Lower bounds on the summatory function of the Möbius function along infinite subsequences

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Abstract

The Mertens function, $M(x) = \sum_{n \leqslant x} \mu(n)$, is classically defined as the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture states that $|M(x)| < C \cdot \sqrt{x}$ with come absolute $C > 0$ for all $x \geqslant 1$. It has a well-known disproof due to Odlyzko and té Riele given in the early 1980's by computation of non-trivial zeta function zeros in conjunction with integral formulas expressing $M(x)$. It is conjectured that $M(x)/\sqrt{x}$ changes sign infinitely often and grows unbounded in the direction of both $\pm\infty$ along infinite subsequences of positive integers. We prove the unboundedness of $|M(x)|/\sqrt{x}$ by showing that

$$\limsup_{x \to \infty} \frac{|M(x)|(\log\log x)^{\frac{11}{4}} (\log\log\log x)^2}{\sqrt{x} \cdot (\log x)^{\frac{1}{2}}} > 0.$$

There is a distinct stylistic flavor and new element of combinatorial analysis to our proof peppered in with the standard methods from analytic, additive and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

Keywords and Phrases: *Möbius function; Mertens function; summatory function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting functions; Dirichlet generating function; asymptotic lower bounds; Mertens conjecture.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; and 11N64.*

Glossary of special notation and conventions

Symbol	**Definition**
$\approx$	We adopt the convention that $f(x) \approx g(x)$ if $\lvert f(x) - g(x)\rvert = O(1)$ as $x \rightarrow \infty$.
$\mathbb{E}[f(x)], \overset{\mathbb{E}}{\sim}$	We use the expectation notation $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \overset{\mathbb{E}}{\sim} h(x)$, to denote that f has a so-called *average order* growth rate of $h(x)$. What this means is that $\frac{1}{x}\sum_{n \leqslant x} f(n) \sim h(x)$, or equivalently that $$\lim_{x\rightarrow\infty} \frac{\frac{1}{x}\sum_{n\leqslant x} f(n)}{h(x)} = 1.$$
B	The absolute constant $B \approx 0.2614972128476427837554$ from the statement of Mertens theorem.
$o(f), O_\alpha(g)$	We write that $f = o(g)$ if $$\lim_{x\rightarrow\infty} \frac{f(x)}{g(x)} = 0.$$ We sometimes adapt the standard big-O notation, writing $f = O_{\alpha_1,\ldots,\alpha_k}(g)$ for some parameters $\alpha_1, \ldots, \alpha_k$ that do not depend on x, if $f(x) = O(g(x))$ subject only to the stated upper bound on f having an implicit dependence only on x (as usual) and on the α_i for $1 \leqslant i \leqslant k$.
$C_k(n)$	These auxiliary functions are defined recursively for $n \geqslant 1$ and $1 \leqslant k \leqslant \Omega(n)$ according to the formula $$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum_{d\mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geqslant 1. \end{cases}$$
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n\geqslant 0}$.
DGF	Given a sequence $\{f(n)\}_{n\geqslant 1}$, its *Dirichlet generating function* (DGF) is defined by $D_f(s) := \sum_{n\geqslant 1} f(n) n^{-s}$ subject to suitable constraints on the real part of the parameter $s \in \mathbb{C}$ that guarantee convergence of $D_f(s)$.
$d(n)$	The divisor function, $d(n) := \sum_{d\mid n} 1$, for $n \geqslant 1$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) = \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (defined below).
$f * g$	The Dirichlet convolution of f and g, $(f * g)(n) := \sum_{d\mid n} f(d) g(n/d)$, where the sum is taken over the divisors d of n for $n \geqslant 1$.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ for $n \geqslant 1$ with $f^{-1}(1) = 1/f(1)$. The Dirichlet inverse of f exists if and only if $f(1) \neq 0$. This inverse function, provided it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$.

Symbol	Definition
$\lfloor x \rfloor$,$\lceil x \rceil$	The floor function is defined as $\lfloor x \rfloor := x - \{x\}$ where $0 \leqslant \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$. The floor function is sometimes also written as $[x] \equiv \lfloor x \rfloor$. The corresponding ceiling function $\lceil x \rceil$ denotes the smallest integer $m \geqslant x$.
$\gg, \ll$	We write $f \gg g$ and $h \ll r$ provided that there are constants $C, D \geqslant 1$ such that whenever x is sufficiently large we have that $C \cdot f(x) \geqslant g(x)$ and $h(x) \leqslant D \cdot r(x)$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$.
H_n	The *first-order harmonic numbers*, $H_n := \sum_{k=1}^{n} \frac{1}{k}$, satisfy the limiting asymptotic relation $$\lim_{n \to \infty} \left[H_n - \log(n)\right] = \gamma,$$ where $\gamma \approx 0.577216$ denotes Euler's gamma constant.
$\mathbb{1}_{\mathbb{S}}$, $\chi_{\mathtt{cond}(x)}$	We use the notation $\mathbb{1}_{\mathbb{S}}, \chi : \mathbb{N} \to \{0, 1\}$ to denote indicator, or characteristic functions of a set. In paticular, $\mathbb{1}_{\mathbb{S}}(n) = 1$ if and only if $n \in \mathbb{S}$, and $\chi_{\mathtt{cond}}(n) = 1$ if and only if n satisfies the boolean-valued condition `cond`.
$[n = k]_{\delta}$,$[\mathtt{cond}]_{\delta}$	The symbol $[n = k]_{\delta}$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise. For a boolean-valued conditions, `cond`, $[\mathtt{cond}]_{\delta}$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention.*
$\lambda(n)$	The Liouville lambda function, $\lambda(n) := (-1)^{\Omega(n)}$, denotes the signed parity of $\Omega(n)$, the number of distinct prime factors of n counting their multiplicity. That is, $\lambda(n) \in \{\pm 1\}$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree, i.e., n has no prime power divisors with exponent greater than one.
$M(x)$	The Mertens function is the summatory function over $\mu(n)$ defined for all integers $x \geqslant 1$ by $M(x) := \sum\limits_{n \leqslant x} \mu(n)$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^{\alpha} \vert\vert n$ (or when p^{α} exactly divides n) for p prime and $n \geqslant 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p \vert n} 1$ and the completely additive function $\Omega(n) := \sum_{p^{\alpha} \vert\vert n} \alpha$. Equivalently, if the prime factorization of $n \geqslant 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leqslant n \leqslant x$ for $x > 1$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leqslant x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leqslant x : \Omega(n) = k\}$ for $x \geqslant 2$.
$\sum_{p \leqslant x}, \prod_{p \leqslant x}$	Unless otherwise specified by context, we use the index variable p to denote that the summation (product) is to be taken only over prime values within the summation bounds.
$P(s)$	For complex s with $\mathrm{Re}(s) > 1$, we define the *prime zeta function* to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$. For $\mathrm{Re}(s) > 1$, the prime zeta function is related to $\zeta(s)$ according to the formula $P(s) = \sum_{k \geqslant 1} \frac{\mu(k)}{k} \log[\zeta(ks)]$.

Symbol	**Definition**
$Q(x)$	For $x \geqslant 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leqslant x$. More precisely, this function is summed and identified with its limiting asymptotic formula as $x \to \infty$ in the following form: $Q(x) := \sum_{n \leqslant x} \mu^2(n) \sim \frac{6}{\pi^2} x + O(\sqrt{x})$.
$\sim$	We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x \to \infty} \frac{A(x)}{B(x)} = 1$.
$\zeta(s)$	The Riemann zeta function, defined by $\zeta(s) := \sum_{n \geqslant 1} n^{-s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation on the entire complex plane with the exception of a simple pole at $s = 1$.

1 Introduction

1.1 Definitions

Suppose that $n \geqslant 2$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ so that $r = \omega(n)$. We define the *Möbius function* to be the signed indicator function of the squarefree integers as follows:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1,\ \forall 1 \leqslant i \leqslant k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many other variants and special properties of the Möbius function and its generalizations [14, *cf.* §2]. A crucial role of the classical $\mu(n)$ forms an inversion relation for arithmetic functions convolved with one by *Möbius inversion*:

$$g(n) = (f * 1)(n) \iff f(n) = (g * \mu)(n), \forall n \geqslant 1.$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined as

$$M(x) = \sum_{n \leqslant x} \mu(n), x \geqslant 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as [15, A002321]

$$\{M(x)\}_{x \geqslant 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2, \ldots\}$$

Clearly, a positive integer $n \geqslant 1$ is *squarefree*, or contains no (prime power) divisors which are squares, if and only if $\mu^2(n) = 0$. A related summatory function which counts the number of *squarefree* integers $n \leqslant x$ then satisfies [3, §18.6] [15, A013928]

$$Q(x) = \sum_{n \leqslant x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \to \infty$:

$$\mu_+(x) = \frac{\#\{1 \leqslant n \leqslant x : \mu(n) = +1\}}{Q(x)} \overset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leqslant n \leqslant x : \mu(n) = -1\}}{Q(x)} \xrightarrow{x \to \infty} \frac{3}{\pi^2}.$$

1.2 Properties

One conventional approach to evaluating the behavior of $M(x)$ for large $x \to \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is expressed given the inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \cdot \int_1^\infty \frac{M(x)}{x^{s+1}} dx,$$

we obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T - \imath\infty}^{T + \imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation, along with the standard Euler product representation for the reciprocal zeta function cited in the first equation above, leads us to the exact expression for $M(x)$ for any real $x > 0$ given by the next theorem due to Titchmarsh.

Theorem 1.1 (Analytic Formula for $M(x)$)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k\geqslant 1}$ satisfying $k \leqslant T_k \leqslant k+1$ for each k such that for any real $x > 0$*

$$M(x) = \lim_{k\rightarrow\infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |\operatorname{Im}(\rho)|<T_k}} \frac{x^\rho}{\rho\cdot\zeta'(\rho)} - 2 + \sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n\cdot(2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2}\left[x\in\mathbb{Z}^+\right]_\delta .$$

A historical unconditional bound on the Mertens function due to Walfisz (1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp\left(-C\cdot\log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan more recently proved new updated estimates bounding $M(x)$ for large x in the following forms [16]:

$$\begin{aligned} M(x) &\ll \sqrt{x}\cdot\exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x}\cdot\exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right),\ \forall\epsilon > 0. \end{aligned}$$

1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C\cdot\sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens for $C = 1$ and all $x < 10000$. Since its beginnings in 1897, the Mertens conjecture has been disproven by computation of non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele from the early 1980's [11]. Since the truth of the conjecture would have implied the RH, more recent attempts at bounding $M(x)$ consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound towards both $\pm\infty$ along infinite subsequences.

In fact, one of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ actually grows without bound on the natural numbers. A precise statement of this problem is to produce an affirmative answer whether $\limsup_{x\rightarrow\infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x\rightarrow\infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there are infinite subsequences of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound towards either $\pm\infty$ along the subsequence. We cite that prior to this point it is only known by computation that [13, *cf.* §4.1] [15, *cf.* A051400; A051401]

$$\limsup_{x\rightarrow\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geqslant 1.826054),$$

and

$$\liminf_{x\rightarrow\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leqslant -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [11, 6, 7, 4]. Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies [10]

$$\limsup_{x\rightarrow\infty} \frac{|M(x)|}{\sqrt{x}\cdot(\log\log x)^{5/4}} = O(1).$$

2 An overview of the core logical steps and components to the proof

We offer an initial step-by-step summary overview of the core components to our proof outlined in the next. As our proof methodology is new and relies on non-standard elements compared to more traditional methods of bounding $M(x)$, we hope that this sketch of the logical components to this argument makes the article easier to parse.

2.1 Step-by-step overview

(1) We prove a matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). See Theorem 3.1 in Section 4.

(2) This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the seemingly unconnected prime counting function and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is stated in (1).

The strong additivity of $\omega(n)$ imparts the characteristic signedness of $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ for all $n \geqslant 1$, which is weighted according to the parity of $\Omega(n)$. The link relating (1) to canonical additive functions and their distributions then lends a recent distinguishing element to the success of the methods in our proof.

(3) We tighten an updated result from [9, §7] providing uniform asymptotic formulas for the summatory functions, $\widehat{\pi}_k(x)$, that indicate the parity of $\Omega(n)$ (sign of $\lambda(n)$) for $n \leqslant x$ and $1 \leqslant k \leqslant \log\log x$. These formulas are proved using expansions of more combinatorially motivated Dirichlet series (see Theorem 3.7). We use this result to bound sums of the form $\sum_{n \leqslant x} \lambda(n) f(n)$ for particular non-negative arithmetic functions f when x is large.

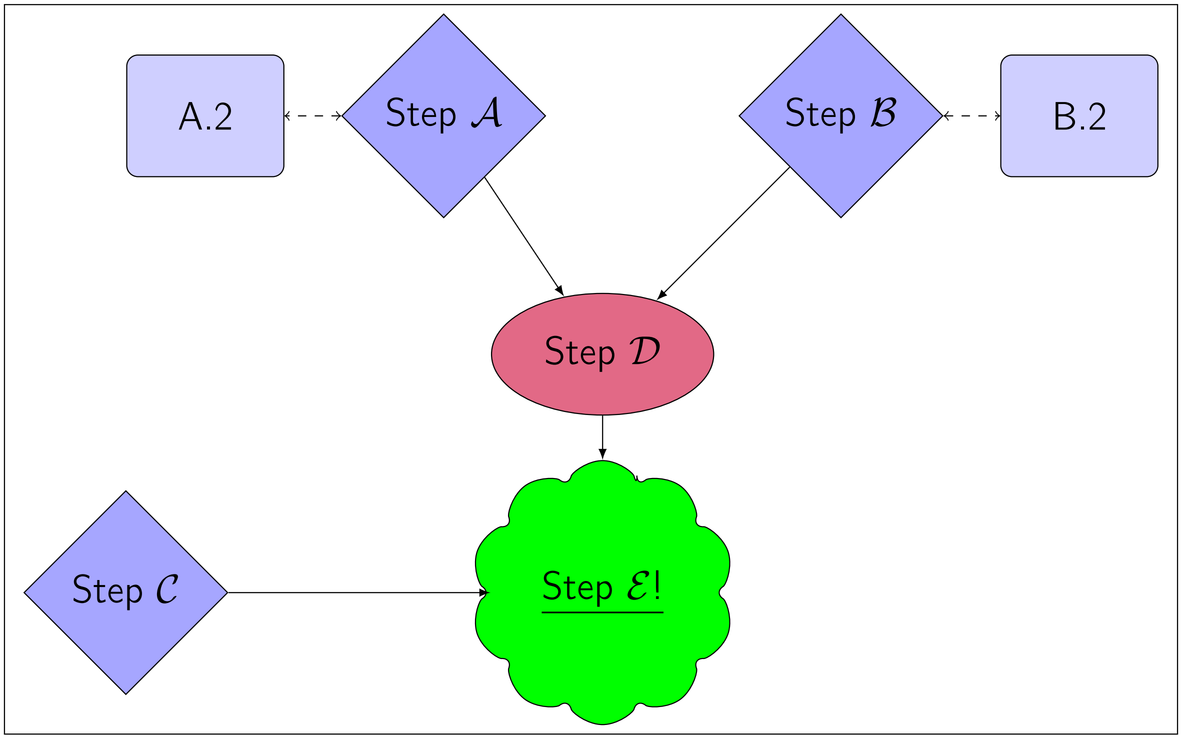
(4) We then turn to bounding the asymptotics of the quasi-periodic functions, $g^{-1}(n)$, by estimating this inverse function's limiting order for large $n \leqslant x$ as $x \rightarrow \infty$ in Section 6. We eventually use these estimates to prove a substantially unique new lower bound formula for the summatory function $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$ along certain asymptotically large infinite subsequences (see Theorem 7.6).

(5) We spend some interim time in Section 7.2 carefully working out a rigorous justification for why the limiting lower bounds we obtain from average order case analysis of our arithmetic function approximations to $g^{-1}(n)$ are sufficient to prove the corollary on the unboundedness of $M(x)$ below.

(6) When we return to step **(2)** with our new lower bounds at hand, we have a new unconditional proof of the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. What we recover is a quick, and rigorous, proof of Corollary 3.8 given in Section 7.3.

2.2 Schematic flowchart of the proof logic

The next flowchart diagramed below shows how the seemingly disparate components of the proof are organized.



Legend to the diagram stages:

▶ **Step A:** *Citations and re-statements of existing theorems proved elsewhere.*

A.A: Key results and constructions:
- Theorem 3.6
- Corollary 5.5
- The results, lemmas, and facts cited in Section 4.3

A.2: Lower bounds on the Abel summation based formula for $G^{-1}(x)$:
- Theorem 3.7 (on page 18)
- Proposition 5.6
- Theorem 7.6

▶ **Step B:** *Constructions of an exact formula for $M(x)$.*

B.B: Key results and constructions:
- Corollary 3.3 (follows from Theorem 3.1 proved on page 12)
- Proposition 4.1

B.2: Asymptotics for the component functions $g^{-1}(n)$ and $G^{-1}(x)$:
- Theorem 6.3 (on page 21)
- Lemma 6.4

▶ **Step C:** *A justification for why lower bounds obtained roughly "on average" suffice.*
- The results proved in Section 7.2

▶ **Step D:** *Interpreting the exact formula for $M(x)$.*
- Proposition 7.1
- Theorem 7.6

▶ **Step E:** *The Holy Grail.* Proving that $\frac{|M(x)|}{\sqrt{x}}$ grows without bound in the limit supremum sense.
- Corollary 3.8 (on page 36)

3 A concrete new approach for bounding $M(x)$ from below

3.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 3.1 (Summatory functions of Dirichlet convolutions). *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leqslant x} f(n)$ and $H(x) := \sum_{n \leqslant x} h(n)$ denote the summatory functions of f, h, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse f^{-1} of f. Then we have the following equivalent expressions for the summatory function of $f * h$ for all integers $x \geqslant 1$:*

$$\begin{aligned}
\pi_{f*h}(x) &:= \sum_{n \leqslant x} \sum_{d \mid n} f(d) h(n/d) \\
&= \sum_{d \leqslant x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].
\end{aligned}$$

Moreover, we can invert the linear system determining the coefficients of $H(k)$ for $1 \leqslant k \leqslant x$ naturally to express $H(x)$ as a linear combination of the original left-hand-side summatory function as follows:

$$\begin{aligned}
H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\
&= \sum_{n=1}^{x} f^{-1}(n) \pi_{f*h}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).
\end{aligned}$$

Corollary 3.2 (Convolutions Arising From Möbius Inversion). *Suppose that g is an arithmetic function on the positive integers such that $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n \leqslant x} (g * \mu)(n)$. Then the Mertens function equals*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geqslant 1.$$

Corollary 3.3 (A motivating special case). *We have exactly that for all $x \geqslant 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

3.2 An exact expression for $M(x)$ in terms of strongly additive functions

From this point on, we fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and denote its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute the Dirichlet inverse of $g(n)$ exactly for the first few sequence values as (see Table T.1 of the appendix section)

$$\{g^{-1}(n)\}_{n \geqslant 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is given by $\operatorname{sgn}(g^{-1}(n)) = \frac{g^{-1}(n)}{|g^{-1}(n)|} = \lambda(n)$ (see Proposition 4.1). This useful property is inherited from the distinctly additive nature of the component function $\omega(n)$ [A].

There does not appear to be an easy, nor subtle direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences. However, the distribution of distinct sets of prime exponents is fairly regular so that $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of $g^{-1}(n)$ over $n \geqslant 2$:

[A] Indeed, for any non-negative additive arithmetic function $a(n)$, $(a+1)^{-1}(n)$ has leading sign given by $\lambda(n)$ for any $n \geqslant 1$. For multiplicative f, we obtain a related condition that $\operatorname{sgn}(f(n)) = (-1)^{\omega(n)}$ for all $n \geqslant 1$.

Heuristic 3.4 (Symmetry in $g^{-1}(n)$ in the exponents in the prime factorization of n)**.** Suppose that $n_1, n_2 \geqslant 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for some $r \geqslant 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly two, three, and so on prime factors (compare with the numerical data in Table T.1 starting on page 39).

Conjecture 3.5. *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) $g^{-1}(1) = 1$*;*

(B) *For all $n \geqslant 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*

(C) *For all squarefree integers $n \geqslant 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!.$$

We illustrate parts (B)–(C) of the conjecture clearly using Table T.1. The realization that the beautiful and remarkably simple combinatorial form of property (C) in Conjecture 3.5 holds for all squarefree $n \geqslant 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ expressed by sums of auxiliary sequences of arithmetic functions [B] (see Section 6).

For natural numbers $n \geqslant 1, k \geqslant 0$, let

$$C_k(n) := \begin{cases} \varepsilon(n) = \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geqslant 1. \end{cases}$$

For any $n \geqslant 1$, we can prove that (see Lemma 6.4)

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{2}$$

In light of the fact that (see Proposition 7.1)

$$M(x) \approx G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)},$$

the formula in (2) implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by appropriate estimates of the summatory function $G^{-1}(x)$ [C].

3.3 Uniform asymptotics from enumerative counting DGFs in Mongomery and Vaughan

Our inspiration for the new bounds found in the last sections of this article allows us to sum non-negative arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. We utilize a somewhat more general hybrid generating function and enumerative DGF method under which we are able to recover "good enough" asymptotics about the summatory functions that encapsulate the parity of $\Omega(n)$ (or sign of $\lambda(n)$) through the summatory tally functions $\widehat{\pi}_k(x)$ (see Section 5.1).

[B] A proof of this property is not difficult to give using Lemma 6.4 stated on page 21.

[C] We can also prove that

$$M(x) = G^{-1}(x) + \sum_{p \leqslant x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right),$$

by inversion since

$$G^{-1}(x) = \sum_{d \leqslant x} (g^{-1} * 1)(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right),$$

with $(g^{-1} * 1)^{-1} = g * \mu = \chi_{\mathbb{P}} + \varepsilon$ defined such that $\chi_{\mathbb{P}}$ is the characteristic function of the primes.

Theorem 3.6 (Montgomery and Vaughan)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leqslant x : \Omega(n) = k\}.$$

For $R < 2$ we have that

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

uniformly for $1 \leqslant k \leqslant R\log\log x$ where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, z \geqslant 0.$$

The proof of the next result is combinatorially motivated in so much as it interprets lower bounds on a key infinite product factor of $\mathcal{G}(z)$ defined in Theorem 3.6 as corresponding to an ordinary generating function of certain homogeneous symmetric polynomials involving reciprocals of the primes.

Theorem 3.7. *For almost every large x we have uniformly for $1 \leqslant k \leqslant \log\log x$ that*

$$\widehat{\pi}_k(x) \gg \frac{4}{3\sqrt{\pi}} \frac{x}{\log x} \left(\frac{\log 2}{\log x}\right)^{\frac{k-1}{\log\log x}+2} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O\left(\frac{k}{(\log\log x)^2}\right)\right).$$

3.4 Cracking the classical unboundedness barrier

In Section 7, we are able to state what forms the bridge between the results we carefully build up to in the proofs established in prior sections of the article. What we eventually obtain at the conclusion of the section is the next important summary corollary that resolves the classical question of the unboundedness of the scaled function Mertens function $q(x) := |M(x)|/\sqrt{x}$ in the limit supremum sense.

Corollary 3.8 (Unboundedness of the the Mertens function, $q(x)$)**.** *We have that*

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

In establishing the rigorous proof of Corollary 3.8 based on our new methods, we not only show unboundedness of $q(x)$, but also set a minimal rate (along a large infinite subsequence) at which this form of the scaled Mertens function grows without bound.

4 Preliminary proofs of new results

4.1 Establishing the summatory function properties and inversion identities

We will first prove Theorem 3.1 using matrix methods and similarity transforms by shift matrices. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95].

Proof of Theorem 3.1. Let h, g be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leqslant x} h(n)$ and $G(x) = \sum_{n \leqslant x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h: $g*h$. Then we can readily see that the following initial formulas hold for all $x \geqslant 1$:

$$\begin{aligned}\pi_{g*h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] H(i).\end{aligned}$$

We form the matrix of coefficients associated with this linear system defining $H(n)$ for all $n \leqslant x$. We then invert the system to express an exact solution for $H(x)$ at any $x \geqslant 1$. Let the matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right), \forall 1 \leqslant j \leqslant x.$$

The matrix we must invert in this problem is lower triangular, with ones on its diagonals, and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

Here, U is a square matrix with finite dimensions whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$\left[(I - U^T)^{-1}\right]_{i,j} = [j \leqslant i]_{\delta}.$$

It is a useful fact that if we take successive differences in x of the floor of certain fractions, $\left\lfloor \frac{x}{j} \right\rfloor$, we get non-zero behavior at the divisors of x:

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

We use this property to shift the matrix $\hat{G}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of g in the following form:

$$\left[(I - U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_{\delta}\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_{\delta}\right).$$

Now we can express the inverse of the target matrix,

$$(g_{x,j}) = (I - U^T)^{-1}\left(g\left(\frac{x}{j}\right)[j|x]_{\delta}\right)(I - U^T),$$

using a similarity transformation conjugated by shift operators as follows:

$$(g_{x,j})^{-1} = (I - U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_{\delta}\right)(I - U^T)$$

$$= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k)\right)(I - U^T)$$

$$= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} g^{-1}(k)\right).$$

Hence, the summatory function $H(x)$ is exactly expressed for any $x \geqslant 1$ by a vector product with the inverse matrix from the previous equation in the form of

$$\begin{aligned} H(x) &= \sum_{k=1}^{x} g_{x,k}^{-1} \cdot \pi_{g*h}(k) \\ &= \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \cdot \pi_{g*h}(k). \end{aligned}$$

□

4.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n. Then we can easily prove that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu. \tag{3}$$

When combined with Corollary 3.2 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 3.3.

Proposition 4.1 (The signedness property of $g^{-1}(n)$)**.** *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)| + [h(n)=0]_{\delta}} \in \{0, \pm 1\}$ *denote the sign of the arithmetic function h at integers $n \geqslant 1$. For the Dirichlet invertible function, $g(n) := \omega(n) + 1$, we have that* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ *for all $n \geqslant 1$.*

Proof. The function $D_f(s) := \sum_{n \geqslant 1} f(n) n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ for σ_f the abcissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = 1/\zeta(s)$ and $D_\omega(s) = P(s)\zeta(s)$. Then by (3) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any invertible arithmetic function f, for all $\operatorname{Re}(s) > 1$ we have

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{4}$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. From this fact, it follows that $\operatorname{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make this intuition rigorous.

By the standard recurrence relation that defines the Dirichlet inverse function of any arithmetic function h such that $h(1) = 1$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d \mid n \\ d > 1}} h(d) h^{-1}(n/d), & n \geqslant 2. \end{cases} \tag{5}$$

For $n \geqslant 2$, the summands in (5) can be simply indexed over the primes $p|n$ given our definition of h from above. This observation yields that we can inductively expand these sums into nested divisor sums provided the depth of the sums does not exceed the capacity to index summations over the primes dividing n. Namely, notice that for $n \geqslant 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p|n} h^{-1}(n/p), && \text{if } \Omega(n) \geqslant 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right), && \text{if } \Omega(n) \geqslant 2 \\ &= -\sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1 p_2}} h^{-1}\left(\frac{n}{p_1 p_2 p_3}\right), && \text{if } \Omega(n) \geqslant 3. \end{aligned}$$

Then by induction, again with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1, n \geqslant 2. \tag{6}$$

If for $n \geqslant 2$ we write the prime factorization of n as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{\omega(n)}^{\alpha_{\omega(n)}}$ where the exponents $\alpha_i \geqslant 1$ for all $1 \leqslant i \leqslant \omega(n)$, we can see that [A]

$$\begin{aligned} |h^{-1}(n)| &\geqslant (\omega(n))! && =: h_\ell^{-1}(n), n \geqslant 2, \\ |h^{-1}(n)| &\leqslant (\Omega(n))!^{\max(\alpha_1, \alpha_2, \ldots, \alpha_{\omega(n)})} && =: h_u^{-1}(n), n \geqslant 2, \end{aligned} \tag{7}$$

with equality precisely at squarefree $n \geqslant 2$. The bounding functions $h_\ell^{-1}(n), h_u^{-1}(n) > 0$ are clearly positive for all $n \geqslant 1$. What these bounds show is that for all $n \geqslant 1$ (with $\lambda(1) = 1$) the following property holds:

$$\operatorname{sgn}(h^{-1}(n)) = \lambda(n).$$

Since λ is completely multiplicative, and since $\mu(n) = \lambda(n)$ whenever n is squarefree, we obtain that

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geqslant 1.$$

Finally, since $|h^{-1}(n)| > 0$ for all $n \geqslant 1$ by the bounds we proved in (7), the previous equation implies our result. □

4.3 Statements of other facts and known limiting asymptotics

Theorem 4.2 (Mertens theorem)**.** *For all $x \geqslant 2$ we have that*

$$P_1(x) := \sum_{p \leqslant x} \frac{1}{p} = \log\log x + B + o(1), \text{ as } x \to \infty,$$

where $B \approx 0.2614972128476427837554$ is an absolute constant [B].

[A] In fact, we recover that

$$\lambda(n) h^{-1}(n) = \frac{(\alpha_1 + \cdots + \alpha_{\omega(n)})!}{\alpha_1! \alpha_2! \cdots \alpha_{\omega(n)}!},$$

so that since $h^{-1} = g^{-1} * 1$ by the DGF above, when $n \geqslant 1$ is squarefree, we recover another proof of property (C) stated in Conjecture 3.5.

[B] Exactly, we have that the *Mertens constant* is defined by

$$B = \gamma + \sum_{m \geqslant 2} \frac{\mu(m)}{m} \log\left[\zeta(m)\right],$$

where $\gamma \approx 0.577215664902$ is Euler's gamma constant.

Corollary 4.3 (Product form of Mertens theorem)**.** *We have that for all sufficiently large $x \gg 2$*

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) = \frac{e^{-B}}{\log x} \left(1 + o(1)\right), \text{ as } x \to \infty,$$

where the notation for the absolute constant $0 < B < 1$ coincides with the definition of Mertens constant from Theorem 4.2. Hence, for any real $z \geqslant 0$ we obtain that

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right)^z = \frac{e^{-Bz}}{(\log x)^z} \left(1 + o(1)\right)^z \sim \frac{e^{-Bz}}{(\log x)^z}, \text{ as } x \to \infty.$$

Proofs of Theorem 4.2 and Corollary 4.3 are found in [3, §22.7; §22.8].

Facts 4.4 (Exponential integrals and the incomplete gamma function)**.** Two variants of the *exponential integral function* are defined by the integral next representations [12, §8.19].

$$\begin{aligned} \operatorname{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, \\ E_1(z) &:= \int_{1}^{\infty} \frac{e^{-tz}}{t} dt, \operatorname{Re}(z) \geqslant 0 \end{aligned}$$

These functions are related by $\operatorname{Ei}(-kz) = -E_1(kz)$ for real $k, z > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $\operatorname{Ei}(\pm x)$ for all real $x > 0$:

$$\begin{aligned} \gamma + \log x - x \leqslant &\operatorname{Ei}(-x) \leqslant \gamma + \log x - x + \frac{x^2}{4}, \\ 1 + \gamma + \log x - \frac{3}{4}x \leqslant &\operatorname{Ei}(x) \quad \leqslant 1 + \gamma + \log x - \frac{3}{4}x + \frac{11}{36}x^2. \end{aligned} \tag{8a}$$

The (upper) *incomplete gamma function* is defined by [12, §8.4]

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt, \operatorname{Re}(s) > 0.$$

We have that the following properties of $\Gamma(s, x)$ hold:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, \tag{8b}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, \text{ as } x \to \infty. \tag{8c}$$

5 Components to the asymptotic analysis of lower bounds for sums of arithmetic functions weighted by $\lambda(n)$

5.1 A discussion of the results proved by Montgomery and Vaughan

Remark 5.1 (Intuition and constructions in Theorem 3.6)**.** For $|z| < 2$ and $\operatorname{Re}(s) > 1$, let

$$F(s,z) := \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \tag{9}$$

and define the DGF coefficients, $a_z(n)$ for $n \geqslant 1$, by the relation

$$\zeta(s)^z \cdot F(s,z) := \sum_{n \geqslant 1} \frac{a_z(n)}{n^s}, \operatorname{Re}(s) > 1.$$

Suppose that $A_z(x) := \sum_{n \leqslant x} a_z(n)$ for $x \geqslant 1$. Then for the choice of the function $F(s,z)$ defined in (9), we obtain the generating function like identity [A]

$$A_z(x) = \sum_{n \leqslant x} z^{\Omega(n)} = \sum_{k \geqslant 0} \widehat{\pi}_k(x) z^k.$$

Thus for $r < 2$, by Cauchy's integral formula we have

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz.$$

Selecting $r := \frac{k-1}{\log\log x}$ leads to the uniform asymptotic formulas for $\widehat{\pi}_k(x)$ given in Theorem 3.6.

We also require the next theorems reproduced from [9, §7.4] that handle the relative scarcity of the distribution of the $\Omega(n)$ for $n \leqslant x$ such that $\Omega(n) > \log\log x$.

Theorem 5.2 (Upper bounds on exceptional values of $\Omega(n)$ for large n)**.** *Let*

$$A(x,r) := \#\left\{n \leqslant x : \Omega(n) \leqslant r \cdot \log\log x\right\},$$
$$B(x,r) := \#\left\{n \leqslant x : \Omega(n) \geqslant r \cdot \log\log x\right\}.$$

If $0 < r \leqslant 1$ *and* $x \geqslant 2$*, then*

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

If $1 \leqslant r \leqslant R < 2$ *and* $x \geqslant 2$*, then*

$$B(x,r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

Theorem 5.3 (Exact bounds on exceptional values of $\Omega(n)$ for large n)**.** *We have that uniformly*

$$\#\left\{3 \leqslant n \leqslant x : \Omega(n) - \log\log n \leqslant 0\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

[A] In fact, we have more generally that

$$\prod_p \left(1 - \frac{z}{p^s}\right)^{-1} = \sum_{n \geqslant 1} \frac{z^{\Omega(n)}}{n^s}, \operatorname{Re}(s) > 1.$$

It is also not a difficult exercise to prove that for any additive arithmetic function $a(n)$, characterized by the property that $a(n) = \sum_{p^\alpha || n} a(p^\alpha)$ for all $n \geqslant 2$, we have that [5, *cf.* §1.7]

$$\sum_{n \geqslant 1} \frac{z^{a(n)}}{n^s} = \prod_p \left(1 + \sum_{m \geqslant 1} \frac{z^{a(p^m)}}{p^{ms}}\right), \operatorname{Re}(s) > 1.$$

Remark 5.4. The proofs of Theorem 5.2 and Theorem 5.3 are found in Chapter 7 of Montgomery and Vaughan. The key interpretation we need to take away from these statements is the result proved as the next corollary. The precise way in which the bound stated in the previous theorem depends on the indeterminate paramater R can be reviewed for reference in the proof algebra and relations cited in the reference [9, §7]. The role of the parameter R involved in stating the previous theorem is a critical bound as the scalar factor in the upper bound on $k \leqslant R \log\log x$ in Theorem 3.6. This parameter determines the precise range of k up to which we obtain the valid uniform bounds in x on the exact asymptotic formulas for $\widehat{\pi}_k(x)$.

We have a discrepancy to work out in so much as we can only form summatory functions over the $\widehat{\pi}_k(x)$ for $1 \leqslant k \leqslant R \log\log x$ using the precisely formulated asymptotic formulas guaranteed by Theorem 3.6. In contrast, for $n \geqslant 2$ we can actually have contributions from values distributed throughout the range $1 \leqslant \Omega(n) \leqslant \log_2(n)$ infinitely often. It is then crucial that we can show that the dominant growth of the asymptotic formulas we obtain for these summatory functions is captured by summing only over k in the truncated range where the uniform bounds hold.

Corollary 5.5. *Using the notation for $A(x,r)$ and $B(x,r)$ from Theorem 5.2, we have that for $\delta > 0$,*

$$o(1) \leqslant \left| \frac{B(x,1+\delta)}{A(x,1)} \right| \ll 2, \text{ as } \delta \to 0^+, x \to \infty.$$

Proof. The lower bound stated above should be clear. To show that the asymptotic upper bound is correct, we compute using Theorem 5.2 and Theorem 5.3 that

$$\left| \frac{B(x,1+\delta)}{A(x,1)} \right| \ll \left| \frac{x \cdot (\log x)^{\delta - \delta \log(1+\delta)}}{O(1) + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right)} \right| \sim \left| \frac{(\log x)^{\delta - \delta\log(1+\delta)}}{\frac{1}{2} + o(1)} \right| \xrightarrow{\delta \to 0^+} 2,$$

as $x \to \infty$. Notice that since $\mathbb{E}[\Omega(n)] = \log\log n + B$, with $0 < B < 1$ the absolute constant from Mertens theorem, when we denote the range of $k > \log\log x$ as holding in the form of $k > (1+\delta)\log\log x$ for $\delta > 0$ at large x, we can assume that $\delta \to 0^+$ as $x \to \infty$. This provides a limiting constant-valued upper bound on the ratios defined above. □

5.2 New results based on refinements of Theorem 3.6

What the enumeratively flavored result in Theorem 3.6 allows us to do is get a sufficient lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. We seek to approximate $\mathcal{G}(z)$ defined in this theorem by only taking finite products of the primes in the factor $\prod_p (1 - z/p)^{-1}$ defining this function for $p \leqslant x$, e.g., indexing the component products only over those primes $p \in \{2, 3, 5, \ldots, x\}$ as $x \to \infty$. We can extend the argument behind the constructions sketched in Remark 5.1 to justify that it suffices to consider only the contributions from these finite products to obtain a lower bound on $\widehat{\pi}_k(x)$.

Proposition 5.6. *For real $s \geqslant 1$, let*

$$P_s(x) := \sum_{p \leqslant x} p^{-s}, x \geqslant 2.$$

When $s := 1$, we have the asymptotic formula from Mertens theorem (see Theorem 4.2). For all integers $s \geqslant 2$ there is an absolutely defined bounding function $\gamma_0(s,x)$ such that

$$\gamma_0(s,x) + o(1) \leqslant P_s(x), \text{ as } x \to \infty.$$

It suffices to take the bound in the previous equation as as the quasi-polynomial function of s, x given by

$$\gamma_0(s,x) = s \log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4} s(s-1)^2 \log^2(2).$$

Proof. Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$, and where our target function smooth function is $f(t) = t^{-s}$ so that $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= \mathrm{Ei}(-(s-1)\log x) - \mathrm{Ei}(-(s-1)\log 2) + o(1), \text{ as } x \to \infty. \end{aligned}$$

Now using the inequalities in Facts 4.4, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned} \frac{P_s(x)}{s} &\geqslant \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}(s-1)^2 \log^2(2) \\ \frac{P_s(x)}{s} &\leqslant \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) + \frac{1}{4}(s-1)^2 \log^2(x). \end{aligned}$$

This completes the proof of the bound cited above in this lemma. □

Proof of Theorem 3.7. Notice that for real $0 \leqslant z < 2$ and any prime $p \geqslant 2$, we have that $(1 - z/p)^{-1} \geqslant 1$ with equality if and only if $z := 0$. For $0 \leqslant z < 2$ and integers $x \geqslant 2$, when we define the function

$$\widehat{P}(z, x) := \prod_{p \leqslant x} \left(1 - \frac{z}{p}\right)^{-1},$$

the right-hand-side product is finite as $x \to \infty$. Moreover, for $x \geqslant 2$ the product function $\widehat{P}(z, x)$ is a non-decreasing function of x when $0 \leqslant z < 2$. Moreover, for fixed, finite $x \geqslant 2$ let

$$\mathbb{P}_x := \{n \geqslant 1 : \text{all prime factors } p|n \text{ satisfy } p \leqslant x\}.$$

Then we can see as in the constructions from Montgomery and Vaughan sketeched in Remark 5.1 that

$$\prod_{p \leqslant x} \left(1 - \frac{z}{p^s}\right)^{-1} = \sum_{n \in \mathbb{P}_x} \frac{z^{\Omega(n)}}{n^s}, x \geqslant 2. \tag{10}$$

By extending the argument we employed in the remark summarizing the proof given in [9, §7.4], we have that the formulas for

$$A_z(x) := \sum_{n \leqslant x} z^{\Omega(n)} = \sum_{k \geqslant 0} \widehat{\pi}_k(x) z^k,$$

that depend on approximations (or inputs) to $\mathcal{G}(z)$ still contain all of the relevant terms, or powers of z, after taking the finite products in (10). In particular, this happens since the products of all non-negative integral powers of the primes $p \leqslant x$ generate the integers $\{1 \leqslant n \leqslant x\}$ as a subset. Thus, if we form an approximation to the function $\mathcal{G}(z)$ in Theorem 3.6 by truncating the product factor as $\widehat{P}(z, x)$ defined above by indexing only for primes $0 \leqslant p \leqslant x$, then what we obtain in the formula for $\widehat{\pi}_k(x)$ guaranteed by the theorem is indeed a *lower bound* for the summatory function.

We have for all integers $0 \leqslant m < +\infty$, and any sequence $\{f(n)\}_{n \geqslant 1}$ with bounded partial sums, that [8, §2]

$$[z^m] \prod_{i \geqslant 1} (1 - f(i)z)^{-1} = [z^m] \exp\left(\sum_{j \geqslant 1} \left(\sum_{i=1}^m f(i)^j\right) \frac{z^j}{j}\right), |z| < 1. \tag{11}$$

In our case we have that $f(i)$ denotes the i^{th} prime in the generating function expansion of (11). We must now find effective bounds on the truncated products in (10) that are both meaningful and still simple enough to use in our new formulas [B].

[B] Exactly, what we should obtain here is the formula

$$\prod_p \left(1 - \frac{z}{p^s}\right)^{-1} = \exp\left(\sum_{j \geqslant 1} P(js) \frac{z^j}{j}\right) = \exp\left(\sum_{j \geqslant 1} \sum_{k \geqslant 2} \frac{\mu(k)}{k} \log \zeta(jks) \times \frac{z^j}{j}\right) = \prod_{n \geqslant 2} \zeta(n)^{\frac{1}{n} \sum_{d|n} \mu(d) z^{\frac{n}{d}}}.$$

It follows from Proposition 5.6 that for $0 \leqslant z < 1$ we obtain

$$\begin{aligned}\log\left[\prod_{p \leqslant x}\left(1-\frac{z}{p}\right)^{-1}\right] &\geqslant (B+\log\log x)z + \sum_{j\geqslant 2}\left[a(x)-b(x)(j-1)-c(x)(j-1)^2\right]z^j \\ &= (B+\log\log x)z - a(x)\left(1-\frac{1}{1-z}+z\right) \\ &\qquad + b(x)\left(1-\frac{2}{1-z}+\frac{1}{(1-z)^2}\right) \\ &\qquad + c(x)\left(1-\frac{4}{1-z}+\frac{5}{(1-z)^2}-\frac{2}{(1-z)^3}\right) \\ &=: \widehat{\mathcal{B}}(x;z). \end{aligned} \tag{12}$$

In the previous equations, the lower bounds formed by the functions $(a,b,c) \equiv (a_\ell, b_\ell, c_\ell)$ evaluated at x are given by the corresponding lower bounds from Proposition 5.6 as

$$(a_\ell, b_\ell, c_\ell) := \left(\log\left(\frac{\log x}{\log 2}\right), \log\left(\frac{x}{2}\right), \frac{1}{4}\log^2 2\right).$$

We have the uniform bound parameter $1 < R < 2$ so that

$$z \equiv z(k,x) = \frac{k-1}{\log\log x} \in [0, R),$$

in the notation of Theorem 3.6 for all $x \gg e+1$ as $x \to \infty$. To perform analysis on the extremal values of the contributions from each weight term (in z) on the functions (a_ℓ, b_ℓ, c_ℓ) given above, we have to rule out one slice of exceptional cases that can make the bound infinite at large x. Namely, we observe that $z \equiv z(k,x)$ satisfies $z = 1$ precisely if $x = \exp\left(e^{k-1}\right)$ for x, k both integers. This case happens almost nowhere for the positive integers $x \in \mathbb{Z}^+ \cap \{n > e : n \in \mathbb{Z}\}$. So for almost every sufficiently large $x \in \mathbb{Z}^+$, we have by elementary calculus that the extremal bounds of the coefficient weight functions in z happen either for $z := 0$ or as $z \to 2^-$.

This observation implies that

$$c_\ell(x)\left(1+\frac{4}{z-1}+\frac{5}{(z-1)^2}+\frac{2}{(z-1)^3}\right) \geqslant 0.$$

The function defined by

$$f(z) := 1 - \frac{2}{1-z} + \frac{1}{(1-z)^2}, z \in [0,2),$$

is also almost always bounded by $f(z) \in [0,4)$. In the former case, we obtain that the limiting exponent is zero in order to obtain a lower bound on (12). We can similarly conclude that almost everywhere

$$1 + z - \frac{1}{1-z} \geqslant z, z \in [z, z+2).$$

Then we obtain the next effective lower bounds on the function from (12) for almost every sufficiently large $x \in \mathbb{Z}^+$:

$$\frac{e^{-Bz}}{(\log x)^z} \times \exp\left(\widehat{\mathcal{B}}(u,x;z)\right) \gg \left(\frac{\log 2}{\log x}\right)^{z+2} =: \widehat{\mathcal{C}}(u,x;z) \tag{13}$$

Essentially, what we are arguing in excluding the case where $z = 1$ is that the possibility the function becomes infinite at x when $z \equiv z(k,x)$ depends on this x is negligible. In fact, we choose to exclude this possibility by just removing any possible cases where this can happen from our consideration in our later sums involving the new lower bounds on $\widehat{\pi}_k(x)$.

Finally, to finish our proof of the new form of the lower bound on $\mathcal{G}(z)$ in Theorem 3.6, we need to bound the reciprocal factor of $\Gamma(z+1)$. Since $z \equiv z(k,x) = \frac{k-1}{\log\log x}$ and $k \in [1, R\log\log x]$, or with $z \in [0, R)$, we obtain for minimal k and all large enough $x \gg 1$ that $\Gamma(z+1) \approx 1$, and for k towards the upper range of its interval that $\Gamma(z+1) \leqslant \Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$. □

6 Average case analysis of bounds on the Dirichlet inverse functions, $g^{-1}(n)$

The property in (C) of Conjecture 3.5 along squarefree $n \geqslant 1$ captures an important characteristic of $g^{-1}(n)$ that holds more globally for all $n \geqslant 1$. In particular, these functions can be expressed via more simple formulas than inspection of the first few initial values of the repetitive, quasi-periodic sequence otherwise suggests. The pages of tabular data given as Table T.1 in the appendix section (refer to page 39) are intended to provide clear insight into why we arrived at the convenient approximations to $g^{-1}(n)$ proved in this section. The table offers illustrative numerical data formed by examining the approximate behavior at hand for the cases of $1 \leqslant n \leqslant 500$ with *Mathematica*.

6.1 Definitions and basic properties of component function sequences

We define the following sequence for integers $n \geqslant 1, k \geqslant 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geqslant 1. \end{cases} \tag{14}$$

The sequence of important semi-diagonals of these functions begins as [15, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geqslant 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \ -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

Remark 6.1 (An effective range of k depending on a fixed large n)**.** Notice that by expanding the recursively-based definition in (14) out to its maximal depth by nested divisor sums, for fixed n, $C_k(n)$ is seen to only ever possibly be non-zero for $k \leqslant \Omega(n)$. Thus, the effective range of k for fixed n is restricted by the conditions of $C_0(n) = \delta_{n,1}$ and that $C_k(n) = 0$, $\forall k > \Omega(n)$ whenever $n \geqslant 2$.

Example 6.2 (Special cases of the functions $C_k(n)$ for small k)**.** We cite the following special cases which are verified by explicit computation using (14) [15, A066922] [A]:

$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= d(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} - \gcd\left(\Omega(n), \omega(n)\right). \end{aligned}$$

6.2 Uniform asymptotics of $C_k(n)$ for large all n and fixed, bounded k

The next theorem makes precise what these formulas already suggest about the main terms of the growth rates of $C_k(n)$ as functions of k, n for limiting cases of n large and fixed k which is necessarily bounded in n, but still taken as an independent parameter.

Theorem 6.3 (Asymptotics for the functions $C_k(n)$)**.** *For $k := 0$, we have by definition that $C_0(n) = \delta_{n,1}$. For all sufficiently large $n > 1$ and any fixed $1 \leqslant k \leqslant \Omega(n)$ taken independently of n, we obtain that the dominant asymptotic term for $C_k(n)$ is given uniformly by* ***(TODO)***

$$\mathbb{E}[C_k(n)] \gg (\log\log n)^{2k-1}, \text{ as } n \to \infty.$$

[A] For all $n, k \geqslant 2$, we have the following recurrence relation satisfied by $C_k(n)$ between successive values of k:

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=0}^{\nu_p(n)-1} C_{k-1}\left(dp^i\right), n \geqslant 1.$$

Proof. We prove our bounds by induction on k. We can see by Example 6.2 that $C_1(n)$ satsfies the formula we must establish when $k := 1$ since $\mathbb{E}[\omega(n)] = \log\log n$. Suppose that $k \geqslant 2$ and let the inductive assumption state that for all $1 \leqslant m < k$

$$\mathbb{E}[C_m(n)] \gg (\log\log n)^{2m-1}.$$

Now using the recursive formula we used to define the sequences of $C_k(n)$ in (14), we have that as $n \to \infty$ [B]

$$\begin{aligned}
\mathbb{E}[C_k(n)] &= \mathbb{E}\left[\sum_{d|n} \omega(n/d) C_{k-1}(d)\right] \\
&= \frac{1}{n} \times \sum_{d \leqslant n} C_{k-1}(d) \times \sum_{r=1}^{\left\lfloor \frac{n}{d} \right\rfloor} \omega(r) \\
&\sim \sum_{d \leqslant n} C_{k-1}(d) \left[\frac{\log\log(n/d) \left[d \leqslant \frac{n}{e}\right]_\delta}{d} + \frac{B}{d}\right] \\
&\sim \sum_{d \leqslant \frac{n}{e}} \left[\sum_{m<d} \frac{\mathbb{E}[C_{k-1}(m)]}{m} \log\log\left(\frac{n}{m}\right) + B \cdot \mathbb{E}[C_{k-1}(d)] + B \cdot \sum_{m<d} \frac{\mathbb{E}[C_{k-1}(m)]}{m}\right] \\
&\gg B\left[n \log n \cdot (\log\log n)^{2k-3} - \log n \cdot (\log\log n)^{2k-3}\right] \times \left(1 + \frac{\log n}{2}\right) \\
&\gg (\log\log n)^{2k-1}.
\end{aligned}$$

In transitioning to the last equation from the previous step, we have used that $\frac{Bn}{2} \cdot (\log n)^2 \gg (\log\log n)^2$ as $n \to \infty$. We have also used that for large $n \to \infty$ and fixed m, we have by an approximation to the incomplete gamma function given by

$$\int_e^n \frac{(\log\log t)^m}{t} \sim (\log n)(\log\log n)^m, \text{ as } n \to \infty.$$

Thus the claim holds by mathematical induction whenever $n \to \infty$ is large and $1 \leqslant k \leqslant \Omega(n)$. ☐

6.3 Relating the auxiliary functions $C_k(n)$ to formulas approximating $g^{-1}(n)$

Lemma 6.4 (An exact formula for $g^{-1}(n)$)**.** *For all $n \geqslant 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d) + 1) g^{-1}(n/d) \quad \implies \quad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n).$$

Now by repeatedly expanding the right-hand-side, and removing corner cases in the nested sums with $\omega(1) = 0$, we find inductively that

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n).$$

The statement then follows by Möbius inversion applied to each side of the last equation. ☐

[B] For all large $x \gg 2$ the summatory function of $\omega(n)$ satisfies [3, §22.10]

$$\sum_{n \leqslant x} \omega(n) = x \log\log x + Bx + O\left(\frac{x}{\log x}\right).$$

More to this point, since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ in the notation of the proof Proposition 4.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for squarefree $n \geqslant 1$. A proof of part (C) of Conjecture 3.5 follows as an immediate consequence.

Corollary 6.5. *For all squarefree integers $n \geqslant 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \tag{15}$$

Proof. Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geqslant 2$ and that n is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where p_i is prime for all $1 \leqslant i \leqslant \omega(n)$. So we can transform the exact divisor sum guaranteed for all n in Lemma 6.4 as follows:

$$\begin{aligned} g^{-1}(n) &= \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i}(-1)^i \cdot C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \end{aligned}$$

The signed contributions in the first of the previous equations is justified by noting that $\lambda(n) = (-1)^{\omega(n)}$ whenever n is squarefree, and that for d squarefree with $\omega(d) = i$, $\Omega(d) = i$. ☐

Corollary 6.6. *We have that*

$$\frac{6}{\pi^2}(\log n)(\log\log n) \ll \mathbb{E}|g^{-1}(n)| \leqslant \mathbb{E}\left[\sum_{d|n} C_{\Omega(d)}(d)\right].$$

Proof. To prove the lower bound, first notice that by Lemma 6.4, Proposition 4.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain that

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{16}$$

In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $n \geqslant 1$ we have $\mu(n) = (-1)^{\omega(n)} = \lambda(n)$, Lemma 6.4 implies

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \end{aligned}$$

Notice in the above equation that $\lambda(n^2) = +1$ for all $n \geqslant 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is necessarily even.

Recall from the introduction that the summatory function of the squarefree integers is given by

$$Q(x) := \sum_{n \leqslant x} \mu^2(n) = \frac{6}{\pi^2}x + O(\sqrt{x}).$$

Then since $C_{\Omega(d)}(d) \geqslant 1$ for all $d \geqslant 1$, and since $\mathbb{E}[C_k(d)]$ is minimized when $k := 1$, we obtain by summing over (16) that as $x \to \infty$

$$\begin{aligned}
\frac{1}{x} \times \sum_{n \leqslant x} |g^{-1}(n)| &= \frac{1}{x} \times \sum_{d \leqslant x} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&\sim \sum_{d \leqslant x} C_{\Omega(d)}(d) \left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dx}}\right)\right] \\
&\geqslant \sum_{d \leqslant x} \left[\frac{6 \cdot C_{\Omega(d)}(d)}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dx}}\right)\right] \\
&= \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(x)}(x)] + \sum_{d<x} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O\left(\frac{1}{\sqrt{x}} \times \int_0^x t^{-1/2} dt\right) \\
&\gg \frac{6}{\pi^2}\left[\sum_{e \leqslant d \leqslant x} \frac{\log\log d}{d}\right] + O(1) \\
&\sim \frac{6}{\pi^2} \times \int_e^x \frac{\log\log t}{t} dt + O(1) \\
&\gg \frac{6}{\pi^2}(\log x)(\log\log x).
\end{aligned}$$

To prove the upper bound, notice that by Lemma 6.4 and Corollary 6.5,

$$|g^{-1}(n)| \leqslant \sum_{d|n} C_{\Omega(d)}(d).$$

Now since both of the above quantities are positive for all $n \geqslant 1$, we must obtain the upper bound on the average order of $|g^{-1}(n)|$ stated above. □

6.3.1 The relation to the distribution of the primes moving forward

Remark 6.7 (Combinatorial relations between the inverse functions and prime factorizations of n)**.** The combinatorial complexity of relating $g^{-1}(n)$ to the distribution of the primes motivates us to consider the properties of this sequences beyond that which the rudimentary bounds we require so far have revealed. While the magnitudes and dispersion of the primes $p \leqslant x$ certainly restricts the repeating of these values we can see in the contributions to $G^{-1}(x)$, the following statement is clear about the relation of the weights $|g^{-1}(n)|$ to the prime numbers: The value of $|g^{-1}(n)|$ is entirely dependent on the pattern of the *exponents* (viewed as multisets) of the distinct prime factors of $n \geqslant 2$. In short, the primes involved in invoking this property of semi-regularity in the distribution of $g^{-1}(n)$ are relevant only as placeholders to the action of the additive functions that operate on their exponents.

Observe that we also have a natural extremal behavior with respect to $\Omega(n)$ corresponding to squarefree integers, and prime powers. Namely, if for $k \geqslant 1$ we set the values of M_k and m_k to be the minimal positive integers

$$M_k := \left\{n \geqslant 2 : |g^{-1}(n)| = \sup_{\substack{j \geqslant 2 \\ \Omega(j)=k}} |g^{-1}(j)|\right\},$$

$$m_k := \left\{n \geqslant 2 : |g^{-1}(n)| = \inf_{\substack{j \geqslant 2 \\ \Omega(j)=k}} |g^{-1}(j)|\right\},$$

then any element of M_k is squarefree and any element of m_k is a prime power. In particular, we have that for any $N_k \in M_k$ and $n_k \in m_k$

$$N_k = \sum_{j=0}^{k} \binom{k}{j} j!, \qquad n_k = 2 \cdot (-1)^k.$$

Moreover, using the definition of the function $h^{-1}(n) = (g^{-1} * 1)(n)$ as in the proof of Proposition 4.1, we can express an exact formula for $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n. Namely, for $n \geqslant 2$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z \cdot \nu_p(n)) = [z^k] \prod_{p^\alpha || n} (1 + \alpha z).$$

Then we have essentially shown using (16) that we can write

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, n \geqslant 2.$$

The combinatorial formula for $h^{-1}(n) = \lambda(n) \cdot (\Omega(n))! \times \prod_{p^\alpha || n} (\alpha!)^{-1}$ we derived in the proof of the key signedness proposition in Section 4 then suggests further patterns and more regularity in the contributions of the distinct weighted terms for $G^{-1}(x)$ when we sum over all of the possible prime exponent patterns that factorize $n \leqslant x$. The relation of the repitition of the distinct values of $|g^{-1}(n)|$ as weights to the signed summatory function $G^{-1}(x)$ makes a clear tie to $M(x)$ through Proposition 7.1 proved in the next section.

7 Lower bounds for $M(x)$ along infinite subsequences

7.1 The culmination of what we have done so far

Proposition 7.1. *For all sufficiently large x, we have that*

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt. \tag{17}$$

Proof. We know by applying Corollary 3.3 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k)+1) \\ &\approx G^{-1}(x) + \sum_{k=1}^{x} g^{-1}(k)\pi(x/k), \end{aligned} \tag{18}$$

We can replace the asymptotically unnecessary floored integer-valued arguments to $\pi(x)$ in (18) using its approximation by the monotone non-decreasing asymptotic order, $\pi(x) \sim \frac{x}{\log x}$. Moreover, we can always bound

$$\frac{Ax}{\log x} \leqslant \pi(x) \leqslant \frac{Bx}{\log x},$$

for suitably defined absolute constants, $A, B > 0$ whenever $x \geqslant 2$. Therefore the approximation obtained by replacing $\pi(x)$ by the dominant term in its limting asymptotic formula is actually valid for all $x > 1$ up to at most a small constant difference.

What we require to sum and simplify the right-hand-side terms from (18) essentially follows from summation by parts [A]. In particular, we argue that for sufficiently large $x \geqslant 2$ we can approximate [B]

$$\begin{aligned} \sum_{k=1}^{x} g^{-1}(k)\pi(x/k) &= G^{-1}(x)\pi(1) - \sum_{k=1}^{x-1} G^{-1}(k)\left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)\right] \\ &= -\sum_{k=1}^{x/2} G^{-1}(k)\left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)\right] \\ &\approx -\sum_{k=1}^{x/2} G^{-1}(k)\left[\frac{x}{k \cdot \log(x/k)} - \frac{x}{(k+1)\cdot\log(x/k)}\right] \qquad (19a) \\ &\approx -\sum_{k=1}^{x/2} G^{-1}(k)\frac{x}{k^2 \cdot \log(x/k)}. \qquad (19b) \end{aligned}$$

Indeed, we can justify that step (19a) is correct by writing

$$\begin{aligned} \frac{x}{(k+1)\log\left(\frac{x}{k+1}\right)} &= \frac{x}{k+1} \cdot \frac{1}{\left[\log\left(\frac{x}{k}\right) + \log\left(1 - \frac{1}{k+1}\right)\right]} = \frac{x}{(k+1)\log\left(\frac{x}{k}\right)} \cdot \frac{1}{1 + \frac{\log\left(1-\frac{1}{k+1}\right)}{\log x\left[1 - \frac{\log k}{\log x}\right]}} \\ &\sim \frac{x}{(k+1)\log\left(\frac{x}{k}\right)}, \text{ as } x \to \infty. \end{aligned}$$

[A] For any arithmetic functions, u_n, v_n, with $U_j := u_1 + u_2 + \cdots + u_j$ for $j \geqslant 1$, we have that [12, §2.10(ii)]

$$\sum_{j=1}^{n-1} u_j \cdot v_j = U_{n-1}v_n + \sum_{j=1}^{n-1} U_j\left(v_j - v_{j+1}\right), n \geqslant 2.$$

[B] Since $\pi(1) = 0$, the actual range of summation corresponds to $k \in \left[1, \frac{x}{2}\right]$.

Moreover, the correctness of the transition from step (19a) to (19b) is verified by seeing that for $\operatorname{Re}(s) > 1$, we have that

$$\infty > \left|\frac{1}{s \cdot (P(s)+1)\zeta(s)}\right| = \left|\int_1^\infty \frac{G^{-1}(x)}{x^{s+1}} dx\right| = \left|\sum_{k \geqslant 1} \frac{G^{-1}(k)}{k^{s+1}}\right|.$$

In particular, when $s := \frac{3}{2}$, we obtain that

$$0 \leqslant \left|\sum_{k \geqslant 1} \frac{G^{-1}(k)}{k^2(k+1)}\right| \leqslant \left|\sum_{k \geqslant 1} \frac{G^{-1}(k)}{k^{\frac{5}{2}}}\right| < \infty,$$

so that we have the difference of the terms is bounded above and below by absolute constants as

$$\left|\sum_{k=1}^{\frac{x}{2}} G^{-1}(k) \left[\frac{1}{k^2} - \frac{1}{k(k+1)}\right]\right| \leqslant \left|\sum_{k=1}^{\frac{x}{2}} \frac{G^{-1}(k)}{k^2(k+1)}\right| = O(1).$$

Now since for x large enough the summand factor $\frac{x}{k^2 \cdot \log(x/k)}$ is monotonic as k ranges over $k \in [1, x/2]$ in ascending order, since this summand factor is a smooth function of k (and x), and since $G^{-1}(x)$ is a summatory function with jumps only at the positive integers, we can approximate $M(x)$ for any finite $x \geqslant 2$ by

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt.$$

We will later only use unsigned lower bound approximations to this function in the next theorems so that the signedness of the summatory function term in the integral formula above as $x \to \infty$ is a moot point entirely. $\square$

7.2 Establishing initial lower bounds on the summatory functions $G^{-1}(x)$

Let the summatory function $G_E^{-1}(x)$ be defined for $x \geqslant 1$ by [C]

$$G_E^{-1}(x) := \sum_{n \leqslant (\log x)^{\frac{3}{2}} (\log\log x)} \lambda(n) \times \sum_{\substack{d \mid n \\ d > e}} \frac{(\log d)^{\frac{1}{4}}}{\log\log d}. \tag{20}$$

Theorem 7.2. *For almost all sufficiently large integers $x \to \infty$, we have that*

$$|G^{-1}(x)| \gg |G_E^{-1}(x)|.$$

Proof. We bound the magnitude of each respective function respectively below and above in the worst cases of the cancellation imparted by the signage on the otherwise positive terms when weighted by $\lambda(n)$. First, consider the following upper bound on $|G_E^{-1}(x)|$:

$$\begin{aligned}
|G_E^{-1}(x)| &= \left|\sum_{e \leqslant n \leqslant (\log x)^{\frac{3}{2}} (\log\log x)} \lambda(n) \times \sum_{\substack{d \mid n \\ d > e}} \frac{(\log d)^{\frac{1}{4}}}{\log\log d}\right| \\
&\ll \sum_{e < d \leqslant (\log x)^{\frac{3}{2}} (\log\log x)} \frac{(\log d)^{\frac{1}{4}}}{\log\log d} \cdot \left\lfloor \frac{(\log x)^{\frac{3}{2}} (\log\log x)}{d} \right\rfloor \\
&\approx (\log x)^{\frac{3}{2}} (\log\log x) \times \int_e^{(\log x)^{\frac{3}{2}} (\log\log x)} \frac{(\log t)^{\frac{1}{4}}}{t \cdot \log\log t} dt
\end{aligned}$$

[C] The subscript of E (as in to be evaluated in expectation) on the function $G_E^{-1}(x)$ is purely for formality of notation and does not correspond to an actual parameter or any implicit dependence on E in the formula that defines this function.

$$= (\log x)^{\frac{3}{2}}(\log\log x) \times \operatorname{Ei}\left(\frac{5}{4}\log\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right)\right)$$
$$\ll \frac{25}{64}\cdot(\log x)^{\frac{3}{2}}(\log\log x)(\log\log\log x)^2.$$

Next, we bound the summatory function $|G^{-1}(x)|$ from below. Notice that in applying the lower bound for $\mathbb{E}|g^{-1}(n)|$ from Corollary 6.6, we obtain that for large $n\to\infty$ and any fixed positive constant $C>0$, we obtain that

$$\mathbb{E}|g^{-1}(Cn)| \gg \frac{6}{\pi^2}\left[(\log n)(\log\log n)+\log(C)(\log\log n)\right]. \tag{21}$$

We define the following densities for large $x\geqslant 2$:

$$\mathcal{L}_+(x) := \frac{1}{n}\cdot\#\{n\leqslant x:\lambda(n)=+1\}$$
$$\mathcal{L}_-(x) := \frac{1}{n}\cdot\#\{n\leqslant x:\lambda(n)=-1\}.$$

We know that [17, *cf.* §1]

$$\lim_{x\to\infty}\mathcal{L}_+(x) = \lim_{x\to\infty}\mathcal{L}_-(x) = \frac{1}{2},$$

so that as $x\to\infty$, the two densities $\mathcal{L}_+(x),\mathcal{L}_-(x)$ may fluctuate, but cannot grow too far apart over extended intervals.

Then we compute that for almost every sufficiently large $n\to\infty$:

$$\frac{|G^{-1}(x)|}{x} = \frac{1}{n}\times\left|\sum_{\substack{d\leqslant x\\ \lambda(d)=+1}}|g^{-1}(d)| - \sum_{\substack{d\leqslant x\\ \lambda(d)=-1}}|g^{-1}(d)|\right|$$
$$\gg \left|\mathbb{E}|g^{-1}\left(\mathcal{L}_+(x)x\right)| - \mathbb{E}|g^{-1}\left((1-\mathcal{L}_+(x))x\right)|\right|.$$

So by applying the lower bound on the average order expectations from Corollary 6.6 along with (21), we obtain for almost every large enough x (i.e., with the exception of x on a set of asymptotic density zero) that

$$\begin{aligned}|G^{-1}(x)| \gg \frac{6x}{\pi^2}\Bigg|&\mathcal{L}_+(x)(\log x)(\log\log x)+(1-\mathcal{L}_+(x))(\log x)(\log\log x)\\ &+\left(\mathcal{L}_+(x)\log\left[\mathcal{L}_+(x)\right]+(1-\mathcal{L}_+(x))\log\left[1-\mathcal{L}_+(x)\right]\right)(\log\log n)\Bigg|\\ = \frac{6x}{\pi^2}\Bigg|&(\log x)(\log\log x)+\left(\log\left[1-\mathcal{L}_+(x)\right]+\mathcal{L}_+(x)\cdot\log\left[\frac{\mathcal{L}_+(x)}{1-\mathcal{L}_+(x)}\right]\right)(\log\log x)\Bigg|, \text{ as } x\to\infty.\end{aligned} \tag{22}$$

Note for verification, that the largest $n\leqslant 500$ such that we have $G^{-1}(n)=0$ in Table T.1 (see page 45) is given by $n:=426$. In this case, we have that $\mathcal{L}_+(n)\approx 0.485915$, $\mathcal{L}_-(n)\approx 0.514845$, and that our lower bound on the right-hand-side of (22) gives approximately that $|G^{-1}(n)|\gtrsim 5.86971$, where it happens that $G^{-1}(427)=5$.

Finally, since for all sufficiently large $x\to\infty$, we have that

$$\frac{6x}{\pi^2}(\log x)(\log\log x) \gg \frac{25}{64}\cdot(\log x)^{\frac{3}{2}}(\log\log x)(\log\log\log x)^2,$$

we have that our claimed relation between the two key summatory functions holds. □

Note that the only cases we need to be wary of in the *almost everywhere* clause to applying the statement of Theorem 7.2 happen when $G^{-1}(x)=0$. This singularity in the distribution of the signed summatory function occurs at most as frequently as on an asymptotically thin subset of the positive integers. It suffices to assume that $G^{-1}(x)\neq 0$ on a dense subset of the integers for the bounds we require to prove Corllary 3.8 in the last subsection.

7.2.1 A few more necessary results

We now use the superscript and subscript notation of (ℓ) not to denote a formal parameter to the functions we define below, but instead to denote that these functions form *lower bound* (rather than exact) approximations to other forms of the functions without the scripted (ℓ).

Lemma 7.3. *Suppose that $\widehat{\pi}_k(x) \geqslant \widehat{\pi}_k^{(\ell)}(x) \geqslant 0$ for $\widehat{\pi}_k^{(\ell)}(x)$ a monotone real-valued function of x for all integers $k \geqslant 1$ whenever $x \geqslant 2$ is sufficiently large. Let*

$$\begin{aligned} A_{\Omega}^{(\ell)}(x) &:= \sum_{k \leqslant \log\log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x) \\ A_{\Omega}(x) &:= \sum_{k \leqslant \log\log x} (-1)^k \widehat{\pi}_k(x). \end{aligned}$$

Then for all sufficiently large x, we have that

$$|A_{\Omega}(x)| \gg |A_{\Omega}^{(\ell)}(x)|.$$

Proof. Given an explicit smooth lower bounding function, $\widehat{\pi}_k^{(\ell)}(x)$, we define the similarly smooth and monotone residual terms in approximating $\widehat{\pi}_k(x)$ using the following notation:

$$\widehat{\pi}_k(x) = \widehat{\pi}_k^{(\ell)}(x) + \widehat{E}_k(x).$$

Then we can form the ordinary exact form of the summatory function as

$$\begin{aligned} |A_{\Omega}(x)| &\gg \left| \sum_{k \leqslant \frac{\log\log x}{2}} \left[\widehat{\pi}_{2k}(x) - \widehat{\pi}_{2k-1}(x)\right] \right| \\ &\geqslant \left| A_{\Omega}^{(\ell)}(x) - \sum_{k \leqslant \frac{\log\log x}{2}} \left[\widehat{E}_{2k}(x) - \widehat{E}_{2k-1}(x)\right] \right| \\ &\geqslant \left| A_{\Omega}^{(\ell)}(x) \right| - \left| \sum_{k \leqslant \frac{\log\log x}{2}} \left[\widehat{E}_{2k}(x) - \widehat{E}_{2k-1}(x)\right] \right|. \end{aligned}$$

If the latter sum, denoted

$$\mathrm{ES}(x) := \left| \sum_{k \leqslant \frac{\log\log x}{2}} \left[\widehat{E}_{2k}(x) - \widehat{E}_{2k-1}(x)\right] \right| \rightarrow \infty,$$

as $x \rightarrow \infty$, then we can always find some absolute $C_0 > 0$ (by monotonicity) such that $\mathrm{ES}(x) \leqslant C_0 \cdot A_{\Omega}(x)$:

$$\mathrm{ES}(x) = \left| A_{\Omega}(x) - A_{\Omega}^{(\ell)}(x) \right| \leqslant \left| |A_{\Omega}(x)| + \left| A_{\Omega}^{(\ell)}(x) \right| \right| \ll 2\,|A_{\Omega}(x)|\,.$$

If on the other hand this sum becomes constant, or is bounded as $x \rightarrow +\infty$, then we also clearly have another absolute $C_1 > 0$ such that $|A_{\Omega}(x)| \geqslant C_1 \cdot |A_{\Omega}^{(\ell)}(x)|$. In either case, the claimed result holds for all large enough x. ∎

Lemma 7.4. *Suppose that $f(n)$ is an arithmetic functions such that $f(n) > 0$ for all $n > u_0$ where $f(n) \gg \widehat{\tau}_{\ell}(n)$ as $n \rightarrow \infty$. Assume that the bounding function $\widehat{\tau}_{\ell}(t)$ is a non-negative continuously differentiable function of t for all large enough $t \gg u_0$. We define the λ-sign-scaled summatory function of f as follows:*

$$F_{\lambda}(x) := \sum_{u_0 < n \leqslant x} \lambda(n) \cdot f(n).$$

Let

$$\begin{aligned} A_{\Omega}^{(\ell)}(t) &:= \sum_{k=1}^{[\log\log t]} (-1)^k \widehat{\pi}_k^{(\ell)}(t), \\ A_{\Omega}(t) &:= \sum_{k=1}^{[\log\log t]} (-1)^k \widehat{\pi}_k(t), \end{aligned}$$

where $\widehat{\pi}_k(x) \geqslant \widehat{\pi}_k^{(\ell)}(x) \geqslant 0$ *for* $\widehat{\pi}_k^{(\ell)}(t)$ *some smooth monotone function of* t *at all sufficiently large* $t \to \infty$. *Then we have that*

$$|F_\lambda(x)| \gg \left| \left| A_{\Omega}^{(\ell)}(x)\widehat{\tau}_\ell(x)\right| - \left| \int_{u_0}^x A_{\Omega}^{(\ell)}(t)\widehat{\tau}_\ell'(t)dt\right|\right|. \tag{23}$$

Proof. We can form an accurate $C^1(\mathbb{R})$ approximation by the smoothness of $\widehat{\pi}_k^{(\ell)}(x)$ that allows us to apply the Abel summation formula using the summatory function $A_{\Omega}^{(\ell)}(t)$ for t on any bounded connected subinterval of $[1,\infty)$. The stated lower bound formula for $F_\lambda(x)$ in (23) above is valid by Abel summation and by applying Lemma 7.3. In particular, whenever

$$0 \leqslant \left| \frac{\sum\limits_{\log\log t < k \leqslant \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_{\Omega}(t)} \right| \ll 2, \text{ as } t \to \infty.$$

the asymptotically dominant terms indicating the parity of $\lambda(n)$ are captured up to a constant factor by the terms in the range over k summed by $A_{\Omega}(t)$ for sufficiently large $t \to \infty$. In other words, taking the sum over the summands that defines $A_{\Omega}(x)$ only over the truncated range of $k \in [1, \log\log x]$ does not non-trivially change the limiting asymptotically dominant terms in the lower bound obtained from using this form of the summatory function in conjunction with the claimed Abel summation formula. This property holds even when we should technically index over all $k \in [1, \log_2(x)]$ to obtain an exact formula for the summatory weight function. By Corollary 5.5, we have that the assertion above holds as $t \to \infty$.

Secondly, observe that provided sufficiently smoothness (differentiability) of close approximations to $A_{\Omega}(t)$ (to $f(t)$) on (u_0, x), we have that

$$\begin{aligned} |F_\lambda(x)| &\geqslant \left| |A_{\Omega}(x)f(x)| - \int_{u_0}^x |A_{\Omega}(t)f'(t)|dt \right| \\ &\gg \left| \left|A_{\Omega}^{(\ell)}(x)\widehat{\tau}_\ell(x)\right| - \int_{u_0}^x \left|A_{\Omega}^{(\ell)}(t)\widehat{\tau}_\ell'(t)\right| dt \right| \\ &\gg \left| \left|A_{\Omega}^{(\ell)}(x)\widehat{\tau}_\ell(x)\right| - \left|\int_{u_0}^x A_{\Omega}^{(\ell)}(t)\widehat{\tau}_\ell'(t)dt\right| \right|. \end{aligned}$$

The previous equations follow from the ordinary Abel summation method by applying the argument in Lemma 7.3 and using the triangle inequality. ◻

Corollary 7.5. *We have that for almost every sufficiently large* x, *that as* $x \to \infty$

$$\left|G_E^{-1}(x)\right| \gg \frac{2\sqrt{2}e\log 2}{3\pi} \times \frac{(\log x)^{\frac{3}{2}}(\log\log x)}{(\log\log x)\sqrt{\log\log\log x}} \times \left| \sum_{e < d \leqslant \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d \cdot \log\log d} \right|.$$

Proof. Using the definition in (20), we obtain on average that [D]

$$\left|G_E^{-1}(x)\right| = \left|\sum_{n \leqslant (\log x)^{\frac{3}{2}}(\log\log x)} \lambda(n) \times \sum_{\substack{d \mid n \\ d > e}} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d}\right|$$

$$= \left|\sum_{e < d \leqslant (\log x)^{\frac{3}{2}}(\log\log x)} \frac{(\log d)^{\frac{1}{4}}}{\log\log d} \times \sum_{n=1}^{\left\lfloor \frac{\log x}{d} \right\rfloor} \lambda(dn)\right|.$$

We see that by complete additivity of $\Omega(n)$ (complete multiplicativity of $\lambda(n)$) that

$$\sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn) = \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(d) \times \lambda(n) = \lambda(d) \times \sum_{n \leqslant \left\lfloor \frac{x}{d} \right\rfloor} \lambda(n).$$

Now using Theorem 3.7 and Lemma 7.3, we can establish that [E]

$$\left|\sum_{n \leqslant x} \lambda(n)\right| \gg \left|\sum_{k \leqslant \log\log x} (-1)^k \cdot \widehat{\pi}_k(x)\right| \tag{24}$$

$$\gg \frac{2\sqrt{2}e(\log 2)^3}{3\pi} \cdot \frac{x}{(\log x)^3\sqrt{\log\log x}} =: \widehat{L}_0(x), \text{ as } x \to \infty.$$

The sign of the sum obtained by taking the right-hand-side of (24) without the absolute value operation is given by $(-1)^{\lfloor \log\log x \rfloor}$. The precise formula for the limiting lower bound stated above for $\widehat{L}_0(x)$ is computed by symbolic summation in *Mathematica* using the new bounds on $\widehat{\pi}_k(x)$ guaranteed by the theorem, and then by applying subsequent standard asymptotic estimates to the resulting formulas for large $x \to \infty$, e.g., in the form of (8c) and Stirling's formula. It follows that

$$|G_E^{-1}(x)| \gg \left|\sum_{e < d \leqslant (\log x)^{\frac{3}{2}}(\log\log x)} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\left\lfloor \log\log\left(\frac{(\log x)^{\frac{3}{2}}(\log\log x)}{d}\right)\right\rfloor} \cdot \widehat{L}_0\left(\frac{(\log x)^{\frac{3}{2}}(\log\log x)}{d}\right)\right|. \tag{25}$$

Outline for the remainder of the proof. We sketch the following core sections remaining to prove our claimed lower bound on $|G_E^{-1}(x)|$:

- **(A)** We identify an initial subinterval of our full bounds on the summation defined by (20). On this subinterval we prove that we can expect constant sign term contributions resulting from the inputs to the function $\widehat{L}_0$ involving (a priori) both d, x for x large and d on this subinterval. This consideration keeps the sign imparted by $\lambda(d)$ intact in the resulting formula. What we are looking for here is a method to discard the local signedness from the otherwise easily bounded function $\widehat{L}_0$ evaluated as the bivariate function of d, x from the equation above.
- **(B)** We then factor out easily bounded terms from the expansion of the monotone $\widehat{L}_0$ on this interval.
- **(C)** We define and determine additional characteristic formulas we will refer to in later sections for the resulting lower bounds that are formed by restricting the range of d in (25) to just this initial range.

[D] For any arithmetic functions f, h, we have that [1, *cf.* §3.10; §3.12]

$$\sum_{n \leqslant x} h(n) \times \sum_{d \mid n} f(d) = \sum_{d \leqslant x} f(d) \times \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} h(dn).$$

[E] This definition actually provides a lower bound on the famous special case of the summatory function of the Liouville lambda function, $L_0(x)$, for almost every sufficiently large x.

(D) Finally, being absolutely rigorous and careful with this approach, we must argue precisely that the oscillatory, signed terms from the upper end of the deleted interval cannot generate trivial bounds by cancellation with the stated lower bounds.

The arguments used to establish the form of the lower bounds stated in this corollary are longer, and are certainly more involved than the proofs of our previous results given in this section so far. Since further disassembly of these distinct parts to proving the stated bound into smaller external lemmas makes the overall logic to the article harder to interpret, we will continue to label subsections of the remaining proof components corresponding to the headings itemized as above to maintain clarity.

Part A. We will simplify (25) using an appeal to accessible contiguous ranges of consecutive integers over which we obtain effectively constant sign contributions from the function $\widehat{L}_0((\log x)^{\frac{3}{2}}(\log\log x)/d)$ as a function of both x, d. An initial contiguous interval is not difficult to extract for large x, though for general $d \in \left(e, (\log x)^{\frac{3}{2}}(\log\log x)\right)$, the sign contributions from this weight function are muddled by a dual dependence on the fluctuations of both the fractional part of logarithmic functions of x and on the precise location of d within the interval. The idea is to identify this initial accesible interval case, and then prove that we can form a lower bound on $G_E^{-1}(x)$ by truncating and summing only over the d in this range.

In particular, consider that

$$\log\log\left(\frac{(\log x)^{\frac{3}{2}}(\log\log x)}{d}\right) = \log\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right) + \log\left(1 - \frac{\log d}{(\log x)^{\frac{3}{2}}(\log\log x)\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right)}\right), \text{ as } x \to \infty.$$

If we take $d \in (e, \log x] =: \mathcal{R}_x$, we have that

$$\frac{\log d}{(\log x)^{\frac{3}{2}}(\log\log x)\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right)} = o(1) \to 0,$$

as $x \to \infty$. So it stands to reason that for d taken within $\mathcal{R}_x$, we expect that for almost every x there are at most a handful of negligible cases of comparitively small order $d \leqslant d_0(x)$ such that

$$\left\lfloor \log\log\left(\frac{(\log x)^{\frac{3}{2}}(\log\log x)}{d}\right)\right\rfloor \sim \left\lfloor \log\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right) + o(1)\right\rfloor,$$

changes in parity transitioning from $d_0(x)-1$ to $d_0(x)$. An argument making this assertion precise brings leads us to two primary cases that rely on the distribution of the fractional parts of $\left\{(\log x)^{\frac{3}{2}}(\log\log x)\right\}$ within $[0,1)$ for large integers $x \to \infty$. Consider the following points justifying that we obtain the desired (almost) constant sign property for all sufficiently large fixed x and any $\log d \in \mathcal{R}_x$:

(1) If the fractional part $\left\{\log\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right)\right\} = 0$, then

$$\left\lfloor \log\log\left(\frac{(\log x)^{\frac{3}{2}}(\log\log x)}{d}\right)\right\rfloor = \left\lfloor \log\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right)\right\rfloor + \left\lfloor -\frac{\log d}{(\log x)^{\frac{3}{2}}(\log\log x)\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right)}\right\rfloor.$$

This implies that provided that

$$-1 \leqslant -\frac{\log d}{(\log x)^{\frac{3}{2}}(\log\log x)\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right)} < 0,$$

we obtain a constant sign term for $\operatorname{sgn}\left[\widehat{L}_0\left(\frac{(\log x)^{\frac{3}{2}}(\log\log x)}{d}\right)\right]$. Since d is positive and maximized at $\log x$, this condition clearly happens for all sufficiently large x.

(2) If the fractional part $\{\log\log\log\log x\} \in (0,1)$, then

$$\left\lfloor \log\log\left(\frac{(\log x)^{\frac{3}{2}}(\log\log x)}{d}\right)\right\rfloor = \left\lfloor \log\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right)\right\rfloor + \left\lfloor \left\{\log\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right)\right\} - \frac{\log d}{(\log x)^{\frac{3}{2}}(\log\log x)\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right)}\right\rfloor.$$

Let the shorthands $f_x := \left\{\log\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right)\right\}$ and $\mathcal{B}(x) := (\log x)^{\frac{3}{2}}(\log\log x)\log\left((\log x)^{\frac{3}{2}}(\log\log x)\right)$. We require that

$$-1 \leqslant f_x - \frac{\log d}{\mathcal{B}(x)} < 0 \iff (1+f_x)\cdot\mathcal{B}(x) \geqslant \log d > 0,$$

which is similarly clearly attained as $x \to \infty$.

In either case, we obtain the constant sign term on the contribution from $\widehat{L}_0$ for d on this subinterval, $\mathcal{R}_x$.

Part B. Then provided that the sign term involving both d and x from (25) does not change for d within our new interval, $\mathcal{R}_x$, we can factor out the dependence of the sign on the monotonically decreasing function $\widehat{L}_0\left((\log x)^{\frac{3}{2}}(\log\log x)/d\right)$ in the variable d as we sum along the lower interval $\mathcal{R}_x$. We can see that this function is decreasing for $d \in \mathcal{R}_x$ by computing its partial derivative with respect to d and evaluating the asymptotically dominant terms with leading negative sign as $x \to \infty$. So we determine that we should select $d := \log x$ in (25) to obtain a global lower bound on $|G_E^{-1}(x)|$ if we truncate the sum defined by (20) to include only the indices $d \in \mathcal{R}_x$.

Part C. Let the magnitudes of the oscillatory remainder term sums be defined for all sufficiently large x by

$$R_E(x) := \left|\sum_{\log x < d < \frac{(\log x)^{\frac{3}{2}}(\log\log x)}{e}} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\left\lfloor \log\log\left(\frac{(\log x)^{\frac{3}{2}}(\log\log x)}{d}\right)\right\rfloor} \cdot \widehat{L}_0\left(\frac{(\log x)^{\frac{3}{2}}(\log\log x)}{d}\right)\right|.$$

Next, let the function $T_E(x)$ correspond to the easily factored dependence of the less simply integrable factors in $\widehat{L}_0$ when we set $d := \log x$. It is defined for all large enough x as

$$T_E(x) := \frac{1}{\log\left[(\log x)^{\frac{3}{2}}\right]^3\sqrt{\log\log\left[(\log x)^{\frac{3}{2}}\right]}} \gg \frac{1}{(\log\log x)^3\sqrt{\log\log\log x}}. \tag{26}$$

Then, as we argued before, we see that as $x \to \infty$

$$\begin{aligned} S_{E,1}(x) &:= \left|\sum_{e<d\leqslant(\log x)^{\frac{3}{2}}(\log\log x)} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\left\lfloor \log\log\left(\frac{(\log x)^{\frac{3}{2}}(\log\log x)}{d}\right)\right\rfloor}\widehat{L}_0\left(\frac{(\log x)^{\frac{3}{2}}(\log\log x)}{d}\right)\right| \\ &\gg \frac{2\sqrt{2}e\log 2}{3\pi} \times (\log x)^{\frac{3}{2}}(\log\log x)T_E(x) \times \left|\sum_{e<d\leqslant\log x}\frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d\cdot\log\log d}\right| \\ &\gg \frac{2\sqrt{2}e\log 2}{3\pi} \times (\log x)^{\frac{3}{2}}(\log\log x)T_E(x) \times \left|A_\Omega^{(\ell)}(\log x)\widehat{\tau}_0(\log x) - \int_e^{\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_0'(t)dt\right|, \end{aligned} \tag{27}$$

where we select the functions $\widehat{\tau}_0(t) := \frac{(\log t)^{1/4}}{t\cdot\log\log t}$ and $-\widehat{\tau}_0'(t) \gg \frac{(\log t)^{1/4}}{t^2\cdot\log\log t}$ in the notation of Lemma 7.4.

What we then obtain from (25) and (27) is the following lower bound by the triangle inequality that holds for all sufficiently large x:

$$|G_E^{-1}(x)| \gg \left| S_{E,1}(x) - R_E(x) \right| \gg S_{E,1}(x), \text{ as } x \to \infty. \tag{28}$$

We have claimed that in fact we can drop the sum terms over upper range of d and still obtain the asymptotic lower bound on $|G_E^{-1}(x)|$ as $x \to \infty$ on the right-hand-side of (28). To justify this step in the proof, we will provide limiting lower bounds on $R_E(x)$ that show that the contribution from these terms in absolute value exceeds the magnitude of the corresponding sums over $d \in \mathcal{R}_x$ when x is large.

Part D. In Theorem 7.6 stated in the next section below, we prove lower bounds on the sums we used to define $S_{E,1}(x)$ above of the form

$$S_{E,1}(x) \gg \frac{(\log x)^{\frac{3}{2}}}{2 \cdot (\log\log x)^{\frac{3}{4}} (\log\log\log x)^2},$$

where the lower bounds on the right-hand-side of the previous equation are clearly $o\left((\log x)^{\frac{3}{2}}\right)$, though still grows without bound as $x \to \infty$. In contrast, we can bound from below to show that the contribution from $R_E(x)$ is at least on the order of a constant times $(\log x)^{\frac{3}{2}}$. To obtain this lower bound, consider that since $\frac{(\log d)^{\frac{1}{4}}}{d \cdot \log\log d}$ is monotone decreasing for all large enough $d > e$, we obtain the smallest possible magnitude on the sum by alternating signs on consecutive terms in the sum. We can then bound the sum as $x \to \infty$ by

$$\begin{aligned}
\frac{R_E(x)}{(\log x)^{\frac{3}{2}} (\log\log x)} &\gg \left| o(1) + \sum_{\log x < d < \frac{(\log x)^{\frac{3}{2}} (\log\log x)}{2e}} \frac{\log(2d)^{1/4}}{2d \cdot \log\log(2d)} - \frac{\log(2d+1)^{1/4}}{(2d+1)\log\log(2d+1)} \right| \\
&\sim \left| o(1) + \sum_{\log x < d < \frac{(\log x)^{\frac{3}{2}} (\log\log x)}{2e}} \frac{\log(2d)^{1/4}}{2d \cdot \log\log(2d)} - \frac{1}{(2d+1)} \frac{\left(\log(2d) + \frac{1}{2d}\right)^{1/4}}{\left(\log\log(2d) + \frac{1}{2d \cdot \log(2d)}\right)} \right| \\
&\approx \left| o(1) + \sum_{\log x < d < \frac{(\log x)^{\frac{3}{2}} (\log\log x)}{2e}} \frac{\log(2d)^{1/4}}{\log\log(2d)} \left[\frac{1}{2d} - \frac{\left(1 + \frac{1}{2d \cdot \log(2d)}\right)^{1/4}}{(2d+1)\left(1 + \frac{1}{2d \cdot \log(2d)\log\log(2d)}\right)} \right] \right|.
\end{aligned}$$

Then by an appeal to binomial and geometric series expansions, we obtain that the significant terms in the above sum are given by [F]

$$\frac{R_E(x)}{(\log x)^{\frac{3}{2}} (\log\log x)} \gg \left| o(1) + \sum_{\log x < d < \frac{(\log x)^{\frac{3}{2}} (\log\log x)}{2e}} O\left(\frac{\log(2d)^{1/4}}{2d(2d+1) \cdot \log\log(2d)} \right) \right| = O(1).$$

What we obtain from the previous several caclulations is that the magnitude of $R_E(x)$ always exceeds that of the lower bound we establish in Theorem 7.6 for the sums over $d \in \mathcal{R}_x$ as $x \to \infty$. In total, we obtain the lower bounds on $G_E(x)$ that correspond to the smaller order terms resulting from the first summation ranges above to be bounded by the functions stated in Theorem 7.6 below. □

[F] In particular, with $|2d\log(2d)|^{-1}, |2d\log(2d)\log\log(2d)|^{-1} < 1$ we can compute that

$$\left(1 + \frac{1}{2d \cdot \log(2d)}\right)^{\frac{1}{4}} \times \left(1 + \frac{1}{2d \cdot \log(2d) \cdot \log\log(2d)}\right)^{-1} = 1 + \frac{1}{8d \cdot \log(2d)} - \frac{1}{2d \cdot \log(2d) \cdot \log\log(2d)} + O\left(\frac{1}{d^2}\right).$$

7.2.2 The proof of a central lower bound on the magnitude of $G_E^{-1}(x)$

The next central theorem is the last barrier required to prove Corollary 3.8 in the next subsection. Combined with Theorem 7.2 proved in the last section, the new lower bounds we establish below provide us with a sufficient mechanism to bound the formula from Proposition 7.1. Since these lower bounds tend to $+\infty$ as $x \to \infty$ (along an infinite subsequence of positive integers), this is a sufficient condition to guarantee the unboundedness of the scaled Mertens function of the form claimed in the corollary.

Theorem 7.6 (Asymptotics and bounds for the summatory function $G^{-1}(x)$)**.** *We define a lower summatory function, $G_\ell^{-1}(x)$, to provide bounds on the magnitude of $G_E^{-1}(x)$ such that*

$$|G_E^{-1}(x)| \gg |G_\ell^{-1}(x)|,$$

for all sufficiently large $x > e$. Let $C_{\ell,1} > 0$ be the absolute constant defined by

$$C_{\ell,1} = \frac{8e^2 \log^6(2)}{9\pi^2} \approx 0.0738056.$$

We obtain the following limiting estimate for the bounding function $G_\ell^{-1}(x)$ as $x \to \infty$:

$$\left|G_\ell^{-1}(x)\right| \gg \frac{C_{\ell,1} \cdot (\log x)^{\frac{3}{2}}}{2 \cdot (\log\log x)^{\frac{11}{4}} (\log\log\log x)^2}.$$

Proof. Recall from our proof of Corollary 3.7 that a lower bound on the variant prime form counting function, $\widehat{\pi}_k(x)$, is given by

$$\widehat{\pi}_k(x) \gg \frac{4}{3\sqrt{\pi}} \frac{x}{\log x} \left(\frac{\log 2}{\log x}\right)^{\frac{k-1}{\log\log x}+2} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O\left(\frac{k}{(\log\log x)^2}\right)\right), \text{ as } x \to \infty.$$

We can then form a lower summatory function indicating the signed contributions over the distinct parity of $\Omega(n)$ for all $n \leqslant x$ as follows by applying (8b) and Stirling's approximation as already noted in the proof of Corollary 7.5 given above:

$$\begin{aligned} \left|A_\Omega^{(\ell)}(t)\right| &= \left|\sum_{k \leqslant \log\log t} (-1)^k \widehat{\pi}_k(t)\right| \\ &\gg \frac{2\sqrt{2}e(\log 2)^3}{3\pi} \cdot \frac{t}{(\log t)^3 \sqrt{\log\log t}} \left(1 + O\left(\frac{1}{\log\log t}\right)\right) \\ &\gg \frac{2\sqrt{2}e(\log 2)^3}{3\pi} \cdot \frac{t}{(\log t)^3 \sqrt{\log\log t}}, \text{ as } t \to \infty. \end{aligned} \tag{29}$$

The actual sign on this function is given by $\operatorname{sgn}(A_\Omega^{(\ell)}(t)) = (-1)^{\lfloor \log\log t \rfloor}$ (see Lemma 7.3). By Lemma 7.4 we know that this summatory function forms a lower bound in absolute value for the actual weight of the signed terms indicated by $\lambda(n)$.

As we determined in (27) from the proof of Corollary 7.5, we take the function $\widehat{\tau}_0(t) = \frac{(\log t)^{1/4}}{t \cdot \log\log t}$ that satisfies

$$-\widehat{\tau}_0'(t) = -\frac{d}{dt}\left[\frac{(\log t)^{\frac{1}{4}}}{t \cdot \log\log t}\right] \gg \frac{(\log t)^{1/4}}{t^2 \cdot \log\log t}.$$

Moreover, we have using the notation from the proof above that we can select the initial form of the lower bound function $G_\ell^{-1}(x)$ to be defined as follows:

$$G_\ell^{-1}(x) := \frac{2\sqrt{2}e\log 2}{3\pi} \cdot (\log x)^{\frac{3}{2}} (\log\log x) \cdot T_E(x) \times \left|A_\Omega^{(\ell)}(\log x)\widehat{\tau}_0(\log x) - \int_e^{\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_0'(t)dt\right|. \tag{30}$$

The inner integral term on the rightmost side of (30) is summed approximately by splitting the terms weighted by $(-1)^{\lfloor \log\log t\rfloor}$ in the form of [G]

$$
\begin{aligned}
\frac{2\sqrt{2}e\log 2}{3\pi} \times \left|\int_e^{\log x} A_{\Omega}^{(\ell)}(t)\widehat{\tau}_0'(t)dt\right| & \gg \frac{2\sqrt{2}e\log 2}{3\pi} \times \left|\sum_{k=e+1}^{\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]} \left[I_\ell\left(e^{e^{2k+1}}\right)e^{e^{2k+1}} - I_\ell\left(e^{e^{2k}}\right)e^{e^{2k}}\right]\right| \\
& \gg \frac{2\sqrt{2}e\log 2}{3\pi} \times \left|\int_{\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]-\frac{1}{2}}^{\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]} I_\ell\left(e^{e^{2k}}\right)e^{e^{2k}}dk\right|. \qquad (31)
\end{aligned}
$$

We express the integrand function,

$$
I_\ell(t) := \frac{2\sqrt{2}e(\log 2)^3}{3\pi} \times \widehat{\tau}_0'(t)A_{\Omega}^{(\ell)}(t),
$$

defined implicitly as in (31) as the following function of k:

$$
I_\ell\left(e^{e^{2k}}\right)e^{e^{2k}} \gg \frac{2\sqrt{2}e^2\log^6(2)}{9\pi^2} \cdot \frac{e^{-\frac{11k}{2}}}{k^{\frac{3}{2}}} =: \widehat{I}_\ell(k). \qquad (32)
$$

So upon input of the upper bound on the range of integration in (31), at the point $k := \frac{\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]}{2}$, we find from the mean value theorem with the monotone function from (32) that

$$
\begin{aligned}
\frac{2\sqrt{2}e(\log 2)^3}{3\pi} & \times (\log x)^{\frac{3}{2}}(\log\log x) \times T_E(x) \times \left|\int_{\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]-\frac{1}{2}}^{\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]} I_\ell\left(e^{e^{2k}}\right)e^{e^{2k}}dk\right| \\
& \gg \frac{2\sqrt{2}e(\log 2)^3}{3\pi} \times (\log x)^{\frac{3}{2}}(\log\log x) \times T_E(x) \times \left|\widehat{I}_\ell\left(\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]\right)\right| \\
& \gg \frac{C_{\ell,1}\cdot(\log x)^{\frac{3}{2}}}{2\cdot(\log\log x)^{\frac{11}{4}}(\log\log\log x)^2}. \qquad (33)
\end{aligned}
$$

Similarly, by evaluating $\widehat{I}_\ell(t)$ at the lower bound on the integral above with $k := \frac{\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]-1}{2}$, we can similarly conclude that

$$
\begin{aligned}
\frac{2\sqrt{2}e(\log 2)^3}{3\pi} & \times (\log x)^{\frac{3}{2}}(\log\log x) \times T_E(x) \times \left|\int_{\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]-\frac{1}{2}}^{\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]} I_\ell\left(e^{e^{2k}}\right)e^{e^{2k}}dk\right| \\
& \ll \frac{e^{\frac{19}{4}}\cdot C_{\ell,1}\cdot(\log x)^{\frac{3}{2}}}{2\cdot(\log\log x)^{\frac{11}{4}}(\log\log\log x)^2}. \qquad (34)
\end{aligned}
$$

[G] That is, we form the disjoint union of the range of integration into subintervals along which the signedness of the integrands are constant according to

$$
\left\{e^e \leqslant t \leqslant (\log x)^{\frac{3}{2}}(\log\log x) : (-1)^{\lfloor \log\log t\rfloor} = +1\right\} = \left(\bigcup_{k=1}^{\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]} \left[e^{e^{2k}}, e^{e^{2k+1}}\right)\right)\bigcup \mathcal{S}_{0,+},
$$

where $|\mathcal{S}_{0,+}| \leqslant \frac{1}{2}$. We can similarly split the interval of integration corresponding to the negatively biased terms on the unsigned integrand functions for $t \in \left[e^e, (\log x)^{\frac{3}{2}}(\log\log x)\right]$.

To make it clear which terms in (30) the limiting lower bounds correspond to, consider the following expansion for the leading term in the Abel summation formula from (30) for comparison with (33):

$$\begin{aligned}\frac{2\sqrt{2}e(\log 2)^3}{3\pi} &\times (\log x)^{\frac{3}{2}}(\log\log x) \times T_E(x) \times \left|\widehat{\tau}_0(\log x)A_{\Omega}^{(\ell)}(\log x)\right| \\ &\gg \frac{C_{\ell,1}\cdot(\log x)^{\frac{3}{2}}}{(\log\log x)^{\frac{11}{4}}(\log\log\log x)^2}. \end{aligned} \tag{35}$$

Hence, by Lemma 7.3 and the triangle inequality, we conclude that we can take $\left|G_{\ell}^{-1}(x)\right|$ bounded below by the term in (33). □

Remark 7.7. What is key to observe about the distinct lower bounds obtained in the proof of the previous theorem is that each of them scaled by $(\log x)^{-1}$ monotonically increases without bound as $x \to \infty$. In particular, the remaining factor after rescaling dominates the asymptotics of the reciprocal powers of iterated logarithms. It is fortunate, and an indication of correct calculations, that up to distinct constant factors, the asymptotic orders of each of (33), (34) and (35) match identically. We expect this correspondence to be somewhat of a rarity that still coincides in these cases even though one of these terms is formed by a product of component functions, where the other two correspond to distinct particular values of another separate product of related functions over which we perform a definite integral operation.

7.3 Proof of the unboundedness of the scaled Mertens function

We finally address the main conclusion of our arguments given so far with the following proof:

Proof of Corollary 3.8. We break up the integral term in Proposition 7.1 over $t \in [u_0, x/2]$ into two pieces: one that is easily bounded from $u_0 \leqslant t \leqslant \sqrt{x}$, and then another that will conveniently give us our slow-growing tendency towards infinity along the subsequence when evaluated using Theorem 7.6. Given a fixed large infinitely tending x, we have some (at least one) point $x_0 \in \left[\sqrt{x}, \frac{x}{2}\right]$ defined such that $|G^{-1}(t)|$ is minimal and non-vanishing as

$$\left|G^{-1}(x_0)\right| := \min_{\substack{\sqrt{x}\leqslant t\leqslant \frac{x}{2} \\ G^{-1}(t)\neq 0}} |G^{-1}(t)|.$$

We can then apply Proposition 7.1 to bound

$$\begin{aligned}\frac{|M(x)|}{\sqrt{x}} &= \frac{1}{\sqrt{x}}\left|G^{-1}(x) - x\cdot\int_1^{x/2}\frac{G^{-1}(t)}{t^2\cdot\log(x/t)}dt\right| \\ &\gg \left|\left|\frac{G^{-1}(x)}{\sqrt{x}}\right| - \sqrt{x}\int_1^{x/2}\frac{|G^{-1}(t)|}{t^2\cdot\log(x/t)}dt\right| \qquad (36) \\ &\gg \sqrt{x}\times\int_{\sqrt{x}}^{x/2}\frac{|G^{-1}(t)|}{t^2\cdot\log(x/t)}dt \\ &\gg \left(\min_{\substack{\sqrt{x}\leqslant t\leqslant \frac{x}{2} \\ G^{-1}(t)\neq 0}} |G^{-1}(t)|\right)\times\int_{\sqrt{x_0}}^{\frac{x}{2}}\frac{2\sqrt{x_0}}{t^2\cdot\log(x_0)}dt \\ &\gg \frac{2\left|G^{-1}(x_0)\right|}{\log(x_0)}. \qquad (37)\end{aligned}$$

In the second to last step, we observe that $G^{-1}(x) = 0$ for x on a set of asymptotic density *at least* bounded below by $\frac{1}{2}$, so that our claim is accurate as the integral does not vanish for all large enough x.

To see the complete logic to the bound we arrived at in (37), observe that the difference of terms we have in (36) is dominated first by our result in (22) from the proof of Theorem 7.2 as

$$\frac{|G^{-1}(x)|}{\sqrt{x}} \gg \frac{6\sqrt{x}}{\pi^2}(\log x)(\log\log x), \text{ for a.e. } x \rightarrow \infty.$$

Secondly, suppose that we have a smooth approximation for $|G^{-1}(t)|$ so that by the mean value theorem for some $c_0 \in [1, \sqrt{x}]$ and $c_1 \in \left[\sqrt{x}, \frac{x}{2}\right]$ we have

$$\begin{aligned}
&\sqrt{x}\left|\int_1^{x/2} \frac{|G^{-1}(t)|}{t^2 \cdot \log(x/t)} dt\right| \\
&\gg \left|\frac{\sqrt{x}\cdot|G^{-1}(c_0)|}{c_0}\int_1^{\sqrt{x}} \frac{dt}{t\log(x/t)} + \sqrt{x}\cdot|G^{-1}(c_1)|\int_{\sqrt{x}}^{x/2}\frac{dt}{t^2\log(x)}\right| \\
&\gg \left|\left(\min_{\substack{1 \leqslant c \leqslant \sqrt{x} \\ G^{-1}(c) \neq 0}} |G^{-1}(c)|\right)\log\log x + \left(\min_{\substack{\sqrt{x} \leqslant c \leqslant \frac{x}{2} \\ G^{-1}(c) \neq 0}} |G^{-1}(c)|\right)\left(\frac{1}{\log x} + o\left(\frac{1}{\log x}\right)\right)\right|.
\end{aligned}$$

Since $G^{-1}(x)$ changes only at $x \in \mathbb{Z}^+$, what we in fact exactly arrive at is a close variant of this mean value theorem type argument. The statements within the last few equations based on the smoothness approximation assumption for the function make it clear without complications what the tendencies are to bounding these growth rates.

By Theorem 7.2 proved in Section 7.2, the result in (37) implies that

$$\frac{|M(x)|}{\sqrt{x}} \gg \frac{2\left|G_E^{-1}(x_0)\right|}{\log(x_0)}. \tag{38}$$

Define the infinite increasing subsequence, $\{x_{0,y}\}_{y \geqslant Y_0}$, by $x_{0,y} := e^{2e^{e^{2y}}}$ for some sufficiently large finite integer $Y_0 \gg 1$. When we assume that $x \mapsto x_{0,y}$ is taken along this subsequence, we can transform the bound in the last equation into a statement about a lower bound for $|M(x)|\log x/\sqrt{x}$ along an infinitely tending subsequence in the following form by applying Theorem 7.6 to (38):

$$\frac{|M(x_{0,y})|}{\sqrt{x_{0,y}}} \gg \frac{C_{\ell,1}\cdot(\log\sqrt{x_{0,y}})^{\frac{1}{2}}}{(\log\log\sqrt{x_{0,y}})^{\frac{11}{4}}(\log\log\log\sqrt{x_{0,y}})^2}, \text{ as } y \rightarrow \infty. \tag{39}$$

Notice by a small, but insightful, technicality in stating (39), we are not actually asserting that $|M(x)|\log x/\sqrt{x}$ grows unbounded along the precise subsequence of $x \mapsto x_{0,y}$. Rather, we are asserting that the unboundedness of this function can be witnessed along some subsequence whose points are taken within an interval as $\hat{x}_{0,y} \in \left[\sqrt{x_{0,y}}, \frac{x_{0,y}}{2}\right]$. We choose to state the lower bound given on the right-hand-side of (39) using the monotonicity of the lower bound on $|G_E^{-1}(x)|$ we proved in Theorem 7.6 without the need for a conditionally defined asymptotic growth rate. We also can verify that for sufficiently large $y \rightarrow \infty$, this infinitely tending subsequence is well defined as $\hat{x}_{0,y+1} > \hat{x}_{0,y}$ for all sufficiently large $y \geqslant Y_0$.

Finally, we evaluate the following limit to show unboundedness:

$$\lim_{x\rightarrow\infty}\left[\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^{\frac{11}{4}}(\log\log\log x)^2}\right] = +\infty.$$

The scaled Mertens function is then unbounded in the limit supremum sense, as we have claimed, since the right-hand-side of (39) tends to positive infinity as $x_{0,y} \rightarrow \infty$, or equivalently as $y \rightarrow \infty$. □

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T.1 Table: The Dirichlet inverse function $g^{-1}(n)$ and the distribution of its summatory function

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_{+}(n)$	$\mathcal{L}_{-}(n)$	$G^{-1}(n)$	$G_{+}^{-1}(n)$	$G_{-}^{-1}(n)$
1	1^1	Y	N	1	0	1.0000000	1.000000	0.000000	1	1	0
2	2^1	Y	Y	-2	0	1.0000000	0.500000	0.500000	-1	1	-2
3	3^1	Y	Y	-2	0	1.0000000	0.333333	0.666667	-3	1	-4
4	2^2	N	Y	2	0	1.5000000	0.500000	0.500000	-1	3	-4
5	5^1	Y	Y	-2	0	1.0000000	0.400000	0.600000	-3	3	-6
6	$2^1 3^1$	Y	N	5	0	1.0000000	0.500000	0.500000	2	8	-6
7	7^1	Y	Y	-2	0	1.0000000	0.428571	0.571429	0	8	-8
8	2^3	N	Y	-2	0	2.0000000	0.375000	0.625000	-2	8	-10
9	3^2	N	Y	2	0	1.5000000	0.444444	0.555556	0	10	-10
10	$2^1 5^1$	Y	N	5	0	1.0000000	0.500000	0.500000	5	15	-10
11	11^1	Y	Y	-2	0	1.0000000	0.454545	0.545455	3	15	-12
12	$2^2 3^1$	N	N	-7	2	1.2857143	0.416667	0.583333	-4	15	-19
13	13^1	Y	Y	-2	0	1.0000000	0.384615	0.615385	-6	15	-21
14	$2^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-1	20	-21
15	$3^1 5^1$	Y	N	5	0	1.0000000	0.466667	0.533333	4	25	-21
16	2^4	N	Y	2	0	2.5000000	0.500000	0.500000	6	27	-21
17	17^1	Y	Y	-2	0	1.0000000	0.470588	0.529412	4	27	-23
18	$2^1 3^2$	N	N	-7	2	1.2857143	0.444444	0.555556	-3	27	-30
19	19^1	Y	Y	-2	0	1.0000000	0.421053	0.578947	-5	27	-32
20	$2^2 5^1$	N	N	-7	2	1.2857143	0.400000	0.600000	-12	27	-39
21	$3^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-7	32	-39
22	$2^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-2	37	-39
23	23^1	Y	Y	-2	0	1.0000000	0.434783	0.565217	-4	37	-41
24	$2^3 3^1$	N	N	9	4	1.5555556	0.458333	0.541667	5	46	-41
25	5^2	N	Y	2	0	1.5000000	0.480000	0.520000	7	48	-41
26	$2^1 13^1$	Y	N	5	0	1.0000000	0.500000	0.500000	12	53	-41
27	3^3	N	Y	-2	0	2.0000000	0.481481	0.518519	10	53	-43
28	$2^2 7^1$	N	N	-7	2	1.2857143	0.464286	0.535714	3	53	-50
29	29^1	Y	Y	-2	0	1.0000000	0.448276	0.551724	1	53	-52
30	$2^1 3^1 5^1$	Y	N	-16	0	1.0000000	0.433333	0.566667	-15	53	-68
31	31^1	Y	Y	-2	0	1.0000000	0.419355	0.580645	-17	53	-70
32	2^5	N	Y	-2	0	3.0000000	0.406250	0.593750	-19	53	-72
33	$3^1 11^1$	Y	N	5	0	1.0000000	0.424242	0.575758	-14	58	-72
34	$2^1 17^1$	Y	N	5	0	1.0000000	0.441176	0.558824	-9	63	-72
35	$5^1 7^1$	Y	N	5	0	1.0000000	0.457143	0.542857	-4	68	-72
36	$2^2 3^2$	N	N	14	9	1.3571429	0.472222	0.527778	10	82	-72
37	37^1	Y	Y	-2	0	1.0000000	0.459459	0.540541	8	82	-74
38	$2^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	13	87	-74
39	$3^1 13^1$	Y	N	5	0	1.0000000	0.487179	0.512821	18	92	-74
40	$2^3 5^1$	N	N	9	4	1.5555556	0.500000	0.500000	27	101	-74
41	41^1	Y	Y	-2	0	1.0000000	0.487805	0.512195	25	101	-76
42	$2^1 3^1 7^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	9	101	-92
43	43^1	Y	Y	-2	0	1.0000000	0.465116	0.534884	7	101	-94
44	$2^2 11^1$	N	N	-7	2	1.2857143	0.454545	0.545455	0	101	-101
45	$3^2 5^1$	N	N	-7	2	1.2857143	0.444444	0.555556	-7	101	-108
46	$2^1 23^1$	Y	N	5	0	1.0000000	0.456522	0.543478	-2	106	-108
47	47^1	Y	Y	-2	0	1.0000000	0.446809	0.553191	-4	106	-110
48	$2^4 3^1$	N	N	-11	6	1.8181818	0.437500	0.562500	-15	106	-121

Table T.1: ***Computations with*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for*** $\mathbf{1 \leqslant n \leqslant 500}$**.**

- ► *The column labeled* `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled* `Sqfree` *and* `PPower`*, respectively, list inclusion of n in the sets of squarefree integers and the prime powers.*
- ► *The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.*
- ► *The last several columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{n} \cdot \#\{n \leqslant x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_{+}^{-1}(x) + G_{-}^{-1}(x)$ where $G_{+}^{-1}(x) > 0$ and $G_{-}^{-1}(x) < 0$ for all $x \geqslant 1$.*

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\hat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
49	7^2	N	Y	2	0	1.5000000	0.448980	0.551020	−13	108	−121
50	$2^1 5^2$	N	N	−7	2	1.2857143	0.440000	0.560000	−20	108	−128
51	$3^1 17^1$	Y	N	5	0	1.0000000	0.450980	0.549020	−15	113	−128
52	$2^2 13^1$	N	N	−7	2	1.2857143	0.442308	0.557692	−22	113	−135
53	53^1	Y	Y	−2	0	1.0000000	0.433962	0.566038	−24	113	−137
54	$2^1 3^3$	N	N	9	4	1.5555556	0.444444	0.555556	−15	122	−137
55	$5^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−10	127	−137
56	$2^3 7^1$	N	N	9	4	1.5555556	0.464286	0.535714	−1	136	−137
57	$3^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	4	141	−137
58	$2^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	9	146	−137
59	59^1	Y	Y	−2	0	1.0000000	0.474576	0.525424	7	146	−139
60	$2^2 3^1 5^1$	N	N	30	14	1.1666667	0.483333	0.516667	37	176	−139
61	61^1	Y	Y	−2	0	1.0000000	0.475410	0.524590	35	176	−141
62	$2^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	40	181	−141
63	$3^2 7^1$	N	N	−7	2	1.2857143	0.476190	0.523810	33	181	−148
64	2^6	N	Y	2	0	3.5000000	0.484375	0.515625	35	183	−148
65	$5^1 13^1$	Y	N	5	0	1.0000000	0.492308	0.507692	40	188	−148
66	$2^1 3^1 11^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	24	188	−164
67	67^1	Y	Y	−2	0	1.0000000	0.477612	0.522388	22	188	−166
68	$2^2 17^1$	N	N	−7	2	1.2857143	0.470588	0.529412	15	188	−173
69	$3^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	20	193	−173
70	$2^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.471429	0.528571	4	193	−189
71	71^1	Y	Y	−2	0	1.0000000	0.464789	0.535211	2	193	−191
72	$2^3 3^2$	N	N	−23	18	1.4782609	0.458333	0.541667	−21	193	−214
73	73^1	Y	Y	−2	0	1.0000000	0.452055	0.547945	−23	193	−216
74	$2^1 37^1$	Y	N	5	0	1.0000000	0.459459	0.540541	−18	198	−216
75	$3^1 5^2$	N	N	−7	2	1.2857143	0.453333	0.546667	−25	198	−223
76	$2^2 19^1$	N	N	−7	2	1.2857143	0.447368	0.552632	−32	198	−230
77	$7^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−27	203	−230
78	$2^1 3^1 13^1$	Y	N	−16	0	1.0000000	0.448718	0.551282	−43	203	−246
79	79^1	Y	Y	−2	0	1.0000000	0.443038	0.556962	−45	203	−248
80	$2^4 5^1$	N	N	−11	6	1.8181818	0.437500	0.562500	−56	203	−259
81	3^4	N	Y	2	0	2.5000000	0.444444	0.555556	−54	205	−259
82	$2^1 41^1$	Y	N	5	0	1.0000000	0.451220	0.548780	−49	210	−259
83	83^1	Y	Y	−2	0	1.0000000	0.445783	0.554217	−51	210	−261
84	$2^2 3^1 7^1$	N	N	30	14	1.1666667	0.452381	0.547619	−21	240	−261
85	$5^1 17^1$	Y	N	5	0	1.0000000	0.458824	0.541176	−16	245	−261
86	$2^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	−11	250	−261
87	$3^1 29^1$	Y	N	5	0	1.0000000	0.471264	0.528736	−6	255	−261
88	$2^3 11^1$	N	N	9	4	1.5555556	0.477273	0.522727	3	264	−261
89	89^1	Y	Y	−2	0	1.0000000	0.471910	0.528090	1	264	−263
90	$2^1 3^2 5^1$	N	N	30	14	1.1666667	0.477778	0.522222	31	294	−263
91	$7^1 13^1$	Y	N	5	0	1.0000000	0.483516	0.516484	36	299	−263
92	$2^2 23^1$	N	N	−7	2	1.2857143	0.478261	0.521739	29	299	−270
93	$3^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	34	304	−270
94	$2^1 47^1$	Y	N	5	0	1.0000000	0.489362	0.510638	39	309	−270
95	$5^1 19^1$	Y	N	5	0	1.0000000	0.494737	0.505263	44	314	−270
96	$2^5 3^1$	N	N	13	8	2.0769231	0.500000	0.500000	57	327	−270
97	97^1	Y	Y	−2	0	1.0000000	0.494845	0.505155	55	327	−272
98	$2^1 7^2$	N	N	−7	2	1.2857143	0.489796	0.510204	48	327	−279
99	$3^2 11^1$	N	N	−7	2	1.2857143	0.484848	0.515152	41	327	−286
100	$2^2 5^2$	N	N	14	9	1.3571429	0.490000	0.510000	55	341	−286
101	101^1	Y	Y	−2	0	1.0000000	0.485149	0.514851	53	341	−288
102	$2^1 3^1 17^1$	Y	N	−16	0	1.0000000	0.480392	0.519608	37	341	−304
103	103^1	Y	Y	−2	0	1.0000000	0.475728	0.524272	35	341	−306
104	$2^3 13^1$	N	N	9	4	1.5555556	0.480769	0.519231	44	350	−306
105	$3^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	28	350	−322
106	$2^1 53^1$	Y	N	5	0	1.0000000	0.481132	0.518868	33	355	−322
107	107^1	Y	Y	−2	0	1.0000000	0.476636	0.523364	31	355	−324
108	$2^2 3^3$	N	N	−23	18	1.4782609	0.472222	0.527778	8	355	−347
109	109^1	Y	Y	−2	0	1.0000000	0.467890	0.532110	6	355	−349
110	$2^1 5^1 11^1$	Y	N	−16	0	1.0000000	0.463636	0.536364	−10	355	−365
111	$3^1 37^1$	Y	N	5	0	1.0000000	0.468468	0.531532	−5	360	−365
112	$2^4 7^1$	N	N	−11	6	1.8181818	0.464286	0.535714	−16	360	−376
113	113^1	Y	Y	−2	0	1.0000000	0.460177	0.539823	−18	360	−378
114	$2^1 3^1 19^1$	Y	N	−16	0	1.0000000	0.456140	0.543860	−34	360	−394
115	$5^1 23^1$	Y	N	5	0	1.0000000	0.460870	0.539130	−29	365	−394
116	$2^2 29^1$	N	N	−7	2	1.2857143	0.456897	0.543103	−36	365	−401
117	$3^2 13^1$	N	N	−7	2	1.2857143	0.452991	0.547009	−43	365	−408
118	$2^1 59^1$	Y	N	5	0	1.0000000	0.457627	0.542373	−38	370	−408
119	$7^1 17^1$	Y	N	5	0	1.0000000	0.462185	0.537815	−33	375	−408
120	$2^3 3^1 5^1$	N	N	−48	32	1.3333333	0.458333	0.541667	−81	375	−456
121	11^2	N	Y	2	0	1.5000000	0.462810	0.537190	−79	377	−456
122	$2^1 61^1$	Y	N	5	0	1.0000000	0.467213	0.532787	−74	382	−456
123	$3^1 41^1$	Y	N	5	0	1.0000000	0.471545	0.528455	−69	387	−456
124	$2^2 31^1$	N	N	−7	2	1.2857143	0.467742	0.532258	−76	387	−463

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d \mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
125	5^3	N	Y	−2	0	2.0000000	0.464000	0.536000	−78	387	−465
126	$2^1 3^2 7^1$	N	N	30	14	1.1666667	0.468254	0.531746	−48	417	−465
127	127^1	Y	Y	−2	0	1.0000000	0.464567	0.535433	−50	417	−467
128	2^7	N	Y	−2	0	4.0000000	0.460938	0.539062	−52	417	−469
129	$3^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	−47	422	−469
130	$2^1 5^1 13^1$	Y	N	−16	0	1.0000000	0.461538	0.538462	−63	422	−485
131	131^1	Y	Y	−2	0	1.0000000	0.458015	0.541985	−65	422	−487
132	$2^2 3^1 11^1$	N	N	30	14	1.1666667	0.462121	0.537879	−35	452	−487
133	$7^1 19^1$	Y	N	5	0	1.0000000	0.466165	0.533835	−30	457	−487
134	$2^1 67^1$	Y	N	5	0	1.0000000	0.470149	0.529851	−25	462	−487
135	$3^3 5^1$	N	N	9	4	1.5555556	0.474074	0.525926	−16	471	−487
136	$2^3 17^1$	N	N	9	4	1.5555556	0.477941	0.522059	−7	480	−487
137	137^1	Y	Y	−2	0	1.0000000	0.474453	0.525547	−9	480	−489
138	$2^1 3^1 23^1$	Y	N	−16	0	1.0000000	0.471014	0.528986	−25	480	−505
139	139^1	Y	Y	−2	0	1.0000000	0.467626	0.532374	−27	480	−507
140	$2^2 5^1 7^1$	N	N	30	14	1.1666667	0.471429	0.528571	3	510	−507
141	$3^1 47^1$	Y	N	5	0	1.0000000	0.475177	0.524823	8	515	−507
142	$2^1 71^1$	Y	N	5	0	1.0000000	0.478873	0.521127	13	520	−507
143	$11^1 13^1$	Y	N	5	0	1.0000000	0.482517	0.517483	18	525	−507
144	$2^4 3^2$	N	N	34	29	1.6176471	0.486111	0.513889	52	559	−507
145	$5^1 29^1$	Y	N	5	0	1.0000000	0.489655	0.510345	57	564	−507
146	$2^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	62	569	−507
147	$3^1 7^2$	N	N	−7	2	1.2857143	0.489796	0.510204	55	569	−514
148	$2^2 37^1$	N	N	−7	2	1.2857143	0.486486	0.513514	48	569	−521
149	149^1	Y	Y	−2	0	1.0000000	0.483221	0.516779	46	569	−523
150	$2^1 3^1 5^2$	N	N	30	14	1.1666667	0.486667	0.513333	76	599	−523
151	151^1	Y	Y	−2	0	1.0000000	0.483444	0.516556	74	599	−525
152	$2^3 19^1$	N	N	9	4	1.5555556	0.486842	0.513158	83	608	−525
153	$3^2 17^1$	N	N	−7	2	1.2857143	0.483660	0.516340	76	608	−532
154	$2^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.480519	0.519481	60	608	−548
155	$5^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	65	613	−548
156	$2^2 3^1 13^1$	N	N	30	14	1.1666667	0.487179	0.512821	95	643	−548
157	157^1	Y	Y	−2	0	1.0000000	0.484076	0.515924	93	643	−550
158	$2^1 79^1$	Y	N	5	0	1.0000000	0.487342	0.512658	98	648	−550
159	$3^1 53^1$	Y	N	5	0	1.0000000	0.490566	0.509434	103	653	−550
160	$2^5 5^1$	N	N	13	8	2.0769231	0.493750	0.506250	116	666	−550
161	$7^1 23^1$	Y	N	5	0	1.0000000	0.496894	0.503106	121	671	−550
162	$2^1 3^4$	N	N	−11	6	1.8181818	0.493827	0.506173	110	671	−561
163	163^1	Y	Y	−2	0	1.0000000	0.490798	0.509202	108	671	−563
164	$2^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	101	671	−570
165	$3^1 5^1 11^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	85	671	−586
166	$2^1 83^1$	Y	N	5	0	1.0000000	0.487952	0.512048	90	676	−586
167	167^1	Y	Y	−2	0	1.0000000	0.485030	0.514970	88	676	−588
168	$2^3 3^1 7^1$	N	N	−48	32	1.3333333	0.482143	0.517857	40	676	−636
169	13^2	N	Y	2	0	1.5000000	0.485207	0.514793	42	678	−636
170	$2^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.482353	0.517647	26	678	−652
171	$3^2 19^1$	N	N	−7	2	1.2857143	0.479532	0.520468	19	678	−659
172	$2^2 43^1$	N	N	−7	2	1.2857143	0.476744	0.523256	12	678	−666
173	173^1	Y	Y	−2	0	1.0000000	0.473988	0.526012	10	678	−668
174	$2^1 3^1 29^1$	Y	N	−16	0	1.0000000	0.471264	0.528736	−6	678	−684
175	$5^2 7^1$	N	N	−7	2	1.2857143	0.468571	0.531429	−13	678	−691
176	$2^4 11^1$	N	N	−11	6	1.8181818	0.465909	0.534091	−24	678	−702
177	$3^1 59^1$	Y	N	5	0	1.0000000	0.468927	0.531073	−19	683	−702
178	$2^1 89^1$	Y	N	5	0	1.0000000	0.471910	0.528090	−14	688	−702
179	179^1	Y	Y	−2	0	1.0000000	0.469274	0.530726	−16	688	−704
180	$2^2 3^2 5^1$	N	N	−74	58	1.2162162	0.466667	0.533333	−90	688	−778
181	181^1	Y	Y	−2	0	1.0000000	0.464088	0.535912	−92	688	−780
182	$2^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.461538	0.538462	−108	688	−796
183	$3^1 61^1$	Y	N	5	0	1.0000000	0.464481	0.535519	−103	693	−796
184	$2^3 23^1$	N	N	9	4	1.5555556	0.467391	0.532609	−94	702	−796
185	$5^1 37^1$	Y	N	5	0	1.0000000	0.470270	0.529730	−89	707	−796
186	$2^1 3^1 31^1$	Y	N	−16	0	1.0000000	0.467742	0.532258	−105	707	−812
187	$11^1 17^1$	Y	N	5	0	1.0000000	0.470588	0.529412	−100	712	−812
188	$2^2 47^1$	N	N	−7	2	1.2857143	0.468085	0.531915	−107	712	−819
189	$3^3 7^1$	N	N	9	4	1.5555556	0.470899	0.529101	−98	721	−819
190	$2^1 5^1 19^1$	Y	N	−16	0	1.0000000	0.468421	0.531579	−114	721	−835
191	191^1	Y	Y	−2	0	1.0000000	0.465969	0.534031	−116	721	−837
192	$2^6 3^1$	N	N	−15	10	2.3333333	0.463542	0.536458	−131	721	−852
193	193^1	Y	Y	−2	0	1.0000000	0.461140	0.538860	−133	721	−854
194	$2^1 97^1$	Y	N	5	0	1.0000000	0.463918	0.536082	−128	726	−854
195	$3^1 5^1 13^1$	Y	N	−16	0	1.0000000	0.461538	0.538462	−144	726	−870
196	$2^2 7^2$	N	N	14	9	1.3571429	0.464286	0.535714	−130	740	−870
197	197^1	Y	Y	−2	0	1.0000000	0.461929	0.538071	−132	740	−872
198	$2^1 3^2 11^1$	N	N	30	14	1.1666667	0.464646	0.535354	−102	770	−872
199	199^1	Y	Y	−2	0	1.0000000	0.462312	0.537688	−104	770	−874
200	$2^3 5^2$	N	N	−23	18	1.4782609	0.460000	0.540000	−127	770	−897

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \hat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
201	$3^1 67^1$	Y	N	5	0	1.0000000	0.462687	0.537313	−122	775	−897
202	$2^1 101^1$	Y	N	5	0	1.0000000	0.465347	0.534653	−117	780	−897
203	$7^1 29^1$	Y	N	5	0	1.0000000	0.467980	0.532020	−112	785	−897
204	$2^2 3^1 17^1$	N	N	30	14	1.1666667	0.470588	0.529412	−82	815	−897
205	$5^1 41^1$	Y	N	5	0	1.0000000	0.473171	0.526829	−77	820	−897
206	$2^1 103^1$	Y	N	5	0	1.0000000	0.475728	0.524272	−72	825	−897
207	$3^2 23^1$	N	N	−7	2	1.2857143	0.473430	0.526570	−79	825	−904
208	$2^4 13^1$	N	N	−11	6	1.8181818	0.471154	0.528846	−90	825	−915
209	$11^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	−85	830	−915
210	$2^1 3^1 5^1 7^1$	Y	N	65	0	1.0000000	0.476190	0.523810	−20	895	−915
211	211^1	Y	Y	−2	0	1.0000000	0.473934	0.526066	−22	895	−917
212	$2^2 53^1$	N	N	−7	2	1.2857143	0.471698	0.528302	−29	895	−924
213	$3^1 71^1$	Y	N	5	0	1.0000000	0.474178	0.525822	−24	900	−924
214	$2^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−19	905	−924
215	$5^1 43^1$	Y	N	5	0	1.0000000	0.479070	0.520930	−14	910	−924
216	$2^3 3^3$	N	N	46	41	1.5000000	0.481481	0.518519	32	956	−924
217	$7^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	37	961	−924
218	$2^1 109^1$	Y	N	5	0	1.0000000	0.486239	0.513761	42	966	−924
219	$3^1 73^1$	Y	N	5	0	1.0000000	0.488584	0.511416	47	971	−924
220	$2^2 5^1 11^1$	N	N	30	14	1.1666667	0.490909	0.509091	77	1001	−924
221	$13^1 17^1$	Y	N	5	0	1.0000000	0.493213	0.506787	82	1006	−924
222	$2^1 3^1 37^1$	Y	N	−16	0	1.0000000	0.490991	0.509009	66	1006	−940
223	223^1	Y	Y	−2	0	1.0000000	0.488789	0.511211	64	1006	−942
224	$2^5 7^1$	N	N	13	8	2.0769231	0.491071	0.508929	77	1019	−942
225	$3^2 5^2$	N	N	14	9	1.3571429	0.493333	0.506667	91	1033	−942
226	$2^1 113^1$	Y	N	5	0	1.0000000	0.495575	0.504425	96	1038	−942
227	227^1	Y	Y	−2	0	1.0000000	0.493392	0.506608	94	1038	−944
228	$2^2 3^1 19^1$	N	N	30	14	1.1666667	0.495614	0.504386	124	1068	−944
229	229^1	Y	Y	−2	0	1.0000000	0.493450	0.506550	122	1068	−946
230	$2^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.491304	0.508696	106	1068	−962
231	$3^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.489177	0.510823	90	1068	−978
232	$2^3 29^1$	N	N	9	4	1.5555556	0.491379	0.508621	99	1077	−978
233	233^1	Y	Y	−2	0	1.0000000	0.489270	0.510730	97	1077	−980
234	$2^1 3^2 13^1$	N	N	30	14	1.1666667	0.491453	0.508547	127	1107	−980
235	$5^1 47^1$	Y	N	5	0	1.0000000	0.493617	0.506383	132	1112	−980
236	$2^2 59^1$	N	N	−7	2	1.2857143	0.491525	0.508475	125	1112	−987
237	$3^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	130	1117	−987
238	$2^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.491597	0.508403	114	1117	−1003
239	239^1	Y	Y	−2	0	1.0000000	0.489540	0.510460	112	1117	−1005
240	$2^4 3^1 5^1$	N	N	70	54	1.5000000	0.491667	0.508333	182	1187	−1005
241	241^1	Y	Y	−2	0	1.0000000	0.489627	0.510373	180	1187	−1007
242	$2^1 11^2$	N	N	−7	2	1.2857143	0.487603	0.512397	173	1187	−1014
243	3^5	N	Y	−2	0	3.0000000	0.485597	0.514403	171	1187	−1016
244	$2^2 61^1$	N	N	−7	2	1.2857143	0.483607	0.516393	164	1187	−1023
245	$5^1 7^2$	N	N	−7	2	1.2857143	0.481633	0.518367	157	1187	−1030
246	$2^1 3^1 41^1$	Y	N	−16	0	1.0000000	0.479675	0.520325	141	1187	−1046
247	$13^1 19^1$	Y	N	5	0	1.0000000	0.481781	0.518219	146	1192	−1046
248	$2^3 31^1$	N	N	9	4	1.5555556	0.483871	0.516129	155	1201	−1046
249	$3^1 83^1$	Y	N	5	0	1.0000000	0.485944	0.514056	160	1206	−1046
250	$2^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	169	1215	−1046
251	251^1	Y	Y	−2	0	1.0000000	0.486056	0.513944	167	1215	−1048
252	$2^2 3^2 7^1$	N	N	−74	58	1.2162162	0.484127	0.515873	93	1215	−1122
253	$11^1 23^1$	Y	N	5	0	1.0000000	0.486166	0.513834	98	1220	−1122
254	$2^1 127^1$	Y	N	5	0	1.0000000	0.488189	0.511811	103	1225	−1122
255	$3^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.486275	0.513725	87	1225	−1138
256	2^8	N	Y	2	0	4.5000000	0.488281	0.511719	89	1227	−1138
257	257^1	Y	Y	−2	0	1.0000000	0.486381	0.513619	87	1227	−1140
258	$2^1 3^1 43^1$	Y	N	−16	0	1.0000000	0.484496	0.515504	71	1227	−1156
259	$7^1 37^1$	Y	N	5	0	1.0000000	0.486486	0.513514	76	1232	−1156
260	$2^2 5^1 13^1$	N	N	30	14	1.1666667	0.488462	0.511538	106	1262	−1156
261	$3^2 29^1$	N	N	−7	2	1.2857143	0.486590	0.513410	99	1262	−1163
262	$2^1 131^1$	Y	N	5	0	1.0000000	0.488550	0.511450	104	1267	−1163
263	263^1	Y	Y	−2	0	1.0000000	0.486692	0.513308	102	1267	−1165
264	$2^3 3^1 11^1$	N	N	−48	32	1.3333333	0.484848	0.515152	54	1267	−1213
265	$5^1 53^1$	Y	N	5	0	1.0000000	0.486792	0.513208	59	1272	−1213
266	$2^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.484962	0.515038	43	1272	−1229
267	$3^1 89^1$	Y	N	5	0	1.0000000	0.486891	0.513109	48	1277	−1229
268	$2^2 67^1$	N	N	−7	2	1.2857143	0.485075	0.514925	41	1277	−1236
269	269^1	Y	Y	−2	0	1.0000000	0.483271	0.516729	39	1277	−1238
270	$2^1 3^3 5^1$	N	N	−48	32	1.3333333	0.481481	0.518519	−9	1277	−1286
271	271^1	Y	Y	−2	0	1.0000000	0.479705	0.520295	−11	1277	−1288
272	$2^4 17^1$	N	N	−11	6	1.8181818	0.477941	0.522059	−22	1277	−1299
273	$3^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−38	1277	−1315
274	$2^1 137^1$	Y	N	5	0	1.0000000	0.478102	0.521898	−33	1282	−1315
275	$5^2 11^1$	N	N	−7	2	1.2857143	0.476364	0.523636	−40	1282	−1322
276	$2^2 3^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−10	1312	−1322
277	277^1	Y	Y	−2	0	1.0000000	0.476534	0.523466	−12	1312	−1324

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\hat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
278	$2^1 139^1$	Y	N	5	0	1.0000000	0.478417	0.521583	−7	1317	−1324
279	$3^2 31^1$	N	N	−7	2	1.2857143	0.476703	0.523297	−14	1317	−1331
280	$2^3 5^1 7^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−62	1317	−1379
281	281^1	Y	Y	−2	0	1.0000000	0.473310	0.526690	−64	1317	−1381
282	$2^1 3^1 47^1$	Y	N	−16	0	1.0000000	0.471631	0.528369	−80	1317	−1397
283	283^1	Y	Y	−2	0	1.0000000	0.469965	0.530035	−82	1317	−1399
284	$2^2 71^1$	N	N	−7	2	1.2857143	0.468310	0.531690	−89	1317	−1406
285	$3^1 5^1 19^1$	Y	N	−16	0	1.0000000	0.466667	0.533333	−105	1317	−1422
286	$2^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.465035	0.534965	−121	1317	−1438
287	$7^1 41^1$	Y	N	5	0	1.0000000	0.466899	0.533101	−116	1322	−1438
288	$2^5 3^2$	N	N	−47	42	1.7659574	0.465278	0.534722	−163	1322	−1485
289	17^2	N	Y	2	0	1.5000000	0.467128	0.532872	−161	1324	−1485
290	$2^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.465517	0.534483	−177	1324	−1501
291	$3^1 97^1$	Y	N	5	0	1.0000000	0.467354	0.532646	−172	1329	−1501
292	$2^2 73^1$	N	N	−7	2	1.2857143	0.465753	0.534247	−179	1329	−1508
293	293^1	Y	Y	−2	0	1.0000000	0.464164	0.535836	−181	1329	−1510
294	$2^1 3^1 7^2$	N	N	30	14	1.1666667	0.465986	0.534014	−151	1359	−1510
295	$5^1 59^1$	Y	N	5	0	1.0000000	0.467797	0.532203	−146	1364	−1510
296	$2^3 37^1$	N	N	9	4	1.5555556	0.469595	0.530405	−137	1373	−1510
297	$3^3 11^1$	N	N	9	4	1.5555556	0.471380	0.528620	−128	1382	−1510
298	$2^1 149^1$	Y	N	5	0	1.0000000	0.473154	0.526846	−123	1387	−1510
299	$13^1 23^1$	Y	N	5	0	1.0000000	0.474916	0.525084	−118	1392	−1510
300	$2^2 3^1 5^2$	N	N	−74	58	1.2162162	0.473333	0.526667	−192	1392	−1584
301	$7^1 43^1$	Y	N	5	0	1.0000000	0.475083	0.524917	−187	1397	−1584
302	$2^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−182	1402	−1584
303	$3^1 101^1$	Y	N	5	0	1.0000000	0.478548	0.521452	−177	1407	−1584
304	$2^4 19^1$	N	N	−11	6	1.8181818	0.476974	0.523026	−188	1407	−1595
305	$5^1 61^1$	Y	N	5	0	1.0000000	0.478689	0.521311	−183	1412	−1595
306	$2^1 3^2 17^1$	N	N	30	14	1.1666667	0.480392	0.519608	−153	1442	−1595
307	307^1	Y	Y	−2	0	1.0000000	0.478827	0.521173	−155	1442	−1597
308	$2^2 7^1 11^1$	N	N	30	14	1.1666667	0.480519	0.519481	−125	1472	−1597
309	$3^1 103^1$	Y	N	5	0	1.0000000	0.482201	0.517799	−120	1477	−1597
310	$2^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.480645	0.519355	−136	1477	−1613
311	311^1	Y	Y	−2	0	1.0000000	0.479100	0.520900	−138	1477	−1615
312	$2^3 3^1 13^1$	N	N	−48	32	1.3333333	0.477564	0.522436	−186	1477	−1663
313	313^1	Y	Y	−2	0	1.0000000	0.476038	0.523962	−188	1477	−1665
314	$2^1 157^1$	Y	N	5	0	1.0000000	0.477707	0.522293	−183	1482	−1665
315	$3^2 5^1 7^1$	N	N	30	14	1.1666667	0.479365	0.520635	−153	1512	−1665
316	$2^2 79^1$	N	N	−7	2	1.2857143	0.477848	0.522152	−160	1512	−1672
317	317^1	Y	Y	−2	0	1.0000000	0.476341	0.523659	−162	1512	−1674
318	$2^1 3^1 53^1$	Y	N	−16	0	1.0000000	0.474843	0.525157	−178	1512	−1690
319	$11^1 29^1$	Y	N	5	0	1.0000000	0.476489	0.523511	−173	1517	−1690
320	$2^6 5^1$	N	N	−15	10	2.3333333	0.475000	0.525000	−188	1517	−1705
321	$3^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−183	1522	−1705
322	$2^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.475155	0.524845	−199	1522	−1721
323	$17^1 19^1$	Y	N	5	0	1.0000000	0.476780	0.523220	−194	1527	−1721
324	$2^2 3^4$	N	N	34	29	1.6176471	0.478395	0.521605	−160	1561	−1721
325	$5^2 13^1$	N	N	−7	2	1.2857143	0.476923	0.523077	−167	1561	−1728
326	$2^1 163^1$	Y	N	5	0	1.0000000	0.478528	0.521472	−162	1566	−1728
327	$3^1 109^1$	Y	N	5	0	1.0000000	0.480122	0.519878	−157	1571	−1728
328	$2^3 41^1$	N	N	9	4	1.5555556	0.481707	0.518293	−148	1580	−1728
329	$7^1 47^1$	Y	N	5	0	1.0000000	0.483283	0.516717	−143	1585	−1728
330	$2^1 3^1 5^1 11^1$	Y	N	65	0	1.0000000	0.484848	0.515152	−78	1650	−1728
331	331^1	Y	Y	−2	0	1.0000000	0.483384	0.516616	−80	1650	−1730
332	$2^2 83^1$	N	N	−7	2	1.2857143	0.481928	0.518072	−87	1650	−1737
333	$3^2 37^1$	N	N	−7	2	1.2857143	0.480480	0.519520	−94	1650	−1744
334	$2^1 167^1$	Y	N	5	0	1.0000000	0.482036	0.517964	−89	1655	−1744
335	$5^1 67^1$	Y	N	5	0	1.0000000	0.483582	0.516418	−84	1660	−1744
336	$2^4 3^1 7^1$	N	N	70	54	1.5000000	0.485119	0.514881	−14	1730	−1744
337	337^1	Y	Y	−2	0	1.0000000	0.483680	0.516320	−16	1730	−1746
338	$2^1 13^2$	N	N	−7	2	1.2857143	0.482249	0.517751	−23	1730	−1753
339	$3^1 113^1$	Y	N	5	0	1.0000000	0.483776	0.516224	−18	1735	−1753
340	$2^2 5^1 17^1$	N	N	30	14	1.1666667	0.485294	0.514706	12	1765	−1753
341	$11^1 31^1$	Y	N	5	0	1.0000000	0.486804	0.513196	17	1770	−1753
342	$2^1 3^2 19^1$	N	N	30	14	1.1666667	0.488304	0.511696	47	1800	−1753
343	7^3	N	Y	−2	0	2.0000000	0.486880	0.513120	45	1800	−1755
344	$2^3 43^1$	N	N	9	4	1.5555556	0.488372	0.511628	54	1809	−1755
345	$3^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.486957	0.513043	38	1809	−1771
346	$2^1 173^1$	Y	N	5	0	1.0000000	0.488439	0.511561	43	1814	−1771
347	347^1	Y	Y	−2	0	1.0000000	0.487032	0.512968	41	1814	−1773
348	$2^2 3^1 29^1$	N	N	30	14	1.1666667	0.488506	0.511494	71	1844	−1773
349	349^1	Y	Y	−2	0	1.0000000	0.487106	0.512894	69	1844	−1775
350	$2^1 5^2 7^1$	N	N	30	14	1.1666667	0.488571	0.511429	99	1874	−1775

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\hat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
351	$3^3 13^1$	N	N	9	4	1.5555556	0.490028	0.509972	108	1883	−1775
352	$2^5 11^1$	N	N	13	8	2.0769231	0.491477	0.508523	121	1896	−1775
353	353^1	Y	Y	−2	0	1.0000000	0.490085	0.509915	119	1896	−1777
354	$2^1 3^1 59^1$	Y	N	−16	0	1.0000000	0.488701	0.511299	103	1896	−1793
355	$5^1 71^1$	Y	N	5	0	1.0000000	0.490141	0.509859	108	1901	−1793
356	$2^2 89^1$	N	N	−7	2	1.2857143	0.488764	0.511236	101	1901	−1800
357	$3^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.487395	0.512605	85	1901	−1816
358	$2^1 179^1$	Y	N	5	0	1.0000000	0.488827	0.511173	90	1906	−1816
359	359^1	Y	Y	−2	0	1.0000000	0.487465	0.512535	88	1906	−1818
360	$2^3 3^2 5^1$	N	N	145	129	1.3034483	0.488889	0.511111	233	2051	−1818
361	19^2	N	Y	2	0	1.5000000	0.490305	0.509695	235	2053	−1818
362	$2^1 181^1$	Y	N	5	0	1.0000000	0.491713	0.508287	240	2058	−1818
363	$3^1 11^2$	N	N	−7	2	1.2857143	0.490358	0.509642	233	2058	−1825
364	$2^2 7^1 13^1$	N	N	30	14	1.1666667	0.491758	0.508242	263	2088	−1825
365	$5^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	268	2093	−1825
366	$2^1 3^1 61^1$	Y	N	−16	0	1.0000000	0.491803	0.508197	252	2093	−1841
367	367^1	Y	Y	−2	0	1.0000000	0.490463	0.509537	250	2093	−1843
368	$2^4 23^1$	N	N	−11	6	1.8181818	0.489130	0.510870	239	2093	−1854
369	$3^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	232	2093	−1861
370	$2^1 5^1 37^1$	Y	N	−16	0	1.0000000	0.486486	0.513514	216	2093	−1877
371	$7^1 53^1$	Y	N	5	0	1.0000000	0.487871	0.512129	221	2098	−1877
372	$2^2 3^1 31^1$	N	N	30	14	1.1666667	0.489247	0.510753	251	2128	−1877
373	373^1	Y	Y	−2	0	1.0000000	0.487936	0.512064	249	2128	−1879
374	$2^1 11^1 17^1$	Y	N	−16	0	1.0000000	0.486631	0.513369	233	2128	−1895
375	$3^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	242	2137	−1895
376	$2^3 47^1$	N	N	9	4	1.5555556	0.489362	0.510638	251	2146	−1895
377	$13^1 29^1$	Y	N	5	0	1.0000000	0.490716	0.509284	256	2151	−1895
378	$2^1 3^3 7^1$	N	N	−48	32	1.3333333	0.489418	0.510582	208	2151	−1943
379	379^1	Y	Y	−2	0	1.0000000	0.488127	0.511873	206	2151	−1945
380	$2^2 5^1 19^1$	N	N	30	14	1.1666667	0.489474	0.510526	236	2181	−1945
381	$3^1 127^1$	Y	N	5	0	1.0000000	0.490814	0.509186	241	2186	−1945
382	$2^1 191^1$	Y	N	5	0	1.0000000	0.492147	0.507853	246	2191	−1945
383	383^1	Y	Y	−2	0	1.0000000	0.490862	0.509138	244	2191	−1947
384	$2^7 3^1$	N	N	17	12	2.5882353	0.492188	0.507812	261	2208	−1947
385	$5^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.490909	0.509091	245	2208	−1963
386	$2^1 193^1$	Y	N	5	0	1.0000000	0.492228	0.507772	250	2213	−1963
387	$3^2 43^1$	N	N	−7	2	1.2857143	0.490956	0.509044	243	2213	−1970
388	$2^2 97^1$	N	N	−7	2	1.2857143	0.489691	0.510309	236	2213	−1977
389	389^1	Y	Y	−2	0	1.0000000	0.488432	0.511568	234	2213	−1979
390	$2^1 3^1 5^1 13^1$	Y	N	65	0	1.0000000	0.489744	0.510256	299	2278	−1979
391	$17^1 23^1$	Y	N	5	0	1.0000000	0.491049	0.508951	304	2283	−1979
392	$2^3 7^2$	N	N	−23	18	1.4782609	0.489796	0.510204	281	2283	−2002
393	$3^1 131^1$	Y	N	5	0	1.0000000	0.491094	0.508906	286	2288	−2002
394	$2^1 197^1$	Y	N	5	0	1.0000000	0.492386	0.507614	291	2293	−2002
395	$5^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	296	2298	−2002
396	$2^2 3^2 11^1$	N	N	−74	58	1.2162162	0.492424	0.507576	222	2298	−2076
397	397^1	Y	Y	−2	0	1.0000000	0.491184	0.508816	220	2298	−2078
398	$2^1 199^1$	Y	N	5	0	1.0000000	0.492462	0.507538	225	2303	−2078
399	$3^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.491228	0.508772	209	2303	−2094
400	$2^4 5^2$	N	N	34	29	1.6176471	0.492500	0.507500	243	2337	−2094
401	401^1	Y	Y	−2	0	1.0000000	0.491272	0.508728	241	2337	−2096
402	$2^1 3^1 67^1$	Y	N	−16	0	1.0000000	0.490050	0.509950	225	2337	−2112
403	$13^1 31^1$	Y	N	5	0	1.0000000	0.491315	0.508685	230	2342	−2112
404	$2^2 101^1$	N	N	−7	2	1.2857143	0.490099	0.509901	223	2342	−2119
405	$3^4 5^1$	N	N	−11	6	1.8181818	0.488889	0.511111	212	2342	−2130
406	$2^1 7^1 29^1$	Y	N	−16	0	1.0000000	0.487685	0.512315	196	2342	−2146
407	$11^1 37^1$	Y	N	5	0	1.0000000	0.488943	0.511057	201	2347	−2146
408	$2^3 3^1 17^1$	N	N	−48	32	1.3333333	0.487745	0.512255	153	2347	−2194
409	409^1	Y	Y	−2	0	1.0000000	0.486553	0.513447	151	2347	−2196
410	$2^1 5^1 41^1$	Y	N	−16	0	1.0000000	0.485366	0.514634	135	2347	−2212
411	$3^1 137^1$	Y	N	5	0	1.0000000	0.486618	0.513382	140	2352	−2212
412	$2^2 103^1$	N	N	−7	2	1.2857143	0.485437	0.514563	133	2352	−2219
413	$7^1 59^1$	Y	N	5	0	1.0000000	0.486683	0.513317	138	2357	−2219
414	$2^1 3^2 23^1$	N	N	30	14	1.1666667	0.487923	0.512077	168	2387	−2219
415	$5^1 83^1$	Y	N	5	0	1.0000000	0.489157	0.510843	173	2392	−2219
416	$2^5 13^1$	N	N	13	8	2.0769231	0.490385	0.509615	186	2405	−2219
417	$3^1 139^1$	Y	N	5	0	1.0000000	0.491607	0.508393	191	2410	−2219
418	$2^1 11^1 19^1$	Y	N	−16	0	1.0000000	0.490431	0.509569	175	2410	−2235
419	419^1	Y	Y	−2	0	1.0000000	0.489260	0.510740	173	2410	−2237
420	$2^2 3^1 5^1 7^1$	N	N	−155	90	1.1032258	0.488095	0.511905	18	2410	−2392
421	421^1	Y	Y	−2	0	1.0000000	0.486936	0.513064	16	2410	−2394
422	$2^1 211^1$	Y	N	5	0	1.0000000	0.488152	0.511848	21	2415	−2394
423	$3^2 47^1$	N	N	−7	2	1.2857143	0.486998	0.513002	14	2415	−2401
424	$2^3 53^1$	N	N	9	4	1.5555556	0.488208	0.511792	23	2424	−2401
425	$5^2 17^1$	N	N	−7	2	1.2857143	0.487059	0.512941	16	2424	−2408

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \hat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
426	$2^1 3^1 71^1$	Y	N	−16	0	1.0000000	0.485915	0.514085	0	2424	−2424
427	$7^1 61^1$	Y	N	5	0	1.0000000	0.487119	0.512881	5	2429	−2424
428	$2^2 107^1$	N	N	−7	2	1.2857143	0.485981	0.514019	−2	2429	−2431
429	$3^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	−18	2429	−2447
430	$2^1 5^1 43^1$	Y	N	−16	0	1.0000000	0.483721	0.516279	−34	2429	−2463
431	431^1	Y	Y	−2	0	1.0000000	0.482599	0.517401	−36	2429	−2465
432	$2^4 3^3$	N	N	−80	75	1.5625000	0.481481	0.518519	−116	2429	−2545
433	433^1	Y	Y	−2	0	1.0000000	0.480370	0.519630	−118	2429	−2547
434	$2^1 7^1 31^1$	Y	N	−16	0	1.0000000	0.479263	0.520737	−134	2429	−2563
435	$3^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.478161	0.521839	−150	2429	−2579
436	$2^2 109^1$	N	N	−7	2	1.2857143	0.477064	0.522936	−157	2429	−2586
437	$19^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	−152	2434	−2586
438	$2^1 3^1 73^1$	Y	N	−16	0	1.0000000	0.477169	0.522831	−168	2434	−2602
439	439^1	Y	Y	−2	0	1.0000000	0.476082	0.523918	−170	2434	−2604
440	$2^3 5^1 11^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−218	2434	−2652
441	$3^2 7^2$	N	N	14	9	1.3571429	0.476190	0.523810	−204	2448	−2652
442	$2^1 13^1 17^1$	Y	N	−16	0	1.0000000	0.475113	0.524887	−220	2448	−2668
443	443^1	Y	Y	−2	0	1.0000000	0.474041	0.525959	−222	2448	−2670
444	$2^2 3^1 37^1$	N	N	30	14	1.1666667	0.475225	0.524775	−192	2478	−2670
445	$5^1 89^1$	Y	N	5	0	1.0000000	0.476404	0.523596	−187	2483	−2670
446	$2^1 223^1$	Y	N	5	0	1.0000000	0.477578	0.522422	−182	2488	−2670
447	$3^1 149^1$	Y	N	5	0	1.0000000	0.478747	0.521253	−177	2493	−2670
448	$2^6 7^1$	N	N	−15	10	2.3333333	0.477679	0.522321	−192	2493	−2685
449	449^1	Y	Y	−2	0	1.0000000	0.476615	0.523385	−194	2493	−2687
450	$2^1 3^2 5^2$	N	N	−74	58	1.2162162	0.475556	0.524444	−268	2493	−2761
451	$11^1 41^1$	Y	N	5	0	1.0000000	0.476718	0.523282	−263	2498	−2761
452	$2^2 113^1$	N	N	−7	2	1.2857143	0.475664	0.524336	−270	2498	−2768
453	$3^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−265	2503	−2768
454	$2^1 227^1$	Y	N	5	0	1.0000000	0.477974	0.522026	−260	2508	−2768
455	$5^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476923	0.523077	−276	2508	−2784
456	$2^3 3^1 19^1$	N	N	−48	32	1.3333333	0.475877	0.524123	−324	2508	−2832
457	457^1	Y	Y	−2	0	1.0000000	0.474836	0.525164	−326	2508	−2834
458	$2^1 229^1$	Y	N	5	0	1.0000000	0.475983	0.524017	−321	2513	−2834
459	$3^3 17^1$	N	N	9	4	1.5555556	0.477124	0.522876	−312	2522	−2834
460	$2^2 5^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−282	2552	−2834
461	461^1	Y	Y	−2	0	1.0000000	0.477223	0.522777	−284	2552	−2836
462	$2^1 3^1 7^1 11^1$	Y	N	65	0	1.0000000	0.478355	0.521645	−219	2617	−2836
463	463^1	Y	Y	−2	0	1.0000000	0.477322	0.522678	−221	2617	−2838
464	$2^4 29^1$	N	N	−11	6	1.8181818	0.476293	0.523707	−232	2617	−2849
465	$3^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.475269	0.524731	−248	2617	−2865
466	$2^1 233^1$	Y	N	5	0	1.0000000	0.476395	0.523605	−243	2622	−2865
467	467^1	Y	Y	−2	0	1.0000000	0.475375	0.524625	−245	2622	−2867
468	$2^2 3^2 13^1$	N	N	−74	58	1.2162162	0.474359	0.525641	−319	2622	−2941
469	$7^1 67^1$	Y	N	5	0	1.0000000	0.475480	0.524520	−314	2627	−2941
470	$2^1 5^1 47^1$	Y	N	−16	0	1.0000000	0.474468	0.525532	−330	2627	−2957
471	$3^1 157^1$	Y	N	5	0	1.0000000	0.475584	0.524416	−325	2632	−2957
472	$2^3 59^1$	N	N	9	4	1.5555556	0.476695	0.523305	−316	2641	−2957
473	$11^1 43^1$	Y	N	5	0	1.0000000	0.477801	0.522199	−311	2646	−2957
474	$2^1 3^1 79^1$	Y	N	−16	0	1.0000000	0.476793	0.523207	−327	2646	−2973
475	$5^2 19^1$	N	N	−7	2	1.2857143	0.475789	0.524211	−334	2646	−2980
476	$2^2 7^1 17^1$	N	N	30	14	1.1666667	0.476891	0.523109	−304	2676	−2980
477	$3^2 53^1$	N	N	−7	2	1.2857143	0.475891	0.524109	−311	2676	−2987
478	$2^1 239^1$	Y	N	5	0	1.0000000	0.476987	0.523013	−306	2681	−2987
479	479^1	Y	Y	−2	0	1.0000000	0.475992	0.524008	−308	2681	−2989
480	$2^5 3^1 5^1$	N	N	−96	80	1.6666667	0.475000	0.525000	−404	2681	−3085
481	$13^1 37^1$	Y	N	5	0	1.0000000	0.476091	0.523909	−399	2686	−3085
482	$2^1 241^1$	Y	N	5	0	1.0000000	0.477178	0.522822	−394	2691	−3085
483	$3^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−410	2691	−3101
484	$2^2 11^2$	N	N	14	9	1.3571429	0.477273	0.522727	−396	2705	−3101
485	$5^1 97^1$	Y	N	5	0	1.0000000	0.478351	0.521649	−391	2710	−3101
486	$2^1 3^5$	N	N	13	8	2.0769231	0.479424	0.520576	−378	2723	−3101
487	487^1	Y	Y	−2	0	1.0000000	0.478439	0.521561	−380	2723	−3103
488	$2^3 61^1$	N	N	9	4	1.5555556	0.479508	0.520492	−371	2732	−3103
489	$3^1 163^1$	Y	N	5	0	1.0000000	0.480573	0.519427	−366	2737	−3103
490	$2^1 5^1 7^2$	N	N	30	14	1.1666667	0.481633	0.518367	−336	2767	−3103
491	491^1	Y	Y	−2	0	1.0000000	0.480652	0.519348	−338	2767	−3105
492	$2^2 3^1 41^1$	N	N	30	14	1.1666667	0.481707	0.518293	−308	2797	−3105
493	$17^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	−303	2802	−3105
494	$2^1 13^1 19^1$	Y	N	−16	0	1.0000000	0.481781	0.518219	−319	2802	−3121
495	$3^2 5^1 11^1$	N	N	30	14	1.1666667	0.482828	0.517172	−289	2832	−3121
496	$2^4 31^1$	N	N	−11	6	1.8181818	0.481855	0.518145	−300	2832	−3132
497	$7^1 71^1$	Y	N	5	0	1.0000000	0.482897	0.517103	−295	2837	−3132
498	$2^1 3^1 83^1$	Y	N	−16	0	1.0000000	0.481928	0.518072	−311	2837	−3148
499	499^1	Y	Y	−2	0	1.0000000	0.480962	0.519038	−313	2837	−3150
500	$2^2 5^3$	N	N	−23	18	1.4782609	0.480000	0.520000	−336	2837	−3173