Lower bounds on the Mertens function $M(x)$ for $x \gg 2.3315 \times 10^{1656520}$

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Abstract

The Mertens function, $M(x) = \sum_{n \leq x} \mu(n)$, is classically defined to be the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture which stated that $|M(x)| < C \cdot \sqrt{x}$ for all $x \geq 1$ has a well-known disproof due to Odlyzko et. al. given in the early 1980's by computation of non-trivial zeta function zeros in conjunction with integral formulas expressing $M(x)$. It is conjectured and widely believed that $M(x)/\sqrt{x}$ changes sign infinitely often and grows unbounded in the direction of both $\pm\infty$ along subsequences of integers $x \geq 1$. Our proof of this property of $q(x) \equiv M(x)/\sqrt{x}$ is not based on standard estimates of $M(x)$ by Mellin inversion, which are intimately tied to the distribution of the non-trivial zeros of the Riemann zeta function. There is a distinct stylistic flavor and element of combinatorial analysis peppered in with the standard methods from analytic number theory which distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

Keywords and Phrases: *Möbius function sums; Mertens function; summatory function; arithmetic functions; Dirichlet inverse; Liouville lambda function; prime omega functions; prime counting functions; Dirichlet series and DGFs; asymptotic lower bounds; Mertens conjecture.*

Primary Math Subject Classifications (2010): *11N37; 11A25; 11N60; 11N64; and 11-04.*

Reference on special notation and other conventions

Symbol	**Definition**
$o(g), O_\alpha(h)$	Using standard notation, we write that $f = o(g)$ if $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$ We adapt the stock big-Oh notation, writing $f = O_{\alpha_1,\ldots,\alpha_k}(g)$ for some parameters $\alpha_1,\ldots,\alpha_k$ if $f = O(g)$ subject only to some potentially fluctuating parameters that depend on the fixed α_i.
$\lceil x \rceil$	The ceiling function $\lceil x \rceil := x + 1 - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$.
$C_k(n)$	Auxillary component functions in obtaining asymptotic bounds on $g^{-1}(n)$. These functions are defined for $k \geq 0$ and $n \geq 1$ by the formula $$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum_{d \mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero.
DGF	*Dirichlet generating function (or DGF).* Given a sequence $\{f(n)\}_{n\geq 0}$, its DGF enumerates the sequence in a different way than formal generating functions in an auxiliary variable. Namely, for $\lvert s \rvert < \sigma_a$, the abcissa of absolute convergence of the series, the DGF $D_f(s)$ constitutes an analytic function of s given by: $D_f(s) := \sum_{n\geq 1} f(n)/n^s$. The DGF is alternately called the *Dirichlet series* of an arithmetic function f. The DGF of f can be inverted via a contour-based integral formula representation to solve for $f(n)$. It is also closely related to the Mellin transform of the summatory function of f at $-s$. type
$\sigma_0(n), d(n)$	The ordinary divisor function, $d(n) := \sum_{d\mid n} 1$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) = \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$.
$f * g$	The Dirichlet convolution of f and g, $f * g(n) := \sum_{d\mid n} f(d) g(n/d)$, for $n \geq 1$. This symbol for the discrete convolution of two arithmetic functions is the only notion of convolution of functions we employ within the article that is not explicitly expanded by the definition of another relvant convolution operation, e.g., integral formula or summation with exactly specified indices as input to the functions at hand.

Symbol	**Definition**
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ provided that $f(1) \neq 0$. The inverse function, when it exists, is unique and satisfies the relations that $f^{-1} * f = f * f^{-1} = \varepsilon$.
$\lfloor x \rfloor$	The floor function $\lfloor x \rfloor := x - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega+1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$. This function definition is the key to unraveling our new bounds proved in the article, and so, is henceforth taken as standard notation moving on.
$\mathrm{Id}_k(n)$	The power-scaled identity function, $\mathrm{Id}_k(n) := n^k$ for $n \geq 1$.
$\mathbb{1}_{\mathbb{S}}$, $\chi_{\mathtt{cond}(x)}$	We use the notation $\mathbb{1}, \chi : \mathbb{N} \to \{0,1\}$ to denote indicator, or characteristic functions. In paticular, $\mathbb{1}_{\mathbb{S}}(n) = 1$ if and only if $n \in \mathbb{S}$, and $\chi_{\mathtt{cond}}(n) = 1$ if and only if n satisfies the condition `cond`.
$\log_*^m(x)$	The iterated logarithm function defined recursively for integers $m \geq 0$ by

$$\log_*^m(x) := \begin{cases} x, & \text{if } m = 0; \\ \log x, & \text{if } m = 1; \\ \log\left[\log_*^{m-1}(x)\right], & \text{if } m \geq 2. \end{cases}$$

$[n = k]_\delta$	Synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and zero otherwise.
$[\mathtt{cond}]_\delta$	For a boolean-valued `cond`, $[\mathtt{cond}]_\delta$ evaluates to one precisely when `cond` is true, and zero otherwise.
$\lambda(n)$	The Liouville lambda function, $\lambda(n) := (-1)^{\Omega(n)}$, denotes the parity of $\Omega(n)$, the number of distinct prime factors of n counting multiplicity. That is, $\lambda(n) \in \{\pm 1\}$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$. Notice that if n is squarefree, then $\lambda(n) = \mu(n)$, where more generally we have that $\lambda(n) = \sum_{d^2 \mid n} \mu\left(\frac{n}{d^2}\right)$. This function is Dirichlet invertible with inverse function given by $\lambda^{-1}(n) = \mu^2(n)$, the (unsigned) characteristic function of the squarefree integers.
$\gcd(m,n); (m,n)$	The greatest common divisor of m and n. Both notations for the GCD are used interchangably within the article.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree, i.e., has no prime power divisors with exponent greater than one.
$M(x)$	The Mertens function which is the summatory function over $\mu(n)$, $M(x) := \sum_{n \leq x} \mu(n)$.

Symbol	Definition
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^\alpha \vert\vert n$ (p^α exactly divides n) for p prime and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the distinct prime factor counting functions as $\omega(n) := \sum_{p \vert n} 1$ and $\Omega(n) := \sum_{p^\alpha \vert\vert n} \alpha$. Equivalently, if the factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we define that $\omega(1) = \Omega(1) = 0$.
$\sum_{p \leq x}, \prod_{p \leq x}$	Unless otherwise specified by context, we use the index variable p to denote that the summation (product) is to be taken only over prime values within the summation bounds.
$P(s)$	For complex s with $\Re(s) > 1$, we define the prime zeta function to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$. This function has an analytic continuation to $\Re(s) \in (0, 1)$ with a logarithmic singularity near $s := 1$: $P(1 + \varepsilon) = -\log \varepsilon + C + O(\varepsilon)$.
$\sigma_\alpha(n)$	The generalized sum-of-divisors function, $\sigma_\alpha(n) := \sum\limits_{d \vert n} d^\alpha$, for any $n \geq 1$ and $\alpha \in \mathbb{C}$.
$\left[{n \atop k}\right]$	The unsigned Stirling numbers of the first kind, $\left[{n \atop k}\right] = (-1)^{n-k} \cdot s(n, k)$.
$\sim, \approx, \gtrsim, \lesssim$	We say that two functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $$\lim_{x \to \infty} \frac{A(x)}{B(x)} = 1.$$ We also sometimes express the *average order* of an arithmetic function $f \sim h$ that may actually oscillate, or say have value of one infinitely often, in the cases that $\frac{1}{x} \cdot \sum_{n \leq x} f(n) \sim h(x)$ (for example, we often would write that $\Omega(n) \sim \log \log n$, even though technically, $1 \leq \Omega(n) \leq \frac{\log n}{\log 2}$). We write that $f(x) \approx g(x)$ if $\lvert f(x) - g(x) \rvert = O(1)$. We say that $h(x) \gtrsim r(x)$ if $h \gg r$ as $x \to \infty$, and define the relation $\lesssim$ similarly as $h(x) \lesssim r(x)$ if $h \ll r$ as $x \to \infty$. When applying these relations we still consider leading constants to be meaningful terms that are preserved.
$\sum_{n \leq x}'$	We denote by $\sum_{n \leq x}' f(n)$ the summatory function of f at x minus $\frac{f(x)}{2}$ if $x \in \mathbb{Z}$.
$\tau_m(n)$	Let $\tau_m(n) \equiv \mathbb{1}_{*_m}(n)$ denote the m-fold Dirichlet convolution of one with itself at n. Note that $\tau_2(n)$ yields the divisor function, $d(n) \equiv \sigma_0(n)$.
$\zeta(s)$	The Riemann zeta function, defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\Re(s) > 1$, and by analytic continuation to the entire complex plane with the exception of a simple pole at $s = 1$.

1 Introduction

1.1 Preface on notation: Unconventional notions of asymptotics

The notation of $\gtrsim, \lesssim$ defined in the prior section on notation employed in the article is convenient for expressing upper and lower bounds on functions given by asymptotically dominant main terms in the expansion of more complicated symbolic expansions. For example, suppose that exactly

$$f(x) \geq -(\log\log\log x)^2 + 3 \times 10^{1000000} \cdot (\log\log\log x)^{1.999999999} + E(x),$$

where $E(x) = o\left((\log\log\log x)^2\right)$ and the unusually complicated expression for $E(x)$ requires more than 100000 ascii characters to typeset accurately. Then naturally, we prefer to work with only the expression for the asymptotically dominant main term in the lower bounds stated above. Note that since this main term contribution does not dominate the bound until x is very large, so that replacing the right-hand-side expression with just this term yields an invalid inequality except for in limiting cases. In this instance, we prefer to write

$$f(x) \gtrsim -(\log\log\log x)^2,$$

which indicates that this substantially simplifed form of the lower bound on f holds as $x \rightarrow \infty$.

Hence, we use these new symbols, $\gtrsim, \lesssim$, as asymptotic relations defined to simplify our results by dropping expressions involving more precise, exact terms that are nonetheless asymptotically insignificant, to obtain accurate statements in limiting cases of large x that hold as $x \rightarrow \infty$ that capture the more simple essence of the bound as we choose to view it. This notation is particularly powerful and is utilized in this article when we express many lower bound estimates for functions that would otherwise require literally pages of typeset symbols to state exactly, but which have simple enough formulae when considered as bounds that hold in this type of limiting asymptotic context.

To distinguish between classical and modern usages of the notation $\sim$, we will write $A(x) \sim B(x)$ to denote that $A(x)/B(x) \xrightarrow{x\rightarrow\infty} 1$. In place of the use of $\sim$ to denote the *average order* of an arithmetic function as in [7], we will use a modern, probabilistically themed expectation symbol as follows:

$$\mathbb{E}[f(x)] = g(x) \iff \frac{1}{x}\sum_{n\leq x} f(x) \sim g(x).$$

We note that these subtle distinctions in usage of traditional notation for asymptotic relations are key to understanding our choices of upper and lower bound expressions given throughout the article.

1.2 The Mertens function – definition, properties, known results and conjectures

Suppose that $n \geq 1$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. We define the *Möebius function* to be the signed indicator function of the squarefree integers:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leq i \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many known variants and special properties of the Möebius function and its generalizations [15, *cf.* §2], however, for our purposes we seek to explore the properties and asymptotics of weighted summatory functions over $\mu(n)$. The Mertens summatory function, or *Mertens function*, is defined as [17, A002321]

$$\begin{aligned} M(x) &= \sum_{n\leq x}\mu(n),\ x\geq 1, \\ &\longmapsto \{1,0,-1,-1,-2,-1,-2,-2,-2,-1,-2,-2,-3,-2,-1,-1,-2,-2,-3,-3,-2,-1,-2,-2\} \end{aligned}$$

A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as [17, A013928]

$$Q(n) = \sum_{n\leq x}|\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x\to\infty$:

$$\mu_+(x) = \frac{\#\{1\leq n\leq x: \mu(n)=+1\}}{Q(x)} = \mu_-(x) = \frac{\#\{1\leq n\leq x: \mu(n)=-1\}}{Q(x)} \xrightarrow[n\to\infty]{} \frac{3}{\pi^2}.$$

While this limiting law suggests an even bias for the Mertens function, in practice $M(x)$ has a noted negative bias in its values, and the actual local oscillations between the approximate densities of the sets $\mu_\pm(x)$ lend an unpredictable nature to the function and its characteistic oscillatory sawtooth shaped plot.

1.2.1 Properties

The well-known approach to evaluating the behavior of $M(x)$ for large $x\to\infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x>0$. This formula is easily expressed via an inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since by Perron's formula we have

$$\frac{1}{\zeta(s)} = \int_1^\infty \frac{s\cdot M(x)}{x^{s+1}}dx,$$

we then obtain that

$$M(x) = \lim_{T\to\infty}\frac{1}{2\pi\imath}\int_{T-\imath\infty}^{T+\imath\infty}\frac{x^s}{s\cdot\zeta(s)}ds.$$

This representation along with the standard Euler product representation for the reciprocal zeta function leads us to the exact expression for $M(x)$ when $x>0$ given by the next theorem.

Theorem 1.1 (Analytic Formula for $M(x)$)**.** *Assuming the RH, we can show that there exists an infinite sequence $\{T_k\}_{k\geq 1}$ satisfying $k\leq T_k\leq k+1$ for each k such that for any $x\in\mathbb{R}_{>0}$*

$$M(x) = \lim_{k\to\infty}\sum_{\substack{\rho:\zeta(\rho)=0 \\ |\Im(\rho)|<T_k}}\frac{x^\rho}{\rho\cdot\zeta'(\rho)} - 2 + \sum_{n\geq 1}\frac{(-1)^{n-1}}{n\cdot(2n)!\zeta(2n+1)}\left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2}\left[x\in\mathbb{Z}^+\right]_\delta.$$

An unconditional bound on the Mertens function due to Walfisz (1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp\left(-C \cdot \log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan in 2009 proved new updated estimates bounding $M(x)$ for large x of the following forms [18]:

$$\begin{aligned} M(x) &\ll \sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right), \ \forall \epsilon > 0. \end{aligned}$$

Other explicit bounds due to the article by Kotnik include the following simpler estimates for the Mertens function when x is sufficiently large:

$$\begin{aligned} |M(x)| &< \frac{x}{4345}, \ \forall x > 2160535, \\ |M(x)| &< \frac{0.58782 \cdot x}{\log^{11/9}(x)}, \ \forall x > 685. \end{aligned}$$

1.2.2 Conjectures

The *Riemann Hypothesis* (RH) is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. It is still unresolved whether

$$\limsup_{x\to\infty} |M(x)|/\sqrt{x} = \infty,$$

although computational evidence suggests that this is a likely conjecture [10, 8]. There is a rich history to the original statement of the *Mertens conjecture* which states that

$$|M(x)| < c \cdot x^{1/2}, \quad \text{some constant } c > 0,$$

which was first verified by Mertens for $c = 1$ and $x < 10000$, although since its beginnings in 1897 has since been disproved by computation by Odlyzko and té Riele in the early 1980's.

There are a number of other interesting unsolved and at least somewhat accessible open problems related to the asymptotic behavior of $M(x)$ at large x. It is believed that the sign of $M(x)$ changes infinitely often. That is to say that it is widely believed that $M(x)$ is ocsillatory and exhibits a negative bias insomuch as $M(x) < 0$ more frequently than $M(x) > 0$ over all $x \in \mathbb{N}$. One of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ is unbounded on the natural numbers. In particular, the precise statement of this problem is to produce an affirmative answer whether $\limsup_{x\to\infty} |M(x)|/\sqrt{x} = +\infty$, or equivalently whether there is an infinite sequence of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that $M(x_i)x_i^{-1/2}$ grows without bound along this subsequence.

Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies that

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}(\log\log x)^{5/4}},$$

corresponds to some bounded constant. To date an exact rigorous proof that $M(x)/\sqrt{x}$ is unbounded still remains elusive, though there is suggestive probabilistic evidence of this property established by Ng in 2008. We cite that prior to this point it is known that [14, *cf.* §4.1]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad \text{(now } 1.826054\text{)},$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad \text{(now } -1.837625\text{)},$$

although based on work by Odlyzyko and te Riele it seems probable that each of these limits should be $\pm\infty$, respectively [12, 9, 10, 8]. It is also known that $M(x) = \Omega_{\pm}(\sqrt{x})$ and $M(x)/\sqrt{x} = \Omega_{\pm}(1)$.

1.3 A new approach to bounding $M(x)$ from below

1.3.1 Summing series over Dirichlet convolutions

Theorem 1.2 (Summatory functions of Dirichlet convolutions)**.** *Let $f, g : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n\leq x} f(n)$ and $G(x) := \sum_{n\leq x} g(n)$ denote the summatory functions of f, g, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse $f^{-1}(n)$ of f, i.e., the unique arithmetic function such that $f * f^{-1} = \varepsilon$ where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity with respect to Dirichlet convolution. Then, letting the counting function $\pi_{f*g}(x)$ be defined as in the first equation below, we have the following equivalent expressions for the summatory function of $f * g$ for integers $x \geq 1$:*

$$\begin{aligned}
\pi_{f*g}(x) &= \sum_{n\leq x}\sum_{d|n} f(d)g(n/d) \\
&= \sum_{d\leq x} f(d) G\left(\left\lfloor \frac{x}{d}\right\rfloor\right) \\
&= \sum_{k=1}^{x} G(k)\left[F\left(\left\lfloor \frac{x}{k}\right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1}\right\rfloor\right)\right].
\end{aligned}$$

Moreover, we can invert the linear system determining the coefficients of $G(k)$ for $1 \leq k \leq x$ naturally to express $G(x)$ as a linear combination of the original left-hand-side summatory function as:

$$\begin{aligned}
G(x) &= \sum_{j=1}^{x} \pi_{f*g}(j)\left[F^{-1}\left(\left\lfloor \frac{x}{j}\right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1}\right\rfloor\right)\right] \\
&= \sum_{n=1}^{x} f^{-1}(n)\pi_{f*g}\left(\left\lfloor \frac{x}{n}\right\rfloor\right).
\end{aligned}$$

Corollary 1.3 (Convolutions Arising From Möbius Inversion)**.** *Suppose that g is an arithmetic function with $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n\leq x}(g*\mu)(n)$. Then the Mertens function equals*

$$M(x) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right)\widetilde{G}(k), \forall x \geq 1.$$

1.3.2 A motivating special case

Using $\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu$, where $\chi_{\mathbb{P}}$ is the characteristic function of the primes, we have that $\widetilde{G}(x) = \pi(x) + 1$ in Corollary 1.3. In particular, the corollary implies that

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

We can compute the first few terms for the Dirichlet inverse sequence of $g(n) := \omega(n) + 1$ numerically for the first few sequence values as

$$\{g^{-1}(n)\}_{n\geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is given by $\lambda(n) = \frac{g^{-1}(n)}{|g^{-1}(n)|}$ (see Proposition 2.3). Note that since the DGF of $\omega(n)$ is given by $D_\omega(s) = P(s)\zeta(s)$ where $P(s)$ is the *prime zeta function*, we do have a Dirichlet series for the inverse functions to invert coefficient-wise using more classical contour integral methods*. Our new methods do not rely on typical constructions for bounding $M(x)$ based on estimates of the non-trivial zeros of the Riemann zeta function that have so far to date been employed to bound the Mertens function from above. We will instead take a more combinatorial tack to investigating bounds on this inverse function sequence in the coming sections.

Consider the following motivating conjecture: A01-2020-04-26

Conjecture 1.4. *Suppose that $n \geq 1$ is a squarefree integer. We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n) = (\omega + 1)^{-1}(n)$ over these integers:*

(A) $g^{-1}(1) = 1$;

(B) $\operatorname{sgn}(g^{-1}(n)) = \mu(n) \equiv \lambda(n)$;

(C) We can write the inverse function at squarefree n as

$$g^{-1}(n) = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!.$$

We illustrate parts (B)–(C) of this conjecture clearly using Table T.1 given on page 41 of the appendix section. A table of the first several explicit values of $(f+1)^{-1}(n)$ for $f(1) = 0$ and symbolic additive f are also given in Table T.2 on page 42.

The realization that the beautiful and remarkably simple form of property (C) in Conjecture 1.4 holds for all squarefree $n \geq 1$ motivates our pursuit of formulas for the inverse functions $g^{-1}(n)$ based on the configuration of the exponents in the prime factorization of any $n \geq 2$. In Section 4 we consider expansions of these inverse functions recursively, starting from a few first exact cases of an auxillary function, $C_k(n)$, that depends on the precise exponents in the prime factorization of n. We then prove limiting asymptotics for

*E.g., using [1, §11]

$$f(n) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{n^{\sigma+\imath t}}{\zeta(\sigma+\imath t)(P(\sigma+\imath t)+1)}, \sigma > 1.$$

Fröberg has also previously done some preliminary investigation as to the properties of the inversion to find the coefficients of $(1 + P(s))^{-1}$ in [4].

these functions and assemble the main terms in the expansion of $g^{-1}(n)$ using artifacts from combinatorial analysis and elementary number theory. The summation methods we employ to weight sums of our arithmetic functions according to the sign of $\lambda(n)$ (or parity of $\Omega(n)$) is reminiscent of the combinatorially motivated sieve methods in [3, §17]. The identity in (1) provides us with a powerful new method to bound $M(x)$ from below. We will sketch the key results and formulation to the construction we actually use to prove the new lower bounds on $M(x)$ next.

1.3.3 Fixing an exact expression for $M(x)$ by special arithmetic functions

From this point on, we fix the Dirichlet invertible function $g(n) := \omega(n) + 1$ and denote its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. For natural numbers $n \geq 1, k \geq 0$, let

$$C_k(n) := \begin{cases} \varepsilon(n) = \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$

By Möbius inversion (see Lemma 4.2), we have that

$$(g^{-1} * 1)(n) = \lambda(n) \cdot C_{\Omega(n)}(n), \forall n \geq 1. \tag{2}$$

We have limiting asymptotics on these functions given by the following theorem:

Theorem 1.5 (Asymptotics for the functions $C_k(n)$)**.** *The function $\sigma_0 * \tau_m$ is multiplicative with values at prime powers given by*

$$(\sigma_0 * \tau_m)(p^\alpha) = \binom{\alpha + m + 1}{m + 1}.$$

We have the following asymptotic base cases for the functions $C_k(n)$:

$$\begin{aligned} C_1(n) &\sim \log\log n \\ C_2(n) &\sim \frac{\sigma_0(n) n}{\log n} + O(\log\log n) \\ C_3(n) &\sim -\frac{(\sigma_0 * 1)(n) n^2}{\log n} + O\left(n \cdot \log\log n\right). \end{aligned}$$

For all $k \geq 4$, we obtain that the dominant asymptotic term and the error bound terms for $C_k(n)$ are given by

$$C_k(n) \sim (\sigma_0 * \tau_{k-2})(n) \times \frac{(-1)^k n^{k-1}}{(\log n)^{k-1}(k-1)!} + O_k\left(\frac{n^{k-2}}{(k-2)!} \cdot \frac{(\log\log n)^{k-2}}{(\log n)^{k-2}}\right), \text{ as } n \rightarrow \infty.$$

Then we can prove (see Corollary 4.8) that

$$g^{-1}(n) \sim \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

Notice that this formula is substantially easier to evaluate than the corresponding sums in (2) given directly by Möbius inversion – and hence, we prefer to work with bounds on it we prove as new results rather than the more complicated exact formula from the cited equation above. The last result in turn implies that

$$G^{-1}(x) \gtrsim \sum_{n \leq x} \lambda(n) \cdot C_{\Omega(n)}(n) \times \sum_{d=1}^{\left\lfloor \frac{x}{n} \right\rfloor} \lambda(d). \tag{3}$$

In light of the fact that (by an integral-based interpretation of integer convolution using summation by parts)

$$M(x) \sim G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)},$$

the formula in (3) implies that we can establish new *lower bounds* on $M(x)$ by appropriate estimates of the summatory function $G^{-1}(x)$ where trivially we have the bounded inner sums $L_0(x) := \sum_{n \leq x} \lambda(n) \in [-x, x]$ for all $x \geq 2$.

As explicit lower bounds for $M(x)$ along subsequences are not obvious, and are historically ellusive and non-trivial to obtain as we expect sign changes of this function infinitely often, we find this approach to be an effective one. Now, having motivated why we must carefully estimate the $G^{-1}(x)$ bounds using our new methods, we will require the bounds suggested in the next section to work at summing the summatory functions, $G^{-1}(x)$, for large x as $x \to \infty$.

1.3.4 Some enumerative (or counting) DGFs from Mongomery and Vaughan

Our inspiration for the new bounds found in the last sections of this article allows us to sum non-negative arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. In particular, it uses a hybrid generating function and DGF method under which we are able to recover "good enough" asymptotics about the summatory functions that encapsulate the parity of $\lambda(n)$:

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}, k \geq 1.$$

The precise statement of the theorem that we transform for these new bounds is re-stated as follows:

Theorem 1.6 (Montgomery and Vaughan, §7.4)**.** *Let $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$. For $R < 2$ we have that*

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

uniformly for $1 \leq k \leq R \log\log x$ where

$$\mathcal{G}(z) := \frac{F(1,z)}{\Gamma(z+1)} = \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z.$$

The precise formulations of the inverse function asymptotics proved in Section 4 depend on being able to form significant lower bounds on the summatory functions of an always positive arithmetic function weighted by $\lambda(n)$. The next theorem, proved carefully in Section 3, is the primary starting point for our new asymptotic lower bounds.

Theorem 1.7 (Generating functions of symmetric functions)**.** *We obtain lower bounds of the following form for $A_0 > 0$ an absolute constant, and $C_0(x)$ a function only of x where we take $z \geq 0$ to be a real-valued parameter uniformly bounded in $x \geq 2$:*

$$\mathcal{G}(z) \geq A_0 \cdot C_0(x)^z$$

It suffices to take

$$\begin{aligned} A_0 &= \frac{4}{3 \cdot \log 2 \cdot 2^{27/4} \cdot \Gamma\left(\frac{5}{2}\right)} \approx 0.0134439 \\ C_0(x) &= \frac{4}{3 \log 2}. \end{aligned}$$

1.4 Cracking the classical unboundedness result, so to speak

In Section 5, we provide the culmination of what we build up to in the proofs established in prior sections of the article. Namely, we prove the form of an explicit limiting lower bound for the summatory function, $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$, along a specific subsequence over which the parity of both $\left\lfloor \frac{1}{2}\log\log\log\log x\right\rfloor$ and $\left\lfloor \frac{3}{2}\log\log\log\log x\right\rfloor$ are predictably signed. What we obtain is the following important summary corollary verifying the unboundedness of the scaled function $|M(x)|/\sqrt{x}$ in the limit supremum sense:

Corollary 1.8 (Bounds for the classically scaled Mertens function)**.** *Let $u_0 := e^{e^{e^e}}$ and define the infinite increasing subsequence, $\{x_n\}_{n\geq 1}$, by $x_n := e^{e^{e^{e^{6n}}}}$. We have that along the increasing subsequence x_y for large $y \geq \max\left(\left\lceil e^{e^{e^e}}\right\rceil, u_0+1\right)$:*

$$\frac{|M(x_y)|}{\sqrt{x_y}} \gtrsim 2C_{\ell,1}\cdot(\log\log\sqrt{x_y})(\log\log\log\sqrt{x_y})^{4+\frac{3}{\log 2}-\frac{3}{\log 3}} + o(1),$$

as $y\to\infty$. In the previous equation, we adopt the notation for the absolute constant $C_{\ell,1}>0$ defined more precisely by

$$C_{\ell,1} := \frac{1}{36\cdot 2^{3/4}\sqrt{\pi}\cdot\log 2} \approx 0.000183209.$$

This is all to say that in establishing the rigorous proof of Corollary 1.8 based on our new methods, we not only show that

$$\limsup_{x\to\infty}\frac{|M(x)|}{\sqrt{x}} = +\infty,$$

but also set a minimal rate (along a large subsequence) at which the scaled Mertens function grows without bound.

1.5 Outline: Core components to the proof

We offer another brief step-by-step summary overview of the critical components to our proof outlined in the introduction above, and then which are proved piece-by-piece in the next sections of the article below. This outline is provided to help the reader see our logic and proof methodology as easily and quickly as possible.

(1) We prove an apparently yet undiscovered matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1)\neq 0$). Namely, a careful matrix and symmetry transformation based proof of Theorem 1.2 is given in Section 2.

(2) This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the seemingly unconnected prime counting function, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n)+1$. This formula is already stated in (1) expanded above.

  (i) The average order, $\mathbb{E}[\omega(n)] = \log\log n$, imparts an iterated logarithmic structure to our expansions, which many have conjectured we should see in limiting bounds on $M(x)$, but which are practically elusive in most non-conjectural known formulas I have seen proved rigorously in print.

(ii) The additivity of $\omega(n)$ dictates that the sign of $g^{-1}(n) = (\omega+1)^{-1}(n)$ is $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ (see Proposition 2.3). The corresponding weighted summatory functions of $\lambda(n)$ have more established predictable properties, such as known sign biases and upper bounds. These summatory functions are generally speaking more regular and easier to work with than traditional approaches to summing $M(x)$ and its complicating summand terms of the Möbius function. Note that our proof is essentially much different than what is known about sums of consecutive values of $\mu(n)$ over short intervals, both in interpretation and methodology.

(3) We tighten a result from [11, §7] providing summatory functions that indicate the parity of $\lambda(n)$ using elementary arguments and more combinatorially flavored expansions of Dirichlet series in our proof of Theorem 1.7. Our motivations are different than in the reference for exploiting the unique properties of this construction. Namely, we are not after a CLT-like statement for the functions $\Omega(n)$ and $\omega(n)$. Rather, we seek to sum $\sum_{n\leq x}\lambda(n)f(n)$ for general non-negative arithmetic functions f using Abel summation when x is large.

(4) We then turn to the asymptotics if the quasi-periodic $g^{-1}(n)$, estimating this inverse function's limiting asymptotics for large n (or $n \leq x$ when x is very large) in Section 4. We eventually use these estimates to prove a substantially unique new lower bound formula for the summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$ along prescribed asymptotically large infinite subsequences that tend to $+\infty$ (see Theorem 5.2).

(5) When we return to (2) with our new lower bounds, and bootstrap, we recover "magic" in the form of showing the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. What we recover is a quick, and rigorous, proof of Corollary 1.8.

(6) We remark that while this technique and approach to the classical problem at hand is certainly new, it is not just novel, and its discovery will invariably lead to similar applications given careful study of limsup-type bounds on the summatory functions of other special signed arithmetic function sequences.

Note that in these cases, if f is multiplicative and $f(n) > 0$ for all $n \geq 1$, then $\operatorname{sgn}(f^{-1}(n)) = (-1)^{\omega(n)}$. This variation in signedness tends to complicate, but still closely parallel our argument involving the parity of $\lambda(n) = (-1)^{\Omega(n)}$ for the Mertens function case.

2 Preliminary proofs of lemmas and new results

2.1 Establishing the summatory function inversion identities

There are a vast number of Dirichlet convolution identities for special number theoretic functions over which we can form summatory functions and perform inversion via Theorem 1.2. For example, we have notable identities of the forms [5, 16]

$$(f * 1)(n) = [q^n] \sum_{m \geq 1} f(m) q^m / (1 - q^m),$$

and

$$\sigma_k = \mathrm{Id}_k * 1, \mathrm{Id}_1 = \phi * \sigma_0, \mathrm{Id}_k = J_k * 1, \log = \Lambda * 1, 2^{\omega} = \mu^2 * 1.$$

We will go ahead and prove this useful theorem, a crucial component to our new more combinatorial formulations used to bound $M(x)$ in later sections, before moving on.

Remark 2.1 (Proving related inversion formula statements)**.** The proof given below follows from a straightforward matrix inversion procedure involving standard *shift matrix* operations, and similarity transformations of these operators. Notice that this proof provides a natural analog (and alternate expansion) to the already well known, established inversion relations for general arithmetic-function-weighted sums of summatory functions as typically cited in [1, §2.14]. The classical formulation we have to work with is re-stated as follows for any Dirichlet invertible function $\alpha(1) \neq 0$ and any function F convolved with such an arithmetic function:

$$G(x) := \sum_{n \leq x} \alpha(n) \cdot F\left(\frac{x}{n}\right) \iff F(x) = \sum_{n \leq x} \alpha^{-1}(n) \cdot G\left(\frac{x}{n}\right).$$

Related results for summations of Dirichlet convolutions appear in [1, §3.10; §3.12; *cf.* §4.9, p. 95].

Proof of Theorem 1.2. Let h, g be arithmetic functions where $g(1) \neq 0$ necessarily has a Dirichlet inverse. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h: $g * h$. Then we can easily see that the following expansions hold:

$$\begin{aligned} \pi_{g*h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n) h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] H(i). \end{aligned}$$

We form the matrix of coefficients associated with this system for $H(x)$, and proceed to invert it to express an exact solution for this function over all $x \geq 1$. Let the ordinary (initial, non-inverse) matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1}.$$

The matrix we must invert in this problem is lower triangular, with ones on its diagonals – and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T); \qquad U = (\delta_{i,j+1}), (I - U^T)^{-1} = ([j \leq i]_{\delta}).$$

Here, U is the $N \times N$ matrix whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$.

It is a useful fact that if we take successive differences of floor functions, we get non-zero behavior at divisors:

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

We use this property to shift the matrix $\hat{G}$, and then invert the result, to obtain a matrix involving the Dirichlet inverse of g:

$$\left[(I - U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Now we can express the inverse of the target matrix $(g_{x,j})$ in terms of these Dirichlet inverse functions as follows:

$$\begin{aligned}
(g_{x,j}) &= (I - U^T)^{-1}\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I - U^T) \\
(g_{x,j})^{-1} &= (I - U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I - U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k)\right)(I - U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} g^{-1}(k)\right).
\end{aligned}$$

Thus the summatory function H is exactly expressed by the inverse vector product of the form

$$\begin{aligned}
H(x) &= \sum_{k=1}^{x} g_{x,k}^{-1} \cdot \pi_{g*h}(k) \\
&= \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor+1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j)\right) \cdot \pi_{g*h}(k). \square
\end{aligned}$$

2.2 Proving the crucial signedness property from the conjecture

Proposition 2.2 (The characteristic function of the primes). *Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the incompletely additive function that counts the number of distinct prime factors of n. Then we have the convolution identity given by*

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu.$$

The summatory function of the left-hand-side of the previous equation is clearly $\widetilde{G}(x) = \pi(x) + 1$ for all $x \geq 1$.

Proof. The core is to prove that for all $n \geq 1$, $\chi_{\mathbb{P}}(n) = (\mu * \omega)(n)$ – our essential claim. We notice that the Mellin transform of $\pi(x)$ – the summatory function of $\chi_{\mathbb{P}}(n)$ – at $-s$ is given by

$$\begin{aligned} s \cdot \int_1^{\infty} \frac{\pi(x)}{x^{s+1}} dx &= \sum_{n\geq 1} \frac{\nabla[\pi](n-1)}{n^s} \\ &= \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s} = P(s), \end{aligned}$$

where $\nabla[f](n) := f(n+1) - f(n)$ denotes the standard *forward difference operator* used to express a discrete derivative type operation on arithmetic functions. This is typical construction which more generally relates the Mellin transform $s \cdot \mathcal{M}[S_f](-s)$ to the DGF of the sequence $f(n)$ as cited, for example, in [1, §11]. Now we consider the DGF of the right-hand-side function, $f(n) := (\mu * \omega)(n)$, as

$$D_f(s) = \frac{1}{\zeta(s)} \times \sum_{n \geq 1} \frac{\omega(n)}{n^s} = P(s).$$

Thus for any $\Re(s) > 1$, the DGFs of each side of the claimed equation coincide. So by uniqueness of Dirichlet series, we see that in fact the claim holds. To obtain the full result, we add to each side of this equation a term of $\varepsilon(n) \equiv (\mu * 1)(n)$, and then factor the resulting convolution identity. □

Proposition 2.3 (The key sign function property of $g^{-1}(n)$)**.** *For the Dirichlet invertible function, $g(n) := \omega(n) + 1$ defined such that $g(1) = 1$, at any $n \geq 1$, we have that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$. Here, the notation for the operation given by $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)| + [h(n)=0]_{\delta}} \in \{0, \pm 1\}$ denotes the sign, or signed parity, of the arithmetic function h at n.*

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Proof. Let $D_f(s) := \sum_{n \geq 1} f(n) n^{-s}$ denote the Dirichlet generating function (DGF) of $f(n)$. Then we have that

$$D_{(\omega+1)^{-1}}(s) = \frac{D_{\lambda}(s)}{(P(s)+1)\zeta(2s)}.$$

Let $h^{-1}(n) := (\omega * \mu + \varepsilon)^{-1}(n) = [n^{-s}](P(s)+1)^{-1}$. Then we have that

$$\begin{aligned} (h^{-1} * 1)(n) &= -\sum_{p_1 | n} h^{-1}\left(\frac{n}{p_1}\right) = \lambda(n) \times \sum_{p_1 | n} \sum_{p_2 | \frac{n}{p_1}} \cdots \sum_{p_{\Omega(n)} | \frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1 \\ &= \begin{cases} \lambda(n) \times (\Omega(n)-1)!, & n \geq 2; \\ \lambda(n), & n = 1. \end{cases} \end{aligned}$$

So by Möbius inversion

$$h^{-1}(n) = \lambda(n) \left[\sum_{\substack{d | n \\ d < n}} \lambda(d) \mu(d) (\Omega(n/d) - 1)! + 1 \right] = \lambda(n) \left[\sum_{\substack{d | n \\ d < n}} \mu^2(d) (\Omega(n/d) - 1)! + 1 \right].$$

Then we finally have that

$$(\omega+1)^{-1}(n) = \lambda(n) \times \sum_{d|n} \lambda(d) \left[\sum_{\substack{r|\frac{n}{d} \\ r<\frac{n}{d}}} \mu^2(r)(\Omega\left(\frac{n}{dr}\right) - 1)! + 1 \right] \chi_{\mathrm{sq}}(d)\mu(\sqrt{d}),$$

where χ_{sq} is the characteristic function of the squares. In either case of $\lambda(n) = \pm 1$, there are positive constants $C_{1,n}, C_{2,n} > 0$ such that

$$\lambda(n)C_{1,n} \times \sum_{d^2|n} \lambda(d^2)\mu(d) \leq g^{-1}(n) \leq \lambda(n)C_{1,n} \times \sum_{d^2|n} \lambda(d^2)\mu(d),$$

where $\sum_{d^2|n} \lambda(d^2)\mu(d) = \sum_{d^2|n} \mu^2(n) > 0$. This proves the result. □

More generally, we have that for f a non-negative additive arithmetic function that vanishes at one, $\operatorname{sgn}((f+1)^{-1}) = \lambda(n) = (-1)^{\Omega(n)}$. We can state similar properties for the common case of multiplicative f in the form of the following result: If $f(n) > 0$ for all $n \geq 1$ and f is multiplicative, then $\operatorname{sgn}(f^{-1}(n)) = (-1)^{\omega(n)}$.

2.3 Other facts and listings of results we will need in our proofs

Theorem 2.4 (Mertens theorem)**.**

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log\log x + B + o(1),$$

where $B \approx 0.2614972128476427837554$ *is an absolute constant.*

Corollary 2.5. *We have that for sufficiently large* $x \gg 1$

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-B}}{\log x} \left(1 + o(1)\right).$$

Hence, for $1 < |z| < R < 2$ *we obtain that*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^z = \frac{e^{-Bz}}{(\log x)^z} \left(1 + o(1)\right)^z.$$

Facts 2.6 (Exponential Integrals and Incomplete Gamma Functions)**.** The following two variants of the *exponential integral function* are defined by [13, §8.19]

$$\begin{aligned} \mathrm{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, \\ E_1(z) &:= \int_1^{\infty} \frac{e^{-tz}}{t} dt, \Re(z) \geq 0, \end{aligned}$$

where $\mathrm{Ei}(-kz) = -E_1(kz)$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $E_1(z)$:

$$1 - \frac{3}{4}z \leq E_1(z) - \gamma - \log z \leq 1 - \frac{3}{4}z + \frac{11}{36}z^2. \tag{4a}$$

A related function is the (upper) *incomplete gamma function* defined by [13, §8.4]

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt, \Re(s) > 0.$$

We have the following properties of $\Gamma(s, x)$:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, \tag{4b}$$

$$\Gamma(s+1, 1) = e^{-1} \left\lfloor \frac{s!}{e} \right\rfloor, s \in \mathbb{Z}^+, \tag{4c}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, |x| \to +\infty. \tag{4d}$$

3 Summing arithmetic functions weighted by $\lambda(n)$

3.1 Discussion: The enumerative DGF result in Theorem 1.6 from Montgomery and Vaughan

What this enumeratively-flavored result of Montgomery and Vaughan allows us to do is get a "good enough" lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. For comparison, we already have known, more classical bounds due to Erdös (or earlier) that state for

$$\pi_k(x) := \#\{n \leq x : \omega(n) = k\},$$

we have tightly that [2, 11]

$$\pi_k(x) = (1 + o(1)) \cdot \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

We seek to approximate the right-hand-side of $\mathcal{G}(z)$ by only taking the products of the primes $p \leq u$, e.g., $p \in \{2, 3, 5, \ldots, u\}$, of which the last element in this set has average order of $\log\left[\frac{u}{\log u}\right]$ for some minimal $u \geq 2$ as $x \to \infty$. We also state the following theorem reproduced from [11, Thm. 7.20] that handles the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \frac{3}{2} \log\log x$. This allows us later to show that taking just $k \in [1, \frac{3}{2} \log\log x]$ and summing over such k in Theorem 1.6 captures the asymptotically relevant, dominant behavior of the values of $\pi_k(x)$ for $k \leq \frac{\log x}{\log 2}$ (where $\Omega(n) \leq \frac{\log n}{\log 2}$ for all $n \geq 2$).

Theorem 3.1 (Bounds on exceptional values of $\Omega(n)$ for large n, MV 7.20)**.** *Let*

$$B(x, r) := \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\}.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x, r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \textit{as } x \to \infty.$$

In particular, we have that for $r \in \left(\frac{3}{2}, 2\right)$,

$$\left|1 - \frac{B(x, r)}{B(x, 3/2)}\right| \xrightarrow{x \to \infty} 1.$$

The proof of Theorem 3.1 is found in the cited reference as Chapter 7 of Montgomery and Vaughan.

3.2 The key new results utilizing Theorem 1.6

Corollary 3.2. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \gg 2.$$

When $s := 1$, we have the known bound in Mertens theorem. For $s > 1$, we obtain that

$$P_s(x) \approx E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1).$$

It follows that

$$\gamma_0(s, x) + o(1) \leq P_s(x) \leq \gamma_1(s, x) + o(1),$$

where it suffices to take

$$\gamma_0(z,x) = -s\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}s(s-1)\log(x/2) - \frac{11}{36}s(s-1)^2\log^2(x)$$
$$\gamma_1(z,x) = -s\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}s(s-1)\log(x/2) + \frac{11}{36}s(s-1)^2\log^2(2).$$

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Proof. Let $s > 1$ be real-valued. By Abel summation where our summatory function is given by $A(x) = \pi(x) \sim \frac{x}{\log x}$ and our function $f(t) = t^{-s}$ so that $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1), |x| \to \infty. \end{aligned}$$

Now using the inequalities in Facts 2.6, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\frac{P_s(x)}{s} \geq -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1)\log(x/2) - \frac{11}{36}(s-1)^2\log^2(x)$$
$$\frac{P_s(x)}{s} \leq -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1)\log(x/2) + \frac{11}{36}(s-1)^2\log^2(2).$$

This completes the proof of the bounds cited above in the statement of this lemma. □

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Proof of Theorem 1.7. We have that for all integers $0 \leq k \leq m$

$$[z^k]\prod_{1\leq i\leq m}(1-f(i)z)^{-1} = [z^k]\exp\left(\sum_{j\geq 1}\left(\sum_{i=1}^m f(i)^j\right)\frac{z^j}{j}\right).$$

In our case we have that $f(i)$ denotes the i^{th} prime. Hence, summing over all $p \leq x$ in place of $0 \leq k \leq m$ in the previous formula applied in tandem with Corollary 3.2, we obtain that the logarithm of the generating function series we are after when we sum over all $p \leq u$ for some $u \geq 2$ corresponds to

$$\begin{aligned} \log\left[\prod_{p\leq u}\left(1-\frac{z}{p}\right)^{-1}\right] &\geq (B+\log\log u)z + \sum_{j\geq 2}\left[a(u)+b(u)(j-1)+c(u)(j-1)^2\right]z^j \\ &= (B+\log\log u)z - a(u)\left(1+\frac{1}{z-1}+z\right) + b(u)\left(1+\frac{2}{z-1}+\frac{1}{(z-1)^2}\right) \\ &\qquad - c(u)\left(1+\frac{4}{z-1}+\frac{5}{(z-1)^2}+\frac{2}{(z-1)^3}\right). \end{aligned}$$

In the previous equations, the lower bounds formed by the functions (a,b,c) are given by the corresponding upper bounds from Corollary 3.2 due to the leading sign on the following expansions:

$$(a_\ell, b_\ell, c_\ell) := \left(-\log\left(\frac{\log u}{\log 2}\right), \frac{3}{4}\log\left(\frac{u}{2}\right), \frac{11}{36}\log^2 2\right).$$

Then we simplify to obtain that

$$\frac{e^{-Bz}}{(\log u)^z} \times \prod_{p \leq u} \left(1 - \frac{z}{p}\right)^{-1} \geq e^{Bz} (\log u)^z \left(\frac{\log u}{\log 2}\right)^{1+\frac{1}{z-1}+z} \left(\frac{u}{2}\right)^{\frac{3}{4}\left(1+\frac{2}{z-1}+\frac{1}{(z-1)^2}\right)} \times \qquad (5)$$
$$\times \exp\left(-\frac{11}{36}\left(1 + \frac{4}{z-1} + \frac{5}{(z-1)^2} + \frac{2}{(z-1)^3}\right)(\log 2)^2\right).$$

Now we need to determine which values of $u \geq 2$ minimize the expression in (5). For this we will use introdutory calculus in the form of the second derivative test with respect to u. Let $f_z(u)$ denote the right-hand-side expression in the last equation. Then

$$f_z'(u_0) = 0 \implies u_0 = e^{-\frac{4(z-1)}{3}}.$$

When we substitute this outstanding parameterized value of u into the second derivative of the same function we obtain

$$f_z''(u_0) = 2^{\frac{8z^3-27z^2+32z-16}{4(z-1)^2}} 3^{-z+\frac{1}{1-z}+1}(1-z)^{z+\frac{1}{z-1}-2} z^2 (\log 2)^{\frac{z^2}{1-z}} \times$$
$$\times \exp\left(B + \frac{60z^4 + z^3\left(11\log^2(2) - 312\right) + z^2\left(540 + 11\log^2(2)\right) - 384z + 96}{36(z-1)^3}\right) > 0,$$

which leads to a minimum value, or lower bound, at this u_0. Now we make a practical decision to set the uniform bound parameter to a middle ground value of $1 < R < 2$ as $R := \frac{3}{2}$ so that

$$z \equiv z(k,x) = \frac{k-1}{\log\log x} \in [0, R),$$

for $x \gg 1$ very large. Thus $(z-1)^{-m} \in [(-1)^m, 2^m]$ for integers $m \geq 1$, and we can then form the lower bound stated above. Since $z \equiv z(k,x) = \frac{k-1}{\log\log x}$ and $k \in [1, R\log\log x]$, we obtain that for small k and $x \gg 1$ large $\Gamma(z+1) \approx 1$, and for k towards the upper range of its interval that $\Gamma(z+1) \approx \Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$. In total, what we get out of these formulas is stated as in the theorem bounds. □

4 Precisely enumerating and bounding the Dirichlet inverse functions, $g^{-1}(n) := (\omega+1)^{-1}(n)$

Conjecture 1.4 is not the most accurate way to express the limiting behavior of the Dirichlet inverse functions $g^{-1}(n)$, though it does capture an important characteristic – namely, that these functions can be expressed via more simple formulas than inspection of the initial sequence properties might otherwise suggest. The key idea is that by having information about $g^{-1}(n)$ in terms of its prime factorization exponents for $n \leq x$, we should be able to extrapolate what we need which is information about the average behavior of the summatory functions, $G^{-1}(x)$, from the proofs above. The following observation that is suggestive of the semi-periodicity at play with the distinct values of $g^{-1}(n)$ distributed over $n \geq 2$.

Heuristic 4.1 (Symmetry in $g^{-1}(n)$ in the exponents in the prime factorization of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly two, three, and so on prime factors. There does not appear to be an easy, nor subtle direct recursion between the distinct g^{-1} values, except through auxiliary function sequences. We will settle for an asymptotically accurate main term approximation to $g^{-1}(n)$ for large n as $n \to \infty$ in the average case.

With all of this in mind, we define the following sequence for integers $n \geq 1, k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{6}$$

We will illustrate by example the first few cases of these functions for small k after we prove the next lemma. The sequence of important semi-diagonals of these functions begins as [17, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \ -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

Lemma 4.2 (An exact formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu(n/d) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$\begin{aligned} g^{-1}(n) &= -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) f^{-1}(n/d) & \implies \\ (g^{-1} * 1)(n) &= -(\omega * g^{-1})(n). \end{aligned}$$

Now by repeatedly expanding the right-hand-side, and removing corner cases in the nested sums since $\omega(1) = 0$ by convention, we find that

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n).$$

The statement follows by Möbius inversion applied to each side of the last equation. $\square$

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Example 4.3 (Special cases of the functions $C_k(n)$ for small k)**.** We cite the following special cases which should be easy enough to see on paper:

$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= \sigma_0(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} - \gcd\left(\Omega(n), \omega(n)\right). \end{aligned}$$

We also can see a recurrence relation between successive $C_k(n)$ values over k of the form

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} C_{k-1}\left(d \cdot p^i\right). \tag{7}$$

Thus we can work out further cases of the $C_k(n)$ for a while until we are able to understand the general trends of its asymptotic behaviors. In particular, we can compute the main term of $C_3(n)$ as follows where we use the notation that p, q are prime indices:

$$\begin{aligned} C_3(n) &\sim \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \sum_{q|dp^i} \frac{\nu_q(dp^i)}{\nu_q(dp^i)+1} \sigma_0(d)(i+1) \\ &= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \left[\sum_{q|d} \frac{\nu_q(d)}{\nu_q(d)+1} \sigma_0(d)(i+1) + \sum_{j=1}^{i} \frac{j}{(j+1)} \sigma_0(d)(i+1)\right] \\ &= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{q|d} \sigma_0(d) \left[\frac{\nu_p(n)(\nu_p(n)+3)}{2} \frac{\nu_q(d)}{\nu_q(d)+1} + \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2)\left(4\nu_p(n)+9-6H_{\nu_p(n)+2}^{(1)}\right)\right]. \end{aligned}$$

We will break the two key component sums into separate calculations. First, we compute that*

$$\begin{aligned} C_{3,1}(n) &= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \frac{\nu_p(n)(\nu_p(n)+3)}{2} \times \sum_{q|d} \frac{\nu_q(d)}{\nu_q(d)+1} \sigma_0(d) \\ &= \sum_{\substack{p,q|n \\ p \neq q}} \sum_{d|\frac{n}{p^{\nu_p(n)} q^{\nu_q(n)}}} \frac{\nu_p(n)(\nu_p(n)+3)}{2} \times \sum_{i=1}^{\nu_q(n)} \frac{\nu_q(dq^i)}{\nu_q(dq^i)+1} \sigma_0(dq^i) \\ &= \sum_{\substack{p,q|n \\ p \neq q}} \sum_{d|\frac{n}{p^{\nu_p(n)} q^{\nu_q(n)}}} \frac{\nu_p(n)(\nu_p(n)+3)\nu_q(n)(\nu_q(n)+3)}{4} \sigma_0(d) \\ &= (\sigma_0 * 1)(n) \times \sum_{\substack{p,q|n \\ p \neq q}} \frac{\nu_p(n)(\nu_p(n)+3)\nu_q(n)(\nu_q(n)+3)}{(\nu_p(n)+1)(\nu_p(n)+2)(\nu_q(n)+1)(\nu_q(n)+2)}. \end{aligned}$$

*Here, the arithmetic function $\sigma_0 * 1$ is multiplicative. Its value at prime powers can be computed as

$$(\sigma_0 * 1)(p^\alpha) = \sum_{i=0}^{\alpha} (i+1) = \frac{(\alpha+1)(\alpha+2)}{2},$$

where $\sigma_0(p^\beta) = \beta + 1$.

Next, we have that

$$\begin{aligned}
C_{3,2}(n) &= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{q|d} \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2)\left(4\nu_p(n)+9-6H_{\nu_p(n)+2}^{(1)}\right)\sigma_0(d) \\
&= \sum_{\substack{p,q|n \\ p\neq q}} \sum_{d|\frac{n}{p^{\nu_p(n)}q^{\nu_q(n)}}} \sum_{i=1}^{\nu_q(n)} \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2)\left(4\nu_p(n)+9-6H_{\nu_p(n)+2}^{(1)}\right)\sigma_0(d)(i+1) \\
&= (\sigma_0 * 1)(n) \times \sum_{\substack{p,q|n \\ p\neq q}} \frac{1}{6} \frac{\nu_q(n)(\nu_q(n)+3)}{(\nu_q(n)+1)(\nu_q(n)+2)}\left(4\nu_p(n)+9-6H_{\nu_p(n)+2}^{(1)}\right).
\end{aligned}$$

Now to roughly bound the error term, e.g., the GCD of prime omega functions from the exact formula for $C_3(n)$, we observe that the divisor function has average order of the form:

$$\mathbb{E}[d(n)] = \log n + (2\gamma - 1) + O\left(\frac{1}{\sqrt{n}}\right).$$

Then using that $\mathbb{E}[\omega(n)], \mathbb{E}[\Omega(n)] = \log\log n$ (except in rare cases when n is primorial, a power of 2, etc. †), as discussed in the next remarks, we bound the error as

$$\begin{aligned}
C_{3,3}(n) &= -\sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \gcd\left(\Omega(d)+i, \omega(d)+1\right) \\
&= \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} O\left(\sigma_0(n) \cdot \log\log n\right) \\
&= O\left(\pi(n) \cdot \log n \cdot \log\log n\right) \\
&= O\left(n \cdot \log\log n\right).
\end{aligned}$$

In total, we obtain that

$$\begin{aligned}
C_3(n) = (\sigma_0 * 1)(n) &\times \sum_{\substack{p,q|n \\ p\neq q}} \frac{1}{6} \frac{\nu_q(n)(\nu_q(n)+3)}{(\nu_q(n)+1)(\nu_q(n)+2)}\left(4\nu_p(n)+9-6H_{\nu_p(n)+2}^{(1)}\right) \\
&+ \sigma_0(n) \times \sum_{\substack{p,q|n \\ q\neq p}} \frac{2^{\nu_q(n)}\nu_p(n)(\nu_p(n)+3)}{4(\nu_p(n)+1)(\nu_q(n)+1)} \\
&+ O\left(n \cdot \log\log n\right).
\end{aligned} \tag{8}$$

For the next cases, we would use similar techniques. The key is to compute enough small cases that we can see the dominant asymptotic terms in these expansions. We will expand more on this below.

Remark 4.4 (Recursive growth of the functions $C_k(n)$ in the average case)**.** We note that the average order of $\mathbb{E}[\Omega(n)] = \log\log n$, so that for large $x \gg 1$ tending to infinity, we can expect (on average) that for $p|n$, $1 \leq \nu_p(n)$ (for large $p|x, p \sim \frac{x}{\log x}$) and $\nu_p(n) \approx \log\log n$.

†In this context, we write $\mathbb{E}[\Omega(n)] = \log\log n$ to denote the *average order* of this arithmetic function – even though its actual values may fluctuate non-uniformly infinitely often, e.g., $\Omega(2^m) = m$ and for primes p we have that $\Omega(p) = 1$, but most of the time the asymptotic relation holds when we sum, or average over all possible $n \leq x$. What this notation corresponds to, or means in practice, is that the average of the summatory function satisfies: $\frac{1}{x} \cdot \sum_{n \leq x} \Omega(n) \sim \log\log x$.

However, if x is primorial, we can have $\mathbb{E}[\Omega(x)] = \frac{\log x}{\log\log x}$. There is, however, a duality with the size of $\Omega(x)$ and the rate of growth of the $\nu_p(x)$ exponents. That is to say that on average, even though $\mathbb{E}[\nu_p(x)] = \log\log n$ for most $p|x$, if $\Omega(x) = m \approx O(1)$ is small, then

$$\nu_p(x) \approx \log_{\sqrt[m]{\frac{x}{\log x}}}(x) = \frac{m \log x}{\log\left(\frac{x}{\log x}\right)}.$$

Since we will be essentially averaging the inverse functions, $g^{-1}(n)$, via their summatory functions over the range $n \leq x$ for x large, we tend not to worry about bounding anything but by the average order case, which wins out when we sum (i.e., average) and tend to infinity. Given these observations, we can use the function $C_3(n)$ we just computed exactly as an asymptotic benchmark to build further approximations. In particular, the dominant order terms in $C_3(n)$ are given by

$$C_3(n) \sim \frac{(\sigma_0 * 1)(n) n^2}{\log^2 n} - \frac{(\sigma_0 * 1)(n) n^2}{\log n} + O\left(n \cdot \log\log n\right).$$

We will leave the terms involving the divisor function $\sigma_0(n)$ and convolutions involving it unevaluated because of how much their growth can fluctuate depending on prime factorizations for now.

Summary 4.5 (Asymptotics of the $C_k(n)$)**.** We have the following asymptotic relations relations for the growth of small cases of the functions $C_k(n)$:

$$\begin{aligned}
C_1(n) &\sim \log\log n \\
C_2(n) &\sim \frac{\sigma_0(n) n}{\log n} + O(\log\log n) \\
C_3(n) &\sim \frac{(\sigma_0 * 1)(n) n^2}{\log^2 n} - \frac{(\sigma_0 * 1)(n) n^2}{\log n} + O\left(n \cdot \log\log n\right).
\end{aligned}$$

Theorem 1.5 proved next makes precise what these formulas suggest about the growth rates of $C_k(n)$.

Proof of Theorem 1.5. We showed how to compute the formulas for the base cases in the preceeding examples discussed above in Example 4.3. We can also see that $C_3(n)$ satsfies the formula we must establish when $k := 3$. Let's proceed by using induction with the recurrence formula from (7) relating $C_k(n)$ to $C_{k-1}(n)$ for all $k \geq 1$. The strategy is to precisely evaluate the sums recursively. The strategy is to drop asymptotically insignificant terms that, indeed make the formulas most precise, but which will along the way contribute negligible weight to our target bounds. What results after performing this procedure is a main term formula that is precise for sufficiently large n as we let $n \to \infty$. We will compute the main term formula first, then complete the proof by bounding the easier big-Oh error term calculations to wrap up our induction.

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Main term formula inductive proof. Suppose that $k \geq 4$. By the recurrence formula for $C_k(n)$, we have that

$$C_k(n) \sum_{p|n} \sum_{d|np^{-\nu_p(n)}} \sum_{i=1}^{\nu_p(n)} -\frac{(dp^i)^{k-1}}{(\log(dp^i))^{k-1}} \binom{i+k-1}{k-1} (\sigma_0 * \tau_{k-2})(d).$$

Now to handle the inner sum, we bound by setting $\alpha \equiv \nu_p(n)$ and invoking *Mathematica* in the form of

$$\mathrm{IC}_k(n) = \sum_{i=1}^{\alpha} -\frac{(dp^i)^{k-1}}{(\log(dp^i))^{k-1}} \binom{i+k-1}{k-1}$$

$$\begin{aligned}
&\approx \int -\frac{(dp^{\alpha})^{k-1}}{(\log(dp^{\alpha}))^{k-1}} \binom{\alpha+k-1}{k-1} \\
&\sim \frac{1}{(k-1)!\log^k p} \left(\mathrm{Ei}((k-2)\log(dp^{\alpha}))\left[\log^{k-1}(d) - (k-1)!\log^{k-1}(p)\right]\right) \\
&\qquad - \frac{1}{(k-2)(k-1)!\log^k p} \left(\log^{k-2}(d) + \alpha^{k-2}\log^{k-2}(p)\right).
\end{aligned}$$

We now simplify somewhat again by setting

$$p \mapsto \left(\frac{n}{e}\right)^{\frac{1}{\log\log n}}, \alpha \mapsto \log\log n, \log p \mapsto \frac{\log n}{\log\log n}.$$

Also, since $p \gg_n d$, we obtain the dominant asymptotic growth terms of

$$\begin{aligned}
\mathrm{IC}_k(n) &\sim \frac{\alpha^{k-2}}{(k-2)(k-1)!\log^2 p} \\
&\approx \frac{(\log\log n)^k}{(k-2)(k-1)!\log^2 n}.
\end{aligned}$$

Now, as we did in the previous example work, we handle the sums by pulling out a factor of the inner divisor sum depending only on n (and k):

$$\begin{aligned}
C_k(n) &= \sum_{p|n} (\sigma_0 * \tau_{k-1})(n) \binom{p^{\nu_p(n)}+k}{k}^{-1} \times \mathrm{IC}_k(n) \\
&= (\sigma_0 * \tau_{k-1})(n) \binom{p^{\nu_p(n)}+k}{k}^{-1} \cdot \pi(n) \times \mathrm{IC}_k(n)
\end{aligned}$$

Combining with the remaining terms we get by induction a proof of our target bounds for $C_k(n)$.

Establishing the error term bound inductively. To bound the error terms, again suppose inductively that $k \geq 4$. We compute the big-O bounds as follows letting $\alpha \equiv \nu_p(n)$:

$$\begin{aligned}
\mathrm{ET}_k(n) &= \sum_{i=1}^{\nu_p(n)} n^{k-2} \cdot \frac{(\log\log n)^{k-2}}{(\log n)^{k-2}} \\
&\approx \int (dp^{\alpha})^{k-2} \log\log(dp^{\alpha}) d\alpha \\
&= -\frac{\mathrm{Ei}((k-2)\log(dp^{\alpha}))}{(k-2)\log p} + \frac{d^{k-2}p^{(k-2)\alpha}}{(k-2)\log p}\log(dp^{\alpha}) \\
&\sim \frac{d^{k-2}p^{(k-2)\alpha}}{(k-2)\log p}\log(dp^{\alpha}).
\end{aligned}$$

In the last expansion, we have dropped the exponential integral terms since they provide at most polynomial powers of the logarithm of their inputs.

To evaluate the outer divisor sum from the recurrence relation for $C_k(n)$, we will require the following bound providing an average order on the *generalized sum-of-divisors functions*, $\sigma_\alpha(n) := \sum_{d|n} d^{\alpha}$. In particular, we have that for integers $\alpha \geq 2$ [13, §27.11]:

$$\mathbb{E}[\sigma_\alpha(n)] = \frac{\zeta(\alpha+1)}{\alpha+1}x^{\alpha} + O(x^{\alpha-1}).$$

Approximating the number of terms in the prime divisor sum by $\pi(x) = \frac{x}{\log x}$, we thus obtain

$$\mathrm{ET}_k(n) \approx \frac{(\log\log n)^{k-1} e^{k-2}}{(k-1)(k-2)} x^{(k-2)\left(1-\frac{1}{\log\log x}\right)+1+\log\log x} \zeta(k-1).$$

So up to what is effectively constant in k, and dropping lower order terms for a slightly suboptimal, but still sufficient for our purposes, error bound formula, we have completed the proof by induction. □

Corollary 4.6 (Asymptotics for a very special case of the functions $C_k(n)$)**.** *For $k \gg 1$ sufficiently large, we have that*

$$C_{\Omega(n)}(n) \sim (\sigma_0 * \tau_{\log\log n - 2})(n) \times \lambda(n) \frac{n^{\log\log n - 1}}{(\log n)^{\log\log -1} \Gamma(\log\log n)}.$$

Moreover, by considering the average orders of the function $\nu_p(n)$ for p large and tending to infinity, we have bounds on the aysmptotic behavior of these functions of the form

$$\lambda(n) \widehat{\tau}_0(n) \precsim C_{\Omega(n)}(n) \precsim \lambda(n) \widehat{\tau}_1(n).$$

It suffices to take the functions

$$\begin{aligned}
\widehat{\tau}_0(n) &:= \frac{1}{\log 2} \cdot \frac{\log n}{(\log n)^{\log\log n}} \cdot \frac{n^{\log\log n - 1}}{\Gamma(\log\log n)} \\
\widehat{\tau}_1(n) &:= \frac{1}{2e\log 2} \cdot \frac{(\log n)^2}{(\log n)^{\log\log n}} \cdot \frac{n^{\log\log n}}{\Gamma(\log\log n)}.
\end{aligned}$$

Proof. The first stated formula follows from Theorem 1.5 by setting $k := \Omega(n) \sim \log\log n$ and simplifying. We evaluate the Dirichlet convolution functions and approximate as follows:

$$\begin{aligned}
(\sigma_0 * \tau_{\log\log n - 2})(n) &= \sum_{p|n} \binom{\nu_p(n) + \log\log n - 1}{\log\log n - 1} \\
&\geq \sum_{p|n} \frac{(\nu_p(n) + \log\log n - 1)^{\log\log n - 1}}{(\log\log n)^{\log\log n - 1}} \\
&\sim \frac{n}{\log 2} \\
(\sigma_0 * \tau_{\log\log n - 2})(n) &\leq \left(\frac{(\nu_p(n) + \log\log n - 1)e}{\log\log n - 1}\right)^{\log\log n - 1} \\
&\sim (2e)^{\log\log n - 1} \\
&= \frac{n \cdot \log n}{2e\log 2}.
\end{aligned}$$

The upper and lower bounds are obtained from the next well known binomial coefficient approximations using Stirling's formula.

$$\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{n^k}{k!} < \left(\frac{ne}{k}\right)^k$$ □

Using Lemma 4.2 directly is problematic since forming the summatory function of the exact $g^{-1}(n)$ that obey this formula leads to a nested recurrence relation involving $M(x)$ –

e.g., more in-order sums of consecutive Möbius function terms yet again. Some suggestive numerical experiments suggest that this implicit recursive dependence of our new formulas for $M(x)$ can be more simply avoided by using an inexact, but still asymptotically sufficient in form expression for $g^{-1}(n)$. The next corollary provides the inexact, but asymptotically accurate formula for these inverse functions we have in mind.

Also, for the same somewhat intangible "niceness" properties many researchers in this area have found in working with bounds for sums of the function $\lambda(n)$ in place of $\mu(n)$, we prefer expressions obtained by summation by parts that divide the summatory functions of g^{-1} into cases of the parity of the completely additive $\Omega(n)$ for $n \leq x$. This preference is according to my heuristic anthropomorphizing the disposition of certain signed arithmetic functions that says such sums are "better behaved" in attitude towards us and happen to readily offer easier bounds that we can establish explicitly.

Corollary 4.7 (Computing the inverse functions)**.** *In contrast to the complicated formulation given by Lemma 4.2, we have that for all $n \geq 2$*

$$g^{-1}(n) \sim \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

Moreover, we can bound the error terms to ensure that

$$\left| \frac{\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)}{g^{-1}(n)} \right| \xrightarrow{n\rightarrow\infty} 0.$$

Proof. Using Lemma 4.2, it suffices to show that the squarefree divisors $d|n$ such that $\operatorname{sgn}(\mu(d)\lambda(n/d)) = -1$ have an order of magnitude smaller magnitude (in the little-o notation sense) than the corresponding cases of positive sign on the terms in the divisor sum from the lemma. This is because the sign of the terms in the Möbius inversion sum from the lemma will have already matched exactly that of the terms in $\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)$ except possibly in these comparitively rare cases when $\operatorname{sgn}(\mu(d)\lambda(n/d)) = -1$. Thus, we need only compute a reasonable bound of when such innacuracies between the exact formula from Lemma 4.2 differs from the approximation we claim works when we take the divisor sum over the unsigned $C_{\Omega(d)}(d)$ terms and weight by an overall factor of $\lambda(n)$. It is obvious that if we can show that the difference (ratio) in these two formulas is asymptotically negligible as we let $n \rightarrow \infty$, then we can use the substantially easier to evaluate approximation to calculate accurate new bounds on the summatory function of $g^{-1}(n)$ later on in the results found below (following the conclusion of this proof).

Initial strategy diverging from Möbius inversion. Let n have m_1 prime factors p_1 such that $v_{p_1}(n) = 1$, m_2 such that $v_{p_2}(n) = 2$, and the remaining $m_3 := \Omega(n) - m_1 - 2m_2$ prime factors of higher-order exponentation. We have a few cases to consider after re-writing the sum from the lemma in the following form:

$$g^{-1}(n) = \lambda(n)C_{\Omega(n)}(n) + \sum_{i=1}^{\omega(n)} \left\{ \sum_{\substack{d|n \\ \omega(d)=\Omega(d)=i \\ \#\{p|d:\nu_p(d)=1\}=k_1 \\ \#\{p|d:\nu_p(d)=2\}=k_2 \\ \#\{p|d:\nu_p(d)\geq 3\}=k_3}} \mu(d)\lambda(n/d)C_{\Omega(n/d)}(n/d) \right\}.$$

We obtain the following cases of the squarefree divisors contributing to the signage on the terms in the above sum:

- The sign of $\mu(d)$ is $(-1)^i = (-1)^{k_1+k_2+k_3}$;
- If $m_3 < \#\{p|n : \nu_p(n) \geq 3\}$, then $\lambda(n/d) = 1$ (since $\mu(n/d) = 0$);
- Given (k_1, k_2, k_3) as above, since $\lambda(n) = (-1)^{\Omega(n)}$, we have that $\mu(d) \cdot \lambda(n/d) = (-1)^{i-k_1-k_2}\lambda(n)$.

Thus we define the following sums, parameterized in the $(m_1, m_2, m_3; n)$, which corresponds to a change in expected parity transitioning from the Möbius inversion sum from Lemma 4.2 to the sum approximating $g^{-1}(n)$ defined at the start of this result:

$$\widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) := \sum_{i=1}^{\omega(n)/2} \sum_{k_1=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{i}{2} \rfloor - k_1} \left[\binom{m_1}{2k_1+1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1}\binom{2m_2}{2k_2}\right] [i - k_1 - k_2 = k_3 \equiv m_3]_\delta$$

$$\widetilde{S}_{\mathrm{even}}(m_1, m_2, m_3; n) := \sum_{i=1}^{\omega(n)/2} \sum_{k_1=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{i}{2} \rfloor - k_1} \left[\binom{m_1}{2k_1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1+1}\binom{2m_2}{2k_2}\right] [i - k_1 - k_2 = k_3 \equiv m_3]_\delta \,.$$

Part I (Lower bounds on the inner sums of the count functions). We claim that

$$\begin{aligned} \widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) &\gg \binom{m_1}{i+1} + \binom{m_1}{\frac{i}{2}}\binom{2m_2-1}{\frac{i}{2}+1} \\ \widetilde{S}_{\mathrm{even}}(m_1, m_2, m_3; n) &\gg \binom{m_1}{i+1} + \binom{m_1}{\frac{i}{2}-1}\binom{2m_2}{\frac{i}{2}+1}. \end{aligned} \tag{9}$$

To prove (9) we have to provide a straightforward bound that represents the maximums of the terms in m_1, m_2. In particular, observe that for

$$\begin{aligned} \widetilde{S}_{\mathrm{odd}}(m_1, m_2; u) &= \sum_{k_1=0}^{u} \sum_{k_2=0}^{u-k_1} \left[\binom{m_1}{2k_1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1+1}\binom{2m_2}{2k_2}\right] \\ \widetilde{S}_{\mathrm{even}}(m_1, m_2; u) &= \sum_{k_1=0}^{u} \sum_{k_2=0}^{u-k_1} \left[\binom{m_1}{2k_1+1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1}\binom{2m_2}{2k_2}\right], \end{aligned}$$

we have that

$$\begin{aligned} \widetilde{S}_{\mathrm{odd}}(m_1, m_2; u) &\gtrsim \binom{m_1}{2u+1} + \max_{1 \leq k_1 \leq u} \binom{m_1}{2k_1+1}\binom{2m_2}{2u+1-2k_1} \\ &= \binom{m_1}{2u+1} + \binom{m_1}{2k_1+1}\binom{2m_2}{2u+1-2k_1}\Bigg|_{k_1=\frac{u}{2}} \\ &= \binom{m_1}{2u+1} + \binom{m_1}{u+1}\binom{2m_2}{u+1} \\ \widetilde{S}_{\mathrm{even}}(m_1, m_2; u) &\gtrsim \binom{m_1}{2u+1} + \max_{1 \leq k_1 \leq u} \binom{m_1}{2k_1}\binom{2m_2}{2u+1-2k_1} \\ &= \binom{m_1}{2u+1} + \binom{m_1}{u-1}\binom{2m_2}{u+1}. \end{aligned}$$

The lower bounds in (9) then follow by setting $u \equiv \lfloor \frac{i}{2} \rfloor$.

Part II (Bounding m_1, m_2, m_3 and effective (i, k_1, k_2) contributing to the count). We thus have to determine the asymptotic growth rate of $\widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) + \widetilde{S}_{\mathrm{even}}(m_1, m_2, m_3; n)$, and show that it is of comparatively small order. First, we bound the count of non-zero

m_3 for $n \leq x$ from below. For the cases where we expect differences in signage, it's the last Iverson convention term that kills the order of growth, e.g., we expect differences when the parameter m_3 is larger than the usual configuration. We know that

$$\pi_k(x) \sim \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

Using the formula for $\pi_k(x)$, we can count the average orders of m_1, m_2 as

$$N_{m_1}(x) \approx \frac{1}{x}\#\{n \leq x : \omega(n) = 1\} \sim \frac{\log\log x}{\log x}$$
$$N_{m_2}(x) \approx \frac{1}{x}\#\{n \leq x : \omega(n) = 2\} \sim \frac{(\log\log x)^2}{\log x}.$$

Additionally, in Corollary 1.7 on page 11 we will prove a lower bound on $\widehat{\pi}_k(x)$. We use this result immediately below without proof. A09-2020.04-26

When we have parameters with respect to some $n \geq 1$ such that $m_3 > 0$, it must be the case that

$$\Omega(n) - \omega(n) > \begin{cases} 0, & \text{if } \omega(n) \geq 2; \\ 1, & \text{if } \omega(n) = 1. \end{cases}$$

To count the number of cases $n \leq x$ where this happens, we form the sums

$$\begin{aligned} N_{m_3}(x) &\gg \pi_1(x) \times \sum_{k=3}^{\frac{3}{2}\log\log x} \widehat{\pi}_k(x) + \sum_{k=2}^{\frac{3}{2}\log\log x} \sum_{j=k+1}^{\frac{3}{2}\log\log x} \pi_k(x)\widehat{\pi}_j(x) \\ &\gtrsim \frac{4A_0x^2}{3\sqrt{\pi}\log^{3/2}(2)} \cdot \frac{(\log x)^{-\frac{1}{2}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}}{\sqrt{\log\log x}} - \frac{\sqrt{2}A_0x^2}{\pi\log^{3/2}(2)} \cdot \frac{(\log x)^{1+3\log 2-3\log 3}}{\log\log x} \\ &\gtrsim \frac{4A_0x^2}{3\sqrt{\pi}\log^{3/2}(2)} \cdot \frac{(\log x)^{-\frac{1}{2}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}}{\sqrt{\log\log x}} \\ &= \frac{x^2}{36 \cdot 2^{3/4}\sqrt{\pi}\log 2} \cdot \frac{(\log x)^{-\frac{1}{2}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}}{\sqrt{\log\log x}}. \end{aligned}$$

Now in practice, we are not summing up $n \leq x$, but rather $n \leq \log\log x$. So the above function evaluates to

$$N_{m_3}(\log\log x) \gg \frac{(\log\log x)^2}{36 \cdot 2^{3/4}\sqrt{\pi}\log 2} \cdot \frac{(\log\log\log x)^{-\frac{1}{2}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}}{\sqrt{\log\log\log\log x}}.$$

Next, we go about solving the subproblem of finding when $i - k_1 - k_2 = m_3$. Clearly, since $k_1, k_2 \geq 0$, we have that $i \leq N_{m_3}(\log\log x)$. Moreover, since $2 \leq k_1 + k_2 \leq i/2$, when x is large, we actually obtain a number of soultions less than the order of

$$\mathcal{S}_0(x) := \frac{N_{m_3}(\log\log x)^2}{2} \gtrsim \frac{(\log\log x)^4}{2592 \cdot 2^{3/2} \cdot \pi\log^2 2} \cdot \frac{(\log\log\log x)^{\frac{3}{\log 2}-1-\frac{3}{\log 3}}}{\log\log\log\log x}.$$

Part III (Putting it all together). Using the binomial coefficient inequality

$$\binom{n}{k} \geq \frac{n^k}{k^k},$$

we can work out carefully on paper using (9) that

$$\widetilde{S}_{\rm odd}(m_1, m_2, m_3; x) \gtrsim \mathcal{S}_0(x)\left(\frac{\log\log\log x}{2\log x}\right)^{\frac{2\log\log x}{\log\log\log x}+1}\left[1 + \frac{(\log\log\log x)^2}{\log^2 x}(4\log\log x \cdot \log\log\log x)^{\frac{\log\log x}{2\log\log\log x}+1}\right]$$

$$\widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; x) \gtrsim \mathcal{S}_0(x) \left(\frac{\log\log\log x}{2\log x}\right)^{\frac{2\log\log x}{\log\log\log x}+1} \left[1 + \left(\frac{\log x}{2\log\log\log x}\right)(8\log\log x)^{\frac{\log\log x}{\log\log\log x}+1}\right].$$

Thus, citing Corollary 4.6, we easily see the conclusion of this result that we stated above. In other words, the divisor sum in the corollary statement accuarately approximates the main term and sign of $g^{-1}(n)$ as $n \to \infty$. □

Corollary 4.8. *We have that for sufficiently large x, as $x \to \infty$ that*

$$G^{-1}(x) \gtrsim \widehat{L}_0(\log\log x) \times \sum_{n \leq \log\log x} \lambda(n) \cdot C_{\Omega(n)}(n),$$

where the function

$$\widehat{L}_0(\log\log x) := (-1)^{\left\lfloor \frac{3}{2}\log\log\log\log x \right\rfloor} \left\{\frac{2A_0}{\sqrt{\pi}\log^{\frac{3}{2}}(2)}\right\} \cdot (\log\log x)(\log\log\log x)^{\frac{1}{2}+\frac{3}{2\log 2}-\frac{3}{2\log 3}} \sqrt{\log\log\log\log x},$$

with the exponent $\frac{1}{2} + \frac{3}{2\log 2} - \frac{3}{2\log 3} \approx 1.29868$.

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Proof. Using Corollary 4.7, we have that

$$\begin{aligned} G^{-1}(x) &\approx \sum_{n \leq x} \lambda(n) \cdot (g^{-1} * 1)(n) \\ &= \sum_{d \leq \log\log x} C_{\Omega(d)}(d) \times \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn). \end{aligned}$$

Now we see that by complete additivity (multiplicativity) of $\Omega(n)$ (as indicated by the sign of $\lambda(n)$) that

$$\sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn) = \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(d)\lambda(n) = \lambda(d) \sum_{n \leq \left\lfloor \frac{x}{d} \right\rfloor} \lambda(n).$$

Borrowing a result from the next sections (proved in Section 3), we can establish that

$$\sum_{n \leq x} \lambda(n) \gg \sum_{n \leq \frac{3}{2}\log\log x} (-1)^k \cdot \widehat{\pi}_k(x) =: \widehat{L}_0(x).$$

Then since for large enough x and $d \leq x$,

$$\log(x/d) \sim \log x, \log\log(x/d) \sim \log\log x,$$

we can obtain the stated result, e.g., so that $\widehat{L}_0(x) \sim \widehat{L}_0(x/d)$ for large $x \to \infty$. □

5 Key applications: Establishing lower bounds for $M(x)$ by cases along infinite subsequences

5.1 The culmination of what we have done so far

As noted before in the previous subsections, we cannot hope to evaluate functions weighted by $\lambda(n)$ except for on average using Abel summation. For this task, we need to know the bounds on $\widehat{\pi}_k(x)$ we developed in the proof of Corollary 1.7. A summation by parts argument shows that* †

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k)+1) \\ &\approx G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k)\cdot \frac{x}{k^2\log(x/k)} \\ &\approx G^{-1}(x) - x\cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2\cdot\log(x/t)}dt. \end{aligned} \tag{10}$$

The result proved in Lemma 5.1 is key to justifying the asymptotics obtained next in Theorem 5.2.

Lemma 5.1. *Suppose that $f_k(n)$ is a sequence of arithmetic functions such that $f_k(n)>0$ for all $n\geq 1$, $f_0(n)=\delta_{n,1}$, and $f_{\Omega(n)}(n) \gtrsim \widehat{\tau}_\ell(n)$ as $n\to\infty$ where $\widehat{\tau}_\ell(t)$ is a continuously differentiable function of t for all large enough $t\gg 1$. We define the λ-sign-scaled summatory function of f as follows:*

$$F_\lambda(x) := \sum_{\substack{n\leq x \\ \Omega(n)\leq x}} \lambda(n)\cdot f_{\Omega(n)}(n).$$

Let

$$A_\Omega^{(\ell)}(t) := \sum_{k=1}^{\left\lfloor \frac{3}{2}\log\log t\right\rfloor} (-1)^k \widehat{\pi}_k(t).$$

Then we have that

$$F_\lambda(\log\log x) \gtrsim A_\Omega^{(\ell)}(\log\log x)\widehat{\tau}_\ell(\log\log x) - \int_1^{\log\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_\ell'(t)dt.$$

Proof. The formula for $F_\lambda(x)$ is valid by Abel summation provided that

$$\left|\frac{\sum\limits_{\frac{3}{2}\log\log t<k\leq\frac{\log t}{\log 2}} (-1)^k\widehat{\pi}_k(t)}{A_\Omega^{(\ell)}(t)}\right| = o(1),$$

*Here, we drop the unnecessary floored integer-valued arguments to $\pi(x)$ in place of its approximation by $\pi(x)\sim\frac{x}{\log x}$. In fact, since we can always bound

$$\frac{Ax}{\log x}\leq \pi(x)\leq \frac{Bx}{\log x},$$

for suitably defined absolute constants, $A,B>0$, we are not losing any precision asymptotically by making this small leap in approximation from exact summation (in the first formula) to the integral formula representing convolution (in the second formula below).

†Since $\pi(1)=0$, the actual range of summation corresponds to $k\in\left[1,\frac{x}{2}\right]$.

e.g., the asymptotically dominant terms indicating the parity of $\lambda(n)$ are encompassed by the terms summed by $A_{\Omega}^{(\ell)}(t)$ for sufficiently large t as $t \to \infty$. Using the arguments in Montgomery and Vaughan [11, §7; Thm. 7.20] (see Theorem 3.1), we can see that uniformly in x

$$\left|\frac{\sum_{k\leq x} \pi_k(x)}{B\left(x, \frac{3}{2}\right)}\right| \sim 1, \tag{11}$$

as $x \to \infty$ where $B(x, r)$ is defined as in the cited theorem re-stated on page 19 from the reference. Thus we have captured the asymptotically dominant main order terms in our formula as $x \to \infty$. $\square$

To simplify notation, for integers $m \geq 1$, let the *iterated logarithm function* (not to be confused with powers of $\log x$) be defined for $x > 0$ by

$$\log_*^m(x) := \begin{cases} x, & \text{if } m = 0; \\ \log x, & \text{if } m = 1; \\ \log\left(\log_*^{m-1}(x)\right), & \text{if } m \geq 2. \end{cases}$$

So $\log_*^2(x) = \log\log x$, $\log_*^3(x) = \log\log\log x$, $\log_*^4(x) = \log\log\log\log x$, $\log_*^5(x) = \log\log\log\log\log x$, and so on. This notation will come in handy to abbreviate the dominant asymptotic terms we find in Theorem 5.2 proved below.

We use the result of Corollary 4.6 and Corollary 1.7 to prove the following central theorem:

Theorem 5.2 (Asymptotics and bounds for the summatory functions $G^{-1}(x)$)**.** *We define the lower summatory function, $G_{\ell}^{-1}(x)$, to provide bounds on the magnitude of $G^{-1}(x)$:*

$$|G_{\ell}^{-1}(x)| \ll |G^{-1}(x)|,$$

for all sufficiently large $x \gg 1$. We have the next asymptotic approximations for the lower summatory function where $C_{\ell,1}$ is the absolute constant defined by

$$C_{\ell,1} = \frac{4A_0^2}{\sqrt{3}\pi^2\log^4(2)} = \frac{1}{36 \cdot 2^{3/4}\sqrt{\pi}\cdot\log 2} \approx 0.000183209.$$

That is, we have

$$\left|G_{\ell}^{-1}(x)\right| \gtrsim \left|C_{\ell,1}\cdot(\log x)(\log\log x)(\log\log\log x)^{4+\frac{3}{\log 2}-\frac{3}{\log 3}}\right|.$$

The exponent in the previous equation is numerically approximated as $4 + \frac{3}{\log 2} - \frac{3}{\log 3} \approx 5.59737$.

Proof Sketch: Logarithmic scaling to the accurate order of the inverse functions. For the sums given by

$$S_{g^{-1}}(x) := \sum_{n\leq x} \lambda(n)\cdot C_{\Omega(n)}(n),$$

we notice that using the asymptotic bounds (rather than the exact formulas) for the functions $C_{\Omega(n)}(n)$, we have over-summed by quite a bit. In particular, following from the intent behind the constructions in the last sections, we are really summing only over all

$n \leq x$ with $\Omega(n) \leq x$. Since $\Omega(n) \leq \lfloor \log_2 n \rfloor = \left\lfloor \frac{\log n}{\log 2} \right\rfloor$, many of the terms in the previous equation are actually zero (recall that $C_0(n) = \delta_{n,1}$). So we are actually only summing to the average order of $\mathbb{E}[\Omega(n)] = \log\log n$ in practice, or to the slightly larger bound if the leading sign term on $G_\ell^{-1}(x)$ is negative. Hence, the sum (in general) that we are really interested in bounding is bounded below in magnitude by $S_{g^{-1}}(\log\log x)$ or $S_{g^{-1}}(\log_2(x))$, where we can now safely apply the asymptotic formulas for the $C_k(n)$ functions from Corollary 4.6 that hold once we have verified these constraints. $\square$

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Proof. Recall from our proof of Corollary 1.7 that a lower bound is given by

$$\widehat{\pi}_k(x) \gtrsim \frac{A_0 x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(\frac{4}{3\log 2}\right)^{\frac{k-1}{\log\log x}}.$$

Thus we can form a lower summatory function indicating the parity of all $\Omega(n)$ for $n \leq x$ as

$$\begin{aligned} A_\Omega^{(\ell)}(t) &= \sum_{k \leq \frac{3}{2}\log\log t} (-1)^k \widehat{\pi}_k(x) \\ &\sim (-1)^{\left\lfloor \frac{3\log\log t}{2} \right\rfloor} \cdot \frac{2A_0 t}{\sqrt{\pi}\log^{3/2}(2)} (\log t)^{\frac{1}{2}+\frac{3}{2\log 2}-\frac{3}{2\log 3}} \sqrt{\log\log\log t}. \end{aligned} \tag{12}$$

From Corollary 4.6 we recover from the approximation to the *polygamma function*, $\psi^{(0)}(x) \sim \log x$, that

$$\begin{aligned} \widehat{\tau}_0'(t) &= \frac{d}{dx}\left[\frac{1}{\log 2 \cdot \Gamma(\log\log t)} \frac{t^{\log\log t - 1}}{(\log t)^{\log\log t - 1}}\right] \\ &\gtrsim \frac{\log\log t}{\sqrt{2\pi}\log 2 \cdot \Gamma(\log\log t)} \cdot \frac{t^{\log\log t - 2}}{(\log t)^{\log\log t - 1}}. \end{aligned}$$

Next, as in Lemma 5.1, we apply Abel summation to obtain that

$$G_\ell^{-1}(x) = \widehat{\tau}_0(\log\log x) A_\Omega^{(\ell)}(\log\log x) - \widehat{\tau}_0(u_0) A_\Omega^{(\ell)}(u_0) - \int_{u_0}^{\log\log x} \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t) dt. \tag{13}$$

The integration term in (13) is summed approximately as follows:

$$\begin{aligned} \int_{u_0-1}^{\log\log x} \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t) dt &\sim \sum_{k=u_0+1}^{\frac{3}{4}\log\log\log\log x} \left(I_\ell\left(e^{e^{\frac{4k+2}{3}}}\right) - I_\ell\left(e^{e^{\frac{4k}{3}}}\right)\right) \frac{e^{e^{\frac{4k}{3}}}}{\Gamma\left(\frac{4k}{3}\right)} \\ &\approx C_0(u_0) + (-1)^{\left\lfloor \frac{\log\log\log\log x}{2} \right\rfloor} \times \int_{\frac{3\log\log\log\log x}{4}-\frac{1}{2}}^{\frac{3\log\log\log\log x}{4}} \frac{I_\ell\left(e^{e^{\frac{4k}{3}}}\right)}{\Gamma\left(\frac{4k}{3}\right)} e^{e^{\frac{4k}{3}}} dk, \end{aligned} \tag{14}$$

where we define the integrand function, $I_\ell(t) := \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t)$, with some limiting simplifications for $k \in \left[\frac{3\log\log\log\log x}{4} - \frac{1}{2}, \frac{3\log\log\log\log x}{4}\right]$ as ‡

$$I_\ell\left(e^{e^{\frac{4k}{3}}}\right) \frac{e^{e^{\frac{4k}{3}}}}{\Gamma\left(\frac{4k}{3}\right)} \gtrsim \sqrt{\frac{2}{3}} \cdot \frac{8A_0}{3\pi\log^{5/2}(2)} \cdot \frac{(\log\log x)^{\frac{4k}{3}}}{\Gamma\left(\frac{4k}{3}+1\right)} (\log\log\log x)^2. \tag{15}$$

‡Here, we have used that for sufficiently large x,

$$(\log\log\log x)^2 \precsim \exp\left(-(\log\log\log\log x)^2\right).$$

So using the lower bound on the integrand in (15), and applying Stirling's formula, we find that

$$\int_{\frac{3\log\log\log\log x}{4}-\frac{1}{2}}^{\frac{3\log\log\log\log x}{4}} I_\ell\left(e^{e^{\frac{4k}{3}}}\right)\frac{e^{e^{\frac{4k}{3}}}}{\Gamma\left(\frac{4k}{3}+1\right)}dk \tag{16}$$
$$\gtrsim \frac{2A_0}{\sqrt{3}\pi^{3/2}\log^{5/2}(2)}\cdot\left(\frac{\log\log x}{\log\log\log\log x}\right)^{\log\log\log\log x}\frac{(\log\log\log x)^{\frac{7}{2}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}}{\sqrt{\log\log\log\log x}}.$$

Finally, we need to expand the core components of the leading terms in (13) as

$$\widehat{\tau}_0(\log\log x) \gtrsim \frac{\sqrt{\log\log\log\log x}}{2\pi\cdot\log 2}\left(\frac{\log\log\log x}{\log\log x}\right)^2\left(\frac{\log\log x}{\log\log\log x\cdot\log\log\log\log x}\right)^{\log\log\log\log x}$$
$$A_\Omega^{(\ell)}(\log\log x) \gtrsim (-1)^{\left\lfloor\frac{3\log\log\log\log x}{2}\right\rfloor}\cdot\frac{2A_0(\log\log x)}{\sqrt{\pi}\log^{3/2}(2)}(\log\log\log x)^{\frac{1}{2}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}\sqrt{\log\log\log\log\log x}.$$

These last formulas imply the forms of the stated bounds when we drop the lower-order constant term and multiply through by the bounds for the function $\widehat{L}_0(\log\log x)$ given by

$$\widehat{L}_0(\log\log x) := (-1)^{\left\lfloor\frac{3}{2}\log\log\log\log x\right\rfloor}\left\{\frac{2A_0}{\sqrt{\pi}\log^{\frac{3}{2}}(2)}\right\}(\log\log x)(\log\log\log x)^{\frac{1}{2}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}\sqrt{\log\log\log\log x},$$

as proved in Corollary 4.8. Moreover, it is clear that the asymptotically dominant behavior of the lower bound for $|G_\ell^{-1}(x)|$ comes from the integral term calculated in (16). We simplify the term involved in that bound by writing

$$\left(\frac{\log\log x}{\log\log\log\log x}\right)^{\log\log\log\log x} = (\log\log\log x)^{\frac{\log\log x}{\log\log\log\log x}} = \exp\left(\log\log x\right) = \log x.$$

Thus our otherwise large-order looking bounds obtained above actually just constribute a masked logarithmically growing factor to the main term of our bound. □

5.2 Lower bounds on the scaled Mertens function along an infinite subsequence

Proof of Corollary 1.8. It suffices to take $u_0 = e^{e^{e^e}}$. Now, we break up the integral over $t\in[u_0, x/2]$ into two pieces: one that is easily bounded from $u_0 \leq t \leq \sqrt{x}$, and then another that will conveniently give us our logarithmically slow-growing tendency towards infinity along the subsequence.

First, since $\pi(j) = \pi(\sqrt{x})$ for all $\sqrt{x} \leq j < x$, we can take the first chunk of the interval of integration and bound it using (10) as

$$-\int_{u_0}^{\sqrt{x}} \frac{2\sqrt{x}}{t^2\log(x)}G_\ell^{-1}(t)dt \gtrsim B_{\ell,2}\times\frac{2\sqrt{x}}{\log(x)}\cdot\left(\max_{u_0\leq t\leq\sqrt{x}} G_\ell^{-1}(t)\right)$$
$$= o(\sqrt{x}),$$

where $B_{\ell,2}$ can be taken as an indefinite, but still absolute constant with respect to u_0. The maximum in the previous equation is clearly attained by taking $t := \sqrt{x}$. The bound follows, and will be good enough to dispense with this term as of the negligible limiting form of $o(1)$ when we scale the function as $|M(x)|/\sqrt{x}$.

Next, we have to prove a related bound on the second portion of the interval from $\sqrt{x} \leq t \leq x/2$:

$$-\int_{\sqrt{x}}^{x/2} \frac{2x}{t^2 \log(x)} \cdot G_{\ell}^{-1}(t)dt \gtrsim \frac{2}{\log x} \cdot \left(\max_{\sqrt{x} \leq t \leq x/2} G_{\ell}^{-1}(t) \right)$$
$$= 2C_{\ell,1} \cdot (\log\log\sqrt{x})(\log\log\log\sqrt{x})^{4+\frac{3}{\log 2}-\frac{3}{\log 3}} + o(1).$$

Finally, since $G_{\ell}^{-1}(x) = o(\sqrt{x})$, we obtain in total that as $x \to \infty$ along this infinite subsequence:

$$\frac{|M(x)|}{\sqrt{x}} \gtrsim 2C_{\ell,1} \cdot (\log\log\sqrt{x})(\log\log\log\sqrt{x})^{4+\frac{3}{\log 2}-\frac{3}{\log 3}} + o(1),$$

The above expression tends to $+\infty$ as $x \to \infty$, however, only extremly slowly and along the defined infinite subsequence of asymptotically very large x. $\square$

6 Conclusions

6.1 Summary

- Using average order bounds, summatory functions, and the $\precsim$-type relations for lower bounds.
- Somewhat oddly, we did not need substantially improved bounds on $L_0(x) := \sum_{n\leq x} \lambda(n)$ than what is already known in upper bound form to obtain our new bounds on the Mertens function, aka, summatory function of the "testier" Möbius function.

6.2 Future research and work that still needs to be done

- Refinements of these bounds to find the tightest possible lower (limit supremum) bounds, e.g., proofs of an optimal version of Gonek's original conjecture.
- Generalizations to weighted Mertens functions of the form $M_\alpha(x) := \sum_{n\leq x} \mu(n)n^{-\alpha}$.
- Indications of sign changes and exceptionally small, or zero values of $M(x)$.
- What our more combinatorial approach to bounding $M(x)$ effectively suggests about necessary, but unproved, zeta zero bounds that have historically formed the basis for arguments bounding $M(x)$ using Mellin inversion.
- Evaluate alternate strategies and approaches using different Dirichlet convolution functions besides g and $g^{-1}(n)$ (corresponding to $\pi(x)$) with Theorem 1.2.

6.3 Motivating a general technique towards bounding the summatory functions of arbitrary arithmetic f

6.4 The general construction using Theorem 1.2

6.4.1 A proposed generalization

For each $n \geq 1$, let $A(n) \subseteq \{d : 1 \leq d \leq n, d|n\}$ be a subset of the divisors of n. We say that a natural number $n \geq 1$ is A-*primitive* if $A(n) = \{1, n\}$. Under a list of assumptions so that the resulting A-convolutions are *regular convolutions*, we get a generalized multiplicative Möbius function [15, §2.2]:

$$\mu_A(p^\alpha) = \begin{cases} 1, & \alpha = 0; \\ -1, & p^\alpha > 1 \text{ is } A\text{-primitive}; \\ 0, & \text{otherwise}. \end{cases}$$

We also define the functions $\omega_A(n) := \#\{d|n : \mathrm{d\ is\ an\ A-primitive\ factor\ of\ n}\}$ and $\Omega_A(n) := \#\{p^\alpha||n : \mathrm{p\ is\ an\ A-primitive\ factor\ of\ n}\}$. Then the characteristic function of the set $A := \cup_{n\geq 1} A(n)$ is given by $\chi_A(n) = [n \in A]_\delta$. By Möbius inversion, we have that $\chi_A = \omega_A *_A \mu_A$. Moreover, for the A-counting function, $\pi_A(x)$, defined by

$$\pi_A(x) := \#\{n \leq x : n \in A\},$$

we can define a corresponding notion of a generalized A-Mertens function, $M_A(x) := \sum_{n \leq x} \mu_A(n)$. This function then satisfies (by Theorem 1.2) the relation that

$$M_A(x) = \sum_{k=1}^{x} (\omega_A + 1)^{-1}(k) \cdot \pi_A(x/k),$$

where the inverse function, $(\omega_A + 1)^{-1}(n)$, is defined with respect to A-convolution. We conjecture, but do not prove here, that $\operatorname{sgn}((\omega_A + 1)^{-1}(n)) = \lambda_A(n) =: (-1)^{\Omega_A(n)}$.

Using formulas similar in construction to (10), we can differentiate to find expressions for $\pi_A(x)$. The significance of this is that provided we can prove sufficiently large bounds for $M_A(x)$ along the same lines as we have done for $M(x)$, the resulting formula may be able to speak towards the density, or even infinitude in special cases, of the set A.

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A Appendix: Supplementary tables and data

T.1 Table: Computations with a highly signed Dirichlet inverse function

n	**Primes**	–	**Sqfree**	**PPower**	$\bar{\mathbb{S}}$	–	$g^{-1}(n)$	$\lambda(n)\,\mathrm{sgn}(g^{-1}(n))$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_2(n)$	–	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1^1	–	Y	N	N	–	1	1	0	0	–	1	1	0
2	2^1	–	Y	Y	N	–	−2	1	0	0	–	−1	1	−2
3	3^1	–	Y	Y	N	–	−2	1	0	0	–	−3	1	−4
4	2^2	–	N	Y	N	–	2	1	0	−1	–	−1	3	−4
5	5^1	–	Y	Y	N	–	−2	1	0	0	–	−3	3	−6
6	$2^1 3^1$	–	Y	N	N	–	5	1	0	−1	–	2	8	−6
7	7^1	–	Y	Y	N	–	−2	1	0	0	–	0	8	−8
8	2^3	–	N	Y	N	–	−2	1	0	−2	–	−2	8	−10
9	3^2	–	N	Y	N	–	2	1	0	−1	–	0	10	−10
10	$2^1 5^1$	–	Y	N	N	–	5	1	0	−1	–	5	15	−10
11	11^1	–	Y	Y	N	–	−2	1	0	0	–	3	15	−12
12	$2^2 3^1$	–	N	N	Y	–	−7	1	2	−2	–	−4	15	−19
13	13^1	–	Y	Y	N	–	−2	1	0	0	–	−6	15	−21
14	$2^1 7^1$	–	Y	N	N	–	5	1	0	−1	–	−1	20	−21
15	$3^1 5^1$	–	Y	N	N	–	5	1	0	−1	–	4	25	−21
16	2^4	–	N	Y	N	–	2	1	0	−3	–	6	27	−21
17	17^1	–	Y	Y	N	–	−2	1	0	0	–	4	27	−23
18	$2^1 3^2$	–	N	N	Y	–	−7	1	2	−2	–	−3	27	−30
19	19^1	–	Y	Y	N	–	−2	1	0	0	–	−5	27	−32
20	$2^2 5^1$	–	N	N	Y	–	−7	1	2	−2	–	−12	27	−39
21	$3^1 7^1$	–	Y	N	N	–	5	1	0	−1	–	−7	32	−39
22	$2^1 11^1$	–	Y	N	N	–	5	1	0	−1	–	−2	37	−39
23	23^1	–	Y	Y	N	–	−2	1	0	0	–	−4	37	−41
24	$2^3 3^1$	–	N	N	Y	–	9	1	4	−3	–	5	46	−41
25	5^2	–	N	Y	N	–	2	1	0	−1	–	7	48	−41
26	$2^1 13^1$	–	Y	N	N	–	5	1	0	−1	–	12	53	−41
27	3^3	–	N	Y	N	–	−2	1	0	−2	–	10	53	−43
28	$2^2 7^1$	–	N	N	Y	–	−7	1	2	−2	–	3	53	−50
29	29^1	–	Y	Y	N	–	−2	1	0	0	–	1	53	−52
30	$2^1 3^1 5^1$	–	Y	N	N	–	−16	1	0	−4	–	−15	53	−68
31	31^1	–	Y	Y	N	–	−2	1	0	0	–	−17	53	−70
32	2^5	–	N	Y	N	–	−2	1	0	−4	–	−19	53	−72
33	$3^1 11^1$	–	Y	N	N	–	5	1	0	−1	–	−14	58	−72
34	$2^1 17^1$	–	Y	N	N	–	5	1	0	−1	–	−9	63	−72
35	$5^1 7^1$	–	Y	N	N	–	5	1	0	−1	–	−4	68	−72
36	$2^2 3^2$	–	N	N	Y	–	14	1	9	1	–	10	82	−72
37	37^1	–	Y	Y	N	–	−2	1	0	0	–	8	82	−74
38	$2^1 19^1$	–	Y	N	N	–	5	1	0	−1	–	13	87	−74
39	$3^1 13^1$	–	Y	N	N	–	5	1	0	−1	–	18	92	−74
40	$2^3 5^1$	–	N	N	Y	–	9	1	4	−3	–	27	101	−74
41	41^1	–	Y	Y	N	–	−2	1	0	0	–	25	101	−76
42	$2^1 3^1 7^1$	–	Y	N	N	–	−16	1	0	−4	–	9	101	−92
43	43^1	–	Y	Y	N	–	−2	1	0	0	–	7	101	−94
44	$2^2 11^1$	–	N	N	Y	–	−7	1	2	−2	–	0	101	−101
45	$3^2 5^1$	–	N	N	Y	–	−7	1	2	−2	–	−7	101	−108
46	$2^1 23^1$	–	Y	N	N	–	5	1	0	−1	–	−2	106	−108
47	47^1	–	Y	Y	N	–	−2	1	0	0	–	−4	106	−110
48	$2^4 3^1$	–	N	N	Y	–	−11	1	6	−4	–	−15	106	−121

Table T.1: ***Computations of*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for small*** $\mathbf{1 \leq n \leq 48}$.
The column labeled `Primes` *provides the prime factorization of each* n *so that the values of* $\omega(n)$ *and* $\Omega(n)$ *are easily extracted. The columns labeled, respectively,* `Sqfree`, `PPower` *and* $\bar{\mathbb{S}}$ *list inclusion of* n *in the sets of squarefree integers, prime powers, and the set* $\bar{\mathbb{S}}$ *that denotes the positive integers* n *which are neither squarefree nor prime powers. The next two columns provide the explicit values of the inverse function* $g^{-1}(n)$ *and indicate that the sign of this function at* n *is given by* $\lambda(n) = (-1)^{\Omega(n)}$.
The next two columns show the small-ish magnitude differences between the unsigned magnitude of $g^{-1}(n)$ *and the summations* $\widehat{f}_1(n) := \sum_{k\geq 0} \binom{\omega(n)}{k} \cdot k!$ *and* $\widehat{f}_2(n) := \sum_{k \geq 0} \binom{\omega(n)}{k} \cdot \#\{d|n : \omega(d) = k\}$*. Finally, the last three columns show the summatory function of* $g^{-1}(n)$, $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$*, deconvolved into its respective positive and negative components:* $G_+^{-1}(x) := \sum_{n \leq x} g^{-1}(n) \left[g^{-1}(n) > 0\right]_\delta$ *and* $G_-^{-1}(x) := \sum_{n\leq x} g^{-1}(n) \left[g^{-1}(n) < 0\right]_\delta$.

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T.2 Table: Dirichlet inverse functions of $(f+1)(n)$ for f additive and vanishing at one

n	$\lambda(n)$	$(f+1)^{-1}(n)$
1	1	1
2	-1	$-f(2)-1$
3	-1	$-f(3)-1$
4	1	$f(2)^2+2f(2)-f(4)$
5	-1	$-f(5)-1$
6	1	$2f(3)f(2)+f(2)+f(3)+1$
7	-1	$-f(7)-1$
8	-1	$-f(2)^3-3f(2)^2+2f(4)f(2)-f(2)+2f(4)-f(8)$
9	1	$f(3)^2+2f(3)-f(9)$
10	1	$2f(5)f(2)+f(2)+f(5)+1$
11	-1	$-f(11)-1$
12	-1	$-3f(3)f(2)^2-f(2)^2-4f(3)f(2)-2f(2)+2f(3)f(4)+f(4)$
13	-1	$-f(13)-1$
14	1	$2f(7)f(2)+f(2)+f(7)+1$
15	1	$2f(5)f(3)+f(3)+f(5)+1$
16	1	$f(2)^4+4f(2)^3-3f(4)f(2)^2+3f(2)^2-6f(4)f(2)+2f(8)f(2)+f(4)^2-f(4)+2f(8)-f(16)$
17	-1	$-f(17)-1$
18	-1	$-3f(2)f(3)^2-f(3)^2-4f(2)f(3)-2f(3)+2f(2)f(9)+f(9)$
19	-1	$-f(19)-1$
20	-1	$-3f(5)f(2)^2-f(2)^2-4f(5)f(2)-2f(2)+f(4)+2f(4)f(5)$
21	1	$2f(7)f(3)+f(3)+f(7)+1$
22	1	$2f(11)f(2)+f(2)+f(11)+1$
23	-1	$-f(23)-1$
24	1	$4f(3)f(2)^3+f(2)^3+9f(3)f(2)^2+3f(2)^2+2f(3)f(2)-6f(3)f(4)f(2)-2f(4)f(2)+f(2)-4f(3)f(4)-2f(4)+2f(3)f(8)+f(8)$
25	1	$f(5)^2+2f(5)-f(25)$
26	1	$2f(13)f(2)+f(2)+f(13)+1$
27	-1	$-f(3)^3-3f(3)^2+2f(9)f(3)-f(3)+2f(9)-f(27)$
28	-1	$-3f(7)f(2)^2-f(2)^2-4f(7)f(2)-2f(2)+f(4)+2f(4)f(7)$
29	-1	$-f(29)-1$
30	-1	$-2f(3)f(2)-6f(3)f(5)f(2)-2f(5)f(2)-f(2)-f(3)-2f(3)f(5)-f(5)-1$
31	-1	$-f(31)-1$

Table T.2: ***Dirichlet inverse functions of additive arithmetic functions.*** *The table provides a list of the Dirichlet inverse functions of $(f+1)(n)$ for f additive such that $f(1)=0$.*

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T.3 Table: Dirichlet inverse functions of $g(n)$ for g multiplicative

n	$(-1)^{\omega(n)}$	$f^{-1}(n)$
2	-1	$-g(2)$
3	-1	$-g(3)$
4	-1	$g(2)^2 - g(4)$
5	-1	$-g(5)$
6	1	$g(2)g(3)$
7	-1	$-g(7)$
8	-1	$-g(2)^3 + 2g(4)g(2) - g(8)$
9	-1	$g(3)^2 - g(9)$
10	1	$g(2)g(5)$
11	-1	$-g(11)$
12	1	$g(3)g(4) - g(2)^2g(3)$
13	-1	$-g(13)$
14	1	$g(2)g(7)$
15	1	$g(3)g(5)$
16	-1	$g(2)^4 - 3g(4)g(2)^2 + 2g(8)g(2) + g(4)^2 - g(16)$
17	-1	$-g(17)$
18	1	$g(2)g(9) - g(2)g(3)^2$
19	-1	$-g(19)$
20	1	$g(4)g(5) - g(2)^2g(5)$
21	1	$g(3)g(7)$
22	1	$g(2)g(11)$
23	-1	$-g(23)$
24	1	$g(3)g(2)^3 - 2g(3)g(4)g(2) + g(3)g(8)$
25	-1	$g(5)^2 - g(25)$
26	1	$g(2)g(13)$
27	-1	$-g(3)^3 + 2g(9)g(3) - g(27)$
28	1	$g(4)g(7) - g(2)^2g(7)$
29	-1	$-g(29)$
30	-1	$-g(2)g(3)g(5)$
31	-1	$-g(31)$
32	-1	$-g(2)^5 + 4g(4)g(2)^3 - 3g(8)g(2)^2 - 3g(4)^2g(2) + 2g(16)g(2) + 2g(4)g(8) - g(32)$
33	1	$g(3)g(11)$
34	1	$g(2)g(17)$

Table T.3: ***Dirichlet inverse functions of multiplicative arithmetic functions.*** *The table provides a list of the Dirichlet inverse functions of $g(n)$ for g multiplicative such that $g(1) = 1$.*

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