

# New characterizations of partial sums of the Möbius function

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## Abstract

The Mertens function,  $M(x) := \sum_{n \leq x} \mu(n)$ , is defined as the summatory function of the classical Möbius function for  $x \geq 1$ . The inverse function  $g^{-1}(n) := (\omega + 1)^{-1}(n)$  with respect to Dirichlet convolution is defined in terms of the strongly additive function  $\omega(n)$  that counts the number of distinct prime factors of any integer  $n \geq 2$  without multiplicity. For large  $x$  and  $n \leq x$ , we associate a natural combinatorial significance to the magnitude of the distinct values of the function  $g^{-1}(n)$  that depends directly on the exponent patterns in the prime factorizations of the integers in  $\{2, 3, \dots, x\}$  viewed as multisets. We prove an Erdős-Kac theorem analog for the distribution of the unsigned sequence  $|g^{-1}(n)|$  over  $n \leq x$  with a limiting tendency towards standard normal as  $x \rightarrow \infty$ .

For all  $x \geq 1$ , discrete convolutions of the summatory function  $G^{-1}(x) := \sum_{n \leq x} \lambda(n) |g^{-1}(n)|$  with the prime counting function  $\pi(x)$  determine exact formulas and new characterizations of asymptotic bounds for  $M(x)$ . In this way, we prove another concrete link to the distribution of  $L(x) := \sum_{n \leq x} \lambda(n)$  with the Mertens function and connect these classical summatory functions with an explicit normal tending probability distribution at large  $x$ . The last sections consider asymptotics of  $G^{-1}(x)$  and their relation to the new formulas we establish for  $M(x)$  involving this function.

**Keywords and Phrases:** *Möbius function; Mertens function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function; Erdős-Kac theorem; strongly additive function.*

**Math Subject Classifications (MSC 2010):** *11N37; 11A25; 11N60; 11N64; and 11-04.*

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## Notation and conventions

The next listing provides a glossary of common notation, conventions and abbreviations used throughout the article.

| Symbols                      | Definition  |
|------------------------------|---|
| $\approx, \sim$              | We write that $f(x) \approx g(x)$ if $ f(x) - g(x)  \ll 1$ as $x \rightarrow \infty$ . Two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} = 1$ .   |
| $\chi_{\mathbb{P}}(n), P(s)$ | The indicator function of the primes equals one if and only if $n \in \mathbb{Z}^+$ is prime, and is zero-valued otherwise. For any $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$ , we define the prime zeta function to be the Dirichlet generating function (DGF) defined by $P(s) = \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s}$ . The function $P(s)$ has an analytic continuation to the half-plane $\operatorname{Re}(s) > 0$ by the formula $P(s) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log \zeta(ks)$ with poles at the reciprocal of each positive integer and a natural boundary at the line $\operatorname{Re}(s) = 0$ . |
| $C_k(n), C_{\Omega(n)}(n)$   | The sequence is defined recursively for integers $n \geq 1$ and $k \geq 0$ as follows: $C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum_{d n} \omega(d) C_{k-1}\left(\frac{n}{d}\right), & \text{if } k \geq 1. \end{cases}$ It represents the multiple ( $k$ -fold) convolution of the function $\omega(n)$ with itself. The function $C_{\Omega(n)}(n)$ has the DGF $(1 - P(s))^{-1}$ for $\operatorname{Re}(s) > 1$ .  |
| $[q^n]F(q)$                  | The coefficient of $q^n$ in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function (OGF) of some sequence, $\{f_n\}_{n \geq 0}$ . Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n \geq 0} f_n q^n$ .  |
| $\varepsilon(n)$             | The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$ , defined such that for any arithmetic function $f$ we have that $f * \varepsilon = \varepsilon * f = f$ where the operation $*$ denotes Dirichlet convolution (see definition below).  |
| $f * g$                      | The Dirichlet convolution of $f$ and $g$ is denoted by the divisor sum $(f * g)(n) := \sum_{d n} f(d)g\left(\frac{n}{d}\right)$ for $n \geq 1$ .  |
| $f^{-1}(n)$                  | The Dirichlet inverse $f^{-1}$ of any arithmetic function $f$ exists if and only if $f(1) \neq 0$ . The Dirichlet inverse of any $f$ such that $f(1) \neq 0$ is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \times \sum_{\substack{d n \\ d > 1}} f(d) f^{-1}\left(\frac{n}{d}\right)$ for $n \geq 2$ with $f^{-1}(1) = f(1)^{-1}$ . When it exists, this inverse function is unique and satisfies the characteristic relations that $f^{-1} * f = f * f^{-1} = \varepsilon$ .  |
| $\gg, \ll, \asymp$           | For functions $A, B$ , the notation $A \ll B$ implies that $A = O(B)$ . Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$ . When we have that $A, B \geq 0$ , $A \ll B$ and $B \ll A$ , we write $A \asymp B$ .  |
| $g^{-1}(n), G^{-1}(x)$       | The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ for $x \geq 1$ .  |

| Symbols                                | Definition   |
|--|--|
| $[n = k]_\delta, [\text{cond}]_\delta$ | The symbol $[n = k]_\delta$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$ , and is zero otherwise. For boolean-valued conditions, <b>cond</b> , the symbol $[\text{cond}]_\delta$ evaluates to one precisely when <b>cond</b> is true, and to zero otherwise. This notation is called <i>Iverson's convention</i> .  |
| $\lambda(n), L(x)$                     | The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$ . Its summatory function is defined by $L(x) := \sum_{n \leq x} \lambda(n)$ for all $x \geq 1$ .   |
| $\mu(n), M(x)$                         | The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers $n \geq 1$ where $\mu(n) = (-1)^{\omega(n)}$ whenever $n$ is squarefree. The Mertens function is the summatory function defined for all integers $x \geq 1$ by $M(x) := \sum_{n \leq x} \mu(n)$ .  |
| $\Phi(z)$                              | For $z \in \mathbb{R}$ , we use the notation for the cumulative density function of the standard normal distribution given by $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^z e^{-\frac{t^2}{2}} dt$ .   |
| $\nu_p(n)$                             | The valuation function that extracts the maximal exponent of $p$ in the prime factorization of $n$ , e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^\alpha \parallel n$ for $p \geq 2$ prime, $\alpha \geq 1$ and $n \geq 2$ .  |
| $\omega(n), \Omega(n)$                 | We define the strongly additive function $\omega(n) := \sum_{p n} 1$ and the completely additive function $\Omega(n) := \sum_{p^\alpha \parallel n} \alpha$ . This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$ , then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$ . By convention we set $\omega(1) = \Omega(1) = 0$ . |
| $\pi_k(x), \widehat{\pi}_k(x)$         | For integers $k \geq 1$ , the prime counting function variant $\pi_k(x)$ denotes the number of $2 \leq n \leq x$ with exactly $k$ distinct prime factors: $\pi_k(x) := \#\{2 \leq n \leq x : \omega(n) = k\}$ . Similarly, the function $\widehat{\pi}_k(x) := \#\{2 \leq n \leq x : \Omega(n) = k\}$ for $x \geq 2$ and fixed $k \geq 1$ .  |
| $Q(x)$                                 | For $x \geq 1$ , we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$ . That is, $Q(x) := \sum_{n \leq x} \mu^2(n)$ .   |
| $W(x)$                                 | For $x, y \in \mathbb{R}_{\geq 0}$ , we write that $x = W(y)$ if and only if $xe^x = y$ . This function denotes the principal branch of the multi-valued Lambert $W$ function defined on the non-negative reals.   |
| $\zeta(s)$                             | The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}$ when $\text{Re}(s) > 1$ , and by analytic continuation $s \in \mathbb{C}$ with the exception of a simple pole at $s = 1$ of residue one.   |

# 1 Introduction

The *Mertens function*, or the summatory function of  $\mu(n)$ , is defined on the positive integers by the partial sums

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The first several values of this summatory function begin as follows [27, [A008683](#); [A002321](#)]:

$$\{M(x)\}_{x \geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \dots\}.$$

The Mertens function is related to the partial sums of the Liouville lambda function, denoted by  $L(x) := \sum_{n \leq x} \lambda(n)$  [27, [A008836](#)], via the relation [10, 16]

$$L(x) = \sum_{d \leq \sqrt{x}} M\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), x \geq 1.$$

The main interpretation to take away from the article is our new characterizations of  $M(x)$  through two primary auxiliary unsigned sequences and their summatory functions. This characterization is formed by constructing the combinatorially motivated sequences related to the distribution of the primes by convolutions of strongly additive functions. The methods in this article stem from a curiosity about an elementary identity from the list of exercises in [1, §2; cf. §11]. In particular, the indicator function of the primes is given by Möbius inversion as the Dirichlet convolution  $\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu$ . We form partial sums of  $(\omega + 1) * \mu(n)$  over  $n \leq x$  for any  $x \geq 1$  and then apply classical inversion type theorems to relate  $M(x)$  to the partial sums of  $g^{-1}(n) := (\omega + 1)^{-1}(n)$ .

## 1.1 Motivation

There is a natural relationship of  $g^{-1}(n)$  with the auxiliary function  $C_{\Omega(n)}(n)$ , or the  $\Omega(n)$ -fold Dirichlet convolution of  $\omega(n)$  with itself at  $n$ , which we prove by elementary methods in Section 3. These identities inspire the deep connection between the unsigned inverse function and additive prime counting combinatorics we find in Section 3.3. In this sense, the new results stated within this article diverge from the proof templates typified by previous methods to bound  $M(x)$  cited in the references. The function  $C_{\Omega(n)}(n)$  was considered under alternate notation by Fröberg (circa 1968) in his work on the series expansions of the *prime zeta function*,  $P(s)$ , e.g., the prime sums defined as the *Dirichlet generating function* (DGF) of  $\chi_{\mathbb{P}}(n)$ . The clear interpretation of the function  $C_{\Omega(n)}(n)$  in connection with  $M(x)$  is unique to our work to establish the properties of this auxiliary sequence stated in the next sections. References to uniform asymptotics for restricted partial sums of  $C_{\Omega(n)}(n)$  and the features of the limiting distribution of this function are missing in surrounding literature (cf. Corollary 4.4; Proposition 4.5; and Theorem 4.7).

We utilize the results in [17, §7.4; §2.4] that apply traditional analytic methods to formulate limiting asymptotics and to directly prove an Erdős-Kac theorem characterizing key properties of the distribution of the completely additive function  $\Omega(n)$ . Adaptations of the key ideas from the exposition in the reference provide a foundation for analytic proofs of several limiting properties of, asymptotic formulae for restricted partial sums involving, and a limiting Erdős-Kac type theorem for  $C_{\Omega(n)}(n)$  and  $|g^{-1}(n)|$ . The sequence  $g^{-1}(n)$  and its partial sums defined by  $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$  are linked to canonical examples of strongly and completely additive functions, e.g., in relation to  $\omega(n)$  and  $\Omega(n)$ , respectively. The definitions of the sequences we define, and the proof methods given in the spirit of Montgomery and Vaughan's work, allow us to reconcile the property of strong additivity with the signed partial sums of a multiplicative function. We leverage the connection of  $C_{\Omega(n)}(n)$  and  $|g^{-1}(n)|$  with the canonical number theoretic additive functions to obtain the results proved in Section 4.

Since we prove that  $\text{sgn}(g^{-1}(n)) = \lambda(n)$  for all  $n \geq 1$ , we have a probabilistic perspective from which to express our intuition about features of the distribution of  $G^{-1}(x)$  via the properties of its summands. The partial sums defined by  $G^{-1}(x)$  are precisely related to the properties of  $|g^{-1}(n)|$  and asymptotics for  $L(x)$ . Our new results then relate the distribution of  $L(x)$ , an explicitly identified normal tending probability distribution, and  $M(x)$  as  $x \rightarrow \infty$ . Formalizing the properties of the distribution of  $L(x)$  is still viewed as a problem that is equally as difficult as understanding the properties of  $M(x)$  well at large  $x$  or along infinite subsequences.

Our characterizations of  $M(x)$  by the summatory function of the signed inverse sequence,  $G^{-1}(x)$ , is suggestive of new approaches to bounding the Mertens function. These results motivate future work to state upper (and possibly lower) bounds on  $M(x)$  in terms of the additive combinatorial properties of the repeated distinct values of the sign weighted summands of  $G^{-1}(x)$ . We also expect that an outline of the method behind the collective proofs we provide with respect to the Mertens function can be generalized to identify associated additive functions with the same role of  $\omega(n)$  in this paper to express asymptotics for partial sums of other signed multiplicative functions.

## 1.2 Preliminaries

An approach to evaluating the limiting asymptotic behavior of  $M(x)$  for large  $x \rightarrow \infty$  considers an inverse Mellin transform of the reciprocal of the Riemann zeta function. In particular, since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \times \int_1^\infty \frac{M(x)}{x^{s+1}} dx, \text{Re}(s) > 1,$$

we obtain that

$$M(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \times \int_{T-i\infty}^{T+i\infty} \frac{x^s}{s\zeta(s)} ds.$$

The previous two formulas lead to the exact expression of  $M(x)$  for any  $x > 0$  given by the next theorem.

**Theorem 1.1** (Titchmarsh). *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence  $\{T_k\}_{k \geq 1}$  satisfying  $k \leq T_k \leq k+1$  for each integer  $k \geq 1$  such that for any real  $x > 0$*

$$M(x) = \lim_{k \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < |\text{Im}(\rho)| < T_k}} \text{Re} \left( \frac{x^\rho}{\rho \zeta'(\rho)} \right) - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n(2n)! \zeta(2n+1)} \left( \frac{2\pi}{x} \right)^{2n} + \frac{\mu(x)}{2} [x \in \mathbb{Z}^+]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant  $C_1 > 0$  such that

$$M(x) \ll x \times \exp \left( -C_1 \log^{\frac{3}{5}}(x) (\log \log x)^{-\frac{1}{5}} \right).$$

Under the assumption of the RH, Soundararajan and Humphries, respectively, improved estimates bounding  $M(x)$  from above for large  $x$  in the following forms [28, 10]:

$$\begin{aligned} M(x) &\ll \sqrt{x} \exp \left( (\log x)^{\frac{1}{2}} (\log \log x)^{14} \right), \\ M(x) &\ll \sqrt{x} \exp \left( (\log x)^{\frac{1}{2}} (\log \log x)^{\frac{5}{2} + \epsilon} \right), \text{ for all } \epsilon > 0. \end{aligned}$$

The RH is equivalent to showing that  $M(x) = O \left( x^{\frac{1}{2} + \epsilon} \right)$  for any  $0 < \epsilon < \frac{1}{2}$ . There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C_2 \sqrt{x}, \text{ for some absolute constant } C_2 > 0.$$

The conjecture was first verified by F. Mertens himself for  $C_2 = 1$  and all  $x < 10000$  without the benefit of modern computation. Since its beginnings in 1897, the Mertens conjecture was disproved by computational methods involving non-trivial simple zeta function zeros with comparatively small imaginary parts in the famous paper by Odlyzko and te Riele [22]. More recent attempts at bounding  $M(x)$  naturally consider determining the rates at which the function  $q(x) := M(x)x^{-\frac{1}{2}}$  grows with or without bound along infinite subsequences, i.e., considering the asymptotics of  $q(x)$  in the limit supremum and limit infimum senses.

It is verified by computation that [25, cf. §4.1] [27, cf. [A051400](#); [A051401](#)]

$$\limsup_{x \rightarrow \infty} \frac{M(x)}{\sqrt{x}} > 1.060 \quad (\text{more recently } \geq 1.826054),$$

and

$$\liminf_{x \rightarrow \infty} \frac{M(x)}{\sqrt{x}} < -1.009 \quad (\text{more recently } \leq -1.837625).$$

Based on the work by Odlyzko and te Riele, it is likely that each of these limiting bounds evaluates to  $\pm\infty$ , respectively [22, 14, 15, 11]. A conjecture due to Gonek asserts that in fact  $M(x)$  satisfies [21]

$$\limsup_{x \rightarrow \infty} \frac{|M(x)|}{\sqrt{x}(\log \log \log x)^{\frac{5}{4}}} = C_3,$$

for an absolute constant  $C_3 > 0$ .

### 1.3 A concrete new approach to characterizing $M(x)$

#### 1.3.1 Summatory functions of Dirichlet convolutions of arithmetic functions

We prove the formulas in the next inversion theorem by matrix methods in Section 2.1.

**Theorem 1.2** (Summatory functions of Dirichlet convolutions). *Let  $f, h : \mathbb{Z}^+ \rightarrow \mathbb{C}$  be any arithmetic functions such that  $f(1) \neq 0$ . Suppose that  $F(x) := \sum_{n \leq x} f(n)$  and  $H(x) := \sum_{n \leq x} h(n)$  denote the summatory functions of  $f$  and  $h$ , respectively, and that  $F^{-1}(x) := \sum_{n \leq x} f^{-1}(n)$  denotes the summatory function of the Dirichlet inverse of  $f$  for any  $x \geq 1$ . We have the following exact expressions for the summatory function of the convolution  $f * h$  for all integers  $x \geq 1$ :*

$$\begin{aligned} \pi_{f*h}(x) &:= \sum_{n \leq x} \sum_{d|n} f(d)h\left(\frac{n}{d}\right) \\ &= \sum_{d \leq x} f(d)H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{k=1}^x H(k) \left( F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right). \end{aligned}$$

Moreover, for all  $x \geq 1$

$$\begin{aligned} H(x) &= \sum_{j=1}^x \pi_{f*h}(j) \left( F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \right) \\ &= \sum_{k=1}^x f^{-1}(k) \pi_{f*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right). \end{aligned}$$

Two key consequences of Theorem 1.2 as it applies to the summatory function  $M(x)$  are stated as the next corollaries.

**Corollary 1.3** (Applications of Möbius inversion). *Suppose that  $h$  is an arithmetic function such that  $h(1) \neq 0$ . Define the summatory function of the convolution of  $h$  with  $\mu$  by  $\tilde{H}(x) := \sum_{n \leq x} (h * \mu)(n)$ . Then the Mertens function is expressed by the sum*

$$M(x) = \sum_{k=1}^x \left( \sum_{j=\lfloor \frac{x}{k+1} \rfloor + 1}^{\lfloor \frac{x}{k} \rfloor} h^{-1}(j) \right) \tilde{H}(k), \forall x \geq 1.$$

**Corollary 1.4** (Key Identity). *We have that for all  $x \geq 1$*

$$M(x) = \sum_{k=1}^x (\omega + 1)^{-1}(k) \left( \pi \left( \left\lfloor \frac{x}{k} \right\rfloor \right) + 1 \right). \quad (1)$$

### 1.3.2 An exact expression for $M(x)$ via strongly additive functions

Fix the notation for the Dirichlet invertible function  $g(n) := \omega(n) + 1$  and define its inverse with respect to Dirichlet convolution by  $g^{-1}(n)$  [27, A341444]. We compute the first several values of this sequence exactly as

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \dots\}.$$

There is not a simple direct recursion between the distinct values of  $g^{-1}(n)$  that holds for all  $n \geq 1$ . The distribution of distinct sets of prime exponents is still quite regular since  $\omega(n)$  and  $\Omega(n)$  play a crucial role in the repetition of common values of  $g^{-1}(n)$ . The next observation is suggestive of the quasi-periodicity of the distribution of distinct values of this inverse function over  $n \geq 2$ .

**Observation 1.5** (Additive symmetry in  $g^{-1}(n)$  from the prime factorizations of  $n \leq x$ ). Suppose that  $n_1, n_2 \geq 2$  are such that their factorizations into distinct primes are given by  $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$  and  $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ . If  $\{\alpha_1, \dots, \alpha_r\} \equiv \{\beta_1, \dots, \beta_r\}$  as multisets of prime exponents, then  $g^{-1}(n_1) = g^{-1}(n_2)$ . For example,  $g^{-1}$  has the same values on the squarefree integers with exactly one, two, three (and so on) prime factors. Hence, there is clear additive structure underneath the sequence  $\{g^{-1}(n)\}_{n \geq 2}$ .

**Proposition 1.6.** *We have the following properties of the Dirichlet inverse function  $g^{-1}(n)$ :*

(A) *For all  $n \geq 1$ ,  $\text{sgn}(g^{-1}(n)) = \lambda(n)$ ;*

(B) *For all squarefree integers  $n \geq 1$ , we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \times m!;$$

(C) *If  $n \geq 2$  and  $\Omega(n) = k$  for some  $k \geq 1$ , then*

$$2 \leq |g^{-1}(n)| \leq \sum_{j=0}^k \binom{k}{j} \times j!.$$

The signedness property in (A) is proved precisely in Proposition 2.1. A proof of (B) follows from Lemma 3.1 stated on page 15. The realization that the beautiful and remarkably simple combinatorial form of property (B) in Proposition 1.6 holds for all squarefree  $n \geq 1$  motivates our pursuit of simpler formulas for the inverse functions  $g^{-1}(n)$  through the sums of auxiliary subsequences  $C_k(n)$  with  $k := \Omega(n)$  defined in Section 3. That is, we observe a familiar formula for  $g^{-1}(n)$  on an asymptotically dense infinite subset of integers (with density  $\frac{6}{\pi^2}$ ), e.g., that holds for all squarefree  $n \geq 2$ , and then seek to extrapolate by proving there are regular tendencies of the distribution of this sequence viewed more generally over any  $n \geq 2$ .



An exact expression for  $g^{-1}(n)$  is given by

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d), n \geq 1,$$

where the sequence  $\lambda(n)C_{\Omega(n)}(n)$  has the DGF  $(1 + P(s))^{-1}$  and  $C_{\Omega(n)}(n)$  has the DGF  $(1 - P(s))^{-1}$  for  $\text{Re}(s) > 1$  (see Proposition 2.1). The function  $C_{\Omega(n)}(n)$  has been previously considered in [8] with its exact formula given by [12, cf. §3]

$$C_{\Omega(n)}(n) = \begin{cases} 1, & \text{if } n = 1; \\ (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}, & \text{if } n \geq 2. \end{cases}$$

In Corollary 4.4, we use the result proved in Theorem 4.2 to show uniformly for  $1 \leq k \leq 2 \log \log x$  that there is an absolute constant  $A_0 > 0$  such that

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) = \frac{A_0 \sqrt{2\pi x}}{\log x} \times \widehat{G}\left(\frac{k-1}{\log \log x}\right) \frac{(\log \log x)^{k-\frac{1}{2}}}{(k-1)!} \left(1 + O\left(\frac{1}{\log \log x}\right)\right), \text{ as } x \rightarrow \infty,$$

where we define  $\widehat{G}(z) := \frac{\zeta(2)^{-z}}{\Gamma(1+z)\Gamma(1+P(2)z)}$  for  $0 \leq |z| < P(2)^{-1}$ . In Proposition 4.5, we use an adaptation of the asymptotic formulas for the summations proved in the appendix combined with the form of *Rankin's method* from [17, Thm. 7.20] to show that there is an absolute constant  $B_0 > 0$  such that the average order of the sequence satisfies

$$\frac{1}{n} \times \sum_{k \leq n} C_{\Omega(k)}(k) = B_0 (\log n) \sqrt{\log \log n} \left(1 + O\left(\frac{1}{\log \log n}\right)\right) \text{ as } n \rightarrow \infty.$$

In Corollary 4.6, we prove that the average order of the unsigned inverse sequence is

$$\frac{1}{n} \times \sum_{k \leq n} |g^{-1}(k)| = \frac{6B_0 (\log n)^2 \sqrt{\log \log n}}{\pi^2} \left(1 + O\left(\frac{1}{\log \log n}\right)\right), \text{ as } n \rightarrow \infty.$$

In Section 4.3, we prove a variant of the Erdős-Kac theorem that characterizes the distribution of the sequence  $C_{\Omega(n)}(n)$ . The theorem leads the conclusion of the following statement for any fixed  $Y > 0$ , with  $\mu_x(C) := \log \log x - \log\left(\frac{\sqrt{2\pi A_0}}{\zeta(2)(1+P(2))}\right)$  and  $\sigma_x(C) := \sqrt{\log \log x}$ , that holds uniformly for any  $-Y \leq y \leq Y$  as  $x \rightarrow \infty$  (see Corollary 4.8):

$$\begin{aligned} & \frac{1}{x} \times \# \left\{ 2 \leq n \leq x : \frac{|g^{-1}(n)|}{(\log n) \sqrt{\log \log n}} - \frac{6}{\pi^2 n (\log n) \sqrt{\log \log n}} \times \sum_{k \leq n} |g^{-1}(k)| \leq y \right\} \\ &= \Phi\left(\frac{\frac{\pi^2 y}{6} - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{1}{\sqrt{\log \log x}}\right). \end{aligned}$$

The regularity and quasi-periodicity we have alluded to in the remarks above are then quantifiable inasmuch as  $|g^{-1}(n)|$  tends to a multiple of its average order with a normal tendency. If  $x$  is sufficiently large and if we pick any integer  $n \in [2, x]$  uniformly at random, then the following statement also holds:

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2 n} \times \sum_{k \leq n} |g^{-1}(k)| \leq \frac{6}{\pi^2} (\log n) \sqrt{\log \log n} (\alpha \sigma_x(C) + \mu_x(C))\right) = \Phi(\alpha) + o(1), \alpha \in \mathbb{R}. \quad (\text{D})$$

It follows from the last property that for almost all sufficiently large in integers  $n$ , we have that

$$|g^{-1}(n)| \leq \frac{6}{\pi^2 n} \times \sum_{k \leq n} |g^{-1}(k)| (1 + o(1)).$$

### 1.3.3 Formulas illustrating the new characterizations of $M(x)$

Let the summatory function  $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$  for integers  $x \geq 1$  [27, A341472]. We prove that (see Proposition 5.1)

$$M(x) = G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} G^{-1}(k) \left( \pi \left( \left\lfloor \frac{x}{k} \right\rfloor \right) - \pi \left( \left\lfloor \frac{x}{k+1} \right\rfloor \right) \right), x \geq 1, \quad (2)$$

and that (cf. Section 3.2)

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1} \left( \left\lfloor \frac{x}{p} \right\rfloor \right), x \geq 1.$$

These formulas imply that we can establish asymptotic bounds on  $M(x)$  along infinite subsequences by sharply bounding the summatory function  $G^{-1}(x)$  along those points. We also have the identification of  $G^{-1}(x)$  with  $L(x)$  given by

$$G^{-1}(x) = L(x)|g^{-1}(x)| - \sum_{n < x} L(n) (|g^{-1}(n+1)| - |g^{-1}(n)|),$$

where the distribution of  $|g^{-1}(n)|$  is characterized by Corollary 4.8.

In Section 5.2, we prove that for  $\sigma_1 \approx 1.39943$  the unique solution to  $P(\sigma) = 1$  on  $(1, \infty)$

$$\limsup_{x \rightarrow \infty} \frac{\log |G^{-1}(x)|}{\log x} \geq \sigma_1.$$

Hence, for any  $\epsilon > 0$ , Corollary 5.3 proves that there are arbitrarily large  $x$  such that

$$|G^{-1}(x)| > x^{\sigma_1 - \epsilon}.$$

Nonetheless, we still expect substantial local cancellation in the terms involving  $G^{-1}(x)$  in our new formulas for  $M(x)$  characterized by this summatory function cited above at almost every large  $x$  (see Section 5.3).

## 2 Initial elementary proofs of new results

### 2.1 Establishing the summatory function properties and inversion identities

We will offer a proof of Theorem 1.2 suggested by an intuitive construction through matrix based methods in this section. Related results on summations of Dirichlet convolutions and their inversion appear in [1, §2.14; §3.10; §3.12; cf. §4.9, p. 95]. It is similarly not difficult to prove the identity that

$$\sum_{n \leq x} h(n)(q * r)(n) = \sum_{n \leq x} q(n) \times \sum_{k \leq \lfloor \frac{x}{n} \rfloor} r(k)h(kn).$$

*Proof of Theorem 1.2.* Let  $h, r$  be arithmetic functions such that  $r(1) \neq 0$ . Denote the summatory functions of  $h$  and  $r$ , respectively, by  $H(x) = \sum_{n \leq x} h(n)$  and  $R(x) = \sum_{n \leq x} r(n)$ . We define  $\pi_{r * h}(x)$  to be the summatory function of the Dirichlet convolution of  $r$  with  $h$ . We have that the following formulas hold for all  $x \geq 1$ :

$$\begin{aligned} \pi_{r * h}(x) &:= \sum_{n=1}^x \sum_{d|n} r(n)h\left(\frac{n}{d}\right) = \sum_{d=1}^x r(d)H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^x \left( R\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - R\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right) H(i). \end{aligned} \quad (3)$$

The first formula above is well known in the references. The second formula is justified directly using summation by parts as [23, §2.10(ii)]

$$\begin{aligned} \pi_{r * h}(x) &= \sum_{d=1}^x h(d)R\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i \leq x} \left( \sum_{j \leq i} h(j) \right) \times \left( R\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - R\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right). \end{aligned}$$

We form the invertible matrix of coefficients  $\hat{R}$  associated with this linear system defining  $H(j)$  for all  $1 \leq j \leq x$  in (3) by setting

$$r_{x,j} := R\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - R\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv R_{x,j} - R_{x,j+1},$$

where

$$R_{x,j} := R\left(\left\lfloor \frac{x}{j} \right\rfloor\right), 1 \leq j \leq x.$$

Since  $r_{x,x} = R(1) = r(1)$  and  $r_{x,j} = 0$  for all  $j > x$ , the matrix  $\hat{R}$  we have defined in this problem is lower triangular with a non-zero constant on its diagonals, and is hence invertible. If we let  $\hat{R} := (R_{x,j})$ , then this matrix is expressed by applying an invertible shift operation as

$$(r_{x,j}) = \hat{R}(I - U^T).$$

The square matrix  $U$  of sufficiently large finite dimensions  $N \times N$  has  $(i, j)^{th}$  entries for all  $1 \leq i, j \leq N$  that are defined by  $(U)_{i,j} = \delta_{i+1,j}$  such that

$$[(I - U^T)^{-1}]_{i,j} = [j \leq i]_{\delta}.$$

Observe that

$$\left\lfloor \frac{x}{j} \right\rfloor - \left\lfloor \frac{x-1}{j} \right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

The previous property implies that

$$R\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - R\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} r\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

We use the last property in (4) to shift the matrix  $\hat{R}$ , and then invert the result to obtain a matrix involving the Dirichlet inverse of  $g$  in the following forms:

$$[(I - U^T)\hat{R}]^{-1} = \left(r\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(r^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

In particular, our target matrix in the inversion problem is defined by

$$(r_{x,j}) = (I - U^T) \left(r\left(\frac{x}{j}\right)[j|x]_\delta\right) (I - U^T)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators as

$$\begin{aligned} (r_{x,j})^{-1} &= (I - U^T)^{-1} \left(r^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right) (I - U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} r^{-1}(k)\right) (I - U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} r^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} r^{-1}(k)\right). \end{aligned}$$

Hence, the summatory function  $H(x)$  is given exactly for any integers  $x \geq 1$  by a vector product with the inverse matrix from the previous equation by

$$H(x) = \sum_{k=1}^x \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor+1}^{\left\lfloor \frac{x}{k} \right\rfloor} r^{-1}(j)\right) \times \pi_{r \ast h}(k).$$

We can prove another inversion formula providing the coefficients of the summatory function  $R^{-1}(j)$  for  $1 \leq j \leq x$  from the last equation by adapting our argument to prove (3) above. This leads to the following equivalent identity expressing  $H(x)$ :

$$H(x) = \sum_{k=1}^x r^{-1}(x) \pi_{r \ast h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right). \quad \square$$

## 2.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let  $\chi_{\mathbb{P}}$  denote the characteristic function of the primes, let  $\varepsilon(n) = \delta_{n,1}$  be the multiplicative identity with respect to Dirichlet convolution, and denote by  $\omega(n)$  the strongly additive function that counts the number of distinct prime factors of  $n$  (without multiplicity). We can see using elementary methods that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) \ast \mu. \quad (5)$$

In particular, since  $\mu \ast 1 = \varepsilon$  and

$$\omega(n) = \sum_{p|n} 1 = \sum_{d|n} \chi_{\mathbb{P}}(d), n \geq 1,$$

the result in (5) follows by Möbius inversion. When combined with Corollary 1.3 this convolution identity yields the exact formula for  $M(x)$  stated in (1) of Corollary 1.4.

**Proposition 2.1** (The signedness of  $g^{-1}(n)$ ). *Let the operator  $\text{sgn}(h(n)) = \frac{h(n)}{|h(n)| + [h(n)=0]_\delta} \in \{0, \pm 1\}$  denote the sign of the arithmetic function  $h$  at integers  $n \geq 1$ . For the Dirichlet invertible function  $g(n) := \omega(n) + 1$ , we have that  $\text{sgn}(g^{-1}(n)) = \lambda(n)$  for all  $n \geq 1$ .*

*Proof.* The function  $D_f(s) := \sum_{n \geq 1} f(n)n^{-s}$  defines the *Dirichlet generating function* (DGF) of any arithmetic function  $f(n)$  which is convergent for all  $s \in \mathbb{C}$  satisfying  $\text{Re}(s) > \sigma_f$  with  $\sigma_f$  the abscissa of convergence of the series. Recall that  $D_1(s) = \zeta(s)$ ,  $D_\mu(s) = \zeta(s)^{-1}$  and  $D_\omega(s) = P(s)\zeta(s)$  for  $\text{Re}(s) > 1$ . Then by (5) and the known property that whenever  $f(1) \neq 0$ , the DGF of  $f^{-1}(n)$  is the reciprocal of the DGF of the arithmetic function  $f$ , we have for all  $\text{Re}(s) > 1$  that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{\zeta(s)(1+P(s))}. \quad (6)$$

It follows that  $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$  when we take  $h := \chi_{\mathbb{P}} + \varepsilon$ . We first show that  $\text{sgn}(h^{-1}) = \lambda$ . This observation implies that  $\text{sgn}(h^{-1} * \mu) = \lambda$ .

By a combinatorial argument related to multinomial coefficient expansions of these sums, we recover exactly that [8, cf. §2]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ \lambda(n)(\Omega(n))! \times \prod_{p^\alpha \parallel n} \frac{1}{\alpha!}, & n \geq 2. \end{cases} \quad (7)$$

In particular, by expanding the DGF of  $h^{-1}$  in powers of  $P(s)$  we count that

$$\begin{aligned} \frac{1}{1+P(s)} &= \sum_{n \geq 1} \frac{h^{-1}(n)}{n^s} = \sum_{k \geq 0} (-1)^k P(s)^k \\ &= \sum_{\substack{n \geq 1 \\ n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}}} \frac{(-1)^{\alpha_1 + \alpha_2 + \dots + \alpha_k}}{n^s} \times \binom{\alpha_1 + \alpha_2 + \dots + \alpha_k}{\alpha_1, \alpha_2, \dots, \alpha_k} = \sum_{\substack{n \geq 1 \\ n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}}} \frac{\lambda(n)}{n^s} \times \binom{\Omega(n)}{\alpha_1, \alpha_2, \dots, \alpha_k}. \end{aligned}$$

Since  $\lambda$  is completely multiplicative we have that  $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$  for all divisors  $d|n$  when  $n \geq 1$ . We also know that  $\mu(n) = \lambda(n)$  whenever  $n$  is squarefree, so that we obtain the following result:

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1. \quad \square$$

The conclusion of the proof of Proposition 2.1 implies a stronger result that

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d).$$

We have adopted the notation that for  $n \geq 2$ ,  $C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha \parallel n} (\alpha!)^{-1}$ , where the same function,  $C_0(1)$ , is taken to be one for  $n := 1$  (see Section 3). The scaled functions  $f_1(n) := \frac{C_{\Omega(n)}(n)}{(\Omega(n))!}$  and  $f_2(n) := \frac{\lambda(n)C_{\Omega(n)}(n)}{(\Omega(n))!}$  are multiplicative.

### 2.3 The distributions of $\omega(n)$ and $\Omega(n)$

The next theorems reproduced from [17, §7.4] characterize the relative scarcity of the distributions of  $\omega(n)$  and  $\Omega(n)$  for  $n \leq x$  such that  $\omega(n), \Omega(n) > \log \log x$ . Since  $\frac{1}{n} \times \sum_{k \leq n} \omega(k) = \log \log n + B_1$  and  $\frac{1}{n} \times \sum_{k \leq n} \Omega(k) = \log \log n + B_2$  for  $B_1, B_2 \in (0, 1)$  absolute constants in each case, these results imply a regular tendency of these additive arithmetic functions towards their respective average orders.

**Theorem 2.2** (Upper bounds on exceptional values of  $\Omega(n)$  for large  $n$ ). *For  $x \geq 2$  and  $r > 0$ , let*

$$\begin{aligned} A(x, r) &:= \#\{n \leq x : \Omega(n) \leq r \log \log x\}, \\ B(x, r) &:= \#\{n \leq x : \Omega(n) \geq r \log \log x\}. \end{aligned}$$

*If  $0 < r \leq 1$  and  $x \geq 2$ , then*

$$A(x, r) \ll x(\log x)^{r-1-r \log r}, \quad \text{as } x \rightarrow \infty.$$

*If  $1 \leq r \leq R < 2$  and  $x \geq 2$ , then*

$$B(x, r) \ll_R x(\log x)^{r-1-r \log r}, \quad \text{as } x \rightarrow \infty.$$

Theorem 2.3 is a special case analog to the celebrated Erdős-Kac theorem typically stated for the normally distributed values of the function  $\frac{\omega(n) - \log \log n}{\sqrt{\log \log n}}$  over  $n \leq x$  as  $x \rightarrow \infty$  [17, cf. Thm. 7.21] [13, cf. §1.7].

**Theorem 2.3.** *We have that as  $x \rightarrow \infty$*

$$\#\{3 \leq n \leq x : \Omega(n) \leq \log \log n\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log \log x}}\right).$$

**Theorem 2.4** (Montgomery and Vaughan). *Recall that for integers  $k \geq 1$  and  $x \geq 2$  we have defined*

$$\widehat{\pi}_k(x) := \#\{2 \leq n \leq x : \Omega(n) = k\}.$$

*For  $0 < R < 2$  we have uniformly for all  $1 \leq k \leq R \log \log x$  that*

$$\widehat{\pi}_k(x) = \frac{x}{\log x} \times \mathcal{G}\left(\frac{k-1}{\log \log x}\right) \frac{(\log \log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log \log x)^2}\right)\right),$$

*where we define*

$$\mathcal{G}(z) := \frac{1}{\Gamma(1+z)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, \quad 0 \leq |z| < R.$$

**Remark 2.5.** We can extend the work in [17] on the distribution of  $\Omega(n)$  to find analogous results for the distribution of  $\omega(n)$ . In particular, we have for  $0 < R < 2$  that as  $x \rightarrow \infty$

$$\pi_k(x) = \frac{x}{\log x} \times \widetilde{\mathcal{G}}\left(\frac{k-1}{\log \log x}\right) \frac{(\log \log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log \log x)^2}\right)\right), \quad (8)$$

uniformly for  $1 \leq k \leq R \log \log x$ . The analogous function to express these bounds for  $\omega(n)$  is defined by  $\widetilde{\mathcal{G}}(z) := \widetilde{F}(1, z) \times \Gamma(1+z)^{-1}$  where we take

$$\widetilde{F}(s, z) := \prod_p \left(1 + \frac{z}{p^s - 1}\right) \left(1 - \frac{1}{p^s}\right)^z, \quad \operatorname{Re}(s) > \frac{1}{2}; |z| \leq R < 2.$$

Let the functions

$$\begin{aligned} C(x, r) &:= \#\{n \leq x : \omega(n) \leq r \log \log x\}, \\ D(x, r) &:= \#\{n \leq x : \omega(n) \geq r \log \log x\}. \end{aligned}$$

Then we have upper bounds given by

$$\begin{aligned} C(x, r) &\ll x(\log x)^{r-1-r \log r}, \quad \text{uniformly for } 0 < r \leq 1, \\ D(x, r) &\ll_R x(\log x)^{r-1-r \log r}, \quad \text{uniformly for } 1 \leq r \leq R < 2. \end{aligned}$$

### 3 Auxiliary sequences related to the inverse function $g^{-1}(n)$

The computational data given as Table B in the appendix section is intended to provide clear insight into why we eventually arrived at the stated formulas for  $g^{-1}(n)$  proved in this section. The table provides illustrative numerical data by examining the approximate behavior at hand for the cases of  $1 \leq n \leq 500$  with *Mathematica* [26].

#### 3.1 Definitions and properties of triangular component function sequences

We define the following sequence for integers  $n \geq 1$  and  $k \geq 0$ :

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum_{d|n} \omega(d) C_{k-1}\left(\frac{n}{d}\right), & \text{if } k \geq 1. \end{cases} \quad (9)$$

The Dirichlet inverse  $f^{-1}(n)$  of any arithmetic function  $f$  such that  $f(1) \neq 0$  is computed exactly by an  $\Omega(n)$ -fold convolution of  $f$  with itself. The motivation for considering the auxiliary sequence representing the  $k$ -fold Dirichlet convolution of  $\omega(n)$  with itself follows from our definition of  $g^{-1}(n) := (\omega + 1)^{-1}(n)$ . We prove a few precise relations of the function  $C_{\Omega(n)}(n)$  to the inverse sequence  $g^{-1}(n)$  that result in the next subsections. In fact,  $h^{-1}(n) \equiv \lambda(n)C_{\Omega(n)}(n)$  is the same function given by the formula in (7) of Proposition 2.1.

By recursively expanding the definition of  $C_k(n)$  at any fixed  $n \geq 2$ , we see that we can form a chain of at most  $\Omega(n)$  iterated (or nested) divisor sums by unfolding the definition of (9) inductively. By the same argument, we see that at fixed  $n$ , the function  $C_k(n)$  is seen to be non-zero only for positive integers  $k \leq \Omega(n)$  whenever  $n \geq 2$ . A sequence of relevant signed semi-diagonals of the functions  $C_k(n)$  begins as follows [27, A008480]:

$$\{\lambda(n)C_{\Omega(n)}(n)\}_{n \geq 1} = \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, -3, 2, 2, -1, 4, 1, 2, \dots\}.$$

We see by (7) that  $C_{\Omega(n)}(n) \leq (\Omega(n))!$  for all  $n \geq 1$  with equality precisely at the squarefree integers where  $(\Omega(n))! = (\omega(n))!$ .

#### 3.2 Relating the function $C_{\Omega(n)}(n)$ to exact formulas for $g^{-1}(n)$

**Lemma 3.1.** *For all  $n \geq 1$ , we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

*Proof.* We first expand the recurrence relation for the Dirichlet inverse when  $g^{-1}(1) = g(1)^{-1} = 1$  as

$$g^{-1}(n) = - \sum_{\substack{d|n \\ d>1}} (\omega(d) + 1) g^{-1}\left(\frac{n}{d}\right) \implies (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \quad (10)$$

We argue that for  $1 \leq m \leq \Omega(n)$ , we can inductively expand the implication on the right-hand-side of (10) in the form of  $(g^{-1} * 1)(n) = F_m(n)$  where  $F_m(n) := (-1)^m (C_m(-) * g^{-1})(n)$ , so that

$$F_m(n) = - \begin{cases} \sum_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & 2 \leq m \leq \Omega(n), \\ (\omega * g^{-1})(n), & m = 1. \end{cases}$$

By repeatedly applying the right-hand-side of the previous equation, we find that when  $m := \Omega(n)$  (e.g., with the expansions in the previous equation taken to a maximal depth) we get the relation

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \quad (11)$$

The formula for  $g^{-1}(n)$  then follows from (11) by Möbius inversion.  $\square$

**Corollary 3.2.** *For all positive integers  $n \geq 1$ , we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \quad (12)$$

*Proof.* By applying Lemma 3.1, Proposition 2.1 and the complete multiplicativity of  $\lambda(n)$ , we easily obtain the stated result. In particular, since  $\mu(n)$  is non-zero only at squarefree integers and since at any squarefree  $d \geq 1$  we have  $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$ , Lemma 3.1 implies

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \end{aligned}$$

We see that that  $\lambda(n^2) = +1$  for all  $n \geq 1$  since the number of distinct prime factors (counting multiplicity) of any square integer is even.  $\square$

Since  $C_{\Omega(n)}(n) = |h^{-1}(n)|$  using the notation defined in the the proof of Proposition 2.1, we can see that  $C_{\Omega(n)}(n) = (\omega(n))!$  for all squarefree  $n \geq 1$ . We also have that whenever  $n \geq 1$  is squarefree

$$|g^{-1}(n)| = \sum_{d|n} C_{\Omega(d)}(d).$$

A proof of part (B) of Proposition 1.6 follows as an immediate consequence.

**Remark 3.3.** Combined with the signedness property of  $g^{-1}(n)$  guaranteed by Proposition 2.1, Corollary 3.2 shows that the summatory function of this sequence satisfies

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_{\Omega(d)}(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Equation (5) implies that

$$\lambda(d) C_{\Omega(d)}(d) = (g^{-1} * 1)(d) = (\chi_{\mathbb{P}} + \varepsilon)^{-1}(d).$$

We recover by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1. \quad (13)$$

### 3.3 Combinatorial connections to the distribution of the primes

The combinatorial properties of  $g^{-1}(n)$  are deeply tied to the distribution of the primes  $p \leq n$  as  $n \rightarrow \infty$ . The magnitudes of and spacings between the primes  $p \leq n$  certainly restricts the repeating of these distinct sequence values. We can see that the following is still clear about the relation of the weight functions  $|g^{-1}(n)|$  to the distribution of the primes: The value of  $|g^{-1}(n)|$  is entirely dependent only on the pattern of the exponents (viewed as multisets) of the distinct prime factors of  $n \geq 2$ , rather than on the prime factor weights themselves (*cf.* Observation 1.5). This property implies that  $|g^{-1}(n)|$  has an inherently additive, rather than multiplicative, structure underneath the distribution of its distinct values over  $n \leq x$ .



**Example 3.4.** There is a natural extremal behavior with respect to the distinct values of  $\Omega(n)$  at squarefree integers and prime powers. For integers  $k \geq 1$  we define the infinite sets  $M_k$  and  $m_k$  to correspond to the maximal (minimal) sets of positive integers such that

$$M_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

$$m_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+.$$

Any element of  $M_k$  is squarefree and any element of  $m_k$  is a prime power. Moreover, for any fixed  $k \geq 1$  we have that for any  $N_k \in M_k$  and  $n_k \in m_k$

$$(-1)^k g^{-1}(N_k) = \sum_{j=0}^k \binom{k}{j} \times j!, \quad \text{and} \quad (-1)^k g^{-1}(n_k) = 2.$$

**Remark 3.5.** The formula for the function  $h^{-1}(n) = (g^{-1} * 1)(n)$  defined in the proof of Proposition 2.1 implies that we can express an exact formula for  $g^{-1}(n)$  in terms of symmetric polynomials in the exponents of the prime factorization of  $n$ . For  $n \geq 2$  and  $0 \leq k \leq \omega(n)$  let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z\nu_p(n)) = [z^k] \prod_{p^\alpha || n} (1 + \alpha z).$$

Then we can prove using (7) and (12) that we can expand exact formulas for the signed inverse sequence in the following form:

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, \quad n \geq 2.$$

The combinatorial formula for  $h^{-1}(n) = \lambda(n)(\Omega(n))! \times \prod_{p^\alpha || n} (\alpha!)^{-1}$  we discovered in the proof of the proposition from Section 2<sup>1</sup> suggests additional patterns and regularity in the contributions of the distinct sign weighted terms in the summands of  $G^{-1}(x)$ .

<sup>1</sup>This sequence is also considered using a different motivation based on the DGFs  $(1 \pm P(s))^{-1}$  in [8, §2].

## 4 The distributions of $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$

We observed the intuition in the introduction that the relation of the component functions,  $g^{-1}(n)$  and  $C_{\Omega(n)}(n)$ , to the canonically additive functions  $\omega(n)$  and  $\Omega(n)$  leads to the regular properties of these functions illustrated by the numerical data from Table B. Each of  $\omega(n)$  and  $\Omega(n)$  satisfies an Erdős-Kac theorem that provides a central limiting distribution for each of these functions over  $n \leq x$  as  $x \rightarrow \infty$  [7, 2, 24] (cf. [12]). In the remainder of this section, we use analytic methods in the spirit of [17, §7.4] to prove new properties that characterize the distributions of the auxiliary functions in analogous ways.

### 4.1 Analytic proofs extending bivariate DGF methods for additive functions

**Theorem 4.1.** *Let the bivariate DGF  $\widehat{F}(s, z)$  be defined in terms of the prime zeta function,  $P(s)$ , for  $\operatorname{Re}(s) > 1$  and  $|z| < |P(s)|^{-1}$  by*

$$\widehat{F}(s, z) := \frac{1}{1 + P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

*For  $|z| < P(2)^{-1}$ , the summatory function of the coefficients of  $\widehat{F}(s, z)\zeta(s)^z$  corresponds to*

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

*We have for all sufficiently large  $x$  and any  $|z| < P(2)^{-1}$  that*

$$\widehat{A}_z(x) = \frac{x \widehat{F}(2, z)}{\Gamma(z)} \times (\log x)^{z-1} + O_z \left( x (\log x)^{\operatorname{Re}(z)-2} \right).$$

*Proof.* We can generate exponentially scaled forms of the function  $C_{\Omega(n)}(n)$  by product identity of the following form using the results from the proof of Proposition 2.1:

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left( 1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! p^{rs}} \right)^{-1} = \exp(-zP(s)); \operatorname{Re}(s) \geq 2 \wedge \operatorname{Re}(P(s)z) > -1.$$

This Euler-type product based expansion is related to the construction of the parameterized bivariate DGFs in [17, §7.4]. By computing a termwise Laplace transform applied to the right-hand-side of the above equation, we obtain that

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n) (-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp(-tzP(s)) dt = \frac{1}{1 + P(s)z}; \operatorname{Re}(s) > 1 \wedge \operatorname{Re}(P(s)z) > -1.$$

It follows that

$$\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times \widehat{F}(s, z); \operatorname{Re}(s) > 1 \wedge |z| < |P(s)|^{-1}.$$

Since  $\widehat{F}(s, z)$  is an analytic function of  $s$  for all  $\operatorname{Re}(s) \geq 2$  whenever the parameter  $|z| < |P(s)|^{-1}$ , if the sequence  $\{b_z(n)\}_{n \geq 1}$  indexes the coefficients in the DGF expansion of  $\widehat{F}(s, z)\zeta(s)^z$ , then the series

$$\left| \sum_{n \geq 1} \frac{b_z(n) (\log n)^{2R+1}}{n^s} \right| < +\infty,$$

is uniformly bounded for all  $\operatorname{Re}(s) \geq 2$  and  $|z| \leq R < |P(s)|^{-1} < +\infty$ . This fact follows by repeated termwise differentiation of the series  $\lceil 2R + 1 \rceil$  times with respect to  $s$ .

For fixed  $0 < |z| < 2$ , let the sequence  $d_z(n)$  be generated as the coefficients of the DGF

$$\zeta(s)^z = \sum_{n \geq 1} \frac{d_z(n)}{n^s}, \operatorname{Re}(s) > 1,$$

with corresponding summatory function defined by  $D_z(x) := \sum_{n \leq x} d_z(n)$ . The theorem proved in the reference [17, Thm. 7.17; §7.4] shows that for any  $0 < |z| < 2$  and all integers  $x \geq 2$

$$D_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O_z\left(x(\log x)^{\operatorname{Re}(z)-2}\right).$$

We set  $b_z(n) := (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$ , define the convolution  $a_z(n) := \sum_{d|n} b_z(d) d_z\left(\frac{n}{d}\right)$ , and take its summatory function as  $A_z(x) := \sum_{n \leq x} a_z(n)$ . Then we have that

$$\begin{aligned} A_z(x) &= \sum_{m \leq \frac{x}{2}} b_z(m) D_z\left(\frac{x}{m}\right) + \sum_{\frac{x}{2} < m \leq x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \times \sum_{m \leq \frac{x}{2}} \frac{b_z(m)}{m^2} \times m \log\left(\frac{x}{m}\right)^{z-1} + O\left(\sum_{m \leq x} \frac{x|b_z(m)|}{m^2} \times m \log\left(\frac{2x}{m}\right)^{\operatorname{Re}(z)-2}\right). \end{aligned} \quad (14)$$

We can sum the coefficients  $\frac{b_z(m)}{m}$  for integers  $m \leq u$  with  $u \geq 2$  taken sufficiently large as follows:

$$\sum_{m \leq u} \frac{b_z(m)}{m} = (\widehat{F}(2, z) + O_z(u^{-2})) u - \int_1^u (\widehat{F}(2, z) + O_z(t^{-2})) dt = \widehat{F}(2, z) + O_z(u^{-1}).$$

Suppose that  $0 < |z| \leq R < P(2)^{-1} \approx 2.21118$ . For large  $x$ , the error term in (14) satisfies

$$\begin{aligned} \sum_{m \leq x} \frac{x|b_z(m)|}{m^2} \times m \log\left(\frac{2x}{m}\right)^{\operatorname{Re}(z)-2} &\ll x(\log x)^{\operatorname{Re}(z)-2} \times \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m} \\ &\quad + x(\log x)^{-(R+2)} \times \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m} (\log m)^{2R} \\ &= O_z\left(x(\log x)^{\operatorname{Re}(z)-2}\right), \end{aligned}$$

whenever  $0 < |z| \leq R$ . When  $m \leq \sqrt{x}$  we have

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\operatorname{Re}(z)-2}\right).$$

A related upper bound is obtained for the left-hand-side of the previous equation when  $\sqrt{x} < m < x$  and  $0 < |z| < R$ . The combined sum over the interval  $m \leq \frac{x}{2}$  corresponds to bounding the sum components when we take  $0 < |z| \leq R$  as follows:

$$\begin{aligned} \sum_{m \leq \frac{x}{2}} b_z(m) D_z\left(\frac{x}{m}\right) &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \times \sum_{m \leq \frac{x}{2}} \frac{b_z(m)}{m} \\ &\quad + O_R\left(x(\log x)^{\operatorname{Re}(z)-2} \times \sum_{m \leq \sqrt{x}} \frac{|b_z(m)| \log m}{m} + x(\log x)^{R-1} \times \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m}\right) \\ &= \frac{x \widehat{F}(2, z)}{\Gamma(z)} (\log x)^{z-1} + O_R\left(x(\log x)^{\operatorname{Re}(z)-2} \times \sum_{m \geq 1} \frac{b_z(m) (\log m)^{2R+1}}{m^2}\right) \\ &= \frac{x \widehat{F}(2, z)}{\Gamma(z)} (\log x)^{z-1} + O_R\left(x(\log x)^{\operatorname{Re}(z)-2}\right). \end{aligned} \quad \square$$

**Theorem 4.2.** For large  $x$  and integers  $k \geq 1$ , let

$$\widehat{C}_{k,*}(x) := \sum_{\substack{n \leq x \\ \Omega(n)=k}} (-1)^{\omega(n)} C_k(n)$$

Let the function  $\widehat{G}(z) := \widehat{F}(2, z) \times \Gamma(1+z)^{-1}$  for  $0 \leq |z| < P(2)^{-1}$  where  $\widehat{F}(s, z)$  is defined as in Theorem 4.1 for  $\operatorname{Re}(s) > 1$ . As  $x \rightarrow +\infty$ , we have uniformly that for any  $1 \leq k \leq 2 \log \log x$

$$\widehat{C}_{k,*}(x) = -\widehat{G}\left(\frac{k-1}{\log \log x}\right) \frac{x}{\log x} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \left(1 + O\left(\frac{k}{(\log \log x)^2}\right)\right).$$

*Proof.* When  $k = 1$ , we have that  $\Omega(n) = \omega(n)$  for all  $n \leq x$  such that  $\Omega(n) = k$ . The  $n \leq x$  that satisfy this requirement are precisely the primes  $p \leq x$ . Thus we get that the formula is satisfied as

$$\sum_{p \leq x} (-1)^{\omega(p)} C_1(p) = -\sum_{p \leq x} 1 = -\frac{x}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Since  $O((\log x)^{-1}) = O((\log \log x)^{-2})$  as  $x \rightarrow \infty$ , we obtain the required error term bound at  $k = 1$ .

For  $2 \leq k \leq 2 \log \log x$ , we will apply the error estimate from Theorem 4.1 at  $r := \frac{k-1}{\log \log x}$  in the formula

$$\widehat{C}_{k,*}(x) = \frac{1}{2\pi i} \times \int_{|v|=r} \frac{\widehat{A}_{-v}(x)}{v^{k+1}} dv.$$

At large  $x$ , the error term in the formula contributes terms that are bounded from above by

$$\begin{aligned} \left| x(\log x)^{\operatorname{Re}(v)-2} v^{-(k+1)} \right| &\ll x(\log x)^{r-2} r^{-(k+1)} \ll \frac{x}{(\log x)^2} \cdot \frac{(\log \log x)^{k+1}}{(k-1)^{k+1}} \cdot e^{k-1} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log \log x)^{k+1}}{(k-1)^{\frac{1}{2}}(k-1)!} \ll \frac{x}{\log x} \cdot \frac{k(\log \log x)^{k-5}}{(k-1)!}, \text{ as } x \rightarrow \infty. \end{aligned}$$

We next find an asymptotically accurate approximation to the main term of the coefficients of the following contour integral for  $r \in [0, z_{\max}] \subseteq [0, P(2)^{-1}]$ :

$$\widehat{C}_{k,*}(x) \sim \frac{(-1)^k x}{\log x} \times \int_{|v|=r} \frac{(\log x)^{-v} \zeta(2)^v}{\Gamma(1-v) v^k (1-P(2)v)} dv. \quad (15)$$

The main term for the sums  $\widehat{C}_{k,*}(x)$  is given by  $-\frac{x}{\log x} \times I_k(r, x)$ , where we take

$$\begin{aligned} I_k(r, x) &= \frac{(-1)^{k-1}}{2\pi i} \times \int_{|v|=r} \frac{\widehat{G}(-v)(\log x)^{-v}}{v^k} dv \\ &=: I_{1,k}(r, x) + I_{2,k}(r, x). \end{aligned}$$

Taking  $r = \frac{k-1}{\log \log x}$ , the first of the component integrals in the last equation is defined to be

$$I_{1,k}(r, x) := \frac{(-1)^{k-1} \widehat{G}(-r)}{2\pi i} \times \int_{|v|=r} \frac{(\log x)^{-v}}{v^k} dv = \widehat{G}(-r) \times \frac{(\log \log x)^{k-1}}{(k-1)!}.$$

The second component integral,  $I_{2,k}(r, x)$ , corresponds to error terms in our approximation that we must bound. This function is defined by

$$I_{2,k}(r, x) := \frac{(-1)^{k-1}}{2\pi i} \times \int_{|v|=r} (\widehat{G}(-r) - \widehat{G}(-v)) \times \frac{(\log x)^{-v}}{v^k} dv.$$

After integrating by parts [17, cf. Thm. 7.19; §7.4], we have that

$$I_{2,k}(r, x) := \frac{(-1)^{k-1}}{2\pi i} \times \int_{|v|=r} (\widehat{G}(-r) - \widehat{G}(-v) - \widehat{G}'(-r)(v+r)) (\log x)^{-v} v^{-k} dv.$$

We find that

$$|\widehat{G}(-r) - \widehat{G}(-v) - \widehat{G}'(-r)(v+r)| = \left| \int_{-r}^{-v} (v+w) \widehat{G}''(-w) dw \right| \ll |v+r|^2.$$

With the parameterization  $v = -re^{2\pi i\theta}$  for real  $\theta \in [-\frac{1}{2}, \frac{1}{2}]$  and with  $r := \frac{k-1}{\log \log x}$  we get that

$$|I_{2,k}(r, x)| \ll r^{3-k} \int_{-\frac{1}{2}}^{\frac{1}{2}} (\sin \pi\theta)^2 e^{(k-1)\cos(2\pi\theta)} d\theta.$$

Since  $|\sin x| \leq |x|$  for all  $|x| < 1$  and  $\cos(2\pi\theta) \leq 1 - 8\theta^2$  whenever  $-\frac{1}{2} \leq \theta \leq \frac{1}{2}$ , we obtain bounds of the next forms by again setting  $r = \frac{k-1}{\log \log x}$  for any  $1 \leq k \leq 2 \log \log x$ .

$$\begin{aligned} |I_{2,k}(r, x)| &\ll r^{3-k} e^{k-1} \times \int_0^\infty \theta^2 e^{-8(k-1)\theta^2} d\theta \\ &\ll \frac{r^{3-k} e^{k-1}}{(k-1)^{\frac{3}{2}}} \ll \frac{(\log \log x)^{k-3} e^{k-1}}{(k-1)^{\frac{3}{2}} (k-1)^{k-3}} \ll \frac{k(\log \log x)^{k-3}}{(k-1)!}. \end{aligned}$$

Finally, we see that whenever  $1 \leq k \leq 2 \log \log x$

$$1 = \widehat{G}(0) \geq \widehat{G}\left(\frac{k-1}{\log \log x}\right) = \frac{1}{\Gamma\left(1 + \frac{k-1}{\log \log x}\right)} \cdot \frac{\zeta(2)^{\frac{1-k}{\log \log x}}}{\left(1 + \frac{P(2)(k-1)}{\log \log x}\right)} \geq \widehat{G}(2) \approx 0.097027.$$

In particular, the function satisfies  $\widehat{G}\left(\frac{k-1}{\log \log x}\right) \gg 1$  for all  $1 \leq k \leq 2 \log \log x$ . This implies the result of the theorem.  $\square$

**Lemma 4.3.** *As  $x \rightarrow \infty$ , we have that*

$$\left| \sum_{n \leq x} (-1)^{\omega(n)} \right| = \frac{(-1)^{\lfloor \log \log x \rfloor} x}{A_0 \sqrt{2\pi \log \log x}} + O\left(\frac{x}{(\log \log x)}\right).$$

*Proof.* An adaptation of the proof of Lemma A.3 from the appendix shows that for any  $a \in (1, 1.76322)$  we have

$$\begin{aligned} S_a(x) &:= \frac{x}{\log x} \times \left| \sum_{k=1}^{\lfloor a \log \log x \rfloor} \frac{(-1)^k (\log \log x)^{k-1}}{(k-1)!} \right| \\ &= \frac{\sqrt{a}x}{\sqrt{2\pi}(a+1)a^{\{a \log \log x\}}} \cdot \frac{(\log x)^{a-1-a \log a}}{\sqrt{\log \log x}} \left( 1 + O\left(\frac{1}{\log \log x}\right) \right), \end{aligned} \quad (16)$$

where  $\{x\} = x - \lfloor x \rfloor \in [0, 1)$  is the *fractional part* of  $x$ . Suppose that we take  $a := \frac{3}{2}$  so that  $a - 1 - a \log a = \frac{1}{2} \left(1 - 3 \log\left(\frac{3}{2}\right)\right) \approx -0.108198$ . We define the summatory function

$$L_{**}(x) := \sum_{n \leq x} (-1)^{\omega(n)} = \sum_{k \leq \log \log x} 2 \cdot (-1)^k \pi_k(x) + S_{\frac{3}{2}}(x) + O\left(\#\left\{n \leq x : \omega(n) \geq \frac{3}{2} \log \log x\right\}\right).$$

We can show that for any  $1 < k \leq \log \log x$ , the function  $\widetilde{\mathcal{G}}(z)$  defined in Remark 2.5 is decreasing with  $\widetilde{\mathcal{G}}(0) = 1$  and

$$\widetilde{\mathcal{G}}\left(\frac{k-1}{\log \log x}\right) \geq \widetilde{\mathcal{G}}\left(1 - \frac{1}{\log \log x}\right) \geq \widetilde{\mathcal{G}}(1) = 1.$$

By applying the uniform asymptotics for  $\pi_k(x)$  as  $x \rightarrow \infty$  when  $1 \leq k \leq R \log \log x$  for  $1 \leq R < 2$ , we have by Lemma A.3 and (16) that at large  $x$  there is an absolute constant  $A_0 > 0$  such that

$$L_{**}(x) = \frac{(-1)^{\lfloor \log \log x \rfloor} x}{A_0 \sqrt{2\pi \log \log x}} + O\left(E_\omega(x) + \frac{x}{(\log x)^{0.108198} \sqrt{\log \log x}} + \#\left\{n \leq x : \omega(x) \geq \frac{3}{2} \log \log x\right\}\right).$$

The error term in the previous equation corresponds to the bounds on the next sum as  $x \rightarrow \infty$ . The following estimate is obtained from Stirling's formula, (26a) and (26c):

$$\begin{aligned} E_\omega(x) &\ll \frac{x}{\log x} \times \sum_{1 \leq k \leq \log \log x} \frac{(\log \log x)^{k-1}}{(k-1)!} \\ &= \frac{x \cdot \Gamma(\log \log x, \log \log x)}{\Gamma(\log \log x)} \sim \frac{x}{2 \log \log x} \left(1 + O\left(\frac{1}{\sqrt{\log \log x}}\right)\right). \end{aligned}$$

Similarly, by applying the second set of results stated in Remark 2.5, we see that

$$\#\left\{n \leq x : \omega(x) \geq \frac{3}{2} \log \log x\right\} \ll \frac{x}{(\log x)^{0.108198}}. \quad \square$$

**Corollary 4.4.** *We have at large  $x$  that uniformly for  $1 \leq k \leq 2 \log \log x$*

$$\widehat{C}_k(x) := \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) = A_0 \sqrt{2\pi x} \cdot \widehat{G}\left(\frac{k-1}{\log \log x}\right) \frac{(\log \log x)^{k-\frac{1}{2}}}{(k-1)!} \left(1 + O\left(\frac{1}{\log \log x}\right)\right).$$

*Proof.* Suppose that  $h(t)$  and  $\sum_{n \leq t} \lambda_*(n)$  are any piecewise sufficiently smooth and differentiable functions on  $\mathbb{R}^+$ . We have integral formulas that result by applying Abel summation and integration by parts in the next equations.

$$\sum_{n \leq x} \lambda_*(n) h(n) = \left(\sum_{n \leq x} \lambda_*(n)\right) h(x) - \int_1^x \left(\sum_{n \leq t} \lambda_*(n)\right) h'(t) dt \quad (17a)$$

$$\sim \int_1^x \frac{d}{dt} \left[\sum_{n \leq t} \lambda_*(n)\right] h(t) dt \quad (17b)$$

Let the signed left-hand-side summatory function of our function  $\lambda_*(n)$  in (17a) when  $h(n) := C_{\Omega(n)}(n)$  be defined precisely for large  $x$  and any integers  $1 \leq k \leq 2 \log \log x$  by

$$\begin{aligned} \widehat{C}_{k,*}(x) &:= \sum_{\substack{n \leq x \\ \Omega(n)=k}} (-1)^{\omega(n)} C_{\Omega(n)}(n) \\ &= -\frac{x}{\log x} \times \widehat{G}\left(\frac{k-1}{\log \log x}\right) \frac{(\log \log x)^{k-1}}{(k-1)!} \left(1 + O\left(\frac{1}{\log \log x}\right)\right). \end{aligned}$$

The second equation above follows from Theorem 4.2 for large  $x$ .

We transform our previous results for the partial sums over the signed sequences  $(-1)^{\omega(n)} C_{\Omega(n)}(n)$  such that  $\Omega(n) = k$  to approximate the same sums over only the unsigned summands  $C_{\Omega(n)}(n)$ . In particular, since  $1 \leq k \leq 2 \log \log x$ , we have that

$$\widehat{C}_{k,*}(x) = \sum_{\substack{n \leq x \\ \Omega(n)=k}} (-1)^{\omega(n)} C_{\Omega(n)}(n) = \sum_{n \leq x} (-1)^{\omega(n)} [\omega(n) \leq 2 \log \log x]_\delta \times C_{\Omega(n)}(n) [\Omega(n) = k]_\delta.$$

We have by Lemma 4.3 that as  $t \rightarrow \infty$

$$L_*(t) := \sum_{\substack{n \leq t \\ \omega(n) \leq 2 \log \log t}} (-1)^{\omega(n)} = \sum_{k=1}^{2 \log \log t} (-1)^k \pi_k(t)$$

$$= \frac{(-1)^{\lfloor \log \log t \rfloor} t}{A_0 \sqrt{2\pi \log \log t}} \left( 1 + O\left( \frac{1}{\sqrt{\log \log t}} \right) \right). \quad (18)$$

Except for  $t$  within a subset of  $(0, \infty)$  with measure zero where  $L_*(t)$  changes sign, the main term of the derivative of this summatory function is given by

$$L'_*(t) \sim \frac{(-1)^{\lfloor \log \log t \rfloor}}{A_0 \sqrt{2\pi \log \log t}}.$$

We apply the formula from (17b), to deduce that as  $x \rightarrow \infty$

$$\begin{aligned} \widehat{C}_{k,*}(x) &\sim \sum_{j=1}^{\log \log x - 1} \frac{2 \cdot (-1)^j}{A_0 \sqrt{2\pi}} \times \int_{e^{e^j}}^{e^{e^{j+1}}} \frac{C_{\Omega(t)}(t) [\Omega(t) = k]_\delta}{\sqrt{\log \log t}} dt \\ &\sim \int_1^{\frac{\log \log x}{2}} \int_{e^{e^{2s-1}}}^{e^{e^{2s}}} \frac{2C_{\Omega(t)}(t) [\Omega(t) = k]_\delta}{A_0 \sqrt{2\pi \log \log t}} dt ds. \end{aligned}$$

For fixed large  $x$ ,  $\widehat{C}_{k,*}(x)$  is a monotone function of  $k$  on  $[1, 2 \log \log x]$ . Hence, we have that

$$A_0 \sqrt{2\pi} x (\log x) \widehat{C}'_{k,*}(x) \sim \int_{x^{e^{-1}}}^x \frac{C_{\Omega(t)}(t) [\Omega(t) = k]_\delta}{\sqrt{\log \log t}} dt.$$

For large  $x$ , since  $(\log \log t)^{-\frac{1}{2}}$  is continuous and monotone decreasing on  $[x^{e^{-1}}, x]$ , and since

$$\frac{1}{\sqrt{\log \log x}} - \frac{1}{\sqrt{\log \log (x^{e^{-1}})}} = \frac{1}{\sqrt{\log \log x}} \left( 1 + O\left( \frac{1}{\log \log x} \right) \right),$$

we find

$$\widehat{C}_k(x) = A_0 \sqrt{2\pi} x (\log x) \sqrt{\log \log x} \cdot \widehat{C}'_{k,*}(x) \left( 1 + O\left( \frac{1}{\log \log x} \right) \right). \quad (19)$$

Note that because  $C_{\Omega(n)}(n) \geq 1$ , we have that  $\widehat{C}_k(x) \gg \widehat{C}_k(x^{e^{-1}})$  and that this function is bounded below by the right-hand-side of (19) by the asymptotics for  $\widehat{\pi}_k(x)$  stated in Theorem 2.4.  $\square$

## 4.2 Average orders of the unsigned sequences

**Proposition 4.5.** *We have that there is an absolute constant  $B_0 > 0$  such that as  $n \rightarrow \infty$  we have*

$$\frac{1}{n} \times \sum_{k \leq n} C_{\Omega(k)}(k) = B_0 (\log n) \sqrt{\log \log n} \left( 1 + O\left( \frac{1}{\log \log n} \right) \right).$$

*Proof.* By Corollary 4.4 and Proposition A.2 with  $\lambda = \frac{1}{2}$ , we have that

$$\begin{aligned} \sum_{k=1}^{2 \log \log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) &\asymp \sum_{k=1}^{2 \log \log x} \frac{x (\log \log x)^{k-\frac{1}{2}}}{(k-1)!} \left( 1 + O\left( \frac{1}{\log \log x} \right) \right) \\ &= \frac{x (\log x) \sqrt{\log \log x} \cdot \Gamma(2 \log \log x, \log \log x)}{\Gamma(2 \log \log x)} \left( 1 + O\left( \frac{1}{\log \log x} \right) \right) \\ &= x (\log x) \sqrt{\log \log x} \left( 1 + O\left( \frac{1}{(\log x)^{0.386294} \sqrt{\log \log x}} \right) \right) \left( 1 + O\left( \frac{1}{\log \log x} \right) \right). \end{aligned}$$

For  $0 \leq z \leq 2$ , the function  $\widehat{G}(z)$  is a decreasing function of  $z$  with  $\widehat{G}(0) = 1$ . Then we see that there is an absolute constant  $B_0 > 0$  such that

$$\sum_{k=1}^{2 \log \log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) = B_0 x (\log x) \sqrt{\log \log x} \left( 1 + O\left(\frac{1}{\log \log x}\right) \right).$$

We claim that

$$\begin{aligned} \frac{1}{x} \times \sum_{n \leq x} C_{\Omega(n)}(n) &= \frac{1}{x} \times \sum_{k \geq 1} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \\ &= \frac{1}{x} \times \sum_{k=1}^{2 \log \log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) (1 + o(1)), \text{ as } x \rightarrow \infty. \end{aligned}$$

To prove the claim it suffices to show that

$$\frac{1}{x} \times \sum_{\substack{n \leq x \\ \Omega(n) > 2 \log \log x}} C_{\Omega(n)}(n) = o\left((\log x) \sqrt{\log \log x}\right). \quad (20)$$

We proved in Theorem 4.1 that for all sufficiently large  $x$

$$\sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)} = \frac{x \widehat{F}(2, z)}{\Gamma(z)} (\log x)^{z-1} + O\left(x (\log x)^{\operatorname{Re}(z)-2}\right).$$

By Lemma 4.3, we have that the summatory function

$$\left| \sum_{n \leq x} (-1)^{\omega(n)} \right| \leq \frac{x}{A_0 \sqrt{2\pi \log \log x}} \left( 1 + O\left(\frac{1}{\sqrt{\log \log x}}\right) \right),$$

where  $\frac{d}{dx} \left[ \frac{x}{\sqrt{\log \log x}} \right] = \frac{1}{\sqrt{\log \log x}} + o(1)$ . We can argue as in the proof of Corollary 4.4 using integration by parts with the Abel summation formula that whenever  $1 < |z| < P(2)^{-1}$  and when  $x$  is sufficiently large we have that

$$\begin{aligned} \sum_{n \leq x} C_{\Omega(n)}(n) z^{\Omega(n)} &\ll \frac{\widehat{F}(2, z) x (\log x) \sqrt{\log \log x}}{\Gamma(z)} \times \frac{\partial}{\partial x} [x (\log x)^{z-1}] \\ &\ll \frac{\widehat{F}(2, z) x \sqrt{\log \log x}}{\Gamma(z)} (\log x)^z. \end{aligned} \quad (21)$$

For large  $x$  and any fixed  $0 < r < P(2)^{-1}$ , we define

$$\widehat{B}(x, r) := \sum_{\substack{n \leq x \\ \Omega(n) \geq r \log \log x}} C_{\Omega(n)}(n).$$

We adapt the proof from the reference [17, cf. Thm. 7.20; §7.4] by applying (21) when  $1 \leq r < P(2)^{-1}$ . Since  $r \cdot \widehat{F}(2, r) = \frac{r \cdot \zeta(2)^{-r}}{1 + P(2)^r} \ll 1$  for  $r \in [1, P(2)^{-1})$ , and similarly since we have that  $\frac{1}{\Gamma(1+r)} \gg 1$  for  $r$  taken within this same range, we find that

$$x \sqrt{\log \log x} (\log x)^r \gg \sum_{\substack{n \leq x \\ \Omega(n) \geq r \log \log x}} C_{\Omega(n)}(n) r^{\Omega(n)} \gg \sum_{\substack{n \leq x \\ \Omega(n) \geq r \log \log x}} C_{\Omega(n)}(n) r^{r \log \log x},$$

for all  $1 \leq r < P(2)^{-1}$ . This implies that for  $r := 2$  we have

$$\widehat{B}(x, r) \ll x (\log x)^{r-r \log r} \sqrt{\log \log x} = O\left(x (\log x)^{0.613706} \sqrt{\log \log x}\right) \quad (22)$$



We need to evaluate the limiting asymptotics of the sum

$$S_2(x) := \frac{1}{x} \times \sum_{k > 2 \log \log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \ll \frac{1}{x} \times \widehat{B}(x, 2).$$

We have shown that  $S_2(x) = O((\log x)^{0.613706} \sqrt{\log \log x})$  as  $x \rightarrow \infty$ . This implies that (20) holds.  $\square$

**Corollary 4.6.** *We have that as  $n \rightarrow \infty$*

$$\frac{1}{n} \times \sum_{k \leq n} |g^{-1}(k)| = \frac{6B_0(\log n)^2 \sqrt{\log \log n}}{\pi^2} \left( 1 + O\left(\frac{1}{\log \log n}\right) \right).$$

*Proof.* We use the formula from Proposition 4.5 to sum the average order of  $C_{\Omega(n)}(n)$  as  $n \rightarrow \infty$ . This result and (23) imply that for all sufficiently large  $t \rightarrow +\infty$

$$\begin{aligned} \int \frac{\sum_{n \leq t} C_{\Omega(n)}(n)}{t^2} dt &= B_0(\log t)^2 \sqrt{\log \log t} - \frac{1}{4} \sqrt{\frac{\pi}{2}} \operatorname{erfi}\left(\sqrt{2 \log \log t}\right) \\ &= B_0(\log t)^2 \sqrt{\log \log t} \left( 1 + O\left(\frac{1}{\log \log t}\right) \right). \end{aligned}$$

The error term in the last equation is obtained from the asymptotic series in (23). In particular, as  $|z| \rightarrow \infty$ , the *imaginary error function*,  $\operatorname{erfi}(z)$ , has the following asymptotic expansion [23, §7.12]:

$$\operatorname{erfi}(z) := \frac{2}{\sqrt{\pi} \cdot i} \times \int_0^{iz} e^{t^2} dt = \frac{e^{z^2}}{\sqrt{\pi}} \left( z^{-1} + \frac{z^{-3}}{2} + \frac{3z^{-5}}{4} + \frac{15z^{-7}}{8} + O(z^{-9}) \right). \quad (23)$$

The summatory function that counts the number of *squarefree* integers  $n \leq x$  satisfies [9, §18.6] [27, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) = \frac{6x}{\pi^2} + O(\sqrt{x}), \text{ as } x \rightarrow \infty.$$

Therefore summing over the formula from (12) we find that

$$\begin{aligned} \frac{1}{n} \times \sum_{k \leq n} |g^{-1}(k)| &= \frac{1}{n} \times \sum_{d \leq n} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{n}{d} \right\rfloor\right) \\ &\sim \sum_{d \leq n} C_{\Omega(d)}(d) \left[ \frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dn}}\right) \right] \\ &= \frac{6}{\pi^2} \left[ \frac{1}{n} \times \sum_{k \leq n} C_{\Omega(k)}(k) + \sum_{d < n} \sum_{k \leq d} \frac{C_{\Omega(k)}(k)}{d^2} \right] + O(1). \end{aligned} \quad \square$$

### 4.3 Erdős-Kac theorem analogs for the distributions of the unsigned auxiliary functions

**Theorem 4.7.** *For large  $x > e$ , let the mean and variance parameter analogs be defined by*

$$\mu_x(C) := \log \log x - \log\left(\sqrt{2\pi} A_0 \widehat{G}(1)\right), \quad \text{and} \quad \sigma_x(C) := \sqrt{\log \log x}.$$

*Let  $Y > 0$  be fixed. We have uniformly for all  $-Y \leq z \leq Y$  that*

$$\frac{1}{x} \times \# \left\{ 2 \leq n \leq x : \frac{\frac{C_{\Omega(n)}(n)}{(\log n) \sqrt{\log \log n}} - \mu_x(C)}{\sigma_x(C)} \leq z \right\} = \Phi(z) + O\left(\frac{1}{\sqrt{\log \log x}}\right), \text{ as } x \rightarrow \infty.$$

*Proof.* Fix any  $Y > 0$  and set  $z \in [-Y, Y]$ . For large  $x$  and  $2 \leq n \leq x$ , define the following auxiliary variables:

$$\alpha_n := \frac{C_{\Omega(n)}(n) - \mu_n(C)}{\sigma_n(C)}, \quad \text{and} \quad \beta_{n,x} := \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)}.$$

Let the corresponding densities be defined by the functions

$$\Phi_1(x, z) := \frac{1}{x} \times \#\{n \leq x : \alpha_n \leq z\},$$

and

$$\Phi_2(x, z) := \frac{1}{x} \times \#\{n \leq x : \beta_{n,x} \leq z\}.$$

We assert that it suffices to show that  $\Phi_2(x, z) = \Phi(x) + O\left(\frac{1}{\sqrt{\log \log x}}\right)$  as  $x \rightarrow \infty$  in place of considering the distribution of  $\Phi_1(x, z)$  to obtain the conclusion. The differences of the normalizing terms  $\sigma_n(C)$  and  $\sigma_x(C)$  and their reciprocals are small for  $\sqrt{x} \leq n \leq x$ . For  $n \in [\sqrt{x}, x]$  as  $x \rightarrow \infty$  we see that

$$\left| \frac{\mu_n(C)}{\sigma_n(C)} - \frac{\mu_x(C)}{\sigma_x(C)} \right| \leq \frac{\log 2}{\sqrt{\log \log x}}(1 + o(1)),$$

and

$$\left| \frac{1}{\sigma_n(C)} - \frac{1}{\sigma_x(C)} \right| \leq \frac{2 \log 2}{(\log x) \sqrt{\log \log x}}(1 + o(1)).$$

By applying (22), when  $\Omega(n) > 2\mu_x(C)$

$$\lim_{x \rightarrow \infty} \frac{1}{x} \times \left\{ \sqrt{x} \leq n \leq x : (\log x)^{0.613706} \sqrt{\log \log x} = o\left(\frac{C_{\Omega(n)}(n)}{(\log x) \sqrt{\log \log x}}\right) \right\} = 0.$$

Similarly, using Proposition 4.5 when  $\Omega(n) \leq 2\mu_x(C)$

$$\lim_{x \rightarrow \infty} \frac{1}{x} \times \left\{ \sqrt{x} \leq n \leq x : 2 \log(2) = o\left(\frac{C_{\Omega(n)}(n)}{(\log x) \sqrt{\log \log x}}\right) \right\} = 0.$$

It follows that when  $\sqrt{x} \leq n \leq x$  we have  $|\alpha_n - \beta_{n,x}| = o(1)$ . Thus it suffices to replace  $\alpha_n$  by  $\beta_{n,x}$  and estimate the limiting densities corresponding to these alternate terms as  $x \rightarrow \infty$  to obtain the stated result. The rest of our argument closely parallels the method from the proof of the related theorem in [17, Thm. 7.21; §7.4] that gives an analytic proof of their analog to the Erdős-Kac theorem for the distribution of  $\Omega(n)$ . We will also require a probabilistic ansatz that the functions  $X_{n,k} := \frac{C_{\Omega(n)}(n)}{(\log n) \sqrt{\log \log n}}$  for distinct  $n \leq x$  such that  $\Omega(n) = k$  when  $1 \leq k \leq 2 \log \log x$  can be viewed as independent random variables with finite first and second moments much like in the first heuristics established for the Erdős-Kac theorem (cf. [2]).

Abel summation applied to Corollary 4.4 implies that uniformly for  $1 \leq k \leq 2 \log \log x$

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} \frac{C_{\Omega(n)}(n)}{(\log n) \sqrt{\log \log n}} = \frac{A_0 \sqrt{2\pi} x}{\log x} \times \widehat{G}\left(\frac{k-1}{\log \log x}\right) \frac{(\log \log x)^{k-1}}{(k-1)!} \left(1 + O\left(\frac{1}{\log \log x}\right)\right), \text{ as } x \rightarrow \infty.$$

For  $k \in [1, \log \log x]$ , the main term in the last equation is an increasing function of  $k$  so that the maximum bound on this term is given by  $(\log \log x)^{-\frac{1}{2}}$ . Then by Chebyshev's inequality, for  $\Omega(n) = k$  with  $1 \leq k \leq 2 \log \log x$  and any  $\varepsilon > 0$  we have that

$$\mathbb{P}\left(\left|\frac{C_{\Omega(n)}(n)}{(\log n) \sqrt{\log \log n}} - \frac{A_0 \sqrt{2\pi}}{\log x} \times \widehat{G}\left(\frac{k-1}{\log \log x}\right) \frac{(\log \log x)^{k-1}}{(k-1)!}\right| \geq \varepsilon\right) \ll \frac{1}{\varepsilon^2 x \sqrt{\log \log x}}. \quad (24)$$

The upper bound in the previous equation is formed by asserting a known inequality relating the 1-norm of a  $x$ -dimensional vector  $v_x$  to  $\sqrt{x}\|v_x\|_2$  where we have assumed that the scaled terms over the  $n \leq x$  have finite mean and variance.

Let  $k \geq 1$  be a natural number such that  $k := t + \mu_x(C)$ . For fixed large  $x$ , we define the small parameter  $\delta_{t,x} := \frac{t}{\mu_x(C)}$  for the target probability density function of the standard normal distribution in the variable  $t \in \mathbb{R}$ . When  $|t| \leq \frac{1}{2}\mu_x(C)$ , we have by Stirling's formula and the corollary that

$$\begin{aligned} \frac{1}{x} \times \sum_{\substack{n \leq x \\ \Omega(n)=k}} \frac{C_{\Omega(n)}(n)}{(\log n)\sqrt{\log \log n}} &\sim \frac{A_0\sqrt{2\pi}}{\log x} \times \widehat{G}\left(\frac{k-1}{\log \log x}\right) \frac{(\log \log x)^{k-1}}{(k-1)!} \\ &\sim \frac{e^t(1+\delta_{t,x})^{\frac{1}{2}-t-\mu_x(C)}}{\sqrt{2\pi \log \log x} \cdot \widehat{G}(1)} \times \widehat{G}\left(\frac{k-1}{\log \log x}\right). \end{aligned}$$

We have the uniform estimate that  $\log(1+\delta_{t,x}) = \delta_{t,x} - \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)$  whenever  $|\delta_{t,x}| \leq \frac{1}{2}$ . Then we can expand the factor involving  $\delta_{t,x}$  from the previous equation as follows:

$$\begin{aligned} (1+\delta_{t,x})^{\frac{1}{2}-\mu_x(C)(1+\delta_{t,x})} &= \exp\left(\left(\frac{1}{2} - \mu_x(C)(1+\delta_{t,x})\right) \times \left(\delta_{t,x} - \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)\right)\right) \\ &= \exp\left(-t + \frac{t-t^2}{2\mu_x(C)} - \frac{t^2}{4\mu_x(C)^2} + O\left(\frac{|t|^3}{\mu_x(C)^2}\right)\right). \end{aligned}$$

For both  $|t| \leq \mu_x(C)^{\frac{1}{2}}$  and  $\mu_x(C)^{\frac{1}{2}} < |t| \leq \mu_x(C)^{\frac{2}{3}}$ , we can see that

$$\frac{t}{\mu_x(C)} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Similarly, for both  $|t| \leq 1$  and  $|t| > 1$ , we have that

$$\frac{t^2}{\mu_x(C)^2} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Let the corresponding error terms in  $x$  and  $t$  be denoted by

$$\widetilde{E}(x, t) := O\left(\frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}\right).$$

Combining these estimates with the previous computations, we deduce that uniformly for  $|t| \leq \mu_x(C)^{\frac{2}{3}}$  we have

$$\begin{aligned} \frac{A_0\sqrt{2\pi}}{\log x} \times \widehat{G}\left(\frac{k-1}{\log \log x}\right) \frac{(\log \log x)^{k-1}}{(k-1)!} \\ \sim \frac{1}{\sqrt{2\pi\sigma_x(C)}} \times \widehat{G}\left(\frac{k-1}{\log \log x}\right) \times \exp\left(-\frac{t^2}{2\sigma_x(C)^2}\right) \times (1 + \widetilde{E}(x, t)). \end{aligned}$$

For  $|t| \leq \mu_x(C)^{\frac{2}{3}}$ , we have that

$$\widehat{G}\left(\frac{k-1}{\log \log x}\right) = \widehat{G}(1) + O\left(\frac{1+|t|}{\log \log x}\right),$$

where  $\widehat{G}(1) = \zeta(2)^{-1}(1+P(2))^{-1} \approx 0.418611$  is the DGF of  $g^{-1}(n)$  evaluated at  $s = 2$ . Hence, by Corollary 4.4 we have that

$$\frac{1}{x} \times \sum_{\substack{n \leq x \\ \Omega(n)=k}} \frac{C_{\Omega(n)}(n)}{(\log n)\sqrt{\log \log n}} = \frac{1}{\sqrt{2\pi\sigma_x(C)}} \times \exp\left(-\frac{(k-\mu_x(C))^2}{2\sigma_x(C)^2}\right) \times (1 + \widetilde{E}(x, |k-\mu_x(C)|))$$

By the same argument given in the proof of Proposition 4.5, we see that the contributions of these summatory functions for  $k \leq \mu_x(C) - \mu_x(C)^{\frac{2}{3}}$  is negligible.

We then sum over a corresponding range of

$$\mu_x(C) - \mu_x(C)^{\frac{2}{3}} \leq k \leq \mu_x(C) + z\sigma_x(C),$$

to approximate the stated normalized densities. As  $x \rightarrow \infty$  the three terms that result (one main term and two error terms, respectively) can be considered to each correspond to a Riemann sum for an associated integral whose limiting formula corresponds to a main term given by the standard normal cumulative density function,  $\Phi(z)$ . Finally, using (24) with  $\varepsilon = O(1)$ , we see that

$$\frac{1}{x} \times \# \left\{ 2 \leq n \leq x : \frac{\frac{C_{\Omega(n)}(n)}{(\log n)\sqrt{\log \log n}} - \mu_x(C)}{\sigma_x(C)} \leq z \right\} = \Phi(z) + O\left(\frac{1}{\sqrt{\log \log x}} + \frac{\sqrt{\log \log x}}{x}\right). \quad \square$$

**Corollary 4.8.** *Suppose that  $\mu_x(C)$  and  $\sigma_x(C)$  are defined as in Theorem 4.7 for large  $x$ . Let  $Y > 0$ . For  $Y > 0$  and we have uniformly for all  $-Y \leq y \leq Y$  that as  $x \rightarrow \infty$*

$$\frac{1}{x} \cdot \# \left\{ 2 \leq n \leq x : \frac{|g^{-1}(n)|}{(\log n)\sqrt{\log \log n}} - \frac{6}{\pi^2 n (\log n)\sqrt{\log \log n}} \times \sum_{k \leq n} |g^{-1}(k)| \leq y \right\} = \Phi\left(\frac{\frac{\pi^2 y}{6} - \mu_x(C)}{\sigma_x(C)}\right) + o(1).$$

*Proof.* We claim that

$$|g^{-1}(n)| - \frac{6}{\pi^2 n} \times \sum_{k \leq n} |g^{-1}(k)| \sim \frac{6}{\pi^2} C_{\Omega(n)}(n), \text{ as } n \rightarrow \infty.$$

As in the proof of Corollary 4.6, we obtain that

$$\frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| = \frac{6}{\pi^2} \left( \frac{1}{x} \times \sum_{n \leq x} C_{\Omega(n)}(n) \right) + \sum_{d < x} \sum_{k \leq d} \frac{C_{\Omega(k)}(k)}{d^2} + O(1).$$

Let the *backwards difference operator* with respect to  $x$  be defined for  $x \geq 2$  and any arithmetic function  $f$  as  $\Delta_x(f(x)) := f(x) - f(x-1)$ . We see that for large  $n$  (cf. last lines in the proof of Corollary 4.6)

$$\begin{aligned} |g^{-1}(n)| &= \Delta_n \left( \sum_{k \leq n} g^{-1}(k) \right) \sim \frac{6}{\pi^2} \Delta_n \left( \sum_{d \leq n} C_{\Omega(d)}(d) \cdot \frac{n}{d} \right) \\ &= \frac{6}{\pi^2} \left( C_{\Omega(n)}(n) + \sum_{d < n} C_{\Omega(d)}(d) \frac{n}{d} - \sum_{d < n} C_{\Omega(d)}(d) \frac{(n-1)}{d} \right) \\ &\sim \frac{6}{\pi^2} \left( C_{\Omega(n)}(n) + \frac{1}{n-1} \times \sum_{k < n} |g^{-1}(k)| \right), \text{ as } n \rightarrow \infty. \end{aligned}$$

Since  $\frac{1}{n-1} \times \sum_{k < n} |g^{-1}(k)| \sim \frac{1}{n} \times \sum_{k \leq n} |g^{-1}(k)|$  for all sufficiently large  $n$ , the result follows by a normalization of Theorem 4.7.  $\square$

#### 4.4 Probabilistic interpretations

**Lemma 4.9.** *Let  $\mu_x(C)$  and  $\sigma_x(C)$  be defined as in Theorem 4.7. For all  $x$  sufficiently large, if we pick any integer  $n \in [2, x]$  uniformly at random, then each of the following statements holds for  $\alpha \in \mathbb{R}$ :*

$$\mathbb{P} \left( \frac{|g^{-1}(n)|}{(\log n)\sqrt{\log \log n}} - \frac{6}{\pi^2 n (\log n)\sqrt{\log \log n}} \times \sum_{k \leq n} |g^{-1}(k)| \leq \frac{6}{\pi^2} \mu_x(C) \right) = \frac{1}{2} + o(1) \quad (\text{A})$$

$$\mathbb{P} \left( \frac{|g^{-1}(n)|}{(\log n)\sqrt{\log \log n}} - \frac{6}{\pi^2 n (\log n)\sqrt{\log \log n}} \times \sum_{k \leq n} |g^{-1}(k)| \leq \frac{6}{\pi^2} (\alpha \sigma_x(C) + \mu_x(C)) \right) = \Phi(\alpha) + o(1). \quad (\text{B})$$

*Proof.* Each of these results is a consequence of Corollary 4.8. The result in (A) follows since  $\Phi(0) = \frac{1}{2}$  by taking

$$y = \frac{6}{\pi^2} (\alpha \sigma_x(C) + \mu_x(C)),$$

in Corollary 4.8 for  $\alpha = 0$ . □

As  $\alpha \rightarrow +\infty$ , we get that the right-hand-side of (B) in Lemma 4.9 tends to one for large  $x \rightarrow \infty$ . It follows from Lemma 4.9 and Corollary 4.6 that when  $x$  is large relative to a fixed  $\delta > 0$ , for almost every sufficiently large integer in  $x^\delta \leq n \leq x$  we recover that

$$|g^{-1}(n)| \leq \frac{6}{\pi^2 n} \times \sum_{k \leq n} |g^{-1}(k)| (1 + o(1)), \text{ as } n \rightarrow \infty.$$

## 5 New formulas and limiting relations characterizing $M(x)$

### 5.1 Formulas relating $M(x)$ to the summatory function $G^{-1}(x)$

**Proposition 5.1.** *For all sufficiently large  $x$ , we have that the Mertens function satisfies*

$$M(x) = G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} G^{-1}(k) \left( \pi \left( \left\lfloor \frac{x}{k} \right\rfloor \right) - \pi \left( \left\lfloor \frac{x}{k+1} \right\rfloor \right) \right). \quad (25)$$

*Proof.* We know by applying Corollary 1.4 that

$$\begin{aligned} M(x) &= \sum_{k=1}^x g^{-1}(k) \left( \pi \left( \left\lfloor \frac{x}{k} \right\rfloor \right) + 1 \right) \\ &= G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} g^{-1}(k) \pi \left( \left\lfloor \frac{x}{k} \right\rfloor \right) \\ &= G^{-1}(x) + G^{-1} \left( \left\lfloor \frac{x}{2} \right\rfloor \right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k) \left( \pi \left( \left\lfloor \frac{x}{k} \right\rfloor \right) - \pi \left( \left\lfloor \frac{x}{k+1} \right\rfloor \right) \right). \end{aligned}$$

The upper bound on the sum is truncated to  $k \in [1, \frac{x}{2}]$  in the second equation above due to the fact that  $\pi(1) = 0$ . The third formula above follows directly by (ordinary) summation by parts.  $\square$

By the result from (13) proved in Section 3.2, we similarly recall that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1} \left( \left\lfloor \frac{x}{p} \right\rfloor \right), \text{ for all } x \geq 1.$$

Summation by parts implies that we can also express  $G^{-1}(x)$  in terms of the summatory function  $L(x)$  and differences of the unsigned sequence whose distribution is given by Corollary 4.8 as follows:

$$G^{-1}(x) = \sum_{n \leq x} \lambda(n) |g^{-1}(n)| = L(x) |g^{-1}(x)| - \sum_{n < x} L(n) (|g^{-1}(n+1)| - |g^{-1}(n)|), \text{ for all } x \geq 1.$$

### 5.2 Asymptotics of $G^{-1}(x)$

The following results are due to Professor Bob Vaughan and his suggestions on upper bounds for the growth of  $|G^{-1}(x)|$  as  $x \rightarrow \infty$ . The ideas at the crux of the proof of the next theorem are found in Davenport and Heilbronn [3, 4] and date back to the work of Hans Bohr [29, cf. §11] (Second Edition, 1986).

**Theorem 5.2.** *Let  $\sigma_1$  denote the unique solution to the equation  $P(\sigma) = 1$  for  $\sigma > 1$ . There are complex  $s$  with  $\operatorname{Re}(s)$  arbitrarily close to  $\sigma_1$  such that  $1 + P(s) = 0$ .*

*Proof.* The function  $P(\sigma)$  is decreasing on  $(1, \infty)$ , tends to  $+\infty$  as  $\sigma \rightarrow 1^+$ , and tends to zero as  $\sigma \rightarrow \infty$ . Thus we find that the equation  $P(\sigma) = 1$  has a unique solution for  $\sigma > 1$ , which we denote by  $\sigma = \sigma_1 \approx 1.39943$ . Let  $\delta > 0$  be chosen small enough that  $|1 - P(z)| > 0$  for all  $z$  such that  $|z - \sigma_1| = \delta$ . Set

$$\eta = \min_{\substack{z \in \mathbb{C} \\ |z - \sigma_1| = \delta}} |1 - P(z)|.$$

Since  $P(z)$  is continuous whenever  $\operatorname{Re}(z) > 1$ , we have that  $\eta > 0$ . Let  $X \geq 2$  be a sufficiently large integer so that

$$\sum_{p > X} p^{\delta - \sigma_1} < \frac{\eta}{4}.$$

Kronecker's theorem provides a fixed  $t$  such that the following inequality holds [9, §XXIII]:

$$\max_{2 < p \leq X} \min_{n \in \mathbb{Z}} \left| \frac{t \log p}{2\pi} - n - \frac{1}{2} \right| < \delta \eta.$$

Thus we have that

$$\sum_{p > 2} p^{\delta - \sigma_1} |p^{it} + 1| < \frac{\eta}{2}.$$

Hence, for all  $z$  such that  $|z - \sigma_1| = \delta$ , we have

$$|P(z + it) + P(z)| < \frac{\eta}{2}.$$

We apply Rouché's theorem to see that the functions  $1 - P(z)$  and  $1 - P(z) + P(z + it) + P(z)$  have the same number of zeros in the disk  $\mathcal{D}_\delta = \{z \in \mathbb{C} : |z - \sigma_1| < \delta\}$ . Since  $1 - P(z)$  has at least one zero within  $\mathcal{D}_\delta$ , we must have that  $1 + P(w)$  has at least one zero with  $|w - \sigma_1 - it| < \delta$ . Since we can take  $\delta$  as small as necessary, there are zeros of the function  $1 + P(s)$  that are arbitrarily close to the line  $s = \sigma_1$ .  $\square$

**Corollary 5.3.** *Let  $\sigma_1 > 1$  be defined as in Theorem 5.2. For any  $\epsilon > 0$ , there are arbitrarily large  $x$  such that*

$$|G^{-1}(x)| > x^{\sigma_1 - \epsilon}.$$

*Proof.* We have by (6) that

$$D_{g^{-1}}(s) := \sum_{n \geq 1} \frac{g^{-1}(n)}{n^s} = \frac{1}{\zeta(s)(1 + P(s))}, \operatorname{Re}(s) > 1.$$

Theorem 5.2 implies that  $D_{g^{-1}}(s)$  has singularities for  $s \in \mathbb{C}$  such that  $\operatorname{Re}(s)$  is arbitrarily close to  $\sigma_1$ . By applying [17, Cor. 1.2; §1.2], we have that any Dirichlet series is locally uniformly convergent in its half-plane of convergence, that is for  $\operatorname{Re}(s) > \sigma_c$ , and is hence analytic in this half-plane. It follows that the abscissa of convergence for the DGF  $D_{g^{-1}}(s)$  is given by  $\sigma_c \geq \sigma_1 > 1$ , i.e., the abscissa of convergence of this DGF cannot be smaller than  $\sigma_1$ . The result proved in [17, Thm. 1.3; §1.2] then provides that

$$\limsup_{x \rightarrow \infty} \frac{\log |G^{-1}(x)|}{\log x} = \sigma_c \geq \sigma_1. \quad \square$$

### 5.3 Local cancellation of $G^{-1}(x)$ in the new formulas for $M(x)$

**Lemma 5.4.** *Suppose that  $p_n$  denotes the  $n^{\text{th}}$  prime for  $n \geq 1$  [27, A000040]. Let  $\mathcal{P}_\#$  denote the set of positive primorial integers as [27, A002110]*

$$\mathcal{P}_\# = \{n\#\}_{n \geq 1} = \left\{ \prod_{k=1}^n p_k : n \geq 1 \right\} = \{2, 6, 30, 210, 2310, 30030, \dots\}.$$

We have that as  $m \rightarrow \infty$

$$\begin{aligned} -G^{-1}((4m+1)\#) &\asymp (4m+1)!, \\ G^{-1}\left(\frac{(4m+1)\#}{p_k}\right) &\asymp (4m)!, \text{ for all } 1 \leq k \leq 4m+1. \end{aligned}$$

*Proof.* We have by part (B) of Proposition 1.6 that for all squarefree integers  $n \geq 1$

$$|g^{-1}(n)| = \sum_{j=0}^{\omega(n)} \binom{\omega(n)}{j} \times j! = (\omega(n))! \times \sum_{j=0}^{\omega(n)} \frac{1}{j!}$$

$$= (\omega(n))! \times \left( e + O\left(\frac{1}{(\omega(n) + 1)!}\right) \right).$$

Let  $m$  be a large positive integer. Then we obtain that

$$\begin{aligned} G_U^{-1}((4m+1)\#) &:= \sum_{\substack{n \leq (4m+1)\# \\ \omega(n) = \Omega(n)}} \lambda(n) |g^{-1}(n)| \\ &= \sum_{0 \leq k \leq 4m+1} \binom{4m+1}{k} (-1)^k k! \left( e + O\left(\frac{1}{(k+1)!}\right) \right) \\ &= -(4m+1)! + O(1). \end{aligned}$$

We argue that the corresponding sums over the non-squarefree  $n \leq (4m+1)\#$  contribute strictly less than the order of  $G_U^{-1}((4m+1)\#)$  for the main term of the summatory function  $G^{-1}((4m+1)\#)$ . Suppose that  $2 \leq n \leq (4m+1)\#$  is not squarefree. We have the next largest growth cases of the sequence along those  $n$  with  $|g^{-1}(n)| \leq |g^{-1}(p_s^2 t)|$  for some  $1 \leq s \leq 4m+1$  and  $t$  squarefree. If  $s = 1$  so that  $p_s = 2$  we have that the largest possible squarefree part  $t$  satisfies  $t \leq p_3 p_4 \cdots p_{4m+1}$ . A corresponding  $t$  with  $\omega(t) = 4m-1$  that attains the same distinct bounding value on  $|g^{-1}(n)|$  is obtained by taking any (unordered) rearrangement of the distinct prime factors bounding  $t$  from above by the previous product. By Lemma 3.1, we have that

$$\begin{aligned} \left| \sum_{k=2}^{\log_2((4m+1)\#)} \sum_{\substack{1 \leq t \leq \frac{(4m+1)\#}{p_1^k} \\ \omega(t) = 4m-1}} g^{-1}(p_1^k t) \right| &\leq \left| \sum_{k=2}^{\log_2((4m+1)\#)} \sum_{\substack{1 \leq t \leq \frac{(4m+1)\#}{p_1^k} \\ \omega(t) = 4m-1}} (-1)^{\omega(t)+k-1} \cdot (t+1)! \left(1 - \frac{1}{k}\right) \right| \\ &\leq \left| \sum_{k=2}^{\log_2((4m+1)\#)} (-1)^k \left(1 - \frac{1}{k}\right) \right| \times \binom{4m}{4m-1} (4m)! \\ &\leq (2 - \log 2 + o(1)) (4m) (4m)!. \end{aligned}$$

We consider the contributions from subsequent leading powers of the other  $p_s \leq (4m+1)\#$  when  $2 \leq s \leq 4m+1$ . If we have that  $|g^{-1}(n)| \leq |g^{-1}(p_s^2 t)|$  for  $p_s \geq 3$  and  $t \leq p_{r+1} p_{r+2} \cdots p_{4m+1}$  squarefree, we get that

$$\begin{aligned} \left| \sum_{k=2}^{\log_{p_s}((4m+1)\#)} \sum_{\substack{1 \leq t \leq \frac{(4m+1)\#}{p_1^k} \\ \omega(t) = 4m+1-r}} g^{-1}(p_s^k t) \right| &\leq (2 - \log 2 + o(1)) (4m+1-r)! \binom{4m}{4m-r} \\ &= (2 - \log 2 + o(1)) (4m+1-r) \times \frac{(4m)!}{r!}. \end{aligned}$$

For any fixed  $p_s$  with  $2 \leq s \leq 4m+1$ , we bound the lower index  $r$  according to  $p_s^2(1+o(1)) \leq r \log r$  by the prime number theorem. This equation requires that

$$r \geq e^{W_0(p_s^2(1+o(1)))} = e^{2 \log p_s - \log \log(p_s^2) + o(1)} \sim p_s^2 - 2 \log p_s.$$

Then the corresponding sums  $G_L^{-1}((4m+1)\#)$  are bounded from above by

$$G_L^{-1}((4m+1)\#) := \left| \sum_{\substack{n \leq (4m+1)\# \\ \omega(n) < \Omega(n)}} g^{-1}(n) \right|$$



$$\begin{aligned}
 &\leq \sum_{r=2}^{4m+1} \frac{(2 - \log 2 + o(1))(4m+1-r)(4m)!}{r!} \\
 &= (2 - \log 2 + o(1))((4e-8)m-1)(4m)! + O(1) \\
 &< (4m+1)!(1+o(1)).
 \end{aligned}$$

Hence, we see that  $-G^{-1}((4m+1)\#) \asymp (4m+1)!$  as  $m \rightarrow \infty$ . Similarly, we can show that for any  $1 \leq k \leq 4m+1$

$$G^{-1}\left(\frac{(4m+1)\#}{p_k}\right) \sim \sum_{0 \leq k \leq 4m} \binom{4m}{k} (-1)^k k! \left(e + O\left(\frac{1}{(k+1)!}\right)\right) \asymp (4m)!. \quad \square$$

The analysis of the maximal bounds from below on  $G^{-1}(x)$  as  $x \rightarrow \infty$  guaranteed by Corollary 5.3 complicate the interpretation of Proposition 5.1 in forming new asymptotics for  $M(x)$ . Even though we get comparatively large order growth of  $G^{-1}(x)$  infinitely often, we expect that there is usually (almost always) a large amount of cancellation between the successive values of this summatory function given by (13). Lemma 5.4 demonstrates this phenomenon well along the asymptotically large infinite subsequence of  $x$  taken along the primorial integers, or integers  $x = (4m+1)\#$  that are formed precisely as the product of the first  $4m+1$  primes.

Since we have that for sufficiently large  $n$  [5, 6]

$$n\# \sim e^{\vartheta(p_n)} \asymp n^n (\log n)^n e^{-n(1+o(1))}, \text{ as } n \rightarrow \infty,$$

the RH requires that the leading constants with opposing signs on the asymptotics for the functions from the lemma match. This observation follows from the fact that if we obtain a contrary result, equation (13) would imply that

$$\frac{M((4m+1)\#)}{\sqrt{(4m+1)\#}} \gg [(4m+1)\#]^{\delta_0}, \text{ as } m \rightarrow \infty,$$

for some fixed absolute  $\delta_0 > 0$ . The formula in (13) then shows that under the RH we witness the expected substantial cancellation from the summatory function terms involving  $G^{-1}(x)$  in the formula for  $M(x)$  along this subsequence. In fact, for sufficiently large  $m$ , we get that the following properties holds:

- (i)  $\operatorname{sgn}(G^{-1}((4m+1)\#)) = -\operatorname{sgn}\left(\sum_{p \leq (4m+1)\#} G^{-1}\left(\frac{(4m+1)\#}{p}\right)\right);$
- (ii)  $\lim_{m \rightarrow \infty} \frac{-G^{-1}((4m+1)\#)}{\sum_{p \leq (4m+1)\#} G^{-1}\left(\frac{(4m+1)\#}{p}\right)} = 1;$
- (iii)  $M((4m+1)\#) \gg \sum_{\substack{n \leq (4m+1)\# \\ \omega(n) = \Omega(n)}} g^{-1}(n) \left(1 + \pi\left(\frac{(4m+1)\#}{n}\right)\right).$

Hence, along this subsequence, the contributions of the local maxima for the absolute values of  $|g^{-1}(n)|$  at the squarefree integers cancel nearly completely and do not contribute the main term for the limiting asymptotic expansion of  $M(x)$  along  $x = (4m+1)\#$  as  $m \rightarrow \infty$ .

## 6 Conclusions

We have identified a new sequence,  $\{g^{-1}(n)\}_{n \geq 1}$ , that is the Dirichlet inverse of the shifted strongly additive function  $\omega(n)$ . Section 3.3, shows that there is a natural combinatorial interpretation to the distribution of distinct values of  $|g^{-1}(n)|$  for  $n \leq x$  involving the distribution of the primes  $p \leq x$  at large  $x$ . In particular, the magnitude of  $g^{-1}(n)$  depends only on the pattern of the exponents of the prime factorization of  $n$ . The signedness of  $g^{-1}(n)$  is given by  $\lambda(n)$  for all  $n \geq 1$ . This leads to a new relations of the summatory function  $G^{-1}(x)$  that characterize the distribution of  $M(x)$  to the distribution of the summatory function  $L(x)$ .

We emphasize that our new work on the Mertens function proved within this article is significant in providing a new window through which we can view bounding  $M(x)$  through asymptotics of auxiliary sequences and partial sums. The computational data generated in Table B of the appendix section suggests numerically that the distribution of  $G^{-1}(x)$  may be easier to work with than that of  $M(x)$  or  $L(x)$ . The additively combinatorial relation of the distinct (and repetition of) values of  $|g^{-1}(n)|$  for  $n \leq x$  are suggestive towards bounding main terms for  $G^{-1}(x)$  along infinite subsequences in future work.

One topic that we do not explore in the article is to consider what correlation (if any) exists between  $\lambda(n)$  and the unsigned sequence of  $|g^{-1}(n)|$  with the limiting distribution proved in Corollary 4.8. Much in the same way that variants of the famous Erdős-Kac theorem are historically established by defining random variables related to  $\omega(n)$ , we also suggest an analysis of the summatory function  $G^{-1}(x)$  by scaling the explicitly distributed  $|g^{-1}(n)|$  for  $n \leq x$  as  $x \rightarrow \infty$  by its signed weight of  $\lambda(n)$  using an initial heuristic along these lines for future work.

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## A Appendix: Asymptotic formulas

We thank Gergő Nemes from the Alfréd Rényi Institute of Mathematics for his careful notes on the limiting asymptotics for the sums identified in this section. We have adapted the communication of his proofs to establish the next few lemmas.

**Facts A.1** (The incomplete gamma function). The (upper) *incomplete gamma function* is defined by [23, §8.4]

$$\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt, a \in \mathbb{R}, |\arg z| < \pi.$$

The function  $\Gamma(a, z)$  can be continued to an analytic function of  $z$  on the universal covering of  $\mathbb{C} \setminus \{0\}$ . For  $a \in \mathbb{Z}^+$ , the function  $\Gamma(a, z)$  is an entire function of  $z$ . The following properties of  $\Gamma(a, z)$  hold [23, §8.4; §8.11(i)]:

$$\Gamma(a, z) = (a-1)! e^{-z} \times \sum_{k=0}^{a-1} \frac{z^k}{k!}, \text{ for } a \in \mathbb{Z}^+, z \in \mathbb{C}; \quad (26a)$$

$$\Gamma(a, z) \sim z^{a-1} e^{-z}, \text{ for fixed } a \in \mathbb{C}, \text{ as } z \rightarrow +\infty. \quad (26b)$$

Moreover, for real  $z > 0$ , as  $z \rightarrow +\infty$  we have that [18]

$$\Gamma(z, z) = \sqrt{\frac{\pi}{2}} z^{z-\frac{1}{2}} e^{-z} + O(z^{z-1} e^{-z}), \quad (26c)$$

If  $z, a \rightarrow \infty$  with  $z = \lambda a$  for some  $\lambda > 1$  such that  $(\lambda - 1)^{-1} = o(\sqrt{|a|})$ , then [18]

$$\Gamma(a, z) = z^a e^{-z} \times \sum_{n \geq 0} \frac{(-a)^n b_n(\lambda)}{(z-a)^{2n+1}}, \quad (26d)$$

where the sequence  $b_n(\lambda)$  satisfies the characteristic relation that  $b_0(\lambda) = 1$  and<sup>2</sup>

$$b_n(\lambda) = \lambda(1-\lambda)b'_{n-1}(\lambda) + \lambda(2n-1)b_{n-1}(\lambda), n \geq 1.$$

**Proposition A.2.** Let  $a, z, \lambda$  be positive real parameters such that  $z = \lambda a$ . If  $\lambda \in (0, 1)$ , then as  $z \rightarrow +\infty$

$$\Gamma(a, z) = \Gamma(a) + O_\lambda(z^{a-1} e^{-z}).$$

If  $\lambda > 1$ , then as  $z \rightarrow +\infty$

$$\Gamma(a, z) = \frac{z^{a-1} e^{-z}}{1-\lambda^{-1}} + O_\lambda(z^{a-2} e^{-z}).$$

If  $\lambda > 0.567142 > W(1)$  where  $W(x)$  denotes the principal branch of the Lambert  $W$ -function for  $x \geq 0$ , then as  $z \rightarrow +\infty$

$$\Gamma(a, ze^{\pm\pi i}) = -e^{\pm\pi i a} \frac{z^{a-1} e^z}{1+\lambda^{-1}} + O_\lambda(z^{a-2} e^z).$$

The first two asymptotic estimates are only useful when  $\lambda$  is bounded away from the transition point at 1. We cannot write the last expansion above as  $\Gamma(a, -z)$  directly unless  $a \in \mathbb{Z}^+$  as the incomplete gamma function has a branch point at the origin with respect to its second variable. This function becomes a single-valued analytic function of its second input by continuation on the universal covering of  $\mathbb{C} \setminus \{0\}$ .

---

<sup>2</sup>An exact formula for  $b_n(\lambda)$  is given in terms of the *second-order Eulerian number triangle* [27, A008517] as follows:

$$b_n(\lambda) = \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \lambda^{k+1}.$$

*Proof.* The first asymptotic estimate follows directly from the following asymptotic series expansion that holds as  $z \rightarrow +\infty$  [20, Eq. (2.1)]:

$$\Gamma(a, z) \sim \Gamma(a) + z^a e^{-z} \times \sum_{k \geq 0} \frac{(-a)^k b_k(\lambda)}{(z-a)^{2k+1}}.$$

Using the notation from (26d) and [19], we have that

$$\Gamma(a, z) = \frac{z^{a-1} e^{-z}}{1-\lambda^{-1}} + z^a e^{-z} R_1(a, \lambda).$$

From the bounds in [19, §3.1], we get that

$$|z^a e^{-z} R_1(a, \lambda)| \leq z^a e^{-z} \times \frac{a \cdot b_1(\lambda)}{(z-a)^3} = \frac{z^{a-2} e^{-z}}{(1-\lambda^{-1})^3}$$

Note that the main and error terms in the previous equation can also be seen by applying the asymptotic series in (26d) directly.

The proof of the third equation above follows from the following asymptotics [18, Eq. (1.1)]

$$\Gamma(-a, z) \sim z^{-a} e^{-z} \times \sum_{n \geq 0} \frac{a^n b_n(-\lambda)}{(z+a)^{2n+1}},$$

by setting  $a \mapsto a e^{\pm \pi i}$  and  $z \mapsto z e^{\pm \pi i}$  with  $\lambda = \frac{z}{a} > 0.567142 > W(1)$ . The restriction on the range of  $\lambda$  over which the third formula holds is made to ensure that the last formula from the reference is valid at negative real  $a$ .  $\square$

**Lemma A.3.** *For  $x \rightarrow +\infty$ , we have that*

$$S_1(x) := \frac{x}{\log x} \times \left| \sum_{1 \leq k \leq \lfloor \log \log x \rfloor} \frac{(-1)^k (\log \log x)^{k-1}}{(k-1)!} \right| = \frac{x}{2\sqrt{2\pi \log \log x}} + O\left(\frac{x}{(\log \log x)^{\frac{3}{2}}}\right).$$

*Proof.* We have for  $n \geq 1$  and any  $t > 0$  by (26a) that

$$\sum_{1 \leq k \leq n} \frac{(-1)^k t^{k-1}}{(k-1)!} = -e^{-t} \times \frac{\Gamma(n, -t)}{(n-1)!}.$$

Suppose that  $t = n + \xi$  with  $\xi = O(1)$  (e.g., so we can formally take the floor of the input  $n$  to truncate the last sum). By the third formula in Proposition A.2 with the parameters  $(a, z, \lambda) \mapsto (n, t, 1 + \frac{\xi}{n})$ , we deduce that as  $n, t \rightarrow +\infty$ .

$$\Gamma(n, -t) = (-1)^{n+1} \times \frac{t^n e^t}{t+n} + O\left(\frac{nt^n e^t}{(t+n)^3}\right) = (-1)^{n+1} \frac{t^n e^t}{2n} + O\left(\frac{t^{n-1} e^t}{n}\right). \quad (27)$$

Accordingly, we see that

$$\sum_{1 \leq k \leq n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{t^n}{2n!} + O\left(\frac{t^{n-1}}{n!}\right).$$

By the variant of Stirling's formula in [23, cf. Eq. (5.11.8)], we have

$$n! = \Gamma(1+t-\xi) = \sqrt{2\pi} t^{t-\xi+\frac{1}{2}} e^{-t} (1 + O(t^{-1})) = \sqrt{2\pi} t^{n+\frac{1}{2}} e^{-t} (1 + O(t^{-1})).$$

Hence, as  $n \rightarrow +\infty$  with  $t := n + \xi$  and  $\xi = O(1)$ , we obtain that

$$\sum_{k=1}^n \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{e^t}{2\sqrt{2\pi} t} + O\left(e^t t^{-\frac{3}{2}}\right).$$

The conclusion follows by taking  $n := \lfloor \log \log x \rfloor$ ,  $t := \log \log x$  and applying the triangle inequality to obtain the result.  $\square$

**B Table: Computations involving  $g^{-1}(n)$  and  $G^{-1}(x)$  for  $1 \leq n, x \leq 500$** 

| $n$ | <b>Primes</b>                                | <b>Sqfree</b> | <b>PPower</b> | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum d n C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|--|---------------|---------------|-------------|--|---|--------------------|--------------------|-------------|---------------|---------------|
| 1   | 1 <sup>1</sup>                               | Y             | N             | 1           | 0  | 1.000000  | 1.000000           | 0.000000           | 1           | 1             | 0             |
| 2   | 2 <sup>1</sup>                               | Y             | Y             | -2          | 0  | 1.000000  | 0.500000           | 0.500000           | -1          | 1             | -2            |
| 3   | 3 <sup>1</sup>                               | Y             | Y             | -2          | 0  | 1.000000  | 0.333333           | 0.666667           | -3          | 1             | -4            |
| 4   | 2 <sup>2</sup>                               | N             | Y             | 2           | 0  | 1.500000  | 0.500000           | 0.500000           | -1          | 3             | -4            |
| 5   | 5 <sup>1</sup>                               | Y             | Y             | -2          | 0  | 1.000000  | 0.400000           | 0.600000           | -3          | 3             | -6            |
| 6   | 2 <sup>1</sup> 3 <sup>1</sup>                | Y             | N             | 5           | 0  | 1.000000  | 0.500000           | 0.500000           | 2           | 8             | -6            |
| 7   | 7 <sup>1</sup>                               | Y             | Y             | -2          | 0  | 1.000000  | 0.428571           | 0.571429           | 0           | 8             | -8            |
| 8   | 2 <sup>3</sup>                               | N             | Y             | -2          | 0  | 2.000000  | 0.375000           | 0.625000           | -2          | 8             | -10           |
| 9   | 3 <sup>2</sup>                               | N             | Y             | 2           | 0  | 1.500000  | 0.444444           | 0.555556           | 0           | 10            | -10           |
| 10  | 2 <sup>1</sup> 5 <sup>1</sup>                | Y             | N             | 5           | 0  | 1.000000  | 0.500000           | 0.500000           | 5           | 15            | -10           |
| 11  | 11 <sup>1</sup>                              | Y             | Y             | -2          | 0  | 1.000000  | 0.454545           | 0.545455           | 3           | 15            | -12           |
| 12  | 2 <sup>2</sup> 3 <sup>1</sup>                | N             | N             | -7          | 2  | 1.2857143                                       | 0.416667           | 0.583333           | -4          | 15            | -19           |
| 13  | 13 <sup>1</sup>                              | Y             | Y             | -2          | 0  | 1.000000  | 0.384615           | 0.615385           | -6          | 15            | -21           |
| 14  | 2 <sup>1</sup> 7 <sup>1</sup>                | Y             | N             | 5           | 0  | 1.000000  | 0.428571           | 0.571429           | -1          | 20            | -21           |
| 15  | 3 <sup>1</sup> 5 <sup>1</sup>                | Y             | N             | 5           | 0  | 1.000000  | 0.466667           | 0.533333           | 4           | 25            | -21           |
| 16  | 2 <sup>4</sup>                               | N             | Y             | 2           | 0  | 2.500000  | 0.500000           | 0.500000           | 6           | 27            | -21           |
| 17  | 17 <sup>1</sup>                              | Y             | Y             | -2          | 0  | 1.000000  | 0.470588           | 0.529412           | 4           | 27            | -23           |
| 18  | 2 <sup>1</sup> 3 <sup>2</sup>                | N             | N             | -7          | 2  | 1.2857143                                       | 0.444444           | 0.555556           | -3          | 27            | -30           |
| 19  | 19 <sup>1</sup>                              | Y             | Y             | -2          | 0  | 1.000000  | 0.421053           | 0.578947           | -5          | 27            | -32           |
| 20  | 2 <sup>2</sup> 5 <sup>1</sup>                | N             | N             | -7          | 2  | 1.2857143                                       | 0.400000           | 0.600000           | -12         | 27            | -39           |
| 21  | 3 <sup>1</sup> 7 <sup>1</sup>                | Y             | N             | 5           | 0  | 1.000000  | 0.428571           | 0.571429           | -7          | 32            | -39           |
| 22  | 2 <sup>1</sup> 11 <sup>1</sup>               | Y             | N             | 5           | 0  | 1.000000  | 0.454545           | 0.545455           | -2          | 37            | -39           |
| 23  | 23 <sup>1</sup>                              | Y             | Y             | -2          | 0  | 1.000000  | 0.434783           | 0.565217           | -4          | 37            | -41           |
| 24  | 2 <sup>3</sup> 3 <sup>1</sup>                | N             | N             | 9           | 4  | 1.555556  | 0.458333           | 0.541667           | 5           | 46            | -41           |
| 25  | 5 <sup>2</sup>                               | N             | Y             | 2           | 0  | 1.500000  | 0.480000           | 0.520000           | 7           | 48            | -41           |
| 26  | 2 <sup>1</sup> 13 <sup>1</sup>               | Y             | N             | 5           | 0  | 1.000000  | 0.500000           | 0.500000           | 12          | 53            | -41           |
| 27  | 3 <sup>3</sup>                               | N             | Y             | -2          | 0  | 2.000000  | 0.481481           | 0.518519           | 10          | 53            | -43           |
| 28  | 2 <sup>2</sup> 7 <sup>1</sup>                | N             | N             | -7          | 2  | 1.2857143                                       | 0.464286           | 0.535714           | 3           | 53            | -50           |
| 29  | 29 <sup>1</sup>                              | Y             | Y             | -2          | 0  | 1.000000  | 0.448276           | 0.551724           | 1           | 53            | -52           |
| 30  | 2 <sup>1</sup> 3 <sup>1</sup> 5 <sup>1</sup> | Y             | N             | -16         | 0  | 1.000000  | 0.433333           | 0.566667           | -15         | 53            | -68           |
| 31  | 31 <sup>1</sup>                              | Y             | Y             | -2          | 0  | 1.000000  | 0.419355           | 0.580645           | -17         | 53            | -70           |
| 32  | 2 <sup>5</sup>                               | N             | Y             | -2          | 0  | 3.000000  | 0.406250           | 0.593750           | -19         | 53            | -72           |
| 33  | 3 <sup>1</sup> 11 <sup>1</sup>               | Y             | N             | 5           | 0  | 1.000000  | 0.424242           | 0.575758           | -14         | 58            | -72           |
| 34  | 2 <sup>1</sup> 17 <sup>1</sup>               | Y             | N             | 5           | 0  | 1.000000  | 0.441176           | 0.558824           | -9          | 63            | -72           |
| 35  | 5 <sup>1</sup> 7 <sup>1</sup>                | Y             | N             | 5           | 0  | 1.000000  | 0.457143           | 0.542857           | -4          | 68            | -72           |
| 36  | 2 <sup>2</sup> 3 <sup>2</sup>                | N             | N             | 14          | 9  | 1.3571429                                       | 0.472222           | 0.527778           | 10          | 82            | -72           |
| 37  | 37 <sup>1</sup>                              | Y             | Y             | -2          | 0  | 1.000000  | 0.459459           | 0.540541           | 8           | 82            | -74           |
| 38  | 2 <sup>1</sup> 19 <sup>1</sup>               | Y             | N             | 5           | 0  | 1.000000  | 0.473684           | 0.526316           | 13          | 87            | -74           |
| 39  | 3 <sup>1</sup> 13 <sup>1</sup>               | Y             | N             | 5           | 0  | 1.000000  | 0.487179           | 0.512821           | 18          | 92            | -74           |
| 40  | 2 <sup>3</sup> 5 <sup>1</sup>                | N             | N             | 9           | 4  | 1.555556  | 0.500000           | 0.500000           | 27          | 101           | -74           |
| 41  | 41 <sup>1</sup>                              | Y             | Y             | -2          | 0  | 1.000000  | 0.487805           | 0.512195           | 25          | 101           | -76           |
| 42  | 2 <sup>1</sup> 3 <sup>1</sup> 7 <sup>1</sup> | Y             | N             | -16         | 0  | 1.000000  | 0.476190           | 0.523810           | 9           | 101           | -92           |
| 43  | 43 <sup>1</sup>                              | Y             | Y             | -2          | 0  | 1.000000  | 0.465116           | 0.534884           | 7           | 101           | -94           |
| 44  | 2 <sup>2</sup> 11 <sup>1</sup>               | N             | N             | -7          | 2  | 1.2857143                                       | 0.454545           | 0.545455           | 0           | 101           | -101          |
| 45  | 3 <sup>2</sup> 5 <sup>1</sup>                | N             | N             | -7          | 2  | 1.2857143                                       | 0.444444           | 0.555556           | -7          | 101           | -108          |
| 46  | 2 <sup>1</sup> 23 <sup>1</sup>               | Y             | N             | 5           | 0  | 1.000000  | 0.456522           | 0.543478           | -2          | 106           | -108          |
| 47  | 47 <sup>1</sup>                              | Y             | Y             | -2          | 0  | 1.000000  | 0.446809           | 0.553191           | -4          | 106           | -110          |
| 48  | 2 <sup>4</sup> 3 <sup>1</sup>                | N             | N             | -11         | 6  | 1.8181818                                       | 0.437500           | 0.562500           | -15         | 106           | -121          |

**Table B:** Computations involving  $g^{-1}(n) \equiv (\omega + 1)^{-1}(n)$  and  $G^{-1}(x)$  for  $1 \leq n \leq 500$ .

- The column labeled **Primes** provides the prime factorization of each  $n$  so that the values of  $\omega(n)$  and  $\Omega(n)$  are easily extracted. The columns labeled **Sqfree** and **PPower**, respectively, list inclusion of  $n$  in the sets of squarefree integers and the prime powers.
- The next three columns provide the explicit values of the inverse function  $g^{-1}(n)$  and compare its explicit value with other estimates. We define the function  $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \times k!$ .
- The last columns indicate properties of the summatory function of  $g^{-1}(n)$ . The notation for the densities of the sign weight of  $g^{-1}(n)$  is defined as  $\mathcal{L}_{\pm}(x) := \frac{1}{n} \times \#\{n \leq x : \lambda(n) = \pm 1\}$ . The last three columns then show the explicit components to the signed summatory function,  $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ , decomposed into its respective positive and negative magnitude sum contributions:  $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$  where  $G_+^{-1}(x) > 0$  and  $G_-^{-1}(x) < 0$  for all  $x \geq 1$ . That is, the component functions  $G_{\pm}^{-1}(x)$  displayed in the last two columns of the table correspond to the summatory function  $G^{-1}(x)$  with summands that are positive and negative, respectively.

| $n$ | Primes  | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum_{d n} C_{\Omega(d)}^{(d)}}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|---|--------|--------|-------------|--|--|--------------------|--------------------|-------------|---------------|---------------|
| 49  | 7 <sup>2</sup>                                | N      | Y      | 2           | 0  | 1.5000000  | 0.448980           | 0.551020           | -13         | 108           | -121          |
| 50  | 2 <sup>1</sup> 5 <sup>2</sup>                 | N      | N      | -7          | 2  | 1.2857143  | 0.440000           | 0.560000           | -20         | 108           | -128          |
| 51  | 3 <sup>1</sup> 7 <sup>1</sup>                 | Y      | N      | 5           | 0  | 1.0000000  | 0.450980           | 0.549020           | -15         | 113           | -128          |
| 52  | 2 <sup>2</sup> 13 <sup>1</sup>                | N      | N      | -7          | 2  | 1.2857143  | 0.442308           | 0.557692           | -22         | 113           | -135          |
| 53  | 53 <sup>1</sup>                               | Y      | Y      | -2          | 0  | 1.0000000  | 0.433962           | 0.566038           | -24         | 113           | -137          |
| 54  | 2 <sup>1</sup> 3 <sup>3</sup>                 | N      | N      | 9           | 4  | 1.5555556  | 0.444444           | 0.555556           | -15         | 122           | -137          |
| 55  | 5 <sup>1</sup> 11 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.454545           | 0.545455           | -10         | 127           | -137          |
| 56  | 2 <sup>3</sup> 7 <sup>1</sup>                 | N      | N      | 9           | 4  | 1.5555556  | 0.464286           | 0.535714           | -1          | 136           | -137          |
| 57  | 3 <sup>1</sup> 19 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.473684           | 0.526316           | 4           | 141           | -137          |
| 58  | 2 <sup>1</sup> 29 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.482759           | 0.517241           | 9           | 146           | -137          |
| 59  | 59 <sup>1</sup>                               | Y      | Y      | -2          | 0  | 1.0000000  | 0.474576           | 0.525424           | 7           | 146           | -139          |
| 60  | 2 <sup>2</sup> 3 <sup>1</sup> 5 <sup>1</sup>  | N      | N      | 30          | 14                                       | 1.1666667  | 0.483333           | 0.516667           | 37          | 176           | -139          |
| 61  | 61 <sup>1</sup>                               | Y      | Y      | -2          | 0  | 1.0000000  | 0.475410           | 0.524590           | 35          | 176           | -141          |
| 62  | 2 <sup>1</sup> 31 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.483871           | 0.516129           | 40          | 181           | -141          |
| 63  | 3 <sup>2</sup> 7 <sup>1</sup>                 | N      | N      | -7          | 2  | 1.2857143  | 0.476190           | 0.523810           | 33          | 181           | -148          |
| 64  | 2 <sup>6</sup>                                | N      | Y      | 2           | 0  | 3.5000000  | 0.484375           | 0.515625           | 35          | 183           | -148          |
| 65  | 5 <sup>1</sup> 13 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.492308           | 0.507692           | 40          | 188           | -148          |
| 66  | 2 <sup>1</sup> 3 <sup>1</sup> 11 <sup>1</sup> | Y      | N      | -16         | 0  | 1.0000000  | 0.484848           | 0.515152           | 24          | 188           | -164          |
| 67  | 67 <sup>1</sup>                               | Y      | Y      | -2          | 0  | 1.0000000  | 0.477612           | 0.522388           | 22          | 188           | -166          |
| 68  | 2 <sup>2</sup> 17 <sup>1</sup>                | N      | N      | -7          | 2  | 1.2857143  | 0.470588           | 0.529412           | 15          | 188           | -173          |
| 69  | 3 <sup>1</sup> 23 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.478261           | 0.521739           | 20          | 193           | -173          |
| 70  | 2 <sup>1</sup> 5 <sup>1</sup> 7 <sup>1</sup>  | Y      | N      | -16         | 0  | 1.0000000  | 0.471429           | 0.528571           | 4           | 193           | -189          |
| 71  | 71 <sup>1</sup>                               | Y      | Y      | -2          | 0  | 1.0000000  | 0.464789           | 0.535211           | 2           | 193           | -191          |
| 72  | 2 <sup>3</sup> 3 <sup>2</sup>                 | N      | N      | -23         | 18                                       | 1.4782609  | 0.458333           | 0.541667           | -21         | 193           | -214          |
| 73  | 73 <sup>1</sup>                               | Y      | Y      | -2          | 0  | 1.0000000  | 0.452055           | 0.547945           | -23         | 193           | -216          |
| 74  | 2 <sup>1</sup> 37 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.459459           | 0.540541           | -18         | 198           | -216          |
| 75  | 3 <sup>1</sup> 5 <sup>2</sup>                 | N      | N      | -7          | 2  | 1.2857143  | 0.453333           | 0.546667           | -25         | 198           | -223          |
| 76  | 2 <sup>2</sup> 19 <sup>1</sup>                | N      | N      | -7          | 2  | 1.2857143  | 0.447368           | 0.552632           | -32         | 198           | -230          |
| 77  | 7 <sup>1</sup> 11 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.454545           | 0.545455           | -27         | 203           | -230          |
| 78  | 2 <sup>1</sup> 3 <sup>1</sup> 13 <sup>1</sup> | Y      | N      | -16         | 0  | 1.0000000  | 0.448718           | 0.551282           | -43         | 203           | -246          |
| 79  | 79 <sup>1</sup>                               | Y      | Y      | -2          | 0  | 1.0000000  | 0.443038           | 0.556962           | -45         | 203           | -248          |
| 80  | 2 <sup>4</sup> 5 <sup>1</sup>                 | N      | N      | -11         | 6  | 1.8181818  | 0.437500           | 0.562500           | -56         | 203           | -259          |
| 81  | 3 <sup>4</sup>                                | N      | Y      | 2           | 0  | 2.5000000  | 0.444444           | 0.555556           | -54         | 205           | -259          |
| 82  | 2 <sup>1</sup> 41 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.451220           | 0.548780           | -49         | 210           | -259          |
| 83  | 83 <sup>1</sup>                               | Y      | Y      | -2          | 0  | 1.0000000  | 0.445783           | 0.554217           | -51         | 210           | -261          |
| 84  | 2 <sup>2</sup> 3 <sup>1</sup> 7 <sup>1</sup>  | N      | N      | 30          | 14                                       | 1.1666667  | 0.452381           | 0.547619           | -21         | 240           | -261          |
| 85  | 5 <sup>1</sup> 17 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.458824           | 0.541176           | -16         | 245           | -261          |
| 86  | 2 <sup>1</sup> 43 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.465116           | 0.534884           | -11         | 250           | -261          |
| 87  | 3 <sup>1</sup> 29 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.471264           | 0.528736           | -6          | 255           | -261          |
| 88  | 2 <sup>3</sup> 11 <sup>1</sup>                | N      | N      | 9           | 4  | 1.5555556  | 0.477273           | 0.522727           | 3           | 264           | -261          |
| 89  | 89 <sup>1</sup>                               | Y      | Y      | -2          | 0  | 1.0000000  | 0.471910           | 0.528090           | 1           | 264           | -263          |
| 90  | 2 <sup>1</sup> 3 <sup>2</sup> 5 <sup>1</sup>  | N      | N      | 30          | 14                                       | 1.1666667  | 0.477778           | 0.522222           | 31          | 294           | -263          |
| 91  | 7 <sup>1</sup> 13 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.483516           | 0.516484           | 36          | 299           | -263          |
| 92  | 2 <sup>2</sup> 23 <sup>1</sup>                | N      | N      | -7          | 2  | 1.2857143  | 0.478261           | 0.521739           | 29          | 299           | -270          |
| 93  | 3 <sup>1</sup> 31 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.483871           | 0.516129           | 34          | 304           | -270          |
| 94  | 2 <sup>1</sup> 47 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.489362           | 0.510638           | 39          | 309           | -270          |
| 95  | 5 <sup>1</sup> 19 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.494737           | 0.505263           | 44          | 314           | -270          |
| 96  | 2 <sup>5</sup> 3 <sup>1</sup>                 | N      | N      | 13          | 8  | 2.0769231  | 0.500000           | 0.500000           | 57          | 327           | -270          |
| 97  | 97 <sup>1</sup>                               | Y      | Y      | -2          | 0  | 1.0000000  | 0.494845           | 0.505155           | 55          | 327           | -272          |
| 98  | 2 <sup>1</sup> 7 <sup>2</sup>                 | N      | N      | -7          | 2  | 1.2857143  | 0.489796           | 0.510204           | 48          | 327           | -279          |
| 99  | 3 <sup>2</sup> 11 <sup>1</sup>                | N      | N      | -7          | 2  | 1.2857143  | 0.484848           | 0.515152           | 41          | 327           | -286          |
| 100 | 2 <sup>2</sup> 5 <sup>2</sup>                 | N      | N      | 14          | 9  | 1.3571429  | 0.490000           | 0.510000           | 55          | 341           | -286          |
| 101 | 101 <sup>1</sup>                              | Y      | Y      | -2          | 0  | 1.0000000  | 0.485149           | 0.514851           | 53          | 341           | -288          |
| 102 | 2 <sup>1</sup> 3 <sup>1</sup> 17 <sup>1</sup> | Y      | N      | -16         | 0  | 1.0000000  | 0.480392           | 0.519608           | 37          | 341           | -304          |
| 103 | 103 <sup>1</sup>                              | Y      | Y      | -2          | 0  | 1.0000000  | 0.475728           | 0.524272           | 35          | 341           | -306          |
| 104 | 2 <sup>3</sup> 13 <sup>1</sup>                | N      | N      | 9           | 4  | 1.5555556  | 0.480769           | 0.519231           | 44          | 350           | -306          |
| 105 | 3 <sup>1</sup> 5 <sup>1</sup> 7 <sup>1</sup>  | Y      | N      | -16         | 0  | 1.0000000  | 0.476190           | 0.523810           | 28          | 350           | -322          |
| 106 | 2 <sup>1</sup> 53 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.481132           | 0.518868           | 33          | 355           | -322          |
| 107 | 107 <sup>1</sup>                              | Y      | Y      | -2          | 0  | 1.0000000  | 0.476636           | 0.523364           | 31          | 355           | -324          |
| 108 | 2 <sup>2</sup> 3 <sup>3</sup>                 | N      | N      | -23         | 18                                       | 1.4782609  | 0.472222           | 0.527778           | 8           | 355           | -347          |
| 109 | 109 <sup>1</sup>                              | Y      | Y      | -2          | 0  | 1.0000000  | 0.467890           | 0.532110           | 6           | 355           | -349          |
| 110 | 2 <sup>1</sup> 5 <sup>1</sup> 11 <sup>1</sup> | Y      | N      | -16         | 0  | 1.0000000  | 0.463636           | 0.536364           | -10         | 355           | -365          |
| 111 | 3 <sup>1</sup> 37 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.468468           | 0.531532           | -5          | 360           | -365          |
| 112 | 2 <sup>4</sup> 7 <sup>1</sup>                 | N      | N      | -11         | 6  | 1.8181818  | 0.464286           | 0.535714           | -16         | 360           | -376          |
| 113 | 113 <sup>1</sup>                              | Y      | Y      | -2          | 0  | 1.0000000  | 0.460177           | 0.539823           | -18         | 360           | -378          |
| 114 | 2 <sup>1</sup> 3 <sup>1</sup> 19 <sup>1</sup> | Y      | N      | -16         | 0  | 1.0000000  | 0.456140           | 0.543860           | -34         | 360           | -394          |
| 115 | 5 <sup>1</sup> 23 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.460870           | 0.539130           | -29         | 365           | -394          |
| 116 | 2 <sup>2</sup> 29 <sup>1</sup>                | N      | N      | -7          | 2  | 1.2857143  | 0.456897           | 0.543103           | -36         | 365           | -401          |
| 117 | 3 <sup>2</sup> 13 <sup>1</sup>                | N      | N      | -7          | 2  | 1.2857143  | 0.452991           | 0.547009           | -43         | 365           | -408          |
| 118 | 2 <sup>1</sup> 59 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.457627           | 0.542373           | -38         | 370           | -408          |
| 119 | 7 <sup>1</sup> 17 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.462185           | 0.537815           | -33         | 375           | -408          |
| 120 | 2 <sup>3</sup> 3 <sup>1</sup> 5 <sup>1</sup>  | N      | N      | -48         | 32                                       | 1.3333333  | 0.458333           | 0.541667           | -81         | 375           | -456          |
| 121 | 11 <sup>2</sup>                               | N      | Y      | 2           | 0  | 1.5000000  | 0.462810           | 0.537190           | -79         | 377           | -456          |
| 122 | 2 <sup>1</sup> 61 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.467213           | 0.532787           | -74         | 382           | -456          |
| 123 | 3 <sup>1</sup> 41 <sup>1</sup>                | Y      | N      | 5           | 0  | 1.0000000  | 0.471545           | 0.528455           | -69         | 387           | -456          |
| 124 | 2 <sup>2</sup> 31 <sup>1</sup>                | N      | N      | -7          | 2  | 1.2857143  | 0.467742           | 0.532258           | -76         | 387           | -463          |



| $n$ | Primes         | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum_{d n} C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|----------------|--------|--------|-------------|--|---|--------------------|--------------------|-------------|---------------|---------------|
| 125 | $5^3$          | N      | Y      | -2          | 0  | 2.0000000   | 0.464000           | 0.536000           | -78         | 387           | -465          |
| 126 | $2^1 3^2 7^1$  | N      | N      | 30          | 14                                       | 1.1666667   | 0.468254           | 0.531746           | -48         | 417           | -465          |
| 127 | $127^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.464567           | 0.535433           | -50         | 417           | -467          |
| 128 | $2^7$          | N      | Y      | -2          | 0  | 4.0000000   | 0.460938           | 0.539062           | -52         | 417           | -469          |
| 129 | $3^1 43^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.465116           | 0.534884           | -47         | 422           | -469          |
| 130 | $2^1 5^1 13^1$ | Y      | N      | -16         | 0  | 1.0000000   | 0.461538           | 0.538462           | -63         | 422           | -485          |
| 131 | $131^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.458015           | 0.541985           | -65         | 422           | -487          |
| 132 | $2^2 3^1 11^1$ | N      | N      | 30          | 14                                       | 1.1666667   | 0.462121           | 0.537879           | -35         | 452           | -487          |
| 133 | $7^1 19^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.466165           | 0.533835           | -30         | 457           | -487          |
| 134 | $2^1 67^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.470149           | 0.529851           | -25         | 462           | -487          |
| 135 | $3^3 5^1$      | N      | N      | 9           | 4  | 1.5555556   | 0.474074           | 0.525926           | -16         | 471           | -487          |
| 136 | $2^3 17^1$     | N      | N      | 9           | 4  | 1.5555556   | 0.477941           | 0.522059           | -7          | 480           | -487          |
| 137 | $137^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.474453           | 0.525547           | -9          | 480           | -489          |
| 138 | $2^1 3^1 23^1$ | Y      | N      | -16         | 0  | 1.0000000   | 0.471014           | 0.528986           | -25         | 480           | -505          |
| 139 | $139^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.467626           | 0.532374           | -27         | 480           | -507          |
| 140 | $2^2 5^1 7^1$  | N      | N      | 30          | 14                                       | 1.1666667   | 0.471429           | 0.528571           | 3           | 510           | -507          |
| 141 | $3^1 47^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.475177           | 0.524823           | 8           | 515           | -507          |
| 142 | $2^1 71^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.478873           | 0.521127           | 13          | 520           | -507          |
| 143 | $11^1 13^1$    | Y      | N      | 5           | 0  | 1.0000000   | 0.482517           | 0.517483           | 18          | 525           | -507          |
| 144 | $2^4 3^2$      | N      | N      | 34          | 29                                       | 1.6176471   | 0.486111           | 0.513889           | 52          | 559           | -507          |
| 145 | $5^1 29^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.489655           | 0.510345           | 57          | 564           | -507          |
| 146 | $2^1 73^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.493151           | 0.506849           | 62          | 569           | -507          |
| 147 | $3^1 7^2$      | N      | N      | -7          | 2  | 1.2857143   | 0.489796           | 0.510204           | 55          | 569           | -514          |
| 148 | $2^2 37^1$     | N      | N      | -7          | 2  | 1.2857143   | 0.486486           | 0.513514           | 48          | 569           | -521          |
| 149 | $149^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.483221           | 0.516779           | 46          | 569           | -523          |
| 150 | $2^1 3^1 5^2$  | N      | N      | 30          | 14                                       | 1.1666667   | 0.486667           | 0.513333           | 76          | 599           | -523          |
| 151 | $151^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.483444           | 0.516556           | 74          | 599           | -525          |
| 152 | $2^3 19^1$     | N      | N      | 9           | 4  | 1.5555556   | 0.486842           | 0.513158           | 83          | 608           | -525          |
| 153 | $3^2 17^1$     | N      | N      | -7          | 2  | 1.2857143   | 0.483660           | 0.516340           | 76          | 608           | -532          |
| 154 | $2^1 7^1 11^1$ | Y      | N      | -16         | 0  | 1.0000000   | 0.480519           | 0.519481           | 60          | 608           | -548          |
| 155 | $5^1 31^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.483871           | 0.516129           | 65          | 613           | -548          |
| 156 | $2^2 3^1 13^1$ | N      | N      | 30          | 14                                       | 1.1666667   | 0.487179           | 0.512821           | 95          | 643           | -548          |
| 157 | $157^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.484076           | 0.515924           | 93          | 643           | -550          |
| 158 | $2^1 79^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.487342           | 0.512658           | 98          | 648           | -550          |
| 159 | $3^1 53^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.490566           | 0.509434           | 103         | 653           | -550          |
| 160 | $2^5 5^1$      | N      | N      | 13          | 8  | 2.0769231   | 0.493750           | 0.506250           | 116         | 666           | -550          |
| 161 | $7^1 23^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.496894           | 0.503106           | 121         | 671           | -550          |
| 162 | $2^1 3^4$      | N      | N      | -11         | 6  | 1.8181818   | 0.493827           | 0.506173           | 110         | 671           | -561          |
| 163 | $163^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.490798           | 0.509202           | 108         | 671           | -563          |
| 164 | $2^2 41^1$     | N      | N      | -7          | 2  | 1.2857143   | 0.487805           | 0.512195           | 101         | 671           | -570          |
| 165 | $3^1 5^1 11^1$ | Y      | N      | -16         | 0  | 1.0000000   | 0.484848           | 0.515152           | 85          | 671           | -586          |
| 166 | $2^1 83^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.487952           | 0.512048           | 90          | 676           | -586          |
| 167 | $167^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.485030           | 0.514970           | 88          | 676           | -588          |
| 168 | $2^3 3^1 7^1$  | N      | N      | -48         | 32                                       | 1.3333333   | 0.482143           | 0.517857           | 40          | 676           | -636          |
| 169 | $13^2$         | N      | Y      | 2           | 0  | 1.5000000   | 0.485207           | 0.514793           | 42          | 678           | -636          |
| 170 | $2^1 5^1 17^1$ | Y      | N      | -16         | 0  | 1.0000000   | 0.482353           | 0.517647           | 26          | 678           | -652          |
| 171 | $3^2 19^1$     | N      | N      | -7          | 2  | 1.2857143   | 0.479532           | 0.520468           | 19          | 678           | -659          |
| 172 | $2^2 43^1$     | N      | N      | -7          | 2  | 1.2857143   | 0.476744           | 0.523256           | 12          | 678           | -666          |
| 173 | $173^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.473988           | 0.526012           | 10          | 678           | -668          |
| 174 | $2^1 3^1 29^1$ | Y      | N      | -16         | 0  | 1.0000000   | 0.471264           | 0.528736           | -6          | 678           | -684          |
| 175 | $5^2 7^1$      | N      | N      | -7          | 2  | 1.2857143   | 0.468571           | 0.531429           | -13         | 678           | -691          |
| 176 | $2^4 11^1$     | N      | N      | -11         | 6  | 1.8181818   | 0.465909           | 0.534091           | -24         | 678           | -702          |
| 177 | $3^1 59^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.468927           | 0.531073           | -19         | 683           | -702          |
| 178 | $2^1 89^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.471910           | 0.528090           | -14         | 688           | -702          |
| 179 | $179^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.469274           | 0.530726           | -16         | 688           | -704          |
| 180 | $2^2 3^2 5^1$  | N      | N      | -74         | 58                                       | 1.2162162   | 0.466667           | 0.533333           | -90         | 688           | -778          |
| 181 | $181^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.464088           | 0.535912           | -92         | 688           | -780          |
| 182 | $2^1 7^1 13^1$ | Y      | N      | -16         | 0  | 1.0000000   | 0.461538           | 0.538462           | -108        | 688           | -796          |
| 183 | $3^1 61^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.464481           | 0.535519           | -103        | 693           | -796          |
| 184 | $2^3 23^1$     | N      | N      | 9           | 4  | 1.5555556   | 0.467391           | 0.532609           | -94         | 702           | -796          |
| 185 | $5^1 37^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.470270           | 0.529730           | -89         | 707           | -796          |
| 186 | $2^1 3^1 31^1$ | Y      | N      | -16         | 0  | 1.0000000   | 0.467742           | 0.532258           | -105        | 707           | -812          |
| 187 | $11^1 17^1$    | Y      | N      | 5           | 0  | 1.0000000   | 0.470588           | 0.529412           | -100        | 712           | -812          |
| 188 | $2^2 47^1$     | N      | N      | -7          | 2  | 1.2857143   | 0.468085           | 0.531915           | -107        | 712           | -819          |
| 189 | $3^3 7^1$      | N      | N      | 9           | 4  | 1.5555556   | 0.470899           | 0.529101           | -98         | 721           | -819          |
| 190 | $2^1 5^1 19^1$ | Y      | N      | -16         | 0  | 1.0000000   | 0.468421           | 0.531579           | -114        | 721           | -835          |
| 191 | $191^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.465969           | 0.534031           | -116        | 721           | -837          |
| 192 | $2^6 3^1$      | N      | N      | -15         | 10                                       | 2.3333333   | 0.463542           | 0.536458           | -131        | 721           | -852          |
| 193 | $193^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.461140           | 0.538860           | -133        | 721           | -854          |
| 194 | $2^1 97^1$     | Y      | N      | 5           | 0  | 1.0000000   | 0.463918           | 0.536082           | -128        | 726           | -854          |
| 195 | $3^1 5^1 13^1$ | Y      | N      | -16         | 0  | 1.0000000   | 0.461538           | 0.538462           | -144        | 726           | -870          |
| 196 | $2^2 7^2$      | N      | N      | 14          | 9  | 1.3571429   | 0.464286           | 0.535714           | -130        | 740           | -870          |
| 197 | $197^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.461929           | 0.538071           | -132        | 740           | -872          |
| 198 | $2^1 3^2 11^1$ | N      | N      | 30          | 14                                       | 1.1666667   | 0.464646           | 0.535354           | -102        | 770           | -872          |
| 199 | $199^1$        | Y      | Y      | -2          | 0  | 1.0000000   | 0.462312           | 0.537688           | -104        | 770           | -874          |
| 200 | $2^3 5^2$      | N      | N      | -23         | 18                                       | 1.4782609   | 0.460000           | 0.540000           | -127        | 770           | -897          |

| $n$ | Primes            | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum_{d n} C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|-------------------|--------|--------|-------------|--|---|--------------------|--------------------|-------------|---------------|---------------|
| 201 | $3^1 67^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.462687           | 0.537313           | -122        | 775           | -897          |
| 202 | $2^1 101^1$       | Y      | N      | 5           | 0  | 1.0000000   | 0.465347           | 0.534653           | -117        | 780           | -897          |
| 203 | $7^1 29^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.467980           | 0.532020           | -112        | 785           | -897          |
| 204 | $2^2 3^1 17^1$    | N      | N      | 30          | 14                                       | 1.1666667   | 0.470588           | 0.529412           | -82         | 815           | -897          |
| 205 | $5^1 41^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.473171           | 0.526829           | -77         | 820           | -897          |
| 206 | $2^1 103^1$       | Y      | N      | 5           | 0  | 1.0000000   | 0.475728           | 0.524272           | -72         | 825           | -897          |
| 207 | $3^2 23^1$        | N      | N      | -7          | 2  | 1.2857143   | 0.473430           | 0.526570           | -79         | 825           | -904          |
| 208 | $2^4 13^1$        | N      | N      | -11         | 6  | 1.8181818   | 0.471154           | 0.528846           | -90         | 825           | -915          |
| 209 | $11^1 19^1$       | Y      | N      | 5           | 0  | 1.0000000   | 0.473684           | 0.526316           | -85         | 830           | -915          |
| 210 | $2^1 3^1 5^1 7^1$ | Y      | N      | 65          | 0  | 1.0000000   | 0.476190           | 0.523810           | -20         | 895           | -915          |
| 211 | $211^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.473934           | 0.526066           | -22         | 895           | -917          |
| 212 | $2^2 53^1$        | N      | N      | -7          | 2  | 1.2857143   | 0.471698           | 0.528302           | -29         | 895           | -924          |
| 213 | $3^1 71^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.474178           | 0.525822           | -24         | 900           | -924          |
| 214 | $2^1 107^1$       | Y      | N      | 5           | 0  | 1.0000000   | 0.476636           | 0.523364           | -19         | 905           | -924          |
| 215 | $5^1 43^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.479070           | 0.520930           | -14         | 910           | -924          |
| 216 | $2^3 3^3$         | N      | N      | 46          | 41                                       | 1.5000000   | 0.481481           | 0.518519           | 32          | 956           | -924          |
| 217 | $7^1 31^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.483871           | 0.516129           | 37          | 961           | -924          |
| 218 | $2^1 109^1$       | Y      | N      | 5           | 0  | 1.0000000   | 0.486239           | 0.513761           | 42          | 966           | -924          |
| 219 | $3^1 73^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.488584           | 0.511416           | 47          | 971           | -924          |
| 220 | $2^2 5^1 11^1$    | N      | N      | 30          | 14                                       | 1.1666667   | 0.490909           | 0.509091           | 77          | 1001          | -924          |
| 221 | $13^1 17^1$       | Y      | N      | 5           | 0  | 1.0000000   | 0.493213           | 0.506787           | 82          | 1006          | -924          |
| 222 | $2^1 3^1 37^1$    | Y      | N      | -16         | 0  | 1.0000000   | 0.490991           | 0.509009           | 66          | 1006          | -940          |
| 223 | $223^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.488789           | 0.511211           | 64          | 1006          | -942          |
| 224 | $2^5 7^1$         | N      | N      | 13          | 8  | 2.0769231   | 0.491071           | 0.508929           | 77          | 1019          | -942          |
| 225 | $3^2 5^2$         | N      | N      | 14          | 9  | 1.3571429   | 0.493333           | 0.506667           | 91          | 1033          | -942          |
| 226 | $2^1 113^1$       | Y      | N      | 5           | 0  | 1.0000000   | 0.495575           | 0.504425           | 96          | 1038          | -942          |
| 227 | $227^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.493392           | 0.506608           | 94          | 1038          | -944          |
| 228 | $2^2 3^1 19^1$    | N      | N      | 30          | 14                                       | 1.1666667   | 0.495614           | 0.504386           | 124         | 1068          | -944          |
| 229 | $229^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.493450           | 0.506550           | 122         | 1068          | -946          |
| 230 | $2^1 5^1 23^1$    | Y      | N      | -16         | 0  | 1.0000000   | 0.491304           | 0.508696           | 106         | 1068          | -962          |
| 231 | $3^1 7^1 11^1$    | Y      | N      | -16         | 0  | 1.0000000   | 0.489177           | 0.510823           | 90          | 1068          | -978          |
| 232 | $2^3 29^1$        | N      | N      | 9           | 4  | 1.5555556   | 0.491379           | 0.508621           | 99          | 1077          | -978          |
| 233 | $233^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.489270           | 0.510730           | 97          | 1077          | -980          |
| 234 | $2^1 3^2 13^1$    | N      | N      | 30          | 14                                       | 1.1666667   | 0.491453           | 0.508547           | 127         | 1107          | -980          |
| 235 | $5^1 47^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.493617           | 0.506383           | 132         | 1112          | -980          |
| 236 | $2^2 59^1$        | N      | N      | -7          | 2  | 1.2857143   | 0.491525           | 0.508475           | 125         | 1112          | -987          |
| 237 | $3^1 79^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.493671           | 0.506329           | 130         | 1117          | -987          |
| 238 | $2^1 7^1 17^1$    | Y      | N      | -16         | 0  | 1.0000000   | 0.491597           | 0.508403           | 114         | 1117          | -1003         |
| 239 | $239^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.489540           | 0.510460           | 112         | 1117          | -1005         |
| 240 | $2^4 3^1 5^1$     | N      | N      | 70          | 54                                       | 1.5000000   | 0.491667           | 0.508333           | 182         | 1187          | -1005         |
| 241 | $241^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.489627           | 0.510373           | 180         | 1187          | -1007         |
| 242 | $2^1 11^2$        | N      | N      | -7          | 2  | 1.2857143   | 0.487603           | 0.512397           | 173         | 1187          | -1014         |
| 243 | $3^5$             | N      | Y      | -2          | 0  | 3.0000000   | 0.485597           | 0.514403           | 171         | 1187          | -1016         |
| 244 | $2^2 61^1$        | N      | N      | -7          | 2  | 1.2857143   | 0.483607           | 0.516393           | 164         | 1187          | -1023         |
| 245 | $5^1 7^2$         | N      | N      | -7          | 2  | 1.2857143   | 0.481633           | 0.518367           | 157         | 1187          | -1030         |
| 246 | $2^1 3^1 41^1$    | Y      | N      | -16         | 0  | 1.0000000   | 0.479675           | 0.520325           | 141         | 1187          | -1046         |
| 247 | $13^1 19^1$       | Y      | N      | 5           | 0  | 1.0000000   | 0.481781           | 0.518219           | 146         | 1192          | -1046         |
| 248 | $2^3 31^1$        | N      | N      | 9           | 4  | 1.5555556   | 0.483871           | 0.516129           | 155         | 1201          | -1046         |
| 249 | $3^1 83^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.485944           | 0.514056           | 160         | 1206          | -1046         |
| 250 | $2^1 5^3$         | N      | N      | 9           | 4  | 1.5555556   | 0.488000           | 0.512000           | 169         | 1215          | -1046         |
| 251 | $251^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.486056           | 0.513944           | 167         | 1215          | -1048         |
| 252 | $2^2 3^2 7^1$     | N      | N      | -74         | 58                                       | 1.2162162   | 0.484127           | 0.515873           | 93          | 1215          | -1122         |
| 253 | $11^1 23^1$       | Y      | N      | 5           | 0  | 1.0000000   | 0.486166           | 0.513834           | 98          | 1220          | -1122         |
| 254 | $2^1 127^1$       | Y      | N      | 5           | 0  | 1.0000000   | 0.488189           | 0.511811           | 103         | 1225          | -1122         |
| 255 | $3^1 5^1 17^1$    | Y      | N      | -16         | 0  | 1.0000000   | 0.486275           | 0.513725           | 87          | 1225          | -1138         |
| 256 | $2^8$             | N      | Y      | 2           | 0  | 4.5000000   | 0.488281           | 0.511719           | 89          | 1227          | -1138         |
| 257 | $257^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.486381           | 0.513619           | 87          | 1227          | -1140         |
| 258 | $2^1 3^1 43^1$    | Y      | N      | -16         | 0  | 1.0000000   | 0.484496           | 0.515504           | 71          | 1227          | -1156         |
| 259 | $7^1 37^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.486486           | 0.513514           | 76          | 1232          | -1156         |
| 260 | $2^2 5^1 13^1$    | N      | N      | 30          | 14                                       | 1.1666667   | 0.488462           | 0.511538           | 106         | 1262          | -1156         |
| 261 | $3^2 29^1$        | N      | N      | -7          | 2  | 1.2857143   | 0.486590           | 0.513410           | 99          | 1262          | -1163         |
| 262 | $2^1 131^1$       | Y      | N      | 5           | 0  | 1.0000000   | 0.488550           | 0.511450           | 104         | 1267          | -1163         |
| 263 | $263^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.486692           | 0.513308           | 102         | 1267          | -1165         |
| 264 | $2^3 3^1 11^1$    | N      | N      | -48         | 32                                       | 1.3333333   | 0.484848           | 0.515152           | 54          | 1267          | -1213         |
| 265 | $5^1 53^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.486792           | 0.513208           | 59          | 1272          | -1213         |
| 266 | $2^1 7^1 19^1$    | Y      | N      | -16         | 0  | 1.0000000   | 0.484962           | 0.515038           | 43          | 1272          | -1229         |
| 267 | $3^1 89^1$        | Y      | N      | 5           | 0  | 1.0000000   | 0.486891           | 0.513109           | 48          | 1277          | -1229         |
| 268 | $2^2 67^1$        | N      | N      | -7          | 2  | 1.2857143   | 0.485075           | 0.514925           | 41          | 1277          | -1236         |
| 269 | $269^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.483271           | 0.516729           | 39          | 1277          | -1238         |
| 270 | $2^1 3^3 5^1$     | N      | N      | -48         | 32                                       | 1.3333333   | 0.481481           | 0.518519           | -9          | 1277          | -1286         |
| 271 | $271^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.479705           | 0.520295           | -11         | 1277          | -1288         |
| 272 | $2^4 17^1$        | N      | N      | -11         | 6  | 1.8181818   | 0.477941           | 0.522059           | -22         | 1277          | -1299         |
| 273 | $3^1 7^1 13^1$    | Y      | N      | -16         | 0  | 1.0000000   | 0.476190           | 0.523810           | -38         | 1277          | -1315         |
| 274 | $2^1 137^1$       | Y      | N      | 5           | 0  | 1.0000000   | 0.478102           | 0.521898           | -33         | 1282          | -1315         |
| 275 | $5^2 11^1$        | N      | N      | -7          | 2  | 1.2857143   | 0.476364           | 0.523636           | -40         | 1282          | -1322         |
| 276 | $2^2 3^1 23^1$    | N      | N      | 30          | 14                                       | 1.1666667   | 0.478261           | 0.521739           | -10         | 1312          | -1322         |
| 277 | $277^1$           | Y      | Y      | -2          | 0  | 1.0000000   | 0.476534           | 0.523466           | -12         | 1312          | -1324         |

| $n$ | Primes             | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum d n C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|--------------------|--------|--------|-------------|--|---|--------------------|--------------------|-------------|---------------|---------------|
| 278 | $2^1 139^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.478417           | 0.521583           | -7          | 1317          | -1324         |
| 279 | $3^2 31^1$         | N      | N      | -7          | 2  | 1.2857143                                       | 0.476703           | 0.523297           | -14         | 1317          | -1331         |
| 280 | $2^3 5^1 7^1$      | N      | N      | -48         | 32                                       | 1.3333333                                       | 0.475000           | 0.525000           | -62         | 1317          | -1379         |
| 281 | $281^1$            | Y      | Y      | -2          | 0  | 1.0000000                                       | 0.473310           | 0.526690           | -64         | 1317          | -1381         |
| 282 | $2^1 3^1 47^1$     | Y      | N      | -16         | 0  | 1.0000000                                       | 0.471631           | 0.528369           | -80         | 1317          | -1397         |
| 283 | $283^1$            | Y      | Y      | -2          | 0  | 1.0000000                                       | 0.469965           | 0.530035           | -82         | 1317          | -1399         |
| 284 | $2^2 71^1$         | N      | N      | -7          | 2  | 1.2857143                                       | 0.468310           | 0.531690           | -89         | 1317          | -1406         |
| 285 | $3^1 5^1 19^1$     | Y      | N      | -16         | 0  | 1.0000000                                       | 0.466667           | 0.533333           | -105        | 1317          | -1422         |
| 286 | $2^1 11^1 13^1$    | Y      | N      | -16         | 0  | 1.0000000                                       | 0.465035           | 0.534965           | -121        | 1317          | -1438         |
| 287 | $7^1 41^1$         | Y      | N      | 5           | 0  | 1.0000000                                       | 0.466899           | 0.533101           | -116        | 1322          | -1438         |
| 288 | $2^5 3^2$          | N      | N      | -47         | 42                                       | 1.7659574                                       | 0.465278           | 0.534722           | -163        | 1322          | -1485         |
| 289 | $17^2$             | N      | Y      | 2           | 0  | 1.5000000                                       | 0.467128           | 0.532872           | -161        | 1324          | -1485         |
| 290 | $2^1 5^1 29^1$     | Y      | N      | -16         | 0  | 1.0000000                                       | 0.465517           | 0.534483           | -177        | 1324          | -1501         |
| 291 | $3^1 97^1$         | Y      | N      | 5           | 0  | 1.0000000                                       | 0.467354           | 0.532646           | -172        | 1329          | -1501         |
| 292 | $2^2 73^1$         | N      | N      | -7          | 2  | 1.2857143                                       | 0.465753           | 0.534247           | -179        | 1329          | -1508         |
| 293 | $293^1$            | Y      | Y      | -2          | 0  | 1.0000000                                       | 0.464164           | 0.535836           | -181        | 1329          | -1510         |
| 294 | $2^1 3^1 7^2$      | N      | N      | 30          | 14                                       | 1.1666667                                       | 0.465986           | 0.534014           | -151        | 1359          | -1510         |
| 295 | $5^1 59^1$         | Y      | N      | 5           | 0  | 1.0000000                                       | 0.467797           | 0.532203           | -146        | 1364          | -1510         |
| 296 | $2^3 37^1$         | N      | N      | 9           | 4  | 1.5555556                                       | 0.469595           | 0.530405           | -137        | 1373          | -1510         |
| 297 | $3^3 11^1$         | N      | N      | 9           | 4  | 1.5555556                                       | 0.471380           | 0.528620           | -128        | 1382          | -1510         |
| 298 | $2^1 149^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.473154           | 0.526846           | -123        | 1387          | -1510         |
| 299 | $13^1 23^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.474916           | 0.525084           | -118        | 1392          | -1510         |
| 300 | $2^2 3^1 5^2$      | N      | N      | -74         | 58                                       | 1.2162162                                       | 0.473333           | 0.526667           | -192        | 1392          | -1584         |
| 301 | $7^1 43^1$         | Y      | N      | 5           | 0  | 1.0000000                                       | 0.475083           | 0.524917           | -187        | 1397          | -1584         |
| 302 | $2^1 151^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.476821           | 0.523179           | -182        | 1402          | -1584         |
| 303 | $3^1 101^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.478548           | 0.521452           | -177        | 1407          | -1584         |
| 304 | $2^4 19^1$         | N      | N      | -11         | 6  | 1.8181818                                       | 0.476974           | 0.523026           | -188        | 1407          | -1595         |
| 305 | $5^1 61^1$         | Y      | N      | 5           | 0  | 1.0000000                                       | 0.478689           | 0.521311           | -183        | 1412          | -1595         |
| 306 | $2^1 3^2 17^1$     | N      | N      | 30          | 14                                       | 1.1666667                                       | 0.480392           | 0.519608           | -153        | 1442          | -1595         |
| 307 | $307^1$            | Y      | Y      | -2          | 0  | 1.0000000                                       | 0.478827           | 0.521173           | -155        | 1442          | -1597         |
| 308 | $2^2 7^1 11^1$     | N      | N      | 30          | 14                                       | 1.1666667                                       | 0.480519           | 0.519481           | -125        | 1472          | -1597         |
| 309 | $3^1 103^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.482201           | 0.517799           | -120        | 1477          | -1597         |
| 310 | $2^1 5^1 31^1$     | Y      | N      | -16         | 0  | 1.0000000                                       | 0.480645           | 0.519355           | -136        | 1477          | -1613         |
| 311 | $311^1$            | Y      | Y      | -2          | 0  | 1.0000000                                       | 0.479100           | 0.520900           | -138        | 1477          | -1615         |
| 312 | $2^3 3^1 13^1$     | N      | N      | -48         | 32                                       | 1.3333333                                       | 0.477564           | 0.522436           | -186        | 1477          | -1663         |
| 313 | $313^1$            | Y      | Y      | -2          | 0  | 1.0000000                                       | 0.476038           | 0.523962           | -188        | 1477          | -1665         |
| 314 | $2^1 157^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.477707           | 0.522293           | -183        | 1482          | -1665         |
| 315 | $3^2 5^1 7^1$      | N      | N      | 30          | 14                                       | 1.1666667                                       | 0.479365           | 0.520635           | -153        | 1512          | -1665         |
| 316 | $2^2 79^1$         | N      | N      | -7          | 2  | 1.2857143                                       | 0.477848           | 0.522152           | -160        | 1512          | -1672         |
| 317 | $317^1$            | Y      | Y      | -2          | 0  | 1.0000000                                       | 0.476341           | 0.523659           | -162        | 1512          | -1674         |
| 318 | $2^1 3^1 53^1$     | Y      | N      | -16         | 0  | 1.0000000                                       | 0.474843           | 0.525157           | -178        | 1512          | -1690         |
| 319 | $11^1 29^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.476489           | 0.523511           | -173        | 1517          | -1690         |
| 320 | $2^6 5^1$          | N      | N      | -15         | 10                                       | 2.3333333                                       | 0.475000           | 0.525000           | -188        | 1517          | -1705         |
| 321 | $3^1 107^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.476636           | 0.523364           | -183        | 1522          | -1705         |
| 322 | $2^1 7^1 23^1$     | Y      | N      | -16         | 0  | 1.0000000                                       | 0.475155           | 0.524845           | -199        | 1522          | -1721         |
| 323 | $17^1 19^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.476780           | 0.523220           | -194        | 1527          | -1721         |
| 324 | $2^2 3^4$          | N      | N      | 34          | 29                                       | 1.6176471                                       | 0.478395           | 0.521605           | -160        | 1561          | -1721         |
| 325 | $5^2 13^1$         | N      | N      | -7          | 2  | 1.2857143                                       | 0.476923           | 0.523077           | -167        | 1561          | -1728         |
| 326 | $2^1 163^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.478528           | 0.521472           | -162        | 1566          | -1728         |
| 327 | $3^1 109^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.480122           | 0.519878           | -157        | 1571          | -1728         |
| 328 | $2^3 41^1$         | N      | N      | 9           | 4  | 1.5555556                                       | 0.481707           | 0.518293           | -148        | 1580          | -1728         |
| 329 | $7^1 47^1$         | Y      | N      | 5           | 0  | 1.0000000                                       | 0.483283           | 0.516717           | -143        | 1585          | -1728         |
| 330 | $2^1 3^1 5^1 11^1$ | Y      | N      | 65          | 0  | 1.0000000                                       | 0.484848           | 0.515152           | -78         | 1650          | -1728         |
| 331 | $331^1$            | Y      | Y      | -2          | 0  | 1.0000000                                       | 0.483384           | 0.516616           | -80         | 1650          | -1730         |
| 332 | $2^2 83^1$         | N      | N      | -7          | 2  | 1.2857143                                       | 0.481928           | 0.518072           | -87         | 1650          | -1737         |
| 333 | $3^2 37^1$         | N      | N      | -7          | 2  | 1.2857143                                       | 0.480480           | 0.519520           | -94         | 1650          | -1744         |
| 334 | $2^1 167^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.482036           | 0.517964           | -89         | 1655          | -1744         |
| 335 | $5^1 67^1$         | Y      | N      | 5           | 0  | 1.0000000                                       | 0.483582           | 0.516418           | -84         | 1660          | -1744         |
| 336 | $2^4 3^1 7^1$      | N      | N      | 70          | 54                                       | 1.5000000                                       | 0.485119           | 0.514881           | -14         | 1730          | -1744         |
| 337 | $337^1$            | Y      | Y      | -2          | 0  | 1.0000000                                       | 0.483680           | 0.516320           | -16         | 1730          | -1746         |
| 338 | $2^1 13^2$         | N      | N      | -7          | 2  | 1.2857143                                       | 0.482249           | 0.517751           | -23         | 1730          | -1753         |
| 339 | $3^1 113^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.483776           | 0.516224           | -18         | 1735          | -1753         |
| 340 | $2^2 5^1 17^1$     | N      | N      | 30          | 14                                       | 1.1666667                                       | 0.485294           | 0.514706           | 12          | 1765          | -1753         |
| 341 | $11^1 31^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.486804           | 0.513196           | 17          | 1770          | -1753         |
| 342 | $2^1 3^2 19^1$     | N      | N      | 30          | 14                                       | 1.1666667                                       | 0.488304           | 0.511696           | 47          | 1800          | -1753         |
| 343 | $7^3$              | N      | Y      | -2          | 0  | 2.0000000                                       | 0.486880           | 0.513120           | 45          | 1800          | -1755         |
| 344 | $2^3 43^1$         | N      | N      | 9           | 4  | 1.5555556                                       | 0.488372           | 0.511628           | 54          | 1809          | -1755         |
| 345 | $3^1 5^1 23^1$     | Y      | N      | -16         | 0  | 1.0000000                                       | 0.486957           | 0.513043           | 38          | 1809          | -1771         |
| 346 | $2^1 173^1$        | Y      | N      | 5           | 0  | 1.0000000                                       | 0.488439           | 0.511561           | 43          | 1814          | -1771         |
| 347 | $347^1$            | Y      | Y      | -2          | 0  | 1.0000000                                       | 0.487032           | 0.512968           | 41          | 1814          | -1773         |
| 348 | $2^2 3^1 29^1$     | N      | N      | 30          | 14                                       | 1.1666667                                       | 0.488506           | 0.511494           | 71          | 1844          | -1773         |
| 349 | $349^1$            | Y      | Y      | -2          | 0  | 1.0000000                                       | 0.487106           | 0.512894           | 69          | 1844          | -1775         |
| 350 | $2^1 5^2 7^1$      | N      | N      | 30          | 14                                       | 1.1666667                                       | 0.488571           | 0.511429           | 99          | 1874          | -1775         |

| $n$ | Primes             | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\Sigma d n C_{\Omega(d)}^{(d)}}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|--------------------|--------|--------|-------------|--|--|--------------------|--------------------|-------------|---------------|---------------|
| 351 | $3^3 13^1$         | N      | N      | 9           | 4  | 1.5555556  | 0.490028           | 0.509972           | 108         | 1883          | -1775         |
| 352 | $2^5 11^1$         | N      | N      | 13          | 8  | 2.0769231  | 0.491477           | 0.508523           | 121         | 1896          | -1775         |
| 353 | $353^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.490085           | 0.509915           | 119         | 1896          | -1777         |
| 354 | $2^1 3^1 59^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.488701           | 0.511299           | 103         | 1896          | -1793         |
| 355 | $5^1 71^1$         | Y      | N      | 5           | 0  | 1.0000000  | 0.490141           | 0.509859           | 108         | 1901          | -1793         |
| 356 | $2^2 89^1$         | N      | N      | -7          | 2  | 1.2857143  | 0.488764           | 0.511236           | 101         | 1901          | -1800         |
| 357 | $3^1 7^1 17^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.487395           | 0.512605           | 85          | 1901          | -1816         |
| 358 | $2^1 179^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.488827           | 0.511173           | 90          | 1906          | -1816         |
| 359 | $359^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.487465           | 0.512535           | 88          | 1906          | -1818         |
| 360 | $2^3 3^2 5^1$      | N      | N      | 145         | 129                                      | 1.3034483  | 0.488889           | 0.511111           | 233         | 2051          | -1818         |
| 361 | $19^2$             | N      | Y      | 2           | 0  | 1.5000000  | 0.490305           | 0.509695           | 235         | 2053          | -1818         |
| 362 | $2^1 181^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.491713           | 0.508287           | 240         | 2058          | -1818         |
| 363 | $3^1 11^2$         | N      | N      | -7          | 2  | 1.2857143  | 0.490358           | 0.509642           | 233         | 2058          | -1825         |
| 364 | $2^2 7^1 13^1$     | N      | N      | 30          | 14                                       | 1.1666667  | 0.491758           | 0.508242           | 263         | 2088          | -1825         |
| 365 | $5^1 73^1$         | Y      | N      | 5           | 0  | 1.0000000  | 0.493151           | 0.506849           | 268         | 2093          | -1825         |
| 366 | $2^1 3^1 61^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.491803           | 0.508197           | 252         | 2093          | -1841         |
| 367 | $367^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.490463           | 0.509537           | 250         | 2093          | -1843         |
| 368 | $2^4 23^1$         | N      | N      | -11         | 6  | 1.8181818  | 0.489130           | 0.510870           | 239         | 2093          | -1854         |
| 369 | $3^2 41^1$         | N      | N      | -7          | 2  | 1.2857143  | 0.487805           | 0.512195           | 232         | 2093          | -1861         |
| 370 | $2^1 5^1 37^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.486486           | 0.513514           | 216         | 2093          | -1877         |
| 371 | $7^1 53^1$         | Y      | N      | 5           | 0  | 1.0000000  | 0.487871           | 0.512129           | 221         | 2098          | -1877         |
| 372 | $2^2 3^1 31^1$     | N      | N      | 30          | 14                                       | 1.1666667  | 0.489247           | 0.510753           | 251         | 2128          | -1877         |
| 373 | $373^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.487936           | 0.512064           | 249         | 2128          | -1879         |
| 374 | $2^1 11^1 17^1$    | Y      | N      | -16         | 0  | 1.0000000  | 0.486631           | 0.513369           | 233         | 2128          | -1895         |
| 375 | $3^1 5^3$          | N      | N      | 9           | 4  | 1.5555556  | 0.488000           | 0.512000           | 242         | 2137          | -1895         |
| 376 | $2^3 47^1$         | N      | N      | 9           | 4  | 1.5555556  | 0.489362           | 0.510638           | 251         | 2146          | -1895         |
| 377 | $13^1 29^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.490716           | 0.509284           | 256         | 2151          | -1895         |
| 378 | $2^1 3^3 7^1$      | N      | N      | -48         | 32                                       | 1.3333333  | 0.489418           | 0.510582           | 208         | 2151          | -1943         |
| 379 | $379^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.488127           | 0.511873           | 206         | 2151          | -1945         |
| 380 | $2^2 5^1 19^1$     | N      | N      | 30          | 14                                       | 1.1666667  | 0.489474           | 0.510526           | 236         | 2181          | -1945         |
| 381 | $3^1 127^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.490814           | 0.509186           | 241         | 2186          | -1945         |
| 382 | $2^1 191^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.492147           | 0.507853           | 246         | 2191          | -1945         |
| 383 | $383^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.490862           | 0.509138           | 244         | 2191          | -1947         |
| 384 | $2^7 3^1$          | N      | N      | 17          | 12                                       | 2.5882353  | 0.492188           | 0.507812           | 261         | 2208          | -1947         |
| 385 | $5^1 7^1 11^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.490909           | 0.509091           | 245         | 2208          | -1963         |
| 386 | $2^1 193^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.492228           | 0.507772           | 250         | 2213          | -1963         |
| 387 | $3^2 43^1$         | N      | N      | -7          | 2  | 1.2857143  | 0.490956           | 0.509044           | 243         | 2213          | -1970         |
| 388 | $2^2 97^1$         | N      | N      | -7          | 2  | 1.2857143  | 0.489691           | 0.510309           | 236         | 2213          | -1977         |
| 389 | $389^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.488432           | 0.511568           | 234         | 2213          | -1979         |
| 390 | $2^1 3^1 5^1 13^1$ | Y      | N      | 65          | 0  | 1.0000000  | 0.489744           | 0.510256           | 299         | 2278          | -1979         |
| 391 | $17^1 23^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.491049           | 0.508951           | 304         | 2283          | -1979         |
| 392 | $2^3 7^2$          | N      | N      | -23         | 18                                       | 1.4782609  | 0.489796           | 0.510204           | 281         | 2283          | -2002         |
| 393 | $3^1 131^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.491094           | 0.508906           | 286         | 2288          | -2002         |
| 394 | $2^1 197^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.492386           | 0.507614           | 291         | 2293          | -2002         |
| 395 | $5^1 79^1$         | Y      | N      | 5           | 0  | 1.0000000  | 0.493671           | 0.506329           | 296         | 2298          | -2002         |
| 396 | $2^2 3^2 11^1$     | N      | N      | -74         | 58                                       | 1.2162162  | 0.492424           | 0.507576           | 222         | 2298          | -2076         |
| 397 | $397^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.491184           | 0.508816           | 220         | 2298          | -2078         |
| 398 | $2^1 199^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.492462           | 0.507538           | 225         | 2303          | -2078         |
| 399 | $3^1 7^1 19^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.491228           | 0.508772           | 209         | 2303          | -2094         |
| 400 | $2^4 5^2$          | N      | N      | 34          | 29                                       | 1.6176471  | 0.492500           | 0.507500           | 243         | 2337          | -2094         |
| 401 | $401^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.491272           | 0.508728           | 241         | 2337          | -2096         |
| 402 | $2^1 3^1 67^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.490050           | 0.509950           | 225         | 2337          | -2112         |
| 403 | $13^1 31^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.491315           | 0.508685           | 230         | 2342          | -2112         |
| 404 | $2^2 101^1$        | N      | N      | -7          | 2  | 1.2857143  | 0.490099           | 0.509901           | 223         | 2342          | -2119         |
| 405 | $3^4 5^1$          | N      | N      | -11         | 6  | 1.8181818  | 0.488889           | 0.511111           | 212         | 2342          | -2130         |
| 406 | $2^1 7^1 29^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.487685           | 0.512315           | 196         | 2342          | -2146         |
| 407 | $11^1 37^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.488943           | 0.511057           | 201         | 2347          | -2146         |
| 408 | $2^3 3^1 17^1$     | N      | N      | -48         | 32                                       | 1.3333333  | 0.487745           | 0.512255           | 153         | 2347          | -2194         |
| 409 | $409^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.486553           | 0.513447           | 151         | 2347          | -2196         |
| 410 | $2^1 5^1 41^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.485366           | 0.514634           | 135         | 2347          | -2212         |
| 411 | $3^1 137^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.486618           | 0.513382           | 140         | 2352          | -2212         |
| 412 | $2^2 103^1$        | N      | N      | -7          | 2  | 1.2857143  | 0.485437           | 0.514563           | 133         | 2352          | -2219         |
| 413 | $7^1 59^1$         | Y      | N      | 5           | 0  | 1.0000000  | 0.486683           | 0.513317           | 138         | 2357          | -2219         |
| 414 | $2^1 3^2 23^1$     | N      | N      | 30          | 14                                       | 1.1666667  | 0.487923           | 0.512077           | 168         | 2387          | -2219         |
| 415 | $5^1 83^1$         | Y      | N      | 5           | 0  | 1.0000000  | 0.489157           | 0.510843           | 173         | 2392          | -2219         |
| 416 | $2^5 13^1$         | N      | N      | 13          | 8  | 2.0769231  | 0.490385           | 0.509615           | 186         | 2405          | -2219         |
| 417 | $3^1 139^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.491607           | 0.508393           | 191         | 2410          | -2219         |
| 418 | $2^1 11^1 19^1$    | Y      | N      | -16         | 0  | 1.0000000  | 0.490431           | 0.509569           | 175         | 2410          | -2235         |
| 419 | $419^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.489260           | 0.510740           | 173         | 2410          | -2237         |
| 420 | $2^2 3^1 5^1 7^1$  | N      | N      | -155        | 90                                       | 1.1032258  | 0.488095           | 0.511905           | 18          | 2410          | -2392         |
| 421 | $421^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.486936           | 0.513064           | 16          | 2410          | -2394         |
| 422 | $2^1 211^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.488152           | 0.511848           | 21          | 2415          | -2394         |
| 423 | $3^2 47^1$         | N      | N      | -7          | 2  | 1.2857143  | 0.486998           | 0.513002           | 14          | 2415          | -2401         |
| 424 | $2^3 53^1$         | N      | N      | 9           | 4  | 1.5555556  | 0.488208           | 0.511792           | 23          | 2424          | -2401         |
| 425 | $5^2 17^1$         | N      | N      | -7          | 2  | 1.2857143  | 0.487059           | 0.512941           | 16          | 2424          | -2408         |

| $n$ | Primes             | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum d n C_{\Omega(d)}^{(d)}}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|--------------------|--------|--------|-------------|--|--|--------------------|--------------------|-------------|---------------|---------------|
| 426 | $2^1 3^1 71^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.485915           | 0.514085           | 0           | 2424          | -2424         |
| 427 | $7^1 61^1$         | Y      | N      | 5           | 0  | 1.0000000  | 0.487119           | 0.512881           | 5           | 2429          | -2424         |
| 428 | $2^2 107^1$        | N      | N      | -7          | 2  | 1.2857143  | 0.485981           | 0.514019           | -2          | 2429          | -2431         |
| 429 | $3^1 11^1 13^1$    | Y      | N      | -16         | 0  | 1.0000000  | 0.484848           | 0.515152           | -18         | 2429          | -2447         |
| 430 | $2^1 5^1 43^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.483721           | 0.516279           | -34         | 2429          | -2463         |
| 431 | $431^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.482599           | 0.517401           | -36         | 2429          | -2465         |
| 432 | $2^4 3^3$          | N      | N      | -80         | 75                                       | 1.5625000  | 0.481481           | 0.518519           | -116        | 2429          | -2545         |
| 433 | $433^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.480370           | 0.519630           | -118        | 2429          | -2547         |
| 434 | $2^1 7^1 31^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.479263           | 0.520737           | -134        | 2429          | -2563         |
| 435 | $3^1 5^1 29^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.478161           | 0.521839           | -150        | 2429          | -2579         |
| 436 | $2^2 109^1$        | N      | N      | -7          | 2  | 1.2857143  | 0.477064           | 0.522936           | -157        | 2429          | -2586         |
| 437 | $19^1 23^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.478261           | 0.521739           | -152        | 2434          | -2586         |
| 438 | $2^1 3^1 73^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.477169           | 0.522831           | -168        | 2434          | -2602         |
| 439 | $439^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.476082           | 0.523918           | -170        | 2434          | -2604         |
| 440 | $2^3 5^1 11^1$     | N      | N      | -48         | 32                                       | 1.3333333  | 0.475000           | 0.525000           | -218        | 2434          | -2652         |
| 441 | $3^2 7^2$          | N      | N      | 14          | 9  | 1.3571429  | 0.476190           | 0.523810           | -204        | 2448          | -2652         |
| 442 | $2^1 13^1 17^1$    | Y      | N      | -16         | 0  | 1.0000000  | 0.475113           | 0.524887           | -220        | 2448          | -2668         |
| 443 | $443^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.474041           | 0.525959           | -222        | 2448          | -2670         |
| 444 | $2^2 3^1 37^1$     | N      | N      | 30          | 14                                       | 1.1666667  | 0.475225           | 0.524775           | -192        | 2478          | -2670         |
| 445 | $5^1 89^1$         | Y      | N      | 5           | 0  | 1.0000000  | 0.476404           | 0.523596           | -187        | 2483          | -2670         |
| 446 | $2^1 223^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.477578           | 0.522422           | -182        | 2488          | -2670         |
| 447 | $3^1 149^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.478747           | 0.521253           | -177        | 2493          | -2670         |
| 448 | $2^6 7^1$          | N      | N      | -15         | 10                                       | 2.3333333  | 0.477679           | 0.522321           | -192        | 2493          | -2685         |
| 449 | $449^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.476615           | 0.523385           | -194        | 2493          | -2687         |
| 450 | $2^1 3^2 5^2$      | N      | N      | -74         | 58                                       | 1.2162162  | 0.475556           | 0.524444           | -268        | 2493          | -2761         |
| 451 | $11^1 41^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.476718           | 0.523282           | -263        | 2498          | -2761         |
| 452 | $2^2 113^1$        | N      | N      | -7          | 2  | 1.2857143  | 0.475664           | 0.524336           | -270        | 2498          | -2768         |
| 453 | $3^1 151^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.476821           | 0.523179           | -265        | 2503          | -2768         |
| 454 | $2^1 227^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.477974           | 0.522026           | -260        | 2508          | -2768         |
| 455 | $5^1 7^1 13^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.476923           | 0.523077           | -276        | 2508          | -2784         |
| 456 | $2^3 3^1 19^1$     | N      | N      | -48         | 32                                       | 1.3333333  | 0.475877           | 0.524123           | -324        | 2508          | -2832         |
| 457 | $457^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.474836           | 0.525164           | -326        | 2508          | -2834         |
| 458 | $2^1 229^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.475983           | 0.524017           | -321        | 2513          | -2834         |
| 459 | $3^3 17^1$         | N      | N      | 9           | 4  | 1.5555556  | 0.477124           | 0.522876           | -312        | 2522          | -2834         |
| 460 | $2^2 5^1 23^1$     | N      | N      | 30          | 14                                       | 1.1666667  | 0.478261           | 0.521739           | -282        | 2552          | -2834         |
| 461 | $461^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.477223           | 0.522777           | -284        | 2552          | -2836         |
| 462 | $2^1 3^1 7^1 11^1$ | Y      | N      | 65          | 0  | 1.0000000  | 0.478355           | 0.521645           | -219        | 2617          | -2836         |
| 463 | $463^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.477322           | 0.522678           | -221        | 2617          | -2838         |
| 464 | $2^4 29^1$         | N      | N      | -11         | 6  | 1.8181818  | 0.476293           | 0.523707           | -232        | 2617          | -2849         |
| 465 | $3^1 5^1 31^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.475269           | 0.524731           | -248        | 2617          | -2865         |
| 466 | $2^1 233^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.476395           | 0.523605           | -243        | 2622          | -2865         |
| 467 | $467^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.475375           | 0.524625           | -245        | 2622          | -2867         |
| 468 | $2^2 3^2 13^1$     | N      | N      | -74         | 58                                       | 1.2162162  | 0.474359           | 0.525641           | -319        | 2622          | -2941         |
| 469 | $7^1 67^1$         | Y      | N      | 5           | 0  | 1.0000000  | 0.475480           | 0.524520           | -314        | 2627          | -2941         |
| 470 | $2^1 5^1 47^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.474468           | 0.525532           | -330        | 2627          | -2957         |
| 471 | $3^1 157^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.475584           | 0.524416           | -325        | 2632          | -2957         |
| 472 | $2^3 59^1$         | N      | N      | 9           | 4  | 1.5555556  | 0.476695           | 0.523305           | -316        | 2641          | -2957         |
| 473 | $11^1 43^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.477801           | 0.522199           | -311        | 2646          | -2957         |
| 474 | $2^1 3^1 79^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.476793           | 0.523207           | -327        | 2646          | -2973         |
| 475 | $5^2 19^1$         | N      | N      | -7          | 2  | 1.2857143  | 0.475789           | 0.524211           | -334        | 2646          | -2980         |
| 476 | $2^2 7^1 17^1$     | N      | N      | 30          | 14                                       | 1.1666667  | 0.476891           | 0.523109           | -304        | 2676          | -2980         |
| 477 | $3^2 53^1$         | N      | N      | -7          | 2  | 1.2857143  | 0.475891           | 0.524109           | -311        | 2676          | -2987         |
| 478 | $2^1 239^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.476987           | 0.523013           | -306        | 2681          | -2987         |
| 479 | $479^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.475992           | 0.524008           | -308        | 2681          | -2989         |
| 480 | $2^5 3^1 5^1$      | N      | N      | -96         | 80                                       | 1.6666667  | 0.475000           | 0.525000           | -404        | 2681          | -3085         |
| 481 | $13^1 37^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.476091           | 0.523909           | -399        | 2686          | -3085         |
| 482 | $2^1 241^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.477178           | 0.522822           | -394        | 2691          | -3085         |
| 483 | $3^1 7^1 23^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.476190           | 0.523810           | -410        | 2691          | -3101         |
| 484 | $2^2 11^2$         | N      | N      | 14          | 9  | 1.3571429  | 0.477273           | 0.522727           | -396        | 2705          | -3101         |
| 485 | $5^1 97^1$         | Y      | N      | 5           | 0  | 1.0000000  | 0.478351           | 0.521649           | -391        | 2710          | -3101         |
| 486 | $2^1 3^5$          | N      | N      | 13          | 8  | 2.0769231  | 0.479424           | 0.520576           | -378        | 2723          | -3101         |
| 487 | $487^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.478439           | 0.521561           | -380        | 2723          | -3103         |
| 488 | $2^3 61^1$         | N      | N      | 9           | 4  | 1.5555556  | 0.479508           | 0.520492           | -371        | 2732          | -3103         |
| 489 | $3^1 163^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.480573           | 0.519427           | -366        | 2737          | -3103         |
| 490 | $2^1 5^1 7^2$      | N      | N      | 30          | 14                                       | 1.1666667  | 0.481633           | 0.518367           | -336        | 2767          | -3103         |
| 491 | $491^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.480652           | 0.519348           | -338        | 2767          | -3105         |
| 492 | $2^2 3^1 41^1$     | N      | N      | 30          | 14                                       | 1.1666667  | 0.481707           | 0.518293           | -308        | 2797          | -3105         |
| 493 | $17^1 29^1$        | Y      | N      | 5           | 0  | 1.0000000  | 0.482759           | 0.517241           | -303        | 2802          | -3105         |
| 494 | $2^1 13^1 19^1$    | Y      | N      | -16         | 0  | 1.0000000  | 0.481781           | 0.518219           | -319        | 2802          | -3121         |
| 495 | $3^2 5^1 11^1$     | N      | N      | 30          | 14                                       | 1.1666667  | 0.482828           | 0.517172           | -289        | 2832          | -3121         |
| 496 | $2^4 31^1$         | N      | N      | -11         | 6  | 1.8181818  | 0.481855           | 0.518145           | -300        | 2832          | -3132         |
| 497 | $7^1 71^1$         | Y      | N      | 5           | 0  | 1.0000000  | 0.482897           | 0.517103           | -295        | 2837          | -3132         |
| 498 | $2^1 3^1 83^1$     | Y      | N      | -16         | 0  | 1.0000000  | 0.481928           | 0.518072           | -311        | 2837          | -3148         |
| 499 | $499^1$            | Y      | Y      | -2          | 0  | 1.0000000  | 0.480962           | 0.519038           | -313        | 2837          | -3150         |
| 500 | $2^2 5^3$          | N      | N      | -23         | 18                                       | 1.4782609  | 0.480000           | 0.520000           | -336        | 2837          | -3173         |