generally relate the Mellin transform $s \cdot \mathcal{M}[S_f](-s)$ to the DGF of the sequence $f(n)$, cited, for example, as in [1, §11]. In essence, what the previous equation says is that the DGF of $\chi_{\mathbb{P}}$ is $P(s)$.

Now to show the equivalence of the prime indicator function with the Dirichlet convolution based expression, $\omega * \mu$, we consider the DGF of the right-hand-side function, $f(n) := (\mu * \omega)(n)$, as

$$D_f(s) = \frac{1}{\zeta(s)} \times \sum_{n \geq 1} \frac{\omega(n)}{n^s} = P(s),$$

where it is not difficult to prove that the DGF of $\omega(n)$ is $P(s) \cdot \zeta(s)$.

Thus for any $\Re(s) > 1$, the DGFs of each side of the claimed equation coincide. So by uniqueness of Dirichlet series, we see that in fact the claim holds. To obtain the full result, we add to each side of this equation a term of $\varepsilon(n) \equiv (\mu * 1)(n)$, and then factor the resulting convolution identity. □

When combined with Corollary 4.2, the proof of Proposition 5.1 yields the crucial starting point providing an exact formula for $M(x)$ stated in (1) of Corollary 4.3. Thus, while the formula in (1) is a key component utilized in our proof moving forward, we do not need to explicitly show that it holds for all $x \geq 1$ from this point.

Proposition 5.2 (The key signedness property of $g^{-1}(n)$)**.** *For the Dirichlet invertible function, $g(n) := \omega(n)+1$ defined such that $g(1) = 1$, at any $n \geq 1$, we have that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$. The notation for the operation given by $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0, \pm 1\}$ denotes the sign of the arithmetic function h at n.*

Proof. Let $D_f(s) := \sum_{n \geq 1} f(n) n^{-s}$ denote the Dirichlet generating function (DGF) of any arithmetic function $f(n)$ convergent for $\Re(s) > \sigma_f$. Using Proposition 5.1 and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of the original arithmetic function f, we can express the DGF of our particular $g^{-1}(n)$ explicitly as an analytic function of s for $\Re(s) > 1$. For all $\Re(s) > 1$, expanding the DGF for the function $g^{-1}(n)$ yields

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{4}$$

Let $h^{-1}(n) := (\omega * \mu + \varepsilon)^{-1}(n) = [n^{-s}](P(s)+1)^{-1}$. By the standard recurrence relation we used to define the Dirichlet inverse function of any arithmetic function f such that $f(1) \neq 0$ in the initial listing of notation for the article, we have that

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d \mid n \\ d > 1}} h(d) h^{-1}(n/d), & n \geq 2. \end{cases} \tag{5}$$

We also can see by the definition of $h^{-1}(n)$ we derived from the partial DGF term above, that with $h = \omega * \mu + \varepsilon$, we obtain

$$h(n) = \begin{cases} 1, & n = 1; \\ \chi_{\mathbb{P}}(n), & n \geq 2. \end{cases}$$

So for $n \geq 2$, the summands in (5) can be simply indexed over the primes $p|n$. This observation yields that we can inductively expand these sums into nested divisor sums provided the depth of the sums does not exceed the capacity to index summations over the primes dividing n. In particular, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p \mid n} h^{-1}(n/p) \cancel{= -\sum_{p \mid n} h^{-1}(p)}, & \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1 \mid n} \sum_{p_2 \mid \frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right) = \sum_{p_1 \mid n} \sum_{p_2 \mid \frac{n}{p_1}} h^{-1}(p_2), & \text{if } \Omega(n) \geq 2 \end{aligned}$$

??

$$h^{-1}(n) = -\sum_{p|n} h^{-1}(p/n)$$

implies $\quad h^{-1}(p^k) = -h^{-1}(p^{k-1}) = (-1)^k.$

$$\boxed{h^{-1}(n) = -\sum_{p|n} h^{-1}(p) \quad \text{implies} \quad h^{-1}(p) = -h^{-1}(p).}$$

$$h^{-1}(p^2) = -h^{-1}(p) = 1$$

$$h^{-1}(p^3) = -h^{-1}(p^2) = (-1)$$

$$h^{-1}(pq) = -\left(h^{-1}(p) + h^{-1}(q)\right) = 2$$

$$h^{-1}(p^k q) = -h^{-1}(p^{k-1}q) - h^{-1}(p^k)$$

$$= (-1)^{k+1} - h^{-1}(p^{k-1}q)$$

$$\vdots$$

$$= \Omega(n)\ \lambda(n)\ ??$$

Argument seems too complicated.

Recall that $D_1 = \zeta$, $D_\mu = \frac{1}{\zeta}$ & $D_\omega = P:\zeta$

Then

$$D_{(1+\omega)^{-1}} = \frac{1}{(1+P)\zeta}$$

It follows that

$$(1+\omega)^{-1} = \mu * h^{-1}$$

when $h = 1 + \chi_P$.

We show that $\mathrm{sgn}(h^{-1}) = \mathrm{sgn}(\lambda)$.

From this, it follows from inspection that

$$\mathrm{sgn}(\mu * h^{-1}) = \mathrm{sgn}(\lambda).$$

We have $h^{-1}(1)=1$. And from the standard formula for h^{-1} for $n>1$ there holds

$$h^{-1}(n) = -\sum_{\substack{d \mid n \\ d>1}} h(d)\, h^{-1}(n/d)$$

$$= -\sum \chi_p(d)\, h^{-1}(n/d)$$

$$= -\sum_{p \mid n} h^{-1}(n/p).$$

Repeat this for all the prime divisors of n to conclude that

$$h^{-1}(n) = \lambda(n)\, \Omega(n).$$