Lower bounds on the summatory function of the Möbius function along infinite subsequences

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Last Revised: Monday 20^{th} July, 2020 @ 18:20:57 – Compiled with LaTeX2e

Abstract

The Mertens function, $M(x) = \sum_{n \leq x} \mu(n)$, is classically defined as the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture states that $|M(x)| < C \cdot \sqrt{x}$ for some absolute $C > 0$ for all $x \geq 1$. This classical conjecture has a well-known disproof due to Odlyzko and té Riele by computation of non-trivial zeta function zeros in conjunction with integral formulas expressing $M(x)$. We prove the unboundedness of $|M(x)|/\sqrt{x}$ using new methods by showing that

$$\limsup_{x \rightarrow \infty} \frac{|M(x)| \cdot (\log\log x)^{\frac{3}{2}} (\log\log\log x)^2}{\sqrt{x} \cdot (\log x)^{\frac{1}{4}}} > 0.$$

There is a distinct stylistic flavor and new element of combinatorial analysis to our proof combined with the standard methods from analytic, additive and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

Keywords and Phrases: *Möbius function; Mertens function; summatory function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting functions; Dirichlet generating function; asymptotic lower bounds; Mertens conjecture.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; and 11N64.*

Glossary of special notation and conventions

Symbol — **Definition**

$\approx$ — We write that $f(x) \approx g(x)$ if $|f(x) - g(x)| = O(1)$ as $x \to \infty$.

$\mathbb{E}[f(x)], \overset{\mathbb{E}}{\sim}$ — We adapt the expectation notation $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \overset{\mathbb{E}}{\sim} h(x)$, to denote that f has an *average order* growth rate of $h(x)$. This means that $\frac{1}{x}\sum_{n\leq x} f(n) \sim h(x)$, or equivalently that

$$\lim_{x\to\infty} \frac{\frac{1}{x}\sum_{n\leq x} f(n)}{h(x)} = 1.$$

B — The absolute constant $B \approx 0.2614972$ from the statement of Mertens theorem.

$C_k(n)$ — The sequence is defined recursively for $n \geq 1$ as follows where we assume that $1 \leq k \leq \Omega(n)$:

$$C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$

$[q^n]F(q)$ — The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n\geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n\geq 0} f_n q^n$.

d_k — For non-negative integers $k \geq 0$, we define the densities d_k of the distinct values of the differences of the prime omega functions by $d_k := \lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leq x : \Omega(n) - \omega(n) = k\}$.

$\varepsilon(n)$ — The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (see below).

$f * g$ — The Dirichlet convolution of f and g, $(f * g)(n) := \sum\limits_{d|n} f(d) g(n/d)$, where the sum is taken over the divisors d of n for $n \geq 1$.

$f^{-1}(n)$ — The Dirichlet inverse of f with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d|n \\ d>1}} f(d) f^{-1}(n/d)$ for $n \geq 2$ with $f^{-1}(1) = 1/f(1)$. The Dirichlet inverse of f exists if and only if $f(1) \neq 0$. This inverse function, denoted by f^{-1} when it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$.

$\gg, \ll, \asymp$ — For functions A, B in x, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A \ll B$ and $B \gg A$, we write $A \asymp B$.

$g^{-1}(n), G^{-1}(x)$ — The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$.

H_n — The *first-order harmonic numbers*, $H_n := \sum_{k=1}^{n} \frac{1}{k}$, satisfy the limiting asymptotic relation

$$\lim_{n\to\infty} [H_n - \log(n)] = \gamma,$$

where $\gamma \approx 0.5772157$ denotes Euler's gamma constant.

Symbol	**Definition**
$[n=k]_{\delta}$,$[\mathtt{cond}]_{\delta}$	The symbol $[n=k]_{\delta}$ is a synonym for $\delta_{n,k}$ which is one if and only if $n=k$, and is zero otherwise. For boolean-valued conditions, `cond`, $[\mathtt{cond}]_{\delta}$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention.*
$\lambda_*(n)$	For positive integers $n \geq 2$, we define the next variant of the Liouville lambda function, $\lambda(n)$, as follows: $\lambda_*(n) := (-1)^{\Omega(n)-\omega(n)} = \lambda(n)(-1)^{\omega(n)}$. We have the initial condition that $\lambda_*(1) = 1$.
$\lambda(n)$	The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. That is, $\lambda(n) \in \{\pm 1\}$ for all integers $n \geq 1$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree.
$M(x)$	The Mertens function is the summatory function of $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum\limits_{n \leq x} \mu(n)$.
$\Phi(z)$	For $x \in \mathbb{R}$, we define the function $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{z} e^{-t^2/2} dt$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^{\alpha} \vert\vert n$ (or when p^{α} exactly divides n) for p prime, $\alpha \geq 1$ and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p \vert n} 1$ and the completely additive function $\Omega(n) := \sum_{p^{\alpha} \vert\vert n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leq n \leq x$ for $x > 1$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$ for $x \geq 2$.
$P(s)$	For complex s with $\mathrm{Re}(s) > 1$, we define the *prime zeta function* to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$. For $\mathrm{Re}(s) > 1$, the prime zeta function is related to $\zeta(s)$ according to the formula $P(s) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log[\zeta(ks)]$.
$Q(x)$	For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. More precisely, this function is summed and identified with its limiting asymptotic formula as $x \rightarrow \infty$ in the following form: $Q(x) := \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O(\sqrt{x})$.
$\sim$	We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} = 1$.
$\zeta(s)$	The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation on the rest of the complex plane with the exception of a simple pole at $s = 1$ of residue one.

1 Introduction

1.1 Definitions

We define the *Möbius function* to be the signed indicator function of the squarefree integers in the form of [19, A008683]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \text{ and } n \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

There are many variants and special properties of the Möbius function and its generalizations [18, *cf.* §2]. One crucial role of the classical $\mu(n)$ is that the function forms an inversion relation for the divisor sums formed by arithmetic functions convolved with one through *Möbius inversion*:

$$g(n) = (f * 1)(n) \iff f(n) = (g * \mu)(n), \forall n \geq 1.$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined on the positive integers as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as follows [19, A002321]:

$$\{M(x)\}_{x\geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \ldots\}.$$

Clearly, a positive integer $n \geq 1$ is *squarefree*, or contains no (prime power) divisors which are squares, if and only if $\mu^2(n) = 1$. A related summatory function which counts the number of *squarefree* integers $n \leq x$ satisfies [5, §18.6] [19, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers characterized by the sign of the Möbius function are in fact equal as $x \to \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{x} \overset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{x} \xrightarrow{x\to\infty} \frac{3}{\pi^2}.$$

1.2 Properties

A conventional approach to evaluating the limiting asymptotic behavior of $M(x)$ for large $x \to \infty$ results by considering an inverse Mellin transformation of the reciprocal of the Riemann zeta function. In particular, since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \cdot \int_1^\infty \frac{M(x)}{x^{s+1}} dx, \operatorname{Re}(s) > 1,$$

we obtain that

$$M(x) = \lim_{T\to\infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

The previous two representations lead us to the exact expression of $M(x)$ for any real $x > 0$ given by the next theorem due to Titchmarsh.

Theorem 1.1 (Analytic Formula for $M(x)$)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k\geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any real $x > 0$*

$$M(x) = \lim_{k\to\infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |\operatorname{Im}(\rho)|<T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n\geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp\left(-C \cdot \log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan more recently proved new updated estimates bounding $M(x)$ from above for large x in the following forms [20]:

$$\begin{aligned} M(x) &\ll \sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right), \ \forall \epsilon > 0. \end{aligned}$$

1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C \cdot \sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens for $C = 1$ and all $x < 10000$. Since its beginnings in 1897, the Mertens conjecture has been disproven by computation of non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele [13]. Since the truth of the conjecture would have implied the RH, more recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound along infinite subsequences, e.g., considering the asymptotics of the function in the limit supremum and limit infimum senses.

A precise statement of this problem is to produce an unconditional proof of whether $\limsup_{x\to\infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x\to\infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there are infinite subsequences of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound towards either $\pm\infty$ along the subsequence. We cite that it is only known by computation that [16, *cf.* §4.1] [19, *cf.* A051400; A051401]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leq -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [13, 8, 9, 6]. Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies [12]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log\log\log x)^{5/4}} = O(1).$$

2 An overview of the core components to the proof

We offer an initial step-by-step summary overview of the core components to our proof outlined in the next points. We hope that this sketch of the logical components to this argument makes the article easier to parse.

(1) We prove a matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). See Theorem 3.1 in Section 4.

(2) This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the prime counting function, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is stated in (1). The link relating our new formula in (1) to canonical additive functions and their distributions lends a recent distinguishing element to the success of the methods in our proof.

(3) We tighten bounds from a less classical result proved in [11, §7] providing uniform asymptotic formulas for the summatory functions, $\widehat{\pi}_k(x)$, large $x \gg e$ and $1 \leq k \leq \log\log x$ (see Theorem 3.7). We use this result to bound sums of the form $\sum_{n \leq x} \lambda(n) f(n)$ from below for particular positive arithmetic functions f when x is large.

(4) We then turn to estimating the limiting asymptotics of the quasi-periodic function, $|g^{-1}(n)|$, by proving several formulas bounding its average order as $x \to \infty$ in Section 6. We eventually use these estimates to prove a substantially unique new lower bound formulas for the summatory function $G^{-1}(x) := \sum_{n \leq x} \lambda(n) |g^{-1}(n)|$ along certain asymptotically large infinite subsequences (see Theorem 8.4).

(5) In Section 7, we prove new expectation formulas for $|g^{-1}(n)|$ and the related component sequences $C_{\Omega(n)}(n)$ by proving an Erdös-Kac like theorem satisfied by $C_{\Omega(n)}(n)$. This allows us to prove new asymptotic lower bounds on $|G^{-1}(x)|$ when x is large.

(6) When we return to step **(2)** with our new lower bounds at hand, we are led to a new unconditional proof of the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. In fact, we recover a quick and rigorous proof of Theorem 3.9 given at the conclusion of Section 8.2.

3 A concrete new approach to bounding $M(x)$ from below

3.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 3.1 (Summatory functions of Dirichlet convolutions)**.** *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of f and h, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse of f. We have the following exact expressions for the summatory function of $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned}
\pi_{f*h}(x) &:= \sum_{n \leq x} \sum_{d|n} f(d) h(n/d) \\
&= \sum_{d \leq x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].
\end{aligned}$$

Moreover, for all $x \geq 1$

$$\begin{aligned}
H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\
&= \sum_{n=1}^{x} f^{-1}(n) \pi_{f*h}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).
\end{aligned}$$

Corollary 3.2 (Convolutions arising from Möbius inversion)**.** *Suppose that g is an arithmetic function such that $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n \leq x} (g * \mu)(n)$. The Mertens function is expressed by the sum*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geq 1.$$

Corollary 3.3 (A motivating special case)**.** *We have exactly that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

3.2 An exact expression for $M(x)$ in terms of strongly additive functions

Fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute exactly that (see Table T.1 starting on page 48 of the appendix section)

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these positive terms is given by $\operatorname{sgn}(g^{-1}(n)) = \frac{g^{-1}(n)}{|g^{-1}(n)|} = \lambda(n)$ for all $n \geq 1$ (see Proposition 4.1).

There is not an easy, nor subtle direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences. The distribution of distinct sets of prime exponents is still fairly regular so that $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of $g^{-1}(n)$ over $n \geq 2$:

Heuristic 3.4 (Symmetry in $g^{-1}(n)$ in the prime factorizations of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for $= \omega(n_i) \geq 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly one, two, three, and so on prime factors.

Conjecture 3.5. *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) $g^{-1}(1) = 1$*;*

(B) *For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*

(C) *For all squarefree integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!;$$

(D) *If $n \geq 2$ and $\Omega(n) = k$, then*

$$2 \leq |g^{-1}(n)| \leq \sum_{m=0}^{k} \binom{k}{m} \cdot m!.$$

We illustrate parts (B)–(D) of the conjecture clearly using the computation of initial values of this inverse sequence in Table T.1. A proof of (C) in fact follows from Lemma 6.3 stated on page 22. The realization that the beautiful and remarkably simple combinatorial form of property (C) in Conjecture 3.5 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through sums of auxiliary sequences of arithmetic functions (see Section 6).

We prove that (see Proposition 8.1)

$$M(x) = G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{\sqrt{x}} G^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].$$

This formula implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by bounding appropriate estimates of the summatory function $G^{-1}(x)$. The regularity of $|g^{-1}(n)|$ is useful to our argument in formally bounding $G^{-1}(x)$ from below.

The regularity and quasi-periodicity we alluded to in the previous remarks are actually quantifiable in so much as $|g^{-1}(n)|$ for $n \leq x$ tends to its average order with a skew normal tendency depending on x as $x \to \infty$. In Section 7, we prove the next variant of an Erdös-Kac theorem like analog for a component sequence closely related to $g^{-1}(n) = \lambda(n) \cdot |g^{-1}(n)|$. What results is the following statement for $\mu_x(C) := \log\log x + \hat{a}$, $\sigma_x(C) := \sqrt{\mu_x(C)}$, $\hat{a} \approx -1.37662$ an absolute constant, and any $y \in \mathbb{R}$ (see Corollary 7.9):

$$\#\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq y\} = x \cdot \Phi\left(\frac{\frac{\pi^2}{6} y - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{x}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

3.3 Uniform asymptotics from enumerative bivariate DGFs from Mongomery and Vaughan

Theorem 3.6 (Montgomery and Vaughan)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $R < 2$ we have that uniformly for all $1 \leq k \leq R \log\log x$

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right],$$

where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, 0 \leq |z| \leq R.$$

The proof of the next result is combinatorially motivated in so much as it interprets lower bounds on a key infinite product factor of $\mathcal{G}(z)$ defined in Theorem 3.6 as corresponding to an ordinary generating function of certain homogeneous symmetric polynomials involving the primes. This interpretation allows us to recover the following uniform lower bounds on $\widehat{\pi}_k(x)$ as $x \to \infty$:

Theorem 3.7. *For all sufficiently large x we have uniformly for $1 \leq k \leq \log\log x$ that*

$$\widehat{\pi}_k(x) \gg \frac{x^{1/4}}{(\log x)^2} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^3}\right)\right].$$

3.3.1 Applications of the new uniform lower bound estimates

Our inspiration for the new bounds found in the last sections of this article allows us to approximate finite partial sums of certain bounded non-negative arithmetic functions weighted by the Liouville lambda function $\lambda(n)$.

Lemma 3.8. *Suppose that $f(n)$ is an arithmetic function defined such that $f(n) > 0$ for all $n > u_0$ where $f(n) \gg \widehat{\tau}_\ell(n) > 0$ whenever $n > u_0$ as $n \to \infty$. Assume also that the bounding function $\widehat{\tau}_\ell(t)$ is a continuously differentiable function of t for all large enough $t \gg u_0$. We define the λ-sign-scaled summatory function of f as follows:*

$$F_\lambda(x) := \sum_{u_0 < n \leq x} \lambda(n) f(n).$$

Let the summatory weight function be defined as

$$A_\Omega(t) := \sum_{k=1}^{\lfloor \log\log t \rfloor} (-1)^k \widehat{\pi}_k(t).$$

Suppose that $|A_\Omega(t)| \gg |A_\Omega^{(\ell)}(t)|$ as $t \to \infty$. Then we have that for sufficiently large $x > e$

$$|F_\lambda(x)| \gg \left| \left| A_\Omega^{(\ell)}(x) \widehat{\tau}_\ell(x) \right| - \int_{\frac{\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log x}{2}} \left| A_\Omega^{(\ell)}\left(e^{e^{2t}}\right) \widehat{\tau}_\ell'\left(e^{e^{2t}}\right) \right| e^{e^{2t}} dt \right|. \tag{2}$$

3.3.2 Remarks

We emphasize the relevant recency of the method demonstrated by Montgomery and Vaughan in constructing a proof of Theorem 3.6. To the best of our knowledge, this textbook reference is one of the first clear-cut applications documenting something of a hybrid DGF-and-OGF approach to enumerating sequences of arithmetic functions and their summatory functions. The hybrid method is motivated by the fact that it does not require a direct appeal to traditional highly oscillatory DGF-only inversions and integral formulas involving the Riemmann zeta function. This newer interpretaion of certain bivariate DGFs offers a window into the best of both generating function series worlds: It combines the additivity implicit to the coefficients indexed by a formal power series variable formed by multiplication of these structures, while coordinating the distinct DGF-best property of the multiplicativity of distinct prime powers invoked by taking powers of an Euler product.

3.4 Cracking the classical unboundedness barrier

In Section 8, we are able to state what forms a bridge between the results we carefully prove up to that point the article. What we obtain at the conclusion of the section is the next summary theorem that unconditionally resolves the classical question of the unboundedness of the scaled function Mertens function $q(x) := |M(x)|/\sqrt{x}$ in the limit supremum sense.

Theorem 3.9 (Unboundedness of the the Mertens function, $q(x)$)**.** *We have that*

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

In establishing the rigorous proof of Theorem 3.9 based on our new methods, we not only show unboundedness of $q(x)$, but also set a minimal rate (along a large infinite subsequence) at which this form of the scaled Mertens function grows without bound.

4 Preliminary proofs of new results

4.1 Establishing the summatory function properties and inversion identities

We will offer a proof of Theorem 3.1 suggested by an intuitive construction through matrix methods. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95].

Proof of Theorem 3.1. Let h, g be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h. We have that the following formulas hold for all $x \geq 1$:

$$\begin{aligned}\pi_{g*h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right] H(i). \end{aligned} \tag{3}$$

The first formula above is well known. The second formula is justified directly using summation by parts as[A]

$$\begin{aligned}\pi_{g*h}(x) &= \sum_{d=1}^{x} h(d) G\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i \leq x} \left(\sum_{j \leq i} h(j)\right) \times \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right].\end{aligned}$$

We next form the invertible matrix of coefficients associated with this linear system defining $H(j)$ for all $1 \leq j \leq x$ in (3) by defining

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right), 1 \leq j \leq x.$$

Since $g_{x,x} = G(1) = g(1)$ and $g_{x,j} = 0$ for all $j > x$, the matrix we must invert in this problem is lower triangular with a non-zero constant on its diagonals, and is hence invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressed by applying an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

Here, U is a square matrix with sufficiently large finite dimensions whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$\left[(I - U^T)^{-1}\right]_{i,j} = [j \leq i]_\delta.$$

Observe that

$$\left\lfloor \frac{x}{j} \right\rfloor - \left\lfloor \frac{x-1}{j} \right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.}\end{cases}$$

The previous property implies that

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

[A] For any arithmetic functions, u_n, v_n, with $U_j := u_1 + u_2 + \cdots + u_j$ for $j \geq 1$, we have that [14, §2.10(ii)]

$$\sum_{j=1}^{n-1} u_j \cdot v_j = U_{n-1} v_n + \sum_{j=1}^{n-1} U_j \left(v_j - v_{j+1}\right), n \geq 2.$$

We use the last property in (4) to shift the matrix $\hat{G}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of g in the following form:

$$\left[(I-U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Our target matrix in the inversion problem is defined by

$$(g_{x,j}) = (I-U^T)\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators as follows:

$$\begin{aligned}(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k)\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1}\right\rfloor} g^{-1}(k)\right).\end{aligned}$$

Hence, the summatory function $H(x)$ is given exactly for any $x \geq 1$ by a vector product with the inverse matrix from the previous equation in the next form.

$$H(x) = \sum_{k=1}^{x} g_{x,k}^{-1} \cdot \pi_{g*h}(k) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right)\cdot \pi_{g*h}(k)$$

We can prove an inversion formula providing the coefficients of $G^{-1}(i)$ for $1 \leq i \leq x$ given by the last equation by adapting our argument to prove (3) above. This leads to the identity that

$$H(x) = \sum_{k=1}^{x} g^{-1}(x)\pi_{g*h}\left(\left\lfloor \frac{x}{k}\right\rfloor\right). \qquad \square$$

4.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n. Then we can easily prove using DGFs that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu. \tag{5}$$

When combined with Corollary 3.2 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 3.3.

Proposition 4.1 (The signedness property of $g^{-1}(n)$)**.** *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0, \pm 1\}$ *denote the sign of the arithmetic function h at integers $n \geq 1$. For the Dirichlet invertible function, $g(n) := \omega(n) + 1$, we have that* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ *for all $n \geq 1$.*

Proof. The function $D_f(s) := \sum_{n\geq 1} f(n)n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ for σ_f the abcissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = 1/\zeta(s)$ and $D_\omega(s) = P(s)\zeta(s)$ for $\operatorname{Re}(s) > 1$. Then by (5) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any arithmetic function f such that $f(1) \neq 0$, we have for all $\operatorname{Re}(s) > 1$ that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{6}$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. This observation implies that $\operatorname{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make our claims rigorous.

By the recurrence relation that defines the Dirichlet inverse function of any arithmetic function h such that $h(1) = 1$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d \mid n \\ d>1}} h(d) h^{-1}(n/d), & n \geq 2. \end{cases} \tag{7}$$

For $n \geq 2$, the summands in (7) can be simply indexed over the primes $p|n$ given our definition of h from above. This observation yields that we can inductively unfold these sums into nested divisor sums provided the depth of the expanded divisor sums does not exceed the capacity to index summations over the primes dividing n. Namely, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p|n} h^{-1}\left(\frac{n}{p}\right), && \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right), && \text{if } \Omega(n) \geq 2 \\ &= -\sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1 p_2}} h^{-1}\left(\frac{n}{p_1 p_2 p_3}\right), && \text{if } \Omega(n) \geq 3. \end{aligned}$$

Then by induction with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1, n \geq 2. \tag{8}$$

In fact, by a combinatorial argument we recover exactly that

$$h^{-1}(n) = \lambda(n) \frac{(\alpha_1 + \cdots + \alpha_{\omega(n)})!}{\alpha_1! \alpha_2! \cdots \alpha_{\omega(n)}!} = \lambda(n) \binom{\Omega(n)}{\alpha_1, \alpha_2, \ldots, \alpha_{\omega(n)}}. \tag{9}$$

These expansions imply that the following property holds for all $n \geq 1$:

$$\operatorname{sgn}(h^{-1}(n)) = \lambda(n).$$

Since λ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$ for all $d|n$ and $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever n is squarefree, so that we obtain

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1. \qquad \square$$

4.3 Statements of known limiting asymptotics

Theorem 4.2 (Mertens theorem)**.** *For all $x \geq 2$ we have that*

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log\log x + B + o(1), \text{ as } x \to \infty,$$

where $B \approx 0.2614972128476427837554$ *is an absolute constant*[B].

Corollary 4.3 (Product form of Mertens theorem)**.** *We have that for all sufficiently large* $x \gg 2$

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\left(1 + o(1)\right), \text{ as } x \rightarrow \infty,$$

where the notation for the absolute constant $0 < B < 1$ *coincides with the definition of Mertens constant from Theorem 4.2. Hence, for any real* $z \geq 0$ *we obtain that*

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right)^z \sim \frac{e^{-\gamma z}}{(\log x)^z}, \text{ as } x \rightarrow \infty.$$

Proofs of Theorem 4.2 and Corollary 4.3 are given in [5, §22.7; §22.8]. We have a related analog of Corollary 4.3 that is justified using the Euler product representation for the Riemann zeta function:

$$\prod_{p \leq x}\left(1 + \frac{1}{p}\right) = \prod_{p \leq x}\frac{(1-p^{-2})}{(1-p^{-1})} = \zeta(2)e^{\gamma}(\log x)(1 + o(1)), \text{ as } x \rightarrow \infty.$$

Facts 4.4 (Exponential integrals and the incomplete gamma function)**.** Two variants of the *exponential integral function* are defined by the integral next representations [14, §8.19].

$$\begin{aligned} \mathrm{Ei}(x) &:= \int_{-x}^{\infty}\frac{e^{-t}}{t}dt, x \in \mathbb{R} \\ E_1(z) &:= \int_1^{\infty}\frac{e^{-tz}}{t}dt, \mathrm{Re}(z) \geq 0 \end{aligned}$$

These functions are related by $\mathrm{Ei}(-kz) = -E_1(kz)$ for real $k, z > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $\mathrm{Ei}(\pm x)$ for all real $x > 0$:

$$\begin{aligned} \gamma + \log x - x \leq \mathrm{Ei}(-x) &\leq \gamma + \log x - x + \frac{x^2}{4}, \qquad (10a) \\ 1 + \gamma + \log x - \frac{3}{4}x \leq \mathrm{Ei}(x) \quad &\leq 1 + \gamma + \log x - \frac{3}{4}x + \frac{11}{36}x^2. \end{aligned}$$

The (upper) *incomplete gamma function* is defined by [14, §8.4]

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1}e^{-t}dt, \mathrm{Re}(s) > 0.$$

The following properties of $\Gamma(s, x)$ hold:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1}\frac{x^k}{k!}, s \in \mathbb{Z}^+, x > 0, \qquad (10b)$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, s > 0, \text{ as } x \rightarrow \infty. \qquad (10c)$$

[B] Precisely, we have that the *Mertens constant* is defined by [19, A077761]

$$B = \gamma + \sum_{m \geq 2}\frac{\mu(m)}{m}\log\left[\zeta(m)\right].$$

5 Components to the asymptotic analysis of lower bounds for sums of arithmetic functions weighted by $\lambda(n)$

5.1 A discussion of the results proved by Montgomery and Vaughan

Remark 5.1 (Intuition and constructions in Theorem 3.6)**.** For $|z| < 2$ and $\operatorname{Re}(s) > 1$, let

$$F(s,z) := \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \tag{11}$$

and define the DGF coefficients, $a_z(n)$ for $n \geq 1$, by the product

$$\zeta(s)^z \cdot F(s,z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s}, \operatorname{Re}(s) > 1.$$

Suppose that $A_z(x) := \sum_{n \leq x} a_z(n)$ for $x \geq 1$. Then we obtain the next generating function like identity in z enumerating the $\widehat{\pi}_k(x)$ for $1 \leq k \leq \log\log x$ [A]

$$A_z(x) = \sum_{n \leq x} z^{\Omega(n)} = \sum_{k \geq 0} \widehat{\pi}_k(x) z^k \tag{12}$$

Thus for $r < 2$, by Cauchy's integral formula we have

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz.$$

Selecting $r := \frac{k-1}{\log\log x}$ for $1 \leq k < 2\log\log x$ leads to the uniform asymptotic formulas for $\widehat{\pi}_k(x)$ given in Theorem 3.6. Montgomery and Vaughan then consider individual analysis of the main and error terms for $A_z(x)$ to prove that

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

We will require estimates of $A_{-z}(x)$ from below to form summatory functions that weight the terms of $\lambda(n)$ in our new formulas derived in the next sections.

5.2 New uniform asymptotics based on refinements of Theorem 3.6

Proposition 5.2. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \geq 2.$$

When $s := 1$, we have the asymptotic formula from Mertens theorem (see Theorem 4.2). For all integers $s \geq 2$ there is are absolutely defined quasi-polynomial bounding functions $\gamma_0(s,x)$ and $\gamma_1(s,x)$ in s, x such that

$$\gamma_0(s,x) + o(1) \leq P_s(x) \leq \gamma_1(s,x) + o(1), \text{ as } x \to \infty.$$

It suffices to define the bounds in the previous equation by the functions

$$\begin{aligned}\gamma_0(s,x) &= s\log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}s(s-1)^2\log^2(2)\\ \gamma_1(s,x) &= s\log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) + \frac{1}{4}s(s-1)^2\log^2(x).\end{aligned}$$

[A] In fact, for any additive arithmetic function $a(n)$, characterized by the property that $a(n) = \sum_{p^\alpha || n} a(p^\alpha)$ for all $n \geq 2$, we have that [7, *cf.* §1.7]

$$\prod_p \left(1 - \sum_{m \geq 1} \frac{z^{a(p^m)}}{p^{ms}}\right)^{-1} = \sum_{n \geq 1} \frac{z^{a(n)}}{n^s}, \operatorname{Re}(s) > 1.$$

Proof. Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$, and where our target function smooth function is $f(t) = t^{-s}$ so that $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= \mathrm{Ei}(-(s-1)\log x) - \mathrm{Ei}(-(s-1)\log 2) + o(1), \text{ as } x \to \infty. \end{aligned}$$

Now using the inequalities in Facts 4.4, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned} \frac{P_s(x)}{s} &\geq \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}(s-1)^2 \log^2(2) + o(1) \\ \frac{P_s(x)}{s} &\leq \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) + \frac{1}{4}(s-1)^2 \log^2(x) + o(1). \end{aligned}$$ □

We will first prove the stated form of the lower bound on $\mathcal{G}(-z)$ for $z := \frac{k-1}{\log\log x}$. Then we will discuss the technical adaptations to Montgomery and Vaughan's proof of Theorem 3.6 in Remark 5.3 to justify the new asymptotic lower bounds on $\widehat{\pi}_k(x)$ that hold uniformly for all $1 \leq k \leq \log\log x$.

Proof of Theorem 3.7. For $0 \leq z < 2$ and integers $x \geq 2$, the right-hand-side of the following product is finite.

$$\widehat{P}(z,x) := \prod_{p \leq x} \left(1 - \frac{z}{p}\right)^{-1}.$$

For fixed, finite $x \geq 2$ let

$$\mathbb{P}_x := \{n \geq 1 : \text{all prime divisors } p|n \text{ satisfy } p \leq x\}.$$

Then we can see that

$$\prod_{p \leq x} \left(1 - \frac{z}{p^s}\right)^{-1} = \sum_{n \in \mathbb{P}_x} \frac{z^{\Omega(n)}}{n^s}, x \geq 2. \tag{13}$$

By extending the argument in the proof given in [11, §7.4], we have that the formulas

$$A_{-z}(x) := \sum_{n \leq x} \lambda(n) z^{\Omega(n)} = \sum_{k \geq 0} \widehat{\pi}_k(x)(-z)^k,$$

If we let $a_n(z,x)$ be defined by the DGF

$$\widehat{P}(z,x) := \sum_{n \geq 1} \frac{a_n(z,x)}{n^s},$$

then we show that

$$\sum_{n \leq x} a_n(-z,x) = \sum_{n \leq x} \lambda(n) z^{\Omega(n)} = \sum_{k=0}^{\log_2(x)} \widehat{\pi}_k(x)(-z)^k + \sum_{k > \log_2(x)} e_k(x)(-z)^k.$$

This assertion if correct since the products of all non-negative integral powers of the primes $p \leq x$ generate the integers $\{1 \leq n \leq x\}$ as a subset. Thus we capture all of the relevant terms needed to express $(-1)^k \cdot \widehat{\pi}_k(x)$ via the Cauchy integral formula representation over $A_{-z}(x)$ by replacing the corresponding infinite product terms with $\widehat{P}(-z,x)$ in the definition of $\mathcal{G}(-z)$.

Now we must argue that

$$\mathcal{G}(-z) \gg \prod_{p \leq x} \left(1 + \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{-z}, 0 \leq z < 1, x \geq 2.$$

For $0 \leq z < 1$ and $x \geq 2$, we see that

$$\begin{aligned}
\mathcal{G}(-z) &= \exp\left(-\sum_p \left[\log\left(1+\frac{z}{p}\right) + \log\left(1-\frac{1}{p}\right)\right]\right) \\
&\gg \exp\left(-z \times \sum_{p>x}\left[\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right] - \sum_{p\leq x}\left[\log\left(1+\frac{z}{p}\right)+\log\left(1-\frac{1}{p}\right)\right]\right) \\
&= \widehat{P}(-z,x) \times \exp\left(-z(B+o(1))\right) \gg_z \widehat{P}(-z,x), \text{ as } x\rightarrow\infty.
\end{aligned}$$

Next, we have for all integers $0 \leq k \leq m < \infty$, and any sequence $\{f(n)\}_{n\geq 1}$ with sufficiently bounded partial power sums, that [10, §2]

$$[z^k] \prod_{1\leq i\leq m}(1-f(i)z)^{-1} = [z^k]\exp\left(\sum_{j\geq 1}\left(\sum_{i=1}^m f(i)^j\right)\frac{z^j}{j}\right), |z|<1. \tag{14}$$

In our case we have that $f(i)$ denotes the reciprocal of the i^{th} prime in the generating function expansion of (14). It follows from Proposition 5.2 that for any real $0 \leq z < 1$ we obtain **(TODO) ...**

$$\begin{aligned}
\log\left[\prod_{p\leq x}\left(1+\frac{z}{p}\right)^{-1}\right] &\geq -(B+\log\log x)z + \sum_{j\geq 0}\left[\log\left(\frac{\log x}{\log 2}\right) - (2j+1)\log\left(\frac{x}{2}\right) - (2j+1)^2\frac{\log^2 2}{4}\right] z^{2j+2} \\
&\qquad - \sum_{j\geq 0}\left[\log\left(\frac{\log x}{\log 2}\right) - (2j+2)\log\left(\frac{x}{2}\right) + (2j+2)^2\frac{\log^2 x}{4}\right] z^{2j+3} \\
&= -(B+\log\log x)z + z^2 \times \sum_{j\geq 0}\left[-\log\left(\frac{\log x}{\log 2}\right) + (j+1)\log\left(\frac{x}{2}\right)\right](-z)^j \\
&\qquad - \frac{z^2}{4} \times \sum_{j\geq 0}\left[\log^2 2 + \log^2 x\right](j+1)^2 z^j \\
&= -(B+\log\log x)z - z^2\left[\log\left(\frac{\log x}{\log 2}\right)\frac{1}{1+z} - \log\left(\frac{x}{2}\right)\frac{1}{(1+z)^2}\right] \\
&\qquad + \left(\log^2 2 + \log^2 x\right)\frac{z^2(1+z)}{4\cdot(1-z)^3} \\
&=: \widehat{\mathcal{B}}(x;z).
\end{aligned} \tag{15}$$

We adjust the uniform bound parameter R so that

$$z \equiv z(k,x) = \frac{k-1}{\log\log x} \in [0,1),$$

whenever $1 \leq k \leq \log\log x$ in the notation of Theorem 3.6. This implies that $(1+z)^{-1} \in \left(\frac{1}{2}, 1\right]$, and so

$$\begin{aligned}
-\frac{z^2}{1+z} &= 1 - z - \frac{1}{1+z} \geq -1 \\
\frac{z^2}{(1+z)^2} &= 1 - \frac{2}{(1+z)} + \frac{1}{(1+z)^2} \geq -\frac{3}{4}.
\end{aligned}$$

Then we have from (15) that

$$\widehat{\mathcal{B}}(x;z) \gg \left(\frac{\log 2}{\log x}\right)\cdot\left(\frac{2}{x}\right)^{\frac{3}{4}}\cdot\exp\left(\frac{z^2(1+z)}{4\cdot(1-z)^3}\cdot\log^2 x\right) \gg \frac{1}{x^{3/4}\cdot(\log x)}.$$

In summary, we have arrived at a proof that as $x \to \infty$

$$\frac{e^{\gamma z}}{(\log x)^{-z}} \times \exp\left(\widehat{\mathcal{B}}(u, x; z)\right) \gg \frac{1}{x^{3/4} \cdot (\log x)}. \tag{16}$$

Finally, to finish our proof of the new form of the lower bound on $\mathcal{G}(-z)$, we need to bound the reciprocal factor of $\Gamma(1-z)$. Since $z \equiv z(k, x) = \frac{k-1}{\log\log x}$ and $k \in [1, \log\log x]$, or again with $z \in [0, 1)$, we obtain for minimal k and all large enough $x \gg 1$ that $\Gamma(1-z) = \Gamma(1) = 1$, and for k towards the upper range of its interval that

$$\Gamma(1-z) \approx \Gamma\left(\frac{1}{\log\log x}\right) = \frac{1}{\log\log x}\Gamma\left(1 + \frac{1}{\log\log x}\right) \approx \frac{1}{\log\log x}. \qquad \square$$

Remark 5.3 (Technical adjustments in the proof of Theorem 3.7)**.** We now discuss the differences between our construction and that in the technical proof of Theorem 3.6 in the reference when we bound $\mathcal{G}(-z)$ from below as in Theorem 3.7. The reference proves that for real $0 \leq z < 2$

$$A_{-z}(x) = -\frac{zF(1, -z)}{\Gamma(1-z)} \cdot x(\log x)^{-(z+1)} + O\left(x(\log x)^{-\operatorname{Re}(z)-2}\right). \tag{17}$$

Recall that for $r < 2$ we have by Cauchy's integral formula that

$$(-1)^k \widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|z|=r} \frac{A_{-z}(x)}{z^{k+1}} dz. \tag{18}$$

We first claim that uniformly for large x and $1 \leq k \leq \log\log x$ we have

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{1-k}{\log\log x}\right) \times \frac{x(\log\log x)^{k-1}}{(\log x)(k-1)!}\left[1 + O\left(\frac{k}{(\log\log x)^3}\right)\right]. \tag{19}$$

Then since we have proved in Theorem 3.6 above that

$$\mathcal{G}\left(\frac{1-k}{\log\log x}\right) \gg \frac{1}{x^{3/4} \cdot (\log x)} \cdot \frac{(k-1)}{\log\log x},$$

the result in (19) implies our stated uniform asymptotic bound. Namely, we obtain that

$$\widehat{\pi}_k(x) \gg \frac{x^{1/4}}{(\log x)^2} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O\left(\frac{k}{(\log\log x)^3}\right)\right].$$

We have to provide analogs to the two separate bounds corresponding to the error and main terms of our estimate according to (17) and (18). The error term estimate is simpler, so we tackle it first in the next argument. The second part of our proof establishing the main term in (19) requires us to duplicate and adjust significant parts of the fine-tuned reasoning given in the reference.

Error Term Bound. To prove that the error term bound holds, we estimate that

$$\begin{aligned}\left|\frac{1}{2\pi\imath} \int_{|z|=r} \frac{x}{(\log x)^2} \frac{(\log x)^{-\operatorname{Re}(z)}}{z^{k+1}}\right| &\ll x(\log x)^{-(r+2)} r^{-(k+1)} \ll \frac{x}{(\log x)^2} \frac{(\log\log x)^{k+1}}{e^{k-1}(k-1)^{k+1}} \\ &\ll \frac{x}{(\log x)^2} \frac{(\log\log x)^{k+1}}{e^{2(k-1)}(k-1)!(k-1)} \ll \frac{x}{(\log x)^2} \frac{(\log\log x)^{k+1}}{(k-1)!} \\ &\ll \frac{x}{\log x} \frac{(\log\log x)^{k-4}}{(k-1)!}.\end{aligned} \tag{20}$$

We can calculate that for $0 \leq z < 1$

$$\prod_p \left(1 + \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{-z} = \exp\left(-\sum_p \left[\log\left(1 + \frac{z}{p}\right) + z\log\left(1 - \frac{1}{p}\right)\right]\right)$$

$$\begin{aligned}&\sim \exp\left(-o(z) \times \sum_p \frac{1}{p^2}\right)\\&\gg \exp\left(-o(z)\frac{\pi^2}{6}\right) \gg_z 1.\end{aligned}$$

In other words, we have that $\mathcal{G}\left(\frac{1-k}{\log\log x}\right) \gg 1$. So the error term in (20) is majorized by taking $O\left(\frac{k}{(\log\log x)^3}\right)$ as our upper bound.

Main Term Bounds. Notice that the main term estimate corresponfing to (17) and (18) is given by $\frac{x}{\log x} I$, where

$$I := \frac{(-1)^{k-1}}{2\pi\imath} \int_{|z|=r} G(-z)(\log x)^{-z} z^{-k} dz.$$

In particular, we can write $I = I_1 + I_2$ where we define

$$\begin{aligned}I_1 &:= \frac{(-1)^{k-1} G(-r)}{2\pi\imath} \int_{|z|=r} (\log x)^{-z} z^{-k} dz\\&= \frac{G(-r)(\log\log x)^{k-1}}{(k-1)!}\\I_2 &:= \frac{(-1)^{k-1}}{2\pi\imath} \int_{|z|=r} (G(-z) - G(-r))(\log x)^{-z} z^{-k} dz\\&= \frac{(-1)^{k-1}}{2\pi\imath} \int_{|z|=r} (G(-z) - G(-r) + G'(-r)(z+r))(\log x)^{-z} z^{-k} dz.\end{aligned}$$

We have by a power series expansion of $G''(-w)$ about $-z$ and integrating the resulting series termwise with respect to w that

$$\left|G(-z) - G(-r) + G'(-r)(z+r)\right| = \left|\int_{-r}^{z} (z+w) G''(-w) dw\right| \ll G''(-r) \times |z+r|^2 \ll |z+r|^2.$$

Now we parameterize the curve in the contour for I_2 by writing $z = re^{2\pi\imath t}$ for $t \in [-1/2, 1/2]$. This leads us to the bounds

$$\begin{aligned}|I_2| &= r^{3-k} \times \int_{-1/2}^{1/2} |e^{2\pi\imath t} + 1|^2 \cdot (\log x)^{re^{2\pi\imath t}} \cdot e^{2\pi\imath t} dt\\&\ll r^{3-k} \times \int_{-1/2}^{1/2} \sin^2(\pi t) \cdot e^{(1-k)\cos(2\pi t)} dt.\end{aligned}$$

Whenever $|x| \leq 1$, we know that $|\sin x| \leq |x|$. We can construct bounds on $-\cos(2\pi t)$ for $t \in [-1/2, 1/2]$ by writing $\cos(2x) = 1 - 2\sin^2 x$ for $|x| < 1/2$. Then by the alternating Taylor series expansions of the sine function

$$\begin{aligned}1 - 2\sin^2(2\pi t) &\geq 1 - 2\left(1 - \frac{\pi t}{3}\right)^2 \geq -1 - \frac{2\pi^2 t^2}{9} \qquad \Longrightarrow\\-\cos(2\pi t) &\leq 1 + \frac{2\pi^2 t^2}{9} \leq \left(4 + \frac{2\pi^2}{9}\right) t^2 \leq 1 + 3t^2.\end{aligned}$$

So it follows that

$$\begin{aligned}|I_2| &\ll r^{3-k} e^{k-1} \times \left|\int_0^\infty t^2 e^{3(k-1)t^2} dt\right|\\&\ll \frac{r^{3-k} e^{k-1}}{(k-1)^{3/2}} = \frac{(\log\log x)^{k-3} e^{k-1}}{(k-1)^{k-3/2}}\\&\ll \frac{k \cdot (\log\log x)^{k-3}}{(k-1)!}.\end{aligned}$$

Thus the contribution from the term $|I_2|$ can then be asborbed into the error term bound in (19).

5.3 The distribution of exceptional values of $\Omega(n)$

The next theorems reproduced from [11, §7.4] characterize the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \log\log x$. The tendency of this canonical completely additive function to not deviate substantially from its average order is an extraordinary property that allows us to prove asymptotic relations on summatory functions that are weighted by its parity without having to account for significant local oscillations when we average over a large interval.

Theorem 5.4 (Upper bounds on exceptional values of $\Omega(n)$ for large n)**.** *Let*

$$\begin{aligned} A(x,r) &:= \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\}, \\ B(x,r) &:= \#\left\{n \leq x : \Omega(n) \geq r \cdot \log\log x\right\}. \end{aligned}$$

If $0 < r \leq 1$ and $x \geq 2$, then

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x \rightarrow \infty.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x,r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \rightarrow \infty.$$

Theorem 5.5 is an analog to the celebrated Erdös-Kac theorem typically stated for the normally distributed values of the scaled-shifted $\omega(n)$ function over $n \leq x$ as $x \rightarrow \infty$.

Theorem 5.5 (Exact bounds on exceptional values of $\Omega(n)$ for large n)**.** *We have that as $x \rightarrow \infty$*

$$\#\left\{3 \leq n \leq x : \Omega(n) - \log\log n \leq 0\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Remark 5.6. The key interpretation we need to take away from the statements of Theorem 5.4 and Theorem 5.5 is the result proved in the next corollary. The role of the parameter R involved in stating the previous theorem is a critical bound as the scalar factor in the upper bound on $k \leq R\log\log x$ in Theorem 3.6 up to which our uniform bounds given by Theorem 3.7 hold. In contrast, for $n \geq 2$ we can actually have contributions from values distributed throughout the range $1 \leq \Omega(n) \leq \log_2(n)$ infinitely often. It is then crucial that we can show that the main term in the asymptotic formulas we obtain for these summatory functions is captured by summing only over the truncated range of $k \in [1, \log\log x]$ where the uniform bounds hold.

Corollary 5.7. *Using the notation for $A(x,r)$ and $B(x,r)$ from Theorem 5.4, we have that for $x \geq 2$ and $\delta > 0$,*

$$o(1) \leq \frac{B(x, 1+\delta)}{A(x,1)} \ll 2, \text{ as } \delta \rightarrow 0^+, x \rightarrow \infty.$$

Proof. The lower bound stated above is clear. To show that the asymptotic upper bound is correct, we compute using Theorem 5.4 and Theorem 5.5 that

$$\frac{B(x,1+\delta)}{A(x,1)} \ll \frac{x \cdot (\log x)^{\delta - \delta\log(1+\delta)}}{O(1) + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right)} \sim o_\delta(1),$$

as $x \rightarrow \infty$. Notice that since $\mathbb{E}[\Omega(n)] = \log\log n + B$, with $0 < B < 1$ the absolute constant from Mertens theorem, when we denote the range of $k > \log\log x$ as holding in the form of $k > (1+\delta)\log\log x$ for $\delta > 0$ at large x, we can assume that $\delta \rightarrow 0^+$ as $x \rightarrow \infty$. In particular, this holds since $k > \log\log x$ implies that

$$\lfloor \log\log x \rfloor + 1 \geq (1+\delta)\log\log x \qquad \Longrightarrow \qquad \delta \leq \frac{1 + \{\log\log x\}}{\log\log x} = o(1), \text{ as } x \rightarrow \infty.$$

The key consequence is that $B(x, 1+\delta)$ is at most a bounded constant multiple of $A(x,1)$ for all large x. ☐

6 Average case analysis of bounds on the Dirichlet inverse functions, $g^{-1}(n)$

The pages of tabular data given as Table T.1 in the appendix section (refer to page 48) are intended to provide clear insight into why we arrived at the approximations to $g^{-1}(n)$ proved in this section. The table provides illustrative numerical data by examining the approximate behavior at hand for the cases of $1 \leq n \leq 500$ with *Mathematica*.

6.1 Definitions and basic properties of component function sequences

We define the following auxiliary coefficient sequence for integers $n \geq 1, k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{21}$$

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (21) inductively. By the same argument, we see that at fixed n, the function $C_k(n)$ is seen to be non-zero only for positive integers $k \leq \Omega(n)$ whenever $n \geq 2$. A sequence of relevant signed semi-diagonals of the functions $C_k(n)$ begins as [19, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \; -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

Example 6.1 (Special cases of the functions $C_k(n)$ for small k)**.** We cite the following special cases which are verified by explicit computation using (21) [19, A066922][A]:

$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= d(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} - \gcd\left(\Omega(n), \omega(n)\right). \end{aligned}$$

The connection between the functions $C_k(n)$ and the inverse sequence $g^{-1}(n)$ is clarified precisely in Section 6.3. Before we can prove explicit bounds on $|g^{-1}(n)|$ through its relation to these functions, we will require a perspective on the lower asymptotic order of $C_k(n)$ for fixed k when n is large.

6.2 Uniform asymptotics of $C_k(n)$ for large all n and fixed k

The next theorem formally proves a minimal growth rate of the class of functions $C_k(n)$ as functions of fixed k and $n \to \infty$. In the statement of the result that follows, we view k as a fixed variable which is necessarily bounded in n, but is still taken as an independent parameter of n.

Theorem 6.2 (Asymptotics of the functions $C_k(n)$)**.** *For $k := 0$, we have by definition that $C_0(n) = \delta_{n,1}$. For all sufficiently large $n > 1$ and any fixed $1 \leq k \leq \Omega(n)$ taken independently of n, we obtain that the asymptotic main term for the expected order of $C_k(n)$ is bounded uniformly from below as*

$$\mathbb{E}[C_k(n)] \gg (\log\log n)^{2k-1}, \text{ as } n \to \infty.$$

[A] For all $n, k \geq 2$, we have the following recurrence relation satisfied by $C_k(n)$ between successive values of k:

$$C_k(n) = \sum_{p|n} \sum_{d | \frac{n}{p^{\nu_p(n)}}} \sum_{i=0}^{\nu_p(n)-1} C_{k-1}\left(dp^i\right), n \geq 1.$$

Proof. We prove our bounds by induction on k. We can see by Example 6.1 that $C_1(n)$ satsfies the formula we must establish when $k := 1$ since $\mathbb{E}[\omega(n)] = \log\log n$. Suppose that $k \geq 2$ and let our inductive assumption provide that for all $1 \leq m < k$ and $n \geq 2$

$$\mathbb{E}[C_m(n)] \gg (\log\log n)^{2m-1}.$$

For all large $x > e$, we cite that the summatory function of $\omega(n)$ satisfies [5, §22.10]

$$\sum_{n \leq x} \omega(n) = x \log\log x + Bx + O\left(\frac{x}{\log x}\right).$$

Now using the recursive formula we used to define the sequences of $C_k(n)$ in (21), we have that as $n \to \infty$

$$\begin{aligned}
\mathbb{E}[C_k(n)] &= \mathbb{E}\left[\sum_{d|n} \omega(n/d) C_{k-1}(d)\right] \\
&= \frac{1}{n} \times \sum_{d \leq n} C_{k-1}(d) \times \sum_{r=1}^{\left\lfloor \frac{n}{d} \right\rfloor} \omega(r) \\
&\sim \sum_{d \leq n} C_{k-1}(d) \left[\frac{\log\log(n/d) \left[d \leq \frac{n}{e}\right]_\delta}{d} + \frac{B}{d} + o(1)\right] \\
&\sim \sum_{d \leq \frac{n}{e}} \left[\sum_{m<d} \frac{\mathbb{E}[C_{k-1}(m)]}{m} \log\log\left(\frac{n}{m}\right) + B \cdot \mathbb{E}[C_{k-1}(d)] + B \cdot \sum_{m<d} \frac{\mathbb{E}[C_{k-1}(m)]}{m}\right] \qquad (22) \\
&\gg \sum_{d \leq \frac{n}{e}} \frac{\mathbb{E}[C_{k-1}(m)]}{m} \\
&\gg (\log n)(\log\log n)^{2k-3}.
\end{aligned}$$

In transitioning from the previous step, we have used that $(\log n) \gg (\log\log n)^2$ as $n \to \infty$. We have also used that for large n and fixed m, by an asymptotic approximation to the incomplete gamma function we have that

$$\int_e^n \frac{(\log\log t)^m}{t} dt \sim (\log n)(\log\log n)^m, \text{ as } n \to \infty.$$

Hence, the claim follows by mathematical induction for large $n \to \infty$ whenever $1 \leq k \leq \Omega(n)$. ☐

6.3 Relating the auxiliary functions $C_k(n)$ to formulas approximating $g^{-1}(n)$

Lemma 6.3 (An exact formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) g^{-1}(n/d) \quad \Longrightarrow \quad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \qquad (23)$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (23) in the form of $(g^{-1} * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m \cdot (C_m(-) * g^{-1})(n)$, or so that

$$F_m(n) = -\begin{cases} \sum\limits_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum\limits_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & m \geq 2, \\ (\omega * g^{-1})(n), & m = 1. \end{cases}$$

By repeatedly expanding the right-hand-side of the previous equation, we find that for $m := \Omega(n)$

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \tag{24}$$

The formula then follows from (24) by Möbius inversion applied to each side of the last equation. $\square$

Corollary 6.4. *For all squarefree integers $n \geq 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \tag{25}$$

Proof. Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geq 2$ and that n is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where p_i is prime for all $1 \leq i \leq \omega(n)$. Since all divisors of any squarefree n are necessarily also squarefree, we can transform the exact divisor sum guaranteed for all n in Lemma 6.3 into a sum that partitions the divisors according to the number of distinct prime factors:

$$\begin{aligned} g^{-1}(n) &= \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i} (-1)^i \cdot C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \end{aligned}$$

The signed contributions in the first of the previous equations is justified by noting that $\lambda(n) = (-1)^{\omega(n)}$ whenever n is squarefree, and that for $d \geq 1$ squarefree we have the correspondence $\omega(d) = k \implies \Omega(d) = k$ for $1 \leq k \leq \log_2(d)$. $\square$

Since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ using the notation defined in the the proof of Proposition 4.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for squarefree $n \geq 1$. A proof of part (C) of Conjecture 3.5 follows as an immediate consequence.

Lemma 6.5. *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{26}$$

Proof. By applying Lemma 6.3, Proposition 4.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$. Lemma 6.3 implies

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \end{aligned}$$

In the last equation, we see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. $\square$

Combined with the signedness property of $g^{-1}(n)$ guaranteed by Proposition 4.1, Lemma 6.5 shows that the summatory function is expressed as

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_{\Omega(d)}(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Since $\lambda(d)C_{\Omega(d)}(d) = (g^{-1} * 1)^{-1}(d) = (\chi_{\mathbb{P}} + \varepsilon)(d)$ where $\chi_{\mathbb{P}}$ denotes the characteristic function of the primes, we also clearly recover by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1.$$

Corollary 6.6. *We have that*

$$(\log n)(\log\log n) \ll \mathbb{E}|g^{-1}(n)| \leq \mathbb{E}\left[\sum_{d|n} C_{\Omega(d)}(d)\right].$$

Proof. To prove the lower bound, recall from the introduction that the summatory function of the squarefree integers is given by

$$Q(x) := \sum_{n \leq x} \mu^2(n) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Then since $C_{\Omega(d)}(d) \geq 1$ for all $d \geq 1$, and since $\mathbb{E}[C_k(d)]$ is minimized when $k := 1$ according to Theorem 6.2, we obtain by summing over (26) that

$$\begin{aligned}
\frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| &= \frac{1}{x} \times \sum_{d \leq x} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&\sim \sum_{d \leq x} C_{\Omega(d)}(d)\left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dx}}\right)\right] \\
&= \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(x)}(x)] + \sum_{d<x} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O\left(\frac{1}{\sqrt{x}} \times \int_0^x t^{-1/2} dt\right) \\
&\gg \left[\sum_{e \leq d \leq x} \frac{\log\log d}{d}\right] + O(1) \\
&\sim \times \int_e^x \frac{\log\log t}{t} dt + O(1) \\
&\gg (\log x)(\log\log x), \text{ as } x \to \infty.
\end{aligned}$$

To prove the upper bound, notice that by Lemma 6.3 and Corollary 6.4,

$$|g^{-1}(n)| \leq \sum_{d|n} C_{\Omega(d)}(d), n \geq 1.$$

Now since both of the above quantities are positive for all $n \geq 1$, we clearly obtain the upper bound stated above when we average over $n \leq x$ for all large x. $\square$

6.3.1 A connection to the distribution of the primes

Remark 6.7. The combinatorial complexity of $g^{-1}(n)$ is deeply tied to the distribution of the primes $p \leq n$ as $n \to \infty$. While the magnitudes and dispersion of the primes $p \leq x$ certainly restricts the repeating of these distinct sequence values we can see in the contributions to $G^{-1}(x)$, the following statement is still clear about the relation of the weight functions $|g^{-1}(n)|$ to the distribution of the primes: The value of $|g^{-1}(n)|$ is entirely dependent on the pattern of the *exponents* (viewed as multisets) of the distinct prime factors of $n \geq 2$. The relation of the repitition of the distinct values of $|g^{-1}(n)|$ in forming bounds on $G^{-1}(x)$ makes another clear tie to $M(x)$ through Proposition 8.1 in the next section.

Example 6.8 (Combinatorial significance to the distribution of $g^{-1}(n)$)**.** We have a natural extremal behavior with respect to distinct values of $\Omega(n)$ corresponding to squarefree integers, and prime powers. Namely, if for $k \geq 1$ we define the infinite sets M_k and m_k to correspond to the maximal (minimal) positive integers such that

$$M_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\},$$

$$m_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\},$$

then any element of M_k is squarefree and any element of m_k is a prime power. In particular, we have that for any $N_k \in M_k$ and $n_k \in m_k$

$$N_k = \sum_{j=0}^{k} \binom{k}{j} \cdot j!, \qquad \text{and} \qquad n_k = 2 \cdot (-1)^k.$$

The formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 4.1 implies that we can express an exact formula for $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n. Namely, for $n \geq 2$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z \cdot \nu_p(n)) = [z^k] \prod_{p^{\alpha}||n} (1 + \alpha z), 0 \leq k \leq \omega(n).$$

Then we have essentially shown using (9) and (26) that we can expand

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, n \geq 2.$$

The combinatorial formula for $h^{-1}(n) = \lambda(n) \cdot (\Omega(n))! \times \prod_{p^{\alpha}||n} (\alpha!)^{-1}$ we derived in the proof of the key signedness proposition in Section 4 suggests further patterns and more regularity in the contributions of the distinct weighted terms for $G^{-1}(x)$ when we sum over all of the distinct prime exponent patterns that factorize $n \leq x$.

7 New formulas and bounds for $g^{-1}(n)$ and its summatory function

7.1 Exact probabilistic bounds on the distributions of component sequences

We have remarked already in the introduction that the relation of the component functions, $g^{-1}(n)$ and $C_k(n)$, to the canonical additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties of these functions witnessed in Table T.1. In particular, each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdös-Kac theorem that shows that a shifted and scaled variant of each of the sets of these function values can be expressed through a limiting normal distribution as $n \to \infty$. This extremely regular tendency of these functions towards their average order is inherited by the component function sequences we are summing in the approximation of $M(x)$ stated by Proposition 8.1. In the remainder of this section we establish more technical analytic proofs of related properties of our key sequences, again in the spirit of Montgomery and Vaughan's reference.

Proposition 7.1. *For $|z| < P(2)^{-1}$, let the summatory function be defined as*

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

Let the function $F(s,z)$ is defined for $\operatorname{Re}(s) > 1$ and $|z| < 2$ in terms of the prime zeta function by

$$F(s,z) := \frac{1}{1 - P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

Then we have that for large x

$$\widehat{A}_z(x) = \frac{x}{\Gamma(z)} \cdot F(2,z) \cdot (\log x)^{z-1} + O_z\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right), |z| < P(2)^{-1}.$$

Proof. We know from the proof of Proposition 4.1 that for $n \geq 2$

$$C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}.$$

Then we can generate the denominator terms by the Dirichlet series

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! \cdot p^{rs}}\right)^{-1} = \exp\left(z \cdot P(s)\right), \operatorname{Re}(s) > 1, z \in \mathbb{C}.$$

By computing a Laplace transform on the right-hand-side of the above with respect to the variable z, we obtain

$$\sum_{n \geq 1} C_{\Omega(n)}(n) \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp\left(tz \cdot P(s)\right) dt = \frac{1}{1 - P(s)z}, \operatorname{Re}(s) > 1, |z| < |P(s)|^{-1}.$$

It follows that

$$\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times F(s,z),$$

where

$$F(s,z) := \frac{1}{1 - P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z, \operatorname{Re}(s) > 1, |z| < |P(s)|^{-1}.$$

Since $F(s,z)$ is convergent as an analytic function of s for all $\operatorname{Re}(s) > 1$ whenever $|z| < 2$, if $b_z(n)$ are the coefficients of the DGF $F(s,z)$, then

$$\left| \sum_{n \geq 1} \frac{b_z(n)(\log n)^{2R+1}}{n^s} \right| < +\infty,$$

is uniformly bounded for $|z| \leq R$. We must adapt the details to the case where the next proof method arises in the first application from [11, §7.4; Thm. 7.18] so that we can sum over our modified function depending on $\Omega(n)$. In particular, we cannot guarantee convergence of $F(s,z)$ by setting $s := 1$, so we modify the proof to show that we can in fact set $s := 2$ in this function to obtain a related result.

Let the function $d_z(n)$ be generated as the coefficients of the DGF $\zeta(s)^z$ for $\mathrm{Re}(s) > 1$, with corresponding summatory function $D_z(x) := \sum_{n \leq x} d_z(n)$. Adopting the notation from the reference, we set $b_z(n) := (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$, let the convolution $a_z(n) := \sum_{d|n} b_z(d) d_z(n/d)$, and define the summatory function $A_z(x) := \sum_{n \leq x} a_z(n)$. The theorem in [11, Thm. 7.17; §7.4] implies that for any $z \in \mathbb{C}$ and $x \geq 2$

$$D_z(x) = \frac{x (\log x)^{z-1}}{\Gamma(z)} + O\left(x \cdot (\log x)^{\mathrm{Re}(z)-2}\right).$$

Then we have that

$$\begin{aligned} A_z(x) &= \sum_{m \leq x/2} b_z(m) D_z(x/m) + \sum_{x/2 < m \leq x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \times \sum_{m \leq x/2} \frac{b_z(m)}{m^2} \times m \cdot \log\left(\frac{x}{m}\right)^{z-1} + O\left(x \sum_{m \leq x} \frac{|b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2}\right). \end{aligned} \tag{27}$$

We can sum the coefficients for $u \geq e$ large as

$$\sum_{m \leq u} \frac{b_z(m)}{m} = (F(2,z) + O(u^{-2}))u - \int_1^u (F(2,z) + O(t^{-2}))dt = F(2,z) + O(u^{-1}).$$

The error term in (27) satisfies

$$\begin{aligned} x \sum_{m \leq x} \frac{|b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2} &\ll x (\log x)^{\mathrm{Re}(z)-2} \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m} + x (\log x)^{-(R+2)} \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m} (\log m)^{2R} \\ &\ll x (\log x)^{\mathrm{Re}(z)-2} \cdot F(2,z) = O_z\left(x \cdot (\log x)^{\mathrm{Re}(z)-2}\right), |z| \leq R. \end{aligned}$$

In the main term estimate for $A_z(x)$ from (27), when $m \leq \sqrt{x}$ we have

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\mathrm{Re}(z)-2}\right).$$

The total main term sum over the interval $m \leq x/2$ then corresponds to bounding

$$\begin{aligned} \sum_{m \leq x/2} b_z(m) D_z(x/m) &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \sum_{m \leq x/2} \frac{b_z(m)}{m} \\ &\quad + O_z\left(x (\log x)^{\mathrm{Re}(z)-2} \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m} + x (\log x)^{R-1} \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} F(2,z) + O_z\left(x (\log x)^{\mathrm{Re}(z)-2} \sum_{m \geq 1} \frac{b_z(m) (\log m)^{2R+1}}{m^2}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} F(2,z) + O_z\left(x (\log x)^{\mathrm{Re}(z)-2}\right). \end{aligned}$$

□

Theorem 7.2. *We have uniformly for $1 \leq k < \log\log x$ that as $x \to \infty$*

$$\widehat{C}_k(x) := \sum_{\substack{n \leq x \\ \Omega(n) = k}} \lambda(n) (-1)^{\omega(n)} C_k(n) \asymp \frac{x}{\log x} \cdot \frac{(-1)^{k-1} (\log\log x - \log\zeta(2))^k}{k!} \left[1 + O\left(\frac{k}{(\log\log x)^3}\right)\right].$$

Proof. The proof is a similar adaptation of the method of Montgomery and Vaughan we cited in Remark 5.3 to prove our variant of Theorem 3.7. We begin by bounding a contour integral over the error term for fixed large x for $r := \frac{k-1}{\log\log x}$ with $r < 2$:

$$\begin{aligned}
\left|\int_{|z|=r} \frac{x\cdot(\log x)^{-(\mathrm{Re}(z)+2)}}{z^{k+1}}dz\right| &\ll x(\log x)^{-(r+2)}r^{-(k+1)} \ll \frac{x}{(\log x)^2}\cdot\frac{(\log\log x)^{k+1}}{(k-1)^{k+1}}\cdot\frac{1}{e^{k-1}} \\
&\ll \frac{x}{(\log x)^2}\cdot\frac{(\log\log x)^{k+1}}{(k-1)^{3/2}}\cdot\frac{1}{e^{2k}(k-1)!} \\
&\ll \frac{x}{(\log x)^2}\cdot\frac{(\log\log x)^{k-1}}{(k-1)!} \ll \frac{x}{\log x}\cdot\frac{k\cdot(\log\log x)^{k-5}}{(k-1)!}.
\end{aligned}$$

We must find an asymptotically accurate main term approximation to the coefficients of the following contour integral for $r \in [0, z_{\max}]$ where $z_{\max} < 2$:

$$\widetilde{A}_r(x) := -\int_{|z|=r} \frac{x\cdot(\log x)^{-z}\zeta(2)^z}{(\log x)\Gamma(1+z)\cdot z^{k+1}(1+P(2)z)}dz. \tag{28}$$

Finding an exact formula for the derivatives of the function that is implicit to the Cauchy integral formula (CIF) for (28) is complicated significantly by the need to differentiate $\Gamma(1+z)^{-1}$ up to integer order k in the formula. We can show that provided a restriction on the uniform bound parameter to $1 \leq r < 1$, we can approximate the contour integral in (28) using a sane bounding procedure where the resulting main term is accurate up to a bounded constant factor.

We observe that for $r := 1$, the function $|\Gamma(1+re^{2\pi\imath t})|$ has a singularity (pole) when $t := \frac{1}{2}$. Thus we restrict the range of $|z| = r$ so that $0 \leq r < 1$ to necessarily avoid this problematic value of t when we parameterize $z = re^{2\pi\imath t}$ as a real integral over $t \in [0,1]$. Then we can compute the finite extremal values as

$$\begin{aligned}
\min_{\substack{0\leq r<1\\0\leq t\leq 1}} |\Gamma(1+re^{2\pi\imath t})| &= |\Gamma(1+re^{2\pi\imath t})|\Big|_{(r,t)\approx(1,0.740592)} \approx 0.520089 \\
\max_{\substack{0\leq r<1\\0\leq t\leq 1}} |\Gamma(1+re^{2\pi\imath t})| &= |\Gamma(1+re^{2\pi\imath t})|\Big|_{(r,t)\approx(1,0.999887)} \approx 1.
\end{aligned}$$

This shows that

$$\widetilde{A}_r(x) \asymp -\int_{|z|=r} \frac{x\cdot\exp(-P(2)z)(\log x)^{-z}}{(\log x)\cdot z^{k+1}(1+P(2)z)}dz, \tag{29}$$

where as $x\to\infty$

$$\frac{\widetilde{A}_r(x)}{-\int_{|z|=r}\frac{x(\log x)^{-z}\zeta(2)^z}{(\log x)\cdot z^{k+1}(1+P(2)z)}dz} \in [1, 1.92275].$$

In particular, this argument holds by an analog to the mean value theorem for real integrals based on sufficient continuity conditions on the parameterized path and the smoothness of the integrand viewed as a function of z.

By induction we can compute the remaining coefficients $[z^k]\Gamma(1+z)\times\widehat{A}_z(x)$ with respect to x for fixed $k \leq \log\log x$ using the CIF. Namely, it is not difficult to see that for any integer $m \geq 0$, we have the m^{th} partial derivative of the integrand with respect to z has the following expansion:

$$\begin{aligned}
\frac{1}{m!}\times\frac{\partial^{(m)}}{\partial z^{(m)}}\left[\frac{(\log x)^{-z}\zeta(2)^z}{1+P(2)z}\right]\Bigg|_{z=0} &= \sum_{j=0}^{m}\frac{(-1)^m P(2)^j(\log\log x-\log\zeta(2))^{m-j}}{(m-j)!} \\
&= \frac{(-P(2))^m(\log x)^{\frac{1}{P(2)}}\zeta(2)^{-\frac{1}{P(2)}}}{m!}\times\Gamma\left(m+1, 1+\frac{\log\log x}{P(2)}\right)
\end{aligned}$$

$$\sim \frac{(-1)^m (\log\log x - \log\zeta(2))^m}{m!}.$$

Now by parameterizing the countour around $|z| = r := \frac{k-1}{\log\log x} < 1$ we deduce that the the main term of our approximation corresponds to

$$-\int_{|z|=r} \frac{x \cdot (\log x)^{-z} \zeta(2)^z}{(\log x) z^{k+1}(1+P(2)z)} dz \asymp \frac{x}{\log x} \cdot \frac{(-1)^{k-1}(\log\log x - \log\zeta(2))^k}{k!}. \qquad \square$$

Remark 7.3. An exact DGF expression for $\lambda(n) C_{\Omega(n)}(n)$ is in fact very much complicated by the need to estimate the asymptotics of the coefficients of the right-hand-side products

$$\begin{aligned}\sum_{n\geq 1} \frac{\lambda(n) C_{\Omega(n)}(n) z^{\Omega(n)}}{(\Omega(n))! \cdot n^s} &= \prod_p \left(2 - \exp\left(-z \cdot p^{-s}\right)\right)^{-1}, \operatorname{Re}(s) > 1, |z| < \log 2 \\ &= \exp\left(\sum_{j\geq 1}\sum_p \left(e^{-zp^{-s}} - 1\right)^j \frac{1}{j}\right).\end{aligned}$$

It is unclear how to exactly, and effectively, bound the coefficients of powers of z in the DGF expansion defined by the last equation. We use an alternate method in Corollary 7.5 to obtain the asymptotics for the actual summatory functions on which we require tight average case bounds.

Remark 7.4 (A standard simplifying assumption)**.** For $m \leq \omega_{\max}$ and $k \leq \Omega_{\max}$, as $n \to \infty$ we expect

$$\mathbb{P}\left(\omega(n) = m | \Omega(n) = k\right) \approx \frac{\omega_{\max} + 1 - k}{\omega_{\max}},$$

so that the conditional distribution of $\omega(n), \Omega(n)$ is not uniform over its bounded range. However, we do as is standard fare in proofs of the more traditional Erdös-Kac theorems require the simplifying assumption that as $n \to \infty$, we expect independently that $\omega(n), \Omega(n)$ are approximately equally likely to assume any values in some bounded $[1, M]$. This means we can treat the difference $\Omega(n) - \omega(n)$ as being approximately randomly distributed over some bounded range of its possible values. For a more rigorous treatment of this underlying principle see [4, 2, 15].

Let the constant $\widehat{c} \approx 0.378647$ be defined explicitly as the product of primes

$$\widehat{c} := \frac{1}{4} \times \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right)^{-1}.$$

This constant is related to expressions of the asymptotic densities of the following sets for integers $k \geq 0$ [11, §2.4]:

$$\begin{aligned}N_k(x) &= \{n \leq x : \Omega(n) - \omega(n) = k\} \\ &= d_k x + O\left(\left(\frac{3}{4}\right)^k \sqrt{x} (\log x)^{4/3}\right), \qquad (30a)\end{aligned}$$

For each natural number $k \geq 0$, $d_k > 0$ is an absolute constant that satisfies

$$d_k = \frac{\widehat{c}}{2^k} + O\left(5^{-k}\right). \qquad (30b)$$

A hybrid DGF generating function for these densities is given by

$$\sum_{k\geq 0} d_k z^k = \prod_p \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p-z}\right). \qquad (30c)$$

The limiting distribution of $\Omega(n) - \omega(n)$ is utilized in the proof of Theorem 7.2.

Corollary 7.5 (Summatory functions of the unsigned component sequences)**.** *We have that for large $x \geq 2$ and $1 \leq k \leq \log\log x$*

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp \frac{3}{2\hat{c}} \cdot \frac{x}{\log x} \cdot \frac{(\log\log x - \log\zeta(2))^k}{k!} \left[1 + O\left(\frac{1}{(\log\log x)^2}\right)\right].$$

Proof. We handle transforming our previous results for the sum over the unsigned sequence $C_{\Omega(n)}(n)$ such that $\Omega(n) = k$. The argument basically boils down to approximating the smooth summatory function of $\lambda_*(n) := (-1)^{\Omega(n)-\omega(n)}$ using the weighted densities defined by (30). We then have an integral formula involving the non-sign-weighted sequence that results by again applying ordinary Abel summation (and integrating by parts) in the form of

$$\begin{aligned} \sum_{n \leq x} \lambda_*(n) h(n) &= \left(\sum_{n \leq x} \lambda_*(n)\right) h(x) - \int_1^x \left(\sum_{n \leq t} \lambda_*(n)\right) h'(t) dt \qquad (31) \\ &\asymp \left\{ \begin{array}{ll} u_t = L_*(t) & v_t' = h'(t)dt \\ u_t' = L_*'(t)dt & v_t = h(t) \end{array} \right\} \int_1^x \frac{d}{dt}\left[\sum_{n \leq t} \lambda_*(n)\right] h(t) dt. \end{aligned}$$

Let the signed left-hand-side summatory function in (31) for our function be defined by

$$\begin{aligned} \widehat{C}_{k,*}(x) &:= \left| \sum_{\substack{n \leq x \\ \Omega(n)=k}} \lambda(n)(-1)^{\omega(n)} C_{\Omega(n)}(n) \right| \\ &= \frac{x}{\log x} \cdot \frac{(\log\log x - \log\zeta(2))^k}{k!} \left[1 + O\left(\frac{1}{(\log\log x)^2}\right)\right], \end{aligned}$$

where the second equation follows from the proof of Theorem 7.2. Then by differentiating the formula we engineered well for ourselves in (31), and then summing over the uniform range of $1 \leq k \leq \log\log x$, we can recover an approximation to the unsigned summatory function for the sequence we need to bound in later results proved in this section.

We handle the sign weighted terms by defining and approximating the asymptotic main term of the following summatory function (*cf.* Table T.2 starting on page 55):

$$\begin{aligned} L_*(t) := \sum_{n \leq t} \lambda(n)(-1)^{\omega(n)} &= \sum_{j=0}^{\log_2(t)} (-1)^j \cdot \#\{n \leq t : \Omega(n) - \omega(n) = j\} \\ &\sim \sum_{j=0}^{\log_2(t)} \cdot \frac{\hat{c} \cdot t(-1)^j}{2^j} = \frac{2\hat{c} \cdot t}{3} + o(1), \text{ as } t \to \infty. \end{aligned}$$

The approximation to the densities d_k for the difference of the prime omega functions is cited from (30) [11, §2.4]. After applying the formula from (31), we deduce that the unsigned summatory function variant satisfies

$$\begin{aligned} \widehat{C}_{k,*}(x) = \int_1^x L_*'(t) C_{\Omega(t)}(t) dt \qquad &\implies C_{\Omega(x)}(x) \asymp \frac{\widehat{C}_{k,*}'(x)}{L_*'(x)} \\ C_{\Omega(x)}(x) \asymp \frac{3}{2\hat{c}} \left[\frac{(\log\log x - \log\zeta(2)^k}{(\log x)k!}\left(1 - \frac{1}{\log x}\right) + \frac{(\log\log x - \log\zeta(2))^{k-1}}{(\log x)^2(k-1)!}\right] &=: \widehat{C}_{k,**}(x). \end{aligned}$$

So again applying the Abel summation formula, we obtain that

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp \left| \int \widehat{C}_{k,**}(x) dx \right|$$

$$= \frac{3}{2\hat{c}} \cdot \frac{x}{\log x} \cdot \frac{(\log\log x - \log\zeta(2))^k}{k!}\left[1 + O\left(\frac{1}{(\log\log x)^2}\right)\right].$$

This proves the stated formula, and it similarly holds uniformly for all $1 \leq k \leq \log\log x$ when x is large. □

Lemma 7.6. *We have that as* $x \to \infty$

$$\mathbb{E}\left[\sum_{n \leq x} C_{\Omega(n)}(n)\right] \asymp \frac{3}{2\hat{c}\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\log\log n}}\left[1 + O\left(\frac{1}{\log\log n}\right)\right].$$

Proof. We claim that

$$\sum_{n \leq x} C_{\Omega(n)}(n) = \sum_{k=1}^{\log_2(x)} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp \sum_{k=1}^{\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n). \tag{32}$$

To prove (32), it suffices to show that

$$\frac{\sum\limits_{\log\log x < k \leq \log_2(x)} \sum\limits_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n)}{\sum\limits_{k=1}^{\log\log x} \sum\limits_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n)} = o(1), \text{ as } x \to \infty. \tag{33}$$

We first compute the absolute value of the following summatory function by applying Corollary 7.5 for large $x \to \infty$:

$$\begin{aligned}\sum_{k=1}^{\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) &\asymp \sum_{k=1}^{\log\log x} \frac{3}{2\hat{c}} \cdot \frac{x}{\log x} \cdot \frac{(\log\log x - \log\zeta(2))^k}{k!} \times \left[1 + O\left(\frac{1}{\log\log x}\right)\right] \\ &\asymp \frac{3}{2\hat{c}\sqrt{2\pi}} \cdot \frac{x}{\sqrt{\log\log x}}\left[1 + O\left(\frac{1}{\log\log x}\right)\right].\end{aligned} \tag{34}$$

We define the following component sums for large x and $0 < \varepsilon < 1$ so that $(\log\log x)^{\varepsilon \frac{\log\log x}{\log\log\log x}} = o(\log x)$:

$$S_{2,\varepsilon}(x) := \sum_{\log\log x < k \leq \log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n).$$

Then

$$\sum_{k=1}^{\log_2(x)} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \gg S_{2,\varepsilon}(x),$$

with equality as $\varepsilon \to 1$ so that the upper bound of summation tends to $\log x$. To show that (33) holds, observe that whenever $\Omega(n) = k$, we have that $C_{\Omega(n)}(n) \leq k!$. We can bound the sum defined above using Theorem 5.4 for large $x \to \infty$ as

$$\begin{aligned} S_{2,\varepsilon}(x) &\leq \sum_{\log\log x \leq k \leq \log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \ll \sum_{k=\log\log x}^{(\log\log x)^{\varepsilon \frac{\log\log x}{\log\log\log x}}} \frac{\widehat{\pi}_k(x)}{x} \cdot k! \\ &\ll \sum_{k=\log\log x}^{(\log\log x)^{\varepsilon \frac{\log\log x}{\log\log\log x}}} (\log x)^{\frac{k}{\log\log x} - 1 - \frac{k}{\log\log x}(\log k - \log\log\log x)} \cdot \left(\frac{k}{e}\right)^k \sqrt{2\pi k}\end{aligned}$$

$$\begin{aligned}
&\ll \sum_{k=\log\log x}^{\varepsilon \frac{\log\log x}{\log\log\log x}} (\log x)^{k \frac{\log\log\log x}{\log\log x} - 1} \sqrt{k} \ll \frac{1}{(\log x)} \times \int_{\log\log x}^{\varepsilon \frac{\log\log x}{\log\log\log x}} (\log\log x)^t \sqrt{t} \cdot dt \\
&\ll \frac{1}{(\log x)} \sqrt{\frac{\varepsilon \cdot \log\log x}{\log\log\log x}} (\log\log x)^{\frac{\varepsilon \cdot \log\log x}{\log\log\log x}} = o(x),
\end{aligned}$$

where $\lim_{x\to\infty} (\log x)^{\frac{1}{\log\log x}} = e$. By (34) this form of the ratio in (33) clearly tends to zero. If we have a contribution from the terms $\widehat{\pi}_k(x)$ as $\varepsilon \to 1$, e.g., if x is a power of two, then $C_{\Omega(x)}(x) = 1$ by the formula in (9), so that the contribution from this upper-most indexed term is negligible:

$$x = 2^k \implies \Omega(x) = k \implies C_{\Omega(x)}(x) = \frac{(\Omega(x))!}{k!} = 1.$$

The formula for the expectation claimed in the statement of this lemma above then follows from (34) by scaling by $\frac{1}{x}$ and dropping the asymptotically lesser error terms in the bound. $\square$

Corollary 7.7 (Expectation formulas)**.** *We have that as $n \to \infty$*

$$\mathbb{E}|g^{-1}(n)| \asymp \frac{9}{\pi^2 \hat{c}\sqrt{2\pi}} \cdot \frac{(\log n)}{\sqrt{\log\log n}}.$$

Proof. We use the formula from Corollary 7.5 to find $\mathbb{E}[C_{\Omega(n)}(n)]$ up to a small bounded multiplicative constant factor as $n \to \infty$. This implies that for large x

$$\begin{aligned}
\int \frac{\mathbb{E}[C_{\Omega(x)}(x)]}{x} dx &\asymp \frac{3}{2\sqrt{2}\hat{c}} \cdot \operatorname{erfi}\left(\sqrt{\log\log x}\right) \\
&\asymp \frac{3}{2\hat{c}\sqrt{2\pi}} \frac{(\log x)}{\sqrt{\log\log x}}.
\end{aligned}$$

In the previous equation, we have used a known asymptotic expansion of the function $\operatorname{erfi}(z)$ about infinity in the form of [3, §3.2]

$$\operatorname{erfi}(z) = \frac{e^{z^2}}{\sqrt{\pi}} \left(z^{-1} + \frac{1}{2} z^{-3} + \frac{3}{4} z^{-5} + \cdots\right), \text{ as } |z| \to \infty.$$

Therefore, citing the formula we derived in the proof of Corollary 6.6, we find that

$$\begin{aligned}
\mathbb{E}|g^{-1}(n)| &= \frac{6}{\pi^2} \left[\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d<n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O(1) \\
&\asymp \frac{9}{\pi^2 \hat{c}\sqrt{2\pi}} \cdot \frac{(\log n)}{\sqrt{\log\log n}}.
\end{aligned}$$

This proves the claimed formula for the expectation of our key inverse sequence. $\square$

Theorem 7.8. *Let the mean and variance analogs be denoted by*

$$\mu_x(C) := \log\log x + \hat{a}, \qquad \text{and} \qquad \sigma_x(C) := \sqrt{\mu_x(C)},$$

where the absolute constant $\hat{a} := \log\left(\frac{2\hat{c}}{3}\right) \approx -1.37662$. Set $Y > 0$ and suppose that $z \in [-Y, Y]$. Then we have uniformly for all $-Y \leq z \leq Y$ that

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)} \leq z\right\} = \Phi(z) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

Proof. For large x and $n \leq x$, define the following auxiliary variables:

$$\alpha_n := \frac{C_{\Omega(n)}(n) - \mu_n(C)}{\sigma_n(C)}, \quad \beta_{n,x} := \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)}.$$

Let the corresponding densities (whose limiting distributions we must verify) be defined by the functions

$$\Phi_1(x,z) := \frac{1}{x} \cdot \#\{n \leq x : \alpha_n \leq z\},$$

and

$$\Phi_2(x,z) := \frac{1}{x} \cdot \#\{n \leq x : \beta_{n,x} \leq z\}.$$

We first argue that it suffices to consider the distribution of $\Phi_2(x,z)$ as $x \to \infty$ in place of $\Phi_1(x,z)$ to obtain our desired result statement. In particular, the difference of the two auxiliary variables is neglibible as $x \to \infty$ for n, x taken over the ranges that contribute the non-trivial weight to the main term of each density function. We have for $\sqrt{x} \leq n \leq x$ and $C_{\Omega(n)}(n) \leq 2 \cdot \mu_x(C)$ that

$$|\alpha_n - \beta_{n,x}| \ll \frac{1}{\sigma_x(C)} \xrightarrow{x\to\infty} 0.$$

So we naturally prefer to estimate the easier forms of the distribution function $\Phi_2(x,z)$ when x is large, and for any fixed $z \in \mathbb{R}$. That is, we replace α_n by $\beta_{n,x}$ and estimate the limiting densities corresponding to these terms. The rest of our argument follows closely along with the method in the proof of the related theorem in [11, Thm. 7.21; §7.4].

We use the formula proved in Corollary 7.5, which holds uniformly for x large when $1 \leq k \leq \log\log x$, to estimate the densities claimed within the ranges bounded by z as $x \to \infty$. Let $k \geq 1$ be a natural number defined by $k := t + \log\log x + \hat{a}$. We write the small parameter $\delta_{t,x} := \frac{t}{\log\log x + \hat{a}}$. When $|t| \leq \frac{1}{2}(\log\log x + \hat{a})$, we have by Stirling's formula that

$$\begin{aligned}\frac{3}{2\hat{c}} \cdot \frac{x}{\log x} \frac{(\log\log x + P(2))^k}{k!} &\sim \frac{3}{2\hat{c}\sqrt{2\pi}} \cdot \frac{x \cdot e^{\hat{a}+t}(\log\log x - \log\zeta(2))^{\mu_x(C)(1+\delta_{t,x})}}{\sigma_x(C) \cdot \mu_x(C)^{\mu_x(C)(1+\delta_{t,x})}(1+\delta_{t,x})^{\mu_x(C)(1+\delta_{t,x})+\frac{1}{2}}} \\ &\sim \frac{x \cdot e^t}{\sqrt{2\pi} \cdot \sigma_x(C)} (1+\delta_{t,x})^{-\left(\mu_x(C)(1+\delta_{t,x})+\frac{1}{2}\right)},\end{aligned}$$

since $\log\log x - \log\zeta(2) = \mu_x(C)(1+o(1))$ as $x \to \infty$.

We have the uniform estimate $\log(1+\delta_{t,x}) = \delta_{t,x} - \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)$ whenever $|\delta_{t,x}| \leq \frac{1}{2}$. Then we can expand the factor involving $\delta_{t,x}$ in the previous equation as follows:

$$\begin{aligned}(1+\delta_{t,x})^{-\mu_x(C)(1+\delta_{t,x})-\frac{1}{2}} &= \exp\left(\left(\frac{1}{2} + \mu_x(C)(1+\delta_{t,x})\right) \times \left(-\delta_{t,x} + \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)\right)\right) \\ &= \exp\left(-t + \frac{t-t^2}{2\mu_x(C)} - \frac{t^2}{4\mu_x(C)^2} + O\left(\frac{|t|^3}{\mu_x(C)^2}\right)\right).\end{aligned}$$

For both $|t| \leq \mu_x(C)^{1/2}$ and $\mu_x(C)^{1/2} < |t| \leq \mu_x(C)^{2/3}$, we see that

$$\frac{t}{\mu_x(C)} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Similarly, for $|t| \leq 1$ and $|t| > 1$, we see that both

$$\frac{t^2}{\mu_x(C)^2} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Let the error terms in (x,t) be denoted by

$$\widetilde{E}(x,t) := O\left(\frac{1}{\sigma_x(C)}\right) + O\left(\frac{|t|^3}{\mu_x(C)^2}\right).$$

Combining these estimates with the previous computations, we can deduce that uniformly for $|t| \leq \mu_x(C)^{2/3}$

$$\frac{3}{2\hat{c}} \cdot \frac{x}{\log x} \frac{(\log\log x + P(2))^k}{k!} \sim \frac{x}{\sqrt{2\pi} \cdot \sigma_x(C)} \cdot \exp\left(-\frac{t^2}{2\sigma_x(C)^2}\right) \times \left[1 + \widetilde{E}(x,t)\right].$$

By the argument in the proof of Lemma 7.6, we see that the contributions of these summatory functions for $k \leq \mu_x(C) - \mu_x(C)^{2/3}$ is negligible. We also require that $k \leq \log\log x$ as we have worked out in Theorem 7.2. So we sum over a corresponding range of

$$\mu_x(C) - \mu_x(C)^{2/3} \leq k \leq R \cdot \mu_x(C) + z \cdot \sigma_x(C),$$

for $R := 1 - \frac{z}{\sigma_x(C)}$ to approximate the stated normalized densities. Then finally as $x \to \infty$, the three terms that result (one main term, two error terms) can be considered to correspond to a Riemann sum for an associated integral. □

Corollary 7.9. *Let $Y > 0$. Then uniformly for all $-Y \leq y \leq Y$ we have that*

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq y\right\} = \Phi\left(\frac{\frac{\pi^2}{6}y - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

Proof. We compute using the argument sketched in the proof of Corollary 6.6 from Section 6.3 that

$$|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2} C_{\Omega(n)}(n).$$

In particular, let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) := f(x) - f(x-1)$. Then from the proof of the initial corollary, we see that for large n

$$\begin{aligned}
|g^{-1}(n)| = \Delta_n(n \cdot \mathbb{E}|g^{-1}(n)|) &\sim \Delta_n\left(\sum_{d \leq n} \frac{6}{\pi^2} \cdot C_{\Omega(d)}(d) \cdot \frac{x}{d}\right) \\
&= \frac{6}{\pi^2}\left[C_{\Omega(n)}(n) + \sum_{d<n} C_{\Omega(d)}(d)\frac{n}{d} - \sum_{d<n} C_{\Omega(d)}(d)\frac{(n-1)}{d}\right] \\
&= \frac{6}{\pi^2} C_{\Omega(n)}(n) + \frac{6}{\pi^2}\mathbb{E}[C_{\Omega(n)}(n)] \\
&= \frac{6}{\pi^2} C_{\Omega(n)}(n) + o(1), \text{ as } n \to \infty,
\end{aligned}$$

where the last step is a consequence of Lemma 7.6. The result finally follows from Theorem 7.8. □

7.2 Establishing initial lower bounds on the summatory functions $G^{-1}(x)$

Definition 7.10. Let the summatory function $G_E^{-1}(x)$ be defined for $x \geq 1$ by

$$G_E^{-1}(x) := \sum_{n \leq (\log x)^5 (\log\log x)} \lambda(n) \times \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log\log d}. \tag{35}$$

The subscript of E is a formality of notation that does not correspond to an actual parameter or any implicit dependence on E in the function defined above.

Theorem 7.11. *For all sufficiently large integers $x \to \infty$, we have that*

$$|G^{-1}(x)| \gg |G_E^{-1}(x)|.$$

Proof. First, consider the following upper bound on $|G_E^{-1}(x)|$:

$$\begin{aligned}
|G_E^{-1}(x)| &= \left| \sum_{e \leq n \leq (\log x)^5 (\log\log x)} \lambda(n) \times \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{3}{4}}}{\log\log d} \right| \\
&\ll \sum_{e<d\leq(\log x)^5(\log\log x)} \frac{(\log d)^{\frac{1}{4}}}{\log\log d} \cdot \left\lfloor \frac{(\log x)^5(\log\log x)}{d} \right\rfloor \\
&\ll (\log x)^5 (\log\log x) \times \int_e^{(\log x)^5(\log\log x)} \frac{(\log t)^{\frac{1}{4}}}{t \cdot \log\log t} dt \\
&= (\log x)^5(\log\log x) \times \operatorname{Ei}\left(\frac{5}{4}\log\log\left((\log x)^5(\log\log x)\right)\right) \\
&\ll (\log x)^5(\log\log x)(\log\log\log x)^2. \qquad (36)
\end{aligned}$$

We need a couple of observations to sum $G^{-1}(x)$ in absolute value and bound it from below. We will use a lower bound approximating the summatory function of $\lambda(n)$ for $n \leq t$ and t large by summing over the uniform asymptotic bounds proved in Theorem 3.7. To be careful about the expected sign of this summatory function, we first appeal to the original approximation to the functions $\widehat{\pi}_k(x)$ given by Theorem 3.6. As noted in [11, §7.4], the function $\mathcal{G}(z)$ from Theorem 3.6 satisfies

$$\mathcal{G}\left(\frac{k-1}{\log\log x}\right) = 1 + O(1), k \leq \log\log x,$$

so that uniformly for $1 \leq k \leq \log\log x$ we can write

$$\widehat{\pi}_k(x) \asymp \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{1}{\log\log x}\right)\right].$$

By Corollary 5.7, the following summatory function represents the asymptotic main term in the summation $\sum_{n\leq x} \lambda(n)$ as $x \to \infty$:

$$\widehat{L}_2(x) = \sum_{k=1}^{\log\log x} (-1)^k \widehat{\pi}_k(x) = -\frac{x}{(\log x)^2} \cdot \Gamma(\log\log x, -\log\log x) \sim \frac{(-1)^{\lceil \log\log x \rceil} \cdot x}{\sqrt{2\pi}\sqrt{\log\log x}}$$

So we expect the sign of our summatory function approximation to be approximately given by $(-1)^{\lceil \log\log x \rceil}$ for large x. We now find a lower bound on the unsigned magnitude of these summatory functions. In particular, using Theorem 3.7, we have that $\widehat{\pi}_k(x) \gg \widehat{\pi}_k^{(\ell)}(x)$ where

$$\widehat{\pi}_k^{(\ell)}(x) := \frac{x^{1/4}}{(\log x)^2} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^3}\right)\right].$$

So we define our lower bound by

$$\widehat{L}_0(x) := \left| \sum_{k=1}^{\log\log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x) \right| \asymp \frac{x^{\frac{1}{4}}}{(\log x)\sqrt{\log\log x}},$$

where

$$\widehat{L}_0'(x) \asymp \frac{1}{x^{3/4} \cdot (\log x)\sqrt{\log\log x}}.$$

Then by Abel summation and integration by parts, we obtain

$$\begin{aligned}|G^{-1}(x)| &\gg \left|\int_2^x \widehat{L}_0'(t)|g^{-1}(t)|dt\right| \\ &\gg \left|\sum_{k=1}^{\frac{\log\log x}{2}} \left[\widehat{L}_2'\left(e^{e^{2k}}\right)\left|g^{-1}\left(e^{e^{2k}}\right)\right| d\left(e^{e^{2k}}\right) - \widehat{L}_0'\left(e^{e^{2k+1}}\right)\left|g^{-1}\left(e^{e^{2k+1}}\right)\right| d\left(e^{e^{2k+1}}\right)\right]\right| \\ &\gg \left|\int_{\frac{\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log x}{2}} \widehat{L}_0'\left(e^{e^{2t}}\right)\left|g^{-1}\left(e^{e^{2t}}\right)\right| d\left(e^{e^{2t}}\right) dt\right|. \end{aligned} \tag{37}$$

We can compute using Corollary 7.9 that if we select a random $n \in [2, x]$ then

$$\begin{aligned}\mathbb{P}\left(0 \leq |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\log\log x\right) &\sim \frac{1}{2} - \Phi\left(-\sigma_x(C)\right) \\ &\sim \frac{1}{2} - \frac{1}{(\log x)\sqrt{\log\log x}} = \frac{1}{2} + o(1), \text{ as } x \to \infty.\end{aligned}$$

This implies that for asymptotically half of the values of $|g^{-1}(n)|$ for $n \leq x$, we have that

$$|g^{-1}(n)| \geq \mathbb{E}|g^{-1}(n)| \asymp \frac{\log n}{\sqrt{\log\log n}},$$

by Corollary 7.7. So we will approximate $|g^{-1}(n)|$ in (37) as having a lower bound of $\frac{\log n}{\sqrt{\log\log n}}$ on the lower half of the interval for $t \in \left[\frac{\log\log x - 1}{2}, \frac{\log\log x}{2} - \frac{1}{4}\right]$. We see that

$$\begin{aligned}|G^{-1}(x)| &\gg \int_{\frac{\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log x}{2}-\frac{1}{4}} \left(\frac{1}{u^{3/4}\cdot(\log\log u)}\right)\Bigg|_{u=e^{e^{2t}}} \times d\left(e^{e^{2t}}\right) dt \\ &= \int_{\frac{\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log x}{2}-\frac{1}{4}} \exp\left(\frac{1}{4}e^{2t} + 2t\right)\frac{dt}{t} \\ &\ll \frac{1}{\log\log x} \times \int_{\frac{\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log x}{2}-\frac{1}{4}} \exp\left(\frac{1}{4}e^{2t} + 2t\right) dt \\ &\ll \frac{e^{\frac{1}{4}e^{2t}}}{\log\log x}\Bigg|_{t=\frac{\log\log x - 1}{2}} = \frac{x^{\frac{e}{4}}}{\log\log x}.\end{aligned}$$

Naturally from (36) we have concluded that as $x \to \infty$, $|G^{-1}(x)| \gg |G_E^{-1}(x)|$. $\square$

Corollary 7.12. *We have that for almost every sufficiently large x, that as $x \to \infty$*

$$\left|G_E^{-1}(x)\right| \gg \frac{(\log x)^{\frac{5}{4}}}{(\log\log x)^{\frac{3}{4}}\sqrt{\log\log\log x}} \times \left|\sum_{e<d\leq \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d^{1/4}\cdot\log\log d}\right|.$$

Proof. Using the definition in (35), we obtain on average that[A]

$$\left|G_E^{-1}(x)\right| = \left|\sum_{n\leq(\log x)^5(\log\log x)} \lambda(n) \times \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log\log d}\right|$$

[A] For any arithmetic functions f, h, we have that [1, *cf.* §3.10; §3.12]

$$\sum_{n\leq x} h(n) \times \sum_{d|n} f(d) = \sum_{d\leq x} f(d) \times \sum_{n=1}^{\left\lfloor\frac{x}{d}\right\rfloor} h(dn).$$

$$= \left| \sum_{e<d\leq(\log x)^5(\log\log x)} \frac{(\log d)^{\frac{1}{4}}}{\log\log d} \times \sum_{n=1}^{\left\lfloor \frac{(\log x)^5(\log\log x)}{d} \right\rfloor} \lambda(dn) \right|.$$

We see that by complete additivity of $\Omega(n)$ (complete multiplicativity of $\lambda(n)$) that

$$\sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn) = \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(d) \times \lambda(n) = \lambda(d) \times \sum_{n\leq \left\lfloor \frac{x}{d} \right\rfloor} \lambda(n).$$

From Theorem 3.7 and Lemma 8.2 (see below), we can establish that

$$\left| \sum_{k\leq\log\log x} (-1)^k \cdot \widehat{\pi}_k(x) \right| \gg \frac{x^{\frac{1}{4}}}{(\log x)\sqrt{\log\log x}} =: \widehat{L}_0(x), \text{ as } x\rightarrow\infty. \tag{38}$$

The sign of the sum obtained by taking the right-hand-side of (38) without the absolute value operation is given by $(-1)^{1+\lfloor \log\log x\rfloor}$. The precise formula for the limiting lower bound stated above for $\widehat{L}_0(x)$ is computed by symbolic summation in *Mathematica* using the new bounds on $\widehat{\pi}_k(x)$ guaranteed by the theorem, and then by applying subsequent standard asymptotic estimates to the resulting formulas for large $x\rightarrow\infty$ in the form of (10c) and Stirling's formula. It follows that

$$|G_E^{-1}(x)| \gg \left| \sum_{e<d\leq(\log x)^5(\log\log x)} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor} \cdot \widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right) \right|. \tag{39}$$

Outline for the remainder of the proof. We sketch the following steps remaining to prove our claimed lower bound on $|G_E^{-1}(x)|$:

- **(A)** We identify an initial subinterval $\mathcal{R}_x$ where we can expect constant sign term contributions resulting from the inputs to the function $\widehat{L}_0$ involving both (d,x) for x large and d on this smaller subinterval.
- **(B)** We factor out easily bounded terms from the expansion of the monotone $\widehat{L}_0$ on this interval.
- **(C)** We determine additional asymptotic formulas we will refer to in later sections for the resulting lower bounds on $|G_E^{-1}(x)|$ that are formed by restricting the range of d in (39) to $\mathcal{R}_x$.
- **(D)** We argue that the sums of oscillatory terms on the upper end of the deleted interval for $d\in \left(e,(\log x)^{7/3}(\log\log x)\right]\setminus\mathcal{R}_x$ cannot generate trivial bounds by cancellation with the new lower bounds.

Part A. We will simplify (39) by proving that there are ranges of consecutive integers over which we obtain essentially constant sign contributions from the function $\widehat{L}_0((\log x)^{7/3}(\log\log x)/d)$ as $x\rightarrow\infty$. In particular, consider that

$$\begin{aligned}\log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right) &= \log\log\left((\log x)^5(\log\log x)\right) \\ &\quad + \log\left(1-\frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)}\right), \text{ as } x\rightarrow\infty.\end{aligned}$$

If we take $d\in(e,\log x] =: \mathcal{R}_x$, we have that

$$\frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)} = o(1)\rightarrow 0, \text{ as } x\rightarrow\infty.$$

For d within $\mathcal{R}_x$, we expect that for almost every x there are at most a handful of negligible cases of comparitively small order $d\leq d_{0,x}$ such that

$$\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor \sim \left\lfloor \log\log\left((\log x)^5(\log\log x)\right)+o(1)\right\rfloor,$$

changes in parity transitioning from $d \mapsto d+1$. An argument making this assertion precise brings leads us to two primary cases that rely on the small-order distribution of the fractional parts $f_x := \left\{\log\log\left((\log x)^5(\log\log x)\right)\right\}$ within $[0,1)$ for large $x \to \infty$ and any $\log d \in \mathcal{R}_x$:

(1) If the fractional part $f_x = 0$, then

$$\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor = \left\lfloor \log\log\left((\log x)^5(\log\log x)\right)\right\rfloor + \left\lfloor -\frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)}\right\rfloor.$$

This implies that provided that

$$-1 \leq -\frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)} < 0,$$

we obtain a constant multplier as $\operatorname{sgn}\left(\widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right)$ whenever $d \in \mathcal{R}_x$. Since d is positive and maximized at $\log x$, this condition clearly happens for any sufficiently large x.

(2) If the fractional part $f_x \in (0,1)$, then

$$\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor = \left\lfloor \log\log\left((\log x)^5(\log\log x)\right)\right\rfloor + \left\lfloor \left\{\log\log\left((\log x)^5(\log\log x)\right)\right\} - \frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)}\right\rfloor.$$

Define shorthand notation for the function $\mathcal{B}(x) := (\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)$. We require that

$$-1 \leq f_x - \frac{\log d}{\mathcal{B}(x)} < 0 \iff (1+f_x)\cdot\mathcal{B}(x) \geq \log d > 0.$$

This property is similarly clearly attained for $d \in \mathcal{R}_x$ since $(1+f_x)\cdot\mathcal{B}(x) \geq \mathcal{B}(x)$ as $x \to \infty$.

Part B. Provided that the sign term involving both d and x from (39) does not change for $d \in \mathcal{R}_x$, we can remove any oscillations in the sums due to sign changes in the monotonically decreasing function $\widehat{L}_0(d,x) := \widehat{L}_0\left((\log x)^5(\log\log x)/d\right)$. The function $\widehat{L}_0(d,x)$ is monotone decreasing in the variable d for fixed x as we sum along the subinterval $\mathcal{R}_x$ in ascending order. We can see that this function is decreasing in d by computing its partial derivative and evaluating the asymptotic main terms as having a leading negative sign for all large x. Thus we should select $d := \log x$ in (39) to obtain a global lower bound on $|G_E^{-1}(x)|$ if we truncate the sum to range only over the subset of original indices $d \in \mathcal{R}_x$.

Part C. Let the magnitudes of the signed remainder term sums be defined for all sufficiently large x by

$$R_E(x) := \left|\sum_{\log x < d \leq \frac{(\log x)^5(\log\log x)}{e^2}} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor} \cdot \widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right|.$$

Set the function $T_E(x)$ to correspond to the easily factored dependence of the less simply integrable factors in $\widehat{L}_0(d,x)$ when we set $d := \log x$ on $\mathcal{R}_x$. This function is defined for all large enough x as

$$T_E(x) \gg \frac{\log\left[(\log x)^4(\log\log x)\right]^{-1}}{\sqrt{\log\log\left[(\log x)^4(\log\log x)\right]}} \gg \frac{1}{(\log\log x)\sqrt{\log\log\log x}}. \tag{40}$$

Then in limiting cases the lower bounding function satisfies

$$S_{E,1}(x) := \left|\sum_{e < d \leq (\log x)^5(\log\log x)} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor} \widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right|$$

$$\gg (\log x)^{\frac{5}{4}} (\log\log x)^{\frac{1}{4}} T_E(x) \times \left| \sum_{e<d\leq \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d^{1/4}\cdot \log\log d} \right| \tag{41}$$

$$\gg \frac{(\log x)^{\frac{5}{4}}}{(\log\log x)^{\frac{3}{4}}\sqrt{\log\log\log x}} \times \left| \sum_{e<d\leq \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d^{1/4}\cdot \log\log d} \right|.$$

The formulas in (39) and (41) imply the following lower bound by the triangle inequality as $x\to\infty$:

$$|G_E^{-1}(x)| \gg \left| S_{E,1}(x) - R_E(x) \right| \gg S_{E,1}(x), \text{ as } x\to\infty. \tag{42}$$

We have claimed that we can in fact drop the sum terms over upper range of $d\notin \mathcal{R}_x$ and still obtain the asymptotic lower bound on $|G_E^{-1}(x)|$ stated in (42). To justify this step in the proof, we will provide limiting lower bounds on $R_E(x)$ that show that the contribution from the deleted interval in absolute value exceeds the magnitude of the corresponding sums over $d\in\mathcal{R}_x$ defined by $S_{E,1}(x)$ when x is large.

Part D. We want to arrange the signed weight coefficients $\varepsilon_{x,d}\mapsto\{\pm 1\}$ so that the function

$$M_{\pm}(x) := \min_{\varepsilon_{x,d}=\pm 1} \left| \sum_{\log x<d\leq \frac{(\log x)^5(\log\log x)}{e^2}} \frac{\varepsilon_{x,d}\cdot\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times \widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right) \right|,$$

is minimal. We need to prove that this minimal sum exceeds the bound for $S_E(x)$ given in (41) in asymptotic order. That is, we prove that

$$S_{E,1}^{(\ell)}(x) := \frac{(\log x)^{\frac{5}{4}}}{(\log\log x)^{\frac{3}{4}}\sqrt{\log\log\log x}} \times \left| \sum_{e<d\leq \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d^{1/4}\cdot \log\log d} \right| = o\left(M_{\pm}(x)\right), \text{ as } x\to\infty.$$

Notice that by considering the sum term in the previous definition as being unsigned, we have that

$$\begin{aligned} S_{E,1}^{(\ell)}(x) &\ll \frac{(\log x)^{\frac{5}{4}}}{(\log\log x)^{\frac{3}{4}}\sqrt{\log\log\log x}} \times \int_e^{\log x} \frac{(\log t)^{1/4}}{t\cdot(\log\log t)} dt \\ &\ll \frac{(\log x)^{\frac{5}{4}}}{(\log\log x)^{\frac{3}{4}}\sqrt{\log\log\log x}} \times \mathrm{Ei}\left(\frac{5}{4}\log\log\log x\right) \\ &\ll \frac{(\log x)^{\frac{5}{4}}(\log\log\log x)^{\frac{3}{2}}}{(\log\log x)^{\frac{3}{4}}}. \end{aligned} \tag{43}$$

We need to show that $M_{\pm}(x)$ always exceeds this bound. Since the function $L_0(x,d)$ is decreasing in d for $d\in\left(\log x, \frac{(\log x)^5(\log\log x)}{e^2}\right] =: \overline{\mathcal{R}}_x$, we obtain that

$$\widehat{L}_0\left(\frac{(\log x)^{5/4}(\log\log x)}{d}\right) \asymp \frac{(\log x)^{\frac{5}{4}}}{d^{1/4}\cdot(\log\log x)^{\frac{3}{4}}\sqrt{\log\log\log x}}, d\in\overline{\mathcal{R}}_x.$$

So we need to find a global lower bound on the sum

$$S_{\pm}(x) := \frac{(\log x)^{5/4}}{(\log\log x)^{3/4}\sqrt{\log\log\log x}} \times \left| \sum_{\log x<d\leq \frac{(\log x)^5(\log\log x)}{e^2}} \frac{\varepsilon_{x,d}\cdot\lambda(d)(\log d)^{\frac{1}{4}}}{d^{1/4}\cdot\log\log d} \right|,$$

that holds for any choice of the signed weights $\varepsilon_{x,d}$. Notice that for any $d>\log x$ and $\delta\geq 1$, by an expansion of convergent geometric and binomial series, the next difference of terms satisfies

$$\left| \frac{(\log d)^{\frac{1}{4}}}{d^{\frac{1}{4}}\cdot(\log\log d)} - \frac{\log(d+\delta)^{\frac{1}{4}}}{(d+\delta)^{\frac{1}{4}}\cdot\log\log(d+\delta)} \right|$$

$$\sim \frac{\log(d+\delta)^{\frac{1}{4}}}{(d+\delta)^{\frac{1}{4}} \cdot \log\log(d+\delta)} \times \left| \frac{\left(1 - \frac{\delta}{\log(d+\delta)}\right)^{\frac{1}{4}}}{\left(1 - \frac{\delta}{(d+\delta)}\right)^{\frac{1}{4}} \left(1 - \frac{\delta}{(d+\delta)\log(d+\delta)\log\log(d+\delta)}\right)} - 1 \right|$$
$$\gg \frac{\delta}{(d+\delta)^{\frac{1}{4}} \log^{\frac{3}{4}}(d+\delta) \log\log(d+\delta)}.$$

Let the number of sign changes of the terms in our sum on the interval $\overline{\mathcal{R}}_x$ be defined by

$$N_x := \#\left\{d \in \overline{\mathcal{R}}_x : \varepsilon_{x,d+1}\lambda(d+1) = -\varepsilon_{x,d}\lambda(d)\right\}.$$

Define the maximum (minimum) number of consecutively signed terms on this interval to be $\delta_{\max}(x), \delta_{\min}(x) \geq 1$. Then by the difference property we noted above, we have that for $t_k(x) := \log x + (2k+2)\delta_{\max}(x)$

$$\begin{aligned}
\frac{S_{\pm}(x)\sqrt{\log\log\log x}(\log\log x)^{3/4}}{(\log x)^{5/4}} &\gg \left| \sum_{k=0}^{\frac{N_x}{2}} \frac{\delta_{\min}(x)}{t_k(x)^{\frac{3}{4}} \log(t_k(x))^{\frac{3}{4}} \log\log(t_k(x))} - \frac{\log^{\frac{1}{4}}(N_x)}{N_x^{\frac{1}{4}} \log\log(N_x)} \right| \\
&\gg \left| \sum_{k=0}^{\frac{N_x}{2}} \frac{t_0(x)^{\frac{1}{4}} \cdot \delta_{\min}(x)}{t_k(x) \log(t_k(x))^{\frac{3}{4}} \log\log(t_k(x))} - \frac{\log^{\frac{1}{4}}(N_x)}{N_x^{\frac{1}{4}} \log\log(N_x)} \right| \\
&\gg (\log x + 2\delta_{\max}(x))^{3/4} \times \int_0^{\frac{N_x}{2}} \frac{dk}{t_k(x) \log(t_k(x))^{\frac{3}{4}} \log\log(t_k(x))} \\
&\gg (\log x + 2\delta_{\max}(x))^{3/4} \times \frac{\log\log\log(t_{\frac{N_x}{2}}(x))}{2\delta_{\max}(x)}.
\end{aligned}$$

Now because $\delta_{\max}(x) \in [1, u_x - N_x]$ for $u_x := (\log x)^{7/3}(\log\log x)$, we have that

$$\frac{S_{\pm}(x)\sqrt{\log\log\log x}(\log\log x)^{3/4}}{(\log x)^{5/4}} \gg \left[(\log x)^{3/4} \cdot (u_x - \delta_{\max}(x))\right] \times \frac{|\log\log\log\left[(u_x - \delta_{\max}(x))\delta_{\max}(x)\right]|}{(u_x - \delta_{\max}(x))\delta_{\max}(x)} \tag{44}$$

By differentiating the function

$$f(x) := \frac{\log\log\log\left[(u_x - \delta_{\max}(x))\delta_{\max}(x)\right]}{(u_x - \delta_{\max}(x))\delta_{\max}(x)},$$

with the intermediate function $d(x) := u_x - \delta_{\max}(x)$, setting the derivative equal to zero, and solving a differential equation for $\delta_{\max}(x)$ in terms of $d(x)$ and $d'(x)$, we see that a lower bound occurs when $\delta_{\max}(x)d(x) = C_1$ is constant. We can then see that the right-hand-side of (44) is non-negligible, and does not cancel with the bound for $S_{E,1}(x)$, as

$$S_{\pm}(x) \gg \frac{(\log x)^2}{(\log\log x)^{3/4}\sqrt{\log\log\log x}}.$$

This property clearly implies that $S_{\pm}(x)$ is asymptotically larger than the maximum order bound on $S_{E,1}^{(\ell)}(x)$ we proved above in (43). $\square$

8 Lower bounds for $M(x)$ along infinite subsequences

8.1 Expanding the new formula for $M(x)$

Proposition 8.1. *For all sufficiently large x, we have that*

$$M(x) = G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{\sqrt{x}} G^{-1}(k)\left[\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor\frac{x}{k+1}\right\rfloor\right)\right]. \tag{45}$$

Proof. We know by applying Corollary 3.3 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)\left(\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) + 1\right) \\ &= G^{-1}(x) + \sum_{k=1}^{x/2} g^{-1}(k)\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) && (46) \\ &= G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{x/2-1} G^{-1}(k)\left[\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor\frac{x}{k+1}\right\rfloor\right)\right] && (47) \end{aligned}$$

where the upper bound on the sum is truncated by the fact that $\pi(1) = 0$. We see that

$$\frac{x}{k} - \frac{x}{k+1} = \frac{x}{k(k+1)} \sim \frac{x}{k^2},$$

so that $\frac{x}{k^2} \geq 1 \implies k \leq \sqrt{x}$. Thus we can re-write the latter sum to obtain

$$M(x) = G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{\sqrt{x}} G^{-1}(k)\left[\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor\frac{x}{k+1}\right\rfloor\right)\right].$$

We will require more assumptions and information about the behavior of the summatory functions, $G^{-1}(x)$, before we can further bound and simplify this expression for $M(x)$. □

8.1.1 A few more necessary results

We now use the superscript and subscript notation of (ℓ) not to denote a formal parameter to the functions we define below, but instead to denote that these functions form *lower bound* (rather than exact) approximations to other forms of the functions without the scripted (ℓ).

Lemma 8.2. *Suppose that $\widehat{\pi}_k^{(\ell)}(x) = o\left(\widehat{\pi}_k(x)\right)$ where $\widehat{\pi}_k^{(\ell)}(x) \geq 1$ for all integers $1 \leq k \leq \log\log x$ as $x \to \infty$. Let the weighted summatory functions be defined as*

$$\begin{aligned} A_{\Omega}^{(\ell)}(x) &:= \sum_{k \leq \log\log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x) \\ A_{\Omega}(x) &:= \sum_{k \leq \log\log x} (-1)^k \widehat{\pi}_k(x). \end{aligned}$$

Futhermore, suppose that $|A_{\Omega}(x)| \not\to 0$ as $x \to \infty$ and that

$$\begin{aligned} \liminf_{k\to\infty} \frac{\widehat{\pi}_k^{(\ell)}(x)}{\widehat{\pi}_k(x)} &\geq x^{-\rho_0} \\ \limsup_{k\to\infty} \frac{\widehat{\pi}_k^{(\ell)}(x)}{\widehat{\pi}_k(x)} &\leq x^{-\rho_1}, \end{aligned}$$

as $x \to \infty$ for some $\rho_0, \rho_1 > 0$. Then for all sufficiently large x, we have that

$$|A_{\Omega}(x)| \gg |A_{\Omega}^{(\ell)}(x)|.$$

Proof. By the second conditions above, we find that

$$\left|A_\Omega(x) - A_\Omega^{(\ell)}(x)\right| \leq |A_\Omega(x)| \left(1 - \inf_{1 \leq k \leq \log\log x} \frac{\widehat{\pi}_k^{(\ell)}(x)}{\widehat{\pi}_k(x)}\right) = |A_\Omega(x)|(1+o(1))$$
$$\left|A_\Omega(x) - A_\Omega^{(\ell)}(x)\right| \geq |A_\Omega(x)| \left(1 - \sup_{1 \leq k \leq \log\log x} \frac{\widehat{\pi}_k^{(\ell)}(x)}{\widehat{\pi}_k(x)}\right) = |A_\Omega(x)|(1+o(1)).$$

Similarly, we can see that

$$|A_\Omega(x)|(1+o(1)) \leq \left|A_\Omega(x) + A_\Omega^{(\ell)}(x)\right| \leq |A_\Omega(x)|(1+o(1)).$$

This implies that

$$|A_\Omega(x)|(1+o(1)) \ll \left||A_\Omega(x)| \pm |A_\Omega^{(\ell)}(x)|\right| \ll |A_\Omega(x)|(1+o(1)), \text{ as } x \rightarrow \infty.$$

Because we have that $|A_\Omega(x)| \nrightarrow 0$, the previous equation shows that $|A_\Omega^{(\ell)}(x)|$ is bounded above and below by a constant times $|A_\Omega(x)|$. In other words, $|A_\Omega(x)| \gg |A_\Omega^{(\ell)}(x)|$ whenever x is sufficiently large. □

Proof of Lemma 3.8. We can form an accurate $C^1(\mathbb{R})$ approximation by the smoothness of $\widehat{\pi}_k^{(\ell)}(x)$ that allows us to apply the Abel summation formula using the summatory function $A_\Omega(t)$ for t on any bounded connected subinterval of $[1, \infty)$. Namely, we obtain

$$\begin{aligned} |F_\lambda(x)| &\gg \left|A_\Omega(x) f(x) - \int_{u_0}^x A_\Omega(t) f'(t) dt\right| \\ &\gg \left||A_\Omega(x) f(x)| - \int_{u_0}^x |A_\Omega(t) f'(t)| dt\right| \\ &\gg \left||A_\Omega^{(\ell)}(x) \widehat{\tau}_\ell(x)| - \int_{u_0}^x |A_\Omega(t) f'(t)| dt\right|. \end{aligned} \tag{48}$$

The stated lower bound formula for $|F_\lambda(x)|$ in (48) above is valid whenever

$$0 \leq \left|\frac{\sum\limits_{\log\log t < k \leq \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_\Omega(t)}\right| \ll 2, \text{ as } t \rightarrow \infty,$$

Indeed, by Corollary 5.7, we have that the assertion above holds as $t \rightarrow \infty$.
Let the function

$$\widehat{I}_\ell(x) := \int_{\frac{\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log x}{2}} \left|A_\Omega^{(\ell)}\left(e^{e^{2t}}\right) \widehat{\tau}_\ell'\left(e^{e^{2t}}\right)\right| e^{e^{2t}} dt.$$

We have to argue that following property of this function holds as $x \rightarrow \infty$:

$$\int_{u_0}^x |A_\Omega(t) f'(t)| dt \gg \widehat{I}_\ell(x).$$

To prove the property in the previous equation, observe that by hypothesis since $|A_\Omega(x)| \gg |A_\Omega^{(\ell)}(x)|$ as $x \rightarrow \infty$, we have that

$$\int_{u_0}^x |A_\Omega(t) f'(t)| dt \gg \int_{u_0}^x |A_\Omega(t) \widehat{\tau}_\ell'(t)| dt$$

$$\begin{aligned}
&\gg \left| \sum_{k=u_0}^{\log\log x} (-1)^k \left| A_\Omega\left(e^{e^k}\right) \widehat{\tau}_\ell'\left(e^{e^k}\right)\right| \cdot \left(e^{e^k} - e^{e^{k-1}}\right)\right| \\
&\gg \left| \sum_{k=u_0}^{\frac{\log\log x}{2}} \left[\left| A_\Omega\left(e^{e^{2k}}\right) \widehat{\tau}_\ell'\left(e^{e^{2k}}\right)\right| \cdot e^{e^{2k}} - \left| A_\Omega\left(e^{e^{2k-1}}\right) \widehat{\tau}_\ell'\left(e^{e^{2k-1}}\right)\right| \cdot e^{e^{2k-1}}\right]\right| \\
&\gg \int_{\frac{\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log x}{2}} \left| A_\Omega\left(e^{e^{2t}}\right) \widehat{\tau}_\ell'\left(e^{e^{2t}}\right)\right| e^{e^{2t}} dt \\
&\gg \int_{\frac{\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log x}{2}} \left| A_\Omega^{(\ell)}\left(e^{e^{2t}}\right) \widehat{\tau}_\ell'\left(e^{e^{2t}}\right)\right| e^{e^{2t}} dt. \qquad \square
\end{aligned}$$

Corollary 8.3. *Let the smooth bounding functions be defined for large $t \gg e$ as*

$$\begin{aligned}
\widehat{\tau}_\ell(t) &:= \frac{(\log t)^{\frac{1}{4}}}{t^{\frac{1}{4}} \cdot (\log\log t)}, \\
A_\Omega^{(\ell)}(t) &:= \frac{t^{\frac{1}{4}}}{(\log t)\sqrt{\log\log t}}.
\end{aligned}$$

Then we have that as $x \to \infty$

$$|G_E^{-1}(x)| \gg \frac{(\log x)^{5/4}}{(\log\log x)^{3/4}\sqrt{\log\log\log x}} \times \left| A_\Omega^{(\ell)}(\log x)\widehat{\tau}_\ell(\log x) - \int_{\frac{\log\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log\log x}{2}} A_\Omega^{(\ell)}\left(e^{e^{2t}}\right)\widehat{\tau}_\ell\left(e^{e^{2t}}\right) e^{e^{2t}} dt\right|.$$

Proof. By Corollary 7.12, we have that

$$|G_E^{-1}(x)| \gg \frac{(\log x)^{5/4}}{(\log\log x)^{3/4}\sqrt{\log\log\log x}} \times \left| \sum_{e<d\leq \log x} \frac{\lambda(d)(\log d)^{1/4}}{d^{1/4}\cdot \log\log d}\right|, \text{ as } x \to \infty. \tag{49}$$

The crux of the remainder of the proof boils down to checking hypotheses in Lemma 8.2 and Lemma 3.8. We first apply Lemma 8.2 with the lower bound function resulting from Theorem 3.7 as follows:

$$\widehat{\pi}_k^{(\ell)}(x) \asymp \frac{x^{\frac{1}{4}}}{(\log x)^2} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

This shows that the necessary hypotheses on the function $A_\Omega^{(\ell)}(t)$ required by Lemma 3.8 are satisfied according to the sums for the function approximated by (38) for large t. This argument proves that all of the requirements in Lemma 3.8 on our choice of $\widehat{\tau}_\ell(t)$ are also satisfied. So the stated result follows from (49) and Lemma 3.8. $\square$

8.1.2 The proof of a central lower bound on the magnitude of $G_E^{-1}(x)$

The next central theorem is the last barrier required to prove Theorem 3.9 in the next subsection. Combined with Theorem 7.11 proved in the last section, the new lower bounds we establish below provide us with a sufficient mechanism to bound the formula from Proposition 8.1.

Theorem 8.4 (Asymptotics and bounds for the summatory function $G^{-1}(x)$)**.** *We obtain the following limiting estimate for the bounding function $G_E^{-1}(x)$ as $x \to \infty$:*

$$\left|G_E^{-1}(x)\right| \gg \frac{(\log x)^{5/4}}{(\log\log x)^{3/2}(\log\log\log x)^2}.$$

Proof. We can form a lower summatory function indicating the signed contributions over the distinct parity of $\Omega(n)$ for all $n \leq x$ as follows by applying (10b) and Stirling's approximation as already noted in the proof of Corollary 7.12:

$$\left|A_{\Omega}^{(\ell)}(t)\right| = \left|\sum_{k \leq \log\log t} (-1)^k \widehat{\pi}_k^{(\ell)}(t)\right| \gg \frac{t^{\frac{1}{4}}}{(\log t)\sqrt{\log\log t}}, \text{ as } t \to \infty. \tag{50}$$

We select the functions $\widehat{\tau}_0(t) := \frac{(\log t)^{1/4}}{t^{1/4} \cdot \log\log t}$ and compute the main term of its derivative in the form of the next equation using the notation in Corollary 8.3.

$$-\widehat{\tau}_0'(t) = -\frac{d}{dt}\left[\frac{(\log t)^{\frac{1}{4}}}{t^{\frac{1}{4}}(\log\log t)}\right] \gg \frac{(\log t)^{1/4}}{t^{\frac{5}{4}}(\log\log t)} \tag{51}$$

Moreover, we have that we can select the initial form of the lower bound to be defined as follows:

$$\begin{aligned} G_E^{-1}(x) \gg & \frac{(\log x)^{5/4}}{(\log\log x)^{3/4}\sqrt{\log\log\log x}} \times \\ & \times \left| A_{\Omega}^{(\ell)}(\log x)\widehat{\tau}_0(\log x) - \int_{\frac{\log\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log\log x}{2}} \left|A_{\Omega}^{(\ell)}\left(e^{e^{2t}}\right)\widehat{\tau}_0'\left(e^{e^{2t}}\right)\right| e^{e^{2t}} dt \right|. \end{aligned} \tag{52}$$

We express the integrand function as the following function of t:

$$\widehat{I}_{\ell}(t) := \left|A_{\Omega}^{(\ell)}\left(e^{e^{2t}}\right)\widehat{\tau}_0'\left(e^{e^{2t}}\right)\right| e^{e^{2t}} \asymp \frac{e^{-3t/2}}{t^{3/2}}. \tag{53}$$

We find from the mean value theorem applied to the monotone function from (53) that

$$\frac{(\log x)^{5/4}}{(\log\log x)^{3/4}\sqrt{\log\log\log x}} \times \int_{\frac{\log\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log\log x}{2}} \widehat{I}_{\ell}(t)dt \asymp \widehat{I}_{\ell}\left(\frac{\log\log\log x}{2} - \frac{1}{2}\right) \asymp \frac{(\log x)^{5/4}}{(\log\log x)^{3/2}(\log\log\log x)^2}. \tag{54}$$

Consider the following expansion for the leading term in the Abel summation formula from (52) for comparison with (54):

$$\frac{(\log x)^{5/4}}{(\log\log x)^{3/4}\sqrt{\log\log\log x}} \times \left|A_{\Omega}^{(\ell)}(\log x)\widehat{\tau}_0(\log x)\right| \asymp \frac{(\log x)^{5/4}}{(\log\log x)^{3/2}(\log\log\log x)^2} \tag{55}$$

Hence, we conclude that we can take $\left|G_E^{-1}(x)\right|$ bounded below by the difference terms in (54) and (55). ☐

8.2 Proof of the unboundedness of the scaled Mertens function

Lemma 8.5. *For sufficiently large x, $k \in [1, \sqrt{x}]$ and integers $m \geq 0$, we have that*

$$\frac{x}{k \cdot \log^m\left(\frac{x}{k}\right)} - \frac{x}{(k+1) \cdot \log^m\left(\frac{x}{k+1}\right)} \asymp \frac{x}{(\log x)^m \cdot k(k+1)}, \tag{A}$$

and

$$\sum_{k=1}^{\sqrt{x}} \frac{x}{k(k+1)} = \sum_{k=1}^{\sqrt{x}} \frac{x}{k^2} + O(1). \tag{B}$$

Proof. The proof of (A) is obvious since $\log(x/k_0) \asymp \log(x)$ for all $k_0 \in [1, \sqrt{x}+1]$ when x is large. In particular, for $k_0 \in [1, \sqrt{x}+1]$ we have that

$$\frac{1}{2}\log(x)(1+o(1)) \leq \log(x/k_0) \leq \log(x).$$

To prove (B), notice that

$$\frac{x}{k(k+1)} - \frac{x}{k^2} = -\frac{x}{k^2(k+1)}.$$

Then we see that

$$\left|\int_e^{\sqrt{x}} \frac{x}{t^2(t+1)} dt\right| \leq \left|\int_e^{\sqrt{x}} \frac{x}{t^3} dt\right| \asymp \left|\frac{x}{2(\sqrt{x})^2)}\right| = \frac{1}{2}. \qquad \square$$

We finally address the main conclusion of our arguments given so far with the following proof:

Proof of Theorem 3.9. Define the infinite increasing subsequence, $\{x_{0,y}\}_{y\geq Y_0}$, by $x_{0,y} := e^{2e^{e^{2y+1}}}$ for the sequence indices y starting at some sufficiently large finite integer $Y_0 \gg 1$. We can verify that for sufficiently large $y \to \infty$, this infinitely tending subsequence is well defined as $x_{0,y+1} > x_{0,y}$ whenever $y \geq Y_0$. Given a fixed large infinitely tending y, we have some (at least one) point $\widehat{x}_0 \in \left[\sqrt{x}, \frac{x}{2}\right]$ defined such that $|G^{-1}(t)|$ is minimal and non-vanishing on the interval $\mathbb{X}_y := (\sqrt{x_{0,y}}, \sqrt{x_{0,y+1}}]$ in the form of

$$\left|G^{-1}(\widehat{x}_0)\right| := \min_{\substack{\sqrt{x_{0,y}}<t\leq\sqrt{x_{0,y+1}} \\ G^{-1}(t)\neq 0}} |G^{-1}(t)|.$$

Let the shorthand notation $|G^{-1}_{\min}(x_y)| := |G^{-1}(\widehat{x_0})|$. In the last step, we observe that $G^{-1}(x) = 0$ for x on a set of asymptotic density *at least* bounded below by $\frac{1}{2}$, so that our claim is accurate as the integrand lower bound on this interval does not trivially vanish at large y. This happens since the sequence $g^{-1}(n)$ is non-zero for all $n \geq 1$, so that if we do encounter a zero of the summatory function at x, we find a non-zero function value at $x+1$.

We need to bound the prime counting function differences in the formula given by Proposition 8.1 in tandem with enforcing minimal values of the absolute value of $G^{-1}(k)$ for $k \in \mathbb{X}_y$. We will require the following known bounds on the prime counting function due to Rosser and Schoenfeld [17, Thm. 1] for large $x \gg 59$:

$$\frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right). \tag{56}$$

Let the component function $U_M(y)$ be defined for all large y as

$$U_M(y) := -\sum_{k=1}^{\sqrt{\hat{x}_{0,y+1}}} |G^{-1}(k)| \left[\pi\left(\frac{\hat{x}_{0,y+1}}{k}\right) - \pi\left(\frac{\hat{x}_{0,y+1}}{k+1}\right)\right].$$

Combined with Lemma 8.5, these estimates on $\pi(x)$ lead to the following approximations that hold on the increasing sequences taken within the subintervals defined by $\widehat{x}_0$:

$$\begin{aligned}
U_M(y) &\gg -\sum_{k=1}^{\sqrt{\hat{x}_{0,y+1}}} |G^{-1}(k)| \left[\frac{\hat{x}_{0,y+1}}{k\cdot\log\left(\frac{\hat{x}_{0,y+1}}{k}\right)} + \frac{\hat{x}_{0,y+1}}{2k\cdot\log^2\left(\frac{\hat{x}_{0,y+1}}{k}\right)} - \frac{\hat{x}_{0,y+1}}{(k+1)\cdot\log\left(\frac{\hat{x}_{0,y+1}}{k+1}\right)} - \frac{3\hat{x}_{0,y+1}}{2(k+1)\cdot\log^2\left(\frac{\hat{x}_{0,y+1}}{k+1}\right)}\right] \\
&\gg -\sum_{k=\sqrt{\hat{x}_{0,y}}}^{\sqrt{\hat{x}_{0,y+1}}} \frac{\hat{x}_{0,y+1}\cdot|G^{-1}_{\min}(\hat{x_0})|}{k^2}\left[\frac{1}{\log(\hat{x}_{0,y+1})} + \frac{1}{2\log^2(\hat{x}_{0,y+1})}\right] \\
&\gg -\hat{x}_{0,y+1}|G^{-1}_{\min}(\hat{x_0})|\left(\frac{1}{\log(\hat{x}_{0,y+1})} + \frac{1}{2\log^2(\hat{x}_{0,y+1})}\right) \times \int_{\sqrt{\hat{x}_{0,y}}}^{\sqrt{\hat{x}_{0,y+1}}} \frac{dt}{t^2} \\
&\gg \sqrt{\hat{x}_{0,y+1}} \times \frac{|G^{-1}_{\min}(\hat{x_0})|}{\log(\hat{x}_{0,y+1})} \times \left(1 + \frac{1}{\log(\hat{x}_{0,y+1})}\right).
\end{aligned}$$

Now by applying the lower bounds proved in Theorem 7.11, we can see that in fact the following is true:

$$U_M(y) \gg \sqrt{\hat{x}_{0,y+1}} \times \frac{|G_{\min}^{-1}(\hat{x}_0)|}{\log(\hat{x}_{0,y+1})} + o(1), \text{ as } y \to \infty.$$

Now we need to assemble this bound on the summation term in the formula for $M(x)$ from Proposition 8.1 with the leading terms involving the summatory function G^{-1}. In particular, we need to argue that we can effectively drop these leading terms to obtain a lower bound. Then we succeed by applying Theorem 7.11 since the remaining terms given by the function $U_M(y)$ are infinitely tending as $y \to \infty$.

Namely, we clearly see from Theorem 7.11 and the proposition that

$$\begin{aligned}
\frac{|M(\hat{x}_{0,y+1})|}{\sqrt{\hat{x}_{0,y+1}}} &\gg \frac{1}{\sqrt{\hat{x}_{0,y+1}}} \times \left| \left| G^{-1}(\hat{x}_{0,y+1}) + G^{-1}\left(\frac{\hat{x}_{0,y+1}}{2}\right)\right| + U_M(y) \right| \\
&\gg \frac{1}{\sqrt{\hat{x}_{0,y+1}}} \times |U_M(y)| \\
&\gg \frac{\log\left(\sqrt{\hat{x}_{0,y+1}}\right)^{1/4}}{\log\log\left(\sqrt{\hat{x}_{0,y+1}}\right)^{3/2} \log\log\log\left(\sqrt{\hat{x}_{0,y+1}}\right)^{2}}.
\end{aligned} \tag{57}$$

Finally, we evaluate the following limit to conclude unboundedness where $\sqrt{x_{0,y}} \to +\infty$ as $y \to +\infty$:

$$\lim_{x\to\infty} \left[\frac{(\log x)^{\frac{1}{4}}}{(\log\log x)^{\frac{3}{2}} (\log\log\log x)^2} \right] = +\infty.$$

Remarks on this lower bound construction. There is a small, but nonetheless insightful point to explain about a technicality in stating (57). Namely, we are not asserting that $|M(x)|/\sqrt{x}$ grows unbounded along the precise subsequence of $x \mapsto \hat{x}_{0,y+1}$ itself as $y \to \infty$. Rather, we are asserting that the unboundedness of this function can be witnessed along some subsequence whose points are taken within a large interval window of $x \in \left(\sqrt{\hat{x}_{0,y}}, \sqrt{\hat{x}_{0,y+1}}\right]$ as $y \to \infty$. We choose to state the lower bound given on the right-hand-side of (57) using the nicely formulated monotone lower bound on $|G_E^{-1}(x)|$ we proved in Theorem 8.4 with $\hat{x}_0 \geq \sqrt{\hat{x}_{0,y}}$ for all $y \geq Y_0$. □

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T.1 Table: The Dirichlet inverse function $g^{-1}(n)$ and the distribution of its summatory function

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1^1	Y	N	1	0	1.0000000	1.000000	0.000000	1	1	0
2	2^1	Y	Y	-2	0	1.0000000	0.500000	0.500000	-1	1	-2
3	3^1	Y	Y	-2	0	1.0000000	0.333333	0.666667	-3	1	-4
4	2^2	N	Y	2	0	1.5000000	0.500000	0.500000	-1	3	-4
5	5^1	Y	Y	-2	0	1.0000000	0.400000	0.600000	-3	3	-6
6	$2^1 3^1$	Y	N	5	0	1.0000000	0.500000	0.500000	2	8	-6
7	7^1	Y	Y	-2	0	1.0000000	0.428571	0.571429	0	8	-8
8	2^3	N	Y	-2	0	2.0000000	0.375000	0.625000	-2	8	-10
9	3^2	N	Y	2	0	1.5000000	0.444444	0.555556	0	10	-10
10	$2^1 5^1$	Y	N	5	0	1.0000000	0.500000	0.500000	5	15	-10
11	11^1	Y	Y	-2	0	1.0000000	0.454545	0.545455	3	15	-12
12	$2^2 3^1$	N	N	-7	2	1.2857143	0.416667	0.583333	-4	15	-19
13	13^1	Y	Y	-2	0	1.0000000	0.384615	0.615385	-6	15	-21
14	$2^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-1	20	-21
15	$3^1 5^1$	Y	N	5	0	1.0000000	0.466667	0.533333	4	25	-21
16	2^4	N	Y	2	0	2.5000000	0.500000	0.500000	6	27	-21
17	17^1	Y	Y	-2	0	1.0000000	0.470588	0.529412	4	27	-23
18	$2^1 3^2$	N	N	-7	2	1.2857143	0.444444	0.555556	-3	27	-30
19	19^1	Y	Y	-2	0	1.0000000	0.421053	0.578947	-5	27	-32
20	$2^2 5^1$	N	N	-7	2	1.2857143	0.400000	0.600000	-12	27	-39
21	$3^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-7	32	-39
22	$2^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-2	37	-39
23	23^1	Y	Y	-2	0	1.0000000	0.434783	0.565217	-4	37	-41
24	$2^3 3^1$	N	N	9	4	1.5555556	0.458333	0.541667	5	46	-41
25	5^2	N	Y	2	0	1.5000000	0.480000	0.520000	7	48	-41
26	$2^1 13^1$	Y	N	5	0	1.0000000	0.500000	0.500000	12	53	-41
27	3^3	N	Y	-2	0	2.0000000	0.481481	0.518519	10	53	-43
28	$2^2 7^1$	N	N	-7	2	1.2857143	0.464286	0.535714	3	53	-50
29	29^1	Y	Y	-2	0	1.0000000	0.448276	0.551724	1	53	-52
30	$2^1 3^1 5^1$	Y	N	-16	0	1.0000000	0.433333	0.566667	-15	53	-68
31	31^1	Y	Y	-2	0	1.0000000	0.419355	0.580645	-17	53	-70
32	2^5	N	Y	-2	0	3.0000000	0.406250	0.593750	-19	53	-72
33	$3^1 11^1$	Y	N	5	0	1.0000000	0.424242	0.575758	-14	58	-72
34	$2^1 17^1$	Y	N	5	0	1.0000000	0.441176	0.558824	-9	63	-72
35	$5^1 7^1$	Y	N	5	0	1.0000000	0.457143	0.542857	-4	68	-72
36	$2^2 3^2$	N	N	14	9	1.3571429	0.472222	0.527778	10	82	-72
37	37^1	Y	Y	-2	0	1.0000000	0.459459	0.540541	8	82	-74
38	$2^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	13	87	-74
39	$3^1 13^1$	Y	N	5	0	1.0000000	0.487179	0.512821	18	92	-74
40	$2^3 5^1$	N	N	9	4	1.5555556	0.500000	0.500000	27	101	-74
41	41^1	Y	Y	-2	0	1.0000000	0.487805	0.512195	25	101	-76
42	$2^1 3^1 7^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	9	101	-92
43	43^1	Y	Y	-2	0	1.0000000	0.465116	0.534884	7	101	-94
44	$2^2 11^1$	N	N	-7	2	1.2857143	0.454545	0.545455	0	101	-101
45	$3^2 5^1$	N	N	-7	2	1.2857143	0.444444	0.555556	-7	101	-108
46	$2^1 23^1$	Y	N	5	0	1.0000000	0.456522	0.543478	-2	106	-108
47	47^1	Y	Y	-2	0	1.0000000	0.446809	0.553191	-4	106	-110
48	$2^4 3^1$	N	N	-11	6	1.8181818	0.437500	0.562500	-15	106	-121

Table T.1: ***Computations with*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for*** $\mathbf{1 \leq n \leq 500}$**.**

- ► *The column labeled* `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled* `Sqfree` *and* `PPower`*, respectively, list inclusion of n in the sets of squarefree integers and the prime powers.*
- ► *The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.*
- ► *The last several columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{n} \cdot \#\{n \leq x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geq 1$.*

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
49	7^2	N	Y	2	0	1.5000000	0.448980	0.551020	−13	108	−121
50	$2^1 5^2$	N	N	−7	2	1.2857143	0.440000	0.560000	−20	108	−128
51	$3^1 17^1$	Y	N	5	0	1.0000000	0.450980	0.549020	−15	113	−128
52	$2^2 13^1$	N	N	−7	2	1.2857143	0.442308	0.557692	−22	113	−135
53	53^1	Y	Y	−2	0	1.0000000	0.433962	0.566038	−24	113	−137
54	$2^1 3^3$	N	N	9	4	1.5555556	0.444444	0.555556	−15	122	−137
55	$5^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−10	127	−137
56	$2^3 7^1$	N	N	9	4	1.5555556	0.464286	0.535714	−1	136	−137
57	$3^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	4	141	−137
58	$2^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	9	146	−137
59	59^1	Y	Y	−2	0	1.0000000	0.474576	0.525424	7	146	−139
60	$2^2 3^1 5^1$	N	N	30	14	1.1666667	0.483333	0.516667	37	176	−139
61	61^1	Y	Y	−2	0	1.0000000	0.475410	0.524590	35	176	−141
62	$2^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	40	181	−141
63	$3^2 7^1$	N	N	−7	2	1.2857143	0.476190	0.523810	33	181	−148
64	2^6	N	Y	2	0	3.5000000	0.484375	0.515625	35	183	−148
65	$5^1 13^1$	Y	N	5	0	1.0000000	0.492308	0.507692	40	188	−148
66	$2^1 3^1 11^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	24	188	−164
67	67^1	Y	Y	−2	0	1.0000000	0.477612	0.522388	22	188	−166
68	$2^2 17^1$	N	N	−7	2	1.2857143	0.470588	0.529412	15	188	−173
69	$3^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	20	193	−173
70	$2^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.471429	0.528571	4	193	−189
71	71^1	Y	Y	−2	0	1.0000000	0.464789	0.535211	2	193	−191
72	$2^3 3^2$	N	N	−23	18	1.4782609	0.458333	0.541667	−21	193	−214
73	73^1	Y	Y	−2	0	1.0000000	0.452055	0.547945	−23	193	−216
74	$2^1 37^1$	Y	N	5	0	1.0000000	0.459459	0.540541	−18	198	−216
75	$3^1 5^2$	N	N	−7	2	1.2857143	0.453333	0.546667	−25	198	−223
76	$2^2 19^1$	N	N	−7	2	1.2857143	0.447368	0.552632	−32	198	−230
77	$7^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−27	203	−230
78	$2^1 3^1 13^1$	Y	N	−16	0	1.0000000	0.448718	0.551282	−43	203	−246
79	79^1	Y	Y	−2	0	1.0000000	0.443038	0.556962	−45	203	−248
80	$2^4 5^1$	N	N	−11	6	1.8181818	0.437500	0.562500	−56	203	−259
81	3^4	N	Y	2	0	2.5000000	0.444444	0.555556	−54	205	−259
82	$2^1 41^1$	Y	N	5	0	1.0000000	0.451220	0.548780	−49	210	−259
83	83^1	Y	Y	−2	0	1.0000000	0.445783	0.554217	−51	210	−261
84	$2^2 3^1 7^1$	N	N	30	14	1.1666667	0.452381	0.547619	−21	240	−261
85	$5^1 17^1$	Y	N	5	0	1.0000000	0.458824	0.541176	−16	245	−261
86	$2^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	−11	250	−261
87	$3^1 29^1$	Y	N	5	0	1.0000000	0.471264	0.528736	−6	255	−261
88	$2^3 11^1$	N	N	9	4	1.5555556	0.477273	0.522727	3	264	−261
89	89^1	Y	Y	−2	0	1.0000000	0.471910	0.528090	1	264	−263
90	$2^1 3^2 5^1$	N	N	30	14	1.1666667	0.477778	0.522222	31	294	−263
91	$7^1 13^1$	Y	N	5	0	1.0000000	0.483516	0.516484	36	299	−263
92	$2^2 23^1$	N	N	−7	2	1.2857143	0.478261	0.521739	29	299	−270
93	$3^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	34	304	−270
94	$2^1 47^1$	Y	N	5	0	1.0000000	0.489362	0.510638	39	309	−270
95	$5^1 19^1$	Y	N	5	0	1.0000000	0.494737	0.505263	44	314	−270
96	$2^5 3^1$	N	N	13	8	2.0769231	0.500000	0.500000	57	327	−270
97	97^1	Y	Y	−2	0	1.0000000	0.494845	0.505155	55	327	−272
98	$2^1 7^2$	N	N	−7	2	1.2857143	0.489796	0.510204	48	327	−279
99	$3^2 11^1$	N	N	−7	2	1.2857143	0.484848	0.515152	41	327	−286
100	$2^2 5^2$	N	N	14	9	1.3571429	0.490000	0.510000	55	341	−286
101	101^1	Y	Y	−2	0	1.0000000	0.485149	0.514851	53	341	−288
102	$2^1 3^1 17^1$	Y	N	−16	0	1.0000000	0.480392	0.519608	37	341	−304
103	103^1	Y	Y	−2	0	1.0000000	0.475728	0.524272	35	341	−306
104	$2^3 13^1$	N	N	9	4	1.5555556	0.480769	0.519231	44	350	−306
105	$3^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	28	350	−322
106	$2^1 53^1$	Y	N	5	0	1.0000000	0.481132	0.518868	33	355	−322
107	107^1	Y	Y	−2	0	1.0000000	0.476636	0.523364	31	355	−324
108	$2^2 3^3$	N	N	−23	18	1.4782609	0.472222	0.527778	8	355	−347
109	109^1	Y	Y	−2	0	1.0000000	0.467890	0.532110	6	355	−349
110	$2^1 5^1 11^1$	Y	N	−16	0	1.0000000	0.463636	0.536364	−10	355	−365
111	$3^1 37^1$	Y	N	5	0	1.0000000	0.468468	0.531532	−5	360	−365
112	$2^4 7^1$	N	N	−11	6	1.8181818	0.464286	0.535714	−16	360	−376
113	113^1	Y	Y	−2	0	1.0000000	0.460177	0.539823	−18	360	−378
114	$2^1 3^1 19^1$	Y	N	−16	0	1.0000000	0.456140	0.543860	−34	360	−394
115	$5^1 23^1$	Y	N	5	0	1.0000000	0.460870	0.539130	−29	365	−394
116	$2^2 29^1$	N	N	−7	2	1.2857143	0.456897	0.543103	−36	365	−401
117	$3^2 13^1$	N	N	−7	2	1.2857143	0.452991	0.547009	−43	365	−408
118	$2^1 59^1$	Y	N	5	0	1.0000000	0.457627	0.542373	−38	370	−408
119	$7^1 17^1$	Y	N	5	0	1.0000000	0.462185	0.537815	−33	375	−408
120	$2^3 3^1 5^1$	N	N	−48	32	1.3333333	0.458333	0.541667	−81	375	−456
121	11^2	N	Y	2	0	1.5000000	0.462810	0.537190	−79	377	−456
122	$2^1 61^1$	Y	N	5	0	1.0000000	0.467213	0.532787	−74	382	−456
123	$3^1 41^1$	Y	N	5	0	1.0000000	0.471545	0.528455	−69	387	−456
124	$2^2 31^1$	N	N	−7	2	1.2857143	0.467742	0.532258	−76	387	−463

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
125	5^3	N	Y	-2	0	2.0000000	0.464000	0.536000	-78	387	-465
126	$2^1 3^2 7^1$	N	N	30	14	1.1666667	0.468254	0.531746	-48	417	-465
127	127^1	Y	Y	-2	0	1.0000000	0.464567	0.535433	-50	417	-467
128	2^7	N	Y	-2	0	4.0000000	0.460938	0.539062	-52	417	-469
129	$3^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	-47	422	-469
130	$2^1 5^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-63	422	-485
131	131^1	Y	Y	-2	0	1.0000000	0.458015	0.541985	-65	422	-487
132	$2^2 3^1 11^1$	N	N	30	14	1.1666667	0.462121	0.537879	-35	452	-487
133	$7^1 19^1$	Y	N	5	0	1.0000000	0.466165	0.533835	-30	457	-487
134	$2^1 67^1$	Y	N	5	0	1.0000000	0.470149	0.529851	-25	462	-487
135	$3^3 5^1$	N	N	9	4	1.5555556	0.474074	0.525926	-16	471	-487
136	$2^3 17^1$	N	N	9	4	1.5555556	0.477941	0.522059	-7	480	-487
137	137^1	Y	Y	-2	0	1.0000000	0.474453	0.525547	-9	480	-489
138	$2^1 3^1 23^1$	Y	N	-16	0	1.0000000	0.471014	0.528986	-25	480	-505
139	139^1	Y	Y	-2	0	1.0000000	0.467626	0.532374	-27	480	-507
140	$2^2 5^1 7^1$	N	N	30	14	1.1666667	0.471429	0.528571	3	510	-507
141	$3^1 47^1$	Y	N	5	0	1.0000000	0.475177	0.524823	8	515	-507
142	$2^1 71^1$	Y	N	5	0	1.0000000	0.478873	0.521127	13	520	-507
143	$11^1 13^1$	Y	N	5	0	1.0000000	0.482517	0.517483	18	525	-507
144	$2^4 3^2$	N	N	34	29	1.6176471	0.486111	0.513889	52	559	-507
145	$5^1 29^1$	Y	N	5	0	1.0000000	0.489655	0.510345	57	564	-507
146	$2^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	62	569	-507
147	$3^1 7^2$	N	N	-7	2	1.2857143	0.489796	0.510204	55	569	-514
148	$2^2 37^1$	N	N	-7	2	1.2857143	0.486486	0.513514	48	569	-521
149	149^1	Y	Y	-2	0	1.0000000	0.483221	0.516779	46	569	-523
150	$2^1 3^1 5^2$	N	N	30	14	1.1666667	0.486667	0.513333	76	599	-523
151	151^1	Y	Y	-2	0	1.0000000	0.483444	0.516556	74	599	-525
152	$2^3 19^1$	N	N	9	4	1.5555556	0.486842	0.513158	83	608	-525
153	$3^2 17^1$	N	N	-7	2	1.2857143	0.483660	0.516340	76	608	-532
154	$2^1 7^1 11^1$	Y	N	-16	0	1.0000000	0.480519	0.519481	60	608	-548
155	$5^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	65	613	-548
156	$2^2 3^1 13^1$	N	N	30	14	1.1666667	0.487179	0.512821	95	643	-548
157	157^1	Y	Y	-2	0	1.0000000	0.484076	0.515924	93	643	-550
158	$2^1 79^1$	Y	N	5	0	1.0000000	0.487342	0.512658	98	648	-550
159	$3^1 53^1$	Y	N	5	0	1.0000000	0.490566	0.509434	103	653	-550
160	$2^5 5^1$	N	N	13	8	2.0769231	0.493750	0.506250	116	666	-550
161	$7^1 23^1$	Y	N	5	0	1.0000000	0.496894	0.503106	121	671	-550
162	$2^1 3^4$	N	N	-11	6	1.8181818	0.493827	0.506173	110	671	-561
163	163^1	Y	Y	-2	0	1.0000000	0.490798	0.509202	108	671	-563
164	$2^2 41^1$	N	N	-7	2	1.2857143	0.487805	0.512195	101	671	-570
165	$3^1 5^1 11^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	85	671	-586
166	$2^1 83^1$	Y	N	5	0	1.0000000	0.487952	0.512048	90	676	-586
167	167^1	Y	Y	-2	0	1.0000000	0.485030	0.514970	88	676	-588
168	$2^3 3^1 7^1$	N	N	-48	32	1.3333333	0.482143	0.517857	40	676	-636
169	13^2	N	Y	2	0	1.5000000	0.485207	0.514793	42	678	-636
170	$2^1 5^1 17^1$	Y	N	-16	0	1.0000000	0.482353	0.517647	26	678	-652
171	$3^2 19^1$	N	N	-7	2	1.2857143	0.479532	0.520468	19	678	-659
172	$2^2 43^1$	N	N	-7	2	1.2857143	0.476744	0.523256	12	678	-666
173	173^1	Y	Y	-2	0	1.0000000	0.473988	0.526012	10	678	-668
174	$2^1 3^1 29^1$	Y	N	-16	0	1.0000000	0.471264	0.528736	-6	678	-684
175	$5^2 7^1$	N	N	-7	2	1.2857143	0.468571	0.531429	-13	678	-691
176	$2^4 11^1$	N	N	-11	6	1.8181818	0.465909	0.534091	-24	678	-702
177	$3^1 59^1$	Y	N	5	0	1.0000000	0.468927	0.531073	-19	683	-702
178	$2^1 89^1$	Y	N	5	0	1.0000000	0.471910	0.528090	-14	688	-702
179	179^1	Y	Y	-2	0	1.0000000	0.469274	0.530726	-16	688	-704
180	$2^2 3^2 5^1$	N	N	-74	58	1.2162162	0.466667	0.533333	-90	688	-778
181	181^1	Y	Y	-2	0	1.0000000	0.464088	0.535912	-92	688	-780
182	$2^1 7^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-108	688	-796
183	$3^1 61^1$	Y	N	5	0	1.0000000	0.464481	0.535519	-103	693	-796
184	$2^3 23^1$	N	N	9	4	1.5555556	0.467391	0.532609	-94	702	-796
185	$5^1 37^1$	Y	N	5	0	1.0000000	0.470270	0.529730	-89	707	-796
186	$2^1 3^1 31^1$	Y	N	-16	0	1.0000000	0.467742	0.532258	-105	707	-812
187	$11^1 17^1$	Y	N	5	0	1.0000000	0.470588	0.529412	-100	712	-812
188	$2^2 47^1$	N	N	-7	2	1.2857143	0.468085	0.531915	-107	712	-819
189	$3^3 7^1$	N	N	9	4	1.5555556	0.470899	0.529101	-98	721	-819
190	$2^1 5^1 19^1$	Y	N	-16	0	1.0000000	0.468421	0.531579	-114	721	-835
191	191^1	Y	Y	-2	0	1.0000000	0.465969	0.534031	-116	721	-837
192	$2^6 3^1$	N	N	-15	10	2.3333333	0.463542	0.536458	-131	721	-852
193	193^1	Y	Y	-2	0	1.0000000	0.461140	0.538860	-133	721	-854
194	$2^1 97^1$	Y	N	5	0	1.0000000	0.463918	0.536082	-128	726	-854
195	$3^1 5^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-144	726	-870
196	$2^2 7^2$	N	N	14	9	1.3571429	0.464286	0.535714	-130	740	-870
197	197^1	Y	Y	-2	0	1.0000000	0.461929	0.538071	-132	740	-872
198	$2^1 3^2 11^1$	N	N	30	14	1.1666667	0.464646	0.535354	-102	770	-872
199	199^1	Y	Y	-2	0	1.0000000	0.462312	0.537688	-104	770	-874
200	$2^3 5^2$	N	N	-23	18	1.4782609	0.460000	0.540000	-127	770	-897

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\|n} C_{\Omega(d)}(d)}{\|g^{-1}(n)\|}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
201	$3^1 67^1$	Y	N	5	0	1.0000000	0.462687	0.537313	−122	775	−897
202	$2^1 101^1$	Y	N	5	0	1.0000000	0.465347	0.534653	−117	780	−897
203	$7^1 29^1$	Y	N	5	0	1.0000000	0.467980	0.532020	−112	785	−897
204	$2^2 3^1 17^1$	N	N	30	14	1.1666667	0.470588	0.529412	−82	815	−897
205	$5^1 41^1$	Y	N	5	0	1.0000000	0.473171	0.526829	−77	820	−897
206	$2^1 103^1$	Y	N	5	0	1.0000000	0.475728	0.524272	−72	825	−897
207	$3^2 23^1$	N	N	−7	2	1.2857143	0.473430	0.526570	−79	825	−904
208	$2^4 13^1$	N	N	−11	6	1.8181818	0.471154	0.528846	−90	825	−915
209	$11^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	−85	830	−915
210	$2^1 3^1 5^1 7^1$	Y	N	65	0	1.0000000	0.476190	0.523810	−20	895	−915
211	211^1	Y	Y	−2	0	1.0000000	0.473934	0.526066	−22	895	−917
212	$2^2 53^1$	N	N	−7	2	1.2857143	0.471698	0.528302	−29	895	−924
213	$3^1 71^1$	Y	N	5	0	1.0000000	0.474178	0.525822	−24	900	−924
214	$2^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−19	905	−924
215	$5^1 43^1$	Y	N	5	0	1.0000000	0.479070	0.520930	−14	910	−924
216	$2^3 3^3$	N	N	46	41	1.5000000	0.481481	0.518519	32	956	−924
217	$7^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	37	961	−924
218	$2^1 109^1$	Y	N	5	0	1.0000000	0.486239	0.513761	42	966	−924
219	$3^1 73^1$	Y	N	5	0	1.0000000	0.488584	0.511416	47	971	−924
220	$2^2 5^1 11^1$	N	N	30	14	1.1666667	0.490909	0.509091	77	1001	−924
221	$13^1 17^1$	Y	N	5	0	1.0000000	0.493213	0.506787	82	1006	−924
222	$2^1 3^1 37^1$	Y	N	−16	0	1.0000000	0.490991	0.509009	66	1006	−940
223	223^1	Y	Y	−2	0	1.0000000	0.488789	0.511211	64	1006	−942
224	$2^5 7^1$	N	N	13	8	2.0769231	0.491071	0.508929	77	1019	−942
225	$3^2 5^2$	N	N	14	9	1.3571429	0.493333	0.506667	91	1033	−942
226	$2^1 113^1$	Y	N	5	0	1.0000000	0.495575	0.504425	96	1038	−942
227	227^1	Y	Y	−2	0	1.0000000	0.493392	0.506608	94	1038	−944
228	$2^2 3^1 19^1$	N	N	30	14	1.1666667	0.495614	0.504386	124	1068	−944
229	229^1	Y	Y	−2	0	1.0000000	0.493450	0.506550	122	1068	−946
230	$2^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.491304	0.508696	106	1068	−962
231	$3^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.489177	0.510823	90	1068	−978
232	$2^3 29^1$	N	N	9	4	1.5555556	0.491379	0.508621	99	1077	−978
233	233^1	Y	Y	−2	0	1.0000000	0.489270	0.510730	97	1077	−980
234	$2^1 3^2 13^1$	N	N	30	14	1.1666667	0.491453	0.508547	127	1107	−980
235	$5^1 47^1$	Y	N	5	0	1.0000000	0.493617	0.506383	132	1112	−980
236	$2^2 59^1$	N	N	−7	2	1.2857143	0.491525	0.508475	125	1112	−987
237	$3^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	130	1117	−987
238	$2^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.491597	0.508403	114	1117	−1003
239	239^1	Y	Y	−2	0	1.0000000	0.489540	0.510460	112	1117	−1005
240	$2^4 3^1 5^1$	N	N	70	54	1.5000000	0.491667	0.508333	182	1187	−1005
241	241^1	Y	Y	−2	0	1.0000000	0.489627	0.510373	180	1187	−1007
242	$2^1 11^2$	N	N	−7	2	1.2857143	0.487603	0.512397	173	1187	−1014
243	3^5	N	Y	−2	0	3.0000000	0.485597	0.514403	171	1187	−1016
244	$2^2 61^1$	N	N	−7	2	1.2857143	0.483607	0.516393	164	1187	−1023
245	$5^1 7^2$	N	N	−7	2	1.2857143	0.481633	0.518367	157	1187	−1030
246	$2^1 3^1 41^1$	Y	N	−16	0	1.0000000	0.479675	0.520325	141	1187	−1046
247	$13^1 19^1$	Y	N	5	0	1.0000000	0.481781	0.518219	146	1192	−1046
248	$2^3 31^1$	N	N	9	4	1.5555556	0.483871	0.516129	155	1201	−1046
249	$3^1 83^1$	Y	N	5	0	1.0000000	0.485944	0.514056	160	1206	−1046
250	$2^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	169	1215	−1046
251	251^1	Y	Y	−2	0	1.0000000	0.486056	0.513944	167	1215	−1048
252	$2^2 3^2 7^1$	N	N	−74	58	1.2162162	0.484127	0.515873	93	1215	−1122
253	$11^1 23^1$	Y	N	5	0	1.0000000	0.486166	0.513834	98	1220	−1122
254	$2^1 127^1$	Y	N	5	0	1.0000000	0.488189	0.511811	103	1225	−1122
255	$3^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.486275	0.513725	87	1225	−1138
256	2^8	N	Y	2	0	4.5000000	0.488281	0.511719	89	1227	−1138
257	257^1	Y	Y	−2	0	1.0000000	0.486381	0.513619	87	1227	−1140
258	$2^1 3^1 43^1$	Y	N	−16	0	1.0000000	0.484496	0.515504	71	1227	−1156
259	$7^1 37^1$	Y	N	5	0	1.0000000	0.486486	0.513514	76	1232	−1156
260	$2^2 5^1 13^1$	N	N	30	14	1.1666667	0.488462	0.511538	106	1262	−1156
261	$3^2 29^1$	N	N	−7	2	1.2857143	0.486590	0.513410	99	1262	−1163
262	$2^1 131^1$	Y	N	5	0	1.0000000	0.488550	0.511450	104	1267	−1163
263	263^1	Y	Y	−2	0	1.0000000	0.486692	0.513308	102	1267	−1165
264	$2^3 3^1 11^1$	N	N	−48	32	1.3333333	0.484848	0.515152	54	1267	−1213
265	$5^1 53^1$	Y	N	5	0	1.0000000	0.486792	0.513208	59	1272	−1213
266	$2^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.484962	0.515038	43	1272	−1229
267	$3^1 89^1$	Y	N	5	0	1.0000000	0.486891	0.513109	48	1277	−1229
268	$2^2 67^1$	N	N	−7	2	1.2857143	0.485075	0.514925	41	1277	−1236
269	269^1	Y	Y	−2	0	1.0000000	0.483271	0.516729	39	1277	−1238
270	$2^1 3^3 5^1$	N	N	−48	32	1.3333333	0.481481	0.518519	−9	1277	−1286
271	271^1	Y	Y	−2	0	1.0000000	0.479705	0.520295	−11	1277	−1288
272	$2^4 17^1$	N	N	−11	6	1.8181818	0.477941	0.522059	−22	1277	−1299
273	$3^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−38	1277	−1315
274	$2^1 137^1$	Y	N	5	0	1.0000000	0.478102	0.521898	−33	1282	−1315
275	$5^2 11^1$	N	N	−7	2	1.2857143	0.476364	0.523636	−40	1282	−1322
276	$2^2 3^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−10	1312	−1322
277	277^1	Y	Y	−2	0	1.0000000	0.476534	0.523466	−12	1312	−1324

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\|n} C_{\Omega(d)}(d)}{\|g^{-1}(n)\|}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
278	2^1139^1	Y	N	5	0	1.0000000	0.478417	0.521583	−7	1317	−1324
279	3^231^1	N	N	−7	2	1.2857143	0.476703	0.523297	−14	1317	−1331
280	$2^35^17^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−62	1317	−1379
281	281^1	Y	Y	−2	0	1.0000000	0.473310	0.526690	−64	1317	−1381
282	$2^13^147^1$	Y	N	−16	0	1.0000000	0.471631	0.528369	−80	1317	−1397
283	283^1	Y	Y	−2	0	1.0000000	0.469965	0.530035	−82	1317	−1399
284	2^271^1	N	N	−7	2	1.2857143	0.468310	0.531690	−89	1317	−1406
285	$3^15^119^1$	Y	N	−16	0	1.0000000	0.466667	0.533333	−105	1317	−1422
286	$2^111^113^1$	Y	N	−16	0	1.0000000	0.465035	0.534965	−121	1317	−1438
287	7^141^1	Y	N	5	0	1.0000000	0.466899	0.533101	−116	1322	−1438
288	2^53^2	N	N	−47	42	1.7659574	0.465278	0.534722	−163	1322	−1485
289	17^2	N	Y	2	0	1.5000000	0.467128	0.532872	−161	1324	−1485
290	$2^15^129^1$	Y	N	−16	0	1.0000000	0.465517	0.534483	−177	1324	−1501
291	3^197^1	Y	N	5	0	1.0000000	0.467354	0.532646	−172	1329	−1501
292	2^273^1	N	N	−7	2	1.2857143	0.465753	0.534247	−179	1329	−1508
293	293^1	Y	Y	−2	0	1.0000000	0.464164	0.535836	−181	1329	−1510
294	$2^13^17^2$	N	N	30	14	1.1666667	0.465986	0.534014	−151	1359	−1510
295	5^159^1	Y	N	5	0	1.0000000	0.467797	0.532203	−146	1364	−1510
296	2^337^1	N	N	9	4	1.5555556	0.469595	0.530405	−137	1373	−1510
297	3^311^1	N	N	9	4	1.5555556	0.471380	0.528620	−128	1382	−1510
298	2^1149^1	Y	N	5	0	1.0000000	0.473154	0.526846	−123	1387	−1510
299	13^123^1	Y	N	5	0	1.0000000	0.474916	0.525084	−118	1392	−1510
300	$2^23^15^2$	N	N	−74	58	1.2162162	0.473333	0.526667	−192	1392	−1584
301	7^143^1	Y	N	5	0	1.0000000	0.475083	0.524917	−187	1397	−1584
302	2^1151^1	Y	N	5	0	1.0000000	0.476821	0.523179	−182	1402	−1584
303	3^1101^1	Y	N	5	0	1.0000000	0.478548	0.521452	−177	1407	−1584
304	2^419^1	N	N	−11	6	1.8181818	0.476974	0.523026	−188	1407	−1595
305	5^161^1	Y	N	5	0	1.0000000	0.478689	0.521311	−183	1412	−1595
306	$2^13^217^1$	N	N	30	14	1.1666667	0.480392	0.519608	−153	1442	−1595
307	307^1	Y	Y	−2	0	1.0000000	0.478827	0.521173	−155	1442	−1597
308	$2^27^111^1$	N	N	30	14	1.1666667	0.480519	0.519481	−125	1472	−1597
309	3^1103^1	Y	N	5	0	1.0000000	0.482201	0.517799	−120	1477	−1597
310	$2^15^131^1$	Y	N	−16	0	1.0000000	0.480645	0.519355	−136	1477	−1613
311	311^1	Y	Y	−2	0	1.0000000	0.479100	0.520900	−138	1477	−1615
312	$2^33^113^1$	N	N	−48	32	1.3333333	0.477564	0.522436	−186	1477	−1663
313	313^1	Y	Y	−2	0	1.0000000	0.476038	0.523962	−188	1477	−1665
314	2^1157^1	Y	N	5	0	1.0000000	0.477707	0.522293	−183	1482	−1665
315	$3^25^17^1$	N	N	30	14	1.1666667	0.479365	0.520635	−153	1512	−1665
316	2^279^1	N	N	−7	2	1.2857143	0.477848	0.522152	−160	1512	−1672
317	317^1	Y	Y	−2	0	1.0000000	0.476341	0.523659	−162	1512	−1674
318	$2^13^153^1$	Y	N	−16	0	1.0000000	0.474843	0.525157	−178	1512	−1690
319	11^129^1	Y	N	5	0	1.0000000	0.476489	0.523511	−173	1517	−1690
320	2^65^1	N	N	−15	10	2.3333333	0.475000	0.525000	−188	1517	−1705
321	3^1107^1	Y	N	5	0	1.0000000	0.476636	0.523364	−183	1522	−1705
322	$2^17^123^1$	Y	N	−16	0	1.0000000	0.475155	0.524845	−199	1522	−1721
323	17^119^1	Y	N	5	0	1.0000000	0.476780	0.523220	−194	1527	−1721
324	2^23^4	N	N	34	29	1.6176471	0.478395	0.521605	−160	1561	−1721
325	5^213^1	N	N	−7	2	1.2857143	0.476923	0.523077	−167	1561	−1728
326	2^1163^1	Y	N	5	0	1.0000000	0.478528	0.521472	−162	1566	−1728
327	3^1109^1	Y	N	5	0	1.0000000	0.480122	0.519878	−157	1571	−1728
328	2^341^1	N	N	9	4	1.5555556	0.481707	0.518293	−148	1580	−1728
329	7^147^1	Y	N	5	0	1.0000000	0.483283	0.516717	−143	1585	−1728
330	$2^13^15^111^1$	Y	N	65	0	1.0000000	0.484848	0.515152	−78	1650	−1728
331	331^1	Y	Y	−2	0	1.0000000	0.483384	0.516616	−80	1650	−1730
332	2^283^1	N	N	−7	2	1.2857143	0.481928	0.518072	−87	1650	−1737
333	3^237^1	N	N	−7	2	1.2857143	0.480480	0.519520	−94	1650	−1744
334	2^1167^1	Y	N	5	0	1.0000000	0.482036	0.517964	−89	1655	−1744
335	5^167^1	Y	N	5	0	1.0000000	0.483582	0.516418	−84	1660	−1744
336	$2^43^17^1$	N	N	70	54	1.5000000	0.485119	0.514881	−14	1730	−1744
337	337^1	Y	Y	−2	0	1.0000000	0.483680	0.516320	−16	1730	−1746
338	2^113^2	N	N	−7	2	1.2857143	0.482249	0.517751	−23	1730	−1753
339	3^1113^1	Y	N	5	0	1.0000000	0.483776	0.516224	−18	1735	−1753
340	$2^25^117^1$	N	N	30	14	1.1666667	0.485294	0.514706	12	1765	−1753
341	11^131^1	Y	N	5	0	1.0000000	0.486804	0.513196	17	1770	−1753
342	$2^13^219^1$	N	N	30	14	1.1666667	0.488304	0.511696	47	1800	−1753
343	7^3	N	Y	−2	0	2.0000000	0.486880	0.513120	45	1800	−1755
344	2^343^1	N	N	9	4	1.5555556	0.488372	0.511628	54	1809	−1755
345	$3^15^123^1$	Y	N	−16	0	1.0000000	0.486957	0.513043	38	1809	−1771
346	2^1173^1	Y	N	5	0	1.0000000	0.488439	0.511561	43	1814	−1771
347	347^1	Y	Y	−2	0	1.0000000	0.487032	0.512968	41	1814	−1773
348	$2^23^129^1$	N	N	30	14	1.1666667	0.488506	0.511494	71	1844	−1773
349	349^1	Y	Y	−2	0	1.0000000	0.487106	0.512894	69	1844	−1775
350	$2^15^27^1$	N	N	30	14	1.1666667	0.488571	0.511429	99	1874	−1775

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
351	$3^3 13^1$	N	N	9	4	1.5555556	0.490028	0.509972	108	1883	−1775
352	$2^5 11^1$	N	N	13	8	2.0769231	0.491477	0.508523	121	1896	−1775
353	353^1	Y	Y	−2	0	1.0000000	0.490085	0.509915	119	1896	−1777
354	$2^1 3^1 59^1$	Y	N	−16	0	1.0000000	0.488701	0.511299	103	1896	−1793
355	$5^1 71^1$	Y	N	5	0	1.0000000	0.490141	0.509859	108	1901	−1793
356	$2^2 89^1$	N	N	−7	2	1.2857143	0.488764	0.511236	101	1901	−1800
357	$3^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.487395	0.512605	85	1901	−1816
358	$2^1 179^1$	Y	N	5	0	1.0000000	0.488827	0.511173	90	1906	−1816
359	359^1	Y	Y	−2	0	1.0000000	0.487465	0.512535	88	1906	−1818
360	$2^3 3^2 5^1$	N	N	145	129	1.3034483	0.488889	0.511111	233	2051	−1818
361	19^2	N	Y	2	0	1.5000000	0.490305	0.509695	235	2053	−1818
362	$2^1 181^1$	Y	N	5	0	1.0000000	0.491713	0.508287	240	2058	−1818
363	$3^1 11^2$	N	N	−7	2	1.2857143	0.490358	0.509642	233	2058	−1825
364	$2^2 7^1 13^1$	N	N	30	14	1.1666667	0.491758	0.508242	263	2088	−1825
365	$5^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	268	2093	−1825
366	$2^1 3^1 61^1$	Y	N	−16	0	1.0000000	0.491803	0.508197	252	2093	−1841
367	367^1	Y	Y	−2	0	1.0000000	0.490463	0.509537	250	2093	−1843
368	$2^4 23^1$	N	N	−11	6	1.8181818	0.489130	0.510870	239	2093	−1854
369	$3^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	232	2093	−1861
370	$2^1 5^1 37^1$	Y	N	−16	0	1.0000000	0.486486	0.513514	216	2093	−1877
371	$7^1 53^1$	Y	N	5	0	1.0000000	0.487871	0.512129	221	2098	−1877
372	$2^2 3^1 31^1$	N	N	30	14	1.1666667	0.489247	0.510753	251	2128	−1877
373	373^1	Y	Y	−2	0	1.0000000	0.487936	0.512064	249	2128	−1879
374	$2^1 11^1 17^1$	Y	N	−16	0	1.0000000	0.486631	0.513369	233	2128	−1895
375	$3^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	242	2137	−1895
376	$2^3 47^1$	N	N	9	4	1.5555556	0.489362	0.510638	251	2146	−1895
377	$13^1 29^1$	Y	N	5	0	1.0000000	0.490716	0.509284	256	2151	−1895
378	$2^1 3^3 7^1$	N	N	−48	32	1.3333333	0.489418	0.510582	208	2151	−1943
379	379^1	Y	Y	−2	0	1.0000000	0.488127	0.511873	206	2151	−1945
380	$2^2 5^1 19^1$	N	N	30	14	1.1666667	0.489474	0.510526	236	2181	−1945
381	$3^1 127^1$	Y	N	5	0	1.0000000	0.490814	0.509186	241	2186	−1945
382	$2^1 191^1$	Y	N	5	0	1.0000000	0.492147	0.507853	246	2191	−1945
383	383^1	Y	Y	−2	0	1.0000000	0.490862	0.509138	244	2191	−1947
384	$2^7 3^1$	N	N	17	12	2.5882353	0.492188	0.507812	261	2208	−1947
385	$5^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.490909	0.509091	245	2208	−1963
386	$2^1 193^1$	Y	N	5	0	1.0000000	0.492228	0.507772	250	2213	−1963
387	$3^2 43^1$	N	N	−7	2	1.2857143	0.490956	0.509044	243	2213	−1970
388	$2^2 97^1$	N	N	−7	2	1.2857143	0.489691	0.510309	236	2213	−1977
389	389^1	Y	Y	−2	0	1.0000000	0.488432	0.511568	234	2213	−1979
390	$2^1 3^1 5^1 13^1$	Y	N	65	0	1.0000000	0.489744	0.510256	299	2278	−1979
391	$17^1 23^1$	Y	N	5	0	1.0000000	0.491049	0.508951	304	2283	−1979
392	$2^3 7^2$	N	N	−23	18	1.4782609	0.489796	0.510204	281	2283	−2002
393	$3^1 131^1$	Y	N	5	0	1.0000000	0.491094	0.508906	286	2288	−2002
394	$2^1 197^1$	Y	N	5	0	1.0000000	0.492386	0.507614	291	2293	−2002
395	$5^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	296	2298	−2002
396	$2^2 3^2 11^1$	N	N	−74	58	1.2162162	0.492424	0.507576	222	2298	−2076
397	397^1	Y	Y	−2	0	1.0000000	0.491184	0.508816	220	2298	−2078
398	$2^1 199^1$	Y	N	5	0	1.0000000	0.492462	0.507538	225	2303	−2078
399	$3^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.491228	0.508772	209	2303	−2094
400	$2^4 5^2$	N	N	34	29	1.6176471	0.492500	0.507500	243	2337	−2094
401	401^1	Y	Y	−2	0	1.0000000	0.491272	0.508728	241	2337	−2096
402	$2^1 3^1 67^1$	Y	N	−16	0	1.0000000	0.490050	0.509950	225	2337	−2112
403	$13^1 31^1$	Y	N	5	0	1.0000000	0.491315	0.508685	230	2342	−2112
404	$2^2 101^1$	N	N	−7	2	1.2857143	0.490099	0.509901	223	2342	−2119
405	$3^4 5^1$	N	N	−11	6	1.8181818	0.488889	0.511111	212	2342	−2130
406	$2^1 7^1 29^1$	Y	N	−16	0	1.0000000	0.487685	0.512315	196	2342	−2146
407	$11^1 37^1$	Y	N	5	0	1.0000000	0.488943	0.511057	201	2347	−2146
408	$2^3 3^1 17^1$	N	N	−48	32	1.3333333	0.487745	0.512255	153	2347	−2194
409	409^1	Y	Y	−2	0	1.0000000	0.486553	0.513447	151	2347	−2196
410	$2^1 5^1 41^1$	Y	N	−16	0	1.0000000	0.485366	0.514634	135	2347	−2212
411	$3^1 137^1$	Y	N	5	0	1.0000000	0.486618	0.513382	140	2352	−2212
412	$2^2 103^1$	N	N	−7	2	1.2857143	0.485437	0.514563	133	2352	−2219
413	$7^1 59^1$	Y	N	5	0	1.0000000	0.486683	0.513317	138	2357	−2219
414	$2^1 3^2 23^1$	N	N	30	14	1.1666667	0.487923	0.512077	168	2387	−2219
415	$5^1 83^1$	Y	N	5	0	1.0000000	0.489157	0.510843	173	2392	−2219
416	$2^5 13^1$	N	N	13	8	2.0769231	0.490385	0.509615	186	2405	−2219
417	$3^1 139^1$	Y	N	5	0	1.0000000	0.491607	0.508393	191	2410	−2219
418	$2^1 11^1 19^1$	Y	N	−16	0	1.0000000	0.490431	0.509569	175	2410	−2235
419	419^1	Y	Y	−2	0	1.0000000	0.489260	0.510740	173	2410	−2237
420	$2^2 3^1 5^1 7^1$	N	N	−155	90	1.1032258	0.488095	0.511905	18	2410	−2392
421	421^1	Y	Y	−2	0	1.0000000	0.486936	0.513064	16	2410	−2394
422	$2^1 211^1$	Y	N	5	0	1.0000000	0.488152	0.511848	21	2415	−2394
423	$3^2 47^1$	N	N	−7	2	1.2857143	0.486998	0.513002	14	2415	−2401
424	$2^3 53^1$	N	N	9	4	1.5555556	0.488208	0.511792	23	2424	−2401
425	$5^2 17^1$	N	N	−7	2	1.2857143	0.487059	0.512941	16	2424	−2408

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\|n} C_{\Omega(d)}(d)}{\|g^{-1}(n)\|}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
426	$2^1 3^1 71^1$	Y	N	−16	0	1.0000000	0.485915	0.514085	0	2424	−2424
427	$7^1 61^1$	Y	N	5	0	1.0000000	0.487119	0.512881	5	2429	−2424
428	$2^2 107^1$	N	N	−7	2	1.2857143	0.485981	0.514019	−2	2429	−2431
429	$3^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	−18	2429	−2447
430	$2^1 5^1 43^1$	Y	N	−16	0	1.0000000	0.483721	0.516279	−34	2429	−2463
431	431^1	Y	Y	−2	0	1.0000000	0.482599	0.517401	−36	2429	−2465
432	$2^4 3^3$	N	N	−80	75	1.5625000	0.481481	0.518519	−116	2429	−2545
433	433^1	Y	Y	−2	0	1.0000000	0.480370	0.519630	−118	2429	−2547
434	$2^1 7^1 31^1$	Y	N	−16	0	1.0000000	0.479263	0.520737	−134	2429	−2563
435	$3^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.478161	0.521839	−150	2429	−2579
436	$2^2 109^1$	N	N	−7	2	1.2857143	0.477064	0.522936	−157	2429	−2586
437	$19^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	−152	2434	−2586
438	$2^1 3^1 73^1$	Y	N	−16	0	1.0000000	0.477169	0.522831	−168	2434	−2602
439	439^1	Y	Y	−2	0	1.0000000	0.476082	0.523918	−170	2434	−2604
440	$2^3 5^1 11^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−218	2434	−2652
441	$3^2 7^2$	N	N	14	9	1.3571429	0.476190	0.523810	−204	2448	−2652
442	$2^1 13^1 17^1$	Y	N	−16	0	1.0000000	0.475113	0.524887	−220	2448	−2668
443	443^1	Y	Y	−2	0	1.0000000	0.474041	0.525959	−222	2448	−2670
444	$2^2 3^1 37^1$	N	N	30	14	1.1666667	0.475225	0.524775	−192	2478	−2670
445	$5^1 89^1$	Y	N	5	0	1.0000000	0.476404	0.523596	−187	2483	−2670
446	$2^1 223^1$	Y	N	5	0	1.0000000	0.477578	0.522422	−182	2488	−2670
447	$3^1 149^1$	Y	N	5	0	1.0000000	0.478747	0.521253	−177	2493	−2670
448	$2^6 7^1$	N	N	−15	10	2.3333333	0.477679	0.522321	−192	2493	−2685
449	449^1	Y	Y	−2	0	1.0000000	0.476615	0.523385	−194	2493	−2687
450	$2^1 3^2 5^2$	N	N	−74	58	1.2162162	0.475556	0.524444	−268	2493	−2761
451	$11^1 41^1$	Y	N	5	0	1.0000000	0.476718	0.523282	−263	2498	−2761
452	$2^2 113^1$	N	N	−7	2	1.2857143	0.475664	0.524336	−270	2498	−2768
453	$3^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−265	2503	−2768
454	$2^1 227^1$	Y	N	5	0	1.0000000	0.477974	0.522026	−260	2508	−2768
455	$5^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476923	0.523077	−276	2508	−2784
456	$2^3 3^1 19^1$	N	N	−48	32	1.3333333	0.475877	0.524123	−324	2508	−2832
457	457^1	Y	Y	−2	0	1.0000000	0.474836	0.525164	−326	2508	−2834
458	$2^1 229^1$	Y	N	5	0	1.0000000	0.475983	0.524017	−321	2513	−2834
459	$3^3 17^1$	N	N	9	4	1.5555556	0.477124	0.522876	−312	2522	−2834
460	$2^2 5^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−282	2552	−2834
461	461^1	Y	Y	−2	0	1.0000000	0.477223	0.522777	−284	2552	−2836
462	$2^1 3^1 7^1 11^1$	Y	N	65	0	1.0000000	0.478355	0.521645	−219	2617	−2836
463	463^1	Y	Y	−2	0	1.0000000	0.477322	0.522678	−221	2617	−2838
464	$2^4 29^1$	N	N	−11	6	1.8181818	0.476293	0.523707	−232	2617	−2849
465	$3^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.475269	0.524731	−248	2617	−2865
466	$2^1 233^1$	Y	N	5	0	1.0000000	0.476395	0.523605	−243	2622	−2865
467	467^1	Y	Y	−2	0	1.0000000	0.475375	0.524625	−245	2622	−2867
468	$2^2 3^2 13^1$	N	N	−74	58	1.2162162	0.474359	0.525641	−319	2622	−2941
469	$7^1 67^1$	Y	N	5	0	1.0000000	0.475480	0.524520	−314	2627	−2941
470	$2^1 5^1 47^1$	Y	N	−16	0	1.0000000	0.474468	0.525532	−330	2627	−2957
471	$3^1 157^1$	Y	N	5	0	1.0000000	0.475584	0.524416	−325	2632	−2957
472	$2^3 59^1$	N	N	9	4	1.5555556	0.476695	0.523305	−316	2641	−2957
473	$11^1 43^1$	Y	N	5	0	1.0000000	0.477801	0.522199	−311	2646	−2957
474	$2^1 3^1 79^1$	Y	N	−16	0	1.0000000	0.476793	0.523207	−327	2646	−2973
475	$5^2 19^1$	N	N	−7	2	1.2857143	0.475789	0.524211	−334	2646	−2980
476	$2^2 7^1 17^1$	N	N	30	14	1.1666667	0.476891	0.523109	−304	2676	−2980
477	$3^2 53^1$	N	N	−7	2	1.2857143	0.475891	0.524109	−311	2676	−2987
478	$2^1 239^1$	Y	N	5	0	1.0000000	0.476987	0.523013	−306	2681	−2987
479	479^1	Y	Y	−2	0	1.0000000	0.475992	0.524008	−308	2681	−2989
480	$2^5 3^1 5^1$	N	N	−96	80	1.6666667	0.475000	0.525000	−404	2681	−3085
481	$13^1 37^1$	Y	N	5	0	1.0000000	0.476091	0.523909	−399	2686	−3085
482	$2^1 241^1$	Y	N	5	0	1.0000000	0.477178	0.522822	−394	2691	−3085
483	$3^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−410	2691	−3101
484	$2^2 11^2$	N	N	14	9	1.3571429	0.477273	0.522727	−396	2705	−3101
485	$5^1 97^1$	Y	N	5	0	1.0000000	0.478351	0.521649	−391	2710	−3101
486	$2^1 3^5$	N	N	13	8	2.0769231	0.479424	0.520576	−378	2723	−3101
487	487^1	Y	Y	−2	0	1.0000000	0.478439	0.521561	−380	2723	−3103
488	$2^3 61^1$	N	N	9	4	1.5555556	0.479508	0.520492	−371	2732	−3103
489	$3^1 163^1$	Y	N	5	0	1.0000000	0.480573	0.519427	−366	2737	−3103
490	$2^1 5^1 7^2$	N	N	30	14	1.1666667	0.481633	0.518367	−336	2767	−3103
491	491^1	Y	Y	−2	0	1.0000000	0.480652	0.519348	−338	2767	−3105
492	$2^2 3^1 41^1$	N	N	30	14	1.1666667	0.481707	0.518293	−308	2797	−3105
493	$17^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	−303	2802	−3105
494	$2^1 13^1 19^1$	Y	N	−16	0	1.0000000	0.481781	0.518219	−319	2802	−3121
495	$3^2 5^1 11^1$	N	N	30	14	1.1666667	0.482828	0.517172	−289	2832	−3121
496	$2^4 31^1$	N	N	−11	6	1.8181818	0.481855	0.518145	−300	2832	−3132
497	$7^1 71^1$	Y	N	5	0	1.0000000	0.482897	0.517103	−295	2837	−3132
498	$2^1 3^1 83^1$	Y	N	−16	0	1.0000000	0.481928	0.518072	−311	2837	−3148
499	499^1	Y	Y	−2	0	1.0000000	0.480962	0.519038	−313	2837	−3150
500	$2^2 5^3$	N	N	−23	18	1.4782609	0.480000	0.520000	−336	2837	−3173

T.2 Table: Approximations of the summatory functions of $\lambda(n)$ and $\lambda_*(n)$

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$	x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$
50000	−67	1	0.0098	−24.	50000	22169	1.8	50045	−88	1	0.014	−32.	50045	22170	1.8
50001	−68	1	0.0098	−24.	50001	22170	1.8	50046	−87	1	0.014	−32.	50046	22171	1.8
50002	−69	1	0.0098	−24.	50002	22171	1.8	50047	−88	1	0.014	−32.	50047	22172	1.8
50003	−68	1	0.0098	−24.	50003	22172	1.8	50048	−92	1	0.014	−32.	50048	22176	1.8
50004	−64	1	0.0098	−24.	50004	22168	1.8	50049	−90	1	0.014	−32.	50049	22174	1.8
50005	−65	1	0.0098	−24.	50005	22169	1.8	50050	−84	1	0.014	−30.	50050	22168	1.8
50006	−66	1	0.0098	−24.	50006	22170	1.8	50051	−85	1	0.014	−30.	50051	22169	1.8
50007	−67	1	0.0098	−24.	50007	22171	1.8	50052	−87	1	0.014	−32.	50052	22167	1.8
50008	−68	1	0.0098	−24.	50008	22170	1.8	50053	−88	1	0.014	−32.	50053	22168	1.8
50009	−67	1	0.0098	−24.	50009	22171	1.8	50054	−89	1	0.014	−32.	50054	22169	1.8
50010	−64	1	0.0098	−24.	50010	22174	1.8	50055	−88	1	0.014	−32.	50055	22170	1.8
50011	−63	1	0.0098	−22.	50011	22175	1.8	50056	−87	1	0.014	−32.	50056	22171	1.8
50012	−64	1	0.0098	−24.	50012	22174	1.8	50057	−86	1	0.014	−30.	50057	22172	1.8
50013	−66	1	0.0098	−24.	50013	22172	1.8	50058	−89	1	0.014	−32.	50058	22175	1.8
50014	−67	1	0.0098	−24.	50014	22173	1.8	50059	−88	1	0.014	−32.	50059	22176	1.8
50015	−68	1	0.0098	−24.	50015	22174	1.8	50060	−86	1	0.014	−30.	50060	22174	1.8
50016	−71	1	0.011	−24.	50016	22177	1.8	50061	−85	1	0.014	−30.	50061	22175	1.8
50017	−70	1	0.0098	−24.	50017	22178	1.8	50062	−84	1	0.014	−30.	50062	22176	1.8
50018	−71	1	0.011	−24.	50018	22179	1.8	50063	−83	1	0.014	−30.	50063	22177	1.8
50019	−70	1	0.0098	−24.	50019	22180	1.8	50064	−85	1	0.014	−30.	50064	22175	1.8
50020	−72	1	0.011	−26.	50020	22178	1.8	50065	−84	1	0.014	−30.	50065	22176	1.8
50021	−73	1	0.011	−26.	50021	22179	1.8	50066	−83	1	0.014	−30.	50066	22177	1.8
50022	−76	1	0.011	−26.	50022	22176	1.8	50067	−85	1	0.014	−30.	50067	22175	1.8
50023	−77	1	0.012	−26.	50023	22177	1.8	50068	−86	1	0.014	−30.	50068	22174	1.8
50024	−74	1	0.011	−26.	50024	22174	1.8	50069	−87	1	0.014	−32.	50069	22175	1.8
50025	−77	1	0.012	−26.	50025	22171	1.8	50070	−84	1	0.014	−30.	50070	22178	1.8
50026	−76	1	0.011	−26.	50026	22172	1.8	50071	−85	1	0.014	−30.	50071	22179	1.8
50027	−75	1	0.011	−26.	50027	22173	1.8	50072	−86	1	0.014	−30.	50072	22180	1.8
50028	−76	1	0.011	−26.	50028	22172	1.8	50073	−85	1	0.014	−30.	50073	22181	1.8
50029	−79	1	0.012	−30.	50029	22169	1.8	50074	−84	1	0.014	−30.	50074	22182	1.8
50030	−80	1	0.012	−30.	50030	22170	1.8	50075	−87	1	0.014	−32.	50075	22179	1.8
50031	−82	1	0.012	−30.	50031	22172	1.8	50076	−85	1	0.014	−30.	50076	22181	1.8
50032	−80	1	0.012	−30.	50032	22170	1.8	50077	−86	1	0.014	−30.	50077	22182	1.8
50033	−81	1	0.012	−30.	50033	22171	1.8	50078	−89	1	0.014	−32.	50078	22185	1.8
50034	−80	1	0.012	−30.	50034	22172	1.8	50079	−88	1	0.014	−32.	50079	22186	1.8
50035	−79	1	0.012	−30.	50035	22173	1.8	50080	−91	1	0.014	−32.	50080	22189	1.8
50036	−78	1	0.012	−26.	50036	22172	1.8	50081	−90	1	0.014	−32.	50081	22190	1.8
50037	−79	1	0.012	−30.	50037	22173	1.8	50082	−89	1	0.014	−32.	50082	22191	1.8
50038	−80	1	0.012	−30.	50038	22174	1.8	50083	−90	1	0.014	−32.	50083	22192	1.8
50039	−79	1	0.012	−30.	50039	22175	1.8	50084	−89	1	0.014	−32.	50084	22191	1.8
50040	−87	1	0.014	−32.	50040	22167	1.8	50085	−87	1	0.014	−32.	50085	22193	1.8
50041	−86	1	0.014	−30.	50041	22168	1.8	50086	−88	1	0.014	−32.	50086	22194	1.8
50042	−87	1	0.014	−32.	50042	22169	1.8	50087	−89	1	0.014	−32.	50087	22195	1.8
50043	−88	1	0.014	−32.	50043	22170	1.8	50088	−91	1	0.014	−32.	50088	22197	1.8
50044	−89	1	0.014	−32.	50044	22169	1.8	50089	−90	1	0.014	−32.	50089	22198	1.8

Table T.2: ***Approximations to the summatory functions of*** $\lambda(n)$ ***and*** $\lambda_*(n)$.

- *We define the exact summatory functions over these sequences by* $L(x) := \sum_{n \leq x} \lambda(n)$ *and* $L_*(n) := \sum_{n \leq x} \lambda_*(n)$.
- *Let the expected sign ratio function be defined by* $R_{\pm}(x) := \frac{\operatorname{sgn}(L(x))}{(-1)^{\lfloor \log\log x \rfloor}}$.
- *We compare the ratios of the following two functions with* $L(x)$*:* $L_{\approx,1}(x) := \sum_{k=1}^{\log\log x} \frac{x}{\log x} \cdot \frac{(-\log\log x)^{k-1}}{(k-1)!}$ *and* $L_{\approx,2}(x) := \frac{x^{1/4}}{\sqrt{\log x}\sqrt{\log\log x}}$.
- *Finally, we compare the approximations (very accurate) to* $L_*(x)$ *by the summatory function* $\sum_{k \leq x} \widehat{c}(-1)^k \cdot 2^{-k}$ *using the approximation* $L_{\approx}^*(x) := \frac{2\widehat{c}}{3} x$.

We are expecting to see and verify numerically that for sufficiently large x *the following properties:*

- *Almost always we have that* $R_{\pm}(x) = 1$.
- *The ratio* $\frac{L(x)}{L_{\approx,1}(x)}$ *should be bounded by a constant approximately equal to one.*
- *The ratio* $\frac{L(x)}{L_{\approx,2}(x)}$ *should be at least one.*
- *The ratio* $\frac{L_*(x)}{L_{\approx}^*(x)}$ *tends towards an absolute constant.*

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$	x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$
50090	−91	1	0.014	−32.	50090	22199	1.8	50165	−74	1	0.011	−26.	50165	22232	1.8
50091	−92	1	0.014	−32.	50091	22200	1.8	50166	−76	1	0.011	−26.	50166	22234	1.8
50092	−91	1	0.014	−32.	50092	22199	1.8	50167	−77	1	0.012	−26.	50167	22235	1.8
50093	−92	1	0.014	−32.	50093	22200	1.8	50168	−76	1	0.011	−26.	50168	22236	1.8
50094	−82	1	0.012	−30.	50094	22210	1.8	50169	−77	1	0.012	−26.	50169	22237	1.8
50095	−83	1	0.014	−30.	50095	22211	1.8	50170	−76	1	0.011	−26.	50170	22238	1.8
50096	−81	1	0.012	−30.	50096	22209	1.8	50171	−75	1	0.011	−26.	50171	22239	1.8
50097	−80	1	0.012	−30.	50097	22210	1.8	50172	−77	1	0.012	−26.	50172	22237	1.8
50098	−81	1	0.012	−30.	50098	22211	1.8	50173	−76	1	0.011	−26.	50173	22238	1.8
50099	−82	1	0.012	−30.	50099	22212	1.8	50174	−75	1	0.011	−26.	50174	22239	1.8
50100	−75	1	0.011	−26.	50100	22219	1.8	50175	−80	1	0.012	−30.	50175	22244	1.8
50101	−76	1	0.011	−26.	50101	22220	1.8	50176	160	−1	−0.023	60.	50176	22484	1.8
50102	−75	1	0.011	−26.	50102	22221	1.8	50177	−63	1	0.0098	−22.	50177	22707	1.8
50103	−73	1	0.011	−26.	50103	22219	1.8	50178	−64	1	0.0098	−24.	50178	22708	1.8
50104	−72	1	0.011	−26.	50104	22220	1.8	50179	−68	1	0.0098	−24.	50179	22704	1.8
50105	−73	1	0.011	−26.	50105	22221	1.8	50180	−70	1	0.0098	−24.	50180	22702	1.8
50106	−71	1	0.011	−24.	50106	22223	1.8	50181	−71	1	0.011	−24.	50181	22703	1.8
50107	−70	1	0.0098	−24.	50107	22224	1.8	50182	−72	1	0.011	−26.	50182	22704	1.8
50108	−71	1	0.011	−24.	50108	22223	1.8	50183	−73	1	0.011	−26.	50183	22705	1.8
50109	−70	1	0.0098	−24.	50109	22224	1.8	50184	−78	1	0.012	−26.	50184	22700	1.8
50110	−71	1	0.011	−24.	50110	22225	1.8	50185	−77	1	0.012	−26.	50185	22701	1.8
50111	−72	1	0.011	−26.	50111	22226	1.8	50186	−78	1	0.012	−26.	50186	22702	1.8
50112	−59	1	0.0098	−22.	50112	22213	1.8	50187	−77	1	0.012	−26.	50187	22703	1.8
50113	−58	1	0.0078	−22.	50113	22214	1.8	50188	−78	1	0.012	−26.	50188	22702	1.8
50114	−57	1	0.0078	−20.	50114	22215	1.8	50189	−77	1	0.012	−26.	50189	22703	1.8
50115	−56	1	0.0078	−20.	50115	22216	1.8	50190	−79	1	0.012	−30.	50190	22705	1.8
50116	−57	1	0.0078	−20.	50116	22215	1.8	50191	−78	1	0.012	−26.	50191	22706	1.8
50117	−56	1	0.0078	−20.	50117	22216	1.8	50192	−80	1	0.012	−30.	50192	22704	1.8
50118	−57	1	0.0078	−20.	50118	22217	1.8	50193	−75	1	0.011	−26.	50193	22699	1.8
50119	−58	1	0.0078	−22.	50119	22218	1.8	50194	−74	1	0.011	−26.	50194	22700	1.8
50120	−58	1	0.0078	−22.	50120	22218	1.8	50195	−73	1	0.011	−26.	50195	22701	1.8
50121	−60	1	0.0098	−22.	50121	22216	1.8	50196	−75	1	0.011	−26.	50196	22699	1.8
50122	−61	1	0.0098	−22.	50122	22217	1.8	50197	−76	1	0.011	−26.	50197	22700	1.8
50123	−62	1	0.0098	−22.	50123	22218	1.8	50198	−77	1	0.012	−26.	50198	22701	1.8
50124	−60	1	0.0098	−22.	50124	22216	1.8	50199	−78	1	0.012	−26.	50199	22702	1.8
50125	−57	1	0.0078	−20.	50125	22219	1.8	50200	−74	1	0.011	−26.	50200	22698	1.8
50126	−58	1	0.0078	−22.	50126	22220	1.8	50201	−73	1	0.011	−26.	50201	22699	1.8
50127	−61	1	0.0098	−22.	50127	22217	1.8	50202	−71	1	0.011	−24.	50202	22697	1.8
50128	−59	1	0.0098	−22.	50128	22215	1.8	50203	−70	1	0.0098	−24.	50203	22698	1.8
50129	−60	1	0.0098	−22.	50129	22216	1.8	50204	−71	1	0.011	−24.	50204	22697	1.8
50130	−63	1	0.0098	−22.	50130	22213	1.8	50205	−72	1	0.011	−26.	50205	22698	1.8
50131	−64	1	0.0098	−24.	50131	22214	1.8	50206	−73	1	0.011	−26.	50206	22699	1.8
50132	−63	1	0.0098	−22.	50132	22213	1.8	50207	−74	1	0.011	−26.	50207	22700	1.8
50133	−64	1	0.0098	−24.	50133	22214	1.8	50208	−77	1	0.012	−26.	50208	22703	1.8
50134	−65	1	0.0098	−24.	50134	22215	1.8	50209	−78	1	0.012	−26.	50209	22704	1.8
50135	−66	1	0.0098	−24.	50135	22216	1.8	50210	−79	1	0.012	−30.	50210	22705	1.8
50136	−68	1	0.0098	−24.	50136	22218	1.8	50211	−77	1	0.012	−26.	50211	22703	1.8
50137	−67	1	0.0098	−24.	50137	22219	1.8	50212	−78	1	0.012	−26.	50212	22702	1.8
50138	−66	1	0.0098	−24.	50138	22220	1.8	50213	−77	1	0.012	−26.	50213	22703	1.8
50139	−69	1	0.0098	−24.	50139	22217	1.8	50214	−78	1	0.012	−26.	50214	22704	1.8
50140	−71	1	0.011	−24.	50140	22215	1.8	50215	−76	1	0.011	−26.	50215	22702	1.8
50141	−70	1	0.0098	−24.	50141	22216	1.8	50216	−75	1	0.011	−26.	50216	22703	1.8
50142	−69	1	0.0098	−24.	50142	22217	1.8	50217	−76	1	0.011	−26.	50217	22704	1.8
50143	−68	1	0.0098	−24.	50143	22218	1.8	50218	−75	1	0.011	−26.	50218	22705	1.8
50144	−66	1	0.0098	−24.	50144	22220	1.8	50219	−74	1	0.011	−26.	50219	22706	1.8
50145	−67	1	0.0098	−24.	50145	22221	1.8	50220	−69	1	0.0098	−24.	50220	22711	1.8
50146	−66	1	0.0098	−24.	50146	22222	1.8	50221	−70	1	0.0098	−24.	50221	22712	1.8
50147	−67	1	0.0098	−24.	50147	22223	1.8	50222	−69	1	0.0098	−24.	50222	22713	1.8
50148	−62	1	0.0098	−22.	50148	22228	1.8	50223	−68	1	0.0098	−24.	50223	22714	1.8
50149	−63	1	0.0098	−22.	50149	22229	1.8	50224	−66	1	0.0098	−24.	50224	22712	1.8
50150	−66	1	0.0098	−24.	50150	22226	1.8	50225	−74	1	0.011	−26.	50225	22720	1.8
50151	−67	1	0.0098	−24.	50151	22227	1.8	50226	−73	1	0.011	−26.	50226	22721	1.8
50152	−66	1	0.0098	−24.	50152	22228	1.8	50227	−74	1	0.011	−26.	50227	22722	1.8
50153	−67	1	0.0098	−24.	50153	22229	1.8	50228	−73	1	0.011	−26.	50228	22721	1.8
50154	−66	1	0.0098	−24.	50154	22230	1.8	50229	−75	1	0.011	−26.	50229	22719	1.8
50155	−67	1	0.0098	−24.	50155	22231	1.8	50230	−76	1	0.011	−26.	50230	22720	1.8
50156	−68	1	0.0098	−24.	50156	22230	1.8	50231	−77	1	0.012	−26.	50231	22721	1.8
50157	−70	1	0.0098	−24.	50157	22228	1.8	50232	−83	1	0.014	−30.	50232	22727	1.8
50158	−71	1	0.011	−24.	50158	22229	1.8	50233	−82	1	0.012	−30.	50233	22728	1.8
50159	−72	1	0.011	−26.	50159	22230	1.8	50234	−81	1	0.012	−30.	50234	22729	1.8
50160	−71	1	0.011	−24.	50160	22229	1.8	50235	−80	1	0.012	−30.	50235	22730	1.8
50161	−70	1	0.0098	−24.	50161	22230	1.8	50236	−79	1	0.012	−30.	50236	22729	1.8
50162	−71	1	0.011	−24.	50162	22231	1.8	50237	−78	1	0.012	−26.	50237	22730	1.8
50163	−72	1	0.011	−26.	50163	22232	1.8	50238	−76	1	0.011	−26.	50238	22728	1.8
50164	−73	1	0.011	−26.	50164	22231	1.8	50239	−75	1	0.011	−26.	50239	22729	1.8

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L^*_{\approx}(x)}$	x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L^*_{\approx}(x)}$
50240	−70	1	0.0098	−24.	50240	22724	1.8	50315	−73	1	0.011	−26.	50315	22775	1.8
50241	−69	1	0.0098	−24.	50241	22725	1.8	50316	−76	1	0.011	−26.	50316	22772	1.8
50242	−68	1	0.0098	−24.	50242	22726	1.8	50317	−75	1	0.011	−26.	50317	22773	1.8
50243	−67	1	0.0098	−24.	50243	22727	1.8	50318	−76	1	0.011	−26.	50318	22774	1.8
50244	−69	1	0.0098	−24.	50244	22725	1.8	50319	−78	1	0.012	−26.	50319	22772	1.8
50245	−70	1	0.0098	−24.	50245	22726	1.8	50320	−81	1	0.012	−30.	50320	22769	1.8
50246	−69	1	0.0098	−24.	50246	22727	1.8	50321	−82	1	0.012	−30.	50321	22770	1.8
50247	−67	1	0.0098	−24.	50247	22729	1.8	50322	−83	1	0.014	−30.	50322	22771	1.8
50248	−68	1	0.0098	−24.	50248	22730	1.8	50323	−80	1	0.012	−30.	50323	22768	1.8
50249	−67	1	0.0098	−24.	50249	22731	1.8	50324	−79	1	0.012	−30.	50324	22767	1.8
50250	−62	1	0.0098	−22.	50250	22736	1.8	50325	−82	1	0.012	−30.	50325	22764	1.8
50251	−61	1	0.0098	−22.	50251	22737	1.8	50326	−81	1	0.012	−30.	50326	22765	1.8
50252	−60	1	0.0098	−22.	50252	22736	1.8	50327	−80	1	0.012	−30.	50327	22766	1.8
50253	−61	1	0.0098	−22.	50253	22737	1.8	50328	−86	1	0.014	−30.	50328	22772	1.8
50254	−60	1	0.0098	−22.	50254	22738	1.8	50329	−87	1	0.014	−32.	50329	22773	1.8
50255	−56	1	0.0078	−20.	50255	22734	1.8	50330	−86	1	0.014	−30.	50330	22774	1.8
50256	−65	1	0.0098	−24.	50256	22743	1.8	50331	−87	1	0.014	−32.	50331	22775	1.8
50257	−64	1	0.0098	−24.	50257	22744	1.8	50332	−88	1	0.014	−32.	50332	22774	1.8
50258	−65	1	0.0098	−24.	50258	22745	1.8	50333	−89	1	0.014	−32.	50333	22775	1.8
50259	−66	1	0.0098	−24.	50259	22746	1.8	50334	−90	1	0.014	−32.	50334	22776	1.8
50260	−68	1	0.0098	−24.	50260	22744	1.8	50335	−89	1	0.014	−32.	50335	22777	1.8
50261	−69	1	0.0098	−24.	50261	22745	1.8	50336	−81	1	0.012	−30.	50336	22769	1.8
50262	−70	1	0.0098	−24.	50262	22746	1.8	50337	−83	1	0.014	−30.	50337	22767	1.8
50263	−71	1	0.011	−24.	50263	22747	1.8	50338	−82	1	0.012	−30.	50338	22768	1.8
50264	−72	1	0.011	−26.	50264	22748	1.8	50339	−81	1	0.012	−30.	50339	22769	1.8
50265	−70	1	0.0098	−24.	50265	22746	1.8	50340	−86	1	0.014	−30.	50340	22764	1.8
50266	−71	1	0.011	−24.	50266	22747	1.8	50341	−87	1	0.014	−32.	50341	22765	1.8
50267	−72	1	0.011	−26.	50267	22748	1.8	50342	−86	1	0.014	−30.	50342	22766	1.8
50268	−74	1	0.011	−26.	50268	22746	1.8	50343	−87	1	0.014	−32.	50343	22767	1.8
50269	−73	1	0.011	−26.	50269	22747	1.8	50344	−88	1	0.014	−32.	50344	22766	1.8
50270	−72	1	0.011	−26.	50270	22748	1.8	50345	−87	1	0.014	−32.	50345	22767	1.8
50271	−73	1	0.011	−26.	50271	22749	1.8	50346	−85	1	0.014	−30.	50346	22765	1.8
50272	−71	1	0.011	−24.	50272	22751	1.8	50347	−86	1	0.014	−30.	50347	22766	1.8
50273	−72	1	0.011	−26.	50273	22752	1.8	50348	−85	1	0.014	−30.	50348	22765	1.8
50274	−78	1	0.012	−26.	50274	22746	1.8	50349	−86	1	0.014	−30.	50349	22766	1.8
50275	−81	1	0.012	−30.	50275	22743	1.8	50350	−89	1	0.014	−32.	50350	22763	1.8
50276	−82	1	0.012	−30.	50276	22742	1.8	50351	−88	1	0.014	−32.	50351	22764	1.8
50277	−81	1	0.012	−30.	50277	22743	1.8	50352	−85	1	0.014	−30.	50352	22761	1.8
50278	−82	1	0.012	−30.	50278	22744	1.8	50353	−84	1	0.014	−30.	50353	22762	1.8
50279	−81	1	0.012	−30.	50279	22745	1.8	50354	−85	1	0.014	−30.	50354	22763	1.8
50280	−76	1	0.011	−26.	50280	22750	1.8	50355	−87	1	0.014	−32.	50355	22765	1.8
50281	−77	1	0.012	−26.	50281	22751	1.8	50356	−88	1	0.014	−32.	50356	22764	1.8
50282	−78	1	0.012	−26.	50282	22752	1.8	50357	−87	1	0.014	−32.	50357	22765	1.8
50283	−76	1	0.011	−26.	50283	22750	1.8	50358	−86	1	0.014	−30.	50358	22764	1.8
50284	−75	1	0.011	−26.	50284	22749	1.8	50359	−87	1	0.014	−32.	50359	22765	1.8
50285	−76	1	0.011	−26.	50285	22750	1.8	50360	−89	1	0.014	−32.	50360	22767	1.8
50286	−78	1	0.012	−26.	50286	22748	1.8	50361	−88	1	0.014	−32.	50361	22768	1.8
50287	−79	1	0.012	−30.	50287	22749	1.8	50362	−84	1	0.014	−30.	50362	22764	1.8
50288	−79	1	0.012	−30.	50288	22749	1.8	50363	−85	1	0.014	−30.	50363	22765	1.8
50289	−78	1	0.012	−26.	50289	22750	1.8	50364	−89	1	0.014	−32.	50364	22769	1.8
50290	−77	1	0.012	−26.	50290	22751	1.8	50365	−90	1	0.014	−32.	50365	22770	1.8
50291	−78	1	0.012	−26.	50291	22752	1.8	50366	−89	1	0.014	−32.	50366	22771	1.8
50292	−75	1	0.011	−26.	50292	22755	1.8	50367	−90	1	0.014	−32.	50367	22772	1.8
50293	−74	1	0.011	−26.	50293	22756	1.8	50368	−94	1	0.014	−36.	50368	22768	1.8
50294	−73	1	0.011	−26.	50294	22757	1.8	50369	−95	1	0.014	−36.	50369	22769	1.8
50295	−72	1	0.011	−26.	50295	22758	1.8	50370	−98	1	0.015	−36.	50370	22772	1.8
50296	−71	1	0.011	−24.	50296	22759	1.8	50371	−97	1	0.015	−36.	50371	22773	1.8
50297	−72	1	0.011	−26.	50297	22760	1.8	50372	−102	1	0.015	−36.	50372	22778	1.8
50298	−71	1	0.011	−24.	50298	22761	1.8	50373	−100	1	0.015	−36.	50373	22776	1.8
50299	−70	1	0.0098	−24.	50299	22762	1.8	50374	−101	1	0.015	−36.	50374	22777	1.8
50300	−74	1	0.011	−26.	50300	22766	1.8	50375	−104	1	0.015	−36.	50375	22780	1.8
50301	−71	1	0.011	−24.	50301	22769	1.8	50376	−106	1	0.016	−36.	50376	22782	1.8
50302	−72	1	0.011	−26.	50302	22770	1.8	50377	−107	1	0.016	−36.	50377	22783	1.8
50303	−73	1	0.011	−26.	50303	22771	1.8	50378	−106	1	0.016	−36.	50378	22784	1.8
50304	−78	1	0.012	−26.	50304	22776	1.8	50379	−107	1	0.016	−36.	50379	22785	1.8
50305	−77	1	0.012	−26.	50305	22777	1.8	50380	−109	1	0.016	−40.	50380	22783	1.8
50306	−76	1	0.011	−26.	50306	22778	1.8	50381	−108	1	0.016	−36.	50381	22784	1.8
50307	−77	1	0.012	−26.	50307	22779	1.8	50382	−105	1	0.016	−36.	50382	22781	1.8
50308	−78	1	0.012	−26.	50308	22778	1.8	50383	−106	1	0.016	−36.	50383	22782	1.8
50309	−77	1	0.012	−26.	50309	22779	1.8	50384	−104	1	0.015	−36.	50384	22780	1.8
50310	−68	1	0.0098	−24.	50310	22770	1.8	50385	−105	1	0.016	−36.	50385	22781	1.8
50311	−69	1	0.0098	−24.	50311	22771	1.8	50386	−104	1	0.015	−36.	50386	22782	1.8
50312	−70	1	0.0098	−24.	50312	22772	1.8	50387	−105	1	0.016	−36.	50387	22783	1.8
50313	−71	1	0.011	−24.	50313	22773	1.8	50388	−102	1	0.015	−36.	50388	22780	1.8
50314	−72	1	0.011	−26.	50314	22774	1.8	50389	−101	1	0.015	−36.	50389	22781	1.8

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$	x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$
50390	−102	1	0.015	−36.	50390	22782	1.8	50465	−38	1	0.0054	−13.	50465	22824	1.8
50391	−100	1	0.015	−36.	50391	22780	1.8	50466	−37	1	0.0054	−13.	50466	22825	1.8
50392	−99	1	0.015	−36.	50392	22781	1.8	50467	−36	1	0.0054	−13.	50467	22826	1.8
50393	−100	1	0.015	−36.	50393	22782	1.8	50468	−37	1	0.0054	−13.	50468	22825	1.8
50394	−99	1	0.015	−36.	50394	22783	1.8	50469	−36	1	0.0054	−13.	50469	22826	1.8
50395	−98	1	0.015	−36.	50395	22784	1.8	50470	−39	1	0.0059	−13.	50470	22823	1.8
50396	−97	1	0.015	−36.	50396	22783	1.8	50471	−38	1	0.0054	−13.	50471	22824	1.8
50397	−98	1	0.015	−36.	50397	22784	1.8	50472	−32	1	0.0049	−12.	50472	22818	1.8
50398	−99	1	0.015	−36.	50398	22785	1.8	50473	−31	1	0.0049	−11.	50473	22819	1.8
50399	−98	1	0.015	−36.	50399	22786	1.8	50474	−30	1	0.0049	−11.	50474	22820	1.8
50400	−77	1	0.012	−26.	50400	22807	1.8	50475	−27	1	0.0039	−9.0	50475	22817	1.8
50401	−76	1	0.011	−26.	50401	22808	1.8	50476	−28	1	0.0039	−10.	50476	22816	1.8
50402	−75	1	0.011	−26.	50402	22809	1.8	50477	−27	1	0.0039	−9.0	50477	22817	1.8
50403	−76	1	0.011	−26.	50403	22810	1.8	50478	−26	1	0.0037	−9.0	50478	22818	1.8
50404	−77	1	0.012	−26.	50404	22809	1.8	50479	−27	1	0.0039	−9.0	50479	22819	1.8
50405	−78	1	0.012	−26.	50405	22810	1.8	50480	−24	1	0.0037	−9.0	50480	22816	1.8
50406	−77	1	0.012	−26.	50406	22811	1.8	50481	−22	1	0.0034	−8.0	50481	22814	1.8
50407	−78	1	0.012	−26.	50407	22812	1.8	50482	−23	1	0.0034	−8.0	50482	22815	1.8
50408	−77	1	0.012	−26.	50408	22813	1.8	50483	−22	1	0.0034	−8.0	50483	22816	1.8
50409	−75	1	0.011	−26.	50409	22815	1.8	50484	−25	1	0.0037	−9.0	50484	22813	1.8
50410	−73	1	0.011	−26.	50410	22813	1.8	50485	−26	1	0.0037	−9.0	50485	22814	1.8
50411	−74	1	0.011	−26.	50411	22814	1.8	50486	−25	1	0.0037	−9.0	50486	22815	1.8
50412	−72	1	0.011	−26.	50412	22812	1.8	50487	−24	1	0.0037	−9.0	50487	22816	1.8
50413	−71	1	0.011	−24.	50413	22813	1.8	50488	−23	1	0.0034	−8.0	50488	22817	1.8
50414	−67	1	0.0098	−24.	50414	22817	1.8	50489	−22	1	0.0034	−8.0	50489	22818	1.8
50415	−68	1	0.0098	−24.	50415	22818	1.8	50490	−15	1	0.0024	−5.5	50490	22811	1.8
50416	−66	1	0.0098	−24.	50416	22816	1.8	50491	−14	1	0.0020	−5.0	50491	22812	1.8
50417	−67	1	0.0098	−24.	50417	22817	1.8	50492	−13	1	0.0018	−4.5	50492	22811	1.8
50418	−65	1	0.0098	−24.	50418	22815	1.8	50493	−12	1	0.0018	−4.5	50493	22812	1.8
50419	−64	1	0.0098	−24.	50419	22816	1.8	50494	−11	1	0.0017	−4.0	50494	22813	1.8
50420	−62	1	0.0098	−22.	50420	22814	1.8	50495	−10	1	0.0015	−3.8	50495	22814	1.8
50421	−56	1	0.0078	−20.	50421	22820	1.8	50496	−5	1	0.00073	−1.9	50496	22809	1.8
50422	−57	1	0.0078	−20.	50422	22821	1.8	50497	−6	1	0.00092	−2.2	50497	22810	1.8
50423	−58	1	0.0078	−20.	50423	22822	1.8	50498	−7	1	0.00098	−2.5	50498	22811	1.8
50424	−57	1	0.0078	−20.	50424	22823	1.8	50499	−5	1	0.00073	−1.9	50499	22809	1.8
50425	−60	1	0.0098	−22.	50425	22820	1.8	50500	−1	1	0.00015	−0.38	50500	22805	1.8
50426	−61	1	0.0098	−22.	50426	22821	1.8	50501	0	0	0.00	0.00	50501	22806	1.8
50427	−59	1	0.0098	−22.	50427	22819	1.8	50502	1	−1	−0.00015	0.38	50502	22807	1.8
50428	−58	1	0.0078	−20.	50428	22818	1.8	50503	0	0	0.00	0.00	50503	22808	1.8
50429	−57	1	0.0078	−20.	50429	22819	1.8	50504	−1	1	0.00015	−0.38	50504	22809	1.8
50430	−60	1	0.0098	−22.	50430	22816	1.8	50505	−2	1	0.00031	−0.75	50505	22810	1.8
50431	−61	1	0.0098	−22.	50431	22817	1.8	50506	−1	1	0.00015	−0.38	50506	22811	1.8
50432	−65	1	0.0098	−24.	50432	22813	1.8	50507	0	0	0.00	0.00	50507	22812	1.8
50433	−64	1	0.0098	−24.	50433	22814	1.8	50508	4	−1	−0.00061	1.5	50508	22816	1.8
50434	−65	1	0.0098	−24.	50434	22815	1.8	50509	5	−1	−0.00073	1.9	50509	22817	1.8
50435	−64	1	0.0098	−24.	50435	22816	1.8	50510	4	−1	−0.00061	1.5	50510	22818	1.8
50436	−60	1	0.0098	−22.	50436	22812	1.8	50511	3	−1	−0.00046	1.1	50511	22819	1.8
50437	−59	1	0.0098	−22.	50437	22813	1.8	50512	3	−1	−0.00046	1.1	50512	22819	1.8
50438	−58	1	0.0078	−20.	50438	22814	1.8	50513	2	−1	−0.00031	0.75	50513	22820	1.8
50439	−57	1	0.0078	−20.	50439	22815	1.8	50514	1	−1	−0.00015	0.38	50514	22821	1.8
50440	−55	1	0.0078	−20.	50440	22817	1.8	50515	2	−1	−0.00031	0.75	50515	22822	1.8
50441	−56	1	0.0078	−20.	50441	22818	1.8	50516	3	−1	−0.00046	1.1	50516	22821	1.8
50442	−54	1	0.0078	−18.	50442	22820	1.8	50517	5	−1	−0.00073	1.9	50517	22823	1.8
50443	−53	1	0.0078	−18.	50443	22821	1.8	50518	6	−1	−0.00092	2.2	50518	22824	1.8
50444	−54	1	0.0078	−18.	50444	22820	1.8	50519	3	−1	−0.00046	1.1	50519	22821	1.8
50445	−56	1	0.0078	−20.	50445	22818	1.8	50520	8	−1	−0.0012	3.0	50520	22826	1.8
50446	−57	1	0.0078	−20.	50446	22819	1.8	50521	9	−1	−0.0013	3.2	50521	22827	1.8
50447	−56	1	0.0078	−20.	50447	22820	1.8	50522	10	−1	−0.0015	3.8	50522	22828	1.8
50448	−53	1	0.0078	−18.	50448	22817	1.8	50523	9	−1	−0.0013	3.2	50523	22829	1.8
50449	−52	1	0.0073	−18.	50449	22818	1.8	50524	10	−1	−0.0015	3.8	50524	22828	1.8
50450	−49	1	0.0073	−18.	50450	22815	1.8	50525	13	−1	−0.0018	4.5	50525	22825	1.8
50451	−50	1	0.0073	−18.	50451	22816	1.8	50526	10	−1	−0.0015	3.8	50526	22822	1.8
50452	−51	1	0.0073	−18.	50452	22815	1.8	50527	9	−1	−0.0013	3.2	50527	22823	1.8
50453	−50	1	0.0073	−18.	50453	22816	1.8	50528	11	−1	−0.0017	4.0	50528	22825	1.8
50454	−48	1	0.0073	−18.	50454	22814	1.8	50529	12	−1	−0.0018	4.5	50529	22826	1.8
50455	−47	1	0.0068	−18.	50455	22815	1.8	50530	13	−1	−0.0018	4.5	50530	22827	1.8
50456	−48	1	0.0073	−18.	50456	22814	1.8	50531	17	−1	−0.0024	6.0	50531	22831	1.8
50457	−46	1	0.0068	−16.	50457	22812	1.8	50532	19	−1	−0.0027	6.5	50532	22829	1.8
50458	−45	1	0.0068	−16.	50458	22813	1.8	50533	20	−1	−0.0029	7.5	50533	22830	1.8
50459	−46	1	0.0068	−16.	50459	22814	1.8	50534	19	−1	−0.0027	6.5	50534	22831	1.8
50460	−38	1	0.0054	−13.	50460	22822	1.8	50535	21	−1	−0.0034	7.5	50535	22829	1.8
50461	−39	1	0.0059	−13.	50461	22823	1.8	50536	22	−1	−0.0034	8.0	50536	22830	1.8
50462	−40	1	0.0059	−15.	50462	22824	1.8	50537	23	−1	−0.0034	8.0	50537	22831	1.8
50463	−37	1	0.0054	−13.	50463	22821	1.8	50538	22	−1	−0.0034	8.0	50538	22832	1.8
50464	−39	1	0.0059	−13.	50464	22823	1.8	50539	21	−1	−0.0034	7.5	50539	22833	1.8

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L^*_{\approx}(x)}$
50540	24	−1	−0.0037	9.0	50540	22836	1.8
50541	23	−1	−0.0034	8.0	50541	22837	1.8
50542	22	−1	−0.0034	8.0	50542	22838	1.8
50543	21	−1	−0.0034	7.5	50543	22839	1.8
50544	31	−1	−0.0049	11.	50544	22829	1.8
50545	30	−1	−0.0049	11.	50545	22830	1.8
50546	29	−1	−0.0039	10.	50546	22831	1.8
50547	30	−1	−0.0049	11.	50547	22832	1.8
50548	29	−1	−0.0039	10.	50548	22831	1.8
50549	28	−1	−0.0039	10.	50549	22832	1.8
50550	23	−1	−0.0034	8.0	50550	22827	1.8
50551	22	−1	−0.0034	8.0	50551	22828	1.8
50552	21	−1	−0.0034	7.5	50552	22829	1.8
50553	23	−1	−0.0034	8.0	50553	22827	1.8
50554	24	−1	−0.0037	9.0	50554	22828	1.8
50555	25	−1	−0.0037	9.0	50555	22829	1.8
50556	24	−1	−0.0037	9.0	50556	22828	1.8
50557	25	−1	−0.0037	9.0	50557	22829	1.8
50558	24	−1	−0.0037	9.0	50558	22830	1.8
50559	23	−1	−0.0034	8.0	50559	22831	1.8
50560	26	−1	−0.0037	9.0	50560	22828	1.8
50561	25	−1	−0.0037	9.0	50561	22829	1.8
50562	19	−1	−0.0027	6.5	50562	22835	1.8
50563	20	−1	−0.0029	7.5	50563	22836	1.8
50564	19	−1	−0.0027	6.5	50564	22835	1.8
50565	18	−1	−0.0027	6.5	50565	22836	1.8
50566	17	−1	−0.0024	6.0	50566	22837	1.8
50567	18	−1	−0.0027	6.5	50567	22838	1.8
50568	15	−1	−0.0024	5.5	50568	22835	1.8
50569	16	−1	−0.0024	6.0	50569	22836	1.8
50570	17	−1	−0.0024	6.0	50570	22837	1.8
50571	19	−1	−0.0027	6.5	50571	22839	1.8
50572	20	−1	−0.0029	7.5	50572	22838	1.8
50573	21	−1	−0.0034	7.5	50573	22839	1.8
50574	20	−1	−0.0029	7.5	50574	22840	1.8
50575	11	−1	−0.0017	4.0	50575	22849	1.8
50576	13	−1	−0.0018	4.5	50576	22847	1.8
50577	12	−1	−0.0018	4.5	50577	22848	1.8
50578	10	−1	−0.0015	3.8	50578	22850	1.8
50579	11	−1	−0.0017	4.0	50579	22851	1.8
50580	17	−1	−0.0024	6.0	50580	22857	1.8
50581	16	−1	−0.0024	6.0	50581	22858	1.8
50582	15	−1	−0.0024	5.5	50582	22859	1.8
50583	14	−1	−0.0020	5.0	50583	22860	1.8
50584	15	−1	−0.0024	5.5	50584	22861	1.8
50585	14	−1	−0.0020	5.0	50585	22862	1.8
50586	13	−1	−0.0018	4.5	50586	22863	1.8
50587	12	−1	−0.0018	4.5	50587	22864	1.8
50588	11	−1	−0.0017	4.0	50588	22863	1.8
50589	9	−1	−0.0013	3.2	50589	22861	1.8
50590	8	−1	−0.0012	3.0	50590	22862	1.8
50591	7	−1	−0.00098	2.5	50591	22863	1.8
50592	5	−1	−0.00073	1.9	50592	22861	1.8
50593	4	−1	−0.00061	1.5	50593	22862	1.8
50594	3	−1	−0.00046	1.1	50594	22863	1.8
50595	2	−1	−0.00031	0.75	50595	22864	1.8
50596	−2	1	0.00031	−0.75	50596	22860	1.8
50597	−1	1	0.00015	−0.38	50597	22861	1.8
50598	−3	1	0.00046	−1.1	50598	22863	1.8
50599	−4	1	0.00061	−1.5	50599	22864	1.8
50600	−10	1	0.0015	−3.8	50600	22858	1.8
50601	−11	1	0.0017	−4.0	50601	22859	1.8
50602	−10	1	0.0015	−3.8	50602	22860	1.8
50603	−9	1	0.0013	−3.2	50603	22861	1.8
50604	−7	1	0.00098	−2.5	50604	22859	1.8
50605	−8	1	0.0012	−3.0	50605	22860	1.8
50606	−7	1	0.00098	−2.5	50606	22861	1.8
50607	−9	1	0.0013	−3.2	50607	22859	1.8
50608	−11	1	0.0017	−4.0	50608	22857	1.8
50609	−12	1	0.0018	−4.5	50609	22858	1.8
50610	−14	1	0.0020	−5.0	50610	22860	1.8
50611	−15	1	0.0024	−5.5	50611	22861	1.8
50612	−16	1	0.0024	−6.0	50612	22860	1.8
50613	−15	1	0.0024	−5.5	50613	22861	1.8
50614	−14	1	0.0020	−5.0	50614	22862	1.8

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L^*_{\approx}(x)}$
50615	−15	1	0.0024	−5.5	50615	22863	1.8
50616	−22	1	0.0034	−8.0	50616	22856	1.8
50617	−25	1	0.0037	−9.0	50617	22853	1.8
50618	−24	1	0.0037	−9.0	50618	22854	1.8
50619	−25	1	0.0037	−9.0	50619	22855	1.8
50620	−23	1	0.0034	−8.0	50620	22853	1.8
50621	−22	1	0.0034	−8.0	50621	22854	1.8
50622	−23	1	0.0034	−8.0	50622	22855	1.8
50623	−24	1	0.0037	−9.0	50623	22856	1.8
50624	−27	1	0.0039	−9.0	50624	22859	1.8
50625	208	−1	−0.029	72.	50625	23094	1.8
50626	−16	1	0.0024	−6.0	50626	23318	1.9
50627	−17	1	0.0024	−6.0	50627	23319	1.9
50628	−15	1	0.0024	−5.5	50628	23317	1.9
50629	−14	1	0.0020	−5.0	50629	23318	1.9
50630	−13	1	0.0018	−4.5	50630	23319	1.9
50631	−14	1	0.0020	−5.0	50631	23320	1.9
50632	−13	1	0.0018	−4.5	50632	23321	1.9
50633	−12	1	0.0018	−4.5	50633	23322	1.9
50634	−14	1	0.0020	−5.0	50634	23320	1.9
50635	−13	1	0.0018	−4.5	50635	23321	1.9
50636	−14	1	0.0020	−5.0	50636	23320	1.9
50637	−13	1	0.0018	−4.5	50637	23321	1.9
50638	−14	1	0.0020	−5.0	50638	23322	1.9
50639	−13	1	0.0018	−4.5	50639	23323	1.9
50640	−18	1	0.0027	−6.5	50640	23318	1.9
50641	−17	1	0.0024	−6.0	50641	23319	1.9
50642	−16	1	0.0024	−6.0	50642	23320	1.9
50643	−14	1	0.0020	−5.0	50643	23318	1.9
50644	−13	1	0.0018	−4.5	50644	23317	1.9
50645	−14	1	0.0020	−5.0	50645	23318	1.9
50646	−13	1	0.0018	−4.5	50646	23319	1.9
50647	−14	1	0.0020	−5.0	50647	23320	1.9
50648	−15	1	0.0024	−5.5	50648	23321	1.9
50649	−14	1	0.0020	−5.0	50649	23322	1.9
50650	−11	1	0.0017	−4.0	50650	23319	1.9
50651	−12	1	0.0018	−4.5	50651	23320	1.9
50652	−17	1	0.0024	−6.0	50652	23315	1.9
50653	−20	1	0.0029	−7.5	50653	23318	1.9
50654	−19	1	0.0027	−6.5	50654	23319	1.9
50655	−18	1	0.0027	−6.5	50655	23320	1.9
50656	−16	1	0.0024	−6.0	50656	23322	1.9
50657	−15	1	0.0024	−5.5	50657	23323	1.9
50658	−16	1	0.0024	−6.0	50658	23324	1.9
50659	−15	1	0.0024	−5.5	50659	23325	1.9
50660	−17	1	0.0024	−6.0	50660	23323	1.9
50661	−15	1	0.0024	−5.5	50661	23321	1.9
50662	−16	1	0.0024	−6.0	50662	23322	1.9
50663	−15	1	0.0024	−5.5	50663	23323	1.9
50664	−17	1	0.0024	−6.0	50664	23325	1.9
50665	−16	1	0.0024	−6.0	50665	23326	1.9
50666	−19	1	0.0027	−6.5	50666	23323	1.9
50667	−18	1	0.0027	−6.5	50667	23324	1.9
50668	−17	1	0.0024	−6.0	50668	23323	1.9
50669	−16	1	0.0024	−6.0	50669	23324	1.9
50670	−19	1	0.0027	−6.5	50670	23321	1.9
50671	−20	1	0.0029	−7.5	50671	23322	1.9
50672	−22	1	0.0034	−8.0	50672	23320	1.9
50673	−21	1	0.0034	−7.5	50673	23321	1.9
50674	−22	1	0.0034	−8.0	50674	23322	1.9
50675	−25	1	0.0037	−9.0	50675	23319	1.9
50676	−27	1	0.0039	−9.0	50676	23317	1.9
50677	−28	1	0.0039	−10.	50677	23318	1.9
50678	−27	1	0.0039	−9.0	50678	23319	1.9
50679	−25	1	0.0037	−9.0	50679	23321	1.9
50680	−25	1	0.0037	−9.0	50680	23321	1.9
50681	−24	1	0.0037	−9.0	50681	23322	1.9
50682	−25	1	0.0037	−9.0	50682	23323	1.9
50683	−26	1	0.0037	−9.0	50683	23324	1.9
50684	−27	1	0.0039	−9.0	50684	23323	1.9
50685	−26	1	0.0037	−9.0	50685	23324	1.9
50686	−25	1	0.0037	−9.0	50686	23325	1.9
50687	−26	1	0.0037	−9.0	50687	23326	1.9
50688	−3	1	0.00046	−1.1	50688	23303	1.9
50689	−2	1	0.00031	−0.75	50689	23304	1.9

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L^*_{\approx}(x)}$	x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L^*_{\approx}(x)}$
50690	−1	1	0.00015	−0.38	50690	23305	1.9	50765	9	−1	−0.0013	3.2	50765	23339	1.9
50691	−2	1	0.00031	−0.75	50691	23306	1.9	50766	8	−1	−0.0012	3.0	50766	23340	1.9
50692	−3	1	0.00046	−1.1	50692	23305	1.9	50767	7	−1	−0.00098	2.5	50767	23341	1.9
50693	−2	1	0.00031	−0.75	50693	23306	1.9	50768	9	−1	−0.0013	3.2	50768	23339	1.9
50694	−4	1	0.00061	−1.5	50694	23308	1.9	50769	7	−1	−0.00098	2.5	50769	23337	1.9
50695	−3	1	0.00046	−1.1	50695	23309	1.9	50770	6	−1	−0.00085	2.2	50770	23338	1.9
50696	−2	1	0.00031	−0.75	50696	23310	1.9	50771	7	−1	−0.00098	2.5	50771	23339	1.9
50697	0	0	0.00	0.00	50697	23308	1.9	50772	9	−1	−0.0013	3.2	50772	23337	1.9
50698	1	−1	−0.00015	0.38	50698	23309	1.9	50773	8	−1	−0.0012	3.0	50773	23338	1.9
50699	−1	1	0.00015	−0.38	50699	23307	1.9	50774	7	−1	−0.00098	2.5	50774	23339	1.9
50700	−13	1	0.0018	−4.5	50700	23295	1.9	50775	10	−1	−0.0015	3.8	50775	23336	1.9
50701	−12	1	0.0018	−4.5	50701	23296	1.9	50776	9	−1	−0.0013	3.2	50776	23337	1.9
50702	−13	1	0.0018	−4.5	50702	23297	1.9	50777	8	−1	−0.0012	3.0	50777	23338	1.9
50703	−12	1	0.0018	−4.5	50703	23298	1.9	50778	14	−1	−0.0020	5.0	50778	23332	1.9
50704	−14	1	0.0020	−5.0	50704	23296	1.9	50779	13	−1	−0.0018	4.5	50779	23333	1.9
50705	−13	1	0.0018	−4.5	50705	23297	1.9	50780	15	−1	−0.0024	5.5	50780	23331	1.9
50706	−10	1	0.0015	−3.8	50706	23294	1.9	50781	16	−1	−0.0024	6.0	50781	23332	1.9
50707	−11	1	0.0017	−4.0	50707	23295	1.9	50782	17	−1	−0.0024	6.0	50782	23333	1.9
50708	−10	1	0.0015	−3.8	50708	23294	1.9	50783	18	−1	−0.0027	6.5	50783	23334	1.9
50709	−9	1	0.0013	−3.2	50709	23295	1.9	50784	31	−1	−0.0049	11.	50784	23321	1.9
50710	−8	1	0.0012	−3.0	50710	23296	1.9	50785	30	−1	−0.0049	11.	50785	23322	1.9
50711	−9	1	0.0013	−3.2	50711	23297	1.9	50786	29	−1	−0.0039	10.	50786	23323	1.9
50712	−11	1	0.0017	−4.0	50712	23299	1.9	50787	26	−1	−0.0037	9.0	50787	23326	1.9
50713	−12	1	0.0018	−4.5	50713	23300	1.9	50788	25	−1	−0.0037	9.0	50788	23325	1.9
50714	−11	1	0.0017	−4.0	50714	23301	1.9	50789	24	−1	−0.0034	9.0	50789	23326	1.9
50715	−4	1	0.00061	−1.5	50715	23308	1.9	50790	27	−1	−0.0039	9.0	50790	23329	1.9
50716	−3	1	0.00046	−1.1	50716	23307	1.9	50791	28	−1	−0.0039	10.	50791	23330	1.9
50717	−2	1	0.00031	−0.75	50717	23308	1.9	50792	29	−1	−0.0039	10.	50792	23329	1.9
50718	−1	1	0.00015	−0.38	50718	23309	1.9	50793	30	−1	−0.0049	11.	50793	23330	1.9
50719	0	0	0.00	0.00	50719	23310	1.9	50794	29	−1	−0.0039	10.	50794	23331	1.9
50720	−3	1	0.00046	−1.1	50720	23313	1.9	50795	30	−1	−0.0049	11.	50795	23332	1.9
50721	−2	1	0.00031	−0.75	50721	23314	1.9	50796	33	−1	−0.0049	12.	50796	23335	1.9
50722	−3	1	0.00046	−1.1	50722	23315	1.9	50797	34	−1	−0.0049	12.	50797	23336	1.9
50723	−4	1	0.00061	−1.5	50723	23316	1.9	50798	33	−1	−0.0049	12.	50798	23337	1.9
50724	−8	1	0.0012	−3.0	50724	23320	1.9	50799	34	−1	−0.0049	12.	50799	23338	1.9
50725	−11	1	0.0017	−4.0	50725	23317	1.9	50800	25	−1	−0.0037	9.0	50800	23347	1.9
50726	−12	1	0.0018	−4.5	50726	23318	1.9	50801	26	−1	−0.0037	9.0	50801	23348	1.9
50727	−13	1	0.0018	−4.5	50727	23319	1.9	50802	25	−1	−0.0037	9.0	50802	23349	1.9
50728	−14	1	0.0020	−5.0	50728	23320	1.9	50803	26	−1	−0.0037	9.0	50803	23350	1.9
50729	−13	1	0.0018	−4.5	50729	23321	1.9	50804	27	−1	−0.0039	9.0	50804	23349	1.9
50730	−16	1	0.0024	−6.0	50730	23324	1.9	50805	29	−1	−0.0039	10.	50805	23347	1.9
50731	−15	1	0.0024	−5.5	50731	23325	1.9	50806	30	−1	−0.0049	11.	50806	23348	1.9
50732	−14	1	0.0020	−5.0	50732	23324	1.9	50807	25	−1	−0.0037	9.0	50807	23343	1.9
50733	−12	1	0.0018	−4.5	50733	23326	1.9	50808	27	−1	−0.0039	9.0	50808	23345	1.9
50734	−11	1	0.0017	−4.0	50734	23327	1.9	50809	26	−1	−0.0037	9.0	50809	23346	1.9
50735	−12	1	0.0018	−4.5	50735	23328	1.9	50810	25	−1	−0.0037	9.0	50810	23347	1.9
50736	−14	1	0.0020	−5.0	50736	23326	1.9	50811	26	−1	−0.0037	9.0	50811	23348	1.9
50737	−13	1	0.0018	−4.5	50737	23327	1.9	50812	25	−1	−0.0037	9.0	50812	23347	1.9
50738	−14	1	0.0020	−5.0	50738	23328	1.9	50813	28	−1	−0.0039	10.	50813	23344	1.9
50739	−15	1	0.0024	−5.5	50739	23329	1.9	50814	26	−1	−0.0037	9.0	50814	23346	1.9
50740	−17	1	0.0024	−6.0	50740	23327	1.9	50815	27	−1	−0.0039	9.0	50815	23347	1.9
50741	−18	1	0.0027	−6.5	50741	23328	1.9	50816	31	−1	−0.0049	11.	50816	23351	1.9
50742	−16	1	0.0024	−6.0	50742	23326	1.9	50817	30	−1	−0.0049	11.	50817	23352	1.9
50743	−17	1	0.0024	−6.0	50743	23327	1.9	50818	31	−1	−0.0049	11.	50818	23353	1.9
50744	−16	1	0.0024	−6.0	50744	23328	1.9	50819	32	−1	−0.0049	12.	50819	23354	1.9
50745	−15	1	0.0024	−5.5	50745	23329	1.9	50820	25	−1	−0.0037	9.0	50820	23361	1.9
50746	−14	1	0.0020	−5.0	50746	23330	1.9	50821	24	−1	−0.0034	9.0	50821	23362	1.9
50747	−13	1	0.0018	−4.5	50747	23331	1.9	50822	25	−1	−0.0037	9.0	50822	23363	1.9
50748	−11	1	0.0017	−4.0	50748	23329	1.9	50823	23	−1	−0.0034	8.0	50823	23361	1.9
50749	−10	1	0.0015	−3.8	50749	23330	1.9	50824	24	−1	−0.0034	9.0	50824	23362	1.9
50750	−7	1	0.00098	−2.5	50750	23333	1.9	50825	27	−1	−0.0039	9.0	50825	23359	1.9
50751	−9	1	0.0013	−3.2	50751	23331	1.9	50826	28	−1	−0.0039	10.	50826	23360	1.9
50752	−5	1	0.00073	−1.9	50752	23327	1.9	50827	27	−1	−0.0039	9.0	50827	23361	1.9
50753	−6	1	0.00085	−2.2	50753	23328	1.9	50828	28	−1	−0.0039	10.	50828	23360	1.9
50754	−5	1	0.00073	−1.9	50754	23329	1.9	50829	29	−1	−0.0039	10.	50829	23361	1.9
50755	−4	1	0.00061	−1.5	50755	23330	1.9	50830	28	−1	−0.0039	10.	50830	23362	1.9
50756	−5	1	0.00073	−1.9	50756	23329	1.9	50831	29	−1	−0.0039	10.	50831	23363	1.9
50757	−6	1	0.00085	−2.2	50757	23330	1.9	50832	20	−1	−0.0029	7.5	50832	23372	1.9
50758	−7	1	0.00098	−2.5	50758	23331	1.9	50833	19	−1	−0.0027	6.5	50833	23373	1.9
50759	−6	1	0.00085	−2.2	50759	23332	1.9	50834	18	−1	−0.0027	6.5	50834	23374	1.9
50760	2	−1	−0.00031	0.75	50760	23340	1.9	50835	17	−1	−0.0024	6.0	50835	23375	1.9
50761	3	−1	−0.00043	1.1	50761	23341	1.9	50836	18	−1	−0.0027	6.5	50836	23374	1.9
50762	2	−1	−0.00031	0.75	50762	23342	1.9	50837	19	−1	−0.0027	6.5	50837	23375	1.9
50763	3	−1	−0.00043	1.1	50763	23343	1.9	50838	20	−1	−0.0029	7.5	50838	23376	1.9
50764	8	−1	−0.0012	3.0	50764	23338	1.9	50839	19	−1	−0.0027	6.5	50839	23377	1.9

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L^*_{\approx}(x)}$	x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L^*_{\approx}(x)}$
50840	20	−1	−0.0029	7.5	50840	23378	1.9	50915	−11	1	0.0017	−4.0	50915	23417	1.9
50841	18	−1	−0.0027	6.5	50841	23380	1.9	50916	−9	1	0.0013	−3.2	50916	23415	1.9
50842	17	−1	−0.0024	6.0	50842	23381	1.9	50917	−8	1	0.0012	−3.0	50917	23416	1.9
50843	18	−1	−0.0027	6.5	50843	23382	1.9	50918	−9	1	0.0013	−3.2	50918	23417	1.9
50844	16	−1	−0.0024	6.0	50844	23380	1.9	50919	−10	1	0.0015	−3.8	50919	23418	1.9
50845	17	−1	−0.0024	6.0	50845	23381	1.9	50920	−11	1	0.0017	−4.0	50920	23417	1.9
50846	18	−1	−0.0027	6.5	50846	23382	1.9	50921	−10	1	0.0015	−3.8	50921	23418	1.9
50847	17	−1	−0.0024	6.0	50847	23383	1.9	50922	−8	1	0.0012	−3.0	50922	23420	1.9
50848	17	−1	−0.0024	6.0	50848	23383	1.9	50923	−9	1	0.0013	−3.2	50923	23421	1.9
50849	16	−1	−0.0024	6.0	50849	23384	1.9	50924	−8	1	0.0012	−3.0	50924	23420	1.9
50850	27	−1	−0.0039	9.0	50850	23395	1.9	50925	−11	1	0.0017	−4.0	50925	23417	1.9
50851	28	−1	−0.0039	10.	50851	23396	1.9	50926	−10	1	0.0015	−3.8	50926	23418	1.9
50852	27	−1	−0.0039	9.0	50852	23395	1.9	50927	−9	1	0.0013	−3.2	50927	23419	1.9
50853	28	−1	−0.0039	10.	50853	23396	1.9	50928	−6	1	0.00085	−2.2	50928	23416	1.9
50854	27	−1	−0.0039	9.0	50854	23397	1.9	50929	−7	1	0.00098	−2.5	50929	23417	1.9
50855	26	−1	−0.0037	9.0	50855	23398	1.9	50930	−6	1	0.00085	−2.2	50930	23418	1.9
50856	26	−1	−0.0037	9.0	50856	23398	1.9	50931	−8	1	0.0012	−3.0	50931	23416	1.9
50857	25	−1	−0.0037	9.0	50857	23399	1.9	50932	−9	1	0.0013	−3.2	50932	23415	1.9
50858	24	−1	−0.0034	9.0	50858	23400	1.9	50933	−15	1	0.0024	−5.5	50933	23409	1.9
50859	22	−1	−0.0034	8.0	50859	23398	1.9	50934	−14	1	0.0020	−5.0	50934	23410	1.9
50860	24	−1	−0.0034	9.0	50860	23396	1.9	50935	−15	1	0.0024	−5.5	50935	23411	1.9
50861	25	−1	−0.0037	9.0	50861	23397	1.9	50936	−14	1	0.0020	−5.0	50936	23412	1.9
50862	21	−1	−0.0034	7.5	50862	23393	1.9	50937	−13	1	0.0018	−4.5	50937	23413	1.9
50863	22	−1	−0.0034	8.0	50863	23394	1.9	50938	−12	1	0.0017	−4.5	50938	23414	1.9
50864	15	−1	−0.0024	5.5	50864	23401	1.9	50939	−13	1	0.0018	−4.5	50939	23415	1.9
50865	14	−1	−0.0020	5.0	50865	23402	1.9	50940	−7	1	0.00098	−2.5	50940	23421	1.9
50866	13	−1	−0.0018	4.5	50866	23403	1.9	50941	−9	1	0.0013	−3.2	50941	23419	1.9
50867	12	−1	−0.0017	4.5	50867	23404	1.9	50942	−8	1	0.0012	−3.0	50942	23420	1.9
50868	5	−1	−0.00073	1.9	50868	23411	1.9	50943	−7	1	0.00098	−2.5	50943	23421	1.9
50869	9	−1	−0.0013	3.2	50869	23407	1.9	50944	−11	1	0.0017	−4.0	50944	23417	1.9
50870	8	−1	−0.0012	3.0	50870	23408	1.9	50945	−12	1	0.0017	−4.5	50945	23418	1.9
50871	7	−1	−0.00098	2.5	50871	23409	1.9	50946	−10	1	0.0015	−3.8	50946	23420	1.9
50872	8	−1	−0.0012	3.0	50872	23410	1.9	50947	−9	1	0.0013	−3.2	50947	23421	1.9
50873	7	−1	−0.00098	2.5	50873	23411	1.9	50948	−8	1	0.0012	−3.0	50948	23420	1.9
50874	8	−1	−0.0012	3.0	50874	23412	1.9	50949	−5	1	0.00073	−1.9	50949	23417	1.9
50875	5	−1	−0.00073	1.9	50875	23415	1.9	50950	−2	1	0.00031	−0.75	50950	23414	1.9
50876	4	−1	−0.00061	1.5	50876	23414	1.9	50951	−3	1	0.00043	−1.1	50951	23415	1.9
50877	2	−1	−0.00031	0.75	50877	23412	1.9	50952	−2	1	0.00031	−0.75	50952	23416	1.9
50878	3	−1	−0.00043	1.1	50878	23413	1.9	50953	−3	1	0.00043	−1.1	50953	23417	1.9
50879	4	−1	−0.00061	1.5	50879	23414	1.9	50954	−4	1	0.00061	−1.5	50954	23418	1.9
50880	−4	1	0.00061	−1.5	50880	23406	1.9	50955	−3	1	0.00043	−1.1	50955	23419	1.9
50881	−5	1	0.00073	−1.9	50881	23407	1.9	50956	−4	1	0.00061	−1.5	50956	23418	1.9
50882	−4	1	0.00061	−1.5	50882	23408	1.9	50957	−5	1	0.00073	−1.9	50957	23419	1.9
50883	−5	1	0.00073	−1.9	50883	23409	1.9	50958	−5	1	0.00073	−1.9	50958	23419	1.9
50884	−6	1	0.00085	−2.2	50884	23408	1.9	50959	−4	1	0.00061	−1.5	50959	23420	1.9
50885	−5	1	0.00073	−1.9	50885	23409	1.9	50960	4	−1	−0.00061	1.5	50960	23428	1.9
50886	−7	1	0.00098	−2.5	50886	23407	1.9	50961	5	−1	−0.00073	1.9	50961	23429	1.9
50887	−6	1	0.00085	−2.2	50887	23408	1.9	50962	4	−1	−0.00061	1.5	50962	23430	1.9
50888	−5	1	0.00073	−1.9	50888	23409	1.9	50963	3	−1	−0.00043	1.1	50963	23431	1.9
50889	−4	1	0.00061	−1.5	50889	23410	1.9	50964	1	−1	−0.00015	0.38	50964	23429	1.9
50890	−3	1	0.00043	−1.1	50890	23411	1.9	50965	2	−1	−0.00031	0.75	50965	23430	1.9
50891	−4	1	0.00061	−1.5	50891	23412	1.9	50966	1	−1	−0.00015	0.38	50966	23431	1.9
50892	−2	1	0.00031	−0.75	50892	23410	1.9	50967	3	−1	−0.00043	1.1	50967	23429	1.9
50893	−3	1	0.00043	−1.1	50893	23411	1.9	50968	2	−1	−0.00031	0.75	50968	23430	1.9
50894	−2	1	0.00031	−0.75	50894	23412	1.9	50969	1	−1	−0.00015	0.38	50969	23431	1.9
50895	0	0	0.00	0.00	50895	23414	1.9	50970	4	−1	−0.00061	1.5	50970	23434	1.9
50896	−2	1	0.00031	−0.75	50896	23412	1.9	50971	3	−1	−0.00043	1.1	50971	23435	1.9
50897	−3	1	0.00043	−1.1	50897	23413	1.9	50972	2	−1	−0.00031	0.75	50972	23434	1.9
50898	−2	1	0.00031	−0.75	50898	23414	1.9	50973	1	−1	−0.00015	0.38	50973	23435	1.9
50899	−1	1	0.00015	−0.38	50899	23415	1.9	50974	2	−1	−0.00031	0.75	50974	23436	1.9
50900	−5	1	0.00073	−1.9	50900	23419	1.9	50975	−1	1	0.00015	−0.38	50975	23433	1.9
50901	−1	1	0.00015	−0.38	50901	23415	1.9	50976	−10	1	0.0015	−3.8	50976	23442	1.9
50902	−2	1	0.00031	−0.75	50902	23416	1.9	50977	−9	1	0.0013	−3.2	50977	23443	1.9
50903	−1	1	0.00015	−0.38	50903	23417	1.9	50978	−10	1	0.0015	−3.8	50978	23444	1.9
50904	−6	1	0.00085	−2.2	50904	23412	1.9	50979	−9	1	0.0013	−3.2	50979	23445	1.9
50905	−5	1	0.00073	−1.9	50905	23413	1.9	50980	−7	1	0.00098	−2.5	50980	23443	1.9
50906	−4	1	0.00061	−1.5	50906	23414	1.9	50981	−6	1	0.00085	−2.2	50981	23444	1.9
50907	−5	1	0.00073	−1.9	50907	23415	1.9	50982	−5	1	0.00073	−1.9	50982	23445	1.9
50908	−6	1	0.00085	−2.2	50908	23414	1.9	50983	−4	1	0.00061	−1.5	50983	23446	1.9
50909	−7	1	0.00098	−2.5	50909	23415	1.9	50984	−3	1	0.00043	−1.1	50984	23447	1.9
50910	−4	1	0.00061	−1.5	50910	23418	1.9	50985	−5	1	0.00073	−1.9	50985	23445	1.9
50911	−7	1	0.00098	−2.5	50911	23415	1.9	50986	−4	1	0.00061	−1.5	50986	23446	1.9
50912	−9	1	0.0013	−3.2	50912	23417	1.9	50987	−3	1	0.00043	−1.1	50987	23447	1.9
50913	−11	1	0.0017	−4.0	50913	23415	1.9	50988	−6	1	0.00085	−2.2	50988	23444	1.9
50914	−10	1	0.0015	−3.8	50914	23416	1.9	50989	−7	1	0.00098	−2.5	50989	23445	1.9