EXACT FORMULAS FOR THE GENERALIZED SUM-OF-DIVISORS FUNCTIONS

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Abstract. We prove new exact formulas for the generalized sum-of-divisors functions. The formulas for $\sigma_\alpha(x)$ when $\alpha \in \mathbb{C}$ is fixed and $x \geq 1$ involves a finite sum over all of the prime factors $n \leq x$ and terms involving the r-order harmonic number sequences. The generalized harmonic number sequences correspond to the partial sums of the Riemann zeta function when $r > 1$ and are related to the generalized Bernoulli numbers when $r \leq 0$ is integer-valued. A key part of our expansions of the Lambert series generating functions for the generalized divisor functions is formed by taking logarithmic derivatives of the cyclotomic polynomials, $\Phi_n(q)$, which completely factorize the Lambert series terms $(1-q^n)^{-1}$ into irreducible polynomials in q. We also consider applications of our new results to asymptotic approximations for sums over these divisor functions and to the forms of perfect numbers defined by the special case of the divisor function, $\sigma(n)$, when $\alpha := 1$.

Concerned with sequences: A000005; A000203; A000040; A001008; A001157–A001160; A002805; A000396; A013954–A013972; A027642; A027641.

1. Introduction

1.1. **Lambert series generating functions.** We begin our search for interesting formulas for the *generalized sum-of-divisors* functions, $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ for $\alpha \in \mathbb{C}$, by expanding the partial sums of the Lambert series generating these functions defined by [4, §17.10] [10, §27.7]

$$\widetilde{L}_\alpha(q) := \sum_{n\geq 1} \frac{n^\alpha q^n}{1-q^n} = \sum_{m \geq 1} \sigma_\alpha(m) q^m, \ |q| < 1. \tag{1}$$

In this article, we arrive at new expansions of the partial sums of Lambert series generating functions in (1) which generate our special arithmetic sequences as

$$\sigma_\alpha(x) = [q^x] \left(\sum_{n=1}^{x} \frac{n^\alpha q^n}{1-q^n}\right) = \sum_{d|x} d^\alpha, \ \alpha \in \mathbb{Z}^+.$$

In the references [12] we used analogous, and not unrelated, expansions of the terms in the Lambert series (1) and their corresponding higher-order derivatives to obtain new identities and formulas relating the generalized sum-of-divisors functions, $\sigma_\alpha(n)$, to divisor functions, $\sigma_\beta(n)$, for differing orders β and a class of bounded-divisor divisor functions defined naturally from the derivatives of the series considered there.

1.2. **Factoring partial sums into irreducibles.** The main difference in our technique in this article is that instead of differentiating these series to find new identities, we expand by repeated and heavy use of

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the properties of the well-known sequence of *cyclotomic polynomials*, $\Phi_n(q)$, defined by [3, §3] [7, §13.2]

$$\Phi_n(q) := \prod_{\substack{1\leq k\leq n \\ \gcd(k,n)=1}} \left(q - e^{2\pi\imath \frac{k}{n}}\right). \tag{2}$$

In particular, we see that for each integer $n \geq 1$ we have the factorizations

$$q^n - 1 = \prod_{d|n} \Phi_d(q), \tag{3}$$

or equivalently that

$$\Phi_n(x) = \prod_{d|n} (1-q^d)^{\mu(n/d)}. \tag{4}$$

If $n = p^m r$ with p prime and $\gcd(p,r) = 1$, we have an identity that $\Phi_n(q) = \Phi_{pr}(q^{p^{m-1}})$. In later results stated in the article, we are implicitly using the known expansions of the cyclotomic polynomials which condense the order n of the polynomials by exponentiation of the indeterminate q when n contains a factor of a prime power given by

$$\Phi_{2p}(q) = \Phi_p(-q), \Phi_{p^k}(q) = \Phi_p\left(q^{p^{k-1}}\right), \Phi_{p^k r}(q) = \Phi_{pr}\left(q^{p^{k-1}}\right), \Phi_{2^k}(q) = q^{2^{k-1}} + 1, \tag{5}$$

for p and odd prime, $k \geq 1$, and where $p \not| r$. We will require the next definitions to expand our Lambert series generating functions further by factoring its terms by the cyclotomic polynomials[1].

Definition 1.1 (Notation and Logatithmic Derivatives)**.** For $n \geq 2$ and any fixed indeterminate q, we define the following rational functions related to the logarithmic derivatives of the cyclotomic polynomials:

$$\begin{aligned} \Pi_n(q) &:= \sum_{j=0}^{n-2} \frac{(n-1-j)q^j(1-q)}{(1-q^n)} = \frac{(n-1)+nq-q^n}{(1-q)(1-q^n)} \\ \widetilde{\Phi}_n(q) &:= \frac{1}{q} \cdot \frac{d}{dw}\left[\log \Phi_n(w)\right]\Big|_{w\rightarrow \frac{1}{q}}. \end{aligned} \tag{6}$$

For any natural number $n \geq 2$ and prime p, we use $\nu_p(n)$ to denote the largest power of p dividing n. That is, if $p \not| n$, then $\nu_p(n) = 0$ and if $n = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k}$ is the prime factorization of n then $\nu_{p_i}(n) = \gamma_i$. Additionally, we define the function $\widetilde{\chi}_{\mathrm{PP}}(n)$ to denote the indicator function of the positive natural numbers n which are not of the form $n = p^k, 2p^k$ for any primes p and exponents $k \geq 1$. In the notation that follows, we consider sums indexed by p to be summed over only the primes p by convention unless specified otherwise.

1.3. **Factored Lambert series expansions.**

To provide some intuition to the factorizations of the terms in our Lambert series generating functions defined above, the listings in Table 1.1 provide the first several expansions of the right-hand-sides of the next equations which form the key component terms of our new exact formula expansions. In particular, we see that we may write the expansions of the individual Lambert series terms as

$$\frac{nq^n}{1-q^n} + n - \frac{1}{1-q} = \sum_{\substack{d|n \\ d>1}} \widetilde{\Phi}_d(q),$$

where we can reduce the index orders of the cyclotomic polynomials, $\Phi_n(q)$, and their logarithmic derivatives, $\widetilde{\Phi}_d(q)$, in lower-indexed cyclotomic polynomials with q transformed into powers of q to powers of

[1] Special Notation: *Iverson's convention* compactly specifies boolean-valued conditions and is equivalent to the *Kronecker delta function*, $\delta_{i,j}$, as $[n=k]_\delta \equiv \delta_{n,k}$. Similarly, $[\mathtt{cond} = \mathtt{True}]_\delta \equiv \delta_{\mathtt{cond},\mathtt{True}} \in \{0,1\}$, which is 1 if and only if `cond` is true, in the remainder of the article.

n	**Lambert Series Expansions** $\left(\frac{nq^n}{1-q^n}+n-\frac{1}{1-q}\right)$	**Formula Expansions**	**Reduced-Index Formula**
2	$\frac{1}{1+q}$	$\widetilde{\Phi}_2(q)$	$\widetilde{\Phi}_2(q)$
3	$\frac{2+q}{1+q+q^2}$	$\widetilde{\Phi}_3(q)$	$\widetilde{\Phi}_3(q)$
4	$\frac{1}{1+q}+\frac{2}{1+q^2}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_4(q)$	$\widetilde{\Phi}_2(q)+2\widetilde{\Phi}_2\left(q^2\right)$
5	$\frac{4+3q+2q^2+q^3}{1+q+q^2+q^3+q^4}$	$\widetilde{\Phi}_5(q)$	$\widetilde{\Phi}_5(q)$
6	$\frac{1}{1+q}+\frac{2-q}{1-q+q^2}+\frac{2+q}{1+q+q^2}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_3(q)+\widetilde{\Phi}_6(q)$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_3(q)+\widetilde{\Phi}_6(q)$
7	$\frac{6+5q+4q^2+3q^3+2q^4+q^5}{1+q+q^2+q^3+q^4+q^5+q^6}$	$\widetilde{\Phi}_7(q)$	$\widetilde{\Phi}_7(q)$
8	$\frac{1}{1+q}+\frac{2}{1+q^2}+\frac{4}{1+q^4}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_4(q)+\widetilde{\Phi}_8(q)$	$\widetilde{\Phi}_2(q)+2\widetilde{\Phi}_2\left(q^2\right)+4\widetilde{\Phi}_2\left(q^4\right)$
9	$\frac{2+q}{1+q+q^2}+\frac{3\left(2+q^3\right)}{1+q^3+q^6}$	$\widetilde{\Phi}_3(q)+\widetilde{\Phi}_9(q)$	$\widetilde{\Phi}_3(q)+3\widetilde{\Phi}_3\left(q^2\right)$
10	$\frac{1}{1+q}+\frac{4-3q+2q^2-q^3}{1-q+q^2-q^3+q^4}+\frac{4+3q+2q^2+q^3}{1+q+q^2+q^3+q^4}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_5(q)+\widetilde{\Phi}_{10}(q)$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_5(q)+\widetilde{\Phi}_{10}(q)$
11	$\frac{10+9q+8q^2+7q^3+6q^4+5q^5+4q^6+3q^7+2q^8+q^9}{1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10}}$	$\widetilde{\Phi}_{11}(q)$	$\widetilde{\Phi}_{11}(q)$
12	$\frac{1}{1+q}+\frac{2}{1+q^2}+\frac{2-q}{1-q+q^2}+\frac{2+q}{1+q+q^2}-\frac{2\left(-2+q^2\right)}{1-q^2+q^4}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_3(q)+\widetilde{\Phi}_4(q)$	$\widetilde{\Phi}_2(q)+2\widetilde{\Phi}_2\left(q^2\right)+\widetilde{\Phi}_3(q)$
		$+\widetilde{\Phi}_6(q)+\widetilde{\Phi}_{12}(q)$	$+\widetilde{\Phi}_6(q)+2\widetilde{\Phi}_6(q)$
13	$\frac{12+11q+10q^2+9q^3+8q^4+7q^5+6q^6+5q^7+4q^8+3q^9+2q^{10}+q^{11}}{1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10}+q^{11}+q^{12}}$	$\widetilde{\Phi}_{13}(q)$	$\widetilde{\Phi}_{13}(q)$
14	$\frac{1}{1+q}+\frac{6-5q+4q^2-3q^3+2q^4-q^5}{1-q+q^2-q^3+q^4-q^5+q^6}+\frac{6+5q+4q^2+3q^3+2q^4+q^5}{1+q+q^2+q^3+q^4+q^5+q^6}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_7(q)+\widetilde{\Phi}_{14}(q)$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_7(q)+\widetilde{\Phi}_{14}(q)$
15	$\frac{2+q}{1+q+q^2}+\frac{4+3q+2q^2+q^3}{1+q+q^2+q^3+q^4}+\frac{8-7q+5q^3-4q^4+3q^5-q^7}{1-q+q^3-q^4+q^5-q^7+q^8}$	$\widetilde{\Phi}_3(q)+\widetilde{\Phi}_5(q)+\widetilde{\Phi}_{15}(q)$	$\widetilde{\Phi}_3(q)+\widetilde{\Phi}_5(q)+\widetilde{\Phi}_{15}(q)$
16	$\frac{1}{1+q}+\frac{2}{1+q^2}+\frac{4}{1+q^4}+\frac{8}{1+q^8}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_4(q)+\widetilde{\Phi}_8(q)+\widetilde{\Phi}_{16}(q)$	$\widetilde{\Phi}_2(q)+2\widetilde{\Phi}_2\left(q^2\right)+4\widetilde{\Phi}_2\left(q^4\right)+8\widetilde{\Phi}_2\left(q^8\right)$

TABLE 1.1. Expansions of Lambert Series Terms by Cyclotomic Polynomial Primitives

primes according to the identities noted in (5) [3, *cf.* §3] [7, *cf.* §13.2]. Then by appealing to logarithmic derivatives of a product of differentiable rational functions and the definition given in (6) of the last definition, we are able to readily prove that for each natural number $n \geq 2$ we have that

$$\frac{q^n}{1-q^n} = -1 + \frac{1}{n(1-q)} + \frac{1}{n} \sum_{\substack{d|n \\ d>1}} \widetilde{\Phi}_d(q). \tag{7}$$

The third and fourth columns of Table 1.1 naturally suggest by computation the exact forms of the (logarithmic derivative) polynomial expansions we are looking for to expand our Lambert series terms. In effect, the observation of these trends in the polynomial expansions of $1-q^n$ led to the intuition motivating our new results within this article. In particular, we introduce the notation in the next definition corresponding to component sums employed to express sums over the previous identity in our key results stated in the next pages of the article.

Definition 1.2 (More Notation for Component Sums)**.** For fixed q and any $n \geq 1$, we define the component sums, $\widetilde{S}_{i,n}(q)$ for $i = 0, 1, 2$ as follows:

$$\begin{aligned}
\widetilde{S}_{0,n}(q) &= \sum_{\substack{d|n \\ d>1 \\ d \neq p^k, 2p^k}} \widetilde{\Phi}_d(q) \\
\widetilde{S}_{1,n}(q) &= \sum_{p|n} \Pi_{p^{\nu_p(n)}}(q) \\
\widetilde{S}_{2,n}(q) &= \sum_{\substack{2p|n \\ p>2}} \Pi_{p^{\nu_p(n)}}(-q).
\end{aligned}$$

1.4. **Statements of key results and characterizations.**

Theorem 1.3 (Exact Formulas for the Generalized Sum-of-Divisors Functions)**.** *For any fixed $\alpha \in \mathbb{C}$ and natural numbers $x \geq 1$, we have the following generating function formula:*

$$\sigma_\alpha(x) = H_x^{(1-\alpha)} + [q^x] \left(\sum_{n=1}^{x} \widetilde{S}_{0,n}(q) n^{\alpha-1} + \widetilde{S}_{1,n}(q) n^{\alpha-1} + \widetilde{S}_{2,n}(q) n^{\alpha-1} \right).$$

Proposition 1.4 (Series Coefficients of the Component Sums)**.** *For any fixed $\alpha \in \mathbb{C}$ and integers $x \geq 1$, we have the following components of the partial sums of the Lambert series generating functions in Theorem 1.3:*

$$[q^x] \sum_{n=1}^{x} \widetilde{S}_{0,n}(q) n^{\alpha-1} := \sum_{d|n} \tau_x^{(\alpha)}(d) \tag{i}$$

$$[q^x] \sum_{n=1}^{x} \widetilde{S}_{1,n}(q) n^{\alpha-1} = \sum_{p \leq x} \sum_{k=1}^{\nu_p(x)+1} p^{\alpha k-1} H_{\left\lfloor \frac{x}{p^k} \right\rfloor}^{(1-\alpha)} \left(p \left\lfloor \frac{x}{p^k} \right\rfloor - p \left\lfloor \frac{x}{p^k} - \frac{1}{p} \right\rfloor - 1 \right) \tag{ii}$$

$$[q^x] \sum_{n=1}^{x} \widetilde{S}_{2,n}(q) n^{\alpha-1} = \sum_{3 \leq p \leq x} \sum_{k=1}^{\nu_p(x)+1} \frac{p^{\alpha k-1}}{2^{1-\alpha}} H_{\left\lfloor \frac{x}{2p^k} \right\rfloor}^{(1-\alpha)} (-1)^{\left\lfloor \frac{x}{p^{k-1}} \right\rfloor} \left(p \left\lfloor \frac{x}{p^k} \right\rfloor - p \left\lfloor \frac{x}{p^k} - \frac{1}{p} \right\rfloor - 1 \right). \tag{iii}$$

1.5. **Remarks.** Before we continue on to the proofs of our new results, we first have a few remarks about symmetry in the identity from the theorem in the context of negative-order divisor functions and a brief overview of the applications we feature in Section 3. In Section 3 we consider the applications of the theorem to a few notable famous problems. Namely, we consider asymptotics of sums over the sum-of-divisors functions and we consider the implications of our new exact formula in the special case where $\alpha := 1$ to determining conditions for an integer to be a *perfect number* [11, §2].

1.5.1. *Symmetric forms of the exact formulas.* For integers $\alpha \in \mathbb{N}$, we can express the "negative-order" harmonic numbers, $H_n^{(-\alpha)}$, in terms of the *generalized Bernoulli numbers* as

$$\sum_{m=1}^{n} m^\alpha = \frac{1}{\alpha+1} \left(B_{\alpha+1}(n+1) - B_{\alpha+1} \right).$$

Then since a convolution formula proves that $\sigma_{-\beta}(n) = \sigma_\beta(n)/n^\beta$ whenever $\beta > 0$, we may also expand the right-hand-side of the theorem in the symmetric form of

$$\sigma_\alpha(x) = x^\alpha \left(H_x^{(\alpha+1)} + \sum_{d|x} \tau_x^{(-\alpha)}(d) + \sum_{n=1}^{x} [q^x] \left(\widetilde{S}_{1,n}(q) + \widetilde{S}_{2,n}(q) \right) n^{1-\alpha} \right), \tag{8}$$

when $\alpha > 0$ is strictly real-valued. We notice that this symmetry identity provides a curious, and necessarily deep, relation between the Bernoulli numbers and the partial sums of the Riemann zeta function involving nested sums over the primes.

2. Proofs of our new results

Example 2.1. We first revisit a computational example of the rational functions defined by the logarithmic derivatives in Definition 1.1 from the table above. We make use of the next variant of the identity in (3) in the proof below which is obtained by Möebius inversion.

$$\Phi_n(q) = \prod_{d|n} (q^d - 1)^{\mu(n/d)} \tag{9}$$

In the case of our modified rational cyclotomic polynomial functions, $\widetilde{\Phi}_n(q)$, when $n := 15$, we use this product to expand the definition of the function as

$$\begin{aligned}
\widetilde{\Phi}_{15}(q) &= \frac{1}{x} \cdot \frac{d}{dq} \left[\log \left(\frac{(1-q^3)(1-q^5)}{(1-q)(1-q^{15})} \right) \right] \Bigg|_{q \to 1/q} \\
&= \frac{3}{1-q^3} + \frac{5}{1-q^5} - \frac{1}{1-q} - \frac{15}{1-q^{15}} \\
&= \frac{8 - 7q + 5q^3 - 4q^4 + 3q^5 - q^7}{1 - q + q^3 - q^4 + q^5 - q^7 + q^8}.
\end{aligned}$$

The procedure for transforming the difficult-looking terms involving the cyclotomic polynomials when the Lambert series terms, $q^n/(1-q^n)$, are expanded in partial fractions as in Table 1.1 is essentially the same as this example for the cases we will encounter here. In general, we have the next simple lemma when n is a positive integer.

Lemma 2.2 (Key Characterizations of the Tau Divisor Sums)**.** *For integers $n \geq 1$ and any indeterminate q, we have the following expansion of the functions in* (6)*:*

$$\widetilde{\Phi}_n(q) = \sum_{d|n} \frac{d \cdot \mu(n/d)}{(1-q^d)}.$$

In particular, we have that

$$\widetilde{S}_{0,n}(q) = \sum_{d|n} \sum_{r|d} \frac{r \cdot \widetilde{\chi}_{\mathrm{PP}}(d) \cdot \mu(d/r)}{(1-q^r)}.$$

Proof. The proof is essentially the same as the example given above. Since we can refer to this illustrative example, we only need to sketch the details to the remainder of the proof. In particular, we notice that since we have the known identity for the cyclotomic polynomials given by

$$\Phi_n(x) = \prod_{d|n} (1 - q^d)^{\mu(n/d)}$$

we can take logarithmic derivatives to obtain that

$$\frac{1}{x} \cdot \frac{d}{dq} \left[\log \left(1 - q^d\right)^{\pm 1} \right] \Bigg|_{q \to 1/q} = \mp \frac{d}{q^d \left(1 - \frac{1}{q^d}\right)} = \pm \frac{d}{1-q^d},$$

which applied inductively leads us to our result. □

Remark 2.3 (Connections to Ramanujan's Sum). First, let the following notation denote a shorthand for the divisor sum terms in Theorem 1.3:

$$\tau_\alpha(x) = \sum_{d|x} \tau_x^{(\alpha+1)}(d).$$

Observe that the contribution of the first (zero-indexed) sums in Theorem 1.3 correspond to the coefficients

$$\begin{aligned}
\tau_\alpha(x) &= \sum_{d|x} \tau_x^{(\alpha+1)}(d) \\
&= [q^x]\left(\sum_{k=1}^{x} \sum_{\substack{d|k \\ d\neq p^k, 2p^k}} \sum_{r|d} \frac{r\cdot\mu(d/r)}{(1-q^r)} k^\alpha\right) \\
&= \sum_{k=1}^{x} \sum_{r|x} \sum_{\substack{d|k \\ d\neq p^k, 2p^k}} r\cdot\mu(d/r)\cdot[r|d]_\delta \cdot k^\alpha \\
&= \sum_{k=1}^{x} \sum_{\substack{d|k \\ d\neq p^k, 2p^k}} \sum_{r|(d,x)} r\cdot\mu(d/r)\cdot k^\alpha.
\end{aligned}$$

We can also expand the right-hand-side of the previous equation as

$$\tau_\alpha(x) = \sum_{\substack{d=1 \\ d\neq p^k, 2p^k}}^{x} \left(\sum_{r|(d,x)} r\mu(d/r)\right) H_{\lfloor \frac{x}{d}\rfloor}^{(-\alpha)},$$

Then we have a deep connection between the divisor sums in Lemma 2.2 and Ramanujan's sum $c_q(n)$ given by

$$\begin{aligned}
\tau_0(x) &= \sum_{k=1}^{x} \sum_{\substack{d|k \\ d\neq p^k, 2p^k}} c_d(x) \\
&= \sum_{k=1}^{x} \sum_{\substack{d|k \\ d\neq p^k, 2p^k}} \mu\left(\frac{d}{(d,x)}\right) \frac{\varphi(d)}{\varphi\left(\frac{d}{(d,x)}\right)},
\end{aligned}$$

where $\varphi(x)$ denotes Euler's totient function. These identities follow by expanding out Ramanujan's sum in the form of [10, §27.10] [9, §A.7] [4, *cf.* §5.6]

$$c_q(n) = \sum_{d|(q,n)} d\cdot\mu(q/d).$$

Ramanujan's sum also satisfies the convenient bound that $|c_q(n)| \leq (n,q)$ for all $n, q \geq 1$, which can be used to obtain asymptotic estimates in the form of upper bounds for these sums when q is not prime or a prime power.

Proof of Theorem 1.3. We begin with a well-known divisor product formula involving the cyclotomic polynomials when $n \geq 1$ and q is fixed:

$$q^n - 1 = \prod_{d|n} \Phi_d(q).$$

Then by logarithmic differentiation we can see that

$$\begin{aligned}
\frac{q^n}{1-q^n} &= -1 + \frac{1}{n(1-q)} + \frac{1}{n}\sum_{\substack{d|n \\ d>1}} \widetilde{\Phi}_d(q) \qquad (10) \\
&= -1 + \frac{1}{n(1-q)} + \frac{1}{n}\left(\widetilde{S}_{0,n}(q) + \widetilde{S}_{1,n}(q) + \widetilde{S}_{2,n}(q)\right).
\end{aligned}$$

The last equation is obtained from the first expansion above by noting the equivalence of the next two sums as

$$\Pi_n(q) = \widetilde{\Phi}_n(1/q) = \sum_{j=0}^{n-2} \frac{(n-1-j)q^j(1-q)}{1-q^n}.$$

Here we are implicitly using the known expansions of the cyclotomic polynomials which condense the order n of the polynomials by exponentiation of the indeterminate q when n contains a factor of a prime power given by (5) in the introduction. Finally, we complete the proof by summing the right-hand-side of (10) over $n \leq x$ times the weight n^α to obtain the x^{th} partial sum of the Lambert series generating function for $\sigma_\alpha(x)$ [4, §17.10] [10, §27.7], which since each term in the summation contains a power of q^n is $(x+1)$-order accurate to the terms in the infinite series. ☐

Proof of Proposition 1.4. The identity in (i) follows from Lemma 2.2. Since $\Phi_{2p}(q) = \Phi_p(-q)$ for any prime p, we are essentially in the same case with the two component sums in (ii) and (iii). We outline the proof of our expansion for the first sum, $\widetilde{S}_{1,n}(q)$, and note the small changes necessary along the way to adapt the proof to the second sum case. By the properties of the cyclotomic polynomials expanded in (5) , we may factor the denominators of $\Pi_{p^{\varepsilon_p(n)}}(q)$ into smaller irreducible factors of the same polynomial, $\Phi_p(q)$, with inputs varying as special prime-power powers of q. More precisely, we may expand

$$\widetilde{S}_{1,n}(q) = \sum_{p \leq n} \sum_{k=1}^{\varepsilon_p(n)} \underbrace{\frac{\sum_{j=0}^{p-2}(p-1-j)q^{p^{k-1}j}}{\sum_{i=0}^{p-1} q^{p^{k-1}i}}}_{:=Q_{p,k}^{(n)}(q)} \cdot p^{k-1}.$$

In performing the sum $\sum_{n \leq x} Q_{p,k}^{(n)}(q) p^{k-1} n^{\alpha-1}$, these terms of the $Q_{p,k}^{(n)}(q)$ occur again, or have a repeat coefficient, every p^k terms, so we form the coefficient sums for these terms as

$$\sum_{i=i}^{\left\lfloor \frac{x}{p^k} \right\rfloor} \left(ip^k\right)^{\alpha-1} \cdot p^{k-1} = p^{k\alpha-1} \cdot H_{\left\lfloor \frac{x}{p^k} \right\rfloor}^{(1-\alpha)}.$$

We can also compute the inner sums in the previous equations exactly for any fixed t as

$$\sum_{j=0}^{p-2}(p-1-j)t^j = \frac{(p-1)+pt-t^p}{(1-t)^2},$$

where the corresponding paired denominator sums in these terms are given by $1+t+t^2+\cdots+t^{p-1} = (1-t^p)/(1-t)$. We now assemble the full sum over $n \leq x$ we are after in this proof as follows:

$$\sum_{n \leq x} \widetilde{S}_{1,n}(q) \cdot n^{\alpha-1} = \sum_{p \leq x} \sum_{k=1}^{\varepsilon_p(x)} p^{k\alpha-1} H_{\left\lfloor \frac{x}{p^k} \right\rfloor}^{(1-\alpha)} \frac{(p-1)-pq^{p^{k-1}}+q^{p^k}}{(1-q^{p^{k-1}})(1-q^{p^k})}.$$

The corresponding result for the second sums is obtained similarly with the exception of sign changes on the coefficients of the powers of q in the last expansion.

We compute the series coefficients of one of the three cases in the previous equation to show our method of obtaining the full formula. In particular, the right-most term in these expansions leads to the double sum

$$\begin{aligned} C_{3,x,p} &:= [q^x] \frac{q^{p^k}}{(1 \mp q^{p^{k-1}})(1 \mp q^{p^k})} \\ &= [q^x] \sum_{n,j \geq 0} (\pm 1)^{n+j} q^{p^{k-1}(n+p+jp)}. \end{aligned}$$

Thus we must have that $p^{k-1}|x$ in order to have a non-zero coefficient and for $n := x/p^{k-1} - jp - p$ with $0 \leq j \leq x/p^k - 1$ we can compute these coefficients explicitly as

$$C_{3,x,p} := (\pm 1)^{\lfloor x/p^{k-1} \rfloor} \times \sum_{j=0}^{\lfloor x/p^k - 1 \rfloor} 1 = (\pm 1)^{\lfloor x/p^{k-1} \rfloor} \left\lfloor \frac{x}{p^k} - 1 \right\rfloor + 1 = (\pm 1)^{\lfloor x/p^{k-1} \rfloor} \left\lfloor \frac{x}{p^k} \right\rfloor .$$

With minimal simplifications we have arrived at our claimed result in the proposition. □

3. Applications

3.1. **Asymptotics of sums of the divisor functions.** We can use the new exact formula proved by the theorem to asymptotically estimate partial sums, or *average orders* of the respective arithmetic functions, of the following form for integers $x \geq 1$:

$$\Sigma_x^{(\alpha,\beta)} := \sum_{n \leq x} \frac{\sigma_\alpha(n)}{n^\beta}.$$

In the special cases where $\alpha := 0, 1$, we restate a few more famous formulas providing well-known classically (and newer) established asymptotic bounds for sums of this form as follows where $\gamma \approx 0.577216$ is Euler's gamma constant, $d(n) \equiv \sigma_0(n)$ denotes the (Dirichlet) *divisor function*, and $\sigma(n) \equiv \sigma_1(n)$ the (ordinary) *sum-of-divisors function* [6] [10, *cf.* §27.11]:

$$\begin{aligned} \sum_{n \leq x} d(n) &= x \log x + (2\gamma - 1)x + O\left(x^{\frac{131}{416}}\right) \qquad (11) \\ \sum_{n \leq x} \frac{d(n)}{n} &= \frac{1}{2}(\log x)^2 + 2\gamma \log x + O(1) \\ \sum_{n \leq x} \sigma(n) &= \frac{\pi^2}{12} x^2 + O(x \log x). \end{aligned}$$

For the most part, we suggest tackling potential improvements to these asymptotic formulas through our new results given in the theorem and in the symmetric identity (8) as a highly suggested topic for future research, especially since we cannot give the topic a fair and detailed treatment within the context of this article. Moreover, we surmise that more sophisticated estimates of these sums are possible than those given as examples in this section below by combining these results with other asymptotic formulas related to sums over primes.

Example 3.1 (Average Order of the Divisor Function)**.** For comparison with the leading terms in the first of the previous expansions, we can prove the next formula using summation by parts for integers $r \geq 1$.

$$\sum_{j=1}^{n} H_j^{(r)} = (n+1)H_n^{(r)} - H_n^{(r-1)}$$

Then using inexact approximations for the summation terms in the theorem, we are able to evaluate the leading non-error term in the following sum for large integers $t \geq 2$ since $H_n^{(1)} \sim \log n + \gamma$:

$$\begin{aligned} \Sigma_t^{(0,0)} &= -t + (t+1)H_t^{(1)} + O(t \cdot \log^3(t)) \\ &\sim (t+1)\log t + (\gamma - 1)t + \gamma + O(t \cdot \log^3(t)). \end{aligned}$$

It is similarly not difficult to obtain a related estimate for the second famous divisor sum, $\Sigma_t^{(0,1)}$, using the symmetric identity in (8) of the introduction.

3.2. **Perfect numbers.** We finally turn our attention to an immediate application of our new results which is perhaps one of the most famous unresolved problems in number theory: that of determining the form and infinitude of the perfect numbers. A *perfect number* p is a positive integer such that $\sigma(p) = 2p$. The first few perfect numbers are given by the sequence $\{6, 28, 496, 8128, 33550336, \ldots\}$. It currently is not known whether there are infinitely-many perfect numbers, or whether there exist odd perfect numbers. References to work on the distribution of the perfect number counting function, $V(x) := \#\{n \text{ perfect} : n \leq x\}$ are found in [11, §2.7]. Since we now have a fairly simple exact formula for the sum-of-divisors function, $\sigma(n)$, we briefly attempt to formulate conditions for an integer to be perfect within the scope of this article.

It is well known that given a *Mersenne prime* of the form $q = 2^p - 1$ for prime p, then we have an even perfect number P of the corresponding form $P = 2^{p-1}(2^p - 1)$ [11, §2.7] [8]. We suppose that the positive integer P has the form $P = 2^{p-1}(2^p - 1)$ for some (prime) integer $p \geq 2$, and consider the expansion of the sum-of-divisors function on this input to our new exact formulas. Suppose that $R := 2^p - 1 = r_1^{\gamma_1} r_2^{\gamma_2} \cdots r_k^{\gamma_k}$ is the prime factorization of this factor R of P where $\gcd(2, r_i) = 1$ for all $1 \leq i \leq k$ and that $R_s := R/s^{\nu_s(R)}$. Then by the formulas derived in Proposition 1.4 we have by Theorem 1.3 that

$$\begin{aligned}
\sigma(P) = \frac{(p+1)}{2} P &+ \frac{P}{R} \left\lfloor \frac{R}{2} \right\rfloor \left(2 \left\lfloor \frac{R}{2} \right\rfloor - 2 \left\lfloor \frac{R-1}{2} \right\rfloor - 1 \right) + \sum_{d|P} \tau_P^{(1)}(d) \\
&+ \sum_{\substack{3 \leq s \leq P \\ s \text{ prime}}} \frac{3}{2} \frac{(s-1)}{s} P \cdot \nu_s(R) \\
&+ \sum_{\substack{3 \leq s \leq P \\ s \text{ prime}}} s^{\nu_s(R)} \left(\left\lfloor \frac{2^{p-1} R_s}{s} \right\rfloor + \left\lfloor \frac{2^{p-2} R_s}{s} \right\rfloor \right) \times \\
&\times \left(s \left\lfloor \frac{2^{p-1} R_s}{s} \right\rfloor - s \left\lfloor \frac{2^{p-1} R_s - 1}{s} \right\rfloor - 1 \right).
\end{aligned}$$

If we set $\sigma(P) = 2P$, i.e., construct ourselves a perfect number P by assumption to work with, and then finally solve for the linear equation in P from the last equation, we obtain that P is perfect implies that either of the following conditions hold where $\{x\} = x - \lfloor x \rfloor$ denotes the *fractional part of* x for $x \in \mathbb{R}$:

$$P = -\frac{\sum_{d|P} \tau_P^{(1)}(d) + \sum_{\substack{3 \leq s \leq P \\ s \text{ prime}}} s^{\nu_s(R)} \left(\left\lfloor \frac{2^{p-1} R_s}{s} \right\rfloor + \left\lfloor \frac{2^{p-2} R_s}{s} \right\rfloor \right) \left(s \left\lfloor \frac{2^{p-1} R_s}{s} \right\rfloor - s \left\lfloor \frac{2^{p-1} R_s - 1}{s} \right\rfloor - 1 \right)}{\frac{(p-3)}{2} + \frac{1}{R} \left\lfloor \frac{R}{2} \right\rfloor \left(2 \left\lfloor \frac{R}{2} \right\rfloor - 2 \left\lfloor \frac{R-1}{2} \right\rfloor - 1 \right) + \sum_{\substack{3 \leq s \leq P \\ s \text{ prime}}} \frac{3(s-1)}{2s} \cdot \nu_s(R)}.$$

Miraculously, this formula does produce not only integers, but the perfect even integers of our prescribed form, which is easy to verify computationally for the first several known perfect numbers.

Variants. There are endless other variants of the perfect number criteria for the integers that we may only touch on within this article. For example, another related problem is that given an integer $k \geq 3$, determine the forms of all *multiperfect numbers* defined such that $\sigma(n) = kn$. The sequence of elements corresponding to these positive integers such that $n | \sigma(n)$ are sometimes also called *multiply-perfect numbers*, i.e., the positive integers whose *abundancy*, $\sigma(n)/n$, is integer-valued and whose first few ordered entries are given by the sequence $\{1, 6, 28, 120, 496, 672, 8128, 30240, 32760, \ldots\}$.

4. Comparisons to other exact formulas for partition and divisor functions

4.1. **Exact formulas for the divisor and sums-of-divisor functions.**

4.1.1. *Finite sums and trigonometric series identities.* We first remark that there is an obvious finite sum identity which generates partial sums of the generalized sum-of-divisors functions in the following forms [1, *cf.* §7] [2]:

$$\begin{aligned}
\Sigma_\alpha(x) &:= \sum_{n\leq x}\sigma_\alpha(n) = \sum_{d\leq x}\left\lfloor\frac{x}{d}\right\rfloor\cdot d^\alpha \\
&= \sum_{m=0}^{\left\lfloor\frac{\log x}{\log 2}\right\rfloor}\sum_{d=1}^{\left\lfloor\frac{x}{2^m}\right\rfloor-\left\lfloor\frac{x}{2^{m+1}}\right\rfloor}\left\lfloor\frac{x}{d+\left\lfloor x2^{-(m+1)}\right\rfloor}\right\rfloor\left(d+\left\lfloor\frac{x}{2^{m+1}}\right\rfloor\right)^k \\
&= \sum_{m=0}^{\left\lfloor\frac{\log x}{\log p}\right\rfloor}\sum_{d=1}^{\left\lfloor\frac{x}{p^m}\right\rfloor-\left\lfloor\frac{x}{p^{m+1}}\right\rfloor}\left\lfloor\frac{x}{d+\left\lfloor xp^{-(m+1)}\right\rfloor}\right\rfloor\left(d+\left\lfloor\frac{x}{p^{m+1}}\right\rfloor\right)^k,\ p\in\mathbb{Z}, p\geq 2.
\end{aligned}$$

Since the sums $\sum_{d=1}^m (d+a)^k$ are readily expanded by the Bernoulli polynomials, we may approach summing the last finite sum identity by parts [10, §24.4(iii)]. The relation of sums of this type corresponding to the divisor function case where $\alpha := 0$ are considered in the context of the Dirichlet divisor problem in [2] as are the evaluations of several sums involving the floor function such as we have in the statement of Theorem 1.3.

There is another infinite series for the ordinary sums-of-divisors function, $\sigma(n)$, due to Ramanujan in the form of [5, §9, p. 141]

$$\sigma(n) = \frac{n\pi^2}{6}\left[1+\frac{(-1)^n}{2^2}+\frac{2\cos\left(\frac{2}{3}n\pi\right)}{3^2}+\frac{2\cos\left(\frac{2}{5}n\pi\right)+2\cos\left(\frac{4}{5}n\pi\right)}{5^2}+\cdots\right] \tag{12}$$

In similar form, we have a corresponding infinite sum providing an exact formula for the divisor function expanded in terms of the functions $c_q(n)$ defined in [5, §9] of the form (*cf.* Remark 2.3)

$$d(n) = -\sum_{k\geq 1}\frac{c_k(n)}{k}\log(k) = -\frac{c_2(n)}{2}\log 2-\frac{c_3(n)}{3}\log 3-\frac{c_4(n)}{4}\log 4-\cdots. \tag{13}$$

Recurrence relations between the generalized sum-of-divisors functions are proved in the references [12, 13]. There are also a number of known convolution sum identities involving the sum-of-divisors functions which are derived from their relations to Lambert series and Eisenstein series.

4.1.2. *Exact formulas for sums of the divisor function.* Exact formulas for the divisor function, $d(n)$, of a much different characteristic nature are expanded in the results of [1]. First, we compare our finite sum results with the infinite sums in the (weighted) Voronoi formulas for the partial sums over the divisor function expanded as

$$\frac{x^{\nu-1}}{\Gamma(\nu)}\sum_{n\leq x}\left(1-\frac{n}{x}\right)^{\nu-1}d(n) = \frac{x^{\nu-1}}{4\Gamma(\nu)}+\frac{x^\nu\left(\log x+\gamma-\psi(1+\nu)\right)}{\Gamma(\nu+1)}-2\pi x^\nu\sum_{n\geq 1}d(n)F_\nu\left(4\pi\sqrt{nx}\right)$$

$$\begin{aligned}
\sum_{n\leq x}d(n) &= \frac{1}{4}+\left(\log x+2\gamma-1\right)x \\
&\quad -\frac{2\sqrt{x}}{\pi}\sum_{n\geq 1}\frac{d(n)}{\sqrt{n}}\left(K_1\left(4\pi\sqrt{nx}\right)+\frac{\pi}{2}Y_1\left(4\pi\sqrt{nx}\right)\right),
\end{aligned}$$

where $F_\nu(z)$ is some linear combination of the *Bessel functions*, $K_\nu(z)$ and $Y_\nu(z)$, and $\psi(z)$ is the *digamma function.* A third identity for the partial sums, or average order, of the divisor function is expanded directly in terms of the Riemann zeta function, $\zeta(s)$, and its non-trivial zeros ρ in the next equation.

$$\begin{aligned}
\sum_{n\leq x}d(n) &= -\frac{\pi^2}{12}+(\log x+2\gamma-1)x+\frac{\pi^2}{3}\sum_{\substack{\rho:\zeta(\rho)=0\\ \rho\neq -2,-4,-6,\ldots}}\frac{\zeta(\rho/2)^2}{\rho\zeta'(\rho)}x^{\rho/2} \\
&\quad -\frac{\pi^2}{6}\sum_{n\geq 0}\frac{\zeta(-(2n+1))x^{-(2n+1)}}{(2n+1)\zeta'(-(2n+1))}
\end{aligned}$$

While our new exact sum formulas in Theorem 1.3 are deeply tied to the prime numbers $2 \leq p \leq x$ for any x, we once again observe that the last three infinite sum expansions of the partial sums over the divisor function are distinctly much different in character than our new exact finite sum formulas proved by the theorem.

4.2. Comparisons with other exact formulas for special functions.

4.2.1. *Rademacher's formula for the partition function $p(n)$.* Rademacher's famous exact formula for the partition function $p(n)$ when $n \geq 1$ is stated as [14]

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k\geq 1} A_k(n)\sqrt{k}\frac{d}{dn}\left[\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right],$$

where

$$A_k(n) := \sum_{\substack{0\leq h<k \\ \gcd(h,k)=1}} e^{\pi\imath s(h,k)-2\pi\imath nh/k},$$

is a *Kloosterman-like sum* and $s(h,k)$ and $\omega(h,k)$ are the (exponential) *Dedekind sums* defined by

$$\begin{aligned} s(h,k) &:= \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k}\right\rfloor - \frac{1}{2}\right) \\ \omega(h,k) &:= \exp\left(\pi\imath \cdot s(h,k)\right). \end{aligned} \tag{14}$$

In comparison to other not entirely unrelated formulas for special functions, we note that unlike Rademacher's series for the partition function, $p(n)$, expressed in terms of finite Kloosterman-like sums, our expansions require only a sum over finitely-many primes $p \leq x$ to evaluate the special function $\sigma_\alpha(x)$ at x. For comparison with the previous section, we also note that the partition function $p(n)$ is related to the sums-of-divisor function $\sigma(n)$ through the convolution identity

$$np(n) = \sum_{k=1}^{n} \sigma(n-k)p(k).$$

There are multiple recurrence relations that can be given for $p(n)$ including the following expansions:

$$\begin{aligned} p(n) &= p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)-p(n-22)-\cdots \\ &= \sum_{\substack{k=\left\lfloor -(\sqrt{24n+1}+1)/6\right\rfloor \\ k\neq 0}}^{\left\lfloor (\sqrt{24n+1}-1)/6\right\rfloor} (-1)^{k+1} p\left(n-\frac{k(3k+1)}{2}\right). \end{aligned}$$

4.2.2. *Rademacher-type infinite sums for other partition functions.*

Example 4.1 (Overpartitions)**.** An *overpartition* of an integer n is defined to be a representation of n as a sum of positive integers with non-increasing summands such that the last instance of a given summand in the overpartition may or may not have an overline bar associated with it. The total number of overpartitions of n, $\bar{p}(n)$, is generated by

$$\sum_{n\geq 0} \bar{p}(n)q^n = \prod_{m\geq 1} \frac{1+q^m}{1-q^m} = 1+2q+4q^2+8q^3+14q^4+24q^5+40q^6+\cdots.$$

A convergent Rademacher-type infinite series providing an exact formula for the partition function $\bar{p}(n)$ is given by

$$\bar{p}(n) = \sum_{\substack{k\geq 1 \\ 2\nmid k}} \sum_{\substack{0\leq h<k \\ \gcd(h,k)=1}} \frac{\sqrt{k}}{2\pi} \frac{\omega(h,k)^2}{\omega(2h,k)} e^{-2\pi\imath nh/k} \frac{d}{dn}\left[\frac{\sinh\left(\frac{\pi\sqrt{n}}{k}\right)}{\sqrt{n}}\right].$$

Example 4.2 (Partitions Where No Odd Part is Repeated)**.** Let the function $p_{\mathrm{od}}(n)$ denote the number of partitions of a non-negative integer n where no odd component appears more than once. This partition function variant is generated by the infinite product

$$\sum_{n\geq 0} p_{\mathrm{od}}(n)q^n = \prod_{m\geq 1}\frac{1+q^{2m-1}}{1-q^{2m}} = 1+x+x^2+2x^3+3x^4+4x^5+5x^6+7x^7+10x^8+\cdots.$$

We have a known Rademacher-type sum exactly generating $p_{\mathrm{od}}(n)$ for each $n\geq 0$ expanded in the form of

$$p_{\mathrm{od}}(n) = \frac{2}{\pi}\sum_{k\geq 1}\sqrt{k\left(1+(-1)^{k+1}+\left\lfloor\frac{\gcd(k,4)}{4}\right\rfloor\right)}\times$$
$$\times \sum_{\substack{0\leq h<k\\ \gcd(h,k)=1}} \frac{\omega(h,k)\omega\left(\frac{4h}{\gcd(k,4)},\frac{k}{\gcd(k,4)}\right)}{\omega\left(\frac{2h}{\gcd(k,2)},\frac{k}{\gcd(k,2)}\right)} e^{-2\pi\imath nh/k}\frac{d}{dn}\left[\frac{\sinh\left(\frac{\pi\sqrt{\gcd(k,4)(8n-1)}}{4k}\right)}{\sqrt{8n-1}}\right].$$

Each of these three partition function variants are special cases of the partition function $p_r(n)$ which is generated by the infinite product $\prod_{m\geq 1}(1+x^m)/(1-x^{2^r m})$. Another more general, and more complicated Rademacher-like sum for this r partition function is given in [14, §2], though we do not restate this result in this section. As we remarked in the previous section, these infinite expansions are still unlike the prime-related exact finite sum formulas we have proved as new results in the article.

4.2.3. *Sierpinski's Voronoi-type formula for the sum-of-squares function.* The *Gauss circle problem* asks for the count of the number of lattice points, denoted $N(R)$, inside a circle of radius R centered at the origin. It turns out that we have exact formulas for $N(R)$ expanded as both a sum over the sum-of-squares function and in the form of

$$N(R) = 1+4\lfloor R\rfloor + 4\sum_{i=1}^{\lfloor R\rfloor}\left\lfloor\sqrt{R^2-i^2}\right\rfloor.$$

The Gauss circle problem is closely-related to the Dirichlet divisor problem concerning sums over the divisor function, $d(n)$ [1, §1.4] [2]. It so happens that the sum-of-squares function, $r_2(n)$, satisfies the next Sierpinski formula which is similar and of Voronoi type to the first two identities for the divisor function given in the previous subsection. However, we note that the Voronoi-type formulas in the previous section are expanded in terms of certain Bessel functions, where our identity here requires the alternate special function

$$J_\nu(z) = \sum_{n\geq 0}\frac{(-1)^n}{n!\cdot\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{\nu+2n},$$

which is an unconditionally convergent series for all $0<|z|<\infty$ and any selection of the parameter $\nu\in\mathbb{C}$.

$$\sum_{n\leq x} r_2(n) = \pi x + \sqrt{x}\sum_{n\geq 1}\frac{r_2(n)}{\sqrt{n}}J_1\left(2\pi\sqrt{nx}\right)$$

As in the cases of the weighted Voronoi formulas involving the divisor function stated in the last section, we notice that this known exact formula for the special function $r_2(n)$ has a much different nature to its expansion than our new exact finite sum formulas for $\sigma_\alpha(n)$. We do, however, have an analog to the infinite series for the classical divisor functions in (12) and (13) which is expanded in terms of Ramanujan's sum functions

$$c_q(n) = \sum_{\substack{\ell m=q\\ m\mid n}}\mu(\ell)m,$$

as follows [5, §9]:

$$r_2(n) = \pi\left[c_1(n)-\frac{c_3(n)}{3}+\frac{c_5(n)}{5}-\cdots\right].$$

5. Conclusions

In this article, we again began by considering the building blocks of the Lambert series generating functions for the sum-of-divisors functions in (1). The new exact formulas for these special arithmetic functions are obtained in this case by our observation of the expansions of the series terms, $q^n/(1-q^n)$, by cyclotomic polynomials and their logarithmic derivatives. We note that in general it is hard to evaluate the series coefficients of $\widetilde{\Phi}_n(q)$ without forming the divisor sum employed in the proof of part (i) of Proposition 1.4. We employed the established, or at least easy to derive, key formulas for the logarithmic derivatives of the cyclotomic polynomials along with known formulas for reducing cyclotomic polynomials of the form $\Phi_{p^r m}(q)$ when $p \nmid m$ to establish the Lambert series term expansions in (7). The expansions of our new exact formulas for the generalized sum-of-divisors functions are deeply related to the prime numbers and the distribution of the primes $2 \leq p \leq x$ for any $x \geq 2$ through this construction.

One of the other interesting results cited in the introduction provides new relations between the r-order harmonic numbers, $H_n^{(r)}$, when $r > 0$ and the Bernoulli polynomials when $r \leq 0$ in the form of (8). We applied these new exact functions for $\sigma_\alpha(n)$, $\sigma(n)$, and $d(n)$ to formulate asymptotic formulas for the partial sums of the divisor function, or its *average order*, which match more famous known asymptotic formulas for these sums. The primary application of the new exact formulas for $\sigma(n)$ provided us with a new necessary and sufficient condition characterizing the prefectness of a positive integer n in Section 3.2. Finally, in Section 4 we compared the new results proved within this article to other known exact formulas for the sum-of-divisors functions, partition functions, and other special arithmetic sequences.

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