Characterizations of partial sums of the Möbius function by signed sums of additively structured auxiliary unsigned sequences

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Abstract

The Mertens function, $M(x) \coloneqq \sum_{n \leq x} \mu(n)$, is defined as the summatory function of the classical Möbius function for $x \geq 1$. The inverse function $g^{-1}(n) \coloneqq (\omega+1)^{-1}(n)$ taken with respect to Dirichlet convolution is defined in terms of the strongly additive function $\omega(n)$ that counts the number of distinct prime factors of the integers $n \geq 2$ without multiplicity. For large x and $n \leq x$, we associate a natural combinatorial significance to the magnitude of the distinct values of $|g^{-1}(n)|$ that depends directly on the exponent patterns in the prime factorizations of the integers $2 \leq n \leq x$ viewed as multisets. That is, the distinct values of the unsigned inverse function are repeated at any $n \geq 2$ with the precise additive configuration of the exponents in the prime factorization of n regardless of the multiplicative products of primes that serve as the placeholders for the exponents of the distinct prime factors. We conjecture two forms of deterministic Erdős-Kac theorem analogs that characterize the distributions of each of the unsigned sequences

$$C_{\Omega}(n) \coloneqq (\Omega(n))! \times \prod_{p^{\alpha} \| n} \frac{1}{\alpha!}, n \geq 2,$$

and $|g^{-1}(n)|$ over $n \leq x$ as $x \to \infty$. Discrete convolutions of the partial sums

$$G^{-1}(x) \coloneqq \sum_{n \leq x} \lambda(n) |g^{-1}(n)|,$$

with the prime counting function $\pi(x)$ determine exact formulas and new characterizations of the asymptotic behavior of $M(x)$. In this way, we prove another characteristic link of the Mertens function to the distribution of the partial sums $L(x) \coloneqq \sum_{n \leq x} \lambda(n)$ and connect these two classical summatory functions with explicit non-centrally normal probability distributions at large x.

Keywords and Phrases: *Möbius function; Mertens function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function; prime zeta function; Erdős-Kac theorem; strongly additive function.*

Math Subject Classifications (2010): *11N37; 11A25; 11N60; 11N64; and 11-04.*

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1 Introduction

1.1 Definitions and preliminary results on the Mertens function

The *Mertens function* is the summatory function of $\mu(n)$ defined for any positive integer $x \geq 1$ by the partial sums [28, A008683; A002321]

$$M(x) = \sum_{n \leq x} \mu(n), \text{ for } x \geq 1.$$

The Mertens function is related to the partial sums of the Liouville lambda function, denoted by $L(x) := \sum_{n \leq x} \lambda(n)$, via the relation [11, 17] [28, A008836; A002819]

$$L(x) = \sum_{d \leq \sqrt{x}} M\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), \text{ for } x \geq 1.$$

Stating tight bounds on the properties of the distribution of $L(x)$ is still viewed as a problem that is equally as difficult as understanding the properties of $M(x)$ well at large x or along infinite subsequences.

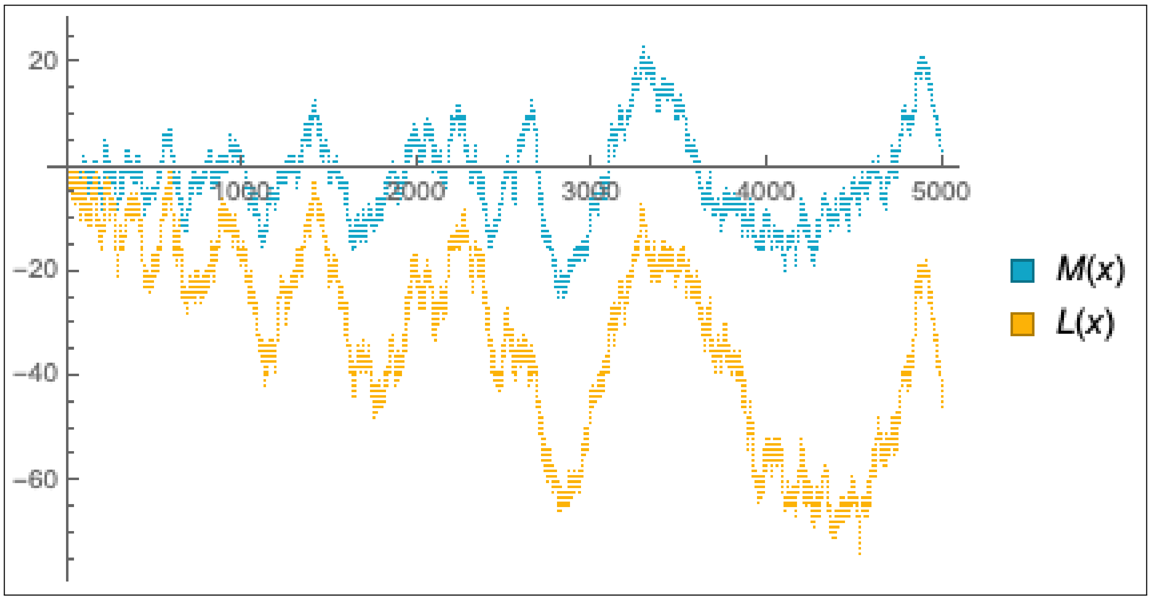


Figure 1.1: *A comparison of the values of the summatory functions $M(x)$ and $L(x)$ at integers $1 \leq x \leq 5000$.*

An approach to evaluating the behavior of $M(x)$ for large $x \to \infty$ considers an inverse Mellin transform of the reciprocal of the Riemann zeta function given by

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \times \int_1^\infty \frac{M(x)}{x^{s+1}} dx, \text{ for } \operatorname{Re}(s) > 1.$$

We then obtain the following contour integral representation of $M(x)$ for $x \geq 1$:

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \times \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s\zeta(s)} ds.$$

The previous formulas lead to the exact expression of $M(x)$ for any $x > 0$ given by the next theorem.

Theorem 1.1 (Titchmarsh)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each integer $k \geq 1$ such that for any real $x > 0$*

$$M(x) = \lim_{k \to \infty} \sum_{\substack{\rho : \zeta(\rho) = 0 \\ 0 < |\operatorname{Im}(\rho)| < T_k}} \frac{x^\rho}{\rho \zeta'(\rho)} + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n(2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} [x \in \mathbb{Z}^+]_\delta - 2.$$

An unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C_1 > 0$ such that

$$M(x) \ll x \times \exp\left(-C_1 \log^{\frac{3}{5}}(x)(\log\log x)^{-\frac{1}{5}}\right).$$

Under the assumption of the RH, Soundararajan and Humphries, respectively, improved estimates bounding $M(x)$ from above for large x in the following forms [29, 11]:

$$M(x) \ll \sqrt{x} \times \exp\left(\sqrt{\log x}(\log\log x)^{14}\right),$$

$$M(x) \ll \sqrt{x} \times \exp\left(\sqrt{\log x}(\log\log x)^{\frac{5}{2}+\epsilon}\right), \text{ for all } \epsilon > 0.$$

The RH is equivalent to showing that

$$M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right), \text{ for all } 0 < \epsilon < \frac{1}{2}. \tag{1.1}$$

There is a rich history to the original statement of the *Mertens conjecture* which asserts that $|M(x)| < C_2\sqrt{x}$ for some absolute constant $C_2 > 0$. The conjecture was first verified by F. Mertens himself for $C_2 = 1$ and all $x < 10^4$ without the benefit of modern computation. Since its beginnings in 1897, the Mertens conjecture was disproved by computational methods involving non-trivial simple zeta function zeros with comparatively small imaginary parts in the famous paper by Odlyzko and te Riele [23].

More recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $M(x)x^{-\frac{1}{2}}$ grows with or without bound along infinite subsequences, e.g., considering the asymptotics of the function in the limit supremum and limit infimum senses. It is verified by computation that [26, *cf.* §4.1] [28, *cf.* A051400; A051401]

$$\overline{L} := \limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{more recently } \overline{L} \geq 1.826054),$$

and

$$\underline{L} := \liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{more recently } \underline{L} \leq -1.837625).$$

Computational tractability has so far been a significant barrier to proving better bounds on these two limiting quantities on modern computers. Based on the work by Odlyzko and te Riele (circa 1985), it is still widely believed that these limiting bounds evaluate to $\pm\infty$, respectively [23, 15, 16, 12]. A conjecture due to Gonek asserts that in fact $M(x)$ satisfies [22]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}(\log\log\log x)^{\frac{5}{4}}} = C_3,$$

for $C_3 > 0$ an absolute constant.

1.2 New characterizations of $M(x)$ by partial sums of auxillary unsigned functions

The main interpretation to take away from the article is the characterization of $M(x)$ through two primary new auxiliary unsigned sequences and their summatory functions, namely, the functions $C_\Omega(n)$ and $|g^{-1}(n)|$, and their partial sums. We fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by

$$g^{-1}(n) := (\omega + 1\!\!1)^{-1}(n), \text{ for } n \geq 1. \tag{1.2}$$

The Dirichlet inverse function defined in (1.2) exists and is unique because $g(1) = 1 \neq 0$. We compute the first several values of this signed inverse sequence as follows [28, A341444]:

$$\{g^{-1}(n)\}_{n\geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

An exact expression for $g^{-1}(n)$ is given by (see Lemma 3.2; and Corollary 3.3)

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_\Omega(d), n \geq 1, \tag{1.3}$$

where the sequence $\lambda(n)C_\Omega(n)$ has the DGF $(1+P(s))^{-1}$ and $C_\Omega(n)$ has DGF $(1-P(s))^{-1}$ for $\mathrm{Re}(s) > 1$ (see Proposition 2.3). The function $C_\Omega(n)$ was considered in [9] with an exact formula given by [13, *cf.* §3]

$$C_\Omega(n) = \begin{cases} 1, & \text{if } n = 1; \\ (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}, & \text{if } n \geq 2. \end{cases}$$

Let $\chi_{\mathbb{P}}(n)$ denote the characteristic function of the primes, let $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n (without multiplicity). We can see using elementary methods that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + \mathbb{1}) * \mu. \tag{1.4}$$

Namely, since $\mu * 1 = \varepsilon$ and

$$\omega(n) = \sum_{p|n} 1 = \sum_{d|n} \chi_{\mathbb{P}}(d), \text{ for } n \geq 1,$$

the result in (1.4) follows by Möbius inversion. The shift by the constant one in the form of $(\omega + \mathbb{1}) * \mu$ in the right-hand-side convolution in (1.4) is selected so that the resulting arithmetic function we convolve with $\mu(n)$ in constructing these summatory functions is Dirichlet invertible with $(\omega + \mathbb{1})(1) \neq 0$ where $\omega(1) := 0$ (by convention).

Based on the convolution identity given in (1.4) above, we can prove the next formulas in (1.6) as special cases of Theorem 2.1 stated in Section 2.1 (see Corollary 2.2). We define the partial sums $G^{-1}(x)$ for integers $x \geq 1$ as follows [28, A341472]:

$$G^{-1}(x) := \sum_{n \leq x} g^{-1}(n) = \sum_{n \leq x} \lambda(n) |g^{-1}(n)|. \tag{1.5}$$

Then we have that for all $x \geq 1$ (see Proposition 6.1; *cf.* Section 3.2)

$$M(x) = \sum_{1 \leq k \leq x} g^{-1}(k) \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right), \tag{1.6a}$$

$$M(x) = G^{-1}(x) + \sum_{1 \leq k \leq \frac{x}{2}} G^{-1}(k) \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right), \tag{1.6b}$$

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right). \tag{1.6c}$$

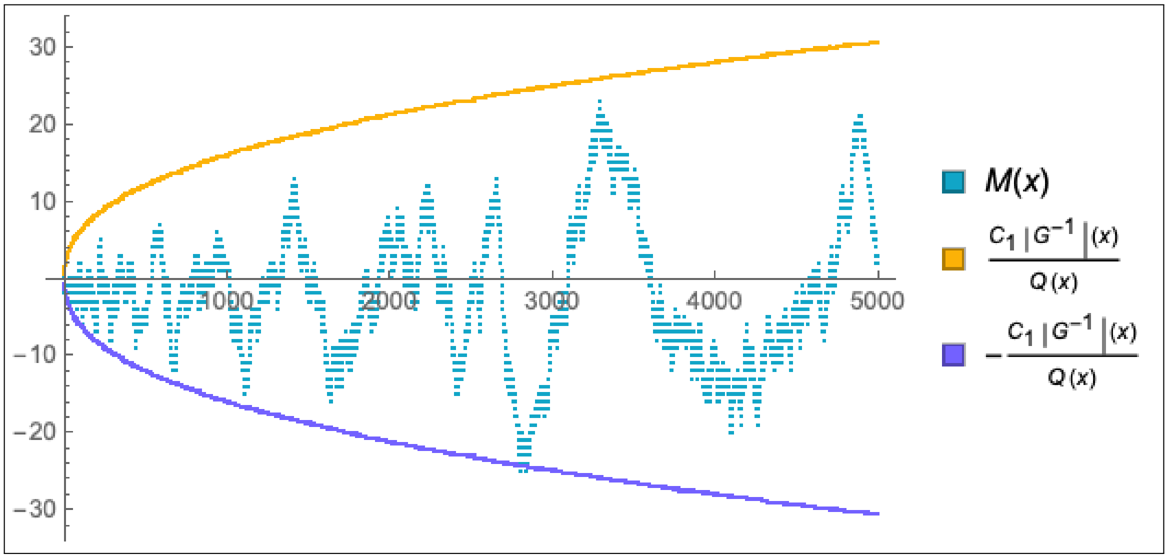
We have an identification of $G^{-1}(x)$ with the summatory function $L(x)$ given by

$$G^{-1}(x) = L(x)|g^{-1}(x)| - \sum_{n<x} L(n) \left(|g^{-1}(n+1)| - |g^{-1}(n)|\right).$$

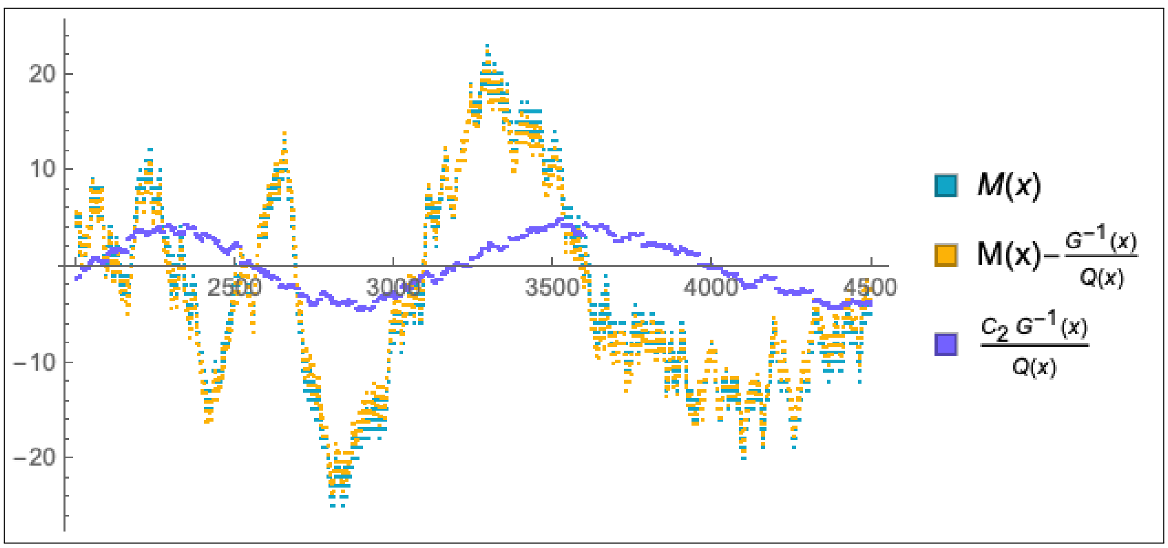
We expect substantial local cancellation in the terms involving $G^{-1}(x)$ in our new formulas for $M(x)$ at almost every large x (see Section 6.2). Suppose that the unsigned partial sums are defined for $x \geq 1$ by

$$|G^{-1}|(x) := \sum_{n \leq x} |g^{-1}(n)|. \tag{1.7}$$

Bounds on the partial sums over the unsigned inverse functions in (1.7) provide some local information on $G^{-1}(x)$ through its connection with $|G^{-1}|(x)$. The plots shown in Figure 1.2 and Figure 1.3 compare the



(a) *Bounded envelopes for the local extremum of $M(x)$ expressed in terms of the partial sums of the unsigned inverse function. The value of the scaling factor C_1 is chosen to be the absolute constant $C_1 := \frac{1}{\zeta(2)}$.*



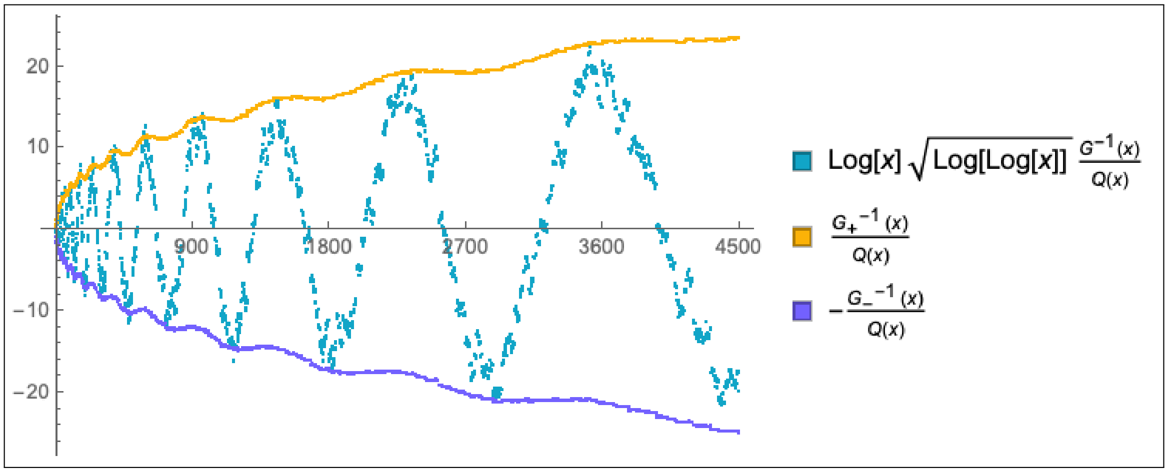
(b) *The displayed jagged plots show that the difference between $M(x)$ and $G^{-1}(x)$ is usually small (cf. (1.6c)). The third more smoothly oscillatory plot shown in purple provides a baseline of the scaled values of $G^{-1}(x)$ where the absolute constant $C_2^{-1} := 1 - \frac{1}{\zeta(2)}$ is arbitrarily chosen so as to keep a display scale that is commensurate with the first two plots.*

Figure 1.2: *Discrete plots displaying comparisons of the growth of $M(x)$ to the new auxillary partial sums for $x \leq 5000$. The scaling function $Q(x) := \sum_{n \leq x} \mu^2(n)$ counts the number of squarefree integers $n \leq x$ for any $x \geq 1$. Numerical computation suggests that this function is a natural scaling factor to relate the growth of the new partial sums to $M(x)$.*

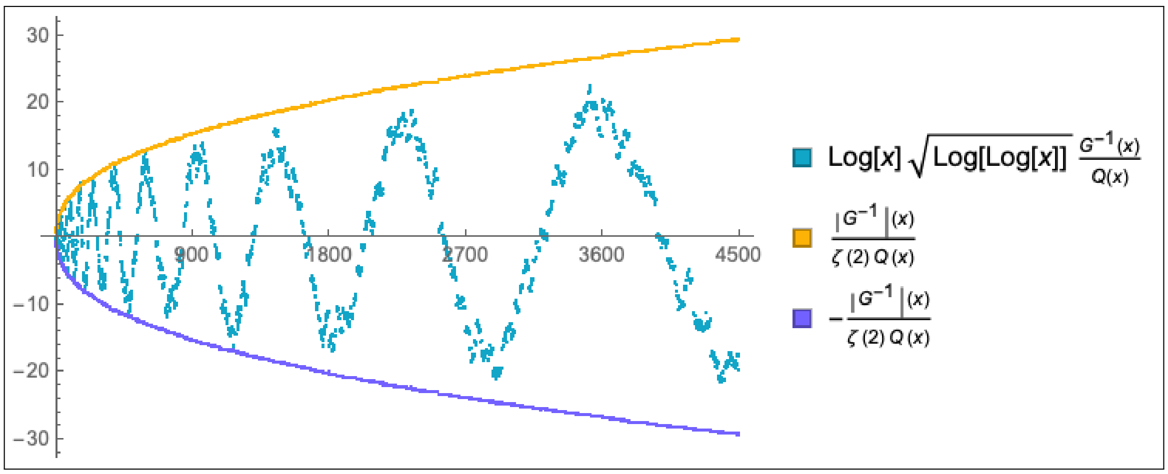
values of $M(x)$ and $G^{-1}(x)$ with scaled forms of auxilliary partial sums related to the expansion of the latter summatory function.

There is not a simple direct recursion between the distinct values of $g^{-1}(n)$ that holds for all $n \geq 1$. Nonetheless, the next observation is suggestive of the quasi-periodicity of the distribution of distinct values of this inverse function over $n \geq 2$.

Observation 1.2 (Additive symmetry in $g^{-1}(n)$ from the prime factorizations of $n \leq x$)**.** Suppose that



(a) *Comparisons of a logarithmically scaled form of $G^{-1}(x)$ and envelopes that bound its local extremum given by sign-weighted components that contribute to these partial sums. Namely, we define $G^{-1}(x) \coloneqq G_+^{-1}(x) + G_-^{-1}(x)$ where the functions $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geq 1$ so that these signed component functions denote the unsigned contributions of only those summands $g^{-1}(n)$ for $n \leq x$ such that $\lambda(n) = \pm 1$, respectively.*



(b) *Comparisions of bounded envelopes for the local extremum of the logarithmically scaled values of $G^{-1}(x)$ to the absolute values of the partial sums of the scaled unsigned inverse function multiplied by $\zeta(2)^{-1}$ to show the relative bounds in the displays.*

Figure 1.3: *Discrete plots displaying comparisons of the growth of $M(x)$ to the new auxillary partial sums for $x \leq 5000$. The scaling function $Q(x) \coloneqq \sum_{n \leq x} \mu^2(n)$ counts the number of squarefree integers $n \leq x$ for any $x \geq 1$.*

$n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \times \cdots \times p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \times \cdots \times q_s^{\beta_s}$. If $r = s$ and $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets (i.e., taking into account the multiplicity of distinct elements in each set) of the prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, $g^{-1}(n)$ has the same values on the squarefree integers n with exactly one, two, three (and so on) prime factors, or for example at all n of the form $n = p_1 p_2^2 p_3^4$ when p_1, p_2 and p_3 are distinct primes. Hence, there is an essentially additive structure underneath the sequence $\{g^{-1}(n)\}_{n \geq 2}$.

The results in Proposition 1.3 below proves the sign of $g^{-1}(n)$ for all $n \geq 1$ (see Proposition 2.3) and provides bounds on the extremum of the unsigned inverse function with respect to the local values of maximal $k \coloneqq \Omega(n)$ for $n \leq x$. A proof of (B) follows from Lemma 3.2 by a simple counting argument when all factors $d|n$ are formed by squarefree products of the $\omega(n)$ distinct primes $p|n$. The realization that the beautiful and remarkably simple combinatorial form of property (B) in Proposition 1.3 holds for all squarefree integers motivates our pursuit of simpler formulas for the inverse function $g^{-1}(n)$ through the sums of the unsigned auxiliary sequence $C_{\Omega}(n)$, that is defined in Section 3.

Proposition 1.3. *We have the following properties of the Dirichlet inverse function $g^{-1}(n)$:*

(A) For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;

(B) For squarefree integers $n \geq 1$, we have that

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \times m!;$$

(C) If $n \geq 2$ and $\Omega(n) = k$ for some $k \geq 1$, then

$$2 \leq |g^{-1}(n)| \leq \sum_{j=0}^{k} \binom{k}{j} \times j!.$$

The function $C_{\Omega}(n)$ identified as a key auxiliary sequence in the explicit formula from (1.3) is considered under alternate notation by Fröberg (circa 1968) in his work on the series expansions of the *prime zeta function*, $P(s)$, i.e., the prime sums defined as the Dirichlet generating function (or DGF) of $\chi_{\mathbb{P}}(n)$. The clear connection of the function $C_{\Omega}(n)$ to $M(x)$ is unique to our work to establish the properties of this auxiliary sequence. In Corollary 4.4, we use the result proved in Theorem 4.2 to show that uniformly for $1 \leq k \leq \frac{3}{2}\log\log x$ there is an absolute constant $A_0 > 0$ such that

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega}(n) = \frac{A_0\sqrt{2\pi}x}{\log x} \times \widehat{G}\left(\frac{k-1}{\log\log x}\right) \frac{(\log\log x)^{k-\frac{1}{2}}}{(k-1)!}\left(1 + O\left(\frac{1}{\log\log x}\right)\right), \text{ as } x \to \infty,$$

where $\widehat{G}(z) := \frac{\zeta(2)^{-z}}{\Gamma(1+z)(1+P(2)z)}$ for $0 \leq |z| < P(2)^{-1}$. References to uniform asymptotics for restricted partial sums of $C_{\Omega}(n)$ and the conjectured features of the limiting distribution of this function are missing in surrounding literature (*cf.* Corollary 4.4; Proposition 4.5; and Conjecture 4.6).

In Proposition 4.5, we use an adaptation of the asymptotics for the summations proved in the appendix section of this article combined with the form of *Rankin's method* from [18, Thm. 7.20] to show that there is an absolute constant $B_0 > 0$ such that

$$\frac{1}{n} \times \sum_{k \leq n} C_{\Omega}(k) = B_0(\log n)\sqrt{\log\log n}\left(1 + O\left(\frac{1}{\log\log n}\right)\right), \text{ as } n \to \infty.$$

In Corollary 5.1, we prove that the average order of $|g^{-1}(n)|$ is

$$\frac{1}{n} \times \sum_{k \leq n} |g^{-1}(k)| = \frac{6B_0(\log n)^2\sqrt{\log\log n}}{\pi^2}\left(1 + O\left(\frac{1}{\log\log n}\right)\right), \text{ as } n \to \infty.$$

The next statements provide a complete picture of the distribution of the unsigned inverse sequence whose values are exactly identified by the formulas in Proposition 1.3. Indeed, for $n \geq 1$ off of the squarefree integers, we can characterize the distrubution of the values of $|g^{-1}(n)|$ by the conjectured generalizations of Erdős-Kac theorem type results that lead to Corollary 5.2 (assuming Conjecture 4.6 holds). That is, in Section 4.3, we conjecture the forms of two variants of the Erdős-Kac theorem that characterize the distribution of $C_{\Omega}(n)$. The first proposed deterministic form of the theorem stated in Conjecture 4.6 leads the conclusion of the following result for any fixed $Y > 0$ which holds uniformly for all $-Y \leq y \leq Y$ with $\mu_x(C) := \log\log x - \log\left(\frac{\sqrt{2\pi}A_0}{\zeta(2)(1+P(2))}\right)$ and $\sigma_x(C) := \sqrt{\log\log x}$ (see Corollary 5.2):

$$\begin{aligned}\frac{1}{x}\times\#\left\{3 \leq n \leq x : \frac{|g^{-1}(n)|}{(\log n)\sqrt{\log\log n}} - \frac{6}{\pi^2 n(\log n)\sqrt{\log\log n}} \times \sum_{k\leq n}|g^{-1}(k)| \leq y\right\} \\ = \Phi\left(\frac{\frac{\pi^2 y}{6} - \mu_x(C)}{\sigma_x(C)}\right) + o(1), \text{ as } x \to \infty.\end{aligned}$$

We similarly conjecture that for any real y, as $x \to \infty$

$$\frac{1}{x}\times\#\left\{2 \leq n \leq x : |g^{-1}(n)| - \frac{6}{\pi^2 n}\times\sum_{k\leq n}|g^{-1}(k)| \leq y\right\} = \Phi\left(\frac{\frac{\pi^2 y}{6} - B_0(\log x)\sqrt{\log\log x}}{D_0\sqrt{x}(\log x)\sqrt{\log\log x}}\right) + o(1), \tag{1.8}$$

where $D_0 > 0$ is an absolute constant.

1.3 Significance of the new results and characterizations

The signed inverse sequence $g^{-1}(n)$ and its partial sums defined by $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$ are linked to canonical examples of strongly and completely additive functions, e.g., in relation to $\omega(n)$ and (signed powers of) $\Omega(n)$, respectively. The definitions of the sequences we formulate, and the proof methods given in the spirit of Montgomery and Vaughan's work, allow us to reconcile the property of strong additivity with the signed partial sums of a multiplicative function. We leverage the connection of $C_\Omega(n)$ and $|g^{-1}(n)|$ with additivity to obtain the results proved in Section **??**. We reformulate the proofs of the results in [18, §7.4; §2.4] that apply analytic methods to formulate limiting asymptotics and to prove an Erdős-Kac theorem analog characterizing key properties of the distribution of the completely additive function $\Omega(n)$. Adaptations of the key ideas from the exposition in the reference provide a foundation for analytic proofs of several limiting properties of, asymptotic formulae for restricted partial sums involving, and in part the conjectured deterministic Erdős-Kac type theorems for both $C_\Omega(n)$ and $|g^{-1}(n)|$.

In the process, we also formalize a probabilistic perspective from which to express our intuition about features of the distribution of $G^{-1}(x)$ via the properties of its $\lambda(n)$-sign-weighted summands, $|g^{-1}(n)|$ for $n \leq x$. By Proposition 1.3 and since

$$\lim_{x\to\infty}\frac{(\log\log x)^{\log\log x}}{\sqrt{x}} = 0,$$

we see that if the result in (1.8) holds, the unsigned inverse function $|g^{-1}(n)|$ is centered at its average order for nearly every sufficiently large n with only a negligible error in the approximation. The regularity and quasi-periodicity we alluded to in the previous remarks are then quantifiable insomuch as $|g^{-1}(n)|$ usually tends to a scaled multiple of its average order with a non-central normal tendency (provided that Conjecture 4.6 holds). Hence, the conjectured corollary implies a decidely more regular tendency of the unsigned inverse sequence. In fact, as we see through the lense of the previous remarks, the values of $|g^{-1}(n)|$ for $n \leq x$ at large x are centered at the average order of this unsigned sequence in a way that is only paralleled in the references on the canonically strongly additive number theoretic functions $\omega(n)$ and $\Omega(n)$ (*cf.* Appendix B). Since we prove that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ for all $n \geq 1$ in Proposition 2.3, the partial sums defined by $G^{-1}(x)$ are precisely related to the properties of $|g^{-1}(n)|$ and asymptotics for $L(x)$. Our new results then relate the distribution of $L(x)$, explicitly identified probability distributions, and $M(x)$ as $x \to \infty$. Our characterizations of $M(x)$ by the summatory function of the signed inverse sequence, $G^{-1}(x)$, is hence suggestive of new approaches to bounding the Mertens function. These results motivate future work to state upper (and possibly lower) bounds on $M(x)$ in terms of the additive combinatorial properties of the repeated distinct values of the sign weighted summands of $G^{-1}(x)$.

2 Initial elementary proofs of new results

2.1 Establishing the summatory function properties and inversion identities

We give a proof of the inversion type results in Theorem 2.1 below by matrix methods in this subsection. Related results on summations of Dirichlet convolutions and their inversion appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95].

Theorem 2.1 (Partial sums of Dirichlet convolutions and their inversions)**.** *Let $r, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $r(1) \neq 0$. Suppose that $R(x) := \sum_{n \leq x} r(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of r and h, respectively, and that $R^{-1}(x) := \sum_{n \leq x} r^{-1}(n)$ denotes the summatory function of the Dirichlet inverse of r for any $x \geq 1$. For any $x \geq 1$, let the partial sums of the Dirichlet convolution $r * h$ be defined by*

$$S_{r*h}(x) := \sum_{n \leq x} \sum_{d|n} r(d) h\left(\frac{n}{d}\right).$$

We have that the following exact expressions hold for all integers $x \geq 1$:

$$\begin{aligned} S_{r*h}(x) &= \sum_{d \leq x} r(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ S_{r*h}(x) &= \sum_{k=1}^{x} H(k) \left(R\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - R\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right). \end{aligned}$$

Moreover, for any $x \geq 1$ we have

$$\begin{aligned} H(x) &= \sum_{j=1}^{x} \pi_{r*h}(j) \left(R^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - R^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right) \\ &= \sum_{k=1}^{x} r^{-1}(k) S_{r*h}(x). \end{aligned}$$

Proof of Theorem 2.1. Let h, r be arithmetic functions such that $r(1) \neq 0$. Let the summatory functions of h, r and r^{-1}, respectively, be denoted by $H(x) = \sum_{n \leq x} h(n)$, $R(x) = \sum_{n \leq x} r(n)$, and $R^{-1}(x) = \sum_{n \leq x} r^{-1}(n)$. We define $\pi_{r*h}(x)$ to be the summatory function of the Dirichlet convolution of r with h. The following formulas hold for all $x \geq 1$:

$$\begin{aligned} S_{r*h}(x) := \sum_{n=1}^{x} \sum_{d|n} r(n) h\left(\frac{n}{d}\right) &= \sum_{d=1}^{x} r(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left(R\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - R\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right) H(i). \end{aligned} \tag{2.1}$$

The first formula above is well known from the references cited above. The second formula is justified directly using summation by parts as [24, §2.10(ii)]

$$\begin{aligned} S_{r*h}(x) &= \sum_{d=1}^{x} h(d) R\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i \leq x} \left(\sum_{j \leq i} h(j)\right) \times \left(R\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - R\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right). \end{aligned}$$

We form the invertible matrix of coefficients, denoted by $\hat{R}$ below, associated with the linear system defining $H(j)$ for all $1 \leq j \leq x$ in (2.1) by setting

$$r_{x,j} := R\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - R\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv R_{x,j} - R_{x,j+1},$$

with

$$R_{x,j} := R\left(\left\lfloor \frac{x}{j} \right\rfloor\right), \text{ for } 1 \leq j \leq x.$$

Since $r_{x,x} = R(1) = r(1) \neq 0$ for all $x \geq 1$ and $r_{x,j} = 0$ for all $j > x$, the matrix we have defined in this problem is lower triangular with a non-zero constant on its diagonals, and so is invertible. If we let $\hat{R} := (R_{x,j})$, then the next matrix is expressed by applying an invertible shift operation as

$$(r_{x,j}) = \hat{R}\left(I - U^T\right).$$

The square matrix U of sufficiently large finite dimensions $N \times N$ for $N \geq x$ has $(i,j)^{th}$ entries for all $1 \leq i, j \leq N$ that are defined by $(U)_{i,j} = \delta_{i+1,j}$ so that

$$\left[(I - U^T)^{-1}\right]_{i,j} = [j \leq i]_\delta.$$

We observe that

$$\left\lfloor \frac{x}{j} \right\rfloor - \left\lfloor \frac{x-1}{j} \right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

The previous equation implies that

$$R\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - R\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} r\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

We use the property in (2.2) to shift the matrix $\hat{R}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of r as follows:

$$\left(\left(I - U^T\right)\hat{R}\right)^{-1} = \left(r\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(r^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Our target matrix in the inversion problem is defined by

$$(r_{x,j}) = \left(I - U^T\right)\left(r\left(\frac{x}{j}\right)[j|x]_\delta\right)\left(I - U^T\right)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators in the form of

$$\begin{aligned} (r_{x,j})^{-1} &= \left(I - U^T\right)^{-1}\left(r^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)\left(I - U^T\right) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} r^{-1}(k)\right)\left(I - U^T\right) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} r^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} r^{-1}(k)\right). \end{aligned}$$

The summatory function $H(x)$ is given exactly for any integers $x \geq 1$ by a vector product with the inverse matrix from the previous equation in the form of

$$H(x) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} r^{-1}(j)\right) \times S_{r*h}(k).$$

We can prove a second inversion formula providing the coefficients of the summatory function $R^{-1}(j)$ for $1 \leq j \leq x$ from the last equation by adapting our argument to prove (2.1) above. This leads to the alternate identity expressing $H(x)$ given by

$$H(x) = \sum_{k=1}^{x} r^{-1}(k) \times S_{r*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right). \qquad \square$$

A key consequence of Theorem 2.1 as it applies to $M(x)$ in the special cases where $h(n) := \mu(n)$ for all $n \geq 1$ is stated in the next corollary.

Corollary 2.2 (Applications of Möbius inversion)**.** *Suppose that r is an arithmetic function such that $r(1) \neq 0$. Define the summatory function of the convolution of r with μ by $\widetilde{R}(x) := \sum_{n\leq x}(r * \mu)(n)$. Then the Mertens function is expressed by the partial sums*

$$M(x) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} r^{-1}(j)\right)\widetilde{R}(k), \forall x \geq 1.$$

The formula that provides $\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu$ exactly when evaluated at any $n \geq 1$ from (1.4) implies the key form of (1.6a) and (1.6b) as an immediate consequence of the stated adaptation of the inversion theorem for $M(x)$.

2.2 Proving the characteristic signedness property of $g^{-1}(n)$

Proposition 2.3 (The sign of $g^{-1}(n)$)**.** *For any arithmetic function $r(n)$, let the function*

$$\operatorname{sgn}(r(n)) := \begin{cases} \frac{r(n)}{|r(n)|}, & \text{if } r(n) \neq 0; \\ 0, & \text{otherwise,} \end{cases}$$

mapping onto the set $\{-1, 0, 1\}$ provide the signedness of r at any $n \geq 1$. We have that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ for all $n \geq 1$.

Proof. The series $D_f(s) := \sum_{n\geq 1} f(n)n^{-s}$ defines the Dirichlet generating function (DGF) of any arithmetic function f which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ where σ_f is the abscissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = \zeta(s)^{-1}$ and $D_\omega(s) = P(s)\zeta(s)$ for $\operatorname{Re}(s) > 1$, where $P(s) := \sum_{n\geq 1} \chi_{\mathbb{P}}(n)n^{-s}$ denotes the *prime zeta function* defined as in the glossary on page 30 for any $\operatorname{Re}(s) > 1$ where the series is convergent (*cf.* [9]). By (1.4) and the fact that whenever $f(1) \neq 0$, the DGF of $f^{-1}(n)$ is $D_f(s)^{-1}$ for $\operatorname{Re}(s) > \sigma_f$, we have that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{\zeta(s)(1 + P(s))}, \text{ for } \operatorname{Re}(s) > 1. \tag{2.3}$$

It follows that $(\omega + 1)^{-1}(n) = (h^{-1} * \mu)(n)$ if we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. This observation implies that $\operatorname{sgn}(h^{-1} * \mu) = \lambda$ as we show by the next arguments.

By a combinatorial argument related to multinomial coefficient expansions of these DGFs that are summed over $h^{-1}(n)$ for $n \geq 1$, we recover exactly that [9, *cf.* §2][1]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ \lambda(n)(\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}, & n \geq 2. \end{cases} \tag{2.4}$$

In particular, notice that by expanding the DGF of h^{-1} geometrically in powers of $P(s)$ (where $|P(s)| < 1$ whenever $\operatorname{Re}(s) > 1$) we can count that

$$\frac{1}{1 + P(s)} = \sum_{n\geq 1} \frac{h^{-1}(n)}{n^s} = \sum_{k\geq 0} (-1)^k P(s)^k$$

[1]Beginning in Section 3, we adopt the alternate notation for the Dirichlet inverse function $h^{-1}(n)$ employed in this proof given by $C_\Omega(n)$. See also the remarks following the conclusion of this proof on the function $C_k(n)$.

$$= 1 + \sum_{\substack{n \geq 2 \\ n = p_1^{\alpha_1} p_2^{\alpha_2} \times \cdots \times p_k^{\alpha_k}}} \frac{(-1)^{\alpha_1+\alpha_2+\cdots+\alpha_k}}{n^s} \times \binom{\alpha_1+\alpha_2+\cdots+\alpha_k}{\alpha_1, \alpha_2, \ldots, \alpha_k}$$

$$= 1 + \sum_{\substack{n \geq 2 \\ n = p_1^{\alpha_1} p_2^{\alpha_2} \times \cdots \times p_k^{\alpha_k}}} \frac{\lambda(n)}{n^s} \times \binom{\Omega(n)}{\alpha_1, \alpha_2, \ldots, \alpha_k}.$$

Since λ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$ for all divisors $d|n$ when $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever n is squarefree, so that we obtain the following results:

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1. \qquad \square$$

The conclusion of the proof of Proposition 2.3 implies the stronger result that

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_\Omega(d).$$

We have adopted the notation that for $n \geq 2$, $C_\Omega(n) \equiv |h^{-1}(n)| = (\Omega(n))! \times \prod_{p^\alpha || n} (\alpha!)^{-1}$, where the same function, $C_0(n)$, is taken to be one for $n := 1$ (see Section 3).

3 Auxiliary sequences related to the inverse function $g^{-1}(n)$

The computational data given as Table E in the appendix section is intended to provide clear insight into the significance of the few characteristic formulas for $g^{-1}(n)$ proved in this section. The table provides illustrative numerical data by examining the first cases of $1 \leq n \leq 500$ with *Mathematica* and *SageMath* [27]. The discrete plots of related summatory functions given in the figures from Appendix **??** also serve to visually motivate why we are interested in the distribution of this sequence, and the component function $C_\Omega(n)$ we explicitly connect to it, in the next results.

3.1 Definitions and properties of bivariate semi-triangular component function sequences

We define the following bivariate sequence for integers $n \geq 1$ and $k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}\left(\frac{n}{d}\right), & \text{if } k \geq 1. \end{cases} \tag{3.1}$$

Using the more standardized definitions in [2, §2], we can alternately identify the k-fold convolution of ω with itself in the following notation: $C_0(n) \equiv \omega^{0*}(n)$ and $C_k(n) \equiv \omega^{k*}(n)$ for integers $k \geq 1$ and $n \geq 1$. The special case of (3.1) where $k := \Omega(n)$ occurs frequently in the next sections of the article. To avoid cumbersome notation when referring to this common function variant, we suppress the double appearance of the index n by writing $C_\Omega(n) := C_{\Omega(n)}(n)$ instead.

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (3.1) inductively. By the same argument, we see that at fixed n, the function $C_k(n)$ is non-zero only possibly when $1 \leq k \leq \Omega(n)$ whenever $n \geq 2$. A sequence of signed semi-diagonals of the functions $C_k(n)$ begins as follows [28, A008480]:

$$\{\lambda(n) C_\Omega(n)\}_{n \geq 1} = \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

We see by (2.4) that $C_\Omega(n) \leq (\Omega(n))!$ for all $n \geq 1$ with equality precisely at the squarefree integers so that $(\Omega(n))! = (\omega(n))!$ whenever $\mu^2(n) = +1$.

3.2 Formulas relating the unsigned $C_{\Omega}(n)$ to $g^{-1}(n)$

Remark 3.1 (Motivation for the next elementary results)**.** The formula exactly expanding $C_{\Omega}(n)$ by finite products in (2.4) (using the prior alternate notation of $h^{-1}(n)$ for this function) shows that its values are determined completely by the *exponents* alone in the prime factorization of any $n \geq 2$. We use the next lemma to write the inverse function $g^{-1}(n)$ we are interested in studying as a Dirichlet convolution of the auxiliary function, $C_{\Omega}(n)$, with the square of the Möbius function, $\mu^2(n) = |\mu(n)|$. This result then allows us to see that up to the leading sign weight by $\lambda(n)$ on the values of this function, there is an essentially additive structure beneath its distinct values $g^{-1}(n)$ for $n \leq x$ (see Section D). The formula that connects $g^{-1}(n)$ to the convolutions defined by $C_k(n)$ when $k := \Omega(n)$ in the previous subsection is not trivial to identify without the Möbius inversion procedure we outline in the next proof.

Lemma 3.2. *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega}(d).$$

Proof. We first expand the recurrence relation for the Dirichlet inverse when $g^{-1}(1) = g(1)^{-1} = 1$ as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) g^{-1}\left(\frac{n}{d}\right) \quad \Longrightarrow \quad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \tag{3.2}$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (3.2) in the form of $(g^{-1} * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m (C_m(-) * g^{-1})(n)$, so that

$$F_m(n) = -\begin{cases} (\omega * g^{-1})(n), & m = 1; \\ \sum\limits_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum\limits_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & 2 \leq m \leq \Omega(n); \\ 0, & \text{otherwise.} \end{cases}$$

When $m := \Omega(n)$, i.e., with the expansions in the previous equation taken to a maximal depth, we obtain the relation

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega}(n) = \lambda(n) C_{\Omega}(n). \tag{3.3}$$

The stated formula for $g^{-1}(n)$ follows from (3.3) by Möbius inversion. □

Corollary 3.3. *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega}(d). \tag{3.4}$$

Proof. By applying Lemma 3.2, Proposition 2.3 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and since at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$, Lemma 3.2 and Proposition 2.3 imply that

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega}(d). \end{aligned}$$

We see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. □

The formula in (3.4) shows that the DGF of the unsigned inverse function, $|g^{-1}(n)|$, is given by the meromorphic function $\frac{1}{\zeta(2s)(1-P(s))}$ for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$. This DGF has a known pole to the right of the line at $\operatorname{Re}(s) = 1$ which occurs for the unique real $\sigma \equiv \sigma_1 \approx 1.39943$ such that $P(\sigma) = 1$ on $(1, +\infty)$.

Remark 3.4. The identification of an exact formula for $g^{-1}(n)$ using Lemma 3.2 implies both of the next results when n is squarefree. It also is suggestive of more regularity beneath the distribution of $|g^{-1}(n)|$ which we quantify with precise statements in the conjectures given in Section 4.3. In particular, since $C_\Omega(n) = |h^{-1}(n)|$ using the original notation from the proof of Proposition 2.3, we can see that $C_\Omega(n) = (\omega(n))!$ for all squarefree $n \geq 1$. We also have that whenever $n \geq 1$ is squarefree

$$|g^{-1}(n)| = \sum_{d|n} C_\Omega(d).$$

Since all divisors of a squarefree integer are squarefree, a proof of part (B) of Proposition 1.3 follows by an elementary counting argument as an immediate consequence of the previous equation.

Remark 3.5. Lemma 3.2 shows that the summatory function of this sequence satisfies

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_\Omega(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Equation (1.4) implies that

$$\lambda(d) C_\Omega(d) = (g^{-1} * 1)(d) = (\chi_{\mathbb{P}} + \varepsilon)^{-1}(d).$$

We recover by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1. \tag{3.5}$$

The proof of Corollary 5.1 (below) shows that

$$\sum_{n \leq x} |g^{-1}(n)| = \sum_{d \leq x} C_\Omega(d) Q\left(\left\lfloor \frac{x}{d} \right\rfloor\right), x \geq 1,$$

where $Q(x) := \sum_{n \leq x} \mu^2(n)$ counts the number of squarefree $n \leq x$.

4 The distribution of $C_\Omega(n)$ and its partial sums

We observed an intuition in the introduction that the relation of the unsigned auxiliary functions, $|g^{-1}(n)|$ and $C_\Omega(n)$, to the canonically additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties illustrated in Table E. Each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdős-Kac type theorem that provides a central limiting distribution for each of these functions over $n \leq x$ as $x \to \infty$ [8, 3, 25] (*cf.* [13]). In the remainder of this section and in the corollaries on $|g^{-1}(n)|$ in Section 5, we use analytic methods, primarily in the spirit of [18, §7.4], to prove and conjecture new properties that characterize the distributions of the unsigned auxiliary functions.

4.1 Analytic proofs extending bivariate DGF methods involving additive functions

Theorem 4.1 proves a core bound on the partial sums of certain sign weighted arithmetic functions which are parameterized in the powers $z^{\Omega(n)}$ of a complex-valued indeterminate z. We use this bound to prove uniform asymptotics for the partial sums, $\sum_{n \leq x} (-1)^{\omega(n)} C_\Omega(n)$, uniformly along only those values of $n \leq x$ with $\Omega(n) = k$ for $1 \leq k \leq \frac{3}{2} \log\log x$ when x is large in Theorem 4.2. At the conclusion of this subsection of the article, we use an argument involving Abel summation with the partial sums of $\lambda_*(n) := (-1)^{\omega(n)}$ to turn the uniform asymptotics for the signed sums into bounds we will need on the corresponding unsigned sums of the same functions along $n \leq x$ such that $\Omega(n) = k$ for k within our uniform ranges (see Lemma 4.3 and the conclusion in Corollary 4.4). The arguments given in the next few proofs are new and technical while mimicking as closely as possible the spirit of the proofs we cite inline from the references [18, 30].

Theorem 4.1. *Let the bivariate DGF $\widehat{F}(s,z)$ be defined in terms of the prime zeta function, $P(s)$, for* $\mathrm{Re}(s) > 1$ *and* $|z| < |P(s)|^{-1}$ *by*

$$\widehat{F}(s,z) := \frac{1}{1+P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

Let the partial sums of the coefficients of the DGF $\widehat{F}(s,z)\zeta(s)^z$ be denoted by

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_\Omega(n) z^{\Omega(n)}, x \geq 1.$$

We have for all sufficiently large $x \geq 2$ and any $|z| < P(2)^{-1} \approx 2.21118$ that

$$\widehat{A}_z(x) = \frac{x\widehat{F}(2,z)}{\Gamma(z)}(\log x)^{z-1} + O_z\left(x(\log x)^{\mathrm{Re}(z)-2}\right).$$

Proof. It follows from (2.4) that we can generate exponentially scaled forms of the function $C_\Omega(n)$ by a product identity of the following form:

$$\sum_{n\geq 1} \frac{C_\Omega(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r\geq 1} \frac{z^{\Omega(p^r)}}{r! p^{rs}}\right)^{-1} = \exp\left(-zP(s)\right), \text{ for } \mathrm{Re}(s) > 1 \wedge \mathrm{Re}(P(s)z) > -1.$$

This Euler type product expansion is similar in construction to the parameterized bivariate DGFs defined in [18, §7.4] [30, *cf.* §II.6.1]. By computing a termwise Laplace transform applied to the right-hand-side of the previous equation, we obtain that

$$\sum_{n\geq 1} \frac{C_\Omega(n)(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp\left(-tzP(s)\right) dt = \frac{1}{1+P(s)z}, \text{ for } \mathrm{Re}(s) > 1 \wedge \mathrm{Re}(P(s)z) > -1.$$

It follows from the Euler product representation of $\zeta(s)$, which is convergent for any $\mathrm{Re}(s) > 1$, that

$$\sum_{n\geq 1} \frac{(-1)^{\omega(n)} C_\Omega(n) z^{\Omega(n)}}{n^s} = \widehat{F}(s,z)\zeta(s)^z, \text{ for } \mathrm{Re}(s) > 1 \wedge |z| < |P(s)|^{-1}.$$

The bivariate DGF $\widehat{F}(s,z)$ is an analytic function of s for all $\mathrm{Re}(s) > 1$ whenever the parameter $|z| < |P(s)|^{-1}$. If the sequence $\{b_z(n)\}_{n\geq 1}$ indexes the coefficients in the DGF expansion of $\widehat{F}(s,z)\zeta(s)^z$, then the series

$$\left|\sum_{n\geq 1} \frac{b_z(n)(\log n)^{2R+1}}{n^s}\right| < +\infty.$$

Moreover, the series in the last equation is uniformly bounded for all $\mathrm{Re}(s) \geq 2$ and $|z| \leq R < |P(s)|^{-1}$. This fact follows by repeated termwise differentiation of the series for the original function $\lceil 2R+1 \rceil$ times with respect to s.

For fixed $0 < |z| < 2$, let the sequence $\{d_z(n)\}_{n\geq 1}$ be generated as the coefficients of the DGF

$$\zeta(s)^z = \sum_{n\geq 1} \frac{d_z(n)}{n^s}, \text{ for } \mathrm{Re}(s) > 1.$$

The corresponding summatory function of $d_z(n)$ is defined by $D_z(x) := \sum_{n\leq x} d_z(n)$. The theorem proved by careful contour integration in [18, Thm. 7.17; §7.4] shows that for any $0 < |z| < 2$ and all integers $x \geq 2$ we have

$$D_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O_z\left(x(\log x)^{\mathrm{Re}(z)-2}\right).$$

Set $b_z(n) := (-1)^{\omega(n)} C_\Omega(n) z^{\Omega(n)}$, define the convolution $a_z(n) := \sum_{d|n} b_z(d) d_z\left(\frac{n}{d}\right)$, and take its partial sums to be $A_z(x) := \sum_{n \leq x} a_z(n)$. Then we have that

$$\begin{aligned} A_z(x) &= \sum_{m \leq \frac{x}{2}} b_z(m) D_z\left(\frac{x}{m}\right) + \sum_{\frac{x}{2} < m \leq x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \times \sum_{m \leq \frac{x}{2}} \frac{b_z(m)}{m} \log\left(\frac{x}{m}\right)^{z-1} + O\left(\sum_{m \leq x} \frac{x|b_z(m)|}{m} \times \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2}\right). \end{aligned} \tag{4.1}$$

We can sum the coefficients $\frac{b_z(m)}{m}$ for integers $m \leq u$ when u is taken sufficiently large as follows:

$$\sum_{m \leq u} \frac{b_z(m)}{m^2} \times m = \left(\widehat{F}(2,z) + O_z\left(u^{-2}\right)\right) u - \int_1^u \left(\widehat{F}(2,z) + O_z\left(t^{-2}\right)\right) dt = \widehat{F}(2,z) + O_z\left(u^{-1}\right).$$

Suppose that $0 < |z| \leq R < P(2)^{-1}$. For large x, the error term in (4.1) satisfies

$$\begin{aligned} \sum_{m \leq x} \frac{x|b_z(m)|}{m} \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2} &\ll x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m} \\ &\quad + x(\log x)^{-(R+2)} \times \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m} (\log m)^{2R}, \\ &= O_z\left(x(\log x)^{\mathrm{Re}(z)-2}\right), \end{aligned}$$

whenever $0 < |z| \leq R$. When $m \leq \sqrt{x}$ we have that

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\mathrm{Re}(z)-2}\right).$$

A related upper bound is obtained for the left-hand-side of the previous equation when $\sqrt{x} < m < x$ and $0 < |z| < R$. The combined sum over the interval $m \leq \frac{x}{2}$ corresponds to bounding the sum components when $0 < |z| \leq R$ by

$$\begin{aligned} \sum_{m \leq \frac{x}{2}} b_z(m) D_z\left(\frac{x}{m}\right) &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \times \sum_{m \leq \frac{x}{2}} \frac{b_z(m)}{m} \\ &\quad + O_R\left(x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m \leq \sqrt{x}} \frac{|b_z(m)| \log m}{m} + x(\log x)^{R-1} \times \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m}\right) \\ &= \frac{x\widehat{F}(2,z)}{\Gamma(z)} (\log x)^{z-1} + O_R\left(x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m \geq 1} \frac{b_z(m)(\log m)^{2R+1}}{m^2}\right) \\ &= \frac{x\widehat{F}(2,z)}{\Gamma(z)} (\log x)^{z-1} + O_R\left(x(\log x)^{\mathrm{Re}(z)-2}\right). \end{aligned}$$

□

Theorem 4.2. *For all large $x \geq 3$ and integers $k \geq 1$, let*

$$\widehat{C}_{k,*}(x) := \sum_{\substack{n \leq x \\ \Omega(n) = k}} (-1)^{\omega(n)} C_\Omega(n)$$

Let $\widehat{G}(z) := \widehat{F}(2,z) \times \Gamma(1+z)^{-1}$ when $0 \leq |z| < P(2)^{-1}$ and where $\widehat{F}(s,z)$ is defined as in Theorem 4.1. As $x \to \infty$, we have uniformly for any $1 \leq k \leq 2\log\log x$ that

$$\widehat{C}_{k,*}(x) = -\widehat{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O\left(\frac{k}{(\log\log x)^2}\right)\right).$$

Proof. When $k = 1$, we have that $\Omega(n) = \omega(n)$ for all $n \leq x$ such that $\Omega(n) = k$. The positive integers n that satisfy this requirement are precisely the primes $p \leq x$. Hence, the formula is satisfied as

$$\sum_{p \leq x} (-1)^{\omega(p)} C_1(p) = -\sum_{p \leq x} 1 = -\frac{x}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Since $O\left((\log x)^{-1}\right) = O\left((\log\log x)^{-2}\right)$ as $x \to \infty$, we obtain the required error term for the bound at $k = 1$. For $2 \leq k \leq 2\log\log x$, we will apply the error estimate from Theorem 4.1 with $r := \frac{k-1}{\log\log x}$ in the formula

$$\widehat{C}_{k,*}(x) = \frac{(-1)^{k+1}}{2\pi\imath} \times \int_{|v|=r} \frac{\widehat{A}_{-v}(x)}{v^{k+1}} dv.$$

The error in the formula contributes terms that are bounded by

$$\begin{aligned} \left|x(\log x)^{-(\operatorname{Re}(v)+2)} v^{-(k+1)}\right| &\ll \left|x(\log x)^{-(r+2)} r^{-(k+1)}\right| \ll \frac{x}{(\log x)^{2-\frac{k-1}{\log\log x}}} \cdot \frac{(\log\log x)^k}{(k-1)^k} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{\frac{1}{2}}(k-1)!} \ll \frac{x}{\log x} \cdot \frac{k(\log\log x)^{k-5}}{(k-1)!}, \text{ as } x \to \infty. \end{aligned}$$

We next find the main term for the coefficients of the following contour integral when $r \in [0, z_{\max}] \subseteq [0, P(2)^{-1})$:

$$\widehat{C}_{k,*}(x) \sim \frac{(-1)^k x}{\log x} \times \int_{|v|=r} \frac{(\log x)^{-v}\zeta(2)^v}{\Gamma(1-v)v^k(1-P(2)v)} dv. \tag{4.2}$$

The main term of $\widehat{C}_{k,*}(x)$ is given by $-\frac{x}{\log x} \times I_k(r,x)$, where we define

$$\begin{aligned} I_k(r,x) &= \frac{1}{2\pi\imath} \times \int_{|v|=r} \frac{\widehat{G}(v)(\log x)^v}{v^k} dv \\ &=: I_{1,k}(r,x) + I_{2,k}(r,x). \end{aligned}$$

Taking $r = \frac{k-1}{\log\log x}$, the first of the component integrals is defined to be

$$I_{1,k}(r,x) := \frac{\widehat{G}(r)}{2\pi\imath} \times \int_{|v|=r} \frac{(\log x)^v}{v^k} dv = \widehat{G}(r) \times \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

The second integral, $I_{2,k}(r,x)$, corresponds to another error term in our approximation. This component function is defined by

$$I_{2,k}(r,x) := \frac{1}{2\pi\imath} \times \int_{|v|=r} \left(\widehat{G}(v) - \widehat{G}(r)\right) \frac{(\log x)^v}{v^k} dv.$$

Integrating by parts shows that [18, *cf.* Thm. 7.19; §7.4]

$$\frac{(r-v)}{2\pi\imath} \times \int_{|v|=r} (\log x)^v v^{-k} dv = 0,$$

so that integrating by parts once again we have

$$I_{2,k}(r,x) := \frac{1}{2\pi\imath} \times \int_{|v|=r} \left(\widehat{G}(v) - \widehat{G}(r) - \widehat{G}'(r)(v-r)\right)(\log x)^v v^{-k} dv.$$

We find that

$$\widehat{G}(v) - \widehat{G}(r) - \widehat{G}'(r)(v-r) = \int_r^v (v-w)\widehat{G}''(w) dw \ll |v-r|^2.$$

With the parameterization $v = re^{2\pi\imath\theta}$ for $\theta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ (again selecting $r := \frac{k-1}{\log\log x}$), we obtain

$$|I_{2,k}(r,x)| \ll r^{3-k} \times \int_{-\frac{1}{2}}^{\frac{1}{2}} (\sin \pi\theta)^2 e^{(k-1)\cos(2\pi\theta)} d\theta.$$

Since $|\sin x| \leq |x|$ for all $|x| < 1$ and $\cos(2\pi\theta) \leq 1 - 8\theta^2$ if $-\frac{1}{2} \leq \theta \leq \frac{1}{2}$, we arrive at the next bounds at any $1 \leq k \leq 2\log\log x$ when $r = \frac{k-1}{\log\log x}$.

$$\begin{aligned}|I_{2,k}(r,x)| &\ll r^{3-k} e^{k-1} \times \int_0^\infty \theta^2 e^{-8(k-1)\theta^2} d\theta \\ &\ll \frac{r^{3-k}e^{k-1}}{(k-1)^{\frac{3}{2}}} = \frac{(\log\log x)^{k-3} e^{k-1}}{(k-1)^{k-\frac{3}{2}}} \ll \frac{k(\log\log x)^{k-3}}{(k-1)!}.\end{aligned}$$

Finally, whenever $1 \leq k \leq 2\log\log x$ we have

$$1 = \widehat{G}(0) \geq \widehat{G}\left(\frac{k-1}{\log\log x}\right) = \frac{1}{\Gamma\left(1 + \frac{k-1}{\log\log x}\right)} \times \frac{\zeta(2)^{\frac{1-k}{\log\log x}}}{\left(1 + \frac{P(2)(k-1)}{\log\log x}\right)} \geq \widehat{G}(2) \approx 0.097027.$$

In particular, the function $\widehat{G}\left(\frac{k-1}{\log\log x}\right) \gg 1$ for all $1 \leq k \leq 2\log\log x$. This in turn implies the result of the theorem. □

Lemma 4.3. *For $x \geq 1$, let*

$$L_\omega(x) := \sum_{n \leq x} (-1)^{\omega(n)}.$$

As $x \to \infty$, there is an absolute constant $A_0 > 0$ such that

$$L_\omega(x) = \frac{(-1)^{\lfloor \log\log x \rfloor} x}{A_0 \sqrt{2\pi \log\log x}} + O\left(\frac{x}{\log\log x}\right).$$

Proof. An adaptation of the proof of Lemma C.3 from the appendix provides that for any $a \in (1, 1.76321) \subset \left(1, W(1)^{-1}\right)$

$$\begin{aligned}S_a(x) &:= \frac{x}{\log x} \times \left| \sum_{k=1}^{\lfloor a \log\log x \rfloor} \frac{(-1)^k (\log\log x)^{k-1}}{(k-1)!} \right| \\ &= \frac{\sqrt{a}x}{\sqrt{2\pi}(a+1)a^{\{a\log\log x\}}} \times \frac{(\log x)^{a-1-a\log a}}{\sqrt{\log\log x}} \left(1 + O\left(\frac{1}{\log\log x}\right)\right).\end{aligned} \tag{4.3}$$

Here, we define $\{x\} = x - \lfloor x \rfloor \in [0, 1)$ to be the *fractional part* of x. Suppose that we take $a := \frac{3}{2}$ so that $a - 1 - a\log a \approx -0.108198$. We can then define and expand the next partial sums as

$$L_\omega(x) := \sum_{n \leq x} (-1)^{\omega(n)} = \sum_{k \leq \log\log x} 2(-1)^k \pi_k(x) + O\left(S_{\frac{3}{2}}(x) + \#\left\{n \leq x : \omega(n) \geq \frac{3}{2}\log\log x\right\}\right).$$

The justification for the error term including $S_{\frac{3}{2}}(x)$ is that for $1 \leq k < \frac{3}{2}\log\log x$, we can show that $\widetilde{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) \asymp 1$ where the function $\widetilde{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right)$ is monotone for k within each of the two disjoint intervals $[1, \log\log x] \cup \left(\log\log x, \frac{3}{2}\log\log x\right]$. Moreover, we can show that for any $1 < k \leq \log\log x$, the function $\widetilde{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right)$ from Remark B.4 is decreasing in k for $1 \leq k \leq \log\log x$ with $\widetilde{\mathcal{G}}(0) = 1$. It also satisfies the following inequalities for k taken within the same range:

$$\widetilde{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) \geq \widetilde{\mathcal{G}}\left(1 - \frac{1}{\log\log x}\right) \geq \widetilde{\mathcal{G}}(1) = 1.$$

We apply the uniform asymptotics for $\pi_k(x)$ that hold as $x \to \infty$ when $1 \leq k \leq R\log\log x$ for $1 \leq R < 2$ from Remark B.4. We then see by Lemma C.3 and (4.3) that for all sufficiently large x there is some absolute constant $A_0 > 0$ such that

$$L_\omega(x) = \frac{(-1)^{\lfloor \log\log x \rfloor} x}{A_0\sqrt{2\pi\log\log x}} + O\left(E_\omega(x) + \frac{x}{(\log x)^{0.108198}\sqrt{\log\log x}} + \#\left\{n \leq x : \omega(x) \geq \frac{3}{2}\log\log x\right\}\right).$$

The error term in the previous equation is bounded by the next sum as $x \to \infty$. In particular, the following estimate is obtained from Stirling's formula, and equations (C.1a) and (C.1c) from the appendix:

$$\begin{aligned} E_\omega(x) &\ll \frac{x}{\log x} \times \sum_{1 \leq k \leq \log\log x} \frac{(\log\log x)^{k-2}}{(k-1)!} \\ &= \frac{x\Gamma(\log\log x, \log\log x)}{\Gamma(\log\log x + 1)} \sim \frac{x}{2\log\log x}\left(1 + O\left(\frac{1}{\sqrt{\log\log x}}\right)\right). \end{aligned}$$

By an application of the second set of results in Remark B.4, we finally see that

$$\#\left\{n \leq x : \omega(x) \geq \frac{3}{2}\log\log x\right\} \ll \frac{x}{(\log x)^{0.108198}}. \qquad \square$$

Hence, we have obtained a correct main and error term on the partial sums $L_\omega(x)$.

Corollary 4.4. *We have uniformly for $1 \leq k \leq \frac{3}{2}\log\log x$ that at all sufficiently large x*

$$\widehat{C}_k(x) := \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_\Omega(n) = A_0\sqrt{2\pi}x\widehat{G}\left(\frac{k-1}{\log\log x}\right)\frac{(\log\log x)^{k-\frac{1}{2}}}{(k-1)!}\left(1 + O\left(\frac{1}{\log\log x}\right)\right).$$

Proof. Suppose that $\hat{h}(t)$ and $\sum_{n \leq t} \lambda_*(n)$ are piecewise smooth and differentiable functions of t on $\mathbb{R}^+$. The next integral formulas result by Abel summation and integration by parts.

$$\sum_{n \leq x} \lambda_*(n)\hat{h}(n) = \left(\sum_{n \leq x} \lambda_*(n)\right)\hat{h}(x) - \int_1^x \left(\sum_{n \leq t} \lambda_*(n)\right)\hat{h}'(t)dt \tag{4.4a}$$

$$\sim \int_1^x \frac{d}{dt}\left[\sum_{n \leq t} \lambda_*(n)\right]\hat{h}(t)dt \tag{4.4b}$$

We transform our previous results for the partial sums of $(-1)^{\omega(n)}C_\Omega(n)$ such that $\Omega(n) = k$ from Theorem 4.2 to approximate the corresponding partial sums of only the unsigned function $C_\Omega(n)$. In particular, since $1 \leq k \leq \frac{3}{2}\log\log x$, we have that

$$\widehat{C}_{k,*}(x) = \sum_{\substack{n \leq x \\ \Omega(n)=k}} (-1)^{\omega(n)}C_\Omega(n) = \sum_{n \leq x} (-1)^{\omega(n)}\left[\omega(n) \leq \frac{3}{2}\log\log x\right]_\delta \times C_\Omega(n)\left[\Omega(n) = k\right]_\delta.$$

By the proof of Lemma 4.3, we have that as $t \to \infty$

$$L_*(t) := \sum_{\substack{n \leq t \\ \omega(n) \leq \frac{3}{2}\log\log t}} (-1)^{\omega(n)} = \frac{(-1)^{\lfloor \log\log t \rfloor} t}{A_0\sqrt{2\pi\log\log t}}\left(1 + O\left(\frac{1}{\sqrt{\log\log t}}\right)\right). \tag{4.5}$$

Except for t within a subset of $(0, \infty)$ of measure zero on which $L_*(t)$ changes sign, the main term of the derivative of this summatory function is approximated almost everywhere by

$$L_*'(t) \sim \frac{(-1)^{\lfloor \log\log t \rfloor}}{A_0\sqrt{2\pi\log\log t}}.$$

We apply the formula from (4.4b), to deduce that as $x \to \infty$ whenever $1 \leq k \leq \frac{3}{2}\log\log x$

$$\begin{aligned}
\widehat{C}_{k,*}(x) &\sim \sum_{j=1}^{\log\log x - 1} \frac{2 \cdot (-1)^{j+1}}{A_0\sqrt{2\pi}} \times \int_{e^{e^j}}^{e^{e^{j+1}}} \frac{C_{\Omega(t)}(t)\left[\Omega(t)=k\right]_{\delta}}{\sqrt{\log\log t}} dt \\
&\sim -\int_1^{\frac{\log\log x}{2}} \int_{e^{e^{2s-1}}}^{e^{e^{2s}}} \frac{2C_{\Omega(t)}(t)\left[\Omega(t)=k\right]_{\delta}}{A_0\sqrt{2\pi\log\log t}} dt ds + \frac{1}{A_0\sqrt{2\pi}} \times \int_{e^e}^{x} \frac{C_{\Omega(t)}(t)\left[\Omega(t)=k\right]_{\delta}}{\sqrt{\log\log t}} dt.
\end{aligned}$$

For large x, $(\log\log t)^{-\frac{1}{2}}$ is continuous and monotone decreasing on $\left[x^{e^{-1}}, x\right]$ with

$$\frac{1}{\sqrt{\log\log x}} - \frac{1}{\sqrt{\log\log\left(x^{e^{-1}}\right)}} = O\left(\frac{1}{(\log x)\sqrt{\log\log x}}\right),$$

Hence, we have that

$$-A_0\sqrt{2\pi}x(\log x)\sqrt{\log\log x}\widehat{C}'_{k,*}(x) = \left(\widehat{C}_k(x) - \widehat{C}_k\left(x^{e^{-1}}\right)\right)(1+o(1)) - x(\log x)\widehat{C}'_k(x). \tag{4.6}$$

For $1 \leq k < \frac{3}{2}\log\log x$, we expect contributions from the squarefree integers $n \leq x$ such that $\omega(n) = \Omega(n) = k$ to be on the order of

$$\widehat{C}'_k(x) \gg \widehat{\pi}_k(x) \asymp \frac{x}{\log x} \times \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

The argument used to justify the last equation is that clearly we find

$$\widehat{C}'_1(x) \gg \widehat{\pi}_1(x), \widehat{C}'_2(x) \gg \widehat{\pi}_2(x).$$

Moreover, for any integers $k \geq 3$ we see that

$$\begin{aligned}
\left|\widehat{C}'_k(x)\right| &\gg \sum_{n \leq x} \frac{|C_{\Omega}(n) - C_{\Omega}(n-1)|}{n} \times \left[\Omega(n)=k\right]_{\delta} \\
&\gg \sum_{2 \leq n \leq x} \frac{C_{\Omega}(n)}{n(n-1)} \times \left[\Omega(n)=k\right]_{\delta} \\
&\gg \sum_{n \leq x} \left[\Omega(n)=k\right]_{\delta}.
\end{aligned}$$

We conclude that $\widehat{C}_k\left(x^{e^{-1}}\right) = o\left(\widehat{C}_k(x)\right)$ at sufficiently large x. Equation (4.6) becomes an ordinary differential equation for $\widehat{C}_k(x)$ after this observation. Its solution has the form

$$\widehat{C}_k(x) = A_0\sqrt{2\pi}(\log x) \times \int_3^x \frac{\sqrt{\log\log t}}{\log t} \times \widehat{C}'_{k,*}(t) dt + O(\log x).$$

When we integrate by parts and apply the result from Theorem 4.2, we find that

$$\begin{aligned}
\widehat{C}_k(x) &= \frac{\sqrt{\log\log x}}{\log x} \times \widehat{C}_{k,*}(x) + O\left(x \times \int_3^x \frac{\sqrt{\log\log t} \times \widehat{C}_{k,*}(t)}{t^2(\log t)^2} dt\right) \\
&= \frac{\sqrt{\log\log x}}{\log x} \times \widehat{C}_{k,*}(x) + O\left(\frac{x}{2^k} \times \Gamma\left(k+\frac{1}{2}, 2\log\log x\right)\right).
\end{aligned}$$

Finally, whenever we assume that $1 \leq k \leq \frac{3}{2}\log\log x$ such that $\lambda > 1$ in Proposition C.2 (*cf.* Facts C.1 for k of substantially lesser order in x than this upper bound), Theorem 4.2 implies the conclusion of our corollary. □

4.2 The average order of $C_{\Omega}(n)$

Proposition 4.5. *There is an absolute constant $B_0 > 0$ such that as $n \to \infty$*

$$\frac{1}{n} \times \sum_{k \leq n} C_{\Omega}(k) = B_0 (\log n) \sqrt{\log\log n} \left(1 + O\left(\frac{1}{\log\log n}\right)\right).$$

Proof. By Corollary 4.4 and Proposition C.2 when $\lambda = \frac{2}{3}$, we have that

$$\begin{aligned}
\sum_{k=1}^{\frac{3}{2}\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega}(n) &\asymp \sum_{k=1}^{\frac{3}{2}\log\log x} \frac{x(\log\log x)^{k-\frac{1}{2}}}{(k-1)!} \left(1 + O\left(\frac{1}{\log\log x}\right)\right) \\
&= \frac{x(\log x)\sqrt{\log\log x}\,\Gamma\left(\frac{3}{2}\log\log x, \log\log x\right)}{\Gamma\left(\frac{3}{2}\log\log x\right)} \left(1 + O\left(\frac{1}{\log\log x}\right)\right) \\
&= x(\log x)\sqrt{\log\log x}\left(1 + O\left(\frac{1}{\log\log x}\right)\right).
\end{aligned}$$

For real $0 \leq z \leq 2$, the function $\widehat{G}(z)$ is monotone in z with $\widehat{G}(0) = 1$ and $\widehat{G}(2) \approx 0.303964$. Then we see that there is an absolute constant $B_0 > 0$ such that

$$\frac{1}{x} \times \sum_{k=1}^{\frac{3}{2}\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega}(n) = B_0 (\log x)\sqrt{\log\log x}\left(1 + O\left(\frac{1}{\log\log x}\right)\right).$$

We claim that

$$\begin{aligned}
\frac{1}{x} \times \sum_{n \leq x} C_{\Omega}(n) &= \frac{1}{x} \times \sum_{k \geq 1} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega}(n) \\
&= \frac{1}{x} \times \sum_{k=1}^{\frac{3}{2}\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega}(n)(1 + o(1)), \text{ as } x \to \infty.
\end{aligned}$$

To prove the claim it suffices to show that

$$\frac{1}{x} \times \sum_{\substack{n \leq x \\ \Omega(n) \geq \frac{3}{2}\log\log x}} C_{\Omega}(n) = o\left((\log x)\sqrt{\log\log x}\right). \tag{4.7}$$

We proved in Theorem 4.1 that for all sufficiently large x and $|z| < P(2)^{-1}$

$$\sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega}(n) z^{\Omega(n)} = \frac{x\widehat{F}(2,z)}{\Gamma(z)}(\log x)^{z-1} + O\left(x(\log x)^{\mathrm{Re}(z)-2}\right).$$

By Lemma 4.3, we have that the summatory function

$$\sum_{n \leq x} (-1)^{\omega(n)} = \frac{(-1)^{\lfloor \log\log x \rfloor} x}{A_0\sqrt{2\pi\log\log x}}\left(1 + O\left(\frac{1}{\sqrt{\log\log x}}\right)\right),$$

where $\frac{d}{dx}\left[\frac{x}{\sqrt{\log\log x}}\right] = \frac{1}{\sqrt{\log\log x}} + o(1)$. We can argue as in the proof of Corollary 4.4 that whenever $0 < |z| < P(2)^{-1}$ with x sufficiently large we have

$$\sum_{n \leq x} C_{\Omega}(n) z^{\Omega(n)} \ll \frac{\widehat{F}(2,z)x(\log x)\sqrt{\log\log x}}{\Gamma(z)} \times \frac{\partial}{\partial x}\left[x(\log x)^{z-1}\right]$$

$$\ll \frac{\widehat{F}(2,z)x\sqrt{\log\log x}}{\Gamma(z)}(\log x)^z. \tag{4.8}$$

For large x and any fixed $0 < r < P(2)^{-1}$, we define

$$\widehat{B}(x,r) := \sum_{\substack{n \le x \\ \Omega(n) \ge r\log\log x}} C_\Omega(n).$$

We adapt the proof from the reference [18, *cf.* Thm. 7.20; §7.4] by applying (4.8) when $1 \le r < P(2)^{-1}$. Since $r\widehat{F}(2,r) = \frac{r\zeta(2)^{-r}}{1+P(2)r} \ll 1$ for $r \in [1, P(2)^{-1})$, and similarly since we have that $\frac{1}{\Gamma(1+r)} \gg 1$ for r within the same range, we find that

$$x\sqrt{\log\log x}(\log x)^r \gg \sum_{\substack{n \le x \\ \Omega(n) \ge r\log\log x}} C_\Omega(n) r^{\Omega(n)} \gg \sum_{\substack{n \le x \\ \Omega(n) \ge r\log\log x}} C_\Omega(n) r^{r\log\log x}.$$

This implies that for $r := \frac{3}{2}$ we have

$$\widehat{B}(x,r) \ll x(\log x)^{r - r\log r}\sqrt{\log\log x} = O\left(x(\log x)^{0.891802}\sqrt{\log\log x}\right) \tag{4.9}$$

We evaluate the limiting asymptotics of the sums

$$S_2(x) := \frac{1}{x} \times \sum_{k \ge \frac{3}{2}\log\log x} \sum_{\substack{n \le x \\ \Omega(n) = k}} C_\Omega(n) \ll \frac{1}{x} \times \widehat{B}\left(x, \frac{3}{2}\right) = O\left((\log x)^{0.891802}\sqrt{\log\log x}\right), \text{ as } x \to \infty.$$

This implies that (4.7) holds. □

4.3 Erdős-Kac theorem analogs for the distributions of the unsigned functions

It is not difficult to prove that

$$\sum_{\substack{n \le x \\ \Omega(n) = k}} \frac{C_\Omega(n)}{(\log n)\sqrt{\log\log n}} = \frac{A_0\sqrt{2\pi}x}{\log x} \times \widehat{G}\left(\frac{k-1}{\log\log x}\right)\frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 + O\left(\frac{1}{\log\log x}\right)\right), \text{ as } x \to \infty$$

A modified set of proof mechanics that draw upon the analytic methods in [18, Thm. 7.21; §7.4] suggest that the first result in (A) of the next conjecture should hold. The average order of $C_\Omega(n)$ is given by Proposition 4.5. We can show that there is an absolute constant $D_0 > 0$ such that the second moment type partial sums of the function $C_\Omega(n)$ satisfy

$$\begin{aligned}\frac{1}{n} \times \left(\sum_{k \le n} C_\Omega(k)^2 - \left(\sum_{k \le n} C_\Omega(k)\right)^2\right) &= \frac{2}{n} \times \sum_{1 \le j < k \le n} C_\Omega(j)C_\Omega(k), \\ &= D_0^2 n(\log n)^2(\log\log n)(1 + o(1)), \text{ as } n \to \infty.\end{aligned}$$

This perspective leads to the second conjectured result in part (B) below. The second conjecture is effectively the statement of an Erdős-Kac type central limit theorem with first and second moment statistics (formed as partial sums of powers of the deterministic function $C_\Omega(n)$) that grow without bound as $x \to \infty$. The intuition for why part (B) of the conjecture should hold is thus probabilistic in nature, rather than being based on the analytic proof constructions that led to the conjectured statement in part (A).

Conjecture 4.6 (Deterministic form of the Erdős-Kac theorem analog for $C_\Omega(n)$)**.** *For sufficiently large x, let the mean and variance parameter analogs be defined by*

$$\mu_x(C) := \log\log x - \log\left(\sqrt{2\pi}A_0\widehat{G}(1)\right), \quad \text{and} \quad \sigma_x(C) := \sqrt{\log\log x},$$

where $\widehat{G}(1) \equiv \frac{1}{\zeta(2)(1+P(2))} \approx 0.418611$. *Then we have for any* $z \in (-\infty, +\infty)$ *that*

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : \frac{\frac{C_\Omega(n)}{(\log n)\sqrt{\log\log n}} - \mu_x(C)}{\sigma_x(C)} \leq z\right\} = \Phi(z) + o(1), \text{ as } x \to \infty. \tag{A}$$

Similarly, for any real z *we have that*

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : \frac{C_\Omega(n) - B_0(\log x)\sqrt{\log\log x}}{D_0\sqrt{x}(\log x)\sqrt{\log\log x}} \leq z\right\} = \Phi(z) + o(1), \text{ as } x \to \infty \tag{B}$$

5 The distribution of the unsigned inverse sequence $|g^{-1}(n)|$

Corollary 5.1. *We have that as* $n \to \infty$

$$\frac{1}{n} \times \sum_{k \leq n} |g^{-1}(k)| = \frac{6B_0(\log n)^2\sqrt{\log\log n}}{\pi^2}\left(1 + O\left(\frac{1}{\log\log n}\right)\right).$$

Proof. As $|z| \to \infty$, the *imaginary error function*, $\operatorname{erfi}(z)$, has the following asymptotic expansion [24, §7.12]:

$$\operatorname{erfi}(z) := \frac{2}{\sqrt{\pi}\imath} \times \int_0^{\imath z} e^{t^2} dt = \frac{e^{z^2}}{\sqrt{\pi}}\left(\frac{1}{z} + \frac{1}{2z^3} + \frac{3}{4z^5} + \frac{15}{8z^7} + O\left(\frac{1}{z^9}\right)\right). \tag{5.1}$$

We use the formula from Proposition 4.5 to sum the average order of $C_\Omega(n)$. The proposition and error terms obtained from (5.1) imply that for all sufficiently large $t \to \infty$

$$\begin{aligned}\int \frac{\sum_{n \leq t} C_\Omega(n)}{t^2} dt &= B_0(\log t)^2\sqrt{\log\log t} - \frac{1}{4}\sqrt{\frac{\pi}{2}}\operatorname{erfi}\left(\sqrt{2\log\log t}\right) \\ &= B_0(\log t)^2\sqrt{\log\log t}\left(1 + O\left(\frac{1}{\log\log t}\right)\right).\end{aligned} \tag{5.2}$$

A classical formula for the summatory function that counts the number of *squarefree* integers $n \leq x$ shows that this function satisfies [10, §18.6] [28, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) = \frac{6x}{\pi^2} + O\left(\sqrt{x}\right), \text{ as } x \to \infty.$$

Therefore, summing over the formula from (3.4) in Section 3.2, we find that

$$\begin{aligned}\frac{1}{n} \times \sum_{k \leq n} |g^{-1}(k)| &= \frac{1}{n} \times \sum_{d \leq n} C_\Omega(d) Q\left(\left\lfloor \frac{n}{d} \right\rfloor\right) \\ &\sim \sum_{d \leq n} C_\Omega(d)\left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dn}}\right)\right] \\ &= \frac{6}{\pi^2}\left[\frac{1}{n} \times \sum_{k \leq n} C_\Omega(k) + \sum_{d < n}\sum_{k \leq d} \frac{C_\Omega(k)}{d^2}\right] + O(1).\end{aligned}$$

The latter sum in the previous equation forms the main term that we approximate using the asymptotics for the integral in (5.2) for all large enough t as $t \to \infty$. □

Corollary 5.2. *Suppose that Conjecture 4.6 is true and that* $\mu_x(C)$ *and* $\sigma_x(C)$ *are defined as in the conjecture for sufficiently large* x. *Let* $Y > 0$. *We have uniformly for all* $-Y \leq y \leq Y$ *that as* $x \to \infty$

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : \frac{|g^{-1}(n)|}{(\log n)\sqrt{\log\log n}} - \frac{6}{\pi^2 n(\log n)\sqrt{\log\log n}} \times \sum_{k \leq n} |g^{-1}(k)| \leq y\right\} = \Phi\left(\frac{\frac{\pi^2 y}{6} - \mu_x(C)}{\sigma_x(C)}\right) + o(1).$$

Moreover, we have that for any real y, as $x \to \infty$

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : |g^{-1}(n)| - \frac{6}{\pi^2 n} \times \sum_{k \leq n} |g^{-1}(k)| \leq y\right\} = \Phi\left(\frac{\frac{\pi^2 y}{6} - B_0(\log x)\sqrt{\log\log x}}{D_0\sqrt{x}(\log x)\sqrt{\log\log x}}\right) + o(1).$$

Proof. We claim that

$$|g^{-1}(n)| - \frac{6}{\pi^2 n} \times \sum_{k \leq n} |g^{-1}(k)| \sim \frac{6}{\pi^2} C_\Omega(n), \text{ as } n \to \infty.$$

As in the proof of Corollary 5.1, we obtain that

$$\frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| = \frac{6}{\pi^2}\left(\frac{1}{x} \times \sum_{n \leq x} C_\Omega(n) + \sum_{d < x} \sum_{k \leq d} \frac{C_\Omega(k)}{d^2}\right) + O(1).$$

Let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) \coloneqq f(x) - f(x-1)$. We see that for large n

$$\begin{aligned}
|g^{-1}(n)| &= \Delta_n\left(\sum_{k \leq n} g^{-1}(k)\right) \sim \frac{6}{\pi^2} \times \Delta_n\left(\sum_{d \leq n} C_\Omega(d) \cdot \frac{n}{d}\right) \\
&= \frac{6}{\pi^2}\left(C_\Omega(n) + \sum_{d<n} C_\Omega(d)\frac{n}{d} - \sum_{d<n} C_\Omega(d)\frac{(n-1)}{d}\right) \\
&\sim \frac{6}{\pi^2}\left(C_\Omega(n) + \frac{1}{n-1} \times \sum_{k<n} |g^{-1}(k)|\right), \text{ as } n \to \infty.
\end{aligned}$$

Since $\frac{1}{n-1} \times \sum_{k<n} |g^{-1}(k)| \sim \frac{1}{n} \times \sum_{k \leq n} |g^{-1}(k)|$ for all sufficiently large n, the results follow by a re-normalization of Conjecture 4.6. □

6 New formulas and limiting relations characterizing $M(x)$

6.1 Formulas relating $M(x)$ to the summatory function $G^{-1}(x)$

Proposition 6.1. *For all sufficiently large x, we have that*

$$M(x) = G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} G^{-1}(k)\left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right). \tag{6.1}$$

Proof. We know by applying Corollary **??** that

$$\begin{aligned}
M(x) &= \sum_{k=1}^{x} g^{-1}(k)\left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right) \\
&= G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} g^{-1}(k)\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) \\
&= G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right).
\end{aligned}$$

The upper bound on the sum is truncated to $k \in \left[1, \frac{x}{2}\right]$ in the second equation above due to the fact that $\pi(1) = 0$. The third formula above follows directly by (ordinary) summation by parts. □

By the result from (3.5) proved in Section 3.2, we recall that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), \text{ for } x \geq 1.$$

Summation by parts implies that we can also express $G^{-1}(x)$ in terms of the summatory function $L(x)$ and differences of the unsigned sequence whose distribution is given by Corollary 5.2. That is, we have

$$G^{-1}(x) = \sum_{n \leq x} \lambda(n)|g^{-1}(n)| = L(x)|g^{-1}(x)| - \sum_{n<x} L(n)\left(|g^{-1}(n+1)| - |g^{-1}(n)|\right), \text{ for } x \geq 1.$$

6.2 Example: Local cancellation of $G^{-1}(x)$ in the new formulas for $M(x)$

Lemma 6.2. *Suppose that p_n denotes the n^{th} prime for $n \geq 1$ [28, A000040]. Let $\mathcal{P}_{\#}$ denote the set of positive primorial integers given by [28, A002110]*

$$\mathcal{P}_{\#} = \{n\#\}_{n \geq 1} = \left\{\prod_{k=1}^{n} p_k : n \geq 1\right\} = \{2, 6, 30, 210, 2310, 30030, \ldots\}.$$

As $m \to \infty$ we have that

$$-G^{-1}((4m+1)\#) \asymp (4m+1)!, \tag{A}$$

$$G^{-1}\left(\frac{(4m+1)\#}{p_k}\right) \asymp (4m)!, \text{ for any } 1 \leq k \leq 4m+1. \tag{B}$$

Proof. We have by part (B) of Proposition 1.3 that for all squarefree integers $n \geq 1$

$$\begin{aligned} |g^{-1}(n)| &= \sum_{j=0}^{\omega(n)} \binom{\omega(n)}{j} \times j! = (\omega(n))! \times \sum_{j=0}^{\omega(n)} \frac{1}{j!} \\ &= (\omega(n))! \times \left(e + O\left(\frac{1}{(\omega(n)+1)!}\right)\right). \end{aligned}$$

Let m be a large positive integer. We obtain main terms of the form

$$\begin{aligned} \sum_{\substack{n \leq (4m+1)\# \\ \omega(n) = \Omega(n)}} \lambda(n)|g^{-1}(n)| &= \sum_{0 \leq k \leq 4m+1} \binom{4m+1}{k} (-1)^k k! \left(e + O\left(\frac{1}{(k+1)!}\right)\right) \\ &= -(4m+1)! + O(1). \end{aligned}$$

The formula for $C_{\Omega}(n)$ stated in (2.4) then implies the result in (A). We can similarly derive for any $1 \leq k \leq 4m+1$ that

$$G^{-1}\left(\frac{(4m+1)\#}{p_k}\right) \asymp \sum_{0 \leq k \leq 4m} \binom{4m}{k} (-1)^k k! \left(e + O\left(\frac{1}{(k+1)!}\right)\right) \asymp (4m)!. \qquad \square$$

Remark 6.3. Even though we get comparatively large order growth of $|G^{-1}|(x) \geq |G^{-1}(x)|$ infinitely often, we should expect that there is usually (almost always) a large cancellation between the successive values of this summatory function in the form of (3.5). Lemma 6.2 demonstrates the phenomenon well along the infinite subsequence of large x taken along the primorials, or the integers $x = (4m+1)\#$ that are precisely the product of the first $4m+1$ primes for $m \geq 1$. In particular, we have that [6, 7]

$$n\# \sim e^{\vartheta(p_n)} \asymp n^n (\log n)^n e^{-n(1+o(1))}, \text{ as } n \to \infty.$$

The RH then requires that the sums of the leading constants with opposing signs on the asymptotics for the functions from the lemma match. Indeed, this observation follows from the fact that if we obtain a contrary result, equation (3.5) would imply that

$$\frac{M((4m+1)\#)}{\sqrt{(4m+1)\#}} \gg [(4m+1)\#]^{\delta_0}, \text{ as } m \to \infty,$$

for some fixed $\delta_0 > 0$ (*cf.* equation (1.1) of the introduction).

7 Conclusions

We have identified a new sequence, $\{g^{-1}(n)\}_{n\geq 1}$, that is the Dirichlet inverse of the shifted strongly additive function $\omega(n)$. Section D, shows that there is a natural combinatorial interpretation to the distribution of distinct values of $|g^{-1}(n)|$ for $n \leq x$ involving the distribution of the primes $p \leq x$ at large x. In particular, the magnitude of $g^{-1}(n)$ depends only on the pattern of the exponents of the prime factorization of n. The sign of $g^{-1}(n)$ is given by $\lambda(n)$ for all $n \geq 1$. This leads to a new relations of the summatory function $G^{-1}(x)$, which characterizes the distribution of $M(x)$, to the distribution of the classical summatory function $L(x)$. We emphasize that our new work on the Mertens function proved within this article is significant in providing a new window through which we can view bounding $M(x)$ through asymptotics of auxiliary sequences and partial sums. The computational data generated in Table E of the appendix section indicates numerically that the distribution of $G^{-1}(x)$ is easier to work with than that of $M(x)$ or $L(x)$. The additively combinatorial relation of the distinct (and repetition of) values of $|g^{-1}(n)|$ for $n \leq x$ are suggestive towards bounding main terms for $G^{-1}(x)$ along infinite subsequences in future work.

We expect that an outline of the method behind the collective proofs we provide with respect to the Mertens function case can be generalized to identify associated additive functions with the same role of $\omega(n)$ in this paper. Such generalizations can then be used to express asymptotics for partial sums of other Dirichlet inverse functions. As in the Mertens function case, the link between strong additivity and the resulting sequences studied to express the partial sums of signed Dirichlet inverse functions are also computationally useful in more efficiently computing all of the first $x \geq 3$ values of these summatory functions when x is large. The key question in formulating generalizations to the new methods we present in this article is in constructing the relevant strongly additive functions that fulfill the role of $\omega(n)$ in our new constructions that characterize $M(x)$. We expect that while natural extensions exist in connection with the Dirichlet inverse of any arithmetic $f > 0$, pinning down a strongly additive analog to the distinct prime counting function in the most general cases will require deep new theorems tied to the DGFs of the original sequences, $\{f(n)\}_{n\geq 1}$.

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A Glossary of notation and conventions

The next listing provides a mostly comprehensive glossary of common notation, conventions and abbreviations used in the article.

Symbols	**Definition**
$\gg, \ll, \asymp$	For functions A, B, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A, B \geq 0$, $A \ll B$ and $B \ll A$, we write $A \asymp B$.
$\approx, \sim$	We write that $f(x) \approx g(x)$ if $\lvert f(x) - g(x) \rvert \ll 1$ as $x \to \infty$. Two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1$.
$\chi_{\mathbb{P}}(n), P(s)$	The indicator function of the primes equals one if and only if $n \in \mathbb{Z}^+$ is prime and is defined to be zero-valued otherwise. For any $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$, we define the prime zeta function to be the Dirichlet generating function (DGF) defined by $P(s) = \sum_{n\geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s}$. The function $P(s)$ has an analytic continuation to the half-plane $\mathrm{Re}(s) > 0$ with the exception of $s = 1$ through the formula $P(s) = \sum_{k\geq 1} \frac{\mu(k)}{k} \log \zeta(ks)$. The DGF $P(s)$ poles at the reciprocal of each positive integer and a natural boundary at the line $\mathrm{Re}(s) = 0$.
$C_k(n), C_\Omega(n)$	The first sequence is defined recursively for integers $n \geq 1$ and $k \geq 0$ as follows: $$C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum_{d\mid n} \omega(d) C_{k-1}\left(\frac{n}{d}\right), & \text{if } k \geq 1. \end{cases}$$ It represents the multiple (k-fold) convolution of the function $\omega(n)$ with itself. The function $C_\Omega(n) := C_{\Omega(n)}(n)$ has the DGF $(1 - P(s))^{-1}$ for $\mathrm{Re}(s) > 1$.
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function (OGF) of a sequence, $\{f_n\}_{n\geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n\geq 0} f_n q^n$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic function f we have that $f * \varepsilon = \varepsilon * f = f$ where the operation $*$ denotes Dirichlet convolution.
$\mathrm{erf}(z), \mathrm{erfi}(z)$	The function $\mathrm{erf}(z)$ denotes the (ordinary) error function. It is related to the CDF, $\Phi(z)$, of the standard normal distribution for any $z \in (-\infty, +\infty)$ through the relation $\Phi(z) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)\right)$. The imaginary error function is defined as $\mathrm{erfi}(z) = \mathrm{erf}(\imath z) := \frac{1}{\imath\sqrt{\pi}} \times \int_0^{\imath z} e^{t^2} dt$ for $z \in (-\infty, +\infty)$.
$f * g$	The Dirichlet convolution of any two arithmetic functions f and g at n is defined to be the divisor sum $(f * g)(n) := \sum_{d\mid n} f(d) g\left(\frac{n}{d}\right)$ for $n \geq 1$.

Symbols	**Definition**
$f^{-1}(n)$	The Dirichlet inverse f^{-1} of an arithmetic function f exists if and only if $f(1) \neq 0$. The Dirichlet inverse of any f such that $f(1) \neq 0$ is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \times \sum_{\substack{d\mid n \\ d>1}} f(d) f^{-1}\left(\frac{n}{d}\right)$ for $n \geq 2$ with $f^{-1}(1) = f(1)^{-1}$. When it exists, this inverse function is unique and satisfies $f^{-1} * f = f * f^{-1} = \varepsilon$.
$\Gamma(a,z)$	The incomplete gamma function is defined as $\Gamma(a,z) := \int_z^\infty t^{a-1} e^{-t} dt$ by continuation for $a \in \mathbb{R}$ and $\lvert\arg(z)\rvert < \pi$. Asymptotics of this function as both $a, z \to \infty$ independently are discussed in the appendix.
$\mathcal{G}(z), \widetilde{\mathcal{G}}(z)$; $\widehat{F}(s,z), \widehat{\mathcal{G}}(z)$	The functions $\mathcal{G}(z)$ and $\widetilde{\mathcal{G}}(z)$ are defined for $0 \leq \lvert z\rvert \leq R < 2$ on page 33 of Section B. The related constructions used to motivate the definitions of $\widehat{F}(s,z)$ and $\widehat{\mathcal{G}}(z)$ are defined by the infinite products over the primes given on pages 16 and 17 of Section 4.1, respectively.
$g^{-1}(n), G^{-1}(x), \lvert G^{-1}\rvert(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega+1)^{-1}(n)$, has the summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ for $x \geq 1$. We define the partial sums of the unsigned inverse function to be $\lvert G^{-1}\rvert(x) := \sum_{n\leq x} \lvert g^{-1}(n)\rvert$ for $x \geq 1$.
$[n=k]_\delta$,$[\texttt{cond}]_\delta$	The symbol $[n=k]_\delta$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise. For Boolean-valued conditions, `cond`, the symbol $[\texttt{cond}]_\delta$ evaluates to one precisely when `cond` is true or to zero otherwise.
$\lambda(n), L(x)$	The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. Its summatory function is defined by the partial sums $L(x) := \sum_{n\leq x} \lambda(n)$ for $x \geq 1$.
$\mu(n), M(x)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers $n \geq 1$ where $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree. The Mertens function is the summatory function defined for all integers $x \geq 1$ by the partial sums $M(x) := \sum_{n\leq x} \mu(n)$.
$\Phi(z), \mathcal{N}(0,1)$	For $z \in \mathbb{R}$, we take the cumulative density function (CDF) of the standard normal distribution to be denoted by $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt$. A random variable Z whose values are distributed according to the CDF $\Phi(z) = \mathbb{P}[Z \leq z]$ has distribution denoted by $Z \sim \mathcal{N}(0,1)$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^\alpha \| n$ for $p \geq 2$ prime, $\alpha \geq 1$ and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p\mid n} 1$ and the completely additive function $\Omega(n) := \sum_{p^\alpha \| n} \alpha$. This means that if the prime factorization of any $n \geq 2$ is given by $n := p_1^{\alpha_1} \times \cdots \times p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. We set $\omega(1) = \Omega(1) = 0$ by convention.
$\pi_k(x), \widehat{\pi}_k(x)$	For integers $k \geq 1$, the function $\pi_k(x)$ denotes the number of $2 \leq n \leq x$ with exactly k distinct prime factors: $\pi_k(x) := \#\{2 \leq n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{2 \leq n \leq x : \Omega(n) = k\}$ for $x \geq 2$ and fixed $k \geq 1$.

Symbols	Definition
$Q(x)$	For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. That is, $Q(x) := \sum_{n \leq x} \mu^2(n)$ where $Q(x) = \frac{6x}{\pi^2} + O(\sqrt{x})$.
$W(x)$	For $x, y \in [0, +\infty)$, we write that $x = W(y)$ if and only if $xe^x = y$. This function denotes the principal branch of the multi-valued Lambert W function taken over the non-negative reals.
$\zeta(s)$	The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation to any $s \in \mathbb{C}$ with the exception of a simple pole at $s = 1$ of residue one.

B Results characterizing the distributions of $\omega(n)$ and $\Omega(n)$

The next theorems reproduced from [18, §7.4] characterize the relative scarcity of the distributions of $\omega(n)$ and $\Omega(n)$ for $n \leq x$ such that $\omega(n), \Omega(n) < \log\log x$ and $\omega(n), \Omega(n) > \log\log x$. Since $\frac{1}{n} \times \sum_{k \leq n} \omega(k) = \log\log n + B_1 + o(1)$ and $\frac{1}{n} \times \sum_{k \leq n} \Omega(k) = \log\log n + B_2 + o(1)$ for $B_1 \approx 0.261497$ and $B_2 \approx 1.03465$ absolute constants in each case [10, §22.10], these results imply a distinctively regular tendency of these strongly additive arithmetic functions towards their respective average orders.

Theorem B.1 (Upper bounds on exceptional values of $\Omega(n)$ for large n)**.** *For $x \geq 2$ and $r > 0$, let*

$$A(x,r) := \#\left\{n \leq x : \Omega(n) \leq r\log\log x\right\},$$
$$B(x,r) := \#\left\{n \leq x : \Omega(n) \geq r\log\log x\right\}.$$

If $0 < r \leq 1$, then

$$A(x,r) \ll \quad x(\log x)^{r-1-r\log r}, \text{ as } x \to \infty.$$

If $1 \leq r \leq R < 2$, then

$$B(x,r) \ll_R x(\log x)^{r-1-r\log r}, \text{ as } x \to \infty.$$

Theorem B.2 is a special case analog of the Erdős-Kac theorem for the normally distributed values of $\frac{\omega(n)-\log\log n}{\sqrt{\log\log n}}$ over $n \leq x$ as $x \to \infty$ [18, *cf.* Thm. 7.21] [14, *cf.* §1.7].

Theorem B.2. *We have that as $x \to \infty$*

$$\#\left\{3 \leq n \leq x : \Omega(n) \leq \log\log n\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Theorem B.3 (Montgomery and Vaughan)**.** *Recall that for integers $k \geq 1$ and $x \geq 2$ we have defined*

$$\widehat{\pi}_k(x) := \#\{2 \leq n \leq x : \Omega(n) = k\}.$$

For $0 < R < 2$ we have uniformly for all $1 \leq k \leq R\log\log x$ that

$$\widehat{\pi}_k(x) = \frac{x}{\log x} \times \mathcal{G}\left(\frac{k-1}{\log\log x}\right)\frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

where

$$\mathcal{G}(z) := \frac{1}{\Gamma(1+z)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1}\left(1 - \frac{1}{p}\right)^z, 0 \leq |z| < R.$$

Remark B.4. We can extend the work in [18] on the distribution of $\Omega(n)$ to obtain corresponding analogous results for the distribution of $\omega(n)$. For $0 < R < 2$ we have that as $x \to \infty$

$$\pi_k(x) = \frac{x}{\log x} \times \widetilde{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right)\frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right), \tag{B.1}$$

uniformly for any $1 \leq k \leq R\log\log x$. The factors of the function $\widetilde{\mathcal{G}}(z)$ used to express these bounds are defined by $\widetilde{\mathcal{G}}(z) := \widetilde{F}(1,z) \times \Gamma(1+z)^{-1}$ where

$$\widetilde{F}(s,z) := \prod_p \left(1 + \frac{z}{p^s - 1}\right)\left(1 - \frac{1}{p^s}\right)^z, \operatorname{Re}(s) > \frac{1}{2}, |z| \leq R < 2.$$

Let the functions

$$C(x,r) := \#\{n \leq x : \omega(n) \leq r\log\log x\},$$
$$D(x,r) := \#\{n \leq x : \omega(n) \geq r\log\log x\}.$$

We have the following upper bounds that hold as $x \to \infty$:

$$C(x,r) \ll \quad x(\log x)^{r-1-r\log r}, \text{ uniformly for } 0 < r \leq 1,$$
$$D(x,r) \ll_R x(\log x)^{r-1-r\log r}, \text{ uniformly for } 1 \leq r \leq R < 2.$$

C Asymptotic formulas for partial sums involving the incomplete gamma function

We appreciate the correspondence with Gergő Nemes from the Alfréd Rényi Institute of Mathematics and his careful notes on the limiting asymptotics for the sums identified in this section. We have adapted the communication of his proofs to establish the next few lemmas based on his recent work in the references [19, 20, 21].

Facts C.1 (The incomplete gamma function)**.** The (upper) *incomplete gamma function* is defined by [24, §8.4]

$$\Gamma(a,z) = \int_z^{\infty} t^{a-1} e^{-t} dt, a \in \mathbb{R}, |\arg z| < \pi.$$

The function $\Gamma(a,z)$ can be continued to an analytic function of z on the universal covering of $\mathbb{C}\setminus\{0\}$. For $a \in \mathbb{Z}^+$, the function $\Gamma(a,z)$ is an entire function of z. The following properties of $\Gamma(a,z)$ hold [24, §8.4; §8.11(i)]:

$$\Gamma(a,z) = (a-1)! e^{-z} \times \sum_{k=0}^{a-1} \frac{z^k}{k!}, \text{ for } a \in \mathbb{Z}^+, z \in \mathbb{C}, \tag{C.1a}$$

$$\Gamma(a,z) \sim z^{a-1} e^{-z}, \text{ for fixed } a \in \mathbb{C}, \text{ as } z \to +\infty. \tag{C.1b}$$

Moreover, for real $z > 0$, as $z \to +\infty$ we have that [19]

$$\Gamma(z,z) = \sqrt{\frac{\pi}{2}} z^{z-\frac{1}{2}} e^{-z} + O\left(z^{z-1} e^{-z}\right), \tag{C.1c}$$

If $z, a \to \infty$ with $z = \lambda a$ for some $\lambda > 1$ such that $(\lambda-1)^{-1} = o\left(\sqrt{|a|}\right)$, then [19]

$$\Gamma(a,z) \sim z^a e^{-z} \times \sum_{n \geq 0} \frac{(-a)^n b_n(\lambda)}{(z-a)^{2n+1}}. \tag{C.1d}$$

The sequence $b_n(\lambda)$ satisfies the characteristic recurrence relation that $b_0(\lambda) = 1$ and[2]

$$b_n(\lambda) = \lambda(1-\lambda) b_{n-1}'(\lambda) + \lambda(2n-1) b_{n-1}(\lambda), n \geq 1.$$

Proposition C.2. *Let a, z, λ be positive real parameters such that $z = \lambda a$. If $\lambda \in (0,1)$, then as $z \to \infty$*

$$\Gamma(a,z) = \Gamma(a) + O_{\lambda}\left(z^{a-1} e^{-z}\right).$$

If $\lambda > 1$, then as $z \to \infty$

$$\Gamma(a,z) = \frac{z^{a-1} e^{-z}}{1-\lambda^{-1}} + O_{\lambda}\left(z^{a-2} e^{-z}\right).$$

If $\lambda > 0.567142 > W(1)$ where $W(x)$ denotes the principal branch of the Lambert W-function for $x \geq 0$, then as $z \to \infty$

$$\Gamma(a, z e^{\pm \pi \imath}) = -e^{\pm \pi \imath a} \frac{z^{a-1} e^z}{1+\lambda^{-1}} + O_{\lambda}\left(z^{a-2} e^z\right).$$

[2] An exact formula for $b_n(\lambda)$ is given in terms of the *second-order Eulerian number triangle* [28, A008517] as follows:

$$b_n(\lambda) = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \lambda^{k+1}.$$

Note that the first two estimates are only useful when λ is bounded away from the transition point at 1. We cannot write the last expansion above as $\Gamma(a,-z)$ directly unless $a \in \mathbb{Z}^+$ as the incomplete gamma function has a branch point at the origin with respect to its second variable. This function becomes a single-valued analytic function of its second input by continuation on the universal covering of $\mathbb{C} \setminus \{0\}$.

Proof. The first asymptotic estimate follows directly from the following asymptotic series expansion that holds as $z \to +\infty$ [21, Eq. (2.1)]:

$$\Gamma(a,z) \sim \Gamma(a) + z^a e^{-z} \times \sum_{k \geq 0} \frac{(-a)^k b_k(\lambda)}{(z-a)^{2k+1}}.$$

Using the notation from (C.1d) and [20], we have that

$$\Gamma(a,z) = \frac{z^{a-1}e^{-z}}{1-\lambda^{-1}} + z^a e^{-z} R_1(a,\lambda).$$

From the bounds in [20, §3.1], we have that

$$|z^a e^{-z} R_1(a,\lambda)| \leq z^a e^{-z} \times \frac{a \cdot b_1(\lambda)}{(z-a)^3} = \frac{z^{a-2}e^{-z}}{(1-\lambda^{-1})^3}$$

The main and error terms in the previous equation can also be seen by applying the asymptotic series in (C.1d) directly.

The proof of the third equation above follows from the following asymptotics [19, Eq. (1.1)]

$$\Gamma(-a,z) \sim z^{-a} e^{-z} \times \sum_{n \geq 0} \frac{a^n b_n(-\lambda)}{(z+a)^{2n+1}},$$

by setting $(a,z) \mapsto (ae^{\pm \pi \imath}, ze^{\pm \pi \imath})$ so that $\lambda = \frac{z}{a} > 0.567142 > W(1)$. The restriction on the range of λ over which the third formula holds is made to ensure that the last formula from the reference is valid at negative real a. □

Lemma C.3. *For $x \to +\infty$, we have that*

$$S_1(x) := \frac{x}{\log x} \times \left| \sum_{1 \leq k \leq \lfloor \log\log x \rfloor} \frac{(-1)^k (\log\log x)^{k-1}}{(k-1)!} \right| = \frac{x}{2\sqrt{2\pi \log\log x}} + O\left(\frac{x}{(\log\log x)^{\frac{3}{2}}}\right).$$

Proof. We have for $n \geq 1$ and any $t > 0$ by (C.1a) that

$$\sum_{1 \leq k \leq n} \frac{(-1)^k t^{k-1}}{(k-1)!} = -e^{-t} \times \frac{\Gamma(n,-t)}{(n-1)!}.$$

Suppose that $t = n + \xi$ with $\xi = O(1)$, e.g., so we can formally take the floor of the input n to truncate the last sum. By the third formula in Proposition C.2 with the parameters $(a,z,\lambda) \mapsto \left(n, t, 1 + \frac{\xi}{n}\right)$, we deduce that as $n, t \to +\infty$.

$$\Gamma(n,-t) = (-1)^{n+1} \times \frac{t^n e^t}{t+n} + O\left(\frac{nt^n e^t}{(t+n)^3}\right) = (-1)^{n+1} \frac{t^n e^t}{2n} + O\left(\frac{t^{n-1}e^t}{n}\right). \tag{C.2}$$

Accordingly, we see that

$$\sum_{1 \leq k \leq n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{t^n}{2n!} + O\left(\frac{t^{n-1}}{n!}\right).$$

By the variant of Stirling's formula in [24, *cf.* Eq. (5.11.8)], we have

$$n! = \Gamma(1+t-\xi) = \sqrt{2\pi}t^{t-\xi+\frac{1}{2}}e^{-t}\left(1+O\left(t^{-1}\right)\right) = \sqrt{2\pi}t^{n+\frac{1}{2}}e^{-t}\left(1+O\left(t^{-1}\right)\right).$$

Hence, as $n \to +\infty$ with $t := n+\xi$ and $\xi = O(1)$, we obtain that

$$\sum_{k=1}^{n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{e^t}{2\sqrt{2\pi t}} + O\left(e^t t^{-\frac{3}{2}}\right).$$

The conclusion follows by taking $n := \lfloor \log\log x \rfloor$, $t := \log\log x$ and applying the triangle inequality to obtain the result. □

D Combinatorial connections between $g^{-1}(n)$ and the distribution of the primes

The combinatorial properties of $g^{-1}(n)$ are deeply tied to the distribution of the primes $p \leq n$ as $n \to \infty$. The magnitudes of and spacing between the primes $p \leq n$ restricts the repeating of these distinct sequence values. We can see that the following is still clear about the relation of the weight functions $|g^{-1}(n)|$ to the distribution of the primes: The value of $|g^{-1}(n)|$ is entirely dependent only on the pattern of the exponents (viewed as multisets) of the distinct prime factors of $n \geq 2$, rather than on the prime factor weights themselves (*cf.* Observation 1.2). This property shows that $|g^{-1}(n)|$ has an inherently additive, rather than multiplicative, structure underneath the distribution of its distinct values over $n \leq x$. The concrete new links of the partial sums of $g^{-1}(n)$ to exact formulae characterizing $M(x)$ is then somewhat surprising given that the coefficients in the expansion of the Euler product representation of $\zeta(s)^{-1}$, or the DGF of the original signed summands, $\mu(n)$, suggests a fundamentally convolved and hard-to-invert multiplication structure that defines the components to the classical partial sums. In this sense, we surmise that by replacing the known hard analysis involving oscillatory imaginary parts in the complex plane with the reciprocal zeta function to define $M(x)$ we gain new information on the underlying structures that can be more easily recovered by working with the unsigned sequences first.

Remark D.1 (Local extrema of the inverse sequence)**.** There is a natural extremal behavior of $|g^{-1}(n)|$ with respect to the distinct values of $\Omega(n)$ at squarefree integers and prime powers. For integers $k \geq 1$ we define the infinite sets $\overline{M}_k$ and $\underline{m}_k$ to correspond to the maximal (minimal) sets of positive integers such that

$$\overline{M}_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

$$\underline{m}_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+.$$

Any element of $\overline{M}_k$ is squarefree and any element of $\underline{m}_k$ is a prime power. Moreover, for any fixed $k \geq 1$ we have that for any $N_k \in \overline{M}_k$ and $n_k \in \underline{m}_k$

$$(-1)^k g^{-1}(N_k) = \sum_{j=0}^{k} \binom{k}{j} \times j!, \qquad \text{and} \qquad (-1)^k g^{-1}(n_k) = 2,$$

where $\lambda(N_k) = \lambda(n_k) = (-1)^k$.

Example D.2 (Combinatorial formulas involving symmetric polynomials in prime factorization exponents)**.** The formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 2.3 shows that we can express $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n. For $n \geq 2$ and $0 \leq k \leq \omega(n)$ let

$$\widehat{e}_k(n) := [t^k] \prod_{p|n} (1 + t\nu_p(n)) = [t^k] \prod_{p^\alpha \| n} (1 + \alpha t).$$

Suppose that we set $P_k(n) := \sum_{p^\alpha \| n} \alpha^k$ for any $n \geq 2$. Newton's (Newton-Girard) formulas relating general classes of elementary symmetric polynomials to their corresponding power-sum symmetric polynomials (here, in the form of $\widehat{P}_k(n)$ for $n \geq k \geq 1$) show that

$$\widehat{e}_0(n) = 1, \widehat{e}_1(n) = \widehat{P}_1(n), \widehat{e}_2(n) = \frac{1}{2}\left(\widehat{P}_1(n)^2 - \widehat{P}_2(n)\right), \widehat{e}_3(n) = \frac{1}{6}\left(\widehat{P}_1(n)^3 - 3\widehat{P}_1(n)\widehat{P}_2(n) + 2\widehat{P}_3(n)\right), n \geq 2,$$

though general expressions can be given in analog to the special cases in the last equation for $k \geq 4$ as well. We can then prove using (2.4) and (3.4) that the following formula holds:

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \times \frac{\widehat{e}_k(n)}{k!}, n \geq 2.$$

We also notice that both of the functions $f_1(n) := \frac{h^{-1}(n)}{(\Omega(n))!}$ and $f_2(n) := \frac{\lambda(n)h^{-1}(n)}{(\Omega(n))!}$ are both multiplicative, Dirichlet invertible arithmetic functions of $n \geq 1$.

The key combinatorial formula for $h^{-1}(n) = \lambda(n)(\Omega(n))! \times \prod_{p^\alpha || n}(\alpha!)^{-1}$ suggests additional patterns and regularity in the contributions of the distinct sign weighted terms in the summands of $G^{-1}(x)$[3]. Section 6.2 discusses limiting asymptotic properties with respect to the expected substantial local cancellation in the formula for $M(x)$ at most large $x \geq 1$ from (3.5) that is expanded exactly through the auxiliary sums $G^{-1}(x)$ as in (3.5) above.

[3]This sequence is also considered using a different motivation based on the DGFs $(1 \pm P(s))^{-1}$ in [9, §2].

E Tables of computations involving $g^{-1}(n)$ and its partial sums

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$	$\lvert G^{-1}\rvert(n)$
1	1^1	Y	N	1	0	1.0000000	1.00000	0	1	1	0	1
2	2^1	Y	Y	−2	0	1.0000000	0.500000	0.500000	−1	1	−2	3
3	3^1	Y	Y	−2	0	1.0000000	0.333333	0.666667	−3	1	−4	5
4	2^2	N	Y	2	0	1.5000000	0.500000	0.500000	−1	3	−4	7
5	5^1	Y	Y	−2	0	1.0000000	0.400000	0.600000	−3	3	−6	9
6	$2^1 3^1$	Y	N	5	0	1.0000000	0.500000	0.500000	2	8	−6	14
7	7^1	Y	Y	−2	0	1.0000000	0.428571	0.571429	0	8	−8	16
8	2^3	N	Y	−2	0	2.0000000	0.375000	0.625000	−2	8	−10	18
9	3^2	N	Y	2	0	1.5000000	0.444444	0.555556	0	10	−10	20
10	$2^1 5^1$	Y	N	5	0	1.0000000	0.500000	0.500000	5	15	−10	25
11	11^1	Y	Y	−2	0	1.0000000	0.454545	0.545455	3	15	−12	27
12	$2^2 3^1$	N	N	−7	2	1.2857143	0.416667	0.583333	−4	15	−19	34
13	13^1	Y	Y	−2	0	1.0000000	0.384615	0.615385	−6	15	−21	36
14	$2^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	−1	20	−21	41
15	$3^1 5^1$	Y	N	5	0	1.0000000	0.466667	0.533333	4	25	−21	46
16	2^4	N	Y	2	0	2.5000000	0.500000	0.500000	6	27	−21	48
17	17^1	Y	Y	−2	0	1.0000000	0.470588	0.529412	4	27	−23	50
18	$2^1 3^2$	N	N	−7	2	1.2857143	0.444444	0.555556	−3	27	−30	57
19	19^1	Y	Y	−2	0	1.0000000	0.421053	0.578947	−5	27	−32	59
20	$2^2 5^1$	N	N	−7	2	1.2857143	0.400000	0.600000	−12	27	−39	66
21	$3^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	−7	32	−39	71
22	$2^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−2	37	−39	76
23	23^1	Y	Y	−2	0	1.0000000	0.434783	0.565217	−4	37	−41	78
24	$2^3 3^1$	N	N	9	4	1.5555556	0.458333	0.541667	5	46	−41	87
25	5^2	N	Y	2	0	1.5000000	0.480000	0.520000	7	48	−41	89
26	$2^1 13^1$	Y	N	5	0	1.0000000	0.500000	0.500000	12	53	−41	94
27	3^3	N	Y	−2	0	2.0000000	0.481481	0.518519	10	53	−43	96
28	$2^2 7^1$	N	N	−7	2	1.2857143	0.464286	0.535714	3	53	−50	103
29	29^1	Y	Y	−2	0	1.0000000	0.448276	0.551724	1	53	−52	105
30	$2^1 3^1 5^1$	Y	N	−16	0	1.0000000	0.433333	0.566667	−15	53	−68	121
31	31^1	Y	Y	−2	0	1.0000000	0.419355	0.580645	−17	53	−70	123
32	2^5	N	Y	−2	0	3.0000000	0.406250	0.593750	−19	53	−72	125
33	$3^1 11^1$	Y	N	5	0	1.0000000	0.424242	0.575758	−14	58	−72	130
34	$2^1 17^1$	Y	N	5	0	1.0000000	0.441176	0.558824	−9	63	−72	135
35	$5^1 7^1$	Y	N	5	0	1.0000000	0.457143	0.542857	−4	68	−72	140
36	$2^2 3^2$	N	N	14	9	1.3571429	0.472222	0.527778	10	82	−72	154
37	37^1	Y	Y	−2	0	1.0000000	0.459459	0.540541	8	82	−74	156
38	$2^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	13	87	−74	161
39	$3^1 13^1$	Y	N	5	0	1.0000000	0.487179	0.512821	18	92	−74	166
40	$2^3 5^1$	N	N	9	4	1.5555556	0.500000	0.500000	27	101	−74	175
41	41^1	Y	Y	−2	0	1.0000000	0.487805	0.512195	25	101	−76	177
42	$2^1 3^1 7^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	9	101	−92	193
43	43^1	Y	Y	−2	0	1.0000000	0.465116	0.534884	7	101	−94	195
44	$2^2 11^1$	N	N	−7	2	1.2857143	0.454545	0.545455	0	101	−101	202
45	$3^2 5^1$	N	N	−7	2	1.2857143	0.444444	0.555556	−7	101	−108	209
46	$2^1 23^1$	Y	N	5	0	1.0000000	0.456522	0.543478	−2	106	−108	214
47	47^1	Y	Y	−2	0	1.0000000	0.446809	0.553191	−4	106	−110	216
48	$2^4 3^1$	N	N	−11	6	1.8181818	0.437500	0.562500	−15	106	−121	227

Table E: Computations involving $g^{-1}(n) \equiv (\omega+1)^{-1}(n)$ and $G^{-1}(x)$ for $1 \leq n \leq 500$.

- *The column labeled* `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled* `Sqfree` *and* `PPower`*, respectively, list inclusion of n in the sets of squarefree integers and the prime powers.*
- *The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \times k!$.*
- *The last columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the (approximate) densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{n} \times \#\{n \leq x : \lambda(n) = \pm 1\}$. The last three columns then show the sign weighted components to the signed summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geq 1$. That is, the component functions $G_{\pm}^{-1}(x)$ displayed in these second to last two columns of the table correspond to the summatory function $G^{-1}(x)$ with summands that are positive and negative, respectively. The final column of the table provides the partial sums of the absolute value of the unsigned inverse sequence, $|G^{-1}|(n) := \sum_{k \leq n} |g^{-1}(k)|$.*

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$	$\lvert G^{-1}\rvert(n)$
49	7^2	N	Y	2	0	1.5000000	0.448980	0.551020	−13	108	−121	229
50	$2^1 5^2$	N	N	−7	2	1.2857143	0.440000	0.560000	−20	108	−128	236
51	$3^1 17^1$	Y	N	5	0	1.0000000	0.450980	0.549020	−15	113	−128	241
52	$2^2 13^1$	N	N	−7	2	1.2857143	0.442308	0.557692	−22	113	−135	248
53	53^1	Y	Y	−2	0	1.0000000	0.433962	0.566038	−24	113	−137	250
54	$2^1 3^3$	N	N	9	4	1.5555556	0.444444	0.555556	−15	122	−137	259
55	$5^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−10	127	−137	264
56	$2^3 7^1$	N	N	9	4	1.5555556	0.464286	0.535714	−1	136	−137	273
57	$3^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	4	141	−137	278
58	$2^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	9	146	−137	283
59	59^1	Y	Y	−2	0	1.0000000	0.474576	0.525424	7	146	−139	285
60	$2^2 3^1 5^1$	N	N	30	14	1.1666667	0.483333	0.516667	37	176	−139	315
61	61^1	Y	Y	−2	0	1.0000000	0.475410	0.524590	35	176	−141	317
62	$2^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	40	181	−141	322
63	$3^2 7^1$	N	N	−7	2	1.2857143	0.476190	0.523810	33	181	−148	329
64	2^6	N	Y	2	0	3.5000000	0.484375	0.515625	35	183	−148	331
65	$5^1 13^1$	Y	N	5	0	1.0000000	0.492308	0.507692	40	188	−148	336
66	$2^1 3^1 11^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	24	188	−164	352
67	67^1	Y	Y	−2	0	1.0000000	0.477612	0.522388	22	188	−166	354
68	$2^2 17^1$	N	N	−7	2	1.2857143	0.470588	0.529412	15	188	−173	361
69	$3^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	20	193	−173	366
70	$2^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.471429	0.528571	4	193	−189	382
71	71^1	Y	Y	−2	0	1.0000000	0.464789	0.535211	2	193	−191	384
72	$2^3 3^2$	N	N	−23	18	1.4782609	0.458333	0.541667	−21	193	−214	407
73	73^1	Y	Y	−2	0	1.0000000	0.452055	0.547945	−23	193	−216	409
74	$2^1 37^1$	Y	N	5	0	1.0000000	0.459459	0.540541	−18	198	−216	414
75	$3^1 5^2$	N	N	−7	2	1.2857143	0.453333	0.546667	−25	198	−223	421
76	$2^2 19^1$	N	N	−7	2	1.2857143	0.447368	0.552632	−32	198	−230	428
77	$7^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−27	203	−230	433
78	$2^1 3^1 13^1$	Y	N	−16	0	1.0000000	0.448718	0.551282	−43	203	−246	449
79	79^1	Y	Y	−2	0	1.0000000	0.443038	0.556962	−45	203	−248	451
80	$2^4 5^1$	N	N	−11	6	1.8181818	0.437500	0.562500	−56	203	−259	462
81	3^4	N	Y	2	0	2.5000000	0.444444	0.555556	−54	205	−259	464
82	$2^1 41^1$	Y	N	5	0	1.0000000	0.451220	0.548780	−49	210	−259	469
83	83^1	Y	Y	−2	0	1.0000000	0.445783	0.554217	−51	210	−261	471
84	$2^2 3^1 7^1$	N	N	30	14	1.1666667	0.452381	0.547619	−21	240	−261	501
85	$5^1 17^1$	Y	N	5	0	1.0000000	0.458824	0.541176	−16	245	−261	506
86	$2^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	−11	250	−261	511
87	$3^1 29^1$	Y	N	5	0	1.0000000	0.471264	0.528736	−6	255	−261	516
88	$2^3 11^1$	N	N	9	4	1.5555556	0.477273	0.522727	3	264	−261	525
89	89^1	Y	Y	−2	0	1.0000000	0.471910	0.528090	1	264	−263	527
90	$2^1 3^2 5^1$	N	N	30	14	1.1666667	0.477778	0.522222	31	294	−263	557
91	$7^1 13^1$	Y	N	5	0	1.0000000	0.483516	0.516484	36	299	−263	562
92	$2^2 23^1$	N	N	−7	2	1.2857143	0.478261	0.521739	29	299	−270	569
93	$3^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	34	304	−270	574
94	$2^1 47^1$	Y	N	5	0	1.0000000	0.489362	0.510638	39	309	−270	579
95	$5^1 19^1$	Y	N	5	0	1.0000000	0.494737	0.505263	44	314	−270	584
96	$2^5 3^1$	N	N	13	8	2.0769231	0.500000	0.500000	57	327	−270	597
97	97^1	Y	Y	−2	0	1.0000000	0.494845	0.505155	55	327	−272	599
98	$2^1 7^2$	N	N	−7	2	1.2857143	0.489796	0.510204	48	327	−279	606
99	$3^2 11^1$	N	N	−7	2	1.2857143	0.484848	0.515152	41	327	−286	613
100	$2^2 5^2$	N	N	14	9	1.3571429	0.490000	0.510000	55	341	−286	627
101	101^1	Y	Y	−2	0	1.0000000	0.485149	0.514851	53	341	−288	629
102	$2^1 3^1 17^1$	Y	N	−16	0	1.0000000	0.480392	0.519608	37	341	−304	645
103	103^1	Y	Y	−2	0	1.0000000	0.475728	0.524272	35	341	−306	647
104	$2^3 13^1$	N	N	9	4	1.5555556	0.480769	0.519231	44	350	−306	656
105	$3^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	28	350	−322	672
106	$2^1 53^1$	Y	N	5	0	1.0000000	0.481132	0.518868	33	355	−322	677
107	107^1	Y	Y	−2	0	1.0000000	0.476636	0.523364	31	355	−324	679
108	$2^2 3^3$	N	N	−23	18	1.4782609	0.472222	0.527778	8	355	−347	702
109	109^1	Y	Y	−2	0	1.0000000	0.467890	0.532110	6	355	−349	704
110	$2^1 5^1 11^1$	Y	N	−16	0	1.0000000	0.463636	0.536364	−10	355	−365	720
111	$3^1 37^1$	Y	N	5	0	1.0000000	0.468468	0.531532	−5	360	−365	725
112	$2^4 7^1$	N	N	−11	6	1.8181818	0.464286	0.535714	−16	360	−376	736
113	113^1	Y	Y	−2	0	1.0000000	0.460177	0.539823	−18	360	−378	738
114	$2^1 3^1 19^1$	Y	N	−16	0	1.0000000	0.456140	0.543860	−34	360	−394	754
115	$5^1 23^1$	Y	N	5	0	1.0000000	0.460870	0.539130	−29	365	−394	759
116	$2^2 29^1$	N	N	−7	2	1.2857143	0.456897	0.543103	−36	365	−401	766
117	$3^2 13^1$	N	N	−7	2	1.2857143	0.452991	0.547009	−43	365	−408	773
118	$2^1 59^1$	Y	N	5	0	1.0000000	0.457627	0.542373	−38	370	−408	778
119	$7^1 17^1$	Y	N	5	0	1.0000000	0.462185	0.537815	−33	375	−408	783
120	$2^3 3^1 5^1$	N	N	−48	32	1.3333333	0.458333	0.541667	−81	375	−456	831
121	11^2	N	Y	2	0	1.5000000	0.462810	0.537190	−79	377	−456	833
122	$2^1 61^1$	Y	N	5	0	1.0000000	0.467213	0.532787	−74	382	−456	838
123	$3^1 41^1$	Y	N	5	0	1.0000000	0.471545	0.528455	−69	387	−456	843
124	$2^2 31^1$	N	N	−7	2	1.2857143	0.467742	0.532258	−76	387	−463	850

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_{+}(n)$	$\mathcal{L}_{-}(n)$	$G^{-1}(n)$	$G_{+}^{-1}(n)$	$G_{-}^{-1}(n)$	$\lvert G^{-1}\rvert(n)$
125	5^3	N	Y	−2	0	2.0000000	0.464000	0.536000	−78	387	−465	852
126	$2^1 3^2 7^1$	N	N	30	14	1.1666667	0.468254	0.531746	−48	417	−465	882
127	127^1	Y	Y	−2	0	1.0000000	0.464567	0.535433	−50	417	−467	884
128	2^7	N	Y	−2	0	4.0000000	0.460938	0.539062	−52	417	−469	886
129	$3^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	−47	422	−469	891
130	$2^1 5^1 13^1$	Y	N	−16	0	1.0000000	0.461538	0.538462	−63	422	−485	907
131	131^1	Y	Y	−2	0	1.0000000	0.458015	0.541985	−65	422	−487	909
132	$2^2 3^1 11^1$	N	N	30	14	1.1666667	0.462121	0.537879	−35	452	−487	939
133	$7^1 19^1$	Y	N	5	0	1.0000000	0.466165	0.533835	−30	457	−487	944
134	$2^1 67^1$	Y	N	5	0	1.0000000	0.470149	0.529851	−25	462	−487	949
135	$3^3 5^1$	N	N	9	4	1.5555556	0.474074	0.525926	−16	471	−487	958
136	$2^3 17^1$	N	N	9	4	1.5555556	0.477941	0.522059	−7	480	−487	967
137	137^1	Y	Y	−2	0	1.0000000	0.474453	0.525547	−9	480	−489	969
138	$2^1 3^1 23^1$	Y	N	−16	0	1.0000000	0.471014	0.528986	−25	480	−505	985
139	139^1	Y	Y	−2	0	1.0000000	0.467626	0.532374	−27	480	−507	987
140	$2^2 5^1 7^1$	N	N	30	14	1.1666667	0.471429	0.528571	3	510	−507	1017
141	$3^1 47^1$	Y	N	5	0	1.0000000	0.475177	0.524823	8	515	−507	1022
142	$2^1 71^1$	Y	N	5	0	1.0000000	0.478873	0.521127	13	520	−507	1027
143	$11^1 13^1$	Y	N	5	0	1.0000000	0.482517	0.517483	18	525	−507	1032
144	$2^4 3^2$	N	N	34	29	1.6176471	0.486111	0.513889	52	559	−507	1066
145	$5^1 29^1$	Y	N	5	0	1.0000000	0.489655	0.510345	57	564	−507	1071
146	$2^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	62	569	−507	1076
147	$3^1 7^2$	N	N	−7	2	1.2857143	0.489796	0.510204	55	569	−514	1083
148	$2^2 37^1$	N	N	−7	2	1.2857143	0.486486	0.513514	48	569	−521	1090
149	149^1	Y	Y	−2	0	1.0000000	0.483221	0.516779	46	569	−523	1092
150	$2^1 3^1 5^2$	N	N	30	14	1.1666667	0.486667	0.513333	76	599	−523	1122
151	151^1	Y	Y	−2	0	1.0000000	0.483444	0.516556	74	599	−525	1124
152	$2^3 19^1$	N	N	9	4	1.5555556	0.486842	0.513158	83	608	−525	1133
153	$3^2 17^1$	N	N	−7	2	1.2857143	0.483660	0.516340	76	608	−532	1140
154	$2^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.480519	0.519481	60	608	−548	1156
155	$5^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	65	613	−548	1161
156	$2^2 3^1 13^1$	N	N	30	14	1.1666667	0.487179	0.512821	95	643	−548	1191
157	157^1	Y	Y	−2	0	1.0000000	0.484076	0.515924	93	643	−550	1193
158	$2^1 79^1$	Y	N	5	0	1.0000000	0.487342	0.512658	98	648	−550	1198
159	$3^1 53^1$	Y	N	5	0	1.0000000	0.490566	0.509434	103	653	−550	1203
160	$2^5 5^1$	N	N	13	8	2.0769231	0.493750	0.506250	116	666	−550	1216
161	$7^1 23^1$	Y	N	5	0	1.0000000	0.496894	0.503106	121	671	−550	1221
162	$2^1 3^4$	N	N	−11	6	1.8181818	0.493827	0.506173	110	671	−561	1232
163	163^1	Y	Y	−2	0	1.0000000	0.490798	0.509202	108	671	−563	1234
164	$2^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	101	671	−570	1241
165	$3^1 5^1 11^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	85	671	−586	1257
166	$2^1 83^1$	Y	N	5	0	1.0000000	0.487952	0.512048	90	676	−586	1262
167	167^1	Y	Y	−2	0	1.0000000	0.485030	0.514970	88	676	−588	1264
168	$2^3 3^1 7^1$	N	N	−48	32	1.3333333	0.482143	0.517857	40	676	−636	1312
169	13^2	N	Y	2	0	1.5000000	0.485207	0.514793	42	678	−636	1314
170	$2^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.482353	0.517647	26	678	−652	1330
171	$3^2 19^1$	N	N	−7	2	1.2857143	0.479532	0.520468	19	678	−659	1337
172	$2^2 43^1$	N	N	−7	2	1.2857143	0.476744	0.523256	12	678	−666	1344
173	173^1	Y	Y	−2	0	1.0000000	0.473988	0.526012	10	678	−668	1346
174	$2^1 3^1 29^1$	Y	N	−16	0	1.0000000	0.471264	0.528736	−6	678	−684	1362
175	$5^2 7^1$	N	N	−7	2	1.2857143	0.468571	0.531429	−13	678	−691	1369
176	$2^4 11^1$	N	N	−11	6	1.8181818	0.465909	0.534091	−24	678	−702	1380
177	$3^1 59^1$	Y	N	5	0	1.0000000	0.468927	0.531073	−19	683	−702	1385
178	$2^1 89^1$	Y	N	5	0	1.0000000	0.471910	0.528090	−14	688	−702	1390
179	179^1	Y	Y	−2	0	1.0000000	0.469274	0.530726	−16	688	−704	1392
180	$2^2 3^2 5^1$	N	N	−74	58	1.2162162	0.466667	0.533333	−90	688	−778	1466
181	181^1	Y	Y	−2	0	1.0000000	0.464088	0.535912	−92	688	−780	1468
182	$2^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.461538	0.538462	−108	688	−796	1484
183	$3^1 61^1$	Y	N	5	0	1.0000000	0.464481	0.535519	−103	693	−796	1489
184	$2^3 23^1$	N	N	9	4	1.5555556	0.467391	0.532609	−94	702	−796	1498
185	$5^1 37^1$	Y	N	5	0	1.0000000	0.470270	0.529730	−89	707	−796	1503
186	$2^1 3^1 31^1$	Y	N	−16	0	1.0000000	0.467742	0.532258	−105	707	−812	1519
187	$11^1 17^1$	Y	N	5	0	1.0000000	0.470588	0.529412	−100	712	−812	1524
188	$2^2 47^1$	N	N	−7	2	1.2857143	0.468085	0.531915	−107	712	−819	1531
189	$3^3 7^1$	N	N	9	4	1.5555556	0.470899	0.529101	−98	721	−819	1540
190	$2^1 5^1 19^1$	Y	N	−16	0	1.0000000	0.468421	0.531579	−114	721	−835	1556
191	191^1	Y	Y	−2	0	1.0000000	0.465969	0.534031	−116	721	−837	1558
192	$2^6 3^1$	N	N	−15	10	2.3333333	0.463542	0.536458	−131	721	−852	1573
193	193^1	Y	Y	−2	0	1.0000000	0.461140	0.538860	−133	721	−854	1575
194	$2^1 97^1$	Y	N	5	0	1.0000000	0.463918	0.536082	−128	726	−854	1580
195	$3^1 5^1 13^1$	Y	N	−16	0	1.0000000	0.461538	0.538462	−144	726	−870	1596
196	$2^2 7^2$	N	N	14	9	1.3571429	0.464286	0.535714	−130	740	−870	1610
197	197^1	Y	Y	−2	0	1.0000000	0.461929	0.538071	−132	740	−872	1612
198	$2^1 3^2 11^1$	N	N	30	14	1.1666667	0.464646	0.535354	−102	770	−872	1642
199	199^1	Y	Y	−2	0	1.0000000	0.462312	0.537688	−104	770	−874	1644
200	$2^3 5^2$	N	N	−23	18	1.4782609	0.460000	0.540000	−127	770	−897	1667

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$	$\lvert G^{-1}\rvert(n)$
201	$3^1 67^1$	Y	N	5	0	1.0000000	0.462687	0.537313	−122	775	−897	1672
202	$2^1 101^1$	Y	N	5	0	1.0000000	0.465347	0.534653	−117	780	−897	1677
203	$7^1 29^1$	Y	N	5	0	1.0000000	0.467980	0.532020	−112	785	−897	1682
204	$2^2 3^1 17^1$	N	N	30	14	1.1666667	0.470588	0.529412	−82	815	−897	1712
205	$5^1 41^1$	Y	N	5	0	1.0000000	0.473171	0.526829	−77	820	−897	1717
206	$2^1 103^1$	Y	N	5	0	1.0000000	0.475728	0.524272	−72	825	−897	1722
207	$3^2 23^1$	N	N	−7	2	1.2857143	0.473430	0.526570	−79	825	−904	1729
208	$2^4 13^1$	N	N	−11	6	1.8181818	0.471154	0.528846	−90	825	−915	1740
209	$11^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	−85	830	−915	1745
210	$2^1 3^1 5^1 7^1$	Y	N	65	0	1.0000000	0.476190	0.523810	−20	895	−915	1810
211	211^1	Y	Y	−2	0	1.0000000	0.473934	0.526066	−22	895	−917	1812
212	$2^2 53^1$	N	N	−7	2	1.2857143	0.471698	0.528302	−29	895	−924	1819
213	$3^1 71^1$	Y	N	5	0	1.0000000	0.474178	0.525822	−24	900	−924	1824
214	$2^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−19	905	−924	1829
215	$5^1 43^1$	Y	N	5	0	1.0000000	0.479070	0.520930	−14	910	−924	1834
216	$2^3 3^3$	N	N	46	41	1.5000000	0.481481	0.518519	32	956	−924	1880
217	$7^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	37	961	−924	1885
218	$2^1 109^1$	Y	N	5	0	1.0000000	0.486239	0.513761	42	966	−924	1890
219	$3^1 73^1$	Y	N	5	0	1.0000000	0.488584	0.511416	47	971	−924	1895
220	$2^2 5^1 11^1$	N	N	30	14	1.1666667	0.490909	0.509091	77	1001	−924	1925
221	$13^1 17^1$	Y	N	5	0	1.0000000	0.493213	0.506787	82	1006	−924	1930
222	$2^1 3^1 37^1$	Y	N	−16	0	1.0000000	0.490991	0.509009	66	1006	−940	1946
223	223^1	Y	Y	−2	0	1.0000000	0.488789	0.511211	64	1006	−942	1948
224	$2^5 7^1$	N	N	13	8	2.0769231	0.491071	0.508929	77	1019	−942	1961
225	$3^2 5^2$	N	N	14	9	1.3571429	0.493333	0.506667	91	1033	−942	1975
226	$2^1 113^1$	Y	N	5	0	1.0000000	0.495575	0.504425	96	1038	−942	1980
227	227^1	Y	Y	−2	0	1.0000000	0.493392	0.506608	94	1038	−944	1982
228	$2^2 3^1 19^1$	N	N	30	14	1.1666667	0.495614	0.504386	124	1068	−944	2012
229	229^1	Y	Y	−2	0	1.0000000	0.493450	0.506550	122	1068	−946	2014
230	$2^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.491304	0.508696	106	1068	−962	2030
231	$3^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.489177	0.510823	90	1068	−978	2046
232	$2^3 29^1$	N	N	9	4	1.5555556	0.491379	0.508621	99	1077	−978	2055
233	233^1	Y	Y	−2	0	1.0000000	0.489270	0.510730	97	1077	−980	2057
234	$2^1 3^2 13^1$	N	N	30	14	1.1666667	0.491453	0.508547	127	1107	−980	2087
235	$5^1 47^1$	Y	N	5	0	1.0000000	0.493617	0.506383	132	1112	−980	2092
236	$2^2 59^1$	N	N	−7	2	1.2857143	0.491525	0.508475	125	1112	−987	2099
237	$3^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	130	1117	−987	2104
238	$2^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.491597	0.508403	114	1117	−1003	2120
239	239^1	Y	Y	−2	0	1.0000000	0.489540	0.510460	112	1117	−1005	2122
240	$2^4 3^1 5^1$	N	N	70	54	1.5000000	0.491667	0.508333	182	1187	−1005	2192
241	241^1	Y	Y	−2	0	1.0000000	0.489627	0.510373	180	1187	−1007	2194
242	$2^1 11^2$	N	N	−7	2	1.2857143	0.487603	0.512397	173	1187	−1014	2201
243	3^5	N	Y	−2	0	3.0000000	0.485597	0.514403	171	1187	−1016	2203
244	$2^2 61^1$	N	N	−7	2	1.2857143	0.483607	0.516393	164	1187	−1023	2210
245	$5^1 7^2$	N	N	−7	2	1.2857143	0.481633	0.518367	157	1187	−1030	2217
246	$2^1 3^1 41^1$	Y	N	−16	0	1.0000000	0.479675	0.520325	141	1187	−1046	2233
247	$13^1 19^1$	Y	N	5	0	1.0000000	0.481781	0.518219	146	1192	−1046	2238
248	$2^3 31^1$	N	N	9	4	1.5555556	0.483871	0.516129	155	1201	−1046	2247
249	$3^1 83^1$	Y	N	5	0	1.0000000	0.485944	0.514056	160	1206	−1046	2252
250	$2^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	169	1215	−1046	2261
251	251^1	Y	Y	−2	0	1.0000000	0.486056	0.513944	167	1215	−1048	2263
252	$2^2 3^2 7^1$	N	N	−74	58	1.2162162	0.484127	0.515873	93	1215	−1122	2337
253	$11^1 23^1$	Y	N	5	0	1.0000000	0.486166	0.513834	98	1220	−1122	2342
254	$2^1 127^1$	Y	N	5	0	1.0000000	0.488189	0.511811	103	1225	−1122	2347
255	$3^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.486275	0.513725	87	1225	−1138	2363
256	2^8	N	Y	2	0	4.5000000	0.488281	0.511719	89	1227	−1138	2365
257	257^1	Y	Y	−2	0	1.0000000	0.486381	0.513619	87	1227	−1140	2367
258	$2^1 3^1 43^1$	Y	N	−16	0	1.0000000	0.484496	0.515504	71	1227	−1156	2383
259	$7^1 37^1$	Y	N	5	0	1.0000000	0.486486	0.513514	76	1232	−1156	2388
260	$2^2 5^1 13^1$	N	N	30	14	1.1666667	0.488462	0.511538	106	1262	−1156	2418
261	$3^2 29^1$	N	N	−7	2	1.2857143	0.486590	0.513410	99	1262	−1163	2425
262	$2^1 131^1$	Y	N	5	0	1.0000000	0.488550	0.511450	104	1267	−1163	2430
263	263^1	Y	Y	−2	0	1.0000000	0.486692	0.513308	102	1267	−1165	2432
264	$2^3 3^1 11^1$	N	N	−48	32	1.3333333	0.484848	0.515152	54	1267	−1213	2480
265	$5^1 53^1$	Y	N	5	0	1.0000000	0.486792	0.513208	59	1272	−1213	2485
266	$2^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.484962	0.515038	43	1272	−1229	2501
267	$3^1 89^1$	Y	N	5	0	1.0000000	0.486891	0.513109	48	1277	−1229	2506
268	$2^2 67^1$	N	N	−7	2	1.2857143	0.485075	0.514925	41	1277	−1236	2513
269	269^1	Y	Y	−2	0	1.0000000	0.483271	0.516729	39	1277	−1238	2515
270	$2^1 3^3 5^1$	N	N	−48	32	1.3333333	0.481481	0.518519	−9	1277	−1286	2563
271	271^1	Y	Y	−2	0	1.0000000	0.479705	0.520295	−11	1277	−1288	2565
272	$2^4 17^1$	N	N	−11	6	1.8181818	0.477941	0.522059	−22	1277	−1299	2576
273	$3^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−38	1277	−1315	2592
274	$2^1 137^1$	Y	N	5	0	1.0000000	0.478102	0.521898	−33	1282	−1315	2597
275	$5^2 11^1$	N	N	−7	2	1.2857143	0.476364	0.523636	−40	1282	−1322	2604
276	$2^2 3^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−10	1312	−1322	2634
277	277^1	Y	Y	−2	0	1.0000000	0.476534	0.523466	−12	1312	−1324	2636

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_{+}(n)$	$\mathcal{L}_{-}(n)$	$G^{-1}(n)$	$G_{+}^{-1}(n)$	$G_{-}^{-1}(n)$	$\lvert G^{-1}\rvert(n)$
278	$2^1 139^1$	Y	N	5	0	1.0000000	0.478417	0.521583	−7	1317	−1324	2641
279	$3^2 31^1$	N	N	−7	2	1.2857143	0.476703	0.523297	−14	1317	−1331	2648
280	$2^3 5^1 7^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−62	1317	−1379	2696
281	281^1	Y	Y	−2	0	1.0000000	0.473310	0.526690	−64	1317	−1381	2698
282	$2^1 3^1 47^1$	Y	N	−16	0	1.0000000	0.471631	0.528369	−80	1317	−1397	2714
283	283^1	Y	Y	−2	0	1.0000000	0.469965	0.530035	−82	1317	−1399	2716
284	$2^2 71^1$	N	N	−7	2	1.2857143	0.468310	0.531690	−89	1317	−1406	2723
285	$3^1 5^1 19^1$	Y	N	−16	0	1.0000000	0.466667	0.533333	−105	1317	−1422	2739
286	$2^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.465035	0.534965	−121	1317	−1438	2755
287	$7^1 41^1$	Y	N	5	0	1.0000000	0.466899	0.533101	−116	1322	−1438	2760
288	$2^5 3^2$	N	N	−47	42	1.7659574	0.465278	0.534722	−163	1322	−1485	2807
289	17^2	N	Y	2	0	1.5000000	0.467128	0.532872	−161	1324	−1485	2809
290	$2^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.465517	0.534483	−177	1324	−1501	2825
291	$3^1 97^1$	Y	N	5	0	1.0000000	0.467354	0.532646	−172	1329	−1501	2830
292	$2^2 73^1$	N	N	−7	2	1.2857143	0.465753	0.534247	−179	1329	−1508	2837
293	293^1	Y	Y	−2	0	1.0000000	0.464164	0.535836	−181	1329	−1510	2839
294	$2^1 3^1 7^2$	N	N	30	14	1.1666667	0.465986	0.534014	−151	1359	−1510	2869
295	$5^1 59^1$	Y	N	5	0	1.0000000	0.467797	0.532203	−146	1364	−1510	2874
296	$2^3 37^1$	N	N	9	4	1.5555556	0.469595	0.530405	−137	1373	−1510	2883
297	$3^3 11^1$	N	N	9	4	1.5555556	0.471380	0.528620	−128	1382	−1510	2892
298	$2^1 149^1$	Y	N	5	0	1.0000000	0.473154	0.526846	−123	1387	−1510	2897
299	$13^1 23^1$	Y	N	5	0	1.0000000	0.474916	0.525084	−118	1392	−1510	2902
300	$2^2 3^1 5^2$	N	N	−74	58	1.2162162	0.473333	0.526667	−192	1392	−1584	2976
301	$7^1 43^1$	Y	N	5	0	1.0000000	0.475083	0.524917	−187	1397	−1584	2981
302	$2^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−182	1402	−1584	2986
303	$3^1 101^1$	Y	N	5	0	1.0000000	0.478548	0.521452	−177	1407	−1584	2991
304	$2^4 19^1$	N	N	−11	6	1.8181818	0.476974	0.523026	−188	1407	−1595	3002
305	$5^1 61^1$	Y	N	5	0	1.0000000	0.478689	0.521311	−183	1412	−1595	3007
306	$2^1 3^2 17^1$	N	N	30	14	1.1666667	0.480392	0.519608	−153	1442	−1595	3037
307	307^1	Y	Y	−2	0	1.0000000	0.478827	0.521173	−155	1442	−1597	3039
308	$2^2 7^1 11^1$	N	N	30	14	1.1666667	0.480519	0.519481	−125	1472	−1597	3069
309	$3^1 103^1$	Y	N	5	0	1.0000000	0.482201	0.517799	−120	1477	−1597	3074
310	$2^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.480645	0.519355	−136	1477	−1613	3090
311	311^1	Y	Y	−2	0	1.0000000	0.479100	0.520900	−138	1477	−1615	3092
312	$2^3 3^1 13^1$	N	N	−48	32	1.3333333	0.477564	0.522436	−186	1477	−1663	3140
313	313^1	Y	Y	−2	0	1.0000000	0.476038	0.523962	−188	1477	−1665	3142
314	$2^1 157^1$	Y	N	5	0	1.0000000	0.477707	0.522293	−183	1482	−1665	3147
315	$3^2 5^1 7^1$	N	N	30	14	1.1666667	0.479365	0.520635	−153	1512	−1665	3177
316	$2^2 79^1$	N	N	−7	2	1.2857143	0.477848	0.522152	−160	1512	−1672	3184
317	317^1	Y	Y	−2	0	1.0000000	0.476341	0.523659	−162	1512	−1674	3186
318	$2^1 3^1 53^1$	Y	N	−16	0	1.0000000	0.474843	0.525157	−178	1512	−1690	3202
319	$11^1 29^1$	Y	N	5	0	1.0000000	0.476489	0.523511	−173	1517	−1690	3207
320	$2^6 5^1$	N	N	−15	10	2.3333333	0.475000	0.525000	−188	1517	−1705	3222
321	$3^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−183	1522	−1705	3227
322	$2^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.475155	0.524845	−199	1522	−1721	3243
323	$17^1 19^1$	Y	N	5	0	1.0000000	0.476780	0.523220	−194	1527	−1721	3248
324	$2^2 3^4$	N	N	34	29	1.6176471	0.478395	0.521605	−160	1561	−1721	3282
325	$5^2 13^1$	N	N	−7	2	1.2857143	0.476923	0.523077	−167	1561	−1728	3289
326	$2^1 163^1$	Y	N	5	0	1.0000000	0.478528	0.521472	−162	1566	−1728	3294
327	$3^1 109^1$	Y	N	5	0	1.0000000	0.480122	0.519878	−157	1571	−1728	3299
328	$2^3 41^1$	N	N	9	4	1.5555556	0.481707	0.518293	−148	1580	−1728	3308
329	$7^1 47^1$	Y	N	5	0	1.0000000	0.483283	0.516717	−143	1585	−1728	3313
330	$2^1 3^1 5^1 11^1$	Y	N	65	0	1.0000000	0.484848	0.515152	−78	1650	−1728	3378
331	331^1	Y	Y	−2	0	1.0000000	0.483384	0.516616	−80	1650	−1730	3380
332	$2^2 83^1$	N	N	−7	2	1.2857143	0.481928	0.518072	−87	1650	−1737	3387
333	$3^2 37^1$	N	N	−7	2	1.2857143	0.480480	0.519520	−94	1650	−1744	3394
334	$2^1 167^1$	Y	N	5	0	1.0000000	0.482036	0.517964	−89	1655	−1744	3399
335	$5^1 67^1$	Y	N	5	0	1.0000000	0.483582	0.516418	−84	1660	−1744	3404
336	$2^4 3^1 7^1$	N	N	70	54	1.5000000	0.485119	0.514881	−14	1730	−1744	3474
337	337^1	Y	Y	−2	0	1.0000000	0.483680	0.516320	−16	1730	−1746	3476
338	$2^1 13^2$	N	N	−7	2	1.2857143	0.482249	0.517751	−23	1730	−1753	3483
339	$3^1 113^1$	Y	N	5	0	1.0000000	0.483776	0.516224	−18	1735	−1753	3488
340	$2^2 5^1 17^1$	N	N	30	14	1.1666667	0.485294	0.514706	12	1765	−1753	3518
341	$11^1 31^1$	Y	N	5	0	1.0000000	0.486804	0.513196	17	1770	−1753	3523
342	$2^1 3^2 19^1$	N	N	30	14	1.1666667	0.488304	0.511696	47	1800	−1753	3553
343	7^3	N	Y	−2	0	2.0000000	0.486880	0.513120	45	1800	−1755	3555
344	$2^3 43^1$	N	N	9	4	1.5555556	0.488372	0.511628	54	1809	−1755	3564
345	$3^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.486957	0.513043	38	1809	−1771	3580
346	$2^1 173^1$	Y	N	5	0	1.0000000	0.488439	0.511561	43	1814	−1771	3585
347	347^1	Y	Y	−2	0	1.0000000	0.487032	0.512968	41	1814	−1773	3587
348	$2^2 3^1 29^1$	N	N	30	14	1.1666667	0.488506	0.511494	71	1844	−1773	3617
349	349^1	Y	Y	−2	0	1.0000000	0.487106	0.512894	69	1844	−1775	3619
350	$2^1 5^2 7^1$	N	N	30	14	1.1666667	0.488571	0.511429	99	1874	−1775	3649

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_\Omega(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$	$\lvert G^{-1}\rvert(n)$
351	$3^3 13^1$	N	N	9	4	1.5555556	0.490028	0.509972	108	1883	−1775	3658
352	$2^5 11^1$	N	N	13	8	2.0769231	0.491477	0.508523	121	1896	−1775	3671
353	353^1	Y	Y	−2	0	1.0000000	0.490085	0.509915	119	1896	−1777	3673
354	$2^1 3^1 59^1$	Y	N	−16	0	1.0000000	0.488701	0.511299	103	1896	−1793	3689
355	$5^1 71^1$	Y	N	5	0	1.0000000	0.490141	0.509859	108	1901	−1793	3694
356	$2^2 89^1$	N	N	−7	2	1.2857143	0.488764	0.511236	101	1901	−1800	3701
357	$3^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.487395	0.512605	85	1901	−1816	3717
358	$2^1 179^1$	Y	N	5	0	1.0000000	0.488827	0.511173	90	1906	−1816	3722
359	359^1	Y	Y	−2	0	1.0000000	0.487465	0.512535	88	1906	−1818	3724
360	$2^3 3^2 5^1$	N	N	145	129	1.3034483	0.488889	0.511111	233	2051	−1818	3869
361	19^2	N	Y	2	0	1.5000000	0.490305	0.509695	235	2053	−1818	3871
362	$2^1 181^1$	Y	N	5	0	1.0000000	0.491713	0.508287	240	2058	−1818	3876
363	$3^1 11^2$	N	N	−7	2	1.2857143	0.490358	0.509642	233	2058	−1825	3883
364	$2^2 7^1 13^1$	N	N	30	14	1.1666667	0.491758	0.508242	263	2088	−1825	3913
365	$5^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	268	2093	−1825	3918
366	$2^1 3^1 61^1$	Y	N	−16	0	1.0000000	0.491803	0.508197	252	2093	−1841	3934
367	367^1	Y	Y	−2	0	1.0000000	0.490463	0.509537	250	2093	−1843	3936
368	$2^4 23^1$	N	N	−11	6	1.8181818	0.489130	0.510870	239	2093	−1854	3947
369	$3^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	232	2093	−1861	3954
370	$2^1 5^1 37^1$	Y	N	−16	0	1.0000000	0.486486	0.513514	216	2093	−1877	3970
371	$7^1 53^1$	Y	N	5	0	1.0000000	0.487871	0.512129	221	2098	−1877	3975
372	$2^2 3^1 31^1$	N	N	30	14	1.1666667	0.489247	0.510753	251	2128	−1877	4005
373	373^1	Y	Y	−2	0	1.0000000	0.487936	0.512064	249	2128	−1879	4007
374	$2^1 11^1 17^1$	Y	N	−16	0	1.0000000	0.486631	0.513369	233	2128	−1895	4023
375	$3^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	242	2137	−1895	4032
376	$2^3 47^1$	N	N	9	4	1.5555556	0.489362	0.510638	251	2146	−1895	4041
377	$13^1 29^1$	Y	N	5	0	1.0000000	0.490716	0.509284	256	2151	−1895	4046
378	$2^1 3^3 7^1$	N	N	−48	32	1.3333333	0.489418	0.510582	208	2151	−1943	4094
379	379^1	Y	Y	−2	0	1.0000000	0.488127	0.511873	206	2151	−1945	4096
380	$2^2 5^1 19^1$	N	N	30	14	1.1666667	0.489474	0.510526	236	2181	−1945	4126
381	$3^1 127^1$	Y	N	5	0	1.0000000	0.490814	0.509186	241	2186	−1945	4131
382	$2^1 191^1$	Y	N	5	0	1.0000000	0.492147	0.507853	246	2191	−1945	4136
383	383^1	Y	Y	−2	0	1.0000000	0.490862	0.509138	244	2191	−1947	4138
384	$2^7 3^1$	N	N	17	12	2.5882353	0.492188	0.507812	261	2208	−1947	4155
385	$5^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.490909	0.509091	245	2208	−1963	4171
386	$2^1 193^1$	Y	N	5	0	1.0000000	0.492228	0.507772	250	2213	−1963	4176
387	$3^2 43^1$	N	N	−7	2	1.2857143	0.490956	0.509044	243	2213	−1970	4183
388	$2^2 97^1$	N	N	−7	2	1.2857143	0.489691	0.510309	236	2213	−1977	4190
389	389^1	Y	Y	−2	0	1.0000000	0.488432	0.511568	234	2213	−1979	4192
390	$2^1 3^1 5^1 13^1$	Y	N	65	0	1.0000000	0.489744	0.510256	299	2278	−1979	4257
391	$17^1 23^1$	Y	N	5	0	1.0000000	0.491049	0.508951	304	2283	−1979	4262
392	$2^3 7^2$	N	N	−23	18	1.4782609	0.489796	0.510204	281	2283	−2002	4285
393	$3^1 131^1$	Y	N	5	0	1.0000000	0.491094	0.508906	286	2288	−2002	4290
394	$2^1 197^1$	Y	N	5	0	1.0000000	0.492386	0.507614	291	2293	−2002	4295
395	$5^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	296	2298	−2002	4300
396	$2^2 3^2 11^1$	N	N	−74	58	1.2162162	0.492424	0.507576	222	2298	−2076	4374
397	397^1	Y	Y	−2	0	1.0000000	0.491184	0.508816	220	2298	−2078	4376
398	$2^1 199^1$	Y	N	5	0	1.0000000	0.492462	0.507538	225	2303	−2078	4381
399	$3^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.491228	0.508772	209	2303	−2094	4397
400	$2^4 5^2$	N	N	34	29	1.6176471	0.492500	0.507500	243	2337	−2094	4431
401	401^1	Y	Y	−2	0	1.0000000	0.491272	0.508728	241	2337	−2096	4433
402	$2^1 3^1 67^1$	Y	N	−16	0	1.0000000	0.490050	0.509950	225	2337	−2112	4449
403	$13^1 31^1$	Y	N	5	0	1.0000000	0.491315	0.508685	230	2342	−2112	4454
404	$2^2 101^1$	N	N	−7	2	1.2857143	0.490099	0.509901	223	2342	−2119	4461
405	$3^4 5^1$	N	N	−11	6	1.8181818	0.488889	0.511111	212	2342	−2130	4472
406	$2^1 7^1 29^1$	Y	N	−16	0	1.0000000	0.487685	0.512315	196	2342	−2146	4488
407	$11^1 37^1$	Y	N	5	0	1.0000000	0.488943	0.511057	201	2347	−2146	4493
408	$2^3 3^1 17^1$	N	N	−48	32	1.3333333	0.487745	0.512255	153	2347	−2194	4541
409	409^1	Y	Y	−2	0	1.0000000	0.486553	0.513447	151	2347	−2196	4543
410	$2^1 5^1 41^1$	Y	N	−16	0	1.0000000	0.485366	0.514634	135	2347	−2212	4559
411	$3^1 137^1$	Y	N	5	0	1.0000000	0.486618	0.513382	140	2352	−2212	4564
412	$2^2 103^1$	N	N	−7	2	1.2857143	0.485437	0.514563	133	2352	−2219	4571
413	$7^1 59^1$	Y	N	5	0	1.0000000	0.486683	0.513317	138	2357	−2219	4576
414	$2^1 3^2 23^1$	N	N	30	14	1.1666667	0.487923	0.512077	168	2387	−2219	4606
415	$5^1 83^1$	Y	N	5	0	1.0000000	0.489157	0.510843	173	2392	−2219	4611
416	$2^5 13^1$	N	N	13	8	2.0769231	0.490385	0.509615	186	2405	−2219	4624
417	$3^1 139^1$	Y	N	5	0	1.0000000	0.491607	0.508393	191	2410	−2219	4629
418	$2^1 11^1 19^1$	Y	N	−16	0	1.0000000	0.490431	0.509569	175	2410	−2235	4645
419	419^1	Y	Y	−2	0	1.0000000	0.489260	0.510740	173	2410	−2237	4647
420	$2^2 3^1 5^1 7^1$	N	N	−155	90	1.1032258	0.488095	0.511905	18	2410	−2392	4802
421	421^1	Y	Y	−2	0	1.0000000	0.486936	0.513064	16	2410	−2394	4804
422	$2^1 211^1$	Y	N	5	0	1.0000000	0.488152	0.511848	21	2415	−2394	4809
423	$3^2 47^1$	N	N	−7	2	1.2857143	0.486998	0.513002	14	2415	−2401	4816
424	$2^3 53^1$	N	N	9	4	1.5555556	0.488208	0.511792	23	2424	−2401	4825
425	$5^2 17^1$	N	N	−7	2	1.2857143	0.487059	0.512941	16	2424	−2408	4832

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\Sigma_{d\|n} C_{\Omega}(d)}{\|g^{-1}(n)\|}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$	$\|G^{-1}\|(n)$
426	$2^1 3^1 71^1$	Y	N	−16	0	1.0000000	0.485915	0.514085	0	2424	−2424	4848
427	$7^1 61^1$	Y	N	5	0	1.0000000	0.487119	0.512881	5	2429	−2424	4853
428	$2^2 107^1$	N	N	−7	2	1.2857143	0.485981	0.514019	−2	2429	−2431	4860
429	$3^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	−18	2429	−2447	4876
430	$2^1 5^1 43^1$	Y	N	−16	0	1.0000000	0.483721	0.516279	−34	2429	−2463	4892
431	431^1	Y	Y	−2	0	1.0000000	0.482599	0.517401	−36	2429	−2465	4894
432	$2^4 3^3$	N	N	−80	75	1.5625000	0.481481	0.518519	−116	2429	−2545	4974
433	433^1	Y	Y	−2	0	1.0000000	0.480370	0.519630	−118	2429	−2547	4976
434	$2^1 7^1 31^1$	Y	N	−16	0	1.0000000	0.479263	0.520737	−134	2429	−2563	4992
435	$3^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.478161	0.521839	−150	2429	−2579	5008
436	$2^2 109^1$	N	N	−7	2	1.2857143	0.477064	0.522936	−157	2429	−2586	5015
437	$19^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	−152	2434	−2586	5020
438	$2^1 3^1 73^1$	Y	N	−16	0	1.0000000	0.477169	0.522831	−168	2434	−2602	5036
439	439^1	Y	Y	−2	0	1.0000000	0.476082	0.523918	−170	2434	−2604	5038
440	$2^3 5^1 11^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−218	2434	−2652	5086
441	$3^2 7^2$	N	N	14	9	1.3571429	0.476190	0.523810	−204	2448	−2652	5100
442	$2^1 13^1 17^1$	Y	N	−16	0	1.0000000	0.475113	0.524887	−220	2448	−2668	5116
443	443^1	Y	Y	−2	0	1.0000000	0.474041	0.525959	−222	2448	−2670	5118
444	$2^2 3^1 37^1$	N	N	30	14	1.1666667	0.475225	0.524775	−192	2478	−2670	5148
445	$5^1 89^1$	Y	N	5	0	1.0000000	0.476404	0.523596	−187	2483	−2670	5153
446	$2^1 223^1$	Y	N	5	0	1.0000000	0.477578	0.522422	−182	2488	−2670	5158
447	$3^1 149^1$	Y	N	5	0	1.0000000	0.478747	0.521253	−177	2493	−2670	5163
448	$2^6 7^1$	N	N	−15	10	2.3333333	0.477679	0.522321	−192	2493	−2685	5178
449	449^1	Y	Y	−2	0	1.0000000	0.476615	0.523385	−194	2493	−2687	5180
450	$2^1 3^2 5^2$	N	N	−74	58	1.2162162	0.475556	0.524444	−268	2493	−2761	5254
451	$11^1 41^1$	Y	N	5	0	1.0000000	0.476718	0.523282	−263	2498	−2761	5259
452	$2^2 113^1$	N	N	−7	2	1.2857143	0.475664	0.524336	−270	2498	−2768	5266
453	$3^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−265	2503	−2768	5271
454	$2^1 227^1$	Y	N	5	0	1.0000000	0.477974	0.522026	−260	2508	−2768	5276
455	$5^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476923	0.523077	−276	2508	−2784	5292
456	$2^3 3^1 19^1$	N	N	−48	32	1.3333333	0.475877	0.524123	−324	2508	−2832	5340
457	457^1	Y	Y	−2	0	1.0000000	0.474836	0.525164	−326	2508	−2834	5342
458	$2^1 229^1$	Y	N	5	0	1.0000000	0.475983	0.524017	−321	2513	−2834	5347
459	$3^3 17^1$	N	N	9	4	1.5555556	0.477124	0.522876	−312	2522	−2834	5356
460	$2^2 5^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−282	2552	−2834	5386
461	461^1	Y	Y	−2	0	1.0000000	0.477223	0.522777	−284	2552	−2836	5388
462	$2^1 3^1 7^1 11^1$	Y	N	65	0	1.0000000	0.478355	0.521645	−219	2617	−2836	5453
463	463^1	Y	Y	−2	0	1.0000000	0.477322	0.522678	−221	2617	−2838	5455
464	$2^4 29^1$	N	N	−11	6	1.8181818	0.476293	0.523707	−232	2617	−2849	5466
465	$3^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.475269	0.524731	−248	2617	−2865	5482
466	$2^1 233^1$	Y	N	5	0	1.0000000	0.476395	0.523605	−243	2622	−2865	5487
467	467^1	Y	Y	−2	0	1.0000000	0.475375	0.524625	−245	2622	−2867	5489
468	$2^2 3^2 13^1$	N	N	−74	58	1.2162162	0.474359	0.525641	−319	2622	−2941	5563
469	$7^1 67^1$	Y	N	5	0	1.0000000	0.475480	0.524520	−314	2627	−2941	5568
470	$2^1 5^1 47^1$	Y	N	−16	0	1.0000000	0.474468	0.525532	−330	2627	−2957	5584
471	$3^1 157^1$	Y	N	5	0	1.0000000	0.475584	0.524416	−325	2632	−2957	5589
472	$2^3 59^1$	N	N	9	4	1.5555556	0.476695	0.523305	−316	2641	−2957	5598
473	$11^1 43^1$	Y	N	5	0	1.0000000	0.477801	0.522199	−311	2646	−2957	5603
474	$2^1 3^1 79^1$	Y	N	−16	0	1.0000000	0.476793	0.523207	−327	2646	−2973	5619
475	$5^2 19^1$	N	N	−7	2	1.2857143	0.475789	0.524211	−334	2646	−2980	5626
476	$2^2 7^1 17^1$	N	N	30	14	1.1666667	0.476891	0.523109	−304	2676	−2980	5656
477	$3^2 53^1$	N	N	−7	2	1.2857143	0.475891	0.524109	−311	2676	−2987	5663
478	$2^1 239^1$	Y	N	5	0	1.0000000	0.476987	0.523013	−306	2681	−2987	5668
479	479^1	Y	Y	−2	0	1.0000000	0.475992	0.524008	−308	2681	−2989	5670
480	$2^5 3^1 5^1$	N	N	−96	80	1.6666667	0.475000	0.525000	−404	2681	−3085	5766
481	$13^1 37^1$	Y	N	5	0	1.0000000	0.476091	0.523909	−399	2686	−3085	5771
482	$2^1 241^1$	Y	N	5	0	1.0000000	0.477178	0.522822	−394	2691	−3085	5776
483	$3^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−410	2691	−3101	5792
484	$2^2 11^2$	N	N	14	9	1.3571429	0.477273	0.522727	−396	2705	−3101	5806
485	$5^1 97^1$	Y	N	5	0	1.0000000	0.478351	0.521649	−391	2710	−3101	5811
486	$2^1 3^5$	N	N	13	8	2.0769231	0.479424	0.520576	−378	2723	−3101	5824
487	487^1	Y	Y	−2	0	1.0000000	0.478439	0.521561	−380	2723	−3103	5826
488	$2^3 61^1$	N	N	9	4	1.5555556	0.479508	0.520492	−371	2732	−3103	5835
489	$3^1 163^1$	Y	N	5	0	1.0000000	0.480573	0.519427	−366	2737	−3103	5840
490	$2^1 5^1 7^2$	N	N	30	14	1.1666667	0.481633	0.518367	−336	2767	−3103	5870
491	491^1	Y	Y	−2	0	1.0000000	0.480652	0.519348	−338	2767	−3105	5872
492	$2^2 3^1 41^1$	N	N	30	14	1.1666667	0.481707	0.518293	−308	2797	−3105	5902
493	$17^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	−303	2802	−3105	5907
494	$2^1 13^1 19^1$	Y	N	−16	0	1.0000000	0.481781	0.518219	−319	2802	−3121	5923
495	$3^2 5^1 11^1$	N	N	30	14	1.1666667	0.482828	0.517172	−289	2832	−3121	5953
496	$2^4 31^1$	N	N	−11	6	1.8181818	0.481855	0.518145	−300	2832	−3132	5964
497	$7^1 71^1$	Y	N	5	0	1.0000000	0.482897	0.517103	−295	2837	−3132	5969
498	$2^1 3^1 83^1$	Y	N	−16	0	1.0000000	0.481928	0.518072	−311	2837	−3148	5985
499	499^1	Y	Y	−2	0	1.0000000	0.480962	0.519038	−313	2837	−3150	5987
500	$2^2 5^3$	N	N	−23	18	1.4782609	0.480000	0.520000	−336	2837	−3173	6010