Lower bounds on the summatory function of the Möbius function along infinite subsequences

Maxie Dion Schmidt
Georgia Institute of Technology
School of Mathematics

Last Revised: Sunday 28th June, 2020 @ 23:08:40 – Compiled with LaTeX2e

Abstract

The Mertens function, $M(x) = \sum_{n \leq x} \mu(n)$, is classically defined as the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture states that $|M(x)| < C \cdot \sqrt{x}$ with come absolute $C > 0$ for all $x \geq 1$. The classical conjecture has a well-known disproof due to Odlyzko and té Riele given in the early 1980's by computation of non-trivial zeta function zeros in conjunction with integral formulas expressing $M(x)$. We prove the unboundedness of $|M(x)|/\sqrt{x}$ using new methods by showing that

$$\limsup_{x \rightarrow \infty} \frac{|M(x)| (\log\log x)^{\frac{5}{2}} (\log\log\log x)^2}{\sqrt{x} \cdot (\log x)^{\frac{1}{4}}} \geq 0.234145.$$

There is a distinct stylistic flavor and new element of combinatorial analysis to our proof combined with the standard methods from analytic, additive and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

Keywords and Phrases: *Möbius function; Mertens function; summatory function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting functions; Dirichlet generating function; asymptotic lower bounds; Mertens conjecture.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; and 11N64.*

Glossary of special notation and conventions

Symbol	**Definition**
$\approx$	We write that $f(x) \approx g(x)$ if $\lvert f(x) - g(x)\rvert = O(1)$ as $x \rightarrow \infty$.
$\mathbb{E}[f(x)], \overset{\mathbb{E}}{\sim}$	We use the expectation notation $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \overset{\mathbb{E}}{\sim} h(x)$, to denote that f has an *average order* growth rate of $h(x)$. What this means is that $\frac{1}{x}\sum_{n\leq x} f(n) \sim h(x)$, or equivalently that $$\lim_{x\rightarrow\infty} \frac{\frac{1}{x}\sum_{n\leq x} f(n)}{h(x)} = 1.$$
B	The absolute constant $B \approx 0.2614972128476427837554$ from the statement of Mertens theorem.
$C_k(n)$	The sequence is defined recursively for $n \geq 1$ as follows when we assume that $1 \leq k \leq \Omega(n)$: $$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum_{d\mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n\geq 0}$. In particular, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) = \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (defined below).
$f * g$	The Dirichlet convolution of f and g, $(f * g)(n) := \sum_{d\mid n} f(d)g(n/d)$, where the sum is taken over the divisors d of n for $n \geq 1$.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ for $n \geq 2$ with $f^{-1}(1) = 1/f(1)$. The Dirichlet inverse of f exists if and only if $f(1) \neq 0$. This inverse function, denoted by f^{-1} provided it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$.
$\gg, \ll$	For functions A, B in x, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$.
H_n	The *first-order harmonic numbers*, $H_n := \sum_{k=1}^{n} \frac{1}{k}$, satisfy the limiting asymptotic relation $$\lim_{n\rightarrow\infty} [H_n - \log(n)] = \gamma,$$ where $\gamma \approx 0.577216$ denotes Euler's gamma constant.
$[n = k]_\delta, [\mathtt{cond}]_\delta$	The symbol $[n = k]_\delta$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise. For a boolean-valued conditions, `cond`, $[\mathtt{cond}]_\delta$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention*.

Symbol	**Definition**
$\lambda(n)$	The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. That is, $\lambda(n) \in \{\pm 1\}$ for all integers $n \geq 1$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree, i.e., when n has no prime power divisors with exponent greater than one. Necessarily, we have that $\mu(n) = 0$ whenever $n \geq 1$ is not squarefree. We can equivalently characterize this function as having the DGF of $1/\zeta(s)$ for all $\mathrm{Re}(s) > 1$.
$M(x)$	The Mertens function is the summatory function over $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum\limits_{n \leq x} \mu(n)$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^{\alpha} \vert\vert n$ (or when p^{α} exactly divides n) for p prime and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p \vert n} 1$ and the completely additive function $\Omega(n) := \sum_{p^{\alpha} \vert\vert n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leq n \leq x$ for $x > 1$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$ for $x \geq 2$.
$P(s)$	For complex s with $\mathrm{Re}(s) > 1$, we define the *prime zeta function* to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$. For $\mathrm{Re}(s) > 1$, the prime zeta function is related to $\zeta(s)$ according to the formula $P(s) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log[\zeta(ks)]$.
$Q(x)$	For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. More precisely, this function is summed and identified with its limiting asymptotic formula as $x \to \infty$ in the following form: $Q(x) := \sum_{n \leq x} \mu^2(n) \sim \frac{6}{\pi^2} x + O(\sqrt{x})$.
$\sim$	We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x \to \infty} \frac{A(x)}{B(x)} = 1$.
$\zeta(s)$	The Riemann zeta function id defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation on the rest of the complex plane with the exception of a simple pole at $s = 1$ with residue one.

1 Introduction

1.1 Definitions

We define the *Möbius function* to be the signed indicator function of the squarefree integers as [14, A008683]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leq i \leq k; \\ 0, & \text{otherwise}, \end{cases}$$

where for natural numbers $n \geq 2$ we have factorization of n into distinct primes defined by $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ so that $r = \omega(n)$. There are many other variants and special properties of the Möbius function and its generalizations [13, *cf.* §2]. A crucial role of the classical $\mu(n)$ forms an inversion relation for arithmetic functions convolved with one by *Möbius inversion*:

$$g(n) = (f * 1)(n) \iff f(n) = (g * \mu)(n), \forall n \geq 1.$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined on the positive integers as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as [14, A002321]

$$\{M(x)\}_{x \geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2, \ldots\}$$

Clearly, a positive integer $n \geq 1$ is *squarefree*, or contains no (prime power) divisors which are squares, if and only if $\mu^2(n) = 0$. A related summatory function which counts the number of *squarefree* integers $n \leq x$ then satisfies [2, §18.6] [14, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \to \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{x} \overset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{x} \xrightarrow{x \to \infty} \frac{3}{\pi^2}.$$

1.2 Properties

One conventional approach to evaluating the behavior of $M(x)$ for large $x \to \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula results by considering inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \cdot \int_1^\infty \frac{M(x)}{x^{s+1}} dx,$$

we obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

The previous two representations lead us to the exact expression for $M(x)$ for any real $x > 0$ given by the next theorem due to Titchmarsh.

Theorem 1.1 (Analytic Formula for $M(x)$)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence* $\{T_k\}_{k\geq 1}$ *satisfying* $k \leq T_k \leq k+1$ *for each* k *such that for any real* $x > 0$

$$M(x) = \lim_{k\to\infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |\operatorname{Im}(\rho)|<T_k}} \frac{x^\rho}{\rho\cdot\zeta'(\rho)} - 2 + \sum_{n\geq 1} \frac{(-1)^{n-1}}{n\cdot(2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2}\left[x\in\mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x\cdot\exp\left(-C\cdot\log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan more recently proved new updated estimates bounding $M(x)$ from above for large x in the following forms [15]:

$$\begin{aligned} M(x) &\ll \sqrt{x}\cdot\exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x}\cdot\exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right),\ \forall\epsilon>0. \end{aligned}$$

1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C\cdot\sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens for $C = 1$ and all $x < 10000$. Since its beginnings in 1897, the Mertens conjecture has been disproven by computation of non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele from the early 1980's [10]. Since the truth of the conjecture would have implied the RH, more recent attempts at bounding $M(x)$ consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound towards both $\pm\infty$ along infinite subsequences.

In fact, one of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ actually grows without bound on the natural numbers. A precise statement of this problem is to produce an unconditional proof of whether $\limsup_{x\to\infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x\to\infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there are infinite subsequences of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound towards either $\pm\infty$ along the subsequence. We cite that prior to this point it is only known by computation that [12, *cf.* §4.1] [14, *cf.* A051400; A051401]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leq -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [10, 5, 6, 3]. Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies [9]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}\cdot(\log\log\log x)^{5/4}} = O(1).$$

2 An overview of the core logical steps and components to the proof

We offer an initial step-by-step summary overview of the core components to our proof outlined in the next points. As our proof methodology is new and relies on non-standard elements compared to more traditional methods of bounding $M(x)$, we hope that this sketch of the logical components to this argument makes the article easier to parse.

(1) We prove a matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). See Theorem 3.1 in Section 4.

(2) This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the seemingly unconnected prime counting function, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is stated in (1).

The strong additivity of $\omega(n)$ yields the characteristic signedness of $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ for all $n \geq 1$. The link relating our new formula in (1) to canonical additive functions and their distributions lends a recent distinguishing element to the success of the methods in our proof.

(3) We tighten bounds from a more recent result proved in [8, §7] providing uniform asymptotic formulas for the summatory functions, $\widehat{\pi}_k(x)$, large $x \gg e$ and $1 \leq k \leq \log\log x$. These formulas are proved using expansions of more combinatorially motivated Dirichlet series (see Theorem 3.7). We use this result to bound sums of the form $\sum_{n \leq x} \lambda(n) f(n)$ from below for particular non-negative arithmetic functions f when x is large.

(4) We then turn to estimating the limiting asymptotics of the quasi-periodic function, $|g^{-1}(n)|$, by proving several formulas bounding its average order as $x \to \infty$ in Section 6. We eventually use these estimates to prove a substantially unique new lower bound formula for the summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ along certain asymptotically large infinite subsequences (see Theorem 7.7).

(5) When we return to step **(2)** with our new lower bounds at hand, we have a new unconditional proof of the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. In fact, what we recover is a quick, and rigorous, proof of Theorem 3.9 given in Section 7.2.

3 A concrete new approach for bounding $M(x)$ from below

3.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 3.1 (Summatory functions of Dirichlet convolutions). *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of f, h, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse of f. Then we have the following exact expressions for the summatory function of $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned}
\pi_{f*h}(x) &:= \sum_{n \leq x} \sum_{d|n} f(d) h(n/d) \\
&= \sum_{d \leq x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].
\end{aligned}$$

Moreover, we can invert the linear system determining the coefficients of $H(k)$ for $1 \leq k \leq x$ in the previous equation as follows:

$$\begin{aligned}
H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\
&= \sum_{n=1}^{x} f^{-1}(n) \pi_{f*h}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).
\end{aligned}$$

Corollary 3.2 (Convolutions Arising From Möbius Inversion). *Suppose that g is an arithmetic function on the positive integers such that $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n \leq x} (g * \mu)(n)$. Then the Mertens function equals*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geq 1.$$

Corollary 3.3 (A motivating special case). *We have exactly that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

3.2 An exact expression for $M(x)$ in terms of strongly additive functions

From this point on, we fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and denote its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute the Dirichlet inverse of $g(n)$ exactly for the first few sequence values as (see Table T.1 starting on page 40 of the appendix section)

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these positive terms is given by $\operatorname{sgn}(g^{-1}(n)) = \frac{g^{-1}(n)}{|g^{-1}(n)|} = \lambda(n)$ for all $n \geq 1$ (see Proposition 4.1). This useful property is inherited from the distinctly additive nature of the component function $\omega(n)$ [A].

[A]Indeed, for any non-negative additive arithmetic function $a(n)$, $(a+1)^{-1}(n)$ has leading sign given by $\lambda(n)$ for any $n \geq 1$. For multiplicative f, we obtain a related condition that $\operatorname{sgn}(f(n)) = (-1)^{\omega(n)}$ for all $n \geq 1$.

There does not appear to be an easy, nor subtle direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences. The distribution of distinct sets of prime exponents is still fairly regular so that $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of $g^{-1}(n)$ over $n \geq 2$:

Heuristic 3.4 (Symmetry in $g^{-1}(n)$ in the prime factorizations of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for some $r \geq 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly two, three, and so on prime factors.

Conjecture 3.5. *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) $g^{-1}(1) = 1$*;*

(B) *For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*

(C) *For all squarefree integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!;$$

(D) *If $n \geq 2$ and $\Omega(n) = k$, then*

$$2 \leq |g^{-1}(n)| \leq \sum_{m=0}^{k} \binom{k}{m} \cdot m!.$$

We illustrate parts (B)–(D) of the conjecture clearly using the computation of initial values of this inverse sequence in Table T.1. The realization that the beautiful and remarkably simple combinatorial form of property (C) in Conjecture 3.5 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through sums of auxiliary sequences of arithmetic functions [B] (see Section 6).

For natural numbers $n \geq 1$ and $k \geq 0$, let

$$C_k(n) := \begin{cases} \varepsilon(n) = \delta_{n,1}, & \text{if } k = 0; \\ \sum_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$

For any $n \geq 1$, we can prove that (see Lemma 6.3)

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{2}$$

In light of the fact that (see Proposition 7.1)

$$M(x) \approx G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)},$$

the formula in (2) implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by bounding appropriate estimates of the summatory function $G^{-1}(x)$.

[B] A proof of this property is not difficult to give using Lemma 6.3 stated on page 22.

3.3 Uniform asymptotics from enumerative counting DGFs in Mongomery and Vaughan

Our inspiration for the new bounds found in the last sections of this article allows us to approximate sums of certain bounded non-negative arithmetic functions weighted by the Liouville lambda function $\lambda(n)$ taken over all $n \leq x$ well from below as $x \rightarrow \infty$.

Theorem 3.6 (Montgomery and Vaughan)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $R < 2$ we have that

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right],$$

uniformly for $1 \leq k \leq R\log\log x$ where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, 0 \leq |z| \leq R.$$

The proof of the next result is combinatorially motivated in so much as it interprets lower bounds on a key infinite product factor of $\mathcal{G}(z)$ defined in Theorem 3.6 as corresponding to an ordinary generating function of certain homogeneous symmetric polynomials involving the primes. This interpretation allows us to recover the following uniform lower bounds on $\widehat{\pi}_k(x)$ as $x \rightarrow \infty$:

Theorem 3.7. *We have that for all sufficiently large $x \rightarrow \infty$ and $1 \leq k \leq \log\log x$*

$$\left|\mathcal{G}\left(\frac{1-k}{\log\log x}\right)\right| \gg \frac{2^{\frac{3}{4}}(\log 2)^{\frac{1}{2}}}{x^{\frac{3}{4}}(\log x)^{\frac{1}{2}}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \times \frac{k-1}{\log\log x}.$$

Then for all large enough x we have uniformly for $1 \leq k \leq \log\log x$ that

$$\widehat{\pi}_k(x) \gg \frac{\widehat{C}_0 x^{\frac{1}{4}}}{(\log x)^{\frac{3}{2}}} \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^3}\right)\right],$$

where the absolute constant is defined by $\widehat{C}_0 := 2^{\frac{3}{4}} e(\log 2)^{\frac{1}{2}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \approx 2.42584$.

Remark 3.8. We emphasize the relevant recency of the method demonstrated by Montgomery and Vaughan in constructing a proof of Theorem 3.6. To the best of our knowledge, this textbook reference is one of the first clear-cut applications documenting something of a hybrid DGF-and-OGF approach to enumerating sequences of arithmetic functions. forming their summatory functions. This method does not require a direct appeal to traditional highly oscillatory DGF-only inversions and integral formulas involving the Riemmann zeta function.

This newer interpretaion of certain bivariate DGFs offers a window into the best of both generating function series worlds: it combines an additive structure implicit to the coefficients indexed by a formal power series variable formed by multiplication of these structures, while coordinating the distinct DGF-best property of the multiplicativity of prime powers invoked by taking powers of a reciprocal Euler product. Since our key Dirichlet inverse function sequence, $g^{-1}(n)$, is formed by multiplication (convolution) of additive function primitives, this construction is particularly useful in motivating our new arguments.

3.4 Cracking the classical unboundedness barrier

In Section 7, we are able to state what forms a bridge between the results we carefully prove up to that point the article. What we obtain at the conclusion of the section is the next summary theorem that unconditionally resolves the classical question of the unboundedness of the scaled function Mertens function $q(x) := |M(x)|/\sqrt{x}$ in the limit supremum sense.

Theorem 3.9 (Unboundedness of the the Mertens function, $q(x)$)**.** *We have that*

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

In establishing the rigorous proof of Theorem 3.9 based on our new methods, we not only show unboundedness of $q(x)$, but also set a minimal rate (along a large infinite subsequence) at which this form of the scaled Mertens function grows without bound.

4 Preliminary proofs of new results

4.1 Establishing the summatory function properties and inversion identities

We will first prove Theorem 3.1 using an intuitive construction by matrix methods. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95].

Proof of Theorem 3.1. Let h, g be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h. Then we have that the following formulas hold for all $x \geq 1$:

$$\begin{aligned}\pi_{g*h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right] H(i).\end{aligned}$$

In particular, the first formula above is well known. The second formula is justified directly using summation by parts [A].

We next form the invertible matrix of coefficients associated with this linear system defining $H(j)$ for all $1 \leq j \leq x$. Let these matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right), \forall 1 \leq j \leq x.$$

The matrix we must invert in this problem is lower triangular with ones on its diagonals, and is hence invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by a secondary invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

Here, U is a square matrix with sufficiently large finite dimensions whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$\left[(I - U^T)^{-1}\right]_{i,j} = [j \leq i]_{\delta}.$$

It is a useful fact that if we take successive differences in x of the floor of certain fractions, $\left\lfloor \frac{x}{j} \right\rfloor$, in the form of

$$\left\lfloor \frac{x}{j} \right\rfloor - \left\lfloor \frac{x-1}{j} \right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise,} \end{cases}$$

for $1 \leq j \leq x$, we obtain non-zero differences at the indices j taken precisely over the divisors of x. This implies that

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

[A] For any arithmetic functions, u_n, v_n, with $U_j := u_1 + u_2 + \cdots + u_j$ for $j \geq 1$, we have that [11, §2.10(ii)]

$$\sum_{j=1}^{n-1} u_j \cdot v_j = U_{n-1} v_n + \sum_{j=1}^{n-1} U_j (v_j - v_{j+1}), n \geq 2.$$

We use the last property in (3) to shift the matrix $\hat{G}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of g in the following form:

$$\left[(I-U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Now we can express the inverse of our target matrix,

$$(g_{x,j}) = (I-U^T)^{-1}\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T),$$

using a similarity transformation conjugated by shift operators as

$$\begin{aligned}(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k)\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1}\right\rfloor} g^{-1}(k)\right).\end{aligned}$$

Hence, the summatory function $H(x)$ is exactly expressed for any $x \geq 1$ by a vector product with the inverse matrix from the previous equation given by

$$H(x) = \sum_{k=1}^{x} g_{x,k}^{-1} \cdot \pi_{g*h}(k) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right)\cdot \pi_{g*h}(k). \qquad \square$$

4.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n. Then we can easily prove that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega+1) * \mu. \tag{4}$$

When combined with Corollary 3.2 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 3.3.

Proposition 4.1 (The signedness property of $g^{-1}(n)$)**.** *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0, \pm 1\}$ *denote the sign of the arithmetic function h at integers $n \geq 1$. For the Dirichlet invertible function, $g(n) := \omega(n)+1$, we have that* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ *for all $n \geq 1$.*

Proof. The function $D_f(s) := \sum_{n\geq 1} f(n)n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ for σ_f the abcissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = 1/\zeta(s)$ and $D_\omega(s) = P(s)\zeta(s)$. Then by (4) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any arithmetic function f such that $f(1) \neq 0$, we have for all $\operatorname{Re}(s) > 1$ that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{5}$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} \ast \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. This observation implies that $\operatorname{sgn}(h^{-1} \ast \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make our claims rigorous.

By the recurrence relation that defines the Dirichlet inverse function of any arithmetic function h such that $h(1) = 1$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d \mid n \\ d > 1}} h(d) h^{-1}(n/d), & n \geq 2. \end{cases} \tag{6}$$

For $n \geq 2$, the summands in (6) can be simply indexed over the primes $p|n$ given our definition of h from above. This observation yields that we can inductively unfold these sums into nested divisor sums provided the depth of the expanded divisor sums does not exceed the capacity to index summations over the primes dividing n. Namely, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p \mid n} h^{-1}(n/p), && \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1 \mid n} \sum_{p_2 \mid \frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right), && \text{if } \Omega(n) \geq 2 \\ &= -\sum_{p_1 \mid n} \sum_{p_2 \mid \frac{n}{p_1}} \sum_{p_3 \mid \frac{n}{p_1 p_2}} h^{-1}\left(\frac{n}{p_1 p_2 p_3}\right), && \text{if } \Omega(n) \geq 3. \end{aligned}$$

Then by induction, again with $h^{-1}(1) = h(1) = 1$, we we should expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1 \mid n} \sum_{p_2 \mid \frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)} \mid \frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1, n \geq 2. \tag{7}$$

If for $n \geq 2$ we write the prime factorization of n as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{\omega(n)}^{\alpha_{\omega(n)}}$ where the exponents $\alpha_i \geq 1$ for all $1 \leq i \leq \omega(n)$, we can see that [B]

$$\begin{aligned} |h^{-1}(n)| &\geq (\omega(n))! && =: h_\ell^{-1}(n), n \geq 2, \\ |h^{-1}(n)| &\leq (\omega(n))!^{\max(\alpha_1, \alpha_2, \ldots, \alpha_{\omega(n)})} && =: h_u^{-1}(n), n \geq 2, \end{aligned} \tag{8}$$

with equality at each bound precisely when $n \geq 2$ is squarefree. By the positivity of these bounding functions $h_\ell^{-1}(n), h_u^{-1}(n) > 0$, for all $n \geq 1$ (with $\lambda(1) = 1$) the following property holds:

$$\operatorname{sgn}(h^{-1}(n)) = \lambda(n).$$

Since λ is completely multiplicative, and since $\mu(n) = \lambda(n)$ whenever n is squarefree, we obtain that

$$g^{-1}(n) = (h^{-1} \ast \mu)(n) = \lambda(n) \times \sum_{d \mid n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1.$$

The previous equation finally implies our result. ∎

[B] In fact, we recover that

$$\lambda(n) h^{-1}(n) = \frac{(\alpha_1 + \cdots + \alpha_{\omega(n)})!}{\alpha_1! \alpha_2! \cdots \alpha_{\omega(n)}!}.$$

4.3 Statements of other facts and known limiting asymptotics

Theorem 4.2 (Mertens theorem). *For all $x \geq 2$ we have that*

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log\log x + B + o(1), \text{ as } x \to \infty,$$

where $B \approx 0.2614972128476427837554$ is an absolute constant [C].

Corollary 4.3 (Product form of Mertens theorem). *We have that for all sufficiently large $x \gg 2$*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-B}}{\log x} \left(1 + o(1)\right), \text{ as } x \to \infty,$$

where the notation for the absolute constant $0 < B < 1$ coincides with the definition of Mertens constant from Theorem 4.2. Hence, for any real $z \geq 0$ we obtain that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^z = \frac{e^{-Bz}}{(\log x)^z} \left(1 + o(1)\right)^z \sim \frac{e^{-Bz}}{(\log x)^z}, \text{ as } x \to \infty.$$

Proofs of Theorem 4.2 and Corollary 4.3 are given in [2, §22.7; §22.8].

Facts 4.4 (Exponential integrals and the incomplete gamma function). Two variants of the *exponential integral function* are defined by the integral next representations [11, §8.19].

$$\begin{aligned} \mathrm{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, \\ E_1(z) &:= \int_{1}^{\infty} \frac{e^{-tz}}{t} dt, \mathrm{Re}(z) \geq 0 \end{aligned}$$

These functions are related by $\mathrm{Ei}(-kz) = -E_1(kz)$ for real $k, z > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $\mathrm{Ei}(\pm x)$ for all real $x > 0$:

$$\begin{aligned} \gamma + \log x - x \leq \mathrm{Ei}(-x) &\leq \gamma + \log x - x + \frac{x^2}{4}, \\ 1 + \gamma + \log x - \frac{3}{4}x \leq \mathrm{Ei}(x) \quad &\leq 1 + \gamma + \log x - \frac{3}{4}x + \frac{11}{36}x^2. \end{aligned} \tag{9a}$$

The (upper) *incomplete gamma function* is defined by [11, §8.4]

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt, \mathrm{Re}(s) > 0.$$

The following properties of $\Gamma(s, x)$ hold:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, x > 0, \tag{9b}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, s > 0, \text{ as } x \to \infty. \tag{9c}$$

[C] Exactly, we have that the *Mertens constant* is defined by

$$B = \gamma + \sum_{m \geq 2} \frac{\mu(m)}{m} \log\left[\zeta(m)\right],$$

where $\gamma \approx 0.577215664902$ is Euler's gamma constant.

5 Components to the asymptotic analysis of lower bounds for sums of arithmetic functions weighted by $\lambda(n)$

5.1 A discussion of the results proved by Montgomery and Vaughan

Remark 5.1 (Intuition and constructions in Theorem 3.6)**.** For $|z| < 2$ and $\mathrm{Re}(s) > 1$, let

$$F(s,z) := \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \tag{10}$$

and define the DGF coefficients, $a_z(n)$ for $n \geq 1$, by the product

$$\zeta(s)^z \cdot F(s,z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s}, \mathrm{Re}(s) > 1.$$

Suppose that $A_z(x) := \sum_{n \leq x} a_z(n)$ for $x \geq 1$. Then we obtain the next generating function like identity in z. [A]

$$A_z(x) = \sum_{n \leq x} z^{\Omega(n)} = \sum_{k \geq 0} \widehat{\pi}_k(x) z^k \tag{11}$$

Thus for $r < 2$, by Cauchy's integral formula we have

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz.$$

Selecting $r := \frac{k-1}{\log\log x}$ for $1 \leq k < 2\log\log x$ leads to the uniform asymptotic formulas for $\widehat{\pi}_k(x)$ given in Theorem 3.6. We will require estimates of $A_{-z}(x)$ from below to form summatory functions that weight the terms of $\lambda(n)$ in our formulas in the next sections.

5.2 New uniform asymptotics based on refinements of Theorem 3.6

What the enumeratively flavored result in Theorem 3.6 allows us to do is get a sufficient lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. We approximate $\mathcal{G}(z)$ defined in the theorem by only taking finite products of the primes in the factor $\prod_p (1 - z/p)^{-1}$ defining this function for $p \leq x$ as $x \to \infty$. We can extend the argument behind the constructions sketched in Remark 5.1 to justify that it suffices to consider only the contributions from these finite products to obtain a corresponding uniform lower bound on $\widehat{\pi}_k(x)$ for $1 \leq k \leq \log\log x$.

Proposition 5.2. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \geq 2.$$

When $s := 1$, we have the asymptotic formula from Mertens theorem (see Theorem 4.2). For all integers $s \geq 2$ there is an absolutely defined bounding function $\gamma_0(s,x)$ such that

$$\gamma_0(s,x) + o(1) \leq P_s(x), \text{ as } x \to \infty.$$

[A] In fact, we have more generally that

$$\prod_p \left(1 - \frac{z}{p^s}\right)^{-1} = \sum_{n \geq 1} \frac{z^{\Omega(n)}}{n^s}, \mathrm{Re}(s) > 1.$$

For any additive arithmetic function $a(n)$, characterized by the property that $a(n) = \sum_{p^\alpha || n} a(p^\alpha)$ for all $n \geq 2$, we have that [4, *cf.* §1.7]

$$\sum_{n \geq 1} \frac{z^{a(n)}}{n^s} = \prod_p \left(1 - \sum_{m \geq 1} \frac{z^{a(p^m)}}{p^{ms}}\right)^{-1}, \mathrm{Re}(s) > 1.$$

It suffices to define the bound in the previous equation as as the quasi-polynomial function in s and x given by

$$\gamma_0(s,x) = s\log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}s(s-1)^2\log^2(2).$$

Proof. Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$, and where our target function smooth function is $f(t) = t^{-s}$ so that $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= \mathrm{Ei}(-(s-1)\log x) - \mathrm{Ei}(-(s-1)\log 2) + o(1), \text{ as } x \to \infty. \end{aligned}$$

Now using the inequalities in Facts 4.4, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned} \frac{P_s(x)}{s} &\geq \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}(s-1)^2\log^2(2) \\ \frac{P_s(x)}{s} &\leq \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) + \frac{1}{4}(s-1)^2\log^2(x). \end{aligned}$$

This completes the proof of the bound stated above. □

Proof of Theorem 3.7. For $0 \leq z < 2$ and integers $x \geq 2$, the right-hand-side of the following product is finite as $x \to \infty$:

$$\widehat{P}(z,x) := \prod_{p \leq x}\left(1 - \frac{z}{p}\right)^{-1}.$$

Moreover, for fixed, finite $x \geq 2$ let

$$\mathbb{P}_x := \{n \geq 1 : \text{all prime factors } p|n \text{ satisfy } p \leq x\}.$$

Then we can see as in the constructions from Montgomery and Vaughan sketeched in Remark 5.1 that

$$\prod_{p \leq x}\left(1 - \frac{z}{p^s}\right)^{-1} = \sum_{n \in \mathbb{P}_x} \frac{z^{\Omega(n)}}{n^s}, x \geq 2. \tag{12}$$

By extending the argument in the proof given in [8, §7.4], we have that the formulas

$$A_{-z}(x) := \sum_{n \leq x} \lambda(n) z^{\Omega(n)} = \sum_{k \geq 0} \widehat{\pi}_k(x)(-z)^k,$$

depending on approximations (or inputs) to $\mathcal{G}(-z)$ still contain all of the relevant terms, or powers of z, after taking the finite products in (12). This assertion if correct since the products of all non-negative integral powers of the primes $p \leq x$ generate the integers $\{1 \leq n \leq x\}$ as a subset.

We have for all integers $0 \leq m < +\infty$, and any sequence $\{f(n)\}_{n \geq 1}$ with bounded partial sums, that [7, §2]

$$[z^m]\prod_{i \geq 1}(1 - f(i)z)^{-1} = [z^m]\exp\left(\sum_{j \geq 1}\left(\sum_{i=1}^m f(i)^j\right)\frac{z^j}{j}\right), |z| < 1. \tag{13}$$

In our case we have that $f(i)$ denotes the reciprocal of the i^{th} prime in the generating function expansion of (13). We find effective bounds on the truncated products in (12) that are both meaningful and still simple enough in form to use in our new formulas.

It follows from Proposition 5.2 that for real $0 \leq z < 1$ we obtain

$$\begin{aligned}
\log\left[\prod_{p \leq x}\left(1+\frac{z}{p}\right)^{-1}\right] &\geq -(B+\log\log x)z + \sum_{j\geq 2}\left[a(x)-b(x)(j-1)-c(x)(j-1)^2\right](-z)^j \\
&= -(B+\log\log x)z + a(x)\left(z+\frac{1}{1+z}-1\right) \\
&\qquad + b(x)\left(1-\frac{2}{1+z}+\frac{1}{(1+z)^2}\right) \\
&\qquad + c(x)\left(1-\frac{4}{1+z}+\frac{5}{(1+z)^2}-\frac{2}{(1+z)^3}\right) \\
&=: \widehat{\mathcal{B}}(x;z). \qquad (14)
\end{aligned}$$

The lower bounds formed by the functions $(a,b,c) \equiv (a_\ell, b_\ell, c_\ell)$ in (14) evaluated at x are given by the corresponding lower bounds from Proposition 5.2 as

$$(a_\ell, b_\ell, c_\ell) := \left(\log\left(\frac{\log x}{\log 2}\right), \log\left(\frac{x}{2}\right), \frac{1}{4}\log^2 2\right).$$

We adjust the uniform bound parameter so that

$$z \equiv z(k,x) = \frac{k-1}{\log\log x} \in [0,1),$$

whenever $1 \leq k \leq \log\log x$ in the notation of Theorem 3.6. This implies that $(1+z)^{-1} \in [1/2, 1]$. The extremal values of the coefficients of $c_\ell(x)$ contribute the following constant factor to our lower bound:

$$\exp\left(c_\ell(x)\left[1-\frac{4}{1+z}+\frac{5}{(1+z)^2}-\frac{2}{(1+z)^3}\right]\right) \geq \exp\left(-\frac{15}{16}(\log 2)^2\right) \approx 0.637357.$$

We next consider the coefficients of $b_\ell(x)$ in our product expansion:

$$\exp\left(b_\ell(x)\left[1-\frac{2}{1+z}+\frac{1}{(1+z)^2}\right]\right) \geq \left(\frac{x}{2}\right)^{-\frac{3}{4}}.$$

Lastly, we will bound the contributions to the product from the coefficients of $a_\ell(x)$ as follows:

$$\begin{aligned}
\exp\left(-a_\ell(x)\left[1-\frac{1}{1+z}+z\right]\right) &\geq \sqrt{\frac{\log 2}{\log x}}\left(\frac{\log x}{\log 2}\right)^z \\
&\gg \sqrt{\frac{\log 2}{\log x}}e^{k-1} \gg \sqrt{\frac{\log 2}{\log x}}.
\end{aligned}$$

In summary, we have arrived at a proof that as $x \to \infty$

$$\frac{e^{Bz}}{(\log x)^{-z}} \times \exp\left(\widehat{\mathcal{B}}(u,x;z)\right) \gg \frac{2^{\frac{3}{4}}(\log 2)^{\frac{1}{2}}}{x^{\frac{3}{4}}(\log x)^{\frac{1}{2}}}\exp\left(-\frac{15}{16}(\log 2)^2\right), \qquad (15)$$

where the leading constant is numerically approximated by $\widehat{C}_0 := 2^{\frac{3}{4}}\sqrt{\log 2}\exp\left(-\frac{15}{16}(\log 2)^2\right) \approx 0.892418$. Finally, to finish our proof of the new form of the lower bound on $\mathcal{G}(-z)$, we need to bound the reciprocal factor of $\Gamma(1-z)$. Since $z \equiv z(k,x) = \frac{k-1}{\log\log x}$ and $k \in [1, \log\log x]$, or again with $z \in [0,1)$, we obtain for minimal k and all large enough $x \gg 1$ that $\Gamma(1-z) = \Gamma(1) = 1$, and for k towards the upper range of its interval that

$$\Gamma(1-z) \approx \Gamma\left(\frac{1}{\log\log x}\right) = \frac{1}{\log\log x}\Gamma\left(1+\frac{1}{\log\log x}\right) \approx \frac{1}{\log\log x}. \qquad \square$$

Remark 5.3 (Technical adjustments in the proof of Theorem 3.6)**.** We now discuss the differences between our construction and that in the technical proof given by Montgomery and Vaughan when we bound $\mathcal{G}(-z)$ from below as in Theorem 3.6. The reference proves that for real $0 \leq z < 2$

$$A_{-z}(x) = -\frac{zF(1,-z)}{\Gamma(1-z)} \cdot x(\log x)^{-(z+1)} + O\left(x(\log x)^{-\operatorname{Re}(z)-2}\right). \tag{16}$$

Recall that for $r < 2$ we have by Cauchy's integral formula that

$$(-1)^k \widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|z|=r} \frac{A_{-z}(x)}{z^{k+1}} dz. \tag{17}$$

We first claim that uniformly for large x and $1 \leq k \leq \log\log x$ we have

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{1-k}{\log\log x}\right) \times \frac{x(\log\log x)^{k-1}}{(\log x)(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^3}\right)\right]. \tag{18}$$

Then since we have proved in Theorem 3.6 above that

$$\left|\mathcal{G}\left(\frac{1-k}{\log\log x}\right)\right| \gg \frac{\widehat{C}_0}{x^{3/4}(\log x)^{1/2}} \cdot \frac{(k-1)}{\log\log x},$$

the result in (18) implies our stated uniform asymptotic bound. Namely, we obtain that

$$\widehat{\pi}_k(x) \gg \frac{\widehat{C}_0 x^{\frac{1}{4}}}{(\log x)^{\frac{3}{2}}} \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

We have to provide analogs to the two separate bounds corresponding to the error and main terms of our estimate according to (16) and (17).
Error Term Bound. To prove that the error term bound holds, we estimate that

$$\begin{aligned}
\left|\frac{1}{2\pi\imath} \int_{|z|=r} \frac{x}{(\log x)^2} \frac{(\log x)^{-\operatorname{Re}(z)}}{z^{k+1}}\right| &\ll x(\log x)^{-(r+2)} r^{-k} \ll \frac{x}{(\log x)^2} \frac{(\log\log x)^k}{e^{k-1}(k-1)^k} \\
&\ll \frac{x}{(\log x)^2} \frac{(\log\log x)^k}{e^{2(k-1)}(k-1)!} \ll \frac{x}{(\log x)^2} \frac{(\log\log x)^k}{(k-1)!} \\
&\ll \frac{x}{\log x} \frac{(\log\log x)^{k-4}}{(k-1)!}.
\end{aligned} \tag{19}$$

Now we can calculate that for $0 \leq z < 1$

$$\begin{aligned}
\prod_p \left(1 + \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{-z} &= \exp\left(-\sum_p \left[\log\left(1 + \frac{z}{p}\right) + z\log\left(1 - \frac{1}{p}\right)\right]\right) \\
&\sim \exp\left(-o(z) \times \sum_p \frac{1}{p^2}\right) \\
&\gg \exp\left(-o(z)\frac{\pi^2}{6}\right) \gg 1.
\end{aligned}$$

In other words, we have that $\left|\mathcal{G}\left(\frac{1-k}{\log\log x}\right)\right| \gg 1$. Thus the error term in (19) is majorized by taking $O\left(\frac{k}{(\log\log x)^3}\right)$.
Main Term Bounds. Now we have to process a more complicated set of integral-based bounds to justify that the main term holds as stated. Notice that the main term estimate corresponfing to (16) and (17) is given by $\frac{x}{\log x} I$, where

$$I := \frac{1}{2\pi\imath} \int_{|z|=r} G(-z)(\log x)^{-z} z^{-k} dz.$$

In particular, we can write $I = I_1 + I_2$ where we define

$$\begin{aligned} I_1 &:= \frac{G(-r)}{2\pi\imath} \int_{|z|=r} (\log x)^{-z} z^{-k} dz \ = \frac{G(-r)(-\log\log x)^{k-1}}{(k-1)!} \\ I_2 &:= \frac{1}{2\pi\imath} \int_{|z|=r} (G(-z) - G(-r))(\log x)^{-z} z^{-k} dz \\ &= \frac{1}{2\pi\imath} \int_{|z|=r} (G(-z) - G(-r) - G'(-r)(z-r))(\log x)^{-z} z^{-k} dz. \end{aligned}$$

We have that

$$\left|G(-z) - G(-r) - G'(-r)(z-r)\right| = \left|\int_r^z (z-w)G''(w)dw\right| \ll |z-r|^2,$$

arguing by a second-order Taylor series expansion where an extreme maximum value of $|(\log x)^{-z}|$ over $|z| = r$ is obtained when $z = -r$:

$$|(\log x)^{-z}| = e^{-\operatorname{Re}(z)\log\log x} \ll e^{r\log\log x}, |z| = r.$$

Moreover, we require a second-degree Taylor expansion of our integrand because we can see that

$$\left|\int (z-r)^2 (\log x)^{-z} z^{-k} dz\right| \asymp \int |z-r|^2 |(\log x)^{-z} z^{-k}||dz|,$$

where for the first-order case we obtain

$$\left|\int (z-r)(\log x)^{-z} z^{-k} dz\right| = o\left(\int |z-r||(\log x)^{-z} z^{-k}||dz|\right).$$

Now we parameterize the curve in the contour for I_2 by writing $z = re^{2\pi\imath t}$ for $t \in [-1/2, 1/2]$. This leads to the bounds

$$\begin{aligned} |I_2| &= r^{3-k} \times \int_{-1/2}^{1/2} |e^{2\pi\imath t} - 1|^2 \cdot (\log x)^{re^{2\pi\imath t}} \cdot e^{2(1-k)\pi\imath t} dt \\ &\ll r^{3-k} \times \int_{-1/2}^{1/2} \sin^2(\pi t) \cdot e^{(1-k)\cos(2\pi t)} dt. \end{aligned}$$

Whenever $|x| \leq 1$, we know that $|\sin x| \leq |x|$. Similarly, we can construct bounds on $-\cos(2\pi t)$ for $t \in [-1/2, 1/2]$ by writing $\cos(2x) = 1 - 2\sin^2 x$ for $|x| < 1/2$. We have an alternating series bound for the sine function that shows

$$\begin{aligned} 1 - 2\sin^2(2\pi t) \geq 1 - 2\left(1 - \frac{\pi t}{3}\right)^2 \geq -1 - \frac{2\pi^2 t^2}{9} \qquad &\implies \\ -\cos(2\pi t) \leq 1 + \frac{2\pi^2 t^2}{9} \leq \left(4 + \frac{2\pi^2}{9}\right) t^2 \leq 1 + 3t^2. & \end{aligned}$$

So it follows that

$$\begin{aligned} |I_2| &\ll r^{3-k} e^{k-1} \times \left|\int_0^\infty t^2 e^{3(k-1)t^2} dt\right| \\ &\ll \frac{r^{3-k} e^{k-1}}{(k-1)^{3/2}} = \frac{(\log\log x)^{k-3} e^{k-1}}{(k-1)^{k-3/2}} \\ &\ll \frac{k \cdot (\log\log x)^{k-3}}{(k-1)!}. \end{aligned}$$

The contribution from the term $|I_2|$ can then be asborbed into the error term bound in (18). Thus our formula lower bound is then correct.

5.3 The distribution of exceptional values of $\Omega(n)$

The next theorems reproduced from [8, §7.4] characterize the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \log\log x$. The tendency of this canonical completely additive function to not deviate substantially from its average order is an exceptional property that allows us to prove asymptotic relations on summatory functions that are weighted by its parity without having to account for significant local oscillations when we average over a large interval.

Theorem 5.4 (Upper bounds on exceptional values of $\Omega(n)$ for large n)**.** *Let*

$$A(x,r) := \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\},$$
$$B(x,r) := \#\left\{n \leq x : \Omega(n) \geq r \cdot \log\log x\right\}.$$

If $0 < r \leq 1$ and $x \geq 2$, then

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x,r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

Theorem 5.5 is an analog to the celebrated Erdös-Kac theorem typically stated for the similarly normally distributed values of the $\omega(n)$ function over $n \leq x$ as $x \to \infty$.

Theorem 5.5 (Exact bounds on exceptional values of $\Omega(n)$ for large n)**.** *We have that as $x \to \infty$*

$$\#\left\{3 \leq n \leq x : \Omega(n) - \log\log n \leq 0\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Remark 5.6. The key interpretation we need to take away from the statements of Theorem 5.4 and Theorem 5.5 is the result proved as the next corollary. The role of the parameter R involved in stating the previous theorem is a critical bound as the scalar factor in the upper bound on $k \leq R\log\log x$ in Theorem 3.6 up to which our uniform bounds given by Theorem 3.6 hold. In contrast, for $n \geq 2$ we can actually have contributions from values distributed throughout the range $1 \leq \Omega(n) \leq \log_2(n)$ infinitely often. It is then crucial that we can show that the main term in the asymptotic formulas we obtain for these summatory functions is captured by summing only over k in the truncated range where the uniform bounds hold.

Corollary 5.7. *Using the notation for $A(x,r)$ and $B(x,r)$ from Theorem 5.4, we have that for $\delta > 0$,*

$$o(1) \leq \left|\frac{B(x,1+\delta)}{A(x,1)}\right| \ll 2, \text{ as } \delta \to 0^+, x \to \infty.$$

Proof. The lower bound stated above should be clear. To show that the asymptotic upper bound is correct, we compute using Theorem 5.4 and Theorem 5.5 that

$$\left|\frac{B(x,1+\delta)}{A(x,1)}\right| \ll \left|\frac{x \cdot (\log x)^{\delta-\delta\log(1+\delta)}}{O(1) + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right)}\right| \sim \left|\frac{(\log x)^{\delta-\delta\log(1+\delta)}}{\frac{1}{2} + o(1)}\right| \xrightarrow{\delta \to 0^+} 2,$$

as $x \to \infty$. Notice that since $\mathbb{E}[\Omega(n)] = \log\log n + B$, with $0 < B < 1$ the absolute constant from Mertens theorem, when we denote the range of $k > \log\log x$ as holding in the form of $k > (1+\delta)\log\log x$ for $\delta > 0$ at large x, we can assume that $\delta \to 0^+$ as $x \to \infty$ [B]. This provides a limiting constant-valued upper bound on the ratios defined above. ∎

[B] In particular, this holds since $k > \log\log x$ implies that

$$\lfloor \log\log x \rfloor + 1 \geq (1+\delta)\log\log x \quad \implies \quad \delta \leq \frac{1 + \{\log\log x\}}{\log\log x} = o(1), \text{ as } x \to \infty.$$

6 Average case analysis of bounds on the Dirichlet inverse functions, $g^{-1}(n)$

The property in (C) of Conjecture 3.5 along squarefree $n \geq 1$ captures an important characteristic of $g^{-1}(n)$ that holds more globally for all $n \geq 1$. In particular, the asymptotic growth of these functions can be captured by more simple formulas than inspection of the first few initial values of the repetitive, quasi-periodic sequence suggests. The pages of tabular data given as Table T.1 in the appendix section (refer to page 40) are intended to provide clear insight into why we arrived at the convenient approximations to $g^{-1}(n)$ proved in this section. The table offers illustrative numerical data by examining the approximate behavior at hand for the cases of $1 \leq n \leq 500$ with *Mathematica*.

6.1 Definitions and basic properties of component function sequences

We define the following sequence for integers $n \geq 1, k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{20}$$

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (20) inductively. This is to emphasize that the growth of $C_k(n)$ when $n \geq 2$ is fixed corresponds to the convolution ω with itself $\Omega(n)$. By the same argument, we see that at fixed n, the function $C_k(n)$ is seen to only ever possibly be non-zero for $k \leq \Omega(n)$ whenever $n \geq 2$.

The sequence of relevant signed semi-diagonals of the functions $C_k(n)$ begins as [14, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \ -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

Example 6.1 (Special cases of the functions $C_k(n)$ for small k)**.** We cite the following special cases which are verified by explicit computation using (20) [14, A066922] [A]:

$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= d(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} - \gcd\left(\Omega(n), \omega(n)\right). \end{aligned}$$

The connection between the auxiliary functions $C_k(n)$ and the inverse sequence $g^{-1}(n)$ is clarified precisely in Section 6.3. Before we can prove explicit bounds on $|g^{-1}(n)|$ through its relation to these functions, we will require a perspective on the lower asymptotic order of $C_k(n)$ for fixed k when n is large.

6.2 Uniform asymptotics of $C_k(n)$ for large all n and fixed, bounded k

The next theorem formally proves a minimal growth rate of the class of functions $C_k(n)$ as functions of k, n for limiting cases of n large and fixed k. In the statement of the result that follows, we view k as a fixed variable which is necessarily bounded in n, but is still taken as an independent parameter as we let $n \to \infty$.

[A] For all $n, k \geq 2$, we have the following recurrence relation satisfied by $C_k(n)$ between successive values of k:

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=0}^{\nu_p(n)-1} C_{k-1}\left(dp^i\right), n \geq 1.$$

Theorem 6.2 (Asymptotics of the functions $C_k(n)$)**.** *For $k := 0$, we have by definition that $C_0(n) = \delta_{n,1}$. For all sufficiently large $n > 1$ and any fixed $1 \leq k \leq \Omega(n)$ taken independently of n, we obtain that the dominant asymptotic term for $C_k(n)$ is bounded uniformly from below as*

$$\mathbb{E}[C_k(n)] \gg (\log\log n)^{2k-1}, \text{ as } n \to \infty.$$

Proof. We prove our bounds by induction on k. We can see by Example 6.1 that $C_1(n)$ satsfies the formula we must establish when $k := 1$ since $\mathbb{E}[\omega(n)] = \log\log n$. Suppose that $k \geq 2$ and let our inductive assumption provide that for all $1 \leq m < k$

$$\mathbb{E}[C_m(n)] \gg (\log\log n)^{2m-1}.$$

Now using the recursive formula we used to define the sequences of $C_k(n)$ in (20), we have that as $n \to \infty$ [B]

$$\begin{aligned}
\mathbb{E}[C_k(n)] &= \mathbb{E}\left[\sum_{d|n} \omega(n/d) C_{k-1}(d)\right] \\
&= \frac{1}{n} \times \sum_{d \leq n} C_{k-1}(d) \times \sum_{r=1}^{\left\lfloor \frac{n}{d} \right\rfloor} \omega(r) \\
&\sim \sum_{d \leq n} C_{k-1}(d) \left[\frac{\log\log(n/d)\left[d \leq \frac{n}{e}\right]_{\delta}}{d} + \frac{B}{d}\right] \\
&\sim \sum_{d \leq \frac{n}{e}} \left[\sum_{m<d} \frac{\mathbb{E}[C_{k-1}(m)]}{m} \log\log\left(\frac{n}{m}\right) + B \cdot \mathbb{E}[C_{k-1}(d)] + B \cdot \sum_{m<d} \frac{\mathbb{E}[C_{k-1}(m)]}{m}\right] \qquad (21) \\
&\gg \frac{B}{n}\left[n\log n \cdot (\log\log n)^{2k-3} - \log n \cdot (\log\log n)^{2k-3}\right] \times \left(1 + \frac{\log n}{2}\right) \\
&\gg (\log\log n)^{2k-1}.
\end{aligned}$$

In transitioning to the last equation from the previous step, we have used that $\frac{B}{2} \cdot (\log n)^2 \gg (\log\log n)^2$ as $n \to \infty$. We have also used that for large n and fixed m, we have by an asymptotic approximation to the incomplete gamma function that results in

$$\int_e^n \frac{(\log\log t)^m}{t} \sim (\log n)(\log\log n)^m, \text{ as } n \to \infty.$$

Thus the claim holds by mathematical induction for large $n \to \infty$ whenever $1 \leq k \leq \Omega(n)$. ∎

6.3 Relating the auxiliary functions $C_k(n)$ to formulas approximating $g^{-1}(n)$

Lemma 6.3 (An exact formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) g^{-1}(n/d) \quad \Longrightarrow \quad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \qquad (22)$$

[B] For all large $x \gg 2$ the summatory function of $\omega(n)$ satisfies [2, §22.10]

$$\sum_{n \leq x} \omega(n) = x \log\log x + Bx + O\left(\frac{x}{\log x}\right).$$

Now by repeatedly expanding the right-hand-side, and removing corner cases in the nested sums with $\omega(1)=0$, we find inductively that

$$(g^{-1}*1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \tag{23}$$

More precisely, we can argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (22) in the form of $(g^{-1}*1)(n) = F_m(n)$ where $F_m(n) := (-1)^m \cdot (C_m(-) * g^{-1})(n)$, or so that

$$F_m(n) = \begin{cases} -\sum\limits_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum\limits_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & m \geq 2, \\ -(\omega * g^{-1})(n), & m = 1. \end{cases}$$

The statement then follows from (23) by Möbius inversion applied to each side of the last equation. □

Since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ using the notation defined in the the proof of Proposition 4.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for squarefree $n \geq 1$. A proof of part (C) of Conjecture 3.5 then follows as an immediate consequence.

Corollary 6.4. *For all squarefree integers $n \geq 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \tag{24}$$

Proof. Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geq 2$ and that n is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where p_i is prime for all $1 \leq i \leq \omega(n)$. So since all divisors of any squarefree n are necessarily also squarefree, we can transform the exact divisor sum guaranteed for all n in Lemma 6.3 into a sum that partitions the divisors by their number of distinct prime factors:

$$\begin{aligned} g^{-1}(n) &= \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i} (-1)^i \cdot C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \end{aligned}$$

The signed contributions in the first of the previous equations is justified by noting that $\lambda(n) = (-1)^{\omega(n)}$ whenever n is squarefree, and that for $d \geq 1$ squarefree we have the correspondence $\omega(d) = k \implies \Omega(d) = k$ for $1 \leq k \leq \log_2(d)$. □

Lemma 6.5. *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{25}$$

Proof. By applying Lemma 6.3, Proposition 4.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $n \geq 1$ we have $\mu(n) = (-1)^{\omega(n)} = \lambda(n)$, Lemma 6.3 implies

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \end{aligned}$$

$$= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d).$$

In the last equation, we see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. □

Combined with the signedness property of $g^{-1}(n)$ guaranteed by Proposition 4.1, Lemma 6.5 shows that the summatory function is expressed as

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_{\Omega(d)}(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Since $\lambda(d)C_{\Omega(d)}(d) = (g^{-1} * 1)^{-1}(d) = (\chi_{\mathbb{P}} + \varepsilon)(d)$ where $\chi_{\mathbb{P}}$ denotes the characteristic function of the primes, we clearly obtain by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1.$$

Corollary 6.6. *We have that*

$$\frac{6}{\pi^2}(\log n)(\log\log n) \ll \mathbb{E}|g^{-1}(n)| \leq \mathbb{E}\left[\sum_{d|n} C_{\Omega(d)}(d)\right].$$

Proof. To prove the lower bound, recall from the introduction that the summatory function of the squarefree integers is given by

$$Q(x) := \sum_{n \leq x} \mu^2(n) = \frac{6}{\pi^2}x + O(\sqrt{x}).$$

Then since $C_{\Omega(d)}(d) \geq 1$ for all $d \geq 1$, and since $\mathbb{E}[C_k(d)]$ is minimized when $k := 1$ according to Theorem 6.2, we obtain by summing over (25) that

$$\begin{aligned}
\frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| &= \frac{1}{x} \times \sum_{d \leq x} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&\sim \sum_{d \leq x} C_{\Omega(d)}(d) \left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dx}}\right)\right] \\
&\geq \sum_{d \leq x} \left[\frac{6 \cdot C_{\Omega(d)}(d)}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dx}}\right)\right] \\
&= \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(x)}(x)] + \sum_{d<x} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O\left(\frac{1}{\sqrt{x}} \times \int_0^x t^{-1/2} dt\right) \\
&\gg \frac{6}{\pi^2}\left[\sum_{e \leq d \leq x} \frac{\log\log d}{d}\right] + O(1) \\
&\sim \frac{6}{\pi^2} \times \int_e^x \frac{\log\log t}{t} dt + O(1) \\
&\gg \frac{6}{\pi^2}(\log x)(\log\log x), \text{ as } x \to \infty.
\end{aligned}$$

To prove the upper bound, notice that by Lemma 6.3 and Corollary 6.4,

$$|g^{-1}(n)| \leq \sum_{d|n} C_{\Omega(d)}(d), n \geq 1.$$

Now since both of the above quantities are positive for all $n \geq 1$, we clearly obtain the upper bound stated above when we average over $n \leq x$ for all large x. □

6.3.1 A connection to the distribution of the primes

Remark 6.7. The combinatorial complexity of relating $g^{-1}(n)$ to the distribution of the primes motivates us to consider the properties of this sequences beyond that which the bounds we have proved so far reveal. While the magnitudes and dispersion of the primes $p \leq x$ certainly restricts the repeating of these values we can see in the contributions to $G^{-1}(x)$, the following statement is clear about the relation of the weights $|g^{-1}(n)|$ to the prime numbers: The value of $|g^{-1}(n)|$ is entirely dependent on the pattern of the *exponents* (viewed as multisets) of the distinct prime factors of $n \geq 2$. The relation of the repitition of the distinct values of $|g^{-1}(n)|$ as weights to the signed summatory function $G^{-1}(x)$ makes a clear tie to $M(x)$ through Proposition 7.1 proved in the next section. We can also make the relation of the distribution of $|g^{-1}(n)|$ to the prime factorization of n more precise via the points in the next example.

Example 6.8 (Combinatorial significance to the distribution of $g^{-1}(n)$)**.** We have a natural extremal behavior with respect to distinct values of $\Omega(n)$ corresponding to squarefree integers, and prime powers. Namely, if for $k \geq 1$ we define the infinite sets M_k and m_k to correspond to the maximal (minimal) positive integers such that

$$M_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\},$$

$$m_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\},$$

then any element of M_k is squarefree and any element of m_k is a prime power. In particular, we have that for any $N_k \in M_k$ and $n_k \in m_k$

$$N_k = \sum_{j=0}^{k} \binom{k}{j} j!, \qquad \text{and} \qquad n_k = 2 \cdot (-1)^k.$$

Moreover, using the formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 4.1, we can express an exact formula for $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n. Namely, for $n \geq 2$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z \cdot \nu_p(n)) = [z^k] \prod_{p^\alpha || n} (1 + \alpha z), 0 \leq k \leq \omega(n).$$

Then we have essentially shown using (25) that we can expand

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, n \geq 2.$$

The combinatorial formula for $h^{-1}(n) = \lambda(n) \cdot (\Omega(n))! \times \prod_{p^\alpha || n} (\alpha!)^{-1}$ we derived in the proof of the key signedness proposition in Section 4 suggests further patterns and more regularity in the contributions of the distinct weighted terms for $G^{-1}(x)$ when we sum over all of the distinct prime exponent patterns that factorize $n \leq x$.

7 Lower bounds for $M(x)$ along infinite subsequences

Proposition 7.1. *For all sufficiently large x, we have that*

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt. \tag{26}$$

Proof. We know by applying Corollary 3.3 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k)+1) \\ &\approx G^{-1}(x) + \sum_{k=1}^{x} g^{-1}(k)\pi(x/k), \end{aligned} \tag{27}$$

We can replace the floored integer-valued arguments to $\pi(x)$ in (27) using its approximation by the monotone non-decreasing asymptotic order, $\pi(x) \sim \frac{x}{\log x}$. We can always bound

$$\frac{Ax}{\log x} \leq \pi(x) \leq \frac{Bx}{\log x},$$

for suitably defined absolute constants, $A, B > 0$ whenever $x \geq 2$. Therefore the approximation obtained by replacing $\pi(x)$ by the main term in its limting asymptotic formula is actually valid for all $x > 1$ up to at most a small constant difference.

What we require to sum and simplify the right-hand-side terms from (27) follows from the exact summation by parts formula. In particular, we argue that for sufficiently large $x \geq 2$ we can approximate [A]

$$\begin{aligned} \sum_{k=1}^{x} g^{-1}(k)\pi(x/k) &= G^{-1}(x)\pi(1) - \sum_{k=1}^{x-1} G^{-1}(k)\left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)\right] \\ &= -\sum_{k=1}^{x/2} G^{-1}(k)\left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)\right] \\ &\approx -\sum_{k=1}^{x/2} G^{-1}(k)\left[\frac{x}{k \cdot \log(x/k)} - \frac{x}{(k+1)\cdot\log(x/k)}\right] \qquad (28a) \\ &\approx -\sum_{k=1}^{x/2} G^{-1}(k)\frac{x}{k^2 \cdot \log(x/k)}. \qquad (28b) \end{aligned}$$

Indeed, we can justify that step (28a) is correct by writing

$$\begin{aligned} \frac{x}{(k+1)\log\left(\frac{x}{k+1}\right)} &= \frac{x}{k+1} \cdot \frac{1}{\left[\log\left(\frac{x}{k}\right) + \log\left(1 - \frac{1}{k+1}\right)\right]} = \frac{x}{(k+1)\log\left(\frac{x}{k}\right)} \cdot \frac{1}{1 + \frac{\log\left(1-\frac{1}{k+1}\right)}{\log x\left[1-\frac{\log k}{\log x}\right]}} \\ &\sim \frac{x}{(k+1)\log\left(\frac{x}{k}\right)}, \text{ as } x \to \infty. \end{aligned}$$

The correctness of the transition from step (28a) to (28b) is verified by seeing that for $\operatorname{Re}(s) > 1$, we have that

$$\infty > \left|\frac{1}{s \cdot (P(s)+1)\zeta(s)}\right| = \left|\int_1^{\infty} \frac{G^{-1}(x)}{x^{s+1}} dx\right| = \left|\sum_{k\geq 1} \frac{G^{-1}(k)}{k^{s+1}}\right|.$$

[A] Since $\pi(1) = 0$, the actual range of summation corresponds to $k \in \left[1, \frac{x}{2}\right]$.

When $s := \frac{3}{2}$, we obtain that

$$0 \leq \left|\sum_{k \geq 1} \frac{G^{-1}(k)}{k^2(k+1)}\right| \leq \left|\sum_{k\geq 1} \frac{G^{-1}(k)}{k^{\frac{5}{2}}}\right| < \infty.$$

Then the difference of the terms in forming the approximation in this step is bounded above and below by absolute constants as

$$\left|\sum_{k=1}^{\frac{x}{2}} G^{-1}(k)\left[\frac{1}{k^2} - \frac{1}{k(k+1)}\right]\right| \leq \left|\sum_{k=1}^{\frac{x}{2}} \frac{G^{-1}(k)}{k^2(k+1)}\right| = O(1).$$

Now for x large enough the summand factor $\frac{x}{k^2 \cdot \log(x/k)}$ is monotonic as k ranges over $k \in [1, x/2]$ in ascending order. Because this summand factor is a smooth function of k (and x) where $G^{-1}(x)$ is a summatory function with jumps only in steps of the positive integers, we can finally approximate $M(x)$ for any finite $x \geq 2$ as follows:

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt.$$

We will later only use unsigned lower bound approximations to this function in the next theorems so that the signedness of the summatory function term in the integral formula above doe not require further attention in limiting cases as $x \to \infty$. □

7.1 Establishing initial lower bounds on the summatory functions $G^{-1}(x)$

Let the summatory function $G_E^{-1}(x)$ be defined for $x \geq 1$ by [B]

$$G_E^{-1}(x) := \sum_{n \leq (\log x)^5 (\log\log x)} \lambda(n) \times \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log\log d}. \tag{29}$$

Theorem 7.2. *For almost all sufficiently large integers $x \to \infty$, we have that*

$$|G^{-1}(x)| \gg |G_E^{-1}(x)|.$$

Proof. First, consider the following upper bound on $|G_E^{-1}(x)|$:

$$\begin{aligned} |G_E^{-1}(x)| &= \left| \sum_{e \leq n \leq (\log x)^5 (\log\log x)} \lambda(n) \times \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log\log d}\right| \\ &\ll \sum_{e<d\leq (\log x)^5(\log\log x)} \frac{(\log d)^{\frac{1}{4}}}{\log\log d} \cdot \left\lfloor \frac{(\log x)^5 (\log\log x)^{16}}{d}\right\rfloor \\ &\ll (\log x)^5 (\log\log x) \times \int_e^{(\log x)^5(\log\log x)} \frac{(\log t)^{\frac{1}{4}}}{t \cdot \log\log t} dt \\ &= (\log x)^5(\log\log x) \times \mathrm{Ei}\left(\frac{5}{4}\log\log\left((\log x)^5(\log\log x)\right)\right) \\ &\ll \frac{25}{64} \cdot (\log x)^5 (\log\log x)(\log\log\log x)^2. \end{aligned} \tag{30}$$

[B]The subscript of E on the function $G_E^{-1}(x)$ is a formality of notation and does not correspond to an actual parameter or any implicit dependence on E in the definition of this function.

Next, we bound the summatory function $|G^{-1}(x)|$ from below. In particular, we compute that for almost every sufficiently large $x \to \infty$:

$$\frac{|G^{-1}(x)|}{x} = \frac{1}{x} \times \left| \sum_{\substack{d \leq x \\ \lambda(d)=+1}} |g^{-1}(d)| - \sum_{\substack{d \leq x \\ \lambda(d)=-1}} |g^{-1}(d)| \right| \gg \left| \mathbb{E}|g^{-1}(x)| - \frac{2}{x} \times \sum_{\substack{d \leq x \\ \lambda(d)=-1}} |g^{-1}(d)| \right|.$$

Let the indeterminate summation in the previous equation be defined by

$$S_-(x) := \sum_{\substack{d \leq x \\ \lambda(d)=-1}} |g^{-1}(d)|.$$

We will find upper and lower bounds on this sum that show $\mathbb{E}|g^{-1}(x)| \gg \frac{S_-(x)}{x}$. First, for the positive summands to be at their largest, we require that for $d \geq 2$

$$|g^{-1}(d)| = \sum_{j=0}^{\omega(d)} \binom{\omega(d)}{j} j!.$$

Then we have that

$$S_-(x) \ll \sum_{1 \leq k \leq \log_2(x)} \widehat{\pi}_k(x) \times \sum_{j=0}^{k} \binom{k}{j} j!. \tag{31}$$

We can bound the summatory function terms by

$$\widehat{\pi}_k(x) \leq \frac{\widehat{\pi}_k(x) \cdot \pi_k(x)}{\#\left\{n \leq x : \Omega(n) = \omega(n) \wedge \Omega(n) = k\right\}}.$$

By an argument considering conditional probabilities of sets, we then obtain

$$\begin{aligned} \#\left\{n \leq x : \Omega(n) = \omega(n) \wedge \Omega(n) = k\right\} &\geq \frac{1}{x} \cdot \#\left\{n \leq x : n \text{ squarefree} \wedge \mu(n) = (-1)^k\right\} \times \widehat{\pi}_k(x) \\ &= \frac{3}{\pi^2} \widehat{\pi}_k(x), \text{ as } x \to \infty. \end{aligned}$$

So from (31)

$$S_-(x) \ll \sum_{1 \leq k \leq \log_2(x)} \frac{\pi^2}{3} \pi_k(x) \times \sum_{j=0}^{k} \binom{k}{j} j!. \tag{32}$$

We weight by the known asymptotic formula for the summatory functions $\pi_k(x) \sim \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}(1+o(1))$ as $x \to \infty$ to find that

$$\begin{aligned} S_-(x) &\ll \frac{\pi^2}{3} \times \sum_{1 \leq k \leq \log_2(x)} \pi_k(x) \times \sum_{j=0}^{k} \binom{k}{j} j! \\ &\ll \frac{\pi^2}{3} \times \frac{x}{(\log x)(\log\log x)} \times \sum_{k \geq 1} k \cdot (\log\log x)^k \sum_{j=0}^{k} \frac{1}{j!} \\ &\ll \frac{\pi^2}{3} \times \frac{ex}{(\log x)(\log\log x)} \times \sum_{k \geq 1} k \cdot (\log\log x)^k \\ &\ll \frac{\pi^2}{3} \times \frac{ex}{(\log x)(\log\log x)^2}. \end{aligned}$$

Thus, over these choices bounding the $g^{-1}(d)$, we obtain that $\frac{S_-(x)}{x} = o(1)$ as $x \to \infty$.
On the other hand, we can choose the summands to satisfy $|g^{-1}(d)| \geq 2$. We define the following densities for large $x \geq 2$:

$$\mathcal{L}_+(x) := \frac{1}{n} \cdot \#\{n \leq x : \lambda(n) = +1\}$$
$$\mathcal{L}_-(x) := \frac{1}{n} \cdot \#\{n \leq x : \lambda(n) = -1\}.$$

We know that [16, *cf.* §1]

$$\lim_{x\to\infty} \mathcal{L}_+(x) = \lim_{x\to\infty} \mathcal{L}_-(x) = \frac{1}{2}.$$

In general, we can have local fluctuations so that $\mathcal{L}_-(x) \in (0, 1)$ for large x, though these densities should be approximately $\frac{1}{2}$. Now we see that

$$S_-(x) \gg 2 \cdot \min\left(\mathcal{L}_-(x), 1 - \mathcal{L}_-(x)\right) \cdot x.$$

This implies that $\frac{S_-(x)}{x} = O(1)$. In either of these extreme cases, we have by Corollary 6.6 that

$$\frac{|G^{-1}(x)|}{x} \gg \frac{6}{\pi^2}(\log x)(\log\log x).$$

Then naturally from (30) we have proved that as $x \to \infty$, $|G^{-1}(x)| \gg |G_E^{-1}(x)|$. □

Note that the only cases of $x \geq 1$ we need to be wary of in the *almost everywhere* clause to applying the statement of Theorem 7.2 happen when $G^{-1}(x) = 0$. In these cases, the bounds we proved above cannot be conclusively shown to hold. It suffices to assume that $G^{-1}(x) \neq 0$ on a dense subset of the integers for the bounds we require to prove Corllary 3.9 in the last subsection.

7.1.1 A few more necessary results (TODO)

We now use the superscript and subscript notation of (ℓ) not to denote a formal parameter to the functions we define below, but instead to denote that these functions form *lower bound* (rather than exact) approximations to other forms of the functions without the scripted (ℓ).

Lemma 7.3. *Suppose that $\widehat{\pi}_k(x) \geq \widehat{\pi}_k^{(\ell)}(x) \geq 0$ for $\widehat{\pi}_k^{(\ell)}(x)$ a monotone real-valued function of x for all integers $k \geq 1$ and sufficiently large $x \geq 2$. Let*

$$A_\Omega^{(\ell)}(x) := \sum_{k \leq \log\log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x)$$
$$A_\Omega(x) := \sum_{k \leq \log\log x} (-1)^k \widehat{\pi}_k(x).$$

Then for all sufficiently large x, we have that

$$|A_\Omega(x)| \gg |A_\Omega^{(\ell)}(x)|.$$

Proof. Given an explicit smooth lower bounding function, $\widehat{\pi}_k^{(\ell)}(x)$, we define the similarly smooth and monotone residual terms in approximating $\widehat{\pi}_k(x)$ through the following notation:

$$\widehat{\pi}_k(x) = \widehat{\pi}_k^{(\ell)}(x) + \widehat{E}_k(x).$$

Then we can form the ordinary exact form of the summatory function as

$$|A_\Omega(x)| \gg \left| \sum_{k \leq \frac{\log\log x}{2}} \left[\widehat{\pi}_{2k}(x) - \widehat{\pi}_{2k-1}(x)\right] \right|$$

$$\geq \left| A_{\Omega}^{(\ell)}(x) - \sum_{k \leq \frac{\log\log x}{2}} \left[\widehat{E}_{2k}(x) - \widehat{E}_{2k-1}(x) \right] \right|$$

$$\geq \left| A_{\Omega}^{(\ell)}(x) \right| - \left| \sum_{k \leq \frac{\log\log x}{2}} \left[\widehat{E}_{2k}(x) - \widehat{E}_{2k-1}(x) \right] \right|.$$

If the latter sum, denoted

$$\mathrm{ES}(x) := \left| \sum_{k \leq \frac{\log\log x}{2}} \left[\widehat{E}_{2k}(x) - \widehat{E}_{2k-1}(x) \right] \right|,$$

grows without bound as $x \to \infty$, then we can always find some absolute $C_0 > 0$ (by monotonicity) such that $\mathrm{ES}(x) \leq C_0 \cdot A_{\Omega}(x)$:

$$\mathrm{ES}(x) = \left| A_{\Omega}(x) - A_{\Omega}^{(\ell)}(x) \right| \leq \left| |A_{\Omega}(x)| + \left| A_{\Omega}^{(\ell)}(x) \right| \right| \ll 2\,|A_{\Omega}(x)|.$$

If on the other hand this sum becomes constant, or is bounded as $x \to +\infty$, then we clearly have another absolute $C_1 > 0$ such that $|A_{\Omega}(x)| \geq C_1 \cdot |A_{\Omega}^{(\ell)}(x)|$. In either case, the claimed result holds for all large enough x. □

Lemma 7.3 shows that we can use lower bound formulas for summatory functions in conjunction with integral-based Abel summation techniues to similarly recover lower bounds on the target functions.

Lemma 7.4. *Suppose that $f(n)$ is an arithmetic function defined such that $f(n) > 0$ for all $n > u_0$ where $f(n) \gg \widehat{\tau}_{\ell}(n)$ as $n \to \infty$. Assume also that the bounding function $\widehat{\tau}_{\ell}(t)$ is a non-negative continuously differentiable function of t for all large enough $t \gg u_0$. We define the λ-sign-scaled summatory function of f as follows:*

$$F_{\lambda}(x) := \sum_{u_0 < n \leq x} \lambda(n) \cdot f(n).$$

Let the summatory weight functions be defined as

$$A_{\Omega}^{(\ell)}(t) := \sum_{k=1}^{\lfloor \log\log t \rfloor} (-1)^k \widehat{\pi}_k^{(\ell)}(t),$$

$$A_{\Omega}(t) := \sum_{k=1}^{\lfloor \log\log t \rfloor} (-1)^k \widehat{\pi}_k(t),$$

where $\widehat{\pi}_k(x) \geq \widehat{\pi}_k^{(\ell)}(x) \geq 0$ for $\widehat{\pi}_k^{(\ell)}(t)$ a smooth monotone function of t at all sufficiently large $t \to \infty$. Then we have that

$$|F_{\lambda}(x)| \gg \left| \left| A_{\Omega}^{(\ell)}(x) \widehat{\tau}_{\ell}(x) \right| - \left| \int_{u_0}^{x} A_{\Omega}^{(\ell)}(t) \widehat{\tau}_{\ell}'(t) dt \right| \right|. \tag{33}$$

Proof. We can form an accurate $C^1(\mathbb{R})$ approximation by the smoothness of $\widehat{\pi}_k^{(\ell)}(x)$ that allows us to apply the Abel summation formula using the summatory function $A_{\Omega}^{(\ell)}(t)$ for t on any bounded connected subinterval of $[1, \infty)$. The stated lower bound formula for $F_{\lambda}(x)$ in (33) above is valid by Abel summation and by applying Lemma 7.3. In particular, whenever

$$0 \leq \left| \frac{\sum\limits_{\log\log t < k \leq \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_{\Omega}(t)} \right| \ll 2, \text{ as } t \to \infty,$$

we see that the asymptotically dominant terms indicating the parity of $\lambda(n)$ are captured up to a constant factor by the terms in the range over k summed by $A_\Omega(t)$. In other words, taking the sum only over the summands that define $A_\Omega(x)$ on the truncated range of $k \in [1, \log\log x]$ does not non-trivially change the asymptotically dominant terms in the lower bound. This property remarkably holds even when we should technically index over all $k \in [1, \log_2(x)]$ to obtain an exact formula for this summatory weight function. By Corollary 5.7, we have that the assertion above holds as $t \to \infty$.

Secondly, observe that provided sufficiently smoothness (differentiability) of close approximations to $A_\Omega(t)$ (to $f(t)$) on (u_0, x), we have that

$$\begin{aligned} |F_\lambda(x)| &\geq \left| |A_\Omega(x) f(x)| - \int_{u_0}^x |A_\Omega(t) f'(t)| dt \right| \\ &\gg \left| \left| A_\Omega^{(\ell)}(x) \widehat{\tau}_\ell(x) \right| - \int_{u_0}^x \left| A_\Omega^{(\ell)}(t) \widehat{\tau}_\ell'(t) \right| dt \right| \\ &\gg \left| \left| A_\Omega^{(\ell)}(x) \widehat{\tau}_\ell(x) \right| - \left| \int_{u_0}^x A_\Omega^{(\ell)}(t) \widehat{\tau}_\ell'(t) dt \right| \right|. \end{aligned}$$

The previous equations follow from the ordinary Abel summation method by applying the argument in Lemma 7.3 and using the triangle inequality. □

Corollary 7.5. *We have that for almost every sufficiently large x, that as $x \to \infty$*

$$\left|G_E^{-1}(x)\right| \gg \frac{2^{\frac{1}{4}} e (\log 2)^{\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{15}{16} (\log 2)^2\right) \times \frac{1}{(\log x)^{\frac{1}{4}} (\log\log x)^{\frac{1}{4}}} \times \left| \sum_{e < d \leq \log x} \frac{\lambda(d) (\log d)^{\frac{1}{4}}}{d^{1/4} \cdot \log\log d} \right|.$$

Proof. Using the definition in (29), we obtain on average that [C]

$$\begin{aligned} \left|G_E^{-1}(x)\right| &= \left| \sum_{n \leq (\log x)^5 (\log\log x)} \lambda(n) \times \sum_{\substack{d \mid n \\ d > e}} \frac{\lambda(d) (\log d)^{\frac{1}{4}}}{\log\log d} \right| \\ &= \left| \sum_{e < d \leq (\log x)^5 (\log\log x)} \frac{(\log d)^{\frac{1}{4}}}{\log\log d} \times \sum_{n=1}^{\left\lfloor \frac{\log x}{d} \right\rfloor} \lambda(dn) \right|. \end{aligned}$$

We see that by complete additivity of $\Omega(n)$ (complete multiplicativity of $\lambda(n)$) that

$$\sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn) = \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(d) \times \lambda(n) = \lambda(d) \times \sum_{n \leq \left\lfloor \frac{x}{d} \right\rfloor} \lambda(n).$$

Now using Theorem 3.7 and Lemma 7.3, we can establish that

$$\left| \sum_{k \leq \log\log x} (-1)^k \cdot \widehat{\pi}_k(x) \right| \gg \frac{2^{\frac{1}{4}} e (\log 2)^{\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{15}{16} (\log 2)^2\right) \cdot \frac{x^{\frac{1}{4}}}{(\log x)^{\frac{3}{2}} \sqrt{\log\log x}} =: \widehat{L}_0(x), \text{ as } x \to \infty. \tag{34}$$

[C] For any arithmetic functions f, h, we have that [1, *cf.* §3.10; §3.12]

$$\sum_{n \leq x} h(n) \times \sum_{d \mid n} f(d) = \sum_{d \leq x} f(d) \times \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} h(dn).$$

The sign of the sum obtained by taking the right-hand-side of (34) without the absolute value operation is given by $(-1)^{1+\lfloor \log\log x\rfloor}$. The precise formula for the limiting lower bound stated above for $\widehat{L}_0(x)$ is computed by symbolic summation in *Mathematica* using the new bounds on $\widehat{\pi}_k(x)$ guaranteed by the theorem, and then by applying subsequent standard asymptotic estimates to the resulting formulas for large $x \to \infty$, e.g., in the form of (9c) and Stirling's formula. It follows that

$$|G_E^{-1}(x)| \gg \left| \sum_{e<d\leq (\log x)^5(\log\log x)} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor} \cdot \widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right|. \tag{35}$$

Outline for the remainder of the proof. We sketch the following core steps remaining to prove our claimed lower bound on $|G_E^{-1}(x)|$:

- **(A)** We identify an initial subinterval of our full bounds on the summation defined by (29). On this subinterval we prove that we can expect constant sign term contributions resulting from the inputs to the function $\widehat{L}_0$ involving (a priori) both d, x for x large and d on this subinterval. This consideration keeps the sign of $\lambda(d)$ intact in the resulting formula.
- **(B)** We then factor out easily bounded terms from the expansion of the monotone $\widehat{L}_0$ on this interval.
- **(C)** We define and determine additional characteristic formulas we will refer to in later sections for the resulting lower bounds that are formed by restricting the range of d in (35) to just this initial range.
- **(D)** Finally, we must argue precisely that the oscillatory, signed terms from the upper end of the deleted interval cannot generate trivial bounds by cancellation with the stated lower bounds.

Part A. We will simplify (35) by proving that there are ranges of consecutive integers over which we obtain effectively constant sign contributions from the function $\widehat{L}_0((\log x)^5(\log\log x)/d)$ as a function of both x, d. The idea is to identify this initial accesible interval case, and then prove that we can form a lower bound on $G_E^{-1}(x)$ by truncating and summing only over the d in this range.

In particular, consider that

$$\begin{aligned}\log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right) &= \log\log\left((\log x)^5(\log\log x)\right) \\ &\quad + \log\left(1 - \frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)}\right), \text{ as } x \to \infty.\end{aligned}$$

If we take $d \in (e, \log x] =: \mathcal{R}_x$, we have that

$$\frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)} = o(1) \to 0,$$

as $x \to \infty$. For d taken within $\mathcal{R}_x$, we expect that for almost every x there are at most a handful of negligible cases of comparitively small order $d \leq d_0(x)$ such that

$$\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor \sim \left\lfloor \log\log\left((\log x)^5(\log\log x)\right) + o(1)\right\rfloor,$$

changes in parity transitioning from $d_0(x)-1$ to $d_0(x)$. An argument making this assertion precise brings leads us to two primary cases that rely inexactly on the distribution of the fractional parts of $\left\{(\log x)^5(\log\log x)\right\}$ within $[0,1)$ for large integers $x \to \infty$ and any $\log d \in \mathcal{R}_x$:

(1) If the fractional part $\left\{\log\log\left((\log x)^5(\log\log x)\right)\right\} = 0$, then

$$\begin{aligned}\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor &= \left\lfloor \log\log\left((\log x)^5(\log\log x)\right)\right\rfloor \\ &\quad + \left\lfloor -\frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)}\right\rfloor.\end{aligned}$$

This implies that provided that

$$-1 \leq -\frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)} < 0,$$

we obtain a constant sign term for $\operatorname{sgn}\left[\widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right]$. Since d is positive and maximized at $\log x$, this condition clearly happens for all sufficiently large x.

(2) If the fractional part $\{\log\log\left((\log x)^5(\log\log x)\right)\} \in (0,1)$, then

$$\begin{aligned}\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor &= \left\lfloor \log\log\left((\log x)^5(\log\log x)\right)\right\rfloor \\ &\quad + \left\lfloor \left\{\log\log\left((\log x)^5(\log\log x)\right)\right\} - \frac{\log d}{(\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)}\right\rfloor.\end{aligned}$$

Define the next shorthand notation for the fractional parts $f_x := \left\{\log\log\left((\log x)^5(\log\log x)\right)\right\}$ and the function $\mathcal{B}(x) := (\log x)^5(\log\log x)\log\left((\log x)^5(\log\log x)\right)$. We require that

$$-1 \leq f_x - \frac{\log d}{\mathcal{B}(x)} < 0 \iff (1+f_x)\cdot\mathcal{B}(x) \geq \log d > 0,$$

which is similarly clearly attained as $x \to \infty$.

In either case, we obtain the constant sign term on the contribution from $\widehat{L}_0$ for d on this subinterval, $\mathcal{R}_x$.

Part B. Then provided that the sign term involving both d and x from (35) does not change for d within our new interval, $\mathcal{R}_x$, we can factor out the dependence of the sign on the monotonically decreasing function $\widehat{L}_0\left((\log x)^5(\log\log x)/d\right)$ in the variable d as we sum along the lower interval $\mathcal{R}_x$. We can see that this function is decreasing for $d \in \mathcal{R}_x$ by computing its partial derivative with respect to d and evaluating the asymptotically dominant terms with leading negative sign as $x \to \infty$. Then we determine that we should select $d := \log x$ in (35) to obtain a global lower bound on $|G_E^{-1}(x)|$ if we truncate the sum defined by (29) to include only the indices $d \in \mathcal{R}_x$.

Part C. Let the magnitudes of the oscillatory remainder term sums be defined for all sufficiently large x by

$$R_E(x) := \left|\sum_{\log x < d < \frac{(\log x)^5(\log\log x)}{e}} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor} \cdot \widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right|.$$

Set the function $T_E(x)$ to correspond to the easily factored dependence of the less simply integrable factors in $\widehat{L}_0$ when we set $d := \log x$. It is defined for all large enough x as

$$T_E(x) := \frac{1}{\log\left[(\log x)^5(\log\log x)\right]^{\frac{3}{2}}\sqrt{\log\log\left[(\log x)^5(\log\log x)\right]}} \gg \frac{1}{(\log\log x)^{\frac{3}{2}}\sqrt{\log\log\log x}}. \tag{36}$$

Then, as we argued before, we see that as $x \to \infty$

$$\begin{aligned} S_{E,1}(x) &:= \left|\sum_{e < d \leq (\log x)^5(\log\log x)} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\left\lfloor \log\log\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right\rfloor} \widehat{L}_0\left(\frac{(\log x)^5(\log\log x)}{d}\right)\right| \\ &\gg \frac{2^{\frac{1}{4}}e(\log 2)^{\frac{1}{2}}}{\sqrt{\pi}}\exp\left(-\frac{15}{16}(\log 2)^2\right) \times (\log x)^{\frac{5}{4}}(\log\log x)^{\frac{1}{4}}T_E(x) \times \left|\sum_{e < d \leq \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d^{1/4}\cdot\log\log d}\right| \qquad (37) \\ &\gg \frac{2^{\frac{1}{4}}e(\log 2)^{\frac{1}{2}}}{\sqrt{\pi}}\exp\left(-\frac{15}{16}(\log 2)^2\right) \times (\log x)^{\frac{5}{4}}(\log\log x)^{\frac{1}{4}}T_E(x) \times \\ &\quad \times \left|A_\Omega^{(\ell)}(\log x)\widehat{\tau}_0(\log x) - \int_e^{\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_0'(t)dt\right|, \end{aligned}$$

where we select the functions $\widehat{\tau}_0(t) := \frac{(\log t)^{1/4}}{t^{1/4} \cdot \log\log t}$ and $-\widehat{\tau}_0'(t) \gg \frac{(\log t)^{1/4}}{4t^{5/4} \cdot \log\log t}$ in the notation of Lemma 7.4.

What we then obtain from (35) and (37) is the following lower bound by the triangle inequality that holds for all sufficiently large x:

$$|G_E^{-1}(x)| \gg \left| S_{E,1}(x) - R_E(x) \right| \gg S_{E,1}(x), \text{ as } x \rightarrow \infty. \tag{38}$$

We have claimed that in fact we can drop the sum terms over upper range of d and still obtain the asymptotic lower bound on $|G_E^{-1}(x)|$ as $x \rightarrow \infty$ on the right-hand-side of (38). To justify this step in the proof, we will provide limiting lower bounds on $R_E(x)$ that show that the contribution from these terms in absolute value exceeds the magnitude of the corresponding sums over $d \in \mathcal{R}_x$ when x is large.

Part D. In Theorem 7.7 stated in the next section, we prove lower bounds on the sums we used to define $S_{E,1}(x)$ of the form

$$S_{E,1}(x) \gg \frac{e^2 (\log 2)^2 \exp\left(-\frac{15}{16}(\log 2)^2\right) \cdot (\log x)^{\frac{5}{4}}}{4\sqrt{2}\pi \cdot (\log\log x)^{\frac{5}{2}} (\log\log\log x)^2}.$$

The lower bounds on the right-hand-side of the previous equation are clearly $o\left((\log x)^{\frac{5}{4}} (\log\log x)^{\frac{1}{4}}\right)$, though still grow without bound as $x \rightarrow \infty$. In contrast, we can bound from below to show that the contribution from $R_E(x)$ is at least on the order of a constant times $(\log x)^{\frac{5}{4}} (\log\log x)^{\frac{1}{4}}$. To obtain this lower bound, consider that since $\frac{(\log d)^{\frac{1}{4}}}{d^{1/4} \cdot \log\log d}$ is monotone decreasing for all large enough $d > e$, we obtain the smallest possible magnitude on the sum by alternating signs on consecutive terms in the sum. We can then bound the sum as $x \rightarrow \infty$ by

$$\begin{aligned}
\frac{R_E(x)}{(\log x)^{\frac{5}{4}} (\log\log x)^{\frac{1}{4}}} &\gg \left| o(1) + \sum_{\log x < d < \frac{(\log x)^5 (\log\log x)}{2e}} \left[\frac{\log(2d)^{1/4}}{(2d)^{1/4} \cdot \log\log(2d)} - \frac{\log(2d+1)^{1/4}}{(2d+1)^{1/4} \log\log(2d+1)} \right] \right| \\
&\approx \left| \sum_{\log x < d < \frac{(\log x)^5 (\log\log x)}{2e}} \frac{\log(2d)^{1/4}}{(2d)^{1/4} \log\log(2d)} \left[1 - \frac{\left(1 + \frac{1}{2d \cdot \log(2d)}\right)^{1/4}}{\left(1 + \frac{1}{2d}\right)^{1/4} \left(1 + \frac{1}{2d \cdot \log(2d) \log\log(2d)}\right)} \right] \right|.
\end{aligned}$$

Then by an appeal to binomial and geometric series expansions, we obtain that the significant terms in the inner terms of the last sum are bounded by

$$\frac{R_E(x)}{(\log x)^{\frac{5}{4}} (\log\log x)^{\frac{1}{4}}} \gg \left| \sum_{\log x < d < \frac{(\log x)^5 (\log\log x)}{2e}} O\left(\frac{\log(2d)^{1/4}}{(2d)^{5/4} \log\log(2d)} \right) \right| = O(1).$$

What we obtain from the previous several caclulations is that the magnitude of $R_E(x)$ always exceeds that of the lower bound we establish in Theorem 7.7 for the sums over $d \in \mathcal{R}_x$ as $x \rightarrow \infty$. $\square$

Remark 7.6 (Foreword and clarifications in approaching the central theorem proofs)**.** There is a subtle point which we do not belabor in formalizing a key component of our weighted summation method using the previous few results. Although the summatory weight function represented by defining (34) technically corresponds to the highly oscillatory local behavior implicit to summing $L(x) := \sum_{n \leq x} \lambda(n)$, we intend on abstracting localized behavior of the latter function by first averaging over $n \leq x$ and then weighting the results á fortiori according to the parity of k in the regular asymptotics underlying the distribution of $\{n \leq x : \Omega(n) = k\}$ for k uniformly bounded in x.

The assumed denominator weight of the function $\widehat{\tau}_0'(t)$, which is partially factored from the input to the function $\widehat{L}_0(x)$ from (35), in (40) is more suggestive that we are performing a weighted integral operation of the form

$$\sum_{n \leq x} \lambda(n) f(n) \stackrel{\blacktriangle}{=} \int_e^x f(t) dL(t),$$

as expressed in the style of Riemann-Stieltjes integral notation. The classical method we have used to state these results is still based on ordinary Abel summation. The scaling in (40) also has the effect of ensuring that the differential-type sign weight represented in these formulas described above does not contradict known behavior and bounds on the related summatory function of the Liouville lambda function.

7.1.2 The proof of a central lower bound on the magnitude of $G_E^{-1}(x)$

The next central theorem is the last barrier required to prove Theorem 3.9 in the next subsection. Combined with Theorem 7.2 proved in the last section, the new lower bounds we establish below provide us with a sufficient mechanism to bound the formula from Proposition 7.1.

Theorem 7.7 (Asymptotics and bounds for the summatory function $G^{-1}(x)$)**.** *We define a lower summatory function, $G_\ell^{-1}(x)$, to provide bounds on the magnitude of $G_E^{-1}(x)$ such that*

$$|G_E^{-1}(x)| \gg |G_\ell^{-1}(x)|,$$

for all sufficiently large $x > e$. Let $C_{\ell,1} > 0$ be the absolute constant defined by

$$C_{\ell,1} = \frac{e^2(\log 2)^2 \exp\left(-\frac{15}{16}(\log 2)^2\right)}{2\sqrt{2}\pi} \approx 0.234145.$$

We obtain the following limiting estimate for the bounding function $G_\ell^{-1}(x)$ as $x \to \infty$:

$$\left|G_\ell^{-1}(x)\right| \gg \frac{C_{\ell,1} \cdot (\log x)^{\frac{5}{4}}}{2 \cdot (\log\log x)^{\frac{5}{2}} (\log\log\log x)^2}.$$

Proof. Recall from our proof of Corollary 3.7 that a lower bound on the variant prime form counting function, $\widehat{\pi}_k(x)$, is given by

$$\widehat{\pi}_k(x) \gg \frac{2^{\frac{3}{4}} e(\log 2)^{\frac{1}{2}} \exp\left(-\frac{15}{16}(\log 2)^2\right) x^{\frac{1}{4}}}{(\log x)^{\frac{5}{2}}} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O\left(\frac{1}{(\log x)(\log\log x)}\right)\right), \text{ as } x \to \infty.$$

We can then form a lower summatory function indicating the signed contributions over the distinct parity of $\Omega(n)$ for all $n \leq x$ as follows by applying (9b) and Stirling's approximation as already noted in the proof of Corollary 7.5 given above:

$$\left|A_\Omega^{(\ell)}(t)\right| = \left|\sum_{k \leq \log\log t} (-1)^k \widehat{\pi}_k(t)\right| \gg \frac{2^{\frac{1}{4}} e(\log 2)^{\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \cdot \frac{x^{\frac{1}{4}}}{(\log x)^{\frac{3}{2}} \sqrt{\log\log x}}, \text{ as } t \to \infty. \tag{39}$$

The actual sign on this function is given by $\operatorname{sgn}(A_\Omega^{(\ell)}(t)) = (-1)^{1+\lfloor \log\log t \rfloor}$ (see Lemma 7.3). By Lemma 7.4 we know that this summatory function forms a lower bound in absolute value for the actual weight of the signed terms indicated by $\lambda(n)$.

As we determined in (37) from the proof of Corollary 7.5, we take the function $\widehat{\tau}_0(t) = \frac{(\log t)^{1/4}}{t^{1/4} \cdot \log\log t}$ that satisfies

$$-\widehat{\tau}_0'(t) = -\frac{d}{dt}\left[\frac{(\log t)^{\frac{1}{4}}}{t^{\frac{1}{4}} \cdot \log\log t}\right] \gg \frac{(\log t)^{1/4}}{4t^{\frac{5}{4}} \cdot \log\log t}. \tag{40}$$

Moreover, we have using the notation from the proof above that we can select the initial form of the lower bound function $G_\ell^{-1}(x)$ to be defined as follows:

$$G_\ell^{-1}(x) := \frac{2^{\frac{1}{4}} e(\log 2)^{\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \cdot (\log x)^{\frac{5}{4}} (\log\log x)^{\frac{1}{4}} \cdot T_E(x) \times \tag{41}$$

$$\times \left| A_{\Omega}^{(\ell)}(\log x)\widehat{\tau}_0(\log x) - \int_e^{\log x} A_{\Omega}^{(\ell)}(t)\widehat{\tau}_0'(t)dt \right|.$$

The inner integral term on the rightmost side of (41) is summed approximately by splitting the terms weighted by $(-1)^{\lfloor \log\log t \rfloor}$ in the form of [D]

$$\begin{aligned}
&\frac{2^{\frac{1}{4}} e(\log 2)^{\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \times \left| \int_e^{\log x} A_{\Omega}^{(\ell)}(t)\widehat{\tau}_0'(t)dt \right| \\
&\qquad \gg \frac{2^{\frac{1}{4}} e(\log 2)^{\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \times \left| \sum_{k=e+1}^{\frac{1}{2}\log\log\left[(\log x)^5(\log\log x)\right]} \left[I_\ell\left(e^{e^{2k+1}}\right) e^{e^{2k+1}} - I_\ell\left(e^{e^{2k}}\right) e^{e^{2k}} \right] \right| \\
&\qquad \gg \frac{2^{\frac{1}{4}} e(\log 2)^{\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \times \left| \int_{\frac{1}{2}\log\log[(\log x)^5(\log\log x)]-\frac{1}{2}}^{\frac{1}{2}\log\log\left[(\log x)^5(\log\log x)\right]} I_\ell\left(e^{e^{2k}}\right) e^{e^{2k}} dk \right|.
\end{aligned} \tag{42}$$

We express the integrand function,

$$I_\ell(t) := \frac{2^{\frac{1}{4}} e(\log 2)^{\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \times \widehat{\tau}_0'(t) A_{\Omega}^{(\ell)}(t),$$

defined implicitly as in (42) as the following function of k:

$$I_\ell\left(e^{e^{2k}}\right) e^{e^{2k}} \gg \frac{e^2(\log 2)}{8\sqrt{2\pi}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \cdot \frac{e^{-\frac{11k}{2}}}{k^2} =: \widehat{I}_\ell(k). \tag{43}$$

When we input upper bound on the range of integration in (42), at the point $k := \frac{\log\log\left[(\log x)^5(\log\log x)\right]}{2}$, we find from the mean value theorem with the monotone function from (43) that

$$\begin{aligned}
&\frac{2^{\frac{1}{4}} e(\log 2)^{\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \times (\log x)^{\frac{5}{4}}(\log\log x)^{\frac{1}{4}} \times T_E(x) \times \left| \int_{\frac{1}{2}\log\log[(\log x)^5(\log\log x)]-\frac{1}{2}}^{\frac{1}{2}\log\log\left[(\log x)^5(\log\log x)\right]} I_\ell\left(e^{e^{2k}}\right) e^{e^{2k}} dk \right| \\
&\quad \gg \frac{2^{\frac{1}{4}} e(\log 2)^{\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \times (\log x)^{\frac{5}{4}}(\log\log x)^{\frac{1}{4}} \times T_E(x) \times \left| \widehat{I}_\ell\left(\frac{1}{2}\log\log\left[(\log x)^5(\log\log x)\right]\right) \right| \\
&\quad \gg \frac{C_{\ell,1} \cdot (\log x)^{\frac{5}{4}}}{2 \cdot (\log\log x)^{\frac{5}{2}}(\log\log\log x)^2}.
\end{aligned} \tag{44}$$

Similarly, by evaluating $\widehat{I}_\ell(t)$ at the lower bound on the integral above with $k := \frac{\log\log\left[(\log x)^5(\log\log x)\right]-1}{2}$, we can similarly conclude that

$$\begin{aligned}
&\frac{2^{\frac{1}{4}} e(\log 2)^{\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \times (\log x)^{\frac{5}{4}}(\log\log x)^{\frac{1}{4}} \times T_E(x) \times \left| \int_{\frac{1}{2}\log\log[(\log x)^5(\log\log x)]-\frac{1}{2}}^{\frac{1}{2}\log\log\left[(\log x)^5(\log\log x)\right]} I_\ell\left(e^{e^{2k}}\right) e^{e^{2k}} dk \right| \\
&\qquad \ll \frac{e^{\frac{11}{4}} \cdot C_{\ell,1} \cdot (\log x)^{\frac{5}{4}}}{2 \cdot (\log\log x)^{\frac{5}{2}}(\log\log\log x)^2}.
\end{aligned} \tag{45}$$

[D] That is, we form the disjoint union of the range of integration into subintervals along which the signedness of the integrands are constant according to

$$\left\{ e^{\frac{e}{5}} \leq t \leq (\log x)^5(\log\log x) : (-1)^{\lfloor \log\log t \rfloor} = +1 \right\} = \left(\bigcup_{k=1}^{\frac{1}{2}\log\log\left[(\log x)^5(\log\log x)\right]} \left[e^{e^{2k}}, e^{e^{2k+1}} \right) \right) \bigcup \mathcal{S}_{0,+},$$

where $|\mathcal{S}_{0,+}| \leq \frac{1}{2}$. We can similarly split the interval of integration corresponding to the negatively biased terms on the unsigned integrand functions for $t \in \left[e^{\frac{e}{5}}, (\log x)^5(\log\log x) \right]$.

To make it clear which terms in (41) yield the limiting lower bounds, consider the following expansion for the leading term in the Abel summation formula from (41) for comparison with (44):

$$\begin{aligned} & \frac{2^{\frac{1}{4}} e (\log 2)^{\frac{1}{2}}}{\sqrt{\pi}} \exp\left(-\frac{15}{16}(\log 2)^2\right) \times (\log x)^{\frac{5}{4}} (\log\log x)^{\frac{1}{4}} \times T_E(x) \times \left|\widehat{\tau}_0(\log x) A_{\Omega}^{(\ell)}(\log x)\right| \\ & \qquad \gg \frac{4 C_{\ell,1} \cdot (\log x)^{\frac{5}{4}} (\log\log x)^{\frac{1}{4}}}{(\log\log\log x)^{\frac{11}{4}} (\log\log\log\log x)^2}. \end{aligned} \tag{46}$$

Hence, by Lemma 7.3 and the triangle inequality, we conclude that we can take $\left|G_{\ell}^{-1}(x)\right|$ bounded below by the term in (44). □

7.2 Proof of the unboundedness of the scaled Mertens function

We finally address the main conclusion of our arguments given so far with the following proof:

Proof of Theorem 3.9. We split the interval of integration from Proposition 7.1 over $t \in [u_0, x/2]$ into two subintervals: one that is easily bounded from $u_0 \leq t \leq \sqrt{x}$, and then another that will conveniently give us our slow-growing tendency towards infinity along the subsequence when evaluated using Theorem 7.7. Given a fixed large infinitely tending x, we have some (at least one) point $x_0 \in \left[\sqrt{x}, \frac{x}{2}\right]$ defined such that $|G^{-1}(t)|$ is minimal and non-vanishing as

$$\left|G^{-1}(x_0)\right| := \min_{\substack{\sqrt{x} \leq t \leq \frac{x}{2} \\ G^{-1}(t) \neq 0}} |G^{-1}(t)|.$$

We can then apply Proposition 7.1 to bound the function as follows:

$$\begin{aligned} \frac{|M(x)|}{\sqrt{x}} &= \frac{1}{\sqrt{x}} \left| G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt \right| \\ &\gg \left| \left| \frac{G^{-1}(x)}{\sqrt{x}} \right| - \sqrt{x} \int_1^{x/2} \frac{|G^{-1}(t)|}{t^2 \cdot \log(x/t)} dt \right| \qquad (47) \\ &\gg \sqrt{x} \times \int_{\sqrt{x}}^{x/2} \frac{|G^{-1}(t)|}{t^2 \cdot \log(x/t)} dt \\ &\gg \left(\min_{\substack{\sqrt{x} \leq t \leq \frac{x}{2} \\ G^{-1}(t) \neq 0}} |G^{-1}(t)| \right) \times \int_{\sqrt{x_0}}^{\frac{x}{2}} \frac{2\sqrt{x_0}}{t^2 \cdot \log(x_0)} dt \\ &\gg \frac{2\left|G^{-1}(x_0)\right|}{\log(x_0)}. \qquad (48) \end{aligned}$$

In the second to last step, we observe that $G^{-1}(x) = 0$ for x on a set of asymptotic density *at least* bounded below by $\frac{1}{2}$, so that our claim is accurate as the integral bound does not vanish at large x.

To complete the logic to the bound we arrived at in (48), observe that the difference of terms we have in (47) bounded below as we have seen in the proof of Theorem 7.2 by

$$\frac{|G^{-1}(x)|}{\sqrt{x}} \gg \frac{6\sqrt{x}}{\pi^2} (\log x)(\log\log x), \text{ for a.e. } x \to \infty.$$

Secondly, for the sake of argument, suppose that there is a smooth approximation for $|G^{-1}(t)|$ so that by the mean value theorem for some $c_0 \in [1, \sqrt{x}]$ and $c_1 \in \left[\sqrt{x}, \frac{x}{2}\right]$ we have

$$\sqrt{x} \left| \int_1^{x/2} \frac{|G^{-1}(t)|}{t^2 \cdot \log(x/t)} dt \right|$$

$$\gg \left| \frac{\sqrt{x} \cdot |G^{-1}(c_0)|}{c_0} \int_1^{\sqrt{x}} \frac{dt}{t \log(x/t)} + \sqrt{x} \cdot |G^{-1}(c_1)| \int_{\sqrt{x}}^{x/2} \frac{dt}{t^2 \log(x)} \right|$$

$$\gg \left| \left(\min_{\substack{1 \leq c \leq \sqrt{x} \\ G^{-1}(c) \neq 0}} |G^{-1}(c)| \right) \log\log x + \left(\min_{\substack{\sqrt{x} \leq c \leq \frac{x}{2} \\ G^{-1}(c) \neq 0}} |G^{-1}(c)| \right) \left(\frac{1}{\log x} + o\left(\frac{1}{\log x} \right) \right) \right|.$$

Since $G^{-1}(x)$ changes stepwise only at $x \in \mathbb{Z}^+$, what we in fact exactly arrive at is a close variant of this mean value theorem type observation. The statements within the last few equations based on the smoothness approximation assumption for the function make it clear without more technical complications how we should go about bounding these growth rates.

By Theorem 7.2, the result in (48) implies that

$$\frac{|M(x)|}{\sqrt{x}} \gg \frac{2 \left| G_E^{-1}(x_0) \right|}{\log(x_0)}. \tag{49}$$

Define the infinite increasing subsequence, $\{x_{0,y}\}_{y \geq Y_0}$, by $x_{0,y} := e^{2e^{e^{2y+1}}}$ for sequence indices starting at some sufficiently large finite integer $Y_0 \gg 1$. When we assume that $x \mapsto x_{0,y}$ is taken along this subsequence, we can transform the bound in the last equation into a statement about a lower bound for $|M(x)|/\sqrt{x}$ along an infinitely tending subsequence by applying Theorem 7.7 in the following form to (49):

$$\frac{|M(x_{0,y})|}{\sqrt{x_{0,y}}} \gg \frac{C_{\ell,1} \cdot (\log \sqrt{x_{0,y}})^{\frac{1}{4}}}{(\log\log \sqrt{x_{0,y}})^{\frac{5}{2}} (\log\log\log \sqrt{x_{0,y}})^2}, \text{ as } y \to \infty. \tag{50}$$

Notice that there is a small, but nonetheless insightful point to make about a technicality in stating (50). Namely, we are not actually asserting that $|M(x)|/\sqrt{x}$ grows unbounded along the precise subsequence of $x \mapsto x_{0,y}$ itself. Rather, we are asserting that the unboundedness of this function can be witnessed along some subsequence whose points are taken within a large interval window for $\hat{x}_{0,y} \in \left[\sqrt{x_{0,y}}, \frac{x_{0,y}}{2}\right]$ as $x, y \to \infty$. We choose to state the lower bound given on the right-hand-side of (50) using the monotonicity of the lower bound on $|G_E^{-1}(x)|$ we proved in Theorem 7.7 without the need for a conditionally defined asymptotic growth rate. We also can verify that for sufficiently large $y \to \infty$, this infinitely tending subsequence is well defined as $\hat{x}_{0,y+1} > \hat{x}_{0,y}$ whenever $y \geq Y_0$.

Finally, we evaluate the following limit to conclude unboundedness:

$$\lim_{x \to \infty} \left[\frac{(\log x)^{\frac{1}{4}}}{(\log\log x)^{\frac{5}{2}} (\log\log\log x)^2} \right] = +\infty.$$

The scaled Mertens function is then unbounded in the limit supremum sense, as we have claimed, since the right-hand-side of (50) tends to positive infinity as $x_{0,y} \to \infty$, or equivalently as $y \to \infty$. □

References

[1] T. M. Apostol. *Introduction to Analytic Number Theory.* Springer–Verlag, 1976.

[2] G. H. Hardy and E. M. Wright, editors. *An Introduction to the Theory of Numbers.* Oxford University Press, 2008 (Sixth Edition).

[3] G. Hurst. Computations of the Mertens function and improved bounds on the Mertens conjecture. *https://arxiv.org/pdf/1610.08551/*, 2017.

[4] H. Iwaniec and E. Kowalski. *Analytic Number Theory*, volume 53. AMS Colloquium Publications, 2004.

[5] T. Kotnik and H. té Riele. The Mertens conjecture revisited. *Algorithmic Number Theory*, 7^{th} International Symposium, 2006.

[6] T. Kotnik and J. van de Lune. On the order of the Mertens function. *Exp. Math.*, 2004.

[7] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford: The Clarendon Press, 1995.

[8] H. L. Montgomery and R. C. Vaughan. *Multiplicative Number Theory: I. Classical Theory.* Cambridge, 2006.

[9] N. Ng. The distribution of the summatory function of the Móbius function. *Proc. London Math. Soc.*, 89(3):361–389, 2004.

[10] A. M. Odlyzko and H. J. J. té Riele. Disproof of the Mertens conjecture. *J. REINE ANGEW. MATH*, 1985.

[11] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. *NIST Handbook of Mathematical Functions.* Cambridge University Press, 2010.

[12] P. Ribenboim. *The new book of prime number records.* Springer, 1996.

[13] J. Sándor and B. Crstici. *Handbook of Number Theory II.* Kluwer Academic Publishers, 2004.

[14] N. J. A. Sloane. The Online Encyclopedia of Integer Sequences, 2020.

[15] K. Soundararajan. Partial sums of the Möbius function. *Annals of Mathematics*, 2009.

[16] T. Tao and J. Teräväinen. Value patterns of multiplicative functions and related sequences. *Forum of Mathematics, Sigma*, 7, 2019.

[17] E. C. Titchmarsh. *The theory of the Riemann zeta function.* Clarendon Press, 1951.

T.1 Table: The Dirichlet inverse function $g^{-1}(n)$ and the distribution of its summatory function

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1^1	Y	N	1	0	1.0000000	1.000000	0.000000	1	1	0
2	2^1	Y	Y	-2	0	1.0000000	0.500000	0.500000	-1	1	-2
3	3^1	Y	Y	-2	0	1.0000000	0.333333	0.666667	-3	1	-4
4	2^2	N	Y	2	0	1.5000000	0.500000	0.500000	-1	3	-4
5	5^1	Y	Y	-2	0	1.0000000	0.400000	0.600000	-3	3	-6
6	$2^1 3^1$	Y	N	5	0	1.0000000	0.500000	0.500000	2	8	-6
7	7^1	Y	Y	-2	0	1.0000000	0.428571	0.571429	0	8	-8
8	2^3	N	Y	-2	0	2.0000000	0.375000	0.625000	-2	8	-10
9	3^2	N	Y	2	0	1.5000000	0.444444	0.555556	0	10	-10
10	$2^1 5^1$	Y	N	5	0	1.0000000	0.500000	0.500000	5	15	-10
11	11^1	Y	Y	-2	0	1.0000000	0.454545	0.545455	3	15	-12
12	$2^2 3^1$	N	N	-7	2	1.2857143	0.416667	0.583333	-4	15	-19
13	13^1	Y	Y	-2	0	1.0000000	0.384615	0.615385	-6	15	-21
14	$2^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-1	20	-21
15	$3^1 5^1$	Y	N	5	0	1.0000000	0.466667	0.533333	4	25	-21
16	2^4	N	Y	2	0	2.5000000	0.500000	0.500000	6	27	-21
17	17^1	Y	Y	-2	0	1.0000000	0.470588	0.529412	4	27	-23
18	$2^1 3^2$	N	N	-7	2	1.2857143	0.444444	0.555556	-3	27	-30
19	19^1	Y	Y	-2	0	1.0000000	0.421053	0.578947	-5	27	-32
20	$2^2 5^1$	N	N	-7	2	1.2857143	0.400000	0.600000	-12	27	-39
21	$3^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-7	32	-39
22	$2^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-2	37	-39
23	23^1	Y	Y	-2	0	1.0000000	0.434783	0.565217	-4	37	-41
24	$2^3 3^1$	N	N	9	4	1.5555556	0.458333	0.541667	5	46	-41
25	5^2	N	Y	2	0	1.5000000	0.480000	0.520000	7	48	-41
26	$2^1 13^1$	Y	N	5	0	1.0000000	0.500000	0.500000	12	53	-41
27	3^3	N	Y	-2	0	2.0000000	0.481481	0.518519	10	53	-43
28	$2^2 7^1$	N	N	-7	2	1.2857143	0.464286	0.535714	3	53	-50
29	29^1	Y	Y	-2	0	1.0000000	0.448276	0.551724	1	53	-52
30	$2^1 3^1 5^1$	Y	N	-16	0	1.0000000	0.433333	0.566667	-15	53	-68
31	31^1	Y	Y	-2	0	1.0000000	0.419355	0.580645	-17	53	-70
32	2^5	N	Y	-2	0	3.0000000	0.406250	0.593750	-19	53	-72
33	$3^1 11^1$	Y	N	5	0	1.0000000	0.424242	0.575758	-14	58	-72
34	$2^1 17^1$	Y	N	5	0	1.0000000	0.441176	0.558824	-9	63	-72
35	$5^1 7^1$	Y	N	5	0	1.0000000	0.457143	0.542857	-4	68	-72
36	$2^2 3^2$	N	N	14	9	1.3571429	0.472222	0.527778	10	82	-72
37	37^1	Y	Y	-2	0	1.0000000	0.459459	0.540541	8	82	-74
38	$2^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	13	87	-74
39	$3^1 13^1$	Y	N	5	0	1.0000000	0.487179	0.512821	18	92	-74
40	$2^3 5^1$	N	N	9	4	1.5555556	0.500000	0.500000	27	101	-74
41	41^1	Y	Y	-2	0	1.0000000	0.487805	0.512195	25	101	-76
42	$2^1 3^1 7^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	9	101	-92
43	43^1	Y	Y	-2	0	1.0000000	0.465116	0.534884	7	101	-94
44	$2^2 11^1$	N	N	-7	2	1.2857143	0.454545	0.545455	0	101	-101
45	$3^2 5^1$	N	N	-7	2	1.2857143	0.444444	0.555556	-7	101	-108
46	$2^1 23^1$	Y	N	5	0	1.0000000	0.456522	0.543478	-2	106	-108
47	47^1	Y	Y	-2	0	1.0000000	0.446809	0.553191	-4	106	-110
48	$2^4 3^1$	N	N	-11	6	1.8181818	0.437500	0.562500	-15	106	-121

Table T.1: ***Computations with*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for*** $\mathbf{1 \leq n \leq 500}$.

- *The column labeled* `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled* `Sqfree` *and* `PPower`*, respectively, list inclusion of n in the sets of squarefree integers and the prime powers.*
- *The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.*
- *The last several columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{n} \cdot \#\{n \leq x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geq 1$.*

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
49	7^2	N	Y	2	0	1.5000000	0.448980	0.551020	−13	108	−121
50	$2^1 5^2$	N	N	−7	2	1.2857143	0.440000	0.560000	−20	108	−128
51	$3^1 17^1$	Y	N	5	0	1.0000000	0.450980	0.549020	−15	113	−128
52	$2^2 13^1$	N	N	−7	2	1.2857143	0.442308	0.557692	−22	113	−135
53	53^1	Y	Y	−2	0	1.0000000	0.433962	0.566038	−24	113	−137
54	$2^1 3^3$	N	N	9	4	1.5555556	0.444444	0.555556	−15	122	−137
55	$5^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−10	127	−137
56	$2^3 7^1$	N	N	9	4	1.5555556	0.464286	0.535714	−1	136	−137
57	$3^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	4	141	−137
58	$2^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	9	146	−137
59	59^1	Y	Y	−2	0	1.0000000	0.474576	0.525424	7	146	−139
60	$2^2 3^1 5^1$	N	N	30	14	1.1666667	0.483333	0.516667	37	176	−139
61	61^1	Y	Y	−2	0	1.0000000	0.475410	0.524590	35	176	−141
62	$2^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	40	181	−141
63	$3^2 7^1$	N	N	−7	2	1.2857143	0.476190	0.523810	33	181	−148
64	2^6	N	Y	2	0	3.5000000	0.484375	0.515625	35	183	−148
65	$5^1 13^1$	Y	N	5	0	1.0000000	0.492308	0.507692	40	188	−148
66	$2^1 3^1 11^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	24	188	−164
67	67^1	Y	Y	−2	0	1.0000000	0.477612	0.522388	22	188	−166
68	$2^2 17^1$	N	N	−7	2	1.2857143	0.470588	0.529412	15	188	−173
69	$3^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	20	193	−173
70	$2^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.471429	0.528571	4	193	−189
71	71^1	Y	Y	−2	0	1.0000000	0.464789	0.535211	2	193	−191
72	$2^3 3^2$	N	N	−23	18	1.4782609	0.458333	0.541667	−21	193	−214
73	73^1	Y	Y	−2	0	1.0000000	0.452055	0.547945	−23	193	−216
74	$2^1 37^1$	Y	N	5	0	1.0000000	0.459459	0.540541	−18	198	−216
75	$3^1 5^2$	N	N	−7	2	1.2857143	0.453333	0.546667	−25	198	−223
76	$2^2 19^1$	N	N	−7	2	1.2857143	0.447368	0.552632	−32	198	−230
77	$7^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−27	203	−230
78	$2^1 3^1 13^1$	Y	N	−16	0	1.0000000	0.448718	0.551282	−43	203	−246
79	79^1	Y	Y	−2	0	1.0000000	0.443038	0.556962	−45	203	−248
80	$2^4 5^1$	N	N	−11	6	1.8181818	0.437500	0.562500	−56	203	−259
81	3^4	N	Y	2	0	2.5000000	0.444444	0.555556	−54	205	−259
82	$2^1 41^1$	Y	N	5	0	1.0000000	0.451220	0.548780	−49	210	−259
83	83^1	Y	Y	−2	0	1.0000000	0.445783	0.554217	−51	210	−261
84	$2^2 3^1 7^1$	N	N	30	14	1.1666667	0.452381	0.547619	−21	240	−261
85	$5^1 17^1$	Y	N	5	0	1.0000000	0.458824	0.541176	−16	245	−261
86	$2^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	−11	250	−261
87	$3^1 29^1$	Y	N	5	0	1.0000000	0.471264	0.528736	−6	255	−261
88	$2^3 11^1$	N	N	9	4	1.5555556	0.477273	0.522727	3	264	−261
89	89^1	Y	Y	−2	0	1.0000000	0.471910	0.528090	1	264	−263
90	$2^1 3^2 5^1$	N	N	30	14	1.1666667	0.477778	0.522222	31	294	−263
91	$7^1 13^1$	Y	N	5	0	1.0000000	0.483516	0.516484	36	299	−263
92	$2^2 23^1$	N	N	−7	2	1.2857143	0.478261	0.521739	29	299	−270
93	$3^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	34	304	−270
94	$2^1 47^1$	Y	N	5	0	1.0000000	0.489362	0.510638	39	309	−270
95	$5^1 19^1$	Y	N	5	0	1.0000000	0.494737	0.505263	44	314	−270
96	$2^5 3^1$	N	N	13	8	2.0769231	0.500000	0.500000	57	327	−270
97	97^1	Y	Y	−2	0	1.0000000	0.494845	0.505155	55	327	−272
98	$2^1 7^2$	N	N	−7	2	1.2857143	0.489796	0.510204	48	327	−279
99	$3^2 11^1$	N	N	−7	2	1.2857143	0.484848	0.515152	41	327	−286
100	$2^2 5^2$	N	N	14	9	1.3571429	0.490000	0.510000	55	341	−286
101	101^1	Y	Y	−2	0	1.0000000	0.485149	0.514851	53	341	−288
102	$2^1 3^1 17^1$	Y	N	−16	0	1.0000000	0.480392	0.519608	37	341	−304
103	103^1	Y	Y	−2	0	1.0000000	0.475728	0.524272	35	341	−306
104	$2^3 13^1$	N	N	9	4	1.5555556	0.480769	0.519231	44	350	−306
105	$3^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	28	350	−322
106	$2^1 53^1$	Y	N	5	0	1.0000000	0.481132	0.518868	33	355	−322
107	107^1	Y	Y	−2	0	1.0000000	0.476636	0.523364	31	355	−324
108	$2^2 3^3$	N	N	−23	18	1.4782609	0.472222	0.527778	8	355	−347
109	109^1	Y	Y	−2	0	1.0000000	0.467890	0.532110	6	355	−349
110	$2^1 5^1 11^1$	Y	N	−16	0	1.0000000	0.463636	0.536364	−10	355	−365
111	$3^1 37^1$	Y	N	5	0	1.0000000	0.468468	0.531532	−5	360	−365
112	$2^4 7^1$	N	N	−11	6	1.8181818	0.464286	0.535714	−16	360	−376
113	113^1	Y	Y	−2	0	1.0000000	0.460177	0.539823	−18	360	−378
114	$2^1 3^1 19^1$	Y	N	−16	0	1.0000000	0.456140	0.543860	−34	360	−394
115	$5^1 23^1$	Y	N	5	0	1.0000000	0.460870	0.539130	−29	365	−394
116	$2^2 29^1$	N	N	−7	2	1.2857143	0.456897	0.543103	−36	365	−401
117	$3^2 13^1$	N	N	−7	2	1.2857143	0.452991	0.547009	−43	365	−408
118	$2^1 59^1$	Y	N	5	0	1.0000000	0.457627	0.542373	−38	370	−408
119	$7^1 17^1$	Y	N	5	0	1.0000000	0.462185	0.537815	−33	375	−408
120	$2^3 3^1 5^1$	N	N	−48	32	1.3333333	0.458333	0.541667	−81	375	−456
121	11^2	N	Y	2	0	1.5000000	0.462810	0.537190	−79	377	−456
122	$2^1 61^1$	Y	N	5	0	1.0000000	0.467213	0.532787	−74	382	−456
123	$3^1 41^1$	Y	N	5	0	1.0000000	0.471545	0.528455	−69	387	−456
124	$2^2 31^1$	N	N	−7	2	1.2857143	0.467742	0.532258	−76	387	−463

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
125	5^3	N	Y	-2	0	2.0000000	0.464000	0.536000	-78	387	-465
126	$2^1 3^2 7^1$	N	N	30	14	1.1666667	0.468254	0.531746	-48	417	-465
127	127^1	Y	Y	-2	0	1.0000000	0.464567	0.535433	-50	417	-467
128	2^7	N	Y	-2	0	4.0000000	0.460938	0.539062	-52	417	-469
129	$3^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	-47	422	-469
130	$2^1 5^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-63	422	-485
131	131^1	Y	Y	-2	0	1.0000000	0.458015	0.541985	-65	422	-487
132	$2^2 3^1 11^1$	N	N	30	14	1.1666667	0.462121	0.537879	-35	452	-487
133	$7^1 19^1$	Y	N	5	0	1.0000000	0.466165	0.533835	-30	457	-487
134	$2^1 67^1$	Y	N	5	0	1.0000000	0.470149	0.529851	-25	462	-487
135	$3^3 5^1$	N	N	9	4	1.5555556	0.474074	0.525926	-16	471	-487
136	$2^3 17^1$	N	N	9	4	1.5555556	0.477941	0.522059	-7	480	-487
137	137^1	Y	Y	-2	0	1.0000000	0.474453	0.525547	-9	480	-489
138	$2^1 3^1 23^1$	Y	N	-16	0	1.0000000	0.471014	0.528986	-25	480	-505
139	139^1	Y	Y	-2	0	1.0000000	0.467626	0.532374	-27	480	-507
140	$2^2 5^1 7^1$	N	N	30	14	1.1666667	0.471429	0.528571	3	510	-507
141	$3^1 47^1$	Y	N	5	0	1.0000000	0.475177	0.524823	8	515	-507
142	$2^1 71^1$	Y	N	5	0	1.0000000	0.478873	0.521127	13	520	-507
143	$11^1 13^1$	Y	N	5	0	1.0000000	0.482517	0.517483	18	525	-507
144	$2^4 3^2$	N	N	34	29	1.6176471	0.486111	0.513889	52	559	-507
145	$5^1 29^1$	Y	N	5	0	1.0000000	0.489655	0.510345	57	564	-507
146	$2^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	62	569	-507
147	$3^1 7^2$	N	N	-7	2	1.2857143	0.489796	0.510204	55	569	-514
148	$2^2 37^1$	N	N	-7	2	1.2857143	0.486486	0.513514	48	569	-521
149	149^1	Y	Y	-2	0	1.0000000	0.483221	0.516779	46	569	-523
150	$2^1 3^1 5^2$	N	N	30	14	1.1666667	0.486667	0.513333	76	599	-523
151	151^1	Y	Y	-2	0	1.0000000	0.483444	0.516556	74	599	-525
152	$2^3 19^1$	N	N	9	4	1.5555556	0.486842	0.513158	83	608	-525
153	$3^2 17^1$	N	N	-7	2	1.2857143	0.483660	0.516340	76	608	-532
154	$2^1 7^1 11^1$	Y	N	-16	0	1.0000000	0.480519	0.519481	60	608	-548
155	$5^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	65	613	-548
156	$2^2 3^1 13^1$	N	N	30	14	1.1666667	0.487179	0.512821	95	643	-548
157	157^1	Y	Y	-2	0	1.0000000	0.484076	0.515924	93	643	-550
158	$2^1 79^1$	Y	N	5	0	1.0000000	0.487342	0.512658	98	648	-550
159	$3^1 53^1$	Y	N	5	0	1.0000000	0.490566	0.509434	103	653	-550
160	$2^5 5^1$	N	N	13	8	2.0769231	0.493750	0.506250	116	666	-550
161	$7^1 23^1$	Y	N	5	0	1.0000000	0.496894	0.503106	121	671	-550
162	$2^1 3^4$	N	N	-11	6	1.8181818	0.493827	0.506173	110	671	-561
163	163^1	Y	Y	-2	0	1.0000000	0.490798	0.509202	108	671	-563
164	$2^2 41^1$	N	N	-7	2	1.2857143	0.487805	0.512195	101	671	-570
165	$3^1 5^1 11^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	85	671	-586
166	$2^1 83^1$	Y	N	5	0	1.0000000	0.487952	0.512048	90	676	-586
167	167^1	Y	Y	-2	0	1.0000000	0.485030	0.514970	88	676	-588
168	$2^3 3^1 7^1$	N	N	-48	32	1.3333333	0.482143	0.517857	40	676	-636
169	13^2	N	Y	2	0	1.5000000	0.485207	0.514793	42	678	-636
170	$2^1 5^1 17^1$	Y	N	-16	0	1.0000000	0.482353	0.517647	26	678	-652
171	$3^2 19^1$	N	N	-7	2	1.2857143	0.479532	0.520468	19	678	-659
172	$2^2 43^1$	N	N	-7	2	1.2857143	0.476744	0.523256	12	678	-666
173	173^1	Y	Y	-2	0	1.0000000	0.473988	0.526012	10	678	-668
174	$2^1 3^1 29^1$	Y	N	-16	0	1.0000000	0.471264	0.528736	-6	678	-684
175	$5^2 7^1$	N	N	-7	2	1.2857143	0.468571	0.531429	-13	678	-691
176	$2^4 11^1$	N	N	-11	6	1.8181818	0.465909	0.534091	-24	678	-702
177	$3^1 59^1$	Y	N	5	0	1.0000000	0.468927	0.531073	-19	683	-702
178	$2^1 89^1$	Y	N	5	0	1.0000000	0.471910	0.528090	-14	688	-702
179	179^1	Y	Y	-2	0	1.0000000	0.469274	0.530726	-16	688	-704
180	$2^2 3^2 5^1$	N	N	-74	58	1.2162162	0.466667	0.533333	-90	688	-778
181	181^1	Y	Y	-2	0	1.0000000	0.464088	0.535912	-92	688	-780
182	$2^1 7^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-108	688	-796
183	$3^1 61^1$	Y	N	5	0	1.0000000	0.464481	0.535519	-103	693	-796
184	$2^3 23^1$	N	N	9	4	1.5555556	0.467391	0.532609	-94	702	-796
185	$5^1 37^1$	Y	N	5	0	1.0000000	0.470270	0.529730	-89	707	-796
186	$2^1 3^1 31^1$	Y	N	-16	0	1.0000000	0.467742	0.532258	-105	707	-812
187	$11^1 17^1$	Y	N	5	0	1.0000000	0.470588	0.529412	-100	712	-812
188	$2^2 47^1$	N	N	-7	2	1.2857143	0.468085	0.531915	-107	712	-819
189	$3^3 7^1$	N	N	9	4	1.5555556	0.470899	0.529101	-98	721	-819
190	$2^1 5^1 19^1$	Y	N	-16	0	1.0000000	0.468421	0.531579	-114	721	-835
191	191^1	Y	Y	-2	0	1.0000000	0.465969	0.534031	-116	721	-837
192	$2^6 3^1$	N	N	-15	10	2.3333333	0.463542	0.536458	-131	721	-852
193	193^1	Y	Y	-2	0	1.0000000	0.461140	0.538860	-133	721	-854
194	$2^1 97^1$	Y	N	5	0	1.0000000	0.463918	0.536082	-128	726	-854
195	$3^1 5^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-144	726	-870
196	$2^2 7^2$	N	N	14	9	1.3571429	0.464286	0.535714	-130	740	-870
197	197^1	Y	Y	-2	0	1.0000000	0.461929	0.538071	-132	740	-872
198	$2^1 3^2 11^1$	N	N	30	14	1.1666667	0.464646	0.535354	-102	770	-872
199	199^1	Y	Y	-2	0	1.0000000	0.462312	0.537688	-104	770	-874
200	$2^3 5^2$	N	N	-23	18	1.4782609	0.460000	0.540000	-127	770	-897

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
201	$3^1 67^1$	Y	N	5	0	1.0000000	0.462687	0.537313	−122	775	−897
202	$2^1 101^1$	Y	N	5	0	1.0000000	0.465347	0.534653	−117	780	−897
203	$7^1 29^1$	Y	N	5	0	1.0000000	0.467980	0.532020	−112	785	−897
204	$2^2 3^1 17^1$	N	N	30	14	1.1666667	0.470588	0.529412	−82	815	−897
205	$5^1 41^1$	Y	N	5	0	1.0000000	0.473171	0.526829	−77	820	−897
206	$2^1 103^1$	Y	N	5	0	1.0000000	0.475728	0.524272	−72	825	−897
207	$3^2 23^1$	N	N	−7	2	1.2857143	0.473430	0.526570	−79	825	−904
208	$2^4 13^1$	N	N	−11	6	1.8181818	0.471154	0.528846	−90	825	−915
209	$11^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	−85	830	−915
210	$2^1 3^1 5^1 7^1$	Y	N	65	0	1.0000000	0.476190	0.523810	−20	895	−915
211	211^1	Y	Y	−2	0	1.0000000	0.473934	0.526066	−22	895	−917
212	$2^2 53^1$	N	N	−7	2	1.2857143	0.471698	0.528302	−29	895	−924
213	$3^1 71^1$	Y	N	5	0	1.0000000	0.474178	0.525822	−24	900	−924
214	$2^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−19	905	−924
215	$5^1 43^1$	Y	N	5	0	1.0000000	0.479070	0.520930	−14	910	−924
216	$2^3 3^3$	N	N	46	41	1.5000000	0.481481	0.518519	32	956	−924
217	$7^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	37	961	−924
218	$2^1 109^1$	Y	N	5	0	1.0000000	0.486239	0.513761	42	966	−924
219	$3^1 73^1$	Y	N	5	0	1.0000000	0.488584	0.511416	47	971	−924
220	$2^2 5^1 11^1$	N	N	30	14	1.1666667	0.490909	0.509091	77	1001	−924
221	$13^1 17^1$	Y	N	5	0	1.0000000	0.493213	0.506787	82	1006	−924
222	$2^1 3^1 37^1$	Y	N	−16	0	1.0000000	0.490991	0.509009	66	1006	−940
223	223^1	Y	Y	−2	0	1.0000000	0.488789	0.511211	64	1006	−942
224	$2^5 7^1$	N	N	13	8	2.0769231	0.491071	0.508929	77	1019	−942
225	$3^2 5^2$	N	N	14	9	1.3571429	0.493333	0.506667	91	1033	−942
226	$2^1 113^1$	Y	N	5	0	1.0000000	0.495575	0.504425	96	1038	−942
227	227^1	Y	Y	−2	0	1.0000000	0.493392	0.506608	94	1038	−944
228	$2^2 3^1 19^1$	N	N	30	14	1.1666667	0.495614	0.504386	124	1068	−944
229	229^1	Y	Y	−2	0	1.0000000	0.493450	0.506550	122	1068	−946
230	$2^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.491304	0.508696	106	1068	−962
231	$3^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.489177	0.510823	90	1068	−978
232	$2^3 29^1$	N	N	9	4	1.5555556	0.491379	0.508621	99	1077	−978
233	233^1	Y	Y	−2	0	1.0000000	0.489270	0.510730	97	1077	−980
234	$2^1 3^2 13^1$	N	N	30	14	1.1666667	0.491453	0.508547	127	1107	−980
235	$5^1 47^1$	Y	N	5	0	1.0000000	0.493617	0.506383	132	1112	−980
236	$2^2 59^1$	N	N	−7	2	1.2857143	0.491525	0.508475	125	1112	−987
237	$3^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	130	1117	−987
238	$2^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.491597	0.508403	114	1117	−1003
239	239^1	Y	Y	−2	0	1.0000000	0.489540	0.510460	112	1117	−1005
240	$2^4 3^1 5^1$	N	N	70	54	1.5000000	0.491667	0.508333	182	1187	−1005
241	241^1	Y	Y	−2	0	1.0000000	0.489627	0.510373	180	1187	−1007
242	$2^1 11^2$	N	N	−7	2	1.2857143	0.487603	0.512397	173	1187	−1014
243	3^5	N	Y	−2	0	3.0000000	0.485597	0.514403	171	1187	−1016
244	$2^2 61^1$	N	N	−7	2	1.2857143	0.483607	0.516393	164	1187	−1023
245	$5^1 7^2$	N	N	−7	2	1.2857143	0.481633	0.518367	157	1187	−1030
246	$2^1 3^1 41^1$	Y	N	−16	0	1.0000000	0.479675	0.520325	141	1187	−1046
247	$13^1 19^1$	Y	N	5	0	1.0000000	0.481781	0.518219	146	1192	−1046
248	$2^3 31^1$	N	N	9	4	1.5555556	0.483871	0.516129	155	1201	−1046
249	$3^1 83^1$	Y	N	5	0	1.0000000	0.485944	0.514056	160	1206	−1046
250	$2^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	169	1215	−1046
251	251^1	Y	Y	−2	0	1.0000000	0.486056	0.513944	167	1215	−1048
252	$2^2 3^2 7^1$	N	N	−74	58	1.2162162	0.484127	0.515873	93	1215	−1122
253	$11^1 23^1$	Y	N	5	0	1.0000000	0.486166	0.513834	98	1220	−1122
254	$2^1 127^1$	Y	N	5	0	1.0000000	0.488189	0.511811	103	1225	−1122
255	$3^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.486275	0.513725	87	1225	−1138
256	2^8	N	Y	2	0	4.5000000	0.488281	0.511719	89	1227	−1138
257	257^1	Y	Y	−2	0	1.0000000	0.486381	0.513619	87	1227	−1140
258	$2^1 3^1 43^1$	Y	N	−16	0	1.0000000	0.484496	0.515504	71	1227	−1156
259	$7^1 37^1$	Y	N	5	0	1.0000000	0.486486	0.513514	76	1232	−1156
260	$2^2 5^1 13^1$	N	N	30	14	1.1666667	0.488462	0.511538	106	1262	−1156
261	$3^2 29^1$	N	N	−7	2	1.2857143	0.486590	0.513410	99	1262	−1163
262	$2^1 131^1$	Y	N	5	0	1.0000000	0.488550	0.511450	104	1267	−1163
263	263^1	Y	Y	−2	0	1.0000000	0.486692	0.513308	102	1267	−1165
264	$2^3 3^1 11^1$	N	N	−48	32	1.3333333	0.484848	0.515152	54	1267	−1213
265	$5^1 53^1$	Y	N	5	0	1.0000000	0.486792	0.513208	59	1272	−1213
266	$2^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.484962	0.515038	43	1272	−1229
267	$3^1 89^1$	Y	N	5	0	1.0000000	0.486891	0.513109	48	1277	−1229
268	$2^2 67^1$	N	N	−7	2	1.2857143	0.485075	0.514925	41	1277	−1236
269	269^1	Y	Y	−2	0	1.0000000	0.483271	0.516729	39	1277	−1238
270	$2^1 3^3 5^1$	N	N	−48	32	1.3333333	0.481481	0.518519	−9	1277	−1286
271	271^1	Y	Y	−2	0	1.0000000	0.479705	0.520295	−11	1277	−1288
272	$2^4 17^1$	N	N	−11	6	1.8181818	0.477941	0.522059	−22	1277	−1299
273	$3^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−38	1277	−1315
274	$2^1 137^1$	Y	N	5	0	1.0000000	0.478102	0.521898	−33	1282	−1315
275	$5^2 11^1$	N	N	−7	2	1.2857143	0.476364	0.523636	−40	1282	−1322
276	$2^2 3^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−10	1312	−1322
277	277^1	Y	Y	−2	0	1.0000000	0.476534	0.523466	−12	1312	−1324

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
278	$2^1 139^1$	Y	N	5	0	1.0000000	0.478417	0.521583	−7	1317	−1324
279	$3^2 31^1$	N	N	−7	2	1.2857143	0.476703	0.523297	−14	1317	−1331
280	$2^3 5^1 7^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−62	1317	−1379
281	281^1	Y	Y	−2	0	1.0000000	0.473310	0.526690	−64	1317	−1381
282	$2^1 3^1 47^1$	Y	N	−16	0	1.0000000	0.471631	0.528369	−80	1317	−1397
283	283^1	Y	Y	−2	0	1.0000000	0.469965	0.530035	−82	1317	−1399
284	$2^2 71^1$	N	N	−7	2	1.2857143	0.468310	0.531690	−89	1317	−1406
285	$3^1 5^1 19^1$	Y	N	−16	0	1.0000000	0.466667	0.533333	−105	1317	−1422
286	$2^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.465035	0.534965	−121	1317	−1438
287	$7^1 41^1$	Y	N	5	0	1.0000000	0.466899	0.533101	−116	1322	−1438
288	$2^5 3^2$	N	N	−47	42	1.7659574	0.465278	0.534722	−163	1322	−1485
289	17^2	N	Y	2	0	1.5000000	0.467128	0.532872	−161	1324	−1485
290	$2^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.465517	0.534483	−177	1324	−1501
291	$3^1 97^1$	Y	N	5	0	1.0000000	0.467354	0.532646	−172	1329	−1501
292	$2^2 73^1$	N	N	−7	2	1.2857143	0.465753	0.534247	−179	1329	−1508
293	293^1	Y	Y	−2	0	1.0000000	0.464164	0.535836	−181	1329	−1510
294	$2^1 3^1 7^2$	N	N	30	14	1.1666667	0.465986	0.534014	−151	1359	−1510
295	$5^1 59^1$	Y	N	5	0	1.0000000	0.467797	0.532203	−146	1364	−1510
296	$2^3 37^1$	N	N	9	4	1.5555556	0.469595	0.530405	−137	1373	−1510
297	$3^3 11^1$	N	N	9	4	1.5555556	0.471380	0.528620	−128	1382	−1510
298	$2^1 149^1$	Y	N	5	0	1.0000000	0.473154	0.526846	−123	1387	−1510
299	$13^1 23^1$	Y	N	5	0	1.0000000	0.474916	0.525084	−118	1392	−1510
300	$2^2 3^1 5^2$	N	N	−74	58	1.2162162	0.473333	0.526667	−192	1392	−1584
301	$7^1 43^1$	Y	N	5	0	1.0000000	0.475083	0.524917	−187	1397	−1584
302	$2^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−182	1402	−1584
303	$3^1 101^1$	Y	N	5	0	1.0000000	0.478548	0.521452	−177	1407	−1584
304	$2^4 19^1$	N	N	−11	6	1.8181818	0.476974	0.523026	−188	1407	−1595
305	$5^1 61^1$	Y	N	5	0	1.0000000	0.478689	0.521311	−183	1412	−1595
306	$2^1 3^2 17^1$	N	N	30	14	1.1666667	0.480392	0.519608	−153	1442	−1595
307	307^1	Y	Y	−2	0	1.0000000	0.478827	0.521173	−155	1442	−1597
308	$2^2 7^1 11^1$	N	N	30	14	1.1666667	0.480519	0.519481	−125	1472	−1597
309	$3^1 103^1$	Y	N	5	0	1.0000000	0.482201	0.517799	−120	1477	−1597
310	$2^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.480645	0.519355	−136	1477	−1613
311	311^1	Y	Y	−2	0	1.0000000	0.479100	0.520900	−138	1477	−1615
312	$2^3 3^1 13^1$	N	N	−48	32	1.3333333	0.477564	0.522436	−186	1477	−1663
313	313^1	Y	Y	−2	0	1.0000000	0.476038	0.523962	−188	1477	−1665
314	$2^1 157^1$	Y	N	5	0	1.0000000	0.477707	0.522293	−183	1482	−1665
315	$3^2 5^1 7^1$	N	N	30	14	1.1666667	0.479365	0.520635	−153	1512	−1665
316	$2^2 79^1$	N	N	−7	2	1.2857143	0.477848	0.522152	−160	1512	−1672
317	317^1	Y	Y	−2	0	1.0000000	0.476341	0.523659	−162	1512	−1674
318	$2^1 3^1 53^1$	Y	N	−16	0	1.0000000	0.474843	0.525157	−178	1512	−1690
319	$11^1 29^1$	Y	N	5	0	1.0000000	0.476489	0.523511	−173	1517	−1690
320	$2^6 5^1$	N	N	−15	10	2.3333333	0.475000	0.525000	−188	1517	−1705
321	$3^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−183	1522	−1705
322	$2^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.475155	0.524845	−199	1522	−1721
323	$17^1 19^1$	Y	N	5	0	1.0000000	0.476780	0.523220	−194	1527	−1721
324	$2^2 3^4$	N	N	34	29	1.6176471	0.478395	0.521605	−160	1561	−1721
325	$5^2 13^1$	N	N	−7	2	1.2857143	0.476923	0.523077	−167	1561	−1728
326	$2^1 163^1$	Y	N	5	0	1.0000000	0.478528	0.521472	−162	1566	−1728
327	$3^1 109^1$	Y	N	5	0	1.0000000	0.480122	0.519878	−157	1571	−1728
328	$2^3 41^1$	N	N	9	4	1.5555556	0.481707	0.518293	−148	1580	−1728
329	$7^1 47^1$	Y	N	5	0	1.0000000	0.483283	0.516717	−143	1585	−1728
330	$2^1 3^1 5^1 11^1$	Y	N	65	0	1.0000000	0.484848	0.515152	−78	1650	−1728
331	331^1	Y	Y	−2	0	1.0000000	0.483384	0.516616	−80	1650	−1730
332	$2^2 83^1$	N	N	−7	2	1.2857143	0.481928	0.518072	−87	1650	−1737
333	$3^2 37^1$	N	N	−7	2	1.2857143	0.480480	0.519520	−94	1650	−1744
334	$2^1 167^1$	Y	N	5	0	1.0000000	0.482036	0.517964	−89	1655	−1744
335	$5^1 67^1$	Y	N	5	0	1.0000000	0.483582	0.516418	−84	1660	−1744
336	$2^4 3^1 7^1$	N	N	70	54	1.5000000	0.485119	0.514881	−14	1730	−1744
337	337^1	Y	Y	−2	0	1.0000000	0.483680	0.516320	−16	1730	−1746
338	$2^1 13^2$	N	N	−7	2	1.2857143	0.482249	0.517751	−23	1730	−1753
339	$3^1 113^1$	Y	N	5	0	1.0000000	0.483776	0.516224	−18	1735	−1753
340	$2^2 5^1 17^1$	N	N	30	14	1.1666667	0.485294	0.514706	12	1765	−1753
341	$11^1 31^1$	Y	N	5	0	1.0000000	0.486804	0.513196	17	1770	−1753
342	$2^1 3^2 19^1$	N	N	30	14	1.1666667	0.488304	0.511696	47	1800	−1753
343	7^3	N	Y	−2	0	2.0000000	0.486880	0.513120	45	1800	−1755
344	$2^3 43^1$	N	N	9	4	1.5555556	0.488372	0.511628	54	1809	−1755
345	$3^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.486957	0.513043	38	1809	−1771
346	$2^1 173^1$	Y	N	5	0	1.0000000	0.488439	0.511561	43	1814	−1771
347	347^1	Y	Y	−2	0	1.0000000	0.487032	0.512968	41	1814	−1773
348	$2^2 3^1 29^1$	N	N	30	14	1.1666667	0.488506	0.511494	71	1844	−1773
349	349^1	Y	Y	−2	0	1.0000000	0.487106	0.512894	69	1844	−1775
350	$2^1 5^2 7^1$	N	N	30	14	1.1666667	0.488571	0.511429	99	1874	−1775

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
351	$3^3 13^1$	N	N	9	4	1.5555556	0.490028	0.509972	108	1883	−1775
352	$2^5 11^1$	N	N	13	8	2.0769231	0.491477	0.508523	121	1896	−1775
353	353^1	Y	Y	−2	0	1.0000000	0.490085	0.509915	119	1896	−1777
354	$2^1 3^1 59^1$	Y	N	−16	0	1.0000000	0.488701	0.511299	103	1896	−1793
355	$5^1 71^1$	Y	N	5	0	1.0000000	0.490141	0.509859	108	1901	−1793
356	$2^2 89^1$	N	N	−7	2	1.2857143	0.488764	0.511236	101	1901	−1800
357	$3^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.487395	0.512605	85	1901	−1816
358	$2^1 179^1$	Y	N	5	0	1.0000000	0.488827	0.511173	90	1906	−1816
359	359^1	Y	Y	−2	0	1.0000000	0.487465	0.512535	88	1906	−1818
360	$2^3 3^2 5^1$	N	N	145	129	1.3034483	0.488889	0.511111	233	2051	−1818
361	19^2	N	Y	2	0	1.5000000	0.490305	0.509695	235	2053	−1818
362	$2^1 181^1$	Y	N	5	0	1.0000000	0.491713	0.508287	240	2058	−1818
363	$3^1 11^2$	N	N	−7	2	1.2857143	0.490358	0.509642	233	2058	−1825
364	$2^2 7^1 13^1$	N	N	30	14	1.1666667	0.491758	0.508242	263	2088	−1825
365	$5^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	268	2093	−1825
366	$2^1 3^1 61^1$	Y	N	−16	0	1.0000000	0.491803	0.508197	252	2093	−1841
367	367^1	Y	Y	−2	0	1.0000000	0.490463	0.509537	250	2093	−1843
368	$2^4 23^1$	N	N	−11	6	1.8181818	0.489130	0.510870	239	2093	−1854
369	$3^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	232	2093	−1861
370	$2^1 5^1 37^1$	Y	N	−16	0	1.0000000	0.486486	0.513514	216	2093	−1877
371	$7^1 53^1$	Y	N	5	0	1.0000000	0.487871	0.512129	221	2098	−1877
372	$2^2 3^1 31^1$	N	N	30	14	1.1666667	0.489247	0.510753	251	2128	−1877
373	373^1	Y	Y	−2	0	1.0000000	0.487936	0.512064	249	2128	−1879
374	$2^1 11^1 17^1$	Y	N	−16	0	1.0000000	0.486631	0.513369	233	2128	−1895
375	$3^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	242	2137	−1895
376	$2^3 47^1$	N	N	9	4	1.5555556	0.489362	0.510638	251	2146	−1895
377	$13^1 29^1$	Y	N	5	0	1.0000000	0.490716	0.509284	256	2151	−1895
378	$2^1 3^3 7^1$	N	N	−48	32	1.3333333	0.489418	0.510582	208	2151	−1943
379	379^1	Y	Y	−2	0	1.0000000	0.488127	0.511873	206	2151	−1945
380	$2^2 5^1 19^1$	N	N	30	14	1.1666667	0.489474	0.510526	236	2181	−1945
381	$3^1 127^1$	Y	N	5	0	1.0000000	0.490814	0.509186	241	2186	−1945
382	$2^1 191^1$	Y	N	5	0	1.0000000	0.492147	0.507853	246	2191	−1945
383	383^1	Y	Y	−2	0	1.0000000	0.490862	0.509138	244	2191	−1947
384	$2^7 3^1$	N	N	17	12	2.5882353	0.492188	0.507812	261	2208	−1947
385	$5^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.490909	0.509091	245	2208	−1963
386	$2^1 193^1$	Y	N	5	0	1.0000000	0.492228	0.507772	250	2213	−1963
387	$3^2 43^1$	N	N	−7	2	1.2857143	0.490956	0.509044	243	2213	−1970
388	$2^2 97^1$	N	N	−7	2	1.2857143	0.489691	0.510309	236	2213	−1977
389	389^1	Y	Y	−2	0	1.0000000	0.488432	0.511568	234	2213	−1979
390	$2^1 3^1 5^1 13^1$	Y	N	65	0	1.0000000	0.489744	0.510256	299	2278	−1979
391	$17^1 23^1$	Y	N	5	0	1.0000000	0.491049	0.508951	304	2283	−1979
392	$2^3 7^2$	N	N	−23	18	1.4782609	0.489796	0.510204	281	2283	−2002
393	$3^1 131^1$	Y	N	5	0	1.0000000	0.491094	0.508906	286	2288	−2002
394	$2^1 197^1$	Y	N	5	0	1.0000000	0.492386	0.507614	291	2293	−2002
395	$5^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	296	2298	−2002
396	$2^2 3^2 11^1$	N	N	−74	58	1.2162162	0.492424	0.507576	222	2298	−2076
397	397^1	Y	Y	−2	0	1.0000000	0.491184	0.508816	220	2298	−2078
398	$2^1 199^1$	Y	N	5	0	1.0000000	0.492462	0.507538	225	2303	−2078
399	$3^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.491228	0.508772	209	2303	−2094
400	$2^4 5^2$	N	N	34	29	1.6176471	0.492500	0.507500	243	2337	−2094
401	401^1	Y	Y	−2	0	1.0000000	0.491272	0.508728	241	2337	−2096
402	$2^1 3^1 67^1$	Y	N	−16	0	1.0000000	0.490050	0.509950	225	2337	−2112
403	$13^1 31^1$	Y	N	5	0	1.0000000	0.491315	0.508685	230	2342	−2112
404	$2^2 101^1$	N	N	−7	2	1.2857143	0.490099	0.509901	223	2342	−2119
405	$3^4 5^1$	N	N	−11	6	1.8181818	0.488889	0.511111	212	2342	−2130
406	$2^1 7^1 29^1$	Y	N	−16	0	1.0000000	0.487685	0.512315	196	2342	−2146
407	$11^1 37^1$	Y	N	5	0	1.0000000	0.488943	0.511057	201	2347	−2146
408	$2^3 3^1 17^1$	N	N	−48	32	1.3333333	0.487745	0.512255	153	2347	−2194
409	409^1	Y	Y	−2	0	1.0000000	0.486553	0.513447	151	2347	−2196
410	$2^1 5^1 41^1$	Y	N	−16	0	1.0000000	0.485366	0.514634	135	2347	−2212
411	$3^1 137^1$	Y	N	5	0	1.0000000	0.486618	0.513382	140	2352	−2212
412	$2^2 103^1$	N	N	−7	2	1.2857143	0.485437	0.514563	133	2352	−2219
413	$7^1 59^1$	Y	N	5	0	1.0000000	0.486683	0.513317	138	2357	−2219
414	$2^1 3^2 23^1$	N	N	30	14	1.1666667	0.487923	0.512077	168	2387	−2219
415	$5^1 83^1$	Y	N	5	0	1.0000000	0.489157	0.510843	173	2392	−2219
416	$2^5 13^1$	N	N	13	8	2.0769231	0.490385	0.509615	186	2405	−2219
417	$3^1 139^1$	Y	N	5	0	1.0000000	0.491607	0.508393	191	2410	−2219
418	$2^1 11^1 19^1$	Y	N	−16	0	1.0000000	0.490431	0.509569	175	2410	−2235
419	419^1	Y	Y	−2	0	1.0000000	0.489260	0.510740	173	2410	−2237
420	$2^2 3^1 5^1 7^1$	N	N	−155	90	1.1032258	0.488095	0.511905	18	2410	−2392
421	421^1	Y	Y	−2	0	1.0000000	0.486936	0.513064	16	2410	−2394
422	$2^1 211^1$	Y	N	5	0	1.0000000	0.488152	0.511848	21	2415	−2394
423	$3^2 47^1$	N	N	−7	2	1.2857143	0.486998	0.513002	14	2415	−2401
424	$2^3 53^1$	N	N	9	4	1.5555556	0.488208	0.511792	23	2424	−2401
425	$5^2 17^1$	N	N	−7	2	1.2857143	0.487059	0.512941	16	2424	−2408

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
426	$2^1 3^1 71^1$	Y	N	-16	0	1.0000000	0.485915	0.514085	0	2424	-2424
427	$7^1 61^1$	Y	N	5	0	1.0000000	0.487119	0.512881	5	2429	-2424
428	$2^2 107^1$	N	N	-7	2	1.2857143	0.485981	0.514019	-2	2429	-2431
429	$3^1 11^1 13^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	-18	2429	-2447
430	$2^1 5^1 43^1$	Y	N	-16	0	1.0000000	0.483721	0.516279	-34	2429	-2463
431	431^1	Y	Y	-2	0	1.0000000	0.482599	0.517401	-36	2429	-2465
432	$2^4 3^3$	N	N	-80	75	1.5625000	0.481481	0.518519	-116	2429	-2545
433	433^1	Y	Y	-2	0	1.0000000	0.480370	0.519630	-118	2429	-2547
434	$2^1 7^1 31^1$	Y	N	-16	0	1.0000000	0.479263	0.520737	-134	2429	-2563
435	$3^1 5^1 29^1$	Y	N	-16	0	1.0000000	0.478161	0.521839	-150	2429	-2579
436	$2^2 109^1$	N	N	-7	2	1.2857143	0.477064	0.522936	-157	2429	-2586
437	$19^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	-152	2434	-2586
438	$2^1 3^1 73^1$	Y	N	-16	0	1.0000000	0.477169	0.522831	-168	2434	-2602
439	439^1	Y	Y	-2	0	1.0000000	0.476082	0.523918	-170	2434	-2604
440	$2^3 5^1 11^1$	N	N	-48	32	1.3333333	0.475000	0.525000	-218	2434	-2652
441	$3^2 7^2$	N	N	14	9	1.3571429	0.476190	0.523810	-204	2448	-2652
442	$2^1 13^1 17^1$	Y	N	-16	0	1.0000000	0.475113	0.524887	-220	2448	-2668
443	443^1	Y	Y	-2	0	1.0000000	0.474041	0.525959	-222	2448	-2670
444	$2^2 3^1 37^1$	N	N	30	14	1.1666667	0.475225	0.524775	-192	2478	-2670
445	$5^1 89^1$	Y	N	5	0	1.0000000	0.476404	0.523596	-187	2483	-2670
446	$2^1 223^1$	Y	N	5	0	1.0000000	0.477578	0.522422	-182	2488	-2670
447	$3^1 149^1$	Y	N	5	0	1.0000000	0.478747	0.521253	-177	2493	-2670
448	$2^6 7^1$	N	N	-15	10	2.3333333	0.477679	0.522321	-192	2493	-2685
449	449^1	Y	Y	-2	0	1.0000000	0.476615	0.523385	-194	2493	-2687
450	$2^1 3^2 5^2$	N	N	-74	58	1.2162162	0.475556	0.524444	-268	2493	-2761
451	$11^1 41^1$	Y	N	5	0	1.0000000	0.476718	0.523282	-263	2498	-2761
452	$2^2 113^1$	N	N	-7	2	1.2857143	0.475664	0.524336	-270	2498	-2768
453	$3^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	-265	2503	-2768
454	$2^1 227^1$	Y	N	5	0	1.0000000	0.477974	0.522026	-260	2508	-2768
455	$5^1 7^1 13^1$	Y	N	-16	0	1.0000000	0.476923	0.523077	-276	2508	-2784
456	$2^3 3^1 19^1$	N	N	-48	32	1.3333333	0.475877	0.524123	-324	2508	-2832
457	457^1	Y	Y	-2	0	1.0000000	0.474836	0.525164	-326	2508	-2834
458	$2^1 229^1$	Y	N	5	0	1.0000000	0.475983	0.524017	-321	2513	-2834
459	$3^3 17^1$	N	N	9	4	1.5555556	0.477124	0.522876	-312	2522	-2834
460	$2^2 5^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	-282	2552	-2834
461	461^1	Y	Y	-2	0	1.0000000	0.477223	0.522777	-284	2552	-2836
462	$2^1 3^1 7^1 11^1$	Y	N	65	0	1.0000000	0.478355	0.521645	-219	2617	-2836
463	463^1	Y	Y	-2	0	1.0000000	0.477322	0.522678	-221	2617	-2838
464	$2^4 29^1$	N	N	-11	6	1.8181818	0.476293	0.523707	-232	2617	-2849
465	$3^1 5^1 31^1$	Y	N	-16	0	1.0000000	0.475269	0.524731	-248	2617	-2865
466	$2^1 233^1$	Y	N	5	0	1.0000000	0.476395	0.523605	-243	2622	-2865
467	467^1	Y	Y	-2	0	1.0000000	0.475375	0.524625	-245	2622	-2867
468	$2^2 3^2 13^1$	N	N	-74	58	1.2162162	0.474359	0.525641	-319	2622	-2941
469	$7^1 67^1$	Y	N	5	0	1.0000000	0.475480	0.524520	-314	2627	-2941
470	$2^1 5^1 47^1$	Y	N	-16	0	1.0000000	0.474468	0.525532	-330	2627	-2957
471	$3^1 157^1$	Y	N	5	0	1.0000000	0.475584	0.524416	-325	2632	-2957
472	$2^3 59^1$	N	N	9	4	1.5555556	0.476695	0.523305	-316	2641	-2957
473	$11^1 43^1$	Y	N	5	0	1.0000000	0.477801	0.522199	-311	2646	-2957
474	$2^1 3^1 79^1$	Y	N	-16	0	1.0000000	0.476793	0.523207	-327	2646	-2973
475	$5^2 19^1$	N	N	-7	2	1.2857143	0.475789	0.524211	-334	2646	-2980
476	$2^2 7^1 17^1$	N	N	30	14	1.1666667	0.476891	0.523109	-304	2676	-2980
477	$3^2 53^1$	N	N	-7	2	1.2857143	0.475891	0.524109	-311	2676	-2987
478	$2^1 239^1$	Y	N	5	0	1.0000000	0.476987	0.523013	-306	2681	-2987
479	479^1	Y	Y	-2	0	1.0000000	0.475992	0.524008	-308	2681	-2989
480	$2^5 3^1 5^1$	N	N	-96	80	1.6666667	0.475000	0.525000	-404	2681	-3085
481	$13^1 37^1$	Y	N	5	0	1.0000000	0.476091	0.523909	-399	2686	-3085
482	$2^1 241^1$	Y	N	5	0	1.0000000	0.477178	0.522822	-394	2691	-3085
483	$3^1 7^1 23^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	-410	2691	-3101
484	$2^2 11^2$	N	N	14	9	1.3571429	0.477273	0.522727	-396	2705	-3101
485	$5^1 97^1$	Y	N	5	0	1.0000000	0.478351	0.521649	-391	2710	-3101
486	$2^1 3^5$	N	N	13	8	2.0769231	0.479424	0.520576	-378	2723	-3101
487	487^1	Y	Y	-2	0	1.0000000	0.478439	0.521561	-380	2723	-3103
488	$2^3 61^1$	N	N	9	4	1.5555556	0.479508	0.520492	-371	2732	-3103
489	$3^1 163^1$	Y	N	5	0	1.0000000	0.480573	0.519427	-366	2737	-3103
490	$2^1 5^1 7^2$	N	N	30	14	1.1666667	0.481633	0.518367	-336	2767	-3103
491	491^1	Y	Y	-2	0	1.0000000	0.480652	0.519348	-338	2767	-3105
492	$2^2 3^1 41^1$	N	N	30	14	1.1666667	0.481707	0.518293	-308	2797	-3105
493	$17^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	-303	2802	-3105
494	$2^1 13^1 19^1$	Y	N	-16	0	1.0000000	0.481781	0.518219	-319	2802	-3121
495	$3^2 5^1 11^1$	N	N	30	14	1.1666667	0.482828	0.517172	-289	2832	-3121
496	$2^4 31^1$	N	N	-11	6	1.8181818	0.481855	0.518145	-300	2832	-3132
497	$7^1 71^1$	Y	N	5	0	1.0000000	0.482897	0.517103	-295	2837	-3132
498	$2^1 3^1 83^1$	Y	N	-16	0	1.0000000	0.481928	0.518072	-311	2837	-3148
499	499^1	Y	Y	-2	0	1.0000000	0.480962	0.519038	-313	2837	-3150
500	$2^2 5^3$	N	N	-23	18	1.4782609	0.480000	0.520000	-336	2837	-3173

Appendix A A probabilistic argument on the distribution of $|g^{-1}(n)|$

A.1 Definitions and preliminaries

Recall from the introduction that we define the Dirichlet invertible function $g(n) := \omega(n) + 1$ and denote its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute the Dirichlet inverse of $g(n)$ exactly for the first few sequence values as (see Table T.1 of the appendix section)

$$\{g^{-1}(n)\}_{n\geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is given by $\operatorname{sgn}(g^{-1}(n)) = \frac{g^{-1}(n)}{|g^{-1}(n)|} = \lambda(n)$. This useful property is inherited from the distinctly additive nature of the component function $\omega(n)$.

There does not appear to be an easy, nor subtle direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences. However, the distribution of distinct sets of prime exponents is fairly regular so that $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of $g^{-1}(n)$ over $n \geq 2$:

Heuristic Appendix A.1 (Symmetry in $g^{-1}(n)$ in the exponents in the prime factorization of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for some $r \geq 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly two, three, and so on prime factors (compare with the numerical data in Table T.1 starting on page 40).

Conjecture Appendix A.2. *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) $g^{-1}(1) = 1$*;*

(B) *For all $n \geq 1$,* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$*;*

(C) *For all squarefree integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!.$$

We illustrate parts (B)–(C) of the conjecture clearly using Table T.1. The realization that the beautiful and remarkably simple combinatorial form of property (C) in Conjecture 3.5 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ expressed by sums of auxiliary sequences of arithmetic functions.

For natural numbers $n \geq 1, k \geq 0$, let

$$C_k(n) := \begin{cases} \varepsilon(n) = \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$

For any $n \geq 1$, we can prove that (see Lemma 6.3)

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{51}$$

A.2 Our forms of generalizations to Erdös-Kac for additive functions

We want to actually quantify how precise our intuition in the previous subsection is with respect to the observation that since $|g^{-1}(n)|$ depends so closely on $\omega(n)$, it should similarly behave regularly, and of course not deviate too far from its average order on the set $n \leq x$ as $x \to \infty$. More generally, we obtain limiting normal-variant distributions for the densities of key arithmetic functions within bounded ranges.

The proof of Theorem Appendix A.3 closely parallels the argument for an Erdös-Kac theorem for the function $\Omega(n)$ from [8, §7.4]. We utilize this result to prove a limiting distribution-like property for the densities of $|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)|$ in Corollary Appendix A.4. This result, in particular, offers a new take on bounding the summatory functions $G^{-1}(x)$ from the previous section.

In what follows, let

$$\mu_x(C) := \frac{\pi^2}{6} \log\log x, \sigma_x(C) := \sqrt{\mu_x(C)}, \widehat{c} := \frac{1}{6} \times \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \approx 1.5147.$$

For any $z \in \mathbb{R}$, we use the standard notation $\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$.

Theorem Appendix A.3 (Central limit theorem I)**.** *Let $Y > 0$ and $z \in [-Y, Y]$. Then*

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : \frac{\lambda(n)(-1)^{\omega(n)} C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)} \leq z\right\} = \frac{\widehat{c}}{(\log x)^{1-\zeta(2)}} \cdot \Phi(z) + o(1),$$

uniformly for all $-Y \leq z \leq Y$ as $x \to \infty$.

Corollary Appendix A.4 (Central limit theorem II)**.** *Let $Y > 0$ and $z \in [-Y, Y]$. Set the auxiliary variable*

$$w_x(z) := \frac{\pi^2}{6} \cdot |z + \mu_x(C)|.$$

Then

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : \frac{|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)|}{\sigma_x(C)} \leq z\right\} = \frac{\widehat{c}}{(\log x)^{1-\zeta(2)}} \left[\Phi(w_x(z)) - \Phi(-w_x(z))\right] + o(1),$$

uniformly for all $-Y \leq z \leq Y$ as $x \to \infty$.

A.3 Proofs of the main theorems

Proposition Appendix A.5. *For $|z| < 2$, let the summatory function be defined as*

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

Then

$$\widehat{A}_z(x) \sim \frac{x \cdot F(2,z)}{\Gamma(z)} (\log x)^{z-1},$$

where the function $F(s,z)$ is defined for $\operatorname{Re}(s) > 1$ in terms of the exponential of the prime zeta function by

$$F(s,z) := \exp\left(z \cdot P(s)\right) \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

Proof. We know from the proof of Proposition 4.1 that for $n \geq 2$

$$C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}.$$

Then we can generate the denominator terms by the Dirichlet series

$$\sum_{n\geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r\geq 1} \frac{z^{\Omega(p^r)}}{r! \cdot p^{rs}}\right)^{-1} = \exp\left(z \cdot P(s)\right).$$

So by computing a Laplace transform on the right-hand-side of the above with respect to the variable z, we obtain

$$\sum_{n\geq 1} C_{\Omega(n)}(n) \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp\left(tz \cdot P(s)\right) dt = \frac{1}{1 - P(s)z}.$$

It follows that

$$\sum_{n\geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times F(s, z),$$

where

$$F(s, z) := \frac{1}{1 - P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z, \operatorname{Re}(s) > 1, |z| \leq R < 2.$$

Note that we are unable to sum this result in exactly the same format as in the reference [8, §7.4; Thm. 7.18] by effectively setting $s := 1$. However, since for any $|z| \leq R < 2$ we have that

$$\left|\sum_{n\geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^2}\right| < +\infty,$$

we will adapt the details to the traditional case where this method arises in the reference application so that we can sum over our modified function depending on $\Omega(n)$. In fact, we notice that since $|z|^{\Omega(n)} \leq n$, we have the exact DGF

$$\mathcal{H}(s) := \sum_{n\geq 1} \frac{\lambda(n) C_{\Omega(n)}(n)}{n^s},$$

which is absolutely convergent for $\operatorname{Re}(s) > 1$. The DGF $\mathcal{H}(s)$ is thus an analytic function of s whenever $\operatorname{Re}(s) > 1$, and so we can differentiate it any integer $m \geq 0$ number of times to still obtain an absolutely convergent series of the form

$$\left|\sum_{n\geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) (\log n)^m z^{\Omega(n)}}{n^s}\right| < +\infty, \operatorname{Re}(s) > 2.$$

Let the function $d_z(n)$ have DGF $\zeta(s)^z$ for $\operatorname{Re}(s) > 1$, with corresponding summatory function $D_z(x) := \sum_{n\leq x} d_z(n)$. Furthermore, adopting the notation from the reference, we set $b_z(n) := (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$, let the convolution $a_z(n) := \sum_{d|n} b_z(d) d_z(n/d)$, and let the summatory function $A_z(x) := \sum_{n\leq x} a_z(n)$. Then we have that

$$\begin{aligned}
A_z(x) &= \sum_{m\leq x/2} b_z(m) D_z(x/m) + \sum_{x/2 < m \leq x} b_z(m) \\
&= \frac{x}{\Gamma(z)} \times \sum_{m\leq x/2} \frac{b_z(m)}{m^2} \log\left(\frac{x}{m}\right)^{z-1} + O\left(x \sum_{m\leq x} \frac{|b_z(m)|}{m^2} \times \log\left(\frac{2x}{m}\right)^{\operatorname{Re}(z)-2}\right).
\end{aligned}$$

The error term in the previous equation satisfies

$$x \sum_{m\leq x} \frac{|b_z(m)|}{m^2} \times \log\left(\frac{2x}{m}\right)^{\operatorname{Re}(z)-2} \ll x(\log x)^{\operatorname{Re}(z)-2} \sum_{m\leq\sqrt{x}} \frac{|b_z(m)|}{m^2} + x(\log x)^{-(R+2)} \sum_{m>\sqrt{x}} \frac{|b_z(m)|}{m^2} (\log m)^{2R}$$

$$\ll x(\log x)^{\mathrm{Re}(z)-2}.$$

In the main term estimate for $A_z(x)$, when $m \leq \sqrt{x}$

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\mathrm{Re}(z)-2}\right).$$

Hence, the main term sum over the interval $m \leq x/2$ corresponds to bounding

$$\begin{aligned}\sum_{m\leq x/2} b_z(m)D_z(x/m) &= x(\log x)^{z-1}\sum_{m\leq x/2}\frac{b_z(m)}{m^2} \\ &\quad + O\left(x(\log x)^{\mathrm{Re}(z)-2}\sum_{m\leq\sqrt{x}}\frac{|b_z(m)|}{m^2} + x(\log x)^{R-1}\sum_{m>\sqrt{x}}\frac{|b_z(m)|}{m^2}\right) \\ &= x(\log x)^{z-1}F(2,z) + O\left(x(\log x)^{\mathrm{Re}(z)-2}\sum_{m\geq 1}\frac{b_z(m)(\log m)^{2R+1}}{m^2}\right).\end{aligned}$$

This yields the asymptotic main term on the bound cited above. □

Theorem Appendix A.6. *We have uniformly for* $\log\log x - (\log\log x)^{2/3} \leq k < 2\log\log x$ *as* $x \to \infty$

$$\widehat{C}_k(x) := \sum_{\substack{n\leq x \\ \Omega(n)=k}} \lambda(n)(-1)^{\omega(n)}C_k(n) \asymp \frac{\widehat{c}\cdot x}{6\cdot\log x}\cdot\frac{\zeta(2)^{k-1}\cdot(\log\log x)^{k-1}}{(k-1)!}.$$

Moreover, for sufficiently large x *we have that*

$$\left|\sum_{n\leq x}\lambda(n)(-1)^{\omega(n)}C_{\Omega(n)}(n)\right| \asymp \frac{\widehat{c}}{\sqrt{2}\pi^{5/2}}\cdot\frac{x(\log x)^{2\log\pi-\frac{\log 2}{2}-\log 3}}{\sqrt{\log\log x}},$$

where the constant power is approximated by $2\log\pi - \frac{\log 2}{2} - \log 3 \approx 0.844274$.

Proof. First, by induction we can compute the coefficients of $\widehat{A}_z(x)$ with respect to x using the Cauchy integral formula in the following form for integers $m \geq 0$:

$$\frac{1}{m!}\times\frac{\partial^{(m)}}{\partial z^{(m)}}\left[\frac{(\log x)^z}{1+P(2)z}\right] = \sum_{j=0}^{m}\frac{(-P(2))^{m-j}(\log\log x)^j}{j!} = e^{-\frac{\log\log x}{P(2)}}\frac{(-P(2))^m}{m!}\times\Gamma\left(m+1,-\frac{\log\log x}{P(2)}\right).$$

We have parameterized the contour around $|z| = r := \frac{k}{\log\log x}$. As $x\to\infty$ becomes unbounded and sufficiently large, and when $m\to\infty$ depending on x, we apply our standard asymptotic estimate of the incomplete gamma function to obtain that

$$\widehat{A}_{-z}(x) \sim \frac{(\log\log x)^{k-1}}{(k-1)!}\times\frac{x}{\log x}\cdot\frac{\zeta(2)^{k-1}}{\Gamma\left(1-\frac{k}{\log\log x}\right)} \asymp \frac{x}{\log x}\cdot\frac{\zeta(2)^{k-1}(\log\log x)^{k-1}}{(k-1)!},$$

uniformly for $\log\log x - (\log\log x)^{2/3} \leq k \leq \log\log x$.

We now need something deeper about the distribution of $\Omega(n) - \omega(n)$ so that we can weight the signed terms with leading $\lambda(n)(-1)^{\omega(n)}$. The squarefree case is obvious, and in fact we can draw upon known results proved in [8, §2.4] that guarantee limiting asymptotic densities of the sets

$$d_k := \frac{1}{x}\cdot\#\{n\leq x : \Omega(n)-\omega(n)=k\} \sim \frac{3\widehat{c}}{2}\cdot 2^{-k} + O(5^{-k}),\ \text{as } x\to\infty.$$

The constant c is absolute and corresponds to the infinite prime product we defined earlier. We can assume, as is typical in justifying the canonical forms of the Erdös-Kac theorems for $\Omega(n)$ and $\omega(n)$, that for a random large $n \gg 1$, each of $(\omega(n), \Omega(n), \Omega(n) - \omega(n))$ are uniformly distributed as indicators of the prime factorization that arises for this n. With this assumption, for $1 \leq \omega(n) \leq \Omega(n) =: k$ we expect the sign of the terms $\lambda(n)(-1)^{\omega(n)}$ to obey the approximate density

$$\sum_{m=0}^{k} (-1)^m \cdot 2^{-m} = \widehat{c} + \frac{(-1)^k \cdot \widehat{c}}{4 \cdot 2^k}.$$

The leading unsigned term in the previous expansion is what yields the main term when we sum over powers of k.

Now we bound the magnitude of the following sums that we defined above by applying asymptotics for the incomplete gamma function in combination with Stirling's formula for x large:

$$\begin{aligned} \left| \sum_{k=1}^{\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} \lambda(n)(-1)^{\omega(n)} C_k(n) \right| &\asymp \left| \sum_{k=1}^{\log\log x} \frac{\widehat{c} \cdot x \cdot \zeta(2)^{k-1} (\log\log x)^{k-1}}{(\log x) \cdot (k-1)!} \right| \\ &\asymp \frac{\widehat{c}}{\sqrt{2}\pi^{5/2}} \cdot \frac{x (\log x)^{2\log\pi - \frac{\log 2}{2} - \log 3}}{\sqrt{\log\log x}}. \end{aligned}$$

Notice that our uniform bounds on $\widehat{C}_k(x)$ proved above hold only when k depends on x and tends to infinity as $x \to \infty$. The summands we have used to obtain the bound in the previous formula capture the leading, most asymptotically significant term resulting from the Cauchy integral formula for $k \leq \log\log x - (\log\log x)^{2/3}$ in the initial range. We know from the reference that the significant contributions of the densities of the sets $\{n \leq x : \Omega(n) = k\}$ occur for k over the latter ranges so that the sum above is still accurate as $x \to \infty$. □

Proof of Theorem Appendix A.3. For large x and $n \leq x$, define the following auxiliary variables:

$$\alpha_n := \frac{C_{\Omega(n)}(n) - \mu_n(C)}{\sigma_n(C)}, \quad \beta_{n,x} := \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)}.$$

Let the corresponding densities (whose limiting distributions we must verify) be defined by the functions

$$\Phi_1(x, z) := \frac{1}{x} \cdot \#\{n \leq x : \alpha_n \leq z\},$$

and

$$\Phi_2(x, z) := \frac{1}{x} \cdot \#\{n \leq x : \beta_{n,x} \leq z\}.$$

We first argue that it suffices to consider the distribution of $\Phi_2(x, z)$ as $x \to \infty$ in place of $\Phi_1(x, z)$ to obtain our desired result statement. In particular, the difference of the two auxiliary variables is neglibible as $x \to \infty$ for n, x taken over the ranges that contribute the non-trivial weight to the main term of each density function. We have for $\sqrt{x} \leq n \leq x$ and $C_{\Omega(n)}(n) \leq 2 \cdot \mu_x(C)$ that

$$|\alpha_n - \beta_{n,x}| \ll \frac{1}{\sigma_x(C)} \xrightarrow{x \to \infty} 0.$$

So we naturally prefer to estimate the easier forms of the distribution function $\Phi_2(x, z)$ when x is large, and for any fixed $z \in \mathbb{R}$. Much of the core of the next logic to our proof is adapted from the proof of Theorem 7.21 in the reference [8, §7.4]. In fact, we need to do little besides repeat the highlights of that argument to justify the form of the distribution of our limiting densities. The main adaptation from the proof in the reference is that we must replace $\log\log x \mapsto \frac{\pi^2}{6} \log\log x$.

For positive integers $k \geq 1$, write $k := u + \zeta(2)\log\log x$, set the parameter $\delta_{u,x} := \frac{u}{\zeta(2)\log\log x}$, and suppose initially that $|u| \leq \frac{\zeta(2)}{2}\log\log x$. As $x \to \infty$, we approximate

$$\frac{\zeta(2)^{k-1}(\log\log x)^{k-1}}{(k-1)!} = \frac{e^u(\log x)^{\zeta(2)}}{\sqrt{2\pi\zeta(2)\log\log x}}(1+\delta_{u,x})^{\frac{1}{2}-\zeta(2)\log\log x - u} \times \left(1 + O\left(\frac{1}{\log\log x}\right)\right),$$

We have uniformly for $|\delta| \leq \frac{1}{2}$ that $\log(1+\delta) = \delta - \frac{\delta^2}{2} + O(|\delta|^3)$. So for the $\delta_{u,x} > 0$ defined above satisfying this uniform bound, we obtain

$$(1+\delta_{u,x})^{\frac{1}{2}-\zeta(2)\log\log x - u} = \exp\left(-u + \frac{u-u^2}{2\zeta(2)\log\log x} - \frac{u^2}{4\zeta(2)^2(\log\log x)^2} + O\left(\frac{|u|^3}{(\log\log x)^3}\right)\right).$$

We complete the estimates as in the reference to verify consistent asymptotics for the cases where $\log\log x - (\log\log x)^{2/3} \leq k \leq (\log\log x)^{2/3}$, $|u| \leq 1$, and $|u| > 1$. This leads to

$$\frac{\zeta(2)^{k-1}(\log\log x)^{k-1}}{(k-1)!} = \frac{(\log x)^{\zeta(2)}}{\sqrt{2\pi\zeta(2)\log\log x}}\exp\left(-\frac{u^2}{2\zeta(2)\log\log x}\right)(1+o(1)), \text{ as } x \to \infty.$$

Thus, we see that

$$\frac{\widehat{C}_k(x)}{x} = \frac{\widehat{c}}{(\log x)^{1-\zeta(2)}\sqrt{2\pi\cdot\zeta(2)\log\log x}}\exp\left(-\frac{(k-\zeta(2)\log\log x)^2}{2\zeta(2)\log\log x}\right)(1+o(1)), \text{ as } x \to \infty.$$

Then we conclude the result by summing over the asymptotically weighted range where $\left(\zeta(2)^{-1}\log\log x\right) - (\log\log x)^{2/3} \leq k \leq \zeta(2)^{-1}\log\log x + z\left(\zeta(2)^{-1}\log\log x\right)^{1/2}$. $\square$

Remark Appendix A.7. Note that technically, the multiplier and scaling by the reciprocal power of $\log x$ prevent us from getting a *probability* distribution proper from these estimates in Corollary Appendix A.3. This indicates that our choices for the mean and variance analogs in a central limit theorem from a probabilistic setting, while convenient and natural following from the model proof in the reference, are slightly imprecise in describing the distribution of this particular arithmetic function case, Nonetheless, as $x \to \infty$, the result we do prove provides useful intuition about the values of key arithmetic functions that are components in building the values and distribution of $g^{-1}(n)$ via divisor sums over $n \leq x$.

Proof of Corollary Appendix A.4. We compute using the argument sketched in the proof of Corollary 6.6 from Section 6.3 that

$$|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2}C_{\Omega(n)}(n).$$

Then the result follows from Theorem Appendix A.3. In particular, we shift, scale, and then take the absolute value of the arithmetic functions $\lambda(n)(-1)^{\omega(n)}C_{\Omega(n)}(n)$ that resulted in the known limiting densities in the first theorem. In particular, what we arrive at is the task of estimating

$$\frac{1}{x}\#\left\{n \leq x : -\frac{\pi^2}{6}|z+\mu_x(C)| \leq \frac{\lambda(n)(-1)^{\omega(n)}C_{\Omega(n)}(n)}{\sigma_x(C)} \leq \frac{\pi^2}{6}|z+\mu_x(C)|\right\},$$

as $x \to \infty$ for $w_x(z) := \frac{\pi^2}{6}|z+\mu_x(C)|$ defined as in the statement of the result. $\square$