My Motivations for Research, Graduate Studies, and Future Career Goals

I want a career working on research that is both important and interesting. My particular areas of interest combine Mathematics and Computer Science with emphasis on number theory, enumerative combinatorics, and software development. I intend to pursue challenging problems in my research with cross–disciplinary applications in Mathematics, Computer Science, and other engineering sciences. I also intend to publish the results of my research on these topics in peer–reviewed journals, present the results through talks at professional conferences, and make the research broadly available for educational, teaching, and other purposes in venues such as the web.

In my sabbatical from the University of Illinois (2005–2009) I had medical issues that prevented me from continuing in formal academic studies for an extended period of time. I knew that my interests would eventually lead me to graduate studies and I pursued research offers from professors to work on their projects. Prior to my time away from a campus environment, I had never had so much free time to devote to my own software projects and personal interests in Mathematics research. During this time I was able to focus my efforts almost exclusively on my own work. It was through this experience that I found a strong passion for my own independent and original research. I have decided that I can look forward to giving the same full–time intensity to my studies required by a rigorous Ph.D. program in my fields of study. This realization has strengthened my dedication to a career as a professional researcher.

Since returning to my undergraduate studies at the University of Illinois in 2009 I have completed a dual degree program in both Computer Science in the College of Engineering and in Mathematics in the College of Liberal Arts and Sciences. In May of 2012 I was awarded both a Bachelor of Science degree in Computer Science with institutional honors of *Highest Honors* and a B.S. degree for Mathematics with institutional honors of *Cum Laude* and departmental honors of *Highest Distinction MATH*. In 2010, I received the Barry M. Goldwater scholarship. The scholarship is awarded to current undergraduate students who demonstrate strong research potential and who are committed to a Ph.D. degree in fields of Mathematics, Engineering, and other scientific disciplines. I was awarded an Honorable Mention for the 2013 NSF GRFP and have reapplied for the 2014 fellowship this year.

I am currently a graduate student in Computer Science at the University of Illinois at Urbana–Champaign where I have held a teaching assistant position in the CS 173 *Discrete Structures* course for the past three semesters. I will complete a thesis–based MS program in Computer Science in the Spring of 2014. This preparation will allow me to contribute to my fields of graduate study by providing me with experience and interdisciplinary skills that I can apply to aspects of my future graduate studies. For example, I have recently become involved with several projects in the *Illinois Geometry Lab* facility in the Mathematics department at the University of Illinois as a programming consultant and by assisting with programming tasks for projects within the lab.

I am applying to the Ph.D. program at the University of Illinois for my graduate studies for the university's strong academic programs in both Mathematics and Computer Science. I feel that the world–class research faculty in Number Theory and Combinatorics at the University of Illinois will provide me with the resources to approach new and challenging research problems as I work towards completing my Ph.D. degree in Mathematics and as my professional research career progresses.

Research Experience and Publications

I published an article in the *Journal of Integer Sequences* (JIS) in 2010 based on a significant subset of my independent Mathematics research conducted in my time away from formal academic studies over 2005–2009. The original research contained in the article primarily considers the polynomial expansions of generalized factorial functions through sets of coefficients defined by triangular recurrence relations that generalize the form of the Stirling numbers of the first kind. A review of the 54–page

JIS publication appeared on *MathSciNet* in 2011 as article `MR2659223 (2011h:05009)`. The review states that the research presented in the article is "excellent" and "will be useful for a further study in enumerative combinatorics". I presented a report talk based on a subset of the research from my *JIS* article at the *Young Mathematicians Conference* held at The Ohio State University in the Summer of 2012.

Applications of my *JIS* article research have led me to consider additional topics in my original research including (i) the continued fraction expansions of the generating functions of generalized factorial functions that formed the topic of my honors project for the *Introduction to Mathematical Research* (Math 496) course taught by Bruce Reznick in 2010, (ii) the research for my *Senior Thesis* (CS 499) course in 2011 considering further generalizations of the triangular recurrences given in the *JIS* article in the context of the partial sums of infinite zeta–function–related series of the form $\sum_n t^n/f(n)^p$, and then (iii) to eventually explore the topics of my Summer 2012 *Research Experience for Graduate Students* (REGS) research that form the foundation of my plans for future graduate research outlined in the concluding section of this essay.

I plan on submitting several other articles I have authored for publication in peer–reviewed journals in the near future. I have updated and added several new results to the continued fractions research from the 2010 Math 496 course and plan to submit the article to the *Journal of Integer Sequences.* The research that has grown out of my Summer 2012 REGS project at the University of Illinois is related to the forms of Euler sums and Dirichlet series. I will submit this research for publication in the *Online Journal of Analytic Combinatorics* (OJAC) suggested by Bruce Berndt later on this Fall. I will submit another article containing my research related to so–termed "*square series*" to the OJAC, an outlet which was also suggested as a good fit for the research by Professor Berndt. The article presents new integral representations for the transformation of the generating function for a sequence, $\langle f_n \rangle$, into a series that enumerates the sequence of $\langle q^{n^2} f_n \rangle$ for some fixed $q \in \mathbb{C}$. These results have a number of applications to series for theta functions and for generating functions that arise in enumerative combinatorics and graph theory.

I have also developed two separate open–source–software projects where the program source code is freely available on the project webpages for purposes including educational study. The topic of my Spring 2014 MS thesis at the University of Illinois is closely related to the original aim of the math–related *Math Pattern Hunter* program which attempted to find closed–form formulas and other identities for input integer and rational sequences by means of programmed and configurable brute force.

Current Research Directions and My Plans for Future Graduate Studies

For the concluding section of my personal statement, I want to provide a summary of my recent, active research interests related to properties of the Riemann zeta function, $\zeta(p) := \sum_{n \geq 1} n^{-p}$, evaluated at the odd positive integers. It is known that infinitely–many of the $\zeta(2k+1)$ constants taken over $k \in \mathbb{Z}^+$ are irrational. In 1978, R. Apéry proved that $\zeta(3)$ is irrational, though no attempts at a generalization of Apéry's original method have been successful so far. More recently, W. Zudilin proved that at least one of the constants in the set $\{\zeta(5), \zeta(7), \zeta(9), \zeta(11)\}$ is irrational. However, to date, only the concrete case of Apéry's constant, $\zeta(3)$, has been shown to be irrational. The next sections of the essay outline my prior research related to this topic, my current progress on the problem, and future directions of this research that I want to continue work on in my graduate studies at the University of Illinois.

Background and Prior Work

My research on *square series* generating function transformations starting in 2011 led me to consider the following identity given in terms of the j^{th} derivatives, $F^{(j)}(z)$, of the ordinary generating function of the sequence $\langle f_n \rangle$ and the Stirling numbers of the second kind, $\left\{ {n \atop k} \right\}$, for $k \in \mathbb{N}$:

$$\sum_{n=0}^{\infty} n^k f_n z^n = \sum_{j=0}^{k} \left\{ {k \atop j} \right\} z^j F^{(j)}(z). \tag{1}$$

In my REGS (*Research Experience for Graduate Students*) research at the University of Illinois over the Summer of 2012, I considered an analog to the formula in (1) for series of the form $\sum_n n^{-k} f_n z^n$. The special case of $f_n \equiv 1$ corresponds to the series for the polylogarithm function, $\mathrm{Li}_k(z) := \sum_{n\geq 1} n^{-k} z^n$, given by

$$\mathrm{Li}_k(z) = \sum_{j=1}^{\infty} \left\{ {k+2 \atop j} \right\}_* \frac{z^j j!}{(1-z)^{j+1}}. \tag{2}$$

I was able to find a non–triangular recurrence relation for the terms of the right–hand–side of (2) and the next recursive formula involving the r–order harmonic numbers, $H_n^{(r)} := \sum_{k=1}^{n} k^{-r}$, for computing these coefficients in closed–form at each $k \in \mathbb{N}$:

$$\left\{ {k+2 \atop j} \right\}_* = \sum_{i=0}^{k-1} \frac{H_j^{(i+1)}}{k} \left\{ {k+1-i \atop j} \right\}_* + \left(\frac{(-1)^{j-1}}{j!} \right) \cdot [k=0]_\delta. \tag{3}$$

The following examples of (3) for $k := 3, 4$ are expanded in terms of the Stirling numbers of the first kind and the harmonic number sequences as

$$\begin{aligned} \left\{ {5 \atop j} \right\}_* &= \frac{(-1)^{j-1}}{j!} \left(\left[{j+1 \atop 4} \right] \frac{1}{j!} + H_j H_j^{(2)} \right) \\ &= \frac{(-1)^{j}}{j!} \left(\left[{j+1 \atop 4} \right] \frac{1}{j!} - \frac{1}{3} \left(H_j^3 + 2H_j^{(3)} \right) \right) \\ \left\{ {6 \atop j} \right\}_* &= \frac{(-1)^{j-1}}{j!} \left(\left[{j+1 \atop 5} \right] \frac{1}{j!} + \frac{1}{2} \left(H_j^2 H_j^{(2)} + H_j^{(4)} \right) \right) \\ &= \frac{(-1)^{j}}{j!} \left(\left[{j+1 \atop 5} \right] \frac{1}{j!} - \frac{1}{12} \left(H_j^4 + 3 \left(H_j^{(2)} \right)^2 + 8H_j H_j^{(3)} \right) \right). \end{aligned} \tag{4}$$

The harmonic–number–based formulas obtained from (3) are similar in form to those for the Stirling numbers of the first kind that I first considered as an application in my 2010 *JIS* article.

A Recursive Approach to the Riemann Zeta Function Involving Harmonic Numbers

Let the function $M_{k+1}^{(d)}(n)$ be defined for $k, d, n \in \mathbb{N}$ in terms of the harmonic–number–based expansions from (3) as

$$M_k^{(d)}(n) := \sum_{j=1}^{n} \binom{n}{j} \left[\left\{ {k+2 \atop j} \right\}_* \left(\frac{(-1)^{j-1}}{j!} \right)^{-1} \right] \frac{(-1)^j}{(j+d)} \cdot \frac{(n+d)!}{n!}, \tag{5}$$

and suppose that $\mathcal{C} := \{c_0, c_1, \ldots, c_{r-1}\}$ denotes a fixed set of constants. The motivation for the results in this section is to construct recurrence relations of the form

$$c_0 H_n^{(p)} = c_1 H_n^{(p-1)} + \cdots + c_{r-1} H_n^{(p+1-r)} + \widetilde{c}_r(\mathcal{C}) H_n^{(p-r)} + \sum_{d=2}^{r} \widetilde{b}_d(\mathcal{C}) M_p^{(d)}(n) \tag{6}$$

for $p \geq 3$ and where the coefficients $\widetilde{c}_r(\mathcal{C}), \widetilde{b}_i(\mathcal{C})$ are rational linear combinations of the fixed terms in $\mathcal{C}$. Provided that $p - r \geq 2$, the limiting cases of (6) formed by letting $n \to \infty$ provide new relations between the zeta function constants, $\zeta(p)$, evaluated at positive integral $p \geq 2$, and where $\zeta(2k)$ is already well known in closed–form as a rational multiple of $(2\pi)^{2k}$ in terms of the Bernoulli numbers, B_{2k}, for all $k \in \mathbb{Z}^+$.

The approach to establishing the recurrences in the form of (6) follows by repeatedly differentiating the series in (2) where $H_n^{(r)} \equiv [z^n] \mathrm{Li}_r(z)/(1-z)$. Specific examples of the recurrences I have found using this method include the following results:

$$\begin{aligned} H_n^{(p)} &= H_n^{(p-2)} - 3M_p^{(2)}(n) + M_p^{(3)}(n) \\ 7H_n^{(p)} &= -12H_n^{(p-1)} - 6H_n^{(p-2)} - H_n^{(p-3)} - M_p^{(2)}(n) - M_p^{(4)}(n) \\ H_n^{(p)} &= 2H_n^{(p-2)} + H_n^{(p-3)} + M_p^{(2)}(n) - 4M_p^{(3)}(n) + M_p^{(4)}(n) \\ 4H_n^{(p)} &= 3H_n^{(p-2)} + H_n^{(p-4)} - 24M_p^{(2)}(n) + 28M_p^{(3)}(n) - 10M_p^{(4)}(n) + M_p^{(5)}(n). \end{aligned} \tag{7}$$

By setting $p \in \{5, 7, 9, 11\}$ in (6) and (7), the left–hand–side limits of these formulas correspond to the particular odd–indexed values of $\zeta(p)$ that I aim to study through this recursive approach to the problem.

My Current Progress, Future Plans for Work On the Problem, and Generalizations

For $k, d, n \in \mathbb{N}$ with $k \geq 1$ and $d \geq 2$, let the function $S_k^{(d)}(n)$ be defined as in (8) where the (unsigned) Stirling numbers of the first kind are denoted in the bracket notation of $\left[{n \atop k}\right]$:

$$S_k^{(d)}(n) := \sum_{j=1}^{n} \binom{n}{j} \left[{j+1 \atop k+1}\right] \frac{(-1)^j}{j! \cdot (j+d)} \cdot \frac{(n+d)!}{n!}. \tag{8}$$

As already noted in the special cases given in (4), the coefficients defined recursively by (3) can be expressed in terms of the Stirling numbers of the first kind with remainder terms involving products of harmonic numbers. This observation leads to the definition of (5) through (8) in the form of $M_k^{(d)}(n) := S_k^{(d)}(n) + T_k^{(d)}(n)$. So far, I have been able to find the following generating function for (8):

$$\sum_{n=0}^{\infty} S_k^{(d)}(n) z^n = \sum_{m=0}^{d-1} \frac{\operatorname{Log}(1-z)^{k+1+m-d} \cdot (-1)^{d-1}}{(k+1+m-d)! \cdot (1-z)^d} \times \left(\sum_{i=0}^{m} \left[{d-i \atop d-m}\right] \binom{d-1}{i}^2 (-1)^i \cdot z^{d-1-i} \cdot i! \right). \tag{9}$$

I have also been able to obtain non–linear recurrence relations for the full terms in (5) including

$$\begin{aligned} (2n+3)(n+d+1) M_k^{(d)}(n) - \left(3n^2 + (2d+9)n + 3d + 7\right) M_k^{(d)}(n+1) \\ + (n+2)^2 M_k^{(d)}(n+2) = \frac{(n+d+1)!}{(n+1)!} \left(\frac{(2n+3)}{(n+1)^k} - \frac{1}{(n+2)^{k-1}} \right). \end{aligned} \tag{10}$$

The forms of these equations suggest a number of possible approaches to studying the limiting behavior of the recurrences of the form in (6) by applying various asymptotic methods.

The non–linear recurrence relation in (10) and the sums involving squares of binomial coefficients in (9) share stylistic similarities with the original construction of Apéry's proof of the irrationality of $\zeta(3)$. The approach to the recurrence relations outlined in this essay is also easily generalized through (2) to study the properties and the irrationality of $\zeta(p)$ for other real, non–integral cases of $p > 1$. I plan to continue my research on these topics and the further generalizations of these results as a part of my graduate studies at the University of Illinois.