ASYMPTOTIC BOUNDS FOR THE MERTENS FUNCTION

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Abstract. The Mertens function is defined as the average order of the Möbius function, or as the summatory function $M(x) = \sum_{n \leq x} \mu(n)$, for all $x \geq 1$. There are many open problems are related to determining optimal asymptotic bounds for this function. The famous statement of Mertens' conjecture which says that $|M(x)| < \sqrt{x}$ has been disproved, though is it known that the Riemann Hypothesis is equivalent to showing that $|M(x)| \ll \sqrt{x} \exp\left(B \frac{\log x}{\log \log x}\right)$ for some constant B. Another unresolved problem related to this function is whether $\limsup_{x \to \infty} |M(x)|/\sqrt{x} = \infty$. In this article, we employ the recent construction of new formulas for the generalized sum-of-divisors functions proved by Schmidt to obtain new results which exactly sum the classical Mertens function for all finite x. We state and prove analogous results for the generalized Mertens function which we define to be $M_\alpha^*(x) = \sum_{n \leq x} n^\alpha \mu(n)$ for any fixed $\alpha \in \mathbb{C}$.

1. Introduction

1.1. Mertens summatory functions. The Mertens summatory function, or *Mertens function*, is defined as

$$M(x) = \sum_{n \leq x} \mu(n), \ x \geq 1,$$

where $\mu(n)$ denotes the Möbius function which is in some sense a signed indicator function for the squarefree integers. A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

We define the notion of a *generalized Mertens summatory function* for fixed $\alpha \in \mathbb{C}$ as

$$M_\alpha^*(x) = \sum_{n \leq x} n^\alpha \mu(n), \ x \geq 1,$$

where the special case of $M_0^*(x)$ corresponds to the definition of the classical Mertens function $M(x)$ defined above. The plots shown in Figure 1.1 illustrate the chaotic behavior of the growth of these functions for x in small intervals when $\alpha \in \{-1, 0, 1, 2\}$. In particular, there are many open problems related to bounding $M(x)$ for large x. The Riemann Hypothesis is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. It is still unresolved whether

$$\limsup_{x \to \infty} |M(x)|/\sqrt{x} = \infty,$$

although computational evidence suggests that this is a likely conjecture [5, 4]. We make a newly well-founded attempt to prove that this conjecture is true in Theorem 2.1.

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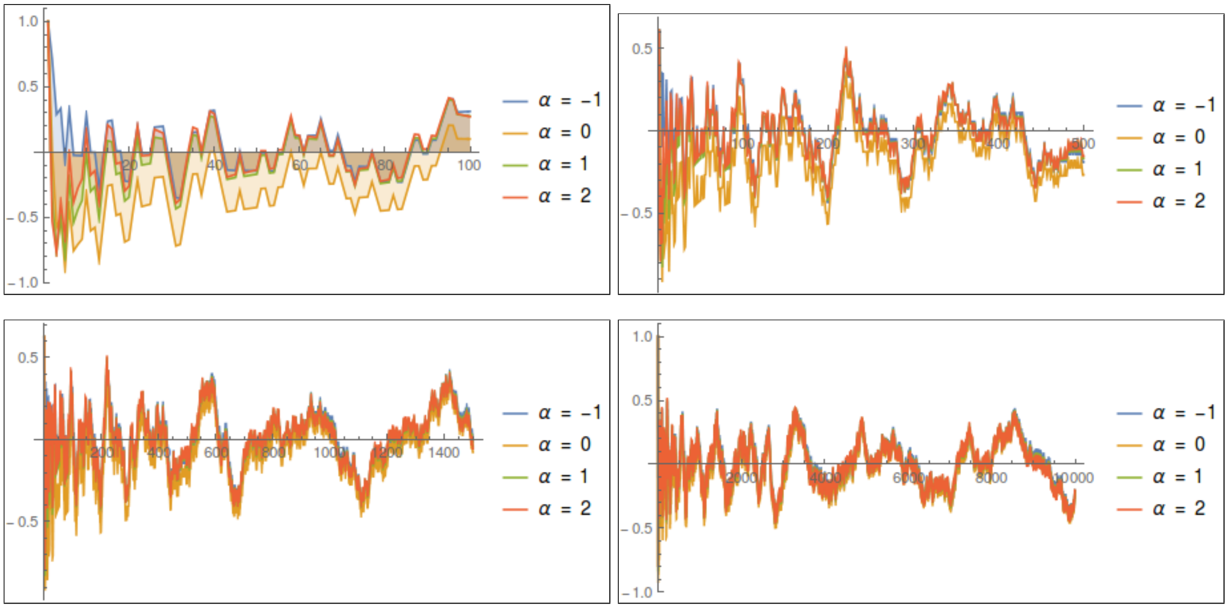


Figure 1.1. *Comparison of the Mertens Summatory Functions $M_\alpha(x)/x^{\frac{1}{2}+\alpha}$ for Small x and α*

1.2. **Exact formulas for the generalized sum-of-divisors functions.** Schmidt has recently proved (2017) several new exact formulas for the *generalized sum-of-divisors functions*, $\sigma_\alpha(x)$, defined for any $x \geq 1$ as

$$\sigma_\alpha(x) = \sum_{d|n} d^\alpha, \ \alpha \in \mathbb{C}.$$

In particular, if we let $H_n^{(r)} = \sum_{k=1}^n k^{-r}$ denote the sequence of *r-order harmonic numbers*, where [8, §2.4(iii)]

$$H_n^{(-t)} = \frac{B_{t+1}(n+1) - B_{t+1}}{(t+1)} = \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + \sum_{k=1}^{t-1} \binom{t}{k} \frac{B_{k+1} n^{t-k}}{(k+1)},$$

is a *Bernoulli polynomial* for any $n \geq 0$ when $t \in \mathbb{N}$, then we can restate the next theorem from [10]. Within this article we assume that an index of summation p denotes that the sum is taken over only prime values of p. We also use the notation that the valuation function

$$\nu_p(x) = m \quad \text{if and only if} \quad p^m \| x,$$

to denote the exact exponent of the prime p dividing x.

Theorem 1.1 (Schmidt, 2017)**.** *For any fixed $\alpha \in \mathbb{C}$ and all $x \geq 1$, we have that*

$$\begin{aligned}\sigma_\alpha(x) = H_x^{(1-\alpha)} + \sum_{d|n} \tau_x^{(\alpha)}(d) + \sum_{2 \leq p \leq x} \sum_{k=1}^{\varepsilon_p(x)+1} p^{\alpha k} H_{\left\lfloor \frac{x}{p^k} \right\rfloor}^{(1-\alpha)} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x - p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right) \\ + \sum_{3 \leq p \leq x} \sum_{k=1}^{\varepsilon_p(x)+1} 2^{\alpha-1} p^{\alpha k} (-1)^{\lfloor x/p^{k-1} \rfloor} H_{\left\lfloor \frac{x}{2p^k} \right\rfloor}^{(1-\alpha)} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x - p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right),\end{aligned}$$

where the divisor sum over the function $\tau_x^{(\alpha)}(d)$ is defined precisely by Lemma 2.4.

Remark 1.2 (Restatement of the Theorem)**.** For $x \geq 1$ and fixed $\alpha \in \mathbb{C}$, we define the sums

$$\begin{aligned} S_1^{(\alpha)}(x) &= \sum_{2 \leq p \leq x} \sum_{k=1}^{\varepsilon_p(x)+1} p^{\alpha k} H_{\left\lfloor \frac{x}{p^k} \right\rfloor}^{(1-\alpha)} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right) \\ S_2^{(\alpha)}(x) &= \sum_{3 \leq p \leq x} \sum_{k=1}^{\varepsilon_p(x)+1} 2^{\alpha-1} p^{\alpha k} (-1)^{\lfloor x/p^{k-1} \rfloor} H_{\left\lfloor \frac{x}{2p^k} \right\rfloor}^{(1-\alpha)} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right). \end{aligned}$$

Then we prefer to work with the next form of Theorem 1.1 stated in terms of our new shorthand sum functions as follows:

$$\left| \sum_{d|x} \tau_x^{(\alpha)}(d) \right| = \left| H_x^{(1-\alpha)} - \sigma_\alpha(x) + S_1^{(\alpha)}(x) + S_2^{(\alpha)}(x) \right|. \tag{1}$$

The statement of the theorem given in (1) is important and significant since it implies deep connections between the sum-of-divisors functions, the generalized Mertens summatory functions, and the partial sums of the Riemann zeta function for real $\alpha < 0$, each related to one another in a convolved formula taken over sums of successive powers of the primes $p \leq x$. Thus we immediately see new relations from the restatement of the key results in [10] above. Moreover, from the previous result, we then obtain our main new results in the article given in the results in the next section as consequences of this restatement in terms of the Mertens functions.

2. New results and proofs of key lemmas

2.1. **Statement of the main theorem.**

Theorem 2.1 (The Limit Supremum of $M(x)$ and Its Values at Large Prime Powers)**.** *Let $x = q^r$ denote a large odd prime power for some $r \geq 4$. Then we have that*

$$\limsup_{\substack{x \to \infty \\ x = q^r}} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

Proof (Sketch). The complete proof of the theorem is given at conclusion of this section. For now, we will elaborate on the key steps in proving the theorem. We begin by noting that

$$\begin{aligned} \left| \sum_{d|x} \tau_x^{(\alpha+1)}(d) \right| &= \left| \sum_{s \in S_x} \sum_{d \mid \gcd(s,x)} s^\alpha \cdot \mu(s/d) \cdot d \right| \\ &= \left| \sum_{m=1}^{x} \mu(m) m^\alpha \underbrace{\left(\sum_{s \in S_x} \sum_{d|(s,x)} d^{\alpha+1} \left[m = \frac{s}{d} \right]_\delta \right)}_{:= d_x^{(\alpha)}(m)} \right| \\ &\leq \left(2 \cdot \sup_{1 \leq i \leq x} |M_\alpha^*(i)| + 1 \right) \times \sum_{m=1}^{x} \left| d_x^{(\alpha)}(m) \right| - 2 \cdot M_\alpha^*(1) d_x^{(\alpha)}(1), \end{aligned}$$

where the upper bound is obtained by summation by parts. We then need to show that infinitely and predictably often at least (and not necessarily for all large x) that we can bound the ratio of the next sums by $x \log \log x$. We consider the cases of large x when $x := q^r$ is a large prime power for some $r \geq 4$ and employ the resulting expansions to complete our proof. The next step in the proof is to show that (1) is approximately

$$\begin{aligned}
\left|\sum_{d|x} \tau_x^{(\alpha+1)}(d)\right| &= \left|H_x^{(1-\alpha)} - \sigma_\alpha(x) + S_1^{(\alpha)}(x) + S_2^{(\alpha)}(x)\right| \\
&\geq \left|\frac{(C_1 - C_2C_4)(x-1)}{\log(x-1)} + \left(\frac{C_2}{2} - 1\right) x \left(\log\log(x-1) + A\right) + \frac{r}{2} x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1\right) \right. \\
&\qquad \left. - \frac{C_3}{4} x + C_3 C_6 - \frac{x-1}{x^{\frac{1}{r}} - 1}\right| \\
&\sim \frac{\widetilde{C}_1 \cdot x}{\log x} + \widetilde{C}_2 \cdot x \log\log x.
\end{aligned}$$

We then combine this asymptotic relation resulting from Theorem 1.1 when $x = q^r$, noting that there are infinitely many such large prime powers from which we may choose, to obtain the bound that

$$0 \leq f_M(x) \leq \sup_{1 \leq i \leq x} |M(i)|,$$

where for large $x = q^r$ of the appropriate form with q and odd prime and $r \geq 4$, we have that

$$\frac{f_M(x)}{\sqrt{x}} \sim \widetilde{C} \cdot x^{\frac{r-4}{2r}} \log\log x.$$

Thus as the lower bound stated in the previous equation increases with x and tends to infinity infinitely often, i.e., whenever we input x as one of our large prime powers, we see that the right-hand-side supremum must tend to infinity infinitely often as well. This is the basic sketch of the argument we will employ when we give the full proof of Theorem 2.1 in the next subsections. For now, we need to develop more machinery and state several lemmas to establish this claim. □

Remark 2.2 (History of the Mertens Conjecture)**.** There is a rich history to the original statement of the *Mertens conjecture* which states that

$$|M(x)| < c \cdot x^{1/2}, \quad \text{some constant } c > 0,$$

which was first verified by Mertens for $c = 1$ and $x < 10000$, although since its beginnings in 1897 has since been disproved by computation. We cite that prior to this point it is known that [9, *cf.* §4.1]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.06,$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009,$$

although based on work by Odlyzyko and te Riele (1985) it seems probable that each of these limits should be $\pm\infty$, respectively. More recently, progress has been made towards (TODO) [2, 4, 5].

2.2. **Key asymptotic bounds and formulas.**

Definition 2.3 (Indexing Sets and Indicator Functions)**.** Let the sets $S_{i,x}$ be defined as in [10, §1], i.e., such that $S_{i,x}$ consists of the integers s in the range $[12, x]$ such that

- **(1)** The set index i divides s: $i|s$;
- **(2)** Either $\nu_2(s) \geq 2$ or there are at least two odd primes dividing s;
- **(3)** The quotient s/i is *squarefree*: $\mu(s/i) \neq 0$; and
- **(4)** If $i = 2^k$ is a power of two, then $s/i > 2$.

We define the auxiliary union set, S_x, to denote

$$\begin{aligned} S_x &= \bigcup_{i=12}^{x} S_{i,i} \\ &= \{12, 15, 20, 21, 24, 28, 30, 33, 35, 36, 39, 40, 42, 44, 45, 48, 51, 52, 55, \ldots\} \cap \{n \in \mathbb{N} : 12 \leq n \leq x\}. \end{aligned}$$

In *words*, the set S_x denotes the natural numbers in the range $12 \leq n \leq x$ that are *not* of the form p^k or $2p^k$ for any primes p, and where the odd prime factorization of a $s \in S_x$ is squarefree (though we do allow multiple powers of 2 to be present in these expansions). As we can see below, the first several non-prime elements of the complement of the limiting set S_∞ correspond to

$$\{n \in \mathbb{N} : n \geq 12\} \setminus S_\infty = \{14, 16, 18, 22, 25, 26, 27, 32, 34, 38, 46, 49, 50, 54, 58, 62, 64, 74, 81, 82, 86, 94, \ldots\}.$$

Next, for $x \geq 1$, let $\chi_{\mathrm{pp}}(x)$ denote the indicator function for prime powers, i.e., the function defined precisely as

$$\chi_{\mathrm{pp}}(x) = \begin{cases} 1, & \text{if } x = p^k \text{ for some prime } p \geq 2 \text{ and } k \geq 1; \\ 0, & \text{otherwise,} \end{cases}$$

and define the composite indicator function for the prime powers $p^k, 2p^k$ as follows where $\chi_{\mathrm{pp}}(x) = 0$ if $x \in \mathbb{Q} \setminus \mathbb{Z}$:

$$\widetilde{\chi}_{\mathrm{pp}}(x) = \chi_{\mathrm{pp}}(x) + \chi_{\mathrm{pp}}\left(\frac{x}{2}\right).$$

Lemma 2.4 (Exact Formulas for the Divisor Sums $\sum_{d|x} \tau_x^{(\alpha)}(d)$)**.** *For $\alpha \in \mathbb{N}$, $m \geq 1$, and $x \geq 12$, let the functions $d_x^{(\alpha)}(m)$ denote the sums*

$$d_x^{(\alpha)}(m) = \sum_{s \in S_x} \sum_{d|(s,x)} d^{\alpha+1} \left[m = \frac{s}{d}\right]_\delta.$$

We see immediately that $d_x^{(\alpha)}(m) \leq x^{\alpha+1}$ for all $m, x \geq 1$. We can expand the divisor sums in Theorem 1.1 exactly in the following forms:

$$\begin{aligned} \sum_{d|x} \tau_x^{(\alpha+1)}(d) &= \sum_{s \in S_x} \sum_{d|\gcd(s,x)} s^\alpha \cdot \mu(s/d) \cdot d \\ &= \sum_{m=1}^{x} \mu(m) m^\alpha \cdot d_x^{(\alpha)}(m). \end{aligned}$$

Moreover, we have a deep connection between these sums and Ramanujan's sum $c_q(n)$ given by

$$\sum_{d|x} \tau_x^{(\alpha+1)}(d) = \sum_{s \in S_x} \mu\left(\frac{s}{\gcd(s,x)}\right) \frac{\varphi(s) \cdot s^\alpha}{\varphi\left(\frac{s}{\gcd(s,x)}\right)},$$

where $\varphi(x)$ denotes Euler's totient function.

Proof. We start with the following formula for computing the divisor sum over $\tau_x^{(\alpha)}(d)$ from [10, §2]:

$$\begin{aligned}\sum_{d|x} \tau_x^{(\alpha+1)}(d) &= [q^x]\left(\sum_{k=1}^{x}\sum_{d|k}\sum_{r|d} \frac{r\cdot \widetilde{\chi}_{\mathrm{pp}}(d)\cdot \mu(d/r)}{(1-q^r)} k^{\alpha}\right) \\ &= \sum_{k=1}^{x}\sum_{r|x}\sum_{d|k} r\cdot \widetilde{\chi}_{\mathrm{pp}}(d)\cdot \mu(d/r)\cdot [r|d]_{\delta} \qquad (2) \\ &= \sum_{s\in S_x}\sum_{d|s} s^{\alpha}\cdot \mu(s/d)\cdot d\,[d|x]_{\delta} \\ &= \sum_{s\in S_x}\sum_{d|\gcd(s,x)} s^{\alpha}\cdot \mu(s/d)\cdot d \qquad (3) \\ &= \sum_{m=1}^{x} \mu(m) m^{\alpha}\left(\sum_{s\in S_x}\sum_{d|(s,x)} d^{\alpha+1}\left[m = \frac{s}{d}\right]_{\delta}\right). \qquad (4)\end{aligned}$$

By (3) above we see that we have factors of Ramanujan's sum, $c_q(n)$, which leads to the identity that [8, §27.10] [6, §A.7] [3, *cf.* §5.6]

$$\sum_{d|x} \tau_x^{(\alpha+1)}(d) = \sum_{s\in S_x} \mu\left(\frac{s}{\gcd(s,x)}\right) \frac{\varphi(s)\cdot s^{\alpha}}{\varphi\left(\frac{s}{\gcd(s,x)}\right)}. \qquad \square$$

In addition to the results proved in Lemma 2.4, we can see by a simple argument that

$$\sum_{m=1}^{x} |d_x^{(\alpha)}(m)| = \sum_{s\in S_x}\sum_{d|(s,x)} d^{\alpha+1}. \qquad (5)$$

The right-hand-side sum in the previous equation is central to the bounds in the next lemma and in the asymptotic formula given later in Lemma 2.7.

Lemma 2.5 (A Lower Bound for the Magnitude of $M(x)$)**.** *For all sufficiently large $x \geq 14$, we have the following bound on the supremum of $|M(i)|$ taken over all $i \leq x$:*

$$\frac{\left|\sum_{d|x} \tau_x^{(\alpha+1)}(d)\right| + 2}{2\cdot \sum_{s\in S_x}\sum_{d|(s,x)} d} - \frac{1}{2} \leq \sup_{1\leq i\leq x} |M(i)|.$$

Proof. We first observe the equivalences of the sums for (1) given in Lemma 2.4. We notice that attempting to bound the absolute value of the sums in (4) is problematic due to the signed nature of the Möbius function terms in the sum and the potential for cancellation if we simply pull out a large factor of the inner divisor sum. For fixed x, we then proceed from here by summation by parts to obtain that

$$\begin{aligned}\left|\sum_{d|x} \tau_x^{(\alpha+1)}(d)\right| &= \left|\sum_{m=1}^{x} \mu(m) m^{\alpha}\cdot d_x^{(\alpha)}(m)\right| \\ &\leq |M_{\alpha}^{*}(x)|\, d_x^{(\alpha)}(x) + \sum_{m=1}^{x-1} |M_{\alpha}^{*}(x)| \left|d_x^{(\alpha)}(m+1) - d_x^{(\alpha)}(m)\right| \\ &\leq 2\sum_{m=1}^{x} |M_{\alpha}^{*}(x)|\, d_x^{(\alpha)}(x) + \sum_{m=1}^{x} m^{\alpha}|\mu(m)|\cdot |d_x^{(\alpha)}(m+1)| - M_{\alpha}^{*}(1) d_x^{(\alpha)}(1).\end{aligned}$$

We then consider the special case of the equivalent expressions for the sums in (5) when $\alpha := 0$ to obtain the correct bound stated above. $\square$

Proposition 2.6 (Asymptotic Bound for the Tau Function Divisor Sum). *Let $x = q^r$ denote a power of the large prime q for some $r \geq 4$. Then when the x tending to infinity of these forms is sufficiently large, we obtain*

$$\left|\sum_{d|x} \tau_x^{(\alpha+1)}(d)\right| \sim \left|\frac{(C_1 - C_2C_4)(x-1)}{\log(x-1)} + \left(\frac{C_2}{2} - 1\right) x \left(\log\log(x-1) + A\right) + \frac{r}{2} x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1\right)\right.$$
$$\left. - \frac{C_3}{4}x + C_3C_6\right| - \frac{x-1}{x^{\frac{1}{r}} - 1},$$

for some constants $C_i > 0$ such that $C_1 - C_2C_4 \neq 0$.

Proof. By the statement of Theorem 1.1 rephrased in (1), we see that

$$\begin{aligned}
\left|\sum_{d|x} \tau_x^{(1)}(d)\right| &= \left|x - \sigma_1(x) + S_1^{(1)}(x) + S_2^{(1)}(x)\right| \\
&= \left|S_1^{(1)}(q^r) + S_2^{(1)}(q^r) - \left(1 + q + \cdots + q^{r-1}\right)\right| \\
&\geq \left|S_1^{(1)}(q^r) + S_2^{(1)}(q^r)\right| - \frac{x-1}{x^{\frac{1}{r}} - 1}.
\end{aligned}$$

We next use the result of *Mertens' theorem* which implies that [6, §6.3] [1, §4.9] [3, §22.8] [8, §27.11]

$$\sum_{p \leq x} \frac{1}{p} = \log\log(x) + A + O\left(\frac{1}{\log x}\right),$$

where A is a limiting constant. In particular, when x is large we can expand the sum for $S_1^{(0)}(x)$ as

$$\begin{aligned}
S_1^{(0)}(x) &= \sum_{2 \leq p < q^r} p \cdot \left\lfloor \frac{q^r}{p} \right\rfloor \left(\left\lfloor \frac{q^r}{p} \right\rfloor - \left\lfloor \frac{q^r}{p} - \frac{1}{p} \right\rfloor - \frac{1}{p}\right) + \sum_{k=1}^{r+1} q^k \left\lfloor q^{r-k} \right\rfloor \left(\left\lfloor q^{r-k} \right\rfloor - \left\lfloor q^{r-k} - \frac{1}{q} \right\rfloor - \frac{1}{q}\right) \\
&= \sum_{2 \leq p < q^r} -\frac{p}{p}\left[\frac{q^r}{p} - \left\{\frac{q^r}{p}\right\}\right] + \sum_{k=1}^{r} q^r \left(1 - \frac{1}{q}\right) - \frac{1}{q} \\
&= C_1\pi(q^r - 1) - q^r \left(\log\log(q^r - 1) + A + O\left(\frac{1}{\log(q^r - 1)}\right)\right) + r \cdot q^{r-1}(q-1) - \frac{1}{q} \\
&\sim \frac{C_1(x-1)}{\log(x-1)} - x\left(\log\log(x-1) + A\right) + r \cdot x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1\right) - \frac{1}{x^{\frac{1}{r}}},
\end{aligned}$$

and similarly, the sum $S_2^{(0)}(x)$ is expanded as

$$\begin{aligned}
S_2^{(0)}(x) &= \sum_{2 \leq p < q^r} C_2 p \left\lfloor \frac{q^r}{2p} \right\rfloor \cdot \frac{1}{p} - \sum_{k=1}^{r} q^k \left\lfloor \frac{q^{r-k}}{2} \right\rfloor \frac{(q-1)}{q} + \frac{1}{q} - 2C_3 \left\lfloor \frac{q^r}{4} \right\rfloor \cdot \frac{1}{2} \\
&= \sum_{2 \leq p < q^r} C_2 \left(\frac{q^r}{2p} - C_4\right) - \sum_{k=1}^{r} q^k \left(\frac{q^{r-k}}{2} - C_5\right) \frac{(q-1)}{q} + \frac{1}{q} - C_3\left(\frac{q^r}{4} - C_6\right) \\
&\sim \frac{C_2}{2} x \left(\log\log(x-1) + A\right) - \frac{C_2C_4(x-1)}{\log(x-1)} - \frac{r}{2} x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1\right) + \frac{1}{x^{\frac{1}{r}}} + C_5(x-1) - \frac{C_3}{4}x + C_3C_6.
\end{aligned}$$

Hence when we add these two sums cancellation of symmetric terms results in

$$\begin{aligned}
S_1^{(0)}(x) + S_2^{(0)}(x) \sim \frac{(C_1 - C_2C_4)(x-1)}{\log(x-1)} &+ \left(\frac{C_2}{2} - 1\right) x \left(\log\log(x-1) + A\right) \\
&+ \frac{r}{2} x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1\right) - \frac{C_3}{4}x + C_3C_6,
\end{aligned}$$

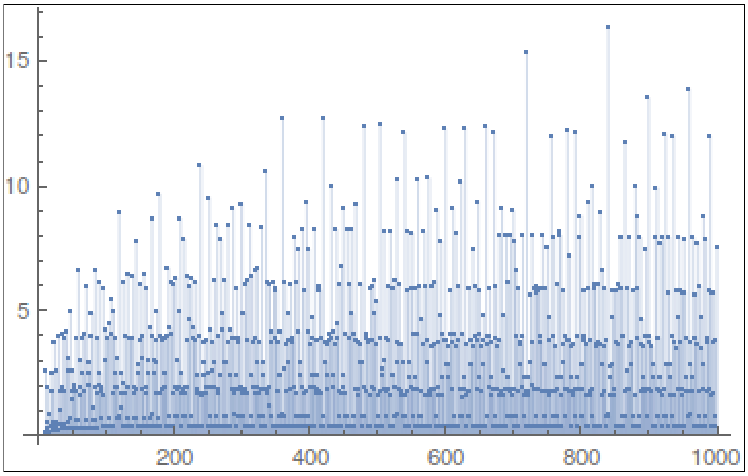


Figure 2.1. *The ratio of the divisor sums* $\sum_{s \in S_x} \sum_{d|(s,x)} d$ *to* $x \log\log x$ *for increasingly large* $x \leq 1000$

which proves our result. $\square$

The last key ingredient to our complete proof of Theorem 2.1 is to bound the divisor sum multipliers in the statement of the bounds for the supremum of $|M(x)|$ for large x implicit to Lemma 2.5. One question to ask is whether the ratios always

$$R_0(x) = \frac{1}{x \log\log x} \left(\sum_{s \in S_x} \sum_{d|(s,x)} d \right) \longrightarrow 0$$

for large x tending to infinity. If this happens to be the case, then we are all set in the proof of Theorem 2.1 given below. What we are able to prove in general through the next lemma is that this ratio tends to zero infinitely often for at least predictable small powers of large primes. The plot shown in Figure 2.1 illustrates the phenomenon in question. The behavior of the limiting ratio of $R_0(x)$ – if indeed it exists – is tied in many cases to the growth of the indexing sets $|S_x|$ for large x. The two side-by-side plots shown in Figure 2.2 compare these set sizes and their relation to the magnitude of the prime counting function $\pi(x) \sim \frac{x}{\log x}$. A basic upper bound for the size of S_x follows from the construction given in Definition 2.3 in which $p \notin S_x$ for any primes p and all x as

$$|S_x| \leq x - \pi(x) \sim x \left(1 - \frac{1}{\log x}\right). \tag{6}$$

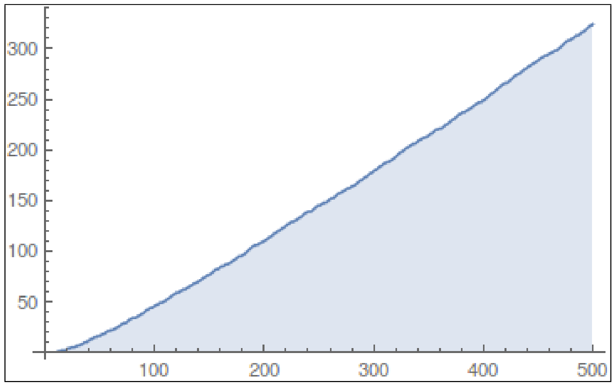
The next lemma employs this bound for $R_0(x)$ when x is taken to be a small power of a large prime. Figure 2.3 demonstrates the bound for the small powers of $r = 2, 3$.

Lemma 2.7 (A Final Divisor Sum Bound)**.** *Suppose that* $x = q^r$ *is a power of a large odd prime* q *for some* $r \geq 4$. *Then we obtain the bound*

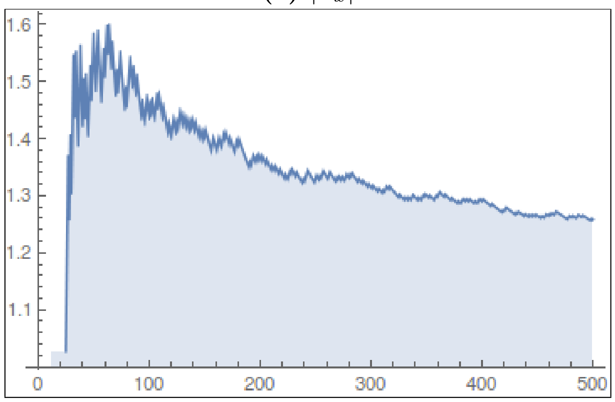
$$\sum_{s \in S_x} \sum_{d|(s,x)} d \leq \frac{x\left(x^{\frac{1}{r}}+1\right)\left(1 - \frac{1}{\log x}\right)}{x^{1-\frac{1}{r}} \xleftarrow{\text{check!}}}.$$

Proof. Since q is an odd prime, for any $s \in S_x$ (any x) we have that either $\gcd(s,x) = 1$ or $q \| s$, i.e., that s has at most one factor of the prime q. Thus if we expand the worst case growth rate of the sums defining the ratios $R_0(x)$, we obtain the bound that

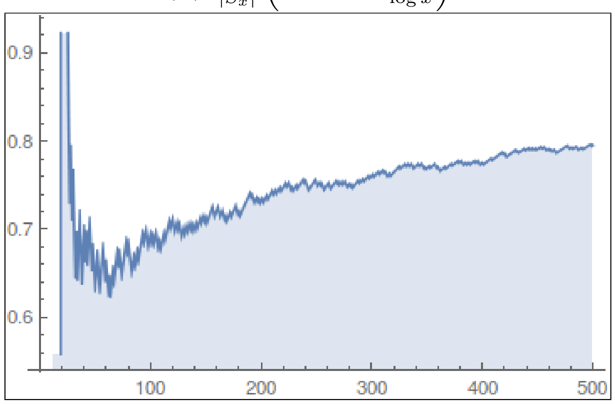
$$\begin{aligned} \sum_{s \in S_x} \sum_{d|(s,x)} d &\leq \sum_{s \in S_x} (1+q) = |S_x| \left(x^{\frac{1}{r}} + 1\right) \\ &\leq x \left(x^{\frac{1}{r}} + 1\right) \left(1 - \frac{1}{\log x}\right). \end{aligned}$$



(a) $|S_x|$



(b) $\frac{1}{|S_x|}\left(x - 12 - \frac{x}{\log x}\right)$



(c) $|S_x|\left(x - 12 - \frac{x}{\log x}\right)^{-1}$

Figure 2.2. *The Growth of the Index Sets* S_x

Now since the elements $s \in S_x$ for any fixed x have at most one prime power of q, we sieve and see that at most one of every q elements in the set can have this distinct prime factor[1]. Hence the stated bound follows. $\square$

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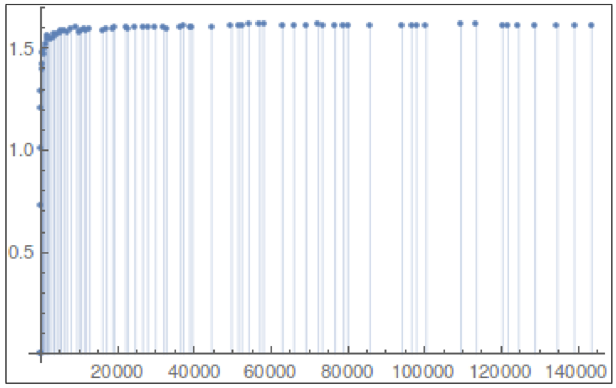
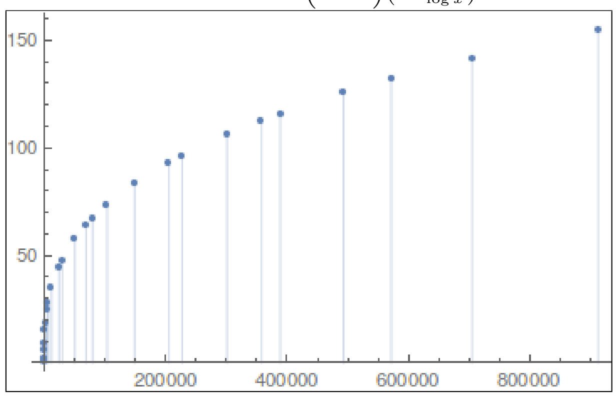
[1] My thought here for the denominator term is this: we want to weed out overcounting all of the multiples of q^k for $2 \leq k \leq \log_q(x)$ in the range $n \leq x$, so we compute the sum

$$\sum_{k=2}^{\log_q(x)} q^k = \frac{q(x-q)}{1-q} = \frac{x^{\frac{1}{r}}\left(x - x^{\frac{1}{r}}\right)}{x^{\frac{1}{r}} - 1},$$

and then divide through by this ratio of overcount. This part is absolutely *crucial* to obtaining a non-trivial lower bound on $|M(x)|$ when $x = q^r$ is large!



(a) $\left(\sum_{s\in S_x}\sum_{d|(s,x)} d\right)\frac{x^{1-\frac{1}{r}}}{x\left(x^{\frac{1}{r}}+1\right)\left(1-\frac{1}{\log x}\right)}$ *for* $x = q^2$



(b) $\left(\sum_{s\in S_x}\sum_{d|(s,x)} d\right)\frac{x^{1-\frac{1}{r}}}{x\left(x^{\frac{1}{r}}+1\right)\left(1-\frac{1}{\log x}\right)}$ *for* $x = q^3$

Figure 2.3. *The Divisor Sum Bound in Lemma 2.7*

2.3. **The complete proof of Theorem 2.1.** We are now at the point where we can assemble the complete results necessary to prove Theorem 2.1. The key idea here is that while the value of $|M(x)|$ is oscillating with x, we can bound the value of $\sup_{1\leq i\leq x}|M(i)|$ below by something increasingly large and tending to infinity infinitely often, i.e., since there are an infinitude of small powers q^r of large primes $q\to\infty$. Then using the lower bound in Lemma 2.5, and combining the bounds in Proposition 2.6 and Lemma 2.7, we see that when $x = q^r$ is large we have

$$\frac{1}{\sqrt{x}}\left(\sup_{1\leq i\leq x}|M(i)|\right) \geq \widetilde{C}_2 \frac{x^{\frac{r-2}{2r}}}{\left(x^{\frac{1}{r}}+1\right)\left(1-\frac{1}{\log x}\right)}\log\log(x-1). \tag{7}$$

Next, for $x = q^r$ a power of a large odd prime q let

$$x_{0,q^r} = \operatorname{argmax}_{1\leq i\leq q^r}|M(i)|.$$

Then we see from (7) that

$$\frac{|M(x_{0,q^r})|}{\sqrt{x_{0,q^r}}} \geq \frac{|M(x_{0,q^r})|}{\sqrt{q^r}} \geq \widetilde{C}_2 \frac{q^{\frac{r-2}{2}}}{(q+1)\left(1-\frac{1}{r\log q}\right)}\log\log(q^r-1).$$

Moreover, since the lower bound in (7) and in the previous equation is increasing with q^r, i.e., as $q\to\infty$, we see that the non-decreasing sequence of x_{0,q^r} must gradually increase with larger and larger q. Thus we see that for any $L>0$, there are infinitely many $x\in\mathbb{N}$ such that $|M(x)|/\sqrt{x} > L$. Hence the result is proved.

2.4. **Generalizations.** We remark that Theorem 2.1 can be effectively generalized to a result of the more general form

$$\limsup_{\substack{x\to\infty \\ x=q^r}} \frac{|M_{\alpha}^{*}(x)|}{(\sqrt{x})^{2\alpha+1}} = +\infty.$$

The only caveat here is that we need to know more precise forms of Mertens' theorem for general sums of the form $\sum_{p\leq x} p^{\alpha k}$ depending on the parameter $\alpha \geq 0$. For now, we will leave the generalizations of our main theorem as an exercise for future research on the generalized Mertens summatory functions $M_{\alpha}^{*}(x)$ defined in the introduction.

3. Appendix: The proof of Theorem 1.1

4. Conclusions

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