Lower bounds on the Mertens function $M(x)$ along infinite subsequences for large $x \gg 2.3315 \times 10^{1656520}$

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Last Revised: Saturday 16th May, 2020 – Compiled with LaTeX2e

Abstract

The Mertens function, $M(x) = \sum_{n \leq x} \mu(n)$, is classically defined to be the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture stating that $|M(x)| < C \cdot \sqrt{x}$ with $C > 0$ for all $x \geq 1$ has a well-known disproof due to Odlyzko and té Riele given in the early 1980's by computation of non-trivial zeta function zeros in conjunction with integral formulas for expressions of $M(x)$. It is conjectured and widely believed that $M(x)/\sqrt{x}$ changes sign infinitely often and grows unbounded in the direction of both $\pm\infty$ along subsequences of integers $x \geq 1$. Our proof of a result close to this property of $M(x)/\sqrt{x}$, e.g., showing that

$$\limsup_{x \to \infty} \frac{|M(x)| \log x}{\sqrt{x}} = +\infty,$$

is not based on standard estimates of $M(x)$ by Mellin inversion, which are intimately tied to the intricate distribution of the non-trivial zeros of the Riemann zeta function. There is a distinct stylistic flavor and new element of combinatorial analysis peppered in with the standard methods from analytic and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

Keywords and Phrases: *Möbius function sums; Mertens function; summatory function; arithmetic functions; Dirichlet inverse; Liouville lambda function; prime omega functions; prime counting functions; Dirichlet series and DGFs; asymptotic lower bounds; Mertens conjecture.*

Primary Math Subject Classifications (2010): *11N37; 11A25; 11N60; 11N64; and 11-04 (TBD).*

Reference on special notation and other conventions

Symbol	**Definition**
$\mathbb{E}[f(x)]$	We use the expectaion notation $\mathbb{E}[f(x)] = h(x)$ to denote that f has a so-called average order growth rate of $h(x)$. What this means is that $\frac{1}{x}\sum_{n\leq x} f(n) \sim h(x)$, or equivalently that $$\lim_{x\to\infty} \frac{\frac{1}{x}\sum_{n\leq x} f(n)}{h(x)} = 1.$$
$o(f), O_\alpha(g)$	Using standard notation, we write that $f = o(g)$ if $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$ We adapt the stock big-O notation, writing $f = O_{\alpha_1,\ldots,\alpha_k}(g)$ for some parameters $\alpha_1,\ldots,\alpha_k$ if $f(x) = O(g(x))$ subject only to the upper bounds having an implicit dependence only on x and the α_i.
$C_k(n)$	Auxillary component functions in obtaining asymptotic bounds on $g^{-1}(n)$. These functions are defined for $k \geq 0$ and $n \geq 1$ by the formula $$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d\mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n\geq 0}$.
DGF	*Dirichlet generating function.* Given a sequence $\{f(n)\}_{n\geq 0}$, its DGF is given by $D_f(s) := \sum_{n\geq 1} f(n)/n^s$ subject to suitable constraints on the real part of the parameter $s \in \mathbb{C}$.
$\sigma_0(n), d(n)$	The ordinary divisor function, $d(n) := \sum_{d\mid n} 1$. The arithmetic functions $d(n) \equiv \sigma_0(n)$ for all $n \geq 1$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) = \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (defined below).
$f * g$	The Dirichlet convolution of f and g, $(f * g)(n) := \sum\limits_{d\mid n} f(d) g(n/d)$, where the sum is taken over the divisors d of n for $n \geq 1$.

Symbol	**Definition**
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d\mid n \\ d>1}} f(d)f^{-1}(n/d)$ for $n \geq 1$ with $f^{-1}(1) = 1/f(1)$ and exists if and only if $f(1) \neq 0$. The inverse function, when it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$.
$\lfloor x \rfloor, \lceil x \rceil$	The floor function is defined as $\lfloor x \rfloor := x - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$. The corresponding ceiling (or greatest integer) function $\lceil x \rceil$ denotes the smallest integer $m \geq x$. The ceiling function is sometimes also written as $[x] \equiv \lceil x \rceil$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega+1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$.
$\mathbb{1}_{\mathbb{S}}$, $\chi_{\mathtt{cond}(x)}$	We use the notation $\mathbb{1}, \chi : \mathbb{N} \rightarrow \{0,1\}$ to denote indicator, or characteristic functions. In paticular, $\mathbb{1}_{\mathbb{S}}(n) = 1$ if and only if $n \in \mathbb{S}$, and $\chi_{\mathtt{cond}}(n) = 1$ if and only if n satisfies the boolean-valued condition `cond`.
$\log_*^m(x)$	The iterated logarithm function defined recursively for integers $m \geq 0$ and any $x > 0$ taken so that the function is non-negative (e.g., with $x \geq e^e$ if $m = 2$, $x \geq e^{e^e}$ if $m = 3$, and so on) by $$\log_*^m(x) := \begin{cases} x, & \text{if } m = 0; \\ \log x, & \text{if } m = 1; \\ \log\left[\log_*^{m-1}(x)\right], & \text{if } m \geq 2. \end{cases}$$
$[n = k]_\delta$	Synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise.
$[\mathtt{cond}]_\delta$	For a boolean-valued conditions, `cond`, $[\mathtt{cond}]_\delta$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is sometimes called *Iverson's convention*.
$\lambda(n)$	The Liouville lambda function, $\lambda(n) := (-1)^{\Omega(n)}$, denotes the parity of $\Omega(n)$, the number of distinct prime factors of n counting their multiplicity. That is, $\lambda(n) \in \{\pm 1\}$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree, i.e., has no prime power divisors with exponent greater than one.
$M(x)$	The Mertens function is the summatory function over $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum_{n \leq x} \mu(n)$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^\alpha \| n$ (p^α exactly divides n) for p prime and $n \geq 2$.

Symbol	**Definition**
$\omega(n)$,$\Omega(n)$	We define these distinct prime factor counting functions as the strongly additive function $\omega(n) := \sum_{p\|n} 1$ and the completely additive function $\Omega(n)$ by $\Omega(n) := \sum_{p^\alpha \|\| n} \alpha$. Equivalently, if the factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leq n \leq x$ for $x > 1$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$ for $x \geq 2$. Montgomery and Vaughan use the alternate notation of $\sigma_k(x)$, which we intentionally avoid due to conflicting notation with other special arithmetic functions used in this article, in place of $\widehat{\pi}_k(x)$.
$\sum_{p \leq x}, \prod_{p \leq x}$	Unless otherwise specified by context, we use the index variable p to denote that the summation (product) is to be taken only over prime values within the summation bounds.
$P(s)$	For complex s with $\Re(s) > 1$, we define the prime zeta function to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$.
$\sigma_\alpha(n)$	The generalized sum-of-divisors function, $\sigma_\alpha(n) := \sum\limits_{d\|n} d^\alpha$, is defined for any $n \geq 1$ and $\alpha \in \mathbb{R}$.
$\sim, \approx, \gtrsim, \lesssim, \gg, \ll$	See the first section of the introduction to the article for clarification of the asymptotic notation we employ in the article including precise definitions of our usage of these limiting asymptotic relation symbols.
$\zeta(s)$	The Riemann zeta function, defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\Re(s) > 1$, and by analytic continuation to the entire complex plane with the exception of a simple pole at $s = 1$.

1 Preface: Explanations of unconventional notions and preconceptions of asymptotics and notation for asymptotic relations

We exphasize that the next itermized careful explanation of the subtle distinctions to our usage of what we consider to be traditional notation for asymptotic relations are key to understanding our choices of upper and lower bound expressions given throughout the article. Thus, to avoid any confusion that may linger as we begin to state our new results and bounds on the functions we work with in this article, we preface the article starting with this section detailing our precise definitions, meanings and assumptions on the uses of certain symbols, operators, and relations. The interpretation of this notation forms the core of how we choose to convey the growth rates of arithmetic functions on their domain of x within this article when x is taken to be very large, and typically tending to infinity [13, *cf.* §2] [3].

1.1 Average order, similarity and approximation of asymptotic growth rates of quantities

1.1.1 Similarity and average order (expectation)

First, we say that two functions $A(x), B(x)$ satisfy the relation $A \sim B$ if

$$\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} = 1.$$

It is sometimes standard to express the *average order* of an arithmetic function f as $f \sim h$, even when the values of $f(n)$ may actually non-monotonically oscillate, or say have value of one infinitely often. What the notation $f \sim h$ means in expressing the average order of f is that $\frac{1}{x} \cdot \sum_{n \leq x} f(n) \sim h(x)$. For example, in the acceptably classic language of [7] we would normally write that $\Omega(n) \sim \log\log n$, even though technically, $1 \leq \Omega(n) \leq \frac{\log n}{\log 2}$. To be absolutely clear about notation, we intentionally do not re-use the $\sim$ relation by instead writing $\mathbb{E}[f(x)] = h(x)$ (as in expectation of f) to denote that f has a limiting average order growing at the rate of h.

A related conception of f having a so-called *normal order* of g holds whenever

$$f(n) = (1 + o(1))g(n), \text{a.e.}$$

1.1.2 Approximation

We choose to adpot the convention to write that $f(x) \approx g(x)$ if $|f(x) - g(x)| = O(1)$. That is, we write $f(x) \approx g(x)$ to denote that f is approximately equal to g at x modulo at most a small constant difference between the functions.

The formula we prefer for the Abel summation variant of summation by parts to express finite sums of a product of two functions is stated as follows [1, *cf.* §4.3] *:

*Compare to the exact formula for *summation by parts* of any arithmetic functions, u_n, v_n, stated as in [13, §2.10(ii)] for $U_j := u_1 + u_2 + \cdots + u_j$ when $j \geq 1$:

$$\sum_{j=1}^{n-1} u_j \cdot v_j = U_{n-1} v_n + \sum_{j=1}^{n-1} U_j \left(v_j - v_{j+1}\right), n \geq 2.$$

Proposition 1.1 (Abel Summation Integral Formula)**.** *Suppose that $t > 0$ is real-valued, and that $A(t) \sim \sum_{n \leq t} a(n)$ for some weighting arithmetic function $a(n)$ with $A(t)$ continuously differentiable on $(0, \infty)$. Furthermore, suppose that $b(n) \sim f(n)$ with f a differentiable function of $n \geq 0$ – that is, $f'(t)$ exists and is smooth for all $t \in (0, \infty)$. Then for $0 \leq y < x$, where we typcially assume that the bounds of summation satisfy $x, y \in \mathbb{Z}^+$, we have that*

$$\sum_{y<n\leq x} a(n)b(n) \sim A(x)b(x) - A(y)b(y) - \int_y^x A(t)f'(t)dt.$$

Remark 1.2. The classical proof of the Abel summation formula given in Apostol's book has an alternate proof method noted in Section 4.3 of this reference. In particular, since $A(x)$ is a step function with jump of $a(n)$ at each integer-valued $n \geq 1$, the integral formula stated in Proposition 1.1 can be expressed in the following Riemann-Stieltjes integral notation:

$$\sum_{y<n\leq x} a(n)b(n) = \int_y^x f(t)dA(t).$$

A notable special case yields an integral approximation to summations we stated above where $[t]$ is the *greatest integer (ceiling) function*:

$$\sum_{y<n\leq x} f(n) = f(x)[x] - f(y)[y] - \int_y^x [t]f'(t)dt.$$

1.1.3 Vinogradov's notation for asymptotics

We use the conventional relations $f(x) \gg g(x)$ and $h(x) \ll r(x)$ to symbolically express that we should expect f to be "substantially" larger than g, and h to be "significantly" smaller, in asymptotic order (e.g., rate of growth when x is large). In practice, we adopt a somewhat looser definition of these symbols which allows $f \gg g$ and $h \ll r$ provided that there are constants $C, D > 0$ such that whenever x is sufficiently large we have that $f(x) \geq C \cdot g(x)$ and $h(x) \leq D \cdot r(x)$. This notation is sometimes called *Vinogradov's asymptotic notation.*

Another way of expressing our precise meaning of these relations is by writing

$$f \gg g \iff g = O(f),$$

and

$$h \ll r \iff r = \Omega(h),$$

using Knuth's well-trodden style of big-O (and Landau notation) and big-Ω (Hardy-Littlewood notation) symbols from the language of theoretical computer science and in the analysis of algorithms.

1.2 An unconventional pair of asymptotic relations employed to drop lower-order terms in upper and lower bounds on arithmetic functions

We define two new defintions of relations for expressing limiting asymptotic bounds on functions by adapting notation for existing operators for clarity of the way we use them here. Namely, we say that $h(x) \overset{\blacktriangle}{\gtrsim} r(x)$ if $h \gg r$ as $x \to \infty$, and define the relation $\overset{\blacktriangle}{\lesssim}$ similarly as $h(x) \overset{\blacktriangle}{\lesssim} r(x)$ if $h \ll r$ as $x \to \infty$. This usage of the notation of $\overset{\blacktriangle}{\gtrsim}, \overset{\blacktriangle}{\lesssim}$ intentionally breaks with the usual conventions for the use of these more standard relations of $\gtrsim, \lesssim$. Our distinct, intentional usage of these relations in our different context is intended to simplify the ways we express otherwise tricky and complicated expressions for upper and lower bounds that hold only exactly in limiting cases where x is large as $x \to \infty$.

The use of the new (modified) notation for $\overset{\blacktriangle}{\gtrsim}$ is intended to capture both that we are conveying a lower bound for the function, and crucially that this lower bound is valid only when x is very large, i.e., in some sense that the lower bound holds in the same sense as the relation $\sim$: for example, entending a notion similar to $|f(x)| \geq g(x)$ with $g(x) \sim h(x)$. This is a subtle distinction that comes into play when we later use it to state lower bounds in our new results.

An demonstratively constructed mock example motivating this usage of these relations that clarifies the point of making this subtle distinction in notation appears below.

Example 1.3. Suppose that exactly for all $x \geq 1$ we have

$$f(x) \geq -(\log\log\log x)^2 + 3 \times 10^{1000000} \cdot (\log\log\log x)^{1.999999999} + E(x),$$

where $E(x) = o\left((\log\log\log x)^2\right)$ and the unusually complicated expression for $E(x)$ requires more than 100000 ascii characters to typeset accurately, e.g., is far too exceedingly complicated to write down and include as a component of our expression for the terms in the primary bound. Then naturally, we prefer to work with only the expression for the asymptotically dominant main term in the lower bounds on $f(x)$ stated above.

Note that in this case the main term contribution does not dominate the bound until x is very large, so that replacing the right-hand-side expression with just this term yields an invalid inequality except for in limiting cases. In this instance, we prefer to write

$$f(x) \overset{\blacktriangle}{\gtrsim} -(\log\log\log x)^2, \text{ as } x \to \infty,$$

or more conventionally applying this notation only to unsigned functions that

$$|f(x)| \overset{\blacktriangle}{\gtrsim} (\log\log\log x)^2, \text{ as } x \to \infty,$$

which indicates that this substantially simplifed form of the lower bound on f holds as $x \to \infty$. In particular, it is problematic to only write that

$$f(x) \geq -(\log\log\log x)^2,$$

since there is a substantial (however, asymptotically negligible) initial range of $x \geq 1$ where this lower bound is invalid as stated in the previous equation.

Remark 1.4 (Emphasizing the rationale of the use of the new notation)**.** Hence, we emphasize that our new uses of the traditional symbols are as asymptotic relations defined to simplify our results by dropping expressions involving more precise, exact terms that are nonetheless asymptotically insignificant, to obtain accurate statements in limiting cases of large x that hold as $x \to \infty$. In principle, this convention allows us to write out simplified bounds that still capture the most simple essence of the upper or lower bound as we choose to view it in this article.

This take on the new meanings denoted by $\overset{\blacktriangle}{\gtrsim}, \overset{\blacktriangle}{\lesssim}$ is particularly powerful and is utilized in this article when we express many lower bound estimates for functions that would otherwise require literally pages of typeset symbols to state exactly, but which have simple enough formulae when considered as bounds that hold in this type of limiting asymptotic context.

1.3 Asymptotic expansions and uniformity

Because a subset of the results we cite that are proved in the references (e.g., borrowed from Chapter 7 of [11]) provide statements of asymptotic bounds that hold *uniformly* for x large, though in a bounded range depending on parameters, we need to briefly make precise what our preconceptions are of this terminology. We introduce the notation for asymptotic expansions of a function $f : \mathbb{R} \to \mathbb{R}$ from [13, §2.1(iii)].

1.3.1 Ordinary asymptotic expansions of a function

Let $\sum_n a_n x^{-n}$ denote a formal power series expansion in x where we ignore any necessary conditions to guarantee convergence of the series. For each integer $n \geq 1$, suppose that

$$f(x) = \sum_{s=0}^{n-1} a_s x^{-s} + O(x^{-n}),$$

as $|x| \to \infty$ where this limiting bound holds for $x \in \mathbb{X}$ in some unbounded set $\mathbb{X} \subseteq \mathbb{R}, \mathbb{C}$. When such a bound holds, we say that $\sum_s a_s x^{-s}$ is a *Poincaré asymptotic expansion*, or just *asymptotic series expansion*, of $f(x)$ as $x \to \infty$ along the fixed set $\mathbb{X}$. The condition in the previous equation is equivalent to writing

$$f(x) \sim a_0 + a_1 x^{-1} + a_2 x^{-2} + \cdots ; x \in \mathbb{X}, \text{ for } |x| \to \infty.$$

The prior two characterizations of an asymptotic expansion for f are also equivalent to the statement that

$$x^n \left(f(x) - \sum_{s=0}^{n-1} a_s x^{-s} \right) \xrightarrow{x \to \infty} a_n.$$

1.3.2 Uniform asymptotic expansions of a function

Let the set $\mathbb{X}$ from the definition in the last subsection correspond to a closed sector of the form

$$\mathbb{X} := \{x \in \mathbb{C} : \alpha \leq \arg(x) \leq \beta\}.$$

Then we say that the asymptotic property

$$f(x) = \sum_{s=0}^{n-1} a_s x^{-s} + O(x^{-n}),$$

from before holds *uniformly* with respect to $\arg(x) \in [\alpha, \beta]$ as $|x| \to \infty$.

Another useful, important notion of uniform asymptotic bounds is taken with respect to some parameter u (or set of parameters, respectively) that ranges over the point set (point sets, respectively) $u \in \mathbb{U}$. In this case, if we have that the u-parameterized expressions

$$\left| x^n \left(f(u, x) - \sum_{s=0}^{n-1} a_s(u) x^{-s} \right) \right|,$$

are bounded for all integers $n \geq 1$ for $x \in \mathbb{X}$ as $|x| \to \infty$, then we say that the asymptotic expansion of f holds *uniformly* for $u \in \mathbb{U}$. Now the function $f \equiv f(u, x)$ and the asymptotic series coefficients $a_s(u)$ may have an implicit dependence on the parameter u. If the previous boundedness condition holds for all positive integers n, we write that

$$f(u, x) \sim \sum_{s=0}^{\infty} a_s(u) x^{-s}; x \in \mathbb{X}, \text{ as } |x| \to \infty,$$

and say that this asymptotic expansion, or bound, holds *uniformly with respect to* $u \in \mathbb{U}$. For u taken outside of $\mathbb{U}$, the stated bound may fail to be valid even for $x \in \mathbb{X}$ as $|x| \to \infty$.

2 An introduction to the Mertens function – definition, properties, known results and conjectures

Suppose that $n \geq 1$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. We define the *Möebius function* to be the signed indicator function of the squarefree integers:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leq i \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many known variants and special properties of the Möebius function and its generalizations [15, *cf.* §2], however, for our purposes we seek to explore the properties and asymptotics of weighted summatory functions over $\mu(n)$. The Mertens summatory function, or *Mertens function*, is defined as [17, A002321]

$$\begin{aligned} M(x) &= \sum_{n \leq x} \mu(n), \ x \geq 1, \\ &\longmapsto \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2, \ldots\} \end{aligned}$$

A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as [17, A013928]

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \rightarrow \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{Q(x)} = \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{Q(x)} \xrightarrow[x \rightarrow \infty]{} \frac{3}{\pi^2}.$$

While this limiting law suggests an even bias for the Mertens function, in practice $M(x)$ has an apparent unproven negative bias in its values, and the actual local oscillations between the approximate densities of the sets $\mu_\pm(x)$ lend an unpredictable nature to the function and the function's characteristic oscillatory sawtooth shaped plot viewed over the positive integers.

2.1 Properties

The conventional approach to evaluating the behavior of $M(x)$ for large $x \rightarrow \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is easily expressed via an inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since

$$\frac{1}{\zeta(s)} = \int_1^\infty \frac{s \cdot M(x)}{x^{s+1}} dx,$$

we then obtain that

$$M(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation along with the standard Euler product representation for the reciprocal zeta function leads us to the exact expression for $M(x)$ when $x > 0$ given by the next theorem.

Theorem 2.1 (Analytic Formula for $M(x)$)**.** *Assuming the Riemann Hypothesis (RH), we can show that there exists an infinite sequence $\{T_k\}_{k\geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any $x \in \mathbb{R}_{>0}$*

$$M(x) = \lim_{k\to\infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |\Im(\rho)|<T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n\geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta .$$

A historical unconditional bound on the Mertens function due to Walfisz (1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp\left(-C \cdot \log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan proved in 2009 new updated estimates bounding $M(x)$ for large x of the following forms [18]:

$$\begin{aligned} M(x) &\ll \sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right), \ \forall \epsilon > 0. \end{aligned}$$

To date, due to the oscillatory nature of $M(x)$ via the signedness of $\mu(n)$, considerably less has been conjectured about explicit lower bounds on $|M(x)|$ along subsequences.

2.2 Conjectures

The RH is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. It is still unresolved whether

$$\limsup_{x\to\infty} |M(x)|/\sqrt{x} = \infty,$$

although computational evidence suggests that this is a likely conjecture [10, 8]. There is a rich history to the original statement of the *Mertens conjecture* which states that

$$|M(x)| < c \cdot x^{1/2}, \quad \text{some absolute constant } c > 0.$$

Mertens conjecture was first verified by Mertens for $c = 1$ and $x < 10000$, although since its beginnings in 1897, the conjecture has since been disproved by computation of low-lying zeta function zeros in a famous paper by Odlyzko and té Riele from the early 1980's.

There are a number of other interesting unsolved and at least somewhat accessible open problems related to the asymptotic behavior of $M(x)$ at large x. It is believed that the sign of $M(x)$ changes infinitely often. That is to say that it is widely believed that $M(x)$ is ocsillatory and exhibits a negative bias insomuch as $M(x) < 0$ more frequently than $M(x) > 0$ over all $x \in \mathbb{N}$.

One of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ is in actuality unbounded on the natural numbers. A precise statement of this problem is to produce an affirmative answer whether $\limsup_{x\to\infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x\to\infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there is an infinite sequence of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound along the subsequence. Currently, an exact rigorous proof that $M(x)/\sqrt{x}$ is unbounded still remains elusive, though there is suggestive probabilistic evidence of this property established by Ng in 2008. We cite that prior to this point it is known that [14, *cf.* §4.1]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad \text{(now 1.826054)},$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } -1.837625),$$

although based on work by Odlyzyko and té Riele it seems probable that each of these limits should be $\pm\infty$, respectively [12, 9, 10, 8].

Extensive computational evidence has produced a conjecture due to Gonek (among attempts on bounds by others) that in fact the limiting behavior of $M(x)$ satisfies that

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}(\log\log x)^{5/4}} = O(1).$$

While it seems to be widely believed that $|M(x)|/\sqrt{x}$ tends to $+\infty$ at a logarithmic rate along subsequences, infinitely tending factors such as the $(\log\log x)^{\frac{5}{4}}$ in Gonek's conjecture do not appear to readily fall out of work on bounds for $M(x)$ by existing methods.

3 Introduction to our new methodology: An concrete approach to bounding $M(x)$ from below

3.1 Summing series over Dirichlet convolutions

Theorem 3.1 (Summatory functions of Dirichlet convolutions). *Let $f, g : \mathbb{Z}^+ \rightarrow \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of f, g, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse $f^{-1}(n)$ of f. Then, letting the counting function $\pi_{f*h}(x)$ be defined as in the first equation below, we have the following equivalent expressions for the summatory function of $f * h$ for integers $x \geq 1$:*

$$\begin{aligned}
\pi_{f*h}(x) &= \sum_{n \leq x} \sum_{d|n} f(d) h(n/d) \\
&= \sum_{d \leq x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].
\end{aligned}$$

Moreover, we can invert the linear system determining the coefficients of $H(k)$ for $1 \leq k \leq x$ naturally to express $H(x)$ as a linear combination of the original left-hand-side summatory function as follows:

$$\begin{aligned}
H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\
&= \sum_{n=1}^{x} f^{-1}(n) \pi_{f*h}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).
\end{aligned}$$

Corollary 3.2 (Convolutions Arising From Möbius Inversion). *Suppose that g is an arithmetic function with $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n \leq x} (g * \mu)(n)$. Then the Mertens function equals*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geq 1.$$

Corollary 3.3 (A motivating special case). *We have exactly that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \qquad (1)$$

3.2 Elaborating on construction behind the motivating special case

We can compute the first few terms for the Dirichlet inverse sequence of the arithmetic function $g(n) := \omega(n) + 1$ from Corollary 3.3 numerically for the first few sequence values as

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is given by $\operatorname{sgn}(g^{-1}(n)) = \frac{g^{-1}(n)}{|g^{-1}(n)|} = \lambda(n)$ (see Proposition 4.2). This useful property is inherited from the distinctly additive nature of the component function $\omega(n)$. We will still require substantially simpler asymptotic formulae for $g^{-1}(n)$ than what complications are suggested by inspection of the initial numerical calculations of this sequence. It does happen that we can find fruitful and combinatorially meaningful ways to express asymptotics for this special inverse sequence (see Theorem 3.6).

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Consider first the following motivating conjecture:

Conjecture 3.4. *Suppose that $n \geq 1$ is a squarefree integer. We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n) = (\omega+1)^{-1}(n)$ over these integers:*

(A) $g^{-1}(1) = 1$*;*

(B) $\operatorname{sgn}(g^{-1}(n)) = \mu(n) \equiv \lambda(n)$*;*

(C) We can write the inverse function at squarefree n as

$$g^{-1}(n) = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!.$$

We illustrate parts (B)–(C) of this conjecture clearly using Table T.1 given on page 43 of the appendix section.

The realization that the beautiful and remarkably simple form of property (C) in Conjecture 3.4 holds for all squarefree $n \geq 1$ motivates our pursuit of formulas for the inverse functions $g^{-1}(n)$ based on the configuration of the exponents in the prime factorization of any $n \geq 2$. The summation methods we employ in Section 6 to weight sums of our arithmetic functions according to the sign of $\lambda(n)$ (or parity of $\Omega(n)$) is also reminiscent of the combinatorially motivated sieve methods in [4, §17].

Remark 3.5 (Comparison to formative methods for bounding $M(x)$)**.** Note that since the DGF of $\omega(n)$ is given by $D_\omega(s) = P(s)\zeta(s)$ where $P(s)$ is the *prime zeta function*, we do have a Dirichlet series for the inverse functions to invert coefficient-wise using more classical contour integral methods*. However, the uniqueness to our new methods is that our new approach does not rely on typical constructions for bounding $M(x)$ based on estimates of the non-trivial zeros of the Riemann zeta function that have so far been employed to bound the Mertens function from above. That is, we will instead take a more combinatorial tack to investigating bounds on this inverse function sequence in the coming sections. By Corollary 3.3, once we have established bounds on this $g^{-1}(n)$ and its summatory function, we will be able to formulate new lower bounds (in the limit supremum sense) on $M(x)$.

3.3 Fixing an exact expression for $M(x)$ through special sequences of arithmetic functions

From this point on, we fix the notation for the Dirichlet invertible function $g(n) := \omega(n)+1$ and denote its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega+1)^{-1}(n)$. For

*E.g., using contour integration or the following integral formula for Dirichlet series inversion [1, §11]:

$$f(n) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{n^{\sigma+\imath t}}{\zeta(\sigma+\imath t)(P(\sigma+\imath t)+1)}, \sigma > 1.$$

Fröberg has also previously done some preliminary investigation as to the properties of the inversion to find the coefficients of $(1+P(s))^{-1}$ in [5].

natural numbers $n \geq 1, k \geq 0$, let

$$C_k(n) := \begin{cases} \varepsilon(n) = \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$

We have limiting asymptotics on these functions given by the following theorem:

Theorem 3.6 (Asymptotics for the functions $C_k(n)$)**.** *For $k := 0$, we have by definition that $C_0(n) = \delta_{n,1}$. For all $k \geq 1$, we obtain that the dominant asymptotic term for $C_k(n)$ is given by*

$$\mathbb{E}[C_k(n)] = (\log\log n)^{2k-1}, \text{ as } n \to \infty.$$

Since we have that

$$(g^{-1} * 1)(n) = \lambda(n) \cdot C_{\Omega(n)}(n), \forall n \geq 1, \tag{2}$$

Möbius inversion provides us with an exact divisor sum based expression for $g^{-1}(n)$ (see Lemma 6.1). Then we can prove (see Corollary 6.5) that we can obtain lower bounds on the magnitude of $g^{-1}(n)$ by approximating it by the simpler divisor sums

$$\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

Notice that this formula is substantially easier to evaluate than the corresponding sums in (2) given directly through Möbius inversion. Hence, we prefer to work with bounds on it that we prove as new results instead rather than with results relying on the more complicated exact formula from the cited equation above.

Specifically, the last result in turn implies that

$$|G^{-1}(x)| \overset{\blacktriangle}{\gtrsim} \left| \sum_{n \leq x} \lambda(n) \cdot C_{\Omega(n)}(n) \times \sum_{d=1}^{\left\lfloor \frac{x}{n} \right\rfloor} \lambda(d) \right|. \tag{3}$$

In light of the fact that (by an integral-based interpretation of integer convolution using summation by parts, see Proposition 7.1)

$$M(x) \sim G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)},$$

the formula in (3) implies that we can establish new *lower bounds* on $M(x)$ by appropriate estimates of the summatory function $G^{-1}(x)$ where trivially we have the bounded inner sums $L_0(x) := \sum_{n \leq x} \lambda(n) \in [-x, x]$ for all $x \geq 2$. As explicit lower bounds for $M(x)$ along particular subsequences are not obvious, and are historically ellusive non-trivial features of the function to obtain as we expect sign changes of this function infinitely often, we find this approach to be an effective one.

3.4 Enumerative (or counting based) DGFs from Mongomery and Vaughan

Now, having motivated why we must carefully estimate the $G^{-1}(x)$ bounds using our new methods, we will require the bounds suggested in the next section to work at bounding the summatory functions, $G^{-1}(x)$, for large x as $x \to \infty$. The precise formulations of the inverse function asymptotics proved in Section 6 depend on being able to form significant lower bounds on the summatory functions of an always positive arithmetic function weighted by $\lambda(n)$. Our inspiration for the new bounds found in the last sections of this article allows us to sum non-negative arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. In particular, it uses a hybrid generating function and enumerative DGF method under which we are able to recover "good enough" asymptotics about the summatory functions that encapsulate the parity of $\lambda(n)$ through the summatory count functions $\widehat{\pi}_k(x)$. The precise statement of the theorem that we transform to state these new bounds is re-stated as Theorem 3.7 below.

Theorem 3.7 (Montgomery and Vaughan, §7.4)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $R < 2$ we have that

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

uniformly for $1 \leq k \leq R\log\log x$ where

$$\mathcal{G}(z) := \frac{F(1,z)}{\Gamma(z+1)} = \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, z \geq 0.$$

The next theorem, proved carefully in Section 5, is the primary starting point for our new asymptotic lower bounds.

Theorem 3.8 (Generating functions of symmetric functions)**.** *We obtain lower bounds of the following form on the function $\mathcal{G}(z)$ from Theorem 3.7 for $A_0 > 0$ an absolute constant, for $C_0(z)$ a strictly linear function only in z, and where we must take $0 \leq z \leq 1$:*

$$\mathcal{G}(z) \geq A_0 \cdot (1-z)^3 \cdot C_0(z)^z.$$

It suffices to take the components to the bound in the previous equation as

$$\begin{aligned} A_0 &= \frac{2^{9/16} \exp\left(-\frac{55}{4}\log^2(2)\right)}{(3e\log 2)^3 \cdot \Gamma\left(\frac{5}{2}\right)} \approx 3.81296 \times 10^{-6} \\ C_0(z) &= \frac{4(1-z)}{3e\log 2}. \end{aligned}$$

In particular, with $0 \leq z \leq 1$ and $z \equiv z(k,x) = \frac{k-1}{\log\log x}$, by Theorem 3.7, we have that

$$\widehat{\pi}_k(x) \overset{\blacktriangle}{\gtrsim} \frac{A_0 \cdot x}{\log x \cdot (\log\log x)^4 \cdot (k-1)!} \cdot \left(\frac{4}{3e\log 2}\right)^k.$$

3.5 Nearly cracking the classical unboundedness barrier

In Section 7, we provide the culmination of what we build up to in the proofs established in prior sections of the article. Namely, we prove the form of an explicit limiting lower bound for the summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, along a specific subsequence. What we obtain is the following important summary corollary that comes close (very close, by a factor of $\log x$) to resolving the classical question of the unboundedness of the scaled function Mertens function $|M(x)|/\sqrt{x}$ in the limit supremum sense:

Corollary 3.9 (Lower Bounds for the Mertens function)**.** *Let $u_0 := e^{e^{e^e}}$ and define the infinite increasing subsequence, $\{x_n\}_{n\geq 1}$, by $x_n := e^{e^{e^{e^{4n}}}}$. We have that along the increasing subsequence x_y for large $y \geq \max\left(\left\lceil e^{e^{e^{e^e}}}\right\rceil, u_0+1\right)$:*

$$\frac{|M(x_y)|}{\sqrt{x_y}} \overset{\blacktriangle}{\gtrsim} 2C_{\ell,1} \cdot \frac{(\log\log\sqrt{x_y})}{\log\sqrt{x_y}} \frac{(\log\log\log\sqrt{x_y})^{2\log 2+\frac{1}{3\log 2}-2}}{(\log\log\log\log\sqrt{x_y})^{\frac{5}{2}}} \cdot \frac{\log_*^5(\sqrt{x_y})^{2\log 2+\frac{1}{3\log 2}}}{\log_*^6(\sqrt{x_y})^{\frac{5}{2}}},$$

as $y \to \infty$. In the previous equation, we adopt the notation for the absolute constant $C_{\ell,1} > 0$ defined more precisely by

$$C_{\ell,1} := \frac{128 \cdot 2^{1/8}}{6561 \cdot e^6 \pi \log^6(2)} \exp\left(-\frac{55}{2}\log^2(2)\right) \approx 2.76631 \times 10^{-10}.$$

This is all to say that in establishing the rigorous proof of Corollary 3.9 based on our new methods, we not only show that

$$\limsup_{x\to\infty} \frac{|M(x)|\log x}{\sqrt{x}} = +\infty,$$

but also set a minimal rate (along a large infinite subsequence) at which this form of the scaled Mertens function grows without bound.

3.6 A summary outline: Listing the core logical steps and critical components to the proof in order of exposition

3.6.1 Step-by-step overview

We offer another brief step-by-step summary overview of the critical components to our proof outlined in the introduction above, and then which are proved piece-by-piece in the next sections of the article below. This outline is provided to help the reader see our logic and proof methodology as easily and quickly as possible.

- **(1)** We prove a matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). See Theorem 3.1 in Section 4.

- **(2)** This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the seemingly unconnected prime counting function, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is already stated in (1) expanded above.

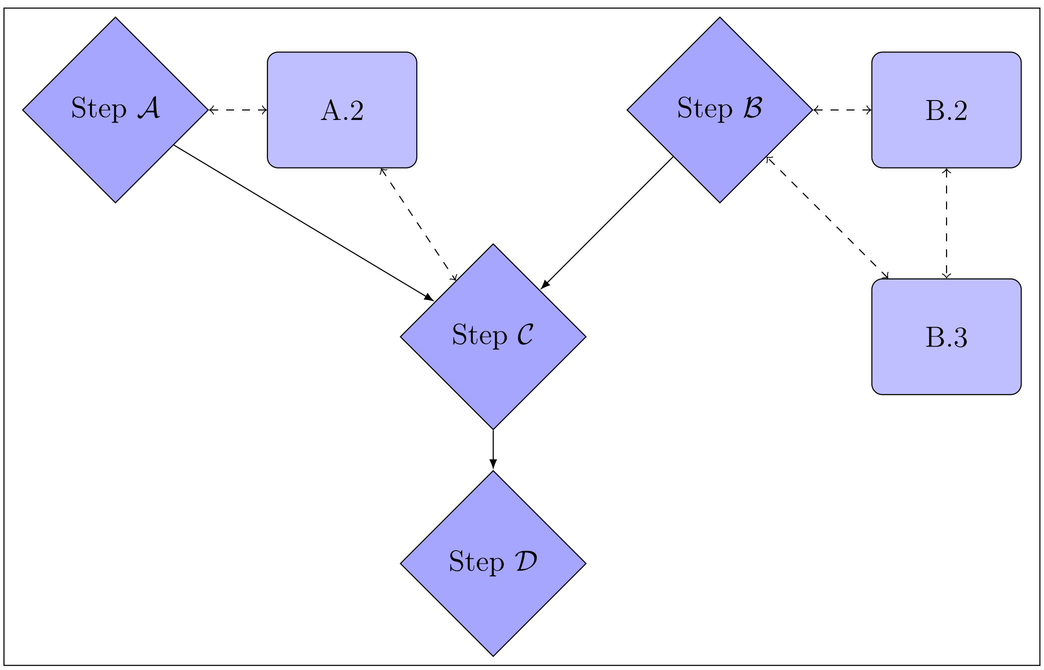
- **(3)** We tighten a result from [11, §7] providing summatory functions that indicate the parity of $\lambda(n)$ using elementary arguments and more combinatorially flavored expansions of Dirichlet series in our proof of Theorem 3.8. We use this result to sum $\sum_{n\leq x} \lambda(n) f(n)$ for particular non-negative arithmetic functions f by Abel summation when x is large.

- **(4)** We then turn to the asymptotics if the quasi-periodic $g^{-1}(n)$, estimating this inverse function's limiting asymptotics for large n (or $n \leq x$ when x is very large) in Section 6. We eventually use these estimates to prove a substantially unique new lower bound formula for the summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$ along prescribed asymptotically large infinite subsequences that tend to $+\infty$ (see Theorem 7.4).

(5) When we return to step **(2)** with our new lower bounds at hand, and bootstrap, we find "magic" in the form of showing the unboundedness of $\frac{|M(x)|\log x}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. What we recover is a quick, and rigorous, proof of Corollary 3.9 given in Section 7.2.

Remark 3.10. Note that much of the effort and length taken in proving the complete set of results in this article is attributed to showing that we can in fact discard asymptotically negligible terms from exact expansions of formulas we require to bound. This machinery is necessary to rigorously justify that the actually most interesting new bounds we obtain here are valid as $x \to \infty$. The preceeding steps have sketched the core components that are central to building our more combinatorially motivated approach to approximating $M(x)$ that we develop in this article. The full listing of components to the proof including "routine", or so-called cut-and-dry, results we need as glue to make our full improvements precise are included in the key to the schematic of our proof logic given in the next subsection.

3.6.2 Diagramatic flowchart of the proof logic with references to results



Key to the diagram stages:

Step A: *Citations and re-statements of existing theorems proved elsewhere*: E.g., statements of non-trivial theorems and key results we need that are proved in the references.

A.A Key results and constructions:
- Theorem 3.7
- Theorem 5.1
- Corollary 5.2
- The results, lemmas, and facts cited in Section 4.3

A.2 Lower bounds on the Abel summation based formula for $G^{-1}(x)$:
- Theorem 3.8
- Corollary 5.3
- Theorem 7.4
- Lemma 7.2

- Lemma 7.3

Step B: *Constructions of an exact formula for $M(x)$*: The exact formula we prove uses special arithmetic functions with particularly "nice" properties and bounds. This choice of the expression from Theorem 3.1 is key to how far we have traveled along the new approaches in this article. In particular, the additivity of $\omega(n)$ and the easily integrable logarithmically weighted bound on $\pi(x)$ for large x are indispensible components to why this proof works well.

B.B Key results and constructions:
- Theorem 3.1
- Corollary 3.2
- Corollary 3.3
- Conjecture 3.4 (to a lesser expository only extent)
- Proposition 4.1
- Proposition 4.2

B.2 Asymptotics for the component functions $g^{-1}(n)$ and $G^{-1}(x)$:
- Theorem 3.6
- Lemma 6.1

B.3 Simplifying the requisite formulas for $g^{-1}(n)$ and $G^{-1}(x)$:
- Corollary 6.4
- Corollary 6.5

Step C: *Re-writing the exact formula for $M(x)$*: Key interpretations used in formulating the lower bounds based on the re-phrased integral formula.
- Proposition 7.1

Step D: *The Holy Grail*: A big leap twoards proving that $\frac{|M(x)|\log x}{\sqrt{x}}$ is unbounded in the limit supremum sense.
- Corollary 3.9

4 Preliminary proofs of lemmas and new results

4.1 Establishing the summatory function inversion identities

There are a vast number of Dirichlet convolution identities for special number theoretic functions over which we can form summatory functions and perform inversion via Theorem 3.1 [6, 16]. This compendia of identities and standard applications suggests that our new methods may be broadly applicable to obtaining new bounds on the summatory functions of other classically special arithmetic functions. We will prove this useful theorem, a crucial component to our new more combinatorial formulations used to bound $M(x)$ in later sections, using matrix methods before moving on. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95].

Proof of Theorem 3.1. Let h, g be arithmetic functions where $g(1) \neq 0$ necessarily has a Dirichlet inverse. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h: $g * h$. Then we can easily see that the following expansions hold:

$$\begin{aligned}
\pi_{g*h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] H(i).
\end{aligned}$$

We form the matrix of coefficients associated with this system for $H(x)$, and proceed to invert it to express an exact solution for this function over all $x \geq 1$. Let the ordinary (initial, non-inverse) matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1}.$$

The matrix we must invert in this problem is lower triangular, with ones on its diagonals – and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T); \qquad U = (\delta_{i,j+1}), (I - U^T)^{-1} = ([j \leq i]_\delta).$$

Here, U is the $N \times N$ matrix whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$.

It is a useful fact that if we take successive differences of floor functions, we get non-zero behavior at divisors:

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

We use this property to shift the matrix $\hat{G}$, and then invert the result, to obtain a matrix involving the Dirichlet inverse of g:

$$\left[(I - U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Now we can express the inverse of the target matrix $(g_{x,j})$ in terms of these Dirichlet inverse functions as follows:

$$(g_{x,j}) = (I - U^T)^{-1}\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I - U^T)$$

$$\begin{aligned}
(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_{\delta}\right)(I-U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k)\right)(I-U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1}\right\rfloor} g^{-1}(k)\right).
\end{aligned}$$

Thus the summatory function H is exactly expressed by the inverse vector product of the form

$$\begin{aligned}
H(x) &= \sum_{k=1}^{x} g_{x,k}^{-1}\cdot \pi_{g*h}(k) \\
&= \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right)\cdot \pi_{g*h}(k). \square
\end{aligned}$$

4.2 Proving the crucial signedness property from the conjecture

Proposition 4.1 (The characteristic function of the primes)**.** *Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n)=\delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the incompletely additive function that counts the number of distinct prime factors of n. Then we have the convolution identity given by*

$$\chi_{\mathbb{P}}+\varepsilon = (\omega+1)*\mu.$$

The summatory function of the left-hand-side of the previous equation is clearly $\widetilde{G}(x)=\pi(x)+1$ in the notation of Corollary 3.2 for all $x\geq 1$.

Proof. The core is to prove that for all $n\geq 1$, $\chi_{\mathbb{P}}(n)=(\mu*\omega)(n)$ – an equivalent form of our essential claim. We notice that the Mellin transform of $\pi(x)$, the summatory function of $\chi_{\mathbb{P}}(n)$, evaluated at the parameter $-s$ is given by

$$\begin{aligned}
s\cdot\int_1^{\infty}\frac{\pi(x)}{x^{s+1}}dx &= \sum_{n\geq 1}\frac{\nabla[\pi](n-1)}{n^s} \\
&= \sum_{n\geq 1}\frac{\chi_{\mathbb{P}}(n)}{n^s} = P(s),
\end{aligned}$$

where $\nabla[f](n):=f(n+1)-f(n)$ denotes the standard *forward difference operator* used to express a discrete derivative type operation on arithmetic functions. This is a typical construction that is used as a tool to more generally relate the Mellin transform $s\cdot\mathcal{M}[S_f](-s)$ to the DGF of the sequence $f(n)$, cited, for example, as in [1, §11]. In essence, what the previous equation says is that the DGF of $\chi_{\mathbb{P}}$ is $P(s)$.

Now to show the equivalence of the prime indicator function with the Dirichlet convolution based expression, $\omega*\mu$, we consider the DGF of the right-hand-side function, $f(n):=(\mu*\omega)(n)$, as

$$D_f(s)=\frac{1}{\zeta(s)}\times\sum_{n\geq 1}\frac{\omega(n)}{n^s}=P(s),$$

where it is not difficult to prove that the DGF of $\omega(n)$ is $P(s) \cdot \zeta(s)$.

Thus for any $\Re(s) > 1$, the DGFs of each side of the claimed equation coincide. So by uniqueness of Dirichlet series, we see that in fact the claim holds. To obtain the full result, we add to each side of this equation a term of $\varepsilon(n) \equiv (\mu * 1)(n)$, and then factor the resulting convolution identity. ☐

When combined with Corollary 3.2, the proof of Proposition 4.1 yields the crucial starting point providing an exact formula for $M(x)$ stated in (1) of Corollary 3.3. Thus, while the formula in (1) is a key component utilized in our proof moving forward, we do not need to explicitly show that it holds for all $x \geq 1$ from this point.

Proposition 4.2 (The key signedness property of $g^{-1}(n)$)**.** *For the Dirichlet invertible function, $g(n) := \omega(n) + 1$ defined such that $g(1) = 1$, at any $n \geq 1$, we have that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$. The notation for the operation given by $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0, \pm 1\}$ denotes the sign of the arithmetic function h at n.* A02-2020-04-26

Proof. Let $D_f(s) := \sum_{n \geq 1} f(n) n^{-s}$ denote the Dirichlet generating function (DGF) of any arithmetic function $f(n)$ convergent for $\Re(s) > \sigma_f$. Using Proposition 4.1 and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of the original arithmetic function f, we can express the DGF of our particular $g^{-1}(n)$ explicitly as an analytic function of s for $\Re(s) > 1$. For all $\Re(s) > 1$, expanding the DGF for the function $g^{-1}(n)$ yields

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{4}$$

Let $h^{-1}(n) := (\omega * \mu + \varepsilon)^{-1}(n) = [n^{-s}](P(s)+1)^{-1}$. Then we have using the standard recurrence relation for the Dirichlet inverse function h^{-1} with $\chi_{\mathbb{P}} = \omega * \mu$ that

$$\begin{aligned} (h^{-1} * 1)(n) &= \sum_{p_1 | n} h^{-1}\left(\frac{n}{p_1}\right) = \lambda(n) \times \sum_{p_1 | n} \sum_{p_2 | \frac{n}{p_1}} \cdots \sum_{p_{\Omega(n)} | \frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1 \\ &= \begin{cases} \lambda(n) \times (\Omega(n)-1)!, & n \geq 2; \\ \lambda(n), & n = 1. \end{cases} \end{aligned} \tag{5}$$

We need to compute the sign of the function $h^{-1} * \mu$, where $g^{-1} = h^{-1} * \mu$ by the DGF in (4) since the DGF of $\mu(n)$ is well-known to be $1/\zeta(s)$ for $\Re(s) > 1$ (and the DGF of a convolution is a product of the component DGFs).

First, by Möbius inversion and the formula for $h^{-1} * 1$ we proved above, for each $n \geq 2$, we have that there exist constants $C_{1,n}, C_{2,n} > 0$ so that

$$C_{1,n} \cdot (\lambda * \mu)(n) \leq h^{-1}(n) \leq C_{2,n} \cdot (\lambda * \mu)(n).$$

This observation follows from the non-negativity of the factorial function in the formula we just proved in (5). Since both λ, μ are multiplicative, $\lambda * \mu$ is multiplicative, where we know that the values of any multiplicative function are uniquely determined by its action at prime powers. So we can compute that for any prime p and integer exponents $\alpha \geq 1$,

$$(\lambda * \mu)(p^\alpha) = \lambda(p^\alpha) - \lambda(p^{\alpha-1}) = 2\lambda(p^\alpha).$$

Then by the multiplicativity of $\lambda(n)$, the previous inequalities are re-stated in the form of

$$2C_{1,n} \cdot \lambda(n) \leq h^{-1}(n) \leq 2C_{2,n} \cdot \lambda(n).$$

Now to bound $h^{-1} * \mu$, we similarly can argue again that by multiplicativity we have

$$4C_{1,n} \cdot \lambda(n) \leq (h^{-1} * \mu)(n) \leq 4C_{2,n} \cdot \lambda(n).$$

Since the absolute constants (for each n) are positive, we clearly recover the signedness of $g^{-1}(n)$ as $\lambda(n)$. $\square$

4.3 Other facts and listings of results we will need in our proofs

Theorem 4.3 (Mertens theorem)**.**

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log\log x + B + o(1),$$

where $B \approx 0.2614972128476427837554$ *is an explicitly defined absolute constant.*

Corollary 4.4. *We have that for sufficiently large* $x \gg 1$

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-B}}{\log x} \left(1 + o(1)\right).$$

Hence, for $1 < |z| < R < 2$ *we obtain that*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^z = \frac{e^{-Bz}}{(\log x)^z} \left(1 + o(1)\right)^z.$$

Facts 4.5 (Exponential Integrals and Incomplete Gamma Functions)**.** The following two variants of the *exponential integral function* are defined by [13, §8.19]

$$\begin{aligned} \mathrm{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, \\ E_1(z) &:= \int_{1}^{\infty} \frac{e^{-tz}}{t} dt, \Re(z) \geq 0, \end{aligned}$$

where $\mathrm{Ei}(-kz) = -E_1(kz)$ for real $k > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $E_1(z)$:

$$1 - \frac{3}{4}z \leq E_1(z) - \gamma - \log z \leq 1 - \frac{3}{4}z + \frac{11}{36}z^2. \tag{6a}$$

A related function is the (upper) *incomplete gamma function* defined by [13, §8.4]

$$\Gamma(s, x) = \int_{x}^{\infty} t^{s-1} e^{-t} dt, \Re(s) > 0.$$

We have the following properties of $\Gamma(s, x)$:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, \tag{6b}$$

$$\Gamma(s+1, 1) = e^{-1} \left\lfloor \frac{s!}{e} \right\rfloor, s \in \mathbb{Z}^+, \tag{6c}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, |x| \to +\infty. \tag{6d}$$

5 Summing arithmetic functions weighted by $\lambda(n)$

5.1 Discussion: The enumerative DGF result in Theorem 3.7 from Montgomery and Vaughan

What the enumeratively-flavored result of Montgomery and Vaughan in Theorem 3.7 allows us to do is get a "good enough" lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. For comparison, we already have known, more classical bounds due to Erdös (and earlier) that we can tightly bound [2, 11]

$$\pi_k(x) = (1+o(1)) \cdot \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

We seek to approximate the right-hand-side of $\mathcal{G}(z)$ by only taking the products of the primes $p \leq u$, e.g., indexing the component products only over those primes $p \in \{2, 3, 5, \ldots, u\}$ for some minimal upper bound u (depending on x) as $x \rightarrow \infty$ (see Remark 5.4). The results proved in Section 5.2 identify a minimal parameter u.

We also state the following theorem reproduced from [11, Thm. 7.20] that handles the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \log\log x$.

Theorem 5.1 (Bounds on exceptional values of $\Omega(n)$ for large n, MV 7.20)**.** *Let*

$$B(x,r) := \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\}.$$

If $1 \leq r \leq R < 2$ *and* $x \geq 2$*, then*

$$B(x,r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \rightarrow \infty.$$

The proof of Theorem 5.2 is found in the cited reference as Chapter 7 of Montgomery and Vaughan. The key interpretation we need is the result stated in the next corollary.

Corollary 5.2. *Using the notation for $B(x,r)$ from Theorem 5.1, we have in particular that for $r \in (1,2)$,*

$$\left|1 - \frac{B(x,r)}{B(x,1)}\right| \xrightarrow{x\rightarrow\infty} 1.$$

We emphasize that Corollary 5.2 implies that for sums involving $\widehat{\pi}_k(x)$ indexed by k, we can capture the dominant asymptotic behavior of these sums by taking k in the truncated range $1 \leq k \leq \log\log x$, e.g., $0 \leq z \leq 1$ in Theorem 3.7. This fact will be important when we prove Theorem 7.4 in Section 7 using a sign-weighted summatory function in Abel summation that depends on these functions (see Lemma 7.2).

5.2 The key new results utilizing Theorem 3.7

We will require a handle on partial sums of integer powers of the reciprocal primes as functions of the integral exponent and the upper summation index x. The next corollary is not a triviality as it comes in handy when we take to the task of proving Theorem 3.8 below. The next statement of Corollary 5.3 effectively generalizes Mertens theorem stated previously as Theorem 4.3 by providing a coarse rate in x below which the reciprocal prime sums tend to absolute constants given by the prime zeta function, $P(s)$.

Corollary 5.3. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \gg 2.$$

When $s := 1$, we have the known bound in Mertens theorem (see Theorem 4.3). For $s > 1$, we obtain that

$$P_s(x) \approx E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1).$$

It follows that for $s \geq 2$ we have that

$$P_s(x) \leq \gamma_1(s,x) + o(1).$$

It suffices to take the bounding function in the previous equation as

$$\gamma_1(s,x) = -s \log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}s(s-1)\log(x/2) + \frac{11}{36}s(s-1)^2\log^2(2).$$

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Proof. Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$ and where our target function $f(t) = t^{-s}$ with $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1), |x| \to \infty. \end{aligned}$$

Now using the inequalities in Facts 4.5, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned} \frac{P_s(x)}{s} &\geq -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1)\log(x/2) - \frac{11}{36}(s-1)^2\log^2(x) \\ \frac{P_s(x)}{s} &\leq -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1)\log(x/2) + \frac{11}{36}(s-1)^2\log^2(2). \end{aligned}$$

This completes the proof of the bounds cited above in the statement of this lemma. $\square$

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Proof of Theorem 3.8. We have that for all integers $0 \leq k \leq m$

$$[z^k] \prod_{1 \leq i \leq m} (1 - f(i)z)^{-1} = [z^k] \exp\left(\sum_{j \geq 1} \left(\sum_{i=1}^m f(i)^j\right) \frac{z^j}{j}\right). \tag{7}$$

In our case we have that $f(i)$ denotes the i^{th} prime. Hence, summing over all $p \leq ux$ in place of $0 \leq k \leq m$ in the previous formula, and in tandem with Corollary 5.3, we obtain that the logarithm of the generating function series we are after when we sum over all $p \leq ux$ for some parameter u that we must next determine corresponds to

$$\begin{aligned} \log\left[\prod_{p \leq ux}\left(1 - \frac{z}{p}\right)^{-1}\right] &\geq (B + \log\log(ux))z + \sum_{j \geq 2}\left[a(ux) + b(ux)(j-1) + c(ux)(j-1)^2\right] z^j \\ &= (B + \log\log(ux))z - a(ux)\left(1 + \frac{1}{z-1} + z\right) \\ &\quad + b(ux)\left(1 + \frac{2}{z-1} + \frac{1}{(z-1)^2}\right) \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad - c(ux)\left(1 + \frac{4}{z-1} + \frac{5}{(z-1)^2} + \frac{2}{(z-1)^3}\right) \\
&=: \widehat{\mathcal{B}}(u, x; z).
\end{aligned}
$$

In the previous equations, the lower bounds formed by the functions (a, b, c) evaluated at ux are given by the corresponding upper bounds from Corollary 5.3 due to the leading sign on the previous expansions as

$$
(a_\ell, b_\ell, c_\ell) := \left(-\log\left(\frac{\log(ux)}{\log 2}\right), \frac{3}{4}\log\left(\frac{ux}{2}\right), \frac{11}{36}\log^2 2\right).
$$

Now we make a practical decision to set the uniform bound parameter to a middle ground value of $1 < R < 2$ at $R := \frac{3}{2}$ (practically, to be truncated and taken as though $R \equiv 1$ in sums) so that

$$
z \equiv z(k, x) = \frac{k}{\log\log x} \in (0, R),
$$

for $x \gg 1$ very large. Thus $(z-1)^{-m} \in [(-1)^m, 2^m]$ for integers $m \geq 1$, and so we can obtain the lower bound stated below. Namely, these bounds on the signed reciprocals of $z-1$ lead to an effective bound of the following form:

$$
\begin{aligned}
\widehat{\mathcal{B}}(u, x; z) \geq (B + \log\log(ux))z - a(ux)\left(1 + \frac{1}{z-1} + z\right) \\
+ b(ux)\left(1 + \frac{2}{z-1} + \frac{1}{(z-1)^2}\right) - 45 \cdot c(ux).
\end{aligned}
$$

Since the function $c(ux)$ is constant, we then also obtain a refined bound of the next form.

$$
\begin{aligned}
\frac{e^{-Bz}}{(\log(ux))^z} \times \exp\left(\widehat{\mathcal{B}}(u, x; z)\right) &\geq \exp\left(-\frac{55}{4}\log^2(2)\right) \times \left(\frac{\log(ux)}{\log 2}\right)^{1+\frac{1}{z-1}+z} \\
&\quad \times \left(\frac{ux}{2}\right)^{\frac{3}{4}\left(1+\frac{2}{z-1}+\frac{1}{(z-1)^2}\right)} \\
&=: \widehat{\mathcal{C}}(u, x; z).
\end{aligned} \tag{8}
$$

Now we need to determine which values of u minimize the expression for the function defined in (8). For this we will use a somewhat weak elementary method from introdutory calculus in the form of the second derivative test with respect to u that immediately discards most of the dependence of (8) on x as we apply it. In particular, we can symbolically invoke the equation solver functionality in *Mathematica* to see that

$$
\frac{\partial}{\partial u}\left[\widehat{\mathcal{C}}(u, x; z)\right]\Bigg|_{u \mapsto u_0} = 0 \implies u_0 \in \left\{\frac{1}{x}, \frac{1}{x}e^{-\frac{4}{3}(z-1)}\right\}.
$$

When we substitute this outstanding parameter value of $u_0 =: \hat{u}_0 \mapsto \frac{1}{x}e^{-\frac{4}{3}(z-1)}$ into the next expression for the second derivative of the same function $\widehat{\mathcal{C}}(u, x; z)$ we obtain

$$
\begin{aligned}
\frac{\partial^2}{\partial u^2}\left[\widehat{\mathcal{C}}(u, x; z)\right]\Bigg|_{u = \hat{u}_0} &= \exp\left(-\frac{55}{4}\log^2(2)\right) x^2 2^{\frac{8z^3-27z^2+32z-16}{4(z-1)^2}} 3^{-z+\frac{1}{1-z}+1} e^{\frac{5z^2-16z+8}{3(z-1)}} \times \\
&\quad \times (1-z)^{z+\frac{1}{z-1}-2} z^2 \log^{\frac{z^2}{1-z}}(2) > 0,
\end{aligned}
$$

provided that $z < 1$, e.g., so that $k \leq \log\log x$ in Theorem 3.7. This restriction on k to note leads to a minimum value on the partial product, or lower bound, at this $u = \hat{u}_0$ since the second derivative is positive at the zero of the first derivative whenever $z < 1$.

After a substitution of $u = \frac{1}{x} e^{-\frac{4}{3}(z-1)}$ into the expression for $\widehat{\mathcal{C}}(u, x; z)$ defined above, we have that

$$\widehat{\mathcal{C}}(u, x; z) \geq \exp\left(-\frac{55}{4} \log^2(2)\right) \cdot 2^{\frac{9}{16}} \left(\frac{1-z}{3e \log 2}\right)^3 \times \left(\frac{4(1-z)}{3e \log 2}\right)^z.$$

Finally, since $z \equiv z(k, x) = \frac{k}{\log \log x}$ and $k \in [0, R \log \log x)$, we obtain that for small k and $x \gg 1$ large $\Gamma(z+1) \approx 1$, and for k towards the upper range of its interval that $\Gamma(z+1) \approx \Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$. In total, what we get out of these formulas is stated as in the theorem bounds. $\square$

Remark 5.4 (Future improvements on the lower bounds for $\mathcal{G}(z)$)**.** In some sense, what we have accomplished in proving the lower bound on $\mathcal{G}(z)$ using elementary calculus in the proof of Theorem 3.8 is sufficient, but still an overall very weak bound. The weakness of this bound contributes to the missing logarithmic factor when we show that

$$\limsup_{x \to \infty} \frac{|M(x)| \log x}{\sqrt{x}} = +\infty, \tag{9}$$

in Corollary 3.9. To actually tighen up the ship of our proof (so to speak) to show what is widely believed, and classically conjectured, as

$$\limsup_{x \to \infty} \frac{|M(x)|}{\sqrt{x}} = +\infty,$$

my best intuition suggests that finding a better (asymptotically higher order) lower bound in this step is crucial. Based on prior calculations, we want to (minimally) in fact show that the parameter u from the previous proof of Theorem 3.8 can allow us to take partial prime products indexed as high as $p \leq x$ (for x as in Theorem 3.7), and have that this scheme still obtains an accurate limiting lower bound on $\mathcal{G}(z)$ for large x. If this assumption is true for all large enough x, then the troublesome factor of $\log x$ in (9) vanishes when we proceed through the subsequent steps remaining in this proof using the new bound type of this improved form.

6 Precisely bounding the Dirichlet inverse functions, $g^{-1}(n)$

Conjecture 3.4 is not the most accurate simple way to express the limiting behavior of the Dirichlet inverse functions $g^{-1}(n)$ we can formulate, though it does capture an important characteristic. Namely, that these functions can be expressed via more simple formulas than inspection of the initial repetitive, quasi-periodic sequence properties might otherwise suggest.

With all of this in mind, we define the following sequence for integers $n \geq 1, k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{10}$$

We will illustrate by example the first few cases of these functions for small k after we prove the next lemma. The sequence of important semi-diagonals of these functions begins as [17, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

Lemma 6.1 (An exact formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu(n/d) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$\begin{aligned} g^{-1}(n) &= -\sum_{\substack{d|n \\ d>1}} (\omega(d) + 1) f^{-1}(n/d) & \implies \\ (g^{-1} * 1)(n) &= -(\omega * g^{-1})(n). \end{aligned}$$

Now by repeatedly expanding the right-hand-side, and removing corner cases in the nested sums since $\omega(1) = 0$ by convention, we find that

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n).$$

The statement follows by Möbius inversion applied to each side of the last equation. $\square$

Example 6.2 (Special cases of the functions $C_k(n)$ for small k)**.** We cite the following special cases which should be easy enough to see on paper by explicit computation using (10):

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$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= \sigma_0(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n) + 1} - \gcd(\Omega(n), \omega(n)). \end{aligned}$$

We have a recurrence relation between successive $C_k(n)$ values over k of the form

$$C_k(n) = \sum_{p|n} \sum_{d | \frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} C_{k-1}\left(d \cdot p^i\right). \tag{11}$$

Summary 6.3 (Asymptotics of the $C_k(n)$)**.** We have the following asymptotic relations relations for the growth of small cases of the functions $C_k(n)$:

$$\begin{aligned}
\mathbb{E}[C_1(n)] &= \log\log n \\
\mathbb{E}[C_2(n)] &= (\log\log n)^3.
\end{aligned}$$

The previous limiting asymptotics are computed from the explicit formulas for small k in Example 6.2 using the average order arguments such that $\mathbb{E}[\nu_p(n)] = \log\log n$ and for $p|n$, $\mathbb{E}[p] = \frac{n}{\log n}$.

Theorem 3.6 from the introduction is proved next. The theorem makes precise what these formulas already suggest about the main terms of the growth rates of $C_k(n)$ as functions of k, n for limiting cases of n large for fixed k. Since we will be essentially averaging the inverse functions, $g^{-1}(n)$, via their summatory functions over the range $n \leq x$ for x large, we tend not to bound any relevant components to obtaining these results but by the average order case, which evens out when we sum (i.e., average) and tend to infinity.

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Proof of Theorem 3.6. We showed how to compute the formulas for the base cases in the preceeding examples discussed above in Example 6.2. We can also see that $C_1(n)$ satsfies the formula we must establish when $k := 1$. Let's proceed by using induction to prove that our asymptotics hold for all $k \geq 1$ using the recurrence formula from (11) relating $C_k(n)$ to $C_{k-1}(n)$ whenever $k \geq 2$. In particular, suppose that $k \geq 2$ and let the inductive assumption for all $1 \leq m < k$ be that

$$\mathbb{E}[C_m(n)] = (\log\log n)^{2m-1}.$$

Now we have by the recursive formula in (11) that

$$\begin{aligned}
C_k(n) &= \sum_{p|n} \sum_{d | \frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} (\log\log(dp^i))^{2k-3} \\
&\sim \sum_{p|n} \sum_{d | \frac{n}{p^{\nu_p(n)}}} \left[\int (\log\log(dp^\alpha))^{2k-3} d\alpha\right]\Bigg|_{\alpha=\nu_p(n)}. \qquad (12)
\end{aligned}$$

The inner integral in the previous equation can be evaluated using the limiting asymptotic expansions for the incomplete gamma function stated in Section 4.3. In particular, for $p|n$ and $n \geq 2$ large, we let the parameters assume average order values of

$$\mathbb{E}[\nu_p(n)] = \log\log n, \mathbb{E}[p] = \frac{n}{\log n}.$$

Then we evaluate the integral from above as

$$\begin{aligned}
\int (\log\log(dp^\alpha))^{2k-3} d\alpha &\sim \alpha\left(\log d + \alpha \cdot \log p\right)^{2k-3} \\
&\sim \alpha\left(\log\alpha + \log\log p + \frac{d}{\alpha\log p}\right)^{2k-3}.
\end{aligned}$$

We know that the average order of the number of primes $p|n$ is given by $\mathbb{E}[\omega(n)] = \log\log n$, so approximating p as the cited function of n initially allows us to take a factor of $\log\log n$

and remove the outer divisor sum in (12). So we obtain that *

$$\begin{aligned}\mathbb{E}[C_k(n)] &= (\log\log n)^2 \left[\log\log\log n + \log\log n + \frac{\pi^2}{12}\frac{n}{\log n}\frac{1}{\left(\frac{n}{\log n}\right)^{\log\log n}}\right]^{2k-3} \\ &\sim (\log\log n)^{2k-1}.\end{aligned}$$

In the previous equation, we have used that the average order of the sum-of-divisors function, $\sigma_1(n)$, is given by $\mathbb{E}[\sigma_1(n)] = \frac{\pi^2 \cdot n}{12}$ [13, §27.11]. Thus by mathematical induction, we have proved that the claimed limiting asymptotic average order behavior holds for $C_k(n)$ whenever $k \geq 1$ as $n \to \infty$. □

Using Lemma 6.1 directly is complicated since forming the summatory function of the exact $g^{-1}(n)$ that obey this formula leads to a nested recurrence relation involving $M(x)$, e.g., more in-order sums of consecutive Möbius function terms that appear yet again in nested form. Some suggestive numerical experiments illustrate that this implicit recursive dependence of our new formulas for $M(x)$ can be avoided simply by using an inexact, but still provably asymptotically sufficient in form expression approximating $g^{-1}(n)$. The next corollary provides the specific inexact, and asymptotically accurate formula for these inverse functions that we have in mind.

What Corollary 6.4 below allows us to do is provide a substantially simpler formula and limiting bound on the summatory functions $G^{-1}(x)$ of $g^{-1}(n)$. The form of this new formula for $G^{-1}(x)$ is established in Corollary 6.5, which is subsequently stated and easily given a short proof immediately after the next result is established. This is an important leap in expressing a workable formula that we can use to bound these summatory functions from below when x is large. We require such a bound to prove the result rigorously justified in Theorem 7.4.

Corollary 6.4 (A simplification in form towards computing the inverse functions)**.** *For $n \geq 2$ as $n \to \infty$ we have that*

$$\mathbb{E}\left[\frac{\sum_{d|n} C_{\Omega(d)}(d)}{|g^{-1}(n)|}\right] \leq 1.$$

Thus if we let

$$\widetilde{G}^{-1}(x) := \sum_{n \leq x} \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d),$$

denote the summatory function defined by approximating $g^{-1}(n)$ by $\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)$, we obtain a lower bound in the form of

$$|G^{-1}(x)| \overset{\blacktriangle}{\gtrsim} \left|\widetilde{G}^{-1}(x)\right|.$$

*Here, we simplify the iterated logarithm expansions as $n \to \infty$ by writing

$$\begin{aligned}\log\log\left(\frac{n}{\log n}\right) &= \log\left[\log n + \log\left(1 + \frac{1}{n\log n}\right)\right] \\ &\sim \log\log n + \frac{1}{n(\log n)^2} \\ &\sim \log\log n.\end{aligned}$$

Proof. Let the approximation to the formula for $g^{-1}(n)$ from Lemma 6.1 be denoted by

$$S_R(n) := \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

The sign on the terms $C_{\Omega(d)}(d)$ in the cited approximation to $g^{-1}(n)$ given by $S_R(n)$ differs from Lemma 6.1 when

$$\operatorname{sgn}\left(\mu(d)\lambda(n/d)\right) = -\lambda(n) \iff \operatorname{sgn}\left(\mu(d)\lambda\left(\frac{n^2}{d}\right)\right) = -1.$$

By a case-by-case analysis of the parity of (n, d), we see that this occurs when one of two cases is met:

(1) n is even, d is even, and $\mu(d) = -1$; or

(2) n is odd, d is odd, and $\mu(d) = +1$.

According to the results on the asymptotic densities of the squarefree integers corresponding to $\mu(n) = \pm 1$ we cited in the preliminaries from Section 2, we obtain the following statements:

(3) The asymptotic density of the integers which are squarefree and satisfy either (1) or (2) is $\frac{3}{\pi^2}$;

(4) The asymptotic density of the integers which are squarefree and satisfy neither (1) nor (2) is $\frac{3}{\pi^2}$.

Moreover, the asymptotic density of the positive integers that are not squarefree is given by $1 - \frac{6}{\pi^2}$. Thus the limiting density of integers such that the sign of the terms in $S_R(n)$ matches those in Lemma 6.1 corresponds to $\frac{3}{\pi^2}$. The divisors d of n in the expression for $S_R(n)$ that do not contribute any weight to the formula for $g^{-1}(n)$ in Lemma 6.1 have asymptotic density $1 - \frac{6}{\pi^2}$ (the density of the non-squarefree integers).

The average order of the divisor function $d(n)$ is known to satisfy [13, §27.11]

$$\mathbb{E}[d(n)] = \log n + 2\gamma - 1 + o(1).$$

When we take the difference between $g^{-1}(n)$ from the lemma and the divisor sum $S_R(n)$, we have to subtract off the terms for the non-squarefree integers that vanish in the exact result due to the Möbius function, and then subtract off (twice) the terms that have leading coefficient of $+1$ where the exact formula weights the term by -1. What this argument leads to is an average order calculation showing that

$$\begin{aligned}
\mathbb{E}\left[\frac{\sum_{d|n} C_{\Omega(d)}(d)}{|g^{-1}(n)|}\right] &= \left|\frac{g^{-1}(n) - \lambda(n)\left(1 - \frac{6}{\pi^2}\right)\mathbb{E}[d(n)] + \lambda(n)\frac{6}{\pi^2}\mathbb{E}[d(n)]}{g^{-1}(n)}\right| \\
&= 1 - \mathbb{E}\left[\frac{\log n + 2\gamma - 1 + o(1)}{|g^{-1}(n)|}\right] \\
&= 1 - \frac{\log n + 2\gamma - 1 + o(1)}{\mathbb{E}\left[|g^{-1}(n)|\right]}. \qquad (13)
\end{aligned}$$

Now, due to the monotonically increasing nature of $C_k(n)$ in $k \geq 1$ for all large enough n, and since the terms in the exact expansion of $g^{-1}(n)$ in Lemma 6.1 are signed, clearly we obtain from Theorem 3.6 that

$$\mathbb{E}\left[|g^{-1}(n)|\right] \leq \mathbb{E}\left[C_{\Omega(n)}(n)\right] = (\log\log n)^{2\log\log n - 1}.$$

Since the right-hand-side bound in the previous equation has an asymptotically larger rate of growth that $\log n$, from (13) we have that the second term in the difference is $o(1)$ as $n \to \infty$. This implies the first bound we have claimed in this corollary. The second conclusion on the bounds satisfied when comparing $G^{-1}(x)$ to the approximate summatory function follows along the same lines as the proof of Lemma 7.3 given in Section 7.1.1. $\square$

Corollary 6.5. *We have that for sufficiently large x, as $x \to \infty$ that*

$$\left|G^{-1}(x)\right| \overset{\blacktriangle}{\gtrsim} \left|\widehat{L}_0(\log\log x) \times \sum_{n \leq \log\log x} \lambda(n) \cdot C_{\Omega(n)}(n)\right|,$$

where the function

$$\left|\widehat{L}_0(x)\right| \overset{\blacktriangle}{\gtrsim} \sqrt{\frac{2}{\pi}} A_0 \cdot x \frac{(\log x)^{2\log 2 + \frac{1}{3\log 2} - 1}}{(\log\log x)^{\frac{5}{2} + \log\log x}},$$

and such that $\operatorname{sgn}(\widehat{L}_0(x)) = (-1)^{\lfloor \log\log x \rfloor}$ *where the exponent* $2\log 2 + \frac{1}{3\log 2} - 1 \approx 0.867193$.

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Proof. Using Corollary 6.4, we have that

$$\begin{aligned}\left|G^{-1}(x)\right| &\overset{\blacktriangle}{\gtrsim} \left|\sum_{n \leq x} \lambda(n) \sum_{d|n} C_{\Omega(d)}(d)\right| \\ &= \left|\sum_{d \leq \log\log x} C_{\Omega(d)}(d) \times \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn)\right|.\end{aligned}$$

Now we see that by complete additivity of $\Omega(n)$ (or corresponding complete multiplicativity of $\lambda(n)$) that

$$\sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn) = \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(d)\lambda(n) = \lambda(d) \sum_{n \leq \left\lfloor \frac{x}{d} \right\rfloor} \lambda(n).$$

Using the result proved in Section 5 (see Theorem 3.8) we can establish that

$$\left|\sum_{n \leq x} \lambda(n)\right| \gg \left|\sum_{k \leq \log\log x} (-1)^k \cdot \widehat{\pi}_k(x)\right| =: \left|\widehat{L}_0(x)\right|.$$

Then since for large enough x and $d \ll x$,

$$\log(x/d) \sim \log x, \log\log(x/d) \sim \log\log x,$$

we can obtain the stated result, e.g., so that $\left|\widehat{L}_0(\log\log x)\right| \sim \left|\widehat{L}_0(\log\log(x/d))\right|$ for $d \leq \log\log x$ and large $x \to \infty$. The limiting lower bound stated for $\widehat{L}_0(x)$ is computed by symbolic summation in *Mathematica* using the new bounds on $\widehat{\pi}_k(x)$ guaranteed by Theorem 3.8. $\square$

7 Establishing the lower bounds for $M(x)$ by cases along infinite subsequences

7.1 The culmination of what we have done so far

As noted before in the previous subsections, we cannot hope to evaluate sums of functions weighted by $\lambda(n)$ that define $G^{-1}(x)$ except for on average using Abel summation. For this task, we need to know the bounds on $\widehat{\pi}_k(x)$ we developed in the proof of Corollary 3.8. A summation by parts argument proves the core component of the next proposition providing an integral formula based expression that approximates $M(x)$ closely through the form of $G^{-1}(x)$.

Proposition 7.1. *For all sufficiently large x, we have that*

$$M(x) \approx G^{-1}(x) + x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt, \tag{14}$$

where $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ is the summatory function of $g^{-1}(n)$.

Proof. We know by applying Corollary 3.3 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k)+1) \\ &= G^{-1}(x) + \sum_{k=1}^{x} g^{-1}(k)\pi(x/k), \end{aligned}$$

where we can drop the asymptotically unnecessary floored integer-valued arguments to $\pi(x)$ in place of its approximation by $\pi(x) \sim \frac{x}{\log x}$. In fact, since we can always bound

$$\frac{Ax}{\log x} \leq \pi(x) \leq \frac{Bx}{\log x},$$

for suitably defined absolute constants, $A, B > 0$, we are not losing any precision asymptotically by making this small leap in approximation from exact summation (by the first formula) to the integral formula case approximating $M(x)$ established below.

What we now require to sum and simplify the right-hand-side summation from the last equation is nothing short of an ordinary summation by parts argument. Namely, we obtain that for sufficiently large $x \geq 2$ *

$$\begin{aligned} \sum_{k=1}^{x} g^{-1}(k)\pi(x/k) &= G^{-1}(x)\pi(1) - \sum_{k=1}^{x-1} G^{-1}(k)\left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)\right] \\ &= -\sum_{k=1}^{x/2} G^{-1}(k)\left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)\right] \\ &\approx -\sum_{k=1}^{x/2} G^{-1}(k)\left[\frac{x}{k \cdot \log(x/k)} - \frac{x}{(k+1) \cdot \log(x/k)}\right] \\ &\approx -\sum_{k=1}^{x/2} G^{-1}(k)\frac{x}{k^2 \cdot \log(x/k)}. \end{aligned}$$

*Since $\pi(1) = 0$, the actual range of summation corresponds to $k \in \left[1, \frac{x}{2}\right]$.

Since for x large enough the summand is monotonic as k ranges in order over $k \in [1, x/2]$, and since the summands in the last equation are smooth functions of k (and x), and also since $G^{-1}(x)$ is a summatory function with jumps at the positive integers (signed in magnitude or not), we can approximate

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt.$$

Moreover, since the bounds of integration are finite, we do not need to dwell on the oscillatory nature of the factor of $G^{-1}(t)$ in the translation from summation to integral representation. We will later only use unsigned lower bound approximations to this function in the next theorems so that the signedness of the summatory function term in the integral formula above is a moot point entirely. $\square$

7.1.1 From the routine: Proofs of a few cut-and-dry lemmas

The results proved next in Lemma 7.2 and Lemma 7.3 are key to completely and carefully rigorously justifying the asymptotic bounds obtained below in Theorem 7.4. Thus these two somewhat more routine results are actually necessary to prove before we can return to the truly interesting matter of the unboundedness of $M(x)\log x/\sqrt{x}$ in the next subsection.

Lemma 7.2. *Suppose that $f_k(n)$ is a sequence of arithmetic functions such that $f_k(n) > 0$ for all $n \geq 1$, $f_0(n) = \delta_{n,1}$, and $f_{\Omega(n)}(n) \overset{\blacktriangle}{\gtrsim} \widehat{\tau}_\ell(n)$ as $n \to \infty$ where $\widehat{\tau}_\ell(t)$ is a continuously differentiable function of t for all large enough $t \gg 1$ †. We define the λ-sign-scaled summatory function of f as follows:*

$$F_\lambda(x) := \sum_{\substack{n \leq x \\ \Omega(n) \leq x}} \lambda(n) \cdot f_{\Omega(n)}(n).$$

Let

$$A_\Omega^{(\ell)}(t) := \sum_{k=1}^{\lfloor \log\log t \rfloor} (-1)^k \widehat{\pi}_k^{(\ell)}(t),$$

where $\widehat{\pi}_k(x) \geq \widehat{\pi}_k^{(\ell)}(x) \geq 0$ and $\widehat{\pi}_k^{(\ell)}(x)$ is a smooth monotone non-decreasing function of x for all x sufficiently large. Then we have that

$$F_\lambda(\log\log x) \overset{\blacktriangle}{\gtrsim} A_\Omega^{(\ell)}(\log\log x)\widehat{\tau}_\ell(\log\log x) - \int_1^{\log\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_\ell'(t) dt.$$

Proof. We first note that we can form an accurate $C^1(\mathbb{R})$ approximation by the smoothness of $\widehat{\pi}_k^{(\ell)}(x)$ that allows us to apply the Abel summation formula using the summatory function $A_\Omega^{(\ell)}(t)$ for t on any connected subinterval of $[1, \infty)$. The second stated formula for $F_\lambda(\log\log x)$ is valid by Abel summation provided that

$$\left| \frac{\sum\limits_{\log\log t < k \leq \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_\Omega^{(\ell)}(t)} \right| = o(1), \text{ as } t \to \infty,$$

† We will require that $\widehat{\tau}_\ell(t) \in C^1(\mathbb{R})$ when we apply the Abel summation formula in the proof of Theorem 7.4. At this point, it is technically an unnecessary condition that is vacously satisfied by assumption (by requirement) and will importantly need to hold only when we specialize to the actual functions employed to form our new bounds in the theorem below.

e.g., the asymptotically dominant terms indicating the parity of $\lambda(n)$ are captured by the terms in the range over k summed by $A_{\Omega}^{(\ell)}(t)$ for sufficiently large t as $t \to \infty$. Using the arguments in Montgomery and Vaughan [11, §7; Thm. 7.20] (see Theorem 5.2), we can see that uniformly in x and for any $r \in (1, 2)$

$$\left| \frac{\sum\limits_{r \cdot \log\log x < k \leq x} (-1)^k \cdot \widehat{\pi}_k(x)}{A_{\Omega}(x)} \right| \stackrel{\blacktriangle}{\gtrsim} \left| \frac{B(x,r)}{\log\log x + B(x,1)} \right| \ll \left| \frac{(\log x)^{r-1-r\log r}}{x(1+o(1))} \right| = o(1), \qquad (15)$$

as $x \to \infty$. Recall that the function $B(x,r)$ is defined and bounded as in the cited theorem from [11] given as it appears in the reference on page 24.

Thus we again emphasize and conclude that we have captured the asymptotically dominant main order terms in our formula as $x \to \infty$ using the definition of $A_{\Omega}(x)$. In other words, taking the sum over the summands that defines $A_{\Omega}(x)$ only over the truncated range of $k \in [1, \log\log x]$ does not affect the limiting asymptotically dominant terms obtained from using this formulation of the summatory function with the Abel summation formula – even when we should technically index over all $k \in [1, \log_2(x)]$ to obtain a precise formula for this function. $\square$

Observe that we use the superscript and subscript of (ℓ) not to denote a formal parameter to the functions we define below, but instead to denote that these functions form *lower bound* approximations to other forms of the functions without the scripted (ℓ).

Lemma 7.3. *Suppose that $\widehat{\pi}_k(x) \geq \widehat{\pi}_k^{(\ell)}(x) \geq 0$ with $\widehat{\pi}_k^{(\ell)}(x)$ a monotone non-decreasing real-valued function for all sufficiently large x. Let*

$$A_{\Omega}^{(\ell)}(x) := \sum_{k \leq \log\log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x)$$
$$A_{\Omega}(x) := \sum_{k \leq \log\log x} (-1)^k \widehat{\pi}_k(x).$$

Then for all sufficiently large x, we have that

$$|A_{\Omega}(x)| \gg |A_{\Omega}^{(\ell)}(x)|.$$

Proof. Given an explicit smooth lower bounding function, $\widehat{\pi}_k^{(\ell)}(x)$, we define the similarly smooth and monotone residual terms in approximating $\widehat{\pi}_k(x)$ using the following notation:

$$\widehat{\pi}_k(x) = \widehat{\pi}_k^{(\ell)}(x) + \widehat{E}_k(x).$$

Then we can form the ordinary form (i.e., the exact, non-lower-bound) on the summatory functions as

$$\begin{aligned}
|A_{\Omega}(x)| &= \left| \sum_{k \leq \frac{\log\log x}{2}} \left[\widehat{\pi}_{2k}(x) - \widehat{\pi}_{2k-1}(x)\right] \right| \\
&\geq \left| A_{\Omega}^{(\ell)}(x) - \sum_{k \leq \frac{\log\log x}{2}} \widehat{E}_{2k-1}(x) \right| \\
&\geq \left| A_{\Omega}^{(\ell)}(x) \right| - \left| \sum_{k \leq \frac{\log\log x}{2}} \widehat{E}_{2k-1}(x) \right|.
\end{aligned}$$

If the latter sum,

$$\mathrm{ES}(x) := \sum_{k \leq \frac{\log\log x}{2}} \widehat{E}_{2k-1}(x) \to \infty,$$

as $x \to \infty$, then we can always find some absolute (by monotonicity) $C_0 > 0$ such that $\mathrm{ES}(x) \leq C_0 \cdot A_\Omega(x)$. If on the other hand this sum becomes constant as $x \to +\infty$, then we also clearly have another absolute $C_1 > 0$ such that $|A_\Omega(x)| \geq C_1 \cdot |A_\Omega^{(\ell)}(x)|$. In either case, the claimed result holds for all large enough x. $\square$

7.1.2 A proof of the key bound from below on $G^{-1}(x)$

We use the result of Corollary 3.8 to prove the following central theorem:

Theorem 7.4 (Asymptotics and bounds for the summatory functions $G^{-1}(x)$)**.** *We define a lower summatory function, $G_\ell^{-1}(x)$, to provide bounds on the magnitude of $G^{-1}(x)$:*

$$|G_\ell^{-1}(x)| \ll |G^{-1}(x)|,$$

for all sufficiently large $x \gg 1$. We have the next asymptotic approximations for the lower summatory function where $C_{\ell,1}$ is the absolute constant defined by

$$C_{\ell,1} = 4A_0^2 = \frac{128 \cdot 2^{1/8}}{6561 \cdot e^6 \pi \log^6(2)} \exp\left(-\frac{55}{2}\log^2(2)\right) \approx 2.76631 \times 10^{-10}.$$

That is, we have

$$\left|G_\ell^{-1}(x)\right| \overset{\blacktriangle}{\gtrsim} C_{\ell,1} \cdot (\log\log x) \frac{(\log\log\log x)^{2\log 2 + \frac{1}{3\log 2} - 2}}{(\log\log\log\log x)^{\frac{5}{2}}} \cdot \frac{\log_*^5(x)^{2\log 2 + \frac{1}{3\log 2}}}{\log_*^6(x)^{\frac{5}{2}}}.$$

The exponent in the previous equation is numerically approximated as $2\log 2 + \frac{1}{3\log 2} - 2 \approx -0.402633 < 0$.

Initial Sketch: Logarithmic scaling of parameters to the accurate order. For the sums given by

$$S_{g^{-1}}(x) := \sum_{n \leq x} \lambda(n) \cdot C_{\Omega(n)}(n),$$

we notice that using the asymptotic bounds (rather than the exact formulas) for the functions $C_{\Omega(n)}(n)$ in Theorem 3.6, we have over-summed on n by quite a bit. In particular, following from the intent behind the constructions in the last sections, we are really summing only over all $n \leq x$ with $\Omega(n) \leq x$. Since $\Omega(n) \leq \lfloor \log_2 n \rfloor$ maximally, many of the terms in the previous equation are actually zero (recall that $C_0(n) = \delta_{n,1}$).

So we are actually only summing on n up to the average order of $\mathbb{E}[\Omega(n)] = \log\log n$ in practice. Hence, the sum that we are really interested in bounding is bounded below in magnitude by $S_{g^{-1}}(\log\log x)$ as bounded in Corollary 3.6. After noting this adjustment, we can then safely apply the asymptotic formulas for the functions $C_k(n)$ from Theorem 3.6 that hold once we have verified these important constraints on k, n, x. $\square$

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Proof. Recall from our proof of Corollary 3.8 that a lower bound on the variant prime form counting function is given by

$$\widehat{\pi}_k(x) \overset{\blacktriangle}{\gtrsim} \frac{A_0 \cdot x}{\log x \cdot (\log\log x)^4 \cdot (k-1)!} \cdot \left(\frac{4}{3e\log 2}\right)^k.$$

So we can then form a lower summatory function indicating the parity of all $\Omega(n)$ for $n \leq x$ as

$$\begin{aligned} \left|A_{\Omega}^{(\ell)}(t)\right| &= \left|\sum_{k \leq \log\log t} (-1)^k \widehat{\pi}_k(x)\right| \qquad (16) \\ &\overset{\blacktriangle}{\gtrsim} \sqrt{\frac{2}{\pi}} A_0 \cdot (\log\log x) \frac{(\log\log\log x)^{2\log 2 + \frac{1}{3\log 2} - 1}}{(\log\log\log\log x)^{\frac{5}{2} + \log\log\log\log x}}, \end{aligned}$$

where the actual sign on this function is given by $\operatorname{sgn}(A_{\Omega}^{(\ell)}(t)) = (-1)^{\lfloor \log\log\log\log x \rfloor}$ (see Lemma 7.3).

Next, by Corollary 3.6 we recover from the main term approximation to $C_k(n)$, denoted here by $\widehat{\tau}_0(t)$, that

$$\widehat{\tau}_0'(t) = \frac{d}{dx}\left[(\log\log t)^{2\log\log t - 1}\right] \overset{\blacktriangle}{\gtrsim} \frac{2(\log\log t)^{2\log\log t - 1}(\log\log\log t)}{t \cdot \log t}.$$

As in Lemma 7.2 and Corollary 6.5, we apply Abel summation to obtain that

$$G_{\ell}^{-1}(x) = \widehat{L}_0(\log\log x)\left[\widehat{\tau}_0(\log\log x) A_{\Omega}^{(\ell)}(\log\log x) - \widehat{\tau}_0(u_0) A_{\Omega}^{(\ell)}(u_0) - \int_{u_0}^{\log\log x} \widehat{\tau}_0'(t) A_{\Omega}^{(\ell)}(t) dt\right]. \qquad (17)$$

The inner integral term on the rightmost side of (17) is summed approximately in the form of

$$\begin{aligned} \int_{u_0}^{\log\log x} \widehat{\tau}_0'(t) A_{\Omega}^{(\ell)}(t) dt &\sim \sum_{k=u_0+1}^{\frac{1}{2}\log\log\log\log x} \left(I_{\ell}\left(e^{e^{2k+1}}\right) - I_{\ell}\left(e^{e^{2k}}\right)\right) e^{e^{2k}} \qquad (18) \\ &\approx C_0(u_0) + (-1)^{\left\lfloor \frac{\log\log\log\log x}{2} \right\rfloor} \times \int_{\frac{\log\log\log\log x}{2} - 1}^{\frac{\log\log\log\log x}{2}} I_{\ell}\left(e^{e^{2k}}\right) e^{e^{2k}} dk. \end{aligned}$$

We define the integrand function, $I_{\ell}(t) := \widehat{\tau}_0'(t) A_{\Omega}^{(\ell)}(t)$, from the previous equations with some limiting simplifications for the $k \in \left[\frac{\log\log\log\log x}{2} - 1, \frac{\log\log\log\log x}{2}\right]$ as

$$I_{\ell}\left(e^{e^{2k}}\right) e^{e^{2k}} \overset{\blacktriangle}{\gtrsim} \frac{2^{3/2} A_0}{\sqrt{\pi}} \left(\frac{2k}{\sqrt{e}}\right)^{4k} \frac{\log(2k)^{2\log 2 + \frac{1}{3\log 2}}}{\log\log(2k)^{\frac{5}{2} + \log\log(2k)}}. \qquad (19)$$

So using the lower bound on the integrand in (19), we find that [‡]

$$\begin{aligned} \widehat{L}_0(\log\log x) &\times \int_{\frac{\log\log\log\log x}{2} - 1}^{\frac{\log\log\log\log x}{2}} I_{\ell}\left(e^{e^{2k}}\right) e^{e^{2k}} dk \qquad (20) \\ &\overset{\blacktriangle}{\gtrsim} \frac{4A_0^2}{\pi} (\log\log x) \frac{(\log\log\log x)^{2\log 2 + \frac{1}{3\log 2} - 2}}{(\log\log\log\log x)^{\frac{5}{2}}} \cdot \frac{\log_*^5(x)^{2\log 2 + \frac{1}{3\log 2}}}{\log_*^6(x)^{\frac{5}{2}}}. \end{aligned}$$

It is clear from our prior computations of the growth of $A_{\Omega}^{(\ell)}(x)$ and $\widehat{\tau}_0(x)$ that the asymptotically dominant behavior of the lower bound for $|G_{\ell}^{-1}(x)|$ comes from the integral term calculated in the last equation of (20). □

[‡] We have invoked the simplification that for sufficiently large x,

$$(\log\log\log\log\log x)^2 \overset{\blacktriangle}{\gtrsim} \exp\left(-(\log\log\log\log\log\log x)^2\right).$$

7.2 Lower bounds on the scaled Mertens function along an infinite subsequence

Proof of Corollary 3.9. It suffices to take $u_0 = e^{e^{e^e}}$. Now, we break up the integral over $t \in [u_0, x/2]$ into two pieces: one that is easily bounded from $u_0 \leq t \leq \sqrt{x}$, and then another that will conveniently give us our slow-growing tendency towards infinity along the subsequence.

First, since $\pi(j) = \pi(\sqrt{x})$ for all $\sqrt{x} \leq j < x$, we can take the first chunk of the interval of integration and bound it using (14) as

$$-\int_{u_0}^{\sqrt{x}} \frac{2\sqrt{x}}{t^2 \log(x)} G_\ell^{-1}(t) dt \overset{\blacktriangle}{\gtrsim} B_{\ell,2} \times \frac{2}{\log(x)} \cdot \left(\min_{u_0 \leq t \leq \sqrt{x}} G_\ell^{-1}(t)\right) = o\left(1\right),$$

where $B_{\ell,2}$ can be taken as an indefinite, but still some absolute constant with respect to u_0. The maximum in the previous equation is clearly attained by taking $t := \sqrt{x}$.

We next have to prove a related bound over the second portion of the interval from $\sqrt{x} \leq t \leq x/2$:

$$\begin{aligned} -\int_{\sqrt{x}}^{x/2} & \frac{2x}{t^2 \log(x)} \cdot G_\ell^{-1}(t) dt \overset{\blacktriangle}{\gtrsim} \frac{2\sqrt{x}}{\log x} \cdot \left(\min_{\sqrt{x} \leq t \leq x/2} G_\ell^{-1}(t)\right) \\ &= 2C_{\ell,1} \cdot \sqrt{x} \cdot \frac{(\log\log\sqrt{x})}{\log x} \frac{(\log\log\log\sqrt{x})^{2\log 2 + \frac{1}{3\log 2} - 2}}{(\log\log\log\log\sqrt{x})^{\frac{5}{2}}} \cdot \frac{\log_*^5(\sqrt{x})^{2\log 2 + \frac{1}{3\log 2}}}{\log_*^6(\sqrt{x})^{\frac{5}{2}}} + o(1). \end{aligned}$$

Finally, since $G_\ell^{-1}(x) = o(\sqrt{x})$, we obtain in total that as $x \to \infty$ along this infinite subsequence:

$$|M(x)| \overset{\blacktriangle}{\gtrsim} 2C_{\ell,1} \cdot \sqrt{x} \cdot \frac{(\log\log\sqrt{x})}{\log x} \frac{(\log\log\log\sqrt{x})^{2\log 2 + \frac{1}{3\log 2} - 2}}{(\log\log\log\log\sqrt{x})^{\frac{5}{2}}} \cdot \frac{\log_*^5(\sqrt{x})^{2\log 2 + \frac{1}{3\log 2}}}{\log_*^6(\sqrt{x})^{\frac{5}{2}}}.$$

Remarks. What we have shown in total is a logarithmically scaled lesser form of the classically conjectured unboundedness property of $M(x)$ in the form of

$$\limsup_{x\to\infty} \frac{|M(x)| \log x}{\sqrt{x}} = +\infty.$$

This statement still comprises a better than previously known rate of the minimal asymptotic tendencies of $|M(x)|/\sqrt{x}$ towards unboundedness along an infinite subsequence, e.g., progress on the problem. □

8 Conclusions

8.1 Summary

- Using average order bounds, summatory functions, and the $\overset{\blacktriangle}{\gtrsim}$-type relations for lower bounds.
- Somewhat oddly, we did not need substantially improved bounds on $L_0(x) := \sum_{n\leq x} \lambda(n)$ than what is already known in upper bound form to obtain our new bounds on the Mertens function, aka, summatory function of the "testier" Möbius function.

8.2 Future research and work that still needs to be done

- Refinements of these bounds to find the tightest possible lower (limit supremum) bounds, e.g., proofs of an optimal version of Gonek's original conjecture.
- Generalizations to weighted Mertens functions of the form $M_\alpha(x) := \sum_{n\leq x} \mu(n)n^{-\alpha}$.
- Indications of sign changes and exceptionally small, or zero values of $M(x)$.
- What our more combinatorial approach to bounding $M(x)$ effectively suggests about necessary, but unproved, zeta zero bounds that have historically formed the basis for arguments bounding $M(x)$ using Mellin inversion.
- Evaluate alternate strategies and approaches using different Dirichlet convolution functions besides g and $g^{-1}(n)$ (corresponding to $\pi(x)$) with Theorem 3.1.

8.3 Motivating a general technique towards bounding the summatory functions of arbitrary arithmetic f

8.4 The general construction using Theorem 3.1

8.4.1 A proposed generalization

For each $n \geq 1$, let $A(n) \subseteq \{d : 1 \leq d \leq n, d|n\}$ be a subset of the divisors of n. We say that a natural number $n \geq 1$ is *A-primitive* if $A(n) = \{1, n\}$. Under a list of assumptions so that the resulting A-convolutions are *regular convolutions*, we get a generalized multiplicative Möbius function [15, §2.2]:

$$\mu_A(p^\alpha) = \begin{cases} 1, & \alpha = 0; \\ -1, & p^\alpha > 1 \text{ is } A\text{-primitive}; \\ 0, & \text{otherwise.} \end{cases}$$

We also define the functions $\omega_A(n) := \#\{d|n : \mathrm{d\ is\ an\ A-primitive\ factor\ of\ n}\}$ and $\Omega_A(n) := \#\{p^\alpha||n : \mathrm{p\ is\ an\ A-primitive\ factor\ of\ n}\}$. Then the characteristic function of the set $A := \cup_{n\geq 1} A(n)$ is given by $\chi_A(n) = [n \in A]_\delta$. By Möbius inversion, we have that $\chi_A = \omega_A *_A \mu_A$. Moreover, for the A-counting function, $\pi_A(x)$, defined by

$$\pi_A(x) := \#\{n \leq x : n \in A\},$$

we can define a corresponding notion of a generalized A-Mertens function, $M_A(x) := \sum_{n \leq x} \mu_A(n)$. This function then satisfies (by Theorem 3.1) the relation that

$$M_A(x) = \sum_{k=1}^{x} (\omega_A + 1)^{-1}(k) \cdot \pi_A(x/k),$$

where the inverse function, $(\omega_A + 1)^{-1}(n)$, is defined with respect to A-convolution. We conjecture, but do not prove here, that $\operatorname{sgn}((\omega_A + 1)^{-1}(n)) = \lambda_A(n) =: (-1)^{\Omega_A(n)}$.

Using formulas similar in construction to (14), we can differentiate to find expressions for $\pi_A(x)$. The significance of this is that provided we can prove sufficiently large bounds for $M_A(x)$ along the same lines as we have done for $M(x)$, the resulting formula may be able to speak towards the density, or even infinitude in special cases, of the set A.

8.5 Working / TODO

(i) The average order, $\mathbb{E}[\omega(n)] = \log\log n$, imparts an iterated logarithmic structure to our expansions, which many have conjectured we should see in limiting bounds on $M(x)$, but which are practically elusive in most non-conjectural known formulas I have seen proved rigorously in print.

(ii) The additivity of $\omega(n)$ dictates that the sign of $g^{-1}(n) = (\omega + 1)^{-1}(n)$ is $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ (see Proposition 4.2). The corresponding weighted summatory functions of $\lambda(n)$ have more established predictable properties, such as known sign biases and upper bounds. These summatory functions are generally speaking more regular and easier to work with than traditional approaches to summing $M(x)$ and its complicating summand terms of the Möbius function. Note that our proof is essentially much different than what is known about sums of consecutive values of $\mu(n)$ over short intervals, both in interpretation and methodology.

- The method easily generalizes to the $\mu^2(n)$ / $Q(x)$ bounds suggested by the form of the function $\mathcal{G}(z)$ suggested in the exercises section of [11].

- Acks: Thanks to Lacey for letting me win the new laptop bet by working on this problem ...

More generally, we have that for f a non-negative additive arithmetic function that vanishes at one, $\operatorname{sgn}((f+1)^{-1}) = \lambda(n) = (-1)^{\Omega(n)}$. We can state similar properties for the common case of multiplicative f in the form of the following result: If $f(n) > 0$ for all $n \geq 1$ and f is multiplicative, then $\operatorname{sgn}(f^{-1}(n)) = (-1)^{\omega(n)}$.

The following observation that is suggestive of the semi-periodicity at play with the distinct values of $g^{-1}(n)$ distributed over $n \geq 2$.

Heuristic 8.1 (Symmetry in $g^{-1}(n)$ in the exponents in the prime factorization of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly two, three, and so on prime factors. There does not appear to be an easy, nor subtle direct recursion between the distinct g^{-1} values, except through auxiliary function sequences. We will settle for an asymptotically accurate main term approximation to $g^{-1}(n)$ for large n as $n \to \infty$ in the average case.

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A Appendix: Supplementary tables and data

T.1 Table: Computations with a signed Dirichlet inverse function and its summatory function

n	Primes	–	Sqfree	PPower	$\bar{\mathbb{S}}$	–	$g^{-1}(n)$	$\lambda(n)\,\mathrm{sgn}(g^{-1}(n))$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_2(n)$	–	$G^{-1}(n)$	$G^{-1}_{+}(n)$	$G^{-1}_{-}(n)$
1	1^1	–	Y	N	N	–	1	1	0	0	–	1	1	0
2	2^1	–	Y	Y	N	–	−2	1	0	0	–	−1	1	−2
3	3^1	–	Y	Y	N	–	−2	1	0	0	–	−3	1	−4
4	2^2	–	N	Y	N	–	2	1	0	−1	–	−1	3	−4
5	5^1	–	Y	Y	N	–	−2	1	0	0	–	−3	3	−6
6	2^13^1	–	Y	N	N	–	5	1	0	−1	–	2	8	−6
7	7^1	–	Y	Y	N	–	−2	1	0	0	–	0	8	−8
8	2^3	–	N	Y	N	–	−2	1	0	−2	–	−2	8	−10
9	3^2	–	N	Y	N	–	2	1	0	−1	–	0	10	−10
10	2^15^1	–	Y	N	N	–	5	1	0	−1	–	5	15	−10
11	11^1	–	Y	Y	N	–	−2	1	0	0	–	3	15	−12
12	2^23^1	–	N	N	Y	–	−7	1	2	−2	–	−4	15	−19
13	13^1	–	Y	Y	N	–	−2	1	0	0	–	−6	15	−21
14	2^17^1	–	Y	N	N	–	5	1	0	−1	–	−1	20	−21
15	3^15^1	–	Y	N	N	–	5	1	0	−1	–	4	25	−21
16	2^4	–	N	Y	N	–	2	1	0	−3	–	6	27	−21
17	17^1	–	Y	Y	N	–	−2	1	0	0	–	4	27	−23
18	2^13^2	–	N	N	Y	–	−7	1	2	−2	–	−3	27	−30
19	19^1	–	Y	Y	N	–	−2	1	0	0	–	−5	27	−32
20	2^25^1	–	N	N	Y	–	−7	1	2	−2	–	−12	27	−39
21	3^17^1	–	Y	N	N	–	5	1	0	−1	–	−7	32	−39
22	2^111^1	–	Y	N	N	–	5	1	0	−1	–	−2	37	−39
23	23^1	–	Y	Y	N	–	−2	1	0	0	–	−4	37	−41
24	2^33^1	–	N	N	Y	–	9	1	4	−3	–	5	46	−41
25	5^2	–	N	Y	N	–	2	1	0	−1	–	7	48	−41
26	2^113^1	–	Y	N	N	–	5	1	0	−1	–	12	53	−41
27	3^3	–	N	Y	N	–	−2	1	0	−2	–	10	53	−43
28	2^27^1	–	N	N	Y	–	−7	1	2	−2	–	3	53	−50
29	29^1	–	Y	Y	N	–	−2	1	0	0	–	1	53	−52
30	$2^13^15^1$	–	Y	N	N	–	−16	1	0	−4	–	−15	53	−68
31	31^1	–	Y	Y	N	–	−2	1	0	0	–	−17	53	−70
32	2^5	–	N	Y	N	–	−2	1	0	−4	–	−19	53	−72
33	3^111^1	–	Y	N	N	–	5	1	0	−1	–	−14	58	−72
34	2^117^1	–	Y	N	N	–	5	1	0	−1	–	−9	63	−72
35	5^17^1	–	Y	N	N	–	5	1	0	−1	–	−4	68	−72
36	2^23^2	–	N	N	Y	–	14	1	9	1	–	10	82	−72
37	37^1	–	Y	Y	N	–	−2	1	0	0	–	8	82	−74
38	2^119^1	–	Y	N	N	–	5	1	0	−1	–	13	87	−74
39	3^113^1	–	Y	N	N	–	5	1	0	−1	–	18	92	−74
40	2^35^1	–	N	N	Y	–	9	1	4	−3	–	27	101	−74
41	41^1	–	Y	Y	N	–	−2	1	0	0	–	25	101	−76
42	$2^13^17^1$	–	Y	N	N	–	−16	1	0	−4	–	9	101	−92
43	43^1	–	Y	Y	N	–	−2	1	0	0	–	7	101	−94
44	2^211^1	–	N	N	Y	–	−7	1	2	−2	–	0	101	−101
45	3^25^1	–	N	N	Y	–	−7	1	2	−2	–	−7	101	−108
46	2^123^1	–	Y	N	N	–	5	1	0	−1	–	−2	106	−108
47	47^1	–	Y	Y	N	–	−2	1	0	0	–	−4	106	−110
48	2^43^1	–	N	N	Y	–	−11	1	6	−4	–	−15	106	−121

Table T.1: *Computations of* $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ *for small* $\mathbf{1 \leq n \leq 48}$.
The column labeled `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled, respectively,* `Sqfree`*,* `PPower` *and $\bar{\mathbb{S}}$ list inclusion of n in the sets of squarefree integers, prime powers, and the set $\bar{\mathbb{S}}$ that denotes the positive integers n which are neither squarefree nor prime powers. The next two columns provide the explicit values of the inverse function $g^{-1}(n)$ and indicate that the sign of this function at n is given by $\lambda(n) = (-1)^{\Omega(n)}$.*
The next two columns show the small-ish magnitude differences between the unsigned magnitude of $g^{-1}(n)$ and the summations $\widehat{f}_1(n) := \sum_{k\geq 0}\binom{\omega(n)}{k}\cdot k!$ and $\widehat{f}_2(n) := \sum_{k \geq 0}\binom{\omega(n)}{k}\cdot \#\{d|n : \omega(d) = k\}$. Finally, the last three columns show the summatory function of $g^{-1}(n)$, $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$, deconvolved into its respective positive and negative components: $G_{+}^{-1}(x) := \sum_{n\leq x} g^{-1}(n)\left[g^{-1}(n) > 0\right]_{\delta}$ and $G_{-}^{-1}(x) := \sum_{n\leq x} g^{-1}(n)\left[g^{-1}(n) < 0\right]_{\delta}$.

A03-2020-04026

A04-2020-04026