1/7 \ \le \ \gamma \ < \ 1, \tag{1}$$

and where

$$\beta \ = \ \frac{d_1}{5} + \frac{d_2}{5^2} + \cdots, \ d_1, ..., d_B \in \{0,1,2\}, \ d_{B+1}, d_{B+2}, ... \in \{0,1,2,3,4\}, \tag{2}$$

and

$$\delta \ = \ \frac{e_1}{5} + \frac{e_2}{5^2} + \cdots, \ e_1, ..., e_C \in \{0,1,2,3\}, \ e_{C+1}, e_{C+2}, ... \in \{0,1,2,3,4,5,6\}. \tag{3}$$

There exists a finite sequence $\{(\alpha_i, \beta_i, \gamma_i, \delta_i)\}_i$ that covers S, such that for each i if

$$\alpha_i \ = \ \frac{d'_1}{5} + \cdots + \frac{d'_k}{5^k}, \ \beta_i \ = \ \frac{e'_1}{5} + \cdots + \frac{e'_k}{5^k}, \ k = k(i),$$

$$\gamma_i \ = \ \frac{f'_1}{7} + \cdots + \frac{f'_\ell}{7^\ell}, \ \delta_i \ = \ \frac{g'_1}{7} + \cdots + \frac{g'_\ell}{7^\ell}, \ \ell = \ell(i),$$

then there exists a sequence of integers

$$n_1 \ < \ n_2 \ < \ n_3 \ < \ \cdots \ < \ n_t, \ t = t(i), \ 3^{n_t} < \min(5^{k-B}, 7^{\ell - C}), \tag{4}$$

such that if

$$\alpha_i \left(\frac{1}{3^{n_1}} + \cdots + \frac{1}{3^{n_t}} \right) + \beta_i \ = \ \frac{r_1}{5} + \frac{r_2}{5^2} + \cdots, \ \text{all } r_j \in \{0,1,2,3,4\}$$

then

$$r_1, r_2, ..., r_{k-1} \ \in \ \{0,1,2\}, \ r_k \in \{0,1\}, \ k = k(i),$$

and if

$$\gamma_i \left(\frac{1}{3^{n_1}} + \cdots + \frac{1}{3^{n_t}} \right) + \delta_i \ = \ \frac{s_1}{7} + \frac{s_2}{7^2} + \cdots, \ \text{all } s_j \in \{0,1,2,3,4,5,6\},$$

then

$$s_1, s_2, ..., s_{\ell-1} \ \in \ \{0,1,2,3\}, \ s_\ell \in \{0,1,2\}, \ \ell = \ell(i).$$

1.2 A corollary of the Hypothesis, that is more useful

The version of the Hypothesis in the previous subsection has lots of subscripts and auxiliary variables to keep track of. In this subsection, we abstract away some of these variables – the version in this subsection is best for verifying the 105 Conjecture (in terms of the amount of notation), and the version in the previous subsection is best for algorithmic verification. The new Hypothesis with parameters B and C is:

Let S denote the set of all vectors $(\alpha, \beta, \gamma, \delta) \in [0,1]^4$ such that (1), (2), and (3) all hold. Then, there exists a sequence of integers

$$0 \ < \ n_1 \ < \ n_2 \ < \ \cdots \ < \ n_t,$$

and integers $k, \ell > 0$, all satisfying

$$3^{n_t} \ < \ \min(5^{k-B}, 7^{\ell-C}) \tag{5}$$

and satisfying

$$\alpha \left(\frac{1}{n_1} + \cdots + \frac{1}{n_t} \right) + \beta \ = \ \frac{r_1}{5} + \frac{r_2}{5^2} + \cdots, \ \ r_1, ..., r_k \in \{0,1,2\}, \tag{6}$$

(All the other r_j's belong to $\{0,1,2,3,4\}$.) and

$$\gamma \left(\frac{1}{3^{n_1}} + \cdots + \frac{1}{3^{n_t}} \right) + \delta \ = \ \frac{s_1}{7} + \frac{s_2}{7^2} + \cdots, \ \ s_1, ..., s_\ell \in \{0,1,2,3\}. \tag{7}$$

(All the other s_j's belong to $\{0,1,2,3,4,5,6\}$.)

Proof that the Hypothesis implies this Corollary. Given that (1), (2), and (3) all hold, we deduce from the Hypothesis the existence of $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ with

$$0 \ < \ \alpha - \alpha_i, \ \beta - \beta_i \ < \ 5^{-k}, \ 0 \ < \ \gamma - \gamma_i, \ \delta - \delta_i \ < \ 7^{-\ell},$$

where $k = k(i)$ and $\ell = \ell(i)$, as in the definition from the beginning of this section.

So,

$$\begin{aligned}
\alpha \left(\frac{1}{3^{n_1}} + \cdots + \frac{1}{3^{n_t}} \right) + \beta \ &= \ \alpha_i \left(\frac{1}{3^{n_1}} + \cdots + \frac{1}{3^{n_t}} \right) + \beta_i \\
&\quad + (\alpha - \alpha_i) \left(\frac{1}{3^{n_1}} + \cdots + \frac{1}{3^{n_t}} \right) + (\beta - \beta_i) \\
&= \ \frac{r_1}{5} + \frac{r_2}{5^2} + \cdots + E,
\end{aligned}$$

with $r_1, ..., r_{k-1} \in \{0,1,2\}$, $r_k \in \{0,1\}$, where the error E satisfies

$$E \ < \ (\alpha - \alpha_i)/2 + (\beta - \beta_i)/2 \ < \ 5^{-k}.$$

This error is so small that it can only affect the term $r_k/5^k$ among the first k terms of $r_1/5 + r_2/5^2 + \cdots$, and can increase it by at most 1. It's clear, then, that (6) must hold.

By similar reasoning,

$$\begin{aligned}
\gamma\left(\frac{1}{3^{n_1}} + \cdots + \frac{1}{3^{n_t}}\right) + \delta \quad &= \quad \gamma_i\left(\frac{1}{3^{n_1}} + \cdots + \frac{1}{3^{n_t}}\right) + \delta_i \\
&\qquad + (\gamma - \gamma_i)\left(\frac{1}{3^{n_1}} + \cdots + \frac{1}{3^{n_t}}\right) + (\delta - \delta_i) \\
&= \quad \frac{s_1}{7} + \frac{s_2}{7^2} + \cdots + F,
\end{aligned}$$

with $s_1, ..., s_{\ell-1} \in \{0,1,2,3\}$, $r_\ell \in \{0,1,2\}$, where the error F satisfies

$$F \ < \ (\gamma - \gamma_i)/2 + (\delta - \delta_i)/2 \ < \ 7^{-\ell}.$$

The error is so small that it can at most affect the term $s_\ell/7^\ell$, and we get then that (7) must hold. This completes the proof.

2 The Hypothesis implies a positive soluton to the 105 problem

2.1 Extending the length of the sequence $1/3^{n_1} + 1/3^{n_2} + \cdots$

Suppose the Hypothesis (corollary in section 1.2) holds for some combination of B, C.

Let $N \geq 1$ be such that the B largest base-5 digits of 3^N are all in the set $\{0,1,2\}$; and that the C largest base-7 digits of 3^N are in the set $\{0,1,2,3\}$. It is easy to show that there are infinitely many $N \geq 1$ such that this holds: find rational numbers k/n and ℓ/n that are good approximations to $(\log 3)/\log 5$ and $(\log 3)/\log 7$, respectively. Then, $n \log 3 - k \log 5$ is close to 0, and positive (which we can ensure by choosing whether k/n is an over or under-approximation); likewise, $n \log 3 - \ell \log 7$ is close to 0. So, 3^n is close to 5^k and 7^ℓ. The leading base-5 and base-7 digits of 3^n are, then, 10000....

Now let $x, y \geq 1$ be such that

$$\alpha = \frac{3^N}{5^x} \in [1/5, 1), \;\; \gamma = \frac{3^N}{7^y} \in [1/7, 1).$$

Let $\beta = \delta = 0$.

Then, $(\alpha, \beta, \gamma, \delta)$ satisfy the conditions of our Hypothesis (corollary); so, there exist $n_1 < n_2 < \cdots < n_t$ as indicated.

Let

$$\beta' \;:=\; \alpha \left(\frac{1}{3^{n_1}} + \cdots + \frac{1}{3^{n_t}} \right), \text{ and } \alpha' \;:=\; \frac{\alpha}{3^{n_t}}.$$

and let

$$\delta' \;:=\; \gamma \left(\frac{1}{3^{n_1}} + \cdots + \frac{1}{3^{n_t}} \right), \text{ and } \gamma' \;:=\; \frac{\gamma}{3^{n_t}}.$$

And then let z be the least integer such that $5^z \alpha' \in [1/5, 1)$; and let w be the least integer such that $7^w \gamma' \in [1/7, 1)$. We note that from (5) we have that

$$z < k - B, \text{ and } w < \ell - C.$$

Set

$$\alpha'' \;:=\; 5^z \alpha'; \;\; \beta'' \;:=\; 5^z \beta' \pmod 1; \;\; \gamma'' \;:=\; 7^w \gamma'; \;\; \delta'' \;:=\; 7^w \delta' \pmod 1.$$

Now we note that the higher B base-5 digits of β'' are all in $\{0, 1, 2\}$, since $n_1 < n_2 < \cdots < n_t$ satisfied the conditions of the corollary to our Hypothesis for $(\alpha, \beta, \gamma, \delta)$. Likewise, the higher C base-7 digits of δ'' are all in $\{0, 1, 2, 3\}$.

Thus, $(\alpha'', \beta'', \gamma'', \delta'')$ satisfy the conditions/assumptions of the Hypothesis; and we can therefore apply it *again* to this 4-tuple, using the same parameters B, C. This, then, will give us that there exist $n_1' < n_2' < \cdots < n_u'$ such that

$$\alpha'' \left(\frac{1}{3^{n_1'}} + \cdots + \frac{1}{3^{n_u'}} \right) + \beta'' \;=\; \frac{r_1'}{5} + \cdots + \frac{r_k'}{5^k} + \cdots$$

and that

$$\gamma'' \left(\frac{1}{3^{n_1'}} + \cdots + \frac{1}{3^{n_u'}} \right) + \delta'' \;=\; \frac{s_1'}{7} + \cdots + \frac{s_\ell'}{7^\ell} + \cdots.$$

The $r_1', ..., r_k' \in \{0, 1, 2\}$; and $s_1', ..., s_\ell' \in \{0, 1, 2, 3\}$.

Now, we combine the two applications of the Hypothesis together as follows:

$$\begin{aligned}
&\alpha\left(\frac{1}{3^{n_1}}+\cdots+\frac{1}{3^{n_t}}+\frac{1}{3^{n_t+n'_1}}+\cdots+\frac{1}{3^{n_t+n'_u}}\right)\\
&\quad=\alpha\left(\frac{1}{3^{n_1}}+\cdots+\frac{1}{3^{n_t}}\right)+\frac{\alpha''}{5^z}\left(\frac{1}{3^{n'_1}}+\cdots+\frac{1}{3^{n'_u}}\right)\\
&\quad=\alpha\left(\frac{1}{3^{n_1}}+\cdots+\frac{1}{3^{n_t}}\right)+\frac{1}{5^z}\left(\frac{r'_1}{5}+\cdots+\frac{r'_k}{5^k}+\cdots\right)-\frac{\beta''}{5^z}.
\end{aligned}$$

Now, using the definition of β'' above (defined in terms of β'), it is easy to see that

$$\alpha\left(\frac{1}{3^{n_1}}+\cdots+\frac{1}{3^{n_t}}\right)-\frac{\beta''}{5^z} \;=\; \frac{r_1}{5}+\cdots+\frac{r_z}{5^z}.$$

(The term $-\beta''/5^z$ is basically subtracting off the lower base-5 digits of $\alpha(1/3^{n_1}+\cdots+1/3^{n_t})$. Thus, we have that

$$\alpha\left(\frac{1}{3^{n_1}}+\cdots+\frac{1}{3^{n_t}}+\frac{1}{3^{n_t+n'_1}}+\cdots+\frac{1}{3^{n_t+n'_u}}\right) \;=\; \frac{r_1}{5}+\cdots\frac{r_z}{5^z}+\frac{r'_1}{5^{z+1}}+\cdots+\frac{r'_k}{5^{z+k}}+\cdots,$$

where $r_1,...,r_z,r'_1,...,r'_k\in\{0,1,2\}$.

Thus, the top $z+k$ base-5 digits of $\alpha(1/3^{n_1}+\cdots+1/3^{n_t+n'_u})$ all belong to the set $\{0,1,2\}$, as desired.

We will get a similar thing to happen for the base-7 digits of $\gamma(1/3^{n_1}+\cdots+1/3^{n_1+n'_u})$.

Now we can apply the Hypothesis *again* and *again*, for example letting $\alpha''':=5^{z'}\alpha/3^{n_t+n'_u}$ for an appropriately chosen z', producing an arbitrarily long sequence $m_1<m_2<m_3<\cdots$ such that

$$\alpha\left(\frac{1}{3^{m_1}}+\frac{1}{3^{m_2}}+\cdots\right) \;=\; \frac{r_1}{5}+\frac{r_2}{5^2}+\cdots, \tag{8}$$

where all the r_i's belong to the set $\{0,1,2\}$; and we can get a similar thing for

$$\gamma\left(\frac{1}{3^{m_1}}+\frac{1}{3^{m_2}}+\cdots\right) \;=\; \frac{s_1}{7}+\frac{s_2}{7^2}+\cdots, \tag{9}$$

in that we can get all the s_i's to be in $\{0,1,2,3\}$.

2.2 How to solve the 105 conjecture

Choose $\alpha = 3^T/5^U \in [1/5, 1)$, where T is as large as you want, such that the first couple digits of α look like $3^T = 5^U + a_{U-1}5^{U-1} + \cdots + a_0$, where $a_{U-1}, ..., a_{U-B+1} \in \{0, 1, 2\}$; and choose $\gamma = 3^T/7^V \in [1/7, 1)$, where the first few digits $3^T = 7^V + b_{V-1}7^{V-1} + \cdots + b_0$, $b_{V-1}, ..., b_{V-C+1} \in \{0, 1, 2, 3\}$. It is easy to prove (via a pigeonhole principle argument) that there are arbitrarily large values of T such that we can force the upper digits of 3^T in bases 5 and 7 to be as we have here (in fact, you can do it with $a_{U-1} = \cdots = a_{U-B+1} = b_{V-1} = \cdots = b_{V-C+1} = 0$).

Now, (8) and (9) imply that

$$s_1 7^{V-1} + s_2 7^{V-2} + \cdots \;=\; 3^{T-m_1} + \cdots + 3^{T-m_u} \;=\; r_1 5^{U-1} + r_2 5^{U-2} + r_3 5^{U-3} + \cdots$$

Thus, we have a number, all of whose base 3, 5 and 7 digits are in $\{0, 1\}$, $\{0, 1, 2\}$ and $\{0, 1, 2, 3\}$, respectively.

3 Verifying the Hypothesis(B,C) computationally

Here, we will explain how to computationally verify the main Hypothesis from section 1. The proposed algorithm is via a branching process:

1. Initialize a list to contain all vectors $(\alpha, \beta, \gamma, \delta, k, \ell)$ where all $k = \ell = 1$, and where $\alpha = a/5$, $a = 1, 2, 3, 4$, $\beta = b/5$, $b = 0, 1, 2$, $\gamma = g/7$, $g = 1, 2, 3, 4, 5, 6$, and $\delta = d/7$, $d = 0, 1, 2, 3$. The initial list will therefore contain $4 \cdot 3 \cdot 6 \cdot 4 = 288$ entries.

2. Now, for each vector in the list, check to see whether there exists a sequence of n_i's satisfying the conclusion of the Hypothesis. Some vectors in the list may have such an associated sequence of n_i's, and some may not. If all the items on the list have an associated set of n_i's, then STOP.

3. If $(\alpha, \beta, \gamma, \delta, k, \ell)$ is a 4-tuple that does *not* have an associated sequence of n_i's that satisfy the Hypothesis, then we remove it and replace it with several other 4-tuples to verify, that have one more digit. Basically, append to the list those vectors $(\alpha', \beta', \gamma, \delta, k + 1, \ell)$ with α' has one

more base-5 digit than α, and β' having one more base-5 digit than β. The way this works is as follows: if $\alpha = a_1/5 + a_2/5^2 + \cdots + a_k/5^k$, then we consider all possibilities for $a_{k+1} = 0, 1, 2, 3, 4$, setting $\alpha' = a_1/5 + \cdots + a_{i+1}/5^{i+1}$; we do a similar thing for β', but range over a different set of restrictions – if $\beta = b_1/5 + \cdots + b_k/5^k$, then we consider all possibilities for $\beta' = b_1/5 + \cdots + b_{k+1}/5^{k+1}$, where b_{k+1} ranges over $0, 1, 2, 3, 4$ if $k \geq B$ and where b_{k+1} ranges over $0, 1, 2$ if $k \leq B - 1$. Then, append to the list $(\alpha, \beta, \gamma', \delta', k, \ell + 1)$ with γ' having one more base-7 digit than γ (the analogous thing we did for α', but base-7) and δ' having one more base-7 digit than δ. The possibilities for δ' are given as follows: if $\delta = d_1/7 + \cdots + d_\ell/7^\ell$, then $\delta' = d_1/7 + \cdots + d_\ell/7^\ell$, where d_ℓ ranges over $0, 1, 2, 3, 4, 5, 6$ if $\ell \geq C$ and ranges over $0, 1, 2, 3$ if $\ell \leq C - 1$. In total, then, we will have removed one vector $(\alpha, \beta, \gamma, \delta, k, \ell)$, and then replaced it with *at most* $5 \cdot 5 + 7 \cdot 7 = 74$ new vectors, and *at least* $5 \cdot 3 + 7 \cdot 4 = 43$ new vectors.

4. Go back to step 2 (you only need to consider whether the new vectors that we've added have an associated sequence of n_i's).

The initial list of 288 vectors resulted in a set of $(\alpha, \beta, \gamma, \delta)$ that covered S; and each time we run step 3 above, we again have a list resulting in $(\alpha, \beta, \gamma, \delta)$ that covers S. So, when the algorithm terminates, we end up with a cover of S, that also happens to satisfy the rest of the Hypothesis.

One can probably speed up the part of the algorithm for finding the n_i's associated to each vector of the form $(\alpha', \beta', \gamma, \delta, k+1, \ell)$ by keeping track of the n_i's that *almost* succeeded in working for $(\alpha, \beta, \gamma, \delta, k, \ell)$. Similarly, one can speed up the seach for n_i's associated to $(\alpha, \beta, \gamma', \delta', k, \ell + 1)$.