using the Abel summation-IBP formula from eqn. #(18) of Corollary 4.3 (on page 22):

$$G^{-1}(x) = \sum_{n \leq x} \lambda(n) |g^{-1}(n)|$$

$$= L(1)|g^{-1}(1)| + \int_1^x L'(x)|g^{-1}(x)| dx$$

$$= O\left(1 + x\, \mathbb{E}|g^{-1}(x)|\right)$$

$$= O\left(1 + \frac{x(\log x)^2}{\sqrt{\log\log x}}\right),$$ "a.e." integer x using page 29.

---

also, you are correct insomuch as what I mean in applying Abel summation is indeed a Riemann-Stieltjes type integral representing

$$G^{-1}(x) = \sum_{N<x} L(N)\left(|g^{-1}(N+1)| - |g^{-1}(N)|\right)$$

Looking back at the proof, what I believe I meant was to apply a MVT (to the integral I had written) to state that

$$G^{-1}(x) = L(c)\left(|g^{-1}(x)| - |g^{-1}(1)|\right)$$

$$= L(c)\left(|g^{-1}(x)| - 1\right)$$

$$\sim L(c)\left(\mathbb{E}|g^{-1}(x)| - 1\right)$$

$$\sim L(c)\,\mathbb{E}|g^{-1}(x)|$$

$$\asymp L(c) \cdot \frac{(\log x)^2}{\sqrt{\log\log x}},$$

for some $c \in (1, x)$

In the notation section at the beginning of the article (starting on page 3), I define

$$\mathbb{E}[f(x)] := \frac{1}{x} \cdot \sum_{n \leq x} f(n)$$

to denote the average order, e.g., in analog to the average, or "expectation" of the arithmetic function f.

Now that you point out that the (mis)use (abuse) of this notation is bad form, I will work on removing it throughout the article. Another mathematician initially suggested it to me since there is some probabilistic interpretation to be made from my new results (i.e., cf. section 4.4 on page 29).

Thank you for pointing this out, as I want the JNT article to go well, and be well received, this year! :)

Do you think it would be better to just remove the statement of the first result from Theorem 5.1, and instead start with:

If the RH is true, then $\forall \varepsilon > 0$ and sufficiently large x,

$$\sigma^{-1}(x) = O\left(\frac{\sqrt{x}\,(\log x)^2}{\sqrt{\log\log x}}\right) *$$

*$\exp\left(\sqrt{\log x}\,(\log\log x)^{5/2+\varepsilon}\right)$?

The first statement (before copyediting again) in Theorem 5.1 is <u>NOT</u> essential, especially since good upper bounds on $L(x)$ are only (approximately) as good as those known for $M(x)$...

Notes on a response to your PDF:

Note that by the arguments given in Section 2.2 (pp. 12–14):

$$\sum_{n\geq 1} \frac{g^{-1}(n)}{n^s} = \frac{1}{\zeta(s)(1+P(s))}, \mathrm{Re}(s) > 1.$$

So for $\mathrm{Re}(s) > 1$:

$$\frac{1}{\zeta(s)(1+P(s))} = \int_1^{\infty} \frac{G^{-1}(x)\,dx}{x^{s+1}} \quad (*)$$

One attribute I might wonder can be considered "elegant" about my efforts to find the distribution of $|g^{-1}(n)|$ (cf. Thm. 4.7 on page 26) and to bound $G^{-1}(x)$, is that

My arguments effectively side step some of the problems that come in with taking the inverse Mellin transform of (*) above.

Perhaps we can talk more about these ideas when we meet over Zoom on Wednesday of next week?

-- MDS