New characterizations of the summatory function of the Möbius function

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Last Revised: Friday 1st January, 2021 @ 10:49:26 – Compiled with LaTeX2e

Abstract

The Mertens function, $M(x) := \sum_{n \leq x} \mu(n)$, is defined as the summatory function of the Möbius function. We prove new bounds on $|M(x)|$ using several modern techniques highlighted by Montgomery and Vaughan. The new methods we draw upon connect formulas and recent Dirichlet generating function (or DGF) series expansions related to the canonically additive functions $\Omega(n)$ and $\omega(n)$. The connection between $M(x)$ and the distribution of these core additive functions we prove at the start of the article in the form of

$$M(x) = \sum_{k=1}^{x} (\omega+1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right],$$

is an indispensible component to the proof. It also leads to regular properties of component sequences in the new formula for $M(x)$ that include generalizations of Erdös-Kac like theorems satisfied by the distributions of these sequences.

Keywords and Phrases: *Möbius function; Mertens function; Dirichlet inverse function; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function; Erdös-Kac theorem; strongly additive functions.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; 11N64; and 11-04.*

TODO: Humphreys JNT citation missing ...
TODO: Acknowledgements section ...
TODO: Funding section from this year ...
TODO: Remove last summary section ...

Glossary of notation and conventions

Symbol	**Definition**
$\approx$	We write that $f(x) \approx g(x)$ if $\lvert f(x)-g(x)\rvert = O(1)$ as $x \rightarrow \infty$.
$\mathbb{E}[f(x)], \overset{\mathbb{E}}{\sim}$	We use the expectation notation of $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \overset{\mathbb{E}}{\sim} h(x)$, to denote that f has an *average order* growth rate of $h(x)$. This means that $\frac{1}{x}\sum_{n\leq x} f(n) \sim h(x)$, or equivalently that $$\lim_{x\rightarrow\infty} \frac{\frac{1}{x}\sum\limits_{n\leq x} f(n)}{h(x)} = 1.$$
B	The absolute constant $B \approx 0.2614972$ from the statement of Mertens theorem.
$\chi_{\mathbb{P}}(n)$	The characteristic (or indicator) function of the primes equals one if and only if $n \in \mathbb{Z}^+$ is prime, and is zero-valued otherwise.
$C_k(n)$	The sequence is defined recursively for $n \geq 1$ as follows: $$C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d\mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$ It represents the multiple, k-fold convolution of the function $\omega(n)$ with itself.
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n\geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n\geq 0} f_n q^n$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (see definition below).
$f * g$	The Dirichlet convolution of f and g, $(f*g)(n) := \sum\limits_{d\mid n} f(d) g(n/d)$, where the sum is taken over the divisors of any $n \geq 1$.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ for $n \geq 2$ with $f^{-1}(1) = 1/f(1)$. The Dirichlet inverse of f exists if and only if $f(1) \neq 0$. This inverse function, denoted by f^{-1} when it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$.
γ	The Euler gamma constant defined by $\gamma := \lim\limits_{n\rightarrow\infty} \left(\sum\limits_{k=1}^{n} \frac{1}{k} - \log n\right) \approx 0.5772157$.
$\gg, \ll, \asymp$	For functions A, B, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A \ll B$ and $B \ll A$, we write $A \asymp B$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega+1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$.

Symbol	**Definition**
$[n=k]_{\delta}$,$[\texttt{cond}]_{\delta}$	The symbol $[n=k]_{\delta}$ is a synonym for $\delta_{n,k}$ which is one if and only if $n=k$, and is zero otherwise. For boolean-valued conditions, `cond`, the symbol $[\texttt{cond}]_{\delta}$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention.*
$\lambda_*(n)$	For positive integers $n \geq 2$, we define the next variant of the Liouville lambda function, $\lambda(n)$, as follows: $\lambda_*(n) := (-1)^{\omega(n)}$. We have the initial condition that $\lambda_*(1) = 1$.
$\lambda(n), L(x)$	The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. Its summatory function is defined by $L(x) := \sum_{n \leq x} \lambda(n)$.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree.
$\mu_x(C), \sigma_x(C)$	We define these analogs to the mean and variance of the function $C_{\Omega(n)}(n)$ in the context of our new Erdös-Kac like theorems as $\mu_x(C) := \log\log x + \hat{a} - \frac{1}{2}\log\log\log x$ and $\sigma_x(C) := \sqrt{\mu_x(C)}$ where $\hat{a} := \log\left(\frac{1}{\sqrt{2\pi}}\right) \approx -0.918939$ is an absolute constant.
$M(x)$	The Mertens function is the summatory function of $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum\limits_{n \leq x} \mu(n)$.
$\Phi(z)$	For $x \in \mathbb{R}$, we define the function giving the normal distribution CDF by $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{z} e^{-t^2/2} dt$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^{\alpha} \vert\vert n$ (or when p^{α} exactly divides n) for p prime, $\alpha \geq 1$ and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p \vert n} 1$ and the completely additive function $\Omega(n) := \sum_{p^{\alpha} \vert\vert n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leq n \leq x$ for $x \geq 2$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$ for $x \geq 2$.
$P(s)$	For complex s with $\mathrm{Re}(s) > 1$, we define the prime zeta function to be the Dirichlet generating function $P(s) = \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s}$.
$Q(x)$	For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. More precisely, this function is summed and identified with its limiting asymptotic formula as $x \to \infty$ in the following form: $Q(x) := \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O(\sqrt{x})$.
$\sim$	We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x \to \infty} \frac{A(x)}{B(x)} = 1$.
$\zeta(s)$	The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation on the rest of the complex plane with the exception of a simple pole at $s = 1$ of residue one.

1 Introduction

1.1 Definitions

We define the *Möbius function* to be the signed indicator function of the squarefree integers in the form of [17, A008683]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \text{ and } n \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined on the positive integers as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as follows [17, A002321]:

$$\{M(x)\}_{x \geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \ldots\}.$$

The Mertens function satisfies that $\sum_{n \leq x} M\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = 1$, and is related to the summatory function $L(x) := \sum_{n \leq x} \lambda(n)$ via the relation [?, 8]

$$L(x) = \sum_{d \leq \sqrt{x}} M\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), x \geq 1.$$

Clearly, a positive integer $n \geq 1$ is *squarefree*, or contains no divisors (other than one) which are squares, if and only if $\mu^2(n) = 1$. A related summatory function which counts the number of *squarefree* integers $n \leq x$ satisfies [4, §18.6] [17, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers characterized by the sign of the Möbius function are in fact equal as $x \to \infty$:

$$\begin{aligned} \mu_+(x) &:= \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{x} \xrightarrow{x \to \infty} \frac{3}{\pi^2} \\ \mu_-(x) &:= \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{x} \xrightarrow{x \to \infty} \frac{3}{\pi^2}. \end{aligned}$$

1.2 Properties

An approach to evaluating the limiting asymptotic behavior of $M(x)$ for large $x \to \infty$ considers an inverse Mellin transformation of the reciprocal of the Riemann zeta function. In particular, since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \cdot \int_1^\infty \frac{M(x)}{x^{s+1}} dx, \operatorname{Re}(s) > 1,$$

we obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T - \imath\infty}^{T + \imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

The previous two representations lead us to the exact expression of $M(x)$ for any real $x > 0$ given by the next theorem.

Theorem 1.1 (Analytic Formula for $M(x)$, Titchmarsh)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence* $\{T_k\}_{k\geq 1}$ *satisfying* $k \leq T_k \leq k+1$ *for each* k *such that for any real* $x > 0$

$$M(x) = \lim_{k\to\infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |\operatorname{Im}(\rho)|<T_k}} \frac{x^\rho}{\rho\cdot\zeta'(\rho)} - 2 + \sum_{n\geq 1} \frac{(-1)^{n-1}}{n\cdot(2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2}\left[x\in\mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x\cdot\exp\left(-C\cdot\log^{\frac{3}{5}}(x)(\log\log x)^{-\frac{3}{5}}\right).$$

Under the assumption of the RH, Soundararajan proved new updated estimates bounding $M(x)$ from above for large x in the following forms [18]:

$$\begin{aligned} M(x) &\ll \sqrt{x}\cdot\exp\left((\log x)^{\frac{1}{2}}(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x}\cdot\exp\left((\log x)^{\frac{1}{2}}(\log\log x)^{\frac{5}{2}+\epsilon}\right)\right),\ \forall\epsilon>0. \end{aligned}$$

1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$ for any $0 < \epsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C\cdot\sqrt{x},\quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens himself for $C = 1$ and all $x < 10000$ without the benefit of modern computation. Since its beginnings in 1897, the Mertens conjecture has been disproven by computational methods with non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele [11]. More recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound along infinite subsequences, e.g., considering the asymptotics of the function in the limit supremum and limit infimum senses.

We cite that it is only known by computation that [14, *cf.* §4.1] [17, *cf.* A051400; A051401]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leq -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [11, 6, 7, 5]. Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies [10]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}\cdot(\log\log\log x)^{\frac{5}{4}}} = O(1).$$

2 A concrete new approach to characterizing $M(x)$

The main interpretation to take away from the article is that we have rigorously motivated an equivalent *alternate characterization* of $M(x)$ by constructing combinatorially relevant sequences related to the distribution of the primes and to standard strongly additive functions that have not yet been studied in the literature surrounding the Mertens function. This new perspective offers equivalent characterizations of $M(x)$ by formulas involving the summatory functions $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ and the prime counting function $\pi(x)$.

The proofs of key properties of these new sequences comes bundled with a scaled normal tending probability distribution for the unsigned magnitude of $|g^{-1}(n)|$ that is similar in many ways to the Erdös-Kac theorems for $\omega(n)$ and $\Omega(n)$. Moreover, since $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$, it follows that we have a new probabilistic perspective from which to express distributional features of the summatory functions $G^{-1}(x)$ as $x \to \infty$ in terms of the properties of $|g^{-1}(n)|$ and $L(x) := \sum_{n \leq x} \lambda(n)$. Note that the distribution of $L(x)$ is typically viewed as a problem on par, or equally as difficult in order, with understanding the distribution of $M(x)$ well as $x \to \infty$. The results in this article concretely connect the distributions of $L(x)$, a well defined scaled normally tending probability distribution, and $M(x)$ as $x \to \infty$.

The new sequence $g^{-1}(n)$ defined precisely below and $G^{-1}(x)$ are crucially tied to standard, canonical examples of strongly and completely additive functions, e.g., $\omega(n)$ and $\Omega(n)$, respectively. As such, it is not surprising that we are able to relate the distributions of these functions by limiting probabilistic normal distributions which are similar to the celebrated results given by the Erdös-Kac theorems for the prime omega function variants. Using the definition of $g^{-1}(n)$, we are able to re-interpret and reconcile exact formulas for $M(x)$ naturally by an easy-to-spot relationship to the distinct primes in the factorizations of $n \leq x$. The prime-related combinatorics at hand here are discussed in more detail by the remarks given in Section 4.3.

2.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 2.1 (Summatory functions of Dirichlet convolutions). *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of f and h, respectively, and that $F^{-1}(x) := \sum_{n \leq x} f^{-1}(n)$ denotes the summatory function of the Dirichlet inverse of f for any $x \geq 1$. We have the following exact expressions for the summatory function of the convolution $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned}
\pi_{f*h}(x) &:= \sum_{n \leq x} \sum_{d|n} f(d) h(n/d) \\
&= \sum_{d \leq x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right].
\end{aligned}$$

Moreover, for all $x \geq 1$

$$\begin{aligned}
H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \right] \\
&= \sum_{k=1}^{x} f^{-1}(k) \cdot \pi_{f*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right).
\end{aligned}$$

Corollary 2.2 (Convolutions arising from Möbius inversion). *Suppose that h is an arithmetic function such that $h(1) \neq 0$. Define the summatory function of the convolution of h with μ by $\widetilde{H}(x) := \sum_{n \leq x} (h * \mu)(n)$. Then the Mertens function is expressed by the sum*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j = \left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} h^{-1}(j) \right) \widetilde{H}(k), \forall x \geq 1.$$

Corollary 2.3 (A motivating special case)**.** *We have that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^{x} (\omega+1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

2.2 An exact expression for $M(x)$ in terms of strongly additive functions

Fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega+1)^{-1}(n)$. We can compute exactly that (see Table T.1 starting on page 38)

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

There is not a simple meaningful direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences whose properties we will discuss in detail. The distribution of distinct sets of prime exponents is still clearly quite regular since $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of this inverse function we notice below over $n \geq 2$:

Heuristic 2.4 (Symmetry in $g^{-1}(n)$ in the prime factorizations of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for $= \omega(n_i) \geq 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly one, two, three, and so on prime factors.

Conjecture 2.5 (Characteristic properties of the inverse sequence)**.** *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) *For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*

(B) *For all squarefree integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!;$$

(C) *If $n \geq 2$ and $\Omega(n) = k$, then*

$$2 \leq |g^{-1}(n)| \leq \sum_{j=0}^{k} \binom{k}{j} \cdot j!.$$

We illustrate the conjecture clearly using the computation of initial values of this inverse sequence in Table T.1. The signedness property in (A) is proved precisely in Proposition 3.1. A proof of (B) in fact follows from Lemma 4.1 stated on page 15. The realization that the beautiful and remarkably simple combinatorial form of property (B) in Conjecture 2.5 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through sums of auxiliary subsequences of arithmetic functions denoted by $C_k(n)$ (see Section 4). That is, we observe a familiar formula for $g^{-1}(n)$ at many integers and then seek to extrapolate and prove more regular tendencies of this sequence more generally at any $n \geq 2$.

An exact expression for $g^{-1}(n)$ through a key semi-diagonal of these subsequences is given by

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d), n \geq 1,$$

where the sequence $\lambda(n) C_{\Omega(n)}(n)$ has DGF $(P(s)+1)^{-1}$ for $\operatorname{Re}(s) > 1$ (see Proposition 3.1). In Corollary 5.5, we prove that the approximate average order mean of the unsigned sequence satisfies

$$\mathbb{E}|g^{-1}(n)| \asymp (\log n)^2 \sqrt{\log \log n}, \text{ as } n \to \infty.$$

In Section 5, we prove the next variant of an Erdös-Kac theorem like analog for a component sequence $C_{\Omega(n)}(n)$. Namely, we prove the following statement for $\mu_x(C) := \log\log x + \hat{a} - \frac{1}{2}\log\log\log x$, $\sigma_x(C) := \sqrt{\mu_x(C)}$, $\hat{a}$ an absolute constant, and any $y \in \mathbb{R}$ (see Corollary 5.7):

$$\frac{1}{x} \cdot \#\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq y\} = \Phi\left(\frac{\frac{\pi^2}{6}y - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

Thus, the regularity and quasi-periodicity we have alluded to in the remarks above are actually quantifiable in so much as $|g^{-1}(n)|$ for $n \leq x$ tends to its average order with a non-central normal tendency depending on x as $x \to \infty$. If x is sufficiently large and we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds:

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq 0\right) = o(1) \tag{A}$$

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\mu_x(C)\right) = \frac{1}{2} + o(1). \tag{B}$$

Moreover, for any positive real $\delta > 0$ we have that

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\mu_x(C)^{1+\delta}\right) = 1 + o_\delta(1), \text{ as } x \to \infty. \tag{C}$$

A consequence of (A) and (C) in the probability estimates above is that for any fixed $\delta > 0$ and $n \in \mathcal{S}_1(\delta)$ taken within a set of asymptotic density one we have that

$$\frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq |g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| + \frac{6}{\pi^2}\mu_x(C)^{\frac{1}{2}+\delta}.$$

Hence when we integrate over a sufficiently spaced set of (e.g., set of wide enough) disjoint consecutive intervals containing large enough integer values, we can assume that an asymptotic lower bound on the contribution of $|g^{-1}(n)|$ is given by the function's average order, and an upper bound is given by the related upper limit above for any fixed $\delta > 0$. In particular, observe that by Corollary 5.7 and Corollary 5.5 we can see that

$$\frac{\pi^2}{6 \cdot \sigma_x(C)} \times \int_{-\infty}^{\infty} z \cdot \Phi'\left(\frac{\frac{\pi^2}{6}z - \mu_x(C)}{\sigma_x(C)}\right) dz = \frac{6}{\pi^2} \cdot \sigma_x(C) = o\left(\mathbb{E}|g^{-1}(x)|\right).$$

Remark 2.6 (Uniform asymptotics from certain bivariate counting DGFs)**.** We emphasize the recency of the method demonstrated by Montgomery and Vaughan in constructing their original proof of Theorem 3.5. To the best of our knowledge, this textbook reference is one of the first clear-cut applications documenting something of a hybrid DGF-and-OGF approach to enumerating sequences of arithmetic functions and their summatory functions. This interpretation of certain bivariate DGFs offers a window into the best of both generating function series worlds. It combines the additivity implicit to the coefficients indexed by a formal power series variable formed by multiplication of these structures, while coordinating the distinct DGF-best property of the multiplicativity of distinct prime powers invoked by taking powers of a reciprocal Euler product. Another set of proofs constructed based on this type of hybrid power series DGF is utilized in Section 5 when we prove the Erdös-Kac theorem like analog that holds for the component sequence $C_{\Omega(n)}(n)$, which is crucially related to $g^{-1}(n)$ by the results in Section 4.

2.2.1 The precise new characterizations of $M(x)$

Let $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ for integers $x \geq 1$. We prove that (see Proposition 6.3)

$$M(x) = G^{-1}(x) + G^{-1}\left(\left\lfloor\frac{x}{2}\right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left[\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor\frac{x}{k+1}\right\rfloor\right)\right] \tag{2}$$

$$= G^{-1}(x) + \sum_{p \leq x} G\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1.$$

This formula implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by bounding appropriate estimates of the summatory function $G^{-1}(x)$. This take on the regularity of $|g^{-1}(n)|$ is imperative to our argument formally bounding the growth of $M(x)$ through its new characterizations by $G^{-1}(x)$. A more combinatorial approach to summing $G^{-1}(x)$ for large x based on the distribution of the primes is outlined in our remarks in Section 4.3.

In the proofs given in Section 6, we begin to use these new equivalent characterizations to relate the distributions of $|g^{-1}(n)|$, $G^{-1}(x)$, $\lambda(n)$ and its often classically studied summatory function $L(x)$, to $M(x)$ as $x \to \infty$. In particular, Proposition 6.1 proves that like the known bound for $M(x)$, we have that $G^{-1}(x) = o(x)$ as $x \to \infty$. The results in Corollary 6.2 prove that for almost every sufficiently large x

$$G^{-1}(x) = O\left(\max_{1 \leq t \leq x} |L(t)| \cdot \mathbb{E}|g^{-1}(x)|\right).$$

Moreover, if the RH is true, then we have the following result for any $\varepsilon > 0$ and almost every integer $x \geq 1$:

$$G^{-1}(x) = O\left(\frac{\sqrt{x} \cdot (\log x)^{\frac{5}{2}}}{(\log\log x)^{2+\varepsilon}} \times \exp\left(\sqrt{\log x} \cdot (\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right).$$

By applying Corollary 6.5, we have that as $x \to \infty$

$$M(x) = O\left(|G^{-1}(x)| + \frac{x}{(\log x)^3} \times \int_1^{\frac{x}{2}} \frac{|G^{-1}(t)|}{t^2} dt\right),$$

and

$$M(x) = O\left(|G^{-1}(x)| + \frac{x}{(\log x)^2} \times \max_{1 \leq t \leq \frac{x}{2}} \frac{|G^{-1}(t)|}{t}\right).$$

Other consequences of the distribution of $G^{-1}(y)$ for $y \leq x$ at large x to bounds and limiting properties of $M(x)$ (like limit supremum, infimum type relations) are discussed in Section 6.3 on page 32. Moving forward, this discussion motivates (and really requires) further study in future work.

3 Preliminary proofs of new results

3.1 Establishing the summatory function properties and inversion identities

We will offer a proof of Theorem 2.1 suggested by an intuitive construction through matrix based methods. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95]. It is not difficult to prove the related identity

$$\sum_{n\leq x} h(n)(f*g)(n) = \sum_{n\leq x} f(n) \times \sum_{k\leq \left\lfloor \frac{x}{n}\right\rfloor} g(k)h(kn).$$

Proof of Theorem 2.1. Let h, g be arithmetic functions such that $g(1)\neq 0$. Denote the summatory functions of h and g, respectively, by $H(x)=\sum_{n\leq x} h(n)$ and $G(x)=\sum_{n\leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h. We have that the following formulas hold for all $x\geq 1$:

$$\begin{aligned} \pi_{g*h}(x) &:= \sum_{n=1}^{x}\sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d}\right\rfloor\right) \\ &= \sum_{i=1}^{x}\left[G\left(\left\lfloor \frac{x}{i}\right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1}\right\rfloor\right)\right] H(i). \end{aligned} \tag{3}$$

The first formula above is well known. The second formula is justified directly using summation by parts as [12, §2.10(ii)]

$$\begin{aligned} \pi_{g*h}(x) &= \sum_{d=1}^{x} h(d) G\left(\left\lfloor \frac{x}{d}\right\rfloor\right) \\ &= \sum_{i\leq x}\left(\sum_{j\leq i} h(j)\right) \times \left[G\left(\left\lfloor \frac{x}{i}\right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1}\right\rfloor\right)\right]. \end{aligned}$$

We next form the invertible matrix of coefficients associated with this linear system defining $H(j)$ for all $1\leq j\leq x$ in (3) by setting

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j}\right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1}\right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j}\right\rfloor\right), 1\leq j\leq x.$$

Since $g_{x,x}=G(1)=g(1)$ and $g_{x,j}=0$ for all $j>x$, the matrix we must work with in this problem is lower triangular with a non-zero constant on its diagonals, and is hence invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressed by applying an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I-U^T).$$

Here, U is a square matrix with sufficiently large finite dimensions whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$\left[(I-U^T)^{-1}\right]_{i,j} = [j\leq i]_{\delta}.$$

Observe that

$$\left\lfloor \frac{x}{j}\right\rfloor - \left\lfloor \frac{x-1}{j}\right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.}\end{cases}$$

The previous property implies that

$$G\left(\left\lfloor \frac{x}{j}\right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j}\right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

We use the last property in (4) to shift the matrix $\hat{G}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of g in the following forms:

$$\left[(I-U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

In particular, our target matrix in the inversion problem is defined by

$$(g_{x,j}) = (I-U^T)\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators as

$$\begin{aligned}(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k)\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1}\right\rfloor} g^{-1}(k)\right).\end{aligned}$$

Hence, the summatory function $H(x)$ is given exactly for any $x \geq 1$ by a vector product with the inverse matrix from the previous equation in the form of

$$H(x) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right)\cdot \pi_{g*h}(k).$$

We can prove an inversion formula providing the coefficients of the summatory function $G^{-1}(i)$ for $1 \leq i \leq x$ given by the last equation by adapting our argument to prove (3) above. This leads to the following identity:

$$H(x) = \sum_{k=1}^{x} g^{-1}(x)\cdot \pi_{g*h}\left(\left\lfloor \frac{x}{k}\right\rfloor\right).$$ □

3.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, let $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n. Then we can easily prove using DGFs that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega+1) * \mu. \tag{5}$$

When combined with Corollary 2.2 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 2.3.

Proposition 3.1 (The signedness property of $g^{-1}(n)$)**.** *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0, \pm 1\}$ *denote the sign of the arithmetic function h at integers $n \geq 1$. For the Dirichlet invertible function $g(n) := \omega(n)+1$, we have that* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ *for all $n \geq 1$.*

Proof. The function $D_f(s) := \sum_{n\geq 1} f(n)n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ for σ_f the abcissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = \zeta(s)^{-1}$ and $D_\omega(s) = P(s)\zeta(s)$ for $\operatorname{Re}(s) > 1$. Then by (5) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any arithmetic function f such that $f(1) \neq 0$ (e.g., this relation between the DGFs of these two functions holds whenever f^{-1} exists), we have for all $\operatorname{Re}(s) > 1$ that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{6}$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. This observation implies that $\operatorname{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make our claims and intuition rigorous.

By the recurrence relation that defines the Dirichlet inverse function of any arithmetic function h such that $h(1) = 1$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d|n \\ d>1}} h(d)h^{-1}(n/d), & n \geq 2. \end{cases} \tag{7}$$

For $n \geq 2$, the summands in (7) can be simply indexed over the primes $p|n$ given our definition of h from above. This observation yields that we can inductively unfold these sums into nested divisor sums provided the depth of the expanded divisor sums does not exceed the capacity to index summations over the primes dividing n. Namely, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p|n} h^{-1}\left(\frac{n}{p}\right), && \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right), && \text{if } \Omega(n) \geq 2 \\ &= -\sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1 p_2}} h^{-1}\left(\frac{n}{p_1 p_2 p_3}\right), && \text{if } \Omega(n) \geq 3. \end{aligned}$$

Then by induction with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1, n \geq 2. \tag{8}$$

In fact, by a combinatorial argument related to multinomial coefficient expansions of the sums in (8), we recover exactly that

$$h^{-1}(n) = \lambda(n)(\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}. \tag{9}$$

The last two expansions imply that the following property holds for all $n \geq 1$:

$$\operatorname{sgn}(h^{-1}(n)) = \lambda(n).$$

Since λ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$ for all divisors $d|n$ when $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever n is squarefree, so that we obtain the following result:

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1. \qquad \square$$

3.3 Statements of known limiting asymptotics

Facts 3.2 (The incomplete gamma function)**.** The (upper) *incomplete gamma function* is defined by [12, §8.4]

$$\Gamma(s,x) = \int_x^{\infty} t^{s-1} e^{-t} dt, \operatorname{Re}(s) > 0.$$

The following properties of $\Gamma(s,x)$ hold:

$$\Gamma(s,x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, x > 0, \tag{10a}$$

$$\Gamma(s,x) \sim x^{s-1} \cdot e^{-x}, s > 0, \text{ as } x \to \infty. \tag{10b}$$

3.4 The distribution of exceptional values of $\omega(n)$ and $\Omega(n)$

The next theorems reproduced from [9, §7.4] characterize the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \log\log x$. Since $\mathbb{E}[\Omega(n)] = \log\log n + B$, these results imply a very regular, normal tendency of this arithmetic function towards its average order.

Theorem 3.3 (Upper bounds on exceptional values of $\Omega(n)$ for large n)**.** *Let*

$$A(x,r) := \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\},$$
$$B(x,r) := \#\left\{n \leq x : \Omega(n) \geq r \cdot \log\log x\right\}.$$

If $0 < r \leq 1$ *and* $x \geq 2$*, then*

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

If $1 \leq r \leq R < 2$ *and* $x \geq 2$*, then*

$$B(x,r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

Theorem 3.4 is a special case analog to the celebrated Erdös-Kac theorem typically stated for the normally distributed values of the scaled-shifted function $\omega(n)$ over $n \leq x$ as $x \to \infty$ [9, *cf.* Thm. 7.21].

Theorem 3.4 (Exact limiting bounds on exceptional values of $\Omega(n)$ for large n)**.** *We have that as* $x \to \infty$

$$\#\left\{3 \leq n \leq x : \Omega(n) - \log\log n \leq 0\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Theorem 3.5 (Montgomery and Vaughan)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $R < 2$ *we have that uniformly for all* $1 \leq k \leq R \cdot \log\log x$

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right],$$

where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, 0 \leq |z| < R.$$

Remark 3.6. We can extend the work in [9] with $\Omega(n)$ to see that for $0 < R < 2$

$$\pi_k(x) = \widehat{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{k!} \left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right], \text{ uniformly for } 1 \leq k \leq R\log\log x.$$

The analogous function to express these bounds for $\omega(n)$ is defined by $\widehat{\mathcal{G}}(z) := \widehat{F}(1,z)/\Gamma(1+z)$ where we take

$$\widehat{F}(s,z) := \prod_p \left(1 + \frac{z}{p^s - 1}\right)\left(1 - \frac{1}{p^s}\right)^z, \operatorname{Re}(s) > \frac{1}{2}; |z| \leq R < 2.$$

Let the functions

$$\begin{aligned} C(x,r) &:= \#\{n \leq x : \omega(n) \leq r \log\log x\} \\ D(x,r) &:= \#\{n \leq x : \omega(n) \geq r \log\log x\}. \end{aligned}$$

Then we have the next uniform upper bounds given by

$$\begin{aligned} C(x,r) &\ll x(\log x)^{r-1-r\log r}, \text{ uniformly for } 0 < r \leq 1, \\ D(x,r) &\ll x(\log x)^{r-1-r\log r}, \text{ uniformly for } 1 \leq r \leq R < 2. \end{aligned}$$

Corollary 3.7. *Suppose that for $x > e$ we define the functions*

$$\begin{aligned} \mathcal{N}_\omega(x) &:= \left|\sum_{k > \log\log x} (-1)^k \pi_k(x)\right| \\ \mathcal{D}_\omega(x) &:= \left|\sum_{k \leq \log\log x} (-1)^k \pi_k(x)\right| \\ \mathcal{A}_\omega(x) &:= \left|\sum_{k \geq 1} (-1)^k \pi_k(x)\right|. \end{aligned}$$

Then as $x \to \infty$, we have that $\mathcal{N}_\omega(x)/\mathcal{D}_\omega(x) = o(1)$ and $\mathcal{A}_\omega(x) \asymp \mathcal{D}_\omega(x)$.

Proof. First, we sum the function $\mathcal{D}_\omega(x)$ exactly, and then apply the limiting asymptotics for the incomplete gamma function from (10b) and Stirling's formula, to obtain that

$$\begin{aligned} \mathcal{D}_\omega(x) &= \left|\sum_{k \leq \log\log x} \frac{(-1)^k \cdot x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\right| \\ &= \left|\frac{x}{(\log x)^2} \cdot \frac{\Gamma(\log\log x, -\log\log x)}{\Gamma(\log\log x)}\right| \\ &\asymp \frac{x}{\log x} \cdot \frac{(\log\log x)^{\log\log x - 1}}{\Gamma(\log\log x)} \\ &\asymp \frac{x}{\sqrt{\log\log x}}. \end{aligned}$$

Next, we notice that for $\delta_{x,k} > 0$ when we define $r \log\log x \leq k := \log\log x + \delta_{x,k}$, we obtain the bounds that $r \leq \frac{\log x}{\log\log x}$. Thus expanding logarithms

$$x(\log x)^{r-1-r\log r} = O\left(\frac{x}{(\log x)^{1+\log x}}\right).$$

Then we see that

$$\frac{\mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)} \ll \frac{\sqrt{\log\log x}}{(\log x)^{1+\log x}} = o(1), \text{ as } x \to \infty.$$

Now consider that we have the next bounds

$$1 + o(1) = \frac{\mathcal{D}_\omega(x) - \mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)} \ll \frac{\mathcal{A}_\omega(x)}{\mathcal{D}_\omega(x)} \ll \frac{\mathcal{N}_\omega(x) - \mathcal{D}_\omega(x)}{\mathcal{D}_\omega(x)} = 1 + o(1).$$

The last equation implies that $\mathcal{A}_\omega(x) \asymp \mathcal{D}_\omega(x)$ as $x \to \infty$. Hence, we can accurately approximate asymptotic order of the sums $\mathcal{A}_\omega(x)$ for large x by only considering the truncated sums $\mathcal{D}_\omega(x)$ where we have the uniform bounds for $1 \leq k \leq \log\log x$. ∎

4 Auxiliary sequences to express the Dirichlet inverse function $g^{-1}(n)$

The pages of tabular data given as Table T.1 in the appendix section (refer to page 38) are intended to provide clear insight into why we eventually arrived at the approximations to $g^{-1}(n)$ initially proved in this section. The table provides illustrative numerical data by examining the approximate behavior at hand for the cases of $1 \leq n \leq 500$ with *Mathematica* [16]. In Section 5, we will use these relations between $g^{-1}(n)$ and $C_{\Omega(n)}(n)$ to prove an Erdös-Kac like analog for the distribution of this function.

4.1 Definitions and properties of triangular component function sequences

We define the following auxiliary coefficient sequence for integers $n \geq 1$ and $k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{11}$$

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (11) inductively. By the same argument, we see that at fixed n, the function $C_k(n)$ is seen to be non-zero only for positive integers $k \leq \Omega(n)$ whenever $n \geq 2$. A sequence of relevant signed semi-diagonals of the functions $C_k(n)$ begins as follows [17, A008480]:

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \; -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

We can see that $C_{\Omega(n)}(n) \leq (\Omega(n))!$ for all $n \geq 1$. In fact, $h^{-1}(n) \equiv \lambda(n) C_{\Omega(n)}(n)$ is the same function given by the formula in (9) from Proposition 3.1. This sequence of semi-diagonals of (11) is precisely related to $g^{-1}(n)$ in the next subsection.

4.2 Relating the function $C_{\Omega(n)}(n)$ to exact formulas for $g^{-1}(n)$

Lemma 4.1 (An exact initial formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) g^{-1}(n/d) \quad \implies \quad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \tag{12}$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (12) in the form of $(g^{-1} * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m \cdot (C_m(-) * g^{-1})(n)$, or so that

$$F_m(n) = -\begin{cases} \sum\limits_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum\limits_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & 2 \leq m \leq \Omega(n), \\ (\omega * g^{-1})(n), & m = 1. \end{cases}$$

By repeatedly expanding the right-hand-side of the previous equation, we find that for $m := \Omega(n)$ (i.e., with the expansions at a maximal depth in the previous equation)

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \tag{13}$$

The formula then follows from (13) by Möbius inversion applied to each side of the last equation. $\square$

Corollary 4.2. *For all squarefree integers $n \geq 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \tag{14}$$

Proof. Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geq 2$ and that n is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where p_i is prime for all $1 \leq i \leq \omega(n)$. Since all divisors of any squarefree n are necessarily also squarefree, we can transform the exact divisor sum guaranteed for all n in Lemma 4.1 into a sum that partitions the divisors according to the number of distinct prime factors as follows:

$$\begin{aligned} g^{-1}(n) &= \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i}(-1)^i \cdot C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \end{aligned}$$

The signed contributions in the first of the previous equations is justified by noting that $\lambda(n) = \mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree, and that for $d \geq 1$ squarefree we have the correspondence $\omega(d) = k \implies \Omega(d) = k$. ☐

Since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ using the notation defined in the the proof of Proposition 3.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for squarefree $n \geq 1$. A proof of part (B) of Conjecture 2.5 follows as an immediate consequence.

Lemma 4.3. *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{15}$$

Proof. By applying Lemma 4.1, Proposition 3.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$, Lemma 4.1 implies

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \end{aligned}$$

In the last equation, we see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. ☐

Remark 4.4. Combined with the signedness property of $g^{-1}(n)$ guaranteed by Proposition 3.1, Lemma 4.3 shows that its summatory function is expressed as

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_{\Omega(d)}(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Additionally, since (5) implies that

$$\lambda(d)C_{\Omega(d)}(d) = (g^{-1} * 1)(d) = (\chi_{\mathbb{P}} + \varepsilon)^{-1}(d),$$

where $\chi_{\mathbb{P}}$ denotes the characteristic function of the primes, we also clearly recover by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1.$$

This connection between the summatory function of $g^{-1}(n)$ and the primes is also relayed by the form of the identity we prove for $M(x)$ in Proposition 6.3 involving the prime counting function, $\pi(x)$.

4.3 A connection to the distribution of the primes

The combinatorial complexity of $g^{-1}(n)$ is deeply tied to the distribution of the primes $p \leq n$ as $n \to \infty$. The magnitudes and dispersion of the primes $p \leq x$ certainly restricts the repeating of these distinct sequence values. Nonetheless, we can see that the following is still clear about the relation of the weight functions $|g^{-1}(n)|$ to the distribution of the primes: The value of $|g^{-1}(n)|$ is entirely dependent on the pattern of the *exponents* (viewed as multisets) of the distinct prime factors of $n \geq 2$ (*cf.* Heuristic 2.4).

Example 4.5 (Combinatorial significance to the distribution of $g^{-1}(n)$)**.** We have a natural extremal behavior with respect to distinct values of $\Omega(n)$ corresponding to squarefree integers and prime powers. Namely, if for $k \geq 1$ we define the infinite sets M_k and m_k to correspond to the maximal (minimal) sets of positive integers such that

$$M_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j) = k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

$$m_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j) = k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

then any element of M_k is squarefree and any element of m_k is a prime power. In particular, we have that for any $N_k \in M_k$ and $n_k \in m_k$

$$(-1)^k \cdot g^{-1}(N_k) = \sum_{j=0}^{k} \binom{k}{j} \cdot j!, \qquad \text{and} \qquad (-1)^k \cdot g^{-1}(n_k) = 2.$$

The formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 3.1 implies that we can express an exact formula for $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n. Namely, for $n \geq 2$ and $0 \leq k \leq \omega(n)$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z \cdot \nu_p(n)) = [z^k] \prod_{p^{\alpha}||n} (1 + \alpha z).$$

Then we have essentially shown using (9) and (15) that we can expand formulas for these inverse functions in the following form:

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, n \geq 2.$$

The combinatorial formula for $h^{-1}(n) = \lambda(n) \cdot (\Omega(n))! \times \prod_{p^{\alpha}||n} (\alpha!)^{-1}$ we derived in the proof of the key signedness proposition in Section 3 suggests further patterns and more regularity in the contributions of the distinct weighted terms for $G^{-1}(x)$.

5 The distributions of the unsigned sequences $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$

We have already suggested in the introduction that the relation of the component functions, $g^{-1}(n)$ and $C_{\Omega(n)}(n)$, to the canonical additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties of these functions witnessed á priori in Table T.1. In particular, each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdös-Kac theorem that shows that the density of a shifted and scaled variant of each of the sets of these function values for $n \leq x$ can be expressed through a limiting normal distribution as $x \to \infty$ [3, 2, 13].

In the remainder of this section we establish more analytical proofs of related properties of these key sequences used to express $G^{-1}(x)$, again in the spirit of Montgomery and Vaughan's reference manual (*cf.* Remark 2.6). By the signedness of the summands proved in Proposition 3.1, asymptotics of $G^{-1}(x)$ are related to the distribution of $|g^{-1}(n)|$ for $n \leq x$ by a close tie to the summatory function $L(x) := \sum_{n \leq x} \lambda(n)$.

Proposition 5.1. *Let the function $\widehat{F}(s,z)$ is defined for $\operatorname{Re}(s) \geq 2$ and $|z| < |P(s)|^{-1}$ in terms of the prime zeta function by*

$$\widehat{F}(s,z) := \frac{1}{1-P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

For $|z| < P(2)^{-1}$, the summatory function of the coefficients of the DGF expansion of $\widehat{F}(s,z)$ are defined as follows:

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

We have that for all sufficiently large x

$$\widehat{A}_z(x) = \frac{x}{\Gamma(z)} \cdot \widehat{F}(2,z) \cdot (\log x)^{z-1} + O_z\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right), |z| < P(2)^{-1}.$$

Proof. We can see by adapting the notation from the proof of Proposition 3.1 that for $n \geq 2$

$$C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}.$$

We can generate scaled forms of these terms through a product identity of the form

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! \cdot p^{rs}}\right)^{-1} = \exp\left(z \cdot P(s)\right), \operatorname{Re}(s) \geq 2, z \in \mathbb{C}.$$

By computing a Laplace transform on the right-hand-side of the above, we obtain

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n) \cdot (-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp\left(tz \cdot P(s)\right) dt = \frac{1}{1-P(s)z}, \operatorname{Re}(s) \geq 2, |z| < |P(s)|^{-1}.$$

It follows that

$$\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times \widehat{F}(s,z), \operatorname{Re}(s) \geq 2, |z| < |P(s)|^{-1}.$$

Since $\widehat{F}(s,z)$ is an analytic function of s for all $\operatorname{Re}(s) > 1$ whenever the parameter $|z| < |P(s)|^{-1}$, if $b_z(n)$ are the coefficients in the DGF expansion of $\widehat{F}(s,z)$ (as above), then

$$\left|\sum_{n \geq 1} \frac{b_z(n)(\log n)^{2R+1}}{n^s}\right| < +\infty,$$

is uniformly bounded for $|z| \leq R$. This fact follows by repeated termwise differentiation with respect to s.

Let the function $d_z(n)$ be generated as the coefficients of the DGF $\zeta(s)^z = \sum_{n\geq 1} \frac{d_z(n)}{n^s}$ for $\mathrm{Re}(s) > 1$, and with corresponding summatory function $D_z(x) := \sum_{n\leq x} d_z(n)$. The theorem in [9, Thm. 7.17; §7.4] implies that for any $z \in \mathbb{C}$ and integers $x \geq 2$

$$D_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\left(x \cdot (\log x)^{\mathrm{Re}(z)-2}\right).$$

We set $b_z(n) \equiv (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$, set the convolution $a_z(n) := \sum_{d|n} b_z(d) d_z(n/d)$, and define its summatory function $A_z(x) := \sum_{n\leq x} a_z(n)$. Then we have that

$$\begin{aligned} A_z(x) &= \sum_{m\leq x/2} b_z(m) D_z(x/m) + \sum_{x/2<m\leq x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \times \sum_{m\leq x/2} \frac{b_z(m)}{m^2} \times m \cdot \log\left(\frac{x}{m}\right)^{z-1} + O\left(\sum_{m\leq x} \frac{x\cdot |b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2}\right). \end{aligned} \tag{16}$$

We can sum the coefficients for $u > e$ large as follows:

$$\sum_{m\leq u} \frac{b_z(m)}{m} = \left(\widehat{F}(2,z) + O(u^{-2})\right) u - \int_1^u \left(\widehat{F}(2,z) + O(t^{-2})\right) dt = \widehat{F}(2,z) + O(1+u^{-1}).$$

Suppose that $|z| \leq R < P(2)^{-1}$. The error term in (16) satisfies

$$\begin{aligned} \sum_{m\leq x} \frac{x\cdot |b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2} &\ll x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m\leq\sqrt{x}} \frac{|b_z(m)|}{m} \\ &\quad + x(\log x)^{-(R+2)} \times \sum_{m>\sqrt{x}} \frac{|b_z(m)|}{m} (\log m)^{2R} \\ &\ll x(\log x)^{\mathrm{Re}(z)-2} \cdot \widehat{F}(2,z) = O_z\left(x \cdot (\log x)^{\mathrm{Re}(z)-2}\right), |z| \leq R. \end{aligned}$$

In the main term estimate for $A_z(x)$ from (16), when $m \leq \sqrt{x}$ we have

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\mathrm{Re}(z)-2}\right).$$

The total sum over the interval $m \leq x/2$ corresponds to bounding the following sum components when we take $|z| \leq R$:

$$\begin{aligned} \sum_{m\leq x/2} b_z(m) D_z(x/m) &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \times \sum_{m\leq x/2} \frac{b_z(m)}{m} \\ &\quad + O_z\left(x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m\leq\sqrt{x}} \frac{|b_z(m)|}{m} + x(\log x)^{R-1} \times \sum_{m>\sqrt{x}} \frac{|b_z(m)|}{m}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \widehat{F}(2,z) + O_z\left(x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m\geq 1} \frac{b_z(m)(\log m)^{2R+1}}{m^2}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \widehat{F}(2,z) + O_z\left(x(\log x)^{\mathrm{Re}(z)-2}\right). \end{aligned}$$

□

Theorem 5.2. *We have uniformly for $1 \leq k < \log\log x$ that as $x \to \infty$*

$$\widehat{C}_k(x) := \sum_{\substack{n\leq x \\ \Omega(n)=k}} (-1)^{\omega(n)} C_k(n) \asymp \frac{x}{\log x} \cdot \frac{(\log\log x + \log\zeta(2))^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

Proof. We begin by bounding a contour integral over the error term for fixed large x when $r := \frac{k-1}{\log\log x}$ with $r < 2$ for $k \geq 2$:

$$\begin{aligned}\left|\int_{|v|=r} \frac{x \cdot (\log x)^{-(\operatorname{Re}(v)+2)}}{v^{k+1}} dv\right| &\ll x(\log x)^{-(r+2)} r^{-(k+1)} \ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{k+1}} \cdot \frac{1}{e^{k-1}} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{3/2}} \cdot \frac{1}{e^{2k}(k-1)!} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \ll \frac{x}{\log x} \cdot \frac{k \cdot (\log\log x)^{k-5}}{(k-1)!}.\end{aligned}$$

When $k = 1$ we have that

$$\begin{aligned}\left|\int_{|v|=r} \frac{x \cdot (\log x)^{-(\operatorname{Re}(v)+2)}}{v^2} dv\right| &= \left|\frac{1}{1!} \times \frac{d}{dv}\left[x \cdot (\log x)^{-(\operatorname{Re}(v)+2)}\right]\right| \\ &\ll \left|\frac{d}{dr}\left[\frac{x}{(\log x)^2} \cdot \exp(-r\log\log x)\right]\right| \\ &\ll \frac{x}{(\log x)(\log\log x)^2}.\end{aligned}$$

We must now find an asymptotically accurate main term approximation to the coefficients of the following contour integral for $r \in [0, z_{\max}]$ where $z_{\max} < P(2)^{-1}$ according to Proposition 5.1:

$$\widetilde{A}_r(x) := \int_{|v|=r} \frac{x \cdot (\log x)^{-v}\zeta(2)^{-v}}{(\log x)\Gamma(1+v) \cdot v^k(1+P(2)v)} dv. \tag{17}$$

We can show that provided a restriction to $1 \leq r < 1$, we can approximate the contour integral in (17) where the resulting main term is accurate up to a bounded constant factor. This procedure removes the gamma function term in the denominator of the integrand by essentially applying a mean value theorem type analog for smoothly parameterized contours. The logic used to justify this type of simplification of form argument is discussed next. We observe that for $r := 1$, the function $|\Gamma(1+re^{2\pi\imath t})|$ has a singularity (pole) when $t := \frac{1}{2}$. We restrict the range of $|v| = r$ so that $0 \leq r < 1$ to necessarily avoid this problematic value of t when we parameterize $v = re^{2\pi\imath t}$ by a real-line integral over $t \in [0, 1]$. We can compute finite extremal values of this function as

$$\min_{\substack{0 \leq r < 1 \\ 0 \leq t \leq 1}} |\Gamma(1+re^{2\pi\imath t})| = |\Gamma(1+re^{2\pi\imath t})|\Big|_{(r,t)\approx(1,0.740592)} \approx 0.520089$$

$$\max_{\substack{0 \leq r < 1 \\ 0 \leq t \leq 1}} |\Gamma(1+re^{2\pi\imath t})| = |\Gamma(1+re^{2\pi\imath t})|\Big|_{(r,t)\approx(1,0.999887)} \approx 1.$$

This shows that

$$\widetilde{A}_r(x) \asymp \int_{|v|=r} \frac{x \cdot (\log x)^{-v}\zeta(2)^{-v}}{(\log x) \cdot v^k(1+P(2)v)} dv, \tag{18}$$

where as $x \to \infty$

$$\frac{\widetilde{A}_r(x)}{\int_{|v|=r} \frac{x(\log x)^{-v}\zeta(2)^{-v}}{(\log x)\cdot v^k(1+P(2)v)} dv} \in [1, 1.92275].$$

By induction we can compute the remaining coefficients $[z^k]\Gamma(1+z) \times \widehat{A}_z(x)$ with respect to x for fixed $k \leq \log\log x$ using the Cauchy integral formula. Namely, it is not difficult to see that for any integer $m \geq 0$, we have the m^{th} partial derivative of the integrand with respect to z has the following limiting expansion by applying (10b):

$$\frac{1}{m!} \times \frac{\partial^{(m)}}{\partial v^{(m)}}\left[\frac{(\log x)^{-v}\zeta(2)^{-v}}{1+P(2)v}\right]\Bigg|_{v=0} = \sum_{j=0}^{m} \frac{(-1)^m P(2)^j (\log\log x + \log\zeta(2))^{m-j}}{(m-j)!}$$

$$= \frac{(-P(2))^m (\log x)^{\frac{1}{P(2)}} \zeta(2)^{\frac{1}{P(2)}}}{m!} \times \Gamma\left(m+1, \frac{\log\log x + \log\zeta(2)}{P(2)}\right)$$
$$\sim \frac{(-1)^m (\log\log x + \log\zeta(2))^m}{m!}.$$

Now by parameterizing the countour around $|z| = r := \frac{k-1}{\log\log x} < 1$ we deduce that the the main term of our approximation corresponds to

$$\int_{|v|=r} \frac{x \cdot (\log x)^{-v} \zeta(2)^{-v}}{(\log x) v^k (1+P(2)v)} dv \asymp \frac{x}{\log x} \cdot \frac{(-1)^{k-1} (\log\log x + \log\zeta(2))^{k-1}}{(k-1)!}. \qquad \square$$

Corollary 5.3. *We have that for large $x \geq 2$ that uniformly for $1 \leq k \leq \log\log x$*

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp 2\sqrt{2\pi} \cdot x \times \frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!}.$$

Proof. We have an integral formula involving the non-sign-weighted sequence that results by applying ordinary Abel summation (and integrating by parts) in the form of

$$\sum_{n \leq x} \lambda_*(n) h(n) = \left(\sum_{n \leq x} \lambda_*(n)\right) h(x) - \int_1^x \left(\sum_{n \leq t} \lambda_*(n)\right) h'(t) dt \tag{19}$$
$$\left\{ \begin{array}{ll} u_t = L_*(t) & v_t' = h'(t)dt \\ u_t' = L_*'(t)dt & v_t = h(t) \end{array} \right\}$$
$$\asymp \int_1^x \frac{d}{dt}\left[\sum_{n \leq t} \lambda_*(n)\right] h(t) dt. \tag{20}$$

Let the signed left-hand-side summatory function for our function corresponding to (19) be defined by

$$\widehat{C}_{k,*}(x) := \sum_{\substack{n \leq x \\ \Omega(n)=k}} (-1)^{\omega(n)} C_{\Omega(n)}(n)$$
$$\asymp \frac{x}{\log x} \cdot \frac{(\log\log x + \log\zeta(2))^{k-1}}{(k-1)!} \left[1 + O\left(\frac{1}{\log\log x}\right)\right]$$
$$\asymp \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{1}{\log\log x}\right)\right]$$

where the second equation above follows from the proof of Theorem 5.2. We adopt the notation that $\lambda_*(n) = (-1)^{\omega(n)}$ for $n \geq 1$.

We next transform our previous results for the partial sums over the signed sequences $\lambda_*(n) \cdot C_{\Omega(n)}(n)$ such that $\Omega(n) = k$. The argument is based on approximating the smooth summatory function of $\lambda_*(n) := (-1)^{\omega(n)}$ using the following known uniform approximation of $\pi_k(x)$ when $1 \leq k \leq \log\log x$ as $x \to \infty$:

$$\pi_k(x) \asymp \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} (1 + o(1)).$$

In particular, we have by an asymptotic approximation to the incomplete gamma function and Corollary 3.7 that

$$L_*(t) := \left|\sum_{n \leq t} (-1)^{\omega(n)}\right| \asymp \left|\sum_{k=1}^{\log\log x} (-1)^k \pi_k(x)\right| \sim \frac{t}{\sqrt{2\pi}\sqrt{\log\log t}}, \text{ as } t \to \infty.$$

The main term for the reciprocal of the derivative of this summatory function is asymptotic to

$$\frac{1}{L_*'(t)} \asymp \sqrt{2\pi} \cdot (\log\log t)^{\frac{1}{2}}.$$

After applying the formula from (19), we thus deduce that the unsigned summatory function variant satisfies

$$\begin{aligned} \widehat{C}_{k,*}(x) = \int_1^x L_*'(t) C_{\Omega(t)}(t) dt \qquad &\implies C_{\Omega(x)}(x) \asymp \frac{\widehat{C}_{k,*}'(x)}{L_*'(x)} \qquad \implies \\ C_{\Omega(x)}(x) \asymp \sqrt{2\pi} \cdot \frac{(\log\log x)^{\frac{1}{2}}}{\log x} &\cdot \left[\frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 - \frac{1}{\log x}\right) + \frac{(\log\log x)^{k-2}}{(\log x)(k-2)!}\right] \\ \asymp \sqrt{2\pi} \cdot \frac{(\log\log x)^{k-\frac{1}{2}}}{(\log x)(k-1)!} &=: \widehat{C}_{k,**}(x). \end{aligned}$$

The ordinary Abel summation formula, and integration by parts, implies that we obtain a main term is given by

$$\begin{aligned} \sum_{\substack{n \le x \\ \Omega(n)=k}} C_{\Omega(n)}(n) &\asymp \int \widehat{C}_{k,**}(x) dx \\ &\asymp 2\sqrt{2\pi} \cdot x \times \frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!}. \end{aligned}$$

□

Lemma 5.4. *We have that as $x \to \infty$*

$$\mathbb{E}\left[C_{\Omega(n)}(n)\right] \asymp 2\sqrt{2\pi} \cdot (\log n)\sqrt{\log\log n}.$$

Proof. We first compute the following summatory function by applying Corollary 5.3:

$$\sum_{k=1}^{\log\log x} \sum_{\substack{n \le x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp 2\sqrt{2\pi} \cdot x \cdot (\log x)\sqrt{\log\log x}. \tag{21}$$

We claim that

$$\sum_{n \le x} C_{\Omega(n)}(n) = \sum_{k=1}^{\log_2(x)} \sum_{\substack{n \le x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp \sum_{k=1}^{\log\log x} \sum_{\substack{n \le x \\ \Omega(n)=k}} C_{\Omega(n)}(n). \tag{22}$$

Then (21) clearly implies our result. To prove (22) it suffices to show that

$$\frac{\sum\limits_{\log\log x < k \le \log_2(x)} \sum\limits_{\substack{n \le x \\ \Omega(n)=k}} C_{\Omega(n)}(n)}{\sum\limits_{k=1}^{\log\log x} \sum\limits_{\substack{n \le x \\ \Omega(n)=k}} C_{\Omega(n)}(n)} = o(1), \text{ as } x \to \infty. \tag{23}$$

We define the following component sums for large x and $0 < \varepsilon < 1$ so that $(\log\log x)^{\frac{\varepsilon \log\log x}{\log\log\log x}} = o(\log x)$:

$$S_{2,\varepsilon}(x) := \sum_{\log\log x < k \le (\log\log x)^{\frac{\varepsilon \log\log x}{\log\log\log x}}} \sum_{\substack{n \le x \\ \Omega(n)=k}} C_{\Omega(n)}(n).$$

Then

$$\sum_{k=\log\log x}^{\log_2(x)} \sum_{\substack{n \le x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \gg S_{2,\varepsilon}(x),$$

with equality as $\varepsilon \to 1$ when the upper bound of summation tends to $\log x$. Observe that whenever $\Omega(n) = k$, we have that $C_{\Omega(n)}(n) \leq k!$. We can then bound the sums defined above using Theorem 3.3 for large $x \to \infty$ as

$$\begin{aligned}
S_{2,\varepsilon}(x) &\leq \sum_{\log\log x \leq k \leq \log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \ll \sum_{k=\log\log x}^{(\log\log x)^{\frac{\varepsilon \log\log x}{\log\log\log x}}} \frac{\widehat{\pi}_k(x)}{x} \cdot k! \\
&\ll \sum_{k=\log\log x}^{(\log\log x)^{\frac{\varepsilon \log\log x}{\log\log\log x}}} (\log x)^{\frac{k}{\log\log x} - 1 - \frac{k}{\log\log x}(\log k - \log\log\log x)} \cdot \left(\frac{k}{e}\right)^k \sqrt{2\pi k} \\
&\ll \sum_{k=\log\log x}^{\frac{\varepsilon \log\log x}{\log\log\log x}} (\log x)^{\frac{2k \cdot \log\log\log x}{\log\log x} - 1} \sqrt{k} \\
&\ll \frac{1}{(\log x)} \times \int_{\log\log x}^{\frac{\varepsilon \log\log x}{\log\log\log x}} (\log\log x)^{2t} \sqrt{t} \cdot dt \\
&\ll \frac{1}{(\log x)} \sqrt{\frac{\varepsilon \cdot \log\log x}{\log\log\log x}} (\log\log x)^{\frac{2\varepsilon \cdot \log\log x}{\log\log\log x}} = o(x),
\end{aligned}$$

where we have a simplification for large x by noticing that $\lim_{x\to\infty} (\log x)^{\frac{1}{\log\log x}} = e$. So by (21) this form of the ratio in (23) clearly tends to zero. ∎

Corollary 5.5. *We have that as $n \to \infty$, the unsigned sequence mean satisfies*

$$\mathbb{E}|g^{-1}(n)| \asymp (\log n)^2 \sqrt{\log\log n}.$$

Proof. We use the formula from Lemma 5.4 to find $\mathbb{E}[C_{\Omega(n)}(n)]$ up to a small bounded multiplicative constant factor as $n \to \infty$. This implies that for large t

$$\begin{aligned}
\int \frac{\mathbb{E}[C_{\Omega(t)}(t)]}{t} dt &\asymp \sqrt{2\pi} \cdot (\log t)^2 \sqrt{\log\log t} - \frac{\pi}{2} \operatorname{erfi}\left(\sqrt{2\log\log t}\right) \\
&\asymp \sqrt{2\pi} \cdot (\log t)^2 \sqrt{\log\log t}.
\end{aligned}$$

Recall from the introduction that the summatory function of the squarefree integers is approximated by

$$Q(x) := \sum_{n \leq x} \mu^2(n) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Therefore summing over (15) we find that

$$\begin{aligned}
\mathbb{E}|g^{-1}(n)| &= \frac{1}{n} \times \sum_{d \leq n} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{n}{d} \right\rfloor\right) \\
&\sim \sum_{d \leq n} C_{\Omega(d)}(d) \left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dn}}\right)\right] \\
&= \frac{6}{\pi^2} \left[\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d<n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O\left(\frac{1}{\sqrt{n}} \times \int_0^n t^{-1/2} dt\right) \\
&= \frac{6}{\pi^2} \left[\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d<n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O(1) \qquad (24) \\
&\asymp \frac{6\sqrt{2}}{\pi^{\frac{3}{2}}} (\log n)^2 \sqrt{\log\log n}.
\end{aligned}$$

∎

Theorem 5.6. *Let the mean and variance analogs be denoted by*

$$\mu_x(C) := \log\log x + \hat{a} - \frac{1}{2} \cdot \log\log\log x, \qquad \text{and} \qquad \sigma_x(C) := \sqrt{\mu_x(C)},$$

where the absolute constant $\hat{a} := \log\left(\frac{1}{\sqrt{2\pi}}\right) \approx -0.918939$*. Set* $Y > 0$ *and suppose that* $z \in [-Y, Y]$*. Then we have uniformly for all* $-Y \leq z \leq Y$ *that*

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)} \leq z\right\} = \Phi(z) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

Proof. For large x and $n \leq x$, define the following auxiliary variables:

$$\alpha_n := \frac{C_{\Omega(n)}(n) - \mu_n(C)}{\sigma_n(C)}, \quad \text{and} \quad \beta_{n,x} := \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)}.$$

Let the corresponding densities (whose limiting distributions we must verify) be defined by the functions

$$\Phi_1(x, z) := \frac{1}{x} \cdot \#\{n \leq x : \alpha_n \leq z\},$$

and

$$\Phi_2(x, z) := \frac{1}{x} \cdot \#\{n \leq x : \beta_{n,x} \leq z\}.$$

We first argue that it suffices to consider the distribution of $\Phi_2(x, z)$ as $x \to \infty$ in place of $\Phi_1(x, z)$ to obtain our desired result. The difference of the two auxiliary variables is neglibible as $x \to \infty$ for (n, x) taken over the ranges that contribute the non-trivial weight to the main term of each density function. In particular, we have for $\sqrt{x} \leq n \leq x$ and $C_{\Omega(n)}(n) \leq 2 \cdot \mu_x(C)$ that the following is true:

$$|\alpha_n - \beta_{n,x}| \ll \frac{1}{\sigma_x(C)} \xrightarrow{x \to \infty} 0.$$

Thus we can replace α_n by $\beta_{n,x}$ and estimate the limiting densities corresponding to the alternate terms. The rest of our argument follows the method in the proof of the related theorem in [9, Thm. 7.21; §7.4] closely. Readers familiar with the methods in the reference will see many parallels to that construction.

We use the formula proved in Corollary 5.3 to estimate the densities claimed within the ranges bounded by z as $x \to \infty$. Let $k \geq 1$ be a natural number defined by $k := t + \mu_x(C)$. We write the small parameter $\delta_{t,x} := \frac{t}{\mu_x(C)}$. When $|t| \leq \frac{1}{2}\mu_x(C)$, we have by Stirling's formula that

$$\begin{aligned} 2\sqrt{2\pi} \cdot x \times \frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!} &\sim \frac{e^{\hat{a}+t}(\log\log x)^{\mu_x(C)(1+\delta_{t,x})}}{\sigma_x(C) \cdot \mu_x(C)^{\mu_x(C)(1+\delta_{t,x})}(1+\delta_{t,x})^{\mu_x(C)(1+\delta_{t,x})+\frac{3}{2}}} \\ &\sim \frac{e^t}{\sqrt{2\pi} \cdot \sigma_x(C)}(1+\delta_{t,x})^{-\left(\mu_x(C)(1+\delta_{t,x})+\frac{3}{2}\right)}, \end{aligned}$$

since $\frac{\mu_x(C)}{\log\log x} = 1 + o(1)$ as $x \to \infty$.

We have the uniform estimate $\log(1 + \delta_{t,x}) = \delta_{t,x} - \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)$ whenever $|\delta_{t,x}| \leq \frac{1}{2}$. Then we can expand the factor involving $\delta_{t,x}$ in the previous equation as follows:

$$\begin{aligned} (1+\delta_{t,x})^{-\mu_x(C)(1+\delta_{t,x})-\frac{1}{2}} &= \exp\left(\left(\frac{1}{2} + \mu_x(C)(1+\delta_{t,x})\right) \times \left(-\delta_{t,x} + \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)\right)\right) \\ &= \exp\left(-t - \frac{3t + t^2}{2\mu_x(C)} + \frac{3t^2}{4\mu_x(C)^2} + O\left(\frac{|t|^3}{\mu_x(C)^2}\right)\right). \end{aligned}$$

For both $|t| \leq \mu_x(C)^{1/2}$ and $\mu_x(C)^{1/2} < |t| \leq \mu_x(C)^{2/3}$, we see that

$$\frac{t}{\mu_x(C)} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Similarly, for $|t| \leq 1$ and $|t| > 1$, we see that both

$$\frac{t^2}{\mu_x(C)^2} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Let the corresponding error terms in (x,t) be denoted by

$$\widetilde{E}(x,t) := O\left(\frac{1}{\sigma_x(C)}\right) + O\left(\frac{|t|^3}{\mu_x(C)^2}\right).$$

Combining these estimates with the previous computations, we can deduce that uniformly for $|t| \leq \mu_x(C)^{2/3}$

$$2\sqrt{2\pi} \cdot x \times \frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!} \sim \frac{1}{\sqrt{2\pi} \cdot \sigma_x(C)} \cdot \exp\left(-\frac{t^2}{2\sigma_x(C)^2}\right) \times \left[1 + \widetilde{E}(x,t)\right].$$

By the same argument utilized in the proof of Lemma 5.4, we see that the contributions of these summatory functions for $k \leq \mu_x(C) - \mu_x(C)^{2/3}$ is negligible. We also require that $k \leq \log\log x$ as we have worked out in Theorem 5.2. So we sum over a corresponding range of

$$\mu_x(C) - \mu_x(C)^{2/3} \leq k \leq R_{z,x} \cdot \mu_x(C) + z \cdot \sigma_x(C),$$

for $R_{z,x} := 1 - \frac{z}{\sigma_x(C)}$ to approximate the stated normalized densities. Then finally as $x \to \infty$, the three terms that result (one main term, two error terms, respectively) can be considered to each correspond to a Riemann sum for an associated integral. □

Corollary 5.7. *Let $Y > 0$. Suppose that $\mu_x(C)$ and $\sigma_x(C)$ are defined as in Theorem 5.6. Uniformly for all $-Y \leq y \leq Y$ we have that*

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : |g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq y\right\} = \Phi\left(\frac{\frac{\pi^2}{6}y - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

Proof. We claim that

$$|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2}C_{\Omega(n)}(n).$$

From (24) we obtain that

$$\frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| = \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(x)}(x)] + \sum_{d<x} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O(1).$$

Let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) := f(x) - f(x-1)$. Then from the proof of Corollary 5.5, we see that for large n

$$\begin{aligned}
|g^{-1}(n)| = \Delta_n(n \cdot \mathbb{E}|g^{-1}(n)|) &\sim \Delta_n\left(\sum_{d \leq n} \frac{6}{\pi^2} \cdot C_{\Omega(d)}(d) \cdot \frac{n}{d}\right) \\
&= \frac{6}{\pi^2}\left[C_{\Omega(n)}(n) + \sum_{d<n} C_{\Omega(d)}(d)\frac{n}{d} - \sum_{d<n} C_{\Omega(d)}(d)\frac{(n-1)}{d}\right] \\
&= \frac{6}{\pi^2}C_{\Omega(n)}(n) + \frac{6}{\pi^2}, \text{ as } n \to \infty.
\end{aligned}$$

Since $\mathbb{E}|g^{-1}(n-1)| \sim \mathbb{E}|g^{-1}(n)|$ by Corollary 5.5, the result finally follows from Theorem 5.6. □

Lemma 5.8. *Suppose that $\mu_x(C)$ and $\sigma_x(C)$ are defined as in Theorem 5.6. If x is sufficiently large and we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds:*

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq 0\right) = o(1) \tag{A}$$

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\mu_x(C)\right) = \frac{1}{2} + o(1). \tag{B}$$

Moreover, for any positive real $\delta > 0$ we have that

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\mu_x(C)^{1+\delta}\right) = 1 + o_\delta(1), \text{ as } x \to \infty. \tag{C}$$

Proof. Each of these results is a consequence of Corollary 5.7. Let the densities $\gamma_z(x)$ be defined for $z \in \mathbb{R}$ and sufficiently large $x > e$ as follows:

$$\gamma_z(x) := \frac{1}{x} \cdot \#\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq z\}.$$

To prove (A), observe that by Corollary 5.7 for $z := 0$ we have that

$$\gamma_0(x) = \Phi\left(-\sigma_x(C)\right) + o(1), \text{ as } x \to \infty.$$

We can see that $\sigma_x(C) \xrightarrow{x\to\infty} +\infty$ where for $z \geq 0$ we have the reflection identity for the normal distribution CDF $\Phi(z) = 1 - \Phi(-z)$. Since we have by an asymptotic approximation to the error function expanded by

$$\begin{aligned}
\Phi(z) &= \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right) \\
&= 1 - \frac{2e^{-z^2/2}}{\sqrt{2\pi}}\left[z^{-1} - z^{-3} + 3z^{-5} - 15z^{-7} + \cdots\right], \text{ as } |z| \to \infty,
\end{aligned}$$

we can see that

$$\gamma_0(x) = \Phi\left(-\sigma_x(C)\right) \asymp \frac{1}{\sigma_x(C)\exp(\mu_x(C)/2)} = o(1).$$

To prove (B), observe setting $z_1 := \frac{6}{\pi^2}\mu_x(C)$ yields that

$$\gamma_{z_1}(x) = \Phi(0) + o(1) = \frac{1}{2} + o(1), \text{ as } x \to \infty.$$

To prove (C), we require that $\mu_x(C)^{\frac{1}{2}+\delta} - \sigma_x(C) \xrightarrow{x\to\infty} +\infty$. Since this happens as $x \to \infty$ for any fixed $\delta > 0$, we have that with $z(\delta) := \frac{6}{\pi^2}\mu_x(C)^{1+\delta}$

$$\begin{aligned}
\gamma_{z(\delta)} &= \Phi\left(\mu_x(C)^{\frac{1}{2}+\delta} - \sigma_x(C)\right) + o(1) \\
&\sim 1 - \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\left(\mu_x(C)^{\frac{1}{2}+\delta} - \sigma_x(C)\right)} \times \exp\left(-\frac{\mu_x(C)}{4} \cdot \left(\mu_x(C)^{\delta} - 1\right)^2\right) \\
&= 1 + o_\delta(1), \text{ as } x \to \infty.
\end{aligned}$$

□

Remark 5.9. A consequence of (A) and (C) in Lemma 5.8 is that for any fixed $\delta > 0$ and $n \in \mathcal{S}_1(\delta)$ taken within a set of asymptotic density one we have that

$$\frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq |g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| + \frac{6}{\pi^2}\mu_x(C)^{\frac{1}{2}+\delta}. \tag{25}$$

Thus when we integrate over a sufficiently spaced set of (e.g., set of wide enough) disjoint consecutive intervals containing large enough integer values, we can assume that an asymptotic lower bound on the contribution of

$|g^{-1}(n)|$ is given by the function's average order, and an upper bound is given by the related upper limit above for any fixed $\delta > 0$. In particular, observe that by Corollary 5.7 and Corollary 5.5 we can see that

$$\frac{\pi^2}{6 \cdot \sigma_x(C)} \times \int_{-\infty}^{\infty} z \cdot \Phi'\left(\frac{\frac{\pi^2}{6}z - \mu_x(C)}{\sigma_x(C)}\right) dz = \frac{6}{\pi^2} \cdot \sigma_x(C) = o\left(\mathbb{E}|g^{-1}(x)|\right).$$

Emphasizing the point above, we can thus again interpret the previous calculation as implying that for n on a large interval, the contribution from $|g^{-1}(n)|$ can be approximated above and below accurately as in the bounds from (25).

6 Proofs of new bounds and limiting relations on $M(x)$

6.1 Establishing initial lower bounds on the summatory function $G^{-1}(x)$

Proposition 6.1. *For all sufficiently large $x \to \infty$, we have that $G^{-1}(x) = o(x)$.*

Proof. We proved in establishing Lemma 4.1 that for all $n \geq 1$

$$(g^{-1} * 1)(n) = \lambda(n) \cdot C_{\Omega(n)}(n).$$

We also know by a direct application of the same lemma that

$$\begin{aligned} G^{-1}(x) &= \sum_{n \leq x} \sum_{d|n} \lambda(d) C_{\Omega(d)}(d) \mu\left(\frac{n}{d}\right) \\ &= \sum_{n \leq x} (g^{-1} * 1)(n) M\left(\left\lfloor \frac{x}{n} \right\rfloor\right). \end{aligned}$$

Now since the previous equation is correct, we can apply one of the identities proved in Theorem 2.1 to see that

$$\begin{aligned} G^{-1}(x) &= \sum_{r=1}^{x} M(r) \left[\sum_{j=1}^{\left\lfloor \frac{x}{r} \right\rfloor} g^{-1}(j) \left\lfloor \frac{x}{rj} \right\rfloor - \sum_{j=1}^{\left\lfloor \frac{x}{r+1} \right\rfloor} g^{-1}(j) \left\lfloor \frac{x}{(r+1)j} \right\rfloor\right] \\ &\asymp \sum_{r=1}^{x} \sum_{j=\left\lfloor \frac{x}{r+1} \right\rfloor}^{\left\lfloor \frac{x}{r} \right\rfloor} M(r) \cdot \frac{g^{-1}(j)}{j} \cdot \frac{x}{r^2}. \end{aligned}$$

A trivial known bound on the Mertens function provides that $M(x) = o(x)$ so that the previous equation shows

$$G^{-1}(x) = o\left(\sum_{r=1}^{x} \frac{x}{r}\left[\frac{r}{x} \cdot G^{-1}\left(\frac{x}{r}\right) - \frac{r+1}{x} \cdot G^{-1}\left(\frac{x}{r+1}\right) + \int_{\frac{x}{r+1}}^{\frac{x}{r}} \frac{G^{-1}(t)}{t^2} dt\right]\right).$$

By an inductive "*bootstrapping method*" we shall assume that $G^{-1}(y) = O(y)$ for all large enough $y \geq 1$. Then clearly we can approximate the last equation above as follows:

$$\begin{aligned} G^{-1}(x) &= O\left(x \times \int_1^x \left[\log\left(\frac{x}{t}\right) - \log\left(\frac{x}{t+1}\right)\right] \frac{dt}{t}\right) \\ &= O\left(x \times \int_1^x \left[\log(t+1) - \log t\right] \frac{dt}{t}\right) \\ &= O\left(x \times \int_1^x \frac{dt}{t^2}\right). \end{aligned}$$

To arrive at the last result, we have used that for $\varepsilon \in [0, 1]$, $\log(1 + \varepsilon) = O(\varepsilon)$. Therefore our claimed result is true when we evaluate the big-O bound in the previous equation at the lower bound of integration, giving that $G^{-1}(x) = O(x)$. □

The most recent best known upper bound on $L(x)$ (assuming the RH) is established by Humphries (2013) based on Soundararajan's result on $M(x)$ cited in Section 1.2 in the following form [?]:

$$L(x) = O\left(\sqrt{x} \cdot \exp\left((\log x)^{\frac{1}{2}} (\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right), \text{ for any } \varepsilon > 0; \text{ as } x \to \infty. \tag{26}$$

Corollary 6.2. *Let the summatory function $L(x) := \sum_{n \leq x} \lambda(n)$. Then we have that for almost every sufficiently large x, the summatory function of $g^{-1}(n)$ is bounded by*

$$G^{-1}(x) = O\left(\max_{1 \leq t \leq x} |L(t)| \cdot \mathbb{E}|g^{-1}(x)|\right).$$

Moreover, if the RH is true, then we have that for any $\varepsilon > 0$ and almost every large integer $x \geq 1$ that

$$G^{-1}(x) = O\left(\frac{\sqrt{x} \cdot (\log x)^{\frac{5}{2}}}{(\log\log x)^{2+\varepsilon}} \times \exp\left(\sqrt{\log x} \cdot (\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right).$$

Proof. Notice that we have the next formulas for $G^{-1}(x)$ by Abel summation and the formula we proved in (19), when $x \geq 1$ is large:

$$\begin{aligned} G^{-1}(x) &= \sum_{n \leq x} \lambda(n) |g^{-1}(n)| \\ &= L(x)|g^{-1}(x)| - \int L(x) \frac{d}{dx} \left|g^{-1}(x)\right| dx \\ &= O\left(\int L'(x)|g^{-1}(x)|\right) \\ &= O\left(\max_{1 \leq t \leq x} |g^{-1}(t)| \times \int L'(x) dx\right). \end{aligned} \tag{27}$$

The last step in the previous equations follows by an application of the mean value theorem. The proof of this result makes an appeal to the material we used to establish the more probabilistic interpretations of the distribution of $|g^{-1}(n)|$ as $n \to \infty$ from Section 5. In particular, by Remark 5.9, we can see that for almost every sufficiently large integer $x \geq 1$

$$\max_{1 \leq t \leq x} |g^{-1}(t)| = O\left(\max_{1 \leq t \leq x} \mathbb{E}|g^{-1}(x)|\right) = O\left((\log x)^2 \sqrt{\log\log x}\right),$$

where the evaluation of the expectation maximum follows from the monotone increasing function given in Corollary 5.5. The last result implies the first claim stated in terms of $L(x)$.

To prove the second claim, we consider the derivative of the asymptotic formula for $L(x)$ cited in (26) as follows:

$$\frac{d}{dx}\left[\sqrt{x} \cdot \exp\left((\log x)^{\frac{1}{2}} (\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right] = O\left(\frac{\exp\left((\log x)^{\frac{1}{2}} (\log\log x)^{\frac{5}{2}+\varepsilon}\right)}{\sqrt{x}}\right).$$

Therefore, we expand the Taylor series for the exponential function, apply the mean value theorem, and integrate the result termwise to obtain the next formulas:

$$\begin{aligned} \int L'(x) dx &= O\left(\sqrt{x} \times \int \sum_{k \geq 0} \frac{(\log x)^{\frac{k}{2}} (\log\log x)^{\left(\frac{5}{2}+\varepsilon\right)k}}{k! \cdot x} dx\right) \\ &= O\left(\sum_{k \geq 0} \frac{\sqrt{x}}{k!} \cdot \frac{(\log x)^{1+\frac{k}{2}}}{(k+2)} \cdot (\log\log x)^{\left(\frac{5}{2}+\varepsilon\right)k}\right) \\ &= O\left(\sqrt{x} \cdot \frac{\sqrt{\log x}}{(\log\log x)^{\frac{5}{2}+\varepsilon}} \times \exp\left((\log x)^{\frac{1}{2}} (\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right). \end{aligned}$$

In conjunction with the first result, and its simplification via the expectation formula we proved in the last section, the previous equations imply that the second result via (27). □

6.2 Bounding $M(x)$ through asymptotics for $G^{-1}(x)$

Proposition 6.3. *For all sufficiently large x, we have that the Mertens function satisfies*

$$M(x) = G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} G^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right]. \tag{28}$$

Proof. We know by applying Corollary 2.3 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right)+1\right] \\ &= G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} g^{-1}(k)\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right) \\ &= G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2}\right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1}\right\rfloor\right)\right]. \end{aligned}$$

The upper bound on the sum is truncated in the second equation above due to the fact that $\pi(1)=0$. □

Lemma 6.4. *For sufficiently large x, integers $k \in \left[\sqrt{x}, \frac{x}{2}\right]$ and $m \geq 0$, we have that*

$$\frac{x}{k \cdot \log^m\left(\frac{x}{k}\right)} - \frac{x}{(k+1)\cdot \log^m\left(\frac{x}{k+1}\right)} \gg \frac{x}{(\log x)^m \cdot k(k+1)}, \tag{A}$$

and

$$\sum_{k=\sqrt{x}}^{\frac{x}{2}} \frac{x}{k(k+1)} = \sum_{k=\sqrt{x}}^{\frac{x}{2}} \frac{x}{k^2} + O(1). \tag{B}$$

Proof. The proof of (A) is obvious since for $k_0 \in \left[\sqrt{x}, \frac{x}{2}\right]$ we have that

$$\log(2)(1+o(1)) \leq \log\left(\frac{x}{k_0}\right) \leq \log(x).$$

To prove (B), notice that

$$\frac{x}{k(k+1)} - \frac{x}{k^2} = -\frac{x}{k^2(k+1)}.$$

Then we see that

$$\left|\int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^2(t+1)}dt\right| \leq \left|\int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^3}dt\right| \asymp 1.$$ □

Corollary 6.5. *We have that as $x \to \infty$,*

$$M(x) = O\left(|G^{-1}(x)| + \frac{x}{(\log x)^3} \times \int_1^{\frac{x}{2}} \frac{|G^{-1}(t)|}{t^2}dt\right),$$

and

$$M(x) = O\left(|G^{-1}(x)| + \frac{x}{(\log x)^2} \times \max_{1 \leq t \leq \frac{x}{2}} \frac{|G^{-1}(t)|}{t}\right).$$

Proof. We need to first bound the prime counting function differences in the formula given by Proposition 6.3. We will require the following known bounds on the prime counting function due to Rosser and Schoenfeld for large $x > 59$ [15, Thm. 1]:

$$\frac{x}{\log x}\left(1+\frac{1}{2\log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1+\frac{3}{2\log x}\right). \tag{29}$$

The result in (29) together with Lemma 6.4 implies that

$$\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1}\right\rfloor\right) = O\left(\frac{x}{k^2 \cdot \log^2\left(\frac{x}{k}\right)}\right).$$

We need to break the summation terms from Proposition 6.3 expressed according to the bound in the previous equation into two sums over disjoint intervals, denoted respectively by $S_1(x)$ and $S_2(x)$. The two component sums are defined as follows:

$$S_1(x) := \sum_{1\leq k\leq\sqrt{x}} \frac{G^{-1}(k)}{k^2\cdot\log^2\left(\frac{x}{k}\right)} \asymp \frac{1}{(\log x)^2} \times \int_1^{\sqrt{x}} \frac{|G^{-1}(t)|}{t^2} dt$$

$$S_2(x) := \sum_{\sqrt{x}\leq k\leq\frac{x}{2}} \frac{G^{-1}(k)}{k^2\cdot\log^2\left(\frac{x}{k}\right)} = O\left(\int_{\sqrt{x}}^{\frac{x}{2}} \frac{|G^{-1}(t)|}{t^2\cdot\log^2\left(\frac{x}{t}\right)}\right).$$

We claim the following two relevant bounds on these sums:

$$x\cdot S_1(x) = O\left(\frac{x}{(\log x)^3} \times \int_1^{\frac{x}{2}} \frac{|G^{-1}(t)|}{t^2} dt\right) \tag{A}$$

$$x\cdot S_2(x) = O\left(\sqrt{x} \times \int_1^{\frac{x}{2}} \frac{|G^{-1}(t)|}{t^2} dt\right). \tag{B}$$

Recall an identity for the Mellin transform $\int_1^\infty \frac{G^{-1}(x)}{x^{s+1}}dx$ that is related to the DGF of $g^{-1}(n)$ [1, *cf.* §11]. We have seen that the DGF of the inverse sequence converges whenever $\mathrm{Re}(s) > 1$. Therefore, it follows that

$$\int_1^{\sqrt{x}} \frac{|G^{-1}(t)|}{t^2} dt = O(1) \quad\implies\quad O(x\cdot S_1(x)) = O\left(\frac{x}{(\log x)^2}\right).$$

By Proposition 6.1, we have that $G^{-1}(x) = o(x)$. Then the mean value theorem implies claim (A). To prove claim (B), define the two sub-integrals by

$$I_{21}(x) := \int_{\sqrt{x}}^{\frac{x}{2}} \frac{dt}{\frac{x}{t}\cdot\log^4\left(\frac{x}{t}\right)}$$

$$I_{22}(x) := \int_{\sqrt{x}}^{\frac{x}{2}} \frac{G^{-1}(t)^2}{t^3 dt}.$$

Using the change of variable $v := \frac{x}{t} \in [2, \sqrt{x}]$ with $dv = -\frac{dt}{x^2}$, the first integral is evaluated as

$$\begin{aligned} I_{21}(x) &= O\left(\frac{1}{x^2} \times \int_2^{\sqrt{x}} \frac{dv}{v\cdot(\log v)^4}\right) \\ &= O\left(x^{-2}\right). \end{aligned}$$

By the mean value theorem and since $\left(\int f(t)^2 dt\right)^{\frac{1}{2}} \leq \int |f(t)| dt$ for f integrable on $[2,\infty)$, we also obtain that

$$\begin{aligned} \sqrt{x}\cdot I_{22}(x) &= O\left(\sqrt{x} \times \int_2^{\sqrt{x}} \frac{|G^{-1}(t)|}{t^{\frac{3}{2}}} dt\right) \\ &= O\left(\sqrt{x} \times \int_1^{\frac{x}{2}} \frac{|G^{-1}(t)|}{t^{\frac{3}{2}}} dt\right) \end{aligned}$$

By a similar Mellin transform to DGF relation to what we considered above, we notice that for all $p > 1$ and any $y \geq 1$

$$\int_1^y \frac{|G^{-1}(t)|}{t^p} dt = O(1).$$

Therefore, we conclude that with $y \mapsto \frac{x}{2}$, the two cases of the previous equation with $p := \frac{3}{2}, 2$ have the same asymptotic upper bound, namely $O(1)$. By inequalities in L^p, we can write that

$$S_2(x) = O\left(\sqrt{x}\cdot I_{21}(x)^{\frac{1}{2}}\cdot I_{22}(x)^{\frac{1}{2}}\right).$$

We then conclude that

$$x \cdot S_2(x) = O\left(\sqrt{x} \times \int_1^{\frac{x}{2}} \frac{|G^{-1}(t)|}{t^2} dt\right).$$

The complete bound on $M(x)$ stated in this corollary follows as a consequence of the formula from Proposition 6.3. □

6.3 New approaches to bounding $M(x)$ along infinite subsequences

Theorem 6.6. *Assume the the RH is true. For $x \geq 1$, define*

$$M_x := \max_{1 \leq t \leq x} \frac{|G^{-1}(t)|}{t}.$$

Let the sets $\mathcal{S}_1(\varepsilon)$ and $\mathcal{S}_2(\varepsilon)$ be defined as follows for any fixed $\varepsilon > 0$:

$$\mathcal{S}_1(\varepsilon) := \left\{x \geq 1 : |g^{-1}(x)| = O((\log x)^2 \sqrt{\log\log x})\right\}$$

$$\mathcal{S}_2(\varepsilon) := \left\{x \geq 1 : |G^{-1}(x)| = O\left(\frac{\sqrt{x} \cdot \sqrt{\log x} \times \exp\left(-\sqrt{\log x}(\log\log x)^{\frac{5}{2}+\varepsilon}\right)}{(\log\log x)^{2+\varepsilon}}\right)\right\}.$$

Then for all $x \in \mathcal{S}_1(\varepsilon) \cap \mathcal{S}_2(\varepsilon)$, we have that

$$M(x) = O\left(\frac{\sqrt{x} \cdot \sqrt{\log x} \times \exp\left(-\sqrt{\log x}(\log\log x)^{\frac{5}{2}+\varepsilon}\right)}{(\log\log x)^{2+\varepsilon}}\right).$$

We first remark that by the discussion in Remark 5.9, for all $\varepsilon > 0$ the set $\mathcal{S}_1(\varepsilon)$ has asymptotic density of one. A sufficient condition that the set $\mathcal{S}_2(\varepsilon)$ is infinite is that $G^{-1}(x) = 0$ for infinitely many positive integers $x \geq 1$. The next proof shows a unique new upper bound on the positive integers x at which these two sets coincide.

Proof. Let $\varepsilon > 0$ be fixed. We first claim that for all $x \in \mathcal{S}_1(\varepsilon)$ we have that

$$M_x = O\left(\frac{(\log x)^{\frac{5}{2}}}{\sqrt{x} \cdot (\log\log x)^{2+\varepsilon}} \times \exp\left(-\sqrt{\log x}(\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right).$$

To prove this claim, we will require the result from Corollary 6.2. Since for all $x \in \mathcal{S}_1(\varepsilon)$ we have that

$$M_x = \max_{1 \leq t \leq x} \frac{|G^{-1}(t)|}{t} \leq \max_{1 \leq t \leq x} |g^{-1}(t)| = \Theta(\mathbb{E}|g^{-1}(x)|),$$

it follows that on $\mathcal{S}_1(\varepsilon)$

$$G^{-1}(x) = O\left(\frac{M_x \cdot \sqrt{x} \cdot \sqrt{\log x} \times \exp\left(\sqrt{\log x}(\log\log x)^{\frac{5}{2}+\varepsilon}\right)}{(\log\log x)^{2+\varepsilon}}\right).$$

Combining the previous equation with the corollary shows that we can write

$$\frac{x \cdot M_x}{(\log x)^2} = O\left(\frac{\sqrt{x}(\log\log x)^{2+\varepsilon}}{(\log x)^{\frac{5}{2}}} \times \exp\left(-\sqrt{\log x}(\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right).$$

We recall from Corollary 6.5 that

$$M(x) = O\left(|G^{-1}(x)| + \frac{x \cdot M_x}{(\log x)^2}\right).$$

Thus if $x \in \mathcal{S}_1(\varepsilon) \cap \mathcal{S}_2(\varepsilon)$, our claimed result follows as

$$M(x) = O\left(\frac{x \cdot M_x}{(\log x)^2}\right).$$

□

Remark 6.7 (Consequences of Theorem 6.6)**.** For fixed $\varepsilon > 0$ and $x \geq 1$, let

$$T_{\varepsilon}(x) := \frac{\sqrt{x} \cdot \sqrt{\log x} \times \exp\left(-\sqrt{\log x}(\log\log x)^{\frac{5}{2}+\varepsilon}\right)}{(\log\log x)^{2+\varepsilon}}.$$

Then we can compute that

$$\lim_{x\to\infty} T_{\varepsilon}(x) = +\infty, \lim_{x\to\infty} \frac{T_{\varepsilon}(x)}{\sqrt{x}} = 0.$$

It follows that if $G^{-1}(x)$ has infinitely many zeros, then

$$\liminf_{x\to\infty} \frac{|M(x)|}{\sqrt{x}} = 0.$$

Moreover, a predictable skew away from the uniform distribution of the zeros of $G^{-1}(x)$ suggests an approach to determining the sign bias of $M(x)$ over large intervals.

We also have the following additional observations to list about conditional asymptotics on $M(x)$ based on the sets identified in Theorem 6.6 for any fixed $\varepsilon > 0$:

- If $|\mathcal{S}_1(\varepsilon) \cap \mathcal{S}_2(\varepsilon)| < \infty$, then for all sufficiently large x, we have that $|M(x)| \sim |G^{-1}(x)|$.
- Suppose that there is an absolute constant $0 < C_U(\varepsilon) \leq 1$ such that for all large enough x

$$\frac{1}{x} \cdot \#\left\{n \leq x : n \in \mathcal{S}_1(\varepsilon) \cap \mathcal{S}_2(\varepsilon)\right\} \leq C_U(\varepsilon).$$

  Then a refernce point for determining the rate of unboundedness corresponding to $|M(x)|$ involves

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}} \geq \limsup_{x\to\infty} \frac{|G^{-1}(x)|}{\sqrt{x}}.$$

- The computational data generated in Table T.1 suggests numerically, especially when compared to the initial values of $M(x)$, that the distribution of $|G^{-1}(x)|$ may be easier to work with that those of $|M(x)|$ or $|L(x)|$. The remarks given in Section 4.3 about the direct relation of the distinct (and repitition of) values of $|g^{-1}(n)|$ for $n \leq x$ to the distribution of the primes and their distinct powers are also suggestive that bounding a main term for $G^{-1}(x)$ should be fairly regular along some infinitely tending subsequences of the integers.

7 Summary and reflections on significance of the new results

7.1 High-level bullet points highlighting key features of the new results

The main interpretation to take away from the article is that we have rigorously motivated an equivalent *alternate characterization* of $M(x)$ by constructing combinatorially relevant sequences related to the distribution of the primes and to standard strongly additive functions that have not yet been studied in the literature surrounding the Mertens function. This new perspective offers equivalent characterizations of $M(x)$ by formulas involving the summatory functions $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ and the prime counting function $\pi(x)$.

The proofs of key properties of these new sequences comes bundled with a scaled normal tending probability distribution for the unsigned magnitude of $|g^{-1}(n)|$ that is similar in many ways to the Erdös-Kac theorems for $\omega(n)$ and $\Omega(n)$. Moreover, since $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$, it follows that we have a new probabilistic perspective from which to express distributional features of the summatory functions $G^{-1}(x)$ as $x \to \infty$ in terms of the properties of $|g^{-1}(n)|$ and $L(x) := \sum_{n \leq x} \lambda(n)$. The distribution of $L(x)$ is typically viewed as a problem on par (equally as difficult in order) with understanding the distribution of $M(x)$ well as $x \to \infty$.

We also emphasize the following key features of these new constructions and the ways they are viewed to characterize $M(x)$:

- The new sequences $g^{-1}(n)$ and $G^{-1}(x)$ for $n, x \geq 1$ are crucially tied to standard, canonical examples of strongly and completely additive functions, e.g., $\omega(n)$ and $\Omega(n)$, respectively. As such, it is not surprising that we are able to relate the distributions of these functions by limiting probabilistic normal distributions which are similar to the celebrated results given by the Erdös-Kac theorems for the prime omega function variants.

- We are able to re-interpret and reconcile exact formulas for $M(x)$ naturally by an easy-to-spot relationship to the distinct primes in the factorizations of $n \leq x$. The prime-related combinatorics at hand here are discussed in more detail by the remarks given in Section 4.3 on page 17.

- We have limiting scaled normal tending distributions for $|g^{-1}(n)|$ over $n \leq x$ as $x \to \infty$. This theorem, combined with our recognized relation of exact formulas for $M(x)$ expressed by the prime counting function $\pi(x)$, suggest some nicer approaches for ways to formulate new bounds on $M(x)$. Additionally, given the easy-to-state bounds on each respective component sequence, $|g^{-1}(n)|$, $G^{-1}(x)$ and $\pi(x)$, we surmise that this new hammer in our toolbox should lead to similarly useful bounds by extended study in future work.

- We can consider new topics like the influence of the zero distribution of $G^{-1}(x)$ on properties that characterize the limiting (including supremum, infimum type) properties of $M(x)$ (see Section 6.3 on page 32 for more precision on this perspective).

7.2 A short detailed summary of the new results and their relevance

7.2.1 A key new sequence of special Dirichlet inverse functions

Fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute exactly that (see Table T.1 starting on page 38)

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

There is not a simple meaningful direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences whose properties we will discuss in detail. The distribution of distinct sets of prime exponents is still clearly quite regular since $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of this inverse function we notice below over $n \geq 2$:

Heuristic 7.1 (Symmetry in $g^{-1}(n)$ in the prime factorizations of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for $= \omega(n_i) \geq 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly one, two, three, and so on prime factors.

Proposition 7.2 (Characteristic properties of the inverse sequence)**.** *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) *For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*

(B) *For all squarefree integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!;$$

(C) *If $n \geq 2$ and $\Omega(n) = k$, then*

$$2 \leq |g^{-1}(n)| \leq \sum_{j=0}^{k} \binom{k}{j} \cdot j!.$$

We illustrate the proposition clearly using the computation of initial values of this inverse sequence in Table T.1. The signedness property in (A) is proved precisely in Proposition 3.1. A proof of (B) in fact follows from Lemma 4.1 stated on page 15. The realization that the beautiful and remarkably simple combinatorial form of property (B) in the last proposition holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through sums of auxiliary subsequences of arithmetic functions denoted by $C_k(n)$ (see Section 4). That is, we observe a familiar formula for $g^{-1}(n)$ at many integers and then seek to extrapolate and prove more regular tendencies of this sequence more generally at any $n \geq 2$.

An exact expression for $g^{-1}(n)$ through a key semi-diagonal of these subsequences is given by

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d), n \geq 1,$$

where the sequence $\lambda(n) C_{\Omega(n)}(n)$ has DGF $(P(s)+1)^{-1}$ for $\operatorname{Re}(s) > 1$ (see Proposition 3.1). In Corollary 5.5, we prove that the approximate average order mean of the unsigned sequence satisfies

$$\mathbb{E}|g^{-1}(n)| \asymp (\log n)^2 \sqrt{\log\log n}, \text{ as } n \to \infty.$$

In Section 5, we prove the next variant of an Erdös-Kac theorem like analog for a component sequence $C_{\Omega(n)}(n)$. Namely, we prove the following statement for $\mu_x(C) := \log\log x + \hat{a} - \frac{1}{2}\log\log\log x$, $\sigma_x(C) := \sqrt{\mu_x(C)}$, $\hat{a}$ an absolute constant, and any $y \in \mathbb{R}$ (see Corollary 5.7):

$$\frac{1}{x} \cdot \#\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq y\} = \Phi\left(\frac{\frac{\pi^2}{6} y - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

Thus, the regularity and quasi-periodicity we have alluded to in the remarks above are actually quantifiable in so much as $|g^{-1}(n)|$ for $n \leq x$ tends to its average order with a non-central normal tendency depending on x as $x \to \infty$. If x is sufficiently large and we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds:

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq 0\right) = o(1) \tag{A}$$

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\mu_x(C)\right) = \frac{1}{2} + o(1). \tag{B}$$

Moreover, for any positive real $\delta > 0$ we have that

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\mu_x(C)^{1+\delta}\right) = 1 + o_\delta(1), \text{ as } x \rightarrow \infty. \tag{C}$$

A consequence of (A) and (C) in the probability estimates above is that for any fixed $\delta > 0$ and $n \in \mathcal{S}_1(\delta)$ taken within a set of asymptotic density one we have that

$$\frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq |g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| + \frac{6}{\pi^2}\mu_x(C)^{\frac{1}{2}+\delta}.$$

Hence when we integrate over a sufficiently spaced set of (e.g., set of wide enough) disjoint consecutive intervals containing large enough integer values, we can assume that an asymptotic lower bound on the contribution of $|g^{-1}(n)|$ is given by the function's average order, and an upper bound is given by the related upper limit above for any fixed $\delta > 0$. In particular, observe that by Corollary 5.7 and Corollary 5.5 we can see that

$$\frac{\pi^2}{6 \cdot \sigma_x(C)} \times \int_{-\infty}^{\infty} z \cdot \Phi'\left(\frac{\frac{\pi^2}{6}z - \mu_x(C)}{\sigma_x(C)}\right) dz = \frac{6}{\pi^2} \cdot \sigma_x(C) = o\left(\mathbb{E}|g^{-1}(x)|\right).$$

7.2.2 The precise new characterizations of $M(x)$

Let $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ for integers $x \geq 1$. We prove that (see Proposition 6.3)

$$\begin{aligned} M(x) &= G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right] \\ &= G^{-1}(x) + \sum_{p \leq x} G\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1. \end{aligned}$$

This formula implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by bounding appropriate estimates of the summatory function $G^{-1}(x)$. This take on the regularity of $|g^{-1}(n)|$ is imperative to our argument formally bounding the growth of $M(x)$ through its new characterizations by $G^{-1}(x)$. A more combinatorial approach to summing $G^{-1}(x)$ for large x based on the distribution of the primes is outlined in our remarks in Section 4.3.

In the proofs given in Section 6, we begin to use these new equivalent characterizations to relate the distributions of $|g^{-1}(n)|$, $G^{-1}(x)$, $\lambda(n)$ and its often classically studied summatory function $L(x)$, to $M(x)$ as $x \rightarrow \infty$. In particular, Proposition 6.1 proves that like the known bound for $M(x)$, we have that $G^{-1}(x) = o(x)$ as $x \rightarrow \infty$. The results in Corollary 6.2 prove that for almost every sufficiently large x

$$G^{-1}(x) = O\left(\max_{1 \leq t \leq x} |L(t)| \cdot \mathbb{E}|g^{-1}(x)|\right).$$

Moreover, if the RH is true, then we have the following result for any $\varepsilon > 0$ and almost every integer $x \geq 1$:

$$G^{-1}(x) = O\left(\frac{\sqrt{x} \cdot (\log x)^{\frac{5}{2}}}{(\log\log x)^{2+\varepsilon}} \times \exp\left(\sqrt{\log x} \cdot (\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right).$$

By applying Corollary 6.5, we have that as $x \rightarrow \infty$

$$M(x) = O\left(|G^{-1}(x)| + \frac{x}{(\log x)^3} \times \int_1^{\frac{x}{2}} \frac{|G^{-1}(t)|}{t^2} dt\right),$$

and

$$M(x) = O\left(|G^{-1}(x)| + \frac{x}{(\log x)^2} \times \max_{1 \leq t \leq \frac{x}{2}} \frac{|G^{-1}(t)|}{t}\right).$$

Other consequences of the distribution of $G^{-1}(y)$ for $y \leq x$ at large x to bounds and limiting properties of $M(x)$ (like limit supremum, infimum type relations) are discussed in Section 6.3 on page 32. Moving forward, this discussion motivates (and really requires) further study in future work.

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T.1 Table: The Dirichlet inverse function $g^{-1}(n)$ and the distribution of its summatory function

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1^1	Y	N	1	0	1.0000000	1.000000	0.000000	1	1	0
2	2^1	Y	Y	-2	0	1.0000000	0.500000	0.500000	-1	1	-2
3	3^1	Y	Y	-2	0	1.0000000	0.333333	0.666667	-3	1	-4
4	2^2	N	Y	2	0	1.5000000	0.500000	0.500000	-1	3	-4
5	5^1	Y	Y	-2	0	1.0000000	0.400000	0.600000	-3	3	-6
6	$2^1 3^1$	Y	N	5	0	1.0000000	0.500000	0.500000	2	8	-6
7	7^1	Y	Y	-2	0	1.0000000	0.428571	0.571429	0	8	-8
8	2^3	N	Y	-2	0	2.0000000	0.375000	0.625000	-2	8	-10
9	3^2	N	Y	2	0	1.5000000	0.444444	0.555556	0	10	-10
10	$2^1 5^1$	Y	N	5	0	1.0000000	0.500000	0.500000	5	15	-10
11	11^1	Y	Y	-2	0	1.0000000	0.454545	0.545455	3	15	-12
12	$2^2 3^1$	N	N	-7	2	1.2857143	0.416667	0.583333	-4	15	-19
13	13^1	Y	Y	-2	0	1.0000000	0.384615	0.615385	-6	15	-21
14	$2^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-1	20	-21
15	$3^1 5^1$	Y	N	5	0	1.0000000	0.466667	0.533333	4	25	-21
16	2^4	N	Y	2	0	2.5000000	0.500000	0.500000	6	27	-21
17	17^1	Y	Y	-2	0	1.0000000	0.470588	0.529412	4	27	-23
18	$2^1 3^2$	N	N	-7	2	1.2857143	0.444444	0.555556	-3	27	-30
19	19^1	Y	Y	-2	0	1.0000000	0.421053	0.578947	-5	27	-32
20	$2^2 5^1$	N	N	-7	2	1.2857143	0.400000	0.600000	-12	27	-39
21	$3^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-7	32	-39
22	$2^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-2	37	-39
23	23^1	Y	Y	-2	0	1.0000000	0.434783	0.565217	-4	37	-41
24	$2^3 3^1$	N	N	9	4	1.5555556	0.458333	0.541667	5	46	-41
25	5^2	N	Y	2	0	1.5000000	0.480000	0.520000	7	48	-41
26	$2^1 13^1$	Y	N	5	0	1.0000000	0.500000	0.500000	12	53	-41
27	3^3	N	Y	-2	0	2.0000000	0.481481	0.518519	10	53	-43
28	$2^2 7^1$	N	N	-7	2	1.2857143	0.464286	0.535714	3	53	-50
29	29^1	Y	Y	-2	0	1.0000000	0.448276	0.551724	1	53	-52
30	$2^1 3^1 5^1$	Y	N	-16	0	1.0000000	0.433333	0.566667	-15	53	-68
31	31^1	Y	Y	-2	0	1.0000000	0.419355	0.580645	-17	53	-70
32	2^5	N	Y	-2	0	3.0000000	0.406250	0.593750	-19	53	-72
33	$3^1 11^1$	Y	N	5	0	1.0000000	0.424242	0.575758	-14	58	-72
34	$2^1 17^1$	Y	N	5	0	1.0000000	0.441176	0.558824	-9	63	-72
35	$5^1 7^1$	Y	N	5	0	1.0000000	0.457143	0.542857	-4	68	-72
36	$2^2 3^2$	N	N	14	9	1.3571429	0.472222	0.527778	10	82	-72
37	37^1	Y	Y	-2	0	1.0000000	0.459459	0.540541	8	82	-74
38	$2^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	13	87	-74
39	$3^1 13^1$	Y	N	5	0	1.0000000	0.487179	0.512821	18	92	-74
40	$2^3 5^1$	N	N	9	4	1.5555556	0.500000	0.500000	27	101	-74
41	41^1	Y	Y	-2	0	1.0000000	0.487805	0.512195	25	101	-76
42	$2^1 3^1 7^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	9	101	-92
43	43^1	Y	Y	-2	0	1.0000000	0.465116	0.534884	7	101	-94
44	$2^2 11^1$	N	N	-7	2	1.2857143	0.454545	0.545455	0	101	-101
45	$3^2 5^1$	N	N	-7	2	1.2857143	0.444444	0.555556	-7	101	-108
46	$2^1 23^1$	Y	N	5	0	1.0000000	0.456522	0.543478	-2	106	-108
47	47^1	Y	Y	-2	0	1.0000000	0.446809	0.553191	-4	106	-110
48	$2^4 3^1$	N	N	-11	6	1.8181818	0.437500	0.562500	-15	106	-121

Table T.1: ***Computations with*** $\mathbf{g^{-1}(n)} \equiv (\omega+\mathbf{1})^{-\mathbf{1}}(\mathbf{n})$ ***for*** $\mathbf{1 \leq n \leq 500}$.

- *The column labeled* `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled* `Sqfree` *and* `PPower`*, respectively, list inclusion of n in the sets of squarefree integers and the prime powers.*
- *The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.*
- *The last columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{n} \cdot \#\{n \leq x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geq 1$.*

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
49	7^2	N	Y	2	0	1.5000000	0.448980	0.551020	−13	108	−121
50	$2^1 5^2$	N	N	−7	2	1.2857143	0.440000	0.560000	−20	108	−128
51	$3^1 17^1$	Y	N	5	0	1.0000000	0.450980	0.549020	−15	113	−128
52	$2^2 13^1$	N	N	−7	2	1.2857143	0.442308	0.557692	−22	113	−135
53	53^1	Y	Y	−2	0	1.0000000	0.433962	0.566038	−24	113	−137
54	$2^1 3^3$	N	N	9	4	1.5555556	0.444444	0.555556	−15	122	−137
55	$5^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−10	127	−137
56	$2^3 7^1$	N	N	9	4	1.5555556	0.464286	0.535714	−1	136	−137
57	$3^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	4	141	−137
58	$2^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	9	146	−137
59	59^1	Y	Y	−2	0	1.0000000	0.474576	0.525424	7	146	−139
60	$2^2 3^1 5^1$	N	N	30	14	1.1666667	0.483333	0.516667	37	176	−139
61	61^1	Y	Y	−2	0	1.0000000	0.475410	0.524590	35	176	−141
62	$2^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	40	181	−141
63	$3^2 7^1$	N	N	−7	2	1.2857143	0.476190	0.523810	33	181	−148
64	2^6	N	Y	2	0	3.5000000	0.484375	0.515625	35	183	−148
65	$5^1 13^1$	Y	N	5	0	1.0000000	0.492308	0.507692	40	188	−148
66	$2^1 3^1 11^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	24	188	−164
67	67^1	Y	Y	−2	0	1.0000000	0.477612	0.522388	22	188	−166
68	$2^2 17^1$	N	N	−7	2	1.2857143	0.470588	0.529412	15	188	−173
69	$3^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	20	193	−173
70	$2^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.471429	0.528571	4	193	−189
71	71^1	Y	Y	−2	0	1.0000000	0.464789	0.535211	2	193	−191
72	$2^3 3^2$	N	N	−23	18	1.4782609	0.458333	0.541667	−21	193	−214
73	73^1	Y	Y	−2	0	1.0000000	0.452055	0.547945	−23	193	−216
74	$2^1 37^1$	Y	N	5	0	1.0000000	0.459459	0.540541	−18	198	−216
75	$3^1 5^2$	N	N	−7	2	1.2857143	0.453333	0.546667	−25	198	−223
76	$2^2 19^1$	N	N	−7	2	1.2857143	0.447368	0.552632	−32	198	−230
77	$7^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−27	203	−230
78	$2^1 3^1 13^1$	Y	N	−16	0	1.0000000	0.448718	0.551282	−43	203	−246
79	79^1	Y	Y	−2	0	1.0000000	0.443038	0.556962	−45	203	−248
80	$2^4 5^1$	N	N	−11	6	1.8181818	0.437500	0.562500	−56	203	−259
81	3^4	N	Y	2	0	2.5000000	0.444444	0.555556	−54	205	−259
82	$2^1 41^1$	Y	N	5	0	1.0000000	0.451220	0.548780	−49	210	−259
83	83^1	Y	Y	−2	0	1.0000000	0.445783	0.554217	−51	210	−261
84	$2^2 3^1 7^1$	N	N	30	14	1.1666667	0.452381	0.547619	−21	240	−261
85	$5^1 17^1$	Y	N	5	0	1.0000000	0.458824	0.541176	−16	245	−261
86	$2^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	−11	250	−261
87	$3^1 29^1$	Y	N	5	0	1.0000000	0.471264	0.528736	−6	255	−261
88	$2^3 11^1$	N	N	9	4	1.5555556	0.477273	0.522727	3	264	−261
89	89^1	Y	Y	−2	0	1.0000000	0.471910	0.528090	1	264	−263
90	$2^1 3^2 5^1$	N	N	30	14	1.1666667	0.477778	0.522222	31	294	−263
91	$7^1 13^1$	Y	N	5	0	1.0000000	0.483516	0.516484	36	299	−263
92	$2^2 23^1$	N	N	−7	2	1.2857143	0.478261	0.521739	29	299	−270
93	$3^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	34	304	−270
94	$2^1 47^1$	Y	N	5	0	1.0000000	0.489362	0.510638	39	309	−270
95	$5^1 19^1$	Y	N	5	0	1.0000000	0.494737	0.505263	44	314	−270
96	$2^5 3^1$	N	N	13	8	2.0769231	0.500000	0.500000	57	327	−270
97	97^1	Y	Y	−2	0	1.0000000	0.494845	0.505155	55	327	−272
98	$2^1 7^2$	N	N	−7	2	1.2857143	0.489796	0.510204	48	327	−279
99	$3^2 11^1$	N	N	−7	2	1.2857143	0.484848	0.515152	41	327	−286
100	$2^2 5^2$	N	N	14	9	1.3571429	0.490000	0.510000	55	341	−286
101	101^1	Y	Y	−2	0	1.0000000	0.485149	0.514851	53	341	−288
102	$2^1 3^1 17^1$	Y	N	−16	0	1.0000000	0.480392	0.519608	37	341	−304
103	103^1	Y	Y	−2	0	1.0000000	0.475728	0.524272	35	341	−306
104	$2^3 13^1$	N	N	9	4	1.5555556	0.480769	0.519231	44	350	−306
105	$3^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	28	350	−322
106	$2^1 53^1$	Y	N	5	0	1.0000000	0.481132	0.518868	33	355	−322
107	107^1	Y	Y	−2	0	1.0000000	0.476636	0.523364	31	355	−324
108	$2^2 3^3$	N	N	−23	18	1.4782609	0.472222	0.527778	8	355	−347
109	109^1	Y	Y	−2	0	1.0000000	0.467890	0.532110	6	355	−349
110	$2^1 5^1 11^1$	Y	N	−16	0	1.0000000	0.463636	0.536364	−10	355	−365
111	$3^1 37^1$	Y	N	5	0	1.0000000	0.468468	0.531532	−5	360	−365
112	$2^4 7^1$	N	N	−11	6	1.8181818	0.464286	0.535714	−16	360	−376
113	113^1	Y	Y	−2	0	1.0000000	0.460177	0.539823	−18	360	−378
114	$2^1 3^1 19^1$	Y	N	−16	0	1.0000000	0.456140	0.543860	−34	360	−394
115	$5^1 23^1$	Y	N	5	0	1.0000000	0.460870	0.539130	−29	365	−394
116	$2^2 29^1$	N	N	−7	2	1.2857143	0.456897	0.543103	−36	365	−401
117	$3^2 13^1$	N	N	−7	2	1.2857143	0.452991	0.547009	−43	365	−408
118	$2^1 59^1$	Y	N	5	0	1.0000000	0.457627	0.542373	−38	370	−408
119	$7^1 17^1$	Y	N	5	0	1.0000000	0.462185	0.537815	−33	375	−408
120	$2^3 3^1 5^1$	N	N	−48	32	1.3333333	0.458333	0.541667	−81	375	−456
121	11^2	N	Y	2	0	1.5000000	0.462810	0.537190	−79	377	−456
122	$2^1 61^1$	Y	N	5	0	1.0000000	0.467213	0.532787	−74	382	−456
123	$3^1 41^1$	Y	N	5	0	1.0000000	0.471545	0.528455	−69	387	−456
124	$2^2 31^1$	N	N	−7	2	1.2857143	0.467742	0.532258	−76	387	−463

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d \mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n) \rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
125	5^3	N	Y	-2	0	2.0000000	0.464000	0.536000	-78	387	-465
126	$2^1 3^2 7^1$	N	N	30	14	1.1666667	0.468254	0.531746	-48	417	-465
127	127^1	Y	Y	-2	0	1.0000000	0.464567	0.535433	-50	417	-467
128	2^7	N	Y	-2	0	4.0000000	0.460938	0.539062	-52	417	-469
129	$3^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	-47	422	-469
130	$2^1 5^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-63	422	-485
131	131^1	Y	Y	-2	0	1.0000000	0.458015	0.541985	-65	422	-487
132	$2^2 3^1 11^1$	N	N	30	14	1.1666667	0.462121	0.537879	-35	452	-487
133	$7^1 19^1$	Y	N	5	0	1.0000000	0.466165	0.533835	-30	457	-487
134	$2^1 67^1$	Y	N	5	0	1.0000000	0.470149	0.529851	-25	462	-487
135	$3^3 5^1$	N	N	9	4	1.5555556	0.474074	0.525926	-16	471	-487
136	$2^3 17^1$	N	N	9	4	1.5555556	0.477941	0.522059	-7	480	-487
137	137^1	Y	Y	-2	0	1.0000000	0.474453	0.525547	-9	480	-489
138	$2^1 3^1 23^1$	Y	N	-16	0	1.0000000	0.471014	0.528986	-25	480	-505
139	139^1	Y	Y	-2	0	1.0000000	0.467626	0.532374	-27	480	-507
140	$2^2 5^1 7^1$	N	N	30	14	1.1666667	0.471429	0.528571	3	510	-507
141	$3^1 47^1$	Y	N	5	0	1.0000000	0.475177	0.524823	8	515	-507
142	$2^1 71^1$	Y	N	5	0	1.0000000	0.478873	0.521127	13	520	-507
143	$11^1 13^1$	Y	N	5	0	1.0000000	0.482517	0.517483	18	525	-507
144	$2^4 3^2$	N	N	34	29	1.6176471	0.486111	0.513889	52	559	-507
145	$5^1 29^1$	Y	N	5	0	1.0000000	0.489655	0.510345	57	564	-507
146	$2^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	62	569	-507
147	$3^1 7^2$	N	N	-7	2	1.2857143	0.489796	0.510204	55	569	-514
148	$2^2 37^1$	N	N	-7	2	1.2857143	0.486486	0.513514	48	569	-521
149	149^1	Y	Y	-2	0	1.0000000	0.483221	0.516779	46	569	-523
150	$2^1 3^1 5^2$	N	N	30	14	1.1666667	0.486667	0.513333	76	599	-523
151	151^1	Y	Y	-2	0	1.0000000	0.483444	0.516556	74	599	-525
152	$2^3 19^1$	N	N	9	4	1.5555556	0.486842	0.513158	83	608	-525
153	$3^2 17^1$	N	N	-7	2	1.2857143	0.483660	0.516340	76	608	-532
154	$2^1 7^1 11^1$	Y	N	-16	0	1.0000000	0.480519	0.519481	60	608	-548
155	$5^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	65	613	-548
156	$2^2 3^1 13^1$	N	N	30	14	1.1666667	0.487179	0.512821	95	643	-548
157	157^1	Y	Y	-2	0	1.0000000	0.484076	0.515924	93	643	-550
158	$2^1 79^1$	Y	N	5	0	1.0000000	0.487342	0.512658	98	648	-550
159	$3^1 53^1$	Y	N	5	0	1.0000000	0.490566	0.509434	103	653	-550
160	$2^5 5^1$	N	N	13	8	2.0769231	0.493750	0.506250	116	666	-550
161	$7^1 23^1$	Y	N	5	0	1.0000000	0.496894	0.503106	121	671	-550
162	$2^1 3^4$	N	N	-11	6	1.8181818	0.493827	0.506173	110	671	-561
163	163^1	Y	Y	-2	0	1.0000000	0.490798	0.509202	108	671	-563
164	$2^2 41^1$	N	N	-7	2	1.2857143	0.487805	0.512195	101	671	-570
165	$3^1 5^1 11^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	85	671	-586
166	$2^1 83^1$	Y	N	5	0	1.0000000	0.487952	0.512048	90	676	-586
167	167^1	Y	Y	-2	0	1.0000000	0.485030	0.514970	88	676	-588
168	$2^3 3^1 7^1$	N	N	-48	32	1.3333333	0.482143	0.517857	40	676	-636
169	13^2	N	Y	2	0	1.5000000	0.485207	0.514793	42	678	-636
170	$2^1 5^1 17^1$	Y	N	-16	0	1.0000000	0.482353	0.517647	26	678	-652
171	$3^2 19^1$	N	N	-7	2	1.2857143	0.479532	0.520468	19	678	-659
172	$2^2 43^1$	N	N	-7	2	1.2857143	0.476744	0.523256	12	678	-666
173	173^1	Y	Y	-2	0	1.0000000	0.473988	0.526012	10	678	-668
174	$2^1 3^1 29^1$	Y	N	-16	0	1.0000000	0.471264	0.528736	-6	678	-684
175	$5^2 7^1$	N	N	-7	2	1.2857143	0.468571	0.531429	-13	678	-691
176	$2^4 11^1$	N	N	-11	6	1.8181818	0.465909	0.534091	-24	678	-702
177	$3^1 59^1$	Y	N	5	0	1.0000000	0.468927	0.531073	-19	683	-702
178	$2^1 89^1$	Y	N	5	0	1.0000000	0.471910	0.528090	-14	688	-702
179	179^1	Y	Y	-2	0	1.0000000	0.469274	0.530726	-16	688	-704
180	$2^2 3^2 5^1$	N	N	-74	58	1.2162162	0.466667	0.533333	-90	688	-778
181	181^1	Y	Y	-2	0	1.0000000	0.464088	0.535912	-92	688	-780
182	$2^1 7^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-108	688	-796
183	$3^1 61^1$	Y	N	5	0	1.0000000	0.464481	0.535519	-103	693	-796
184	$2^3 23^1$	N	N	9	4	1.5555556	0.467391	0.532609	-94	702	-796
185	$5^1 37^1$	Y	N	5	0	1.0000000	0.470270	0.529730	-89	707	-796
186	$2^1 3^1 31^1$	Y	N	-16	0	1.0000000	0.467742	0.532258	-105	707	-812
187	$11^1 17^1$	Y	N	5	0	1.0000000	0.470588	0.529412	-100	712	-812
188	$2^2 47^1$	N	N	-7	2	1.2857143	0.468085	0.531915	-107	712	-819
189	$3^3 7^1$	N	N	9	4	1.5555556	0.470899	0.529101	-98	721	-819
190	$2^1 5^1 19^1$	Y	N	-16	0	1.0000000	0.468421	0.531579	-114	721	-835
191	191^1	Y	Y	-2	0	1.0000000	0.465969	0.534031	-116	721	-837
192	$2^6 3^1$	N	N	-15	10	2.3333333	0.463542	0.536458	-131	721	-852
193	193^1	Y	Y	-2	0	1.0000000	0.461140	0.538860	-133	721	-854
194	$2^1 97^1$	Y	N	5	0	1.0000000	0.463918	0.536082	-128	726	-854
195	$3^1 5^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-144	726	-870
196	$2^2 7^2$	N	N	14	9	1.3571429	0.464286	0.535714	-130	740	-870
197	197^1	Y	Y	-2	0	1.0000000	0.461929	0.538071	-132	740	-872
198	$2^1 3^2 11^1$	N	N	30	14	1.1666667	0.464646	0.535354	-102	770	-872
199	199^1	Y	Y	-2	0	1.0000000	0.462312	0.537688	-104	770	-874
200	$2^3 5^2$	N	N	-23	18	1.4782609	0.460000	0.540000	-127	770	-897

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
201	$3^1 67^1$	Y	N	5	0	1.0000000	0.462687	0.537313	−122	775	−897
202	$2^1 101^1$	Y	N	5	0	1.0000000	0.465347	0.534653	−117	780	−897
203	$7^1 29^1$	Y	N	5	0	1.0000000	0.467980	0.532020	−112	785	−897
204	$2^2 3^1 17^1$	N	N	30	14	1.1666667	0.470588	0.529412	−82	815	−897
205	$5^1 41^1$	Y	N	5	0	1.0000000	0.473171	0.526829	−77	820	−897
206	$2^1 103^1$	Y	N	5	0	1.0000000	0.475728	0.524272	−72	825	−897
207	$3^2 23^1$	N	N	−7	2	1.2857143	0.473430	0.526570	−79	825	−904
208	$2^4 13^1$	N	N	−11	6	1.8181818	0.471154	0.528846	−90	825	−915
209	$11^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	−85	830	−915
210	$2^1 3^1 5^1 7^1$	Y	N	65	0	1.0000000	0.476190	0.523810	−20	895	−915
211	211^1	Y	Y	−2	0	1.0000000	0.473934	0.526066	−22	895	−917
212	$2^2 53^1$	N	N	−7	2	1.2857143	0.471698	0.528302	−29	895	−924
213	$3^1 71^1$	Y	N	5	0	1.0000000	0.474178	0.525822	−24	900	−924
214	$2^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−19	905	−924
215	$5^1 43^1$	Y	N	5	0	1.0000000	0.479070	0.520930	−14	910	−924
216	$2^3 3^3$	N	N	46	41	1.5000000	0.481481	0.518519	32	956	−924
217	$7^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	37	961	−924
218	$2^1 109^1$	Y	N	5	0	1.0000000	0.486239	0.513761	42	966	−924
219	$3^1 73^1$	Y	N	5	0	1.0000000	0.488584	0.511416	47	971	−924
220	$2^2 5^1 11^1$	N	N	30	14	1.1666667	0.490909	0.509091	77	1001	−924
221	$13^1 17^1$	Y	N	5	0	1.0000000	0.493213	0.506787	82	1006	−924
222	$2^1 3^1 37^1$	Y	N	−16	0	1.0000000	0.490991	0.509009	66	1006	−940
223	223^1	Y	Y	−2	0	1.0000000	0.488789	0.511211	64	1006	−942
224	$2^5 7^1$	N	N	13	8	2.0769231	0.491071	0.508929	77	1019	−942
225	$3^2 5^2$	N	N	14	9	1.3571429	0.493333	0.506667	91	1033	−942
226	$2^1 113^1$	Y	N	5	0	1.0000000	0.495575	0.504425	96	1038	−942
227	227^1	Y	Y	−2	0	1.0000000	0.493392	0.506608	94	1038	−944
228	$2^2 3^1 19^1$	N	N	30	14	1.1666667	0.495614	0.504386	124	1068	−944
229	229^1	Y	Y	−2	0	1.0000000	0.493450	0.506550	122	1068	−946
230	$2^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.491304	0.508696	106	1068	−962
231	$3^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.489177	0.510823	90	1068	−978
232	$2^3 29^1$	N	N	9	4	1.5555556	0.491379	0.508621	99	1077	−978
233	233^1	Y	Y	−2	0	1.0000000	0.489270	0.510730	97	1077	−980
234	$2^1 3^2 13^1$	N	N	30	14	1.1666667	0.491453	0.508547	127	1107	−980
235	$5^1 47^1$	Y	N	5	0	1.0000000	0.493617	0.506383	132	1112	−980
236	$2^2 59^1$	N	N	−7	2	1.2857143	0.491525	0.508475	125	1112	−987
237	$3^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	130	1117	−987
238	$2^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.491597	0.508403	114	1117	−1003
239	239^1	Y	Y	−2	0	1.0000000	0.489540	0.510460	112	1117	−1005
240	$2^4 3^1 5^1$	N	N	70	54	1.5000000	0.491667	0.508333	182	1187	−1005
241	241^1	Y	Y	−2	0	1.0000000	0.489627	0.510373	180	1187	−1007
242	$2^1 11^2$	N	N	−7	2	1.2857143	0.487603	0.512397	173	1187	−1014
243	3^5	N	Y	−2	0	3.0000000	0.485597	0.514403	171	1187	−1016
244	$2^2 61^1$	N	N	−7	2	1.2857143	0.483607	0.516393	164	1187	−1023
245	$5^1 7^2$	N	N	−7	2	1.2857143	0.481633	0.518367	157	1187	−1030
246	$2^1 3^1 41^1$	Y	N	−16	0	1.0000000	0.479675	0.520325	141	1187	−1046
247	$13^1 19^1$	Y	N	5	0	1.0000000	0.481781	0.518219	146	1192	−1046
248	$2^3 31^1$	N	N	9	4	1.5555556	0.483871	0.516129	155	1201	−1046
249	$3^1 83^1$	Y	N	5	0	1.0000000	0.485944	0.514056	160	1206	−1046
250	$2^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	169	1215	−1046
251	251^1	Y	Y	−2	0	1.0000000	0.486056	0.513944	167	1215	−1048
252	$2^2 3^2 7^1$	N	N	−74	58	1.2162162	0.484127	0.515873	93	1215	−1122
253	$11^1 23^1$	Y	N	5	0	1.0000000	0.486166	0.513834	98	1220	−1122
254	$2^1 127^1$	Y	N	5	0	1.0000000	0.488189	0.511811	103	1225	−1122
255	$3^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.486275	0.513725	87	1225	−1138
256	2^8	N	Y	2	0	4.5000000	0.488281	0.511719	89	1227	−1138
257	257^1	Y	Y	−2	0	1.0000000	0.486381	0.513619	87	1227	−1140
258	$2^1 3^1 43^1$	Y	N	−16	0	1.0000000	0.484496	0.515504	71	1227	−1156
259	$7^1 37^1$	Y	N	5	0	1.0000000	0.486486	0.513514	76	1232	−1156
260	$2^2 5^1 13^1$	N	N	30	14	1.1666667	0.488462	0.511538	106	1262	−1156
261	$3^2 29^1$	N	N	−7	2	1.2857143	0.486590	0.513410	99	1262	−1163
262	$2^1 131^1$	Y	N	5	0	1.0000000	0.488550	0.511450	104	1267	−1163
263	263^1	Y	Y	−2	0	1.0000000	0.486692	0.513308	102	1267	−1165
264	$2^3 3^1 11^1$	N	N	−48	32	1.3333333	0.484848	0.515152	54	1267	−1213
265	$5^1 53^1$	Y	N	5	0	1.0000000	0.486792	0.513208	59	1272	−1213
266	$2^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.484962	0.515038	43	1272	−1229
267	$3^1 89^1$	Y	N	5	0	1.0000000	0.486891	0.513109	48	1277	−1229
268	$2^2 67^1$	N	N	−7	2	1.2857143	0.485075	0.514925	41	1277	−1236
269	269^1	Y	Y	−2	0	1.0000000	0.483271	0.516729	39	1277	−1238
270	$2^1 3^3 5^1$	N	N	−48	32	1.3333333	0.481481	0.518519	−9	1277	−1286
271	271^1	Y	Y	−2	0	1.0000000	0.479705	0.520295	−11	1277	−1288
272	$2^4 17^1$	N	N	−11	6	1.8181818	0.477941	0.522059	−22	1277	−1299
273	$3^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−38	1277	−1315
274	$2^1 137^1$	Y	N	5	0	1.0000000	0.478102	0.521898	−33	1282	−1315
275	$5^2 11^1$	N	N	−7	2	1.2857143	0.476364	0.523636	−40	1282	−1322
276	$2^2 3^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−10	1312	−1322
277	277^1	Y	Y	−2	0	1.0000000	0.476534	0.523466	−12	1312	−1324

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
278	$2^1 139^1$	Y	N	5	0	1.0000000	0.478417	0.521583	-7	1317	-1324
279	$3^2 31^1$	N	N	-7	2	1.2857143	0.476703	0.523297	-14	1317	-1331
280	$2^3 5^1 7^1$	N	N	-48	32	1.3333333	0.475000	0.525000	-62	1317	-1379
281	281^1	Y	Y	-2	0	1.0000000	0.473310	0.526690	-64	1317	-1381
282	$2^1 3^1 47^1$	Y	N	-16	0	1.0000000	0.471631	0.528369	-80	1317	-1397
283	283^1	Y	Y	-2	0	1.0000000	0.469965	0.530035	-82	1317	-1399
284	$2^2 71^1$	N	N	-7	2	1.2857143	0.468310	0.531690	-89	1317	-1406
285	$3^1 5^1 19^1$	Y	N	-16	0	1.0000000	0.466667	0.533333	-105	1317	-1422
286	$2^1 11^1 13^1$	Y	N	-16	0	1.0000000	0.465035	0.534965	-121	1317	-1438
287	$7^1 41^1$	Y	N	5	0	1.0000000	0.466899	0.533101	-116	1322	-1438
288	$2^5 3^2$	N	N	-47	42	1.7659574	0.465278	0.534722	-163	1322	-1485
289	17^2	N	Y	2	0	1.5000000	0.467128	0.532872	-161	1324	-1485
290	$2^1 5^1 29^1$	Y	N	-16	0	1.0000000	0.465517	0.534483	-177	1324	-1501
291	$3^1 97^1$	Y	N	5	0	1.0000000	0.467354	0.532646	-172	1329	-1501
292	$2^2 73^1$	N	N	-7	2	1.2857143	0.465753	0.534247	-179	1329	-1508
293	293^1	Y	Y	-2	0	1.0000000	0.464164	0.535836	-181	1329	-1510
294	$2^1 3^1 7^2$	N	N	30	14	1.1666667	0.465986	0.534014	-151	1359	-1510
295	$5^1 59^1$	Y	N	5	0	1.0000000	0.467797	0.532203	-146	1364	-1510
296	$2^3 37^1$	N	N	9	4	1.5555556	0.469595	0.530405	-137	1373	-1510
297	$3^3 11^1$	N	N	9	4	1.5555556	0.471380	0.528620	-128	1382	-1510
298	$2^1 149^1$	Y	N	5	0	1.0000000	0.473154	0.526846	-123	1387	-1510
299	$13^1 23^1$	Y	N	5	0	1.0000000	0.474916	0.525084	-118	1392	-1510
300	$2^2 3^1 5^2$	N	N	-74	58	1.2162162	0.473333	0.526667	-192	1392	-1584
301	$7^1 43^1$	Y	N	5	0	1.0000000	0.475083	0.524917	-187	1397	-1584
302	$2^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	-182	1402	-1584
303	$3^1 101^1$	Y	N	5	0	1.0000000	0.478548	0.521452	-177	1407	-1584
304	$2^4 19^1$	N	N	-11	6	1.8181818	0.476974	0.523026	-188	1407	-1595
305	$5^1 61^1$	Y	N	5	0	1.0000000	0.478689	0.521311	-183	1412	-1595
306	$2^1 3^2 17^1$	N	N	30	14	1.1666667	0.480392	0.519608	-153	1442	-1595
307	307^1	Y	Y	-2	0	1.0000000	0.478827	0.521173	-155	1442	-1597
308	$2^2 7^1 11^1$	N	N	30	14	1.1666667	0.480519	0.519481	-125	1472	-1597
309	$3^1 103^1$	Y	N	5	0	1.0000000	0.482201	0.517799	-120	1477	-1597
310	$2^1 5^1 31^1$	Y	N	-16	0	1.0000000	0.480645	0.519355	-136	1477	-1613
311	311^1	Y	Y	-2	0	1.0000000	0.479100	0.520900	-138	1477	-1615
312	$2^3 3^1 13^1$	N	N	-48	32	1.3333333	0.477564	0.522436	-186	1477	-1663
313	313^1	Y	Y	-2	0	1.0000000	0.476038	0.523962	-188	1477	-1665
314	$2^1 157^1$	Y	N	5	0	1.0000000	0.477707	0.522293	-183	1482	-1665
315	$3^2 5^1 7^1$	N	N	30	14	1.1666667	0.479365	0.520635	-153	1512	-1665
316	$2^2 79^1$	N	N	-7	2	1.2857143	0.477848	0.522152	-160	1512	-1672
317	317^1	Y	Y	-2	0	1.0000000	0.476341	0.523659	-162	1512	-1674
318	$2^1 3^1 53^1$	Y	N	-16	0	1.0000000	0.474843	0.525157	-178	1512	-1690
319	$11^1 29^1$	Y	N	5	0	1.0000000	0.476489	0.523511	-173	1517	-1690
320	$2^6 5^1$	N	N	-15	10	2.3333333	0.475000	0.525000	-188	1517	-1705
321	$3^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	-183	1522	-1705
322	$2^1 7^1 23^1$	Y	N	-16	0	1.0000000	0.475155	0.524845	-199	1522	-1721
323	$17^1 19^1$	Y	N	5	0	1.0000000	0.476780	0.523220	-194	1527	-1721
324	$2^2 3^4$	N	N	34	29	1.6176471	0.478395	0.521605	-160	1561	-1721
325	$5^2 13^1$	N	N	-7	2	1.2857143	0.476923	0.523077	-167	1561	-1728
326	$2^1 163^1$	Y	N	5	0	1.0000000	0.478528	0.521472	-162	1566	-1728
327	$3^1 109^1$	Y	N	5	0	1.0000000	0.480122	0.519878	-157	1571	-1728
328	$2^3 41^1$	N	N	9	4	1.5555556	0.481707	0.518293	-148	1580	-1728
329	$7^1 47^1$	Y	N	5	0	1.0000000	0.483283	0.516717	-143	1585	-1728
330	$2^1 3^1 5^1 11^1$	Y	N	65	0	1.0000000	0.484848	0.515152	-78	1650	-1728
331	331^1	Y	Y	-2	0	1.0000000	0.483384	0.516616	-80	1650	-1730
332	$2^2 83^1$	N	N	-7	2	1.2857143	0.481928	0.518072	-87	1650	-1737
333	$3^2 37^1$	N	N	-7	2	1.2857143	0.480480	0.519520	-94	1650	-1744
334	$2^1 167^1$	Y	N	5	0	1.0000000	0.482036	0.517964	-89	1655	-1744
335	$5^1 67^1$	Y	N	5	0	1.0000000	0.483582	0.516418	-84	1660	-1744
336	$2^4 3^1 7^1$	N	N	70	54	1.5000000	0.485119	0.514881	-14	1730	-1744
337	337^1	Y	Y	-2	0	1.0000000	0.483680	0.516320	-16	1730	-1746
338	$2^1 13^2$	N	N	-7	2	1.2857143	0.482249	0.517751	-23	1730	-1753
339	$3^1 113^1$	Y	N	5	0	1.0000000	0.483776	0.516224	-18	1735	-1753
340	$2^2 5^1 17^1$	N	N	30	14	1.1666667	0.485294	0.514706	12	1765	-1753
341	$11^1 31^1$	Y	N	5	0	1.0000000	0.486804	0.513196	17	1770	-1753
342	$2^1 3^2 19^1$	N	N	30	14	1.1666667	0.488304	0.511696	47	1800	-1753
343	7^3	N	Y	-2	0	2.0000000	0.486880	0.513120	45	1800	-1755
344	$2^3 43^1$	N	N	9	4	1.5555556	0.488372	0.511628	54	1809	-1755
345	$3^1 5^1 23^1$	Y	N	-16	0	1.0000000	0.486957	0.513043	38	1809	-1771
346	$2^1 173^1$	Y	N	5	0	1.0000000	0.488439	0.511561	43	1814	-1771
347	347^1	Y	Y	-2	0	1.0000000	0.487032	0.512968	41	1814	-1773
348	$2^2 3^1 29^1$	N	N	30	14	1.1666667	0.488506	0.511494	71	1844	-1773
349	349^1	Y	Y	-2	0	1.0000000	0.487106	0.512894	69	1844	-1775
350	$2^1 5^2 7^1$	N	N	30	14	1.1666667	0.488571	0.511429	99	1874	-1775

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
351	$3^3 13^1$	N	N	9	4	1.5555556	0.490028	0.509972	108	1883	−1775
352	$2^5 11^1$	N	N	13	8	2.0769231	0.491477	0.508523	121	1896	−1775
353	353^1	Y	Y	−2	0	1.0000000	0.490085	0.509915	119	1896	−1777
354	$2^1 3^1 59^1$	Y	N	−16	0	1.0000000	0.488701	0.511299	103	1896	−1793
355	$5^1 71^1$	Y	N	5	0	1.0000000	0.490141	0.509859	108	1901	−1793
356	$2^2 89^1$	N	N	−7	2	1.2857143	0.488764	0.511236	101	1901	−1800
357	$3^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.487395	0.512605	85	1901	−1816
358	$2^1 179^1$	Y	N	5	0	1.0000000	0.488827	0.511173	90	1906	−1816
359	359^1	Y	Y	−2	0	1.0000000	0.487465	0.512535	88	1906	−1818
360	$2^3 3^2 5^1$	N	N	145	129	1.3034483	0.488889	0.511111	233	2051	−1818
361	19^2	N	Y	2	0	1.5000000	0.490305	0.509695	235	2053	−1818
362	$2^1 181^1$	Y	N	5	0	1.0000000	0.491713	0.508287	240	2058	−1818
363	$3^1 11^2$	N	N	−7	2	1.2857143	0.490358	0.509642	233	2058	−1825
364	$2^2 7^1 13^1$	N	N	30	14	1.1666667	0.491758	0.508242	263	2088	−1825
365	$5^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	268	2093	−1825
366	$2^1 3^1 61^1$	Y	N	−16	0	1.0000000	0.491803	0.508197	252	2093	−1841
367	367^1	Y	Y	−2	0	1.0000000	0.490463	0.509537	250	2093	−1843
368	$2^4 23^1$	N	N	−11	6	1.8181818	0.489130	0.510870	239	2093	−1854
369	$3^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	232	2093	−1861
370	$2^1 5^1 37^1$	Y	N	−16	0	1.0000000	0.486486	0.513514	216	2093	−1877
371	$7^1 53^1$	Y	N	5	0	1.0000000	0.487871	0.512129	221	2098	−1877
372	$2^2 3^1 31^1$	N	N	30	14	1.1666667	0.489247	0.510753	251	2128	−1877
373	373^1	Y	Y	−2	0	1.0000000	0.487936	0.512064	249	2128	−1879
374	$2^1 11^1 17^1$	Y	N	−16	0	1.0000000	0.486631	0.513369	233	2128	−1895
375	$3^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	242	2137	−1895
376	$2^3 47^1$	N	N	9	4	1.5555556	0.489362	0.510638	251	2146	−1895
377	$13^1 29^1$	Y	N	5	0	1.0000000	0.490716	0.509284	256	2151	−1895
378	$2^1 3^3 7^1$	N	N	−48	32	1.3333333	0.489418	0.510582	208	2151	−1943
379	379^1	Y	Y	−2	0	1.0000000	0.488127	0.511873	206	2151	−1945
380	$2^2 5^1 19^1$	N	N	30	14	1.1666667	0.489474	0.510526	236	2181	−1945
381	$3^1 127^1$	Y	N	5	0	1.0000000	0.490814	0.509186	241	2186	−1945
382	$2^1 191^1$	Y	N	5	0	1.0000000	0.492147	0.507853	246	2191	−1945
383	383^1	Y	Y	−2	0	1.0000000	0.490862	0.509138	244	2191	−1947
384	$2^7 3^1$	N	N	17	12	2.5882353	0.492188	0.507812	261	2208	−1947
385	$5^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.490909	0.509091	245	2208	−1963
386	$2^1 193^1$	Y	N	5	0	1.0000000	0.492228	0.507772	250	2213	−1963
387	$3^2 43^1$	N	N	−7	2	1.2857143	0.490956	0.509044	243	2213	−1970
388	$2^2 97^1$	N	N	−7	2	1.2857143	0.489691	0.510309	236	2213	−1977
389	389^1	Y	Y	−2	0	1.0000000	0.488432	0.511568	234	2213	−1979
390	$2^1 3^1 5^1 13^1$	Y	N	65	0	1.0000000	0.489744	0.510256	299	2278	−1979
391	$17^1 23^1$	Y	N	5	0	1.0000000	0.491049	0.508951	304	2283	−1979
392	$2^3 7^2$	N	N	−23	18	1.4782609	0.489796	0.510204	281	2283	−2002
393	$3^1 131^1$	Y	N	5	0	1.0000000	0.491094	0.508906	286	2288	−2002
394	$2^1 197^1$	Y	N	5	0	1.0000000	0.492386	0.507614	291	2293	−2002
395	$5^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	296	2298	−2002
396	$2^2 3^2 11^1$	N	N	−74	58	1.2162162	0.492424	0.507576	222	2298	−2076
397	397^1	Y	Y	−2	0	1.0000000	0.491184	0.508816	220	2298	−2078
398	$2^1 199^1$	Y	N	5	0	1.0000000	0.492462	0.507538	225	2303	−2078
399	$3^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.491228	0.508772	209	2303	−2094
400	$2^4 5^2$	N	N	34	29	1.6176471	0.492500	0.507500	243	2337	−2094
401	401^1	Y	Y	−2	0	1.0000000	0.491272	0.508728	241	2337	−2096
402	$2^1 3^1 67^1$	Y	N	−16	0	1.0000000	0.490050	0.509950	225	2337	−2112
403	$13^1 31^1$	Y	N	5	0	1.0000000	0.491315	0.508685	230	2342	−2112
404	$2^2 101^1$	N	N	−7	2	1.2857143	0.490099	0.509901	223	2342	−2119
405	$3^4 5^1$	N	N	−11	6	1.8181818	0.488889	0.511111	212	2342	−2130
406	$2^1 7^1 29^1$	Y	N	−16	0	1.0000000	0.487685	0.512315	196	2342	−2146
407	$11^1 37^1$	Y	N	5	0	1.0000000	0.488943	0.511057	201	2347	−2146
408	$2^3 3^1 17^1$	N	N	−48	32	1.3333333	0.487745	0.512255	153	2347	−2194
409	409^1	Y	Y	−2	0	1.0000000	0.486553	0.513447	151	2347	−2196
410	$2^1 5^1 41^1$	Y	N	−16	0	1.0000000	0.485366	0.514634	135	2347	−2212
411	$3^1 137^1$	Y	N	5	0	1.0000000	0.486618	0.513382	140	2352	−2212
412	$2^2 103^1$	N	N	−7	2	1.2857143	0.485437	0.514563	133	2352	−2219
413	$7^1 59^1$	Y	N	5	0	1.0000000	0.486683	0.513317	138	2357	−2219
414	$2^1 3^2 23^1$	N	N	30	14	1.1666667	0.487923	0.512077	168	2387	−2219
415	$5^1 83^1$	Y	N	5	0	1.0000000	0.489157	0.510843	173	2392	−2219
416	$2^5 13^1$	N	N	13	8	2.0769231	0.490385	0.509615	186	2405	−2219
417	$3^1 139^1$	Y	N	5	0	1.0000000	0.491607	0.508393	191	2410	−2219
418	$2^1 11^1 19^1$	Y	N	−16	0	1.0000000	0.490431	0.509569	175	2410	−2235
419	419^1	Y	Y	−2	0	1.0000000	0.489260	0.510740	173	2410	−2237
420	$2^2 3^1 5^1 7^1$	N	N	−155	90	1.1032258	0.488095	0.511905	18	2410	−2392
421	421^1	Y	Y	−2	0	1.0000000	0.486936	0.513064	16	2410	−2394
422	$2^1 211^1$	Y	N	5	0	1.0000000	0.488152	0.511848	21	2415	−2394
423	$3^2 47^1$	N	N	−7	2	1.2857143	0.486998	0.513002	14	2415	−2401
424	$2^3 53^1$	N	N	9	4	1.5555556	0.488208	0.511792	23	2424	−2401
425	$5^2 17^1$	N	N	−7	2	1.2857143	0.487059	0.512941	16	2424	−2408

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
426	$2^1 3^1 71^1$	Y	N	−16	0	1.0000000	0.485915	0.514085	0	2424	−2424
427	$7^1 61^1$	Y	N	5	0	1.0000000	0.487119	0.512881	5	2429	−2424
428	$2^2 107^1$	N	N	−7	2	1.2857143	0.485981	0.514019	−2	2429	−2431
429	$3^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	−18	2429	−2447
430	$2^1 5^1 43^1$	Y	N	−16	0	1.0000000	0.483721	0.516279	−34	2429	−2463
431	431^1	Y	Y	−2	0	1.0000000	0.482599	0.517401	−36	2429	−2465
432	$2^4 3^3$	N	N	−80	75	1.5625000	0.481481	0.518519	−116	2429	−2545
433	433^1	Y	Y	−2	0	1.0000000	0.480370	0.519630	−118	2429	−2547
434	$2^1 7^1 31^1$	Y	N	−16	0	1.0000000	0.479263	0.520737	−134	2429	−2563
435	$3^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.478161	0.521839	−150	2429	−2579
436	$2^2 109^1$	N	N	−7	2	1.2857143	0.477064	0.522936	−157	2429	−2586
437	$19^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	−152	2434	−2586
438	$2^1 3^1 73^1$	Y	N	−16	0	1.0000000	0.477169	0.522831	−168	2434	−2602
439	439^1	Y	Y	−2	0	1.0000000	0.476082	0.523918	−170	2434	−2604
440	$2^3 5^1 11^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−218	2434	−2652
441	$3^2 7^2$	N	N	14	9	1.3571429	0.476190	0.523810	−204	2448	−2652
442	$2^1 13^1 17^1$	Y	N	−16	0	1.0000000	0.475113	0.524887	−220	2448	−2668
443	443^1	Y	Y	−2	0	1.0000000	0.474041	0.525959	−222	2448	−2670
444	$2^2 3^1 37^1$	N	N	30	14	1.1666667	0.475225	0.524775	−192	2478	−2670
445	$5^1 89^1$	Y	N	5	0	1.0000000	0.476404	0.523596	−187	2483	−2670
446	$2^1 223^1$	Y	N	5	0	1.0000000	0.477578	0.522422	−182	2488	−2670
447	$3^1 149^1$	Y	N	5	0	1.0000000	0.478747	0.521253	−177	2493	−2670
448	$2^6 7^1$	N	N	−15	10	2.3333333	0.477679	0.522321	−192	2493	−2685
449	449^1	Y	Y	−2	0	1.0000000	0.476615	0.523385	−194	2493	−2687
450	$2^1 3^2 5^2$	N	N	−74	58	1.2162162	0.475556	0.524444	−268	2493	−2761
451	$11^1 41^1$	Y	N	5	0	1.0000000	0.476718	0.523282	−263	2498	−2761
452	$2^2 113^1$	N	N	−7	2	1.2857143	0.475664	0.524336	−270	2498	−2768
453	$3^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−265	2503	−2768
454	$2^1 227^1$	Y	N	5	0	1.0000000	0.477974	0.522026	−260	2508	−2768
455	$5^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476923	0.523077	−276	2508	−2784
456	$2^3 3^1 19^1$	N	N	−48	32	1.3333333	0.475877	0.524123	−324	2508	−2832
457	457^1	Y	Y	−2	0	1.0000000	0.474836	0.525164	−326	2508	−2834
458	$2^1 229^1$	Y	N	5	0	1.0000000	0.475983	0.524017	−321	2513	−2834
459	$3^3 17^1$	N	N	9	4	1.5555556	0.477124	0.522876	−312	2522	−2834
460	$2^2 5^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−282	2552	−2834
461	461^1	Y	Y	−2	0	1.0000000	0.477223	0.522777	−284	2552	−2836
462	$2^1 3^1 7^1 11^1$	Y	N	65	0	1.0000000	0.478355	0.521645	−219	2617	−2836
463	463^1	Y	Y	−2	0	1.0000000	0.477322	0.522678	−221	2617	−2838
464	$2^4 29^1$	N	N	−11	6	1.8181818	0.476293	0.523707	−232	2617	−2849
465	$3^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.475269	0.524731	−248	2617	−2865
466	$2^1 233^1$	Y	N	5	0	1.0000000	0.476395	0.523605	−243	2622	−2865
467	467^1	Y	Y	−2	0	1.0000000	0.475375	0.524625	−245	2622	−2867
468	$2^2 3^2 13^1$	N	N	−74	58	1.2162162	0.474359	0.525641	−319	2622	−2941
469	$7^1 67^1$	Y	N	5	0	1.0000000	0.475480	0.524520	−314	2627	−2941
470	$2^1 5^1 47^1$	Y	N	−16	0	1.0000000	0.474468	0.525532	−330	2627	−2957
471	$3^1 157^1$	Y	N	5	0	1.0000000	0.475584	0.524416	−325	2632	−2957
472	$2^3 59^1$	N	N	9	4	1.5555556	0.476695	0.523305	−316	2641	−2957
473	$11^1 43^1$	Y	N	5	0	1.0000000	0.477801	0.522199	−311	2646	−2957
474	$2^1 3^1 79^1$	Y	N	−16	0	1.0000000	0.476793	0.523207	−327	2646	−2973
475	$5^2 19^1$	N	N	−7	2	1.2857143	0.475789	0.524211	−334	2646	−2980
476	$2^2 7^1 17^1$	N	N	30	14	1.1666667	0.476891	0.523109	−304	2676	−2980
477	$3^2 53^1$	N	N	−7	2	1.2857143	0.475891	0.524109	−311	2676	−2987
478	$2^1 239^1$	Y	N	5	0	1.0000000	0.476987	0.523013	−306	2681	−2987
479	479^1	Y	Y	−2	0	1.0000000	0.475992	0.524008	−308	2681	−2989
480	$2^5 3^1 5^1$	N	N	−96	80	1.6666667	0.475000	0.525000	−404	2681	−3085
481	$13^1 37^1$	Y	N	5	0	1.0000000	0.476091	0.523909	−399	2686	−3085
482	$2^1 241^1$	Y	N	5	0	1.0000000	0.477178	0.522822	−394	2691	−3085
483	$3^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−410	2691	−3101
484	$2^2 11^2$	N	N	14	9	1.3571429	0.477273	0.522727	−396	2705	−3101
485	$5^1 97^1$	Y	N	5	0	1.0000000	0.478351	0.521649	−391	2710	−3101
486	$2^1 3^5$	N	N	13	8	2.0769231	0.479424	0.520576	−378	2723	−3101
487	487^1	Y	Y	−2	0	1.0000000	0.478439	0.521561	−380	2723	−3103
488	$2^3 61^1$	N	N	9	4	1.5555556	0.479508	0.520492	−371	2732	−3103
489	$3^1 163^1$	Y	N	5	0	1.0000000	0.480573	0.519427	−366	2737	−3103
490	$2^1 5^1 7^2$	N	N	30	14	1.1666667	0.481633	0.518367	−336	2767	−3103
491	491^1	Y	Y	−2	0	1.0000000	0.480652	0.519348	−338	2767	−3105
492	$2^2 3^1 41^1$	N	N	30	14	1.1666667	0.481707	0.518293	−308	2797	−3105
493	$17^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	−303	2802	−3105
494	$2^1 13^1 19^1$	Y	N	−16	0	1.0000000	0.481781	0.518219	−319	2802	−3121
495	$3^2 5^1 11^1$	N	N	30	14	1.1666667	0.482828	0.517172	−289	2832	−3121
496	$2^4 31^1$	N	N	−11	6	1.8181818	0.481855	0.518145	−300	2832	−3132
497	$7^1 71^1$	Y	N	5	0	1.0000000	0.482897	0.517103	−295	2837	−3132
498	$2^1 3^1 83^1$	Y	N	−16	0	1.0000000	0.481928	0.518072	−311	2837	−3148
499	499^1	Y	Y	−2	0	1.0000000	0.480962	0.519038	−313	2837	−3150
500	$2^2 5^3$	N	N	−23	18	1.4782609	0.480000	0.520000	−336	2837	−3173