# A Appendix: Asymptotic formulas

We thank Gergő Nemes from the Alfréd Rényi Institute of Mathematics for his careful notes on the limiting asymptotics for the sums identified in this section. We have adapted his proofs to establish the next few lemmas.

**Facts A.1** (The incomplete gamma function)**.** The (upper) *incomplete gamma function* is defined by [19, §8.4]

$$\Gamma(a,z) = \int_z^{\infty} t^{a-1} e^{-t} dt, a \in \mathbb{R}, \operatorname{Re}(z) > 0.$$

The following properties of $\Gamma(a,x)$ hold at $z, a > 0$:

$$\Gamma(a,z) = (a-1)! \cdot e^{-z} \times \sum_{k=0}^{a-1} \frac{z^k}{k!}, a \in \mathbb{Z}^+, z > 0, \tag{27a}$$

$$\Gamma(a,z) \sim x^{a-1} \cdot e^{-z}, \text{ for fixed } a > 0, \text{ as } z \to +\infty. \tag{27b}$$

If $z, a \to \infty$ with $z = \lambda a$ for some $\lambda > 0$ such that $(\lambda - 1)^{-1} = o(|a|^{1/2})$, then [14]

$$\Gamma(a,z) = z^a e^{-z} \times \sum_{n \geq 0} \frac{(-a)^n b_n(\lambda)}{(z-a)^{2n+1}}, \tag{27c}$$

where the sequence $b_n(\lambda)$ satisfies the characteristic relation that $b_0(\lambda) = 1$ and[D]

$$b_n(\lambda) = \lambda(1-\lambda) b_{n-1}'(\lambda) + \lambda(2n-1) b_{n-1}(\lambda), n \geq 1.$$

**Proposition A.2.** *Suppose that $a, z, \lambda > 0$ are such that $z = \lambda a$. If $\lambda > 1$, then as $z \to +\infty$*

$$\Gamma(a,z) = \frac{z^{a-1} e^{-z}}{1 - \lambda^{-1}} + O_{\lambda}\left(z^{a-2} e^{-z}\right).$$

*If $\lambda > 0.567142 > W(1)$ where $W(x)$ denotes the principal branch of the Lambert $W$-function, then as $z \to +\infty$*

$$\Gamma(a, z e^{\pm \pi \imath}) = -e^{\pm \pi \imath a} \frac{z^{a-1} e^{z}}{1 + \lambda^{-1}} + O_{\lambda}\left(z^{a-2} e^{z}\right).$$

*Proof.* Using the notation from (27c) and [15], we have that

$$\Gamma(a,z) = \frac{z^{a-1} e^{-z}}{1 - \lambda^{-1}} + z^a e^{-z} R_1(a, \lambda).$$

From the bounds in [15, §3.1], we get

$$|z^a e^{-z} R_1(a,\lambda)| \leq z^a e^{-z} \times \frac{a \cdot b_1(\lambda)}{(z-a)^3} = \frac{z^{a-2} e^{-z}}{(1-\lambda^{-1})^3}$$

The main term in the previous equation can also be seen by applying the asymptotic series in (27c) directly. The proof of the second equation above follows from the following asymptotics [14, Eq. (1.1)]

$$\Gamma(-a,z) \sim z^{-a} e^{-z} \times \sum_{n \geq 0} \frac{a^n b_n(-\lambda)}{(z+a)^{2n+1}},$$

[D] An exact formula for $b_n(\lambda)$ is given in terms of the *second-order Eulerian number triangle* [25, A008517] as follows:

$$b_n(\lambda) = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \lambda^{k+1}.$$

by setting $a \mapsto ae^{\pm\pi\imath}$ and $z \mapsto ze^{\pm\pi\imath}$ with $\lambda = z/a > 0.567142 > W(1)$. Note that we cannot this expansion as $\Gamma(a,-z)$ directly as the incomplete gamma function has branch points with respect to its second variable. This function becomes a single-valued analytic function of its second input on the universal covering of $\mathbb{C}\setminus\{0\}$. The restriction on the range of $\lambda$ over which the second formula holds is made to ensure that the last formula from the reference is valid at negative real $a$. $\square$

**Lemma A.3.** *For $x > e$, we have that*

$$S_1(x) := \frac{x}{\log x} \times \left| \sum_{1 \le k \le \log\log x} \frac{(-1)^k (\log\log x)^{k-1}}{(k-1)!} \right| = \frac{x}{2\sqrt{2\pi \log\log x}} + O\left(\frac{x}{(\log\log x)^{3/2}}\right).$$

*Proof.* We have that for $n \ge 1$ and any $t > 0$ [19, *cf.* §8.4]

$$\sum_{1 \le k \le n} \frac{(-1)^k t^{k-1}}{(k-1)!} = -e^{-t} \times \frac{\Gamma(n,-t)}{(n-1)!}.$$

Suppose that $t = n + \xi$ with $\xi = O(1)$. By the second formula in Proposition A.2 with the parameters $(a,z,\lambda) \mapsto \left(n, t, 1 + \frac{\xi}{n}\right)$, we deduce that as $n, t \to +\infty$.

$$\Gamma(n,-t) = (-1)^{n+1} \times \frac{t^n e^t}{t+n} + O\left(\frac{nt^n e^t}{(t+n)^3}\right) = (-1)^{n+1}\frac{t^n e^t}{2n} + O\left(\frac{t^n e^t}{nt}\right). \tag{28}$$

Accordingly, we see that

$$\sum_{1 \le k \le n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{t^n}{2n!} + O\left(\frac{t^{n-1}}{n!}\right).$$

By a variant of Stirling's formula

$$n! = \Gamma(1+t-\xi) = \sqrt{2\pi} \cdot t^{t-\xi+1/2} e^{-t}\left(1 + O(t^{-1})\right) = \sqrt{2\pi} \cdot t^{n+1/2} e^{-t}\left(1 + O(t^{-1})\right).$$

Hence, as $n \to +\infty$ with $t := n + \xi$ and $\xi = O(1)$, we obtain

$$\sum_{k=1}^{n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{e^t}{2\sqrt{2\pi t}} + O\left(\frac{e^t}{t^{3/2}}\right).$$

The conclusion follows by taking $n, t = \log\log x$ and applying the triangle inequality to obtain the desired result. $\square$

**Lemma A.4.** *For $x > e$, we have that*

$$S_3(x) := \sum_{1 \le k \le \log\log x} \frac{(\log\log x)^{k-1/2}}{(2k-1)(k-1)!} = \frac{(\log x)}{4\sqrt{\log\log x}} + O\left(\frac{\log x}{\log\log x}\right).$$

*Proof.* Notice that we can write

$$\sum_{k \ge 1} \frac{(\log\log x)^{k-1}}{(2k+1)(k-1)!} - \sum_{k \ge 0} \frac{(\log\log x)^k}{(2k+1)k!} \le S_3(x) \le \sum_{k \ge 1} \frac{(\log\log x)^{k-1}}{(2k-1)(k-1)!} - \sum_{k \ge 0} \frac{(\log\log x)^{k-1}}{2 \cdot k!}(1 + o(1)).$$

As $|z| \to \infty$, the *imaginary error function*, denoted by $\operatorname{erfi}(z)$, has the following asymptotic expansion [19, §7.12]:

$$\operatorname{erfi}(z) := \frac{2}{\sqrt{\pi} \cdot \imath} \times \int_0^{\imath z} e^{t^2} dt = \frac{e^{z^2}}{\sqrt{\pi}}\left(z^{-1} + \frac{z^{-3}}{2} + \frac{3z^{-5}}{4} + \frac{15z^{-7}}{8} + O(z^{-9})\right).$$

The symbolic summation procedures in *Mathematica* [24], show that we can arrive at this bound in the form of

$$\frac{\log x}{2\sqrt{\log\log x}} + O\left(\frac{\log x}{\sqrt{\log\log x}}\right) \ll S_3(x) \ll \frac{\log x}{2\sqrt{\log\log x}} + O\left(\frac{\log x}{\log\log x}\right). \tag{29}$$

This implies the bounds given in (29) match the stated formula up to a factor of $\frac{1}{2}$. We will give an exact proof of the result directly using the bounds for the incomplete gamma function established in the recent reference [16]. The reference takes into account the behavior of the incomplete gamma function $\Gamma(a,z)$ near the transition point $\lambda = z/a = 1$ as $a, z \to +\infty$.

For $n \geq 1$ and any $t > 0$, let

$$\widetilde{S}_n(t) := \sum_{1\leq k\leq n} \frac{t^{k-1}}{(2k-1)(k-1)!}.$$

By the formula in (27b) and a change of variable, we get that

$$\begin{aligned}\widetilde{S}_n(t) &= \int_0^1 \left(\sum_{k=1}^n \frac{(s^2t)^{k-1}}{(k-1)!}\right) ds \\ &= \frac{1}{(n-1)!} \times \int_0^1 e^{s^2t}\Gamma(n, s^2t) ds \\ &= \frac{1}{2(n-1)!} \times \int_0^1 e^{xt}\Gamma(n, xt) dx.\end{aligned}$$

Integration by parts performed one time with

$$\left\{\begin{array}{ll} u_x = \Gamma(n, xt) & v_x' = e^{xt} dx \\ u_x' = -t^n x^{n-1} e^{-xt} dx & v_x = \frac{e^{xt}}{t} \end{array}\right\},$$

implies that

$$\begin{aligned}\widetilde{S}_n(t) &= \frac{1}{2(n-1)!} \times \left[\frac{1}{t}\left(\Gamma(n,t)e^t - \Gamma(n,0)\right) + \int_0^t v^{n-1}e^{-v} dv\right] \\ &= \Gamma(n,t)\left(\frac{e^t}{t} - 1\right) + \left(1 - \frac{1}{t}\right)(n-1)!.\end{aligned} \tag{30}$$

Suppose that $t = n + \xi$ where $\xi = O(1)$. According to the reference [16, Eq. (2.4)], we have that as $t \to +\infty$

$$\Gamma(n,t) = t^n e^{-t}\sqrt{\frac{\pi}{2t}}\left(1 + O(t^{-1/2})\right).$$

When we combine each of these estimates for the terms in (30) and let $t \to +\infty$

$$\widetilde{S}_n(t) = \frac{1}{2}\sqrt{\frac{\pi}{2}} \cdot \frac{t^{n-3/2}}{(n-1)!}\left(1 + O\left(\frac{1}{\sqrt{t}}\right)\right).$$

By the variant of Stirling's formula expressed in [19, §5.11(i)], we again have that

$$(n-1)! = \sqrt{2\pi} \cdot t^{n-1/2} e^{-t}\left(1 + O(t^{-1})\right).$$

Whence, with $t = n + O(1)$ and as $n \to +\infty$, we obtain

$$\widetilde{S}_n(t) = \frac{e^t}{4t} + O\left(\frac{e^t}{t^{3/2}}\right).$$

The conclusion follows taking $n = \log\log x$, $t = \log\log x$ and mulitplying $\widetilde{S}_n(t)$ by $(\log\log x)^{1/2}$. □