

Lower bounds on the summatory function of the Möbius function along infinite subsequences

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Last Revised: Thursday 16th July, 2020 @ 18:50:05 – Compiled with L^AT_EX2e

Abstract

The Mertens function, $M(x) = \sum_{n \leq x} \mu(n)$, is classically defined as the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture states that $|M(x)| < C \cdot \sqrt{x}$ for some absolute $C > 0$ for all $x \geq 1$. This classical conjecture has a well-known disproof due to Odlyzko and té Riele by computation of non-trivial zeta function zeros in conjunction with integral formulas expressing $M(x)$. We prove the unboundedness of $|M(x)|/\sqrt{x}$ using new methods by showing that (TODO)

$$\limsup_{x \rightarrow \infty} \frac{|M(x)| \sqrt{\log \log x} \cdot (\log \log \log x)^2}{\sqrt{x} \cdot (\log x)^{\frac{1}{4}}} > 0.$$

There is a distinct stylistic flavor and new element of combinatorial analysis to our proof combined with the standard methods from analytic, additive and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

Keywords and Phrases: *Möbius function; Mertens function; summatory function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting functions; Dirichlet generating function; asymptotic lower bounds; Mertens conjecture.*

Math Subject Classifications (MSC 2010): 11N37; 11A25; 11N60; and 11N64.

Glossary of special notation and conventions

Symbol	Definition
\approx	We write that $f(x) \approx g(x)$ if $ f(x) - g(x) = O(1)$ as $x \rightarrow \infty$.
$\mathbb{E}[f(x)], \sim^{\mathbb{E}}$	We adapt the expectation notation $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \sim^{\mathbb{E}} h(x)$, to denote that f has an <i>average order</i> growth rate of $h(x)$. This means that $\frac{1}{x} \sum_{n \leq x} f(n) \sim h(x)$, or equivalently that $\lim_{x \rightarrow \infty} \frac{\frac{1}{x} \sum_{n \leq x} f(n)}{h(x)} = 1.$
B	The absolute constant $B \approx 0.2614972$ from the statement of Mertens theorem.
$C_k(n)$	The sequence is defined recursively for $n \geq 1$ as follows where we assume that $1 \leq k \leq \Omega(n)$: $C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum_{d n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n \geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n \geq 0} f_n q^n$.
d_k	For non-negative integers $k \geq 0$, we define the densities d_k of the distinct values of the differences of the prime omega functions by $d_k := \lim_{x \rightarrow \infty} \frac{1}{x} \cdot \#\{n \leq x : \Omega(n) - \omega(n) = k\}$. These limiting densities satisfy $d_k = \hat{c} 2^{-k} + O(5^{-k})$ for an absolute constant $\hat{c} \approx 1.5147$. Their generating function corresponds to the prime product DGF expansion given by $d_k = [z^k] \prod_p \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p-z}\right)$ for any $k \geq 0$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (see below).
$f * g$	The Dirichlet convolution of f and g , $(f * g)(n) := \sum_{d n} f(d)g(n/d)$, where the sum is taken over the divisors d of n for $n \geq 1$.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d n \\ d > 1}} f(d) f^{-1}(n/d)$ for $n \geq 2$ with $f^{-1}(1) = 1/f(1)$. The Dirichlet inverse of f exists if and only if $f(1) \neq 0$. This inverse function, denoted by f^{-1} when it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$.
\gg, \ll, \asymp	For functions A, B in x , the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A \ll B$ and $B \gg A$, we write $A \asymp B$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$.

Symbol	Definition
H_n	<p>The <i>first-order harmonic numbers</i>, $H_n := \sum_{k=1}^n \frac{1}{k}$, satisfy the limiting asymptotic relation</p> $\lim_{n \rightarrow \infty} [H_n - \log(n)] = \gamma,$ <p>where $\gamma \approx 0.5772157$ denotes Euler's gamma constant.</p>
$[n = k]_\delta, [\text{cond}]_\delta$	<p>The symbol $[n = k]_\delta$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise. For boolean-valued conditions, cond, $[\text{cond}]_\delta$ evaluates to one precisely when cond is true, and to zero otherwise. This notation is called <i>Iverson's convention</i>.</p>
$\lambda_*(n)$	<p>For positive integers $n \geq 2$, we define the next variant of the Liouville lambda function, $\lambda(n)$, as follows: $\lambda_*(n) := (-1)^{\Omega(n) - \omega(n)} = \lambda(n)(-1)^{\omega(n)}$. We have the initial condition that $\lambda_*(1) = 1$.</p>
$\lambda(n)$	<p>The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. That is, $\lambda(n) \in \{\pm 1\}$ for all integers $n \geq 1$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \pmod{2}$.</p>
$\mu(n)$	<p>The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree.</p>
$M(x)$	<p>The Mertens function is the summatory function of $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum_{n \leq x} \mu(n)$.</p>
$\Phi(z)$	<p>For $x \in \mathbb{R}$, we define the function $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^z e^{-t^2/2} dt$.</p>
$\nu_p(n)$	<p>The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^\alpha \parallel n$ (or when p^α exactly divides n) for p prime, $\alpha \geq 1$ and $n \geq 2$.</p>
$\omega(n), \Omega(n)$	<p>We define the strongly additive function $\omega(n) := \sum_{p n} 1$ and the completely additive function $\Omega(n) := \sum_{p^\alpha \parallel n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$.</p>
$\pi_k(x), \widehat{\pi}_k(x)$	<p>The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leq n \leq x$ for $x > 1$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$ for $x \geq 2$.</p>
$P(s)$	<p>For complex s with $\text{Re}(s) > 1$, we define the <i>prime zeta function</i> to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$. For $\text{Re}(s) > 1$, the prime zeta function is related to $\zeta(s)$ according to the formula $P(s) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log[\zeta(ks)]$.</p>
$Q(x)$	<p>For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. More precisely, this function is summed and identified with its limiting asymptotic formula as $x \rightarrow \infty$ in the following form: $Q(x) := \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O(\sqrt{x})$.</p>
\sim	<p>We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} = 1$.</p>
$\zeta(s)$	<p>The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\text{Re}(s) > 1$, and by analytic continuation on the rest of the complex plane with the exception of a simple pole at $s = 1$ of residue one.</p>

1 Introduction

1.1 Definitions

We define the *Möbius function* to be the signed indicator function of the squarefree integers in the form of [19, A008683]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \text{ and } n \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

There are many variants and special properties of the Möbius function and its generalizations [18, cf. §2]. One crucial role of the classical $\mu(n)$ is that the function forms an inversion relation for the divisor sums formed by arithmetic functions convolved with one through *Möbius inversion*:

$$g(n) = (f * 1)(n) \iff f(n) = (g * \mu)(n), \forall n \geq 1.$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined on the positive integers as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as follows [19, A002321]:

$$\{M(x)\}_{x \geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \dots\}.$$

Clearly, a positive integer $n \geq 1$ is *squarefree*, or contains no (prime power) divisors which are squares, if and only if $\mu^2(n) = 1$. A related summatory function which counts the number of *squarefree* integers $n \leq x$ satisfies [5, §18.6] [19, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O(\sqrt{x}).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers characterized by the sign of the Möbius function are in fact equal as $x \rightarrow \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{x} \underset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{x} \xrightarrow{x \rightarrow \infty} \frac{3}{\pi^2}.$$

1.2 Properties

A conventional approach to evaluating the limiting asymptotic behavior of $M(x)$ for large $x \rightarrow \infty$ results by considering an inverse Mellin transformation of the reciprocal of the Riemann zeta function. In particular, since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \cdot \int_1^\infty \frac{M(x)}{x^{s+1}} dx, \operatorname{Re}(s) > 1,$$

we obtain that

$$M(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{T-i\infty}^{T+i\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

The previous two representations lead us to the exact expression of $M(x)$ for any real $x > 0$ given by the next theorem due to Titchmarsh.

Theorem 1.1 (Analytic Formula for $M(x)$). *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any real $x > 0$*

$$M(x) = \lim_{k \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ |\operatorname{Im}(\rho)| < T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} [x \in \mathbb{Z}^+]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp\left(-C \cdot \log^{3/5}(x)(\log \log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan more recently proved new updated estimates bounding $M(x)$ from above for large x in the following forms [20]:

$$\begin{aligned} M(x) &\ll \sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log \log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log \log x)^{5/2+\epsilon}\right)\right), \quad \forall \epsilon > 0. \end{aligned}$$

1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$ for any $0 < \epsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C \cdot \sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens for $C = 1$ and all $x < 10000$. Since its beginnings in 1897, the Mertens conjecture has been disproven by computation of non-trivial simple zeta function zeros with comparatively small imaginary parts in a famous paper by Odlyzko and té Riele [13]. Since the truth of the conjecture would have implied the RH, more recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound along infinite subsequences, e.g., considering the asymptotics of the function in the limit supremum and limit infimum senses.

A precise statement of this problem is to produce an unconditional proof of whether $\limsup_{x \rightarrow \infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x \rightarrow \infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there are infinite subsequences of natural numbers $\{x_1, x_2, x_3, \dots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound towards either $\pm\infty$ along the subsequence. We cite that it is only known by computation that [16, cf. §4.1] [19, cf. [A051400](#); [A051401](#)]

$$\limsup_{x \rightarrow \infty} \frac{M(x)}{\sqrt{x}} > 1.060 \quad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x \rightarrow \infty} \frac{M(x)}{\sqrt{x}} < -1.009 \quad (\text{now } \leq -1.837625).$$

Based on work by Odlyzko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [13, 8, 9, 6]. Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies [12]

$$\limsup_{x \rightarrow \infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log \log \log x)^{5/4}} = O(1).$$

2 An overview of the core components to the proof

We offer an initial step-by-step summary overview of the core components to our proof outlined in the next points. We hope that this sketch of the logical components to this argument makes the article easier to parse.

- (1) We prove a matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). See Theorem 3.1 in Section 4.
- (2) This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the prime counting function, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is stated in (1). The link relating our new formula in (1) to canonical additive functions and their distributions lends a recent distinguishing element to the success of the methods in our proof.
- (3) We tighten bounds from a less classical result proved in [11, §7] providing uniform asymptotic formulas for the summatory functions, $\widehat{\pi}_k(x)$, large $x \gg e$ and $1 \leq k \leq \log \log x$ (see Theorem 3.7). We use this result to bound sums of the form $\sum_{n \leq x} \lambda(n)f(n)$ from below for particular positive arithmetic functions f when x is large.
- (4) We then turn to estimating the limiting asymptotics of the quasi-periodic function, $|g^{-1}(n)|$, by proving several formulas bounding its average order as $x \rightarrow \infty$ in Section 6. We eventually use these estimates to prove a substantially unique new lower bound formulas for the summatory function $G^{-1}(x) := \sum_{n \leq x} \lambda(n)|g^{-1}(n)|$ along certain asymptotically large infinite subsequences (see Theorem 8.4).
- (5) (TODO)
- (6) When we return to step (2) with our new lower bounds at hand, we are led to a new unconditional proof of the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. In fact, we recover a quick and rigorous proof of Theorem 3.9 given at the conclusion of Section 8.2.

3 A concrete new approach to bounding $M(x)$ from below

3.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 3.1 (Summatory functions of Dirichlet convolutions). *Let $f, h : \mathbb{Z}^+ \rightarrow \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of f and h , respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse of f . We have the following exact expressions for the summatory function of $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned} \pi_{f*h}(x) &:= \sum_{n \leq x} \sum_{d|n} f(d)h(n/d) \\ &= \sum_{d \leq x} f(d)H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{k=1}^x H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right]. \end{aligned}$$

Moreover, for all $x \geq 1$

$$\begin{aligned} H(x) &= \sum_{j=1}^x \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \right] \\ &= \sum_{n=1}^x f^{-1}(n) \pi_{f*h}\left(\left\lfloor \frac{x}{n} \right\rfloor\right). \end{aligned}$$

Corollary 3.2 (Convolutions arising from Möbius inversion). *Suppose that g is an arithmetic function such that $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\tilde{G}(x) := \sum_{n \leq x} (g * \mu)(n)$. The Mertens function is expressed by the sum*

$$M(x) = \sum_{k=1}^x \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \tilde{G}(k), \forall x \geq 1.$$

Corollary 3.3 (A motivating special case). *We have exactly that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^x (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1 \right]. \quad (1)$$

3.2 An exact expression for $M(x)$ in terms of strongly additive functions

Fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute exactly that (see Table T.1 starting on page 47 of the appendix section)

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \dots\}.$$

The sign of these positive terms is given by $\text{sgn}(g^{-1}(n)) = \frac{g^{-1}(n)}{|g^{-1}(n)|} = \lambda(n)$ for all $n \geq 1$ (see Proposition 4.1).

There is not an easy, nor subtle direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences. The distribution of distinct sets of prime exponents is still fairly regular so that $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of $g^{-1}(n)$ over $n \geq 2$:

Heuristic 3.4 (Symmetry in $g^{-1}(n)$ in the prime factorizations of n). Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for $\omega(n_i) \geq 1$. If $\{\alpha_1, \dots, \alpha_r\} \equiv \{\beta_1, \dots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly one, two, three, and so on prime factors.

Conjecture 3.5. *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) $g^{-1}(1) = 1$;

(B) For all $n \geq 1$, $\text{sgn}(g^{-1}(n)) = \lambda(n)$;

(C) For all squarefree integers $n \geq 1$, we have that

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!;$$

(D) If $n \geq 2$ and $\Omega(n) = k$, then

$$2 \leq |g^{-1}(n)| \leq \sum_{m=0}^k \binom{k}{m} \cdot m!.$$

We illustrate parts (B)–(D) of the conjecture clearly using the computation of initial values of this inverse sequence in Table T.1. A proof of (C) in fact follows from Lemma 6.3 stated on page 21. The realization that the beautiful and remarkably simple combinatorial form of property (C) in Conjecture 3.5 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through sums of auxiliary sequences of arithmetic functions (see Section 6).

We prove that (see Proposition 8.1)

$$M(x) = G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{\sqrt{x}} G^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right].$$

This formula implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by bounding appropriate estimates of the summatory function $G^{-1}(x)$.

3.3 Uniform asymptotics from enumerative bivariate DGFs from Montgomery and Vaughan

Theorem 3.6 (Montgomery and Vaughan). *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $R < 2$ we have that uniformly for all $1 \leq k \leq R \log \log x$

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log \log x}\right) \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!} \left[1 + O_R\left(\frac{k}{(\log \log x)^2}\right) \right],$$

where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, \quad 0 \leq |z| \leq R.$$

The proof of the next result is combinatorially motivated in so much as it interprets lower bounds on a key infinite product factor of $\mathcal{G}(z)$ defined in Theorem 3.6 as corresponding to an ordinary generating function of certain homogeneous symmetric polynomials involving the primes. This interpretation allows us to recover the following uniform lower bounds on $\widehat{\pi}_k(x)$ as $x \rightarrow \infty$:

Theorem 3.7. *For all sufficiently large x we have uniformly for $1 \leq k \leq \log \log x$ that*

$$\widehat{\pi}_k(x) \gg \frac{x^{3/4}}{(\log x)^{1/2}} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log \log x)^2}\right) \right].$$

3.3.1 Applications of the new uniform lower bound estimates

Our inspiration for the new bounds found in the last sections of this article allows us to approximate finite partial sums of certain bounded non-negative arithmetic functions weighted by the Liouville lambda function $\lambda(n)$.

Lemma 3.8. (TODO) *Suppose that $f(n)$ is an arithmetic function defined such that $f(n) > 0$ for all $n > u_0$ where $f(n) \gg \hat{\tau}_\ell(n) > 0$ whenever $n > u_0$ as $n \rightarrow \infty$. Assume also that the bounding function $\hat{\tau}_\ell(t)$ is a continuously differentiable function of t for all large enough $t \gg u_0$. We define the λ -sign-scaled summatory function of f as follows:*

$$F_\lambda(x) := \sum_{u_0 < n \leq x} \lambda(n) f(n).$$

Let the summatory weight function be defined as

$$A_\Omega(t) := \sum_{k=1}^{\lfloor \log \log t \rfloor} (-1)^k \hat{\pi}_k(t).$$

Suppose that $|A_\Omega(t)| \gg |A_\Omega^{(\ell)}(t)|$ as $t \rightarrow \infty$. Then we have that for sufficiently large $x > e$

$$|F_\lambda(x)| \gg \left| A_\Omega^{(\ell)}(x) \hat{\tau}_\ell(x) - \int_{\frac{\log \log x}{2} - \frac{1}{2}}^{\frac{\log \log x}{2}} |A_\Omega^{(\ell)}(e^{e^{2t}}) \hat{\tau}_\ell'(e^{e^{2t}})| e^{e^{2t}} dt \right|. \quad (2)$$

3.3.2 Remarks

We emphasize the relevant recency of the method demonstrated by Montgomery and Vaughan in constructing a proof of Theorem 3.6. To the best of our knowledge, this textbook reference is one of the first clear-cut applications documenting something of a hybrid DGF-and-OGF approach to enumerating sequences of arithmetic functions and their summatory functions. The hybrid method is motivated by the fact that it does not require a direct appeal to traditional highly oscillatory DGF-only inversions and integral formulas involving the Riemann zeta function. This newer interpretation of certain bivariate DGFs offers a window into the best of both generating function series worlds: It combines the additivity implicit to the coefficients indexed by a formal power series variable formed by multiplication of these structures, while coordinating the distinct DGF-best property of the multiplicativity of distinct prime powers invoked by taking powers of an Euler product.

3.4 Cracking the classical unboundedness barrier

In Section 8, we are able to state what forms a bridge between the results we carefully prove up to that point the article. What we obtain at the conclusion of the section is the next summary theorem that unconditionally resolves the classical question of the unboundedness of the scaled function Mertens function $q(x) := |M(x)|/\sqrt{x}$ in the limit supremum sense.

Theorem 3.9 (Unboundedness of the the Mertens function, $q(x)$). *We have that*

$$\limsup_{x \rightarrow \infty} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

In establishing the rigorous proof of Theorem 3.9 based on our new methods, we not only show unboundedness of $q(x)$, but also set a minimal rate (along a large infinite subsequence) at which this form of the scaled Mertens function grows without bound.

4 Preliminary proofs of new results

4.1 Establishing the summatory function properties and inversion identities

We will offer a proof of Theorem 3.1 suggested by an intuitive construction through matrix methods. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; cf. §4.9, p. 95].

Proof of Theorem 3.1. Let h, g be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of h and g , respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h . We have that the following formulas hold for all $x \geq 1$:

$$\begin{aligned} \pi_{g*h}(x) &:= \sum_{n=1}^x \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^x g(d)H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^x \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] H(i). \end{aligned} \quad (3)$$

The first formula above is well known. The second formula is justified directly using summation by parts as^A

$$\begin{aligned} \pi_{g*h}(x) &= \sum_{d=1}^x h(d)G\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i \leq x} \left(\sum_{j \leq i} h(j) \right) \times \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right]. \end{aligned}$$

We next form the invertible matrix of coefficients associated with this linear system defining $H(j)$ for all $1 \leq j \leq x$ in (3) by defining

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right), 1 \leq j \leq x.$$

Since $g_{x,x} = G(1) = g(1)$ and $g_{x,j} = 0$ for all $j > x$, the matrix we must invert in this problem is lower triangular with a non-zero constant on its diagonals, and is hence invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressed by applying an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

Here, U is a square matrix with sufficiently large finite dimensions whose $(i, j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$[(I - U^T)^{-1}]_{i,j} = [j \leq i]_{\delta}.$$

Observe that

$$\left\lfloor \frac{x}{j} \right\rfloor - \left\lfloor \frac{x-1}{j} \right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

The previous property implies that

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

^AFor any arithmetic functions, u_n, v_n , with $U_j := u_1 + u_2 + \dots + u_j$ for $j \geq 1$, we have that [14, §2.10(ii)]

$$\sum_{j=1}^{n-1} u_j \cdot v_j = U_{n-1}v_n + \sum_{j=1}^{n-1} U_j (v_j - v_{j+1}), n \geq 2.$$

We use the last property in (4) to shift the matrix \hat{G} , and then invert the result to obtain a matrix involving the Dirichlet inverse of g in the following form:

$$\left[(I - U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Our target matrix in the inversion problem is defined by

$$(g_{x,j}) = (I - U^T) \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right) (I - U^T)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators as follows:

$$\begin{aligned} (g_{x,j})^{-1} &= (I - U^T)^{-1} \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right) (I - U^T) \\ &= \left(\sum_{k=1}^{\lfloor \frac{x}{j} \rfloor} g^{-1}(k)\right) (I - U^T) \\ &= \left(\sum_{k=1}^{\lfloor \frac{x}{j} \rfloor} g^{-1}(k) - \sum_{k=1}^{\lfloor \frac{x}{j+1} \rfloor} g^{-1}(k)\right). \end{aligned}$$

Hence, the summatory function $H(x)$ is given exactly for any $x \geq 1$ by a vector product with the inverse matrix from the previous equation in the next form.

$$H(x) = \sum_{k=1}^x g_{x,k}^{-1} \cdot \pi_{g*h}(k) = \sum_{k=1}^x \left(\sum_{j=\lfloor \frac{x}{k+1} \rfloor + 1}^{\lfloor \frac{x}{k} \rfloor} g^{-1}(j)\right) \cdot \pi_{g*h}(k)$$

We can prove an inversion formula providing the coefficients of $G^{-1}(i)$ for $1 \leq i \leq x$ given by the last equation by adapting our argument to prove (3) above. This leads to the identity that

$$H(x) = \sum_{k=1}^x g^{-1}(x) \pi_{g*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right). \quad \square$$

4.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n . Then we can easily prove using DGFs that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu. \quad (5)$$

When combined with Corollary 3.2 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 3.3.

Proposition 4.1 (The signedness property of $g^{-1}(n)$). *Let the operator $\text{sgn}(h(n)) = \frac{h(n)}{|h(n)| + [h(n)=0]_\delta} \in \{0, \pm 1\}$ denote the sign of the arithmetic function h at integers $n \geq 1$. For the Dirichlet invertible function, $g(n) := \omega(n) + 1$, we have that $\text{sgn}(g^{-1}(n)) = \lambda(n)$ for all $n \geq 1$.*

Proof. The function $D_f(s) := \sum_{n \geq 1} f(n)n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\text{Re}(s) > \sigma_f$ for σ_f the abscissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = 1/\zeta(s)$ and $D_\omega(s) = P(s)\zeta(s)$ for $\text{Re}(s) > 1$. Then by (5) and the

known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any arithmetic function f such that $f(1) \neq 0$, we have for all $\text{Re}(s) > 1$ that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s) + 1)\zeta(s)}. \quad (6)$$

It follows that $(\omega + 1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\text{sgn}(h^{-1}) = \lambda$. This observation implies that $\text{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make our claims rigorous.

By the recurrence relation that defines the Dirichlet inverse function of any arithmetic function h such that $h(1) = 1$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum_{\substack{d|n \\ d>1}} h(d)h^{-1}(n/d), & n \geq 2. \end{cases} \quad (7)$$

For $n \geq 2$, the summands in (7) can be simply indexed over the primes $p|n$ given our definition of h from above. This observation yields that we can inductively unfold these sums into nested divisor sums provided the depth of the expanded divisor sums does not exceed the capacity to index summations over the primes dividing n . Namely, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p|n} h^{-1}\left(\frac{n}{p}\right), & \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right), & \text{if } \Omega(n) \geq 2 \\ &= -\sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1 p_2}} h^{-1}\left(\frac{n}{p_1 p_2 p_3}\right), & \text{if } \Omega(n) \geq 3. \end{aligned}$$

Then by induction with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1, n \geq 2. \quad (8)$$

In fact, by a combinatorial argument we recover exactly that

$$h^{-1}(n) = \lambda(n) \frac{(\alpha_1 + \cdots + \alpha_{\omega(n)})!}{\alpha_1! \alpha_2! \cdots \alpha_{\omega(n)}!} = \lambda(n) \binom{\Omega(n)}{\alpha_1, \alpha_2, \dots, \alpha_{\omega(n)}}. \quad (9)$$

These expansions imply that the following property holds for all $n \geq 1$:

$$\text{sgn}(h^{-1}(n)) = \lambda(n).$$

Since λ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right) \lambda(d) = \lambda(n)$ for all $d|n$ and $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever n is squarefree, so that we obtain

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1. \quad \square$$

4.3 Statements of known limiting asymptotics

Theorem 4.2 (Mertens theorem). *For all $x \geq 2$ we have that*

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log \log x + B + o(1), \text{ as } x \rightarrow \infty,$$

where $B \approx 0.2614972128476427837554$ is an absolute constant^B.

Corollary 4.3 (Product form of Mertens theorem). *We have that for all sufficiently large $x \gg 2$*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} (1 + o(1)), \text{ as } x \rightarrow \infty,$$

where the notation for the absolute constant $0 < B < 1$ coincides with the definition of Mertens constant from Theorem 4.2. Hence, for any real $z \geq 0$ we obtain that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^z \sim \frac{e^{-\gamma z}}{(\log x)^z}, \text{ as } x \rightarrow \infty.$$

Proofs of Theorem 4.2 and Corollary 4.3 are given in [5, §22.7; §22.8]. We have a related analog of Corollary 4.3 that is justified using the Euler product representation for the Riemann zeta function:

$$\prod_{p \leq x} \left(1 + \frac{1}{p}\right) = \prod_{p \leq x} \frac{(1 - p^{-2})}{(1 - p^{-1})} = \zeta(2) e^{\gamma(\log x)} (1 + o(1)), \text{ as } x \rightarrow \infty.$$

Facts 4.4 (Exponential integrals and the incomplete gamma function). Two variants of the *exponential integral function* are defined by the integral next representations [14, §8.19].

$$\begin{aligned} \text{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, x \in \mathbb{R} \\ E_1(z) &:= \int_1^{\infty} \frac{e^{-tz}}{t} dt, \text{Re}(z) \geq 0 \end{aligned}$$

These functions are related by $\text{Ei}(-kz) = -E_1(kz)$ for real $k, z > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $\text{Ei}(\pm x)$ for all real $x > 0$:

$$\begin{aligned} \gamma + \log x - x &\leq \text{Ei}(-x) \leq \gamma + \log x - x + \frac{x^2}{4}, \\ 1 + \gamma + \log x - \frac{3}{4}x &\leq \text{Ei}(x) \leq 1 + \gamma + \log x - \frac{3}{4}x + \frac{11}{36}x^2. \end{aligned} \tag{10a}$$

The (upper) *incomplete gamma function* is defined by [14, §8.4]

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt, \text{Re}(s) > 0.$$

The following properties of $\Gamma(s, x)$ hold:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, x > 0, \tag{10b}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, s > 0, \text{ as } x \rightarrow \infty. \tag{10c}$$

^BPrecisely, we have that the *Mertens constant* is defined by [19, A077761]

$$B = \gamma + \sum_{m \geq 2} \frac{\mu(m)}{m} \log [\zeta(m)].$$

5 Components to the asymptotic analysis of lower bounds for sums of arithmetic functions weighted by $\lambda(n)$

5.1 A discussion of the results proved by Montgomery and Vaughan

Remark 5.1 (Intuition and constructions in Theorem 3.6). For $|z| < 2$ and $\operatorname{Re}(s) > 1$, let

$$F(s, z) := \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \quad (11)$$

and define the DGF coefficients, $a_z(n)$ for $n \geq 1$, by the product

$$\zeta(s)^z \cdot F(s, z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s}, \operatorname{Re}(s) > 1.$$

Suppose that $A_z(x) := \sum_{n \leq x} a_z(n)$ for $x \geq 1$. Then we obtain the next generating function like identity in z enumerating the $\hat{\pi}_k(x)$ for $1 \leq k \leq \log \log x$ ^A

$$A_z(x) = \sum_{n \leq x} z^{\Omega(n)} = \sum_{k \geq 0} \hat{\pi}_k(x) z^k \quad (12)$$

Thus for $r < 2$, by Cauchy's integral formula we have

$$\hat{\pi}_k(x) = \frac{1}{2\pi i} \int_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz.$$

Selecting $r := \frac{k-1}{\log \log x}$ for $1 \leq k < 2 \log \log x$ leads to the uniform asymptotic formulas for $\hat{\pi}_k(x)$ given in Theorem 3.6. Montgomery and Vaughan then consider individual analysis of the main and error terms for $A_z(x)$ to prove that

$$\hat{\pi}_k(x) = \mathcal{G} \left(\frac{k-1}{\log \log x} \right) \frac{x}{\log x} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \left[1 + O \left(\frac{k}{(\log \log x)^2} \right) \right].$$

We will require estimates of $A_{-z}(x)$ from below to form summatory functions that weight the terms of $\lambda(n)$ in our new formulas derived in the next sections.

5.2 New uniform asymptotics based on refinements of Theorem 3.6

Proposition 5.2. For real $s \geq 1$, let

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \geq 2.$$

When $s := 1$, we have the asymptotic formula from Mertens theorem (see Theorem 4.2). For all integers $s \geq 2$ there is absolutely defined quasi-polynomial bounding functions $\gamma_0(s, x)$ and $\gamma_1(s, x)$ in s, x such that

$$\gamma_0(s, x) + o(1) \leq P_s(x) \leq \gamma_1(s, x) + o(1), \text{ as } x \rightarrow \infty.$$

It suffices to define the bounds in the previous equation by the functions

$$\begin{aligned} \gamma_0(s, x) &= s \log \left(\frac{\log x}{\log 2} \right) - s(s-1) \log \left(\frac{x}{2} \right) - \frac{1}{4} s(s-1)^2 \log^2(2) \\ \gamma_1(s, x) &= s \log \left(\frac{\log x}{\log 2} \right) - s(s-1) \log \left(\frac{x}{2} \right) + \frac{1}{4} s(s-1)^2 \log^2(x). \end{aligned}$$

^AIn fact, for any additive arithmetic function $a(n)$, characterized by the property that $a(n) = \sum_{p^\alpha || n} a(p^\alpha)$ for all $n \geq 2$, we have that [7, cf. §1.7]

$$\prod_p \left(1 - \sum_{m \geq 1} \frac{z^{a(p^m)}}{p^{ms}} \right)^{-1} = \sum_{n \geq 1} \frac{z^{a(n)}}{n^s}, \operatorname{Re}(s) > 1.$$

Proof. Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$, and where our target function smooth function is $f(t) = t^{-s}$ so that $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= \text{Ei}(-(s-1) \log x) - \text{Ei}(-(s-1) \log 2) + o(1), \text{ as } x \rightarrow \infty. \end{aligned}$$

Now using the inequalities in Facts 4.4, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned} \frac{P_s(x)}{s} &\geq \log \left(\frac{\log x}{\log 2} \right) - (s-1) \log \left(\frac{x}{2} \right) - \frac{1}{4}(s-1)^2 \log^2(2) + o(1) \\ \frac{P_s(x)}{s} &\leq \log \left(\frac{\log x}{\log 2} \right) - (s-1) \log \left(\frac{x}{2} \right) + \frac{1}{4}(s-1)^2 \log^2(x) + o(1). \end{aligned} \quad \square$$

We will first prove the stated form of the lower bound on $\mathcal{G}(-z)$ for $z := \frac{k-1}{\log \log x}$. Then we will discuss the technical adaptations to Montgomery and Vaughan's proof of Theorem 3.6 in Remark 5.3 to justify the new asymptotic lower bounds on $\widehat{\pi}_k(x)$ that hold uniformly for all $1 \leq k \leq \log \log x$.

Proof of Theorem 3.7. For $0 \leq z < 2$ and integers $x \geq 2$, the right-hand-side of the following product is finite.

$$\widehat{P}(z, x) := \prod_{p \leq x} \left(1 - \frac{z}{p} \right)^{-1}.$$

For fixed, finite $x \geq 2$ let

$$\mathbb{P}_x := \{n \geq 1 : \text{all prime divisors } p|n \text{ satisfy } p \leq x\}.$$

Then we can see that

$$\prod_{p \leq x} \left(1 - \frac{z}{p^s} \right)^{-1} = \sum_{n \in \mathbb{P}_x} \frac{z^{\Omega(n)}}{n^s}, \quad x \geq 2. \quad (13)$$

By extending the argument in the proof given in [11, §7.4], we have that the formulas

$$A_{-z}(x) := \sum_{n \leq x} \lambda(n) z^{\Omega(n)} = \sum_{k \geq 0} \widehat{\pi}_k(x) (-z)^k,$$

If we let $a_n(z, x)$ be defined by the DGF

$$\widehat{P}(z, x) := \sum_{n \geq 1} \frac{a_n(z, x)}{n^s},$$

then we show that

$$\sum_{n \leq x} a_n(-z, x) = \sum_{n \leq x} \lambda(n) z^{\Omega(n)} = \sum_{k=0}^{\log_2(x)} \widehat{\pi}_k(x) (-z)^k + \sum_{k > \log_2(x)} e_k(x) (-z)^k.$$

This assertion is correct since the products of all non-negative integral powers of the primes $p \leq x$ generate the integers $\{1 \leq n \leq x\}$ as a subset. Thus we capture all of the relevant terms needed to express $(-1)^k \cdot \widehat{\pi}_k(x)$ via the Cauchy integral formula representation over $A_{-z}(x)$ by replacing the corresponding infinite product terms with $\widehat{P}(-z, x)$ in the definition of $\mathcal{G}(-z)$.

Now we must argue that

$$\mathcal{G}(-z) \gg \prod_{p \leq x} \left(1 + \frac{z}{p} \right)^{-1} \left(1 - \frac{1}{p} \right)^{-z}, \quad 0 \leq z < 1, x \geq 2.$$

For $0 \leq z < 1$ and $x \geq 2$, we see that

$$\begin{aligned} \mathcal{G}(-z) &= \exp \left(- \sum_p \left[\log \left(1 + \frac{z}{p} \right) + \log \left(1 - \frac{1}{p} \right) \right] \right) \\ &\gg \exp \left(-z \times \sum_{p>x} \left[\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right] - \sum_{p \leq x} \left[\log \left(1 + \frac{z}{p} \right) + \log \left(1 - \frac{1}{p} \right) \right] \right) \\ &= \widehat{P}(-z, x) \times \exp(-z(B + o(1))) \gg_z \widehat{P}(-z, x), \text{ as } x \rightarrow \infty. \end{aligned}$$

Next, we have for all integers $0 \leq k \leq m < \infty$, and any sequence $\{f(n)\}_{n \geq 1}$ with sufficiently bounded partial power sums, that [10, §2]

$$[z^k] \prod_{1 \leq i \leq m} (1 - f(i)z)^{-1} = [z^k] \exp \left(\sum_{j \geq 1} \left(\sum_{i=1}^m f(i)^j \right) \frac{z^j}{j} \right), |z| < 1. \quad (14)$$

In our case we have that $f(i)$ denotes the reciprocal of the i^{th} prime in the generating function expansion of (14). It follows from Proposition 5.2 that for any real $0 \leq z < 1$ we obtain

$$\begin{aligned} \log \left[\prod_{p \leq x} \left(1 + \frac{z}{p} \right)^{-1} \right] &\geq -(B + \log \log x)z + \sum_{j \geq 0} \left[\log \left(\frac{\log x}{\log 2} \right) - (2j+1) \log \left(\frac{x}{2} \right) - (2j+1)^2 \frac{\log^2 2}{4} \right] z^{2j+2} \\ &\quad + \sum_{j \geq 0} \left[\log \left(\frac{\log x}{\log 2} \right) - (2j+2) \log \left(\frac{x}{2} \right) + (2j+2)^2 \frac{\log^2 x}{4} \right] z^{2j+3} \\ &= -(B + \log \log x)z + z^2 \times \sum_{j \geq 0} \left[\log \left(\frac{\log x}{\log 2} \right) - (j+1) \log \left(\frac{x}{2} \right) \right] (-z)^j \\ &\quad - \frac{z^2}{4} \times \sum_{j \geq 0} [\log^2 2 + \log^2 x] (j+1)^2 z^j \\ &= -(B + \log \log x)z + z^2 \left[\log \left(\frac{\log x}{\log 2} \right) \frac{1}{1+z} - \log \left(\frac{x}{2} \right) \frac{1}{(1+z)^2} \right] \\ &\quad + (\log^2 2 + \log^2 x) \frac{z^2(1+z)}{4 \cdot (1-z)^3} \\ &=: \widehat{\mathcal{B}}(x; z). \end{aligned} \quad (15)$$

We adjust the uniform bound parameter R so that

$$z \equiv z(k, x) = \frac{k-1}{\log \log x} \in [0, 1),$$

whenever $1 \leq k \leq \log \log x$ in the notation of Theorem 3.6. This implies that $(1+z)^{-1} \in (\frac{1}{2}, 1]$. Then we have from (15) that

$$\begin{aligned} \widehat{\mathcal{B}}(x; z) &\gg \left(\frac{\log x}{\log 2} \right)^{\frac{z^2}{2}} \cdot \left(\frac{2}{x} \right)^{\frac{1}{4}} \cdot \exp \left(\frac{z^2(1+z)}{4 \cdot (1-z)^3} \cdot \log^2 x \right) \\ &\gg \frac{(\log x)^{1/2}}{x^{1/4}}. \end{aligned}$$

In summary, we have arrived at a proof that as $x \rightarrow \infty$

$$\frac{e^{\gamma z}}{(\log x)^{-z}} \times \exp \left(\widehat{\mathcal{B}}(u, x; z) \right) \gg \frac{(\log x)^{1/2}}{x^{1/4}}. \quad (16)$$

Finally, to finish our proof of the new form of the lower bound on $\mathcal{G}(-z)$, we need to bound the reciprocal factor of $\Gamma(1-z)$. Since $z \equiv z(k, x) = \frac{k-1}{\log \log x}$ and $k \in [1, \log \log x]$, or again with $z \in [0, 1)$, we obtain for minimal k and all large enough $x \gg 1$ that $\Gamma(1-z) = \Gamma(1) = 1$, and for k towards the upper range of its interval that

$$\Gamma(1-z) \approx \Gamma\left(\frac{1}{\log \log x}\right) = \frac{1}{\log \log x} \Gamma\left(1 + \frac{1}{\log \log x}\right) \approx \frac{1}{\log \log x}. \quad \square$$

Remark 5.3 (Technical adjustments in the proof of Theorem 3.7). We now discuss the differences between our construction and that in the technical proof of Theorem 3.6 in the reference when we bound $\mathcal{G}(-z)$ from below as in Theorem 3.7. The reference proves that for real $0 \leq z < 2$

$$A_{-z}(x) = -\frac{zF(1, -z)}{\Gamma(1-z)} \cdot x(\log x)^{-(z+1)} + O\left(x(\log x)^{-\operatorname{Re}(z)-2}\right). \quad (17)$$

Recall that for $r < 2$ we have by Cauchy's integral formula that

$$(-1)^k \widehat{\pi}_k(x) = \frac{1}{2\pi i} \int_{|z|=r} \frac{A_{-z}(x)}{z^{k+1}} dz. \quad (18)$$

We first claim that uniformly for large x and $1 \leq k \leq \log \log x$ we have

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{1-k}{\log \log x}\right) \times \frac{x(\log \log x)^{k-1}}{(\log x)(k-1)!} \left[1 + O\left(\frac{k}{(\log \log x)^3}\right)\right]. \quad (19)$$

Then since we have proved in Theorem 3.6 above that

$$\mathcal{G}\left(\frac{1-k}{\log \log x}\right) \gg \frac{2^{1/4}(\log x)^{1/2}}{\sqrt{\log 2} \cdot x^{1/4}} \cdot \frac{(k-1)}{\log \log x},$$

the result in (19) implies our stated uniform asymptotic bound. Namely, we obtain that

$$\widehat{\pi}_k(x) \gg \frac{2^{1/4}}{\sqrt{\log 2}} \cdot \frac{x^{3/4}}{(\log x)^{1/2}} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log \log x)^2}\right)\right].$$

We have to provide analogs to the two separate bounds corresponding to the error and main terms of our estimate according to (17) and (18). The error term estimate is simpler, so we tackle it first in the next argument. The second part of our proof establishing the main term in (19) requires us to duplicate and adjust significant parts of the fine-tuned reasoning given in the reference.

Error Term Bound. To prove that the error term bound holds, we estimate that

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{x}{(\log x)^2} \frac{(\log x)^{-\operatorname{Re}(z)}}{z^{k+1}} \right| &\ll x(\log x)^{-(r+2)} r^{-(k+1)} \ll \frac{x}{(\log x)^2} \frac{(\log \log x)^{k+1}}{e^{k-1}(k-1)^{k+1}} \\ &\ll \frac{x}{(\log x)^2} \frac{(\log \log x)^{k+1}}{e^{2(k-1)}(k-1)!(k-1)} \ll \frac{x}{(\log x)^2} \frac{(\log \log x)^{k+1}}{(k-1)!} \\ &\ll \frac{x}{\log x} \frac{(\log \log x)^{k-4}}{(k-1)!}. \end{aligned} \quad (20)$$

We can calculate that for $0 \leq z < 1$

$$\begin{aligned} \prod_p \left(1 + \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{-z} &= \exp\left(-\sum_p \left[\log\left(1 + \frac{z}{p}\right) + z \log\left(1 - \frac{1}{p}\right)\right]\right) \\ &\sim \exp\left(-o(z) \times \sum_p \frac{1}{p^2}\right) \end{aligned}$$

$$\gg \exp\left(-o(z)\frac{\pi^2}{6}\right) \gg_z 1.$$

In other words, we have that $\mathcal{G}\left(\frac{1-k}{\log \log x}\right) \gg 1$. So the error term in (20) is majorized by taking $O\left(\frac{k}{(\log \log x)^3}\right)$ as our upper bound.

Main Term Bounds. Notice that the main term estimate corresponding to (17) and (18) is given by $\frac{x}{\log x}I$, where

$$I := \frac{(-1)^{k-1}}{2\pi i} \int_{|z|=r} G(-z)(\log x)^{-z} z^{-k} dz.$$

In particular, we can write $I = I_1 + I_2$ where we define

$$\begin{aligned} I_1 &:= \frac{(-1)^{k-1}G(-r)}{2\pi i} \int_{|z|=r} (\log x)^{-z} z^{-k} dz \\ &= \frac{G(-r)(\log \log x)^{k-1}}{(k-1)!} \\ I_2 &:= \frac{(-1)^{k-1}}{2\pi i} \int_{|z|=r} (G(-z) - G(-r))(\log x)^{-z} z^{-k} dz \\ &= \frac{(-1)^{k-1}}{2\pi i} \int_{|z|=r} (G(-z) - G(-r) + G'(-r)(z+r))(\log x)^{-z} z^{-k} dz. \end{aligned}$$

We have by a power series expansion of $G''(-w)$ about $-z$ and integrating the resulting series termwise with respect to w that

$$|G(-z) - G(-r) + G'(-r)(z+r)| = \left| \int_{-r}^z (z+w)G''(-w)dw \right| \ll G''(-r) \times |z+r|^2 \ll |z+r|^2.$$

Now we parameterize the curve in the contour for I_2 by writing $z = re^{2\pi i t}$ for $t \in [-1/2, 1/2]$. This leads us to the bounds

$$\begin{aligned} |I_2| &= r^{3-k} \times \int_{-1/2}^{1/2} |e^{2\pi i t} + 1|^2 \cdot (\log x)^{re^{2\pi i t}} \cdot e^{2\pi i t} dt \\ &\ll r^{3-k} \times \int_{-1/2}^{1/2} \sin^2(\pi t) \cdot e^{(1-k)\cos(2\pi t)} dt. \end{aligned}$$

Whenever $|x| \leq 1$, we know that $|\sin x| \leq |x|$. We can construct bounds on $-\cos(2\pi t)$ for $t \in [-1/2, 1/2]$ by writing $\cos(2x) = 1 - 2\sin^2 x$ for $|x| < 1/2$. Then by the alternating Taylor series expansions of the sine function

$$\begin{aligned} 1 - 2\sin^2(2\pi t) &\geq 1 - 2\left(1 - \frac{\pi t}{3}\right)^2 \geq -1 - \frac{2\pi^2 t^2}{9} \implies \\ -\cos(2\pi t) &\leq 1 + \frac{2\pi^2 t^2}{9} \leq \left(4 + \frac{2\pi^2}{9}\right)t^2 \leq 1 + 3t^2. \end{aligned}$$

So it follows that

$$\begin{aligned} |I_2| &\ll r^{3-k} e^{k-1} \times \left| \int_0^\infty t^2 e^{3(k-1)t^2} dt \right| \\ &\ll \frac{r^{3-k} e^{k-1}}{(k-1)^{3/2}} = \frac{(\log \log x)^{k-3} e^{k-1}}{(k-1)^{k-3/2}} \\ &\ll \frac{k \cdot (\log \log x)^{k-3}}{(k-1)!}. \end{aligned}$$

Thus the contribution from the term $|I_2|$ can then be asorbed into the error term bound in (19).

5.3 The distribution of exceptional values of $\Omega(n)$

The next theorems reproduced from [11, §7.4] characterize the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \log \log x$. The tendency of this canonical completely additive function to not deviate substantially from its average order is an extraordinary property that allows us to prove asymptotic relations on summatory functions that are weighted by its parity without having to account for significant local oscillations when we average over a large interval.

Theorem 5.4 (Upper bounds on exceptional values of $\Omega(n)$ for large n). *Let*

$$\begin{aligned} A(x, r) &:= \# \{n \leq x : \Omega(n) \leq r \cdot \log \log x\}, \\ B(x, r) &:= \# \{n \leq x : \Omega(n) \geq r \cdot \log \log x\}. \end{aligned}$$

If $0 < r \leq 1$ and $x \geq 2$, then

$$A(x, r) \ll x(\log x)^{r-1-r \log r}, \quad \text{as } x \rightarrow \infty.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x, r) \ll_R x \cdot (\log x)^{r-1-r \log r}, \quad \text{as } x \rightarrow \infty.$$

Theorem 5.5 is an analog to the celebrated Erdős-Kac theorem typically stated for the normally distributed values of the scaled-shifted $\omega(n)$ function over $n \leq x$ as $x \rightarrow \infty$.

Theorem 5.5 (Exact bounds on exceptional values of $\Omega(n)$ for large n). *We have that as $x \rightarrow \infty$*

$$\# \{3 \leq n \leq x : \Omega(n) - \log \log n \leq 0\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log \log x}}\right).$$

Remark 5.6. The key interpretation we need to take away from the statements of Theorem 5.4 and Theorem 5.5 is the result proved in the next corollary. The role of the parameter R involved in stating the previous theorem is a critical bound as the scalar factor in the upper bound on $k \leq R \log \log x$ in Theorem 3.6 up to which our uniform bounds given by Theorem 3.7 hold. In contrast, for $n \geq 2$ we can actually have contributions from values distributed throughout the range $1 \leq \Omega(n) \leq \log_2(n)$ infinitely often. It is then crucial that we can show that the main term in the asymptotic formulas we obtain for these summatory functions is captured by summing only over the truncated range of $k \in [1, \log \log x]$ where the uniform bounds hold.

Corollary 5.7. *Using the notation for $A(x, r)$ and $B(x, r)$ from Theorem 5.4, we have that for $x \geq 2$ and $\delta > 0$,*

$$o(1) \leq \frac{B(x, 1 + \delta)}{A(x, 1)} \ll 2, \quad \text{as } \delta \rightarrow 0^+, x \rightarrow \infty.$$

Proof. The lower bound stated above is clear. To show that the asymptotic upper bound is correct, we compute using Theorem 5.4 and Theorem 5.5 that

$$\frac{B(x, 1 + \delta)}{A(x, 1)} \ll \frac{x \cdot (\log x)^{\delta - \delta \log(1 + \delta)}}{O(1) + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log \log x}}\right)} \sim o_\delta(1),$$

as $x \rightarrow \infty$. Notice that since $\mathbb{E}[\Omega(n)] = \log \log n + B$, with $0 < B < 1$ the absolute constant from Mertens theorem, when we denote the range of $k > \log \log x$ as holding in the form of $k > (1 + \delta) \log \log x$ for $\delta > 0$ at large x , we can assume that $\delta \rightarrow 0^+$ as $x \rightarrow \infty$. In particular, this holds since $k > \log \log x$ implies that

$$\lfloor \log \log x \rfloor + 1 \geq (1 + \delta) \log \log x \quad \implies \quad \delta \leq \frac{1 + \{\log \log x\}}{\log \log x} = o(1), \quad \text{as } x \rightarrow \infty.$$

The key consequence is that $B(x, 1 + \delta)$ is at most a bounded constant multiple of $A(x, 1)$ for all large x . \square

6 Average case analysis of bounds on the Dirichlet inverse functions, $g^{-1}(n)$

The pages of tabular data given as Table T.1 in the appendix section (refer to page 47) are intended to provide clear insight into why we arrived at the approximations to $g^{-1}(n)$ proved in this section. The table provides illustrative numerical data by examining the approximate behavior at hand for the cases of $1 \leq n \leq 500$ with *Mathematica*.

6.1 Definitions and basic properties of component function sequences

We define the following auxiliary coefficient sequence for integers $n \geq 1, k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \quad (21)$$

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (21) inductively. By the same argument, we see that at fixed n , the function $C_k(n)$ is seen to be non-zero only for positive integers $k \leq \Omega(n)$ whenever $n \geq 2$. A sequence of relevant signed semi-diagonals of the functions $C_k(n)$ begins as [19, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, -3, 2, 2, -1, 4, 1, 2, \dots\}.$$

Example 6.1 (Special cases of the functions $C_k(n)$ for small k). We cite the following special cases which are verified by explicit computation using (21) [19, A066922]^A:

$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= d(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n) + 1} - \gcd(\Omega(n), \omega(n)). \end{aligned}$$

The connection between the functions $C_k(n)$ and the inverse sequence $g^{-1}(n)$ is clarified precisely in Section 6.3. Before we can prove explicit bounds on $|g^{-1}(n)|$ through its relation to these functions, we will require a perspective on the lower asymptotic order of $C_k(n)$ for fixed k when n is large.

6.2 Uniform asymptotics of $C_k(n)$ for large all n and fixed k

The next theorem formally proves a minimal growth rate of the class of functions $C_k(n)$ as functions of fixed k and $n \rightarrow \infty$. In the statement of the result that follows, we view k as a fixed variable which is necessarily bounded in n , but is still taken as an independent parameter of n .

Theorem 6.2 (Asymptotics of the functions $C_k(n)$). *For $k := 0$, we have by definition that $C_0(n) = \delta_{n,1}$. For all sufficiently large $n > 1$ and any fixed $1 \leq k \leq \Omega(n)$ taken independently of n , we obtain that the asymptotic main term for the expected order of $C_k(n)$ is bounded uniformly from below as*

$$\mathbb{E}[C_k(n)] \gg (\log \log n)^{2k-1}, \text{ as } n \rightarrow \infty.$$

^AFor all $n, k \geq 2$, we have the following recurrence relation satisfied by $C_k(n)$ between successive values of k :

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p\nu_p(n)}} \sum_{i=0}^{\nu_p(n)-1} C_{k-1}(dp^i), n \geq 1.$$

Proof. We prove our bounds by induction on k . We can see by Example 6.1 that $C_1(n)$ satisfies the formula we must establish when $k := 1$ since $\mathbb{E}[\omega(n)] = \log \log n$. Suppose that $k \geq 2$ and let our inductive assumption provide that for all $1 \leq m < k$ and $n \geq 2$

$$\mathbb{E}[C_m(n)] \gg (\log \log n)^{2m-1}.$$

For all large $x > e$, we cite that the summatory function of $\omega(n)$ satisfies [5, §22.10]

$$\sum_{n \leq x} \omega(n) = x \log \log x + Bx + O\left(\frac{x}{\log x}\right).$$

Now using the recursive formula we used to define the sequences of $C_k(n)$ in (21), we have that as $n \rightarrow \infty$

$$\begin{aligned} \mathbb{E}[C_k(n)] &= \mathbb{E}\left[\sum_{d|n} \omega(n/d) C_{k-1}(d)\right] \\ &= \frac{1}{n} \times \sum_{d \leq n} C_{k-1}(d) \times \sum_{r=1}^{\lfloor \frac{n}{d} \rfloor} \omega(r) \\ &\sim \sum_{d \leq n} C_{k-1}(d) \left[\frac{\log \log(n/d) \lfloor \frac{n}{d} \rfloor_\delta}{d} + \frac{B}{d} + o(1) \right] \\ &\sim \sum_{d \leq \frac{n}{e}} \left[\sum_{m < d} \frac{\mathbb{E}[C_{k-1}(m)]}{m} \log \log \left(\frac{n}{m}\right) + B \cdot \mathbb{E}[C_{k-1}(d)] + B \cdot \sum_{m < d} \frac{\mathbb{E}[C_{k-1}(m)]}{m} \right] \\ &\gg \sum_{d \leq \frac{n}{e}} \frac{\mathbb{E}[C_{k-1}(m)]}{m} \\ &\gg (\log n)(\log \log n)^{2k-3}. \end{aligned} \tag{22}$$

In transitioning from the previous step, we have used that $(\log n) \gg (\log \log n)^2$ as $n \rightarrow \infty$. We have also used that for large n and fixed m , by an asymptotic approximation to the incomplete gamma function we have that

$$\int_e^n \frac{(\log \log t)^m}{t} dt \sim (\log n)(\log \log n)^m, \text{ as } n \rightarrow \infty.$$

Hence, the claim follows by mathematical induction for large $n \rightarrow \infty$ whenever $1 \leq k \leq \Omega(n)$. \square

6.3 Relating the auxiliary functions $C_k(n)$ to formulas approximating $g^{-1}(n)$

Lemma 6.3 (An exact formula for $g^{-1}(n)$). *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$g^{-1}(n) = - \sum_{\substack{d|n \\ d > 1}} (\omega(d) + 1) g^{-1}(n/d) \implies (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \tag{23}$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (23) in the form of $(g^{-1} * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m \cdot (C_m(-) * g^{-1})(n)$, or so that

$$F_m(n) = - \begin{cases} \sum_{\substack{d|n \\ d > 1}} F_{m-1}(d) \times \sum_{\substack{r|\frac{n}{d} \\ r > 1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & m \geq 2, \\ (\omega * g^{-1})(n), & m = 1. \end{cases}$$

By repeatedly expanding the right-hand-side of the previous equation, we find that for $m := \Omega(n)$

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \quad (24)$$

The formula then follows from (24) by Möbius inversion applied to each side of the last equation. \square

Corollary 6.4. *For all squarefree integers $n \geq 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \quad (25)$$

Proof. Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geq 2$ and that n is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where p_i is prime for all $1 \leq i \leq \omega(n)$. Since all divisors of any squarefree n are necessarily also squarefree, we can transform the exact divisor sum guaranteed for all n in Lemma 6.3 into a sum that partitions the divisors according to the number of distinct prime factors:

$$\begin{aligned} g^{-1}(n) &= \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i} (-1)^i \cdot C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \end{aligned}$$

The signed contributions in the first of the previous equations is justified by noting that $\lambda(n) = (-1)^{\omega(n)}$ whenever n is squarefree, and that for $d \geq 1$ squarefree we have the correspondence $\omega(d) = k \implies \Omega(d) = k$ for $1 \leq k \leq \log_2(d)$. \square

Since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ using the notation defined in the the proof of Proposition 4.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for squarefree $n \geq 1$. A proof of part (C) of Conjecture 3.5 follows as an immediate consequence.

Lemma 6.5. *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \quad (26)$$

Proof. By applying Lemma 6.3, Proposition 4.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$. Lemma 6.3 implies

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \end{aligned}$$

In the last equation, we see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. \square

Combined with the signedness property of $g^{-1}(n)$ guaranteed by Proposition 4.1, Lemma 6.5 shows that the summatory function is expressed as

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_{\Omega(d)}(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Since $\lambda(d) C_{\Omega(d)}(d) = (g^{-1} * 1)^{-1}(d) = (\chi_{\mathbb{P}} + \varepsilon)(d)$ where $\chi_{\mathbb{P}}$ denotes the characteristic function of the primes, we also clearly recover by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1.$$

Corollary 6.6. *We have that*

$$(\log n)(\log \log n) \ll \mathbb{E}|g^{-1}(n)| \leq \mathbb{E}\left[\sum_{d|n} C_{\Omega(d)}(d)\right].$$

Proof. To prove the lower bound, recall from the introduction that the summatory function of the squarefree integers is given by

$$Q(x) := \sum_{n \leq x} \mu^2(n) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Then since $C_{\Omega(d)}(d) \geq 1$ for all $d \geq 1$, and since $\mathbb{E}[C_k(d)]$ is minimized when $k := 1$ according to Theorem 6.2, we obtain by summing over (26) that

$$\begin{aligned} \frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| &= \frac{1}{x} \times \sum_{d \leq x} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &\sim \sum_{d \leq x} C_{\Omega(d)}(d) \left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dx}}\right) \right] \\ &= \frac{6}{\pi^2} \left[\mathbb{E}[C_{\Omega(x)}(x)] + \sum_{d < x} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d} \right] + O\left(\frac{1}{\sqrt{x}} \times \int_0^x t^{-1/2} dt\right) \\ &\gg \left[\sum_{e \leq d \leq x} \frac{\log \log d}{d} \right] + O(1) \\ &\sim \times \int_e^x \frac{\log \log t}{t} dt + O(1) \\ &\gg (\log x)(\log \log x), \text{ as } x \rightarrow \infty. \end{aligned}$$

To prove the upper bound, notice that by Lemma 6.3 and Corollary 6.4,

$$|g^{-1}(n)| \leq \sum_{d|n} C_{\Omega(d)}(d), n \geq 1.$$

Now since both of the above quantities are positive for all $n \geq 1$, we clearly obtain the upper bound stated above when we average over $n \leq x$ for all large x . \square

6.3.1 A connection to the distribution of the primes

Remark 6.7. The combinatorial complexity of $g^{-1}(n)$ is deeply tied to the distribution of the primes $p \leq n$ as $n \rightarrow \infty$. While the magnitudes and dispersion of the primes $p \leq x$ certainly restricts the repeating of these distinct sequence values we can see in the contributions to $G^{-1}(x)$, the following statement is still clear about

the relation of the weight functions $|g^{-1}(n)|$ to the distribution of the primes: The value of $|g^{-1}(n)|$ is entirely dependent on the pattern of the *exponents* (viewed as multisets) of the distinct prime factors of $n \geq 2$. The relation of the repetition of the distinct values of $|g^{-1}(n)|$ in forming bounds on $G^{-1}(x)$ makes another clear tie to $M(x)$ through Proposition 8.1 in the next section.

Example 6.8 (Combinatorial significance to the distribution of $g^{-1}(n)$). We have a natural extremal behavior with respect to distinct values of $\Omega(n)$ corresponding to squarefree integers, and prime powers. Namely, if for $k \geq 1$ we define the infinite sets M_k and m_k to correspond to the maximal (minimal) positive integers such that

$$M_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\},$$

$$m_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\},$$

then any element of M_k is squarefree and any element of m_k is a prime power. In particular, we have that for any $N_k \in M_k$ and $n_k \in m_k$

$$N_k = \sum_{j=0}^k \binom{k}{j} \cdot j!, \quad \text{and} \quad n_k = 2 \cdot (-1)^k.$$

The formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 4.1 implies that we can express an exact formula for $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n . Namely, for $n \geq 2$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z \cdot \nu_p(n)) = [z^k] \prod_{p^\alpha || n} (1 + \alpha z), 0 \leq k \leq \omega(n).$$

Then we have essentially shown using (9) and (26) that we can expand

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, n \geq 2.$$

The combinatorial formula for $h^{-1}(n) = \lambda(n) \cdot (\Omega(n))! \times \prod_{p^\alpha || n} (\alpha!)^{-1}$ we derived in the proof of the key signedness proposition in Section 4 suggests further patterns and more regularity in the contributions of the distinct weighted terms for $G^{-1}(x)$ when we sum over all of the distinct prime exponent patterns that factorize $n \leq x$.

7 New formulas and bounds for $g^{-1}(n)$ and its summatory function

7.1 Exact probabilistic bounds on the distributions of component sequences

We have remarked already in the introduction that the relation of the component functions, $g^{-1}(n)$ and $C_k(n)$, to the canonical additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties of these functions witnessed in Table T.1. In particular, each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdős-Kac theorem that shows that a shifted and scaled variant of each of the sets of these function values can be expressed through a limiting normal distribution as $n \rightarrow \infty$. This extremely regular tendency of these functions towards their average order is inherited by the component function sequences we are summing in the approximation of $M(x)$ stated by Proposition 8.1. In the remainder of this section we establish more technical analytic proofs of related properties of our key sequences, again in the spirit of Montgomery and Vaughan's reference.

Proposition 7.1. *For $|z| < 2$, let the summatory function be defined as*

$$\hat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

Let the function $F(s, z)$ is defined for $\operatorname{Re}(s) > 1$ in terms of the exponential of the prime zeta function by

$$F(s, z) := \exp(z \cdot P(s)) \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

Then we have that for large x

$$\hat{A}_z(x) = \frac{x \cdot F(2, z)}{\Gamma(z)} (\log x)^{z-1} + O_{R,z} \left(x \cdot (\log x)^{\operatorname{Re}(z)-2} \right), |z| < P(2)^{-1}.$$

Proof. (TODO) We know from the proof of Proposition 4.1 that for $n \geq 2$

$$C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}.$$

Then we can generate the denominator terms by the Dirichlet series

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! \cdot p^{rs}} \right)^{-1} = \exp(z \cdot P(s)).$$

By computing a Laplace transform on the right-hand-side of the above with respect to the variable z , we obtain

$$\sum_{n \geq 1} C_{\Omega(n)}(n) \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp(tz \cdot P(s)) dt = \frac{1}{1 - P(s)z}, |z| < |P(s)|^{-1}.$$

It follows that

$$\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times F(s, z),$$

where

$$F(s, z) := \frac{1}{1 - P(s)z} \times \prod_p \left(1 - \frac{1}{p^s} \right)^z, \operatorname{Re}(s) > 1, |z| < |P(s)|^{-1}.$$

Note that we are unable to sum this result in exactly the same format as in the reference [11, §7.4; Thm. 7.18] by effectively setting $s := 1$, especially given the weighting action of the Laplace transform on the coefficients of this product in z . However, since for any $|z| \leq R < 2$ we have that

$$\left| \sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^2} \right| < +\infty,$$

we will adapt the details to the traditional case where this method arises in the reference application so that we can sum over our modified function depending on $\Omega(n)$. In fact, we notice that since $|z|^{\Omega(n)} \leq n$, we have the exact DGF

$$\mathcal{H}(s) := \sum_{n \geq 1} \frac{\lambda(n) C_{\Omega(n)}(n)}{n^s},$$

which is absolutely convergent for $\operatorname{Re}(s) > 1$. The DGF $\mathcal{H}(s)$ is thus an analytic function of s whenever $\operatorname{Re}(s) > 1$, and so we can differentiate it any integer $m \geq 0$ number of times to still obtain an absolutely convergent series of the form

$$\left| \sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) (\log n)^m z^{\Omega(n)}}{n^s} \right| < +\infty, \operatorname{Re}(s) > 2, |z| < 2.$$

Let the function $d_z(n)$ have DGF $\zeta(s)^z$ for $\operatorname{Re}(s) > 1$, with corresponding summatory function $D_z(x) := \sum_{n \leq x} d_z(n)$. Adopting the notation from the reference, we set $b_z(n) := (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$, let the convolution $a_z(n) := \sum_{d|n} b_z(d) d_z(n/d)$, and define the summatory function $A_z(x) := \sum_{n \leq x} a_z(n)$. Then we have that

$$\begin{aligned} A_z(x) &= \sum_{m \leq x/2} b_z(m) D_z(x/m) + \sum_{x/2 < m \leq x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \times \sum_{m \leq x/2} \frac{b_z(m)}{m^2} \log \left(\frac{x}{m} \right)^{z-1} + O \left(x \sum_{m \leq x} \frac{|b_z(m)|}{m^2} \times \log \left(\frac{2x}{m} \right)^{\operatorname{Re}(z)-2} \right). \end{aligned}$$

The error term in the previous equation satisfies

$$\begin{aligned} x \sum_{m \leq x} \frac{|b_z(m)|}{m^2} \times \log \left(\frac{2x}{m} \right)^{\operatorname{Re}(z)-2} &\ll x (\log x)^{\operatorname{Re}(z)-2} \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m^2} + x (\log x)^{-(R+2)} \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m^2} (\log m)^{2R} \\ &\ll x (\log x)^{\operatorname{Re}(z)-2}. \end{aligned}$$

In the main term estimate for $A_z(x)$, when $m \leq \sqrt{x}$ we have

$$\log \left(\frac{x}{m} \right)^{z-1} = (\log x)^{z-1} + O \left((\log m) (\log x)^{\operatorname{Re}(z)-2} \right).$$

The main term sum over the interval $m \leq x/2$ corresponds to bounding

$$\begin{aligned} \sum_{m \leq x/2} b_z(m) D_z(x/m) &= x (\log x)^{z-1} \sum_{m \leq x/2} \frac{b_z(m)}{m^2} \\ &\quad + O \left(x (\log x)^{\operatorname{Re}(z)-2} \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m^2} + x (\log x)^{R-1} \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m^2} \right) \\ &= x (\log x)^{z-1} F(2, z) + O \left(x (\log x)^{\operatorname{Re}(z)-2} \sum_{m \geq 1} \frac{b_z(m) (\log m)^{2R+1}}{m^2} \right). \quad \square \end{aligned}$$

Remark 7.2 (A standard simplifying assumption). Let the constant $\hat{c} \approx 1.5147$ be defined explicitly as the product of primes

$$\hat{c} := \frac{1}{6} \times \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2} \right)^{-1}.$$

This constant is related to expressions of the asymptotic densities of the sets

$$N_k(x) := \{n \leq x : \Omega(n) - \omega(n) = k\},$$

for integers $k \geq 0$ in the form of [11, §2.4]

$$N_k(x) = d_k x + O\left(\left(\frac{3}{4}\right)^k \sqrt{x}(\log x)^{4/3}\right), \quad (27a)$$

where for each natural number $k \geq 0$, $d_k > 0$ is an absolute constant that satisfies

$$d_k = \frac{\widehat{c}}{2^k} + O\left(5^{-k}\right). \quad (27b)$$

A hybrid DGF generating function for these densities is given by

$$\sum_{k \geq 0} d_k z^k = \prod_p \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p-z}\right). \quad (27c)$$

The limiting distribution of $\Omega(n) - \omega(n)$ is utilized in the proof of Theorem 7.3.

For $m \leq \omega_{\max}$ and $k \leq \Omega_{\max}$, as $n \rightarrow \infty$ we expect

$$\mathbb{P}(\omega(n) = m | \Omega(n) = k) \approx \frac{\omega_{\max} + 1 - k}{\omega_{\max}},$$

so that the conditional distribution of $\omega(n), \Omega(n)$ is not uniform over its bounded range. However, we do as is standard fare in proofs of the more traditional Erdős-Kac theorems require the simplifying assumption that as $n \rightarrow \infty$, we expect independently that $\omega(n), \Omega(n)$ are approximately equally likely to assume any values in some bounded $[1, M]$. This means we can treat the difference $\Omega(n) - \omega(n)$ as being approximately randomly distributed over some bounded range of its possible values. For a more rigorous treatment of this underlying principle see [4, 2, 15].

Theorem 7.3. *We have uniformly for $1 \leq k < \log \log x$ that as $x \rightarrow \infty$*

$$\widehat{C}_k(x) := \sum_{\substack{n \leq x \\ \Omega(n)=k}} \lambda(n)(-1)^{\omega(n)} C_k(n) \asymp \frac{x}{\log x} \cdot \frac{(-1)^{k-1}(\log \log x + P(2))^k}{k!} \left[1 + O\left(\frac{k}{(\log \log x)^3}\right)\right].$$

Proof. The proof is a similar adaptation of the method of Montgomery and Vaughan we cited in Remark 5.3 to prove our variant of Theorem 3.7. We begin by bounding a contour integral over the error term for fixed large x for $r := \frac{k-1}{\log \log x}$ with $r < P(2)^{-1} \approx 2.21118$:

$$\begin{aligned} \left| \int_{|z|=r} \frac{x \cdot (\log x)^{-(\operatorname{Re}(z)+2)}}{z^{k+1}} dz \right| &\ll x(\log x)^{-(r+2)} r^{-(k+1)} \ll \frac{x}{(\log x)^2} \cdot \frac{(\log \log x)^{k+1}}{(k-1)^{k+1}} \cdot \frac{1}{e^{k-1}} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log \log x)^{k+1}}{(k-1)^{3/2}} \cdot \frac{1}{e^{2k}(k-1)!} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \ll \frac{x}{\log x} \cdot \frac{k \cdot (\log \log x)^{k-5}}{(k-1)!}. \end{aligned}$$

We must find an asymptotically accurate main term approximation to the coefficients of the following contour integral for $r \in [0, z_{\max}]$ where $z_{\max} < P(2)^{-1} \approx 2.21118$:

$$\widetilde{A}_r(x) := - \int_{|z|=r} \frac{x \cdot \exp(-P(2)z)(\log x)^{-z}}{(\log x)\Gamma(1+z) \cdot z^{k+1}(1+P(2)z)} dz. \quad (28)$$

Finding an exact formula for the derivatives of the function that is implicit to the Cauchy integral formula (CIF) for (28) is complicated significantly by the need to differentiate $\Gamma(1+z)^{-1}$ up to integer order k in the formula. What results in this case is a mess of confluent hypergeometric function approximations depending on k and an

extra factor of $(k!)^{-1}$ in the main-most term that *substantially* complicates the formative summation patterns related to the incomplete gamma function in the $\hat{\pi}_k(x)$ cases from Section 5.2. We can show that provided a restriction on the uniform bound parameter to $1 \leq r < 1$, we can approximate the contour integral in (28) using a sane bounding procedure where the resulting main term is accurate up to a bounded constant factor.

We observe that for $r := 1$, the function $|\Gamma(1 + re^{2\pi it})|$ has a singularity (pole) when $t := \frac{1}{2}$. Thus we restrict the range of $|z| = r$ so that $0 \leq r < 1$ to necessarily avoid this problematic value of t when we parameterize $z = re^{2\pi it}$ as a real integral over $t \in [0, 1]$. Then we can compute the finite extremal values as

$$\begin{aligned} \min_{\substack{0 \leq r < 1 \\ 0 \leq t \leq 1}} |\Gamma(1 + re^{2\pi it})| &= |\Gamma(1 + re^{2\pi it})| \Big|_{(r,t) \approx (1, 0.740592)} \approx 0.520089 \\ \max_{\substack{0 \leq r < 1 \\ 0 \leq t \leq 1}} |\Gamma(1 + re^{2\pi it})| &= |\Gamma(1 + re^{2\pi it})| \Big|_{(r,t) \approx (1, 0.999887)} \approx 1. \end{aligned}$$

This shows that

$$\tilde{A}_r(x) \asymp - \int_{|z|=r} \frac{x \cdot \exp(-P(2)z)(\log x)^{-z}}{(\log x) \cdot z^{k+1}(1 + P(2)z)} dz, \quad (29)$$

where as $x \rightarrow \infty$

$$\frac{\tilde{A}_r(x)}{- \int_{|z|=r} \frac{x \cdot \exp(-P(2)z)(\log x)^{-z}}{(\log x) \cdot z^{k+1}(1 + P(2)z)} dz} \in [1, 1.92275].$$

In particular, this argument holds by an analog to the mean value theorem for real integrals based on sufficient continuity conditions on the parameterized path and the smoothness of the integrand viewed as a function of z .

By induction we can compute the remaining coefficients $[z^k]\Gamma(1 + z) \times \hat{A}_z(x)$ with respect to x for fixed $k \leq \log \log x$ using the CIF. Namely, it is not difficult to see that for any integer $m \geq 0$, we have the m^{th} partial derivative of the integrand with respect to z has the following expansion:

$$\begin{aligned} \frac{1}{m!} \times \frac{\partial^{(m)}}{\partial z^{(m)}} \left[\frac{(\log x)^{-z}}{1 + P(2)z} \right] \Big|_{z=0} &= \sum_{j=0}^m \frac{(-1)^m P(2)^j (\log \log x + P(2))^{m-j}}{(m-j)!} \\ &= \frac{e \cdot (-P(2))^m (\log x)^{\frac{1}{P(2)}}}{m!} \times \Gamma \left(m+1, 1 + \frac{\log \log x}{P(2)} \right) \\ &\sim \frac{(-1)^m (\log \log x + P(2))^m}{m!}. \end{aligned}$$

Now by parameterizing the countour around $|z| = r := \frac{k-1}{\log \log x} < 1$ we deduce that the the main term of our approximation corresponds to

$$- \int_{|z|=r} \frac{x \cdot \exp(-P(2)z)(\log x)^{-z}}{(\log x) z^{k+1}(1 + P(2)z)} dz \asymp \frac{x}{\log x} \cdot \frac{(-1)^{k-1} (\log \log x + P(2))^k}{k!}. \quad \square$$

Lemma 7.4. *We have that as $x \rightarrow \infty$*

$$\left| \mathbb{E} \left[\sum_{n \leq x} \lambda(n) (-1)^{\omega(n)} C_{\Omega(n)}(n) \right] \right| \asymp \frac{P(2)}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\log \log x}}.$$

Proof. We observe that

$$\sum_{n \leq x} \lambda(n) (-1)^{\omega(n)} C_{\Omega(n)}(n) = \sum_{k=1}^{\log_2(x)} \sum_{\substack{n \leq x \\ \Omega(n)=k}} \lambda(n) (-1)^{\omega(n)} C_{\Omega(n)}(n) = \sum_{k=1}^{\log_2(x)} \hat{C}_k(x).$$

We claim that

$$\sum_{k=1}^{\log_2(x)} \hat{C}_k(x) \asymp \sum_{k=1}^{\log \log x} \hat{C}_k(x). \quad (30)$$

To prove (30), it suffices to show that

$$\left| \frac{\sum_{\log \log x < k \leq \log_2(x)} \hat{C}_k(x)}{\sum_{k=1}^{\log \log x} \hat{C}_k(x)} \right| = o(1), \text{ as } x \rightarrow \infty. \quad (31)$$

We first compute the absolute value of the following summatory function by applying Theorem 7.3 for large $x \rightarrow \infty$:

$$\begin{aligned} \left| \sum_{k=1}^{\log \log x} \hat{C}_k(x) \right| &\asymp \left| \frac{x}{\log x} - \frac{x \cdot e^{-P(2)} \Gamma(\log \log x, -(\log \log x) + P(2))}{(\log x)^2 \cdot \Gamma(\log \log x)} \right| \left[1 + O\left(\frac{1}{(\log \log x)^2}\right) \right] \\ &\sim \frac{x}{\log x} + \frac{x \cdot \sqrt{\log \log x}}{\sqrt{2\pi}(\log \log x + P(2))} (1 + o(1)). \end{aligned} \quad (32)$$

We define the following component sums for large x and $0 < \varepsilon < 1$ so that $(\log \log x)^{\varepsilon \frac{\log \log x}{\log \log \log x}} = o(\log x)$:

$$S_{2,\varepsilon}(x) := \sum_{P(2)^{-1} \log \log x < k \leq \log \log x} \hat{C}_k(x).$$

Then

$$\sum_{P(2)^{-1} \log \log x < k \leq (\log \log x)^{\varepsilon \frac{\log \log x}{\log \log \log x}}} \hat{C}_k(x) \gg S_{2,\varepsilon}(x),$$

with equality as $\varepsilon \rightarrow 1$ so that the upper bound of summation tends to $\log x$. To show that (31) holds, observe that whenever $\Omega(n) = k$, we have that $C_{\Omega(n)}(n) \leq k!$. We can bound the sum defined above using Theorem 5.4 for large $x \rightarrow \infty$ as

$$\begin{aligned} S_{2,\varepsilon}(x) &\leq \sum_{\log \log x \leq k \leq \log \log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \ll \sum_{k=\log \log x}^{(\log \log x)^{\varepsilon \frac{\log \log x}{\log \log \log x}}} \frac{\hat{\pi}_k(x)}{x} \cdot k! \\ &\ll \sum_{k=\log \log x}^{(\log \log x)^{\varepsilon \frac{\log \log x}{\log \log \log x}}} (\log x)^{\frac{k}{\log \log x} - 1 - \frac{k}{\log \log x} (\log k - \log \log \log x)} \cdot \left(\frac{k}{e}\right)^k \sqrt{2\pi k} \\ &\ll \sum_{k=\log \log x}^{\varepsilon \frac{\log \log x}{\log \log \log x}} (\log x)^{k \frac{\log \log \log x}{\log \log x} - 1} \sqrt{k} \ll \frac{1}{(\log x)} \times \int_{\log \log x}^{\varepsilon \frac{\log \log x}{\log \log \log x}} (\log \log x)^t \sqrt{t} \cdot dt \\ &\ll \frac{1}{(\log x)} \sqrt{\frac{\varepsilon \cdot \log \log x}{\log \log \log x}} (\log \log x)^{\frac{\varepsilon \cdot \log \log x}{\log \log \log x}} = o(x), \end{aligned}$$

where $\lim_{x \rightarrow \infty} (\log x)^{\frac{1}{\log \log x}} = e$. By (32) this form of the ratio in (31) clearly tends to zero. If we have a contribution from the terms $\hat{\pi}_k(x)$ as $\varepsilon \rightarrow 1$, e.g., if x is a power of two, then $C_{\Omega(x)}(x) = 1$ by the formula in (9), so that the contribution from this upper-most indexed term is negligible:

$$x = 2^k \implies \Omega(x) = k \implies C_{\Omega(x)}(x) = \frac{(\Omega(x))!}{k!} = 1.$$

The formula for the expectation claimed in the statement of this lemma above then follows from (32) by scaling by $\frac{1}{x}$ and dropping the asymptotically lesser error terms in the bound. \square

Remark 7.5. The signs of the functions estimated in Theorem 7.3 are dictated by the differences of the prime omega functions as $(-1)^{\Omega(n)-\omega(n)}$. It happens, as we have summarized above, that this distribution is fairly regular with limiting asymptotic densities of the distinct values of the difference between the additive functions. This signedness property, in place of the more natural $\lambda(n)$ weights as appear in Proposition 4.1, is necessary to simplify the DGF expansion we used to obtain the asymptotics for the summatory functions $\widehat{A}_z(x)$ in Proposition 7.1. It also leads to additional cancellation in the corresponding summatory functions and the resulting average order expectations we would obtain from these sums in this raw form.

An exact DGF expression for $\lambda(n)C_{\Omega(n)}(n)$ is in fact very much complicated by the need to estimate the asymptotics of the coefficients of the right-hand-side products

$$\sum_{n \geq 1} \frac{\lambda(n)C_{\Omega(n)}(n)z^{\Omega(n)}}{(\Omega(n))! \cdot n^s} = \prod_p (2 - \exp(-z \cdot p^{-s}))^{-1}, \operatorname{Re}(s) > 1, |z| < \log 2.$$

It is unclear how to exactly, and effectively, bound the coefficients of powers of z in the DGF expansion defined by the last equation. We use an alternate method in the next corollary to obtain the asymptotics for the actual summatory functions on which we require tight average case bounds.

Corollary 7.6 (Summatory functions of unsigned sequences). *We have that for large $x \geq 2$ and $1 \leq k \leq \log \log x$*

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp \frac{3}{2\widehat{c}} \cdot \frac{x \cdot P(2)^{k-1}}{(\log x)^2} \left[\frac{(\log \log x)^{k-1}}{(k-1)!} - \frac{(\log \log x)^{k-2}}{(k-2)!} \right].$$

Proof. We handle transforming our previous results for the sum over the unsigned sequence $C_{\Omega(n)}(n)$ such that $\Omega(n) = k$. The argument basically boils down to approximating the smooth summatory function of $\lambda_*(n) := (-1)^{\Omega(n)-\omega(n)}$ using the weighted densities defined by (27). We then have an integral formula involving the non-sign-weighted sequence that results by again applying ordinary Abel summation (and integrating by parts) in the form of

$$\begin{aligned} \sum_{n \leq x} \lambda_*(n)h(n) &= \left(\sum_{n \leq x} \lambda_*(n) \right) h(x) - \int_1^x \left(\sum_{n \leq t} \lambda_*(n) \right) h'(t) dt \\ &\asymp \left\{ \begin{array}{ll} u_t = L_*(t) & v'_t = h'(t) dt \\ u'_t = L'_*(t) dt & v_t = h(t) \end{array} \right\} \int_1^x \frac{d}{dt} \left[\sum_{n \leq t} \lambda_*(n) \right] h(t) dt. \end{aligned} \quad (33)$$

Let the signed left-hand-side summatory function in (33) for our function be defined by

$$\begin{aligned} \widehat{C}_{k,*}(x) &:= \left| \sum_{\substack{n \leq x \\ \Omega(n)=k}} \lambda(n)(-1)^{\omega(n)} C_{\Omega(n)}(n) \right| \\ &= \frac{x}{P(2)\sqrt{x}} \cdot \frac{(\log x)^{\log P(2)}}{\sqrt{\log \log x}} \left[1 + O\left(\frac{1}{(\log \log x)^2} \right) \right], \end{aligned}$$

where the second equation follows from the proof of Theorem 7.3. Then by differentiating the formula we engineered well for ourselves in (33), and then summing over the uniform range of $1 \leq k \leq \log \log x$, we can recover an approximation to the unsigned summatory function for the sequence we need to bound in later results proved in this section.

We handle the sign weighted terms by defining and approximating the asymptotic main term of the following summatory function (*cf.* Table T.2 starting on page 54):

$$L_*(t) := \sum_{n \leq t} \lambda(n)(-1)^{\omega(n)} = \sum_{j=0}^{\log_2(t)} (-1)^j \cdot \#\{n \leq t : \Omega(n) - \omega(n) = j\}$$

$$\sim \sum_{j=0}^{\log_2(t)} \hat{c}t \cdot \frac{(-1)^j}{2^j} = \frac{2\hat{c}t}{3} + o(1), \text{ as } t \rightarrow \infty.$$

Then applying the formula from (??)

$$\begin{aligned} \hat{C}_*(x) &= \int_1^x L'_*(t) C_{\Omega(t)}(t) dt \implies C_{\Omega(x)}(x) \asymp \frac{\hat{C}'_*(x)}{L'_*(x)} \\ C_{\Omega(x)}(x) &\asymp \frac{3}{4\hat{c}\sqrt{2\pi}P(2)} \cdot \frac{(\log x)^{P(2)-1}}{(\log \log x)^{3/2}} [2(\log x)(\log \log x) + 2 \log P(2)(\log \log x) - 1] \\ &\sim \frac{3(\log x)^{P(2)}}{2\hat{c}\sqrt{2\pi}P(2)\sqrt{\log \log x}} =: \hat{C}_{**}(x). \end{aligned}$$

So by Abel summation (again integrating by parts using the same method), we obtain

$$\begin{aligned} \sum_{n \leq x} C_{\Omega(n)}(n) &\asymp \int \hat{C}_{**}(x) dx = (x-1) \int \frac{\hat{C}_{**}(x)}{x} dx \\ &= \frac{3(x-1)}{2\hat{c}\sqrt{2\pi}P(2)\sqrt{1+P(2)}} \operatorname{erfi}\left(\sqrt{1+P(2)} \cdot \sqrt{\log \log x}\right) \\ &= \frac{3x}{2\hat{c}\sqrt{2\pi}P(2)(1+P(2))} \cdot \frac{(\log x)^{1+P(2)}}{\sqrt{\log \log x}} \left[1 + O\left(\frac{1}{\log \log x}\right)\right]. \end{aligned}$$

In the previous equation, we have used a known asymptotic expansion of the function $\operatorname{erfi}(z)$ about infinity in the form of [3, §3.2]

$$\operatorname{erfi}(z) = \frac{e^{z^2}}{\sqrt{\pi}} \left(z^{-1} + \frac{1}{2}z^{-3} + \frac{3}{4}z^{-5} + \dots \right), \text{ as } |z| \rightarrow \infty.$$

This proves the first formula.

A similar method is used to adapt the asymptotic formulas for the sums $\hat{C}_k(x)$ we proved in Theorem 7.3. Namely, since

$$(x-1) \int \frac{d^2}{dx^2} \left[\frac{x}{\log x} \cdot \frac{P(2)^{k-1}(\log \log x)^{k-1}}{(k-1)!} \right] \cdot x dx = \frac{x \cdot P(1)^{k-1}}{(\log x)^2} \left[\frac{(\log \log x)^{k-2}}{(k-2)!} - \frac{(\log \log x)^{k-1}}{(k-1)!} \right],$$

we obtain the stated result using a similar procedure to what we just proved above. \square

(TODO) ...

Corollary 7.7 (Expectation formulas). *We have that as $n \rightarrow \infty$*

$$\mathbb{E}|g^{-1}(n)| \asymp \frac{9}{\pi^2 \hat{c} \sqrt{2\pi} P(2)} \cdot \frac{(\log x)^{P(2)-1}}{\sqrt{\log \log x}}.$$

Proof. We will use the second formula from Corollary 7.6 to obtain the tightest bound on $\mathbb{E}[C_{\Omega(n)}(n)]$ as $n \rightarrow \infty$:

$$\begin{aligned} \mathbb{E}[C_{\Omega(n)}(n)] &= \sum_{k=1}^{\log_2(x)} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_k(n) \\ &\sim \sum_{k=1}^{P(2) \log \log x} \frac{3 \cdot P(2)^{k-1}}{2\hat{c} \cdot (\log x)^2} \left[\frac{(\log \log x)^{k-1}}{(k-1)!} - \frac{(\log \log x)^{k-2}}{(k-2)!} \right] \\ &\sim \frac{3}{2\hat{c}} \cdot \frac{P(2)^{P(2) \log \log x - 1} (\log \log x)^{P(2) \log \log x - 2}}{(\log x)^2 \Gamma(P(2) \log \log x)} ((P(2) - 1) \log \log x - 1) \end{aligned}$$

$$\sim \frac{3(P(2) - 1)}{2\hat{c}\sqrt{2\pi P(2)}} \cdot \frac{1}{(\log x)^{2-P(2)}\sqrt{\log \log x}} \left[1 + O\left(\frac{1}{\log \log x}\right) \right].$$

This implies that

$$\begin{aligned} \int \frac{\mathbb{E}[C_{\Omega(x)}(x)]}{x} dx &= \frac{3\sqrt{P(2) - 1}}{2\hat{c}\sqrt{2\pi P(2)}} \times \operatorname{erfi}\left(\sqrt{P(2) - 1} \cdot \sqrt{\log \log x}\right) \\ &= \frac{3}{2\hat{c}\sqrt{2\pi P(2)}} \cdot \frac{(\log x)^{P(2)-1}}{\sqrt{\log \log x}} \left[1 + O\left(\frac{1}{\log \log x}\right) \right]. \end{aligned}$$

Therefore, by the formula we derived in the proof of Corollary 6.6 we have

$$\begin{aligned} \mathbb{E}|g^{-1}(n)| &= \frac{6}{\pi^2} \left[\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d < n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d} \right] + O(1) \\ &= \frac{9}{\pi^2 \hat{c} \sqrt{2\pi P(2)}} \cdot \frac{(\log x)^{P(2)-1}}{\sqrt{\log \log x}} \left[1 + O\left(\frac{1}{\log \log x}\right) \right]. \end{aligned} \quad \square$$

Theorem 7.8. *Let the mean and variance analogs be denoted by*

$$\mu_x(C) := P(2) \log \log x, \quad \text{and} \quad \sigma_x(C) := \sqrt{\mu_x(C)}$$

Set $Y > 0$ and suppose that $z \in [-Y, Y]$. Then we have uniformly for all $-Y \leq z \leq Y$ as $x \rightarrow \infty$ that

$$\frac{1}{x} \cdot \# \left\{ 2 \leq n \leq x : \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)} \leq z \right\} = \frac{3}{2\hat{c} \cdot (\log x)^{2-P(2)}} \cdot \Phi(z) + O\left(\frac{1}{(\log x)^{2-P(2)}(\log \log x)^{1/2}}\right).$$

In the previous equation, the denominator exponent is approximately $2 - P(2) \approx 1.54775$.

Proof. For large x and $n \leq x$, define the following auxiliary variables:

$$\alpha_n := \frac{C_{\Omega(n)}(n) - \mu_n(C)}{\sigma_n(C)}, \quad \beta_{n,x} := \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)}.$$

Let the corresponding densities (whose limiting distributions we must verify) be defined by the functions

$$\Phi_1(x, z) := \frac{1}{x} \cdot \#\{n \leq x : \alpha_n \leq z\},$$

and

$$\Phi_2(x, z) := \frac{1}{x} \cdot \#\{n \leq x : \beta_{n,x} \leq z\}.$$

We first argue that it suffices to consider the distribution of $\Phi_2(x, z)$ as $x \rightarrow \infty$ in place of $\Phi_1(x, z)$ to obtain our desired result statement. In particular, the difference of the two auxiliary variables is negligible as $x \rightarrow \infty$ for n, x taken over the ranges that contribute the non-trivial weight to the main term of each density function. We have for $\sqrt{x} \leq n \leq x$ and $C_{\Omega(n)}(n) \leq 2 \cdot \mu_x(C)$ that

$$|\alpha_n - \beta_{n,x}| \ll \frac{1}{\sigma_x(C)} \xrightarrow{x \rightarrow \infty} 0.$$

So we naturally prefer to estimate the easier forms of the distribution function $\Phi_2(x, z)$ when x is large, and for any fixed $z \in \mathbb{R}$. That is, we replace α_n by $\beta_{n,x}$ and estimate the limiting densities corresponding to these terms.

We use the second formula proved in Corollary 7.6, which holds uniformly for x large when $1 \leq k \leq P(2) \log \log x$, to estimate the required densities. Let $k \geq 1$ be a natural number defined by $k := t + P(2) \log \log x$. We write the small parameter $\delta_{t,x} := \frac{t}{P(2) \log \log x}$. When $|t| \leq \frac{1}{2} P(2) \log \log x$, we have by Stirling's formula that

$$P(2)^{k-1} \left[\frac{(\log \log x)^{k-1}}{(k-1)!} - \frac{(\log \log x)^{k-2}}{(k-2)!} \right] = \frac{e^t (\log x)^{P(2)}}{\sqrt{2\pi}} \frac{(1 + \delta_{t,x})^{-P(2) \log \log x + \frac{1}{2}}}{\sqrt{P(2) \log \log x}} \left[1 - \frac{1}{P(2) \log \log x} \right] \times$$

$$\times \left[1 + O\left(\frac{1}{\log \log x}\right) \right].$$

We have the uniform estimate $\log(1 + \delta_{t,x}) = \delta_{t,x} - \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)$ for $|\delta_{t,x}| \leq \frac{1}{2}$. So we can expand the factor in the previous equation as

$$\begin{aligned} (1 + \delta_{t,x})^{-P(2) \log \log x + \frac{1}{2}} &= \exp \left(\left(\frac{1}{2} - P(2) \log \log x \right) \times \left(\delta_{t,x} - \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3) \right) \right) \\ &= \exp \left(-t + \frac{t - t^2}{2P(2) \log \log x} - \frac{t^2 - 2t^3}{4P(2)^2 (\log \log x)^2} + O\left(\frac{|t|^3}{(\log \log x)^2}\right) \right). \end{aligned}$$

For both $|t| \leq (P(2) \log \log x)^{1/2}$ and $(P(2) \log \log x)^{1/2} < |t| \leq (P(2) \log \log x)^{2/3}$, we see that

$$\frac{t}{P(2) \log \log x} \ll \frac{1}{\sqrt{P(2) \log \log x}} + \frac{|t|^3}{(P(2) \log \log x)^2}.$$

Similarly, for $|t| \leq 1$ and $|t| > 1$, we see that

$$\frac{t^2}{(P(2) \log \log x)^2} \ll \frac{1}{\sqrt{P(2) \log \log x}} + \frac{|t|^3}{(P(2) \log \log x)^2}.$$

Combining these estimates with the previous computation, we can deduce that uniformly for $|t| \leq (P(2) \log \log x)^{2/3}$

$$\begin{aligned} P(2)^{k-1} \left[\frac{(\log \log x)^{k-1}}{(k-1)!} - \frac{(\log \log x)^{k-2}}{(k-2)!} \right] \\ = \frac{(\log x)^{P(2)}}{\sqrt{2\pi \log \log x}} (1 + o(1)) \times \exp \left(-\frac{t^2}{2P(2) \log \log x} \right) \times \\ \times \left[1 + O\left(\frac{1}{\sqrt{P(2) \log \log x}}\right) + O\left(\frac{|t|^3}{(P(2) \log \log x)^2}\right) \right]. \end{aligned}$$

In the function defined by $\mathcal{G}(z) = \exp(P(2)z)\zeta(2)^z/\Gamma(1+z)$ from Proposition 7.1 (and Theorem 7.3), by expanding the function about $z = 1$, we obtain that

$$\mathcal{G}\left(\frac{k-1}{\log \log x}\right) = G(1) + O\left(\frac{1+|t|}{\log \log x}\right) = e^{P(2)} \frac{6}{\pi^2} + O\left(\frac{1+|t|}{\log \log x}\right),$$

for t within this range. Hence, by the formula from Corollary 7.6,

$$\begin{aligned} \sum_{\substack{n \leq x \\ \Omega(\bar{n})=k}} C_{\Omega(n)}(n) &= \frac{x}{(\log x)^{2-P(2)} \sqrt{2\pi P(2) \log \log x}} \exp \left(-\frac{(k - P(2) \log \log x)}{2P(2) \log \log x} \right) \times \\ &\times \left[1 + O\left(\frac{1}{\sqrt{P(2) \log \log x}}\right) + O\left(\frac{|k - P(2) \log \log x|^3}{(P(2) \log \log x)^2}\right) \right]. \end{aligned}$$

We have already worked out in as in the proof of Lemma 7.4 that the contributions of these terms for $k \leq P(2) \log \log x - (P(2) \log \log x)^{2/3}$ is negligible using an argument based on Theorem 5.4. As $x \rightarrow \infty$, the three terms that result (one main term, two error terms) can be considered to correspond to a Riemann sum for an associated integral. So the stated result follows. \square

Corollary 7.9. *Let $Y > 0$ and $z \in [-Y, Y]$. Then uniformly for all $-Y \leq z \leq Y$ as $x \rightarrow \infty$ we have that*

$$\begin{aligned} \frac{1}{x} \cdot \#\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq z\} &= \frac{3}{2\widehat{c} \cdot (\log x)^{2-P(2)}} \cdot \Phi\left(\frac{\frac{\pi^2}{6}z - \mu_x(C)}{\sigma_x(C)}\right) \\ &+ O\left(\frac{1}{(\log x)^{2-P(2)} (\log \log x)^{1/2}}\right). \end{aligned}$$

Proof. We compute using the argument sketched in the proof of Corollary 6.6 from Section 6.3 that

$$|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2} C_{\Omega(n)}(n).$$

Then the result follows from Theorem 7.8. \square

Remark 7.10 (Closeness of $|g^{-1}(n)|$ to its average order). We can compute using Corollary 7.9 that

$$\frac{\pi^2}{4\hat{c} \cdot (\log x)^{2-P(2)} \sigma_x(C)} \times \int_{-\infty}^{\infty} z \cdot \Phi' \left(\frac{\frac{\pi^2}{6} z - \mu_x(C)}{\sigma_x(C)} \right) dz = \frac{9}{\pi^2 \hat{c} (\log x)^{2-P(2)} \sqrt{\log \log x}} + o(1) \xrightarrow{x \rightarrow \infty} 0.$$

So we interpret this calculation to mean that the contribution from the sum over $|g^{-1}(n)|$ where $g^{-1}(n)$ is not very close to its average order is essentially negligible. We will use this property in the proof of Theorem 7.12 in the next subsection.

7.2 Establishing initial lower bounds on the summatory functions $G^{-1}(x)$

Definition 7.11. Let the summatory function $G_E^{-1}(x)$ be defined for $x \geq 1$ by

$$G_E^{-1}(x) := \sum_{n \leq (\log x)^{\frac{7}{3}} (\log \log x)} \lambda(n) \times \sum_{\substack{d|n \\ d > e}} \frac{(\log d)^{\frac{3}{4}}}{\log \log d}. \quad (34)$$

The subscript of E is a formality of notation that does not correspond to an actual parameter or any implicit dependence on E in the function defined above.

Theorem 7.12. *For all sufficiently large integers $x \rightarrow \infty$, we have that*

$$|G^{-1}(x)| \gg |G_E^{-1}(x)|.$$

Proof. First, consider the following upper bound on $|G_E^{-1}(x)|$:

$$\begin{aligned} |G_E^{-1}(x)| &= \left| \sum_{e \leq n \leq (\log x)^{7/3} (\log \log x)} \lambda(n) \times \sum_{\substack{d|n \\ d > e}} \frac{(\log d)^{\frac{3}{4}}}{\log \log d} \right| \\ &\ll \sum_{e < d \leq (\log x)^{7/3} (\log \log x)} \frac{(\log d)^{\frac{3}{4}}}{\log \log d} \cdot \left\lfloor \frac{(\log x)^{7/3} (\log \log x)}{d} \right\rfloor \\ &\ll (\log x)^{7/3} (\log \log x) \times \int_e^{(\log x)^{7/3} (\log \log x)} \frac{(\log t)^{\frac{3}{4}}}{t \cdot \log \log t} dt \\ &= (\log x)^{7/3} (\log \log x) \times \text{Ei} \left(\frac{7}{4} \log \log \left((\log x)^{7/3} (\log \log x) \right) \right) \\ &\ll (\log x)^{7/3} (\log \log x) (\log \log \log x)^2. \end{aligned} \quad (35)$$

We need a couple of observations to sum $G^{-1}(x)$ in absolute value and bound it from below. First, as noted in [11, §7.4], the function $\mathcal{G}(z)$ from Theorem 3.6 satisfies

$$\mathcal{G} \left(\frac{k-1}{\log \log x} \right) = 1 + O(1), k \leq \log \log x,$$

so that uniformly for $1 \leq k \leq \log \log x$ we can write

$$\hat{\pi}_k(x) \asymp \frac{x}{\log x} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \left[1 + O \left(\frac{1}{\log \log x} \right) \right].$$

Now by Corollary 5.7, the following summatory function represents the asymptotic main term in the summation $\sum_{n \leq x} \lambda(n)$ as $x \rightarrow \infty$:

$$\begin{aligned} \widehat{L}_2(x) &= \sum_{k=1}^{\log \log x} (-1)^k \widehat{\pi}_k(x) = -\frac{x}{(\log x)^2} \cdot \Gamma(\log \log x, -\log \log x) \\ &\sim \frac{(-1)^{\lfloor \log \log x \rfloor} \cdot x}{\sqrt{2\pi} \sqrt{\log \log x}} \end{aligned}$$

So by Abel summation and integration by parts, we obtain

$$\begin{aligned} |G^{-1}(x)| &= \left| \int_2^x \widehat{L}_2'(t) |g^{-1}(t)| dt \right| \\ &\gg \left| \sum_{k=1}^{\frac{\log \log x}{2}} \left[\widehat{L}_2'(e^{2k}) |g^{-1}(e^{2k})| e^{(1-e^{-1})e^{2k}} - \widehat{L}_2'(e^{2k+1}) |g^{-1}(e^{2k+1})| e^{(1-e^{-1})e^{2k+1}} \right] \right| \\ &\gg \left| \int_{\frac{\log \log x}{2} - \frac{1}{2}}^{\frac{\log \log x}{2}} \widehat{L}_2'(e^{2t}) |g^{-1}(e^{2t})| e^{e^t} dt \right| \gg \frac{e^{(P(2)-1)2t}}{(2t)} e^{e^{2t}} \Big|_{t=\frac{\log \log x}{2} - \frac{1}{2}} \\ &\gg \frac{x \cdot (\log x)^{P(2)-\frac{3}{2}}}{(\log \log x)}. \end{aligned}$$

In the previous equations, we have used the approximation argument to $g^{-1}(n)$ by its average order based on the computation from Remark 7.10. So naturally from (35) we have proved that as $x \rightarrow \infty$, $|G^{-1}(x)| \gg |G_E^{-1}(x)|$. \square

Corollary 7.13. *We have that for almost every sufficiently large x , that as $x \rightarrow \infty$*

$$|G_E^{-1}(x)| \gg \frac{(\log x)^{\frac{7}{4}} (\log \log x)^{\frac{5}{4}}}{\sqrt{\log \log \log x}} \times \left| \sum_{e < d \leq \log x} \frac{\lambda(d) (\log d)^{\frac{3}{4}}}{d^{3/4} \cdot \log \log d} \right|.$$

Proof. Using the definition in (34), we obtain on average that^A

$$\begin{aligned} |G_E^{-1}(x)| &= \left| \sum_{n \leq (\log x)^{7/3} (\log \log x)} \lambda(n) \times \sum_{\substack{d|n \\ d > e}} \frac{(\log d)^{\frac{3}{4}}}{\log \log d} \right| \\ &= \left| \sum_{e < d \leq (\log x)^{7/3} (\log \log x)} \frac{(\log d)^{\frac{3}{4}}}{\log \log d} \times \sum_{n=1}^{\left\lfloor \frac{(\log x)^{7/3} (\log \log x)}{d} \right\rfloor} \lambda(dn) \right|. \end{aligned}$$

We see that by complete additivity of $\Omega(n)$ (complete multiplicativity of $\lambda(n)$) that

$$\sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn) = \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(d) \times \lambda(n) = \lambda(d) \times \sum_{n \leq \left\lfloor \frac{x}{d} \right\rfloor} \lambda(n).$$

^AFor any arithmetic functions f, h , we have that [1, cf. §3.10; §3.12]

$$\sum_{n \leq x} h(n) \times \sum_{d|n} f(d) = \sum_{d \leq x} f(d) \times \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} h(dn).$$

From Theorem 3.7 and Lemma 8.2 (see below), we can establish that

$$\left| \sum_{k \leq \log \log x} (-1)^k \cdot \widehat{\pi}_k(x) \right| \gg \frac{x^{\frac{3}{4}} (\log x)^{\frac{1}{2}}}{\sqrt{\log \log x}} =: \widehat{L}_0(x), \text{ as } x \rightarrow \infty. \quad (36)$$

The sign of the sum obtained by taking the right-hand-side of (36) without the absolute value operation is given by $(-1)^{1+\lfloor \log \log x \rfloor}$. The precise formula for the limiting lower bound stated above for $\widehat{L}_0(x)$ is computed by symbolic summation in *Mathematica* using the new bounds on $\widehat{\pi}_k(x)$ guaranteed by the theorem, and then by applying subsequent standard asymptotic estimates to the resulting formulas for large $x \rightarrow \infty$ in the form of (10c) and Stirling's formula. It follows that

$$|G_E^{-1}(x)| \gg \left| \sum_{e < d \leq (\log x)^{7/3} (\log \log x)} \frac{\lambda(d) (\log d)^{\frac{3}{4}}}{\log \log d} \times (-1)^{\left\lfloor \log \log \left(\frac{(\log x)^{7/3} (\log \log x)}{d} \right) \right\rfloor} \cdot \widehat{L}_0 \left(\frac{(\log x)^{7/3} (\log \log x)}{d} \right) \right|. \quad (37)$$

Outline for the remainder of the proof. We sketch the following steps remaining to prove our claimed lower bound on $|G_E^{-1}(x)|$:

- (A) We identify an initial subinterval \mathcal{R}_x where we can expect constant sign term contributions resulting from the inputs to the function \widehat{L}_0 involving both (d, x) for x large and d on this smaller subinterval.
- (B) We factor out easily bounded terms from the expansion of the monotone \widehat{L}_0 on this interval.
- (C) We determine additional asymptotic formulas we will refer to in later sections for the resulting lower bounds on $|G_E^{-1}(x)|$ that are formed by restricting the range of d in (37) to \mathcal{R}_x .
- (D) We argue that the sums of oscillatory terms on the upper end of the deleted interval for $d \in (e, (\log x)^{7/3} (\log \log x)] \setminus \mathcal{R}_x$ cannot generate trivial bounds by cancellation with the new lower bounds.

Part A. We will simplify (37) by proving that there are ranges of consecutive integers over which we obtain essentially constant sign contributions from the function $\widehat{L}_0((\log x)^{7/3} (\log \log x)/d)$ as $x \rightarrow \infty$. In particular, consider that

$$\begin{aligned} \log \log \left(\frac{(\log x)^{7/3} (\log \log x)}{d} \right) &= \log \log \left((\log x)^{7/3} (\log \log x) \right) \\ &\quad + \log \left(1 - \frac{\log d}{(\log x)^{7/3} (\log \log x) \log \left((\log x)^{7/3} (\log \log x) \right)} \right), \text{ as } x \rightarrow \infty. \end{aligned}$$

If we take $d \in (e, \log x] =: \mathcal{R}_x$, we have that

$$\frac{\log d}{(\log x)^{7/3} (\log \log x) \log \left((\log x)^{7/3} (\log \log x) \right)} = o(1) \rightarrow 0, \text{ as } x \rightarrow \infty.$$

For d within \mathcal{R}_x , we expect that for almost every x there are at most a handful of negligible cases of comparatively small order $d \leq d_{0,x}$ such that

$$\left\lfloor \log \log \left(\frac{(\log x)^{7/3} (\log \log x)}{d} \right) \right\rfloor \sim \left\lfloor \log \log \left((\log x)^{7/3} (\log \log x) \right) + o(1) \right\rfloor,$$

changes in parity transitioning from $d \mapsto d+1$. An argument making this assertion precise brings leads us to two primary cases that rely on the small-order distribution of the fractional parts $f_x := \{ \log \log \left((\log x)^{7/3} (\log \log x) \right) \}$ within $[0, 1)$ for large $x \rightarrow \infty$ and any $\log d \in \mathcal{R}_x$:

(1) If the fractional part $f_x = 0$, then

$$\left\lfloor \log \log \left(\frac{(\log x)^{7/3} (\log \log x)}{d} \right) \right\rfloor = \left\lfloor \log \log \left((\log x)^{7/3} (\log \log x) \right) \right\rfloor + \left\lfloor -\frac{\log d}{(\log x)^{7/3} (\log \log x) \log \left((\log x)^{7/3} (\log \log x) \right)} \right\rfloor.$$

This implies that provided that

$$-1 \leq -\frac{\log d}{(\log x)^{7/3} (\log \log x) \log \left((\log x)^{7/3} (\log \log x) \right)} < 0,$$

we obtain a constant multiplier as $\text{sgn} \left(\widehat{L}_0 \left(\frac{(\log x)^{7/3} (\log \log x)}{d} \right) \right)$ whenever $d \in \mathcal{R}_x$. Since d is positive and maximized at $\log x$, this condition clearly happens for any sufficiently large x .

(2) If the fractional part $f_x \in (0, 1)$, then

$$\left\lfloor \log \log \left(\frac{(\log x)^{7/3} (\log \log x)}{d} \right) \right\rfloor = \left\lfloor \log \log \left((\log x)^{7/3} (\log \log x) \right) \right\rfloor + \left\lfloor \left\{ \log \log \left((\log x)^{7/3} (\log \log x) \right) \right\} - \frac{\log d}{(\log x)^{7/3} (\log \log x) \log \left((\log x)^{7/3} (\log \log x) \right)} \right\rfloor.$$

Define shorthand notation for the function $\mathcal{B}(x) := (\log x)^{7/3} (\log \log x) \log \left((\log x)^{7/3} (\log \log x) \right)$. We require that

$$-1 \leq f_x - \frac{\log d}{\mathcal{B}(x)} < 0 \iff (1 + f_x) \cdot \mathcal{B}(x) \geq \log d > 0.$$

This property is similarly clearly attained for $d \in \mathcal{R}_x$ since $(1 + f_x) \cdot \mathcal{B}(x) \geq \mathcal{B}(x)$ as $x \rightarrow \infty$.

Part B. Provided that the sign term involving both d and x from (37) does not change for $d \in \mathcal{R}_x$, we can remove any oscillations in the sums due to sign changes in the monotonically decreasing function $\widehat{L}_0(d, x) := \widehat{L}_0 \left((\log x)^{7/3} (\log \log x) / d \right)$. The function $\widehat{L}_0(d, x)$ is monotone decreasing in the variable d for fixed x as we sum along the subinterval \mathcal{R}_x in ascending order. We can see that this function is decreasing in d by computing its partial derivative and evaluating the asymptotic main terms as having a leading negative sign for all large x . Thus we should select $d := \log x$ in (37) to obtain a global lower bound on $|G_E^{-1}(x)|$ if we truncate the sum to range only over the subset of original indices $d \in \mathcal{R}_x$.

Part C. Let the magnitudes of the signed remainder term sums be defined for all sufficiently large x by

$$R_E(x) := \left| \sum_{\log x < d \leq \frac{(\log x)^{7/3} (\log \log x)}{e^2}} \frac{\lambda(d) (\log d)^{\frac{3}{4}}}{\log \log d} \times (-1)^{\left\lfloor \log \log \left(\frac{(\log x)^{7/3} (\log \log x)}{d} \right) \right\rfloor} \cdot \widehat{L}_0 \left(\frac{(\log x)^{7/3} (\log \log x)}{d} \right) \right|.$$

Set the function $T_E(x)$ to correspond to the easily factored dependence of the less simply integrable factors in $\widehat{L}_0(d, x)$ when we set $d := \log x$ on \mathcal{R}_x . This function is defined for all large enough x as

$$T_E(x) \gg \frac{\log \left[(\log x)^{4/3} (\log \log x) \right]^{\frac{1}{2}}}{\sqrt{\log \log \left[(\log x)^{4/3} (\log \log x) \right]}} \gg \frac{(\log \log x)^{\frac{1}{2}}}{\sqrt{\log \log \log x}}. \quad (38)$$

Then in limiting cases the lower bounding function satisfies

$$S_{E,1}(x) := \left| \sum_{e < d \leq (\log x)^{7/3} (\log \log x)} \frac{\lambda(d) (\log d)^{\frac{3}{4}}}{\log \log d} \times (-1)^{\left\lfloor \log \log \left(\frac{(\log x)^{7/3} (\log \log x)}{d} \right) \right\rfloor} \widehat{L}_0 \left(\frac{(\log x)^{7/3} (\log \log x)}{d} \right) \right|$$

$$\begin{aligned}
 & \gg (\log x)^{\frac{4}{3}} (\log \log x) T_E(x) \times \left| \sum_{e < d \leq \log x} \frac{\lambda(d) (\log d)^{\frac{3}{4}}}{d^{3/4} \cdot \log \log d} \right| \\
 & \gg \frac{(\log x)^{\frac{4}{3}} (\log \log x)^{\frac{3}{2}}}{\sqrt{\log \log \log x}} \times \left| \sum_{e < d \leq \log x} \frac{\lambda(d) (\log d)^{\frac{3}{4}}}{d^{3/4} \cdot \log \log d} \right|.
 \end{aligned} \tag{39}$$

The formulas in (37) and (39) imply the following lower bound by the triangle inequality as $x \rightarrow \infty$:

$$|G_E^{-1}(x)| \gg |S_{E,1}(x) - R_E(x)| \gg S_{E,1}(x), \text{ as } x \rightarrow \infty. \tag{40}$$

We have claimed that we can in fact drop the sum terms over upper range of $d \notin \mathcal{R}_x$ and still obtain the asymptotic lower bound on $|G_E^{-1}(x)|$ stated in (40). To justify this step in the proof, we will provide limiting lower bounds on $R_E(x)$ that show that the contribution from the deleted interval in absolute value exceeds the magnitude of the corresponding sums over $d \in \mathcal{R}_x$ defined by $S_{E,1}(x)$ when x is large.

Part D. We want to arrange the signed weight coefficients $\varepsilon_{x,d} \mapsto \{\pm 1\}$ so that the function

$$M_{\pm}(x) := \min_{\varepsilon_{x,d} = \pm 1} \left| \sum_{\log x < d \leq \frac{(\log x)^{7/3} (\log \log x)}{e^2}} \frac{\varepsilon_{x,d} \cdot \lambda(d) (\log d)^{\frac{3}{4}}}{\log \log d} \times \widehat{L}_0 \left(\frac{(\log x)^{7/3} (\log \log x)}{d} \right) \right|,$$

is minimal. We need to prove that this minimal sum exceeds the bound for $S_E(x)$ given in (39) in asymptotic order. That is, we prove that

$$S_{E,1}^{(\ell)}(x) := \frac{(\log x)^{\frac{4}{3}} (\log \log x)^{\frac{3}{2}}}{\sqrt{\log \log \log x}} \times \left| \sum_{e < d \leq \log x} \frac{\lambda(d) (\log d)^{\frac{3}{4}}}{d^{3/4} \cdot \log \log d} \right| = o(M_{\pm}(x)), \text{ as } x \rightarrow \infty.$$

Notice that by considering the sum term in the previous definition as being unsigned, we have that

$$\begin{aligned}
 S_{E,1}^{(\ell)}(x) & \ll \frac{(\log x)^{\frac{4}{3}} (\log \log x)^{\frac{3}{2}}}{\sqrt{\log \log \log x}} \times \int_e^{\log x} \frac{(\log t)^{3/4}}{t \cdot (\log \log t)} dt \\
 & \ll \frac{(\log x)^{\frac{19}{12}} (\log \log x)^{\frac{3}{2}}}{\sqrt{\log \log \log x}} \times \text{Ei} \left(\frac{7}{4} \log \log \log x \right) \\
 & \ll (\log x)^{\frac{19}{12}} (\log \log x)^{\frac{3}{2}} (\log \log \log x)^{\frac{3}{2}}.
 \end{aligned} \tag{41}$$

We need to show that $M_{\pm}(x)$ always exceeds this bound. Since the function $L_0(x, d)$ is decreasing in d for $d \in \left(\log x, \frac{(\log x)^{7/3} (\log \log x)}{e^2} \right] =: \overline{\mathcal{R}}_x$, we obtain that

$$\widehat{L}_0 \left(\frac{(\log x)^5 (\log \log x)}{d} \right) \asymp \frac{(\log x) (\log \log x)^{5/4}}{d^{3/4} \cdot \sqrt{\log \log \log x}}, d \in \overline{\mathcal{R}}_x.$$

So we need to find a global lower bound on the sum

$$S_{\pm}(x) := \frac{(\log x) (\log \log x)^{5/4}}{\sqrt{\log \log \log x}} \times \left| \sum_{\log x < d \leq \frac{(\log x)^{7/3} (\log \log x)}{e^2}} \frac{\varepsilon_{x,d} \cdot \lambda(d) (\log d)^{\frac{3}{4}}}{d^{3/4} \cdot \log \log d} \right|,$$

that holds for any choice of the signed weights $\varepsilon_{x,d}$. Notice that for any $d > \log x$ and $\delta \geq 1$, by an expansion of convergent geometric and binomial series, the next difference of terms satisfies

$$\left| \frac{(\log d)^{\frac{3}{4}}}{d^{\frac{3}{4}} \cdot (\log \log d)} - \frac{\log(d + \delta)^{\frac{3}{4}}}{(d + \delta)^{\frac{3}{4}} \cdot \log \log(d + \delta)} \right|$$

$$\begin{aligned}
 &\sim \frac{\log(d+\delta)^{\frac{3}{4}}}{(d+\delta)^{\frac{3}{4}} \cdot \log \log(d+\delta)} \times \left| \frac{\left(1 - \frac{\delta}{\log(d+\delta)}\right)^{\frac{3}{4}}}{\left(1 - \frac{\delta}{(d+\delta)}\right)^{\frac{3}{4}} \left(1 - \frac{\delta}{(d+\delta) \log(d+\delta) \log \log(d+\delta)}\right)} - 1 \right| \\
 &\gg \frac{\delta}{(d+\delta)^{\frac{3}{4}} \log^{\frac{1}{4}}(d+\delta) \log \log(d+\delta)}.
 \end{aligned}$$

Let the number of sign changes of the terms in our sum on the interval $\overline{\mathcal{R}}_x$ be defined by

$$N_x := \# \{d \in \overline{\mathcal{R}}_x : \varepsilon_{x,d+1} \lambda(d+1) = -\varepsilon_{x,d} \lambda(d)\}.$$

Define the maximum (minimum) number of consecutively signed terms on this interval to be $\delta_{\max}(x), \delta_{\min}(x) \geq 1$. Then by the difference property we noted above, we have that for $t_k(x) := \log x + (2k+2)\delta_{\max}(x)$

$$\begin{aligned}
 \frac{S_{\pm}(x) \sqrt{\log \log \log x}}{(\log x)(\log \log x)^{5/4}} &\gg \left| \sum_{k=0}^{\frac{N_x}{2}} \frac{\delta_{\min}(x)}{t_k(x)^{\frac{3}{4}} \log(t_k(x))^{\frac{1}{4}} \log \log(t_k(x))} - \frac{\log^{\frac{3}{4}}(N_x)}{N_x^{\frac{3}{4}} \log \log(N_x)} \right| \\
 &\gg \left| \sum_{k=0}^{\frac{N_x}{2}} \frac{t_0(x)^{\frac{1}{4}} \cdot \delta_{\min}(x)}{t_k(x) \log(t_k(x))^{\frac{1}{4}} \log \log(t_k(x))} - \frac{\log^{\frac{3}{4}}(N_x)}{N_x^{\frac{3}{4}} \log \log(N_x)} \right| \\
 &\gg (\log x + 2\delta_{\max}(x))^{1/4} \times \int_0^{\frac{N_x}{2}} \frac{dk}{t_k(x) \log(t_k(x))^{\frac{1}{4}} \log \log(t_k(x))} \\
 &\gg (\log x + 2\delta_{\max}(x))^{1/4} \times \frac{\log \log \log(t_{\frac{N_x}{2}}(x))}{2\delta_{\max}(x)}.
 \end{aligned}$$

Now because $\delta_{\max}(x) \in [1, u_x - N_x]$ for $u_x := (\log x)^{7/3}(\log \log x)$, we have that

$$\frac{S_{\pm}(x) \sqrt{\log \log \log x}}{(\log x)(\log \log x)^{5/4}} \gg \log x \times \frac{\log \log \log((u_x - \delta_{\max}(x))\delta_{\max}(x))}{(u_x - \delta_{\max}(x))\delta_{\max}(x)} \quad (42)$$

By differentiating the right-hand-side of the previous equation scaled by $(\log x)^{-1}$, setting the derivative equal to zero, and solving a differential equation for $\delta_{\max}(x)$, we see that a lower bound occurs when $\delta_{\max}(x) = C$ or when $\delta_{\max}(x) \approx u_x$. In either case, we see that the right-hand-side of (42) is non-negligible. This property clearly implies that $S_{\pm}(x)$ is asymptotically larger than the maximum order bound on $S_{E,1}^{(\ell)}(x)$ we proved above in (41). \square

8 Lower bounds for $M(x)$ along infinite subsequences

8.1 Expanding the new formula for $M(x)$

Proposition 8.1. *For all sufficiently large x , we have that*

$$M(x) = G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{\sqrt{x}} G^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right]. \quad (43)$$

Proof. We know by applying Corollary 3.3 that

$$\begin{aligned} M(x) &= \sum_{k=1}^x g^{-1}(k) \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1 \right) \\ &= G^{-1}(x) + \sum_{k=1}^{x/2} g^{-1}(k) \pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) \end{aligned} \quad (44)$$

$$= G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{x/2-1} G^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right] \quad (45)$$

where the upper bound on the sum is truncated by the fact that $\pi(1) = 0$. We see that

$$\frac{x}{k} - \frac{x}{k+1} = \frac{x}{k(k+1)} \sim \frac{x}{k^2},$$

so that $\frac{x}{k^2} \geq 1 \implies k \leq \sqrt{x}$. Thus we can re-write the latter sum to obtain

$$M(x) = G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{\sqrt{x}} G^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right].$$

We will require more assumptions and information about the behavior of the summatory functions, $G^{-1}(x)$, before we can further bound and simplify this expression for $M(x)$. \square

8.1.1 A few more necessary results

We now use the superscript and subscript notation of (ℓ) not to denote a formal parameter to the functions we define below, but instead to denote that these functions form *lower bound* (rather than exact) approximations to other forms of the functions without the scripted (ℓ) .

Lemma 8.2. *Suppose that $\widehat{\pi}_k^{(\ell)}(x) = o(\widehat{\pi}_k(x))$ where $\widehat{\pi}_k^{(\ell)}(x) \geq 1$ for all integers $1 \leq k \leq \log \log x$ as $x \rightarrow \infty$. Let the weighted summatory functions be defined as*

$$\begin{aligned} A_{\Omega}^{(\ell)}(x) &:= \sum_{k \leq \log \log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x) \\ A_{\Omega}(x) &:= \sum_{k \leq \log \log x} (-1)^k \widehat{\pi}_k(x). \end{aligned}$$

Furthermore, suppose that $|A_{\Omega}(x)| \nrightarrow 0$ as $x \rightarrow \infty$ and that

$$\begin{aligned} \liminf_{k \rightarrow \infty} \frac{\widehat{\pi}_k^{(\ell)}(x)}{\widehat{\pi}_k(x)} &\geq x^{-\rho_0} \\ \limsup_{k \rightarrow \infty} \frac{\widehat{\pi}_k^{(\ell)}(x)}{\widehat{\pi}_k(x)} &\leq x^{-\rho_1}, \end{aligned}$$

as $x \rightarrow \infty$ for some $\rho_0, \rho_1 > 0$. Then for all sufficiently large x , we have that

$$|A_{\Omega}(x)| \gg |A_{\Omega}^{(\ell)}(x)|.$$

Proof. By the second conditions above, we find that

$$\begin{aligned} |A_\Omega(x) - A_\Omega^{(\ell)}(x)| &\leq |A_\Omega(x)| \left(1 - \inf_{1 \leq k \leq \log \log x} \frac{\widehat{\pi}_k^{(\ell)}(x)}{\widehat{\pi}_k(x)} \right) = |A_\Omega(x)|(1 + o(1)) \\ |A_\Omega(x) - A_\Omega^{(\ell)}(x)| &\geq |A_\Omega(x)| \left(1 - \sup_{1 \leq k \leq \log \log x} \frac{\widehat{\pi}_k^{(\ell)}(x)}{\widehat{\pi}_k(x)} \right) = |A_\Omega(x)|(1 + o(1)). \end{aligned}$$

Similarly, we can see that

$$|A_\Omega(x)|(1 + o(1)) \leq |A_\Omega(x) + A_\Omega^{(\ell)}(x)| \leq |A_\Omega(x)|(1 + o(1)).$$

This implies that

$$|A_\Omega(x)|(1 + o(1)) \ll \left| |A_\Omega(x)| \pm |A_\Omega^{(\ell)}(x)| \right| \ll |A_\Omega(x)|(1 + o(1)), \text{ as } x \rightarrow \infty.$$

Because we have that $|A_\Omega(x)| \nrightarrow 0$, the previous equation shows that $|A_\Omega^{(\ell)}(x)|$ is bounded above and below by a constant times $|A_\Omega(x)|$. In other words, $|A_\Omega(x)| \gg |A_\Omega^{(\ell)}(x)|$ whenever x is sufficiently large. \square

Proof of Lemma 3.8. We can form an accurate $C^1(\mathbb{R})$ approximation by the smoothness of $\widehat{\pi}_k^{(\ell)}(x)$ that allows us to apply the Abel summation formula using the summatory function $A_\Omega(t)$ for t on any bounded connected subinterval of $[1, \infty)$. Namely, we obtain

$$\begin{aligned} |F_\lambda(x)| &\gg \left| A_\Omega(x)f(x) - \int_{u_0}^x A_\Omega(t)f'(t)dt \right| \\ &\gg \left| |A_\Omega(x)f(x)| - \int_{u_0}^x |A_\Omega(t)f'(t)|dt \right| \\ &\gg \left| |A_\Omega^{(\ell)}(x)\widehat{\tau}_\ell(x)| - \int_{u_0}^x |A_\Omega(t)f'(t)|dt \right|. \end{aligned} \tag{46}$$

The stated lower bound formula for $|F_\lambda(x)|$ in (46) above is valid whenever

$$0 \leq \left| \frac{\sum_{\log \log t < k \leq \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_\Omega(t)} \right| \ll 2, \text{ as } t \rightarrow \infty,$$

Indeed, by Corollary 5.7, we have that the assertion above holds as $t \rightarrow \infty$.

Let the function

$$\widehat{I}_\ell(x) := \int_{\frac{\log \log x}{2} - \frac{1}{2}}^{\frac{\log \log x}{2}} \left| A_\Omega^{(\ell)}(e^{e^{2t}}) \widehat{\tau}'_\ell(e^{e^{2t}}) \right| e^{e^{2t}} dt.$$

We have to argue that following property of this function holds as $x \rightarrow \infty$:

$$\int_{u_0}^x |A_\Omega(t)f'(t)|dt \gg \widehat{I}_\ell(x).$$

To prove the property in the previous equation, observe that by hypothesis since $|A_\Omega(x)| \gg |A_\Omega^{(\ell)}(x)|$ as $x \rightarrow \infty$, we have that

$$\int_{u_0}^x |A_\Omega(t)f'(t)|dt \gg \int_{u_0}^x |A_\Omega(t)\widehat{\tau}'_\ell(t)|dt$$

$$\begin{aligned}
 & \gg \left| \sum_{k=u_0}^{\log \log x} (-1)^k \left| A_\Omega \left(e^{e^k} \right) \widehat{\tau}'_\ell \left(e^{e^k} \right) \right| \cdot \left(e^{e^k} - e^{e^{k-1}} \right) \right| \\
 & \gg \left| \sum_{k=u_0}^{\frac{\log \log x}{2}} \left[\left| A_\Omega \left(e^{e^{2k}} \right) \widehat{\tau}'_\ell \left(e^{e^{2k}} \right) \right| \cdot e^{e^{2k}} - \left| A_\Omega \left(e^{e^{2k-1}} \right) \widehat{\tau}'_\ell \left(e^{e^{2k-1}} \right) \right| \cdot e^{e^{2k-1}} \right] \right| \\
 & \gg \int_{\frac{\log \log x}{2} - \frac{1}{2}}^{\frac{\log \log x}{2}} \left| A_\Omega \left(e^{e^{2t}} \right) \widehat{\tau}'_\ell \left(e^{e^{2t}} \right) \right| e^{e^{2t}} dt \\
 & \gg \int_{\frac{\log \log x}{2} - \frac{1}{2}}^{\frac{\log \log x}{2}} \left| A_\Omega^{(\ell)} \left(e^{e^{2t}} \right) \widehat{\tau}'_\ell \left(e^{e^{2t}} \right) \right| e^{e^{2t}} dt. \quad \square
 \end{aligned}$$

Corollary 8.3 (Conditions on our central bounding functions). *Let the smooth bounding functions be defined for large $t \gg e$ as*

$$\begin{aligned}
 \widehat{\tau}_\ell(t) &:= \frac{(\log t)^{\frac{1}{4}}}{t^{\frac{1}{4}} \cdot (\log \log t)}, \\
 A_\Omega^{(\ell)}(t) &:= \frac{\widehat{C}_0}{\sqrt{2\pi}} \cdot \frac{t^{\frac{1}{4}}}{(\log t)^{\frac{1}{2}} \sqrt{\log \log t}}.
 \end{aligned}$$

Then we have that as $x \rightarrow \infty$

$$|G_E^{-1}(x)| \gg \frac{\widehat{C}_0}{2\sqrt{2\pi}} \cdot \frac{(\log x)^{5/4}}{(\log \log x)^{1/4} \sqrt{\log \log \log x}} \times \left| A_\Omega^{(\ell)}(\log x) \widehat{\tau}_\ell(\log x) - \int_{\frac{\log \log \log x}{2} - \frac{1}{2}}^{\frac{\log \log \log x}{2}} A_\Omega^{(\ell)} \left(e^{e^{2t}} \right) \widehat{\tau}_\ell \left(e^{e^{2t}} \right) e^{e^{2t}} dt \right|.$$

Proof. By Corollary 7.13, we have that

$$|G_E^{-1}(x)| \gg \frac{\widehat{C}_0}{2\sqrt{2\pi}} \cdot \frac{(\log x)^{5/4}}{(\log \log x)^{1/4} \sqrt{\log \log \log x}} \times \left| \sum_{e < d \leq \log x} \frac{\lambda(d)(\log d)^{1/4}}{d^{1/4} \cdot \log \log d} \right|, \text{ as } x \rightarrow \infty. \quad (47)$$

The crux of the remainder of the proof boils down to checking hypotheses in Lemma 8.2 and Lemma 3.8.

We first apply Lemma 8.2 with the lower bound function resulting from Theorem 3.7 as follows:

$$\widehat{\pi}_k^{(\ell)}(x) := \frac{\widehat{C}_0 x^{\frac{1}{4}}}{(\log x)^{\frac{3}{2}}} \frac{(\log \log x)^{k-1}}{(k-1)!}.$$

This shows that the necessary hypotheses on the function $A_\Omega^{(\ell)}(t)$ required by Lemma 3.8 are satisfied according to the sums for the function approximated by (36) for large t . This argument proves that all of the requirements in Lemma 3.8 on our choice of $\widehat{\tau}_\ell(t)$ are also satisfied. So the stated result follows from (47) and Lemma 3.8. \square

8.1.2 The proof of a central lower bound on the magnitude of $G_E^{-1}(x)$

The next central theorem is the last barrier required to prove Theorem 3.9 in the next subsection. Combined with Theorem 7.12 proved in the last section, the new lower bounds we establish below provide us with a sufficient mechanism to bound the formula from Proposition 8.1.

Theorem 8.4 (Asymptotics and bounds for the summatory function $G^{-1}(x)$). *Let $C_{\ell,1} > 0$ be the absolute constant defined by*

$$\widehat{C}_{\ell,1} = \frac{\widehat{C}_0^2}{32\pi} = \frac{(\log 2) \cdot \exp \left(-\frac{15}{16} (\log 2)^2 \right)}{8\sqrt{2\pi}} \approx 0.00792203.$$

We obtain the following limiting estimate for the bounding function $G_E^{-1}(x)$ as $x \rightarrow \infty$:

$$|G_E^{-1}(x)| \gg \frac{(8 - e^{1/4}) \hat{C}_{\ell,1} \cdot (\log x)^{5/4}}{\sqrt{\log \log x} \cdot (\log \log \log x)^2}.$$

Proof. We can form a lower summatory function indicating the signed contributions over the distinct parity of $\Omega(n)$ for all $n \leq x$ as follows by applying (10b) and Stirling's approximation as already noted in the proof of Corollary 7.13:

$$\left| A_{\Omega}^{(\ell)}(t) \right| = \left| \sum_{k \leq \log \log t} (-1)^k \hat{\pi}_k(t) \right| \gg \frac{\hat{C}_0}{\sqrt{2\pi}} \times \frac{t^{\frac{1}{4}}}{(\log t)^{\frac{1}{2}} \sqrt{\log \log t}}, \text{ as } t \rightarrow \infty. \quad (48)$$

We select the functions $\hat{\tau}_0(t) := \frac{(\log t)^{1/4}}{t^{1/4} \cdot \log \log t}$ and $-\hat{\tau}'_0(t) \gg \frac{(\log t)^{1/4}}{4t^{5/4} \cdot \log \log t}$ in the form of the next equation using the notation in Corollary 8.3.

$$-\hat{\tau}'_0(t) = -\frac{d}{dt} \left[\frac{(\log t)^{\frac{1}{4}}}{t^{\frac{1}{4}} \cdot \log \log t} \right] \gg \frac{(\log t)^{1/4}}{4t^{\frac{5}{4}} \cdot \log \log t} \quad (49)$$

Moreover, we have that we can select the initial form of the lower bound to be defined as follows:

$$\begin{aligned} G_E^{-1}(x) &\gg \frac{\hat{C}_0}{2\sqrt{2\pi}} \cdot \frac{(\log x)^{5/4}}{(\log \log x)^{1/4} \sqrt{\log \log \log x}} \times \\ &\times \left| A_{\Omega}^{(\ell)}(\log x) \hat{\tau}_0(\log x) - \int_{\frac{\log \log \log x}{2} - \frac{1}{2}}^{\frac{\log \log \log x}{2}} \left| A_{\Omega}^{(\ell)}(e^{e^{2t}}) \hat{\tau}'_0(e^{e^{2t}}) \right| e^{e^{2t}} dt \right|. \end{aligned} \quad (50)$$

We express the integrand function as the following function of t :

$$\hat{I}_{\ell}(t) := \left| A_{\Omega}^{(\ell)}(e^{e^{2t}}) \hat{\tau}'_0(e^{e^{2t}}) \right| e^{e^{2t}} \gg \frac{\hat{C}_0}{16\sqrt{\pi}} \cdot \frac{e^{-t/2}}{t^{3/2}}. \quad (51)$$

We find from the mean value theorem applied to the monotone function from (51) that

$$\begin{aligned} \frac{\hat{C}_0}{2\sqrt{2\pi}} \cdot \frac{(\log x)^{5/4}}{(\log \log x)^{1/4} \sqrt{\log \log \log x}} \times \int_{\frac{\log \log \log x}{2} - \frac{1}{2}}^{\frac{\log \log \log x}{2}} \hat{I}_{\ell}(t) dt &\ll \frac{1}{2} \hat{I}_{\ell} \left(\frac{\log \log \log x}{2} - \frac{1}{2} \right) \\ &= \frac{e^{1/4} \cdot \hat{C}_{\ell,1} \cdot (\log x)^{5/4}}{\sqrt{\log \log x} \cdot (\log \log \log x)^2}. \end{aligned} \quad (52)$$

Consider the following expansion for the leading term in the Abel summation formula from (50) for comparison with (52):

$$\frac{\hat{C}_0}{2\sqrt{2\pi}} \cdot \frac{(\log x)^{5/4}}{(\log \log x)^{1/4} \sqrt{\log \log \log x}} \times \left| A_{\Omega}^{(\ell)}(\log x) \hat{\tau}_0(\log x) \right| \gg \frac{8\hat{C}_{\ell,1} \cdot (\log x)^{5/4}}{\sqrt{\log \log x} \cdot (\log \log \log x)^2} \quad (53)$$

Hence, we conclude that we can take $|G_E^{-1}(x)|$ bounded below by the difference of terms in (53) and (52). \square

8.2 Proof of the unboundedness of the scaled Mertens function

Lemma 8.5. *For sufficiently large x we have that*

$$\sum_{k=1}^{\sqrt{x}} G^{-1}(k) \left[\frac{x}{k \cdot \log\left(\frac{x}{k}\right)} - \frac{x}{(k+1) \cdot \log\left(\frac{x}{k+1}\right)} \right] \sim \sum_{k=1}^{\sqrt{x}} G^{-1}(k) \left[\frac{x}{k \cdot \log\left(\frac{x}{k}\right)} - \frac{x}{(k+1) \cdot \log\left(\frac{x}{k}\right)} \right], \quad (A)$$

and

$$\sum_{k=1}^{\sqrt{x}} G^{-1}(k) \left[\frac{x}{k \cdot \log\left(\frac{x}{k}\right)} - \frac{x}{(k+1) \cdot \log\left(\frac{x}{k+1}\right)} \right] \approx \sum_{k=1}^{\sqrt{x}} G^{-1}(k) \frac{x}{k^2 \cdot \log(x/k)}. \quad (B)$$

Proof of (A). Indeed, this step is justified by writing

$$\begin{aligned} \frac{x}{(k+1) \log \left(\frac{x}{k+1} \right)} &= \frac{x}{k+1} \cdot \frac{1}{\left[\log \left(\frac{x}{k} \right) + \log \left(1 - \frac{1}{k+1} \right) \right]} = \frac{x}{(k+1) \log \left(\frac{x}{k} \right)} \cdot \frac{1}{1 + \frac{\log \left(1 - \frac{1}{k+1} \right)}{\log x \left[1 - \frac{\log k}{\log x} \right]}} \\ &\sim \frac{x}{(k+1) \log \left(\frac{x}{k} \right)}, \text{ as } x \rightarrow \infty. \end{aligned} \quad \square$$

Proof of (B). The correctness of this step is verified by seeing that for $\operatorname{Re}(s) > 1$, we have that

$$\left| \sum_{k \geq 1} \frac{G^{-1}(k)}{k^{s+1}} \right| = \left| \int_1^\infty \frac{G^{-1}(x)}{x^{s+1}} dx \right| = \left| \frac{1}{s \cdot (P(s) + 1) \zeta(s)} \right| < \infty.$$

When $s := \frac{3}{2}$, we obtain that

$$0 \leq \left| \sum_{k \geq 1} \frac{G^{-1}(k)}{k^2(k+1)} \right| \leq \left| \sum_{k \geq 1} \frac{G^{-1}(k)}{k^{\frac{5}{2}}} \right| < \infty.$$

The difference of the terms in forming the approximation in this step is bounded above and below by absolute constants as

$$\left| \sum_{k=1}^{\frac{x}{2}} G^{-1}(k) \left[\frac{1}{k^2} - \frac{1}{k(k+1)} \right] \right| \leq \left| \sum_{k=1}^{\frac{x}{2}} \frac{G^{-1}(k)}{k^2(k+1)} \right| = O(1). \quad \square$$

We finally address the main conclusion of our arguments given so far with the following proof:

Proof of Theorem 3.9. Define the infinite increasing subsequence, $\{x_{0,y}\}_{y \geq Y_0}$, by $x_{0,y} := e^{2e^{2y+1}}$ for the sequence indices y starting at some sufficiently large finite integer $Y_0 \gg 1$. We can verify that for sufficiently large $y \rightarrow \infty$, this infinitely tending subsequence is well defined as $x_{0,y+1} > x_{0,y}$ whenever $y \geq Y_0$. Given a fixed large infinitely tending y , we have some (at least one) point $\widehat{x}_0 \in [\sqrt{x}, \frac{x}{2}]$ defined such that $|G^{-1}(t)|$ is minimal and non-vanishing on the interval $\mathbb{X}_y := (\sqrt{x_{0,y}}, \sqrt{x_{0,y+1}}]$ in the form of

$$|G^{-1}(\widehat{x}_0)| := \min_{\substack{\sqrt{x_{0,y}} < t \leq \sqrt{x_{0,y+1}} \\ G^{-1}(t) \neq 0}} |G^{-1}(t)|.$$

Let the shorthand notation $|G_{\min}^{-1}(x_y)| := |G^{-1}(\widehat{x}_0)|$. In the last step, we observe that $G^{-1}(x) = 0$ for x on a set of asymptotic density *at least* bounded below by $\frac{1}{2}$, so that our claim is accurate as the integrand lower bound on this interval does not trivially vanish at large y . This happens since the sequence $g^{-1}(n)$ is non-zero for all $n \geq 1$, so that if we do encounter a zero of the summatory function at x , we find a non-zero function value at $x+1$.

We need to bound the prime counting function differences in the formula given by Proposition 8.1 in tandem with enforcing minimal values of the absolute value of $G^{-1}(k)$ for $k \in \mathbb{X}_y$. Before, we were unable to produce universal lower bounds, or even asymptotic main terms of bounds with sufficiently bounded error terms when we summed over $k \in [1, \sqrt{x}]$ in the original formula. In order to guarantee “good behavior”, e.g., so that the resulting error terms in the formulas we construct do not swallow the main terms we want to establish for $|M(x)|/\sqrt{x}$, we cannot immediately appeal to known error bounds for $\pi(x)$ when x is large. Unfortunately, the signedness and inherited cancellation of the summatory functions $G^{-1}(x)$ forces an appeal to even stronger conditions on these formulas. We will require the following known bounds on the prime counting function due to Rosser and Schoenfeld [17, Thm. 1] for large $x \gg 59$:

$$\frac{x}{\log x} \left(1 + \frac{1}{2 \log x} \right) \leq \pi(x) \leq \frac{x}{\log x} \left(1 + \frac{3}{2 \log x} \right). \quad (54)$$

Let the component function $U_M(y)$ be defined for all large y as

$$U_M(y) := - \sum_{k=1}^{\sqrt{\hat{x}_{0,y+1}}} |G^{-1}(k)| \left[\pi \left(\frac{\hat{x}_{0,y+1}}{k} \right) - \pi \left(\frac{\hat{x}_{0,y+1}}{k+1} \right) \right].$$

Combined with Lemma 8.5, these estimates on $\pi(x)$ lead to the following approximations that hold on the increasing sequences taken within the subintervals defined by \hat{x}_0 :

$$\begin{aligned} U_M(y) &\gg - \sum_{k=1}^{\sqrt{\hat{x}_{0,y+1}}} |G^{-1}(k)| \left[\frac{\hat{x}_{0,y+1}}{k \cdot \log \left(\frac{\hat{x}_{0,y+1}}{k} \right)} + \frac{\hat{x}_{0,y+1}}{2k \cdot \log^2 \left(\frac{\hat{x}_{0,y+1}}{k} \right)} - \frac{\hat{x}_{0,y+1}}{(k+1) \cdot \log \left(\frac{\hat{x}_{0,y+1}}{k+1} \right)} - \frac{3\hat{x}_{0,y+1}}{2(k+1) \cdot \log^2 \left(\frac{\hat{x}_{0,y+1}}{k+1} \right)} \right] \\ &\sim - \sum_{k=1}^{\sqrt{\hat{x}_{0,y+1}}} |G^{-1}(k)| \left[\frac{\hat{x}_{0,y+1}}{k^2 \cdot \log \left(\frac{\hat{x}_{0,y+1}}{k} \right)} + \frac{\hat{x}_{0,y+1}}{2k^2 \cdot \log^2 \left(\frac{\hat{x}_{0,y+1}}{k} \right)} \right] \\ &\gg - \sum_{k=1}^{\sqrt{\hat{x}_{0,y+1}}} \frac{\hat{x}_{0,y+1} \cdot |G^{-1}(k)|}{k^2} \left[\frac{1}{\log(\hat{x}_{0,y+1})} + \frac{1}{2 \log^2(\hat{x}_{0,y+1})} \right] \\ &\gg - \hat{x}_{0,y+1} |G_{\min}^{-1}(\hat{x}_0)| \left(\frac{1}{\log(\hat{x}_{0,y+1})} + \frac{1}{2 \log^2(\hat{x}_{0,y+1})} \right) \times \int_{\hat{x}_{0,y}}^{\hat{x}_{0,y+1}} \frac{dt}{t^2} \\ &\gg \sqrt{\hat{x}_{0,y+1}} \frac{|G_{\min}^{-1}(\hat{x}_0)|}{\log(\hat{x}_{0,y+1})} \times \left(1 + \frac{1}{\log(\hat{x}_{0,y+1})} \right). \end{aligned}$$

Now by applying the lower bounds proved in Theorem 7.12, we can see that in fact the following is true:

$$U_M(y) \gg \sqrt{\hat{x}_{0,y+1}} \times \frac{|G_{\min}^{-1}(\hat{x}_0)|}{\log(\hat{x}_{0,y+1})} + o(1), \text{ as } y \rightarrow \infty.$$

Now we need to assemble this bound on the summation term in the formula for $M(x)$ from Proposition 8.1 with the leading terms involving the summatory function G^{-1} . In particular, we need to argue that we can effectively drop these leading terms to obtain a lower bound. Then we succeed by applying Theorem 7.12 since the remaining terms given by the function $U_M(y)$ are infinitely tending as $y \rightarrow \infty$.

Namely, we clearly see from Theorem 7.12 and the proposition that

$$\begin{aligned} \frac{|M(\hat{x}_{0,y+1})|}{\sqrt{\hat{x}_{0,y+1}}} &\gg \frac{1}{\sqrt{\hat{x}_{0,y+1}}} \times \left| G^{-1}(\hat{x}_{0,y+1}) + G^{-1} \left(\frac{\hat{x}_{0,y+1}}{2} \right) + U_M(y) \right| \\ &\gg \frac{1}{\sqrt{\hat{x}_{0,y+1}}} \times |U_M(y)| \\ &\gg \text{TODO}. \end{aligned} \tag{55}$$

Finally, we evaluate the following limit to conclude unboundedness where $\sqrt{x_{0,y}} \rightarrow +\infty$ as $y \rightarrow +\infty$: **(TODO)**

...

$$\lim_{x \rightarrow \infty} \left[\frac{(\log x)^{\frac{1}{4}}}{(\log \log x)^{\frac{1}{2}} (\log \log \log x)^2} \right] = +\infty.$$

Remarks on this lower bound construction. There is a small, but nonetheless insightful point to explain about a technicality in stating (55). Namely, we are not asserting that $|M(x)|/\sqrt{x}$ grows unbounded along the precise subsequence of $x \mapsto \hat{x}_{0,y+1}$ itself as $y \rightarrow \infty$. Rather, we are asserting that the unboundedness of this function can be witnessed along some subsequence whose points are taken within a large interval window of $x \in (\sqrt{\hat{x}_{0,y}}, \sqrt{\hat{x}_{0,y+1}}]$ as $y \rightarrow \infty$. We choose to state the lower bound given on the right-hand-side of (55) using the nicely formulated monotone lower bound on $|G_E^{-1}(x)|$ we proved in Theorem 8.4 with $\hat{x}_{0,y+1} \geq \sqrt{\hat{x}_{0,y+1}}$ for all $y \geq Y_0$. \square

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T.1 Table: The Dirichlet inverse function $g^{-1}(n)$ and the distribution of its summatory function

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \hat{f}_1(n)$	$\frac{\sum_{d n} C_{\Omega(d)}^{(d)}}{ g^{-1}(n) }$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1 ¹	Y	N	1	0	1.0000000	1.000000	0.000000	1	1	0
2	2 ¹	Y	Y	-2	0	1.0000000	0.500000	0.500000	-1	1	-2
3	3 ¹	Y	Y	-2	0	1.0000000	0.333333	0.666667	-3	1	-4
4	2 ²	N	Y	2	0	1.5000000	0.500000	0.500000	-1	3	-4
5	5 ¹	Y	Y	-2	0	1.0000000	0.400000	0.600000	-3	3	-6
6	2 ¹ 3 ¹	Y	N	5	0	1.0000000	0.500000	0.500000	2	8	-6
7	7 ¹	Y	Y	-2	0	1.0000000	0.428571	0.571429	0	8	-8
8	2 ³	N	Y	-2	0	2.0000000	0.375000	0.625000	-2	8	-10
9	3 ²	N	Y	2	0	1.5000000	0.444444	0.555556	0	10	-10
10	2 ¹ 5 ¹	Y	N	5	0	1.0000000	0.500000	0.500000	5	15	-10
11	11 ¹	Y	Y	-2	0	1.0000000	0.454545	0.545455	3	15	-12
12	2 ² 3 ¹	N	N	-7	2	1.2857143	0.416667	0.583333	-4	15	-19
13	13 ¹	Y	Y	-2	0	1.0000000	0.384615	0.615385	-6	15	-21
14	2 ¹ 7 ¹	Y	N	5	0	1.0000000	0.428571	0.571429	-1	20	-21
15	3 ¹ 5 ¹	Y	N	5	0	1.0000000	0.466667	0.533333	4	25	-21
16	2 ⁴	N	Y	2	0	2.5000000	0.500000	0.500000	6	27	-21
17	17 ¹	Y	Y	-2	0	1.0000000	0.470588	0.529412	4	27	-23
18	2 ¹ 3 ²	N	N	-7	2	1.2857143	0.444444	0.555556	-3	27	-30
19	19 ¹	Y	Y	-2	0	1.0000000	0.421053	0.578947	-5	27	-32
20	2 ² 5 ¹	N	N	-7	2	1.2857143	0.400000	0.600000	-12	27	-39
21	3 ¹ 7 ¹	Y	N	5	0	1.0000000	0.428571	0.571429	-7	32	-39
22	2 ¹ 11 ¹	Y	N	5	0	1.0000000	0.454545	0.545455	-2	37	-39
23	23 ¹	Y	Y	-2	0	1.0000000	0.434783	0.565217	-4	37	-41
24	2 ³ 3 ¹	N	N	9	4	1.5555556	0.458333	0.541667	5	46	-41
25	5 ²	N	Y	2	0	1.5000000	0.480000	0.520000	7	48	-41
26	2 ¹ 13 ¹	Y	N	5	0	1.0000000	0.500000	0.500000	12	53	-41
27	3 ³	N	Y	-2	0	2.0000000	0.481481	0.518519	10	53	-43
28	2 ² 7 ¹	N	N	-7	2	1.2857143	0.464286	0.535714	3	53	-50
29	29 ¹	Y	Y	-2	0	1.0000000	0.448276	0.551724	1	53	-52
30	2 ¹ 3 ¹ 5 ¹	Y	N	-16	0	1.0000000	0.433333	0.566667	-15	53	-68
31	31 ¹	Y	Y	-2	0	1.0000000	0.419355	0.580645	-17	53	-70
32	2 ⁵	N	Y	-2	0	3.0000000	0.406250	0.593750	-19	53	-72
33	3 ¹ 11 ¹	Y	N	5	0	1.0000000	0.424242	0.575758	-14	58	-72
34	2 ¹ 17 ¹	Y	N	5	0	1.0000000	0.441176	0.558824	-9	63	-72
35	5 ¹ 7 ¹	Y	N	5	0	1.0000000	0.457143	0.542857	-4	68	-72
36	2 ² 3 ²	N	N	14	9	1.3571429	0.472222	0.527778	10	82	-72
37	37 ¹	Y	Y	-2	0	1.0000000	0.459459	0.540541	8	82	-74
38	2 ¹ 19 ¹	Y	N	5	0	1.0000000	0.473684	0.526316	13	87	-74
39	3 ¹ 13 ¹	Y	N	5	0	1.0000000	0.487179	0.512821	18	92	-74
40	2 ³ 5 ¹	N	N	9	4	1.5555556	0.500000	0.500000	27	101	-74
41	41 ¹	Y	Y	-2	0	1.0000000	0.487805	0.512195	25	101	-76
42	2 ¹ 3 ¹ 7 ¹	Y	N	-16	0	1.0000000	0.476190	0.523810	9	101	-92
43	43 ¹	Y	Y	-2	0	1.0000000	0.465116	0.534884	7	101	-94
44	2 ² 11 ¹	N	N	-7	2	1.2857143	0.454545	0.545455	0	101	-101
45	3 ² 5 ¹	N	N	-7	2	1.2857143	0.444444	0.555556	-7	101	-108
46	2 ¹ 23 ¹	Y	N	5	0	1.0000000	0.456522	0.543478	-2	106	-108
47	47 ¹	Y	Y	-2	0	1.0000000	0.446809	0.553191	-4	106	-110
48	2 ⁴ 3 ¹	N	N	-11	6	1.8181818	0.437500	0.562500	-15	106	-121

Table T.1: Computations with $g^{-1}(n) \equiv (\omega + 1)^{-1}(n)$ for $1 \leq n \leq 500$.

- The column labeled **Primes** provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled **Sqfree** and **PPower**, respectively, list inclusion of n in the sets of squarefree integers and the prime powers.
- The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\hat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.
- The last several columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{x} \cdot \#\{n \leq x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geq 1$.

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d n} C_{\Omega(d)}(d)}{ g^{-1}(n) }$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
49	7^2	N	Y	2	0	1.5000000	0.448980	0.551020	-13	108	-121
50	$2^1 5^2$	N	N	-7	2	1.2857143	0.440000	0.560000	-20	108	-128
51	$3^1 17^1$	Y	N	5	0	1.0000000	0.450980	0.549020	-15	113	-128
52	$2^2 13^1$	N	N	-7	2	1.2857143	0.442308	0.557692	-22	113	-135
53	53^1	Y	Y	-2	0	1.0000000	0.433962	0.566038	-24	113	-137
54	$2^1 3^3$	N	N	9	4	1.5555556	0.444444	0.555556	-15	122	-137
55	$5^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-10	127	-137
56	$2^3 7^1$	N	N	9	4	1.5555556	0.464286	0.535714	-1	136	-137
57	$3^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	4	141	-137
58	$2^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	9	146	-137
59	59^1	Y	Y	-2	0	1.0000000	0.474576	0.525424	7	146	-139
60	$2^2 3^1 5^1$	N	N	30	14	1.1666667	0.483333	0.516667	37	176	-139
61	61^1	Y	Y	-2	0	1.0000000	0.475410	0.524590	35	176	-141
62	$2^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	40	181	-141
63	$3^2 7^1$	N	N	-7	2	1.2857143	0.476190	0.523810	33	181	-148
64	2^6	N	Y	2	0	3.5000000	0.484375	0.515625	35	183	-148
65	$5^1 13^1$	Y	N	5	0	1.0000000	0.492308	0.507692	40	188	-148
66	$2^1 3^1 11^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	24	188	-164
67	67^1	Y	Y	-2	0	1.0000000	0.477612	0.522388	22	188	-166
68	$2^2 17^1$	N	N	-7	2	1.2857143	0.470588	0.529412	15	188	-173
69	$3^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	20	193	-173
70	$2^1 5^1 7^1$	Y	N	-16	0	1.0000000	0.471429	0.528571	4	193	-189
71	71^1	Y	Y	-2	0	1.0000000	0.464789	0.535211	2	193	-191
72	$2^3 3^2$	N	N	-23	18	1.4782609	0.458333	0.541667	-21	193	-214
73	73^1	Y	Y	-2	0	1.0000000	0.452055	0.547945	-23	193	-216
74	$2^1 37^1$	Y	N	5	0	1.0000000	0.459459	0.540541	-18	198	-216
75	$3^1 5^2$	N	N	-7	2	1.2857143	0.453333	0.546667	-25	198	-223
76	$2^2 19^1$	N	N	-7	2	1.2857143	0.447368	0.552632	-32	198	-230
77	$7^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-27	203	-230
78	$2^1 3^1 13^1$	Y	N	-16	0	1.0000000	0.448718	0.551282	-43	203	-246
79	79^1	Y	Y	-2	0	1.0000000	0.443038	0.556962	-45	203	-248
80	$2^4 5^1$	N	N	-11	6	1.8181818	0.437500	0.562500	-56	203	-259
81	3^4	N	Y	2	0	2.5000000	0.444444	0.555556	-54	205	-259
82	$2^1 41^1$	Y	N	5	0	1.0000000	0.451220	0.548780	-49	210	-259
83	83^1	Y	Y	-2	0	1.0000000	0.445783	0.554217	-51	210	-261
84	$2^2 3^1 7^1$	N	N	30	14	1.1666667	0.452381	0.547619	-21	240	-261
85	$5^1 17^1$	Y	N	5	0	1.0000000	0.458824	0.541176	-16	245	-261
86	$2^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	-11	250	-261
87	$3^1 29^1$	Y	N	5	0	1.0000000	0.471264	0.528736	-6	255	-261
88	$2^3 11^1$	N	N	9	4	1.5555556	0.477273	0.522727	3	264	-261
89	89^1	Y	Y	-2	0	1.0000000	0.471910	0.528090	1	264	-263
90	$2^1 3^2 5^1$	N	N	30	14	1.1666667	0.477778	0.522222	31	294	-263
91	$7^1 13^1$	Y	N	5	0	1.0000000	0.483516	0.516484	36	299	-263
92	$2^2 23^1$	N	N	-7	2	1.2857143	0.478261	0.521739	29	299	-270
93	$3^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	34	304	-270
94	$2^1 47^1$	Y	N	5	0	1.0000000	0.489362	0.510638	39	309	-270
95	$5^1 19^1$	Y	N	5	0	1.0000000	0.494737	0.505263	44	314	-270
96	$2^5 3^1$	N	N	13	8	2.0769231	0.500000	0.500000	57	327	-270
97	97^1	Y	Y	-2	0	1.0000000	0.494845	0.505155	55	327	-272
98	$2^1 7^2$	N	N	-7	2	1.2857143	0.489796	0.510204	48	327	-279
99	$3^2 11^1$	N	N	-7	2	1.2857143	0.484848	0.515152	41	327	-286
100	$2^2 5^2$	N	N	14	9	1.3571429	0.490000	0.510000	55	341	-286
101	101^1	Y	Y	-2	0	1.0000000	0.485149	0.514851	53	341	-288
102	$2^1 3^1 17^1$	Y	N	-16	0	1.0000000	0.480392	0.519608	37	341	-304
103	103^1	Y	Y	-2	0	1.0000000	0.475728	0.524272	35	341	-306
104	$2^3 13^1$	N	N	9	4	1.5555556	0.480769	0.519231	44	350	-306
105	$3^1 5^1 7^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	28	350	-322
106	$2^1 53^1$	Y	N	5	0	1.0000000	0.481132	0.518868	33	355	-322
107	107^1	Y	Y	-2	0	1.0000000	0.476636	0.523364	31	355	-324
108	$2^2 3^3$	N	N	-23	18	1.4782609	0.472222	0.527778	8	355	-347
109	109^1	Y	Y	-2	0	1.0000000	0.467890	0.532110	6	355	-349
110	$2^1 5^1 11^1$	Y	N	-16	0	1.0000000	0.463636	0.536364	-10	355	-365
111	$3^1 37^1$	Y	N	5	0	1.0000000	0.468468	0.531532	-5	360	-365
112	$2^4 7^1$	N	N	-11	6	1.8181818	0.464286	0.535714	-16	360	-376
113	113^1	Y	Y	-2	0	1.0000000	0.460177	0.539823	-18	360	-378
114	$2^1 3^1 19^1$	Y	N	-16	0	1.0000000	0.456140	0.543860	-34	360	-394
115	$5^1 23^1$	Y	N	5	0	1.0000000	0.460870	0.539130	-29	365	-394
116	$2^2 29^1$	N	N	-7	2	1.2857143	0.456897	0.543103	-36	365	-401
117	$3^2 13^1$	N	N	-7	2	1.2857143	0.452991	0.547009	-43	365	-408
118	$2^1 59^1$	Y	N	5	0	1.0000000	0.457627	0.542373	-38	370	-408
119	$7^1 17^1$	Y	N	5	0	1.0000000	0.462185	0.537815	-33	375	-408
120	$2^3 3^1 5^1$	N	N	-48	32	1.3333333	0.458333	0.541667	-81	375	-456
121	11^2	N	Y	2	0	1.5000000	0.462810	0.537190	-79	377	-456
122	$2^1 61^1$	Y	N	5	0	1.0000000	0.467213	0.532787	-74	382	-456
123	$3^1 41^1$	Y	N	5	0	1.0000000	0.471545	0.528455	-69	387	-456
124	$2^2 31^1$	N	N	-7	2	1.2857143	0.467742	0.532258	-76	387	-463

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d n} C_{\Omega(d)}(d)}{ g^{-1}(n) }$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
125	5 ³	N	Y	-2	0	2.0000000	0.464000	0.536000	-78	387	-465
126	2 ¹ 3 ² 7 ¹	N	N	30	14	1.1666667	0.468254	0.531746	-48	417	-465
127	127 ¹	Y	Y	-2	0	1.0000000	0.464567	0.535433	-50	417	-467
128	2 ⁷	N	Y	-2	0	4.0000000	0.460938	0.539062	-52	417	-469
129	3 ¹ 43 ¹	Y	N	5	0	1.0000000	0.465116	0.534884	-47	422	-469
130	2 ¹ 5 ¹ 13 ¹	Y	N	-16	0	1.0000000	0.461538	0.538462	-63	422	-485
131	131 ¹	Y	Y	-2	0	1.0000000	0.458015	0.541985	-65	422	-487
132	2 ² 3 ¹ 11 ¹	N	N	30	14	1.1666667	0.462121	0.537879	-35	452	-487
133	7 ¹ 19 ¹	Y	N	5	0	1.0000000	0.466165	0.533835	-30	457	-487
134	2 ¹ 67 ¹	Y	N	5	0	1.0000000	0.470149	0.529851	-25	462	-487
135	3 ³ 5 ¹	N	N	9	4	1.5555556	0.474074	0.525926	-16	471	-487
136	2 ³ 17 ¹	N	N	9	4	1.5555556	0.477941	0.522059	-7	480	-487
137	137 ¹	Y	Y	-2	0	1.0000000	0.474453	0.525547	-9	480	-489
138	2 ¹ 3 ¹ 23 ¹	Y	N	-16	0	1.0000000	0.471014	0.528986	-25	480	-505
139	139 ¹	Y	Y	-2	0	1.0000000	0.467626	0.532374	-27	480	-507
140	2 ² 5 ¹ 7 ¹	N	N	30	14	1.1666667	0.471429	0.528571	3	510	-507
141	3 ¹ 47 ¹	Y	N	5	0	1.0000000	0.475177	0.524823	8	515	-507
142	2 ¹ 71 ¹	Y	N	5	0	1.0000000	0.478873	0.521127	13	520	-507
143	11 ¹ 13 ¹	Y	N	5	0	1.0000000	0.482517	0.517483	18	525	-507
144	2 ⁴ 3 ²	N	N	34	29	1.6176471	0.486111	0.513889	52	559	-507
145	5 ¹ 29 ¹	Y	N	5	0	1.0000000	0.489655	0.510345	57	564	-507
146	2 ¹ 73 ¹	Y	N	5	0	1.0000000	0.493151	0.506849	62	569	-507
147	3 ¹ 7 ²	N	N	-7	2	1.2857143	0.489796	0.510204	55	569	-514
148	2 ² 37 ¹	N	N	-7	2	1.2857143	0.486486	0.513514	48	569	-521
149	149 ¹	Y	Y	-2	0	1.0000000	0.483221	0.516779	46	569	-523
150	2 ¹ 3 ¹ 5 ²	N	N	30	14	1.1666667	0.486667	0.513333	76	599	-523
151	151 ¹	Y	Y	-2	0	1.0000000	0.483444	0.516556	74	599	-525
152	2 ³ 19 ¹	N	N	9	4	1.5555556	0.486842	0.513158	83	608	-525
153	3 ² 17 ¹	N	N	-7	2	1.2857143	0.483660	0.516340	76	608	-532
154	2 ¹ 7 ¹ 11 ¹	Y	N	-16	0	1.0000000	0.480519	0.519481	60	608	-548
155	5 ¹ 31 ¹	Y	N	5	0	1.0000000	0.483871	0.516129	65	613	-548
156	2 ² 3 ¹ 13 ¹	N	N	30	14	1.1666667	0.487179	0.512821	95	643	-548
157	157 ¹	Y	Y	-2	0	1.0000000	0.484076	0.515924	93	643	-550
158	2 ¹ 79 ¹	Y	N	5	0	1.0000000	0.487342	0.512658	98	648	-550
159	3 ¹ 53 ¹	Y	N	5	0	1.0000000	0.490566	0.509434	103	653	-550
160	2 ⁵ 5 ¹	N	N	13	8	2.0769231	0.493750	0.506250	116	666	-550
161	7 ¹ 23 ¹	Y	N	5	0	1.0000000	0.496894	0.503106	121	671	-550
162	2 ¹ 3 ⁴	N	N	-11	6	1.8181818	0.493827	0.506173	110	671	-561
163	163 ¹	Y	Y	-2	0	1.0000000	0.490798	0.509202	108	671	-563
164	2 ² 41 ¹	N	N	-7	2	1.2857143	0.487805	0.512195	101	671	-570
165	3 ¹ 5 ¹ 11 ¹	Y	N	-16	0	1.0000000	0.484848	0.515152	85	671	-586
166	2 ¹ 83 ¹	Y	N	5	0	1.0000000	0.487952	0.512048	90	676	-586
167	167 ¹	Y	Y	-2	0	1.0000000	0.485030	0.514970	88	676	-588
168	2 ³ 3 ¹ 7 ¹	N	N	-48	32	1.3333333	0.482143	0.517857	40	676	-636
169	13 ²	N	Y	2	0	1.5000000	0.485207	0.514793	42	678	-636
170	2 ¹ 5 ¹ 17 ¹	Y	N	-16	0	1.0000000	0.482353	0.517647	26	678	-652
171	3 ² 19 ¹	N	N	-7	2	1.2857143	0.479532	0.520468	19	678	-659
172	2 ² 43 ¹	N	N	-7	2	1.2857143	0.476744	0.523256	12	678	-666
173	173 ¹	Y	Y	-2	0	1.0000000	0.473988	0.526012	10	678	-668
174	2 ¹ 3 ¹ 29 ¹	Y	N	-16	0	1.0000000	0.471264	0.528736	-6	678	-684
175	5 ² 7 ¹	N	N	-7	2	1.2857143	0.468571	0.531429	-13	678	-691
176	2 ⁴ 11 ¹	N	N	-11	6	1.8181818	0.465909	0.534091	-24	678	-702
177	3 ¹ 59 ¹	Y	N	5	0	1.0000000	0.468927	0.531073	-19	683	-702
178	2 ¹ 89 ¹	Y	N	5	0	1.0000000	0.471910	0.528090	-14	688	-702
179	179 ¹	Y	Y	-2	0	1.0000000	0.469274	0.530726	-16	688	-704
180	2 ² 3 ² 5 ¹	N	N	-74	58	1.2162162	0.466667	0.533333	-90	688	-778
181	181 ¹	Y	Y	-2	0	1.0000000	0.464088	0.535912	-92	688	-780
182	2 ¹ 7 ¹ 13 ¹	Y	N	-16	0	1.0000000	0.461538	0.538462	-108	688	-796
183	3 ¹ 61 ¹	Y	N	5	0	1.0000000	0.464481	0.535519	-103	693	-796
184	2 ³ 23 ¹	N	N	9	4	1.5555556	0.467391	0.532609	-94	702	-796
185	5 ¹ 37 ¹	Y	N	5	0	1.0000000	0.470270	0.529730	-89	707	-796
186	2 ¹ 3 ¹ 31 ¹	Y	N	-16	0	1.0000000	0.467742	0.532258	-105	707	-812
187	11 ¹ 17 ¹	Y	N	5	0	1.0000000	0.470588	0.529412	-100	712	-812
188	2 ² 47 ¹	N	N	-7	2	1.2857143	0.468085	0.531915	-107	712	-819
189	3 ³ 7 ¹	N	N	9	4	1.5555556	0.470899	0.529101	-98	721	-819
190	2 ¹ 5 ¹ 19 ¹	Y	N	-16	0	1.0000000	0.468421	0.531579	-114	721	-835
191	191 ¹	Y	Y	-2	0	1.0000000	0.465969	0.534031	-116	721	-837
192	2 ⁶ 3 ¹	N	N	-15	10	2.3333333	0.463542	0.536458	-131	721	-852
193	193 ¹	Y	Y	-2	0	1.0000000	0.461140	0.538860	-133	721	-854
194	2 ¹ 97 ¹	Y	N	5	0	1.0000000	0.463918	0.536082	-128	726	-854
195	3 ¹ 5 ¹ 13 ¹	Y	N	-16	0	1.0000000	0.461538	0.538462	-144	726	-870
196	2 ² 7 ²	N	N	14	9	1.3571429	0.464286	0.535714	-130	740	-870
197	197 ¹	Y	Y	-2	0	1.0000000	0.461929	0.538071	-132	740	-872
198	2 ¹ 3 ² 11 ¹	N	N	30	14	1.1666667	0.464646	0.535354	-102	770	-872
199	199 ¹	Y	Y	-2	0	1.0000000	0.462312	0.537688	-104	770	-874
200	2 ³ 5 ²	N	N	-23	18	1.4782609	0.460000	0.540000	-127	770	-897

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum d n C_{\Omega(d)}(d)}{ g^{-1}(n) }$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
201	$3^1 67^1$	Y	N	5	0	1.0000000	0.462687	0.537313	-122	775	-897
202	$2^1 101^1$	Y	N	5	0	1.0000000	0.465347	0.534653	-117	780	-897
203	$7^1 29^1$	Y	N	5	0	1.0000000	0.467980	0.532020	-112	785	-897
204	$2^2 3^1 17^1$	N	N	30	14	1.1666667	0.470588	0.529412	-82	815	-897
205	$5^1 41^1$	Y	N	5	0	1.0000000	0.473171	0.526829	-77	820	-897
206	$2^1 103^1$	Y	N	5	0	1.0000000	0.475728	0.524272	-72	825	-897
207	$3^2 23^1$	N	N	-7	2	1.2857143	0.473430	0.526570	-79	825	-904
208	$2^4 13^1$	N	N	-11	6	1.8181818	0.471154	0.528846	-90	825	-915
209	$11^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	-85	830	-915
210	$2^1 3^1 5^1 7^1$	Y	N	65	0	1.0000000	0.476190	0.523810	-20	895	-915
211	211^1	Y	Y	-2	0	1.0000000	0.473934	0.526066	-22	895	-917
212	$2^2 53^1$	N	N	-7	2	1.2857143	0.471698	0.528302	-29	895	-924
213	$3^1 71^1$	Y	N	5	0	1.0000000	0.474178	0.525822	-24	900	-924
214	$2^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	-19	905	-924
215	$5^1 43^1$	Y	N	5	0	1.0000000	0.479070	0.520930	-14	910	-924
216	$2^3 3^3$	N	N	46	41	1.5000000	0.481481	0.518519	32	956	-924
217	$7^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	37	961	-924
218	$2^1 109^1$	Y	N	5	0	1.0000000	0.486239	0.513761	42	966	-924
219	$3^1 73^1$	Y	N	5	0	1.0000000	0.488584	0.511416	47	971	-924
220	$2^2 5^1 11^1$	N	N	30	14	1.1666667	0.490909	0.509091	77	1001	-924
221	$13^1 17^1$	Y	N	5	0	1.0000000	0.493213	0.506787	82	1006	-924
222	$2^1 3^1 37^1$	Y	N	-16	0	1.0000000	0.490991	0.509009	66	1006	-940
223	223^1	Y	Y	-2	0	1.0000000	0.488789	0.511211	64	1006	-942
224	$2^5 7^1$	N	N	13	8	2.0769231	0.491071	0.508929	77	1019	-942
225	$3^2 5^2$	N	N	14	9	1.3571429	0.493333	0.506667	91	1033	-942
226	$2^1 113^1$	Y	N	5	0	1.0000000	0.495575	0.504425	96	1038	-942
227	227^1	Y	Y	-2	0	1.0000000	0.493392	0.506608	94	1038	-944
228	$2^2 3^1 19^1$	N	N	30	14	1.1666667	0.495614	0.504386	124	1068	-944
229	229^1	Y	Y	-2	0	1.0000000	0.493450	0.506550	122	1068	-946
230	$2^1 5^1 23^1$	Y	N	-16	0	1.0000000	0.491304	0.508696	106	1068	-962
231	$3^1 7^1 11^1$	Y	N	-16	0	1.0000000	0.489177	0.510823	90	1068	-978
232	$2^3 29^1$	N	N	9	4	1.5555556	0.491379	0.508621	99	1077	-978
233	233^1	Y	Y	-2	0	1.0000000	0.489270	0.510730	97	1077	-980
234	$2^1 3^2 13^1$	N	N	30	14	1.1666667	0.491453	0.508547	127	1107	-980
235	$5^1 47^1$	Y	N	5	0	1.0000000	0.493617	0.506383	132	1112	-980
236	$2^2 59^1$	N	N	-7	2	1.2857143	0.491525	0.508475	125	1112	-987
237	$3^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	130	1117	-987
238	$2^1 7^1 17^1$	Y	N	-16	0	1.0000000	0.491597	0.508403	114	1117	-1003
239	239^1	Y	Y	-2	0	1.0000000	0.489540	0.510460	112	1117	-1005
240	$2^4 3^1 5^1$	N	N	70	54	1.5000000	0.491667	0.508333	182	1187	-1005
241	241^1	Y	Y	-2	0	1.0000000	0.489627	0.510373	180	1187	-1007
242	$2^1 11^2$	N	N	-7	2	1.2857143	0.487603	0.512397	173	1187	-1014
243	3^5	N	Y	-2	0	3.0000000	0.485597	0.514403	171	1187	-1016
244	$2^2 61^1$	N	N	-7	2	1.2857143	0.483607	0.516393	164	1187	-1023
245	$5^1 7^2$	N	N	-7	2	1.2857143	0.481633	0.518367	157	1187	-1030
246	$2^1 3^1 41^1$	Y	N	-16	0	1.0000000	0.479675	0.520325	141	1187	-1046
247	$13^1 19^1$	Y	N	5	0	1.0000000	0.481781	0.518219	146	1192	-1046
248	$2^3 31^1$	N	N	9	4	1.5555556	0.483871	0.516129	155	1201	-1046
249	$3^1 83^1$	Y	N	5	0	1.0000000	0.485944	0.514056	160	1206	-1046
250	$2^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	169	1215	-1046
251	251^1	Y	Y	-2	0	1.0000000	0.486056	0.513944	167	1215	-1048
252	$2^2 3^2 7^1$	N	N	-74	58	1.2162162	0.484127	0.515873	93	1215	-1122
253	$11^1 23^1$	Y	N	5	0	1.0000000	0.486166	0.513834	98	1220	-1122
254	$2^1 127^1$	Y	N	5	0	1.0000000	0.488189	0.511811	103	1225	-1122
255	$3^1 5^1 17^1$	Y	N	-16	0	1.0000000	0.486275	0.513725	87	1225	-1138
256	2^8	N	Y	2	0	4.5000000	0.488281	0.511719	89	1227	-1138
257	257^1	Y	Y	-2	0	1.0000000	0.486381	0.513619	87	1227	-1140
258	$2^1 3^1 43^1$	Y	N	-16	0	1.0000000	0.484496	0.515504	71	1227	-1156
259	$7^1 37^1$	Y	N	5	0	1.0000000	0.486486	0.513514	76	1232	-1156
260	$2^2 5^1 13^1$	N	N	30	14	1.1666667	0.488462	0.511538	106	1262	-1156
261	$3^2 29^1$	N	N	-7	2	1.2857143	0.486590	0.513410	99	1262	-1163
262	$2^1 131^1$	Y	N	5	0	1.0000000	0.488550	0.511450	104	1267	-1163
263	263^1	Y	Y	-2	0	1.0000000	0.486692	0.513308	102	1267	-1165
264	$2^3 3^1 11^1$	N	N	-48	32	1.3333333	0.484848	0.515152	54	1267	-1213
265	$5^1 53^1$	Y	N	5	0	1.0000000	0.486792	0.513208	59	1272	-1213
266	$2^1 7^1 19^1$	Y	N	-16	0	1.0000000	0.484962	0.515038	43	1272	-1229
267	$3^1 89^1$	Y	N	5	0	1.0000000	0.486891	0.513109	48	1277	-1229
268	$2^2 67^1$	N	N	-7	2	1.2857143	0.485075	0.514925	41	1277	-1236
269	269^1	Y	Y	-2	0	1.0000000	0.483271	0.516729	39	1277	-1238
270	$2^1 3^3 5^1$	N	N	-48	32	1.3333333	0.481481	0.518519	-9	1277	-1286
271	271^1	Y	Y	-2	0	1.0000000	0.479705	0.520295	-11	1277	-1288
272	$2^4 17^1$	N	N	-11	6	1.8181818	0.477941	0.522059	-22	1277	-1299
273	$3^1 7^1 13^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	-38	1277	-1315
274	$2^1 137^1$	Y	N	5	0	1.0000000	0.478102	0.521898	-33	1282	-1315
275	$5^2 11^1$	N	N	-7	2	1.2857143	0.476364	0.523636	-40	1282	-1322
276	$2^2 3^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	-10	1312	-1322
277	277^1	Y	Y	-2	0	1.0000000	0.476534	0.523466	-12	1312	-1324

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \hat{f}_1(n)$	$\frac{\sum d n C_{\Omega(d)}(d)}{ g^{-1}(n) }$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
278	$2^1 139^1$	Y	N	5	0	1.0000000	0.478417	0.521583	-7	1317	-1324
279	$3^2 31^1$	N	N	-7	2	1.2857143	0.476703	0.523297	-14	1317	-1331
280	$2^3 5^1 7^1$	N	N	-48	32	1.3333333	0.475000	0.525000	-62	1317	-1379
281	281^1	Y	Y	-2	0	1.0000000	0.473310	0.526690	-64	1317	-1381
282	$2^1 3^1 47^1$	Y	N	-16	0	1.0000000	0.471631	0.528369	-80	1317	-1397
283	283^1	Y	Y	-2	0	1.0000000	0.469965	0.530035	-82	1317	-1399
284	$2^2 71^1$	N	N	-7	2	1.2857143	0.468310	0.531690	-89	1317	-1406
285	$3^1 5^1 19^1$	Y	N	-16	0	1.0000000	0.466667	0.533333	-105	1317	-1422
286	$2^1 11^1 13^1$	Y	N	-16	0	1.0000000	0.465035	0.534965	-121	1317	-1438
287	$7^1 41^1$	Y	N	5	0	1.0000000	0.466899	0.533101	-116	1322	-1438
288	$2^5 3^2$	N	N	-47	42	1.7659574	0.465278	0.534722	-163	1322	-1485
289	17^2	N	Y	2	0	1.5000000	0.467128	0.532872	-161	1324	-1485
290	$2^1 5^1 29^1$	Y	N	-16	0	1.0000000	0.465517	0.534483	-177	1324	-1501
291	$3^1 97^1$	Y	N	5	0	1.0000000	0.467354	0.532646	-172	1329	-1501
292	$2^2 73^1$	N	N	-7	2	1.2857143	0.465753	0.534247	-179	1329	-1508
293	293^1	Y	Y	-2	0	1.0000000	0.464164	0.535836	-181	1329	-1510
294	$2^1 3^1 7^2$	N	N	30	14	1.1666667	0.465986	0.534014	-151	1359	-1510
295	$5^1 59^1$	Y	N	5	0	1.0000000	0.467797	0.532203	-146	1364	-1510
296	$2^3 37^1$	N	N	9	4	1.5555556	0.469595	0.530405	-137	1373	-1510
297	$3^3 11^1$	N	N	9	4	1.5555556	0.471380	0.528620	-128	1382	-1510
298	$2^1 149^1$	Y	N	5	0	1.0000000	0.473154	0.526846	-123	1387	-1510
299	$13^1 23^1$	Y	N	5	0	1.0000000	0.474916	0.525084	-118	1392	-1510
300	$2^2 3^1 5^2$	N	N	-74	58	1.2162162	0.473333	0.526667	-192	1392	-1584
301	$7^1 43^1$	Y	N	5	0	1.0000000	0.475083	0.524917	-187	1397	-1584
302	$2^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	-182	1402	-1584
303	$3^1 101^1$	Y	N	5	0	1.0000000	0.478548	0.521452	-177	1407	-1584
304	$2^4 19^1$	N	N	-11	6	1.8181818	0.476974	0.523026	-188	1407	-1595
305	$5^1 61^1$	Y	N	5	0	1.0000000	0.478689	0.521311	-183	1412	-1595
306	$2^1 3^2 17^1$	N	N	30	14	1.1666667	0.480392	0.519608	-153	1442	-1595
307	307^1	Y	Y	-2	0	1.0000000	0.478827	0.521173	-155	1442	-1597
308	$2^2 7^1 11^1$	N	N	30	14	1.1666667	0.480519	0.519481	-125	1472	-1597
309	$3^1 103^1$	Y	N	5	0	1.0000000	0.482201	0.517799	-120	1477	-1597
310	$2^1 5^1 31^1$	Y	N	-16	0	1.0000000	0.480645	0.519355	-136	1477	-1613
311	311^1	Y	Y	-2	0	1.0000000	0.479100	0.520900	-138	1477	-1615
312	$2^3 3^1 13^1$	N	N	-48	32	1.3333333	0.477564	0.522436	-186	1477	-1663
313	313^1	Y	Y	-2	0	1.0000000	0.476038	0.523962	-188	1477	-1665
314	$2^1 157^1$	Y	N	5	0	1.0000000	0.477707	0.522293	-183	1482	-1665
315	$3^2 5^1 7^1$	N	N	30	14	1.1666667	0.479365	0.520635	-153	1512	-1665
316	$2^2 79^1$	N	N	-7	2	1.2857143	0.477848	0.522152	-160	1512	-1672
317	317^1	Y	Y	-2	0	1.0000000	0.476341	0.523659	-162	1512	-1674
318	$2^1 3^1 53^1$	Y	N	-16	0	1.0000000	0.474843	0.525157	-178	1512	-1690
319	$11^1 29^1$	Y	N	5	0	1.0000000	0.476489	0.523511	-173	1517	-1690
320	$2^6 5^1$	N	N	-15	10	2.3333333	0.475000	0.525000	-188	1517	-1705
321	$3^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	-183	1522	-1705
322	$2^1 7^1 23^1$	Y	N	-16	0	1.0000000	0.475155	0.524845	-199	1522	-1721
323	$17^1 19^1$	Y	N	5	0	1.0000000	0.476780	0.523220	-194	1527	-1721
324	$2^2 3^4$	N	N	34	29	1.6176471	0.478395	0.521605	-160	1561	-1721
325	$5^2 13^1$	N	N	-7	2	1.2857143	0.476923	0.523077	-167	1561	-1728
326	$2^1 163^1$	Y	N	5	0	1.0000000	0.478528	0.521472	-162	1566	-1728
327	$3^1 109^1$	Y	N	5	0	1.0000000	0.480122	0.519878	-157	1571	-1728
328	$2^3 41^1$	N	N	9	4	1.5555556	0.481707	0.518293	-148	1580	-1728
329	$7^1 47^1$	Y	N	5	0	1.0000000	0.483283	0.516717	-143	1585	-1728
330	$2^1 3^1 5^1 11^1$	Y	N	65	0	1.0000000	0.484848	0.515152	-78	1650	-1728
331	331^1	Y	Y	-2	0	1.0000000	0.483384	0.516616	-80	1650	-1730
332	$2^2 83^1$	N	N	-7	2	1.2857143	0.481928	0.518072	-87	1650	-1737
333	$3^2 37^1$	N	N	-7	2	1.2857143	0.480480	0.519520	-94	1650	-1744
334	$2^1 167^1$	Y	N	5	0	1.0000000	0.482036	0.517964	-89	1655	-1744
335	$5^1 67^1$	Y	N	5	0	1.0000000	0.483582	0.516418	-84	1660	-1744
336	$2^4 3^1 7^1$	N	N	70	54	1.5000000	0.485119	0.514881	-14	1730	-1744
337	337^1	Y	Y	-2	0	1.0000000	0.483680	0.516320	-16	1730	-1746
338	$2^1 13^2$	N	N	-7	2	1.2857143	0.482249	0.517751	-23	1730	-1753
339	$3^1 113^1$	Y	N	5	0	1.0000000	0.483776	0.516224	-18	1735	-1753
340	$2^2 5^1 17^1$	N	N	30	14	1.1666667	0.485294	0.514706	12	1765	-1753
341	$11^1 31^1$	Y	N	5	0	1.0000000	0.486804	0.513196	17	1770	-1753
342	$2^1 3^2 19^1$	N	N	30	14	1.1666667	0.488304	0.511696	47	1800	-1753
343	7^3	N	Y	-2	0	2.0000000	0.486880	0.513120	45	1800	-1755
344	$2^3 43^1$	N	N	9	4	1.5555556	0.488372	0.511628	54	1809	-1755
345	$3^1 5^1 23^1$	Y	N	-16	0	1.0000000	0.486957	0.513043	38	1809	-1771
346	$2^1 173^1$	Y	N	5	0	1.0000000	0.488439	0.511561	43	1814	-1771
347	347^1	Y	Y	-2	0	1.0000000	0.487032	0.512968	41	1814	-1773
348	$2^2 3^1 29^1$	N	N	30	14	1.1666667	0.488506	0.511494	71	1844	-1773
349	349^1	Y	Y	-2	0	1.0000000	0.487106	0.512894	69	1844	-1775
350	$2^1 5^2 7^1$	N	N	30	14	1.1666667	0.488571	0.511429	99	1874	-1775

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum d n C_{\Omega(d)}(d)}{ g^{-1}(n) }$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
351	$3^3 13^1$	N	N	9	4	1.5555556	0.490028	0.509972	108	1883	-1775
352	$2^5 11^1$	N	N	13	8	2.0769231	0.491477	0.508523	121	1896	-1775
353	353^1	Y	Y	-2	0	1.0000000	0.490085	0.509915	119	1896	-1777
354	$2^1 3^1 59^1$	Y	N	-16	0	1.0000000	0.488701	0.511299	103	1896	-1793
355	$5^1 71^1$	Y	N	5	0	1.0000000	0.490141	0.509859	108	1901	-1793
356	$2^2 89^1$	N	N	-7	2	1.2857143	0.488764	0.511236	101	1901	-1800
357	$3^1 7^1 17^1$	Y	N	-16	0	1.0000000	0.487395	0.512605	85	1901	-1816
358	$2^1 179^1$	Y	N	5	0	1.0000000	0.488827	0.511173	90	1906	-1816
359	359^1	Y	Y	-2	0	1.0000000	0.487465	0.512535	88	1906	-1818
360	$2^3 3^2 5^1$	N	N	145	129	1.3034483	0.488889	0.511111	233	2051	-1818
361	19^2	N	Y	2	0	1.5000000	0.490305	0.509695	235	2053	-1818
362	$2^1 181^1$	Y	N	5	0	1.0000000	0.491713	0.508287	240	2058	-1818
363	$3^1 11^2$	N	N	-7	2	1.2857143	0.490358	0.509642	233	2058	-1825
364	$2^2 7^1 13^1$	N	N	30	14	1.1666667	0.491758	0.508242	263	2088	-1825
365	$5^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	268	2093	-1825
366	$2^1 3^1 61^1$	Y	N	-16	0	1.0000000	0.491803	0.508197	252	2093	-1841
367	367^1	Y	Y	-2	0	1.0000000	0.490463	0.509537	250	2093	-1843
368	$2^4 23^1$	N	N	-11	6	1.8181818	0.489130	0.510870	239	2093	-1854
369	$3^2 41^1$	N	N	-7	2	1.2857143	0.487805	0.512195	232	2093	-1861
370	$2^1 5^1 37^1$	Y	N	-16	0	1.0000000	0.486486	0.513514	216	2093	-1877
371	$7^1 53^1$	Y	N	5	0	1.0000000	0.487871	0.512129	221	2098	-1877
372	$2^2 3^1 31^1$	N	N	30	14	1.1666667	0.489247	0.510753	251	2128	-1877
373	373^1	Y	Y	-2	0	1.0000000	0.487936	0.512064	249	2128	-1879
374	$2^1 11^1 17^1$	Y	N	-16	0	1.0000000	0.486631	0.513369	233	2128	-1895
375	$3^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	242	2137	-1895
376	$2^3 47^1$	N	N	9	4	1.5555556	0.489362	0.510638	251	2146	-1895
377	$13^1 29^1$	Y	N	5	0	1.0000000	0.490716	0.509284	256	2151	-1895
378	$2^1 3^3 7^1$	N	N	-48	32	1.3333333	0.489418	0.510582	208	2151	-1943
379	379^1	Y	Y	-2	0	1.0000000	0.488127	0.511873	206	2151	-1945
380	$2^2 5^1 19^1$	N	N	30	14	1.1666667	0.489474	0.510526	236	2181	-1945
381	$3^1 127^1$	Y	N	5	0	1.0000000	0.490814	0.509186	241	2186	-1945
382	$2^1 191^1$	Y	N	5	0	1.0000000	0.492147	0.507853	246	2191	-1945
383	383^1	Y	Y	-2	0	1.0000000	0.490862	0.509138	244	2191	-1947
384	$2^7 3^1$	N	N	17	12	2.5882353	0.492188	0.507812	261	2208	-1947
385	$5^1 7^1 11^1$	Y	N	-16	0	1.0000000	0.490909	0.509091	245	2208	-1963
386	$2^1 193^1$	Y	N	5	0	1.0000000	0.492228	0.507772	250	2213	-1963
387	$3^2 43^1$	N	N	-7	2	1.2857143	0.490956	0.509044	243	2213	-1970
388	$2^2 97^1$	N	N	-7	2	1.2857143	0.489691	0.510309	236	2213	-1977
389	389^1	Y	Y	-2	0	1.0000000	0.488432	0.511568	234	2213	-1979
390	$2^1 3^1 5^1 13^1$	Y	N	65	0	1.0000000	0.489744	0.510256	299	2278	-1979
391	$17^1 23^1$	Y	N	5	0	1.0000000	0.491049	0.508951	304	2283	-1979
392	$2^3 7^2$	N	N	-23	18	1.4782609	0.489796	0.510204	281	2283	-2002
393	$3^1 131^1$	Y	N	5	0	1.0000000	0.491094	0.508906	286	2288	-2002
394	$2^1 197^1$	Y	N	5	0	1.0000000	0.492386	0.507614	291	2293	-2002
395	$5^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	296	2298	-2002
396	$2^2 3^2 11^1$	N	N	-74	58	1.2162162	0.492424	0.507576	222	2298	-2076
397	397^1	Y	Y	-2	0	1.0000000	0.491184	0.508816	220	2298	-2078
398	$2^1 199^1$	Y	N	5	0	1.0000000	0.492462	0.507538	225	2303	-2078
399	$3^1 7^1 19^1$	Y	N	-16	0	1.0000000	0.491228	0.508772	209	2303	-2094
400	$2^4 5^2$	N	N	34	29	1.6176471	0.492500	0.507500	243	2337	-2094
401	401^1	Y	Y	-2	0	1.0000000	0.491272	0.508728	241	2337	-2096
402	$2^1 3^1 67^1$	Y	N	-16	0	1.0000000	0.490050	0.509950	225	2337	-2112
403	$13^1 31^1$	Y	N	5	0	1.0000000	0.491315	0.508685	230	2342	-2112
404	$2^2 101^1$	N	N	-7	2	1.2857143	0.490099	0.509901	223	2342	-2119
405	$3^4 5^1$	N	N	-11	6	1.8181818	0.488889	0.511111	212	2342	-2130
406	$2^1 7^1 29^1$	Y	N	-16	0	1.0000000	0.487685	0.512315	196	2342	-2146
407	$11^1 37^1$	Y	N	5	0	1.0000000	0.488943	0.511057	201	2347	-2146
408	$2^3 3^1 17^1$	N	N	-48	32	1.3333333	0.487745	0.512255	153	2347	-2194
409	409^1	Y	Y	-2	0	1.0000000	0.486553	0.513447	151	2347	-2196
410	$2^1 5^1 41^1$	Y	N	-16	0	1.0000000	0.485366	0.514634	135	2347	-2212
411	$3^1 137^1$	Y	N	5	0	1.0000000	0.486618	0.513382	140	2352	-2212
412	$2^2 103^1$	N	N	-7	2	1.2857143	0.485437	0.514563	133	2352	-2219
413	$7^1 59^1$	Y	N	5	0	1.0000000	0.486683	0.513317	138	2357	-2219
414	$2^1 3^2 23^1$	N	N	30	14	1.1666667	0.487923	0.512077	168	2387	-2219
415	$5^1 83^1$	Y	N	5	0	1.0000000	0.489157	0.510843	173	2392	-2219
416	$2^5 13^1$	N	N	13	8	2.0769231	0.490385	0.509615	186	2405	-2219
417	$3^1 139^1$	Y	N	5	0	1.0000000	0.491607	0.508393	191	2410	-2219
418	$2^1 11^1 19^1$	Y	N	-16	0	1.0000000	0.490431	0.509569	175	2410	-2235
419	419^1	Y	Y	-2	0	1.0000000	0.489260	0.510740	173	2410	-2237
420	$2^2 3^1 5^1 7^1$	N	N	-155	90	1.1032258	0.488095	0.511905	18	2410	-2392
421	421^1	Y	Y	-2	0	1.0000000	0.486936	0.513064	16	2410	-2394
422	$2^1 211^1$	Y	N	5	0	1.0000000	0.488152	0.511848	21	2415	-2394
423	$3^2 47^1$	N	N	-7	2	1.2857143	0.486998	0.513002	14	2415	-2401
424	$2^3 53^1$	N	N	9	4	1.5555556	0.488208	0.511792	23	2424	-2401
425	$5^2 17^1$	N	N	-7	2	1.2857143	0.487059	0.512941	16	2424	-2408

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \hat{f}_1(n)$	$\frac{\sum d n C_{\Omega(d)}^{(d)}}{ g^{-1}(n) }$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
426	$2^1 3^1 71^1$	Y	N	-16	0	1.0000000	0.485915	0.514085	0	2424	-2424
427	$7^1 61^1$	Y	N	5	0	1.0000000	0.487119	0.512881	5	2429	-2424
428	$2^2 107^1$	N	N	-7	2	1.2857143	0.485981	0.514019	-2	2429	-2431
429	$3^1 11^1 13^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	-18	2429	-2447
430	$2^1 5^1 43^1$	Y	N	-16	0	1.0000000	0.483721	0.516279	-34	2429	-2463
431	431^1	Y	Y	-2	0	1.0000000	0.482599	0.517401	-36	2429	-2465
432	$2^4 3^3$	N	N	-80	75	1.5625000	0.481481	0.518519	-116	2429	-2545
433	433^1	Y	Y	-2	0	1.0000000	0.480370	0.519630	-118	2429	-2547
434	$2^1 7^1 31^1$	Y	N	-16	0	1.0000000	0.479263	0.520737	-134	2429	-2563
435	$3^1 5^1 29^1$	Y	N	-16	0	1.0000000	0.478161	0.521839	-150	2429	-2579
436	$2^2 109^1$	N	N	-7	2	1.2857143	0.477064	0.522936	-157	2429	-2586
437	$19^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	-152	2434	-2586
438	$2^1 3^1 73^1$	Y	N	-16	0	1.0000000	0.477169	0.522831	-168	2434	-2602
439	439^1	Y	Y	-2	0	1.0000000	0.476082	0.523918	-170	2434	-2604
440	$2^3 5^1 11^1$	N	N	-48	32	1.3333333	0.475000	0.525000	-218	2434	-2652
441	$3^2 7^2$	N	N	14	9	1.3571429	0.476190	0.523810	-204	2448	-2652
442	$2^1 13^1 17^1$	Y	N	-16	0	1.0000000	0.475113	0.524887	-220	2448	-2668
443	443^1	Y	Y	-2	0	1.0000000	0.474041	0.525959	-222	2448	-2670
444	$2^2 3^1 37^1$	N	N	30	14	1.1666667	0.475225	0.524775	-192	2478	-2670
445	$5^1 89^1$	Y	N	5	0	1.0000000	0.476404	0.523596	-187	2483	-2670
446	$2^1 223^1$	Y	N	5	0	1.0000000	0.477578	0.522422	-182	2488	-2670
447	$3^1 149^1$	Y	N	5	0	1.0000000	0.478747	0.521253	-177	2493	-2670
448	$2^6 7^1$	N	N	-15	10	2.3333333	0.477679	0.522321	-192	2493	-2685
449	449^1	Y	Y	-2	0	1.0000000	0.476615	0.523385	-194	2493	-2687
450	$2^1 3^2 5^2$	N	N	-74	58	1.2162162	0.475556	0.524444	-268	2493	-2761
451	$11^1 41^1$	Y	N	5	0	1.0000000	0.476718	0.523282	-263	2498	-2761
452	$2^2 113^1$	N	N	-7	2	1.2857143	0.475664	0.524336	-270	2498	-2768
453	$3^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	-265	2503	-2768
454	$2^1 227^1$	Y	N	5	0	1.0000000	0.477974	0.522026	-260	2508	-2768
455	$5^1 7^1 13^1$	Y	N	-16	0	1.0000000	0.476923	0.523077	-276	2508	-2784
456	$2^3 3^1 19^1$	N	N	-48	32	1.3333333	0.475877	0.524123	-324	2508	-2832
457	457^1	Y	Y	-2	0	1.0000000	0.474836	0.525164	-326	2508	-2834
458	$2^1 229^1$	Y	N	5	0	1.0000000	0.475983	0.524017	-321	2513	-2834
459	$3^3 17^1$	N	N	9	4	1.5555556	0.477124	0.522876	-312	2522	-2834
460	$2^2 5^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	-282	2552	-2834
461	461^1	Y	Y	-2	0	1.0000000	0.477223	0.522777	-284	2552	-2836
462	$2^1 3^1 7^1 11^1$	Y	N	65	0	1.0000000	0.478355	0.521645	-219	2617	-2836
463	463^1	Y	Y	-2	0	1.0000000	0.477322	0.522678	-221	2617	-2838
464	$2^4 29^1$	N	N	-11	6	1.8181818	0.476293	0.523707	-232	2617	-2849
465	$3^1 5^1 31^1$	Y	N	-16	0	1.0000000	0.475269	0.524731	-248	2617	-2865
466	$2^1 233^1$	Y	N	5	0	1.0000000	0.476395	0.523605	-243	2622	-2865
467	467^1	Y	Y	-2	0	1.0000000	0.475375	0.524625	-245	2622	-2867
468	$2^2 3^2 13^1$	N	N	-74	58	1.2162162	0.474359	0.525641	-319	2622	-2941
469	$7^1 67^1$	Y	N	5	0	1.0000000	0.475480	0.524520	-314	2627	-2941
470	$2^1 5^1 47^1$	Y	N	-16	0	1.0000000	0.474468	0.525532	-330	2627	-2957
471	$3^1 157^1$	Y	N	5	0	1.0000000	0.475584	0.524416	-325	2632	-2957
472	$2^3 59^1$	N	N	9	4	1.5555556	0.476695	0.523305	-316	2641	-2957
473	$11^1 43^1$	Y	N	5	0	1.0000000	0.477801	0.522199	-311	2646	-2957
474	$2^1 3^1 79^1$	Y	N	-16	0	1.0000000	0.476793	0.523207	-327	2646	-2973
475	$5^2 19^1$	N	N	-7	2	1.2857143	0.475789	0.524211	-334	2646	-2980
476	$2^2 7^1 17^1$	N	N	30	14	1.1666667	0.476891	0.523109	-304	2676	-2980
477	$3^2 53^1$	N	N	-7	2	1.2857143	0.475891	0.524109	-311	2676	-2987
478	$2^1 239^1$	Y	N	5	0	1.0000000	0.476987	0.523013	-306	2681	-2987
479	479^1	Y	Y	-2	0	1.0000000	0.475992	0.524008	-308	2681	-2989
480	$2^5 3^1 5^1$	N	N	-96	80	1.6666667	0.475000	0.525000	-404	2681	-3085
481	$13^1 37^1$	Y	N	5	0	1.0000000	0.476091	0.523909	-399	2686	-3085
482	$2^1 241^1$	Y	N	5	0	1.0000000	0.477178	0.522822	-394	2691	-3085
483	$3^1 7^1 23^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	-410	2691	-3101
484	$2^2 11^2$	N	N	14	9	1.3571429	0.477273	0.522727	-396	2705	-3101
485	$5^1 97^1$	Y	N	5	0	1.0000000	0.478351	0.521649	-391	2710	-3101
486	$2^1 3^5$	N	N	13	8	2.0769231	0.479424	0.520576	-378	2723	-3101
487	487^1	Y	Y	-2	0	1.0000000	0.478439	0.521561	-380	2723	-3103
488	$2^3 61^1$	N	N	9	4	1.5555556	0.479508	0.520492	-371	2732	-3103
489	$3^1 163^1$	Y	N	5	0	1.0000000	0.480573	0.519427	-366	2737	-3103
490	$2^1 5^1 7^2$	N	N	30	14	1.1666667	0.481633	0.518367	-336	2767	-3103
491	491^1	Y	Y	-2	0	1.0000000	0.480652	0.519348	-338	2767	-3105
492	$2^2 3^1 41^1$	N	N	30	14	1.1666667	0.481707	0.518293	-308	2797	-3105
493	$17^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	-303	2802	-3105
494	$2^1 13^1 19^1$	Y	N	-16	0	1.0000000	0.481781	0.518219	-319	2802	-3121
495	$3^2 5^1 11^1$	N	N	30	14	1.1666667	0.482828	0.517172	-289	2832	-3121
496	$2^4 31^1$	N	N	-11	6	1.8181818	0.481855	0.518145	-300	2832	-3132
497	$7^1 71^1$	Y	N	5	0	1.0000000	0.482897	0.517103	-295	2837	-3132
498	$2^1 3^1 83^1$	Y	N	-16	0	1.0000000	0.481928	0.518072	-311	2837	-3148
499	499^1	Y	Y	-2	0	1.0000000	0.480962	0.519038	-313	2837	-3150
500	$2^2 5^3$	N	N	-23	18	1.4782609	0.480000	0.520000	-336	2837	-3173

T.2 Table: Approximations of the summatory functions of $\lambda(n)$ and $\lambda_*(n)$

x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$
210	-10	1	-0.2546	-0.1015	210	98	0.462138	255	-7	1	-0.1521	-0.06098	255	125	0.485439
211	-11	1	-0.2790	-0.1112	211	99	0.464641	256	-6	1	-0.1300	-0.05211	256	124	0.479674
212	-12	1	-0.3032	-0.1209	212	98	0.457778	257	-7	1	-0.1511	-0.06060	257	125	0.481661
213	-11	1	-0.2769	-0.1104	213	99	0.460278	258	-8	1	-0.1722	-0.06905	258	126	0.483632
214	-10	1	-0.2507	-0.1000	214	100	0.462755	259	-7	1	-0.1502	-0.06023	259	127	0.485589
215	-9	1	-0.2248	-0.08968	215	101	0.465208	260	-6	1	-0.1283	-0.05147	260	126	0.479912
216	-8	1	-0.1991	-0.07942	216	102	0.467639	261	-7	1	-0.1492	-0.05987	261	125	0.474279
217	-7	1	-0.1735	-0.06924	217	103	0.470048	262	-6	1	-0.1275	-0.05116	262	126	0.476249
218	-6	1	-0.1482	-0.05914	218	104	0.472434	263	-7	1	-0.1483	-0.05951	263	127	0.478203
219	-5	1	-0.1230	-0.04910	219	105	0.474799	264	-8	1	-0.1690	-0.06781	264	128	0.480143
220	-4	1	-0.09807	-0.03914	220	104	0.46814	265	-7	1	-0.1474	-0.05916	265	129	0.482068
221	-3	1	-0.07328	-0.02925	221	105	0.470502	266	-8	1	-0.1679	-0.06741	266	130	0.483979
222	-4	1	-0.09735	-0.03886	222	106	0.472844	267	-7	1	-0.1465	-0.05881	267	131	0.485875
223	-5	1	-0.1212	-0.04841	223	107	0.475164	268	-8	1	-0.1669	-0.06701	268	130	0.480367
224	-4	1	-0.09664	-0.03859	224	108	0.477464	269	-9	1	-0.1872	-0.07517	269	131	0.482263
225	-3	1	-0.07221	-0.02884	225	109	0.479743	270	-10	1	-0.2073	-0.08328	270	132	0.484144
226	-2	1	-0.04797	-0.01916	226	110	0.482002	271	-11	1	-0.2274	-0.09134	271	133	0.486012
227	-3	1	-0.07170	-0.02864	227	111	0.484241	272	-12	1	-0.2473	-0.09935	272	132	0.480584
228	-2	1	-0.04763	-0.01903	228	110	0.477774	273	-13	1	-0.2671	-0.1073	273	133	0.482451
229	-3	1	-0.07118	-0.02844	229	111	0.480012	274	-12	1	-0.2458	-0.09878	274	134	0.484305
230	-4	1	-0.09458	-0.03779	230	112	0.482231	275	-13	1	-0.2655	-0.1067	275	133	0.478943
231	-5	1	-0.1178	-0.04708	231	113	0.48443	276	-12	1	-0.2444	-0.09822	276	132	0.473619
232	-4	1	-0.09391	-0.03754	232	114	0.486611	277	-13	1	-0.2639	-0.1061	277	133	0.475485
233	-5	1	-0.1170	-0.04676	233	115	0.488772	278	-12	1	-0.2429	-0.09766	278	134	0.477336
234	-4	1	-0.09325	-0.03728	234	114	0.482451	279	-13	1	-0.2624	-0.1055	279	133	0.472076
235	-3	1	-0.06970	-0.02787	235	115	0.484612	280	-14	1	-0.2817	-0.1133	280	134	0.473927
236	-4	1	-0.09261	-0.03704	236	114	0.478363	281	-15	1	-0.3010	-0.1210	281	135	0.475765
237	-3	1	-0.06922	-0.02768	237	115	0.480523	282	-16	1	-0.3201	-0.1288	282	136	0.477589
238	-4	1	-0.09197	-0.03679	238	116	0.482665	283	-17	1	-0.3391	-0.1364	283	137	0.479401
239	-5	1	-0.1146	-0.04584	239	117	0.484789	284	-18	1	-0.3580	-0.1440	284	136	0.474226
240	-4	1	-0.09134	-0.03655	240	116	0.478643	285	-19	1	-0.3768	-0.1516	285	137	0.476037
241	-5	1	-0.1138	-0.04554	241	117	0.480766	286	-20	1	-0.3955	-0.1592	286	138	0.477835
242	-6	1	-0.1361	-0.05447	242	116	0.474687	287	-19	1	-0.3747	-0.1508	287	139	0.47962
243	-7	1	-0.1582	-0.06334	243	117	0.476809	288	-20	1	-0.3933	-0.1583	288	138	0.474516
244	-8	1	-0.1802	-0.07215	244	116	0.470796	289	-19	1	-0.3725	-0.1500	289	137	0.469448
245	-9	1	-0.2021	-0.08091	245	115	0.464832	290	-20	1	-0.3910	-0.1574	290	138	0.471244
246	-10	1	-0.2238	-0.08961	246	116	0.466968	291	-19	1	-0.3704	-0.1492	291	139	0.473028
247	-9	1	-0.2007	-0.08039	247	117	0.469087	292	-20	1	-0.3888	-0.1566	292	138	0.468016
248	-8	1	-0.1779	-0.07123	248	118	0.471189	293	-21	1	-0.4071	-0.1640	293	139	0.469799
249	-7	1	-0.1551	-0.06213	249	119	0.473274	294	-20	1	-0.3866	-0.1558	294	138	0.464832
250	-6	1	-0.1325	-0.05309	250	120	0.475342	295	-19	1	-0.3663	-0.1476	295	139	0.466614
251	-7	1	-0.1541	-0.06174	251	121	0.477393	296	-18	1	-0.3460	-0.1394	296	140	0.468383
252	-8	1	-0.1755	-0.07034	252	122	0.479429	297	-17	1	-0.3259	-0.1313	297	141	0.47014
253	-7	1	-0.1531	-0.06136	253	123	0.481448	298	-16	1	-0.3059	-0.1233	298	142	0.471886
254	-6	1	-0.1308	-0.05243	254	124	0.483451	299	-15	1	-0.2860	-0.1153	299	143	0.473619

Table T.2: Approximations to the summatory functions of $\lambda(n)$ and $\lambda_*(n)$.

- We define the exact summatory functions over these sequences by $L(x) := \sum_{n \leq x} \lambda(n)$ and $L_*(x) := \sum_{n \leq x} \lambda_*(n)$.
- We compare the ratios of the following two functions with $L(x)$: $L_{\approx,1}(x) := \sum_{k=1}^{\log \log x} \frac{x}{\log x} \cdot \frac{(-\log \log x)^{k-1}}{(k-1)!}$ and $L_{\approx,2}(x) := \frac{x^{3/4}(\log x)^{1/2}}{\sqrt{\log \log x}}$.
- Finally, we compare the approximations (very accurate) to $L_*(x)$ by the summatory function $\sum_{k \leq x} \widehat{c}(-1)^k \cdot 2^{-k}$ using the approximation $L_{\approx}^*(x) := \frac{2\widehat{c}}{3}x$.

(TODO) ... Are these ratios inverted, or possibly need to take much, much larger x to demonstrate this point?

x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L(x)}{L_{\approx}(x)}$	x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L(x)}{L_{\approx}(x)}$
300	-16	1	-0.3042	-0.1226	300	144	0.475342	375	-9	1	-0.1422	-0.05787	375	173	0.456856
301	-15	1	-0.2844	-0.1147	301	145	0.477052	376	-8	1	-0.1262	-0.05133	376	174	0.458275
302	-14	1	-0.2647	-0.1067	302	146	0.478752	377	-7	1	-0.1101	-0.04482	377	175	0.459686
303	-13	1	-0.2451	-0.09886	303	147	0.48044	378	-8	1	-0.1256	-0.05112	378	176	0.46109
304	-14	1	-0.2633	-0.1062	304	146	0.475602	379	-9	1	-0.1410	-0.05739	379	177	0.462486
305	-13	1	-0.2438	-0.09835	305	147	0.47729	380	-8	1	-0.1251	-0.05091	380	176	0.458663
306	-12	1	-0.2245	-0.09055	306	146	0.472494	381	-7	1	-0.1092	-0.04445	381	177	0.460058
307	-13	1	-0.2425	-0.09785	307	147	0.47418	382	-6	1	-0.09338	-0.03802	382	178	0.461446
308	-12	1	-0.2233	-0.09009	308	146	0.469426	383	-7	1	-0.1087	-0.04427	383	179	0.462827
309	-11	1	-0.2041	-0.08237	309	147	0.471111	384	-6	1	-0.09298	-0.03787	384	180	0.464201
310	-12	1	-0.2221	-0.08963	310	148	0.472786	385	-7	1	-0.1082	-0.04409	385	181	0.465567
311	-13	1	-0.2399	-0.09686	311	149	0.47445	386	-6	1	-0.09258	-0.03771	386	182	0.466927
312	-14	1	-0.2577	-0.1040	312	150	0.476103	387	-7	1	-0.1078	-0.04391	387	181	0.463161
313	-15	1	-0.2754	-0.1112	313	151	0.477746	388	-8	1	-0.1229	-0.05008	388	180	0.459415
314	-14	1	-0.2563	-0.1035	314	152	0.479379	389	-9	1	-0.1380	-0.05622	389	181	0.46078
315	-13	1	-0.2374	-0.09589	315	151	0.474713	390	-8	1	-0.1224	-0.04988	390	182	0.462138
316	-14	1	-0.2550	-0.1030	316	150	0.470077	391	-7	1	-0.1069	-0.04355	391	183	0.463489
317	-15	1	-0.2725	-0.1101	317	151	0.471718	392	-8	1	-0.1219	-0.04968	392	182	0.45978
318	-16	1	-0.2899	-0.1171	318	152	0.473349	393	-7	1	-0.1064	-0.04338	393	183	0.46113
319	-15	1	-0.2711	-0.1095	319	153	0.474969	394	-6	1	-0.09101	-0.03711	394	184	0.462473
320	-16	1	-0.2884	-0.1166	320	152	0.47039	395	-5	1	-0.07568	-0.03086	395	185	0.463809
321	-15	1	-0.2697	-0.1090	321	153	0.47201	396	-6	1	-0.09063	-0.03696	396	186	0.465139
322	-16	1	-0.2869	-0.1160	322	154	0.473619	397	-7	1	-0.1055	-0.04304	397	187	0.466461
323	-15	1	-0.2683	-0.1085	323	155	0.475219	398	-6	1	-0.09025	-0.03681	398	188	0.467778
324	-14	1	-0.2498	-0.1010	324	156	0.476809	399	-7	1	-0.1051	-0.04287	399	189	0.469087
325	-15	1	-0.2669	-0.1079	325	155	0.472295	400	-6	1	-0.08987	-0.03667	400	190	0.47039
326	-14	1	-0.2485	-0.1005	326	156	0.473884	401	-7	1	-0.1046	-0.04270	401	191	0.471687
327	-13	1	-0.2302	-0.09310	327	157	0.475463	402	-8	1	-0.1193	-0.04870	402	192	0.472977
328	-12	1	-0.2119	-0.08574	328	158	0.477032	403	-7	1	-0.1042	-0.04253	403	193	0.47426
329	-11	1	-0.1938	-0.07840	329	159	0.478592	404	-8	1	-0.1188	-0.04851	404	192	0.470635
330	-10	1	-0.1757	-0.07111	330	160	0.480143	405	-9	1	-0.1334	-0.05447	405	191	0.467028
331	-11	1	-0.1928	-0.07803	331	161	0.481684	406	-10	1	-0.1479	-0.06040	406	192	0.468317
332	-12	1	-0.2098	-0.08492	332	160	0.477251	407	-9	1	-0.1329	-0.05426	407	193	0.469599
333	-13	1	-0.2267	-0.09178	333	159	0.472844	408	-10	1	-0.1473	-0.06017	408	194	0.470876
334	-12	1	-0.2088	-0.08452	334	160	0.474393	409	-11	1	-0.1617	-0.06606	409	195	0.472146
335	-11	1	-0.1909	-0.07730	335	161	0.475933	410	-12	1	-0.1761	-0.07193	410	196	0.473409
336	-10	1	-0.1731	-0.07010	336	160	0.471569	411	-11	1	-0.1611	-0.06581	411	197	0.474667
337	-11	1	-0.1900	-0.07694	337	161	0.473108	412	-12	1	-0.1754	-0.07165	412	196	0.471111
338	-12	1	-0.2067	-0.08373	338	160	0.468779	413	-11	1	-0.1604	-0.06556	413	197	0.472368
339	-11	1	-0.1890	-0.07658	339	161	0.470317	414	-10	1	-0.1456	-0.05948	414	196	0.468835
340	-10	1	-0.1714	-0.06945	340	160	0.466021	415	-9	1	-0.1307	-0.05343	415	197	0.470092
341	-9	1	-0.1539	-0.06236	341	161	0.467559	416	-8	1	-0.1160	-0.04741	416	198	0.471342
342	-8	1	-0.1365	-0.05531	342	160	0.463296	417	-7	1	-0.1013	-0.04140	417	199	0.472587
343	-9	1	-0.1532	-0.06208	343	161	0.464832	418	-8	1	-0.1155	-0.04723	418	200	0.473825
344	-8	1	-0.1358	-0.05505	344	162	0.46636	419	-9	1	-0.1297	-0.05303	419	201	0.475058
345	-9	1	-0.1524	-0.06180	345	163	0.467879	420	-10	1	-0.1438	-0.05881	420	200	0.471569
346	-8	1	-0.1352	-0.05480	346	164	0.469388	421	-11	1	-0.1579	-0.06457	421	201	0.472801
347	-9	1	-0.1517	-0.06151	347	165	0.47089	422	-10	1	-0.1432	-0.05860	422	202	0.474028
348	-8	1	-0.1345	-0.05456	348	164	0.466691	423	-11	1	-0.1573	-0.06433	423	201	0.470566
349	-9	1	-0.1510	-0.06124	349	165	0.468191	424	-10	1	-0.1427	-0.05838	424	202	0.471792
350	-8	1	-0.1339	-0.05431	350	164	0.464024	425	-11	1	-0.1566	-0.06410	425	201	0.468351
351	-7	1	-0.1169	-0.04741	351	165	0.465523	426	-12	1	-0.1705	-0.06979	426	202	0.469577
352	-6	1	-0.09995	-0.04055	352	166	0.467014	427	-11	1	-0.1560	-0.06386	427	203	0.470796
353	-7	1	-0.1163	-0.04720	353	167	0.468497	428	-12	1	-0.1699	-0.06954	428	202	0.467382
354	-8	1	-0.1326	-0.05383	354	168	0.469971	429	-13	1	-0.1837	-0.07519	429	203	0.468601
355	-7	1	-0.1158	-0.04699	355	169	0.471436	430	-14	1	-0.1974	-0.08083	430	204	0.469814
356	-8	1	-0.1320	-0.05359	356	168	0.46733	431	-15	1	-0.2111	-0.08645	431	205	0.471022
357	-9	1	-0.1482	-0.06015	357	169	0.468795	432	-16	1	-0.2248	-0.09204	432	204	0.467639
358	-8	1	-0.1314	-0.05335	358	170	0.470252	433	-17	1	-0.2383	-0.09762	433	205	0.468846
359	-9	1	-0.1475	-0.05989	359	171	0.4717	434	-18	1	-0.2519	-0.1032	434	206	0.470048
360	-8	1	-0.1308	-0.05312	360	170	0.467639	435	-19	1	-0.2654	-0.1087	435	207	0.471244
361	-7	1	-0.1142	-0.04638	361	169	0.463601	436	-20	1	-0.2788	-0.1142	436	206	0.467892
362	-6	1	-0.09765	-0.03967	362	170	0.465056	437	-19	1	-0.2643	-0.1083	437	207	0.469087
363	-7	1	-0.1137	-0.04618	363	169	0.461046	438	-20	1	-0.2777	-0.1138	438	208	0.470277
364	-6	1	-0.09721	-0.03949	364	168	0.457059	439	-21	1	-0.2911	-0.1193	439	209	0.471462
365	-5	1	-0.08082	-0.03284	365	169	0.45852	440	-22	1	-0.3043	-0.1247	440	210	0.472641
366	-6	1	-0.09676	-0.03932	366	170	0.459973	441	-21	1	-0.2900	-0.1189	441	211	0.473815
367	-7	1	-0.1126	-0.04578	367	171	0.461418	442	-22	1	-0.3032	-0.1243	442	212	0.474983
368	-8	1	-0.1284	-0.05221	368	170	0.457473	443	-23	1	-0.3164	-0.1297	443	213	0.476146
369	-9	1	-0.1442	-0.05861	369	169	0.45355	444	-22	1	-0.3020	-0.1239	444	212	0.472844
370	-10	1	-0.1598	-0.06498	370	170	0.455	445	-21	1	-0.2878	-0.1180	445	213	0.474006
371	-9	1	-0.1435	-0.05836	371	171	0.456443	446	-20	1	-0.2736	-0.1122	446	214	0.475164
372	-8	1	-0.1273	-0.05177	372	170	0.452554	447	-19	1	-0.2594	-0.1064	447	215	0.476316
373	-9	1	-0.1429	-0.05811	373	171	0.453996	448	-20	1	-0.2725	-0.1118	448	214	0.473043
374	-10	1	-0.1584	-0.06444	374	172	0.45543	449	-21	1	-0.2856	-0.1172	449	215	0.474195

x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L^*(x)$	$\frac{L(x)}{L_{\approx}(x)}$	x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L^*(x)$	$\frac{L(x)}{L_{\approx}(x)}$
450	-22	1	-0.2987	-0.1226	450	216	0.475342	525	-17	1	-0.2028	-0.08389	525	255	0.481
451	-21	1	-0.2846	-0.1168	451	217	0.476483	526	-16	1	-0.1906	-0.07884	526	256	0.481969
452	-22	1	-0.2976	-0.1221	452	216	0.473238	527	-15	1	-0.1784	-0.07380	527	257	0.482933
453	-21	1	-0.2835	-0.1164	453	217	0.47438	528	-14	1	-0.1662	-0.06878	528	256	0.480143
454	-20	1	-0.2695	-0.1106	454	218	0.475516	529	-13	1	-0.1541	-0.06377	529	255	0.477363
455	-21	1	-0.2825	-0.1160	455	219	0.476648	530	-14	1	-0.1657	-0.06857	530	256	0.478331
456	-22	1	-0.2954	-0.1213	456	220	0.477774	531	-15	1	-0.1773	-0.07336	531	255	0.475565
457	-23	1	-0.3082	-0.1266	457	221	0.478895	532	-14	1	-0.1652	-0.06837	532	254	0.47281
458	-22	1	-0.2943	-0.1209	458	222	0.480012	533	-13	1	-0.1531	-0.06339	533	255	0.473781
459	-21	1	-0.2804	-0.1152	459	223	0.481124	534	-14	1	-0.1647	-0.06817	534	256	0.474748
460	-20	1	-0.2666	-0.1095	460	222	0.477925	535	-13	1	-0.1527	-0.06321	535	257	0.475712
461	-21	1	-0.2794	-0.1148	461	223	0.479036	536	-12	1	-0.1407	-0.05826	536	258	0.476672
462	-20	1	-0.2656	-0.1091	462	224	0.480143	537	-11	1	-0.1288	-0.05333	537	259	0.477628
463	-21	1	-0.2784	-0.1144	463	225	0.481245	538	-10	1	-0.1169	-0.04841	538	260	0.478581
464	-22	1	-0.2911	-0.1196	464	224	0.478074	539	-11	1	-0.1284	-0.05317	539	259	0.475856
465	-23	1	-0.3038	-0.1249	465	225	0.479175	540	-10	1	-0.1165	-0.04827	540	258	0.473141
466	-22	1	-0.2901	-0.1192	466	226	0.480272	541	-11	1	-0.1280	-0.05302	541	259	0.474097
467	-23	1	-0.3027	-0.1244	467	227	0.481364	542	-10	1	-0.1161	-0.04813	542	260	0.475049
468	-24	1	-0.3153	-0.1296	468	228	0.482451	543	-9	1	-0.1044	-0.04325	543	261	0.475998
469	-23	1	-0.3016	-0.1240	469	229	0.483534	544	-8	1	-0.09263	-0.03839	544	262	0.476944
470	-24	1	-0.3142	-0.1292	470	230	0.484612	545	-7	1	-0.08093	-0.03354	545	263	0.477886
471	-23	1	-0.3006	-0.1236	471	231	0.485686	546	-6	1	-0.06926	-0.02871	546	264	0.478824
472	-22	1	-0.2870	-0.1180	472	232	0.486755	547	-7	1	-0.08068	-0.03345	547	265	0.479759
473	-21	1	-0.2734	-0.1125	473	233	0.48782	548	-8	1	-0.09206	-0.03817	548	264	0.477076
474	-22	1	-0.2860	-0.1176	474	234	0.48888	549	-9	1	-0.1034	-0.04288	549	263	0.474404
475	-23	1	-0.2984	-0.1228	475	233	0.485766	550	-8	1	-0.09178	-0.03806	550	262	0.471741
476	-22	1	-0.2850	-0.1173	476	232	0.482665	551	-7	1	-0.08019	-0.03325	551	263	0.472682
477	-23	1	-0.2974	-0.1224	477	231	0.479577	552	-8	1	-0.09150	-0.03795	552	264	0.473619
478	-22	1	-0.2840	-0.1169	478	232	0.480645	553	-7	1	-0.07994	-0.03316	553	265	0.474554
479	-23	1	-0.2963	-0.1220	479	233	0.481709	554	-6	1	-0.06842	-0.02838	554	266	0.475485
480	-24	1	-0.3087	-0.1271	480	234	0.482769	555	-7	1	-0.07970	-0.03307	555	267	0.476412
481	-23	1	-0.2953	-0.1216	481	235	0.483824	556	-8	1	-0.09095	-0.03774	556	266	0.473774
482	-22	1	-0.2820	-0.1161	482	236	0.484875	557	-9	1	-0.1022	-0.04239	557	267	0.474702
483	-23	1	-0.2943	-0.1212	483	237	0.485921	558	-8	1	-0.09067	-0.03763	558	266	0.472076
484	-22	1	-0.2810	-0.1157	484	238	0.486963	559	-7	1	-0.07922	-0.03288	559	267	0.473003
485	-21	1	-0.2678	-0.1103	485	239	0.488001	560	-6	1	-0.06780	-0.02814	560	266	0.47039
486	-20	1	-0.2546	-0.1049	486	240	0.489035	561	-7	1	-0.07898	-0.03279	561	267	0.471317
487	-21	1	-0.2668	-0.1099	487	241	0.490064	562	-6	1	-0.06760	-0.02806	562	268	0.47224
488	-20	1	-0.2537	-0.1045	488	242	0.491089	563	-7	1	-0.07874	-0.03270	563	269	0.473161
489	-19	1	-0.2406	-0.09915	489	243	0.49211	564	-6	1	-0.06739	-0.02799	564	268	0.470566
490	-18	1	-0.2275	-0.09378	490	242	0.489085	565	-5	1	-0.05608	-0.02329	565	269	0.471486
491	-19	1	-0.2398	-0.09883	491	243	0.490105	566	-4	1	-0.04480	-0.01861	566	270	0.472402
492	-18	1	-0.2268	-0.09348	492	242	0.487096	567	-5	1	-0.05591	-0.02322	567	269	0.469823
493	-17	1	-0.2138	-0.08814	493	243	0.488117	568	-4	1	-0.04466	-0.01855	568	270	0.470739
494	-18	1	-0.2260	-0.09318	494	244	0.489134	569	-5	1	-0.05575	-0.02316	569	271	0.471652
495	-17	1	-0.2131	-0.08786	495	243	0.486145	570	-4	1	-0.04453	-0.01850	570	272	0.472562
496	-18	1	-0.2252	-0.09288	496	242	0.483168	571	-5	1	-0.05558	-0.02310	571	273	0.473469
497	-17	1	-0.2124	-0.08758	497	243	0.484189	572	-4	1	-0.04440	-0.01845	572	272	0.47091
498	-18	1	-0.2245	-0.09259	498	244	0.485205	573	-3	1	-0.03325	-0.01382	573	273	0.471816
499	-19	1	-0.2366	-0.09758	499	245	0.486217	574	-4	1	-0.04427	-0.01840	574	274	0.472719
500	-20	1	-0.2486	-0.1026	500	244	0.483264	575	-5	1	-0.05526	-0.02297	575	273	0.470175
501	-19	1	-0.2358	-0.09727	501	245	0.484276	576	-4	1	-0.04414	-0.01835	576	274	0.471078
502	-18	1	-0.2230	-0.09201	502	246	0.485284	577	-5	1	-0.05509	-0.02291	577	275	0.471978
503	-19	1	-0.2350	-0.09697	503	247	0.486288	578	-6	1	-0.06602	-0.02745	578	274	0.469448
504	-18	1	-0.2222	-0.09172	504	246	0.483358	579	-5	1	-0.05493	-0.02285	579	275	0.470347
505	-17	1	-0.2095	-0.08649	505	247	0.484362	580	-4	1	-0.04388	-0.01825	580	274	0.467829
506	-18	1	-0.2215	-0.09144	506	248	0.485362	581	-3	1	-0.03286	-0.01367	581	275	0.468728
507	-19	1	-0.2334	-0.09637	507	247	0.482451	582	-4	1	-0.04376	-0.01820	582	276	0.469624
508	-20	1	-0.2453	-0.1013	508	246	0.479552	583	-3	1	-0.03277	-0.01363	583	277	0.470518
509	-21	1	-0.2571	-0.1062	509	247	0.480556	584	-2	1	-0.02181	-0.009077	584	278	0.471408
510	-20	1	-0.2445	-0.1010	510	248	0.481555	585	-1	1	-0.01089	-0.004532	585	277	0.468909
511	-19	1	-0.2319	-0.09577	511	249	0.482551	586	0	0	0	0	586	278	0.469799
512	-20	1	-0.2437	-0.1007	512	250	0.483543	587	-1	1	-0.01086	-0.004520	587	279	0.470685
513	-19	1	-0.2311	-0.09548	513	251	0.48453	588	-2	1	-0.02169	-0.009028	588	280	0.471569
514	-18	1	-0.2186	-0.09032	514	252	0.485514	589	-1	1	-0.01083	-0.004508	589	281	0.47245
515	-17	1	-0.2061	-0.08517	515	253	0.486494	590	-2	1	-0.02163	-0.009004	590	282	0.473327
516	-16	1	-0.1937	-0.08004	516	252	0.483632	591	-1	1	-0.01080	-0.004496	591	283	0.474202
517	-15	1	-0.1813	-0.07492	517	253	0.484612	592	-2	1	-0.02157	-0.008980	592	282	0.471728
518	-16	1	-0.1930	-0.07979	518	254	0.485589	593	-3	1	-0.03230	-0.01345	593	283	0.472603
519	-15	1	-0.1807	-0.07469	519	255	0.486561	594	-4	1	-0.04301	-0.01791	594	284	0.473474
520	-16	1	-0.1924	-0.07955	520	256	0.48753	595	-5	1	-0.05369	-0.02236	595	285	0.474343
521	-17	1	-0.2041	-0.08440	521	257	0.488495	596	-6	1	-0.06433	-0.02680	596	284	0.471886
522	-16	1	-0.1918	-0.07931	522	256	0.485662	597	-5	1	-0.05353	-0.02230	597	285	0.472754
523	-17	1	-0.2035	-0.08414	523	257	0.486627	598	-6	1	-0.06415	-0.02673	598	286	0.473619
524	-18	1	-0.2151	-0.08896	524	256	0.483808	599	-7	1	-0.07474	-0.03114	599	287	0.474482

x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L^*(x)$	$\frac{L(x)}{L_{\approx}^*(x)}$	x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L^*(x)$	$\frac{L(x)}{L_{\approx}^*(x)}$
600	-6	1	-0.06397	-0.02666	600	286	0.472041	675	-25	1	-0.2413	-0.1013	675	319	0.468006
601	-7	1	-0.07453	-0.03106	601	287	0.472903	676	-24	1	-0.2313	-0.09709	676	320	0.468779
602	-8	1	-0.08505	-0.03545	602	288	0.473762	677	-25	1	-0.2407	-0.1010	677	321	0.469549
603	-9	1	-0.09555	-0.03983	603	287	0.471334	678	-26	1	-0.2500	-0.1049	678	322	0.470317
604	-10	1	-0.1060	-0.04420	604	286	0.468915	679	-25	1	-0.2401	-0.1008	679	323	0.471083
605	-11	1	-0.1165	-0.04855	605	285	0.466503	680	-26	1	-0.2494	-0.1047	680	324	0.471846
606	-12	1	-0.1269	-0.05290	606	286	0.467367	681	-25	1	-0.2395	-0.1006	681	325	0.472608
607	-13	1	-0.1373	-0.05723	607	287	0.468228	682	-26	1	-0.2488	-0.1045	682	326	0.473367
608	-12	1	-0.1265	-0.05276	608	288	0.469087	683	-27	1	-0.2580	-0.1083	683	327	0.474124
609	-13	1	-0.1369	-0.05709	609	289	0.469943	684	-28	1	-0.2672	-0.1122	684	328	0.474878
610	-14	1	-0.1472	-0.06140	610	290	0.470796	685	-27	1	-0.2574	-0.1081	685	329	0.475631
611	-13	1	-0.1365	-0.05694	611	291	0.471646	686	-26	1	-0.2475	-0.1040	686	330	0.476381
612	-14	1	-0.1468	-0.06124	612	292	0.472494	687	-25	1	-0.2377	-0.09986	687	331	0.477129
613	-15	1	-0.1571	-0.06553	613	293	0.473338	688	-26	1	-0.2469	-0.1037	688	330	0.474996
614	-14	1	-0.1464	-0.06108	614	294	0.47418	689	-25	1	-0.2371	-0.09963	689	331	0.475744
615	-15	1	-0.1566	-0.06536	615	295	0.47502	690	-24	1	-0.2274	-0.09554	690	332	0.47649
616	-16	1	-0.1668	-0.06963	616	296	0.475856	691	-25	1	-0.2365	-0.09941	691	333	0.477233
617	-17	1	-0.1770	-0.07389	617	297	0.47669	692	-26	1	-0.2457	-0.1033	692	332	0.475113
618	-18	1	-0.1872	-0.07814	618	298	0.477521	693	-25	1	-0.2360	-0.09918	693	331	0.472998
619	-19	1	-0.1973	-0.08237	619	299	0.478349	694	-24	1	-0.2263	-0.09511	694	332	0.473743
620	-18	1	-0.1867	-0.07794	620	298	0.475981	695	-23	1	-0.2166	-0.09104	695	333	0.474487
621	-17	1	-0.1761	-0.07351	621	299	0.476809	696	-24	1	-0.2257	-0.09489	696	334	0.475228
622	-16	1	-0.1655	-0.06910	622	300	0.477634	697	-23	1	-0.2160	-0.09083	697	335	0.475967
623	-15	1	-0.1549	-0.06470	623	301	0.478457	698	-22	1	-0.2064	-0.08679	698	336	0.476704
624	-14	1	-0.1444	-0.06031	624	300	0.476103	699	-21	1	-0.1968	-0.08275	699	337	0.477438
625	-13	1	-0.1339	-0.05593	625	299	0.473757	700	-22	1	-0.2059	-0.08659	700	338	0.478171
626	-12	1	-0.1234	-0.05157	626	300	0.474582	701	-23	1	-0.2150	-0.09043	701	339	0.478902
627	-13	1	-0.1335	-0.05579	627	301	0.475405	702	-24	1	-0.2241	-0.09425	702	340	0.479663
628	-14	1	-0.1436	-0.06001	628	300	0.473071	703	-23	1	-0.2145	-0.09022	703	341	0.480357
629	-13	1	-0.1332	-0.05565	629	301	0.473893	704	-24	1	-0.2235	-0.09404	704	340	0.478268
630	-14	1	-0.1432	-0.05986	630	300	0.471569	705	-25	1	-0.2326	-0.09785	705	341	0.478994
631	-15	1	-0.1533	-0.06406	631	301	0.472391	706	-24	1	-0.2230	-0.09383	706	342	0.479718
632	-14	1	-0.1429	-0.05971	632	302	0.473211	707	-23	1	-0.2134	-0.08982	707	343	0.48044
633	-13	1	-0.1325	-0.05538	633	303	0.474028	708	-22	1	-0.2039	-0.08582	708	342	0.478363
634	-12	1	-0.1221	-0.05105	634	304	0.474842	709	-23	1	-0.2129	-0.08962	709	343	0.479085
635	-11	1	-0.1118	-0.04674	635	305	0.475654	710	-24	1	-0.2219	-0.09342	710	344	0.479805
636	-10	1	-0.1015	-0.04244	636	304	0.473349	711	-25	1	-0.2309	-0.09720	711	343	0.477737
637	-11	1	-0.1115	-0.04663	637	303	0.471051	712	-24	1	-0.2214	-0.09321	712	344	0.478457
638	-12	1	-0.1215	-0.05080	638	304	0.471865	713	-23	1	-0.2119	-0.08923	713	345	0.479175
639	-13	1	-0.1314	-0.05497	639	303	0.469577	714	-22	1	-0.2025	-0.08526	714	346	0.479891
640	-12	1	-0.1212	-0.05068	640	304	0.47039	715	-23	1	-0.2114	-0.08903	715	347	0.480605
641	-13	1	-0.1311	-0.05483	641	305	0.471201	716	-24	1	-0.2203	-0.09280	716	346	0.47855
642	-14	1	-0.1410	-0.05898	642	306	0.47201	717	-23	1	-0.2109	-0.08884	717	347	0.479264
643	-15	1	-0.1508	-0.06312	643	307	0.472816	718	-22	1	-0.2015	-0.08488	718	348	0.479976
644	-14	1	-0.1406	-0.05884	644	306	0.470544	719	-23	1	-0.2104	-0.08864	719	349	0.480686
645	-15	1	-0.1504	-0.06296	645	307	0.47135	720	-24	1	-0.2193	-0.09240	720	350	0.481393
646	-16	1	-0.1603	-0.06708	646	308	0.472153	721	-23	1	-0.2099	-0.08845	721	351	0.482099
647	-17	1	-0.1701	-0.07118	647	309	0.472954	722	-24	1	-0.2188	-0.09220	722	350	0.48006
648	-18	1	-0.1798	-0.07528	648	308	0.470696	723	-23	1	-0.2094	-0.08826	723	351	0.480766
649	-17	1	-0.1696	-0.07101	649	309	0.471496	724	-24	1	-0.2183	-0.09200	724	350	0.478734
650	-16	1	-0.1594	-0.06675	650	308	0.469248	725	-25	1	-0.2271	-0.09572	725	349	0.476708
651	-17	1	-0.1692	-0.07084	651	309	0.470048	726	-24	1	-0.2178	-0.09180	726	348	0.474687
652	-18	1	-0.1789	-0.07492	652	308	0.467808	727	-25	1	-0.2266	-0.09552	727	349	0.475396
653	-19	1	-0.1886	-0.07898	653	309	0.468608	728	-26	1	-0.2354	-0.09923	728	350	0.476103
654	-20	1	-0.1983	-0.08304	654	310	0.469406	729	-25	1	-0.2261	-0.09531	729	349	0.474092
655	-19	1	-0.1881	-0.07879	655	311	0.470201	730	-26	1	-0.2348	-0.09902	730	350	0.474799
656	-20	1	-0.1977	-0.08284	656	310	0.467975	731	-25	1	-0.2255	-0.09511	731	351	0.475504
657	-21	1	-0.2074	-0.08688	657	309	0.465755	732	-24	1	-0.2163	-0.09120	732	350	0.473502
658	-22	1	-0.2170	-0.09091	658	310	0.466552	733	-25	1	-0.2250	-0.09490	733	351	0.474207
659	-23	1	-0.2265	-0.09493	659	311	0.467347	734	-24	1	-0.2158	-0.09101	734	352	0.47491
660	-24	1	-0.2361	-0.09893	660	310	0.465139	735	-23	1	-0.2065	-0.08712	735	351	0.472916
661	-25	1	-0.2456	-0.1029	661	311	0.465933	736	-22	1	-0.1973	-0.08325	736	352	0.473619
662	-24	1	-0.2355	-0.09870	662	312	0.466725	737	-21	1	-0.1881	-0.07938	737	353	0.47432
663	-25	1	-0.2450	-0.1027	663	313	0.467515	738	-20	1	-0.1790	-0.07552	738	352	0.472336
664	-24	1	-0.2349	-0.09847	664	314	0.468302	739	-21	1	-0.1877	-0.07921	739	353	0.473037
665	-25	1	-0.2444	-0.1024	665	315	0.469087	740	-20	1	-0.1786	-0.07536	740	352	0.471059
666	-24	1	-0.2343	-0.09823	666	314	0.466896	741	-21	1	-0.1873	-0.07904	741	353	0.47176
667	-23	1	-0.2242	-0.09403	667	315	0.467681	742	-22	1	-0.1960	-0.08272	742	354	0.472459
668	-24	1	-0.2337	-0.09800	668	314	0.465498	743	-23	1	-0.2046	-0.08639	743	355	0.473156
669	-23	1	-0.2237	-0.09381	669	315	0.466282	744	-24	1	-0.2133	-0.09005	744	356	0.473851
670	-24	1	-0.2331	-0.09777	670	316	0.467065	745	-23	1	-0.2042	-0.08620	745	357	0.474544
671	-23	1	-0.2231	-0.09359	671	317	0.467844	746	-22	1	-0.1951	-0.08237	746	358	0.475235
672	-24	1	-0.2325	-0.09754	672	318	0.468622	747	-23	1	-0.2037	-0.08602	747	357	0.473274
673	-25	1	-0.2419	-0.1015	673	319	0.469397	748	-22	1	-0.1946	-0.08220	748	356	0.471317
674	-24	1	-0.2319	-0.09732	674	320	0.47017	749	-21	1	-0.1856	-0.07838	749	357	0.47201

x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L^*(x)$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L^*(x)$	$\frac{L(x)}{L_{\approx,2}(x)}$
750	-22	1	-0.1942	-0.08202	750	358	0.472701	825	-15	1	-0.1221	-0.05189	825	391	0.46934
751	-23	1	-0.2028	-0.08566	751	359	0.47339	826	-16	1	-0.1301	-0.05530	826	392	0.469971
752	-24	1	-0.2114	-0.08929	752	358	0.471444	827	-17	1	-0.1381	-0.05870	827	393	0.4706
753	-23	1	-0.2023	-0.08548	753	359	0.472133	828	-18	1	-0.1461	-0.06209	828	394	0.471227
754	-24	1	-0.2109	-0.08911	754	360	0.47282	829	-19	1	-0.1540	-0.06548	829	395	0.471854
755	-23	1	-0.2019	-0.08531	755	361	0.473505	830	-20	1	-0.1620	-0.06886	830	396	0.472478
756	-22	1	-0.1929	-0.08151	756	360	0.471569	831	-19	1	-0.1537	-0.06536	831	397	0.473101
757	-23	1	-0.2014	-0.08513	757	361	0.472254	832	-20	1	-0.1616	-0.06873	832	396	0.471342
758	-22	1	-0.1924	-0.08134	758	362	0.472938	833	-21	1	-0.1695	-0.07210	833	395	0.469588
759	-23	1	-0.2010	-0.08495	759	363	0.473619	834	-22	1	-0.1774	-0.07546	834	396	0.470212
760	-24	1	-0.2095	-0.08855	760	364	0.474299	835	-21	1	-0.1692	-0.07196	835	397	0.470835
761	-25	1	-0.2180	-0.09215	761	365	0.474977	836	-20	1	-0.1610	-0.06847	836	396	0.469087
762	-26	1	-0.2264	-0.09574	762	366	0.475654	837	-19	1	-0.1528	-0.06499	837	397	0.46971
763	-25	1	-0.2175	-0.09196	763	367	0.476328	838	-18	1	-0.1446	-0.06151	838	398	0.470331
764	-26	1	-0.2259	-0.09554	764	366	0.474408	839	-19	1	-0.1525	-0.06487	839	399	0.470951
765	-25	1	-0.2170	-0.09177	765	365	0.472494	840	-18	1	-0.1443	-0.06139	840	400	0.471569
766	-24	1	-0.2081	-0.08801	766	366	0.47317	841	-17	1	-0.1361	-0.05793	841	399	0.469831
767	-23	1	-0.1992	-0.08426	767	367	0.473844	842	-16	1	-0.1280	-0.05447	842	400	0.470449
768	-24	1	-0.2076	-0.08783	768	366	0.471938	843	-15	1	-0.1199	-0.05102	843	401	0.471066
769	-25	1	-0.2160	-0.09139	769	367	0.472612	844	-16	1	-0.1277	-0.05437	844	400	0.469334
770	-24	1	-0.2072	-0.08765	770	368	0.473284	845	-17	1	-0.1356	-0.05771	845	399	0.467607
771	-23	1	-0.1983	-0.08391	771	369	0.473954	846	-16	1	-0.1275	-0.05427	846	398	0.465884
772	-24	1	-0.2067	-0.08747	772	368	0.472058	847	-17	1	-0.1353	-0.05761	847	397	0.464164
773	-25	1	-0.2151	-0.09102	773	369	0.472728	848	-18	1	-0.1431	-0.06094	848	396	0.462449
774	-24	1	-0.2063	-0.08729	774	368	0.470838	849	-17	1	-0.1350	-0.05750	849	397	0.463071
775	-25	1	-0.2146	-0.09084	775	367	0.468953	850	-16	1	-0.1270	-0.05407	850	396	0.461361
776	-24	1	-0.2058	-0.08712	776	368	0.469624	851	-15	1	-0.1189	-0.05064	851	397	0.461983
777	-25	1	-0.2141	-0.09065	777	369	0.470295	852	-14	1	-0.1109	-0.04722	852	396	0.460278
778	-24	1	-0.2053	-0.08694	778	370	0.470963	853	-15	1	-0.1187	-0.05055	853	397	0.460899
779	-23	1	-0.1966	-0.08323	779	371	0.47163	854	-16	1	-0.1265	-0.05387	854	398	0.461519
780	-24	1	-0.2049	-0.08677	780	370	0.469755	855	-15	1	-0.1184	-0.05046	855	397	0.459821
781	-23	1	-0.1962	-0.08307	781	371	0.470422	856	-14	1	-0.1104	-0.04705	856	398	0.460441
782	-24	1	-0.2045	-0.08659	782	372	0.471087	857	-15	1	-0.1182	-0.05036	857	399	0.461059
783	-23	1	-0.1957	-0.08290	783	373	0.47175	858	-14	1	-0.1102	-0.04696	858	400	0.461676
784	-22	1	-0.1870	-0.07922	784	374	0.472411	859	-15	1	-0.1180	-0.05027	859	401	0.462291
785	-21	1	-0.1783	-0.07554	785	375	0.473071	860	-14	1	-0.1100	-0.04688	860	400	0.460602
786	-22	1	-0.1866	-0.07906	786	376	0.473729	861	-15	1	-0.1177	-0.05018	861	401	0.461218
787	-23	1	-0.1949	-0.08257	787	377	0.474385	862	-14	1	-0.1098	-0.04679	862	402	0.461831
788	-24	1	-0.2031	-0.08607	788	376	0.472527	863	-15	1	-0.1175	-0.05009	863	403	0.462444
789	-23	1	-0.1945	-0.08240	789	377	0.473183	864	-14	1	-0.1096	-0.04671	864	404	0.463055
790	-24	1	-0.2027	-0.08590	790	378	0.473837	865	-13	1	-0.1016	-0.04333	865	405	0.463664
791	-23	1	-0.1940	-0.08224	791	379	0.47449	866	-12	1	-0.09373	-0.03996	866	406	0.464272
792	-22	1	-0.1854	-0.07859	792	378	0.472641	867	-13	1	-0.1014	-0.04325	867	405	0.462595
793	-21	1	-0.1768	-0.07494	793	379	0.473294	868	-12	1	-0.09354	-0.03989	868	404	0.460921
794	-20	1	-0.1682	-0.07130	794	380	0.473945	869	-11	1	-0.08566	-0.03653	869	405	0.46153
795	-21	1	-0.1764	-0.07479	795	381	0.474594	870	-10	1	-0.07780	-0.03318	870	406	0.462138
796	-22	1	-0.1846	-0.07828	796	380	0.472754	871	-9	1	-0.06995	-0.02984	871	407	0.462744
797	-23	1	-0.1928	-0.08175	797	381	0.473403	872	-8	1	-0.06212	-0.02650	872	408	0.463349
798	-22	1	-0.1842	-0.07812	798	382	0.474051	873	-9	1	-0.06981	-0.02978	873	407	0.461684
799	-21	1	-0.1757	-0.07450	799	383	0.474697	874	-10	1	-0.07750	-0.03306	874	408	0.462289
800	-22	1	-0.1838	-0.07797	800	382	0.472866	875	-9	1	-0.06968	-0.02973	875	409	0.462892
801	-23	1	-0.1920	-0.08143	801	381	0.471039	876	-8	1	-0.06188	-0.02640	876	408	0.461233
802	-22	1	-0.1834	-0.07782	802	382	0.471687	877	-9	1	-0.06954	-0.02968	877	409	0.461837
803	-21	1	-0.1749	-0.07421	803	383	0.472333	878	-8	1	-0.06176	-0.02635	878	410	0.462438
804	-20	1	-0.1664	-0.07060	804	382	0.470513	879	-7	1	-0.05398	-0.02304	879	411	0.463039
805	-21	1	-0.1745	-0.07406	805	383	0.471159	880	-6	1	-0.04623	-0.01973	880	410	0.461387
806	-22	1	-0.1827	-0.07751	806	384	0.471803	881	-7	1	-0.05388	-0.02300	881	411	0.461988
807	-21	1	-0.1742	-0.07392	807	385	0.472446	882	-8	1	-0.06152	-0.02626	882	412	0.462587
808	-20	1	-0.1657	-0.07033	808	386	0.473087	883	-9	1	-0.06914	-0.02952	883	413	0.463184
809	-21	1	-0.1738	-0.07377	809	387	0.473726	884	-8	1	-0.06140	-0.02621	884	412	0.46154
810	-20	1	-0.1654	-0.07019	810	386	0.471918	885	-9	1	-0.06901	-0.02946	885	413	0.462138
811	-21	1	-0.1734	-0.07363	811	387	0.472558	886	-8	1	-0.06128	-0.02617	886	414	0.462734
812	-20	1	-0.1650	-0.07006	812	386	0.470756	887	-9	1	-0.06887	-0.02941	887	415	0.463329
813	-19	1	-0.1566	-0.06649	813	387	0.471395	888	-10	1	-0.07645	-0.03265	888	416	0.463922
814	-20	1	-0.1647	-0.06992	814	388	0.472033	889	-9	1	-0.06874	-0.02936	889	417	0.464514
815	-19	1	-0.1563	-0.06636	815	389	0.472668	890	-10	1	-0.07631	-0.03259	890	418	0.465105
816	-18	1	-0.1479	-0.06281	816	388	0.470876	891	-11	1	-0.08386	-0.03582	891	417	0.463471
817	-17	1	-0.1395	-0.05926	817	389	0.471511	892	-12	1	-0.09139	-0.03904	892	416	0.461842
818	-16	1	-0.1312	-0.05572	818	390	0.472146	893	-11	1	-0.08370	-0.03576	893	417	0.462433
819	-15	1	-0.1229	-0.05219	819	389	0.47036	894	-12	1	-0.09122	-0.03898	894	418	0.463024
820	-14	1	-0.1145	-0.04866	820	388	0.468579	895	-11	1	-0.08354	-0.03570	895	419	0.463613
821	-15	1	-0.1226	-0.05209	821	389	0.469214	896	-10	1	-0.07587	-0.03242	896	420	0.464201
822	-16	1	-0.1306	-0.05551	822	390	0.469848	897	-11	1	-0.08338	-0.03563	897	421	0.464787
823	-17	1	-0.1387	-0.05892	823	391	0.47048	898	-10	1	-0.07573	-0.03237	898	422	0.465373
824	-16	1	-0.1304	-0.05540	824	392	0.471111	899	-9	1	-0.06809	-0.02910	899	423	0.465956

x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L(x)}{L_{\approx}^*(x)}$	x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L(x)}{L_{\approx}^*(x)}$
900	-8	1	-0.06047	-0.02585	900	422	0.464338	975	-13	1	-0.09177	-0.03944	975	451	0.458075
901	-7	1	-0.05286	-0.02260	901	423	0.464922	976	-14	1	-0.09874	-0.04244	976	450	0.456591
902	-8	1	-0.06035	-0.02580	902	424	0.465505	977	-15	1	-0.1057	-0.04544	977	451	0.457137
903	-9	1	-0.06783	-0.02900	903	425	0.466086	978	-16	1	-0.1126	-0.04843	978	452	0.457682
904	-8	1	-0.06024	-0.02576	904	426	0.466666	979	-15	1	-0.1055	-0.04537	979	453	0.458226
905	-7	1	-0.05266	-0.02252	905	427	0.467244	980	-16	1	-0.1124	-0.04835	980	454	0.458769
906	-8	1	-0.06012	-0.02571	906	428	0.467822	981	-17	1	-0.1194	-0.05133	981	453	0.457292
907	-9	1	-0.06758	-0.02890	907	429	0.468398	982	-16	1	-0.1123	-0.04827	982	454	0.457835
908	-10	1	-0.07501	-0.03209	908	428	0.466791	983	-17	1	-0.1192	-0.05125	983	455	0.458377
909	-11	1	-0.08244	-0.03526	909	427	0.465188	984	-18	1	-0.1261	-0.05422	984	456	0.458917
910	-10	1	-0.07487	-0.03203	910	428	0.465765	985	-17	1	-0.1190	-0.05117	985	457	0.459457
911	-11	1	-0.08228	-0.03520	911	429	0.466341	986	-18	1	-0.1258	-0.05414	986	458	0.459995
912	-10	1	-0.07473	-0.03198	912	428	0.464744	987	-19	1	-0.1327	-0.05710	987	459	0.460532
913	-9	1	-0.06720	-0.02875	913	429	0.465319	988	-18	1	-0.1256	-0.05405	988	458	0.459064
914	-8	1	-0.05967	-0.02554	914	430	0.465894	989	-17	1	-0.1185	-0.05101	989	459	0.459601
915	-9	1	-0.06707	-0.02870	915	431	0.466467	990	-18	1	-0.1254	-0.05396	990	458	0.458137
916	-10	1	-0.07445	-0.03187	916	430	0.464877	991	-19	1	-0.1323	-0.05692	991	459	0.458674
917	-9	1	-0.06695	-0.02865	917	431	0.46545	992	-18	1	-0.1252	-0.05388	992	460	0.459209
918	-10	1	-0.07432	-0.03181	918	432	0.466021	993	-17	1	-0.1181	-0.05085	993	461	0.459744
919	-11	1	-0.08167	-0.03496	919	433	0.466592	994	-18	1	-0.1250	-0.05379	994	462	0.460278
920	-12	1	-0.08901	-0.03811	920	434	0.467161	995	-17	1	-0.1179	-0.05077	995	463	0.460811
921	-11	1	-0.08152	-0.03490	921	435	0.467729	996	-16	1	-0.1109	-0.04774	996	462	0.459354
922	-10	1	-0.07404	-0.03170	922	436	0.468296	997	-17	1	-0.1177	-0.05069	997	463	0.459886
923	-9	1	-0.06657	-0.02851	923	437	0.468861	998	-16	1	-0.1107	-0.04767	998	464	0.460418
924	-10	1	-0.07390	-0.03165	924	436	0.467282	999	-15	1	-0.1037	-0.04465	999	465	0.460948
925	-11	1	-0.08122	-0.03478	925	435	0.465706	1000	-14	1	-0.09671	-0.04164	1000	466	0.461478
926	-10	1	-0.07377	-0.03160	926	436	0.466273	1001	-15	1	-0.1035	-0.04458	1001	467	0.462006
927	-11	1	-0.08107	-0.03473	927	435	0.464702	1002	-16	1	-0.1103	-0.04752	1002	468	0.462533
928	-10	1	-0.07363	-0.03154	928	436	0.465268	1003	-15	1	-0.1034	-0.04451	1003	469	0.463059
929	-11	1	-0.08092	-0.03467	929	437	0.465833	1004	-16	1	-0.1101	-0.04744	1004	468	0.461612
930	-10	1	-0.07350	-0.03149	930	438	0.466397	1005	-17	1	-0.1169	-0.05037	1005	469	0.462138
931	-11	1	-0.08077	-0.03461	931	437	0.464832	1006	-16	1	-0.1100	-0.04737	1006	470	0.462663
932	-12	1	-0.08803	-0.03772	932	436	0.463271	1007	-15	1	-0.1030	-0.04437	1007	471	0.463187
933	-11	1	-0.08062	-0.03455	933	437	0.463836	1008	-16	1	-0.1098	-0.04729	1008	472	0.46371
934	-10	1	-0.07323	-0.03138	934	438	0.4644	1009	-17	1	-0.1165	-0.05021	1009	473	0.464232
935	-11	1	-0.08048	-0.03449	935	439	0.464962	1010	-18	1	-0.1233	-0.05312	1010	474	0.464752
936	-10	1	-0.07309	-0.03133	936	438	0.463407	1011	-17	1	-0.1163	-0.05013	1011	475	0.465272
937	-11	1	-0.08033	-0.03443	937	439	0.46397	1012	-16	1	-0.1094	-0.04715	1012	474	0.463834
938	-12	1	-0.08755	-0.03753	938	440	0.464531	1013	-17	1	-0.1161	-0.05006	1013	475	0.464354
939	-11	1	-0.08018	-0.03438	939	441	0.465091	1014	-16	1	-0.1092	-0.04707	1014	474	0.462919
940	-10	1	-0.07283	-0.03122	940	440	0.463542	1015	-17	1	-0.1159	-0.04998	1015	475	0.463439
941	-11	1	-0.08004	-0.03432	941	441	0.464102	1016	-16	1	-0.1090	-0.04700	1016	476	0.463957
942	-12	1	-0.08724	-0.03741	942	442	0.464661	1017	-17	1	-0.1158	-0.04990	1017	475	0.462527
943	-11	1	-0.07989	-0.03426	943	443	0.465218	1018	-16	1	-0.1089	-0.04693	1018	476	0.463046
944	-12	1	-0.08708	-0.03735	944	442	0.463676	1019	-17	1	-0.1156	-0.04982	1019	477	0.463563
945	-13	1	-0.09425	-0.04042	945	443	0.464234	1020	-18	1	-0.1223	-0.05271	1020	476	0.462138
946	-14	1	-0.1014	-0.04350	946	444	0.46479	1021	-19	1	-0.1289	-0.05560	1021	477	0.462655
947	-15	1	-0.1086	-0.04657	947	445	0.465345	1022	-20	1	-0.1356	-0.05848	1022	478	0.463171
948	-14	1	-0.1012	-0.04343	948	444	0.463809	1023	-21	1	-0.1423	-0.06136	1023	479	0.463687
949	-13	1	-0.09391	-0.04029	949	445	0.464364	1024	-20	1	-0.1354	-0.05839	1024	478	0.462267
950	-12	1	-0.08661	-0.03716	950	444	0.462833	1025	-21	1	-0.1420	-0.06126	1025	477	0.46085
951	-11	1	-0.07932	-0.03404	951	445	0.463387	1026	-22	1	-0.1487	-0.06413	1026	478	0.461366
952	-12	1	-0.08645	-0.03710	952	446	0.463941	1027	-21	1	-0.1418	-0.06117	1027	479	0.461881
953	-13	1	-0.09357	-0.04016	953	447	0.464493	1028	-22	1	-0.1484	-0.06403	1028	478	0.460468
954	-12	1	-0.08630	-0.03704	954	446	0.462968	1029	-21	1	-0.1416	-0.06108	1029	479	0.460983
955	-11	1	-0.07904	-0.03392	955	447	0.46352	1030	-22	1	-0.1482	-0.06394	1030	480	0.461497
956	-12	1	-0.08614	-0.03698	956	446	0.462	1031	-23	1	-0.1548	-0.06679	1031	481	0.46201
957	-13	1	-0.09324	-0.04003	957	447	0.462552	1032	-24	1	-0.1614	-0.06964	1032	482	0.462522
958	-12	1	-0.08599	-0.03692	958	448	0.463103	1033	-25	1	-0.1680	-0.07249	1033	483	0.463032
959	-11	1	-0.07875	-0.03381	959	449	0.463652	1034	-26	1	-0.1745	-0.07533	1034	484	0.463542
960	-10	1	-0.07153	-0.03071	960	448	0.462138	1035	-25	1	-0.1677	-0.07238	1035	483	0.462138
961	-9	1	-0.06432	-0.02762	961	447	0.460626	1036	-24	1	-0.1608	-0.06943	1036	482	0.460736
962	-10	1	-0.07140	-0.03066	962	448	0.461177	1037	-23	1	-0.1540	-0.06649	1037	483	0.461246
963	-11	1	-0.07847	-0.03370	963	447	0.45967	1038	-24	1	-0.1606	-0.06933	1038	484	0.461756
964	-12	1	-0.08553	-0.03674	964	446	0.458166	1039	-25	1	-0.1671	-0.07216	1039	485	0.462265
965	-11	1	-0.07834	-0.03365	965	447	0.458717	1040	-24	1	-0.1603	-0.06922	1040	484	0.460868
966	-10	1	-0.07115	-0.03056	966	448	0.459267	1041	-23	1	-0.1535	-0.06629	1041	485	0.461377
967	-11	1	-0.07820	-0.03359	967	449	0.459816	1042	-22	1	-0.1467	-0.06336	1042	486	0.461884
968	-12	1	-0.08523	-0.03662	968	448	0.458318	1043	-21	1	-0.1399	-0.06043	1043	487	0.462391
969	-13	1	-0.09225	-0.03964	969	449	0.458867	1044	-22	1	-0.1465	-0.06326	1044	488	0.462897
970	-14	1	-0.09926	-0.04265	970	450	0.459415	1045	-23	1	-0.1530	-0.06609	1045	489	0.463401
971	-15	1	-0.1063	-0.04566	971	451	0.459962	1046	-22	1	-0.1462	-0.06317	1046	490	0.463905
972	-16	1	-0.1132	-0.04866	972	450	0.45847	1047	-21	1	-0.1395	-0.06025	1047	491	0.464408
973	-15	1	-0.1061	-0.04559	973	451	0.459017	1048	-20	1	-0.1327	-0.05734	1048	492	0.46491
974	-14	1	-0.09891	-0.04251	974	452	0.459562	1049	-21	1	-0.1392	-0.06016	1049	493	0.46541

x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L(x)}{L_{\approx}(x)}$	x	$L(x)$	$\frac{\text{sgn}(L(x))}{(-1)^{\lfloor \log \log x \rfloor}}$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L(x)}{L_{\approx}(x)}$
1050	-22	1	-0.1458	-0.06298	1050	492	0.464024	1125	-37	1	-0.2311	-0.1003	1125	527	0.463898
1051	-23	1	-0.1523	-0.06579	1051	493	0.464525	1126	-36	1	-0.2246	-0.09756	1126	528	0.464366
1052	-24	1	-0.1587	-0.06860	1052	492	0.463142	1127	-37	1	-0.2307	-0.1002	1127	527	0.463075
1053	-25	1	-0.1652	-0.07141	1053	491	0.461762	1128	-38	1	-0.2368	-0.1028	1128	528	0.463542
1054	-26	1	-0.1717	-0.07421	1054	492	0.462263	1129	-39	1	-0.2428	-0.1055	1129	529	0.464009
1055	-25	1	-0.1650	-0.07130	1055	493	0.462763	1130	-40	1	-0.2488	-0.1081	1130	530	0.464475
1056	-26	1	-0.1714	-0.07410	1056	494	0.463263	1131	-41	1	-0.2549	-0.1107	1131	531	0.46494
1057	-25	1	-0.1647	-0.07120	1057	495	0.463762	1132	-42	1	-0.2609	-0.1133	1132	530	0.463654
1058	-26	1	-0.1711	-0.07399	1058	494	0.462387	1133	-41	1	-0.2545	-0.1106	1133	531	0.464119
1059	-25	1	-0.1644	-0.07109	1059	495	0.462886	1134	-40	1	-0.2481	-0.1078	1134	530	0.462836
1060	-24	1	-0.1577	-0.06820	1060	494	0.461515	1135	-39	1	-0.2417	-0.1050	1135	531	0.463301
1061	-25	1	-0.1642	-0.07098	1061	495	0.462013	1136	-40	1	-0.2477	-0.1076	1136	530	0.462021
1062	-24	1	-0.1575	-0.06809	1062	494	0.460646	1137	-39	1	-0.2413	-0.1049	1137	531	0.462486
1063	-25	1	-0.1639	-0.07088	1063	495	0.461144	1138	-38	1	-0.2350	-0.1021	1138	532	0.46295
1064	-26	1	-0.1703	-0.07366	1064	496	0.461641	1139	-37	1	-0.2286	-0.09937	1139	533	0.463413
1065	-27	1	-0.1767	-0.07644	1065	497	0.462138	1140	-38	1	-0.2346	-0.1020	1140	532	0.462138
1066	-28	1	-0.1831	-0.07921	1066	498	0.462633	1141	-37	1	-0.2283	-0.09923	1141	533	0.462601
1067	-27	1	-0.1764	-0.07632	1067	499	0.463128	1142	-36	1	-0.2219	-0.09648	1142	534	0.463063
1068	-26	1	-0.1698	-0.07344	1068	498	0.461767	1143	-37	1	-0.2279	-0.09910	1143	533	0.461791
1069	-27	1	-0.1762	-0.07621	1069	499	0.462261	1144	-38	1	-0.2339	-0.1017	1144	534	0.462253
1070	-28	1	-0.1825	-0.07898	1070	500	0.462755	1145	-37	1	-0.2276	-0.09896	1145	535	0.462714
1071	-27	1	-0.1759	-0.07610	1071	499	0.461398	1146	-38	1	-0.2336	-0.1016	1146	536	0.463175
1072	-28	1	-0.1822	-0.07886	1072	498	0.460044	1147	-37	1	-0.2273	-0.09882	1147	537	0.463634
1073	-27	1	-0.1756	-0.07599	1073	499	0.460538	1148	-36	1	-0.2209	-0.09609	1148	536	0.462368
1074	-28	1	-0.1820	-0.07875	1074	500	0.461031	1149	-35	1	-0.2146	-0.09336	1149	537	0.462827
1075	-29	1	-0.1883	-0.08150	1075	499	0.459681	1150	-34	1	-0.2084	-0.09063	1150	536	0.461564
1076	-30	1	-0.1946	-0.08425	1076	498	0.458334	1151	-35	1	-0.2143	-0.09323	1151	537	0.462023
1077	-29	1	-0.1880	-0.08138	1077	499	0.458828	1152	-36	1	-0.2203	-0.09583	1152	536	0.460762
1078	-28	1	-0.1814	-0.07852	1078	498	0.457483	1153	-37	1	-0.2262	-0.09842	1153	537	0.461222
1079	-27	1	-0.1748	-0.07566	1079	499	0.457977	1154	-36	1	-0.2200	-0.09570	1154	538	0.46168
1080	-28	1	-0.1811	-0.07840	1080	500	0.45847	1155	-35	1	-0.2137	-0.09297	1155	539	0.462138
1081	-27	1	-0.1745	-0.07555	1081	501	0.458962	1156	-34	1	-0.2074	-0.09026	1156	540	0.462595
1082	-26	1	-0.1679	-0.07270	1082	502	0.459453	1157	-33	1	-0.2012	-0.08754	1157	541	0.463051
1083	-27	1	-0.1742	-0.07544	1083	501	0.458114	1158	-34	1	-0.2071	-0.09013	1158	542	0.463506
1084	-28	1	-0.1805	-0.07818	1084	500	0.456778	1159	-33	1	-0.2009	-0.08742	1159	543	0.463961
1085	-29	1	-0.1868	-0.08091	1085	501	0.45727	1160	-34	1	-0.2068	-0.09001	1160	544	0.464414
1086	-30	1	-0.1931	-0.08364	1086	502	0.457761	1161	-33	1	-0.2006	-0.08731	1161	545	0.464867
1087	-31	1	-0.1994	-0.08636	1087	503	0.458251	1162	-34	1	-0.2065	-0.08989	1162	546	0.465319
1088	-32	1	-0.2056	-0.08909	1088	502	0.456919	1163	-35	1	-0.2124	-0.09247	1163	547	0.465771
1089	-31	1	-0.1991	-0.08624	1089	503	0.457409	1164	-34	1	-0.2062	-0.08977	1164	546	0.46452
1090	-32	1	-0.2053	-0.08896	1090	504	0.457898	1165	-33	1	-0.2000	-0.08707	1165	547	0.464971
1091	-33	1	-0.2116	-0.09167	1091	505	0.458386	1166	-34	1	-0.2059	-0.08965	1166	548	0.465422
1092	-34	1	-0.2178	-0.09438	1092	504	0.457059	1167	-33	1	-0.1997	-0.08695	1167	549	0.465871
1093	-35	1	-0.2240	-0.09709	1093	505	0.457547	1168	-34	1	-0.2056	-0.08953	1168	548	0.464625
1094	-34	1	-0.2175	-0.09425	1094	506	0.458034	1169	-33	1	-0.1994	-0.08684	1169	549	0.465074
1095	-35	1	-0.2237	-0.09695	1095	507	0.45852	1170	-34	1	-0.2053	-0.08941	1170	548	0.463831
1096	-34	1	-0.2171	-0.09411	1096	508	0.459005	1171	-35	1	-0.2112	-0.09198	1171	549	0.46428
1097	-35	1	-0.2233	-0.09681	1097	509	0.45949	1172	-36	1	-0.2171	-0.09454	1172	548	0.463039
1098	-34	1	-0.2168	-0.09398	1098	508	0.458169	1173	-37	1	-0.2229	-0.09710	1173	549	0.463489
1099	-33	1	-0.2103	-0.09115	1099	509	0.458654	1174	-36	1	-0.2167	-0.09441	1174	550	0.463937
1100	-34	1	-0.2165	-0.09384	1100	510	0.459137	1175	-37	1	-0.2226	-0.09697	1175	549	0.4627
1101	-33	1	-0.2099	-0.09102	1101	511	0.459619	1176	-36	1	-0.2164	-0.09429	1176	548	0.461464
1102	-34	1	-0.2161	-0.09371	1102	512	0.460101	1177	-35	1	-0.2103	-0.09161	1177	549	0.461913
1103	-35	1	-0.2223	-0.09640	1103	513	0.460581	1178	-36	1	-0.2161	-0.09416	1178	550	0.462362
1104	-34	1	-0.2158	-0.09358	1104	512	0.459267	1179	-37	1	-0.2220	-0.09671	1179	549	0.46113
1105	-35	1	-0.2220	-0.09626	1105	513	0.459748	1180	-36	1	-0.2158	-0.09404	1180	548	0.4599
1106	-36	1	-0.2281	-0.09894	1106	514	0.460228	1181	-37	1	-0.2216	-0.09659	1181	549	0.460349
1107	-35	1	-0.2216	-0.09612	1107	515	0.460706	1182	-38	1	-0.2275	-0.09913	1182	550	0.460797
1108	-36	1	-0.2278	-0.09880	1108	514	0.459397	1183	-39	1	-0.2333	-0.1017	1183	549	0.459571
1109	-37	1	-0.2339	-0.1015	1109	515	0.459876	1184	-38	1	-0.2271	-0.09900	1184	550	0.460019
1110	-36	1	-0.2274	-0.09866	1110	516	0.460353	1185	-39	1	-0.2329	-0.1015	1185	551	0.460466
1111	-35	1	-0.2209	-0.09585	1111	517	0.46083	1186	-38	1	-0.2268	-0.09887	1186	552	0.460913
1112	-34	1	-0.2145	-0.09305	1112	518	0.461307	1187	-39	1	-0.2326	-0.1014	1187	553	0.461359
1113	-35	1	-0.2206	-0.09572	1113	519	0.461782	1188	-38	1	-0.2265	-0.09874	1188	552	0.460137
1114	-34	1	-0.2141	-0.09292	1114	520	0.462256	1189	-37	1	-0.2203	-0.09608	1189	553	0.460583
1115	-33	1	-0.2077	-0.09012	1115	521	0.46273	1190	-36	1	-0.2142	-0.09342	1190	554	0.461028
1116	-34	1	-0.2138	-0.09279	1116	522	0.463203	1191	-35	1	-0.2081	-0.09076	1191	555	0.461473
1117	-35	1	-0.2199	-0.09545	1117	523	0.463674	1192	-34	1	-0.2020	-0.08811	1192	556	0.461916
1118	-36	1	-0.2260	-0.09811	1118	524	0.464145	1193	-35	1	-0.2078	-0.09064	1193	557	0.462359
1119	-35	1	-0.2196	-0.09531	1119	525	0.464616	1194	-36	1	-0.2136	-0.09317	1194	558	0.462801
1120	-36	1	-0.2257	-0.09797	1120	526	0.465085	1195	-35	1	-0.2075	-0.09052	1195	559	0.463243
1121	-35	1	-0.2192	-0.09518	1121	527	0.465554	1196	-34	1	-0.2015	-0.08788	1196	558	0.462027
1122	-34	1	-0.2128	-0.09240	1122	528	0.466021	1197	-33	1	-0.1954	-0.08524	1197	557	0.460814
1123	-35	1	-0.2189	-0.09505	1123	529	0.466488	1198	-32	1	-0.1893	-0.08260	1198	558	0.461256
1124	-36	1	-0.2250	-0.09769	1124	528	0.465192	1199	-31	1	-0.1833	-0.07997	1199	559	0.461697