# New characterizations of the summatory function of the Möbius function

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**Abstract**

The Mertens function, $M(x) := \sum_{n \leq x} \mu(n)$, is defined as the summatory function of the Möbius function for $x \geq 1$. The inverse function sequence $\{g^{-1}(n)\}_{n \geq 1}$ taken with respect to Dirichlet convolution is defined in terms of the strongly additive function $\omega(n)$ that counts the number of distinct prime factors of any integer $n \geq 2$. For large $x$ and $n \leq x$, we associate a natural combinatorial significance to the magnitude of the distinct values of the function $g^{-1}(n)$ that depends directly on the exponent patterns in the prime factorizations of the integers in $\{2, 3, \ldots, x\}$ viewed as multisets. We prove an Erdös-Kac theorem analog for the distribution of the unsigned sequence $|g^{-1}(n)|$ with a limiting central limit theorem type tendency towards normal as $x \to \infty$. For all $x \geq 1$, discrete convolutions of $G^{-1}(x) := \sum_{n \leq x} \lambda(n)|g^{-1}(n)|$ with the prime counting function $\pi(x)$ determine exact formulas and asymptotic bounds for $M(x)$. In this way, we prove another concrete link between the distributions of both $L(x) := \sum_{n \leq x} \lambda(n)$ and the Mertens function and connect these classical summatory functions with sums weighted by an explicitly identified normal tending probability distribution at large $x$. The proofs of these resulting combinatorially motivated new characterizations of $M(x)$ we give in the article are rigorous and unconditional.

**Keywords and Phrases:** *Möbius function; Mertens function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function (DGF); Erdös-Kac theorem; strongly additive function.*
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