We have a recurrence relation between successive $C_k(n)$ values over k of the form

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} C_{k-1}\left(d \cdot p^i\right). \tag{12}$$

No proof??

We have limiting asymptotics on these functions given by the following theorem:

Theorem 3.6 (Asymptotics for the functions $C_k(n)$)**.** *For $k := 0$, we have by definition that $C_0(n) = \delta_{n,1}$. For all $k \geq 1$, we obtain that the dominant asymptotic term for $C_k(n)$ is given by*

$$\mathbb{E}[C_k(n)] = (\log\log n)^{2k-1}, \text{ as } n \to \infty.$$

Consider case of $n = p^T$, $p \in \mathbb{P}$, T very large. I assert that

$$C_k(p^T) \geq \beta_k T^{k-1}$$

Thm 3.6 suggests that it is no more than $\lesssim (\log T)^{2k-1}$

$\boxed{\text{So there is a contradiction.}}$

Use exact formula (12)

$$C_k(p^T) = \sum_{i_1=1}^{T} C_{k-1}(p^{i_1})$$

$$\vdots \sum_{i_1=1}^{T} \cdots \sum_{i_k=1}^{i_{k-1}} C_0(p^{i_{k-1}})$$

$$= \sum_{i_1=1}^{T} \cdots \sum_{i_{k-1}=1}^{i_{k-2}} 1$$

$$\geq \beta_k \, T^{k-1}.$$