SUPER SQUARE ROOT LOWER BOUNDS ON THE PARITY OF THE PARTITION FUNCTION

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Abstract. We define the function $N_e(x)$ to be the number of times the parity partition function $p(n)$ is even for $n \leq x$. We expect that on average the value of $N_e(x)$ is approximately $x/2$. However, currently the best known lower bound for the function is given by $N_e(x) \geq C \cdot \sqrt{x} \log\log(x)$ for sufficiently large x. We provide several new plausible and numerically supported conjectures which suggest that we can in fact prove significant increases to the known lower bounds for $N_e(x)$ by factors of small powers of x – effectively "*squashing*" (*cf.* language of Croot) the existing square root barrier encountered in known methods for bounding the problem.

1. Introduction

1.1. Euler's partition function.

1.1.1. *Definitions.* The *partition function* $p(n)$ counts the number of distinct partitions of a natural number $n \geq 1$ into parts of size at least one [5, A000041]. For example, the partition number $p(5) = 7$ as the natural number $n := 5$ is decomposed into the following distinct sums of the form $n = \lambda_1 + \cdots + \lambda_k$ for $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ such that each $\lambda_i \in \{1, 2, \ldots, n\}$:

$$5 = 5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

Alternately, we can define $p(n)$ to be the number of distinct non-negative solutions to the equation

$$n = x_1 + 2x_2 + 3x_3 + \cdots + nx_n,$$

for integers $n \geq 1$. The partition function $p(n)$ is generated by the reciprocal of the infinite q–*Pochhammer symbol*, $(q;q)_\infty$, which is expanded via *Euler's pentagonal theorem* as

$$\begin{aligned}(q;q)_\infty &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \\ &= 1 + \sum_{n \geq 1} (-1)^n \left(q^{n(3n-1)/2} + q^{n(3n+1)/2}\right) \\ &= \sum_{j \geq 0} (-1)^{\lceil \frac{j}{2} \rceil} q^{G_j}\end{aligned}$$

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$$= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots,$$

where the exponents of q in the power series expansion in the previous equations are the *pentagonal numbers*, $\{\omega(0), \omega(1), \omega(-1), \omega(2), \omega(-2), \ldots\}$, also denoted by G_j where for $j \geq 0$ [5, A001318]

$$G_j := \frac{1}{2}\left\lceil \frac{j}{2} \right\rceil \left\lceil \frac{3j+1}{2} \right\rceil = \begin{cases} \frac{j(3j+1)}{2}, & \text{if } j \text{ is even;} \\ \frac{j(3j-1)}{2}, & \text{if } j \text{ is odd.} \end{cases}$$

1.1.2. *Properties of the partition function.* The partition function satisfies a number of known basic properties such as the fundamental recurrence relation

$$\sum_{k \geq 0} (-1)^{\lceil k/2 \rceil} p(n - 1 - G_k) = \delta_{n,1}, \tag{1}$$

which holds for all integers $n \geq 1$, as well as properties such as Rademacher's exact formula formula for $p(n)$ which can be proved by more advanced methods. The identity in (1) is the only known *exact* result relating sums of the partition function minus the values of a quadratic function, namely the pentagonal numbers, and moreover, appears to be the only formula which relates sums of the partition function at a sequence of *polynomial* values identically to zero[1]. While this function is well-studied in number theory and of general interest to many mathematicians who have been studying its properties since the time of Euler, many elementary properties of this function remain a mystery to this day. Our particular motivation for studying this function is to determine new lower bounds for the tally of its values $p(n)$ for $n \leq x$ modulo 2 when $x \gg 1$ is large.

1.2. **The parity problem.** One property of the partition function which is an open unresolved problem concerns the distribution of the parity of $p(n)$, or its reduced values in $\{0, 1\}$ modulo 2, for arbitrary n. More precisely, we are concerned with determining the best possible lower bounds for the function[2]

$$\begin{aligned} N_e(x) &:= \#\{n \leq x : p(n) \equiv 0 \bmod 2\} \\ &= \sum_{n \leq x} [p(n) \equiv 0 \bmod 2]_\delta \\ &= x - N_o(x), \end{aligned}$$

for all $x \geq x_0$ sufficiently large where $N_o(x)$ is the corresponding count of odd parity partition numbers $p(n)$ for $n \leq x$ [5, *cf.* A040051]. It is well-known and can be proved by even elementary methods that this tally function satisfies the so-termed *square root barrier* of

$$N_e(x) \geq C \cdot \sqrt{x}, \quad \text{for } x \geq x_0 \text{ sufficiently large},$$

for some constant $C > 0$, for example as shown in [4] circa 1998. Using more advanced methods involving modular forms, the first bound for $N_e(x)$ can improved slightly to [1]

$$N_e(x) \geq 0.069 \cdot \sqrt{x} \log\log(x),$$

[1] Though we point out that a cubic, or even higher-order, polynomial index relation is desirable in that it would likely lead to more elementary proofs of better parity bounds for the partition function as in the methods of Nicolas and Rusza [4].

[2] Notation: *Iverson's convention* compactly specifies boolean-valued conditions and is equivalent to the *Kronecker delta function*, $\delta_{i,j}$, as $[n = k]_\delta \equiv \delta_{n,k}$. Similarly, $[\mathtt{cond} = \mathtt{True}]_\delta \equiv \delta_{\mathtt{cond},\mathtt{True}}$ in the remainder of the article.

for all $x > 1$[3]. Computation and intuition on the problem suggest that $N_e(x) \sim \frac{x}{2}$ for all large x. However, showing even a relatively small power-of-x factor improvement relative to the order of the expected bound in the form of

$$N_e(x) \geq C_3 \cdot x^{\frac{1}{2}+\varepsilon} + o\left(x^{\frac{1}{2}+\varepsilon}\right), \quad \text{when } x \gg 1,$$

for a particular concrete value of $\varepsilon > 0$ would constitute a significant breakthrough on progress to the parity problem.

1.3. **Our aims in the article.** We aim to make just such progress by showing that, or at least presenting a plausible and extensively numerically supported approach to showing that,

$$N_e(x) \geq C \cdot x^{0.51} + o\left(x^{0.51}\right).$$

The construction we suggest in the next sections is also easily generalized to showing significant power-of-x improvements to known (if any) lower bounds on the even parity of other special partition functions whose reciprocal q-series-related generating function satisfies a well-known, or at least identifiable closed-form, expansion as a power series in q. In particular, if our conjectures made within this article do in fact lead to significant progress on the parity problem for $p(n)$, it is likely that we will be able to prove analogous bounds for the partition function $q(n)$, among others, as a straightforward direct extension of our reasoning here (see [2] for comparable results).

2. Constructing a new approach to the parity problem

2.1. **New identities for the Möbius function.** In previous drafts of this article, I proved the following proposition using the theory of so-termed "*Lambert series factorizations*" which I worked on with Mircea Merca over the summer of 2017 as summarized in [3]. It turns out that the recurrence formula stated in (1) combined with a simple application of Möbius inversion is enough to prove this simple variant of a more general result which can be applied in the context of bounds on the parity of other partition functions. As Ernie has pointed out, the result in Proposition 2.1 itself is somewhat trivial on its own. However, it is useful in the constructions below (in particular, given its application in the form of Corollary 2.2) so we prove it first using the straightforward method immediately below.

Proposition 2.1. *For natural numbers $n \geq 1$, we have the identity that*

$$\mu(n) = \sum_{d|n} \left[p(d-1) + \sum_{k=1}^{d} (-1)^{\lceil k/2 \rceil} p(d-1-G_k)\right] \mu(n/d),$$

where $\mu(n)$ is the Möbius function.

Proof (Möbius Inversion). The well-known identity provided in (1) is first restated in the following form:

$$\begin{aligned}\delta_{n,1} &= p(n-1) + \sum_{k=1}^{n} (-1)^{\lceil k/2 \rceil} p(n-1-G_k) \\ &= \sum_{d|n} \mu(d).\end{aligned}$$

[3] Thanks to S. Ahlgren for pointing out references to the most current known parity bounds for $p(n)$.

By Möbius inversion, the last equation holds if and only if the statement of the proposition is correct for all $n \geq 1$. □

Corollary 2.2 (An Exact Identity for the Möbius Function)**.** *For $q \geq 2$ prime, we have that*

$$0 \equiv p(q-1) + \sum_{k=1}^{2\mu_q} p(q-1-G_k) \pmod{2},$$

where

$$\mu_q := \left\lfloor \frac{\sqrt{24(q-1)+1}+1}{6} \right\rfloor.$$

In particular, when q is prime, we can count that at least one of the elements in the set

$$\{p(q-1-G_k) : 1 \leq k \leq 2\mu_q\} \cup \{p(q-1)\},$$

is of even parity.

Proof. We can prove this theorem using the known identities related to sums of multiplicative functions and the partition function $p(n)$ stated in Proposition 2.1. In particular, if the integer n is prime then it has only the divisors $d \in \{1, n\}$ and $\mu(n) = -1$ in the expansion guaranteed by the proposition. Then plugging in these explicit and simple forms of the divisors $d|n$ to the formula implies the stated result above in the form of

$$-1 = p(q-1) - p(0) + \sum_{k=1}^{\mu(n)} p(n-1-G_k) \pmod{2}. \qquad \square$$

Remark 2.3 (Applications to Other Special Partition Functions)**.** As hinted at in the beginning of this section, the identity proved by Möbius inversion in Proposition 2.1, we can state this result in a more general form derived from the so-termed *Lambert series factorization theorems* studied in [3]. In particular, suppose that $P(n)$ denotes any special partition function of interest whose reciprocal generating function, denoted $(q;q)_{P,\infty}$ say, has a power series expansion of the following form:

$$(q;q)_{P,\infty} := 1 + \sum_{j\geq 1} C_{P,G_{P,j}} q^{G_{P,j}}.$$

We can then modify the result in the proposition to the more general statement that

$$\mu(n) = \sum_{d|n} \sum_{k=1}^{n} P(k-1)\mu(n/d) \left([G_{P,d-k} > 0]_\delta \, C_{P,G_{P,d-k}} + [d-k=0]_\delta \right), \tag{2}$$

which as in the corollary above then implies that

$$0 \equiv P(q-1) + \sum_{\substack{k\geq 1 \\ G_{P,k} < q}} C_{P,G_{P,k}} P(q-1-G_{P,k}) \pmod{2}.$$

In other words, if we wish to study the corresponding parity problem for the partition function $q(n) := [q^n](q;q^2)_\infty^{-1}$ counting the number of partitions of n whose parts are *distinct*, or equivalently the number of partitions of n into odd parts, [5, A000009], we can define a pentagonal number analog by the series coefficients of $(q;q^2)_\infty$ [5, A081362] and proceed to consider the parity problem on subsets of the primes such that the total number of terms involving $P(\cdot)$ are odd on the right-hand-side of the last equation. Thus we have an effective generalization of our method described below in the article to form analogous lower bounds on the even-ness tallies of other special partition functions.

2.2. **A Counting Heuristic for Bounding $N_e(x)$.** We can make extensive use – perhaps even of more utility than the proof of Proposition 2.1 itself warrants – of the identity in Corollary 2.2 to finding improved lower bounds on $N_e(x)$ counting the even-ness of the partition function on a subset of primes $q \leq x$ rather than on the entire subset of the integers $n \geq x$ implicit to the definition of this parity counting function. In particular, if we denote the set of primes by $\mathbb{P}$, then we have the following heuristic for bounding $N_e(x)$ from below:

Proposition 2.4. *Suppose that $Q \subseteq \mathbb{P} \cap \{1, 2, \ldots, x\}$ for some $x > 1$ denotes an appropriately chosen bounded subset of the primes $q \leq x$. Then we have the bound*

$$N_e(x) \geq |Q| - I_{e,Q},$$

where the indicator function $I_{e,Q}$ counts the maximum possible overlap of indices $q_1 - G_j = q_2 - G_k$ for $1 \leq k < j \leq 2\mu_{q_1}$ and where $q_1 > q_2$ for $q_1, q_2 \in Q$. More precisely, we have that the stated bound holds where the indicator function $I_{e,Q}$ with respect to the set Q is defined to be

$$\begin{aligned} I_{e,Q} = \sum_{\substack{q_1, q_2 \in Q \\ q_1 > q_2}} \sum_{t=1}^{2\mu_{q_1}} \sum_{k=1}^{\min(t-1, 2\mu_{q_2})} \Big(& \left[2q_1 - 2q_2 - (3t^2 + (6k+1)t) = 0\right]_\delta \\ & + \left[2q_1 - 2q_2 - (3t^2 + (6k-1)t) = 0\right]_\delta \\ & + \left[2q_1 - 2q_2 - (3t^2 + (6k+1)t + 2k) = 0\right]_\delta \\ & + \left[2q_1 - 2q_2 - (3t^2 + (6k-1)t - 2k) = 0\right]_\delta \Big). \end{aligned}$$

Proof. The inequality for a lower bound on $N_e(x)$ follows from a straightforward counting argument based on Corollary 2.2. In particular, if we order the subset $Q := \{q_1, q_2, \ldots, q_r\}$ of primes $q_j \leq x$, we see from the corollary that we should have at least one even value of $p(n)$ corresponding to the formula at each prime $q_j \in Q$. However, it is possible that we are over counting the same partition number index across the multiple prime values in the set Q, i.e., for $(q_1, q_2, j, k) := TODO$ we have that $q_1 - G_j = q_2 - G_k$, so by our náive counting above where we assume nothing more about the parity of $p(n)$ than what is implied by the corollary, if $q_1, q_2 \in Q$ it is possible that we have over counted the same even partition number twice in the main term of $|Q|$! Therefore, if we subtract the count (strictly speaking a náive over count of the duplicate indices) of times where the indices in the formula at q_j is duplicated in the formula corresponding to another $q_{j'}$ then we have arrived at our stated lower bound.

What remains is to prove the second indicator sum formula for the occurrences of these duplicate indices denoted by $I_{e,Q}$. In particular, for distinct $q_1 > q_2$ and

$j > k$ where $q_1 - G_j, q_2 - G_k \geq 0$, then we have the four separate cases below corresponding to $G_j = \frac{j(3j+1)}{2}, \frac{j(3j-1)}{2}$ and similarly $G_k = \frac{k(3k+1)}{2}, \frac{k(3k-1)}{2}$ where we define $t := j - k$ to simplify notation:

$$(G_j, G_k) := \left(\frac{j(3j+1)}{2}, \frac{k(3k+1)}{2}\right): \quad G_j - G_k = \frac{1}{2}(3t^2 + (6k+1)t) \tag{I}$$

$$(G_j, G_k) := \left(\frac{j(3j-1)}{2}, \frac{k(3k-1)}{2}\right): \quad G_j - G_k = \frac{1}{2}(3t^2 + (6k-1)t) \tag{II}$$

$$(G_j, G_k) := \left(\frac{j(3j+1)}{2}, \frac{k(3k-1)}{2}\right): \quad G_j - G_k = \frac{1}{2}(3t^2 + (6k+1)t + 2k) \tag{III}$$

$$(G_j, G_k) := \left(\frac{j(3j-1)}{2}, \frac{k(3k+1)}{2}\right): \quad G_j - G_k = \frac{1}{2}(3t^2 + (6k-1)t - 2k). \tag{IV}$$

We give more refined versions of the indicator sums for $I_{e,Q}$ which are defined for specific subsets $Q_x \subseteq \mathbb{P} \cap \{1, 2, \ldots, x\}$ when we define them in the context below. □

2.3. **Formally defining our construction.** We notice that if we choose our subsets Q in the previous proposition to contain all of the primes $q \leq x$, then $|Q| = \pi(x) \sim x/\log(x)$ whose main term approximation has a fantastically simple improvement to the known bounds for $N_e(x)$ – namely, $x/\log(x) >> C \cdot \sqrt{x} \log\log(x)$. However, with the laws of physics currently in existence on this planet available for us to manipulate at this time, we are not so lucky to obtain our desired bound improvement by this method as $I_{e,Q} \gg \pi(x)$ for all large enough x which we have computed. We must then modify our construction to find prime subsets Q_x such that $|Q_x|$ represents the main term in the lower bound that we aim to obtain and *more importantly* such that the corresponding over count I_{e,Q_x} in our heuristic for the bound satisfies $I_{e,Q_x} = o(|Q_x|)$ for all large x.

We claim based on intuition and an extensive body of computational evidence that we can construct such sets Q_x by taking *subsets* of appropriately spaced "*fortunate*" primes, or rather fortunately chosen primes, from the set $\mathbb{P} \cap \{1, 2, \ldots, x\}$ when $x \gg 1$ is sufficiently large. In our case, this amounts to being able to chose some optimally spaced subset Q_x of the primes $q \leq x$ such that $|Q_x| \geq \lfloor x^{0.51} \rfloor$ and such that the overlap in the distribution of the primes in our subset Q_x minus all possible pentagonal numbers are minimized to be at most $o(x^{0.51})$. We note here that the subproblem this phrasing seems to invoke – namely, considering the distribution of subsets of primes minus quadratic functions over some preset range – is highly non-trivial! After some numerical trial and error and thought on the particular construction we require, we instead make the problem simpler, by choosing suitable primes in logarithmically spaced intervals with respect to powers of our $p := 1.96 = (0.51)^{-1}$. In particular, we make the following subproblem-specific definitions:

Definition 2.5. For a fixed $t \geq 1$ we define the intervals

$$I_p(t) := [t^p, (t+1)^p],$$

for some prescribed $p \in (1, 2)$ (i.e., our chosen $p := 1.96 = (0.51)^{-1}$). For each $t \geq 1$, we define the corresponding "*fortunate*" prime $q_{p,t}$ to be the largest prime in the interval $I_p(t)$ and then our more specific special case prime subsequence of $q_t := q_{1.96,t}$. Finally, for sufficiently large $x > 1$ we define the prime subset

$Q_x \subseteq \mathbb{P} \cap \{1, 2, \ldots, x\}$ as follows:

$$Q_x := \left\{q_t : 1 \leq t \leq \left\lfloor x^{0.51} \right\rfloor \text{ and } q_t \text{ is the } \textit{largest} \text{ prime } q_t \in I_{1.96}(t)\right\}.$$

We notice that for each $t \geq 1$, we can also express the primes q_t in the following form for a fixed sequence of $d_t \geq 0$:

$$q_t := \left\lceil t^{1.96} \right\rceil + d_t.$$

Moreover, for any fixed $i \geq 1$ such that $t-i \geq 1$, we can bound these offset sequence values by

$$|d_t - d_{t-i}| > |\lceil t^p \rceil - \lceil (t-i)^p \rceil| > 0, 1, 2,$$

for all sufficiently large $t \geq t_0$.

Thus based on extensive numerical support of our construction we claim the following:

Claim 2.6 (Existence of Super Square Root Power Bounds)**.** *For all $x > 1$, we have that*

$$|Q_x| = \left\lfloor x^{0.51} \right\rfloor = C_{0.51} \cdot x^{0.51},$$

for an absolute limiting constant $C_{0.51} > 0$. Moreover, we can bound the indicator function sums in our corresponding heuristic bound for $N_e(x)$ as $I_{e,Q_x} = o(x^{0.51})$, where we can expand this duplicate index counting function as follows:

$$\begin{aligned} I_{e,Q} \geq \sum_{s=1}^{\left\lfloor x^{0.51} \right\rfloor} \sum_{i=1}^{s-1} \sum_{t=1}^{2\mu_{q_s}} \sum_{k=1}^{2\mu_{q_{s-i}}} \Big(& \left[2q_s - 2q_{s-i} - (3t^2 + (6k+1)t) = 0\right]_\delta \\ & + \left[2q_s - 2q_{s-i} - (3t^2 + (6k-1)t) = 0\right]_\delta \\ & + \left[2q_s - 2q_{s-i} - (3t^2 + (6k+1)t + 2k) = 0\right]_\delta \\ & + \left[2q_s - 2q_{s-i} - (3t^2 + (6k-1)t - 2k) = 0\right]_\delta \Big). \end{aligned}$$

Conjecture 2.7 (Properties of Indicator Function Solutions)**.** *We claim that if any one (or all) of the following equations are satisfied for $1 \leq i < s \leq \lfloor x^{0.51} \rfloor$, $1 \leq k < t \leq 2\mu_{q_s}$, then we must have that $b \in \{0, 1, 2\}$:*

$$\left\lceil s^{1.96} \right\rceil - \left\lceil (s-i)^{1.96} \right\rceil + b = 3t^2 + (6k+1)t \tag{I}$$
$$\left\lceil s^{1.96} \right\rceil - \left\lceil (s-i)^{1.96} \right\rceil + b = 3t^2 + (6k-1)t \tag{II}$$
$$\left\lceil s^{1.96} \right\rceil - \left\lceil (s-i)^{1.96} \right\rceil + b = 3t^2 + (6k+1)t + 2k \tag{III}$$
$$\left\lceil s^{1.96} \right\rceil - \left\lceil (s-i)^{1.96} \right\rceil + b = 3t^2 + (6k+1)t - 2k. \tag{IV}$$

We also observe that the sequence

$$\{\lceil x^{1.96} \rceil\}_{x \geq 1} = \{1, 4, 9, 16, 24, 34, 46, 59, 75, 92, 110, 131, 153, \ldots\},$$

does not have an entry in the integer sequences database [5]*, and does not have an apparent generating function, though something special must surely be going on with the sequence to support the previous conjecture as we have done numerically.*

2.4. **One approach to evaluating the indicator sums.** Let's first observe that all positive solutions to the equation

$$m = 3t^2 + ct,\ m, c, t \in \mathbb{Z}^+$$

require that

$$6t = -c + \sqrt{c^2 + 12m},$$

i.e., that in order for us to have integer solutions t to the above exact equation, we have the necessary condition that $c^2 + 24m = s^2$ is square where $s \equiv 1 \pmod{6}$. In the first case (I), we need to evaluate the asymptotic behavior of the sums

$$S_1^*(u) := \sum_{s=1}^{u} \sum_{i=1}^{s-1} \# \left\{(k, s) : 12(q_s - q_{s-i}) \equiv (6s+1)^2 - (6k+1)^2\right\}.$$

In order to extend the sums in the previous equation to the cases in (II)–(IV), we define the following modified forms of the sum (difference) of squares function, $r_{2,m}^*(n)$, and their corresponding generating functions expressed through the theta functions as follows:

$$\begin{aligned}
r_{2,1}^*(n) &= \# \left\{(k, s) : n = (6s+1)^2 - (6k+1)^2, k, s \geq 1\right\} \\
&= [q^n] \sum_{m,k \geq 1} q^{(6m+1)^2 - (6k+1)^2} \\
r_{2,2}^*(n) &= \# \left\{(k, s) : n = (6s+1)^2 - (6k-1)^2, k, s \geq 1\right\} \\
&= [q^n] \sum_{m,k \geq 1} q^{(6m+1)^2 - (6k-1)^2} \\
r_{2,3}^*(n) &= \# \left\{(k, s) : n = (6s+1)^2 - (6k+1)^2 - 2k, k, s \geq 1\right\} \\
&= [q^n] \sum_{m,k \geq 1} q^{(6m+1)^2 - (6k+1)^2 - 2k} \\
r_{2,4}^*(n) &= \# \left\{(k, s) : n = (6s+1)^2 - (6k-1)^2 + 2k, k, s \geq 1\right\} \\
&= [q^n] \sum_{m,k \geq 1} q^{(6m+1)^2 - (6k-1)^2 + 2k}.
\end{aligned}$$

Then we have that our indicator sums from Claim 2.6 are given by

$$I_{e,Q_x} = \sum_{s=1}^{\lfloor x^{0.51} \rfloor} \sum_{i=1}^{s-1} \left(r_{2,1}^* + r_{2,2}^* + r_{2,3}^* + r_{2,4}^*\right) (12(q_s - q_{s-i})) \tag{3}$$

Moreover, if we let

$$S^*(u) := \sum_{s=1}^{u} \sum_{i=1}^{s-1} \left(r_{2,1}^* + r_{2,2}^* + r_{2,3}^* + r_{2,4}^*\right) (12(q_s - q_{s-i})),$$

we have the recurrence relation

$$S_p^*(u+1) = S^*(u) + \sum_{i=1}^{u} \left(r_{2,1}^* + r_{2,2}^* + r_{2,3}^* + r_{2,4}^*\right) (12(q_{p,u} - q_{p,u-i})).$$

It is clear that we will need some more detailed knowledge of the properties of the prime differences, $q_s - q_{s-i}$, in order to bound these sums. The question remains as to whether we really need deep knowledge of the distribution of these primes, or whether it will suffice to place them only sufficiently near the end of the intervals $I_p(t)$ to appropriately bound the indicator sums we need by $o(x^{0.51})$.

If we further admit solutions of $s \leq 0$ (with $k \geq 1$) in the count functions, $r_{2,m}^{*}(n)$ defined above and then call the corresponding functions $r_{2,m}^{**}(n)$, we can overestimate the value of I_{e,Q_x} slightly, but with the advantage that we have better generating functions and representations of these functions as sums of partition functions. Let the following shorthand denote the series generated by the Jacobi triple product identity:

$$\begin{aligned} G(x,q) &= \sum_{k=-\infty}^{\infty} x^k q^{k(k+1)/2} \\ &= (q;q)_\infty (-xq;q)_\infty \left(-\frac{q}{x};q\right)_\infty. \end{aligned}$$

Then we can define our new notions of the differences of squares functions, which we again denote by $r_{2,m}^{**}(n)$ for $m = 1,2,3,4$, in terms of their bilateral series generating functions as

$$\begin{aligned} \sum_{n\geq 0} r_{2,1}^{**}(n) q^n &= \sum_{k=1}^{\infty} G\left(q^{24(3k+2)}, q^{72}\right) \\ &= \sum_{k=1}^{\infty} (w^3;w^3)_\infty (-w^{-(3k-1)};w^3)_\infty (-w^{3k+5};w^3)_\infty \\ \sum_{n\geq 0} r_{2,2}^{**}(n) q^n &= \sum_{k=1}^{\infty} q^{24k} \cdot G\left(q^{24(3k+2)}, q^{72}\right) \\ &= \sum_{k=1}^{\infty} w^k \cdot (w^3;w^3)_\infty (-w^{-(3k-1)};w^3)_\infty (-w^{3k+5};w^3)_\infty \\ \sum_{n\geq 0} r_{2,3}^{**}(n) q^n &= \sum_{k=1}^{\infty} q^{2k} \cdot G\left(q^{24(3k+2)}, q^{72}\right) \\ &= \sum_{k=1}^{\infty} w^{\frac{k}{12}} \cdot (w^3;w^3)_\infty (-w^{-(3k-1)};w^3)_\infty (-w^{3k+5};w^3)_\infty \\ \sum_{n\geq 0} r_{2,4}^{**}(n) q^n &= \sum_{k=1}^{\infty} q^{22k} \cdot G\left(q^{24(3k+2)}, q^{72}\right) \\ &= \sum_{k=1}^{\infty} w^{\frac{11k}{12}} \cdot (w^3;w^3)_\infty (-w^{-(3k-1)};w^3)_\infty (-w^{3k+5};w^3)_\infty, \end{aligned}$$

where we define the shorthand of $w := q^{24}$ in the above expansions. We then let $x_{s,i} := 12(q_s - q_{s-i})$ and define the special partition function sums as

$$\widehat{P}_{3,k}(n) := \sum_{j=0}^{n} \sum_{m=0}^{n-72G_j} (-1)^{\lceil j/2 \rceil} P_{3,-(72k+48)}(m) P_{3,72k+48}(n - 72G_j - m),$$

where $P_{\alpha,\beta}(n) := [q^n](q^{\alpha+\beta};q^\alpha)_\infty$ denotes the number of partitions of n into parts strictly of the form $\alpha m + \beta$. Finally, we can give our new upper bound for the function I_{e,Q_x} in the following forms:

$$\begin{aligned} I_{e,Q_x} &\leq \sum_{s=1}^{\lfloor x^{0.51} \rfloor} \sum_{i=1}^{s-1} \left(r_{2,1}^{**} + r_{2,2}^{**} + r_{2,3}^{**} + r_{2,4}^{**}\right)(x_{s,i}) \\ &= \sum_{s=1}^{\lfloor x^{0.51} \rfloor} \sum_{i=1}^{s-1} \sum_{k=1}^{x^{s,i}} \left[\widehat{P}_{3,k}(x_{s,i}) + \widehat{P}_{3,k}(x_{s,i}-24k) + \widehat{P}_{3,k}(x_{s,i}-2k) + \widehat{P}_{3,k}(x_{s,i}-22k)\right]. \end{aligned}$$

We note that the partition function sequences $\{P_{3,\pm(3k+2)}(n)\}_{n\geq 1}$ do not depend on the input of k. Thus we need only list these corresponding sequences in the two cases given specifically below.

$$\begin{aligned} \{P_{3,-(3k+2)}(n)\}_{n\geq 1} &= \{1, 1, 3, 4, 6, 10, 14, 20, 29, 39, 54, 74, 98, 131, 172, 225, \ldots\} \\ \{P_{3,3k+2}(n)\}_{n\geq 1} &= \{0, 1, 2, 3, 5, 9, 12, 18, 26, 37, 50, 70, 93, 125, 165, 218, \ldots\} \end{aligned}$$

The values of the sums involving $\widetilde{P}_{3,k}(n)$ yield non-zero terms only when the input of n corresponds (at minimum) to a multiple of 24, as the generating functions for these sequences suggest.

Now we have a more concrete method for bounding the error terms I_{e,Q_x} using partition functions and possible generating function approaches. Then we should now be close to a position where we can tell how much information we will need about the distribution of the primes q_t to determine our upper bound for the error. This is also the topic of the next subsection.

2.5. **Another approach based on the differences of the primes in Q_x.** In the previous subsection, we suggested a method for summing the indicator sums implicit to the error term I_{e,Q_x} which seemed to naturally beg the question on how much we really need to know about the distribution of the primes q_s in our "*fortunate*" (or fortunately chosen) subset of primes Q_x. Based on some computational observations and noticing that $24 \mid 12(q_s - q_{s-i})$ for all s and i, we have another modified criteria for the differences between the primes in the set Q_x and when the indicators in our sums of interest contribute 1. In particular, in this formulation it becomes necessary to know more information about the positive *differences* between primes defined by the *multiset*

$$D_x := \left\{q_s - q_{s-i} : 1 \leq s \leq \lfloor x^{0.51} \rfloor, 1 \leq i < s\right\},$$

where for convenience, we take $d_{\max,x} := \max(D_x)$. As an example, if we have our prime subset Q_x defined to be

$$Q_x = \{3, 7, 13, 23, 31, 43, 53, 73\},$$

then our corresponding difference multiset of elements of the form (d, count) is given by

$$\begin{aligned} D_x = \big\{&(4,1), (6,1), (8,1), (10,3), (12,1), (16,1), (18,1), (20,3), (22,1), (24,1), \\ &(28,1), (30,3), (36,1), (40,2), (42,1), (46,1), (50,2), (60,1), (66,1), (70,1)\big\}. \end{aligned}$$

Now with this in mind, we have our criteria for evaluating, and hence bounding, the error term I_{e,Q_x}:

$$I_{e,Q_x} = \sum_{j=1}^{d_{\max,x}} \#\left\{d \in D_x : d \equiv j(3j+1) \pmod{6j}\right\} \qquad \text{(Indicator I)}$$

$$+\sum_{j=1}^{d_{\max,x}} \#\left\{d \in D_x : d \equiv 0 \pmod{12j+2}\right\} \qquad \text{(Indicator II)}$$

$$+\sum_{j=1}^{d_{\max,x}} \#\left\{d \in D_x : d \equiv j(3j+1) \pmod{72j-2}\right\} \qquad \text{(Indicator III)}$$

$$+\sum_{j=1}^{d_{\max,x}} \#\left\{d \in D_x : d \equiv j(3j+7) \pmod{72j+26}\right\}. \qquad \text{(Indicator IV)}$$

So it appears that *now* we should proceed with researching the known results for primes in bounded intervals to see if we can use any existing theorems to count the inner terms in the sums in the previous equation. One choice to note (and motivating the selection of the *last*, or one of, if we modify this method later), is that ideally as $t \longrightarrow \infty$ gets increasingly large, the differences between the primes $q_t - q_{t-1}$ should strictly increase so that the elements in the difference multiset can become more manageable (one thought on this).

3. Computational Data

The indicator sums computed in Table 3.1 provide examples that our bounds are correctly bounded above by a constant times $x^{0.51}$.

1 1.5

u	$\lceil u^{1.96} \rceil$ $(\varepsilon = 0.01)$	$S^*_{1.96}(u)$	$\lceil u^{5/3} \rceil$ $(\varepsilon = 0.1)$	$S^*_{5/3}(u)$	$\lceil u^{1.538} \rceil$ $(\varepsilon = 0.15)$	$S^*_{1.538}(u)$

Table 3.1. *Examples of the Indicator Sum Values, $S^*_p(u)$, Compared to the Claimed Bound of $o(x^{1/p})$*

4. Conclusions

4.1. Summary.

4.2. Topics for future research and investigation.

Acknowledgments.

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