Let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) := f(x) - f(x-1)$. We see that for large n

$$\begin{aligned}
|g^{-1}(n)| = \Delta_n\left(\sum_{k\leq n} g^{-1}(k)\right) &\sim \frac{6}{\pi^2} \times \Delta_n\left(\sum_{d\leq n} C_\Omega(d)\cdot\frac{n}{d}\right) \\
&= \frac{6}{\pi^2}\left(C_\Omega(n) + \sum_{d<n} C_\Omega(d)\frac{n}{d} - \sum_{d<n} C_\Omega(d)\frac{(n-1)}{d}\right) \\
&\sim \frac{6}{\pi^2}\left(C_\Omega(n) + \frac{1}{n-1}\times\sum_{k<n}|g^{-1}(k)|\right), \text{ as } n\to\infty.
\end{aligned}$$

Since $\frac{1}{n-1}\times\sum_{k<n}|g^{-1}(k)| \sim \frac{1}{n}\times\sum_{k\leq n}|g^{-1}(k)|$ for all sufficiently large n, the result follows by a re-normalization of Conjecture 4.11. □

5 New formulas and limiting relations characterizing $M(x)$

5.1 Formulas relating $M(x)$ to the summatory function $G^{-1}(x)$

Proposition 5.1. *For all sufficiently large x, we have that*

$$M(x) = G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} G^{-1}(k)\left(\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor\frac{x}{k+1}\right\rfloor\right)\right). \tag{26}$$

Proof. We know by applying Corollary 1.4 that

$$\begin{aligned}
M(x) &= \sum_{k=1}^{x} g^{-1}(k)\left(\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) + 1\right) \\
&= G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} g^{-1}(k)\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) \\
&= G^{-1}(x) + G^{-1}\left(\left\lfloor\frac{x}{2}\right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left(\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor\frac{x}{k+1}\right\rfloor\right)\right).
\end{aligned}$$

The upper bound on the sum is truncated to $k \in \left[1, \frac{x}{2}\right]$ in the second equation above due to the fact that $\pi(1) = 0$. The third formula above follows directly by (ordinary) summation by parts. □

By the result from (13) proved in Section 3.2, we recall that

$$M(x) = G^{-1}(x) + \sum_{p\leq x} G^{-1}\left(\left\lfloor\frac{x}{p}\right\rfloor\right), \text{ for } x \geq 1.$$

Summation by parts implies that we can also express $G^{-1}(x)$ in terms of the summatory function $L(x)$ and differences of the unsigned sequence whose distribution is given by Corollary 4.12. That is, we have

$$G^{-1}(x) = \sum_{n\leq x}\lambda(n)|g^{-1}(n)| = L(x)|g^{-1}(x)| - \sum_{n<x} L(n)\left(|g^{-1}(n+1)| - |g^{-1}(n)|\right), \text{ for } x \geq 1.$$

5.2 Asymptotics of the partial sums of the unsigned inverse sequence

The following proofs are credited to Professor R. C. Vaughan and his suggestions about approaches to upper bounds on $|G^{-1}|(x)$ that are attained along infinite subsequences as $x \to \infty$. The ideas at the crux of the proof of the next theorem are found in the references by Davenport and Heilbronn [4, 5] and are known to date back to the work of Harald Bohr [31, *cf.* §11].

Theorem 5.2. *Let σ_1 denote the unique solution to the equation $P(\sigma) = 1$ for $\sigma > 1$. There are complex s with $\mathrm{Re}(s)$ arbitrarily close to σ_1 such that $1 - P(s) = 0$.*

Proof. The function $P(\sigma)$ is decreasing on $(1, \infty)$, tends to $+\infty$ as $\sigma \to 1^+$, and tends to zero as $\sigma \to \infty$. Thus we find that the equation $P(\sigma) = 1$ has a unique solution for $\sigma > 1$, which we denote by $\sigma = \sigma_1 \approx 1.39943$. Let $\delta > 0$ be chosen small enough that $|1 - P(z)| > 0$ for all z such that $|z - \sigma_1| = \delta$. Set

$$\eta = \min_{\substack{z \in \mathbb{C} \\ |z-\sigma_1|=\delta}} |1 - P(z)|.$$

Since $P(z)$ is continuous whenever $\mathrm{Re}(z) > 1$, we have that $\eta > 0$. Let $X \geq 2$ be a sufficiently large integer so that

$$\sum_{p>X} p^{\delta-\sigma_1} < \frac{\eta}{4}.$$

Kronecker's theorem provides a fixed t such that the following inequality holds [10, §XXIII]:

$$\max_{2<p\leq X} \min_{n\in\mathbb{Z}} \left| \frac{t \log p}{2\pi} - n - \frac{1}{2} \right| < \delta\eta.$$

Thus we have that

$$\sum_{p>2} p^{\delta-\sigma_1} \left| p^{\imath t} + 1 \right| < \frac{\eta}{2}.$$

Hence, for all z such that $|z - \sigma_1| = \delta$, we have

$$|P(z + \imath t) + P(z)| < \frac{\eta}{2}.$$

We apply Rouché's theorem to see that the functions $1 - P(z)$ and $1 - P(z) + P(z + \imath t) + P(z)$ have the same number of zeros in the disk $\mathcal{D}_\delta = \{z \in \mathbb{C} : |z - \sigma_1| < \delta\}$. Since $1 - P(z)$ has at least one zero within $\mathcal{D}_\delta$, we must have that $1 + P(w)$ has at least one zero with $|w - \sigma_1 - \imath t| < \delta$. Since we can take δ as small as necessary, there are zeros of the function $1 + P(s)$ that are arbitrarily close to the line $s = \sigma_1$. ☐

Corollary 5.3. *Suppose that the partial sums of the unsigned inverse sequence are defined as follows:*

$$|G^{-1}|(x) := \sum_{n\leq x} |g^{-1}(n)|, x \geq 1.$$

Let $\sigma_1 > 1$ be defined as in Theorem 5.2. For any $\epsilon > 0$, there are arbitrarily large x such that

$$|G^{-1}|(x) > x^{\sigma_1-\epsilon}.$$

Proof. Since the DGF of the function $C_\Omega(n)$ is given by $(1 - P(s))^{-1}$ for $\mathrm{Re}(s) > 1$, we have that

$$D_{|g^{-1}|}(s) := \sum_{n\geq1} \frac{|g^{-1}(n)|}{n^s} = \frac{1}{\zeta(s)(1 - P(s))}, \text{ for } \mathrm{Re}(s) > 1.$$

Theorem 5.2 implies that $D_{g^{-1}}(s)$ has singularities $s \in \mathbb{C}$ such that the $\mathrm{Re}(s)$ are arbitrarily close to σ_1. By applying [18, Cor. 1.2; §1.2], we have that any Dirichlet series is locally uniformly convergent in its half-plane of convergence, e.g., for $\mathrm{Re}(s) > \sigma_c$, and is hence analytic in this half-plane. It follows that the abscissa of convergence of $D_{g^{-1}}(s)$ is given by $\sigma_c \geq \sigma_1 > 1$. In particular, the abscissa of convergence of this DGF cannot be smaller than σ_1. The result proved in [18, Thm. 1.3; §1.2] then shows that

$$\limsup_{x\to\infty} \frac{\log |G^{-1}|(x)}{\log x} = \sigma_c \geq \sigma_1. \qquad \square$$

Remark 5.4 (Some possible implications for bounds on $M(x)$)**.** Notice that for any $x \geq 1$ we can for the signed partial sums of $g^{-1}(n)$ as

$$G^{-1}(x) = \sum_{n\leq x} \lambda(n)|g^{-1}(n)| \sim \sum_{n\leq x} \lambda(n)\left(\int_{n-1}^{n} \frac{d}{dt}|G^{-1}|(t)dt\right).$$

Thus, we note that it may eventually worthwhile to attempt to extract more precise information about the asymptotics of this summatory function, and its characterization of $M(x)$ through (13), based on limit-supremum type bounds of the type in Corollary 5.3 along infinite subsequences of the positive integers. In particular, an important motivating open problem is to resolve whether it is the case that

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}} = +\infty,$$

and if so, to determine the rate with which the square-root scaled Mertens function becomes unbounded. Extensions of the bounds we have proved in this subsection then formulate one concrete new approach to this problem.

5.3 Local cancellation of $G^{-1}(x)$ in the new formulas for $M(x)$

Lemma 5.5. *Suppose that p_n denotes the n^{th} prime for $n \geq 1$ [28, A000040]. Let $\mathcal{P}_{\#}$ denote the set of positive primorial integers as [28, A002110]*

$$\mathcal{P}_{\#} = \{n\#\}_{n\geq 1} = \left\{\prod_{k=1}^{n} p_k : n \geq 1\right\} = \{2, 6, 30, 210, 2310, 30030, \ldots\}.$$

As $m \to \infty$ we have

$$\begin{aligned} -G^{-1}((4m+1)\#) &= (4m+1)!\left(1 + O\left(\frac{1}{m^2}\right)\right), \\ G^{-1}\left(\frac{(4m+1)\#}{p_k}\right) &= (4m)!\left(1 + O\left(\frac{1}{m^2}\right)\right), \text{ for all } 1 \leq k \leq 4m+1. \end{aligned}$$

Proof. We have by part (B) of Proposition 1.6 that for all squarefree integers $n \geq 1$

$$\begin{aligned} |g^{-1}(n)| &= \sum_{j=0}^{\omega(n)} \binom{\omega(n)}{j} \times j! = (\omega(n))! \times \sum_{j=0}^{\omega(n)} \frac{1}{j!} \\ &= (\omega(n))! \times \left(e + O\left(\frac{1}{(\omega(n)+1)!}\right)\right). \end{aligned}$$

Let m be a large positive integer. We obtain main terms of the form

$$\begin{aligned} G_U^{-1}((4m+1)\#) &:= \sum_{\substack{n\leq(4m+1)\# \\ \omega(n)=\Omega(n)}} \lambda(n)|g^{-1}(n)| \\ &= \sum_{0\leq k\leq 4m+1} \binom{4m+1}{k}(-1)^k k!\left(e + O\left(\frac{1}{(k+1)!}\right)\right) \\ &= -(4m+1)! + O(1). \end{aligned}$$

We argue that the analogous sums over the non-squarefree $n \leq (4m+1)\#$ contribute strictly less than the order of $G_U^{-1}((4m+1)\#)$ to the main term of $G^{-1}((4m+1)\#)$. Suppose that $2 \leq n \leq (4m+1)\#$ is not squarefree. We have the next largest order of growth of the sequence along those n with $|g^{-1}(n)| \leq |g^{-1}(p_s^2 t)|$ for some $1 \leq s \leq 4m+1$ and where t is squarefree. If $s = 1$ so that $p_s = 2$, we have that the largest possible