(1) and (2) /: (These are similar):

For (1) /:

$$\log \prod_p \left(1+\frac{z}{p-1}\right)^{-1} \sim -z \cdot \sum_p \frac{1}{p-1} = -z \cdot \sum_p \frac{1}{p}(1+o(1))$$

$$= -z \cdot \lim_{x \to +\infty} \left(\ell\ell x + B + O\left(\frac{1}{\log x}\right)\right) \quad (*).$$

So for $|z| = r := \frac{k-1}{\ell\ell x}$, $1 \le k \le \ell\ell x$,

$$(*) = -Bz - (k-1) + o_z(1)$$

$$\Rightarrow \prod_p \left(1+\frac{z}{p-1}\right)^{-1} \in e^{o(1)-Bz}\left[1-o(1), 1\right] \asymp 1 \checkmark$$

Similarly, for (2) /: Again, $|z| := \frac{k-1}{\ell\ell x}$:

$$\prod_p \left(1-\frac{1}{p}\right)^{z} \sim \exp(-(k-1)) \cdot (1 \pm o(1))$$

$$\in [1, e] \asymp 1.$$

Again, we write $A(x) \asymp B(x)$ if $A \ll B$, and $B \ge 0$ s.t. $B \ll A$ (same order up to an abs. limiting constant factor).

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(1)+(2)+(3) $\Rightarrow$ The claim: $\hat{G}\left(\frac{k-1}{\ell\ell x}\right) \asymp e^{o(1)} \asymp 1$

$\square$