Ernie,
We are tasked with asserting that the following proof is correct:

Theorem 1.2. *There is an absolute constant $B_0 > 0$ such that*

$$\frac{1}{n} \times \sum_{k \leq n} \log C_\Omega(k) = B_0 \cdot (\log\log n)(\log\log\log n)\left(1 + O\left(\frac{1}{\log\log n}\right)\right), \quad \text{as } n \to \infty.$$

Proof of Theorem 1.2. We first use (1.9) to see that there is an absolute constant $P_0 > 0$ such that

$$\sum_{k \geq 1} \sum_{\substack{n \leq x \\ \Omega(n) = k}} \log C_\Omega(n) = \sum_{k \geq 1} P_0 \times \#\{n \leq x : \Omega(n) = k\} \times \log(k!). \tag{2.2}$$

For $x \geq 3$, consider the following partial sums:

$$L_\Omega(x) := \sum_{1 \leq k \leq \frac{3}{2}\log\log x} \sum_{\substack{n \leq x \\ \Omega(n) = k}} \log C_\Omega(n).$$

For any $z \geq 0$, we cite the following known form of Binet's formula for the log-gamma function [17, §5.9(i)]:

$$\log z! = \left(z + \frac{1}{2}\right)\log(1 + z) - z + O(1).$$

Then provided that (2.2) holds, there is an absolute constant $B_0^* > 0$ such that

$$L_\Omega(x) = \sum_{1 \leq k \leq \frac{3}{2}\log\log x} \frac{B_0^* x (\log\log x)^{k-1}}{(\log x)(k-1)!}\left(\left(k + \frac{1}{2}\right)\log(1 + k) - k\right)\left(1 + O\left(\frac{1}{\log\log x}\right)\right). \tag{2.3}$$

The right-hand-side of (2.3) can be approximated by Abel summation using the functions

$$A_x(u) := \frac{B_0^* x \Gamma(u, \log\log x)}{\Gamma(u)}; f(u) := \frac{(2u+1)}{2}\log(1+u) - \frac{(2u+1)}{2}, f'(u) = \log(1+u) - \frac{1}{2(1+u)}.$$

Then we have by Proposition C.3 that for some absolute constant $B_0 > 0$

$$\begin{aligned} L_\Omega(x) &= A_x\left(\frac{3}{2}\log\log x\right) f\left(\frac{3}{2}\log\log x\right) - \int_0^{\frac{3}{2}} A_x(\alpha\log\log x) f'(\alpha\log\log x)\, d\alpha \\ &= B_0 x (\log\log x)(\log\log\log x)\left(1 + O\left(\frac{1}{\log\log x}\right)\right). \end{aligned}$$

It suffices to show

$$\sum \qquad \log C_\Omega(n) = o(x\log\log x), \quad \text{as } x \to \infty. \tag{2.4}$$

First, some intuition for why to expect the result in Thm 1.2 is correct:

(1) Erdös-Kac type CLT for $\Omega(n)$:

$$\#\left\{n \le x : \frac{\Omega(n) - \ell\ell x}{\sqrt{\ell\ell x}} \le z\right\} = x \cdot \Phi(z) + O\left(\frac{1}{\ell\ell x}\right),$$

here, $\ell\ell x \equiv \log\log x$, and similarly henceforth, $\ell\ell\ell x \equiv \log\log\log x$.

(2) We also have the next result stated in Appendix B. Note that $f(r) := r - 1 - r\log r$ is zero when $r = 1$, and f is decreasing for both $r \in (0,1)$ and $r \in (1,2)$.

Theorem B.1. *For $x \geq 2$ and $r > 0$, let*

$$A(x,r) := \#\{n \leq x : \Omega(n) \leq r \log\log x\},$$
$$B(x,r) := \#\{n \leq x : \Omega(n) \geq r \log\log x\}.$$

If $0 < r \leq 1$, then

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

If $1 \leq r \leq R < 2$, then

$$B(x,r) \ll_R x(\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

- Now, the asymptotic density of the squarefree integers is

$$\frac{1}{x}\sum_{n\leq x}\mu^2(n) = \frac{6}{\pi^2} + O\left(\frac{1}{\sqrt{x}}\right),$$

with $0 < \frac{6}{\pi^2} \simeq 0.61 < 1$.

- Recall that for all squarefree $n \geq 1$, $C_\ell(n) = (\Omega(n))!$.
- Hence, we expect there to be a limiting absolute constant $R_0 > \frac{6}{\pi^2}$ such that

$$\frac{1}{x}\sum_{n\leq x}\log C_\ell(n) = \log\left[R_0\Gamma(\ell\ell x+1)\right](1+o(1))$$
$$= (\ell\ell x)(\ell\ell\ell x)(1+o(1)).$$

That's it for intuition. Now to prove that this result holds formally, let's see what we've got:

- Let $S_\Omega(x) := \sum_{n \leq x} \log C_\Omega(x)$.
- We should expect (pf. to follow) that the main term for $S_\Omega(x)$ comes from the restricted sums

$$L_\Omega(x) := \sum_{\substack{n \leq x \\ \Omega(n) \leq 3\log\log x}} \log C_\Omega(n).$$

- We can evaluate an asymptotic formula for $L_\Omega(x)$ using the following result proved in Appendix C:

Proposition C.3. *Let a, z, ρ be positive real parameters such that $z = \rho a$. If $\rho \in (0,1)$, then as $z \to \infty$*

$$\Gamma(a,z) = \Gamma(a) + O_\rho\left(z^{a-1}e^{-z}\right).$$

If $\rho > 1$, then as $z \to \infty$

$$\Gamma(a,z) = \frac{z^{a-1}e^{-z}}{1-\rho^{-1}} + O_\rho\left(z^{a-2}e^{-z}\right).$$

If $\rho > W(1) \approx 0.56714$, then as $z \to \infty$

$$\Gamma(a, ze^{\pm\pi\imath}) = -e^{\pm\pi\imath a}\frac{z^{a-1}e^{z}}{1+\rho^{-1}} + O_\rho\left(z^{a-2}e^{z}\right).$$

(use cases of $\rho < \frac{2}{3}$, $\rho \geq \frac{2}{3}$).

- To bound the error term for $S_\Omega(x)$, consider the sums defined by

$$L^*_\Omega(x) := \sum_{\substack{n \leq x \\ \Omega(n) \geq \frac{3}{2}\log\log x}} \log(\Omega(n)).$$

- By the argument above showing that the asymptotic density of the squarefree $n \leq x$ is $0 < \frac{6}{\pi^2} < 1$, we see that for "most" $n \leq x$, $C_\Omega(n) = (\Omega(n))!$. Then by Binet's asymptotic expansion for the log-gamma function, for "most" of the $n \leq x$, $\log C_\Omega(n) \ll \Omega(n) \cdot \log \Omega(n)$

- So we can construct a good enough upper bound on $L^*_\Omega(x)$ as follows:

$$L^*_\Omega(x) \ll \sum_{\substack{n \leq x \\ \Omega(n) \geq \frac{3}{2} \log\log x}} \Omega(n) \log \Omega(n) \quad (**)$$

We know that for $\{x_n\}_{n \geq 1} \subseteq \mathbb{R}$ that satisfy $x_n \geq 1$ $\forall n \geq 1$:

$$\sum_{n \leq x} x_n^2 \leq \left(\sum_{n \leq x} x_n\right)^2 \quad (***)$$

Let $L_\Omega^{**}(x) := \frac{1}{x} \sum_{n \le x} \log \Omega(n)$.

By (***) and the Cauchy-Schwarz inequality, we have

$$(**) \ll \left(\frac{1}{x} \sum_{\substack{n \le x \\ \Omega(n) \ge \frac{3}{2} \log\log x}} \Omega(n)^2 \right)^{1/2} \cdot L_\Omega^{**}(x)$$

$$\ll L_\Omega^{**}(x) \cdot \sum_{\substack{n \le x \\ \Omega(n) \ge \frac{3}{2} \log\log x}} \Omega(n)$$

$$\ll x \cdot (\log x)^{\alpha} \cdot L_\Omega^{**}(x),$$

where $\alpha := \frac{3}{2} - 1 - \frac{3}{2} \log\left(\frac{3}{2}\right) \simeq -0.108198 < 0$, by Theorem B.1.

- Finally, we just need to show that the main term of $L_2^{**}(x)$ is sufficiently small.
- Recall a form of the AGM inequality for a sequence of $\{a_n\}_{n\geq 1} \subseteq \mathbb{R}_+$:

$$\left(\prod_{1\leq j\leq x} a_j\right)^{\frac{1}{x}} \leq \frac{1}{x} \cdot \sum_{n\leq x} \log(a_n).$$

- With $a_n := \Omega(n)$ for all $n \geq 1$:

$$\left(\prod_{j\leq x} \Omega(j)\right)^{\frac{1}{x}}$$

$$\ll x \exp\left(\frac{1}{x} \sum_{\substack{n\leq x \\ \Omega(n) \geq \frac{3}{2}\log\log x}} \Omega(n)\right)$$

$$\ll x \cdot \exp\left((\log x)^{1-\beta}\right),$$

where $\beta := 1-\delta$

$\simeq 1.1082 > 1$

$$\ll x^{1+\frac{1}{(\log x)^{\beta}}}$$

$$= x(1+o(1)), \text{ as } x \to +\infty$$

(*4)

- Hence, by (*4), we see that

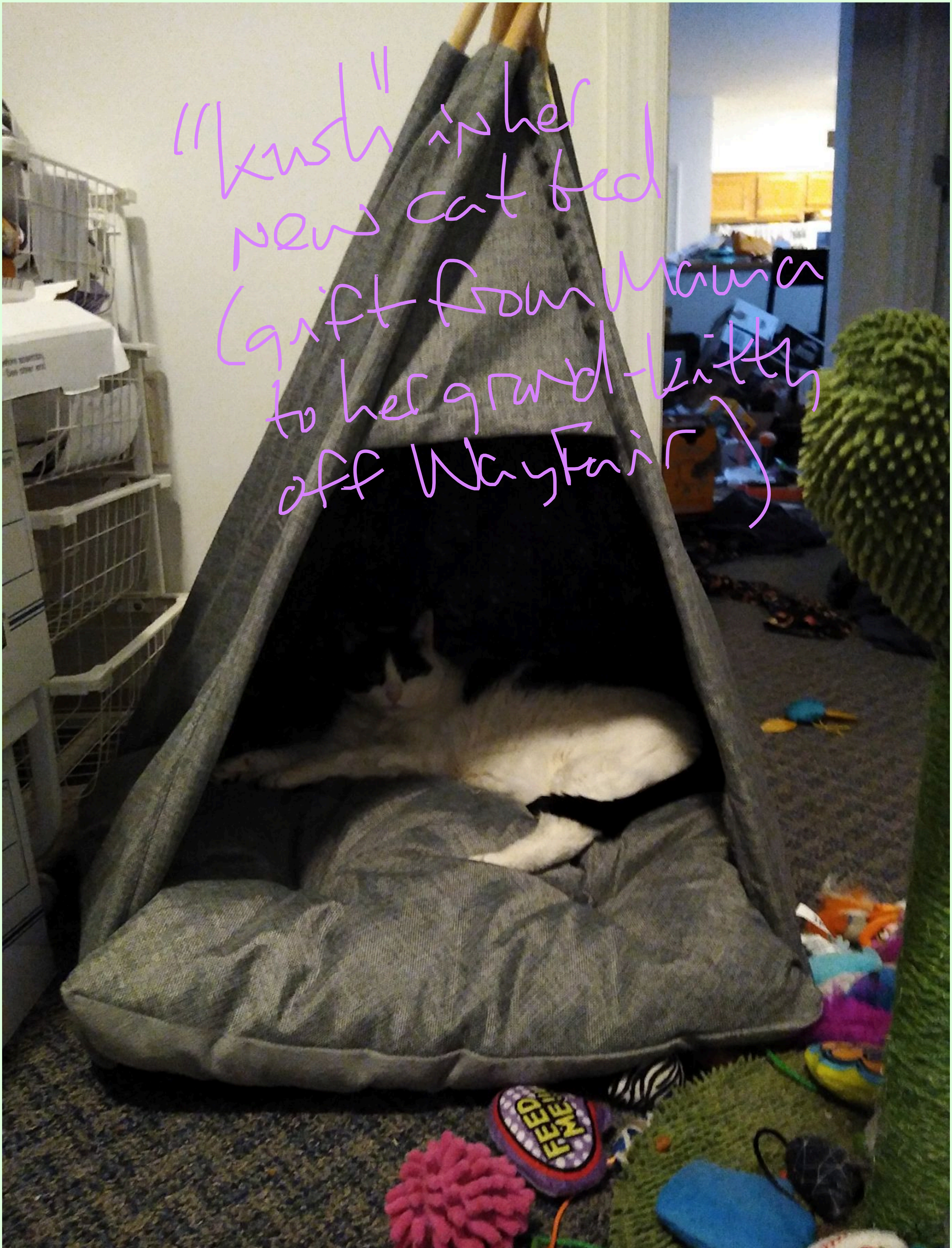
$$L^{**}_{\Omega}(x) = o\left((\ell\ell x)(\ell\ell\ell x)\right).$$

- In total, we find that

$$\frac{1}{x} L^{*}_{\Omega}(x) \ll \frac{(\ell\ell x)(\ell\ell\ell x)}{(\log x)^{1.1081}}$$

- I conclude in response to your last comments over email that the pf. of Thm. 1.2 is cogent.
- May we please now drop bickering over slicker proofs and try to work towards verifying that the rest of the paper is reasonably error free?

— MDS


"Kush" in her
new cat bed
(gift from Mama
to her grand-kitty
off WayFair)
FEED ME