PARTITION FUNCTION TRANSFORMATIONS OF THE SIGNEDNESS OF AN ARITHMETIC FUNCTIONS

HAMED MOUSAVI
MAXIE D. SCHMIDT

ABSTRACT. TODO.

1. INTRODUCTION

The sign changes of an arithmetic function f are often considered in applications where we must estimate the growth of sums depending on f. For any fixed f, we define its summatory function for all positive integers $x \geq 1$ by

$$S_f(x) := \sum_{n \leq x} f(n).$$

Given any two arithmetic functions f and g, we define their *Dirichlet convolution*, $f * g$, to be the divisor sum

$$(f * g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right), \forall n \geq 1.$$

The multiplicative inverse with respect to Dirichlet convolution is defined by $\varepsilon(n) \equiv \delta_{n,1}$ so that $f * \varepsilon = \varepsilon * f = f$ for any arithmetic f. If $f(1) \neq 1$, then it is Dirichlet invertible. That is, there is another arithmetic function $f^{-1}(n)$ such that $f * f^{-1} = f^{-1} * f = \varepsilon$. We find that the signedness of f^{-1} is dictated, or prescribed, by the local sign change patterns of f. Because such locally unpredictable signage is crucial to the understanding of many classical problems and applications, we state the next proposition to clarify the situation for a class of nicely behaved invertible arithmetic $f \geq 0$.

Proposition 1.1 (The Sequence of Signs of the Dirichlet Inverse)**.** *Suppose that* $f(1) := c_f \neq 0$ *and that* $f(n) \geq 0$ *for all* $n \geq 2$*. Then*

1. *If* f *is completely multiplicative then* $\operatorname{sgn}(f^{-1}(n)) = \mu(n)$*;*
2. *If* $c_f = 1$*, then* $\operatorname{sgn}(f^{-1}(n)) = \lambda(n) = (-1)^{\Omega(n)}$*;*
3. *If* $c_f < 1$*, then* $\operatorname{sgn}(f^{-1}(n)) = -1$*.*

The class of non-negative invertible f covered by the formulas above tend to have nicer, more manageable summatory functions as well. For example, consider that whenever $f(n) \geq 0$ for all $n \geq 1$, we have that its summatory function satsfies predictable properties under Riemann differentiation and integration [**?**, *cf.* §13].

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1.1. **Known results on local sign changes of an arithmetic function.** One heuristic that recurs in applications is that if an invertible $f \geq 0$ is non-negative for all $n \geq 1$, then the corresponding sequence of sign changes for its inverse f^{-1} is oscillatory and typically hard to predict. The same is true for invertible non-negative integer matrices: the corresponding inverse matrices as a general measure tend to display semi-random, highly oscillatory, and variably signed behavior. It is known that the analytic properties, poles, and zeros of the Dirichlet generating function, $D_f(s)$, or DGF, of f provide key insights into the sign changes of these functions [**?**, **?**, **?**]:

$$TODO,$$

Note that the DGF of f is related to the Mellin transform of its summatory function at $-s$ by

$$D_f(s) = s \cdot \int_1^{\infty} \frac{S_f(x)}{x^{s+1}} dx, \mathrm{Re}(s) > \sigma_{a,f},$$

where $\sigma_{a,f}$ is the abscissa of absolute convergence of the DGF $D_f(s)$.

1.2. **A smoothing sum of exponentials construction that extracts signedness from a sequence.** Let $\zeta_m := \exp(2\pi\imath/m)$ denote the primitive m^{th} root of unity. Our idea and new construction is to "encode" f in such a way that its signedness is expressed in a predictable way. To this end, we define two invertible transformations on a fixed f formed by a discrete convolution with the special partition number sequences defined above:

$$\begin{aligned} s_{m,k}[f](n) &:= \sum_{j=1}^{n} f(j)\zeta_m^{n-j} \exp\left(\pi\sqrt{k(n-j)}\right), \forall n \geq 1; m \in \mathbb{Z}^+; k \in (0,\infty); \\ t_{m,k}[f](n) &:= \sum_{j|n} f(j)\zeta_m^{n-j} \exp\left(\pi\sqrt{k(n-j)}\right), \forall n \geq 1; m \in \mathbb{Z}^+; k \in (0,\infty). \end{aligned}$$

The transformation in the previous equation is invertible by convolution with another signed sequence of exponential functions. The characteristic limiting behavior we observe in these transformations is typified by the next definitions.

Definition 1.2. We say that a arithmetic sequence $\{f(n)\}_{n\geq 1}$ has *property* $\mathcal{P}_{1,m,k}$ at N if the sign of $\mathrm{Re}\left\{s_{m,k}[f](n) \cdot \zeta_m^{-n}\right\}$ is constant for all $n \geq N$. Similarly, we say that f has *property* $\mathcal{P}_{2,m,k}$ at N if the sign of $\mathrm{Im}\left\{s_{m,k}[f](n) \cdot \zeta_m^{-n}\right\}$ is constant for all $n \geq N$. We define for $i := 1, 2$

$$M_{i,m,k}(f) := \sup\left\{n \geq 1 : f \text{ does not have property } \mathcal{P}_{i,m,k} \text{ at } n\right\}.$$

The above definition shows that if an arithmetic function f satisfies $\mathcal{P}_{m,k}$ at some finite $N \geq 1$, then for all sufficiently large $n \geq N$, the signed magnitude of the real part of the transformation $s_{m,k}[f](n)$ tends to one side of the real line or the other. The definition does not provide the limiting signage of the transformation sequence even if the function f satisfies property $\mathcal{P}_{m,k}$.

Theorem 1.3 (A Sign Smoothing Convolution Operator by Exponential Function Scaling)**.** *For any Dirichlet invertible arithmetic function f which is non-vanishing on the positive integer, any $m \in \mathbb{Z}^+$, and any $k \in (0,\infty)$ such that $f(n) \ll \exp(\pi\sqrt{kn})$, $M_{1,m,k}(f^{-1})$ and $M_{2,m,k}(f^{-1})$ are finite. Moreover,*

$$\lim_{n\to\infty} \left|\mathrm{Re}\left\{\frac{s_{m,k}[f^{-1}](n)}{\zeta_m^n}\right\}\right| = +\infty, \quad \textit{and} \quad \lim_{n\to\infty} \left|\mathrm{Im}\left\{\frac{s_{m,k}[f^{-1}](n)}{\zeta_m^n}\right\}\right| = +\infty$$

Theorem 1.4. *For any Dirichlet invertible f which is non-vanishing on the positive integers, any $m \in \mathbb{Z}^+$, and any $k \in (0, \infty)$, we have that*

$$M_{1,m,k}(f^{-1}) = TODO_{1,m,k}; \tag{A.1}$$

$$M_{2,m,k}(f^{-1}) = TODO_{2,m,k}; \tag{A.2}$$

$$\limsup_{n\to\infty} \left(\operatorname{sgn}\left\{\operatorname{Re}\left[s_{m,k}[f^{-1}](n) \cdot \zeta_m^{-n}\right]\right\}\right) = TODO_{m,k}; \tag{B}$$

$$\limsup_{n\to\infty} \left(\operatorname{sgn}\left\{\operatorname{Im}\left[s_{m,k}[f^{-1}](n) \cdot \zeta_m^{-n}\right]\right\}\right) = TODO_{m,k}. \tag{C}$$

Theorem 1.5 (A Sign Smoothing Convolution Operator by Exponential Function Scaling)**.** *For any Dirichlet invertible arithmetic function f which is non-vanishing on the positive integers, any $m \in \mathbb{Z}^+$, and any $k \in (0, \infty)$ such that $f(n) \ll \exp(\pi\sqrt{kn})$, the sign of $\operatorname{Re}\left[t_{m,k}[f^{-1}](n) \cdot \zeta_m^{-n}\right]$ is eventually constant. That is, there exists a finite $N_f \geq 1$ such that for all $n > N_f$,*

$$\operatorname{sgn}\left\{\operatorname{Re}\left[t_{m,k}[f^{-1}](n)\zeta_m^{-n}\right]\right\} - \operatorname{sgn}\left\{\operatorname{Re}\left[t_{m,k}[f^{-1}](n-1)\zeta_m^{1-n}\right]\right\} = 0.$$

1.3. **Applications.** The idea in applying Theorem 1.3 is simple in that if we have a reliable mechanism for estimating the $s_i[f](n)$ given a known DGF of f, then the signed and locally oscillatory calclulations of the summatory function $S_f(x)$ should become less complicated. Indeed, provided we know $s_i[f](n)$, we can split

$$S_f(x) := S_f^+(x) - S_f^{-1}(x), \forall x \geq 1$$

into a difference of two summatory functions of the same sign, as we do with the positive and negative parts of signed measures. These split summatory functions correspond to the expansions

$$S_f^+(x) = TODO$$
$$S_f^-(x) = TODO.$$

There are many applications ...

1.4. **Generalizations.** Let $V(f, Y)$ denote the number of sign changes of f on the interval $(0, Y]$. More precisely, we have that

$$V(f, Y) := \sup\left\{N : \exists \{x_i\}_{i=1}^N, 0 < x_1 < \cdots < x_N \leq Y, f(x_i) \neq 0, \operatorname{sgn}(f(x_i)) \neq \operatorname{sgn}(f(x_{i+1})), \forall 1 \leq i < N\right\}.$$

Then we have that for any $\mathcal{A}_n \subseteq \{1, 2, \ldots, n\}$, and monotone increasing function $E : \mathbb{N} \to \mathbb{R}$ such that $f(n) \ll E(n)$, that

$$\begin{aligned}\sum_{j\in\mathcal{A}_n} f(j)E(n-j) &= \sum_{j\in\mathcal{A}_n} \left[1 - 2(V(f,j) - V(f,j-1))\right] |f|(j)E(n-j) \\ &= \sum_{j\in\mathcal{A}_n} |f|(j)E(n-j) - 2V(f,n)|f|(n)E(0) \\ &\quad + 2\sum_{\substack{j\in\mathcal{A}_n \\ j<n}} V(f,j)\left[|f|(j+1)E(n-1-j) - |f|(j)E(n-j)\right].\end{aligned}$$

We arrive at a natural question:

Question 1.6 (A Natural Sign Smoothing Function for f)**.** Given that $f(n) = \Theta(U_f(n))$ and $V(f, Y) = \Theta(V_f(Y))$, what is the optimal (minimal) choice of monotone increasing functions E_f, E_f^* such that the signs of

$$\sum_{j|n} f(j)E_f^*(n/j), \sum_{j=1}^{n} f(j)E_f(n-j),$$

are eventually constant? Can we choose the best possible functions E_f, E_f^* so that the signs of the corresponding sequences above are constant for all $n \geq N_f$ with $N_f \geq 1$ minimal over the natural numbers?

Give tables of some interesting example cases ...

2. Proofs of key results

3. Applications

4. Open questions and generalizations

Email address: hmousavi6@gatech.edu, maxieds@gmail.com, mschmidt34@gatech.edu

Georgia Institute of Technology, School of Mathematics, 117 Skiles Building, 686 Cherry Street NW, Atlanta, GA 30332, USA