Lower bounds on the Mertens function along infinite subsequences

Maxie Dion Schmidt
Georgia Institute of Technology
School of Mathematics

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Abstract

The Mertens function, $M(x) = \sum_{n \leqslant x} \mu(n)$, is classically defined as the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture stating that $|M(x)| < C \cdot \sqrt{x}$ with come absolute $C > 0$ for all $x \geqslant 1$ has a well-known disproof due to Odlyzko and té Riele given in the early 1980's by computation of non-trivial zeta function zeros in conjunction with integral formulas expressing $M(x)$. It is conjectured that $M(x)/\sqrt{x}$ changes sign infinitely often and grows unbounded in the direction of both $\pm\infty$ along subsequences of integers $x \geqslant 1$. Our proof this property of $|M(x)|/\sqrt{x}$, e.g., showing that

$$\limsup_{x \to \infty} \frac{|M(x)|}{\sqrt{x}} = +\infty,$$

is not based on standard estimates of $M(x)$ that are intimately tied to the intricate distribution of the non-trivial zeros of the Riemann zeta function. There is a distinct stylistic flavor and new element of combinatorial analysis to our proof peppered in with the standard methods from analytic, additive and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

Keywords and Phrases: *Möbius function; Mertens function; summatory function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting functions; Dirichlet generating function; asymptotic lower bounds; Mertens conjecture.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; and 11N64.*

Glossary of special notation and conventions

Symbol	Definition
$\mathbb{E}[f(x)], \overset{\mathbb{E}}{\sim}$	We use the expectation notation $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \overset{\mathbb{E}}{\sim} h(x)$, to denote that f has a so-called *average order* growth rate of $h(x)$. What this means is that $\frac{1}{x}\sum_{n \leqslant x} f(n) \sim h(x)$, or equivalently that $$\lim_{x\to\infty} \frac{\frac{1}{x}\sum_{n\leqslant x} f(n)}{h(x)} = 1.$$
B	The absolute constant $B \approx 0.2614972128476427837554$ from the statement of Mertens theorem.
$o(f), O_\alpha(g)$	We write that $f = o(g)$ if $$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 0.$$ We sometimes adapt the standard big-O notation, writing $f = O_{\alpha_1,\ldots,\alpha_k}(g)$ for some parameters $\alpha_1, \ldots, \alpha_k$ that do not depend on x, if $f(x) = O(g(x))$ subject only to the stated upper bound on f having an implicit dependence only on x (as usual) and the α_i for $1 \leqslant i \leqslant k$.
$C_k(n)$	Auxillary component functions in obtaining asymptotic bounds on $g^{-1}(n)$. These functions are defined recursively for $n \geqslant 1$ and $1 \leqslant k \leqslant \Omega(n)$ according to the formula $$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d\mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geqslant 1. \end{cases}$$
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n\geqslant 0}$.
DGF	Given a sequence $\{f(n)\}_{n\geqslant 0}$, its *Dirichlet generating function* (DGF) is defined by $D_f(s) := \sum_{n\geqslant 1} f(n)/n^s$ subject to suitable constraints on the real part of the parameter $s \in \mathbb{C}$ that guarantee convergence of $D_f(s)$.
$d(n)$	The divisor function, $d(n) := \sum_{d\mid n} 1$, for $n \geqslant 1$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) = \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (defined below).
$f * g$	The Dirichlet convolution of f and g, $(f * g)(n) := \sum\limits_{d\mid n} f(d)g(n/d)$, where the sum is taken over the divisors d of n for $n \geqslant 1$.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ for $n \geqslant 1$ with $f^{-1}(1) = 1/f(1)$. The Dirichlet inverse of f exists if and only if $f(1) \neq 0$. This inverse function, provided it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$.

Symbol	Definition
$\lfloor x \rfloor$,$\lceil x \rceil$	The floor function is defined as $\lfloor x \rfloor := x - \{x\}$ where $0 \leqslant \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$. The floor function is sometimes also written as $[x] \equiv \lfloor x \rfloor$. The corresponding ceiling function $\lceil x \rceil$ denotes the smallest integer $m \geqslant x$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$.
H_n	The *first-order harmonic numbers*, $H_n \equiv H_n^{(1)}$, satisfy the limiting asymptotic relation $$\lim_{n \to \infty} [H_n - \log(n)] = \gamma,$$ where $\gamma \approx 0.577216$ denotes Euler's gamma constant.
$\mathbb{1}_{\mathbb{S}}$, $\chi_{\mathtt{cond}(x)}$	We use the notation $\mathbb{1}, \chi : \mathbb{N} \to \{0, 1\}$ to denote indicator, or characteristic functions of a set. In paticular, $\mathbb{1}_{\mathbb{S}}(n) = 1$ if and only if $n \in \mathbb{S}$, and $\chi_{\mathtt{cond}}(n) = 1$ if and only if n satisfies the boolean-valued condition `cond`.
$[n = k]_\delta$,$[\mathtt{cond}]_\delta$	The notation $[n = k]_\delta$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise. For a boolean-valued conditions, `cond`, $[\mathtt{cond}]_\delta$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention.*
$\lambda(n)$	The Liouville lambda function, $\lambda(n) := (-1)^{\Omega(n)}$, denotes the signed parity of $\Omega(n)$, the number of distinct prime factors of n counting their multiplicity. That is, $\lambda(n) \in \{\pm 1\}$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree, i.e., n has no prime power divisors with exponent greater than one.
$M(x)$	The Mertens function is the summatory function over $\mu(n)$ defined for all integers $x \geqslant 1$ by $M(x) := \sum\limits_{n \leqslant x} \mu(n)$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^\alpha \vert\vert n$ (or when p^α exactly divides n) for p prime and $n \geqslant 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p \mid n} 1$ and the completely additive function $\Omega(n) := \sum_{p^\alpha \vert\vert n} \alpha$. Equivalently, if the prime factorization of $n \geqslant 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leqslant n \leqslant x$ for $x > 1$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leqslant x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leqslant x : \Omega(n) = k\}$ for $x \geqslant 2$.
$\sum_{p \leqslant x}, \prod_{p \leqslant x}$	Unless otherwise specified by context, we use the index variable p to denote that the summation (product) is to be taken only over prime values within the summation bounds.
$P(s)$	For complex s with $\mathrm{Re}(s) > 1$, we define the *prime zeta function* to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$.
$\left[{n \atop k}\right]$	The unsigned Stirling numbers of the first kind, $\left[{n \atop k}\right] = (-1)^{n-k} \cdot s(n, k)$.
$\zeta(s)$	The Riemann zeta function, defined by $\zeta(s) := \sum_{n \geqslant 1} n^{-s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation on the entire complex plane with the exception of a simple pole at $s = 1$.

1 Preface: Notation to express asymptotic relations

We exphasize that the next careful explanation of the subtle distinctions to our usage of what we consider to be traditional notation for asymptotic relations are key to understanding our choices of bounding expressions given throughout the article. Thus, to avoid any confusion that may linger as we begin to state our new results and bounds on the functions we work with in this article, we preface the article starting with this section detailing our precise definitions, meanings and assumptions on the uses of certain symbols, operators, and relations as $x \to \infty$ [13, *cf.* §2] [2].

1.1 Average order, similarity and approximation of asymptotic growth rates

1.1.1 Similarity and average order (expectation)

We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if

$$\lim_{x \to \infty} \frac{A(x)}{B(x)} = 1.$$

It is conventional to express the *average order* of an arithmetic function f as $f \sim h$, even when the values of $f(n)$ may actually non-monotonically oscillate in magnitude infinitely often. What the notation $f \sim h$ means when using this notation to express the average order of f is that

$$\frac{1}{x} \cdot \sum_{n \leqslant x} f(n) \sim h(x).$$

For example, in the acceptably classic language of [4] we would normally write that $\Omega(n) \sim \log\log n$, even though technically, $1 \leqslant \Omega(n) \leqslant \frac{\log n}{\log 2}$ where $\Omega(n)$ attains values along this entire bounded range in n infinitely often. To be absolutely clear about notation, we intentionally do not re-use the $\sim$ relation by instead writing $\mathbb{E}[f(x)] = h(x)$ (as in expectation of f), or sometimes $\overset{\mathbb{E}}{\sim}$ for convenience, to denote that f has a limiting average order growing at the rate of h.

1.1.2 Abel summation

The formula we prefer for the Abel summation variant of summation by parts to express finite sums of a product of two functions is stated as follows [1, *cf.* §4.3] [A]:

Proposition 1.1 (Abel Summation Integral Formula)**.** *Suppose that $t > 0$ is real-valued, and that $A(t) \sim \sum_{n \leqslant t} a(n)$ for some weighting arithmetic function $a(n)$ with $A(t)$ continuously differentiable on $(0, \infty)$. Furthermore, suppose that $b(n) \sim f(n)$ with f a differentiable function of $n \geqslant 0$. That is, $f'(t)$ exists and is smooth for all $t \in (0, \infty)$. Then for $0 \leqslant y < x$ we have that*

$$\sum_{y < n \leqslant x} a(n)b(n) \sim A(x)b(x) - A(y)b(y) - \int_y^x A(t) f'(t) dt.$$

1.1.3 Approximation

We adopt the convention that $f(x) \approx g(x)$ if $|f(x) - g(x)| = O(1)$ as $x \to \infty$. That is, we write $f(x) \approx g(x)$ to denote that f is approximately equal to g at x modulo at most a small constant difference between the functions when x is large.

[A] Compare to the exact formula for *summation by parts* of any arithmetic functions, u_n, v_n, stated as in [13, §2.10(ii)] for $U_j := u_1 + u_2 + \cdots + u_j$ when $j \geqslant 1$:

$$\sum_{j=1}^{n-1} u_j \cdot v_j = U_{n-1} v_n + \sum_{j=1}^{n-1} U_j \left(v_j - v_{j+1}\right), n \geqslant 2.$$

1.1.4 Vinogradov's notation for asymptotics

We use the relations $f(x) \gg g(x)$ and $h(x) \ll r(x)$ to symbolically express that we should expect f to be "substantially" larger than g, and respectively h to be "significantly" smaller than r, in asymptotic order (e.g., rate of growth when x is large). In practice, we adopt a somewhat looser definition of these symbols which allows $f \gg g$ and $h \ll r$ provided that there are constants $C, D \geqslant 1$ such that whenever x is sufficiently large we have that $C \cdot f(x) \geqslant g(x)$ and $h(x) \leqslant D \cdot r(x)$. This notation is sometimes called *Vinogradov's asymptotic notation*.

1.2 Asymptotic expansions and uniformity

We introduce the notation for asymptotic expansions of a function $f : \mathbb{R} \to \mathbb{R}$ from [13, §2.1(iii)] in the next subsections.

1.2.1 Ordinary asymptotic expansions of a function

Let $\sum_n a_n x^{-n}$ denote a formal power series expansion in x where we ignore any necessary conditions to guarantee convergence of the series. For each integer $n \geqslant 1$, suppose that

$$f(x) = \sum_{s=0}^{n-1} a_s x^{-s} + O(x^{-n}),$$

as $|x| \to \infty$ where this limiting bound holds for $x \in \mathbb{X}$ in some unbounded set $\mathbb{X} \subseteq \mathbb{C}$. When such a bound holds, we say that $\sum_s a_s x^{-s}$ is a *Poincaré asymptotic expansion*, or just an *asymptotic series expansion*, of $f(x)$ as $|x| \to \infty$ along the fixed set $\mathbb{X}$. The condition in the previous equation is equivalent to writing

$$f(x) \sim a_0 + a_1 x^{-1} + a_2 x^{-2} + \cdots ; x \in \mathbb{X}, \text{ as } |x| \to \infty.$$

The prior two characterizations of an asymptotic expansion for f are also equivalent to the statement that

$$x^n \left(f(x) - \sum_{s=0}^{n-1} a_s x^{-s} \right) \xrightarrow{x \to \infty} a_n.$$

1.2.2 Uniform asymptotic expansions of a function

Let the set $\mathbb{X}$ from the definition in the last subsection correspond to a closed sector of the form

$$\mathbb{X} := \{x \in \mathbb{C} : \alpha \leqslant \arg(x) \leqslant \beta\}.$$

Then we say that the asymptotic expansion

$$f(x) = \sum_{s=0}^{n-1} a_s x^{-s} + O(x^{-n}),$$

holds *uniformly* with respect to $\arg(x) \in [\alpha, \beta]$ as $|x| \to \infty$.

Another useful notion of uniform asymptotic bounds is taken with respect to some parameter u (or set of parameters, respectively) that ranges over the point set (point sets, respectively) $u \in \mathbb{U}$. In this case, if we have that the u-parameterized expressions

$$\left| x^n \left(f(u,x) - \sum_{s=0}^{n-1} a_s(u) x^{-s} \right) \right|,$$

are bounded for all integers $n \geqslant 1$ with $x \in \mathbb{X}$ as $|x| \to \infty$, then we say that the asymptotic expansion of f holds *uniformly* for $u \in \mathbb{U}$. Now the function $f \equiv f(u,x)$ and the asymptotic series coefficients $a_s(u)$ may have an implicit dependence on the parameter u. If the previous boundedness condition holds for all positive integers n, we write that

$$f(u,x) \sim \sum_{s=0}^{\infty} a_s(u) x^{-s}; x \in \mathbb{X}, \text{ as } |x| \to \infty,$$

and say that this asymptotic expansion holds *uniformly with respect to* $u \in \mathbb{U}$. For u taken outside of $\mathbb{U}$, the stated limiting bound may fail to be valid even for $x \in \mathbb{X}$ as $|x| \to \infty$.

1.3 Limiting densities of subsets of the integers

In the proofs given in Section 8 of the article, we will require a precise notion of the *asymptotic density* of a set $\mathcal{S} \subseteq \mathbb{Z}^+$. When this limit exists, we denote the asymptotic density of $\mathcal{S}$ by $\alpha_{\mathcal{S}} \in [0,1]$, defined as follows:

$$\alpha_{\mathcal{S}} := \lim_{x \to \infty} \frac{1}{x} \cdot \#\left\{n \leqslant x : n \in \mathcal{S}\right\}.$$

In other words, if the set $\mathcal{S}$ has asymptotic density $\alpha_{\mathcal{S}}$, then for all sufficiently large x

$$\alpha_{\mathcal{S}} + o(1) \leqslant \frac{1}{x} \cdot \#\left\{n \leqslant x : n \in \mathcal{S}\right\} \leqslant \alpha_{\mathcal{S}} + o(1).$$

When the limit definition of $\alpha_{\mathcal{S}}$ does not exist, or if by some pathology of the way $\mathcal{S}$ is defined we cannot express the exact limit, we are often interested in sets of bounded asymptotic density. We say that $\mathcal{S}$ has *bounded asymptotic density* if for all large x there exist constants $0 \leqslant B \leqslant C \leqslant 1$ such that

$$B + o(1) \leqslant \frac{1}{x} \cdot \#\left\{n \leqslant x : n \in \mathcal{S}\right\} \leqslant C + o(1), \text{ as } x \to \infty.$$

Clearly, finite or bounded subsets of the positive integers have limiting asymptotic density of zero. If the asymptotic density of $\mathcal{S}$ is one, and some property $\mathcal{P}(n)$ holds for all $n \in \mathcal{S}$, then we say that $\mathcal{P}(n)$ is true *almost everywhere* (on the integers), also abbreviated as holding "a.e." on the positive integers as $n \to \infty$.

2 An introduction to the Mertens function

2.1 Definitions

Suppose that $n \geqslant 2$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $k = \omega(n)$. We define the *Möebius function* to be the signed indicator function of the squarefree integers as follows:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leqslant i \leqslant k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many other variants and special properties of the Möebius function and its generalizations [15, *cf.* §2]. A crucial role of the classical $\mu(n)$ forms an inversion relation for arithmetic functions convolved with one (e.g., Dirichlet convolutions of the form $g = f * 1$) by *Möbius inversion*:

$$g(n) = (f * 1)(n) \iff f(n) = (g * \mu)(n), \forall n \geqslant 1.$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined as

$$M(x) = \sum_{n \leqslant x} \mu(n), x \geqslant 1.$$

The sequence of the oscillatory values of this summatory function begins as [16, A002321]

$$\{M(x)\}_{x \geqslant 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2, \ldots\}$$

Clearly, a positive integer $n \geqslant 1$ is *squarefree*, or contains no (prime power) divisors which are squares, if and only if $\mu^2(n) = 0$. A related summatory function which counts the number of *squarefree* integers $n \leqslant x$ then satisfies [4, §18.6] [16, A013928]

$$Q(x) = \sum_{n \leqslant x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \to \infty$:

$$\mu_+(x) = \frac{\#\{1 \leqslant n \leqslant x : \mu(n) = +1\}}{Q(x)} \overset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leqslant n \leqslant x : \mu(n) = -1\}}{Q(x)} \xrightarrow{x \to \infty} \frac{3}{\pi^2}.$$

The actual local oscillations between the approximate densities of the sets $\mu_\pm(x)$ lend an unpredictable nature to the function and characterize the oscillatory sawtooth shaped plot of $M(x)$ over the positive integers.

2.2 Properties

One conventional approach to evaluating the behavior of $M(x)$ for large $x \to \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is expressed given the inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \int_1^\infty \frac{s \cdot M(x)}{x^{s+1}} dx,$$

we obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation, along with the standard Euler product representation for the reciprocal zeta function cited in the first equation above, leads us to the exact expression for $M(x)$ for any real $x > 0$ given by the next theorem due to Titchmarsh.

Theorem 2.1 (Analytic Formula for $M(x)$)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k\geqslant 1}$ satisfying $k \leqslant T_k \leqslant k+1$ for each k such that for any real $x > 0$*

$$M(x) = \lim_{k\to\infty} \sum_{\substack{\rho:\zeta(\rho)=0\\ |\operatorname{Im}(\rho)|<T_k}} \frac{x^\rho}{\rho\cdot\zeta'(\rho)} - 2 + \sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n\cdot(2n)!\zeta(2n+1)}\left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2}\left[x\in\mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x\cdot\exp\left(-C\cdot\log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan proved new updated estimates in 2009 bounding $M(x)$ for large x in the following forms [17]:

$$\begin{aligned} M(x) &\ll \sqrt{x}\cdot\exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x}\cdot\exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right),\ \forall\epsilon>0. \end{aligned}$$

2.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which posits that

$$|M(x)| < C\cdot\sqrt{x},\quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens for $C = 1$ and all $x < 10000$. Since its beginnings in 1897, the Mertens conjecture has been disproven by computation of non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele from the early 1980's. Since the truth of the conjecture would have implied the RH, more recent attempts at bounding $M(x)$ consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound towards both $\pm\infty$ along infinite subsequences.

One of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ is in actuality unbounded on the natural numbers. A precise statement of this problem is to produce an affirmative answer whether $\limsup_{x\to\infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x\to\infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there are an infinite subsequences of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound towards either $\pm\infty$ along the subsequence. We cite that prior to this point it is only known by computation that [14, *cf.* §4.1]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geqslant 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leqslant -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [12, 7, 8, 5].

Extensive computational evidence has produced a conjecture due to Gonek (among attempts on limiting bounds by others) that in fact the limiting behavior of $M(x)$ satisfies [11]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}\cdot(\log\log x)^{5/4}} = O(1).$$

While it seems to be widely believed that $|M(x)|/\sqrt{x}$ tends to $+\infty$ at some logarithmically scaled rate along subsequences, the infinitely tending factors such as the $(\log\log x)^{\frac{5}{4}}$ in Gonek's conjecture do not appear to readily fall out of work on unconditional bounds for $M(x)$ by existing methods.

3 An overview of the core logical steps and components to the proof

We offer an initial step-by-step summary overview of the critical components to our proof outlined in the next subsections of the introduction. As our proof methodology is new and relies on non-standard elements compared to more traditional methods of bounding $M(x)$, we hope that this sketch of the logical components to this argument makes the article easier to parse.

3.1 Step-by-step overview

(1) We prove a matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). See Theorem 4.1 in Section 5.

(2) This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the seemingly unconnected prime counting function, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is stated in (1).

(3) We tighten an updated result from [10, §7] providing uniform asymptotic formulas for the summatory functions, $\widehat{\pi}_k(x)$, that indicate the parity of $\Omega(n)$ (sign of $\lambda(n)$) for $n \leqslant x$ and $1 \leqslant k \leqslant \log\log x$. These formulas are proved using expansions of more combinatorially motivated Dirichlet series (see Theorem 4.7). We use this result to sum $\sum_{n \leqslant x} \lambda(n) f(n)$ for particular non-negative arithmetic functions f when x is large.

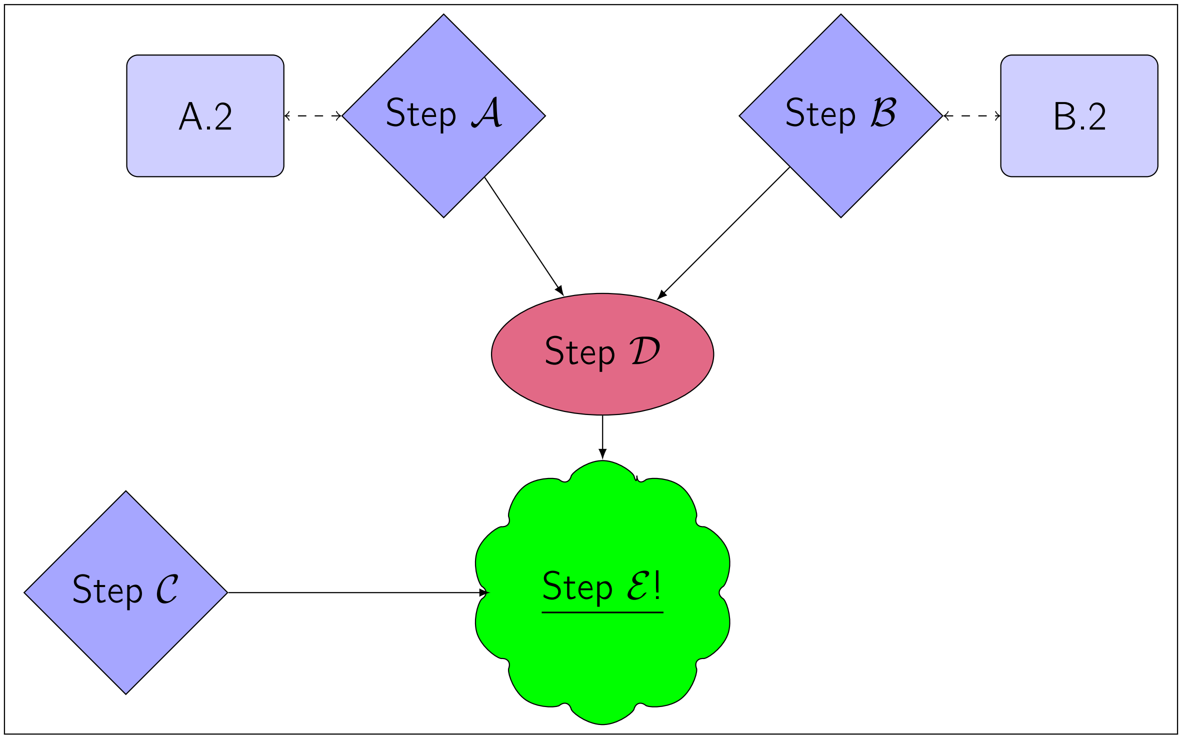
(4) We then turn to the average order asymptotics of the quasi-periodic functions, $g^{-1}(n)$, by estimating this inverse function's limiting asymptotics for large $n \leqslant x$ as $x \to \infty$ in Section 7. We eventually use these estimates to prove a substantially unique new lower bound formula for the summatory function $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$ along certain asymptotically large infinite subsequences (see Theorem 9.5).

(5) We spend some interim time in Section 8 carefully working out a rigorous justification for why the limiting lower bounds we obtain from average order case analysis of our arithmetic function approximations to $g^{-1}(n)$ are sufficient to prove the corollary on the unboundedness of $M(x)$ below.

(6) When we return to step **(2)** with our new lower bounds at hand, we find "magic" in the form of showing the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. What we recover is a quick, and rigorous, proof of Corollary 4.11 given in Section 9.2.

3.2 Schematic flowchart of the proof logic

The next flowchart diagramed below shows how the seemingly disparate components of the proof are organized.



Legend to the diagram stages:

- ▶ **Step A:** *Citations and re-statements of existing theorems proved elsewhere.*
  - **A.A:** Key results and constructions:
    - Theorem 4.6
    - Corollary 6.5
    - The results, lemmas, and facts cited in Section 5.3
  - **A.2:** Lower bounds on the Abel summation based formula for $G^{-1}(x)$:
    - Theorem 4.7 (on page 22)
    - Proposition 6.6
    - Theorem 9.5
- ▶ **Step B:** *Constructions of an exact formula for $M(x)$.*
  - **B.B:** Key results and constructions:
    - Corollary 4.3 (follows from Theorem 4.1 proved on page 16)
    - Proposition 5.1
  - **B.2:** Asymptotics for the component functions $g^{-1}(n)$ and $G^{-1}(x)$:
    - Theorem 4.5 (on page 24)
    - Lemma 7.4
- ▶ **Step C:** *A justification for why lower bounds obtained on average suffice.*
  - Theorem 4.8 (proved on page 28)
  - The lemmas and necessary results we use to build up to a proof that the hypotheses needed to apply the conclusion of Theorem 4.8 are regularly attained for all large x given in Section 8.2.
- ▶ **Step D:** *Interpreting the exact formula for $M(x)$.*
  - Proposition 9.1
  - Theorem 9.5
- ▶ **Step E:** *The Holy Grail.* Proving that $\frac{|M(x)|}{\sqrt{x}}$ grows without bound in the limit supremum sense.
  - Corollary 4.11 (on page 42)

4 A concrete new approach for bounding $M(x)$ from below

4.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 4.1 (Summatory functions of Dirichlet convolutions)**.** *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leqslant x} f(n)$ and $H(x) := \sum_{n \leqslant x} h(n)$ denote the summatory functions of f, h, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse f^{-1} of f. Then, letting the counting function $\pi_{f*h}(x)$ be defined as in the first equation below, we have the following equivalent expressions for the summatory function of $f * h$ for all integers $x \geqslant 1$:*

$$\begin{aligned}
\pi_{f*h}(x) &= \sum_{n \leqslant x} \sum_{d|n} f(d) h(n/d) \\
&= \sum_{d \leqslant x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right].
\end{aligned}$$

Moreover, we can invert the linear system determining the coefficients of $H(k)$ for $1 \leqslant k \leqslant x$ naturally to express $H(x)$ as a linear combination of the original left-hand-side summatory function as follows:

$$\begin{aligned}
H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \right] \\
&= \sum_{n=1}^{x} f^{-1}(n) \pi_{f*h}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).
\end{aligned}$$

Corollary 4.2 (Convolutions Arising From Möbius Inversion)**.** *Suppose that g is an arithmetic function on the positive integers such that $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n \leqslant x} (g * \mu)(n)$. Then the Mertens function equals*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geqslant 1.$$

Corollary 4.3 (A motivating special case)**.** *We have exactly that for all $x \geqslant 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1 \right]. \tag{1}$$

4.2 An exact expression for $M(x)$ in terms of strongly additive functions

From this point on, we fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and denote its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute the first few terms for the Dirichlet inverse of $g(n)$ exactly for the first few sequence values as (see Table T.1 of the appendix section)

$$\{g^{-1}(n)\}_{n \geqslant 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is given by $\operatorname{sgn}(g^{-1}(n)) = \frac{g^{-1}(n)}{|g^{-1}(n)|} = \lambda(n)$ (see Proposition 5.1). This useful property is inherited from the distinctly additive nature of the component function $\omega(n)$ [A].

[A] Indeed, for any non-negative additive arithmetic function $a(n)$, $(a + 1)^{-1}(n)$ has leading sign given by $\lambda(n)$ for any $n \geqslant 1$. For multiplicative f, we obtain a related condition that $\operatorname{sgn}(f(n)) = (-1)^{\omega(n)}$ for all $n \geqslant 1$.

Conjecture 4.4. *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) $g^{-1}(1) = 1$;

(B) *For all $n \geqslant 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*

(C) *For all squarefree integers $n \geqslant 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!.$$

We illustrate parts (B)–(C) of the conjecture more clearly using Table T.1 given starting on page 44. The realization that the beautiful and remarkably simple combinatorial form of property (C) in Conjecture 4.4 holds for all squarefree $n \geqslant 1$ motivates our pursuit of formulas for the inverse functions $g^{-1}(n)$ expressed by sums of auxiliary sequences of arithmetic functions [B].

For natural numbers $n \geqslant 1, k \geqslant 0$, let

$$C_k(n) := \begin{cases} \varepsilon(n) = \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geqslant 1. \end{cases}$$

We have limiting asymptotics on these functions in terms of n and k given by the following theorem:

Theorem 4.5 (Asymptotics for the functions $C_k(n)$)**.** *For $k := 0$, we have by definition that $C_0(n) = \delta_{n,1}$. For all sufficiently large $n > 1$ and any fixed $1 \leqslant k \leqslant \Omega(n)$ taken independently of n, we obtain that the dominant asymptotic term for $C_k(n)$ is given uniformly by*

$$\mathbb{E}[C_k(n)] \gg (\log \log n)^{2k-1}, \text{ as } n \to \infty.$$

For any $n \geqslant 1$, we can prove that (see Lemma 7.4)

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{2}$$

In light of the fact that (see Proposition 9.1)

$$M(x) \approx G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)},$$

the formula in (2) implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by appropriate estimates of the summatory function $G^{-1}(x)$ [C].

[B] A proof of this property is not difficult to give using Lemma 7.4 stated on page 25.

[C] We can also prove that

$$M(x) = G^{-1}(x) + \sum_{p \leqslant x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right),$$

by inversion since

$$G^{-1}(x) = \sum_{d \leqslant x} (g^{-1} * 1)(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right),$$

with $(g^{-1} * 1)^{-1} = g * \mu = \chi_{\mathbb{P}} + \varepsilon$ defined such that $\chi_{\mathbb{P}}$ is the characteristic function of the primes.

4.3 Uniform asymptotics from enumerative counting DGFs in Mongomery and Vaughan

Our inspiration for the new bounds found in the last sections of this article allows us to sum non-negative arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. We utilize a somewhat more general hybrid generating function and enumerative DGF method under which we are able to recover "good enough" asymptotics about the summatory functions that encapsulate the parity of $\Omega(n)$ (or sign of $\lambda(n)$) through the summatory tally functions $\widehat{\pi}_k(x)$. The precise statement of the result that we transform to state these new bounds is provided next in Theorem 4.6 (see Section 6.1).

Theorem 4.6 (Montgomery and Vaughan)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leqslant x : \Omega(n) = k\}.$$

For $R < 2$ we have that

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

uniformly for $1 \leqslant k \leqslant R\log\log x$ where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, z \geqslant 0.$$

The next theorem, proved carefully in Section 6, is the primary starting point for our new asymptotic lower bounds. The proof of this result is combinatorially motivated in so much as it interprets lower bounds on a key infinite product factor of $\mathcal{G}(z)$ defined in Theorem 4.6 as corresponding to an ordinary generating function of certain homogeneous symmetric polynomials involving reciprocals of the primes.

Theorem 4.7. *Suppose that $0 \leqslant z < 1$ is real-valued. We obtain lower bounds of the following form on the function $\mathcal{G}(z)$ from Theorem 4.6 for $A_0 > 0$ an absolute constant and for $C_0(z)$ a strictly linear function only in z:*

$$\mathcal{G}(z) \geqslant A_0 \cdot (1-z)^3 \cdot C_0(z)^z.$$

It suffices to take the components of the bound given in the previous equation as

$$\begin{aligned} A_0 &= \frac{2^{9/16} \exp\left(-\frac{55}{4}\log^2(2)\right)}{(3e\log 2)^3 \cdot \Gamma\left(\frac{5}{2}\right)} \approx 3.81296 \times 10^{-6} \\ C_0(z) &= \frac{4(1-z)}{3e\log 2}. \end{aligned}$$

In particular, for $1 \leqslant k \leqslant \log\log x$ we have that

$$\widehat{\pi}_k(x) \gg \frac{A_0 \cdot x}{\log x \cdot (\log\log x)^4 \cdot (k-1)!} \cdot \left(\frac{4}{3e\log 2}\right)^{\frac{k}{\log\log x}}.$$

4.4 Recovering meaningful asymptotics from an average case analysis of bounds

4.4.1 An average-to-global phenomenon for the average case analysis of our new lower bounds

Theorem 4.8. *Let the summatory function $G_E^{-1}(x)$ be defined for $x \geqslant 1$ by* [D]

$$G_E^{-1}(x) := \sum_{n \leqslant (\log x)^2} \lambda(n) \times \sum_{\substack{d|n \\ d>e}} (\log d)(\log\log d). \tag{3a}$$

[D] The subscript of E (as in expectation) on the function $G_E^{-1}(x)$ is purely for notation and does not correspond to a formal parameter or any implicit dependence on E in the formula that defines this function.

Suppose that $B, C \in (0,1)$ denote some respectively minimally and maximally defined absolute constants such that for a bounded constant $Y \geqslant 0$, we have that as $x \rightarrow \infty$

$$B + o(1) \leqslant \frac{1}{x} \cdot \#\left\{n \leqslant x : |G^{-1}(n)| - |G_E^{-1}(n)| \leqslant Y\right\} \leqslant C + o(1). \tag{3b}$$

That is, if for a bounded constant $Y \geqslant 0$ we have that the set

$$\left\{n \leqslant x : |G^{-1}(n)| - |G_E^{-1}(n)| \leqslant Y\right\},$$

has bounded asymptotic density in $(0,1)$ such that the condition in (3b) holds for all large x, then we take

$$\begin{aligned} B &:= \liminf_{x\rightarrow\infty} \frac{1}{x} \cdot \#\left\{n \leqslant x : |G^{-1}(n)| - |G_E^{-1}(n)| \leqslant Y\right\} \in (0,1) \\ C &:= \limsup_{x\rightarrow\infty} \frac{1}{x} \cdot \#\left\{n \leqslant x : |G^{-1}(n)| - |G_E^{-1}(n)| \leqslant Y\right\} \in (0,1). \end{aligned}$$

If such constants $B, C \in (0,1)$ exist, then there is some $\varepsilon \in (0,1)$ (depending on B, C) with $0 < B - \varepsilon, C + \varepsilon < 1$ such that for all sufficiently large x we have at least one point $x_0 \in [(B-\varepsilon)x, (C+\varepsilon)x]$ such that

$$|G^{-1}(x_0)| \geqslant \left|G_E^{-1}(x_0)\right| + Y.$$

It suffices to take $\varepsilon := \frac{1}{2}\min(B, 1-C)$ to attain the point x_0 within the above interval for all sufficiently large x.

We prove Theorem 4.8, and rigorously justify that its hypotheses are in fact regularly attainable for all large x, in Section 8. This result allows us to express lower bounds based on average case estimates of certain arithmetic functions we have defined to approximate $g^{-1}(n)$ and still recover an infinite subsequence along which we can witness the classical unboundedness property of $|M(x)|/\sqrt{x}$ stated below in Corollary 4.11.

4.4.2 Intuition for average case asymptotics leading to exact estimates near any large x

There does not appear to be an easy, nor subtle direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences. However, the distribution of distinct sets of prime exponents is fairly regular so that $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of $g^{-1}(n)$ over $n \geqslant 2$:

Heuristic 4.9 (Symmetry in $g^{-1}(n)$ in the exponents in the prime factorization of n)**.** Suppose that $n_1, n_2 \geqslant 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for some $r \geqslant 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly two, three, and so on prime factors (compare with Table T.1 starting on page 44).

The next points make clear what our intuition should suggest about the relation of the actual function values to the average case expectation of $g^{-1}(n)$ for all $n \leqslant x$ when x is large.

Remark 4.10. Given that we have chosen to work with a representation for $M(x)$ that depends critically on the distribution of the values of the additive functions, $\omega(n)$ and $\Omega(n)$, there is substantial intuition involved á priori that suggests our sums over these functions behave regularly on average. Stated precisely, when we define the function $\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$, for any real $z \in (-\infty, +\infty)$ we have that [6, §1.7]

$$\#\left\{n \leqslant x : \frac{\omega(n) - \log\log x}{\sqrt{\log\log x}} \leqslant z\right\} = \Phi(z) \cdot x + o(1),$$

and uniformly for $-Z \leqslant z \leqslant Z$ with respect to any $Z > 0$ that [10, §7.4]

$$\#\left\{3 \leqslant n \leqslant x : \frac{\Omega(n) - \log\log n}{\sqrt{\log\log n}} \leqslant z\right\} = \Phi(z) \cdot x + O_Z\left(\frac{x}{\sqrt{\log\log x}}\right).$$

When the bounding parameter in these Erdös-Kac like theorems is set to $z := 0$, we provably expect these sums involving these canonical additive functions and the distribution of their values to tend towards their asymptotic average case behavior infinitely often, and predictably near any large x as in Theorem 4.8. Thus when it comes to recovering globally regular behavior from an average case analysis of bounds of our new arithmetic functions from below, the choice in stating (1) as it depends on the canonical additive function examples we have cited is *absolutely essential* to the success of our proof.

4.5 Cracking the classical unboundedness barrier

In Section 9, we are able to state what forms a culmination of the results we carefully build up to in the proofs established in prior sections of the article. What we eventually obtain at the conclusion of the section is the following important summary corollary that resolves the classical question of the unboundedness of the scaled function Mertens function $q(x) := |M(x)|/\sqrt{x}$ in the limit supremum sense:

Corollary 4.11 (Unboundedness of the the Mertens function)**.** *Define the infinite increasing subsequence,* $\{x_{0,n}\}_{n\geqslant 1}$*, by* $x_{0,n} := e^{2e^{e^{e^{2n}}}}$. *We have that for all sufficiently large* $y \gg \lceil x_{0,1} \rceil + 1$ *the following bound holds:*

$$\frac{|M(x_{0,y})|}{\sqrt{x_{0,y}}} \gg \frac{2C_{\ell,1} \cdot (\log\sqrt{x_{0,y}})(\log\log\log\log\sqrt{x_{0,y}})}{(\log\log\log\sqrt{x_{0,y}})^{\frac{7}{2}}(\log\log\log\log\log\sqrt{x_{0,y}})^{\frac{3}{2}}} \times \exp\left(-2(\log\log\log\sqrt{x_{0,y}})^2\right), \text{ as } y \rightarrow \infty.$$

The constant $C_{\ell,1} > 0$ *in the previous equation is taken as*

$$C_{\ell,1} := \frac{256 \cdot 2^{1/8}}{59049 \cdot \pi^2 e^8 \log^8(2)} \exp\left(-\frac{55}{2}\log^2(2)\right) \approx 5.51187 \times 10^{-12}.$$

This is all to say that in establishing the rigorous proof of Corollary 4.11 based on our new methods, we not only show that

$$\limsup_{x\rightarrow\infty} \frac{|M(x)|}{\sqrt{x}} = +\infty,$$

but also set a minimal rate (along a large infinite subsequence) at which this form of the scaled Mertens function $q(x)$ grows without bound.

5 Preliminary proofs of new results

The purpose of this section is to provide proofs and statements of elementary and otherwise well established facts and results. The proof of Theorem 4.1 allows us to easily justify the exact formula in (1). The strong additivity of $\omega(n)$ provides the characteristic signedness of $g^{-1}(n)$ corresponding to the parity of $\Omega(n)$ which we prove precisely in Proposition 5.1. The link relating (1) to canonical additive functions and their distributions then lends a recent distinguishing element to the success of the methods in our proof. Namely, we can draw upon the properties of strongly additive functions in place of a central reliance on the distributions of characteristically multiplicative structures about which we know (in general) much less about finite sums of their consecutive values over all $n \leqslant x$.

5.1 Establishing the summatory function properties and inversion identities

We will first prove Theorem 4.1 using matrix methods and similarity transforms by shift matrices. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95].

Proof of Theorem 4.1. Let h, g be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leqslant x} h(n)$ and $G(x) = \sum_{n \leqslant x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h: $g*h$. Then we can readily see that the following initial formulas hold for all $x \geqslant 1$:

$$\begin{aligned}\pi_{g*h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] H(i).\end{aligned}$$

We form the matrix of coefficients associated with this linear system defining $H(n)$ for all $n \leqslant x$. We then invert the system to express an exact solution for $H(x)$ at any $x \geqslant 1$. Let the matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right), \forall 1 \leqslant j \leqslant x.$$

The matrix we must invert in this problem is lower triangular, with ones on its diagonals, and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

Here, U is the $N \times N$ matrix whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$\left[(I - U^T)^{-1}\right]_{i,j} = [j \leqslant i]_{\delta}.$$

It is a useful fact that if we take successive differences in x of the floor of certain fractions, $\left\lfloor \frac{x}{j} \right\rfloor$, we get non-zero behavior at the divisors of x:

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.}\end{cases}$$

We use this property to shift the matrix $\hat{G}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of g of the following form:

$$\left[(I - U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_{\delta}\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_{\delta}\right).$$

Now we can express the inverse of the target matrix $(g_{x,j})$ using a similarity transformation conjugated by shift operators as follows:

$$\begin{aligned}
(g_{x,j}) &= (I-U^T)^{-1}\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\
(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k)\right)(I-U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1}\right\rfloor} g^{-1}(k)\right).
\end{aligned}$$

Hence, the summatory function $H(x)$ is exactly expressed for any $x \geqslant 1$ by a vector product with the inverse matrix from the previous equation in the form of

$$\begin{aligned}
H(x) &= \sum_{k=1}^{x} g_{x,k}^{-1} \cdot \pi_{g*h}(k) \\
&= \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right) \cdot \pi_{g*h}(k). \square
\end{aligned}$$

5.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n. Then we can easily prove that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu. \tag{4}$$

When combined with Corollary 4.2 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 4.3.

Proposition 5.1 (The key signedness property of $g^{-1}(n)$)**.** *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0, \pm 1\}$ *denote the sign of the arithmetic function h at n. For the Dirichlet invertible function,* $g(n) := \omega(n) + 1$ *defined such that* $g(1) = 1$*, for all* $n \geqslant 1$*, we have that* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$.

Proof. Recall that $D_f(s) := \sum_{n\geqslant 1} f(n)n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ for σ_f the abcissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = 1/\zeta(s)$ and $D_\omega(s) = P(s)\zeta(s)$. Then by (4) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any invertible arithmetic function f, for all $\operatorname{Re}(s) > 1$ we have

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{5}$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + 1$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. From this fact, it follows by inspection that $\operatorname{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make this intuition rigorous.

By the standard recurrence relation we used to define the Dirichlet inverse function of any arithmetic function h, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d|n \\ d>1}} h(d)h^{-1}(n/d), & n \geqslant 2. \end{cases} \tag{6}$$

For $n \geqslant 2$, the summands in (6) can be simply indexed over the primes $p|n$ given our definition of h from above. This observation yields that we can inductively expand these sums into nested divisor sums provided the depth of the sums does not exceed the capacity to index summations over the primes dividing n. Namely, notice that for $n \geqslant 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p|n} h^{-1}(n/p), && \text{if } \Omega(n) \geqslant 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1p_2}\right), && \text{if } \Omega(n) \geqslant 2 \\ &= -\sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1p_2}} h^{-1}\left(\frac{n}{p_1p_2p_3}\right), && \text{if } \Omega(n) \geqslant 3. \end{aligned}$$

Then by induction, again with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$h^{-1}(n) = \lambda(n) \times \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1p_2\cdots p_{\Omega(n)-1}}} 1, n \geqslant 2.$$

If for $n \geqslant 2$ we write the prime factorization of n as $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{\omega(n)}^{\alpha_{\omega(n)}}$ where the exponents $\alpha_i \geqslant 1$ for all $1 \leqslant i \leqslant \omega(n)$, we can see that

$$\begin{aligned} h^{-1}(n) &\geqslant \lambda(n) \times 1 \cdot 2 \cdot 3 \cdots \omega(n) = \lambda(n) \times (\omega(n))!, && n \geqslant 2 \\ h^{-1}(n) &\leqslant \lambda(n) \times (\omega(n))!^{\max(\alpha_1,\alpha_2,\ldots,\alpha_{\omega(n)})}, && n \geqslant 2. \end{aligned}$$

In other words, what these bounds show is that for all $n \geqslant 1$ (with $\lambda(1) = 1$) the following property holds:

$$\operatorname{sgn}(h^{-1}(n)) = \lambda(n). \tag{7}$$

From (7), we immediately have bounding constants $1 \leqslant C_{1,n}, C_{2,n} < +\infty$ that exist for each $n \geqslant 1$ so that

$$C_{1,n} \cdot (\lambda * \mu)(n) \leqslant (h^{-1} * \mu)(n) \leqslant C_{2,n} \cdot (\lambda * \mu)(n). \tag{8}$$

Since both λ, μ are multiplicative, the convolution $\lambda * \mu$ is multiplicative. We know that the values of any multiplicative function are uniquely determined by its action at prime powers. So we can compute that for any prime p and non-negative integer exponents $\alpha \geqslant 1$ that

$$\begin{aligned} (\lambda * \mu)(p^\alpha) &= \sum_{i=0}^{\alpha} \lambda(p^{\alpha-i})\mu(p^i) \\ &= \lambda(p^\alpha) - \lambda(p^{\alpha-1}) \\ &= (-1)^{\Omega(p^\alpha)} - (-1)^{\Omega(p^{\alpha-1})} = (-1)^\alpha - (-1)^{\alpha-1} = 2\lambda(p^\alpha). \end{aligned}$$

Then by the multiplicativity of $\lambda(n)$, the previous inequalities in (8) are re-stated in the form of

$$2C_{1,n} \cdot \lambda(n) \leqslant h^{-1}(n) \leqslant 2C_{2,n} \cdot \lambda(n).$$

Since the absolute constants $C_{1,n}, C_{2,n}$ are always positive, we clearly recover the signedness of $g^{-1}(n)$ as $\lambda(n)$ for all $n \geqslant 1$. □

5.3 Statements of other facts and known limiting asymptotics

Theorem 5.2 (Mertens theorem). *For all $x \geqslant 2$ we have that*

$$P_1(x) := \sum_{p \leqslant x} \frac{1}{p} = \log\log x + B + o(1),$$

where $B \approx 0.2614972128476427837554$ is an absolute constant [A].

Corollary 5.3 (Product form of Mertens theorem). *We have that for all sufficiently large $x \gg 2$*

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) = \frac{e^{-B}}{\log x} \left(1 + o(1)\right),$$

where the notation for the absolute constant $0 < B < 1$ coincides with the definition of Mertens constant from Theorem 5.2. Hence, for any real $z \geqslant 0$ we obtain that

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right)^z = \frac{e^{-Bz}}{(\log x)^z} \left(1 + o(1)\right)^z \sim \frac{e^{-Bz}}{(\log x)^z}, \text{ as } x \to \infty.$$

Proofs of Theorem 5.2 and Corollary 5.3 are found in [4, §22.7; §22.8].

Facts 5.4 (Exponential integrals and the incomplete gamma function). The following two variants of the *exponential integral function* are defined by the integral representations [13, §8.19]

$$\begin{aligned} \mathrm{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, \\ E_1(z) &:= \int_1^{\infty} \frac{e^{-tz}}{t} dt, \mathrm{Re}(z) \geqslant 0. \end{aligned}$$

These two functions are related by $\mathrm{Ei}(-kz) = -E_1(kz)$ for real $k, z > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $E_1(z)$ for all large real $z \gg e$ [B]:

$$1 - \frac{3}{4}z \leqslant E_1(z) - \gamma - \log z \leqslant 1 - \frac{3}{4}z + \frac{11}{36}z^2. \tag{9a}$$

The (upper) *incomplete gamma function* is defined by [13, §8.4]

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt, \mathrm{Re}(s) > 0.$$

We have the following properties of $\Gamma(s, x)$:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, \tag{9b}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, \text{ as } x \to \infty. \tag{9c}$$

The result in (9b) allows us to form summatory functions over $\widehat{\pi}_k(n)$ for $n \leqslant x$ based on Theorem 4.6 and Theorem 4.7 expressed by the incomplete gamma function (see Corollary 9.4). For large $x \to \infty$, we can approximate these sums by (9c). The limiting asymptotics in (9c) also allow us to simplify certain definite integral formulas involving fixed powers of $\log t$ (and $\log\log t$) for large, infinitely tending upper limits of integration required by the proofs in Section 8.

[A] Exactly, we have that the *Mertens constant* is defined by

$$B = \gamma + \sum_{m \geqslant 2} \frac{\mu(m)}{m} \log\left[\zeta(m)\right],$$

where $\gamma \approx 0.577215664902$ is Euler's gamma constant.

[B] Indeed, these inequalities are usually stated to provide bounds on $\mathrm{Ei}(z)$ when $z > 0$ is real-valued. That the bounds are also valid for $E_1(z)$ at all sufficiently large $z \gg e$ is a matter of convenience we state for reference in this way to arrive at the formulas we see in Proposition 6.6 and subsequently obtain in the proof of Theorem 4.7.

6 Components to the asymptotic analysis of lower bounds for sums of arithmetic functions weighted by $\lambda(n)$

In this section, we re-state a couple of key results proved in [10, §7.4] that we rely on to prove Corollary 6.5 stated below. This corollary is important as it shows that (signed) summatory functions over $\widehat{\pi}(x)$ capture the dominant asymptotics of the full summatory function formed by taking $1 \leqslant k \leqslant \log_2(x)$ when we truncate the range of of the index k and instead sum only up to the uniform bound for $1 \leqslant k \leqslant \log\log x$ guaranteed by applying Theorem 4.6. We also prove Theorem 4.7 in this section. This key theorem allows us to establish a global minimum we can attain on the function $\mathcal{G}(z)$ from Theorem 4.6. This in turn implies the uniform lower bounds on $\widehat{\pi}_k(x)$ guaranteed at the conclusion of the second theorem by a straightforward manipulation of inequalities.

6.1 Results proved by Montgomery and Vaughan

Remark 6.1 (Intuition and constructions in Theorem 4.6)**.** For $|z| < 2$ and $\mathrm{Re}(s) > 1$, let

$$F(s,z) := \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \tag{10}$$

and define the DGF coefficients, $a_z(n)$ for $n \geqslant 1$, by the relation

$$\zeta(s)^z \cdot F(s,z) := \sum_{n \geqslant 1} \frac{a_z(n)}{n^s}, \mathrm{Re}(s) > 1.$$

Suppose that $A_z(x) := \sum_{n \leqslant x} a_z(n)$ for $x \geqslant 1$. Then for the choice of the function $F(s,z)$ defined in (10), we obtain

$$A_z(x) = \sum_{n \leqslant x} z^{\Omega(n)} = \sum_{k \geqslant 0} \widehat{\pi}_k(x) z^k.$$

Thus for $r < 2$, by Cauchy's integral formula we have

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz.$$

Selecting $r := \frac{k-1}{\log\log x}$ leads to the uniform asymptotic formulas for $\widehat{\pi}_k(x)$ given in Theorem 4.6.

We also require the next theorems reproduced from [10, §7.4] that handle the relative scarcity of the distribution of the $\Omega(n)$ for $n \leqslant x$ such that $\Omega(n) > \log\log x$.

Theorem 6.2 (Upper bounds on exceptional values of $\Omega(n)$ for large n)**.** *Let*

$$\begin{aligned} A(x,r) &:= \#\left\{n \leqslant x : \Omega(n) \leqslant r \cdot \log\log x\right\}, \\ B(x,r) &:= \#\left\{n \leqslant x : \Omega(n) \geqslant r \cdot \log\log x\right\}. \end{aligned}$$

If $0 < r \leqslant 1$ *and* $x \geqslant 2$, *then*

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \textit{as } x \to \infty.$$

If $1 \leqslant r \leqslant R < 2$ *and* $x \geqslant 2$, *then*

$$B(x,r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \textit{as } x \to \infty.$$

Theorem 6.3 (Exact bounds on exceptional values of $\Omega(n)$ for large n)**.** *We have that uniformly*

$$\#\left\{3 \leqslant n \leqslant x : \Omega(n) - \log\log n \leqslant 0\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Remark 6.4. The proofs of Theorem 6.2 and Theorem 6.3 are found in §7.4 of Montgomery and Vaughan. The key interpretation we need is the result stated in the next corollary. The precise way in which the bound stated in the previous theorem depends on the indeterminate paramater R can be reviewed for reference in the proof algebra and relations cited in the reference [10, §7]. The role of the parameter R involved in stating the previous theorem is more notably critical as the scalar factor the upper bound on $k \leqslant R \log\log x$ in Theorem 4.6 up to which we obtain the valid uniform bounds in x on the asymptotic formulas for $\widehat{\pi}_k(x)$.

We have a discrepancy to work out in so much as we can only form summatory functions over the $\widehat{\pi}_k(x)$ for $1 \leqslant k \leqslant R \log\log x$ using the asymptotic formulas guaranteed by Theorem 4.6, even though we can actually have contributions from values distributed throughout the range $1 \leqslant \Omega(n) \leqslant \log_2(n)$ infinitely often. It is then crucial that we can show that the dominant growth of the asymptotic formulas we obtain for these summatory functions is captured by summing only over k in the truncated range where the uniform bounds hold. This fact, stated next as in Corollary 6.5, will be relevant when we prove Theorem 9.5 using a sign-weighted summatory function with Abel summation that depends on the new lower bounds for these functions (see Lemma 9.3).

Corollary 6.5. *Using the notation for $A(x,r)$ and $B(x,r)$ from Theorem 6.2, we have that for $\delta > 0$,*

$$o(1) \leqslant \left|\frac{B(x,1+\delta)}{A(x,1)}\right| \ll 2, \text{ as } \delta \to 0^+, x \to \infty.$$

Proof. The lower bound stated above should be clear. To show that the asymptotic upper bound is correct, we compute using Theorem 6.2 and Theorem 6.3 that

$$\left|\frac{B(x,1+\delta)}{A(x,1)}\right| \ll \left|\frac{x \cdot (\log x)^{\delta - \delta\log(1+\delta)}}{O(1) + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right)}\right| \sim \left|\frac{(\log x)^{\delta-\delta\log(1+\delta)}}{\frac{1}{2} + o(1)}\right| \xrightarrow{\delta \to 0^+} 2,$$

as $x \to \infty$. Notice that since $\mathbb{E}[\Omega(n)] = \log\log n + B$, with $0 < B < 1$ the absolute constant from Mertens theorem, when we denote the range of $k > \log\log x$ as holding in the form of $k > (1+\delta)\log\log x$ for $\delta > 0$, we can assume that $\delta \to 0^+$ as $x \to \infty$. □

6.2 New results based on refinements of Theorem 4.6

What the enumeratively flavored result in Theorem 4.6 allows us to do is get a sufficient lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. We seek to approximate $\mathcal{G}(z)$ defined in this theorem by only taking finite products of the primes in the factor of $\prod_p (1 - z/p)^{-1}$ for $p \leqslant ux$, e.g., indexing the component products only over those primes $p \in \{2, 3, 5, \ldots, ux\}$ for some minimal upper bound depending on the parameter u as $x \to \infty$.

We first require a handle on partial sums of integer powers of the reciprocal primes as functions of the integral exponent and the upper summation index x. The next statement of Proposition 6.6 effectively provides a coarse rate in x below which the reciprocal prime sums tend to absolute constants given by the prime zeta function, $P(s)$. We make particular use of the property of finite-degree polynomial dependence of these bounds in s to simplify the computations in the theorem below.

Proposition 6.6. *For real $s \geqslant 1$, let*

$$P_s(x) := \sum_{p \leqslant x} p^{-s}, x \geqslant 2.$$

When $s := 1$, we have the asymptotic formula from Mertens theorem (see Theorem 5.2). For all integers $s \geqslant 2$ there is are absolutely defined bounding functions $\gamma_0(s,x), \gamma_1(s,x)$ such that

$$\gamma_0(s,x) + o(1) \leqslant P_s(x) \leqslant \gamma_1(s,x) + o(1), \text{ as } x \to \infty.$$

It suffices to take the bounds in the previous equation as

$$\gamma_0(z,x) = -s \log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}s(s-1)\log(x/2) - \frac{11}{36}s(s-1)^2\log^2(x)$$

$$\gamma_1(s,x) := -s\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}s(s-1)\log(x/2) + \frac{11}{36}s(s-1)^2\log^2(2).$$

Proof. Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$ and where our target function smooth function is $f(t) = t^{-s}$ with $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1), \text{ as } x \to \infty. \end{aligned}$$

Now using the inequalities in Facts 5.4, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned} \frac{P_s(x)}{s} &\geqslant -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1)\log(x/2) - \frac{11}{36}(s-1)^2\log^2(x) \\ \frac{P_s(x)}{s} &\leqslant -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1)\log(x/2) + \frac{11}{36}(s-1)^2\log^2(2). \end{aligned}$$

This completes the proof of the bounds cited above in the statement of this lemma. □

Proof of Theorem 4.7. We have for all integers $0 \leqslant k \leqslant m$, and any sequence $\{f(n)\}_{n\geqslant 1}$ with bounded partial sums, that [9, §2]

$$[z^k]\prod_{1\leqslant i\leqslant m}(1-f(i)z)^{-1} = [z^k]\exp\left(\sum_{j\geqslant 1}\left(\sum_{i=1}^m f(i)^j\right)\frac{z^j}{j}\right), |z| < 1. \tag{11}$$

In our case we have that $f(i)$ denotes the i^{th} prime in the generating function expansion of (11). Hence, by summing over all primes $p \leqslant ux$ in the previous formula and applying Proposition 6.6, we obtain that the logarithm of the corresponding generating function in z is given by

$$\begin{aligned} \log\left[\prod_{p\leqslant ux}\left(1-\frac{z}{p}\right)^{-1}\right] &\geqslant (B+\log\log(ux))z + \sum_{j\geqslant 2}\left[a(ux)+b(ux)(j-1)+c(ux)(j-1)^2\right]z^j \\ &= (B+\log\log(ux))z - a(ux)\left(1+\frac{1}{z-1}+z\right) \\ &\qquad + b(ux)\left(1+\frac{2}{z-1}+\frac{1}{(z-1)^2}\right) \\ &\qquad - c(ux)\left(1+\frac{4}{z-1}+\frac{5}{(z-1)^2}+\frac{2}{(z-1)^3}\right) \\ &=: \widehat{\mathcal{B}}(u,x;z), \end{aligned}$$

for some minimal parameter u that we will determine below.

In the previous equations, the lower bounds formed by the functions (a,b,c) evaluated at ux are given by the corresponding upper bounds from Proposition 6.6 due to the leading sign on the previous expansions as

$$(a_\ell, b_\ell, c_\ell) := \left(-\log\left(\frac{\log(ux)}{\log 2}\right), \frac{3}{4}\log\left(\frac{ux}{2}\right), \frac{11}{36}\log^2 2\right).$$

Now we make a decision to set the uniform bound parameter to a middle ground value of $1 < R < 2$ at $R := \frac{3}{2}$ so that

$$z \equiv z(k,x) = \frac{k-1}{\log\log x} \in [0,R),$$

for $x \gg 1$ sufficiently large. Thus $(z-1)^{-m} \in [(-1)^m, 2^m]$ for integers $m \geqslant 1$, and so we can obtain the following lower bound:

$$\widehat{\mathcal{B}}(u,x;z) \geqslant (B+\log\log(ux))z - a(ux)\left(1+\frac{1}{z-1}+z\right)$$

$$+ b(ux)\left(1 + \frac{2}{z-1} + \frac{1}{(z-1)^2}\right) - 45 \cdot c(ux).$$

Since the function $c(ux)$ is constant, we then also obtain the next bounds.

$$\begin{aligned} \frac{e^{-Bz}}{(\log(ux))^z} \times \exp\left(\widehat{\mathcal{B}}(u, x; z)\right) &\geqslant \exp\left(-\frac{55}{4}\log^2(2)\right) \times \left(\frac{\log(ux)}{\log 2}\right)^{1+\frac{1}{z-1}+z} \times \left(\frac{ux}{2}\right)^{\frac{3}{4}\left(1+\frac{2}{z-1}+\frac{1}{(z-1)^2}\right)} \\ &=: \widehat{\mathcal{C}}(u, x; z) \end{aligned} \tag{12}$$

Now we need to determine which values of u minimize the expression for the function defined in (12). For this we will use an elementary method from introdutory calculus to determine a global minimum for the product in z. We can symbolically use *Mathematica* to see that

$$\frac{\partial}{\partial u}\left[\widehat{\mathcal{C}}(u, x; z)\right]\Bigg|_{u \mapsto u_0} = 0 \implies u_0 \in \left\{\frac{1}{x}, \frac{1}{x}e^{-\frac{4}{3}(z-1)}\right\}.$$

When we substitute this $u_0 =: \hat{u}_0 \mapsto \frac{1}{x}e^{-\frac{4}{3}(z-1)}$ into the next expression for the second derivative of the function $\widehat{\mathcal{C}}(u, x; z)$ we obtain that

$$\begin{aligned} \frac{\partial^2}{\partial u^2}\left[\widehat{\mathcal{C}}(u, x; z)\right]\Bigg|_{u=\hat{u}_0} &= \exp\left(-\frac{55}{4}\log^2(2)\right) \times 2^{\frac{8z^3-27z^2+32z-16}{4(z-1)^2}} 3^{-z+\frac{1}{1-z}+1} e^{\frac{5z^2-16z+8}{3(z-1)}} \times \\ &\qquad \times (\log 2)^{\frac{z^2}{1-z}} \cdot (xz)^2 (1-z)^{z+\frac{1}{z-1}-2} > 0, \end{aligned}$$

provided that $z < 1$. The restriction to $0 \leqslant z < 1$ is equivalent to requiring that $1 \leqslant k \leqslant \log\log x$ in Theorem 4.6.

After substitution of $u = \frac{1}{x}e^{-\frac{4}{3}(z-1)}$ into the expression for $\widehat{\mathcal{C}}(u, x; z)$ defined above, we have that

$$\widehat{\mathcal{C}}(u, x; z) \geqslant 2^{\frac{9}{16}} \cdot \exp\left(-\frac{55}{4}\log^2(2)\right)\left(\frac{1-z}{3e\log 2}\right)^3 \times \left(\frac{4(1-z)}{3e\log 2}\right)^z.$$

Finally, since $z \equiv z(k, x) = \frac{k-1}{\log\log x}$ and $k \in [0, R\log\log x)$, we obtain for minimal k and all large enough $x \gg 1$ that $\Gamma(z+1) \approx 1$, and for k towards the upper range of its interval that $\Gamma(z+1) \leqslant \Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$. $\square$

7 Average case analysis of bounds on the Dirichlet inverse functions, $g^{-1}(n)$

The property in (C) of Conjecture 4.4 along squarefree $n \geqslant 1$ captures an important characteristic of $g^{-1}(n)$ that holds more globally for all $n \geqslant 1$. In particular, these functions can be expressed via more simple formulas than inspection of the first few initial values of the repetitive, quasi-periodic sequence otherwise suggests. The pages of tabular data given as Table T.1 given in the appendix section starting on page 44 are intended to provide clear insight into why we arrived at the convenient approximations to $g^{-1}(n)$ proved in this section. The table offers illustrative numerical data formed by examining the approximate behavior at hand for the first several cases of $1 \leqslant n \leqslant 500$ with *Mathematica*.

7.1 Definitions and basic properties of component function sequences

We define the following sequence for integers $n \geqslant 1, k \geqslant 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geqslant 1. \end{cases} \tag{13}$$

The sequence of important semi-diagonals of these functions begins as [16, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geqslant 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \; -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

Remark 7.1 (An effective range of k depending on a fixed large n)**.** Notice that by expanding the recursively-based definition in (13) out to its maximal depth by nested divisor sums, for fixed n, $C_k(n)$ is seen to only ever possibly be non-zero for $k \leqslant \Omega(n)$. Thus, the effective range of k for fixed n is restricted by the conditions of $C_0(n) = \delta_{n,1}$ and that $C_k(n) = 0$, $\forall k > \Omega(n)$ whenever $n \geqslant 2$.

Example 7.2 (Special cases of the functions $C_k(n)$ for small k)**.** We cite the following special cases which are verified by explicit computation using (13) [A]:

$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= d(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} - \gcd(\Omega(n), \omega(n)). \end{aligned}$$

7.2 Uniform asymptotics of $C_k(n)$ for large all n and fixed, bounded k

Theorem 4.5 from the introduction is proved next. The theorem makes precise what these formulas already suggest about the main terms of the growth rates of $C_k(n)$ as functions of k, n for limiting cases of n large and fixed k which is necessarily bounded in n, but still taken as an independent parameter.

Proof of Theorem 4.5. We prove our bounds by induction on k. We can see by Example 7.2 that $C_1(n)$ satsfies the formula we must establish when $k := 1$ since $\mathbb{E}[\omega(n)] = \log\log n$. Suppose that $k \geqslant 2$ and let the inductive assumption state that for all $1 \leqslant m < k$

$$\mathbb{E}[C_m(n)] \gg (\log\log n)^{2m-1}.$$

[A]For all $k \geqslant 2$, we have the following recurrence relation satisfied by $C_k(n)$ between successive values of k:

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=0}^{\nu_p(n)-1} C_{k-1}\left(p^i\right), n \geqslant 1.$$

Now using the recursive formula we used to define the sequences of $C_k(n)$ in (13), we have that as $n \rightarrow \infty$ [B]

$$\begin{aligned}
\mathbb{E}[C_k(n)] &= \mathbb{E}\left[\sum_{d|n} \omega(n/d) C_{k-1}(d)\right] \\
&= \frac{1}{n} \times \sum_{d \leqslant n} C_{k-1}(d) \times \sum_{r=1}^{\left\lfloor \frac{n}{d} \right\rfloor} \omega(r) \\
&\sim \sum_{d \leqslant n} C_{k-1}(d) \left[\frac{\log\log(n/d) \left[d \leqslant \frac{n}{e}\right]_{\delta}}{d} + \frac{B}{d}\right] \\
&\sim \sum_{d \leqslant \frac{n}{e}} \left[\sum_{m<d} \frac{\mathbb{E}[C_{k-1}(m)]}{m} \log\log\left(\frac{n}{m}\right) + B \cdot \mathbb{E}[C_{k-1}(d)] + B \cdot \sum_{m<d} \frac{\mathbb{E}[C_{k-1}(m)]}{m}\right] \\
&\gg B\left[n \log n \cdot (\log\log n)^{2k-3} - \log n \cdot (\log\log n)^{2k-3}\right] \times \left(1 + \frac{\log n}{2}\right) \\
&\gg (\log\log n)^{2k-1}.
\end{aligned}$$

In transitioning to the last equation from the previous step, we have used that $\frac{Bn}{2} \cdot (\log n)^2 \gg (\log\log n)^2$ as $n \rightarrow \infty$. We have also used that for large $n \rightarrow \infty$ and fixed m, we have by an approximation to the incomplete gamma function given by

$$\int_e^n \frac{(\log\log t)^m}{t} \sim (\log n)(\log\log n)^m, \text{ as } n \rightarrow \infty.$$

Thus the claim holds by mathematical induction whenever $n \rightarrow \infty$ is large and $1 \leqslant k \leqslant \Omega(n)$. $\square$

Remark 7.3. In Section 8 we show that when $k := \Omega(n)$ depends on n, then

$$\mathbb{E}[C_{\Omega(n)}(n)] \gg (\log n)(\log\log n)^{2\log\log n - 1} \gg \log n \cdot \log\log n.$$

Indeed, for any fixed integral powers $m \geqslant 1$, whenever $n \rightarrow \infty$ is taken large enough we have that

$$\mathbb{E}[C_{\Omega(n)}(n)] \gg (\log n)^m \cdot \log\log n.$$

The estimates above, especially the rightmost lower bound in the previous equation at $m := 1$, are much weaker than the sharpest possible estimate we could have obtained working through the arithmetic in the proof of Theorem 4.5. However, it turns out that these bounds are sufficient to prove the necessary hypotheses of Theorem 4.8 are attainable for all large x in Section 8.2.

7.3 Relating the auxiliary functions $C_k(n)$ to formulas approximating $g^{-1}(n)$

Lemma 7.4 (An exact formula for $g^{-1}(n)$)**.** *For all $n \geqslant 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d) + 1) g^{-1}(n/d) \qquad \Longrightarrow \qquad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n).$$

[B]For all large $x \gg 2$ the summatory function of $\omega(n)$ satisfies [4, §22.10]

$$\sum_{n \leqslant x} \omega(n) = x \log\log x + Bx + O\left(\frac{x}{\log x}\right).$$

Now by repeatedly expanding the right-hand-side, and removing corner cases in the nested sums with $\omega(1) = 0$, we find inductively that

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n).$$

The statement then follows by Möbius inversion applied to each side of the last equation. □

Corollary 7.5. *For all squarefree integers $n \geqslant 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \tag{14}$$

Proof. Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geqslant 2$ and that n is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where p_i is prime for all $1 \leqslant i \leqslant \omega(n)$. So we can transform the exact divisor sum guaranteed for all n in Lemma 7.4 as follows:

$$\begin{aligned} g^{-1}(n) &= \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i}(-1)^i \cdot C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \end{aligned}$$

The signed contributions in the first of the previous equations is justified by noting that $\lambda(n) = (-1)^{\omega(n)}$ whenever n is squarefree, and that for d squarefree with $\omega(d) = i$, $\Omega(d) = i$. □

Corollary 7.6. *We have that*

$$\frac{6}{\pi^2} \log x \ll \mathbb{E}|g^{-1}(n)| \leqslant \mathbb{E}\left[\sum_{d|n} C_{\Omega(d)}(d)\right].$$

Proof. To prove the lower bound, first notice that by Lemma 7.4, Proposition 5.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain that

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{15}$$

In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $n \geqslant 1$ we have $\mu(n) = (-1)^{\omega(n)} = \lambda(n)$, Lemma 7.4 implies

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \end{aligned}$$

Notice in the above equation that $\lambda(n^2) = +1$ for all $n \geqslant 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is necessarily even.

Recall from the introduction that the summatory function of the squarefree integers is given by

$$Q(x) := \sum_{n \leqslant x} \mu^2(n) = \frac{6}{\pi^2} x + O(\sqrt{x}).$$

Then since $C_{\Omega(d)}(d) \geqslant 1$ for all $d \geqslant 1$, we obtain by summing over (15) that as $x \rightarrow \infty$

$$\begin{aligned}
\frac{1}{x} \times \sum_{n \leqslant x} |g^{-1}(n)| &= \frac{1}{x} \times \sum_{d \leqslant x} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&\sim \sum_{d \leqslant x} C_{\Omega(d)}(d) \left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dx}}\right)\right] \\
&\geqslant \sum_{d \leqslant x} \left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dx}}\right)\right] \\
&\sim \frac{6}{\pi^2}\left(\log x + \gamma + O\left(\frac{1}{x}\right)\right) + O\left(\frac{1}{\sqrt{x}} \times \int_0^x t^{-1/2} dt\right) \\
&= \frac{6}{\pi^2} \log x + O(1).
\end{aligned}$$

To prove the upper bound, notice that by Lemma 7.4 and Corollary 7.5,

$$|g^{-1}(n)| \leqslant \sum_{d|n} C_{\Omega(d)}(d).$$

Now since both of the above quantities are positive for all $n \geqslant 1$, we must obtain the upper bound on the average order of $|g^{-1}(n)|$ stated above. $\square$

8 A rigorous justification for applying Theorem 4.8

The point of proving the results in this section before moving onto the core results needed in the next section is to provide a rigorous justification for the intuition we sketched in Section 4.4 of the introduction. That is, we expect our arithmetic functions that are closely tied to the canonical strongly additive functions, $\omega(n)$ and $\Omega(n)$, to similarly behave regularly (and infinitely often) with respect to their values being close to the average case for large x. What we have established so far, and will establish for $G^{-1}(x)$ in Section 9, are lower bound estimates that hold essentially only *on average*. This means that for all sufficiently large $x \to \infty$, we need to show that the expected value lower bounds are achieved in asymptotic order more globally within predictably some small window (interval) near x.

8.1 The proof of our central theorem

Proof of Theorem 4.8. The result is obtained by contradiction. Suppose that x is so large that the inequalities in the hypotheses hold given a satisfactory fixed bounded $0 \leqslant Y < +\infty$. We have assumed that the constants $B, C \in (0,1)$ are the tightest possible bounds on the next set as $x \to \infty$ according to their precise definitions given in the theorem statement. We need to show that a concrete fixed $\varepsilon \in (0,1)$ satisfying the conditions in the theorem exists (depending only on B, C).

Let $x \geqslant 1$ be fixed and sufficiently large. Suppose that for all $\varepsilon \in (0,1)$ satisfying $0 < B - \varepsilon, C + \varepsilon < 1$, we have that

$$|G^{-1}(x_0)| < |G_E^{-1}(x_0)| + Y, \forall x_0 \in [(B-\varepsilon)x, (C+\varepsilon)x]. \tag{16}$$

For large integers $x \gg 1$, we have a disjoint set decomposition of the positive integers $n \leqslant x$ given by

$$\{1 \leqslant n \leqslant x\} = \{1 \leqslant n < (B-\varepsilon)x\} \oplus \{(B-\varepsilon)x \leqslant n \leqslant (C+\varepsilon)x\} \oplus \{(C+\varepsilon)x < n \leqslant x\},$$

where the three disjoint sets above are respectively denoted in increasing left-to-right order by $\mathcal{D}_i(x)$ for $i = 1, 2, 3$. The set decomposition in the previous equation yields that as $x \to \infty$, if (16) is true, then

$$\begin{aligned}
\mathcal{G}_1(x) &:= \frac{1}{x} \cdot \#\left\{n \in \mathcal{D}_1(x) : |G^{-1}(x_0)| - |G_E^{-1}(x_0)| \leqslant Y\right\} \in [(B-\varepsilon)^2 + o(1), (B-\varepsilon)(C+\varepsilon) + o(1)] \\
\mathcal{G}_2(x) &:= \frac{1}{x} \cdot \#\left\{n \in \mathcal{D}_2(x) : |G^{-1}(x_0)| - |G_E^{-1}(x_0)| \leqslant Y\right\} \in [B-\varepsilon, C+\varepsilon] \\
\mathcal{G}_3(x) &:= \frac{1}{x} \cdot \#\left\{n \in \mathcal{D}_3(x) : |G^{-1}(x_0)| - |G_E^{-1}(x_0)| \leqslant Y\right\} \\
&\in [(B-\varepsilon) - (B-\varepsilon)(C+\varepsilon) + o(1), (C+\varepsilon) - (C+\varepsilon)^2 + o(1)].
\end{aligned} \tag{17}$$

For $x \geqslant 1$, let the density of our target set at x be denoted by

$$\mathcal{G}_0(x) := \frac{1}{x} \cdot \#\left\{n \leqslant x : |G^{-1}(x_0)| - |G_E^{-1}(x_0)| \leqslant Y\right\}.$$

Then we obtain by summing the respective upper and lower bounds on the densities for the disjoint sets given in (17) above that

$$(B-\varepsilon)^2 + B - \varepsilon + (B-\varepsilon)(1-C-\varepsilon) + o(1) \leqslant \mathcal{G}_0(x) \leqslant (B-\varepsilon)(C+\varepsilon) + C + \varepsilon + (C+\varepsilon)(1-C-\varepsilon) + o(1).$$

We show that contrary to our assumption, we can in fact pick any $\varepsilon > 0$ that satisfies $B - 2\varepsilon < C, 0 < B - \varepsilon < 1, 0 < C + \varepsilon < 1$, e.g., choosing $\varepsilon := \frac{1}{2}\min(B, 1-C)$ will satisfy our requirements. Indeed, given such a choice of this parameter, we have that

$$C + \varepsilon - [(B-\varepsilon)(C+\varepsilon) + C + \varepsilon + (C+\varepsilon)(1-C-\varepsilon)] = -(C+\varepsilon)(1+B-C-2\varepsilon) < 0.$$

This implies a contradiction to the maximality in the limit supremum sense of our tight upper bound of $C \in (0,1)$. Then we must have that our assumption on x_0 is invalid as $x \to \infty$. More to the point, must be such a fixed $\varepsilon > 0$ and such a $x_0 \in [(B-\varepsilon)x, (C+\varepsilon)x]$ so that $|G^{-1}(x_0)| \geqslant |G_E^{-1}(x_0)| + Y$ whenever x is sufficiently large. $\square$

8.2 Verifying the hypotheses in Theorem 4.8 are achieved for all large x

8.2.1 Building up to a proof of the necessary hypotheses: Preliminary facts and results

Lemma 8.1 (Asymptotic densities of exceptional values of positive arithmetic functions)**.** *Let $F \in C^1(0,\infty)$ be a monotone increasing function such that $F(x) \not\to 0$ as $x \to \infty$. Suppose that f is an arithmetic function such that $f(n) > 0$ for all $n \geqslant 1$ and so that*

$$\sum_{n \leqslant x} f(n) \geqslant x \cdot F(x) + o(xF(x)), \text{ as } x \to \infty.$$

Let the set defined by

$$\mathcal{F}_- := \{n \geqslant 1 : f(n) < F(n)\},$$

have corresponding limiting asymptotic density

$$\gamma_- := \lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{F}_-\}.$$

Then the limit γ_- exists and $\gamma_- = 0$. In other words, for almost every sufficiently large $n \to \infty$, $f(n) \geqslant F(n)$ and $-f(n) \leqslant -F(n)$ where $\mathbb{E}[f(n)] = F(n)$ as $n \to \infty$.

Proof. First, suppose that the limit we used to define γ_- exists with $\gamma_- \in [0,1)$. By the positivity of $f(n)$, we know that $F(x)$ is positive for all sufficiently large x. So we have that as $x \to \infty$

$$\begin{aligned} \sum_{n \leqslant x} f(n) &\leqslant \sum_{\substack{n \leqslant x \\ n \notin \mathcal{F}_-}} f(n) + \sum_{\substack{n \leqslant x \\ n \in \mathcal{F}_-}} F(n) \\ &< \sum_{n \leqslant (1-\gamma_-)x} f(n) + \sum_{(1-\gamma_-)x \leqslant n \leqslant x} F(n) \\ &\sim xF(x) - \int_{(1-\gamma_-)x}^{x} tF'(t)dt \\ &= (1-\gamma_-)x \cdot F((1-\gamma_-)x) + \gamma_- x \cdot F(c), \end{aligned} \tag{18}$$

integrating by parts and for some $c \in [(1-\gamma_-)x, x]$ by the mean value theorem. So by (18), we have that

$$\sum_{n \leqslant x} f(n) < x \cdot F(x), \text{ as } x \to \infty.$$

Unless $\gamma_- = 0$ this property contradicts our hypothesis on the limiting behavior of the summatory function of $f(n)$. Notice also that the limiting density cannot be identically one since if $\gamma_- = 1$, then

$$\begin{aligned} \frac{1}{x} \times \sum_{n \leqslant x} f(n) &< \frac{1}{x} \times \sum_{n \leqslant x} F(n) \\ &\leqslant \max_{1 \leqslant j \leqslant x} F(j) + o(F(x)) = F(x) + o(F(x)), \end{aligned}$$

by the assumption of the monotonicity of F on $(0,\infty)$. Hence, we conclude that $\gamma_- = 0$ provided that the limit exists.

If a limiting value for γ_- does not exist, then for infinitely many large $x \geqslant 2$ within a set $\mathcal{M} \subset \mathbb{Z}^+$ where the density of $\mathcal{M} \cap \{n \leqslant x\}$ is positive, we have that

$$M_x := \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{F}_-\} \in (0,1),$$

where M_x non-monotonically oscillates in value along a subsequence. Using a similar method to what we argued above, in this case there are infinitely many x within a set of eventually always positive limiting densities such that our assumption on the asymptotic lower bound on the summatory function of $f(n)$ also does not hold. This contradiction shows that the limit γ_- must in fact exist, and as we have shown above, is then necessarily zero. ∎

Proposition 8.2. *For sufficiently large $n \to \infty$, we have that*

$$\mathbb{E}[C_{\Omega(n)}(n)] \gg (\log n) \cdot (\log\log n)^{2\log\log n - 1} \gg \log n \cdot \log\log n, \text{ as } n \to \infty.$$

Proof. We must first argue that the set of $n > e$ on which $\Omega(n)$ differs substantially from its average order of $\mathbb{E}[\Omega(n)] = \log\log n$ has asymptotic density zero. For $\delta, \rho > 0$, let

$$\begin{aligned} \Omega_+(\delta, x) &:= \frac{1}{x} \cdot \#\{n \leqslant x : \Omega(n) \geqslant (1+\delta)\log\log x\} \\ \Omega_-(\rho, x) &:= \frac{1}{x} \cdot \#\{n \leqslant x : \Omega(n) \leqslant (1+\rho)\log\log x\}. \end{aligned}$$

We utilize Theorem 6.2 to show each of the following as $x \to \infty$:

$$\begin{aligned} \Omega_+(\delta, x) &\ll (\log x)^{\delta - (1+\delta)\log(1+\delta)} \\ \Omega_-(\rho, x) &\ll (\log x)^{\rho - (1+\rho)\log(1+\rho)}. \end{aligned}$$

Thus for all $\delta, \rho > 0$, where we typically can assume very small values of the parameters $\delta, \rho \approx 0^+$, we have that

$$\Omega_+(\delta, x) = o(1), \Omega_-(\rho, x) = o(1), \text{ as } x \to \infty. \tag{19}$$

The results expanded in (19) show that we can expect the asymptotic density of the positive integers $n \leqslant x$ where $\Omega(n) \not\approx \mathbb{E}[\Omega(n)]$ to be small, and tending to zero as $n \to \infty$.

With our result for fixed $1 \leqslant k \leqslant \Omega(n)$ from Theorem 4.5, we can conclude that

$$\begin{aligned} \mathbb{E}[C_{\Omega(n)}(n)] &\gg \frac{1}{n} \sum_{d \leqslant n} (\log\log d)^{2\Omega(d)-1} \\ &\sim (\log n) \cdot (\log\log n)^{2\log\log n - 1}, \text{ as } n \to \infty. \end{aligned} \tag{20}$$

Hence, we also have that

$$\mathbb{E}[C_{\Omega(n)}(n)] \gg \log n \cdot \log\log n, \text{ as } n \to \infty.$$

To prove that (20) is correct, notice that for any fixed m we have integrating by parts and applying (9c) at large $n \to \infty$ that [A]

$$\begin{aligned} \frac{1}{n} \times \int_e^n (\log\log t)^m dt &= \frac{1}{n}\left[n \cdot (\log n)(\log\log n)^m - (\log n)(\log\log n)^m\right] \\ &\sim (\log n)(\log\log n)^m. \end{aligned} \tag{21}$$

The claimed two implications follow are seen to hold by a perturbed expansion of the binomial series where [3, *cf.* §6]

$$\begin{aligned} \frac{1}{n} \times \int_e^n (\log\log t)^{2\log\log t - 1} dt &\approx \frac{1}{n} \times \int_e^n \frac{(1+\log\log t)^{2\log\log t}}{\log\log t} dt \\ &= \frac{1}{n} \times \int_e^n \sum_{s \geqslant 0} \sum_{k=0}^{s} \left[{s \atop k}\right] (2\log\log t)^k (-1)^{s-k} \times \frac{(\log\log t)^{s-1}}{s!} dt. \end{aligned}$$

Our result then follows from the fact that we can integrate the last equation termwise using the integral formula in (21) from above. □

[A] In particular, we obtain the following definite integral formula exactly for fixed m:

$$\int_e^n \frac{(\log\log t)^m}{t} dt = (-1)^m \cdot \Gamma(m+1, -\log\log n).$$

Proposition 8.3. *For all sufficiently large n on a set of asymptotic density one, we have that*

$$|g^{-1}(n)| \gg \frac{2}{\pi^2}(\log n)^3(\log\log n).$$

Proof. An immediate consequence of Proposition 8.2 is that for all sufficiently large n we have that

$$\mathbb{E}\left[C_{\Omega(n)}(n)\right] \gg (\log n)^2(\log\log n).$$

Recall once again that the summatory function of the squarefree integers satisfies

$$Q(x) := \sum_{n\leqslant x}\mu^2(n) = \frac{6}{\pi^2}x + O(\sqrt{x}).$$

Then by Corollary 7.5 and since

$$|g^{-1}(n)| \leqslant \sum_{d|n} C_{\Omega(d)}(d), \forall n \geqslant 1,$$

we have that as $n \to \infty$

$$\begin{aligned}
\mathbb{E}|g^{-1}(n)| &\geqslant \frac{1}{n} \times \sum_{\substack{m\leqslant n \\ \mu^2(m)=1}} \sum_{d|m} C_{\Omega(d)}(d) \\
&= \frac{1}{n} \times \sum_{d\leqslant n} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{n}{d}\right\rfloor\right) \\
&\sim \frac{1}{n} \times \sum_{d\leqslant n} \mathbb{E}[C_{\Omega(d)}(d)] \cdot d\left(\frac{6}{\pi^2}\frac{n}{d+1} - \frac{6}{\pi^2}\frac{n}{d}\right) \\
&\sim \sum_{d\leqslant n} \mathbb{E}[C_{\Omega(d)}(d)] \cdot \frac{6}{\pi^2 d} \\
&\gg \frac{6}{\pi^2}\int_e^n \frac{(\log t)^2(\log\log t)}{t} dt \\
&= \frac{2}{\pi^2}\left((\log n)^3 \log\log n - \frac{(\log n)^3}{3}\right) \\
&\gg \frac{2}{\pi^2}(\log n)^3 \log\log n.
\end{aligned}$$

So using our observation in Lemma 8.1 where $\mathbb{E}|g^{-1}(n)| \not\to 0$ by Corollary 7.6, we have that our claim holds. □

Corollary 8.4. *For all sufficiently large n on a set of asymptotic density one, we have that*

$$\sum_{\substack{d|n \\ d>e}} (\log d)(\log\log d) - |g^{-1}(n)| \leqslant 0.$$

Proof. First, we see that we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{\substack{d|n \\ d>e}} (\log d)(\log\log d)\right] &= \frac{1}{n} \times \sum_{e<d\leqslant n} (\log d)(\log\log d)\left\lfloor \frac{n}{d}\right\rfloor \\
&\sim \int_e^n \frac{(\log t)(\log\log t)}{t} dt \\
&= \frac{(\log n)^2}{2}(\log\log n) - \frac{(\log n)^2}{4}
\end{aligned}$$

$$\sim \frac{(\log n)^2}{2}(\log\log n).$$

Now on a set of asymptotic density one, we have that since $\mathbb{E}|g^{-1}(n)| \nrightarrow 0$ by Corollary 7.6, Lemma 8.1 implies that

$$\begin{aligned}\mathbb{E}\left[\sum_{\substack{d|n\\d>e}}(\log d)(\log\log d) - |g^{-1}(n)|\right] &= \mathbb{E}\left[\sum_{\substack{d|n\\d>e}}(\log d)(\log\log d)\right] - \mathbb{E}|g^{-1}(n)| \\ &\ll \frac{(\log n)^2}{2}(\log\log n) - \mathbb{E}|g^{-1}(n)|.\end{aligned}$$

So applying Lemma 8.1 and by Proposition 8.3, for all large enough n within a set of asymptotic density one on the integers, we have that

$$\sum_{\substack{d|n\\d>e}}(\log d)(\log\log d) - |g^{-1}(n)| \ll 0, \text{ as } n \to \infty. \qquad \square$$

Proposition 8.5. *Let the set where $G^{-1}(x)$ is non-positive be defined as*

$$\mathcal{G}_- := \left\{n \leqslant x : G^{-1}(x) \leqslant 0\right\}.$$

We claim that for all large $x \to \infty$, the density of this set is always positive and less than one:

$$0 + o(1) < \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{G}_-\} < 1 + o(1).$$

Moreover, if a limiting asymptotic density for $\mathcal{G}_-$ exists, it does not tend to zero as $x \to \infty$:

$$\lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{G}_-\} \neq 0.$$

We will prove Proposition 8.5 after we prove Proposition 9.1 in the next section.

8.2.2 The proof that the necessary hypotheses in Theorem 4.8 are attained for all large x

Proof of the hypotheses of Theorem 4.8. Let $G_E^{-1}(x)$ be defined as in (3a) of the theorem. We need to find some absolute tight limiting constants $B, C \in (0,1)$ such that as $x \to \infty$

$$B + o(1) \leqslant \frac{1}{x} \cdot \#\left\{n \leqslant x : |G^{-1}(n)| - |G_E^{-1}(n)| \leqslant Y\right\} \leqslant C + o(1), \tag{22}$$

for some bounded constant $0 \leqslant Y < +\infty$. By Corollary 8.4, for all n sufficiently large within a set $\mathcal{S}_E$ of asymptotic density one,

$$\sum_{\substack{d|n\\d>e}}(\log d)(\log\log d) - |g^{-1}(n)| \leqslant 0, \forall n \in \mathcal{S}_E, \text{ as } n \to \infty. \tag{23}$$

We aim to sum the functions $G^{-1}(x)$ and $G_E^{-1}(x)$ weighted by the same signs on the terms at each large enough n that satisfy the condition in (23).

Since the sign of $g^{-1}(n)$ is $\lambda(n)$ as given by Proposition 5.1, for all large enough $n \to \infty$ on the set $\mathcal{S}_E$ defined as in (23), we have that both

$$\sum_{\substack{e\leqslant n\leqslant x\\\lambda(n)=+1}} g^{-1}(n) \geqslant \sum_{\substack{e\leqslant n\leqslant x\\\lambda(n)=+1}}\sum_{\substack{d|n\\d>e}}(\log d)(\log\log d) \geqslant \sum_{\substack{e\leqslant n\leqslant(\log x)^2\\\lambda(n)=+1}}\sum_{\substack{d|n\\d>e}}(\log d)(\log\log d)$$

$$\sum_{\substack{e\leqslant n\leqslant x\\ \lambda(n)=-1}} g^{-1}(n) \leqslant -\sum_{\substack{e\leqslant n\leqslant x\\ \lambda(n)=-1}} \sum_{\substack{d|n\\ d>e}} (\log d)(\log\log d) \leqslant -\sum_{\substack{e\leqslant n\leqslant (\log x)^2\\ \lambda(n)=-1}} \sum_{\substack{d|n\\ d>e}} (\log d)(\log\log d).$$

Hence, we have that almost every large $x \in \mathbb{Z}^+$, as $x \to \infty$ the following equation is true for some constant offset $0 \leqslant Y < +\infty$ that is determined by the values of $g^{-1}(n)$ for the small order $n \in \mathcal{S}_E$ where the limiting predicate of (23) does not necessarily hold:

$$G^{-1}(x) \geqslant \sum_{n\leqslant (\log x)^2} \lambda(n) \sum_{\substack{d|n\\ d>e}} (\log d)(\log\log d) + Y. \tag{24}$$

Now we notice that the right-hand-side of (24) corresponds to the definition of the function $G_E^{-1}(x) + Y$. So we see that if $G^{-1}(x) \leqslant 0$ for sufficiently large x where (24) holds, then also $G_E^{-1}(x) \leqslant 0$. Then letting

$$\mathcal{A}_E(Y) := \left\{x \geqslant 1 : G^{-1}(x) \geqslant G_E^{-1}(x) + Y \wedge G^{-1}(x) \leqslant 0\right\},$$

we have that $|G^{-1}(x)| - |G_E^{-1}(x)| \leqslant Y$, $\forall x \in \mathcal{A}_E(Y)$. We still need to show that the density of $\mathcal{A}_E(Y) \cap \{n \leqslant x\}$ can be bounded closely below and above by some respective constants $B, C \in (0,1)$ for large enough $x \to \infty$.

Using Proposition 8.5 and that (24) holds almost everywhere on the sufficiently large positive integers, we can see that there must be some limitingly tight constants $B, C \in (0,1)$ bounding the densities of the infinite set, $\mathcal{A}_E(Y)$, such that the condition $|G^{-1}(x)| - |G_E^{-1}(x)| \leqslant Y$ holds for all large $x \in \mathcal{A}_E(Y)$ with

$$B + o(1) \leqslant \frac{1}{x} \cdot \#\left\{n \leqslant x : n \in \mathcal{A}_E(Y)\right\} \leqslant C + o(1), \text{ as } x \to \infty.$$

That is, for the constant Y defined as in (24), we have seen that we can select

$$\begin{aligned}
B &:= \liminf_{x\to\infty} \frac{1}{x} \cdot \#\left\{n \leqslant x : n \in \mathcal{A}_E(Y)\right\} \in (0,1) \\
C &:= \limsup_{x\to\infty} \frac{1}{x} \cdot \#\left\{n \leqslant x : n \in \mathcal{A}_E(Y)\right\} \in (0,1).
\end{aligned}$$

Hence, we have shown that the necessary conditions in hypotheses of Theorem 4.8 can in fact be achieved for all sufficiently large $x \to \infty$. We have implicitly used the fact that the intersection of a set $\mathcal{S}_1$ of asymptotic density one with another infinite set $\mathcal{S}_2$ of bounded asymptotic density must similarly have bounded limiting densities of order not exceeding the tightest possible bounds on $\mathcal{S}_2$. $\square$

9 Lower bounds for $M(x)$ along infinite subsequences

9.1 The culmination of what we have done so far

Proposition 9.1. *For all sufficiently large x, we have that*

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt. \tag{25}$$

Proof. We know by applying Corollary 4.3 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k) + 1) \\ &= G^{-1}(x) + \sum_{k=1}^{x} g^{-1}(k)\pi(x/k), \end{aligned} \tag{26}$$

where we can drop the asymptotically unnecessary floored integer-valued arguments to $\pi(x)$ in place of its approximation by the monotone non-decreasing asymptotic order, $\pi(x) \sim \frac{x}{\log x}$. Moreover, we can always bound

$$\frac{Ax}{\log x} \leqslant \pi(x) \leqslant \frac{Bx}{\log x},$$

for suitably defined absolute constants, $A, B > 0$. Therefore the approximation obtained by replacing $\pi(x)$ by the dominant term in its limting asymptotic formula is actually valid for all $x > 1$ up to a small constant difference.

What we now require to sum and simplify the right-hand-side summation from (26) is a summation by parts argument. In particular, we obtain that for sufficiently large $x \geqslant 2$ [A]

$$\begin{aligned} \sum_{k=1}^{x} g^{-1}(k)\pi(x/k) &= G^{-1}(x)\pi(1) - \sum_{k=1}^{x-1} G^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)\right] \\ &= -\sum_{k=1}^{x/2} G^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)\right] \\ &\approx -\sum_{k=1}^{x/2} G^{-1}(k) \left[\frac{x}{k \cdot \log(x/k)} - \frac{x}{(k+1) \cdot \log(x/k)}\right] \\ &\approx -\sum_{k=1}^{x/2} G^{-1}(k) \frac{x}{k^2 \cdot \log(x/k)}. \end{aligned}$$

Since for x large enough the summand factor $\frac{x}{k^2 \cdot \log(x/k)}$ is monotonic as k ranges over $k \in [1, x/2]$ in ascending order, this summand factor is a smooth function of k (and x), and since $G^{-1}(x)$ is a summatory function with jumps only at the positive integers, we can approximate $M(x)$ for any finite $x \geqslant 2$ by

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt.$$

We will later only use unsigned lower bound approximations to this function in the next theorems so that the signedness of the summatory function term in the integral formula above as $x \to \infty$ is a moot point entirely. $\square$

[A] Since $\pi(1) = 0$, the actual range of summation corresponds to $k \in \left[1, \frac{x}{2}\right]$.

Proof of Proposition 8.5. Suppose to the contrary that

$$\Gamma_- := \lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{G}_-\} = 0,$$

i.e., assume that $G^{-1}(x) > 0$ almost everywhere for all sufficiently large positive integers x. We will utilize the formula for $M(x)$ frin Proposition 9.1 to derive a contradiction under this assumption. In particular, assuming the above limiting density is zero, we have that

$$\frac{|M(x)|}{x} \approx \left| \int_1^{x/2} \frac{|G^{-1}(t)|}{t^2 \cdot \log(x/t)} dt - \frac{|G^{-1}(x)|}{x} \right|, \text{ a.e., as } x \to \infty. \tag{27}$$

So for almost every sufficiently large $x \to \infty$, we have that

$$\frac{|M(x)|}{x} \gg \left| \int_1^{x/2} \frac{|\mathbb{E}[g^{-1}(t)]|}{t \cdot \log(x/t)} dt - |\mathbb{E}[g^{-1}(x)]| \right|. \tag{28}$$

We can justify (28) by seeing that for any constant $u_0 > 1$, $\int_1^{u_0} \frac{dt}{t^2 \cdot \log(x/t)} = o(1)$ is of lower order growth than the integral contribution over $t \in \left[u_0, \frac{x}{2}\right]$ in (27) as $x \to \infty$.

We also can compute that

$$\int \frac{dt}{t \cdot \log(x/t)} = -\log\log(x/t) + C,$$

So by the signedness of the sequence $g^{-1}(n)$, we can write that minimally $|\mathbb{E}[g^{-1}(n)]| \geqslant o(1) \to 0$ as $n \to \infty$. Combined, it follows at any rate that we can bound the right-hand-side of (28) from below by only the remaining non-integral term as

$$\frac{|M(x)|}{x} \gg \left|\mathbb{E}[g^{-1}(x)]\right|. \tag{29}$$

Since we have assumed that almost everywhere $G^{-1}(x) > 0$ when x is sufficiently large, for infinitely many x we have that

$$\begin{aligned} \left|\mathbb{E}[g^{-1}(x)]\right| &= \frac{1}{x} \times \left[\sum_{\substack{n \leqslant x \\ \lambda(n)=+1}} |g^{-1}(n)| - \sum_{\substack{n \leqslant x \\ \lambda(n)=-1}} |g^{-1}(n)| \right] \\ &\geqslant \frac{1}{x} \times \left[\sum_{n \leqslant \left(\frac{1}{2}+\delta_x\right)x} |g^{-1}(n)| \right] (1+o(1)) \\ &\geqslant \left(\frac{1}{2}+\delta_x\right)^{-1} \cdot \mathbb{E}\left| g^{-1}\left(\left(\frac{1}{2}+\delta_x\right)x\right)\right| (1+o(1)), \end{aligned} \tag{30}$$

with $\delta_x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ for all x. The base factor term of $\frac{1}{2}$ in the upper limit of summation from the previous equation above corresponds to the known fact that [18]

$$\lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leqslant x : \lambda(n) = +1\} = \frac{1}{2}.$$

Thus we expect in fact to take $\delta_x \approx 0$ for almost every large enough x. When we apply Corollary 7.6 to (29) using (30), we must have that for infinitely many $x \to \infty$

$$\frac{|M(x)|}{x} \gg \frac{6}{\pi^2} \log(x)\,(1+o(1)) \xrightarrow{x\to\infty} +\infty.$$

Then we recover a contradiction to the known property that $|M(x)| \leqslant x$ for all $x \geqslant 1$. So $\Gamma_- > 0$ (if the limit exists), or otherwise the limiting densities of $\mathcal{G}_- \cap \{n \leqslant x\}$ in $\{n \leqslant x\}$ are eventually always positive as $x \to \infty$.

A similarly constructed argument shows the corresponding result is true for the set $\mathcal{G}_+$ on which $G^{-1}(x) \geqslant 0$. Thus, we conclude from these two consequences that the limiting densities of $\mathcal{G}_- \cap \{n \leqslant x\}$ are positive, less than one, and in particular cannot tend to zero when $x \to \infty$. □

9.1.1 A few more necessary results

We now use the superscript and subscript notation of (ℓ) not to denote a formal parameter to the functions we define below, but instead to denote that these functions form *lower bound* (rather than exact) approximations to other forms of the functions without the scripted (ℓ).

Lemma 9.2. *Suppose that $\widehat{\pi}_k(x) \geqslant \widehat{\pi}_k^{(\ell)}(x) \geqslant 0$ with $\widehat{\pi}_k^{(\ell)}(x)$ a monotone real-valued function for all sufficiently large x. Let*

$$\begin{aligned} A_{\Omega}^{(\ell)}(x) &:= \sum_{k \leqslant \log\log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x) \\ A_{\Omega}(x) &:= \sum_{k \leqslant \log\log x} (-1)^k \widehat{\pi}_k(x). \end{aligned}$$

Then for all sufficiently large x, we have that

$$|A_{\Omega}(x)| \gg |A_{\Omega}^{(\ell)}(x)|.$$

Proof. Given an explicit smooth lower bounding function, $\widehat{\pi}_k^{(\ell)}(x)$, we define the similarly smooth and monotone residual terms in approximating $\widehat{\pi}_k(x)$ using the following notation:

$$\widehat{\pi}_k(x) = \widehat{\pi}_k^{(\ell)}(x) + \widehat{E}_k(x).$$

Then we can form the ordinary exact form of the summatory function A_{Ω} as

$$\begin{aligned} |A_{\Omega}(x)| &= \left| \sum_{k \leqslant \frac{\log\log x}{2}} \left[\widehat{\pi}_{2k}(x) - \widehat{\pi}_{2k-1}(x)\right] \right| \\ &\geqslant \left| A_{\Omega}^{(\ell)}(x) - \sum_{k \leqslant \frac{\log\log x}{2}} \widehat{E}_{2k-1}(x) \right| \\ &\geqslant \left| A_{\Omega}^{(\ell)}(x) \right| - \left| \sum_{k \leqslant \frac{\log\log x}{2}} \widehat{E}_{2k-1}(x) \right|. \end{aligned}$$

If the latter sum, denoted

$$\mathrm{ES}(x) := \sum_{k \leqslant \frac{\log\log x}{2}} \widehat{E}_{2k-1}(x) \to \infty,$$

as $x \to \infty$, then we can always find some absolute $C_0 > 0$ (by monotonicity) such that $\mathrm{ES}(x) \leqslant C_0 \cdot A_{\Omega}(x)$. If on the other hand this sum becomes constant as $x \to +\infty$, then we also clearly have another absolute $C_1 > 0$ such that $|A_{\Omega}(x)| \geqslant C_1 \cdot |A_{\Omega}^{(\ell)}(x)|$. In either case, the claimed result holds for all large enough x. □

Lemma 9.3. *Suppose that $f_k(n)$ is a sequence of arithmetic functions such that $f_k(n) > 0$ for all $n > u_0$ and $1 \leqslant k \leqslant \Omega(n)$ where $f_{\Omega(n)}(n) \gg \widehat{\tau}_{\ell}(n)$ as $n \to \infty$. We suppose that the bounding function $\widehat{\tau}_{\ell}(t)$ is a continuously differentiable function of t for all large enough $t \gg u_0$ [B]. We define the λ-sign-scaled summatory function of f as follows:*

$$F_{\lambda}(x) := \sum_{u_0 < n \leqslant \log\log\log x} \lambda(n) \cdot f_{\Omega(n)}(n).$$

[B] We will require that $\widehat{\tau}_{\ell}(t) \in C^1(\mathbb{R})$ when we apply the Abel summation formula in the proof of Theorem 9.5. At this point, it is technically an unnecessary condition that is vacously satisfied by assumption (by requirement) and will importantly need to hold only when we specialize to the actual functions employed to form our new bounds in the theorem proof below.

Let

$$\begin{aligned} A_\Omega(t) &:= \sum_{k=1}^{[\log\log t]} (-1)^k \widehat{\pi}_k(t), \\ A_\Omega^{(\ell)}(t) &:= \sum_{k=1}^{[\log\log t]} (-1)^k \widehat{\pi}_k^{(\ell)}(t), \end{aligned}$$

where $\widehat{\pi}_k(x) \geqslant \widehat{\pi}_k^{(\ell)}(x) \geqslant 0$ for $\widehat{\pi}_k^{(\ell)}(t)$ some smooth monotone non-decreasing function of t for all sufficiently large $t \to \infty$. Then we have that

$$|F_\lambda(x)| \gg \left| A_\Omega^{(\ell)}(\log\log\log x)\widehat{\tau}_\ell(\log\log\log x) - \int_{u_0}^{\log\log\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_\ell'(t)dt \right|.$$

Proof. We can form an accurate $C^1(\mathbb{R})$ approximation by the smoothness of $\widehat{\pi}_k^{(\ell)}(x)$ that allows us to apply the Abel summation formula using the summatory function $A_\Omega^{(\ell)}(t)$ for t on any bounded connected subinterval of $[1, \infty)$. The second stated formula for $F_\lambda(x)$ above is valid by Abel summation and by applying Lemma 9.2 whenever

$$0 \leqslant \left| \frac{\sum\limits_{\log\log t < k \leqslant \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_\Omega(t)} \right| \ll 2, \text{ as } t \to \infty.$$

What the last equation implies is that the asymptotically dominant terms indicating the parity of $\lambda(n)$ are captured up to a constant factor by the terms in the range over k summed by $A_\Omega(t)$ for sufficiently large $t \to \infty$. Using Corollary 6.5, we have that the assertion above holds as $t \to \infty$. $\square$

In other words, taking the sum over the summands that defines $A_\Omega(x)$ only over the truncated range of $k \in [1, \log\log x]$ does not non-trivially change the limiting asymptotically dominant terms in the lower bound obtained from using this form of the summatory function in conjunction with the Abel summation formula. This property holds even when we should technically index over all $k \in [1, \log_2(x)]$ to obtain an exact formula for the summatory weight function.

Corollary 9.4. *We have that for almost every sufficiently large x, that "on average"* [C] *as $x \to \infty$*

$$\left|G_E^{-1}(x)\right| \gg \left| \sum_{e < d \leqslant \log\log\log x} \lambda(d)(\log d)(\log\log d) \times \widehat{L}_0\left(\frac{(\log x)^2}{d}\right) \right|,$$

where the function

$$\widehat{L}_0(x) := \sqrt{\frac{2}{\pi}} \cdot \frac{A_0}{3e\log 2} \cdot \frac{x}{(\log\log x)^{\frac{5}{2} + \log\log x}}.$$

Proof. Using the definition in (3a), we obtain on average that [D]

$$\left|G_E^{-1}(x)\right| = \left| \sum_{n \leqslant \log x} \lambda(n) \sum_{\substack{d | n \\ d > e}} (\log d)(\log\log d) \right|$$

[C] This distinction in the statement is necessary since our limiting lower bounds have so far depended on average order estimates of certain sums and arithmetic functions as $n \to \infty$.

[D] For any arithmetic functions f, h, we have that [1, *cf.* §3.10; §3.12]

$$\sum_{n \leqslant x} h(n) \times \sum_{d|n} f(d) = \sum_{d \leqslant x} f(d) \times \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} h(dn).$$

$$= \left| \sum_{e<d \leqslant \log x} \log d \cdot \log\log d \times \sum_{n=1}^{\left\lfloor \frac{\log x}{d} \right\rfloor} \lambda(dn) \right|.$$

We see that by complete additivity of $\Omega(n)$ (multiplicativity of $\lambda(n)$) that

$$\sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn) = \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(d)\lambda(n) = \lambda(d) \sum_{n \leqslant \left\lfloor \frac{x}{d} \right\rfloor} \lambda(n).$$

Now using Theorem 4.7 and Lemma 9.2, we can establish that

$$\begin{aligned} \left| \sum_{n \leqslant x} \lambda(n) \right| &\gg \left| \sum_{k \leqslant \log\log x} (-1)^k \cdot \widehat{\pi}_k(x) \right| \\ &\gg \sqrt{\frac{2}{\pi}} \cdot \frac{A_0}{3e \log 2} \cdot \frac{x}{(\log\log x)^{\frac{5}{2}+\log\log x}} =: \widehat{L}_0(x). \end{aligned} \tag{31}$$

In particular, the precise formula for the limiting lower bound stated above for $\widehat{L}_0(x)$ is computed by symbolic summation in *Mathematica* using the new bounds on $\widehat{\pi}_k(x)$ guaranteed by the theorem, and then by applying subsequent standard asymptotic estimates to the resulting formulas for large $x \to \infty$, e.g., in the form of (9c) and Stirling's formula). The sign of the sum obtained by taking (31) without an absolute value is given by $(-1)^{\lfloor \log\log x \rfloor}$.

It follows that

$$|G_E^{-1}(x)| \gg \left| \sum_{e<d \leqslant (\log x)^2} \lambda(d) \cdot \log d \cdot \log\log d \times (-1)^{\left\lfloor \log\log\left(\frac{(\log x)^2}{d}\right) \right\rfloor} \cdot \widehat{L}_0\left(\frac{(\log x)^2}{d}\right) \right|. \tag{32}$$

We will simplify (32) by modifying the limits of summation on d to be bounded by a suitable function of x so that for almost every large x and with the exception of at most finitely many d on this new interval, we obtain a constant sign from the weight on $\widehat{L}_0\left(\frac{(\log x)^2}{d}\right)$, or equivalently, a constant parity for $\left\lfloor \log\log\left(\frac{(\log x)^2}{d}\right) \right\rfloor$ at such d. Consider that

$$\log\log\left(\frac{(\log x)^2}{d}\right) = \log\log\log x + \log\left(1 - \frac{\log d}{\log\log x}\right), \text{ as } x \to \infty.$$

If we take $d \in [e, \log\log\log x]$, we have that $\frac{\log d}{\log\log x} = o(1)$ as $x \to \infty$. So for d within this range, we expect that for almost every x there are at most a handful of comparitively small order $d \leqslant d_0(x)$ such that

$$\left\lfloor \log\log\left(\frac{(\log x)^2}{d}\right) \right\rfloor \sim \lfloor \log\log\log x + o(1) \rfloor,$$

changes parity at $d_0(x)$. Then provided that the sign term involving both d and x from (32) does not change for d within our new interval, we can factor out the dependence of the sign on the monotonically decreasing function $\widehat{L}_0\left(\frac{(\log x)^2}{d}\right)$ in the variable d. We note that the function $\widehat{L}_0\left(\frac{(\log x)^2}{d}\right)$ is decreasing on $d \in \left[e, \frac{x}{e}\right)$, as can be viewed by computing its first derivative and observing that the sign on this function is negative for all sufficiently large x for d taken in this interval. So we determine that we should select $d := \log\log\log x$ in (32) to obtain a global lower bound on $|G_E^{-1}(x)|$.

Let the function $T_E(x)$ be defined for all large enough x as

$$T_E(x) := \frac{1}{\log\log\left(\frac{(\log x)^2}{\log\log\log x}\right)^{\frac{5}{2}+\log\log\left(\frac{(\log x)^2}{\log\log\log x}\right)}} \gg \frac{1}{(\log\log\log x)^{\frac{5}{2}+\log\log\log x}}. \tag{33}$$

Then we see that as $x \to \infty$

$$\begin{aligned} S_{E,1}(x) &:= \left| \sum_{e<d\leqslant \log\log\log x} \lambda(d)(\log d)(\log\log d) \times \widehat{L}_0\left(\frac{(\log x)^2}{d}\right) \right| \\ &\gg \sqrt{\frac{2}{\pi}} \frac{A_0}{3e\log 2} \times (\log x)^2 T_E(x) \times \left| \sum_{e<d\leqslant \log\log\log x} \frac{\lambda(d)(\log d)(\log\log d)}{d} \right| \\ &\gg \sqrt{\frac{2}{\pi}} \frac{A_0}{3e\log 2} \times (\log x)^2 T_E(x) \times \left| A_\Omega^{(\ell)}(\log\log\log x)\widehat{\tau}_0(\log\log\log x) - \int_e^{\log\log\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_0'(t)dt \right|, \quad (34) \end{aligned}$$

where we select the functions $\widehat{\tau}_0(t) := \frac{\log t \cdot \log\log t}{t}$ and $-\widehat{\tau}_0'(t) \gg \frac{\log t\cdot \log\log t}{t^2}$ in the notation from Lemma 9.3. Let the remainder term sums be defined for all sufficiently large x by

$$R_E(x) := \left| \sum_{\log\log\log x<d<\frac{(\log x)^2}{e}} \lambda(d)\cdot \log d \cdot \log\log d \times (-1)^{\left\lfloor \log\log\left(\frac{\log x}{d}\right)\right\rfloor} \cdot \widehat{L}_0\left(\frac{\log x}{d}\right) \right|.$$

What we then obtain from (32) and (34) is the following lower bound from the triangle inequality that holds for all sufficiently large x:

$$|G_E^{-1}(x)| \gg \left| S_{E,1}(x) - R_E(x) \right| \gg S_{E,1}(x), \text{ as } x \to \infty. \quad (35)$$

We have claimed that in fact we can drop the sum terms over upper range of d and still obtain an asymptotic lower bound on $|G_E^{-1}(x)|$ as $x \to \infty$ on the right-hand-side of (35). To justify this step in the proof, we will provide asymptotic upper and lower bounds on $R_E(x)$ that show that the contribution from these terms in absolute value exceeds the magnitude of the corresponding sums for $d \in [e, \log\log\log x]$ when x is large. In other words, the magnitudes of the terms in the sums defining the difference of the functions in (35) cannot match and cause a zero-tending cancellation of the terms when x is large.

First, observe that

$$\begin{aligned} R_E(x) &\ll (\log x)^2 \times \sum_{\log\log\log x<d\leqslant \frac{(\log x)^2}{e}} \frac{\log d \cdot \log\log d}{d} \\ &\sim (\log x)^2 \times \int_{\log\log\log x}^{(\log x)^2} \frac{(\log t)(\log\log t)}{t} dt \\ &\sim (\log x)^2 \left[\frac{(\log t)^2}{2}\log\log t - \frac{(\log t)^2}{4}\right]\Bigg|_{t=\log\log\log x}^{t=(\log x)^2} \\ &\sim 2\cdot (\log x)^2(\log\log x)^2(\log\log\log x). \end{aligned}$$

Similarly, we can bound below to show that $R_E(x)$ is at least on the order of a constant times $(\log x)^2$. To obtain this lower bound, note that since $\frac{\log d\cdot\log\log d}{d}$ is monotone decreasing for all large enough $d > e$, we obtain the smallest possible magnitude on the sum by alternating signs on consecutive terms in the sum. So we have some constant $C > 0$ such that

$$\begin{aligned} R_E(x) &\gg (\log x)^2 \times \left| \sum_{\log\log\log x<d<\frac{(\log x)^2}{2e}} \frac{\log(2d)\log\log(2d)}{2d} - \frac{\log(2d+1)\log\log(2d+1)}{2d+1} \right| \\ &\sim (\log x)^2 \times \left| \sum_{\log\log\log x<d<\frac{(\log x)^2}{2e}} \frac{\log(2d)\log\log(2d)}{2d} - \frac{1}{2d+1}\left(\log(2d) + \frac{1}{2d}\right)\left(\log\log(2d) + \frac{1}{2d\cdot\log(2d)}\right) \right| \end{aligned}$$

$$\approx (\log x)^2 \times \left| \sum_{\log\log\log x < d < \frac{(\log x)^2}{2e}} \frac{\log(2d)\log\log(2d)}{4d^2} - \frac{1}{2d}\left[\frac{\log(2d)}{2d} + \frac{1}{2d} + \frac{1}{4d^2 \cdot \log(2d)}\right]\right|$$
$$\gg C \times (\log x)^2 + o(1), \text{ as } x \to \infty.$$

In total, what we obtain is that the magnitude of $R_E(x)$ always exceeds that of the lower bound we establish in Theorem 9.5 for the sums over $d \in [e, \log\log\log x]$. So we obtain the lower bounds on $G_E(x)$ that correspond to the smaller order terms resulting from the first sums above. □

9.1.2 The proof of a central lower bound on the magnitude of $G_E^{-1}(x)$

The next central theorem is the last key barrier required to prove Corollary 4.11 in the next subsection.

Theorem 9.5 (Asymptotics and bounds for the summatory functions $G^{-1}(x)$)**.** *We define a lower summatory function, $G_\ell^{-1}(x)$, to provide bounds on the magnitude of $G_E^{-1}(x)$ such that*

$$|G_\ell^{-1}(x)| \leqslant |G_E^{-1}(x)|,$$

for all sufficiently large $x \geqslant e$. Let $C_{\ell,1} > 0$ be the absolute constant defined by

$$C_{\ell,1} = \frac{8A_0^2}{9\pi e^2 \log^2 2} = \frac{256 \cdot 2^{1/8}}{59049 \cdot \pi^2 e^8 \log^8 2} \exp\left(-\frac{55}{2}\log^2 2\right) \approx 5.51187 \times 10^{-12}.$$

We obtain the following limiting estimate for the bounding function $G_\ell^{-1}(x)$ holding as we say "on average" as $x \to \infty$:

$$\left|G_\ell^{-1}(x)\right| \gg \frac{C_{\ell,1} \cdot (\log x)^2 (\log\log\log\log x)}{(\log\log\log x)^{\frac{7}{2}} (\log\log\log\log\log x)^{\frac{3}{2}}} \times \exp\left(-2(\log\log\log x)^2\right).$$

Proof. Recall from our proof of Corollary 4.7 that a lower bound on the variant prime form counting function, $\widehat{\pi}_k(x)$, is given by

$$\widehat{\pi}_k(x) \gg \frac{A_0 \cdot x}{\log x \cdot (\log\log x)^4 \cdot (k-1)!} \cdot \left(\frac{4}{3e\log 2}\right)^{\frac{k}{\log\log x}}, \text{ as } x \to \infty.$$

So we can then form a lower summatory function indicating the signed contributions over the distinct parity of $\Omega(n)$ for all $n \leqslant x$ as follows by applying (9b) and Stirling's approximation:

$$\left|A_\Omega^{(\ell)}(t)\right| = \left|\sum_{k \leqslant \log\log t} (-1)^k \widehat{\pi}_k(t)\right| \gg \sqrt{\frac{2}{\pi}} \cdot \frac{A_0}{3e\log 2} \cdot \frac{t}{(\log\log t)^{\frac{5}{2}+\log\log t}}. \tag{36}$$

The actual sign on this function is given by $\operatorname{sgn}(A_\Omega^{(\ell)}(t)) = (-1)^{\lfloor \log\log t \rfloor}$ (see Lemma 9.2). By Lemma 9.3 we know that this summatory function forms a lower bound in absolute value for the actual weight of the signed terms indicated by $\lambda(n)$.

As we determined in (34) from the proof of Corollary 9.4, we take the function $\widehat{\tau}_0(t) = \frac{\log t \cdot \log\log t}{t}$ that satisfies

$$-\widehat{\tau}_0'(t) = -\frac{d}{dt}\left[\frac{\log t \cdot \log\log t}{t}\right] = \frac{\log t \cdot \log\log t}{t^2} - \frac{\log\log t}{t^2} - \frac{1}{t^2} \gg \frac{\log t \cdot \log\log t}{t^2}.$$

Moreover, we have using the notation from the proof above that we can select the lower bound function $G_\ell^{-1}(x)$ to be defined as follows:

$$G_\ell^{-1}(x) := \sqrt{\frac{2}{\pi}} \frac{A_0}{3e\log 2} \times (\log x)^2 T_E(x) \times \left|A_\Omega^{(\ell)}(\log\log\log x)\widehat{\tau}_0(\log\log\log x) - \int_e^{\log\log\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_0'(t)dt\right|. \tag{37}$$

The inner integral term on the rightmost side of (37) is summed approximately by splitting the terms sign weighted by $(-1)^{\lfloor \log\log t\rfloor}$ in the form of

$$\begin{aligned}\sqrt{\frac{2}{\pi}}\frac{A_0}{3e\log 2}\times\left|\int_e^{\log\log\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_0'(t)dt\right| &\gg \left|\sum_{k=e+1}^{\frac{1}{2}\log\log\log\log\log x}\left[I_\ell\left(e^{e^{2k+1}}\right)-I_\ell\left(e^{e^{2k}}\right)\right]e^{e^{2k}}\right| \\ &\gg \sqrt{\frac{2}{\pi}}\frac{A_0}{3e\log 2}\times\left|\int_{\frac{\log\log\log\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log\log\log\log x}{2}} I_\ell\left(e^{e^{2k}}\right)e^{e^{2k}}dk\right|.\end{aligned} \tag{38}$$

We express the integrand function, $I_\ell(t) := \sqrt{\frac{2}{\pi}}\frac{A_0}{3e\log 2}\times\widehat{\tau}_0'(t)A_\Omega^{(\ell)}(t)$, defined as in the previous equations with some limiting simplifications for the $k\in\left[\frac{\log\log\log\log\log x}{2}-\frac{1}{2},\frac{\log\log\log\log\log x}{2}\right]$ as

$$I_\ell\left(e^{e^{2k}}\right)e^{e^{2k}} \gg \frac{2^{3/2}A_0^2}{9\pi e^2\log^2 2}\cdot\frac{\exp\left(e^{2k}\right)}{2^{2k}\cdot k^{2k+3/2}} =: \widehat{I}_\ell(k). \tag{39}$$

So using the lower bound on the increasing integrand in (39) (the complete terms in x are decreasing), we find from the mean value theorem that

$$\begin{aligned}\sqrt{\frac{2}{\pi}}\frac{A_0}{3e\log 2}&\times(\log x)^2T_E(x)\times\left|\int_{\frac{\log\log\log\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log\log\log\log x}{2}} I_\ell\left(e^{e^{2k}}\right)e^{e^{2k}}dk\right| \\ &\gg \sqrt{\frac{2}{\pi}}\frac{A_0}{3e\log 2}\times(\log x)^2T_E(x)\times\left|\widehat{I}_\ell\left(\frac{\log\log\log x}{2}\right)\right| \\ &\gg \frac{C_{\ell,1}\cdot(\log x)^2(\log\log\log\log x)}{2\cdot(\log\log\log x)^{\frac{5}{2}+\log 2+\log\log\log x+\log\log\log\log x}(\log\log\log\log\log x)^{\frac{3}{2}+\log\log\log\log\log x}}.\end{aligned} \tag{40}$$

Moreover, by evaluating $\widehat{I}_\ell(t)$ at the lower bound on the integral above, we can similarly conclude that

$$\begin{aligned}\sqrt{\frac{2}{\pi}}\frac{A_0}{3e\log 2}&\times(\log x)^2T_E(x)\times\left|\int_{\frac{\log\log\log\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log\log\log\log x}{2}} I_\ell\left(e^{e^{2k}}\right)e^{e^{2k}}dk\right| \\ &\ll \frac{C_{\ell,1}\cdot(\log x)^2(\log\log\log\log x)}{2e\cdot(\log\log\log x)^{\frac{5}{2}+\log 2+\log\log\log x+\log\log\log\log x}(\log\log\log\log\log x)^{\frac{1}{2}+\log\log\log\log\log x}}.\end{aligned}$$

To make it clear which terms in (37) the limiting lower bounds correspond to, consider the following expansion for the leading term in the Abel summation formula from (37) for comparison with (40):

$$\begin{aligned}\sqrt{\frac{2}{\pi}}\frac{A_0}{3e\log 2}&\times(\log x)^2T_E(x)\times\left|\widehat{\tau}_0(\log\log\log x)A_\Omega^{(\ell)}(\log\log\log x)\right| \\ &\gg \frac{C_{\ell,1}\cdot(\log x)^2(\log\log\log\log x)}{(\log\log\log x)^{\frac{7}{2}+\log\log\log x}(\log\log\log\log\log x)^{\frac{3}{2}+\log\log\log\log\log x}}.\end{aligned} \tag{41}$$

Hence, we see that

$$\begin{aligned}\left|G_\ell^{-1}(x)\right| &\gg \frac{C_{\ell,1}\cdot(\log x)^2(\log\log\log\log x)}{(\log\log\log x)^{\frac{7}{2}+\log\log\log x}(\log\log\log\log\log x)^{\frac{3}{2}+\log\log\log\log\log x}} \\ &\gg \frac{C_{\ell,1}\cdot(\log x)^2(\log\log\log\log x)}{(\log\log\log x)^{\frac{7}{2}}(\log\log\log\log\log x)^{\frac{3}{2}}}\times\exp\left\{-\left((\log\log\log x)^2+(\log\log\log\log\log x)^2\right)\right\}.\end{aligned}$$

What is key to observe about these lower bounds is that each of them scaled by $(\log x)^{-1}$ monotonically increases without bound as $x\to\infty$. In particular, the remaining factor of $\log x$ after rescaling dominates the asymptotics of the reciprocal powers of iterated logarithms. $\square$

9.2 Proof of the unboundedness of the scaled Mertens function

We finally address the main conclusion of our arguments given so far with the following proof:

Proof of Corollary 4.11. We break up the integral term in Proposition 9.1 over $t \in [u_0, x/2]$ into two pieces: one that is easily bounded from $u_0 \leqslant t \leqslant \sqrt{x}$, and then another that will conveniently give us our slow-growing tendency towards infinity along the subsequence when evaluated using Theorem 9.5.

We can apply Proposition 9.1 to see that for some $x_0 \in \left[\sqrt{x}, \frac{x}{2}\right]$ such that

$$\left|G^{-1}(x_0)\right| := \min_{\sqrt{x} \leqslant t \leqslant \frac{x}{2}} |G^{-1}(t)|,$$

we can bound

$$\begin{aligned}
\frac{|M(x)|}{\sqrt{x}} &= \frac{1}{\sqrt{x}} \left| G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt \right| \\
&\gg \left| \sqrt{x} \times \int_{\sqrt{x}}^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt \right| \\
&\gg \left| \int_{\sqrt{x_0}}^{\frac{x}{2}} \frac{2\sqrt{x_0}}{t^2 \cdot \log\left(x_0\right)} dt \right| \times \left(\min_{\sqrt{x} \leqslant t \leqslant \frac{x}{2}} |G^{-1}(t)| \right) \\
&\gg \frac{2\left|G^{-1}(x_0)\right|}{\log\left(x_0\right)}. \qquad (42)
\end{aligned}$$

When we assume that $x \mapsto x_y$ is taken along the subsequence defined within the intervals defined above, we can transform the bound in the last equation into a statement about a lower bound for $|M(x)|/\sqrt{x}$ along an infinitely tending subsequence. For sufficiently large y, this subsequence is guaranteed to exist by our proof of Theorem 4.8 using the methods we have developed to establish it and the necessary hypotheses in Section 8.

In particular, the existence of this infinite subsequence shows that there is some x_y for each large enough $y \to \infty$ such that $|G^{-1}(x_y)| \gg |G_E^{-1}(x_y)| \gg |G_\ell^{-1}(x_y)|$ where $x_y \to \infty$ as $y \to \infty$. So

$$\begin{aligned}
\frac{|M(x)|}{\sqrt{x}} &\gg \frac{2\left|G^{-1}(x_0)\right|}{\log\left(x_0\right)} \\
&\gg \frac{2C_{\ell,1} \cdot (\log x_y)(\log\log\log\log x_y)}{(\log\log\log x_y)^{\frac{7}{2}} (\log\log\log\log\log x_y)^{\frac{3}{2}}} \times \exp\left(-2(\log\log\log x_y)^2\right). \qquad (43)
\end{aligned}$$

We want this sequence $\{x_y\}_{y \geqslant Y_0}$ for Y_0 sufficiently large to correspond to $x_y \equiv x_0$ for x_0 as in (42) above. Let $\varepsilon_0 := \frac{1}{2}\min(B, 1-C)$ for $B, C \in (0,1)$ as in the hypotheses of Theorem 4.8. Then we have if we take $x \mapsto x_{0,y}$ with $x_{0,y} := \exp\left(2e^{e^{e^{e^{2y}}}}\right)$, we recover from (43) that

$$\begin{aligned}
\frac{|M(x_{0,y})|}{\sqrt{x_{0,y}}} &\gg \frac{C_{\ell,1} \cdot (\log\left[(B-\varepsilon_0)\sqrt{x_{0,y}}\right])(\log\log\log\log\left[(B-\varepsilon_0)\sqrt{x_{0,y}}\right]) \times \exp\left(-2(\log\log\log\left[(B-\varepsilon_0)\sqrt{x_{0,y}}\right])^2\right)}{(\log\log\log\left[(B-\varepsilon_0)\sqrt{x_{0,y}}\right])^{\frac{7}{2}} (\log\log\log\log\log\left[(B-\varepsilon_0)\sqrt{x_{0,y}}\right])^{\frac{3}{2}}} \\
&\gg \frac{2C_{\ell,1} \cdot (\log\sqrt{x_{0,y}})(\log\log\log\log\sqrt{x_{0,y}})}{(\log\log\log\sqrt{x_{0,y}})^{\frac{7}{2}} (\log\log\log\log\log\sqrt{x_{0,y}})^{\frac{3}{2}}} \times \exp\left(-2(\log\log\log\sqrt{x_{0,y}})^2\right), \qquad (44)
\end{aligned}$$

along this infinitely tending subsequence as $y \to \infty$. The scaled Mertens function is then unbounded in the limit supremum sense, as we have claimed, since the right-hand-side of the previous equation tends to positive infinity as $x_{0,y} \to \infty$. □

References

[1] T. M. Apostol. *Introduction to Analytic Number Theory.* Springer–Verlag, 1976.

[2] P. Flajolet and R. Sedgewick. *Analytic Combinatorics.* Cambridge University Press, 2009 (Third printing 2010).

[3] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science.* Addison-Wesley, 1994.

[4] G. H. Hardy and E. M. Wright, editors. *An Introduction to the Theory of Numbers.* Oxford University Press, 2008 (Sixth Edition).

[5] G. Hurst. Computations of the Mertens function and improved bounds on the Mertens conjecture. https://arxiv.org/pdf/1610.08551/, 2017.

[6] H. Iwaniec and E. Kowalski. *Analytic Number Theory*, volume 53. AMS Colloquium Publications, 2004.

[7] T. Kotnik and H. té Riele. The Mertens conjecture revisited. *Algorithmic Number Theory*, 7^{th} International Symposium, 2006.

[8] T. Kotnik and J. van de Lune. On the order of the Mertens function. *Exp. Math.*, 2004.

[9] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford: The Clarendon Press, 1995.

[10] H. L. Montgomery and R. C. Vaughan. *Multiplicative Number Theory: I. Classical Theory.* Cambridge, 2006.

[11] N. Ng. The distribution of the summatory function of the Móbius function. *Proc. London Math. Soc.*, 89(3):361–389, 2004.

[12] A. M. Odlyzko and H. J. J. té Riele. Disproof of the Mertens conjecture. *J. REINE ANGEW. MATH*, 1985.

[13] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. *NIST Handbook of Mathematical Functions.* Cambridge University Press, 2010.

[14] P. Ribenboim. *The new book of prime number records.* Springer, 1996.

[15] J. Sándor and B. Crstici. *Handbook of Number Theory II.* Kluwer Academic Publishers, 2004.

[16] N. J. A. Sloane. The Online Encyclopedia of Integer Sequences, 2020.

[17] K. Soundararajan. Partial sums of the Möbius function. *Annals of Mathematics*, 2009.

[18] T. Tao and J. Teräväinen. Value patterns of multiplicative functions and related sequences. *Forum of Mathematics, Sigma*, 7, 2019.

[19] E. C. Titchmarsh. *The theory of the Riemann zeta function.* Clarendon Press, 1951.

T.1 Table: The Dirichlet inverse function $g^{-1}(n)$ and the distribution of its summatory function

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1^1	Y	N	1	0	1.0000000	1.00000	0	1	1	0
2	2^1	Y	Y	-2	0	1.0000000	0.500000	0.500000	-1	1	-2
3	3^1	Y	Y	-2	0	1.0000000	0.333333	0.666667	-3	1	-4
4	2^2	N	Y	2	0	1.5000000	0.500000	0.500000	-1	3	-4
5	5^1	Y	Y	-2	0	1.0000000	0.400000	0.600000	-3	3	-6
6	$2^1 3^1$	Y	N	5	0	1.0000000	0.500000	0.500000	2	8	-6
7	7^1	Y	Y	-2	0	1.0000000	0.428571	0.571429	0	8	-8
8	2^3	N	Y	-2	0	2.0000000	0.375000	0.625000	-2	8	-10
9	3^2	N	Y	2	0	1.5000000	0.444444	0.555556	0	10	-10
10	$2^1 5^1$	Y	N	5	0	1.0000000	0.500000	0.500000	5	15	-10
11	11^1	Y	Y	-2	0	1.0000000	0.454545	0.545455	3	15	-12
12	$2^2 3^1$	N	N	-7	2	1.2857143	0.416667	0.583333	-4	15	-19
13	13^1	Y	Y	-2	0	1.0000000	0.384615	0.615385	-6	15	-21
14	$2^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-1	20	-21
15	$3^1 5^1$	Y	N	5	0	1.0000000	0.466667	0.533333	4	25	-21
16	2^4	N	Y	2	0	2.5000000	0.500000	0.500000	6	27	-21
17	17^1	Y	Y	-2	0	1.0000000	0.470588	0.529412	4	27	-23
18	$2^1 3^2$	N	N	-7	2	1.2857143	0.444444	0.555556	-3	27	-30
19	19^1	Y	Y	-2	0	1.0000000	0.421053	0.578947	-5	27	-32
20	$2^2 5^1$	N	N	-7	2	1.2857143	0.400000	0.600000	-12	27	-39
21	$3^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-7	32	-39
22	$2^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-2	37	-39
23	23^1	Y	Y	-2	0	1.0000000	0.434783	0.565217	-4	37	-41
24	$2^3 3^1$	N	N	9	4	1.5555556	0.458333	0.541667	5	46	-41
25	5^2	N	Y	2	0	1.5000000	0.480000	0.520000	7	48	-41
26	$2^1 13^1$	Y	N	5	0	1.0000000	0.500000	0.500000	12	53	-41
27	3^3	N	Y	-2	0	2.0000000	0.481481	0.518519	10	53	-43
28	$2^2 7^1$	N	N	-7	2	1.2857143	0.464286	0.535714	3	53	-50
29	29^1	Y	Y	-2	0	1.0000000	0.448276	0.551724	1	53	-52
30	$2^1 3^1 5^1$	Y	N	-16	0	1.0000000	0.433333	0.566667	-15	53	-68
31	31^1	Y	Y	-2	0	1.0000000	0.419355	0.580645	-17	53	-70
32	2^5	N	Y	-2	0	3.0000000	0.406250	0.593750	-19	53	-72
33	$3^1 11^1$	Y	N	5	0	1.0000000	0.424242	0.575758	-14	58	-72
34	$2^1 17^1$	Y	N	5	0	1.0000000	0.441176	0.558824	-9	63	-72
35	$5^1 7^1$	Y	N	5	0	1.0000000	0.457143	0.542857	-4	68	-72
36	$2^2 3^2$	N	N	14	9	1.3571429	0.472222	0.527778	10	82	-72
37	37^1	Y	Y	-2	0	1.0000000	0.459459	0.540541	8	82	-74
38	$2^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	13	87	-74
39	$3^1 13^1$	Y	N	5	0	1.0000000	0.487179	0.512821	18	92	-74
40	$2^3 5^1$	N	N	9	4	1.5555556	0.500000	0.500000	27	101	-74
41	41^1	Y	Y	-2	0	1.0000000	0.487805	0.512195	25	101	-76
42	$2^1 3^1 7^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	9	101	-92
43	43^1	Y	Y	-2	0	1.0000000	0.465116	0.534884	7	101	-94
44	$2^2 11^1$	N	N	-7	2	1.2857143	0.454545	0.545455	0	101	-101
45	$3^2 5^1$	N	N	-7	2	1.2857143	0.444444	0.555556	-7	101	-108
46	$2^1 23^1$	Y	N	5	0	1.0000000	0.456522	0.543478	-2	106	-108
47	47^1	Y	Y	-2	0	1.0000000	0.446809	0.553191	-4	106	-110
48	$2^4 3^1$	N	N	-11	6	1.8181818	0.437500	0.562500	-15	106	-121

Table T.1: ***Computations with*** $\mathbf{g^{-1}(n)} \equiv (\omega+\mathbf{1})^{-\mathbf{1}}(\mathbf{n})$ ***for*** $\mathbf{1 \leqslant n \leqslant 500}$**.**

- ▶ *The column labeled* `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled* `Sqfree` *and* `PPower`*, respectively, list inclusion of n in the sets of squarefree integers and the prime powers.*
- ▶ *The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.*
- ▶ *The last several columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{n} \cdot \#\{n \leqslant x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geqslant 1$.*

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\hat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
49	7^2	N	Y	2	0	1.5000000	0.448980	0.551020	−13	108	−121
50	$2^1 5^2$	N	N	−7	2	1.2857143	0.440000	0.560000	−20	108	−128
51	$3^1 17^1$	Y	N	5	0	1.0000000	0.450980	0.549020	−15	113	−128
52	$2^2 13^1$	N	N	−7	2	1.2857143	0.442308	0.557692	−22	113	−135
53	53^1	Y	Y	−2	0	1.0000000	0.433962	0.566038	−24	113	−137
54	$2^1 3^3$	N	N	9	4	1.5555556	0.444444	0.555556	−15	122	−137
55	$5^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−10	127	−137
56	$2^3 7^1$	N	N	9	4	1.5555556	0.464286	0.535714	−1	136	−137
57	$3^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	4	141	−137
58	$2^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	9	146	−137
59	59^1	Y	Y	−2	0	1.0000000	0.474576	0.525424	7	146	−139
60	$2^2 3^1 5^1$	N	N	30	14	1.1666667	0.483333	0.516667	37	176	−139
61	61^1	Y	Y	−2	0	1.0000000	0.475410	0.524590	35	176	−141
62	$2^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	40	181	−141
63	$3^2 7^1$	N	N	−7	2	1.2857143	0.476190	0.523810	33	181	−148
64	2^6	N	Y	2	0	3.5000000	0.484375	0.515625	35	183	−148
65	$5^1 13^1$	Y	N	5	0	1.0000000	0.492308	0.507692	40	188	−148
66	$2^1 3^1 11^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	24	188	−164
67	67^1	Y	Y	−2	0	1.0000000	0.477612	0.522388	22	188	−166
68	$2^2 17^1$	N	N	−7	2	1.2857143	0.470588	0.529412	15	188	−173
69	$3^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	20	193	−173
70	$2^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.471429	0.528571	4	193	−189
71	71^1	Y	Y	−2	0	1.0000000	0.464789	0.535211	2	193	−191
72	$2^3 3^2$	N	N	−23	18	1.4782609	0.458333	0.541667	−21	193	−214
73	73^1	Y	Y	−2	0	1.0000000	0.452055	0.547945	−23	193	−216
74	$2^1 37^1$	Y	N	5	0	1.0000000	0.459459	0.540541	−18	198	−216
75	$3^1 5^2$	N	N	−7	2	1.2857143	0.453333	0.546667	−25	198	−223
76	$2^2 19^1$	N	N	−7	2	1.2857143	0.447368	0.552632	−32	198	−230
77	$7^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−27	203	−230
78	$2^1 3^1 13^1$	Y	N	−16	0	1.0000000	0.448718	0.551282	−43	203	−246
79	79^1	Y	Y	−2	0	1.0000000	0.443038	0.556962	−45	203	−248
80	$2^4 5^1$	N	N	−11	6	1.8181818	0.437500	0.562500	−56	203	−259
81	3^4	N	Y	2	0	2.5000000	0.444444	0.555556	−54	205	−259
82	$2^1 41^1$	Y	N	5	0	1.0000000	0.451220	0.548780	−49	210	−259
83	83^1	Y	Y	−2	0	1.0000000	0.445783	0.554217	−51	210	−261
84	$2^2 3^1 7^1$	N	N	30	14	1.1666667	0.452381	0.547619	−21	240	−261
85	$5^1 17^1$	Y	N	5	0	1.0000000	0.458824	0.541176	−16	245	−261
86	$2^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	−11	250	−261
87	$3^1 29^1$	Y	N	5	0	1.0000000	0.471264	0.528736	−6	255	−261
88	$2^3 11^1$	N	N	9	4	1.5555556	0.477273	0.522727	3	264	−261
89	89^1	Y	Y	−2	0	1.0000000	0.471910	0.528090	1	264	−263
90	$2^1 3^2 5^1$	N	N	30	14	1.1666667	0.477778	0.522222	31	294	−263
91	$7^1 13^1$	Y	N	5	0	1.0000000	0.483516	0.516484	36	299	−263
92	$2^2 23^1$	N	N	−7	2	1.2857143	0.478261	0.521739	29	299	−270
93	$3^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	34	304	−270
94	$2^1 47^1$	Y	N	5	0	1.0000000	0.489362	0.510638	39	309	−270
95	$5^1 19^1$	Y	N	5	0	1.0000000	0.494737	0.505263	44	314	−270
96	$2^5 3^1$	N	N	13	8	2.0769231	0.500000	0.500000	57	327	−270
97	97^1	Y	Y	−2	0	1.0000000	0.494845	0.505155	55	327	−272
98	$2^1 7^2$	N	N	−7	2	1.2857143	0.489796	0.510204	48	327	−279
99	$3^2 11^1$	N	N	−7	2	1.2857143	0.484848	0.515152	41	327	−286
100	$2^2 5^2$	N	N	14	9	1.3571429	0.490000	0.510000	55	341	−286
101	101^1	Y	Y	−2	0	1.0000000	0.485149	0.514851	53	341	−288
102	$2^1 3^1 17^1$	Y	N	−16	0	1.0000000	0.480392	0.519608	37	341	−304
103	103^1	Y	Y	−2	0	1.0000000	0.475728	0.524272	35	341	−306
104	$2^3 13^1$	N	N	9	4	1.5555556	0.480769	0.519231	44	350	−306
105	$3^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	28	350	−322
106	$2^1 53^1$	Y	N	5	0	1.0000000	0.481132	0.518868	33	355	−322
107	107^1	Y	Y	−2	0	1.0000000	0.476636	0.523364	31	355	−324
108	$2^2 3^3$	N	N	−23	18	1.4782609	0.472222	0.527778	8	355	−347
109	109^1	Y	Y	−2	0	1.0000000	0.467890	0.532110	6	355	−349
110	$2^1 5^1 11^1$	Y	N	−16	0	1.0000000	0.463636	0.536364	−10	355	−365
111	$3^1 37^1$	Y	N	5	0	1.0000000	0.468468	0.531532	−5	360	−365
112	$2^4 7^1$	N	N	−11	6	1.8181818	0.464286	0.535714	−16	360	−376
113	113^1	Y	Y	−2	0	1.0000000	0.460177	0.539823	−18	360	−378
114	$2^1 3^1 19^1$	Y	N	−16	0	1.0000000	0.456140	0.543860	−34	360	−394
115	$5^1 23^1$	Y	N	5	0	1.0000000	0.460870	0.539130	−29	365	−394
116	$2^2 29^1$	N	N	−7	2	1.2857143	0.456897	0.543103	−36	365	−401
117	$3^2 13^1$	N	N	−7	2	1.2857143	0.452991	0.547009	−43	365	−408
118	$2^1 59^1$	Y	N	5	0	1.0000000	0.457627	0.542373	−38	370	−408
119	$7^1 17^1$	Y	N	5	0	1.0000000	0.462185	0.537815	−33	375	−408
120	$2^3 3^1 5^1$	N	N	−48	32	1.3333333	0.458333	0.541667	−81	375	−456
121	11^2	N	Y	2	0	1.5000000	0.462810	0.537190	−79	377	−456
122	$2^1 61^1$	Y	N	5	0	1.0000000	0.467213	0.532787	−74	382	−456
123	$3^1 41^1$	Y	N	5	0	1.0000000	0.471545	0.528455	−69	387	−456
124	$2^2 31^1$	N	N	−7	2	1.2857143	0.467742	0.532258	−76	387	−463

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\hat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
125	5^3	N	Y	-2	0	2.0000000	0.464000	0.536000	-78	387	-465
126	$2^1 3^2 7^1$	N	N	30	14	1.1666667	0.468254	0.531746	-48	417	-465
127	127^1	Y	Y	-2	0	1.0000000	0.464567	0.535433	-50	417	-467
128	2^7	N	Y	-2	0	4.0000000	0.460938	0.539062	-52	417	-469
129	$3^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	-47	422	-469
130	$2^1 5^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-63	422	-485
131	131^1	Y	Y	-2	0	1.0000000	0.458015	0.541985	-65	422	-487
132	$2^2 3^1 11^1$	N	N	30	14	1.1666667	0.462121	0.537879	-35	452	-487
133	$7^1 19^1$	Y	N	5	0	1.0000000	0.466165	0.533835	-30	457	-487
134	$2^1 67^1$	Y	N	5	0	1.0000000	0.470149	0.529851	-25	462	-487
135	$3^3 5^1$	N	N	9	4	1.5555556	0.474074	0.525926	-16	471	-487
136	$2^3 17^1$	N	N	9	4	1.5555556	0.477941	0.522059	-7	480	-487
137	137^1	Y	Y	-2	0	1.0000000	0.474453	0.525547	-9	480	-489
138	$2^1 3^1 23^1$	Y	N	-16	0	1.0000000	0.471014	0.528986	-25	480	-505
139	139^1	Y	Y	-2	0	1.0000000	0.467626	0.532374	-27	480	-507
140	$2^2 5^1 7^1$	N	N	30	14	1.1666667	0.471429	0.528571	3	510	-507
141	$3^1 47^1$	Y	N	5	0	1.0000000	0.475177	0.524823	8	515	-507
142	$2^1 71^1$	Y	N	5	0	1.0000000	0.478873	0.521127	13	520	-507
143	$11^1 13^1$	Y	N	5	0	1.0000000	0.482517	0.517483	18	525	-507
144	$2^4 3^2$	N	N	34	29	1.6176471	0.486111	0.513889	52	559	-507
145	$5^1 29^1$	Y	N	5	0	1.0000000	0.489655	0.510345	57	564	-507
146	$2^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	62	569	-507
147	$3^1 7^2$	N	N	-7	2	1.2857143	0.489796	0.510204	55	569	-514
148	$2^2 37^1$	N	N	-7	2	1.2857143	0.486486	0.513514	48	569	-521
149	149^1	Y	Y	-2	0	1.0000000	0.483221	0.516779	46	569	-523
150	$2^1 3^1 5^2$	N	N	30	14	1.1666667	0.486667	0.513333	76	599	-523
151	151^1	Y	Y	-2	0	1.0000000	0.483444	0.516556	74	599	-525
152	$2^3 19^1$	N	N	9	4	1.5555556	0.486842	0.513158	83	608	-525
153	$3^2 17^1$	N	N	-7	2	1.2857143	0.483660	0.516340	76	608	-532
154	$2^1 7^1 11^1$	Y	N	-16	0	1.0000000	0.480519	0.519481	60	608	-548
155	$5^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	65	613	-548
156	$2^2 3^1 13^1$	N	N	30	14	1.1666667	0.487179	0.512821	95	643	-548
157	157^1	Y	Y	-2	0	1.0000000	0.484076	0.515924	93	643	-550
158	$2^1 79^1$	Y	N	5	0	1.0000000	0.487342	0.512658	98	648	-550
159	$3^1 53^1$	Y	N	5	0	1.0000000	0.490566	0.509434	103	653	-550
160	$2^5 5^1$	N	N	13	8	2.0769231	0.493750	0.506250	116	666	-550
161	$7^1 23^1$	Y	N	5	0	1.0000000	0.496894	0.503106	121	671	-550
162	$2^1 3^4$	N	N	-11	6	1.8181818	0.493827	0.506173	110	671	-561
163	163^1	Y	Y	-2	0	1.0000000	0.490798	0.509202	108	671	-563
164	$2^2 41^1$	N	N	-7	2	1.2857143	0.487805	0.512195	101	671	-570
165	$3^1 5^1 11^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	85	671	-586
166	$2^1 83^1$	Y	N	5	0	1.0000000	0.487952	0.512048	90	676	-586
167	167^1	Y	Y	-2	0	1.0000000	0.485030	0.514970	88	676	-588
168	$2^3 3^1 7^1$	N	N	-48	32	1.3333333	0.482143	0.517857	40	676	-636
169	13^2	N	Y	2	0	1.5000000	0.485207	0.514793	42	678	-636
170	$2^1 5^1 17^1$	Y	N	-16	0	1.0000000	0.482353	0.517647	26	678	-652
171	$3^2 19^1$	N	N	-7	2	1.2857143	0.479532	0.520468	19	678	-659
172	$2^2 43^1$	N	N	-7	2	1.2857143	0.476744	0.523256	12	678	-666
173	173^1	Y	Y	-2	0	1.0000000	0.473988	0.526012	10	678	-668
174	$2^1 3^1 29^1$	Y	N	-16	0	1.0000000	0.471264	0.528736	-6	678	-684
175	$5^2 7^1$	N	N	-7	2	1.2857143	0.468571	0.531429	-13	678	-691
176	$2^4 11^1$	N	N	-11	6	1.8181818	0.465909	0.534091	-24	678	-702
177	$3^1 59^1$	Y	N	5	0	1.0000000	0.468927	0.531073	-19	683	-702
178	$2^1 89^1$	Y	N	5	0	1.0000000	0.471910	0.528090	-14	688	-702
179	179^1	Y	Y	-2	0	1.0000000	0.469274	0.530726	-16	688	-704
180	$2^2 3^2 5^1$	N	N	-74	58	1.2162162	0.466667	0.533333	-90	688	-778
181	181^1	Y	Y	-2	0	1.0000000	0.464088	0.535912	-92	688	-780
182	$2^1 7^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-108	688	-796
183	$3^1 61^1$	Y	N	5	0	1.0000000	0.464481	0.535519	-103	693	-796
184	$2^3 23^1$	N	N	9	4	1.5555556	0.467391	0.532609	-94	702	-796
185	$5^1 37^1$	Y	N	5	0	1.0000000	0.470270	0.529730	-89	707	-796
186	$2^1 3^1 31^1$	Y	N	-16	0	1.0000000	0.467742	0.532258	-105	707	-812
187	$11^1 17^1$	Y	N	5	0	1.0000000	0.470588	0.529412	-100	712	-812
188	$2^2 47^1$	N	N	-7	2	1.2857143	0.468085	0.531915	-107	712	-819
189	$3^3 7^1$	N	N	9	4	1.5555556	0.470899	0.529101	-98	721	-819
190	$2^1 5^1 19^1$	Y	N	-16	0	1.0000000	0.468421	0.531579	-114	721	-835
191	191^1	Y	Y	-2	0	1.0000000	0.465969	0.534031	-116	721	-837
192	$2^6 3^1$	N	N	-15	10	2.3333333	0.463542	0.536458	-131	721	-852
193	193^1	Y	Y	-2	0	1.0000000	0.461140	0.538860	-133	721	-854
194	$2^1 97^1$	Y	N	5	0	1.0000000	0.463918	0.536082	-128	726	-854
195	$3^1 5^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-144	726	-870
196	$2^2 7^2$	N	N	14	9	1.3571429	0.464286	0.535714	-130	740	-870
197	197^1	Y	Y	-2	0	1.0000000	0.461929	0.538071	-132	740	-872
198	$2^1 3^2 11^1$	N	N	30	14	1.1666667	0.464646	0.535354	-102	770	-872
199	199^1	Y	Y	-2	0	1.0000000	0.462312	0.537688	-104	770	-874
200	$2^3 5^2$	N	N	-23	18	1.4782609	0.460000	0.540000	-127	770	-897

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\hat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
201	$3^1 67^1$	Y	N	5	0	1.0000000	0.462687	0.537313	−122	775	−897
202	$2^1 101^1$	Y	N	5	0	1.0000000	0.465347	0.534653	−117	780	−897
203	$7^1 29^1$	Y	N	5	0	1.0000000	0.467980	0.532020	−112	785	−897
204	$2^2 3^1 17^1$	N	N	30	14	1.1666667	0.470588	0.529412	−82	815	−897
205	$5^1 41^1$	Y	N	5	0	1.0000000	0.473171	0.526829	−77	820	−897
206	$2^1 103^1$	Y	N	5	0	1.0000000	0.475728	0.524272	−72	825	−897
207	$3^2 23^1$	N	N	−7	2	1.2857143	0.473430	0.526570	−79	825	−904
208	$2^4 13^1$	N	N	−11	6	1.8181818	0.471154	0.528846	−90	825	−915
209	$11^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	−85	830	−915
210	$2^1 3^1 5^1 7^1$	Y	N	65	0	1.0000000	0.476190	0.523810	−20	895	−915
211	211^1	Y	Y	−2	0	1.0000000	0.473934	0.526066	−22	895	−917
212	$2^2 53^1$	N	N	−7	2	1.2857143	0.471698	0.528302	−29	895	−924
213	$3^1 71^1$	Y	N	5	0	1.0000000	0.474178	0.525822	−24	900	−924
214	$2^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−19	905	−924
215	$5^1 43^1$	Y	N	5	0	1.0000000	0.479070	0.520930	−14	910	−924
216	$2^3 3^3$	N	N	46	41	1.5000000	0.481481	0.518519	32	956	−924
217	$7^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	37	961	−924
218	$2^1 109^1$	Y	N	5	0	1.0000000	0.486239	0.513761	42	966	−924
219	$3^1 73^1$	Y	N	5	0	1.0000000	0.488584	0.511416	47	971	−924
220	$2^2 5^1 11^1$	N	N	30	14	1.1666667	0.490909	0.509091	77	1001	−924
221	$13^1 17^1$	Y	N	5	0	1.0000000	0.493213	0.506787	82	1006	−924
222	$2^1 3^1 37^1$	Y	N	−16	0	1.0000000	0.490991	0.509009	66	1006	−940
223	223^1	Y	Y	−2	0	1.0000000	0.488789	0.511211	64	1006	−942
224	$2^5 7^1$	N	N	13	8	2.0769231	0.491071	0.508929	77	1019	−942
225	$3^2 5^2$	N	N	14	9	1.3571429	0.493333	0.506667	91	1033	−942
226	$2^1 113^1$	Y	N	5	0	1.0000000	0.495575	0.504425	96	1038	−942
227	227^1	Y	Y	−2	0	1.0000000	0.493392	0.506608	94	1038	−944
228	$2^2 3^1 19^1$	N	N	30	14	1.1666667	0.495614	0.504386	124	1068	−944
229	229^1	Y	Y	−2	0	1.0000000	0.493450	0.506550	122	1068	−946
230	$2^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.491304	0.508696	106	1068	−962
231	$3^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.489177	0.510823	90	1068	−978
232	$2^3 29^1$	N	N	9	4	1.5555556	0.491379	0.508621	99	1077	−978
233	233^1	Y	Y	−2	0	1.0000000	0.489270	0.510730	97	1077	−980
234	$2^1 3^2 13^1$	N	N	30	14	1.1666667	0.491453	0.508547	127	1107	−980
235	$5^1 47^1$	Y	N	5	0	1.0000000	0.493617	0.506383	132	1112	−980
236	$2^2 59^1$	N	N	−7	2	1.2857143	0.491525	0.508475	125	1112	−987
237	$3^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	130	1117	−987
238	$2^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.491597	0.508403	114	1117	−1003
239	239^1	Y	Y	−2	0	1.0000000	0.489540	0.510460	112	1117	−1005
240	$2^4 3^1 5^1$	N	N	70	54	1.5000000	0.491667	0.508333	182	1187	−1005
241	241^1	Y	Y	−2	0	1.0000000	0.489627	0.510373	180	1187	−1007
242	$2^1 11^2$	N	N	−7	2	1.2857143	0.487603	0.512397	173	1187	−1014
243	3^5	N	Y	−2	0	3.0000000	0.485597	0.514403	171	1187	−1016
244	$2^2 61^1$	N	N	−7	2	1.2857143	0.483607	0.516393	164	1187	−1023
245	$5^1 7^2$	N	N	−7	2	1.2857143	0.481633	0.518367	157	1187	−1030
246	$2^1 3^1 41^1$	Y	N	−16	0	1.0000000	0.479675	0.520325	141	1187	−1046
247	$13^1 19^1$	Y	N	5	0	1.0000000	0.481781	0.518219	146	1192	−1046
248	$2^3 31^1$	N	N	9	4	1.5555556	0.483871	0.516129	155	1201	−1046
249	$3^1 83^1$	Y	N	5	0	1.0000000	0.485944	0.514056	160	1206	−1046
250	$2^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	169	1215	−1046
251	251^1	Y	Y	−2	0	1.0000000	0.486056	0.513944	167	1215	−1048
252	$2^2 3^2 7^1$	N	N	−74	58	1.2162162	0.484127	0.515873	93	1215	−1122
253	$11^1 23^1$	Y	N	5	0	1.0000000	0.486166	0.513834	98	1220	−1122
254	$2^1 127^1$	Y	N	5	0	1.0000000	0.488189	0.511811	103	1225	−1122
255	$3^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.486275	0.513725	87	1225	−1138
256	2^8	N	Y	2	0	4.5000000	0.488281	0.511719	89	1227	−1138
257	257^1	Y	Y	−2	0	1.0000000	0.486381	0.513619	87	1227	−1140
258	$2^1 3^1 43^1$	Y	N	−16	0	1.0000000	0.484496	0.515504	71	1227	−1156
259	$7^1 37^1$	Y	N	5	0	1.0000000	0.486486	0.513514	76	1232	−1156
260	$2^2 5^1 13^1$	N	N	30	14	1.1666667	0.488462	0.511538	106	1262	−1156
261	$3^2 29^1$	N	N	−7	2	1.2857143	0.486590	0.513410	99	1262	−1163
262	$2^1 131^1$	Y	N	5	0	1.0000000	0.488550	0.511450	104	1267	−1163
263	263^1	Y	Y	−2	0	1.0000000	0.486692	0.513308	102	1267	−1165
264	$2^3 3^1 11^1$	N	N	−48	32	1.3333333	0.484848	0.515152	54	1267	−1213
265	$5^1 53^1$	Y	N	5	0	1.0000000	0.486792	0.513208	59	1272	−1213
266	$2^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.484962	0.515038	43	1272	−1229
267	$3^1 89^1$	Y	N	5	0	1.0000000	0.486891	0.513109	48	1277	−1229
268	$2^2 67^1$	N	N	−7	2	1.2857143	0.485075	0.514925	41	1277	−1236
269	269^1	Y	Y	−2	0	1.0000000	0.483271	0.516729	39	1277	−1238
270	$2^1 3^3 5^1$	N	N	−48	32	1.3333333	0.481481	0.518519	−9	1277	−1286
271	271^1	Y	Y	−2	0	1.0000000	0.479705	0.520295	−11	1277	−1288
272	$2^4 17^1$	N	N	−11	6	1.8181818	0.477941	0.522059	−22	1277	−1299
273	$3^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−38	1277	−1315
274	$2^1 137^1$	Y	N	5	0	1.0000000	0.478102	0.521898	−33	1282	−1315
275	$5^2 11^1$	N	N	−7	2	1.2857143	0.476364	0.523636	−40	1282	−1322
276	$2^2 3^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−10	1312	−1322
277	277^1	Y	Y	−2	0	1.0000000	0.476534	0.523466	−12	1312	−1324

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
278	$2^1 139^1$	Y	N	5	0	1.0000000	0.478417	0.521583	−7	1317	−1324
279	$3^2 31^1$	N	N	−7	2	1.2857143	0.476703	0.523297	−14	1317	−1331
280	$2^3 5^1 7^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−62	1317	−1379
281	281^1	Y	Y	−2	0	1.0000000	0.473310	0.526690	−64	1317	−1381
282	$2^1 3^1 47^1$	Y	N	−16	0	1.0000000	0.471631	0.528369	−80	1317	−1397
283	283^1	Y	Y	−2	0	1.0000000	0.469965	0.530035	−82	1317	−1399
284	$2^2 71^1$	N	N	−7	2	1.2857143	0.468310	0.531690	−89	1317	−1406
285	$3^1 5^1 19^1$	Y	N	−16	0	1.0000000	0.466667	0.533333	−105	1317	−1422
286	$2^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.465035	0.534965	−121	1317	−1438
287	$7^1 41^1$	Y	N	5	0	1.0000000	0.466899	0.533101	−116	1322	−1438
288	$2^5 3^2$	N	N	−47	42	1.7659574	0.465278	0.534722	−163	1322	−1485
289	17^2	N	Y	2	0	1.5000000	0.467128	0.532872	−161	1324	−1485
290	$2^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.465517	0.534483	−177	1324	−1501
291	$3^1 97^1$	Y	N	5	0	1.0000000	0.467354	0.532646	−172	1329	−1501
292	$2^2 73^1$	N	N	−7	2	1.2857143	0.465753	0.534247	−179	1329	−1508
293	293^1	Y	Y	−2	0	1.0000000	0.464164	0.535836	−181	1329	−1510
294	$2^1 3^1 7^2$	N	N	30	14	1.1666667	0.465986	0.534014	−151	1359	−1510
295	$5^1 59^1$	Y	N	5	0	1.0000000	0.467797	0.532203	−146	1364	−1510
296	$2^3 37^1$	N	N	9	4	1.5555556	0.469595	0.530405	−137	1373	−1510
297	$3^3 11^1$	N	N	9	4	1.5555556	0.471380	0.528620	−128	1382	−1510
298	$2^1 149^1$	Y	N	5	0	1.0000000	0.473154	0.526846	−123	1387	−1510
299	$13^1 23^1$	Y	N	5	0	1.0000000	0.474916	0.525084	−118	1392	−1510
300	$2^2 3^1 5^2$	N	N	−74	58	1.2162162	0.473333	0.526667	−192	1392	−1584
301	$7^1 43^1$	Y	N	5	0	1.0000000	0.475083	0.524917	−187	1397	−1584
302	$2^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−182	1402	−1584
303	$3^1 101^1$	Y	N	5	0	1.0000000	0.478548	0.521452	−177	1407	−1584
304	$2^4 19^1$	N	N	−11	6	1.8181818	0.476974	0.523026	−188	1407	−1595
305	$5^1 61^1$	Y	N	5	0	1.0000000	0.478689	0.521311	−183	1412	−1595
306	$2^1 3^2 17^1$	N	N	30	14	1.1666667	0.480392	0.519608	−153	1442	−1595
307	307^1	Y	Y	−2	0	1.0000000	0.478827	0.521173	−155	1442	−1597
308	$2^2 7^1 11^1$	N	N	30	14	1.1666667	0.480519	0.519481	−125	1472	−1597
309	$3^1 103^1$	Y	N	5	0	1.0000000	0.482201	0.517799	−120	1477	−1597
310	$2^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.480645	0.519355	−136	1477	−1613
311	311^1	Y	Y	−2	0	1.0000000	0.479100	0.520900	−138	1477	−1615
312	$2^3 3^1 13^1$	N	N	−48	32	1.3333333	0.477564	0.522436	−186	1477	−1663
313	313^1	Y	Y	−2	0	1.0000000	0.476038	0.523962	−188	1477	−1665
314	$2^1 157^1$	Y	N	5	0	1.0000000	0.477707	0.522293	−183	1482	−1665
315	$3^2 5^1 7^1$	N	N	30	14	1.1666667	0.479365	0.520635	−153	1512	−1665
316	$2^2 79^1$	N	N	−7	2	1.2857143	0.477848	0.522152	−160	1512	−1672
317	317^1	Y	Y	−2	0	1.0000000	0.476341	0.523659	−162	1512	−1674
318	$2^1 3^1 53^1$	Y	N	−16	0	1.0000000	0.474843	0.525157	−178	1512	−1690
319	$11^1 29^1$	Y	N	5	0	1.0000000	0.476489	0.523511	−173	1517	−1690
320	$2^6 5^1$	N	N	−15	10	2.3333333	0.475000	0.525000	−188	1517	−1705
321	$3^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−183	1522	−1705
322	$2^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.475155	0.524845	−199	1522	−1721
323	$17^1 19^1$	Y	N	5	0	1.0000000	0.476780	0.523220	−194	1527	−1721
324	$2^2 3^4$	N	N	34	29	1.6176471	0.478395	0.521605	−160	1561	−1721
325	$5^2 13^1$	N	N	−7	2	1.2857143	0.476923	0.523077	−167	1561	−1728
326	$2^1 163^1$	Y	N	5	0	1.0000000	0.478528	0.521472	−162	1566	−1728
327	$3^1 109^1$	Y	N	5	0	1.0000000	0.480122	0.519878	−157	1571	−1728
328	$2^3 41^1$	N	N	9	4	1.5555556	0.481707	0.518293	−148	1580	−1728
329	$7^1 47^1$	Y	N	5	0	1.0000000	0.483283	0.516717	−143	1585	−1728
330	$2^1 3^1 5^1 11^1$	Y	N	65	0	1.0000000	0.484848	0.515152	−78	1650	−1728
331	331^1	Y	Y	−2	0	1.0000000	0.483384	0.516616	−80	1650	−1730
332	$2^2 83^1$	N	N	−7	2	1.2857143	0.481928	0.518072	−87	1650	−1737
333	$3^2 37^1$	N	N	−7	2	1.2857143	0.480480	0.519520	−94	1650	−1744
334	$2^1 167^1$	Y	N	5	0	1.0000000	0.482036	0.517964	−89	1655	−1744
335	$5^1 67^1$	Y	N	5	0	1.0000000	0.483582	0.516418	−84	1660	−1744
336	$2^4 3^1 7^1$	N	N	70	54	1.5000000	0.485119	0.514881	−14	1730	−1744
337	337^1	Y	Y	−2	0	1.0000000	0.483680	0.516320	−16	1730	−1746
338	$2^1 13^2$	N	N	−7	2	1.2857143	0.482249	0.517751	−23	1730	−1753
339	$3^1 113^1$	Y	N	5	0	1.0000000	0.483776	0.516224	−18	1735	−1753
340	$2^2 5^1 17^1$	N	N	30	14	1.1666667	0.485294	0.514706	12	1765	−1753
341	$11^1 31^1$	Y	N	5	0	1.0000000	0.486804	0.513196	17	1770	−1753
342	$2^1 3^2 19^1$	N	N	30	14	1.1666667	0.488304	0.511696	47	1800	−1753
343	7^3	N	Y	−2	0	2.0000000	0.486880	0.513120	45	1800	−1755
344	$2^3 43^1$	N	N	9	4	1.5555556	0.488372	0.511628	54	1809	−1755
345	$3^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.486957	0.513043	38	1809	−1771
346	$2^1 173^1$	Y	N	5	0	1.0000000	0.488439	0.511561	43	1814	−1771
347	347^1	Y	Y	−2	0	1.0000000	0.487032	0.512968	41	1814	−1773
348	$2^2 3^1 29^1$	N	N	30	14	1.1666667	0.488506	0.511494	71	1844	−1773
349	349^1	Y	Y	−2	0	1.0000000	0.487106	0.512894	69	1844	−1775
350	$2^1 5^2 7^1$	N	N	30	14	1.1666667	0.488571	0.511429	99	1874	−1775

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
351	$3^3 13^1$	N	N	9	4	1.5555556	0.490028	0.509972	108	1883	−1775
352	$2^5 11^1$	N	N	13	8	2.0769231	0.491477	0.508523	121	1896	−1775
353	353^1	Y	Y	−2	0	1.0000000	0.490085	0.509915	119	1896	−1777
354	$2^1 3^1 59^1$	Y	N	−16	0	1.0000000	0.488701	0.511299	103	1896	−1793
355	$5^1 71^1$	Y	N	5	0	1.0000000	0.490141	0.509859	108	1901	−1793
356	$2^2 89^1$	N	N	−7	2	1.2857143	0.488764	0.511236	101	1901	−1800
357	$3^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.487395	0.512605	85	1901	−1816
358	$2^1 179^1$	Y	N	5	0	1.0000000	0.488827	0.511173	90	1906	−1816
359	359^1	Y	Y	−2	0	1.0000000	0.487465	0.512535	88	1906	−1818
360	$2^3 3^2 5^1$	N	N	145	129	1.3034483	0.488889	0.511111	233	2051	−1818
361	19^2	N	Y	2	0	1.5000000	0.490305	0.509695	235	2053	−1818
362	$2^1 181^1$	Y	N	5	0	1.0000000	0.491713	0.508287	240	2058	−1818
363	$3^1 11^2$	N	N	−7	2	1.2857143	0.490358	0.509642	233	2058	−1825
364	$2^2 7^1 13^1$	N	N	30	14	1.1666667	0.491758	0.508242	263	2088	−1825
365	$5^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	268	2093	−1825
366	$2^1 3^1 61^1$	Y	N	−16	0	1.0000000	0.491803	0.508197	252	2093	−1841
367	367^1	Y	Y	−2	0	1.0000000	0.490463	0.509537	250	2093	−1843
368	$2^4 23^1$	N	N	−11	6	1.8181818	0.489130	0.510870	239	2093	−1854
369	$3^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	232	2093	−1861
370	$2^1 5^1 37^1$	Y	N	−16	0	1.0000000	0.486486	0.513514	216	2093	−1877
371	$7^1 53^1$	Y	N	5	0	1.0000000	0.487871	0.512129	221	2098	−1877
372	$2^2 3^1 31^1$	N	N	30	14	1.1666667	0.489247	0.510753	251	2128	−1877
373	373^1	Y	Y	−2	0	1.0000000	0.487936	0.512064	249	2128	−1879
374	$2^1 11^1 17^1$	Y	N	−16	0	1.0000000	0.486631	0.513369	233	2128	−1895
375	$3^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	242	2137	−1895
376	$2^3 47^1$	N	N	9	4	1.5555556	0.489362	0.510638	251	2146	−1895
377	$13^1 29^1$	Y	N	5	0	1.0000000	0.490716	0.509284	256	2151	−1895
378	$2^1 3^3 7^1$	N	N	−48	32	1.3333333	0.489418	0.510582	208	2151	−1943
379	379^1	Y	Y	−2	0	1.0000000	0.488127	0.511873	206	2151	−1945
380	$2^2 5^1 19^1$	N	N	30	14	1.1666667	0.489474	0.510526	236	2181	−1945
381	$3^1 127^1$	Y	N	5	0	1.0000000	0.490814	0.509186	241	2186	−1945
382	$2^1 191^1$	Y	N	5	0	1.0000000	0.492147	0.507853	246	2191	−1945
383	383^1	Y	Y	−2	0	1.0000000	0.490862	0.509138	244	2191	−1947
384	$2^7 3^1$	N	N	17	12	2.5882353	0.492188	0.507812	261	2208	−1947
385	$5^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.490909	0.509091	245	2208	−1963
386	$2^1 193^1$	Y	N	5	0	1.0000000	0.492228	0.507772	250	2213	−1963
387	$3^2 43^1$	N	N	−7	2	1.2857143	0.490956	0.509044	243	2213	−1970
388	$2^2 97^1$	N	N	−7	2	1.2857143	0.489691	0.510309	236	2213	−1977
389	389^1	Y	Y	−2	0	1.0000000	0.488432	0.511568	234	2213	−1979
390	$2^1 3^1 5^1 13^1$	Y	N	65	0	1.0000000	0.489744	0.510256	299	2278	−1979
391	$17^1 23^1$	Y	N	5	0	1.0000000	0.491049	0.508951	304	2283	−1979
392	$2^3 7^2$	N	N	−23	18	1.4782609	0.489796	0.510204	281	2283	−2002
393	$3^1 131^1$	Y	N	5	0	1.0000000	0.491094	0.508906	286	2288	−2002
394	$2^1 197^1$	Y	N	5	0	1.0000000	0.492386	0.507614	291	2293	−2002
395	$5^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	296	2298	−2002
396	$2^2 3^2 11^1$	N	N	−74	58	1.2162162	0.492424	0.507576	222	2298	−2076
397	397^1	Y	Y	−2	0	1.0000000	0.491184	0.508816	220	2298	−2078
398	$2^1 199^1$	Y	N	5	0	1.0000000	0.492462	0.507538	225	2303	−2078
399	$3^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.491228	0.508772	209	2303	−2094
400	$2^4 5^2$	N	N	34	29	1.6176471	0.492500	0.507500	243	2337	−2094
401	401^1	Y	Y	−2	0	1.0000000	0.491272	0.508728	241	2337	−2096
402	$2^1 3^1 67^1$	Y	N	−16	0	1.0000000	0.490050	0.509950	225	2337	−2112
403	$13^1 31^1$	Y	N	5	0	1.0000000	0.491315	0.508685	230	2342	−2112
404	$2^2 101^1$	N	N	−7	2	1.2857143	0.490099	0.509901	223	2342	−2119
405	$3^4 5^1$	N	N	−11	6	1.8181818	0.488889	0.511111	212	2342	−2130
406	$2^1 7^1 29^1$	Y	N	−16	0	1.0000000	0.487685	0.512315	196	2342	−2146
407	$11^1 37^1$	Y	N	5	0	1.0000000	0.488943	0.511057	201	2347	−2146
408	$2^3 3^1 17^1$	N	N	−48	32	1.3333333	0.487745	0.512255	153	2347	−2194
409	409^1	Y	Y	−2	0	1.0000000	0.486553	0.513447	151	2347	−2196
410	$2^1 5^1 41^1$	Y	N	−16	0	1.0000000	0.485366	0.514634	135	2347	−2212
411	$3^1 137^1$	Y	N	5	0	1.0000000	0.486618	0.513382	140	2352	−2212
412	$2^2 103^1$	N	N	−7	2	1.2857143	0.485437	0.514563	133	2352	−2219
413	$7^1 59^1$	Y	N	5	0	1.0000000	0.486683	0.513317	138	2357	−2219
414	$2^1 3^2 23^1$	N	N	30	14	1.1666667	0.487923	0.512077	168	2387	−2219
415	$5^1 83^1$	Y	N	5	0	1.0000000	0.489157	0.510843	173	2392	−2219
416	$2^5 13^1$	N	N	13	8	2.0769231	0.490385	0.509615	186	2405	−2219
417	$3^1 139^1$	Y	N	5	0	1.0000000	0.491607	0.508393	191	2410	−2219
418	$2^1 11^1 19^1$	Y	N	−16	0	1.0000000	0.490431	0.509569	175	2410	−2235
419	419^1	Y	Y	−2	0	1.0000000	0.489260	0.510740	173	2410	−2237
420	$2^2 3^1 5^1 7^1$	N	N	−155	90	1.1032258	0.488095	0.511905	18	2410	−2392
421	421^1	Y	Y	−2	0	1.0000000	0.486936	0.513064	16	2410	−2394
422	$2^1 211^1$	Y	N	5	0	1.0000000	0.488152	0.511848	21	2415	−2394
423	$3^2 47^1$	N	N	−7	2	1.2857143	0.486998	0.513002	14	2415	−2401
424	$2^3 53^1$	N	N	9	4	1.5555556	0.488208	0.511792	23	2424	−2401
425	$5^2 17^1$	N	N	−7	2	1.2857143	0.487059	0.512941	16	2424	−2408

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
426	$2^1 3^1 71^1$	Y	N	−16	0	1.0000000	0.485915	0.514085	0	2424	−2424
427	$7^1 61^1$	Y	N	5	0	1.0000000	0.487119	0.512881	5	2429	−2424
428	$2^2 107^1$	N	N	−7	2	1.2857143	0.485981	0.514019	−2	2429	−2431
429	$3^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	−18	2429	−2447
430	$2^1 5^1 43^1$	Y	N	−16	0	1.0000000	0.483721	0.516279	−34	2429	−2463
431	431^1	Y	Y	−2	0	1.0000000	0.482599	0.517401	−36	2429	−2465
432	$2^4 3^3$	N	N	−80	75	1.5625000	0.481481	0.518519	−116	2429	−2545
433	433^1	Y	Y	−2	0	1.0000000	0.480370	0.519630	−118	2429	−2547
434	$2^1 7^1 31^1$	Y	N	−16	0	1.0000000	0.479263	0.520737	−134	2429	−2563
435	$3^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.478161	0.521839	−150	2429	−2579
436	$2^2 109^1$	N	N	−7	2	1.2857143	0.477064	0.522936	−157	2429	−2586
437	$19^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	−152	2434	−2586
438	$2^1 3^1 73^1$	Y	N	−16	0	1.0000000	0.477169	0.522831	−168	2434	−2602
439	439^1	Y	Y	−2	0	1.0000000	0.476082	0.523918	−170	2434	−2604
440	$2^3 5^1 11^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−218	2434	−2652
441	$3^2 7^2$	N	N	14	9	1.3571429	0.476190	0.523810	−204	2448	−2652
442	$2^1 13^1 17^1$	Y	N	−16	0	1.0000000	0.475113	0.524887	−220	2448	−2668
443	443^1	Y	Y	−2	0	1.0000000	0.474041	0.525959	−222	2448	−2670
444	$2^2 3^1 37^1$	N	N	30	14	1.1666667	0.475225	0.524775	−192	2478	−2670
445	$5^1 89^1$	Y	N	5	0	1.0000000	0.476404	0.523596	−187	2483	−2670
446	$2^1 223^1$	Y	N	5	0	1.0000000	0.477578	0.522422	−182	2488	−2670
447	$3^1 149^1$	Y	N	5	0	1.0000000	0.478747	0.521253	−177	2493	−2670
448	$2^6 7^1$	N	N	−15	10	2.3333333	0.477679	0.522321	−192	2493	−2685
449	449^1	Y	Y	−2	0	1.0000000	0.476615	0.523385	−194	2493	−2687
450	$2^1 3^2 5^2$	N	N	−74	58	1.2162162	0.475556	0.524444	−268	2493	−2761
451	$11^1 41^1$	Y	N	5	0	1.0000000	0.476718	0.523282	−263	2498	−2761
452	$2^2 113^1$	N	N	−7	2	1.2857143	0.475664	0.524336	−270	2498	−2768
453	$3^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−265	2503	−2768
454	$2^1 227^1$	Y	N	5	0	1.0000000	0.477974	0.522026	−260	2508	−2768
455	$5^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476923	0.523077	−276	2508	−2784
456	$2^3 3^1 19^1$	N	N	−48	32	1.3333333	0.475877	0.524123	−324	2508	−2832
457	457^1	Y	Y	−2	0	1.0000000	0.474836	0.525164	−326	2508	−2834
458	$2^1 229^1$	Y	N	5	0	1.0000000	0.475983	0.524017	−321	2513	−2834
459	$3^3 17^1$	N	N	9	4	1.5555556	0.477124	0.522876	−312	2522	−2834
460	$2^2 5^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−282	2552	−2834
461	461^1	Y	Y	−2	0	1.0000000	0.477223	0.522777	−284	2552	−2836
462	$2^1 3^1 7^1 11^1$	Y	N	65	0	1.0000000	0.478355	0.521645	−219	2617	−2836
463	463^1	Y	Y	−2	0	1.0000000	0.477322	0.522678	−221	2617	−2838
464	$2^4 29^1$	N	N	−11	6	1.8181818	0.476293	0.523707	−232	2617	−2849
465	$3^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.475269	0.524731	−248	2617	−2865
466	$2^1 233^1$	Y	N	5	0	1.0000000	0.476395	0.523605	−243	2622	−2865
467	467^1	Y	Y	−2	0	1.0000000	0.475375	0.524625	−245	2622	−2867
468	$2^2 3^2 13^1$	N	N	−74	58	1.2162162	0.474359	0.525641	−319	2622	−2941
469	$7^1 67^1$	Y	N	5	0	1.0000000	0.475480	0.524520	−314	2627	−2941
470	$2^1 5^1 47^1$	Y	N	−16	0	1.0000000	0.474468	0.525532	−330	2627	−2957
471	$3^1 157^1$	Y	N	5	0	1.0000000	0.475584	0.524416	−325	2632	−2957
472	$2^3 59^1$	N	N	9	4	1.5555556	0.476695	0.523305	−316	2641	−2957
473	$11^1 43^1$	Y	N	5	0	1.0000000	0.477801	0.522199	−311	2646	−2957
474	$2^1 3^1 79^1$	Y	N	−16	0	1.0000000	0.476793	0.523207	−327	2646	−2973
475	$5^2 19^1$	N	N	−7	2	1.2857143	0.475789	0.524211	−334	2646	−2980
476	$2^2 7^1 17^1$	N	N	30	14	1.1666667	0.476891	0.523109	−304	2676	−2980
477	$3^2 53^1$	N	N	−7	2	1.2857143	0.475891	0.524109	−311	2676	−2987
478	$2^1 239^1$	Y	N	5	0	1.0000000	0.476987	0.523013	−306	2681	−2987
479	479^1	Y	Y	−2	0	1.0000000	0.475992	0.524008	−308	2681	−2989
480	$2^5 3^1 5^1$	N	N	−96	80	1.6666667	0.475000	0.525000	−404	2681	−3085
481	$13^1 37^1$	Y	N	5	0	1.0000000	0.476091	0.523909	−399	2686	−3085
482	$2^1 241^1$	Y	N	5	0	1.0000000	0.477178	0.522822	−394	2691	−3085
483	$3^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−410	2691	−3101
484	$2^2 11^2$	N	N	14	9	1.3571429	0.477273	0.522727	−396	2705	−3101
485	$5^1 97^1$	Y	N	5	0	1.0000000	0.478351	0.521649	−391	2710	−3101
486	$2^1 3^5$	N	N	13	8	2.0769231	0.479424	0.520576	−378	2723	−3101
487	487^1	Y	Y	−2	0	1.0000000	0.478439	0.521561	−380	2723	−3103
488	$2^3 61^1$	N	N	9	4	1.5555556	0.479508	0.520492	−371	2732	−3103
489	$3^1 163^1$	Y	N	5	0	1.0000000	0.480573	0.519427	−366	2737	−3103
490	$2^1 5^1 7^2$	N	N	30	14	1.1666667	0.481633	0.518367	−336	2767	−3103
491	491^1	Y	Y	−2	0	1.0000000	0.480652	0.519348	−338	2767	−3105
492	$2^2 3^1 41^1$	N	N	30	14	1.1666667	0.481707	0.518293	−308	2797	−3105
493	$17^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	−303	2802	−3105
494	$2^1 13^1 19^1$	Y	N	−16	0	1.0000000	0.481781	0.518219	−319	2802	−3121
495	$3^2 5^1 11^1$	N	N	30	14	1.1666667	0.482828	0.517172	−289	2832	−3121
496	$2^4 31^1$	N	N	−11	6	1.8181818	0.481855	0.518145	−300	2832	−3132
497	$7^1 71^1$	Y	N	5	0	1.0000000	0.482897	0.517103	−295	2837	−3132
498	$2^1 3^1 83^1$	Y	N	−16	0	1.0000000	0.481928	0.518072	−311	2837	−3148
499	499^1	Y	Y	−2	0	1.0000000	0.480962	0.519038	−313	2837	−3150
500	$2^2 5^3$	N	N	−23	18	1.4782609	0.480000	0.520000	−336	2837	−3173