New characterizations of partial sums of the Möbius function

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Abstract

The Mertens function, $M(x) \coloneqq \sum_{n \leq x} \mu(n)$, is defined as the summatory function of the classical Möbius function for $x \geq 1$. The inverse sequence $\{g^{-1}(n)\}_{n \geq 1}$ taken with respect to Dirichlet convolution is defined in terms of the strongly additive function $\omega(n)$ that counts the number of distinct prime factors of any integer $n \geq 2$ without considering multiplicity. For large x and $n \leq x$, we associate a natural combinatorial significance to the magnitude of the distinct values of the function $g^{-1}(n)$ that depends directly on the exponent patterns in the prime factorizations of the integers in $\{2, 3, \ldots, x\}$ viewed as multisets.

We prove an Erdős-Kac theorem analog for the distribution of the unsigned sequence $|g^{-1}(n)|$ over $n \leq x$ with a central limit theorem tendency towards normal as $x \to \infty$. For all $x \geq 1$, discrete convolutions of the summatory function $G^{-1}(x) \coloneqq \sum_{n \leq x} \lambda(n) |g^{-1}(n)|$ with the prime counting function $\pi(x)$ determine exact formulas and new characterizations of asymptotic bounds for $M(x)$. In this way, we prove another concrete link to the distribution of $L(x) \coloneqq \sum_{n \leq x} \lambda(n)$ with the Mertens function and connect these classical summatory functions with an explicit normal tending probability distribution at large x. The proofs of these resulting combinatorially motivated new characterizations of $M(x)$ are rigorous and unconditional.

Keywords and Phrases: *Möbius function; Mertens function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function; Erdős-Kac theorem; strongly additive function.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; 11N64; and 11-04.*

"It is evident that the primes are randomly distributed but, unfortunately, we do not know what 'random' means." – **R. C. Vaughan**

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1 Introduction

1.1 Preliminaries

1.1.1 Definitions

We define the *Möbius function* to be the signed indicator function of the squarefree integers in the form of [25, A008683]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \wedge n \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined on the positive integers as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as follows [25, A002321]:

$$\{M(x)\}_{x\geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \ldots\}.$$

The Mertens function satisfies that $\sum_{n\leq x} M\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = 1$, and is related to the summatory function $L(x) := \sum_{n\leq x} \lambda(n)$ via the relation [6, 12]

$$L(x) = \sum_{d\leq\sqrt{x}} M\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), x \geq 1.$$

A positive integer $n \geq 1$ is *squarefree*, or contains no divisors which are squares (other than one when $n \geq 2$), if and only if $\mu^2(n) = 1$. The summatory function that counts the number of *squarefree* integers $n \leq x$ satisfies [5, §18.6] [25, A013928]

$$Q(x) = \sum_{n\leq x} \mu^2(n) = \frac{6x}{\pi^2} + O\left(\sqrt{x}\right), \text{ as } x \to \infty.$$

1.1.2 Properties

A conventional approach to evaluating the limiting asymptotic behavior of $M(x)$ for large $x \to \infty$ considers an inverse Mellin transformation of the reciprocal of the Riemann zeta function. In particular, since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \times \int_1^\infty \frac{M(x)}{x^{s+1}} dx, \operatorname{Re}(s) > 1,$$

we obtain that

$$M(x) = \lim_{T\to\infty} \frac{1}{2\pi\imath} \times \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s\zeta(s)} ds.$$

The previous two representations lead us to the exact expression of $M(x)$ for any $x > 0$ given by the next theorem.

Theorem 1.1 (Titchmarsh)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k\geq 1}$ satisfying $k \leq T_k \leq k+1$ for each integer $k \geq 1$ such that for any real $x > 0$*

$$M(x) = \lim_{k\to\infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |\operatorname{Im}(\rho)|<T_k}} \frac{x^\rho}{\rho\zeta'(\rho)} - 2 + \sum_{n\geq 1} \frac{(-1)^{n-1}}{n(2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} [x \in \mathbb{Z}^+]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C_1 > 0$ such that

$$M(x) \ll x \times \exp\left(-C_1 \log^{\frac{3}{5}}(x)(\log\log x)^{-\frac{3}{5}}\right).$$

Under the assumption of the RH, Soundararajan and Humphries, respectively, improved estimates bounding $M(x)$ from above for large x in the following form for any fixed $\epsilon > 0$ [26, 6]:

$$M(x) \ll \sqrt{x} \cdot \exp\left((\log x)^{\frac{1}{2}}(\log\log x)^{14}\right),$$
$$M(x) = O\left(\sqrt{x} \times \exp\left((\log x)^{\frac{1}{2}}(\log\log x)^{\frac{5}{2}+\epsilon}\right)\right).$$

1.1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$ for any $0 < \epsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C_2\sqrt{x}, \quad \text{for some absolute constant } C_2 > 0.$$

The conjecture was first verified by Mertens himself for $C_2 = 1$ and all $x < 10000$ without the benefit of modern computation. Since its beginnings in 1897, the Mertens conjecture was disproved by computational methods with non-trivial simple zeta function zeros with comparatively small imaginary parts in the famous paper by Odlyzko and te Riele [18]. More recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $q(x) := M(x)/\sqrt{x}$ grows with or without bound along infinite subsequences, e.g., considering the asymptotics of $q(x)$ in the limit supremum and limit infimum senses.

It is verified by computation that [21, *cf.* §4.1] [25, *cf.* A051400; A051401]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leq -1.837625).$$

Based on the work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [18, 10, 11, 7]. A famous conjecture due to Gonek asserts that in fact $M(x)$ satisfies [17]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}(\log\log\log x)^{\frac{5}{4}}} = O(1).$$

1.2 A concrete new approach to characterizing $M(x)$

The new characterizations of $M(x)$ by the methods in this article stem from a standing curiosity about an under utilized elementary identity from the list of exercises in [1, §2; *cf.* §11]. We see that the indicator function of the primes is given by Möbius inversion as the Dirichlet convolution $\chi_{\mathbb{P}} = \omega * \mu$, or equivalently as $\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu$. We form the summatory function of $(\omega + 1) * \mu(n)$ over $n \leq x$ for any $x \geq 1$ and then apply classical style inversion theorems to exactly relate $M(x)$ to the partial sums of $g^{-1}(n) := (\omega+1)^{-1}(n)$. There is a natural relationship of $g^{-1}(n)$ with the auxiliary function $C_{\Omega(n)}(n)$ (defined below) that we prove by elementary methods in Section 3. These identities inspire a deep connection between the unsigned inverse function and additive prime counting combinatorics. In this sense, the new results stated within this article diverge from the proof templates typified by previous methods to bound $M(x)$ cited in the references. Our characterizations of $M(x)$ by the summatory function of the signed inverse sequence,

$G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, is suggestive of new approaches to bounding the Mertens function by considering properties of the latter partial sums. Our results motivate future work that will provide new upper (and possibly lower) bounds on $M(x)$ through the additive combinatorial properties of the repeated distinct values of the sign weighted summands of $G^{-1}(x)$ (see Section 3.3).

We are informed by the modern results in [13, §7.4; §2.4] that apply traditional analytic methods to formulate limiting asymptotics and to directly give a rigorous proof of an Erdős-Kac theorem for the completely additive function $\Omega(n)$. Adaptations of the key ideas in the expository proofs from the reference provide us with a foundation for analytic proofs of several limiting properties of, asymptotic formulae for restricted partial sums involving, and a limiting Erdős-Kac type theorem for $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$. The sequence $g^{-1}(n)$ and its summatory function $G^{-1}(x)$ are crucially tied to canonical number theoretic examples of strongly and completely additive functions, e.g., to $\omega(n)$ and $\Omega(n)$, respectively. The definitions of the auxiliary sequences we define, and the proof methods given in the spirit of Montgomery and Vaughan's work, allow us to reconcile the property of strong additivity with the signed partial sums of a multiplicative function. The results we obtain at the conclusion of Section 4 are then á priori more predictable as strong additivity leads to explicit probability measures and central limit type theorems that predict the distributions of functions of this type over $n \leq x$ as $x \to \infty$ [9, *cf.* Thm. 1.5; §1.7]. In effect, we make progress on the initially elementary problem at hand by leveraging the connection of $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$ with the canonically additive functions to obtain our results proved in Section 4. We expect that an outline of the method behind the collective proofs we provide with respect to the Mertens function special case can be generalized to identify another associated additive function with the role of $\omega(n)$ here to express asymptotics for partial sums of other multiplicative functions.

While we primarily rely on the analytic proof methodology presented in [13], and require the statements of several asymptotic formulas and bounds related to $\omega(n)$ and $\Omega(n)$ from the reference, most, if not all, of the results given in this manuscript are new. The function $C_{\Omega(n)}(n)$ was considered under alternate notation by the work of Fröberg (circa 1968) on the series expansions of the *prime zeta function*, $P(s)$, e.g., the prime sums defined as the DGF of $\chi_{\mathbb{P}}(n)$. The clear interpretation of the function in connection with $M(x)$ is unique to our work on the properties this auxiliary function. Explicit references to uniform asymptotics for restricted partial sums of $C_{\Omega(n)}(n)$ and the probabilistic features of the limiting distribution of this function are tenuous in number theoretic literature. We also utilize techniques from recent work over the last few years in the appendix section to evaluate asymptotics of several special partial sums we require in terms of the upper incomplete gamma function $\Gamma(a, z)$ as both parameters a, z simultaneously tend to infinity at a rate proportional to $\log\log x$ [14, 15, 16].

The main interpretation to take away from the article is our rigorous motivation of an equivalent characterization of $M(x)$ using two new number theoretic sequences and their summatory functions. This characterization is formed by constructing combinatorially relevant sequences related to the distribution of the primes through convolutions of strongly additive functions. In particular, our new perspective offers exact formulas for $M(x)$ at any $x \geq 1$ through discrete convolutions of $G^{-1}(x)$ with the prime counting function $\pi(x)$ (see Section 5). Since we prove that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ for all $n \geq 1$, it follows that we have an updated probabilistic perspective from which to express our intuition about features of the distribution of the summatory function $G^{-1}(x)$ at large x. The partial sums defined by $G^{-1}(x)$ are related precisely to the properties of $|g^{-1}(n)|$ and asymptotics for $L(x) := \sum_{n \leq x} \lambda(n)$. The new results in this article then relate the distribution of $L(x)$, an explicitly identified normal tending probability distribution, and $M(x)$ as $x \to \infty$. Formalizing the properties of the distribution of $L(x)$ is still typically viewed as a problem that is equally as difficult as understanding the properties of $M(x)$ well at large x or along infinite subsequences.

1.2.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 1.2 (Summatory functions of Dirichlet convolutions)**.** *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory* *functions of f and h, respectively, and that $F^{-1}(x) := \sum_{n\leq x} f^{-1}(n)$ denotes the summatory function of the Dirichlet inverse of f for any $x \geq 1$. We have the following exact expressions for the summatory function of the convolution $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned}\pi_{f*h}(x) &:= \sum_{n\leq x}\sum_{d|n} f(d)h\left(\frac{n}{d}\right) \\ &= \sum_{d\leq x} f(d) H\left(\left\lfloor \frac{x}{d}\right\rfloor\right) \\ &= \sum_{k=1}^{x} H(k)\left[F\left(\left\lfloor \frac{x}{k}\right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1}\right\rfloor\right)\right].\end{aligned}$$

Moreover, for all $x \geq 1$

$$\begin{aligned}H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j)\left[F^{-1}\left(\left\lfloor \frac{x}{j}\right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1}\right\rfloor\right)\right] \\ &= \sum_{k=1}^{x} f^{-1}(k)\pi_{f*h}\left(\left\lfloor \frac{x}{k}\right\rfloor\right).\end{aligned}$$

Corollary 1.3 (Applications of Möbius inversion)**.** *Suppose that h is an arithmetic function such that $h(1) \neq 0$. Define the summatory function of the convolution of h with μ by $\widetilde{H}(x) := \sum_{n\leq x}(h * \mu)(n)$. Then the Mertens function is expressed by the sum*

$$M(x) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} h^{-1}(j)\right)\widetilde{H}(k), \forall x \geq 1.$$

Corollary 1.4 (Key Identity)**.** *We have that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^{x}(\omega+1)^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right) + 1\right]. \tag{1}$$

1.2.2 An exact expression for $M(x)$ via strongly additive functions

Fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n)$ [25, A341444]. We can compute exactly that (see Table B on page 40)

$$\{g^{-1}(n)\}_{n\geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

There is not a simple direct recursion between the distinct values of $g^{-1}(n)$ that holds for all $n \geq 1$. The distribution of distinct sets of prime exponents is still clearly quite regular since $\omega(n)$ and $\Omega(n)$ play a crucial role in the repetition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of this inverse function over $n \geq 2$:

Heuristic 1.5 (Symmetry in $g^{-1}(n)$ from the prime factorizations of $n \leq x$)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1}\cdots q_r^{\beta_r}$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly one, two, three (and so on) prime factors (*cf.* Section 3.3).

Conjecture 1.6 (Characteristic properties of the inverse sequence)**.** *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

- ***(A)*** *For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*

(B) *For all squarefree integers $n \geq 2$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \times m!;$$

(C) *If $n \geq 2$ and $\Omega(n) = k$ for some $k \geq 1$, then*

$$2 \leq |g^{-1}(n)| \leq \sum_{j=0}^{k} \binom{k}{j} \times j!.$$

The signedness property in (A) is proved precisely in Proposition 2.1. A proof of (B) follows from Lemma 3.1 stated on page 16.

The realization that the beautiful and remarkably simple combinatorial form of property (B) in Conjecture 1.6 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through the sums of auxiliary subsequences $C_k(n)$ with $k := \Omega(n)$ defined in Section 3. That is, we observe a familiar formula for $g^{-1}(n)$ on an asymptotically dense infinite subset of integers (with density $\frac{6}{\pi^2}$), e.g., that holds for all squarefree $n \geq 2$, and then seek to extrapolate by proving there are regular tendencies of the distribution of this sequence viewed more generally over any $n \geq 2$.

An exact expression for $g^{-1}(n)$ is given by

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d), n \geq 1,$$

where the sequence $\lambda(n)C_{\Omega(n)}(n)$ has DGF $(P(s)+1)^{-1}$ for $\operatorname{Re}(s) > 1$ (see Proposition 2.1). The function $C_{\Omega(n)}(n)$ is previously considered in [4] with its exact formula given by (*cf.* [8])

$$C_{\Omega(n)}(n) = \begin{cases} 1, & \text{if } n = 1; \\ (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}, & \text{if } n \geq 2. \end{cases}$$

In Corollary 4.3, we use the result proved in Theorem 4.2 to show uniformly for $1 \leq k \leq 2 \log\log x$ that there is an absolute constant $A_0 > 0$ such that

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \sim \frac{4A_0\sqrt{2\pi} \cdot x}{(2k-1)} \cdot \frac{(\log\log x)^{k-\frac{1}{2}}}{(k-1)!}, \text{ as } x \to \infty.$$

In Proposition 4.5, we use an adaptation of the asymptotic formulas for the summations proved in the appendix combined with the form of *Rankin's method* from [13, Thm. 7.20] to show that

$$\mathbb{E}[C_{\Omega(n)}(n)] \sim \frac{2A_0\sqrt{2\pi}(\log n)}{\sqrt{\log\log n}}, \text{ as } n \to \infty.$$

In Corollary 4.6, we then prove that the average order of the unsigned inverse sequence is

$$\mathbb{E}|g^{-1}(n)| = \frac{12A_0}{\pi} \cdot \frac{(\log n)^2}{\sqrt{\log\log n}}(1+o(1)), \text{ as } n \to \infty.$$

In Section 4.3, we prove a variant of the Erdős-Kac theorem that characterizes the distribution of the sequence $C_{\Omega(n)}(n)$. The theorem leads the conclusion of the following statement for any fixed $Y > 0$, with $\mu_x(C) := \log\log x - \log\left(4A_0\sqrt{2\pi}\right)$ and $\sigma_x(C) := \sqrt{\log\log x}$, that holds uniformly for any $-Y \leq y \leq Y$ as $x \to \infty$ (see Corollary 4.8):

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : |g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq y\right\} = \Phi\left(\frac{\frac{\pi^2 y}{6} - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right).$$

The regularity and quasi-periodicity we have alluded to in the remarks above are then quantifiable in so much as the distribution of $|g^{-1}(n)|$ for $n \leq x$ tends to (a predictable multiple of) its average order with a normal tendency depending on x as $x \to \infty$. That is, if $x > e$ is sufficiently large and if we pick any integer $n \in [2, x]$ uniformly at random, then the following statement holds:

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\left(\alpha\sigma_x(C) + \mu_x(C)\right)\right) = \Phi(\alpha) + o(1), \alpha \in \mathbb{R}. \tag{D}$$

It follows from the last property that as $n \to \infty$,

$$|g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|(1 + o(1)),$$

on an infinite set of the integers with asymptotic density of one over the positive integers.

1.2.3 Formulas illustrating the new characterizations of $M(x)$

Let the summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ for integers $x \geq 1$ [25, A341472]. We prove that (see Proposition 5.2)

$$M(x) = G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right], x \geq 1. \tag{2}$$

This formula implies that we can establish new asymptotic bounds on $M(x)$ along infinite subsequences by sharply bounding the summatory function $G^{-1}(x)$ at those points. The take on the regularity of $|g^{-1}(n)|$ is as such imperative to our arguments that formally bound the growth of $M(x)$ by its new identification with $G^{-1}(x)$. A combinatorial approach to summing $G^{-1}(x)$ for large x based on the distribution of the primes is outlined in our remarks in Section 3.3.

Theorem 5.1 proves that for almost every sufficiently large x there exists some $1 \leq t_0 \leq x$ such that [A]

$$G^{-1}(x) = O\left(L(t_0) \times \mathbb{E}|g^{-1}(x)|\right).$$

If the RH is true, then for every $\epsilon > 0$ and all sufficiently large x we have that

$$G^{-1}(x) = O\left(\frac{\sqrt{x}(\log x)^2}{\sqrt{\log\log x}} \times \exp\left(\sqrt{\log x}(\log\log x)^{\frac{5}{2}+\epsilon}\right)\right).$$

In Corollary 5.4, we also prove that

$$M(x) = O\left(G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) + \frac{x}{\log x} \times \sum_{k \leq \sqrt{x}} \frac{G^{-1}(k)}{k^2} + (\log x)^2\sqrt{\log\log x}\right).$$

A complete discussion of the properties of the summatory functions $G^{-1}(x)$ motivates more study in future work to extend the full range of possibilities for viewing the new structure behind $M(x)$ we identify within this article. The prime-related combinatorics at hand are again discussed in Section 3.3.

1.3 Notation and conventions

The next listing provides a glossary of common notation, conventions and abbreviations used throughout the article.

[A] By the terminology *almost every* large integer x, we mean that the result holds for all large x taken within an infinite subset of $\mathbb{Z}^+$ with asymptotic density one.

Symbol	**Definition**
$\approx, \sim$	We write that $f(x) \approx g(x)$ if $\lvert f(x) - g(x)\rvert = O(1)$ as $x \to \infty$. Two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1$.
$\mathbb{E}[f(x)]$	We use the expectation notation of $\mathbb{E}[f(x)] = h(x)$ to denote that f has an *average order* of $h(x)$. This means that $\frac{1}{x} \times \sum_{n\leq x} f(n) \sim h(x)$.
$\chi_{\mathbb{P}}(n)$	The characteristic (or indicator) function of the primes equals one if and only if $n \in \mathbb{Z}^+$ is prime, and is zero-valued otherwise.
$C_k(n), C_{\Omega(n)}(n)$	The sequence is defined recursively for integers $n \geq 1$ and $k \geq 0$ as follows: $$C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d\mid n} \omega(d) C_{k-1}\left(\frac{n}{d}\right), & \text{if } k \geq 1. \end{cases}$$ It represents the multiple (k-fold) convolution of the function $\omega(n)$ with itself.
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function (OGF) of some sequence, $\{f_n\}_{n\geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n\geq 0} f_n q^n$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic function f we have that $f * \varepsilon = \varepsilon * f = f$ where the operation $*$ denotes Dirichlet convolution (see definition below).
$f * g$	The Dirichlet convolution of f and g is denoted by $(f*g)(n) := \sum\limits_{d\mid n} f(d) g\left(\frac{n}{d}\right)$ where the sum is taken over the divisors of any $n \geq 1$.
$f^{-1}(n)$	The Dirichlet inverse f^{-1} of any arithmetic function f exists if and only if $f(1) \neq 0$. The Dirichlet inverse of any f such that $f(1) \neq 0$ is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ for $n \geq 2$ with $f^{-1}(1) = 1/f(1)$. When it exists, this inverse function is unique and satisfies the characteristic relations that $f^{-1} * f = f * f^{-1} = \varepsilon$.
$\gg, \ll, \asymp$	For functions A, B, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A, B \geq 0$, $A \ll B$ and $B \ll A$, we write $A \asymp B$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$.
$[n = k]_{\delta}, [\mathtt{cond}]_{\delta}$	The symbol $[n = k]_{\delta}$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise. For boolean-valued conditions, `cond`, the symbol $[\mathtt{cond}]_{\delta}$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention*.
$\lambda(n), L(x)$	The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. Its summatory function is defined by $L(x) := \sum_{n\leq x} \lambda(n)$.

Symbol	**Definition**
$\mu(n), M(x)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers $n \geq 1$ where $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree. The Mertens function is the summatory function defined for all integers $x \geq 1$ by $M(x) := \sum\limits_{n \leq x} \mu(n)$.
$\Phi(z)$	For $z \in \mathbb{R}$, we define the CDF of the standard normal distribution to be $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^{\alpha} \| n$ (that is, when p^{α} exactly divides n) for $p \geq 2$ prime, $\alpha \geq 1$ and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p \mid n} 1$ and the completely additive function $\Omega(n) := \sum_{p^{\alpha} \| n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention we set $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	For integers $k \geq 1$, the prime counting function variant $\pi_k(x)$ denotes the number of $2 \leq n \leq x$ with exactly k distinct prime factors: $\pi_k(x) := \#\{2 \leq n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{2 \leq n \leq x : \Omega(n) = k\}$ for $x \geq 2$ and fixed $k \geq 1$.
$P(s)$	For complex s with $\operatorname{Re}(s) > 1$, we define the prime zeta function to be the Dirichlet generating function (DGF) $P(s) = \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s} = \sum_{k \geq 2} \frac{\mu(k)}{k} \log \zeta(ks)$.
$Q(x)$	For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. That is, $Q(x) := \sum_{n \leq x} \mu^2(n)$.
$W(x)$	For $x, y \in \mathbb{R}_{\geq 0}$, we write that $x = W(y)$ if and only if $xe^x = y$. This function denotes the principal branch of the multi-valued Lambert W function defined on the non-negative reals.
$\zeta(s)$	The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\operatorname{Re}(s) > 1$, and by analytic continuation on the rest of the complex plane with the exception of a simple pole at $s = 1$ of residue one.