$$C_{\Omega(x)}(x)\left[\Omega(x)=k\right]_{\delta} \sim 2\sqrt{2\pi\log\log x}\times \widehat{C}'_{k,*}(x)(1+o(1)) =: \widehat{C}_{k,**}(x).$$

We have that

$$\widehat{C}_{k,**}(x) \sim -2\sqrt{2\pi\log\log x}\left[\frac{(\log\log x)^{k-1}}{(\log x)(k-1)!}\left(1-\frac{1}{\log x}\right)+\frac{(\log\log x)^{k-2}}{(\log x)^2(k-2)!}\right].$$

Hence, integration by parts and Proposition A.2 yield the main term

$$\begin{aligned}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n) &\sim \left|\int \widehat{C}_{k,**}(x)dx\right| \qquad (19)\\ &\sim \frac{4\sqrt{2\pi}\cdot x(\log\log x)^{k-1/2}}{(2k-1)(k-1)!}+\frac{2\sqrt{2\pi}\cdot x\Gamma\left(k-\frac{1}{2},\log\log x\right)}{(k-1)!}-\frac{2\sqrt{2\pi}\cdot x\Gamma\left(k-\frac{3}{2},\log\log x\right)}{(k-1)!}\\ &\sim \frac{4\sqrt{2\pi}\cdot x(\log\log x)^{k-1/2}}{(2k-1)(k-1)!}.\end{aligned}$$

□

4.2 Average order of the unsigned sequences

Proposition 4.5. *We have that as $n\to\infty$*

$$\mathbb{E}\left[C_{\Omega(n)}(n)\right]=\frac{\sqrt{2\pi}(\log n)}{\sqrt{\log\log n}}(1+o(1)).$$

Proof. We first compute the following summatory function by applying Corollary 4.4 and Lemma A.5 from the appendix:

$$\sum_{k=1}^{\mathbf{2}\log\log x}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n) \sim \frac{\sqrt{2\pi}\cdot x(\log x)}{\sqrt{\log\log x}}. \qquad (20)$$

We claim that

$$\begin{aligned}\frac{1}{x}\times\sum_{n\leq x} C_{\Omega(n)}(n) &= \frac{1}{x}\times\sum_{k\geq 1}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n)\\ &= \frac{1}{x}\times\sum_{k=1}^{\mathbf{2}\log\log x}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n)(1+o(1)),\text{ as } x\to\infty. \qquad (21)\end{aligned}$$

To prove (21) it suffices to show that

$$\frac{1}{x}\times\sum_{k>2\log\log x}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n) = O\left((\log x)^{\mathbf{0.613706}}\times(\log\log x)\right),\text{ as } x\to\infty. \qquad (22)$$

We proved in Theorem 4.1 that for all sufficiently large x

$$\sum_{n\leq x}(-1)^{\omega(n)}C_{\Omega(n)}(n)z^{\Omega(n)}=\frac{x\widehat{F}(2,z)}{\Gamma(z)}(\log x)^{z-1}+O\left(x(\log x)^{\mathrm{Re}(z)-2}\right).$$

By (18) we have that the summatory function

$$\left|\sum_{n\leq x}(-1)^{\omega(n)}\right| = \frac{x}{2\sqrt{2\pi\log\log x}}+O\left(\frac{x}{(\log\log x)^{3/2}}\right).$$

We can argue as in the proof of Corollary 4.4 using integration by parts with the Abel summation formula that whenever $1 < |z| < P(2)^{-1}$ and $x > e$ is sufficiently large we have

$$\begin{aligned}\sum_{n \leq x} C_{\Omega(n)}(n) z^{\Omega(n)} &\ll \frac{\widehat{F}(2,z)}{\Gamma(z)} \times \int_e^x \frac{\sqrt{\log\log t}}{t} \frac{\partial}{\partial t}\left[t(\log t)^{z-1}\right] dt \\ &\ll \frac{x\widehat{F}(2,z)}{\Gamma(z)}\Bigg[\frac{(\log x)^{z-1}(z+\log x)}{z}\sqrt{\log\log x} - \frac{\sqrt{\pi}}{2\sqrt{z-1}} \operatorname{erfi}\left(\sqrt{(z-1)\log\log x}\right) \\ &\qquad - \frac{\sqrt{\pi}}{2z^{3/2}} \operatorname{erfi}\left(\sqrt{z\log\log x}\right)\Bigg] \\ &\ll \frac{x\widehat{F}(2,z)}{\Gamma(1+z)}(\log x)^z \sqrt{\log\log x}. \end{aligned} \tag{23}$$

The dropped error term in the last formula follows from the asymptotic series for $\operatorname{erfi}(z)$ in (24). Namely, as $|z| \to \infty$, the *imaginary error function*, denoted by $\operatorname{erfi}(z)$, has the following asymptotic expansion [19, §7.12]:

$$\operatorname{erfi}(z) := \frac{2}{\sqrt{\pi} \cdot \imath} \times \int_0^{\imath z} e^{t^2} dt = \frac{e^{z^2}}{\sqrt{\pi}}\left(z^{-1} + \frac{z^{-3}}{2} + \frac{3z^{-5}}{4} + \frac{15z^{-7}}{8} + O\left(z^{-9}\right)\right). \tag{24}$$

For all large enough $x > e$, we define

$$\widehat{B}(x,r) := \sum_{\substack{n \leq x \\ \Omega(n) \geq r\log\log x}} C_{\Omega(n)}(n).$$

We argue as in the proof from the reference [13, *cf.* Thm. 7.20; §7.4] applying (23) that for $1 \leq r < P(2)^{-1}$

$$\begin{aligned}\sum_{\substack{n \leq x \\ \Omega(n) \geq r\log\log x}} C_{\Omega(n)}(n) r^{\Omega(n)} &\ll x(\log x)^{-r\log r} \times \sum_{n \leq x} C_{\Omega(n)}(n) r^{\Omega(n)} \\ &\sim \frac{x\widehat{F}(2,z)}{\Gamma(1+z)}\sqrt{\log\log x}(\log x)^{r-r\log r}.\end{aligned}$$

Since $\widehat{F}(2,r) = \frac{\zeta(2)^{-r}}{1+P(2)r} \ll 1$ for $r \in [1, P(2)^{-1})$, and similarly since we have that $\frac{1}{\Gamma(1+r)} \gg 1$ for r taken within this same range, we obtain

$$\sum_{\substack{n \leq x \\ \Omega(n) \geq r\log\log x}} C_{\Omega(n)}(n) r^{\Omega(n)} \ll x\sqrt{\log\log x} \times (\log x)^{r-r\log r}, \text{ for all } 1 \leq r < P(2)^{-1}.$$

When $1 \leq r < P(2)^{-1}$ we also have

$$x\sqrt{\log\log x}(\log x)^{r-r\log r} \gg \sum_{\substack{n \leq x \\ \Omega(n) \geq r\log\log x}} C_{\Omega(n)}(n) r^{\Omega(n)} \gg \sum_{\substack{n \leq x \\ \Omega(n) \geq r\log\log x}} C_{\Omega(n)}(n) r^{r\log\log x}.$$

This implies that for $\mathbf{r} := \mathbf{2}$ we have

$$\widehat{B}(x,r) \ll x(\log x)^{r-2r\log r}\sqrt{\log\log x} = O\left(x(\log x)^{\mathbf{0.613706}} \times \sqrt{\log\log x}\right) \tag{25}$$

We wish to evaluate the limiting asymptotics of the sum

$$S_2(x) := \frac{1}{x\sqrt{\log\log x}} \times \sum_{k \geq \mathbf{2}\log\log x} \sum_{\substack{n \leq x \\ \Omega(n) = k}} C_{\Omega(n)}(n) \ll \widehat{B}(x,2).$$

We have proved that $S_2(x)\sqrt{\log\log x} = O\left((\log x)^{\mathbf{0.61306}}(\log\log x)\right)$ as $x \to \infty$, as claimed. □