Lower bounds on the Mertens function $M(x)$ for $x \gg 2.3315 \times 10^{1656520}$

New unique lower bounds on $M(x)/\sqrt{x}$ along an asymptotically huge infinite subsequence of reals

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Abstract

The Mertens function, $M(x) = \sum_{n \leq x} \mu(n)$, is classically defined to be the summatory function of the Möbius function $\mu(n)$. In some sense, the Móbius function can be viewed as a signed indicator function of the squarefree integers which have asymptotic density of $6/\pi^2 \approx 0.607927$ and a corresponding well-known asymptotic average order formula. The signed terms in the sums in the definition of the Mertens function introduce complications in the form of semi-randomness and cancellation inherent to the distribution of the Möbius function over the natural numbers. The Mertens conjecture which states that $|M(x)| < C \cdot \sqrt{x}$ for all $x \geq 1$ has a well-known disproof due to Odlyzko et. al. It is widely believed that $M(x)/\sqrt{x}$ is an unbounded function which changes sign infinitely often and exhibits a negative bias over all natural numbers $x \geq 1$.

We focus on obtaining new lower bounds for $M(x)$ by methods that generalize to handle other related cases of special number theoretic summatory functions. The key to our proofs calls upon a known result from the standardized summatory function enumeration by DGFs found in Chapter 7 of Montogomery and Vaughan. There is also a distinct flavor of combinatorial analysis peppered in with the standard methods from analytic number theory which distinguishes our methods.

Keywords and Phrases: *Möbius function sums; Mertens function; summatory function; arithmetic functions; Dirichlet inverse; Liouville lambda function; prime omega functions; prime counting functions; Dirichlet series and DGFs; asymptotic lower bounds; Mertens conjecture; asymptotic methods from the Montgomery and Vaughan textbook.*

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1 Introduction

1.1 The Mertens function – definition, properties, known results and conjectures

Suppose that $n \geq 1$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2 \alpha_2 \cdots p_k^{\alpha_k}$. We define the *Möebius function* to be the signed indicator function of the squarefree integers:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leq i \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many known variants and special properties of the Möebius function and its generalizations [8, *cf.* §2], however, for our purposes we seek to explore the properties and asymptotics of weighted summatory functions over $\mu(n)$. The Mertens summatory function, or *Mertens function*, is defined as

$$\begin{aligned} M(x) &= \sum_{n \leq x} \mu(n), \ x \geq 1, \\ &\longmapsto \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2\} \end{aligned}$$

A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \rightarrow \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{Q(x)} = \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{Q(x)} \xrightarrow[n \rightarrow \infty]{} \frac{3}{\pi^2}.$$

While this limiting law suggests an even bias for the Mertens function, in practice $M(x)$ has a noted negative bias in its values, and the actual local oscillations between the approximate densities of the sets $\mu_\pm(x)$ lend an unpredictable nature to the function and its characteistic oscillatory sawtooth shaped plot – even over asymptotically large and variable intervals.

1.2 Properties

The well-known approach to evaluating the behavior of $M(x)$ for large $x \rightarrow \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is easily expressed via an inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since by Perron's formula we have

$$\frac{1}{\zeta(s)} = \int_1^\infty \frac{s \cdot M(x)}{x^{s+1}} dx,$$

we then obtain that

$$M(x) = \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation along with the standard Euler product representation for the reciprocal zeta function leads us to the exact expression for $M(x)$ when $x > 0$ given by the next theorem.

Theorem 1.1 (Analytic Formula for $M(x)$)**.** *If the RH is true, the there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any $x \in \mathbb{R}_{>0}$*

$$M(x) = \lim_{k \rightarrow \infty} \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |\Im(\rho)| < T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

An unconditional bound on the Mertens function due to Walfisz [?] states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \exp\left(-C \cdot \log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan proved new updated estimates bounding $M(x)$ for large x in 2009 of the following forms:

$$\begin{aligned} M(x) &\ll \sqrt{x} \exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x}\exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right), \ \forall \epsilon > 0. \end{aligned}$$

Other explicit bounds due to the article by Kotnik include the following simpler estimates for the Mertens function when x is sufficiently large:

$$\begin{aligned} |M(x)| &< \frac{x}{4345}, \ \forall x > 2160535, \\ |M(x)| &< \frac{0.58782 \cdot x}{\log^{11/9}(x)}, \ \forall x > 685. \end{aligned}$$

1.3 Conjectures

The Riemann Hypothesis is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. For $\Re(\alpha) < 1$, we know the limiting absolute behavior of these functions as $x \to \infty$ as the Dirichlet generating function

$$\frac{1}{\zeta(\alpha)} = \lim_{x \longrightarrow \infty} M^*_{-\alpha}(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}},$$

which is definitively bounded for all large x. It is still unresolved whether

$$\limsup_{x \to \infty} |M(x)|/\sqrt{x} = \infty,$$

although computational evidence suggests that this is a likely conjecture [?, ?]. There is a rich history to the original statement of the *Mertens conjecture* which states that

$$|M(x)| < c \cdot x^{1/2}, \ \text{ some constant } c > 0,$$

which was first verified by Mertens for $c = 1$ and $x < 10000$, although since its beginnings in 1897 has since been disproved by computation.

There are a number of other interesting unsolved and at least somewhat accessible open problems related to the asymptotic behavior of $M(x)$ at large x. It is believed that the sign of $M(x)$ changes infinitely often. That is to say that it is widely believed that $M(x)$ is ocsillatory and exhibits a negative bias in so much as $M(x) < 0$ more frequently than $M(x) > 0$ over all $x \in \mathbb{N}$[1] One of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ is unbounded on the natural numbers. In particular, the precise statement of this problem is to produce an affirmative answer whether $\limsup_{x\to\infty} |M(x)|/\sqrt{x} = +\infty$, or equivalently whether there is an infinite sequence of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that $M(x_i)x_i^{-1/2}$ grows without bound along this subsequence.

Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies that

$$\limsup_{x \to \infty} \frac{|M(x)|}{\sqrt{x}(\log\log x)^{5/4}},$$

[1]See for example the discussion in the following thread:
https://mathoverflow.net/questions/98174/is-mertens-function-negatively-biased.

corresponds to some bounded constant. A probabilistic proof along these lines has been given by Ng in 2008, though to date an exact rigorous proof (rather than somewhat heuristic argument) that $M(x)/\sqrt{x}$ is unbounded still remains elusive. We cite that prior to this point it is known that [**?**, *cf.* §4.1]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad \text{(now } 1.826054\text{)},$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad \text{(now } -1.837625\text{)},$$

although based on work by Odlyzko and te Riele it seems probable that each of these limits should be $\pm\infty$, respectively [**?**, **?**, **?**, **?**]. It is also known that $M(x) = \Omega_{\pm}(\sqrt{x})$ and $M(x)/\sqrt{x} = \Omega_{\pm}(1)$.

1.4 A new approach to bounding $M(x)$ from below

1.4.1 Summing series over Dirichlet convolutions

Theorem 1.2 (Summatory functions of Dirichlet convolutions)**.** *Let $f, g : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n\leq x} f(n)$ and $G(x) := \sum_{n\leq x} g(n)$ denote the summatory functions of f, g, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse $f^{-1}(n)$ of f, i.e., the unique arithmetic function such that $f * f^{-1} = \varepsilon$ where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity with respect to Dirichlet convolution. Then we have the following equivalent expressions for the summatory function of $f * g$ for integers $x \geq 1$:*

$$\begin{aligned}
\pi_{f*g}(x) &= \sum_{n\leq x}\sum_{d|n} f(d)g(n/d) \\
&= \sum_{d\leq x} f(d) G\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} G(k)\left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].
\end{aligned}$$

Moreover, we can invert the linear system determining the coefficients of $G(k)$ for $1 \leq k \leq x$ naturally to express $G(x)$ as a linear combination of the original left-hand-side summatory function as:

$$G(x) = \sum_{j=1}^{x} \pi_{f*g}(j)\left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right].$$

Corollary 1.3 (Convolutions Arising From Möbius Inversion)**.** *Suppose that g is an arithmetic functions with $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n\leq x}(g * \mu)(n)$. Then*

$$M(x) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor+1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j)\right)\widetilde{G}(k), \forall x \geq 1.$$

1.4.2 A motivating special case

Using $\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu$, where $\chi_{\mathbb{P}}$ is the characteristic function of the primes, we have that $\widetilde{G}(x) = \pi(x) + 1$ in Corollary 1.3. In particular, the corollary implies that

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

We can compute the first few terms for the Dirichlet inverse sequence of $g(n) := \omega(n) + 1$ numerically for the first few sequence values as

$$\{g^{-1}(n)\}_{n\geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is given by $\lambda(n) = (-1)^{\Omega(n)}$ (see Proposition 3.1). Note that since the DGF of $\omega(n)$ is given by $D_\omega(s) = P(s)\zeta(s)$ where $P(s)$ is the *prime zeta function*, we do have a Dirichlet series for the inverse functions to invert coefficient-wise using more classical contour integral methods. In fact, Fröberg has previously done some preliminary investigation as to the properties of the inversion to find the coefficients of $(1 + P(s))^{-1}$ [2]. However, we will take a more combinatorial tact to investigating bounds on this inverse function sequence in the coming sections.

Conjecture 1.4. *Suppose that $n \geq 1$ is a squarefree integer. We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n) = (\omega + 1)^{-1}(n)$ over these integers:*

(A) $g^{-1}(1) = 1$*, which follows immediately by computation;*

(B) $\operatorname{sgn}(g^{-1}(n)) = \mu(n) \equiv \lambda(n)$*;*

(C) If $w(n) = k$, we can write the inverse function at k as

$$g^{-1}(n) = \sum_{m=0}^{k} \binom{k}{m} \cdot m!.$$

The "obvious" interpretation to these sums is that we are counting up the factorizations of n according to the number of prime factors in each term, selecting each k of the primes, and then scaling by a "count" of how many ways there are to arrange k such factors[2].

We illustrate parts (B)–(C) of this conjecture clearly using Table T.1 given on page 35 of the appendix section.

So why exactly is the Dirichlet inverse function, $g^{-1}(n)$, difficult to evaluate? There are mutltiple apparent reasons for this. The first is that the Dirichlet inverse function not only depends on the prime factorization of n in the typical way, involving weighted sums of $\Omega(n)$ terms of the function $\omega(n) + 1$, but also in the additive nature of how we build up and assemble these terms in an essentially non-multiplicative, but instead very additive, way. Note that for primes a, b and positive integers $m, n \geq 1$, the (incomplete) additivity of $\omega(n)$ implies that $\omega(a^m b^n) = \omega(a) + \omega(b)$ where by convention $\omega(1) = 0$. The extra additive factor of $+1$ (that was added to make the function Dirichlet invertible) also does not depend on n in the corresponding expansions of the Dirichlet inverse terms.

1.4.3 Enter the fates: Re-discovering a known result using DGFs from Mongomery and Vaughan

Theorem 1.5 (Montgomery and Vaughan, §7.4)**.** *Let $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$. For $R < 2$ we have that*

$$\widehat{\pi}_k(x) = G\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

uniformly for $1 \leq k \leq R \log\log x$ where

$$G(z) := \frac{F(1,z)}{\Gamma(z+1)} = \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z.$$

[2] *It's a nice thought in so much as these formulas *DO* seem to accuratly approximate the inverse sequence enough of the time to be useful.*

1.5 Caveats and remarks

1.5.1 Comparison to tightness of the best known upper bounds on $M(x)$

The current best *upper bound* for the Mertens function is cited by Soundarajan in his 2009 *Annals* paper [10]. These bounds are recounted in the introduction to $M(x)$ given in Section 1.1. In contrast, the formulas we have seen in the previous subsection allow us to pinpoint cancellation – even if the signedness observed corresponds to taking the difference of two oppositely signed asymptotic formulae.

1.5.2 Contra-wisdom dé Croutons

The more "combinatorial" results found in the proofs here employ routine to semi-routine known techniques from combinatorial analysis and apply them to this more classically analytic number theoretic problem. This new usage with respect to summing bounds for $M(x)$ ought not distract from the significance of what these proofs imply – after all, there is more to the "depth" of results based on the classical Riemann zeta function than the residues involving the Cauchy integral formula or an inverse Mellin transform.

1.6 Organization of this document

1.6.1 Notation and conventions

To make a common source of references to definitions of key functions and sequences defined within the article, we provide a comprehensive list of commonly used notation and conventions. Where possible, we have included page references to local definitions of special sequences and functions where they are given in the text. We have organized the symbols list alphabetically by symbol. This listing of notation and conventions is a useful reference to accompany the article. It is provided in Appendix Appendix A starting on page 30.

2 Preliminary proofs and configuration

Given the interpretation of the summatory functions over an arbitrary Dirichlet convolution (and the vast number of such identities for special number theoretic functions – *cf.* [3, ?]), it is not surprising that this formulation of the first theorem may well provide many fruitful applications, indeed. In addition to those cited in the compendia of the catalog reference, we have notable identities of the form: $(f*1)(n) = [q^n]\sum_{m\geq 1} f(m)q^m/(1-q^m)$, $\sigma_k = \mathrm{Id}_k * 1$, $\mathrm{Id}_1 = \phi * \sigma_0$, $\chi_{\mathrm{sq}} = \lambda * 1$ (see sections below), $\mathrm{Id}_k = J_k * 1$, $\log = \Lambda * 1$, and of course $2^\omega = \mu^2 * 1$. The result in Theorem 1.2 is natural and displays a quite beautiful form of symmetry between the initial matrix terms,

$$t_{x,j}(f) = \sum_{k=\left\lfloor \frac{x}{j+1}\right\rfloor+1}^{\left\lfloor \frac{x}{j}\right\rfloor} f(k),$$

and the corresponding inverse matrix,

$$t_{x,j}^{-1}(f) = \sum_{k=\left\lfloor \frac{x}{j+1}\right\rfloor+1}^{\left\lfloor \frac{x}{j}\right\rfloor} f^{-1}(k),$$

as expressed by the duality of f and its Dirichlet inverse function f^{-1}. Since the recurrence relations for the summatory functions $G(x)$ arise naturally in applications where we have established bounds on sums of Dirichlet convolutions of arithmetic functions, we will go ahead and prove it here before moving along to some motivating examples of the use of this theorem.

Proof of Theorem 1.2. Let h, g be arithmetic functions where $g(1) \neq 1$ has a Dirichlet inverse. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n\leq x} h(n)$ and $G(x) = \sum_{n\leq x} g(n)$. We define $S_{g,h}(x)$ to be the summatory function of the Dirichlet convolution of g with h: $g*h$. Then we can easily see that the following expansions hold:

$$\begin{aligned} S_{g,h}(x) &:= \sum_{n=1}^{x}\sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d}\right\rfloor\right) \\ &= \sum_{i=1}^{x}\left[G\left(\left\lfloor \frac{x}{i}\right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1}\right\rfloor\right)\right] H(i). \end{aligned}$$

Thus we have an implicit statement of a recurrence relation for the summatory function H, weighted by g and G, whose non-homogeneous term is the summatory function, $S_{g,h}(x)$, of the Dirichlet convolutions $g*h$. We form the matrix of coefficients associated with this system for $H(x)$, and proceed to invert it to express an exact solution for this function over all $x \geq 1$. Let the ordinary (initial, non-inverse) matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j}\right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1}\right\rfloor\right) \equiv G_{x,j} - G_{x,j+1}.$$

Then the matrix we must invert in this problem is lower triangular, with ones on its diagonals – and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T); U = (\delta_{i,j+1}), (I-U^T)^{-1} = ([j \leq i]_\delta).$$

It is a nice round fact that if we take successive differences of floor functions, we get non-zero behavior at divisors:

$$G\left(\left\lfloor \frac{x}{j}\right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j}\right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.}\end{cases}$$

We use this property to invertibly shift the matrix $\hat{G}$, and then invert the result, to obtain a matrix involving the Dirichlet inverse of g:

$$\left[(I-U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Now we can express the inverse of the target matrix $(g_{x,j})$ in terms of these Dirichlet inverse functions as follows:

$$\begin{aligned}(g_{x,j}) &= (I-U^T)^{-1}\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\ (g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k)\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1}\right\rfloor} g^{-1}(k)\right).\end{aligned}$$

Thus the summatory function H is exactly expressed by the inverse vector product of the form

$$\begin{aligned}H(x) &= \sum_{k=1}^{x} g_{x,k}^{-1} \cdot S_{g,h}(k) \\ &= \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right)\cdot S_{g,h}(k).\square\end{aligned}$$

Proposition 2.1 (An Antique Divisor Sum Identity). *Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n)=\delta_{n,1}$ be the identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the additive function that counts the number of distinct prime factors of n. Then*

$$\chi_{\mathbb{P}} + \varepsilon = (\omega+1) * \mu.$$

The summatory function of the LHS is $\widetilde{G}(x) = \pi(x)+1$.

Proof. The the crux of the stated identity is to prove that for all $n \geq 1$, $\chi_{\mathbb{P}}(n) = (\mu * \omega)(n)$ – our claim. We notice that the Mellin transform of $\pi(x)$ – the summatory function of $\chi_{\mathbb{P}}(n)$ – at $-s$ is given by

$$\begin{aligned}s\cdot\int_1^\infty \frac{\pi(x)}{x^{s+1}}dx &= \sum_{n\geq 1}\frac{\nabla[\pi](n-1)}{n^s} \\ &= \sum_{n\geq 1}\frac{\chi_{\mathbb{P}}(n)}{n^s} = P(s).\end{aligned}$$

This is typical fodder which more generally relates the Mellin transform $\mathcal{M}[S_f](-s)$ to the DGF of the sequence $f(n)$ as cited, for example, in [1, §11]. Now we consider the DGF of the right-hand-side function, $f(n) := (\mu*\omega)(n)$, as

$$D_f(s) = \frac{1}{\zeta(s)} \times \sum_{n\geq 1}\frac{\omega(n)}{n^s} = P(s),$$

by Lemma **??**. Thus for any $\Re(s) > 1$, the DGFs of each side of the claimed equation coincide. So by uniqueness of Dirichlet series, we see that in fact the claim holds. To obtain the full result, we add to each side of this equation a term of $\varepsilon(n) \equiv (\mu * 1)(n)$, and then factor the resulting convolution identity. $\square$

3 Precisely defining, enumerating, and bounding the Dirichlet inverse functions, $(\omega+1)^{-1}(n)$

3.1 Proving the conjecture over the squarefree integers

Proposition 3.1 (Signage of Dirichlet inverses of positive and bounded arithmetic functions)**.** *Suppose that f is an arithmetic function with $f(1)=1$ and such that $f(n)>1$ for all $n\geq 2$. Then for all $n\geq 1$,*

$$\operatorname{sgn}\left(f^{-1}(n)\right)=\lambda(n).$$

Proof. We begin by using an identity for the Dirichlet inverse of a general arithmetic function with $f(1)=1$ proved in [5, 6]:

$$\begin{aligned}
f^{-1}(n) &= \sum_{j=1}^{\Omega(n)}\left[(f-\varepsilon)_{*2j}(n)-(f-\varepsilon)_{*2j-1}(n)\right] \\
&= \sum_{j=1}^{\Omega(n)}\sum_{r=0}^{j}\binom{j}{r}(-1)^r f_{*r}(n)\varepsilon(n)^{j-r}.
\end{aligned}$$

In the previous equations we have suppressed the long form of the multiple convolution expansions and intend f_{*k} to denote the k-fold convolution of f with itself.

We know that $f^{-1}(1)=\frac{1}{f(1)}$. For $n\geq 2$, the last equation only has non-zero terms when $j-r=0$. Thus for $n\geq 2$, we have that

$$\begin{aligned}
f^{-1}(n) &= \sum_{j=1}^{\Omega(n)}(-1)^j f_{*j}(n) \\
&= (-1)^{\Omega(n)}\times\sum_{j=1}^{\Omega(n)}(-1)^{j+1} f_{*\Omega(n)+1-j}(n)
\end{aligned}$$

TODO ...

□

3.2 Developing an improved conjecture: Proving precise bounds on the inverse functions $g^{-1}(n)$ for all n

Conjecture 1.4 is not the most accurate fomulation of the limiting behavior of the Dirichlet inverse functions $g^{-1}(n)$ that we can prove. We need to come up with better bounds to plug back into the asymptotic analysis from the previous pages. It turns out that these results are related to symmetric functions of the exponents in the prime factorizations of each $n\leq x$. The idea is that by having information about $g^{-1}(n)$ in terms of its prime factorization exponents for $n\leq x$, we should be able to extrapolate what we need which is information about the average behavior of the summatory functions, $G^{-1}(x)$, from the proofs above.

We define the following sequence for integers $n\geq 1, k\geq 0$:

$$C_k(n):=\begin{cases}\varepsilon(n), & \text{if } k=0; \\ \sum\limits_{d|n}\omega(d)C_{k-1}(n/d), & \text{if } k\geq 1.\end{cases} \tag{2}$$

We will illustrate by example the first few cases of these functions for small k after we prove the next lemma. The sequence of important semi-diagonals of these functions begins as [9, A008480]

$$\{\lambda(n)\cdot C_{\Omega(n)}(n)\}_{n\geq 1}\mapsto\{1,-1,-1,1,-1,2,-1,-1,1,2,-1,-3,-1,2,2,1,-1,-3,-1,\ -3,2,2,-1,4,1,2,\ldots\}.$$

Lemma 3.2 (An exact formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu(n/d)\lambda(d)C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$\begin{aligned} g^{-1}(n) &= -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) f^{-1}(n/d) && \implies \\ (g^{-1} * 1)(n) &= -(\omega * g^{-1})(n). \end{aligned}$$

Now by repeatedly expanding the right-hand-side, and removing corner cases in the nested sums since $\omega(1) = 0$ by convention, we find that

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n).$$

The statement follows by Möbius inversion applied to each side of the last equation. □

Notice that this approach, while it definitely has its complications due to the necessary step of Möbius inversion, is somewhat simpler that trying to form the Dirichlet inverse of the sum of $\omega+1$ directly, though this is also a possible approach. We decided against that method as a primary motivator because of the need to estimate convolutions of ω with m-fold convolutions of 1, e.g., the generalized Dirichlet divisor problem – which is known to be hard also. We are already going to need to evaluate iterated convolutions of $\omega(n)$ with itself, so we might as well sidestep the need to understand the Dirichlet divisor function complications in that form.

Example 3.3 (Special cases of the functions $C_k(n)$ for small k)**.** We cite the following special cases which should be easy enough to see on paper:

$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= \sigma_0(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} - \gcd\left(\Omega(n), \omega(n)\right). \end{aligned}$$

We also can see a recurrence relation between successive $C_k(n)$ values over k of the form

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} C_{k-1}\left(d \cdot p^i\right). \tag{3}$$

Thus we can work out further cases of the $C_k(n)$ for a while until we are able to understand the general trends of its asymptotic behaviors. In particular, we can compute the main term of $C_3(n)$ as follows where we use the notation that p, q are prime indices:

$$\begin{aligned} C_3(n) &\sim \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \sum_{q|dp^i} \frac{\nu_q(dp^i)}{\nu_q(dp^i)+1} \sigma_0(d)(i+1) \\ &= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \left[\sum_{q|d} \frac{\nu_q(d)}{\nu_q(d)+1} \sigma_0(d)(i+1) + \sum_{j=1}^{i} \frac{j}{(j+1)} \sigma_0(d)(i+1) \right] \\ &= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{q|d} \sigma_0(d) \left[\frac{\nu_p(n)(\nu_p(n)+3)}{2} \frac{\nu_q(d)}{\nu_q(d)+1} + \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2)\left(4\nu_p(n) + 9 - 6H^{(1)}_{\nu_p(n)+2}\right) \right]. \end{aligned}$$

We will break the two key component sums into separate calculations. First, we compute that[3]

$$\begin{aligned}
C_{3,1}(n) &= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \frac{\nu_p(n)(\nu_p(n)+3)}{2} \times \sum_{q|d} \frac{\nu_q(d)}{\nu_q(d)+1} \sigma_0(d) \\
&= \sum_{\substack{p,q|n \\ p \neq q}} \sum_{d|\frac{n}{p^{\nu_p(n)} q^{\nu_q(n)}}} \frac{\nu_p(n)(\nu_p(n)+3)}{2} \times \sum_{i=1}^{\nu_q(n)} \frac{\nu_q(dq^i)}{\nu_q(dq^i)+1} \sigma_0(dq^i) \\
&= \sum_{\substack{p,q|n \\ p \neq q}} \sum_{d|\frac{n}{p^{\nu_p(n)} q^{\nu_q(n)}}} \frac{\nu_p(n)(\nu_p(n)+3)\nu_q(n)(\nu_q(n)+3)}{4} \sigma_0(d) \\
&= (\sigma_0 * 1)(n) \times \sum_{\substack{p,q|n \\ p \neq q}} \frac{\nu_p(n)(\nu_p(n)+3)\nu_q(n)(\nu_q(n)+3)}{(\nu_p(n)+1)(\nu_p(n)+2)(\nu_q(n)+1)(\nu_q(n)+2)}.
\end{aligned}$$

Next, we have that

$$\begin{aligned}
C_{3,2}(n) &= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{q|d} \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2) \left(4\nu_p(n) + 9 - 6H^{(1)}_{\nu_p(n)+2}\right) \sigma_0(d) \\
&= \sum_{\substack{p,q|n \\ p \neq q}} \sum_{d|\frac{n}{p^{\nu_p(n)} q^{\nu_q(n)}}} \sum_{i=1}^{\nu_q(n)} \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2) \left(4\nu_p(n) + 9 - 6H^{(1)}_{\nu_p(n)+2}\right) \sigma_0(d)(i+1) \\
&= (\sigma_0 * 1)(n) \times \sum_{\substack{p,q|n \\ p \neq q}} \frac{1}{6} \frac{\nu_q(n)(\nu_q(n)+3)}{(\nu_q(n)+1)(\nu_q(n)+2)} \left(4\nu_p(n) + 9 - 6H^{(1)}_{\nu_p(n)+2}\right).
\end{aligned}$$

Now to roughly bound the error term, e.g., the GCD of prime omega functions from the exact formula for $C_3(n)$, we observe that the divisor function has average order of the form:

$$d(n) \sim \log n + (2\gamma - 1) + O\left(\frac{1}{\sqrt{n}}\right).$$

Then using that $\omega(n), \Omega(n) \sim \log\log n$, as discussed in the next remarks, we bound the error as

$$\begin{aligned}
C_{3,3}(n) &= -\sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \gcd\left(\Omega(d) + i, \omega(d) + 1\right) \\
&= \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} O\left(\sigma_0(n) \cdot \log\log n\right) \\
&= O\left(\pi(n) \cdot \log n \cdot \log\log n\right) \\
&= O\left(n \cdot \log\log n\right).
\end{aligned}$$

In total, we obtain that

$$C_3(n) = (\sigma_0 * 1)(n) \times \sum_{\substack{p,q|n \\ p \neq q}} \frac{1}{6} \frac{\nu_q(n)(\nu_q(n)+3)}{(\nu_q(n)+1)(\nu_q(n)+2)} \left(4\nu_p(n) + 9 - 6H^{(1)}_{\nu_p(n)+2}\right)) \qquad (4)$$

[3] Here, the arithmetic function $\sigma_0 * 1$ is multiplicative. It's value at prime powers can be computed as

$$(\sigma_0 * 1)(p^\alpha) = \sum_{i=0}^{\alpha} (i+1) = \frac{(\alpha+1)(\alpha+2)}{2}.$$

$$+\sigma_0(n)\times\sum_{\substack{p,q|n\\ q\neq p}}\frac{2^{\nu_q(n)}\nu_p(n)(\nu_p(n)+3)}{4(\nu_p(n)+1)(\nu_q(n)+1)}$$
$$+O\left(n\cdot\log\log n\right).$$

For the next cases, we would use similar techniques. The key is to compute enough small cases that we can see the dominant asymptotic terms in these expansions. We will expand more on this below.

Remark 3.4 (Recursive growth of the functions $C_k(n)$)**.** We note that the average order of $\Omega(n)\sim\log\log n$, so that for large $x\gg 1$ tending to infinity, we can expect (on average) that for $p|n$, $1\leq\nu_p(n)$ (for large $p|x, p\sim\frac{x}{\log x}$) and $\nu_p(n)\approx\log\log n$. However, if x is primorial, we can have $\Omega(x)\sim\frac{\log x}{\log\log x}$. There is, however, a duality with the size of $\Omega(x)$ and the rate of growth of the $\nu_p(x)$ exponents. That is to say that on average, even though $\nu_p(x)\sim\log\log n$ for most $p|x$, if $\Omega(x)=m\approx O(1)$ is small, then

$$\nu_p(x)\approx\log_{\sqrt[m]{\frac{x}{\log x}}}(x)=\frac{m\log x}{\log\left(\frac{x}{\log x}\right)}.$$

Since we will be essentially averaging the inverse functions, $g^{-1}(n)$, via their summatory functions over the range $n\leq x$ for x large, we tend not to worry about bounding anything but by the average case, which wins when we sum and tend to infinity. Given these observations, we can use the function $C_3(n)$ we just painstakingly computed exactly as an asymptotic benchmark to build further approximations. In particular, the dominant order terms in $C_3(n)$ are given by

$$C_3(n)\sim\frac{(\sigma_0*1)(n)n^2}{\log^2 n}-\frac{(\sigma_0*1)(n)n^2}{\log n}+O\left(n\cdot\log\log n\right).$$

We will leave the terms involving the divisor function $\sigma_0(n)$ and convolutions involving it unevaluated because of how much their growth can fluctuate depending on prime factorizations for now.

Summary 3.5 (Asymptotics of the $C_k(n)$)**.** We have the following asymptotic relations relations for the growth of small cases of the functions $C_k(n)$:

$$\begin{aligned}
C_1(n)&\sim\log\log n\\
C_2(n)&\sim\frac{\sigma_0(n)n}{\log n}+O(\log\log n)\\
C_3(n)&\sim\frac{(\sigma_0*1)(n)n^2}{\log^2 n}-\frac{(\sigma_0*1)(n)n^2}{\log n}+O\left(n\cdot\log\log n\right).
\end{aligned}$$

Theorem 3.7 stated below makes precise what these formulas suggest about the growth rates of $C_k(n)$.

Remark 3.6 (Symmetry in the prime factorizations of the inverse functions)**.** Now that we can see a general order pattern to the growth of the components of the inverse function constructions, we should discuss periodicity of both these functions, $C_{\Omega(n)}(n)$, and then the resulting inverse function formulas approximating $g^{-1}(n)$. Namely, as the exact formula in terms of the prime factorizations of n derived in (4) shows, the values of these functions are symmetric in the prime factorizations of n.

The approximations we made in the last few results and remarks merely allow us to formulate what the behavior should be for large $n\gg 1$. In fact, if two integers n_1,n_2 have the same prime exponent pattern, the values of both $C_{\Omega(n)}(n)$ and $g^{-1}(n)$ are (up to signage) identical regardless of the primes these exponent patterns exponentiate! We may be able to make use of this fact in avoiding duplication, or counting the number of identical values, as we did implicitly when we used the functions $\widehat{\pi}_k(n)$ in the previous subsection.

Theorem 3.7 (Asymptotics for the functions $C_k(n)$)**.** *Let $\mathbb{1}_{*_m}(n)$ denote the m-fold Dirichlet convolution of one with itself at n. The function $\sigma_0*1_{*_m}$ is multiplicative with values at prime powers given by*

$$(\sigma_0*\mathbb{1}_{*_m})(p^\alpha)=\binom{\alpha+m+1}{m+1}.$$

We have the following asymptotic bases cases for the functions $C_k(n)$:

$$\begin{aligned}
C_1(n) &\sim \log\log n \\
C_2(n) &\sim \frac{\sigma_0(n)n}{\log n} + O(\log\log n) \\
C_3(n) &\sim -\frac{(\sigma_0 * 1)(n)n^2}{\log n} + O\left(n \cdot \log\log n\right).
\end{aligned}$$

For all $k \geq 4$, we obtain that the dominant asymptotic term and the error bound terms for $C_k(n)$ are given by

$$C_k(n) \sim (\sigma_0 * \mathbb{1}_{*_{k-2}})(n) \times \frac{(-1)^k n^{k-1}}{(\log n)^{k-1}(k-1)!} + O_k\left(\frac{n^{k-2}}{(k-2)!} \cdot \frac{(\log\log n)^{k-2}}{(\log n)^{k-2}}\right), \text{ as } n \to \infty.$$

Proof. We showed how to compute the formulas for the base cases in the preceeding examples discussed above. We can also see that $C_3(n)$ satsfies the target formula specification. Let's proceed by using induction with the recurrence formula from (3) relating $C_k(n)$ to $C_{k-1}(n)$ for all $k \geq 1$. The strategy is to precisely evaluate the sums recursively, and drop the messy troblesome lower order terms that contribute to the nuances of the full formulas, along the way. What results is precise for sufficiently large $n \gg 1$. We will compute the main term formula first, then complete the proof by bounding the easier error term calculations.

Suppose that $k \geq 4$. By the recurrence formula for $C_k(n)$, we have that

$$C_k(n) \sum_{p|n} \sum_{d|np^{-\nu_p(n)}} \sum_{i=1}^{\nu_p(n)} -\frac{(dp^i)^{k-1}}{(\log(dp^i))^{k-1}} \binom{i+k-1}{k-1} (\sigma_0 * \mathbb{1}_{*_{k-2}})(d).$$

Now to handle the inner sum, we bound by setting $\alpha \equiv \nu_p(n)$ and invoking *Mathematica* in the form of

$$\begin{aligned}
\mathrm{IC}_k(n) &= \sum_{i=1}^{\alpha} -\frac{(dp^i)^{k-1}}{(\log(dp^i))^{k-1}} \binom{i+k-1}{k-1} \\
&\approx \int -\frac{(dp^\alpha)^{k-1}}{(\log(dp^\alpha))^{k-1}} \binom{\alpha+k-1}{k-1} \\
&\sim \frac{1}{(k-1)!\log^k p} \left(\mathrm{Ei}((k-2)\log(dp^\alpha)) \left[\log^{k-1}(d) - (k-1)!\log^{k-1}(p)\right]\right) \\
&\qquad - \frac{1}{(k-2)(k-1)!\log^k p} \left(\log^{k-2}(d) + \alpha^{k-2}\log^{k-2}(p)\right).
\end{aligned}$$

We now simplify somewhat again by setting

$$p \mapsto \left(\frac{n}{e}\right)^{\frac{1}{\log\log n}}, \alpha \mapsto \log\log n, \log p \mapsto \frac{\log n}{\log\log n}.$$

Also, since $p \gg_n d$, we obtain the dominant asymptotic growth terms of

$$\begin{aligned}
\mathrm{IC}_k(n) &\sim \frac{\alpha^{k-2}}{(k-2)(k-1)!\log^2 p} \\
&\approx \frac{(\log\log n)^k}{(k-2)(k-1)!\log^2 n}.
\end{aligned}$$

Now, as we did in the previous example work, we handle the sums by pulling out a factor of the inner divisor sum depending only on n (and k):

$$C_k(n) = \sum_{p|n} (\sigma_0 * \mathbb{1}_{*_{k-1}})(n) \binom{p^{\nu_p(n)}+k}{k}^{-1} \times \mathrm{IC}_k(n)$$

$$= (\sigma_0 * \mathbb{1}_{*_{k-1}})(n) \binom{p^{\nu_p(n)}+k}{k}^{-1} \cdot \pi(n) \times \mathrm{IC}_k(n)$$

Combining with the remaining terms we get by induction, we have proved our target bound holds for $C_k(n)$.

To bound the error terms, again suppose inductively that $k \geq 4$. We compute the big-O bounds as follows letting $\alpha \equiv \nu_p(n)$:

$$\begin{aligned}
\mathrm{ET}_k(n) &= \sum_{i=1}^{\nu_p(n)} n^{k-2} \cdot \frac{(\log\log n)^{k-2}}{(\log n)^{k-2}} \\
&\approx \int (dp^\alpha)^{k-2} \log\log(dp^\alpha) d\alpha \\
&= -\frac{\mathrm{Ei}((k-2)\log(dp^\alpha))}{(k-2)\log p} + \frac{d^{k-2}p^{(k-2)\alpha}}{(k-2)\log p}\log(dp^\alpha) \\
&\sim \frac{d^{k-2}p^{(k-2)\alpha}}{(k-2)\log p}\log(dp^\alpha).
\end{aligned}$$

In the last expansion, we have dropped the exponential integral terms since they provide at most polynomial powers of the logarithm of their inputs. To evaluate the outer divisor sum from the recurrence relation for $C_k(n)$, we will require the followiing bound providing an average order on the *generalied sum-of-divisors functions*, $\sigma_\alpha(n) := \sum_{d|n} d^\alpha$. In particular, we have that for integers $\alpha \geq 2$ [7, §27.11]:

$$\sigma_\alpha(n) \sim \frac{\zeta(\alpha+1)}{\alpha+1} x^\alpha + O(x^{\alpha-1}).$$

Approximating the number of terms in the prime divisor sum by $\pi(x) \sim \frac{x}{\log x}$, we thus obtain

$$\mathrm{ET}_k(n) \approx \frac{(\log\log n)^{k-1} e^{k-2}}{(k-1)(k-2)} x^{(k-2)\left[1-\frac{1}{\log\log x}\right]+1+\log\log x} \zeta(k-1).$$

So up to what is effectively constant in k, and dropping lower order terms for a slightly suboptimal, but still sufficient for our purposes, error bound formula, we have completed the proof by induction. □

Corollary 3.8 (Computing the inverse functions)**.** *In contrast to the complicated formulation given by Lemma 3.2, we have that*

$$g^{-1}(n) \sim \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

This is to say that for all $n \geq 2$

$$\left| 1 - \frac{\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)}{g^{-1}(n)} \right| = o\left(\sum_{d|n} C_{\Omega(d)}(d) \right).$$

Moreover, we can bound the error terms as

$$\left| 1 - \frac{\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)}{g^{-1}(n)} \right| = O\left(TODO \right).$$

Proof. Using Lemma 3.2, it suffices to show that the squarefree divisors $d|n$ such that $\operatorname{sgn}(\mu(d)\lambda(n/d)) = -1$ have an order of magnitude less adbundancy than the corresponding cases of positive sign on the terms in the divisor sum from the lemma. Let n have m_1 prime factors p_1 such that $v_{p_1}(n) = 1$, m_2 such that $v_{p_2}(n) = 2$, and the remaining $m_3 := \Omega(n) - m_1 - 2m_2$ prime factors of higher-order exponentation. We have a few cases to consider after re-writing the sum from the lemma in the following form:

$$g^{-1}(n) = \lambda(n) C_{\Omega(n)}(n) + \sum_{i=1}^{\omega(n)} \left\{ \sum_{\substack{d \mid n \\ \omega(d)=\Omega(d)=i \\ \#\{p \mid d : \nu_p(d)=1\}=k_1 \\ \#\{p \mid d : \nu_p(d)=2\}=k_2 \\ \#\{p \mid d : \nu_p(d)\geq 3\}=k_3}} \mu(d) \lambda(n/d) C_{\Omega(n/d)}(n/d) \right\}.$$

We obtain the following cases of the squarefree divisors contributing to the signage on the terms in the above sum:

- The sign of $\mu(d)$ is $(-1)^i = (-1)^{k_1+k_2+k_3}$;
- If $m_3 < \#\{p \mid n : \nu_p(n) \geq 3\}$, then $\lambda(n/d) = 1$ (since $\mu(n/d) = 0$);
- Given (k_1, k_2, k_3) as above, since $\lambda(n) = (-1)^{\Omega(n)}$, we have that $\mu(d) \cdot \lambda(n/d) = (-1)^{i-k_1-k_2} \lambda(n)$.

Thus we define the following sums, parameterized in the $(m_1, m_2, m_3; n)$, which corresponds to a change in expected parity transitioning from the Möbius inversion sum from Lemma 3.2 to the sum approximating $g^{-1}(n)$ defined at the start of this result:

$$\widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) := \sum_{i=1}^{\omega(n)/2} \sum_{k_1=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{i}{2} \rfloor - k_1} \left[\binom{m_1}{2k_1+1} \binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1} \binom{2m_2}{2k_2} \right] [i - k_1 - k_2 = k_3 \equiv m_3]_\delta$$
$$\widetilde{S}_{\mathrm{even}}(m_1, m_2, m_3; n) := \sum_{i=1}^{\omega(n)/2} \sum_{k_1=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{i}{2} \rfloor - k_1} \left[\binom{m_1}{2k_1} \binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1+1} \binom{2m_2}{2k_2} \right] [i - k_1 - k_2 = k_3 \equiv m_3]_\delta .$$

Part I (Lower bounds on the inner sums of the count functions). We claim that

$$\begin{aligned} \widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) &\gg \binom{m_1}{i+1} + \binom{m_1}{\frac{i}{2}} \binom{2m_2 - 1}{\frac{i}{2}+1} \\ \widetilde{S}_{\mathrm{even}}(m_1, m_2, m_3; n) &\gg \binom{m_1}{i+1} + \binom{m_1}{\frac{i}{2}-1} \binom{2m_2}{\frac{i}{2}+1}. \end{aligned} \tag{5}$$

To prove (5) we have to provide a straightforward bound that represents the maximums of the terms in m_1, m_2. In particular, observe that for

$$\widetilde{S}_{\mathrm{odd}}(m_1, m_2; u) = \sum_{k_1=0}^{u} \sum_{k_2=0}^{u-k_1} \left[\binom{m_1}{2k_1} \binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1+1} \binom{2m_2}{2k_2} \right]$$
$$\widetilde{S}_{\mathrm{even}}(m_1, m_2; u) = \sum_{k_1=0}^{u} \sum_{k_2=0}^{u-k_1} \left[\binom{m_1}{2k_1+1} \binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1} \binom{2m_2}{2k_2} \right],$$

we have that

$$\begin{aligned} \widetilde{S}_{\mathrm{odd}}(m_1, m_2; u) &\gtrsim \binom{m_1}{2u+1} + \max_{1 \leq k_1 \leq u} \binom{m_1}{2k_1+1} \binom{2m_2}{2u+1-2k_1} \\ &= \binom{m_1}{2u+1} + \binom{m_1}{2k_1+1} \binom{2m_2}{2u+1-2k_1} \Bigg|_{k_1 = \frac{u}{2}} \end{aligned}$$

$$
\begin{aligned}
&= \binom{m_1}{2u+1} + \binom{m_1}{u+1}\binom{2m_2}{u+1} \\
\widetilde{S}_{\mathrm{even}}(m_1, m_2; u) &\gtrsim \binom{m_1}{2u+1} + \max_{1\leq k_1 \leq u} \binom{m_1}{2k_1}\binom{2m_2}{2u+1-2k_1} \\
&= \mathit{TODO}.
\end{aligned}
$$

The lower bounds in (5) then follow by setting $u \equiv \left\lfloor \frac{i}{2} \right\rfloor$.

Part II (Bounding m_1, m_2, m_3 and effective (i, k_1, k_2) contributing to the count). We thus have to determine the asymptotic growth rate of $\widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) + \widetilde{S}_{\mathrm{even}}(m_1, m_2, m_3; n)$, and show that it is of comparitively small order. First, we bound the count of non-zero m_3 for $n \leq x$ from below. For the cases where we expect differences in signage, it's the last Iverson convention term that kills the order of growth, e.g., we expect differences when the parameter m_3 is larger than the usual configuration. We know that

$$
\pi_k(x) \sim \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}.
$$

Using the formula for $\pi_k(x)$, we can count the average orders of m_1, m_2 as

$$
\begin{aligned}
N_{m_1}(x) &\approx \frac{1}{x} \#\{n \leq x : \omega(n) = 1\} \sim \frac{\log\log x}{\log x} \\
N_{m_2}(x) &\approx \frac{1}{x} \#\{n \leq x : \omega(n) = 2\} \sim \frac{(\log\log x)^2}{\log x}.
\end{aligned}
$$

Additionally, in Corollary 4.6, we proved a lower bound on $\widehat{\pi}_k(x)$. When we have parameters with respect to some $n \geq 1$ such that $m_3 > 0$, it must be the case that

$$
\Omega(n) - \omega(n) > \begin{cases} 0, & \text{if } \omega(n) \geq 2; \\ 1, & \text{if } \omega(n) = 1. \end{cases}
$$

To count the number of cases $n \leq x$ where this happens, we form the sums

$$
\begin{aligned}
N_{m_3}(x) &\gg \pi_1(x) \times \sum_{k=3}^{\frac{3}{2}\log\log x} \widehat{\pi}_k(x) + \sum_{k=2}^{\frac{3}{2}\log\log x} \sum_{j=k+1}^{\frac{3}{2}\log\log x} \pi_k(x)\widehat{\pi}_j(x) \\
&\gtrsim \frac{Ae}{B} \frac{x}{\log^{\frac{13}{14}}(x)} + \sum_{k=2}^{\log\log x} \pi_k(x) \left[\frac{Ae}{B} \log^{\frac{1}{14}}(x)\right] \\
&\gtrsim \frac{Ae}{B} \frac{x}{\log^{\frac{13}{14}}(x)} + \frac{Ae\sqrt{2}}{2\sqrt{\pi}B} \frac{x}{\log^{\frac{13}{14}}(x)\sqrt{\log\log x}}.
\end{aligned}
$$

Now in practice, we are not summing up $n \leq x$, but rather $n \leq \log\log x$. So the above function evaluates to

$$
N_{m_3}(\log\log x) \gg \frac{\log\log x}{(\log\log\log x)^{13/14}} \gg \frac{\log\log x}{\log\log\log x}.
$$

Next, we go about solving the subproblem of finding when $i - k_1 - k_2 = m_3$. First, we find a lower solution index on i using asymptotics for the *Lambert W-function*, $W_0(x) = \log x - \log\log x + o(1)$:

$$
\begin{aligned}
\frac{i}{2} = \frac{\log\log x}{\log\log\log x} &\iff \log\log x \gtrsim \frac{i}{2}(\log i + \log\log i) \\
&\iff \frac{i}{2} \sim \frac{\log\log x}{\log\log\log x}.
\end{aligned}
$$

Now since $2 \leq k_1 + k_2 \leq i/2$, when x is large, we actually obtain a number of soultions on the order of

$$
\frac{\log\log x}{2} - \frac{\log\log x}{\log\log\log x} = \frac{\log\log x}{2}(1 + o(1)).
$$

Part III (Putting it all together). Using the binomial coefficient inequality

$$\binom{n}{k} \geq \frac{n^k}{k^k},$$

we can work out carefully on paper using (5) that

$$\widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) \gtrsim \frac{\log\log x}{2} \left(\frac{\log\log\log x}{2\log x}\right)^{\frac{2\log\log x}{\log\log\log x}+1} \left[1 + \frac{(\log\log\log x)^2}{\log^2 x}(4\log\log x \cdot \log\log\log x)^{\frac{\log\log x}{2\log\log\log x}+1}\right]$$

$$\widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) \gtrsim \frac{\log\log x}{2} \left(\frac{\log\log\log x}{2\log x}\right)^{\frac{2\log\log x}{\log\log\log x}+1} \left[1 + \left(\frac{\log x}{2\log\log\log x}\right)(8\log\log x)^{\frac{\log\log x}{\log\log\log x}+1}\right].$$

Part IV (Obtaining the rate at which the ratio goes to zero). (TODO) ...

□

Corollary 3.9 (Asymptotics for very special case of the functions $C_k(n)$)**.** *For $k \gg 1$ sufficiently large, we have that*

$$C_{\Omega(n)}(n) \sim (\sigma_0 * \mathbb{1}_{*_{\log\log n - 2}})(n) \times \lambda(n) \frac{n^{\log\log n - 1}}{(\log n)^{\log\log - 1}\Gamma(\log\log n)}.$$

Moreover, by considering the average orders of the function $\nu_p(n)$ for p large and tending to infinity, we have bounds on the aysmptotic behavior of these functions of the form

$$\lambda(n)\widehat{\tau}_0(n) \precsim C_{\Omega(n)}(n) \precsim \lambda(n)\widehat{\tau}_1(n).$$

It suffices to take the functions

$$\widehat{\tau}_0(n) := \frac{1}{\log 2} \cdot \frac{\log n}{(\log n)^{\log\log n}} \cdot \frac{n^{\log\log n - 1}}{\Gamma(\log\log n)}$$

$$\widehat{\tau}_1(n) := \frac{1}{2e\log 2} \cdot \frac{(\log n)^2}{(\log n)^{\log\log n}} \cdot \frac{n^{\log\log n}}{\Gamma(\log\log n)}.$$

Proof. The first stated formula follows from Theorem 3.7 by setting $k := \Omega(n) \sim \log\log n$ and simplifying. We evaluate the Dirichlet convolution functions and approximate as follows:

$$\begin{aligned}
(\sigma_0 * \mathbb{1}_{\log\log n - 2})(n) &= \sum_{p|n} \binom{\nu_p(n) + \log\log n - 1}{\log\log n - 1} \\
&\geq \sum_{p|n} \frac{(\nu_p(n) + \log\log n - 1)^{\log\log n - 1}}{(\log\log n)^{\log\log n - 1}} \\
&\sim \frac{n}{\log 2} \\
(\sigma_0 * \mathbb{1}_{\log\log n - 2})(n) &\leq \left(\frac{(\nu_p(n) + \log\log n - 1)e}{\log\log n - 1}\right)^{\log\log n - 1} \\
&\sim (2e)^{\log\log n - 1} \\
&= \frac{n \cdot \log n}{2e\log 2}.
\end{aligned}$$

The upper and lower bounds are obtained from the next well known binomial coefficient approximations using Stirling's formula.

$$\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{n^k}{k!} < \left(\frac{ne}{k}\right)^k$$

□

4 Summing functions weighted by the Liouville lambda function, $\lambda(n) := (-1)^{\Omega(n)}$: *Borrowing a method of enumeration of summatory functions by Dirichlet series and Euler products from Montgomery and Vaughan, Chapter 7*

4.1 Discussion: The enumerative DGF result in Theorem 1.5 from Montgomery and Vaughan

In the reference we have defined $F(s, z)$ for $\Re(s) > 1$ such that the Dirichlet series coefficients, $a_z(n)$, are defined by

$$\zeta(s)^z F(s, z) := \sum_{n\geq 1} \frac{a_z(n)}{n^s}, \Re(s) > 1.$$

For the function

$$F(s, z) := \prod_p \left(1 - \frac{z}{p^s}\right)\left(1 - \frac{1}{p^s}\right)^z,$$

we obtain in the notation above that $a_z(n) \equiv z^{\Omega(n)}$, and that the summatory function satisfies

$$A_z(x) := \sum_{n\leq x} z^{\Omega(n)} = \sum_{k\geq 0} \widehat{\pi}_k(x) z^k.$$

Hence, by the Cauchy integral formula, for $r < 2$ we get that

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \oint_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz,$$

from which we obtain the stated formula in the theorem.

What this enumeratively-flavored result of Montgomery and Vaughan allows us to do is get a "good enough" lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Lioville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. For comparison, we have known, more classical bounds due to Erdös and TODO that state for

$$\pi_k(x) := \#\{n \leq x : \omega(n) = k\},$$

we have tightly that [?, ?]

$$\pi_k(x) = (1 + o(1)) \cdot \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

4.2 Preliminary results

We seek to approximate the right-hand-side of $G(z)$ by only taking the products of the primes $p \leq x$, e.g., $p \in \{2, 3, 5, \ldots, x\}$. We will require some fairly elementary estimates of products of primes, Mertens theorem on the rate of divergence of the sum of the reciprocals of the primes, and on some generating function techniques involving elementary symmetric functions.

Theorem 4.1 (Mertens theorem)**.**

$$P_1(x) := \sum_{p\leq x} \frac{1}{p} = \log\log x + B + o\left(\frac{1}{\log x}\right),$$

where $B \approx 0.2614972128476427837554$ *is an absolute constant. We actually can bound the left-hand-side more explicitly by*

$$P_1(x) = \log\log x + B + O\left(e^{-(\log x)^{\frac{1}{14}}}\right).$$

Corollary 4.2. *We have that for sufficiently large $x \gg 1$*

$$\prod_{p \leq x}\left(1-\frac{1}{p}\right) = \frac{e^{-B}}{\log x}\left[1-\frac{(\log x)^{1/14}}{B}+o\left((\log x)^{1/14}\right)\right].$$

Hence, for $1 < |z| < R < 2$ we obtain that

$$\prod_{p \leq x}\left(1-\frac{1}{p}\right)^z = \frac{e^{-Bz}}{(\log x)^z}\left[1-\frac{z}{B}(\log x)^{\frac{1}{14}}+o_z\left(z^2\cdot(\log x)^{\frac{1}{14}}\right)\right].$$

Proof. By taking logarithms and using Mertens theorem above, we obtain that

$$\begin{aligned}\log\prod_{p \leq x}\left(1-\frac{1}{p}\right) &= \sum_{p \leq x}\log\left(1-\frac{1}{p}\right) \\ &\approx -\log\log x - B + O\left(e^{-(\log x)^{1/14}}\right).\end{aligned}$$

Hence, the first formula follows by expanding out an alternating series for the exponential function. The second formula follows for $z \notin \mathbb{Z}$ by an application of the generalized binomial series given by

$$\log\prod_{p \leq x}\left(1-\frac{1}{p}\right)^z \approx \frac{e^{-Bz}}{(\log x)^z}\times\sum_{r \geq 0}\binom{z}{r}\frac{(-1)^r}{B^r}(\log x)^{\frac{r}{14}},$$

where for $1 < |z| < 2$, we obtain the next result stated above with $\binom{z}{1} = z$ and $\binom{z}{2} = z(z-1)/2$. □

Facts 4.3 (Exponential Integrals and Incomplete Gamma Functions)**.** The following two variants of the *exponential integral function* are defined by

$$\begin{aligned}\mathrm{Ei}(x) &:= \int_{-x}^{\infty}\frac{e^{-t}}{t}dt, \\ E_1(z) &:= \int_{1}^{\infty}\frac{e^{-tz}}{t}dt, \Re(z) \geq 0,\end{aligned}$$

where $\mathrm{Ei}(-kz) = -E_1(kz)$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $E_1(z)$:

$$1-\frac{3}{4}z \leq E_1(z)-\gamma-\log z \leq 1-\frac{3}{4}z+\frac{11}{36}z^2. \tag{6a}$$

A related function is the (uppper) *incomplete gamma function* defined by

$$\Gamma(s,x) = \int_x^{\infty} t^{s-1}e^{-t}dt, \Re(s) > 0.$$

We have the following properties of $\Gamma(s,x)$:

$$\Gamma(s,x) = (s-1)!\cdot e^{-x}\times\sum_{k=0}^{s-1}\frac{x^k}{k!}, s \in \mathbb{Z}^+, \tag{6b}$$

$$\Gamma(s+1,1) = e^{-1}\left\lfloor\frac{s!}{e}\right\rfloor, s \in \mathbb{Z}^+, \tag{6c}$$

$$\Gamma(s,x) \sim x^{s-1}\cdot e^{-x}, |x| \to +\infty. \tag{6d}$$

Corollary 4.4. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \gg 1.$$

When $s := 1$, we have the known bound in Mertens theorem. For $s > 1$, we obtain that

$$P_s(x) \approx E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1).$$

It follows that

$$\gamma_0(s,x) + o(1) \leq P_s(x) \leq \gamma_1(s,x) + o(1),$$

where it suffices to take

$$\begin{aligned}
\gamma_0(z,x) &= -s\log\left[\frac{\log x}{\log 2}\right] - \frac{3}{4}s(s-1)\log(x/2) - \frac{11}{36}s(s-1)^2\log^2(2) \\
\gamma_1(z,x) &= \quad s\log\left[\frac{\log x}{\log 2}\right] - \frac{3}{4}s(s-1)\log(x/2) + \frac{11}{36}s(s-1)^2\log^2(x).
\end{aligned}$$

Proof. Let $s > 1$ be real-valued. By Abel summation where our summatory function is given by $A(x) = \pi(x) \sim \frac{x}{\log x}$ and our function $f(t) = t^{-s}$ so that $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned}
P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\
&= E_1((s-1)\log x) - E_1((s-1)\log 2) + o(1), |x| \to \infty.
\end{aligned}$$

Now using the inequalities in Facts 4.3, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned}
\frac{P_s(x)}{s} &\geq -\log\left[\frac{\log x}{\log 2}\right] - \frac{3}{4}(s-1)\log(x/2) - \frac{11}{36}(s-1)^2\log^2(2) \\
\frac{P_s(x)}{s} &\leq \quad \log\left[\frac{\log x}{\log 2}\right] - \frac{3}{4}(s-1)\log(x/2) + \frac{11}{36}(s-1)^2\log^2(x).
\end{aligned}$$

This completes the proof of the bounds cited above in the statement of this lemma. □

4.3 The key new results utilizing Theorem 1.5

Theorem 4.5 (Generating Functions of Symmetric Functions). *We have that for all integers $0 \leq k \leq m$*

$$[z^k] \prod_{1 \leq i \leq m} (1 - f(i)z)^{-1} = [z^k] \exp\left(\sum_{j \geq 1} \left(\sum_{i=1}^m f(i)^j\right) \frac{z^j}{j}\right).$$

Thus we obtain upper and lower bounds of the partial prime products of the form

$$\alpha_0(z,x) \leq \prod_{p \leq x} \left(1 - \frac{z}{p}\right)^{-1} \leq \alpha_1(z,x),$$

where it suffices to take

$$\begin{aligned}
\alpha_0(z,x) &= \frac{\exp\left(\frac{55}{4}\log^2 2\right)}{\log^3 2}(\log x)^3 \left(\frac{e^B \log^2 x}{\log 2}\right)^z \\
\alpha_1(z,x) &= \exp\left(\frac{11}{3}\log^2 x\right)\left(e^B \log 2\right)^z.
\end{aligned}$$

Proof. In our case we have that $f(i)$ denotes the i^{th} prime. Hence, summing over all $p \leq x$ in place of $0 \leq k \leq m$ in the previous formula applied in tandem with Corollary 4.4, we obtain that the logarithm of the generating function series we are after corresponds to

$$\log\left[\prod_{p \leq x}\left(1 - \frac{z}{p}\right)^{-1}\right] = (B + \log\log x)z + \sum_{j \geq 2}\left[a(x) + b(x)(j-1) + c(x)(j-1)^2\right] z^j$$

$$= (B + \log\log x)z - a(x)\left(1 + \frac{1}{z-1} + z\right) + b(x)\left(1 + \frac{2}{z-1} + \frac{1}{(z-1)^2}\right)$$
$$- c(x)\left(1 + \frac{4}{z-1} + \frac{5}{(z-1)^2} + \frac{2}{(z-1)^3}\right).$$

In the previous equations, the upper and lower bounds formed by the functions (a, b, c) are given by

$$(a_\ell, b_\ell, c_\ell) := \left(-\log\left[\frac{\log x}{\log 2}\right], \frac{3}{4}\log\left(\frac{x}{2}\right), -\frac{11}{36}\log^2 2\right)$$
$$(a_u, b_u, c_u) := \left(\log\left[\frac{\log x}{\log 2}\right], -\frac{3}{4}\log\left(\frac{x}{2}\right), \frac{11}{36}\log^2 x\right).$$

Now we make a prudent decision to set $R := \frac{3}{2}$ so that

$$z \equiv z(k, x) = \frac{k-1}{\log\log x} \in [0, R),$$

for $x \gg 1$ very large. Thus $(z-1)^{-m} \in [(-1)^m, 2^m]$ for integers $m \geq 1$, and we can then form the upper and lower bounds from above. What we get out of these formulas is stated as in the theorem bounds. □

Corollary 4.6 (Bounds on $G(z)$ from MV)**.** *We have that for the function $G(z) := F(1, z)/\Gamma(z+1)$ from Montgomery and Vaughan, there is a constant A_0 and functions of x only, $B_0(x), C_0(x)$, so that*

$$A_0 \cdot B_0(x) \cdot C_0(x)^z \left(1 - \frac{z}{B}(\log x)^{\frac{1}{14}}\right) \leq G(z).$$

It suffices to take

$$A_0 = \frac{\exp\left(\frac{55}{4}\log^2 2\right)}{\log^3(2) \cdot \Gamma(5/2)} \approx 1670.84511225$$
$$B_0(x) = \log^3 x$$
$$C_0(x) = \frac{\log x}{\log 2}.$$

Proof. This result is a consequence of applying both Corollary 4.2 and Theorem 4.5 to the definition of $G(z)$. In particular, we obtain bounds of the following form from the theorem:

$$\frac{A_0 \cdot B_0(x) \cdot C_0(x)^z}{\Gamma(z+1)} \leq \frac{G(z)}{\prod_p \left(1 - \frac{1}{p}\right)^z} \leq \frac{A_1 \cdot B_1(x) \cdot C_1(x)^z}{\Gamma(z+1)}.$$

Since $z \equiv z(k, x) = \frac{k-1}{\log\log x}$ and $k \in [1, R\log\log x]$, we obtain that for small k and $x \gg 1$ large $\Gamma(z+1) \approx 1$, and for k towards the upper bound of its interval that $\Gamma(z+1) \approx \Gamma(5/2)$ (recall that we set $R := 3/2$ in the preceeding proof of Theorem 4.5). Thus when we expand out the formula given by the corollary in conjunction with these bounds on the gamma function, we obtain the claimed results. □

5 Key applications: Establishing lower bounds for $M(x)$ by cases along infinite subsequences

5.1 A culmination of what we have done so far

As before, in the previous subsection, we cannot hope to evaluate $\lambda(n)$ just yet, but only on average using Abel summation. We need to know the bounds on $\widehat{\pi}_k(x)$ we developed in the proof of Corollary 4.6. As we demonstrated in (**??**), a summation by parts argument shows that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k)+1) \\ &\approx G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)}. \end{aligned} \tag{7}$$

Thus it suffices for us to compute the effective *average order* of $g^{-1}(n)$ by summing its summatory function, $G^{-1}(n)$, including absorbing the signage of $\lambda(n)$ into the parity of the $\Omega(n)$ terms we are summing over.

To simplify notation, for integers $m \geq 1$, let the *iterated logarithm function* (not to be confused with powers of $\log x$) be defined for $x > 0$ by

$$\log_*^m(x) := \begin{cases} x, & \text{if } m = 0; \\ \log x, & \text{if } m = 1; \\ \log\left[\log_*^{m-1}(x)\right], & \text{if } m \geq 2. \end{cases}$$

So $\log_*^2(x) = \log\log x$, $\log_*^3(x) = \log\log\log x$, $\log_*^4(x) = \log\log\log\log x$, $\log_*^5(x) = \log\log\log\log\log x$, and so on. This notation will come in handy to abbreviate the dominant asymptotic terms we find next in Theorem 5.1.

We use the result of Corollary 3.9 and Corollary 4.6 to prove the following theorem:

Theorem 5.1 (Asymptotics and bounds for the summatory functions $G^{-1}(x)$)**.** *We define the upper and lower summatory functions, $G_u^{-1}(x)$ and $G_\ell^{-1}(x)$, respectively, to provide bounds on the magnitude of $G^{-1}(x)$:*

$$|G_\ell^{-1}(x)| \ll |G^{-1}(x)|,$$

for all sufficiently large $x \gg 1$. We have the following asymptotic approximations for the lower and upper summatory functions where C_ℓ is an absolute constant:

$$\begin{aligned} -G_\ell^{-1}(x) \gtrsim{} & -C_\ell - \frac{3A(2e+3)}{\sqrt{2\pi}9e^2 B \log^2 2 (\log\log 2)^{\frac{3}{2}}} (\log\log x)^{\frac{39}{7}} \left(\frac{\log x}{\log\log x}\right)^{\log_*^3(x)-1} \log_*^3(x)^{\frac{\log_*^3(x)}{2}} \\ & - \begin{cases} \frac{A3^{5/2}(2e+3)}{32\pi e^2 \log 2 (\log\log 2)^{\frac{3}{2}}} \left(\frac{2^{25/6}}{3^{4/3}}\right)^{\log_*^3(x)} \log\log x \cdot \log_*^3(x)^{1/3} \log_*^4(x)^{11/7} \frac{\log_*^5(x)^{\frac{3}{2}\log_*^5(x)}}{\log_*^3(x)^{\frac{5}{2}\log_*^3(x)}}, & \textit{if } (-1)^{\lfloor \log_*^4(x) \rfloor} = +1; \\ -\frac{A3^{5/2}(2e+3)}{16\pi e \log^{3/2}(2)} \left(\frac{2^{25/6}}{3^{4/3}}\right)^{\log_*^4(x)} \log_*^3(x)^{4/3} \log_*^5(x)^{11/7} \frac{\log_*^6(x)^{\frac{3}{2}\log_*^6(x)}}{\log_*^4(x)^{\frac{5}{2}\log_*^4(x)}}, & \textit{if } (-1)^{\lfloor \log_*^4(x) \rfloor} = -1; \end{cases} \\ G_\ell^{-1}(x) \gtrsim{} & C_\ell + \frac{3A(2e+3)}{\sqrt{2\pi}4eB(\log\log\log 2)^{\frac{3}{2}}} \log_*^3(x)^{\frac{32}{7}} \left(\frac{\log\log x}{\log_*^3(x)}\right)^{\log_*^4(x)-1} \log_*^4(x)^{\frac{\log_*^4(x)}{2}+1} \\ & + \begin{cases} \frac{A3^{5/2}(2e+3)}{16\pi e \log^{3/2}(2)} \left(\frac{2^{25/6}}{3^{4/3}}\right)^{\log_*^4(x)} \log_*^3(x)^{4/3} \log_*^5(x)^{11/7} \frac{\log_*^6(x)^{\frac{3}{2}\log_*^6(x)}}{\log_*^4(x)^{\frac{5}{2}\log_*^4(x)}}, & \textit{if } (-1)^{\lfloor \log_*^4(x) \rfloor} = +1; \\ -\frac{A3^{5/2}(2e+3)}{32\pi e^2 \log 2 (\log\log 2)^{\frac{3}{2}}} \left(\frac{2^{25/6}}{3^{4/3}}\right)^{\log_*^3(x)} \log\log x \cdot \log_*^3(x)^{1/3} \log_*^4(x)^{11/7} \frac{\log_*^5(x)^{\frac{3}{2}\log_*^5(x)}}{\log_*^3(x)^{\frac{5}{2}\log_*^3(x)}}, & \textit{if } (-1)^{\lfloor \log_*^4(x) \rfloor} = -1. \end{cases} \end{aligned}$$

Proof (Lower Bounds). Recall from our proof of Corollary 4.6 that a lower bound on the function $\widehat{\pi}_k(x)$ is given by $G\left(\frac{k-1}{\log\log x}\right)$ where the function $G(z)$ is bounded below by

$$G(z) \gg A_0 x \frac{(\log\log x)^{k-1}}{(k-1)!} \left(\frac{\log x}{\log 2}\right)^z \log^2 x \left(1 - \frac{z}{B} \log^{\frac{1}{14}}(x)\right).$$

Thus we can form a lower summatory function indicating the parity of all $\Omega(n)$ for $n \leq x$ as

$$\begin{aligned} A_{\Omega}^{(\ell)}(t) &= \sum_{k \leq \frac{3}{2}\log\log t} (-1)^k G\left(\frac{k-1}{\log\log t}\right) \\ &\sim \frac{3A}{4eB\log^{\frac{3}{2}}(2)\Gamma\left(1+\frac{3}{2}\log\log t\right)}\left((2e+3)\log^{\frac{1}{14}}(t) - 2B\right)\log^{\frac{3}{2}}(t)(\log\log t)^{\frac{3}{2}\log\log t}. \end{aligned} \tag{8}$$

We apply Abel summation to obtain that

$$G_{\ell}^{-1}(x) = \widehat{\tau}_0(x)A_{\Omega}^{(\ell)}(x) - \widehat{\tau}_0(u_0)A_{\Omega}^{(\ell)}(u_0) - \int_{u_0}^{x} \widehat{\tau}_0'(t)A_{\Omega}^{(\ell)}(t)dt, \tag{9}$$

where we define the integrand function, $I_{\ell}(t) := (-1)^{1+\left\lfloor \frac{3\log\log t}{2}\right\rfloor} \cdot \widehat{\tau}_0'(t)A_{\Omega}^{(\ell)}(t)$, as (with some limiting simplifications)

$$\begin{aligned} I_{\ell}\left(e^{e^{\frac{4k}{3}}}\right)e^{e^{\frac{4k}{3}}} &= \frac{4A4^{2k-1}9^{-k}k^{2k}\left((3+2e)e^{2k/21} - 2B\right)]\exp\left(-\frac{16k^2}{9} + 2k + e^{4k/3}\left(\frac{4k}{3}-1\right)-1\right)}{3B\log^{\frac{5}{2}}(2)} \times \\ &\times \left(4e^{4k/3}k - 8k - 3\log k - 3\gamma + 6 + 3\log 3 - 6\log 2\right). \end{aligned}$$

There is no hope to be had with traditional symbolic integration software to integrate this function precisely without some limiting and algebraic modifications. The integration term in (9) is summed approximately as follows:

$$\begin{aligned} \int_{u_0-1}^{x} \widehat{\tau}_0'(t)A_{\Omega}^{(\ell)}(t)dt &\sim \sum_{k=u_0+1}^{\log\log\log\log x} \left(\frac{I_{\ell}\left(e^{e^{\frac{4k+2}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!} - \frac{I_{\ell}\left(e^{e^{\frac{4k}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!}\right)e^{e^{\frac{4k}{3}}} \\ &\approx C_0(u_0) + (-1)^{\left\lfloor \frac{\log\log x}{2}\right\rfloor} \times \int_{\frac{\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log x}{2}} \frac{I_{\ell}\left(e^{e^{\frac{4k}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!}e^{e^{\frac{4k}{3}}}dk. \end{aligned}$$

We cannot integrate the integrands we require exactly in terms of known functions, besides by numerical methods, which is clearly a useless endeavor for our purposes. Fortunately, the differences on the upper and lower bounds on each integral in the last equation is small, and in particular $\frac{1}{2} \lll \log\log x$. So we can use a small perturbation of $+1$ in the power terms of $I_{\ell}(t)$ combined with an appeal to the binomial series, the expansion of binomial coefficients by the Stirling numbers of the first kind, and the following exact indefinite integral for $x, z \in \mathbb{R}$:

$$\int t^p e^{ct}dt = \frac{(-1)^p}{c^{p+1}}\Gamma(p+1, -ct) \sim \frac{e^{ct}t^p}{c}.$$

Define the following function of t and note the change of variable $t \mapsto \frac{k-1}{2}$:

$$I_{\ell}\left(e^{e^{\frac{4k}{3}}}\right)e^{e^{\frac{4k}{3}}} = (1+k)^{2k}\exp\left(-\frac{16k^2}{9}\left(\frac{4k}{3}-1\right)e^{\frac{4k}{3}}\right)e^{2k-1}\widehat{f}(t_0).$$

So we take one reciprocal factor in the next integrand, and set the remaining powers of t^p to be t_0^p for t_0 a bound of integration which results in a lower bound on our target integrand from Abel summation. From this perspective, we obtain using the exponential generating functions for the Stirling numbers of the first kind that [4, §7.4]

$$\begin{aligned} \widehat{T}_{\ell}(t_0; t) &= \int \widehat{I}_{\ell}(t)dt \\ &\gg \sum_{m\geq 0}\sum_{n\geq 0}\sum_{q\geq 0}\sum_{j\geq 0}\sum_{r\geq 0} \frac{(-1)^{m+q+j+r}}{m!n!q!j!}\left(\frac{4}{3}\right)^{2m+n}\begin{bmatrix} j \\ r \end{bmatrix}\left\{\int t^{2m+n+j+r}\exp\left(\left(2+\frac{4}{3}(n+q)\right)t\right)dt\right\}\frac{\widehat{f}(t_0)}{e} \end{aligned}$$

$$\gtrsim -\frac{3\widehat{f}(t_0)}{4e}e^{2t}e^{-\frac{16k^2}{9}}\left(\gamma+\frac{e^{te^{\frac{4t}{3}}}}{te^{\frac{4t}{3}}}+\frac{4t}{3}\right)\left(\gamma+\frac{e^{te^{\frac{4t}{3}}}}{ke^{\frac{4t}{3}}}-\frac{4t}{3}\right)t^{2t}$$

In the previous equation, we have used that $(n+q+12)^{-1}\gtrsim\frac{1}{nq}$ and that for large $x\ggg 1$ tending to infinity

$$\sum_{m\geq 1}\frac{(-x)^m}{m\cdot m!}=-\left(\gamma+\Gamma(0,x)+\log x\right)\sim-\left(\gamma+\frac{e^{-x}}{x}+\log x\right).$$

Now we define the coefficient functions, which would have otherwise complicated our integrals, in the form of $\widehat{f}(t_0)=\mathrm{cf}_+(t_0)-\mathrm{cf}_-(t_0)$ as

$$\mathrm{cf}_+(t):=\left(\frac{16}{9}\right)^t\left(2B(8t+3\gamma+6\log 2)+6B\log t+12e^{10t/7}t+8e^{\frac{10t}{7}+1}t+6e^{\frac{2t}{21}+1}(2+\log 3)+9e^{2t/21}(2+\log 3)\right)$$
$$\mathrm{cf}_-(t):=\left(\frac{16}{9}\right)^t\left(2B\left(4e^{4t/3}t+6+3\log 3\right)+(3+2e)e^{2t/21}(8t+3\gamma+6\log 2)+3(3+2e)e^{2t/21}\log t\right).$$

Let

$$\widehat{h}(t):=3\cdot 4^{-t-1}\left(\frac{3}{4}\right)^{\frac{4t}{3}}\frac{\sqrt{3}}{16\pi t^{\frac{10t}{3}+1}}.$$

Thus applying Stirling's formula when x is large we have that

$$\begin{aligned}\widehat{R}_\ell(x)&=(-1)^{\left\lfloor\frac{\log\log x}{2}\right\rfloor}\times\int_{\frac{\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log x}{2}}\frac{I_\ell\left(e^{e^{\frac{4k}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!}e^{e^{\frac{4k}{3}}}\,dk\\&\gtrsim(-1)^{\left\lfloor\frac{x}{2}\right\rfloor}\times\widehat{h}\left(\frac{\log\log x}{2}\right)\left[\widehat{T}\left(\frac{\log\log x}{2};\frac{\log\log x}{2}\right)\left(\mathrm{cf}_+\left(\frac{\log\log x-1}{2}\right)-\mathrm{cf}_-\left(\frac{\log\log x}{2}\right)\right)\right.\\&\quad\left.-\widehat{T}\left(\frac{\log\log x-1}{2};\frac{\log\log x-1}{2}\right)\left(\mathrm{cf}_+\left(\frac{\log\log x}{2}\right)-\mathrm{cf}_-\left(\frac{\log\log x-1}{2}\right)\right)\right].\end{aligned}\tag{10}$$

Since for real $0<s<1$ such that $s\to 0$, we have that $\log(1+s)\sim s$ and $(1+s)^{-1}\sim -s$, we can approximate the differences implies by the last estimate using that for t large tending to infinity we have

$$\log_*^m\left(t-\frac{1}{2}\right)\sim\log_*^m(t)-\frac{1}{2\log^{m-1}t},\, m\geq 1.$$

Then taking the difference in (10) above and removing lower-order terms that do not contribute to the dominant asymptotic terms, we obtain that

$$\int_{u_0}^{x^*}\widehat{\tau}_0'(t)A_\Omega^{(\ell)}(t)\gtrsim C_0(u_0)+\frac{(-1)^{\lfloor\log\log x\rfloor}\cdot A3^{5/2}(2e+3)}{16\pi e\log 3/2(2)}\left(\frac{2^{25/6}}{3^{4/3}}\right)^{\log\log x}\log^{4/3}(x)\log_*^3(x)^{11/7}\frac{\log_*^4(x)^{\frac{3}{2}\log_*^4(x)}}{(\log\log x)^{\frac{5}{2}\log\log x}}.\tag{11}$$

Finally, using Stirling's formula for very large x and (8), we can see that

$$\widehat{\tau}_0(x)\sim\frac{\log^2 x\cdot\log\log x}{\sqrt{2\pi}\cdot x}\left(\frac{x}{\log x\cdot\log\log x}\right)^{\log\log x}$$
$$A_\Omega^{(\ell)}(x)\sim\frac{3A(2e+3)}{4eB\log^{\frac{3}{2}}(2)}\log^{\frac{11}{7}}(x)(\log\log x)^{\frac{3}{2}\log\log x}.$$

So we have that the first term in (9) is given by

$$\widehat{\tau}_0(x)A_\Omega^{(\ell)}(x)\sim\frac{3A(2e+3)}{\sqrt{2\pi}4eB\log^{\frac{3}{2}}(2)\cdot x}\log^{\frac{25}{7}}(x)\left(\frac{x}{\log x}\right)^{\log\log x}(\log\log x)^{\frac{\log\log x}{2}+1}.$$

Now what we have done using the approximations to $C_{\Omega(n)}(n)$ for large n is sum up

$$\sum_{n\leq x} C_\Omega(n) :\gtrsim C_\ell + \frac{3A(2e+3)}{\sqrt{2\pi}4eB\log^{\frac{3}{2}}(2)}\log^{\frac{32}{7}}(x)\left(\frac{x}{\log x}\right)^{\log\log x-1}(\log\log x)^{\frac{\log\log x}{2}+1}$$
$$+\frac{(-1)^{\lfloor\log\log x\rfloor}\cdot A3^{5/2}(2e+3)}{16\pi e\log 3/2(2)}\left(\frac{2^{25/6}}{3^{4/3}}\right)^{\log\log x}\log^{4/3}(x)\log_*^3(x)^{11/7}\frac{\log_*^4(x)^{\frac{3}{2}\log_*^4(x)}}{(\log\log x)^{\frac{5}{2}\log\log x}}.$$

In practice, we have over-summed by quite a bit. Since $\Omega(n)\leq\lfloor\log_2 n\rfloor = \left\lfloor\frac{\log n}{\log 2}\right\rfloor$, many of the terms in the previous equation are actually zero (recall that $C_0(n)=\delta_{n,1}$). So we are actually only going to sum up to the average order of $\Omega(n)\sim\log\log n$ in some cases, and to the maximum order of $\Omega(n)\leq\frac{\log n}{\log 2}$ in others, depending on the signage of the last term.

For the upper bound cases, we form essentially the same argument using the integrand

$$\begin{aligned}\widetilde{I}_\ell(t) &:= A_\Omega^{(\ell)}(t)\widehat{\tau}_1'(t)\\ &= A_\Omega^{(\ell)}(t)\left[\frac{\widehat{\tau}_0(t)}{2e\log t}-\frac{\widehat{\tau}_0(t)}{2e\log^2 t}+\frac{t\cdot\widehat{\tau}_0'(t)}{2e\log t}\right]\\ &\sim\frac{t}{2e\cdot\log t}\widehat{I}_\ell(t).\end{aligned}$$

This completes the proof of the claimed lower bound for x sufficiently large and tending to infinity. □

5.2 Apparent progress on a classical conjecture: Lower bounds on the scaled Mertens function along an infinite subsequence

Corollary 5.2 (Bounds for the classically scaled Mertens function)**.** *Let* $u_0 := e^{e^{e^e}}$ *and define the infinite increasing subsequence,* $\{x_n\}_{n\geq 1}$*, by* $x_n := e^{e^{e^{e^{2n}}}}$*. We have that along the increasing subsequence* x_y *for large* $y\geq\max\left(\left\lceil e^{e^{e^e}}\right\rceil, u_0+1\right), y\gg 3\times 10^{1656520}$*:*

$$\frac{|M(x_{0,y})|}{\sqrt{x_{0,y}}}\gtrsim TODO,\text{ as } y\to\infty.$$

Proof of the Asymptotic Lower Bound. Since $\pi(1)=0$ and $\pi(j)=\pi(\sqrt{x})$ for all $\sqrt{x}\leq j\leq\frac{x}{2}$, we can write (7) in the following form using Abel summation:

$$\begin{aligned}M(x) &\gtrsim G_\ell^{-1}(x)+G_\ell^{-1}(u_0)A_\Omega^{(\ell)}(u_0)-G_\ell^{-1}(x)A_\Omega^{(\ell)}(x)\\ &\quad+\int_{u_0}^{\sqrt{x}}\frac{x}{t^2\log(x/t)}\frac{d}{dt}\left[G_\ell^{-1}(t)\right]dt+\int_{\sqrt{x}}^{x/2}\frac{\sqrt{x}}{t^2\log(\sqrt{x}/t)}\frac{d}{dt}\left[G_\ell^{-1}(t)\right]dt.\end{aligned}\tag{12}$$

Moreover, we have that

$$\left|\int_{\sqrt{x}}^{x/2}\frac{2\pi(\sqrt{x})}{t^2\log x}\frac{d}{dt}\left[G_\ell^{-1}(t)\right]dt\right|\leq\frac{4\pi(\sqrt{x})}{\log x}\left(\frac{2}{x}-\frac{1}{\sqrt{x}}\right)\frac{d}{dx}\left[\max\left(|G_\ell^{-1}(\sqrt{x})|,|G_\ell^{-1}(x/2)|\right)\right]=o(1).$$

We select x and u_0 so that $G_\ell^{-1}(u_0)=0$, as is it's derivative, and $\min(-G_\ell^{-1}(u_0),-G_\ell^{-1}(\sqrt{x}))=-G_\ell^{-1}(\sqrt{x})$. Then (12) becomes

$$\begin{aligned}M(x) &\gg G_\ell^{-1}(x)-G_\ell^{-1}(x)A_\Omega^{(\ell)}(x)+\int_{u_0}^{\sqrt{x}}\frac{x}{t^2\log(x/t)}\frac{d}{dt}\left[G_\ell^{-1}(t)\right]dt+o(1)\\ &\gg G_\ell^{-1}(x)-G_\ell^{-1}(x)A_\Omega^{(\ell)}(x)+\frac{x}{\log(x/u_0)}\times\int_{u_0}^{\sqrt{x}}\frac{1}{t^2}\frac{d}{dt}\left[G_\ell^{-1}(t)\right]dt+o(1)\end{aligned}$$

$$\gg G_\ell^{-1}(x) - G_\ell^{-1}(x)A_\Omega^{(\ell)}(x) - \frac{2x}{\log(x/u_0)}\left(\frac{d}{dt}\left[G_\ell^{-1}(t)\right]\Bigg|_{t=\sqrt{x}}\frac{1}{\sqrt{x}} - \frac{d}{dt}\left[G_\ell^{-1}(t)\right]\Bigg|_{t=u_0}\frac{1}{u_0}\right)$$
$$= G_\ell^{-1}(x) - G_\ell^{-1}(x)A_\Omega^{(\ell)}(x) - \frac{2\sqrt{x}}{\log(x/u_0)}\frac{d}{dt}\left[G_\ell^{-1}(t)\right]\Bigg|_{t=\sqrt{x}}\frac{1}{\sqrt{x}}.$$

To satisfy the conditions above, it suffices to take $u_0 = e^{e^{e^e}}$, and a sufficient requirement on x (using Theorem 5.1) is that $\lfloor \log\log\log\log x \rfloor \equiv 0 \bmod 2$ using the formula for $-G_\ell^{-1}(t)$. Thus the last equation is re-written as

$$\begin{aligned}
M(x) \gg\ & G_\ell^{-1}(x) - G_\ell^{-1}(x)A_\Omega^{(\ell)}(x) \\
& + \frac{2\sqrt{x}}{\log(x/u_0)}\Bigg[C_\ell + \frac{3A(2e+3)}{\sqrt{2\pi}9e^2B\log^2 2(\log\log 2)^{\frac{3}{2}}}(\log\log x)^{\frac{39}{7}}\left(\frac{\log x}{\log\log x}\right)^{\log_*^3(x)-1}\log_*^3(x)^{\frac{\log_*^3(x)}{2}} \\
& - \frac{A3^{5/2}(2e+3)}{32\pi e^2\log 2(\log\log 2)^{\frac{3}{2}}}\left(\frac{2^{25/6}}{3^{4/3}}\right)^{\log_*^3(x)}\log\log x\cdot\log_*^3(x)^{1/3}\log_*^4(x)^{11/7}\frac{\log_*^5(x)^{\frac{3}{2}\log_*^5(x)}}{\log_*^3(x)^{\frac{5}{2}\log_*^3(x)}}\Bigg] + o(1).
\end{aligned}$$

Now when we scale $M(x)$ by a reciprocal of $\sqrt{x}$ and let $x \to \infty$ along this subsequence, we obtain that

$$\frac{M(x)}{\sqrt{x}} \gg \frac{2}{\log(x/u_0)}\left[\frac{3A(2e+3)}{\sqrt{2\pi}9e^2B\log^2 2(\log\log 2)^{\frac{3}{2}}}(\log\log x)^{\frac{39}{7}}\left(\frac{\log x}{\log\log x}\right)^{\log_*^3(x)-1}\log_*^3(x)^{\frac{\log_*^3(x)}{2}}\right] + o(1)$$

And we are done. Notice that the above expression tends to $+\infty$ as $x \to \infty$. $\square$

6 Generalizations to weighted Mertens functions

6.1 A standard notion of generalizing the Mertens function

We define the notion of a *generalized, or weighted, Mertens summatory function* for fixed $\alpha \in \mathbb{C}$ as

$$M_\alpha(x) = \sum_{n \leq x} \frac{\mu(n)}{n^\alpha}, \ x \geq 1, \tag{13}$$

where the special case of $M_0^*(x)$ coincides with the definition of the classical Mertens function $M(x)$. We have in these cases that

$$M_\alpha(x) = \sum_{n \leq x} g^{-1}(x) \left[\pi\left(\frac{x}{n}\right) + \left(\frac{n}{x}\right)^\alpha\right]. \tag{14}$$

The plots shown in Figure 6.1 illustrate the chaotic behavior of the growth of these functions for x in small intervals when $\alpha \in \{1, 0, -1, -2\}$.

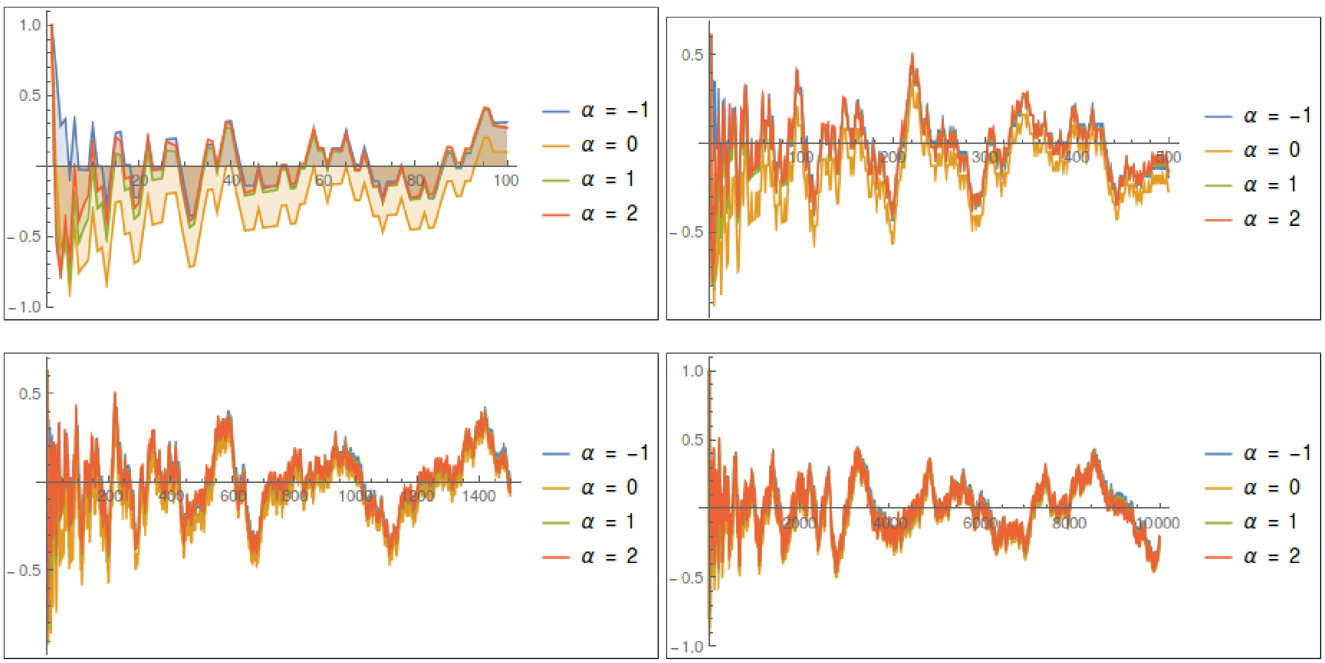

Figure 6.1: *Comparison of the Mertens Summatory Functions $M_\alpha(x)/x^{\frac{1}{2}-\alpha}$ for Small x and α*

Related questions are often posed in relation to the strikingly similar properties of the summatory functions over the Liouville lambda function, $L_\alpha(x) := \sum_{n \leq x} \lambda(n) n^{-\alpha}$ [?, ?]. For example, it is known that (TODO) ...

6.2 Analogous asymptotic lower bounds on $M_\alpha(x)$

TODO ...

7 Conclusions

7.1 Summary

7.2 Remaining conjectures and work left to be done on this method

Note, have done nothing to touch on upper bounds ...

7.3 Future research: Generalizations of this method to arbitrary arithmetic functions

We state the following theorem without proof (exercise for the reader) which effectively generalizes the formula re-stated in Theorem 1.5 from Chapter 7 of Montgomery and Vaughan:

Theorem 7.1. *For arithmetic functions $f_0(n), f(n)$, positive natural numbers $U \geq 1$, and complex-valued z, s, let the DGFs G_0, F_0 be defined such that*

$$G_0(f_0; s)^z := \sum_{n\geq 1} \frac{g_z^{(f_0)}(n)}{n^s}; \Re(s) > 1,$$

$$F_0(f_0; U; s, z) := \prod_p \left(1 + \sum_{1\leq r\leq U} \frac{f_0(p^r)z^r}{p^{rs}}\right) = \sum_{m\geq 1} \frac{f(m)z^{f_0(m)}}{m^s}; \Re(s) \geq 1,$$

where for all primes p, $g_z(p^i) = z^{f_0(p^i)}$ $\forall 1 \leq i \leq U$. Let the summatory function

$$A_z^{(g)}(x) := \sum_{n\leq x} a_z^{(g)}(n),$$

where we define

$$G_0(f_0; s)^z \times F_0(f_0; U; s, z) =: \sum_{n\geq 1} \frac{a_z^{(g)}(n)}{n^s}.$$

Let $F_0(x)$ denote the summatory function of f_0, and define its asymptotic average order to be

$$F_{0,\mathrm{ave}}(x) \sim \frac{F_0(x)}{x}.$$

Suppose that for all $1 \leq i \leq U$, the DGF

$$\sum_{m\geq 1} \frac{a_z^{(g)}(m)}{m^i} \frac{e^{F^i_{0,\mathrm{ave}}(x)}}{\Gamma(z+i)},$$

is uniformly bounded in x for $|z| \leq \max_p p \leq i+1 \equiv R_i$.
Define the function $G_i(f_0; U; z)$ by the formula

$$G_i(f_0; U; z) := \frac{F_0(f_0; U; 1, z)}{\Gamma(z+i)}.$$

Let the value counting function, *$\pi_k^{(f_0,f)}(x)$, be defined as*

$$\pi_k^{(f_0,f)}(x) := \#\left\{n \leq x : f(n) \neq 0, f_0(n) = k\right\}.$$

Then uniformly for all $0 < R \leq R_U$ and $k \leq R \cdot F^U_{0,\mathrm{ave}}(x)$ we have that

$$\pi_k^{(f_0,f)}(x) \underset{x\to\infty}{\sim} \sum_{i=1}^{U} \pi_1^{(f_0,f)}(x) \cdot G_i\left(\frac{k-i}{F^i_{0,\mathrm{ave}}(x)}\right) \binom{k}{i} \frac{\left\{F^i_{0,\mathrm{ave}}(x)\right\}^{k-i}}{k!} \left(1 + O_R\left(\frac{k-i}{F^i_{0,\mathrm{ave}}(x)}\right)\right).$$

References and complete bibliography links

[1] T. M. Apostol. *Introduction to Analytic Number Theory.* Springer–Verlag, 1976.

[2] C. E. Fröberg. On the prime zeta function. *BIT Numerical Mathematics*, 8:87–202, 1968.

[3] H. W. Gould and T. Shonhiwa. A catalog of interesting Dirichlet series. *Missouri J. Math. Sci.*, 20(1):2–18, 2008.

[4] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science.* Addison-Wesley, 1994.

[5] M. Merca and M. D. Schmidt. Factorization theorems for generalized Lambert series and applications. *Ramanujan Journal*, 2019.

[6] H. Mousavi and M. D. Schmidt. Factorization theorems for relatively prime divisor sums, GCD sums and generalized Ramanujan sums, 2020.

[7] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. *NIST Handbook of Mathematical Functions.* Cambridge University Press, 2010.

[8] J. Sándor and B. Crstici. *Handbook of Number Theory II.* Kluwer Academic Publishers, 2004.

[9] N. J. A. Sloane. The Online Encyclopedia of Integer Sequences, 2020.

[10] K. Soundararajan. Partial sums of the Möbius function. *Annals of Mathematics*, 2009.

Appendix A Reference on common abbreviations, special notation and other conventions

Symbol	Definition
$a_k(f,g;n)$	Discrete Fourier coefficients of the periodic divisor sums $s_k(f,g;n)$ defined as symbol $s_k(f,g;n)$ in this glossary. The precise definition of these sums is given by $a_k(f,g;n) = \sum_{d\mid(k,n)} g(d)f(n/d)\frac{d}{k}$.
$a_{k,\ell}$	Sequence of coefficients that are defined explicitly in the discrete Fourier series expansion of the type II sums $L_{f,g,k}(x)$. These coefficients are implicitly defined by Definition **??** by the sums $L_{f,g,k}(n) = \sum_{\ell=0}^{k-1} a_{k,\ell}\cdot e(\ell n/k)$, where $e(x)$ is the shorthand for the complex exponential terms in the exponential sums we define in the article.
$\Delta^k[f](n)$	We denote by the operator $\Delta^k[f]$ at $n \geq 1$ the following: $\Delta^k[f](n) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k f(n-k)$.
$\lceil x \rceil$	The ceiling function $\lceil x\rceil := x+1-\{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$.
$\chi_{1,k}(n)$	The principal Dirichlet character modulo k, i.e., the indicator function of the natural numbers which are relatively prime for $n,k \geq 1$, $\chi_{1,k}(n) = [(n,k)=1]_\delta$.
$C_k(n)$	Sequence of nested k-convolutions of an arithmetic function f with itself. The precise definition of this sequence is given by $$C_k(n) = \begin{cases} \widehat{f}(n) - \widehat{f}(1)\varepsilon(n), & \text{if } k=1; \\ \sum_{d\mid n}\left(\widehat{f}(d)-\widehat{f}(1)\varepsilon(d)\right) C_{k-1}(n/d), & \text{if } k \geq 2, \end{cases}$$ where the symbol $\hat{f}(n)$ is defined in glossary entry $\hat{f}(n)$.
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero.
$\int_{C-\imath T}^{C+\imath T}, \int_{C-\imath\infty}^{C+\imath\infty}$	TODO
$c_q(n)$	Ramanujan's sum, $c_q(n) := \sum_{d\mid(q,n)} d\mu(q/d)$.
DGF	*Dirichlet generating function.* Given a sequence $\{f(n)\}_{n\geq 0}$, its DGF enumerates the sequence in a different way than formal generating functions in an auxiliary variable. Namely, for $\lvert s\rvert < \sigma_a$, the abcissa of absolute convergence of the series, the DGF $D_f(s)$ constitutes an analytic function of s given by: $D_f(s) := \sum_{n\geq 1} f(n)/n^s$. The DGF is alternately called the *Dirichlet series* of an arithmetic function f. type
$D_f(n)$	Function related to the Dirichlet inverse of a function f. More precisely, this function is defined by the sum $D_f(n) := \sum_{j=1}^{n}\frac{\mathrm{ds}_{2j}(f;n)}{\hat{f}(1)^{2j+1}}$, where this definition involves the glossary symbols $\mathrm{ds}_j(f;n)$ and $\hat{f}(n)$. Lemma **??** relates this function to the Dirichlet inverse of the function $\hat{f}(n)$.
$d(n)$	The ordinary divisor function, $d(n) := \sum_{d\mid n} 1$.

Symbol	Definition
$\mathrm{ds}_j(f;n)$	Summands in the formula for the Dirichlet inverse of an arithmetic function. The precise definition of this function is given by $$\mathrm{ds}_j(f;n) = \begin{cases} (-1)^{\delta_{n,1}} \widehat{f}(n), & \text{if } j=1; \\ \sum\limits_{\substack{d\mid n \\ d>1}} \widehat{f}(d)\,\mathrm{ds}_{j-1}\left(f;\frac{n}{d}\right), & \text{if } j\geq 2, \end{cases}$$ where the fixed function $\hat{f}$ is defined by glossary symbol $\hat{f}(n)$.
$\mathrm{DFT}[f](k)$	The discrete Fourier transform (DFT) of f at k. We use this transformation in Section **??** of the article.
$\mathrm{DTFT}[f](k)$	The discrete time Fourier transform (DTFT) of f at k, also denoted by $F[k]$.
EGF	*Exponential generating function.* Given a sequence $\{f_n\}_{n\geq 0}$, its EGF (or sometimes called ordinary power series, EPS) enumerates the sequence by powers of a typically formal variable z: $\widehat{F}(z) := \sum_{n\geq 0} f_n/n! z^n$. For $z\in\mathbb{C}$ within some radius or abcissa of convergence for the series, asymptotic properties can be extracted from the closed-form representation of $\widehat{F}$, and/or the original sequence terms can be recovered by performing an inverse Z-transform on the OGF. type
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n)=\delta_{n,1}$.
$e(x)$	The complex exponential function, $e(x) := \exp(2\pi\imath\cdot x)$.
$f * C_-(m)$	This notation indicates that the index over which we perform the Dirchlet convolution is given by the dash parameter, $(f*C_-(m))(n) := \sum_{d\mid n} f(d) C_{n/d}(m)$.
$f * C_k(-)$	This notation indicates that the index over which we perform the Dirchlet convolution is given by the dash parameter, $(f*C_k(-))(n) := \sum_{d\mid n} f(d) C_k(n/d)$.
$*$; $f*g$	The Dirichlet convolution of f and g, $f*g(n) := \sum\limits_{d\mid n} f(d)g(n/d)$, for $n\geq 1$. This symbol for the discrete convolution of two arithmetic functions is the only notion of convolution of functions we employ within the article.
$\hat{f}(n)$	A shorthand notation for scaled arithmetic function terms $\hat{f}(n) := w^n/(w^n-1)f(n)$ for some indeterminate w.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ provided that $f(1)\neq 0$.
$F[k]$	Discrete Fourier transform coefficients.
$\lfloor x\rfloor$	The floor function $\lfloor x\rfloor := x - \{x\}$ where $0\leq\{x\}<1$ denotes the fractional part of $x\in\mathbb{R}$.
$f_{\pm}(n)$	For any arithmetic function f, we define $f_{\pm}(n) = f(n)\,[n>1]_{\delta} - f(1)\,[n=1]_{\delta}$, i.e., the function that has identical values as f for all $n\geq 2$, and whose initial value is $f_{\pm}(1) := -f(1)$ when $n=1$.
G_j	Denotes the interleaved (or generalized) sequence of pentagonal numbers defined explictly by the formula $G_j := \frac{1}{2}\left\lceil\frac{j}{2}\right\rceil\left\lceil\frac{3j+1}{2}\right\rceil$. The sequence begins as $\{G_j\}_{j\geq 0} = \{0,1,2,5,7,12,15,22,26,35,40,51,\ldots\}$.
$\mathrm{Id}_k(n)$	The power-scaled identity function, $\mathrm{Id}_k(n) := n^k$ for $n\geq 1$.
$\mathbb{1}_{\mathbb{S}}$, $\chi_{\mathtt{cond}(x)}$	We use the notation $\mathbb{1},\chi:\mathbb{N}\to\{0,1\}$ to denote indicator, or characteristic functions. In paticular, $\mathbb{1}_{\mathbb{S}}(n)=1$ if and only if $n\in\mathbb{S}$, and $\chi_{\mathtt{cond}}(n)=1$ if and only if n satisfies the condition `cond`.
$[n=k]_{\delta}$	Synonym for $\delta_{n,k}$ which is one if and only if $n=k$, and zero otherwise.

Symbol	Definition
$[\texttt{cond}]_\delta$	For a boolean-valued $\texttt{cond}$, $[\texttt{cond}]_\delta$ evaluates to one precisely when $\texttt{cond}$ is true, and zero otherwise.
$L_{f,g,k}(x)$	The type II Anderson-Apostol sum over the arithmetic functions f, g, $L_{f,g,k}(x) := \sum\limits_{d\mid(k,x)} f(d)g(x/d)$.
$\gcd(m,n); (m,n)$	The greatest common divisor of m and n. Both notations for the GCD are used interchangably within the article.
$\mu(n)$	The Möbius function.
$\mu_{n,k}$	The corresponding invertible sequence is an analog to the role of the Möbius function in Möbius inversion. In this case these inversion coefficients are defined such that $$g(n) = \sum_{\substack{d=1 \\ (d,n)=1}}^{n} f(d) \quad\Longleftrightarrow\quad f(n) = \sum_{d=1}^{n} g(d+1)\mu_{n,d}.$$ See Proposition **??** and Section **??** for the relation of this sequence (and its inverse) to the factorizations of type I sums.
$\mu_{n,k}^{(-1)}$	Inverse matrix sequence of $\mu_{n,k}$.
$M(x)$	The Mertens function which is the summatory function over $\mu(n)$, $M(x) := \sum\limits_{n\leq x} \mu(n)$.
OGF	*Ordinary generating function.* Given a sequence $\{f_n\}_{n\geq 0}$, its OGF (or sometimes called ordinary power series, OPS) enumerates the sequence by powers of a typically formal variable z: $F(z) := \sum_{n\geq 0} f_n z^n$. For $z \in \mathbb{C}$ within some radius or abcissa of convergence for the series, asymptotic properties can be extracted from the closed-form representation of F, and/or the original sequence terms can be recovered by performing an inverse Z-transform on the OGF.
$\phi_k(n)$	Generalized totient function, $\phi_k(n) := \sum\limits_{\substack{1\leq d\leq n \\ (d,n)=1}} d^k$.
$\phi(n)$	Euler's classical totient function, $\phi(n) := \sum\limits_{\substack{1\leq d\leq n \\ (d,n)=1}} 1$.
$\Phi_n(z)$	The n^{th} cyclotomic polynomial in z defined by $\Phi_n(z) := \prod\limits_{\substack{1\leq k\leq n \\ (k,n)=1}} (z - e^{2\pi\imath k/n})$.
$p(n)$	The partition function generated by $p(n) = [q^n] \prod\limits_{n\geq 1} (1-q^n)^{-1}$.
$\sum_{p\leq x}$	Unless otherwise specified by context, we use the index variable p to denote that the summation is to be taken only over prime values within the summation bounds.
$P(s)$	For complex s with $\Re(s) > 1$, we define $P(s) = \sum_{p \text{ prime}} p^{-s}$.
$(q;q)_\infty$	The infinite q-Pochhammer symbol defined as the product $(q;q)_\infty := \prod\limits_{n\geq 1}(1-q^n)$ for $\lvert q\rvert < 1$.
$\sigma_\alpha(n)$	The generalized sum-of-divisors function, $\sigma_\alpha(n) := \sum\limits_{d\mid n} d^\alpha$, for any $n \geq 1$ and $\alpha \in \mathbb{C}$.

Symbol	Definition
$s_k(f,g;n)$	Shorthand for the periodic (modulo k) divisor sums expanded by the functions listed in $a_k(f,g;n)$ of this glossary. The precise expansion and corresponding finite Fourier series expansion of this function is given by $s_k(f,g;n) = \sum_{d\mid(n,k)} f(d)g(k/d) = \sum_{m=1}^{k} a_k(f,g;m)e^{2\pi\imath\cdot mn/k}$.
$s_{n,k}$	Matrix coefficients in Lambert series type factorizations. These coefficients are defined precisely as the coefficients of the generating function $[q^n](q;q)_\infty q^k/(1-q^k)$ for $k\geq 1$ where $(q;q)_\infty$ is the infinite q-Pochhammer symbol.
$\sum'_{n\leq x}$	We denote by $\sum'_{n\leq x} f(n)$ the summatory function of f at x minus $\frac{f(x)}{2}$ if $x\in\mathbb{Z}$.
$T_f(x)$	The type I sum over an arithmetic function f, $T_f(n) := \sum_{\substack{d\leq x\\(d,x)=1}} f(d)$.
$t_{n,k}$	Matrix sequence involved in the generating function expansions of the type I sums defined as $$T_f(x) = [q^x]\left(\frac{1}{(q;q)_\infty}\sum_{n\geq 2}\sum_{k=1}^{n} t_{n,k}f(k)\cdot q^n + f(1)\cdot q\right)$$
$t_{n,k}^{(-1)}$	Inverse matrix of the sequence $t_{n,k}$.
$\hat{u}_{n,k}(f,w)$	Matrix coefficients defined in terms of an indeterminate parameter w as $\hat{u}_{n,k}(f,w) := (w^k-1)\cdot u_{n,k}(f,w)$.
$u_{n,k}(f,w)$	Matrix sequence defined in the expansion of the generating functions for the type II sums as $$g(x) = [q^x]\left(\frac{1}{(q;q)_\infty}\sum_{n\geq 2}\sum_{k=1}^{n} u_{n,k}(f,w)\left[\sum_{m=1}^{k} L_{f,g,m}(k)w^m\right]\cdot q^n\right),\ w\in\mathbb{C}.$$
$u_{n,k}^{(-1)}(f,w)$	Inverse matrix terms of the sequence $u_{n,k}(f,w)$.
$y_f(n)$	The function $y_f(n)$ denotes the Dirichlet inverse of the function $h(n) := f(n)\phi(n)/n^2$ where $\phi(n)$ is Euler's totient function and f is any invertible arithmetic function such that $f(1)\neq 0$. This function is used to express the result in Corollary **??**. A special case, denoted by $y(n)$, corresponding to the case where $f(n)\equiv n$ is employed in stating Corollary **??** in Section **??**.
$\zeta(s)$	The Riemann zeta function, defined by $\zeta(s) := \sum_{n\geq 1} n^{-s}$ when $\Re(s)>1$, and by analytic continuation to the entire complex plane with the exception of a simple pole at $s=1$.

Appendix B Accompanying software references and examples

An indexed repository has been created on the author's personal GitHub page at the following link to archive the computations referenced in this thesis: https://github.com/maxieds/MertensBoundsPackage. Thus, this software allows the author to provide portable software implementations of operations that are frequently useful in approaching the topics we consider within this document. There are both `Mathematica` and `SageMath` (e.g., Python) packages contained in this repository. Special and/or time consuming computations we use here have also been archived in a separate directory of the repository above.

T.1 Table: Computations with a highly signed Dirichlet inverse function

n	**Primes**		**Sqfree**	**PPower**	$\bar{\mathbb{S}}$		$g^{-1}(n)$	$\lambda(n)\,\mathrm{sgn}(g^{-1}(n))$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_2(n)$		$G^{-1}(n)$	$G_{+}^{-1}(n)$	$G_{-}^{-1}(n)$
1	1^1	–	Y	N	N	–	1	1	0	0	–	1	1	0
2	2^1	–	Y	Y	N	–	−2	1	0	0	–	−1	1	−2
3	3^1	–	Y	Y	N	–	−2	1	0	0	–	−3	1	−4
4	2^2	–	N	Y	N	–	2	1	0	−1	–	−1	3	−4
5	5^1	–	Y	Y	N	–	−2	1	0	0	–	−3	3	−6
6	$2^1 3^1$	–	Y	N	N	–	5	1	0	−1	–	2	8	−6
7	7^1	–	Y	Y	N	–	−2	1	0	0	–	0	8	−8
8	2^3	–	N	Y	N	–	−2	1	0	−2	–	−2	8	−10
9	3^2	–	N	Y	N	–	2	1	0	−1	–	0	10	−10
10	$2^1 5^1$	–	Y	N	N	–	5	1	0	−1	–	5	15	−10
11	11^1	–	Y	Y	N	–	−2	1	0	0	–	3	15	−12
12	$2^2 3^1$	–	N	N	Y	–	−7	1	2	−2	–	−4	15	−19
13	13^1	–	Y	Y	N	–	−2	1	0	0	–	−6	15	−21
14	$2^1 7^1$	–	Y	N	N	–	5	1	0	−1	–	−1	20	−21
15	$3^1 5^1$	–	Y	N	N	–	5	1	0	−1	–	4	25	−21
16	2^4	–	N	Y	N	–	2	1	0	−3	–	6	27	−21
17	17^1	–	Y	Y	N	–	−2	1	0	0	–	4	27	−23
18	$2^1 3^2$	–	N	N	Y	–	−7	1	2	−2	–	−3	27	−30
19	19^1	–	Y	Y	N	–	−2	1	0	0	–	−5	27	−32
20	$2^2 5^1$	–	N	N	Y	–	−7	1	2	−2	–	−12	27	−39
21	$3^1 7^1$	–	Y	N	N	–	5	1	0	−1	–	−7	32	−39
22	$2^1 11^1$	–	Y	N	N	–	5	1	0	−1	–	−2	37	−39
23	23^1	–	Y	Y	N	–	−2	1	0	0	–	−4	37	−41
24	$2^3 3^1$	–	N	N	Y	–	9	1	4	−3	–	5	46	−41
25	5^2	–	N	Y	N	–	2	1	0	−1	–	7	48	−41
26	$2^1 13^1$	–	Y	N	N	–	5	1	0	−1	–	12	53	−41
27	3^3	–	N	Y	N	–	−2	1	0	−2	–	10	53	−43
28	$2^2 7^1$	–	N	N	Y	–	−7	1	2	−2	–	3	53	−50
29	29^1	–	Y	Y	N	–	−2	1	0	0	–	1	53	−52
30	$2^1 3^1 5^1$	–	Y	N	N	–	−16	1	0	−4	–	−15	53	−68
31	31^1	–	Y	Y	N	–	−2	1	0	0	–	−17	53	−70
32	2^5	–	N	Y	N	–	−2	1	0	−4	–	−19	53	−72
33	$3^1 11^1$	–	Y	N	N	–	5	1	0	−1	–	−14	58	−72
34	$2^1 17^1$	–	Y	N	N	–	5	1	0	−1	–	−9	63	−72
35	$5^1 7^1$	–	Y	N	N	–	5	1	0	−1	–	−4	68	−72
36	$2^2 3^2$	–	N	N	Y	–	14	1	9	1	–	10	82	−72
37	37^1	–	Y	Y	N	–	−2	1	0	0	–	8	82	−74
38	$2^1 19^1$	–	Y	N	N	–	5	1	0	−1	–	13	87	−74
39	$3^1 13^1$	–	Y	N	N	–	5	1	0	−1	–	18	92	−74
40	$2^3 5^1$	–	N	N	Y	–	9	1	4	−3	–	27	101	−74
41	41^1	–	Y	Y	N	–	−2	1	0	0	–	25	101	−76
42	$2^1 3^1 7^1$	–	Y	N	N	–	−16	1	0	−4	–	9	101	−92
43	43^1	–	Y	Y	N	–	−2	1	0	0	–	7	101	−94
44	$2^2 11^1$	–	N	N	Y	–	−7	1	2	−2	–	0	101	−101
45	$3^2 5^1$	–	N	N	Y	–	−7	1	2	−2	–	−7	101	−108
46	$2^1 23^1$	–	Y	N	N	–	5	1	0	−1	–	−2	106	−108
47	47^1	–	Y	Y	N	–	−2	1	0	0	–	−4	106	−110
48	$2^4 3^1$	–	N	N	Y	–	−11	1	6	−4	–	−15	106	−121
49	7^2	–	N	Y	N	–	2	1	0	−1	–	−13	108	−121
50	$2^1 5^2$	–	N	N	Y	–	−7	1	2	−2	–	−20	108	−128
51	$3^1 17^1$	–	Y	N	N	–	5	1	0	−1	–	−15	113	−128
52	$2^2 13^1$	–	N	N	Y	–	−7	1	2	−2	–	−22	113	−135
53	53^1	–	Y	Y	N	–	−2	1	0	0	–	−24	113	−137
54	$2^1 3^3$	–	N	N	Y	–	9	1	4	−3	–	−15	122	−137
55	$5^1 11^1$	–	Y	N	N	–	5	1	0	−1	–	−10	127	−137
56	$2^3 7^1$	–	N	N	Y	–	9	1	4	−3	–	−1	136	−137

Table T.1: ***Computations of the first several cases of*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for*** $\mathbf{1 \leq n \leq 56}$.
The column labeled `Primes` provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled, respectively, `Sqfree`, `PPower` and $\bar{\mathbb{S}}$ list inclusion of n in the sets of squarefree integers, prime powers, and the set $\bar{\mathbb{S}}$ that denotes the positive integers n which are neither squarefree nor prime powers. The next two columns provide the explicit values of the inverse function $g^{-1}(n)$ and indicate that the sign of this function at n is given by $\lambda(n) = (-1)^{\Omega(n)}$. Then the next two columns show the small-ish magnitude differences between the unsigned magnitude of $g^{-1}(n)$ and the summations $\widehat{f}_1(n) := \sum_{k \geq 0} \binom{\omega(n)}{k} \cdot k!$ and $\widehat{f}_2(n) := \sum_{k \geq 0} \binom{\omega(n)}{k} \cdot \#\{d|n : \omega(d) = k\}$. Finally, the last three columns show the summatory function of $g^{-1}(n)$, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, borken down into its respective positive and negative components: $G_{+}^{-1}(x) := \sum_{n \leq x} g^{-1}(n) \left[g^{-1}(n) > 0\right]_{\delta}$ and $G_{-}^{-1}(x) := \sum_{n \leq x} g^{-1}(n) \left[g^{-1}(n) < 0\right]_{\delta}$.