Asymptotic Bounds for the Mertens Function:

A Proof That the Mertens Function is Unbounded with Generalizations and Applications to the Liouville Summatory Functions

Maxie D. Schmidt

School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332

maxieds@gmail.com
mschmidt34@gatech.edu

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Abstract. The Mertens function is defined as the average order of the Möbius function, or as the summatory function $M(x) = \sum_{n \leq x} \mu(n)$, for all $x \geq 1$. There are many open problems are related to determining optimal asymptotic bounds for this function. In this article, we employ the recent construction of new formulas for the generalized sum-of-divisors functions proved by the author to obtain new results which exactly sum the classical Mertens function for all finite x. We state and prove analogous results for the generalized Mertens function which we define to be $M_\alpha^*(x) = \sum_{n \leq x} n^\alpha \mu(n)$ for any fixed $\alpha \in \mathbb{C}$. The primary new results in the article prove that the classical Mertens function $|M(x)|/\sqrt{x}$ is unbounded as $x \rightarrow \infty$, i.e., $\limsup_{x \rightarrow \infty} |M(x)|/\sqrt{x} = +\infty$. We are the first to formally prove this result which extends existing work, notably by Odlyzko and te Riele in 1985, among other more recent references, suggesting that this limit should be infinite.

Extensions of the Mertens Function Formulas. A closely-related construction also allows us to define the generalized summatory functions over Liouville's function, $\lambda(n) = (-1)^{\Omega(n)}$, as $L_{-\alpha}(x) = \sum_{n \leq x} n^\alpha \lambda(n)$ and then to obtain almost identical bounds to those we prove in the Mertens function cases. In the second case, we instead obtain the new identity that $|L_0(x)| \geq \frac{3}{2}\sqrt{x} \log\log x$ for infinitely many x. We thus effectively improve the well-known bound by Borwein et. al. from 2008 which shows that $L_0(x) > 0.0618672\sqrt{x}$ and $L_0(x) < -1.3892783\sqrt{x}$ each for infinitely many positive $x \geq 1$. We are the first to prove the improvement of the existing bounds of the form $|L_0(x)| > C\sqrt{x}$ by an increased factor of $\log\log x$.

Keywords: Mertens function; Möbius function; Lioville's function; sum-of-divisors function; Ramanujan's sum.
2010 MSC: 11A25; 11N37; 11N56.
Pages: 1–19.

1 An index of non-standard notation and conventions in the article

1.1 Classical summatory functions and their weighted variants

The primary content of the article focuses on proving new asymptotic bounds for the summatory functions (i.e., average order sums) of the *Möbius function* and the *Liouville lambda function* and their α-weighted variants. In particular, we define the two classical cases of these summatory functions, denoted by $M(x) = M_0^*(x)$ and $L(x) = L_0(x)$, and their corresponding weighted sums, $M_\alpha^*(x)$ and (following as in the conventions from the references) $L_{-\alpha}(x)$, as follows:

- $M(x)$ $M(x) = \sum_{n \leq x} \mu(n)$.
- $M(x)$ $M_\alpha^*(x) = \sum_{n \leq x} n^\alpha \mu(n)$ for $\alpha \in \mathbb{Z}$.
- $L(x)$ $L(x) = \sum_{n \leq x} \lambda(n)$.
- $L(x)$ $L_{-\alpha}(x) = \sum_{n \leq x} n^\alpha \lambda(n)$ for fixed $\alpha \in \mathbb{C}$ where we also commonly denote the class of sums when $\alpha = 1$ by $T(x) = L_1(x)$.

1.2 Special function notation

Additional notation standard for special functions employed within the article includes

- $H_n^{(r)}$ The r-order harmonic numbers, $H_n^{(r)} = \sum_{k \leq n} k^{-r}$, correspond to the sequences of partial sums of the Riemann zeta function, $\zeta(r)$, for $\mathrm{Re}(r) > 1$. For integer-order $r \in \mathbb{Z}$, if $r = 1$, the first-order harmonic numbers satisfy $H_n \sim \log n + \gamma_C$, if $r = 0$ the sequence is given by $H_n^{(0)} = n$, and if $r < 0$ the sequence is express by finite degree polynomials in n whose coefficients are rational multiples of the *Bernoulli numbers*, $B_n = n! \cdot [z^n] \frac{z}{e^z - 1}$.
- B_n, $B_n(x)$ The *Bernoulli polynomials* generalize the Bernoulli numbers, $B_n = B_n(0)$, which are enumerated by the exponential generating function $B_n(x) = n! \cdot [z^n] \frac{ze^{xz}}{e^z - 1}$.
- $\sigma_\alpha(x)$ The generalized *sum-of-divisors functions* are defined by $\sigma_\alpha(x) = \sum_{d \mid x} d^\alpha$ where the sum is taken over all divisors d of x in the range $1 \leq d \leq x$.
- $\Phi_n(x)$ The *cyclotomic polynomials* defined for $n \geq 1$ and indeterminate x as

$$\Phi_n(q) := \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} \left(q - e^{2\pi\imath \frac{k}{n}} \right).$$

1.3 Article-specific definitions of new functions and notation

We provide the following list of references to definitions of several key functions and other non-standard notation in the article to would-be bewildered readers:

- $[d = m]_\delta$ We use *Iverson's convention* $[\texttt{cond} = \texttt{True}]_\delta \equiv \delta_{\texttt{cond},\texttt{True}}$ according to whether the condition `cond` is true or false where $\delta_{n,k}$ denotes Kronecker's delta function.

- $d_x^{(\alpha)}(m)$ Defined in Definition 2.4 on page 6.
- $T^{(\alpha)}(x)$ Defined in Definition 2.4 on page 6.
- $\tau_x^{(\alpha+1)}(d)$ A.k.a., the so-called *tau function* divisor sum function from Theorem 2.1 and in [15]. This function is formally defined as in the reference on the first line of the proof of Lemma 2.6 on page 6.
- $D^{(\alpha)}(x)$ Defined in Definition 2.4 on page 6.
- $S_j^{(\alpha+1)}(x)$ Defined for $j = 1, 2$ on page 5. The key bounds on these sums for x prime given in Proposition 2.9 on page 9 are central to the proof of Theorem 2.3 in Section 2.2.3.

2 | New asymptotics of the Mertens and Liouville summatory functions

2.1 Introduction

2.1.1 Mertens summatory functions

The Mertens summatory function, or *Mertens function*, is defined as

$$M(x) = \sum_{n \leq x} \mu(n), \ x \geq 1,$$

where $\mu(n)$ denotes the Möbius function which is in some sense a signed indicator function for the squarefree integers. A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

We define the notion of a *generalized Mertens summatory function* for fixed $\alpha \in \mathbb{C}$ as

$$M_\alpha^*(x) = \sum_{n \leq x} n^\alpha \mu(n), \ x \geq 1,$$

where the special case of $M_0^*(x)$ corresponds to the definition of the classical Mertens function $M(x)$ defined above. The plots shown in Figure 2.1.1 illustrate the chaotic behavior of the growth of these functions for x in small intervals when $\alpha \in \{-1, 0, 1, 2\}$.

2.1.2 Open problems

There are many open problems related to bounding $M(x)$ for large x. For example, the Riemann Hypothesis is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. For $\mathrm{Re}(\alpha) < 1$, we know the limiting absolute behavior of these functions as $x \to \infty$ as the Dirichlet generating function

$$\frac{1}{\zeta(\alpha)} = \lim_{x \longrightarrow \infty} M_\alpha^*(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{-\alpha}},$$

which is definitively bounded for all large x. It is still unresolved whether

$$\limsup_{x \to \infty} |M(x)|/\sqrt{x} = \infty,$$

although computational evidence suggests that this is a likely conjecture [7, 6]. We make a newly well-founded attempt to prove that this conjecture is true in Theorem 2.3. There is a rich history to the original statement of the *Mertens conjecture* which states that

$$|M(x)| < c \cdot x^{1/2}, \quad \text{some constant } c > 0,$$

which was first verified by Mertens for $c = 1$ and $x < 10000$, although since its beginnings in 1897 has since been disproved by computation. We cite that prior to this point it is known that [13, *cf.* §4.1]

$$\limsup_{x \to \infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad \text{(now 1.826054)},$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } -1.837625),$$

although based on work by Odlyzko and te Riele it seems probable that each of these limits should be $\pm\infty$, respectively [11, 8, 7, 6]. While it is known that $M(x) = \Omega_{\pm}(\sqrt{x})$ and $M(x)/\sqrt{x} = \Omega_{\pm}(1)$, we appear to offer the first complete proof that the function $M(x)/\sqrt{x}$ is in fact *unbounded* in the next sections of this article.

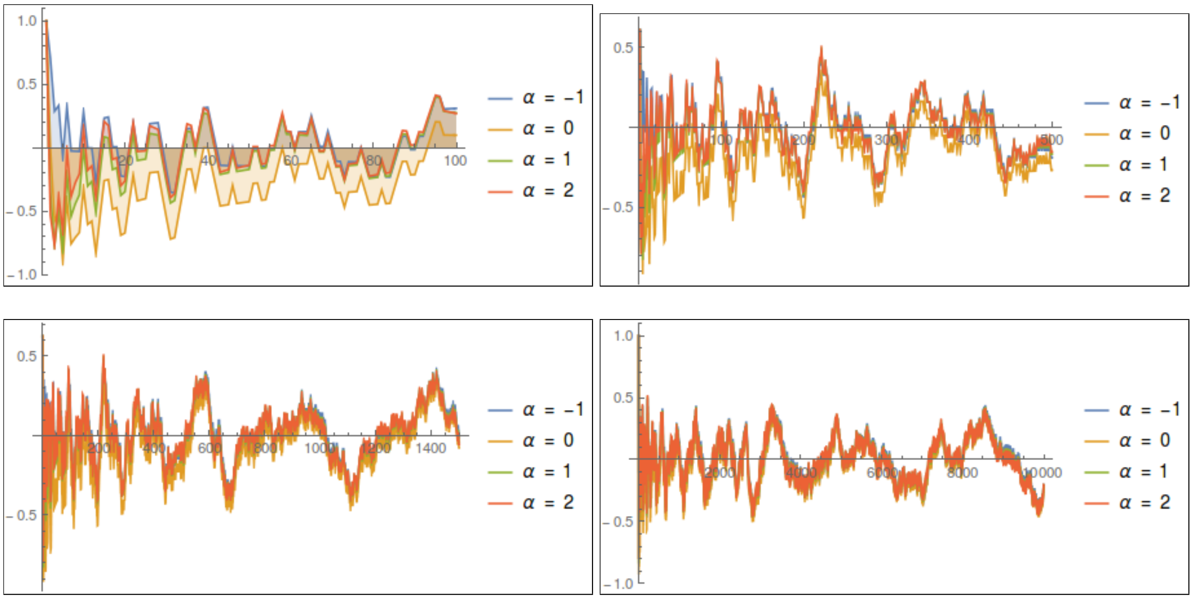


Figure 2.1.1: *Comparison of the Mertens Summatory Functions $M_\alpha(x)/x^{\frac{1}{2}+\alpha}$ for Small x and α*

2.1.3 Exact formulas for the generalized sum-of-divisors functions

The author has recently proved (2017) several new exact formulas for the *generalized sum-of-divisors functions*, $\sigma_\alpha(x)$, defined for any $x \geq 1$ as

$$\sigma_\alpha(x) = \sum_{d|x} d^\alpha, \ \ \alpha \in \mathbb{C}.$$

In particular, if we let $H_n^{(r)} = \sum_{k=1}^n k^{-r}$ denote the sequence of *r-order harmonic numbers*, where [12, §2.4(iii)]

$$H_n^{(-t)} = \frac{B_{t+1}(n+1) - B_{t+1}}{(t+1)} = \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + \sum_{k=1}^{t-1} \binom{t}{k} \frac{B_{k+1} n^{t-k}}{(k+1)}, \tag{2.1}$$

is a *Bernoulli polynomial* for any $n \geq 0$ when $t \in \mathbb{Z}^+$, then we can restate the next theorem from [15]. Within this article we assume that an index of summation p denotes that the sum is taken over only prime values of p. We also use the notation that the valuation function

$$\nu_p(x) = m \quad \text{if and only if} \quad p^m \| x,$$

to denote the exact exponent of the prime p dividing x.

Theorem 2.1 (Schmidt, 2017)**.** *For any fixed $\alpha \in \mathbb{C}$ and all $x \geq 1$, we have that*

$$\begin{aligned} \sigma_\alpha(x) = H_x^{(1-\alpha)} + \sum_{d|n} \tau_x^{(\alpha+1)}(d) + \sum_{2\leq p\leq x} \sum_{k=1}^{\nu_p(x)+1} p^{\alpha k} H^{(1-\alpha)}_{\left\lfloor \frac{x}{p^k} \right\rfloor} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right) \\ + \sum_{3\leq p\leq x} \sum_{k=1}^{\nu_p(x)+1} 2^{\alpha-1} p^{\alpha k} (-1)^{\lfloor x/p^{k-1} \rfloor} H^{(1-\alpha)}_{\left\lfloor \frac{x}{2p^k} \right\rfloor} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right), \end{aligned}$$

where the divisor sum over the function $\tau_x^{(\alpha)}(d)$ is defined precisely by Lemma 2.6.

Remark 2.2 (Restatement of the Theorem)**.** For $x \geq 1$ and fixed $\alpha \in \mathbb{C}$, we define the sums (*cf.* Section 3)

$$S_1^{(\alpha+1)}(x) = \sum_{2 \leq p \leq x} \sum_{k=1}^{\nu_p(x)+1} p^{\alpha k} H_{\left\lfloor \frac{x}{p^k} \right\rfloor}^{(1-\alpha)} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right)$$

$$S_2^{(\alpha+1)}(x) = \sum_{3 \leq p \leq x} \sum_{k=1}^{\nu_p(x)+1} 2^{\alpha-1} p^{\alpha k} (-1)^{\lfloor x/p^{k-1} \rfloor} H_{\left\lfloor \frac{x}{2p^k} \right\rfloor}^{(1-\alpha)} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right).$$

Then we prefer to work with the next form of Theorem 2.1 stated in terms of our new shorthand sum functions as follows:

$$\left| \sum_{d|x} \tau_x^{(\alpha+1)}(d) \right| = \left| H_x^{(1-\alpha)} - \sigma_\alpha(x) + S_1^{(\alpha+1)}(x) + S_2^{(\alpha+1)}(x) \right|. \tag{2.2}$$

The statement of the theorem given in (2.2) is important and significant since it implies deep connections between the sum-of-divisors functions, the generalized Mertens summatory functions, and the partial sums of the Riemann zeta function for real $\alpha < 0$, each related to one another in a convolved formula taken over sums of successive powers of the primes $p \leq x$. Thus we immediately see new relations from the restatement of the key results in [15] above. Moreover, from the previous result, we then obtain our main new results in the article given in the results in the next section as consequences of this restatement in terms of the Mertens functions.

2.2 New results and proofs of key lemmas

2.2.1 Statement of the main theorem

Theorem 2.3 (The Limit Supremum of $M(x)$ and Its Values at Large Primes)**.** *Let $x = q$ denote a large odd prime. Then we have that*

$$\limsup_{\substack{x \to \infty \\ x=q}} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

Proof Sketch. The complete proof of the theorem is given at conclusion of this section. For now, we will elaborate on the key steps in proving the theorem. We begin by noting that

$$\left| \sum_{d|x} \tau_x^{(\alpha+1)}(d) \right| \leq \left| T^{(\alpha)}(x) \right| + \left(2 \cdot \sup_{1 \leq n \leq x} |M_\alpha^*(n)| + x^\alpha \right) \times D^{(\alpha)}(x) - M_\alpha^*(1) d_x^{(\alpha)}(1), \qquad (\textit{Lemma 2.8})$$

where the upper bound is obtained by summation by parts and the corresponding divisor sums denoted by the functions in the previous equation are defined in the next subsection. We then need to show that infinitely and predictably often at least (and not necessarily for all large x) that we can bound the ratio of the next sums by a constant factor of $x \log\log x$. We consider the cases of large x when $x := q$ is a large prime and employ the resulting expansions to complete our proof. The next step in the proof is to show that (2.2) is approximately

$$\begin{aligned} \left| \sum_{d|x} \tau_x^{(\alpha+1)}(d) \right| &= \left| H_x^{(1-\alpha)} - \sigma_\alpha(x) + S_1^{(\alpha+1)}(x) + S_2^{(\alpha+1)}(x) \right| \\ &\geq \widetilde{C} \cdot x \log\log(x-1), \end{aligned} \qquad (\textit{Prop. 2.9})$$

where for sufficiently large x we can take $\widetilde{C} = \frac{3}{8}$. We then prove by a key construction that the effective multiples of the functions $|T^{(0)}(x)| \rightsquigarrow \frac{x}{16} \log\log x$ and $D^{(0)}(x) \leq O(\sqrt{x})$ we can take in our first bound from above when x is a sufficiently large prime. Then for all sufficiently large primes x we have that

$$\frac{1}{\sqrt{x}} \left(\sup_{1 \leq n \leq x} |M(n)| \right) \geq \frac{3}{8C} \log\log(x-1) \longrightarrow \infty \text{ as } x \to \infty. \qquad (\textit{Lemma 2.8, Prop. 2.9})$$

Thus as the lower bound stated in the previous equations increases with x and tends to infinity infinitely often, i.e., whenever we input x as one of our large primes, we see that the right-hand-side supremum must tend to infinity infinitely often as well. This is the basic sketch of the argument we will employ when we complete the full proof of Theorem 2.3 in Section 2.2.3. For now, we need to develop more machinery and state several lemmas to establish this claim. ∎

2.2.2 Key asymptotic bounds and formulas

Definition 2.4 (Divisor Sums Over Bounded Divisors)**.** We construct the non-empty $S_{D,x} = \{d_x\}$ as the singleton set whose only element is defined to be

$$d_x := \max\left(\left\{\frac{1}{8}\sqrt{x} \leq d \leq \frac{7}{8}\sqrt{x} : d \neq p^k, 2p^k \text{ for any primes } p,\ k \geq 1\ \neq 0\right\} \cap \mathbb{Z} \cup \{1\}\right).$$

We then define the corresponding complement set of $S_{D,x}$ to be

$$S_{T,x} = \{1, 2, \ldots, x\} \setminus S_{D,x}.$$

Next, we let our divisor sums over these specially bounded sets be defined as follows for $\alpha \in \mathbb{N}$, $m \geq 1$, and $x \geq 12$:

$$\begin{aligned} d_x^{(\alpha)}(m) &= \sum_{k=1}^{x} \sum_{\substack{d \mid k \\ d \in S_{D,x}}} \sum_{r \mid (d,x)} r^{\alpha+1} \left(\frac{k}{d}\right)^{\alpha} \left[m = \frac{d}{r}\right]_{\delta} \\ D^{(\alpha)}(x) &= \sum_{m=1}^{x} \left|d_x^{(\alpha)}(m)\right| = \sum_{k=1}^{x} \sum_{\substack{d \mid k \\ d \in S_{D,x}}} \sum_{r \mid (d,x)} r^{\alpha+1} \left(\frac{k}{d}\right)^{\alpha} \\ T^{(\alpha)}(x) &= \sum_{k=1}^{x} \sum_{\substack{d \mid k \\ d \in S_{T,x} \\ d \neq p^k, 2p^k}} \sum_{r \mid (d,x)} r \cdot \mu(d/r) \cdot k^{\alpha}. \end{aligned}$$

The implicit bounds we have imposed on each of the last two divisor sums will soon be of importance in our proof of the main theorem in this section.

Remark 2.5 (Bounds for the Divisor Sum Sets)**.** In particular, we argue that the bulk of the prospective elements d in the inner set used to define $S_{D,x}$ by d_x above satisfy $d, d_x = O\left(\frac{3}{4}\sqrt{x}\right)$ when x is sufficiently large. For sufficiently large x, the asymptotic density of the set of non-prime odd squarefree integers $n \leq x$ tends to

$$\frac{1}{2x}\left(\frac{6x}{\pi^2} - \frac{x}{\log x}\right) = \frac{3}{\pi^2} + o(1),$$

where $\frac{3}{\pi^2} \approx 0.303964$ is less than the length of the range of our defined interval over $\sqrt{x}$ as $\frac{7}{8} - \frac{1}{8} = \frac{3}{4}$ suffices. In other words, we expect that we should have approximately

$$\frac{9}{2\pi^2}\sqrt{x} - \frac{6}{\pi^2}\left[\pi\left(\frac{7}{8}\sqrt{x}\right) - \pi\left(\frac{\sqrt{x}}{8}\right) + \pi\left(\frac{7}{16}\sqrt{x}\right) - \pi\left(\frac{\sqrt{x}}{16}\right)\right],$$

choices for our d_x singleton set defined as above. This calculation is a good heuristic to measure the plausibility of the well-defined-ness of our set $S_{D,x}$ of elements of order $O(\sqrt{x})$ since the odd squarefree integers are not uniformly distributed in $1 \leq d \leq x$ and the failure to obtain the element d_x for large x would imply a contradiction to the asymptotic density constant given above.

Lemma 2.6 (Exact Formulas for the Divisor Sums $\sum_{d|x} \tau_x^{(\alpha)}(d)$)**.** *For $\alpha \in \mathbb{N}$, $m \geq 1$, and $x \geq 12$ we can expand the divisor sums in Theorem 2.1 exactly in the following forms:*

$$\sum_{d \mid x} \tau_x^{(\alpha+1)}(d) = T^{(\alpha)}(x) + \sum_{m=1}^{x} \mu(m) m^{\alpha} \cdot d_x^{(\alpha)}(m).$$

Proof. We start with the following formula for computing the divisor sum over the implicitly-defined $\tau_x^{(\alpha)}(d)$ from [15, §2]:

$$\sum_{d \mid x} \tau_x^{(\alpha+1)}(d) = [q^x]\left(\sum_{k=1}^{x} \sum_{\substack{d \mid k \\ d \neq p^k, 2p^k}} \sum_{r \mid d} \frac{r \cdot \mu(d/r)}{(1-q^r)} k^{\alpha}\right)$$

$$= \sum_{k=1}^{x} \sum_{r|x} \sum_{\substack{d|k \\ d \neq p^k, 2p^k}} r \cdot \mu(d/r) \cdot [r|d]_{\delta} \cdot k^{\alpha}$$

$$= \sum_{k=1}^{x} \sum_{\substack{d|k \\ d \neq p^k, 2p^k}} \sum_{r|(d,x)} r \cdot \mu(d/r) \cdot k^{\alpha} \tag{2.3}$$

$$= T_v^{(\alpha)}(x) + \sum_{m=1}^{x} \mu(m) m^{\alpha} \cdot d_x^{(\alpha)}(m). \tag{2.4}$$

We can also expand the right-hand-side of (2.3) as

$$\sum_{d|x} \tau_x^{(\alpha+1)}(d) = \sum_{\substack{d=1 \\ d \neq p^k, 2p^k}}^{x} \left(\sum_{r|(d,x)} r\mu(d/r) \right) H^{(-\alpha)}_{\left\lfloor \frac{x}{d} \right\rfloor},$$

which for x a target prime simplifies substantially to

$$\boxed{\sum_{d|x} \tau_x^{(\alpha+1)}(d) = \sum_{\substack{d=1 \\ d \neq p^k, 2p^k}}^{x} \mu(d) H^{(-\alpha)}_{\left\lfloor \frac{x}{d} \right\rfloor} = \sum_{\substack{d=1 \\ d \neq p^k, 2p^k}}^{x} \mu(d) H^{(-\alpha)}_{\left\lfloor \frac{x}{d} \right\rfloor} + \sum_{2 \leq p \leq x} H^{(-\alpha)}_{\left\lfloor \frac{x}{p} \right\rfloor} - \sum_{3 \leq p \leq x/2} H^{(-\alpha)}_{\left\lfloor \frac{x}{2p} \right\rfloor}.} \tag{2.5}$$

The last two formulas will be key in obtaining the bound we require in Lemma 2.8. □

Remark 2.7 (Connection to Ramanujan's Sum)**.** We have a deep connection between the divisor sums in Lemma 2.6 and Ramanujan's sum $c_q(n)$ given by

$$\sum_{d|x} \tau_x^{(1)}(d) = \sum_{k=1}^{x} \sum_{\substack{d|k \\ d \neq p^k, 2p^k}} c_d(x)$$

$$= \sum_{k=1}^{x} \sum_{\substack{d|k \\ d \neq p^k, 2p^k}} \mu\left(\frac{d}{(d,x)}\right) \frac{\varphi(d)}{\varphi\left(\frac{d}{(d,x)}\right)},$$

where $\varphi(x)$ denotes Euler's totient function. These identities follow by expanding out Ramanujan's sum in the form of [12, §27.10] [10, §A.7] [4, *cf.* §5.6]

$$c_q(n) = \sum_{d|(q,n)} d \cdot \mu(q/d),$$

and then applying the formula in (2.3) from the proof of the lemma above. Ramanujan's sum also satisfies the convenient bound that $|c_q(n)| \leq (n,q)$ for all $n, q \geq 1$, which can be used to obtain asymptotic estimates in the form of upper bounds for these sums when x is not prime or a prime power.

Lemma 2.8 (A Lower Bound for the Magnitude of $M(x)$)**.** *For all sufficiently large primes $x \geq 17$ and the selected partition of the divisors $d \leq x$, we have the following bound on the supremum of $|M(n)|$ taken over all $n \leq x$:*

$$\frac{\left| \sum_{d|x} \tau_x^{(1)}(d) \right| - \frac{1}{8} \left| T^{(0)}(x) \right| + d_x^{(0)}(1)}{2 \cdot D^{(0)}(x)} - \frac{x^{\alpha}}{2} \leq \sup_{1 \leq n \leq x} |M(n)|. \tag{2.6}$$

Proof. We first observe the equivalences of the sums for (2.2) given in Lemma 2.6. For fixed x, we then proceed from here by summation by parts to obtain that

$$\left| \sum_{d|x} \tau_x^{(\alpha+1)}(d) \right| \leq \left| T^{(\alpha)}(x) \right| + \left| \sum_{m=1}^{x} \mu(m) m^{\alpha} \cdot d_x^{(\alpha)}(m) \right|$$

$$\leq \left|T^{(\alpha)}(x)\right| + |M_\alpha^*(x)|\, d_x^{(\alpha)}(x) + \sum_{m=1}^{x-1} |M_\alpha^*(x)| \left|d_x^{(\alpha)}(m+1) - d_x^{(\alpha)}(m)\right|$$
$$\leq \left|T^{(\alpha)}(x)\right| + 2\sum_{m=1}^{x} |M_\alpha^*(x)|\, d_x^{(\alpha)}(m) + \sum_{m=1}^{x} m^\alpha |\mu(m)| \cdot |d_x^{(\alpha)}(m)| - M_\alpha^*(1) d_x^{(\alpha)}(1)$$
$$\leq \left|T^{(\alpha)}(x)\right| + \left(2 \sup_{1 \leq i \leq x} |M_\alpha^*(i)| + x^\alpha\right) \cdot D^{(\alpha)}(x) - d_x^{(\alpha)}(1) \tag{2.7}$$

We then consider the special case when $\alpha := 0$ to obtain a bound for the Mertens function $M(x)$. The inequality in (2.7) in fact holds for *all* large x and not just in the cases where x is prime as we will shortly prove improved bounds for immediately below.

In Proposition 2.9, we prove that $|T^{(0)}(x)| \sim C_1 x \log\log x$ for some constant C_1 approaching $\frac{1}{2}$ as x tends to infinity along the sequence of primes. Since we can argue that $D^{(0)}(x) = O(\sqrt{x})$ by

$$D^{(0)}(x) = \sum_{k=1}^{x} \sum_{\substack{d|k \\ d \in S_{D,x} \\ d \neq p^k, 2p^k}} \sum_{r|(d,x)} r = \sum_{1 \leq k \leq x} \sum_{\substack{d|k \\ d \in S_{D,x}}} 1 \leq \sum_{d \in S_{D,x}} \frac{x}{d} \leq \frac{C}{2}\sqrt{x},$$

the next key difference between the divisor sums in the last equation is small:

$$\left|\sum_{d|x} \tau_x^{(1)}(d)\right| - |T^{(0)}(x)| \approx C_2 \sqrt{x}.$$

Thus in order for us to obtain a non-trivial lower bound on the supremum over the Mertens function, we need to in fact prove a better bound than (2.7) of the form stated above in the lemma. That is, we need to show that the upper bound given in (2.7) is not tight to the precise value of $|T^{(0)}(x)|$ and we have some room to subtract off slightly less that this function (by a factor of $\frac{1}{8}$) and still obtain a valid lower bound for the Mertens function terms in the supremum on the right-hand-side.

To do this, we proceed summing (2.5) by parts when x is prime and $\alpha = 0$ to obtain that

$$\begin{aligned}
\left|\sum_{d|x} \tau_x^{(\alpha+1)}(d)\right| &= \left|\sum_{\substack{k=1 \\ k, k+1 \neq p^k, 2p^k}}^{x-1} M(k) \left(\left\lfloor \frac{x}{k+1} \right\rfloor - \left\lfloor \frac{x}{k} \right\rfloor\right)\right| \\
&\leq \left| 2 \sum_{\substack{k=1 \\ k \neq p^k, 2p^k}}^{x-1} |M(k)| \left\lfloor \frac{x}{k} \right\rfloor - x + \left|\sum_{\substack{k=2 \\ k \neq p^k, 2p^k}} \mu(k) \left\lfloor \frac{x}{k} \right\rfloor\right|\right| \\
&= \left| 2 \sum_{\substack{k=1 \\ k \neq p^k, 2p^k}}^{x-1} |M(k)| \left\lfloor \frac{x}{k} \right\rfloor - x + \frac{1}{8}|T^{(0)}(x)| + \frac{C\sqrt{x}}{8} + \frac{7}{8} \times \left|\sum_{\substack{k=2 \\ k \neq p^k, 2p^k}} \mu(k) \left\lfloor \frac{x}{k} \right\rfloor\right|\right| \\
&\leq \frac{1}{8}|T^{(0)}(x)| + \left|x - \frac{C\sqrt{x}}{8}\right| + 2\left(\sup_{n<x} |M(n)|\right) \left|\sum_{\substack{k=1 \\ k \neq p^k, 2p^k}}^{x-1} \left\lfloor \frac{x}{k} \right\rfloor + \frac{7}{8} \times \left|\sum_{\substack{k=2 \\ k \neq p^k, 2p^k}}^{x-1} \mu(k) \left\lfloor \frac{x}{k} \right\rfloor\right|\right|.
\end{aligned}$$

We evaluate the asymptotic behavior of each of the two remaining sums in the last equation in separate steps. Note that the leftmost sum is given by

$$\begin{aligned}
\sum_{\substack{k=1 \\ k \neq p^k, 2p^k}}^{x-1} \left\lfloor \frac{x}{k} \right\rfloor &= \sum_{k=1}^{x-1} \left\lfloor \frac{x}{k} \right\rfloor - \sum_{p<x} \left\lfloor \frac{x}{p} \right\rfloor - \sum_{p<x/2} \left\lfloor \frac{x}{2p} \right\rfloor \\
&\leq x(\log(x-1) + \gamma) - C_{3,x}(x-1) - x(\log\log x + A) - \frac{x}{2}\left(\log\log\frac{x}{2} + A\right) + C_{4,x}\pi(x) + C_{5,x}\pi\left(\frac{x}{2}\right),
\end{aligned}$$

where for $0 \leq C_{3,x}, C_{4,x}, C_{5,x} < 1$ for all x. Next, we evaluate the second, rightmost sum by first rephrasing a quick and dirty argument taken from Terry Tao's blog post on partial sums of the Möbius function to see that

$$1 = \sum_{k=1}^{x} \delta_{k,1} = \sum_{k=1}^{x} \sum_{d|k} \mu(d) = \sum_{d=1}^{x} \frac{x}{d}\mu(d) - \sum_{d=1}^{x} \left\{\frac{x}{d}\right\} \mu(d),$$

where $|\{x/d\}\,\mu(d)| \leq 1$ for all $1 \leq d \leq x$. Thus we have that

$$\begin{aligned} \sum_{\substack{k=2 \\ k \neq p^k, 2p^k}} \mu(k) \left\lfloor \frac{x}{k} \right\rfloor &= \sum_{k=1}^{x-1} \mu(k) \left\lfloor \frac{x}{k} \right\rfloor + \sum_{p<x} \left\lfloor \frac{x}{p} \right\rfloor - \sum_{3 \leq p < x/2} \left\lfloor \frac{x}{2p} \right\rfloor \\ &= 1 + \left\lfloor \frac{x}{4} \right\rfloor + \frac{x(A-2)}{2} + x \log\log x - \frac{x}{2} \log\log \frac{x}{2} - C_{6,x}\pi(x) + C_{7,x}\pi\left(\frac{x}{2}\right), \end{aligned}$$

where the relative bounded constants depending on x satisfy $0 \leq C_{6,x}, C_{7,x} < 1$ for all large x. Then combining our results we have the inequality

$$\begin{aligned} \left| \sum_{d|x} \tau_x^{(\alpha+1)}(d) \right| &\leq \frac{1}{8}|T^{(0)}(x)| + \left| x - \frac{C\sqrt{x}}{8} \right| \\ + 2\left[\sup_{n<x} |M(n)|\right] &\left| x(\log(x-1) + \gamma) + C_{3,x} + 1 - (A + C_{3,x} + 1)x - x \log\log \frac{x}{2} + C_{46,x}\pi(x) + C_{57,x}\pi\left(\frac{x}{2}\right) \right|, \end{aligned}$$

with $-1 < C_{46,x} < 1$ and $0 \leq C_{57,x} < 2$ for all x. Finally, we must see that we recover enough "lack of tightness" about the $|T^{(0)}(x)|$ term from the first known naíve bound established in (2.7) essentially by showing that the left-hand-side of

$$\boxed{\frac{\left|\sum_{d|x} \tau_x^{(\alpha+1)}(d)\right| - \frac{1}{8}|T^{(0)}(x)| - \left|x - \frac{C\sqrt{x}}{8}\right|}{2\sqrt{x}\log(x-1)} \leq \left(\sup_{n \leq x} |M(n)|\right) \sqrt{x}(1 + o(1)),}$$

is unbounded along an increasing sequence of primes $x \to \infty$. So we are done. We note that the choice of the constant multiplier of $\frac{1}{8}T^{(0)}(x)$ is somewhat arbitrary and is only important in so much as $\frac{1}{8} < \frac{1}{2}$ in order to obtain a non-trivial, strictly positive increasing lower bound on the Mertens function supremum through the next proposition. $\square$

Proposition 2.9 (Asymptotic Bound for the Tau Function Divisor Sum). *Let $x = q^r$ denote a power of the large prime q for some $r \geq 1$. Then when the x tending to infinity of these forms is sufficiently large, we obtain*

$$\left| \sum_{d|x} \tau_x^{(\alpha+1)}(d) \right| \geq \widetilde{C} \cdot x \log\log(x-1),$$

for some absolute constant which we may effectively take as $\widetilde{C} = \frac{1}{2}$ for x sufficiently large.

Proof. By the statement of Theorem 2.1 rephrased in (2.2), we see that

$$\begin{aligned} \left| \sum_{d|x} \tau_x^{(1)}(d) \right| &= \left| x - \sigma_1(x) + S_1^{(1)}(x) + S_2^{(1)}(x) \right| \\ &= \left| S_1^{(1)}(q^r) + S_2^{(1)}(q^r) - \left(1 + q + \cdots + q^{r-1}\right) \right| \\ &\geq \left| S_1^{(1)}(q^r) + S_2^{(1)}(q^r) \right| - \frac{x-1}{x^{\frac{1}{r}} - 1}. \end{aligned}$$

We next use the result of *Mertens' theorem* which implies that [10, §6.3] [1, §4.9] [4, §22.8] [12, §27.11]

$$\sum_{p \leq x} \frac{1}{p} = \log\log(x) + A + O\left(\frac{1}{\log x}\right),$$

where $A \approx 0.2614972128$ is a limiting constant [16, *cf.* §1.3]. In particular, when x is large we can expand the sum for $S_1^{(1)}(x)$ as

$$\begin{aligned}
S_1^{(1)}(x) &= \sum_{\substack{2 \leq p < q^r \\ p \neq q}} p \cdot \left\lfloor \frac{q^r}{p} \right\rfloor \left(\left\lfloor \frac{q^r}{p} \right\rfloor - \left\lfloor \frac{q^r}{p} - \frac{1}{p} \right\rfloor - \frac{1}{p} \right) + \sum_{k=1}^{r+1} q^k \left\lfloor q^{r-k} \right\rfloor \left(\left\lfloor q^{r-k} \right\rfloor - \left\lfloor q^{r-k} - \frac{1}{q} \right\rfloor - \frac{1}{q} \right) \\
&= \sum_{\substack{2 \leq p < q^r \\ p \neq q}} -\frac{p}{p} \left[\frac{q^r}{p} - \left\{ \frac{q^r}{p} \right\} \right] + \sum_{k=1}^{r} q^r \left(1 - \frac{1}{q} \right) \\
&= C_1 \pi(q^r - 1) - q^r \left(\log\log(q^r - 1) + A + O\left(\frac{1}{\log(q^r - 1)} \right) \right) + r \cdot q^{r-1}(q-1) \\
&\sim \frac{C_1(x-1)}{\log(x-1)} - x\left(\log\log(x-1) + A\right) + r \cdot x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1 \right),
\end{aligned}$$

and similarly, the sum $S_2^{(1)}(x)$ is expanded as

$$\begin{aligned}
S_2^{(1)}(x) &= \sum_{\substack{2 \leq p < q^r \\ p \neq q}} p \left\lfloor \frac{q^r}{2p} \right\rfloor \cdot \frac{1}{p} + \sum_{k=1}^{r} q^k \left\lfloor \frac{q^{r-k}}{2} \right\rfloor \frac{(q-1)}{q} + 2C_3 \left\lfloor \frac{q^r}{4} \right\rfloor \cdot \frac{1}{2} \\
&= \sum_{\substack{2 \leq p < q^r \\ p \neq q}} \left(\frac{q^r}{2p} - C_4 \right) + \sum_{k=1}^{r} q^k \left(\frac{q^{r-k}}{2} - C_5 \right) \frac{(q-1)}{q} + C_3 \left(\frac{q^r}{4} - C_6 \right) \\
&\sim \frac{1}{2} x \left(\log\log(x-1) + A\right) + \frac{C_2 C_4 (x-1)}{\log(x-1)} + \frac{r}{2} x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1 \right) - C_5(x-1) + \frac{C_3}{4} x - C_3 C_6.
\end{aligned}$$

Hence when we add these two sums cancellation of symmetric terms results in

$$S_1^{(1)}(x) + S_2^{(1)}(x) \sim \frac{(C_2 C_4 - C_1)(x-1)}{\log(x-1)} + \frac{x}{2}\left(\log\log(x-1) + A\right),$$

which proves our result. In the previous equation, we have that each constant $0 \leq C_i < 1$ since the fractional parts corresponding to the floor function terms in the respective bounds for $S_i^{(\alpha+1)}(x)$ are in this range. $\square$

2.2.3 The complete proof of Theorem 2.3

We are now at the point where we can assemble the complete results necessary to prove Theorem 2.3. The key idea here is that while the value of $|M(x)|$ is oscillating with x, we can bound the value of $\sup_{1 \leq n \leq x} |M(n)|$ below by something increasingly large and tending to infinity infinitely often, i.e., since there are an infinitude of large primes $q \to \infty$. Then using the lower bound in Lemma 2.8, and combining the bounds in Proposition 2.9, we see that when $x = q$ is large we have proved that

$$\frac{1}{\sqrt{x}} \left(\sup_{1 \leq n \leq x} |M(n)| \right) \geq \frac{3}{8} \log\log(x-1). \tag{2.8}$$

Next, for $x = q$ a large odd prime q let

$$x_{0,q} = \operatorname*{argmax}_{1 \leq n \leq q} |M(n)|.$$

Then we see from (2.8) that

$$\frac{|M(x_{0,q})|}{\sqrt{x_{0,q}}} \geq \frac{|M(x_{0,q})|}{\sqrt{q}} \geq \frac{3}{8C} \log\log(q-1) \longrightarrow \infty \text{ as } q \to \infty.$$

Moreover, since the lower bound in (2.8) and in the previous equation is increasing with q, i.e., as $q \to \infty$, we see that the non-decreasing sequence of $x_{0,q}$ must gradually increase with larger and larger q. Thus we see that for any $L > 0$, there are infinitely many $x \in \mathbb{N}$ such that $|M(x)|/\sqrt{x} > L$. More formally, that is to say that given any $L > 0$ for all primes $x > \exp\left(e^{\frac{8L}{3}}\right) + 1$ we have that $|M(x)|/\sqrt{x} > L$, i.e., that

$$\boxed{\limsup_{x\rightarrow\infty} \frac{|M(x)|}{\sqrt{x}} = \lim_{x\rightarrow\infty}\left(\sup_{n\geq x} \frac{|M(n)|}{\sqrt{n}}\right) \geq \lim_{x\rightarrow\infty} \frac{|M(x\log x)|}{\sqrt{x\log x}} = +\infty.}$$

Hence the result is proved.

2.2.4 Generalizations to Higher-Order (Weighted) Mertens Functions

We remark that Theorem 2.3 can be effectively generalized to a conjecture of the more general form

$$\limsup_{\substack{x\rightarrow\infty \\ x=q^r}} \frac{|M_\alpha^*(x)|}{(\sqrt{x})^{\alpha+1}} = +\infty.$$

The only caveat here is that we need to know more precise forms of Mertens' theorem for general sums of the form $\sum_{p\leq x} p^{\alpha k}$ depending on the parameter $\alpha \geq 0$. This generalization is a simple enough jump to make when we consider $\alpha \in \mathbb{Z}^+$. In particular, by the integral form of the summation by parts formula in [16, §2.1], where we take $\pi(x) = x/\log x + O(\sqrt{x})$, we see that

$$\sum_{p\leq x} p^\alpha = \pi(x)x^\alpha - \alpha\,\mathrm{Ei}((\alpha+2)\log x) + O_\alpha\left(x^{\alpha+1/2}\right),$$

where $\mathrm{Ei}(w)$ denotes the *exponential integral function* [12, §6]. The exponential integral satisfies the following asymptotic expansion for any natural number $N \geq 1$:

$$\mathrm{Ei}(w) = \frac{e^{-w}}{w} \times \sum_{n=0}^{N-1} \frac{(-1)^n n!}{w^n} + O\left(N! w^{-N}\right).$$

Then we see that

$$\sum_{p\leq x} p^\alpha = \frac{x^{\alpha+1}}{\log x} - \frac{\alpha}{x^{\alpha+2}\log(x^{\alpha+2})} + O\left(x^{\alpha+1/2}\right),\ \alpha \geq 1.$$

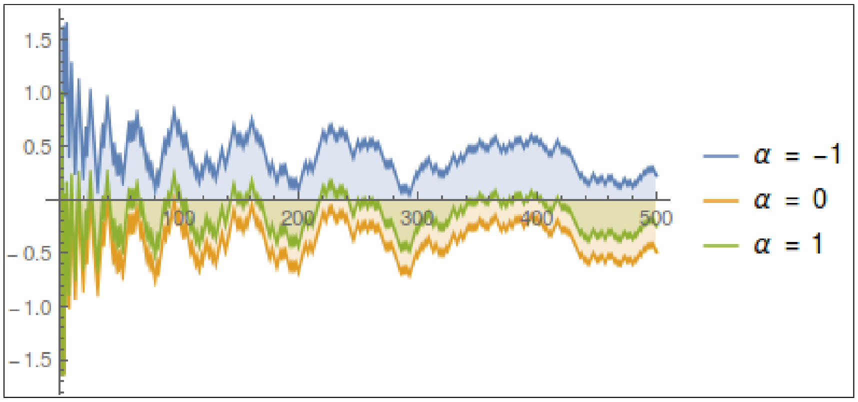
The next key component in formulating the generalized result is the expansion given in (2.5). Finally, for x prime, the functions $D^{(\alpha)}(x)$ and $T^{(\alpha)}(x)$ can be expanded as in in (2.5) in the forms of (TODO: Check)

$$\begin{aligned} D^{(\alpha)}(x) &= \sum_{\substack{d=1 \\ d\in S_{D,x} \\ d\neq p^k, 2p^k}}^{x} d^{-\alpha} H_{\lfloor x/d\rfloor}^{(-\alpha)} \\ T^{(\alpha)}(x) &= \sum_{\substack{d=1 \\ d\in S_{T,x} \\ d\neq p^k, 2p^k}}^{x} \mu(d) H_{\lfloor x/d\rfloor}^{(-\alpha)}, \end{aligned}$$

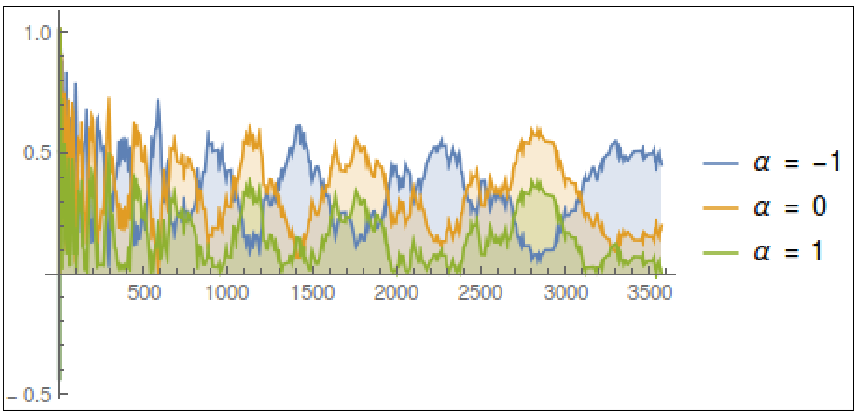
where the generalized harmonic numbers in the previous two equations correspond to the polynomials in (2.1) whose coefficients are rational multiples of the Bernoulli numbers. For now, we will leave the remainder of the generalizations of our main theorem as an exercise for future research on the generalized Mertens summatory functions $M_\alpha^*(x)$ defined in the introduction.

2.3 Applications to Liouville's summatory functions

2.3.1 Liouville's summatory functions



(a) $\frac{L_{-\alpha}(x)}{x^{\alpha+1/2}\log\log x}$



(b) $\frac{|L_{-\alpha}(x)|}{x^{\alpha+1/2}\log\log x}$ *for x prime*

Figure 2.3.1: *Comparison of the Liouville Summatory Functions* $L_{-\alpha}(x)$

For integers $x \geq 1$, we define the *Liouville summatory function* to be

$$L(x) = \sum_{n \leq x} \lambda(n).$$

For fixed $\alpha \in \mathbb{C}$, we follow the conventions employed in [2, 5, 9], and adopt the following notation for the "weighted" Liouville summatory functions defined as

$$L_{-\alpha}(x) = \sum_{n \leq x} n^{\alpha} \lambda(n).$$

Figure 2.3.1 provides scaled plots of these functions for the first few cases of $\alpha \in \{-1, 0, 1\}$. The cases where $\alpha := 0, 1$ are commonly denoted in the alternate, more classical notation of $L(x)$ and $T(x)$, respectively.

As in the Mertens function case, there are many open and famous historical conjectures related to these classes of summatory functions. For instance, *Pólya's conjecture* originally stated in 1919 concerns whether $L(x) \leq 0$ for all $x > 1$, though this conjecture has since been shown to be false by the computation of a large counterexample in 1980. How close the order of $|L(x)|$ is to $\sqrt{x}$ for infinitely many integers x is another often considered open problem. In 2008, Borwein et. al. showed that

$$L(x) > 0.0618672\sqrt{x} \quad \text{and} \quad L(x) < -1.3892783\sqrt{x},$$

each for infinitely positive integers x. This work has recently been improved by Mossinghoff and Trudgian in 2017 to show that for infinitely many x

$$L(x) > 1.0028\sqrt{x} \quad \text{ and } \quad L(x) < -2.3723\sqrt{x}.$$

Similar known bounds exist for the weighted sums $L_1(x)$ which hold at infinitely many integers $x \geq 1$ [9]. We are concerned with bounding the quantities

$$\limsup_{x\rightarrow\infty} \left(\frac{|L(x)|}{\sqrt{x}\log\log x}\right) > 0,$$

in the next subsection.

2.3.2 New bounds for $L(x)$

Since the values of the Möbius function match $\lambda(n)$ whenever n is a positive squarefree integer, we have an identity for the Möbius function given by

$$\mu(n) = |\mu(n)|\lambda(n).$$

In particular, we can extend the results for the divisor sums first given in Lemma 2.6 according to the expansions noted in Lemma 3.2 of the appendix to obtain analogous results to the Mertens function bounds with $L(x)$ in place of $M(x)$. Let the modified functions for this case be defined as follows:

$$\begin{aligned}
\widehat{d}_x^{(\alpha)}(m) &= \sum_{k=1}^{x} \sum_{\substack{d|k \\ d\in S_{D,x} \\ d\neq p^k, 2p^k}} \sum_{r|(d,x)} r^{\alpha+1}\left(\frac{k}{d}\right)^{\alpha} |\mu(d/r)| \left[m = \frac{d}{r}\right]_{\delta} \\
\widehat{D}^{(\alpha)}(x) &= \sum_{m=1}^{x} \left|\widehat{d}_x^{(\alpha)}(m)\right| \\
&= \sum_{k=1}^{x} \sum_{\substack{d|k \\ d\in S_{D,x} \\ d\neq p^k, 2p^k}} \sum_{r|(d,x)} r^{\alpha+1}\left(\frac{k}{d}\right)^{\alpha} |\mu(d/r)|.
\end{aligned}$$

Then by a straightforward adaptation of the proofs of Lemma 2.6 and Lemma 2.8, we can easily see that

$$\sum_{d|x} \tau_x^{(\alpha+1)}(d) = T^{(\alpha)}(x) + \sum_{m=1}^{x} \lambda(m) m^{\alpha} \cdot \widehat{d}_x^{(\alpha)}(m),$$

and consequently by summation by parts that

$$\frac{\left|\sum_{d|x}\tau_x^{(1)}(d)\right| - \frac{1}{8}\left|T^{(0)}(x)\right| + \widehat{d}_x^{(0)}(1)}{2\cdot \widehat{D}^{(0)}(x)} - \frac{x^{\alpha}}{2} \leq \sup_{1\leq n\leq x} |L(n)|. \tag{2.9}$$

Corollary 2.10 (Lower Bounds on Liouville's Summatory Function)**.** *Let x be a large odd prime. Then we have that the following bound holds for some $C \approx \frac{7}{8}$:*

$$|L(x)| \geq \frac{3}{8C}\sqrt{x}\log\log x.$$

Remark 2.11 (Pushing the Power Bounds)**.** Following as in the proof of Lemma 2.8, we conjecture that it may in fact be possible to improve our lower bounds on $|L(x)|$ to the following form for some $p \in (0, \frac{1}{2}]$:

$$|L(x)| \geq Cx^{1-p}\log\log x, \quad \text{for infinitely many } x.$$

More to this point, let

$$d_{x,p,k} := \max\left\{2^{-k}x^{1-p} \leq d \leq (1-2^{-k})x^{1-p} : \widetilde{\chi}_{\mathrm{pp}}(d) \neq 0\right\},$$

and set the corresponding set of plausible p to be

$$\widetilde{P}_{d,k} := \{p \in (0, 1/2] : d_{x,p,k} \text{ is well-defined for all large } x \}.$$

If we then take

$$p := \sup_{k \geq 3} \widetilde{P}_{d,k},$$

we see that we can push the lower bound on $|L(x)|$ to the previously conjectured form for infinitely many x where $C \approx 1/(1-2^{-k}) = \frac{2^k}{2^k-1}$ for some $k \in \mathbb{Z}^+$. Moreover, it appears that it is possible to find an appropriate k such that we can take $p = \frac{1}{4}$ (or even $\varepsilon > 0$). This sort of appeal to the heuristic in the first part of the proof to the proposition requires that

$$1 - 2^{-(k-1)} \geq \frac{3}{\pi^2} \qquad \Longrightarrow \qquad k \geq \left\lfloor \log_2\left(\frac{2\pi^2}{\pi^2-3}\right)\right\rfloor + 1 = 2,$$

and that for all sufficiently large x we have that there is a non-prime squarefree integer in the range $2^{-k}x^{1-p} \leq d \leq (1-2^{-k})x^{1-p}$. Thus our new improvements to the bounds on $|L(x)|$ given in the references [2, 9] actually suggest much more significant orders of growth of this summatory function for infinitely many x than can currently be proved with other methods related to this function.

2.4 Conclusions

2.4.1 Summary

We have proved a "famous" open conjecture showing that along the sequence of sufficiently large primes $q \sim m \log m$, we have that $|M(q)|/\sqrt{q}$ grows increasingly unbounded as $q \to \infty$. We do not consider the local oscillations of the functions between the large odd primes q here, though our exact formulas bounding the Mertens function employed in the proof of Theorem 2.3 certainly suggest an approach to a more complicated analysis of those properties as well. The new key ingredients to interpreting the new formulas from [15] is in (I) recognizing the forms of the $M_\alpha^*(x)$ in the divisor sums from Lemma 2.6, and (II) evaluating the prime sum formulas in Theorem 2.1 for these corresponding cases. In particular, we required both bounds in the proof of the main theorem in the article, one in terms of $|M(x)|$ obtained by summation by parts and the second obtained from bounding the new sum-of-divisors formulas proved by the author in the references. We have also proved new bounds satisfied by the Liouville summatory function, $L(x)$, which improve the constants and form of the previous best known bounds given in [2] by a factor of $\log\log x$ infinitely often.

A more subtle property that we have proved along the way states that

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}\log\log x} \geq \frac{3}{8}.$$

The exact specification of the order of a function $f_0(x)$ such that

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x} f_0(x)} \geq C_M,$$

is constant remains an open question of much potential interest in the references. We have structured our main arguments about the unboundedness of $M(x)/\sqrt{x}$ only at primes x primarily for the convenience of bounding complicated expressions for the sums in (2.2) involving $\nu_p(x)$ for primes p over all large x and in summing the known divisors of (d, x) for any $d \leq x$. However, we note that our approach here defines a more general construction that in fact bounds the order of $|M(x)|$ for all sufficiently large x. Namely, we have the "loose" bound proved in (2.7), along with the two-faceted methods, i.e., leading to constructively different characteristic expansions, for extracting the order of growth of the so-called *tau divisor sums* in Theorem 2.1 and Lemma 2.8.

As already mentioned in Remark 2.7, the deep connection of this class of divisor sums to Ramanujan's well-studied sum, $c_q(n)$, suggests new approaches for the general limiting cases of any subsequence of natural numbers approaching infinity. There is much work that still needs to be done to strategically utilize these new formulas to their fullest extent in finding new asymptotic relations for $M(x)$ and $M_\alpha^*(x)$ when $\alpha \geq 1$ is an integer. We also cannot neglect to acknowlege the deeper analytic connections between these summatory functions over the Möbius function, the Riemann zeta function, and its non-trivial zeros. Section 2.4.3 provides more details of this connection.

2.4.2 Unexpected results

One unexpected result from the start of the project is that the method we employ here to produce the lower bounds on the classical case of $M(x)$ nicely extend to the "weighted" Mertens function cases of $M_{\alpha}^{*}(x)$ defined in the introduction. We believe that we are the first to formally define these generalized analogs to the Mertens' function $M(x)$ and note that these definitions fall naturally out of the component divisor sums in the formulas proved by the author in [15]. Another unexpected bound for the summatory function for Liouville's function follows from the key re-characterization of the divisor sums in Lemma 2.6 according to the identity in (3.2) from Lemma 3.2 in the last appendix summarizing the preliminary theorem from [15]. The realization of this new identity for the Möbius function involving Liouville's function $\lambda(n)$ allows us to effectively reuse the bounds we obtain to prove Theorem 2.3 in Section 2.2 in the new context of the summatory functions $L_{-\alpha}(x)$ which we define and consider in Section 2.3.

2.4.3 Moving forward

Relation to sums over the zeros of the zeta function

Bibliography

[1] T. M. Apostol, *Introduction to analytic number theory*, Springer, 1976.

[2] P. Borwein, R. Ferguson, and M. J. Mossinghoff, Sign Changes in Sums of the Liouville Function, *Mathematics of Computation* **77** (2008), no. 263, pp. 1681–1694.

[3] P. Erdös, On the integers having exactly K prime factors, *Annals of Mathematics*, Vol. 49 (1), 1948, pp. 53–66.

[4] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford University Press, 2008.

[5] P. Humphries, The distribution of weighted sums of the Liouville function and Pólya's conjecture, *Journal of Number Theory*, Volume 133, Issue 2, 2013, pp. 545–582.

[6] G. Hurst, Computations of the Mertens function and improved bounds on the Mertens conjecture, https://arxiv.org/pdf/1610.08551/ (2017).

[7] T. Kotnik and J. van de Lune, On the order of the Mertens function, *Exp. Math.*, 2004.

[8] T. Kotnik and H. te Riele, The Mertens conjecture revisited, in *Algorithmic Number Theory*, 7^{th} International Symposium, Springer-Verlag, 2006.

[9] Mossinghoff M.J., Trudgian T.S. (2017) *The Liouville Function and the Riemann Hypothesis*. In: Montgomery H., Nikeghbali A., Rassias M. (eds) Exploring the Riemann Zeta Function.

[10] M. B. Nathanson, *Additive number theory: the classical bases*, Springer, 1996.

[11] A. M. Odlyzko and H. J. J. te Riele, Disproof of the Mertens conjecture, *J. REINE ANGEW. MATH*, 1985.

[12] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.

[13] P. Ribenboim, *The new book of prime number records*, Springer, 1996.

[14] Y. Saouter and H. te Riele, Improved results on the Mertens conjecture, *Mathematics of Computation* (2014).

[15] M. D. Schmidt, Exact formulas for the generalized sum-of-divisors functions, https://arxiv.org/abs/1705.03488, submitted (2017).

[16] M. B. Villarino, Mertens's proof of Mertens' theorem, 2005, https://arxiv.org/abs/math/0504289.

3 Short Appendix: The proof of Theorem 2.1

3.1 Motivation

Given that the core elements of the proof of the main result in Theorem 2.3 first proved in this article follow from a careful asymptotic treatment of the first results in Theorem 2.1, and its restatement given in (2.2), we provide this appendix on the proof of these first results given by the author in 2017 [15]. The next results present a concise statement of the key results in the proof of that theorem given in the original reference.

Definition 3.1 (Notation)**.** For $n \geq 1$ and any fixed indeterminate q, we define the next rational functions related to the logarithmic derivatives of the cyclotomic polynomials, $\Phi_n(q)$.

$$\Pi_n(q) := \sum_{j=0}^{n-2} \frac{(n-1-j)q^j(1-q)}{(1-q^n)} = \frac{(n-1)+nq-q^n}{(1-q)} \tag{3.1}$$

$$\widetilde{\Phi}_n(q) := \frac{1}{q} \cdot \frac{d}{dw}\left[\log \Phi_n(w)\right]\Big|_{w\rightarrow\frac{1}{q}}.$$

For fixed q and any $n \geq 1$, we define the component sums, $\widetilde{S}_{i,n}(q)$ for $i = 0, 1, 2$ as follows:

$$\begin{aligned}
\widetilde{S}_{0,n}(q) &= \sum_{\substack{d|n \\ d>1 \\ d\neq p^k, 2p^k}} \widetilde{\Phi}_d(q) \\
\widetilde{S}_{1,n}(q) &= \sum_{p|n} \Pi_{p^{\nu_p(n)}}(q) \\
\widetilde{S}_{2,n}(q) &= \sum_{2p|n} \Pi_{p^{\nu_p(n)}}(q).
\end{aligned}$$

3.2 Statements and proof of key components

Lemma 3.2 (Key Characterizations of the Tau Divisor Sums)**.** *For integers $n \geq 1$ and any indeterminate q, we have the following expansion of the functions in* (3.1)*:*

$$\widetilde{\Phi}_n(q) = \sum_{d|n} \frac{d \cdot \mu(n/d)}{(1-q^d)}.$$

In particular, we have that

$$\begin{aligned}
\widetilde{S}_{0,n}(q) &= \sum_{d|n}\sum_{r|d} \frac{r \cdot \widetilde{\chi}_{\mathrm{PP}}(d) \cdot \mu(d/r)}{(1-q^r)} \\
&= \sum_{d|n}\sum_{r|d} \frac{r \cdot \widetilde{\chi}_{\mathrm{PP}}(d) \cdot |\mu(d/r)|\lambda(d/r)}{(1-q^r)}.
\end{aligned}$$

Proof. The proof is essentially the same as the example given in the reference. Since we can refer to this illustrative example, we only need to sketch the details to the remainder of the proof. In particular, we notice that since we have the known identity for the cyclotomic polynomials given by

$$\Phi_n(x) = \prod_{d|n} (1-q^d)^{\mu(n/d)} = \prod_{d|n} (1-q^d)^{|\mu(n/d)|\lambda(n/d)}, \tag{3.2}$$

where $\lambda(n) = (-1)^{\Omega(n)}$ matches the value of $\mu(n)$ exactly when n is squarefree, we can take logarithmic derivatives to obtain that

$$\frac{1}{x} \cdot \frac{d}{dq}\left[\log\left(1-q^d\right)^{\pm 1}\right]\Bigg|_{q \to 1/q} = \mp \frac{d}{q^d\left(1-\frac{1}{q^d}\right)} = \pm \frac{d}{1-q^d},$$

which applied inductively leads us to our result. □

Theorem 3.3 (Exact Formulas for the Generalized Sum-of-Divisors Functions)**.** *For any fixed $\alpha \in \mathbb{C}$ and natural numbers $x \geq 1$, we have the following generating function formula:*

$$\sigma_\alpha(x) = H_x^{(1-\alpha)} + [q^x]\left(\sum_{n=1}^{x} \widetilde{S}_{0,n}(q) n^\alpha + \widetilde{S}_{1,n}(q) n^\alpha + \widetilde{S}_{2,n}(q) n^\alpha\right).$$

Proof. We begin with a well-known divisor product formula involving the cyclotomic polynomials when $n \geq 1$ and q is fixed:

$$q^n - 1 = \prod_{d|n} \Phi_d(q).$$

Then by logarithmic differentiation we can see that

$$\begin{aligned} \frac{q^n}{1-q^n} &= -1 + \frac{1}{n(1-q)} + \sum_{\substack{d|n \\ d>1}} \widetilde{\Phi}_d(q) \\ &= -1 + \frac{1}{n(1-q)} + \widetilde{S}_{0,n} + \widetilde{S}_{1,n} + \widetilde{S}_{2,n}. \end{aligned} \tag{3.3}$$

The last equation is obtained from the first expansion above by noting the equivalence of the next two sums as

$$\Pi_n(1/q) = \widetilde{\Phi}_n(q) = \sum_{j=0}^{n-2} \frac{(n-1-j)q^j(1-q)}{1-q^n},$$

where it is known that

$$\sum_{\substack{d|n \\ d>1}} \widetilde{\Phi}_d(q) = \frac{nq^{n-1}}{q^n-1} - \frac{1}{q-1} = \frac{(n-1)q^{n-2} + (n-2)q^{n-3} + \cdots + 2q + 1}{q^{n-1} + q^{n-2} + \cdots + q + 1}.$$

Here we are implicitly using the known expansions of the cyclotomic polynomials which condense the order n of the polynomials by exponentiation of the indeterminate q when n contains a factor of a prime power given by

$$\Phi_{2p}(q) = \Phi_p(-q), \Phi_{p^k}(q) = \Phi_p\left(q^{p^{k-1}}\right), \Phi_{p^k r}(q) = \Phi_{pr}\left(q^{p^{k-1}}\right), \Phi_{2^k}(q) = q^{2^{k-1}} + 1, \tag{3.4}$$

for p and odd prime, $k \geq 1$, and where $p \not| r$. Finally, we complete the proof by summing the right-hand-side of (3.3) over $n \leq x$ times the weight n^α to obtain the x^{th} partial sum of the Lambert series generating function for $\sigma_\alpha(x)$ [4, §17.10] [12, §27.7], which since each term in the summation contains a power of q^n is $(x+1)$-order accurate to the terms in the infinite series. □

Proposition 3.4 (Series Coefficients of the Component Sums)**.** *For any fixed $\alpha \in \mathbb{C}$ and integers $x \geq 1$, we have the following components of the partial sums of the Lambert series generating functions in Theorem 3.3:*

$$[q^x]\sum_{n=1}^{x} \widetilde{S}_{0,n}(q) n^\alpha = \sum_{d|n} \tau_x^{(\alpha)}(d) \tag{i}$$

$$[q^x]\sum_{n=1}^{x}\widetilde{S}_{1,n}(q)n^{\alpha} = \sum_{p\leq x}\sum_{k=1}^{\varepsilon_p(x)+1} p^{\alpha k-1}H_{\left\lfloor \frac{x}{p^k}\right\rfloor}^{(1-\alpha)}\left(p\left\lfloor \frac{x}{p^k}\right\rfloor - p\left\lfloor \frac{x}{p^k}-\frac{1}{p}\right\rfloor - 1\right) \tag{ii}$$

$$[q^x]\sum_{n=1}^{x}\widetilde{S}_{2,n}(q)n^{\alpha} = \sum_{3\leq p\leq x}\sum_{k=1}^{\varepsilon_p(x)+1} \frac{p^{\alpha k-1}}{2^{1-\alpha}}H_{\left\lfloor \frac{x}{2p^k}\right\rfloor}^{(1-\alpha)}(-1)^{\left\lfloor \frac{x}{p^{k-1}}\right\rfloor}\left(p\left\lfloor \frac{x}{p^k}\right\rfloor - p\left\lfloor \frac{x}{p^k}-\frac{1}{p}\right\rfloor - 1\right). \tag{iii}$$

Proof. The identity in (i) follows from Lemma 3.2. Since $\Phi_{2p}(q) = \Phi_p(-q)$ for any prime p, we are essentially in the same case with the two component sums in (ii) and (iii). We outline the proof of our expansion for the first sum, $\widetilde{S}_{1,n}(q)$, and note the small changes necessary along the way to adapt the proof to the second sum case. By the properties of the cyclotomic polynomials expanded in (3.4) , we may factor the denominators of $\Pi_{p^{\varepsilon_p(n)}}(q)$ into smaller irreducible factors of the same polynomial, $\Phi_p(q)$, with inputs varying as special prime-power powers of q. More precisely, we may expand

$$\widetilde{S}_{1,n}(q) = \sum_{p\leq n}\sum_{k=1}^{\varepsilon_p(n)} \underbrace{\frac{\sum_{j=0}^{p-2}(p-1-j)q^{p^{k-1}j}}{\sum_{i=0}^{p-1}q^{p^{k-1}i}}}_{:=Q_{p,k}^{(n)}(q)} \cdot p^{k-1}.$$

In performing the sum $\sum_{n\leq x}Q_{p,k}^{(n)}(q)p^{k-1}n^{\alpha-1}$, these terms of the $Q_{p,k}^{(n)}(q)$ occur again, or have a repeat coefficient, every p^k terms, so we form the coefficient sums for these terms as

$$\sum_{i=i}^{\left\lfloor \frac{x}{p^k}\right\rfloor}\left(ip^k\right)^{\alpha-1}\cdot p^{k-1} = p^{k\alpha-1}\cdot H_{\left\lfloor \frac{x}{p^k}\right\rfloor}^{(1-\alpha)}.$$

We can also compute the inner sums in the previous equations exactly for any fixed t as

$$\sum_{j=0}^{p-2}(p-1-j)t^j = \frac{(p-1)+pt-t^p}{(1-t)^2},$$

where the corresponding paired denominator sums in these terms are given by $1+t+t^2+\cdots+t^{p-1} = (1-t^p)/(1-t)$. We now assemble the full sum over $n\leq x$ we are after in this proof as follows:

$$\sum_{n\leq x}\widetilde{S}_{1,n}(q)\cdot n^{\alpha-1} = \sum_{p\leq x}\sum_{k=1}^{\varepsilon_p(x)} p^{k\alpha-1}H_{\left\lfloor \frac{x}{p^k}\right\rfloor}^{(1-\alpha)}\frac{(p-1)-pq^{p^{k-1}}+q^{p^k}}{(1-q^{p^{k-1}})(1-q^{p^k})}.$$

The corresponding result for the second sums is obtained similarly with the exception of sign changes on the coefficients of the powers of q in the last expansion.

We compute the series coefficients of one of the three cases in the previous equation to show our method of obtaining the full formula. In particular, the right-most term in these expansions leads to the double sum

$$\begin{aligned} C_{3,x,p} &:= [q^x]\frac{q^{p^k}}{(1\mp q^{p^{k-1}})(1\mp q^{p^k})} \\ &= [q^x]\sum_{n,j\geq 0}(\pm 1)^{n+j}q^{p^{k-1}(n+p+jp)}. \end{aligned}$$

Thus we must have that $p^{k-1}|x$ in order to have a non-zero coefficient and for $n := x/p^{k-1}-jp-p$ with $0\leq j\leq x/p^k-1$ we can compute these coefficients explicitly as

$$C_{3,x,p} := (\pm 1)^{\lfloor x/p^{k-1}\rfloor}\times\sum_{j=0}^{\lfloor x/p^k-1\rfloor}1 = (\pm 1)^{\lfloor x/p^{k-1}\rfloor}\left\lfloor \frac{x}{p^k}-1\right\rfloor + 1 = (\pm 1)^{\lfloor x/p^{k-1}\rfloor}\left\lfloor \frac{x}{p^k}\right\rfloor.$$

With minimal simplifications we have arrived at our claimed result in the proposition. □