Proposition 1. *Let a, z and λ be positive real parameters such that $z = \lambda a$. If $\lambda > 1$, then*

$$\Gamma(a,z) = \frac{z^{a-1}e^{-z}}{1-\lambda^{-1}} + \mathcal{O}_\lambda\left(z^{a-2}e^{-z}\right) \tag{1}$$

as $z \to +\infty$. If $\lambda > 0.57 > W(1)$ (with W being the principal branch of the Lambert-W function), then

$$\Gamma(a, ze^{\pm\pi i}) = -e^{\pm\pi i a}\frac{z^{a-1}e^{z}}{1+\lambda^{-1}} + \mathcal{O}_\lambda\left(z^{a-2}e^{z}\right) \tag{2}$$

as $z \to +\infty$.

Note that we cannot write $\Gamma(a,-z)$ as the incomplete gamma function is a multivalued function of its second variable. It becomes a single-valued analytic function of its second variable on the universal covering of $\mathbb{C} \setminus \{0\}$.

Proof. With the notation of the paper [1],

$$\Gamma(a,z) = \frac{z^{a-1}e^{-z}}{1-\lambda^{-1}} + z^a e^{-z} R_1(a,\lambda)$$

By [1, Subsection 3.1]

$$\left|z^a e^{-z} R_1(a,\lambda)\right| \le z^a e^{-z}\frac{ab_1(\lambda)}{(z-a)^3} = \frac{1}{(1-\lambda^{-1})^3} z^{a-2}e^{-z}.$$

If the explicit dependence of the error term on λ is not needed, (1) follows directly from [1, Eq. (1.1)] and the definition of an asymptotic power series. To obtain (2), we employ [2, Eq. (1)] with

$$a = ae^{\pm\pi i},\ z = ze^{\pm\pi i},\ \lambda > 0.57 > W(1)$$

and refer to the definition of an asymptotic power series. The requirement on λ is made so that [1, Eq. (1.1)] is valid for negative a. □

Proposition 2. *As $x \to +\infty$,*

$$\frac{x}{\log x}\left|\sum_{k=1}^{\lfloor \log\log x\rfloor}(-1)^k\frac{(\log\log x)^{k-1}}{(k-1)!}\right| = \frac{1}{2}\frac{x}{\sqrt{2\pi\log\log x}} + \mathcal{O}\left(\frac{x}{(\log\log x)^{3/2}}\right).$$

Proof. We have for $t > 0$

$$\sum_{k=1}^{n}(-1)^k\frac{t^{k-1}}{(k-1)!} = -e^{-t}\frac{\Gamma(n,te^{\pi i})}{(n-1)!}$$

(cf. [4, Eq. 8.4.8]). Now assume that $t = n+\xi$, $\xi = \mathcal{O}(1)$. Employing (2) with

$$a = n,\ z = t,\ \lambda = 1 + \frac{\xi}{n},$$

we deduce

$$\Gamma(n, te^{\pi i}) = (-1)^{n+1}\frac{t^n e^t}{t+n} + \mathcal{O}\left(\frac{t^n e^t n}{(t+n)^3}\right) = \frac{(-1)^{n+1}}{2}\frac{t^n e^t}{n} + \mathcal{O}\left(\frac{t^n e^t}{nt}\right)$$

as $n \to +\infty$ (or, equivalently, $t \to +\infty$). Accordingly,

$$\sum_{k=1}^{n}(-1)^k \frac{t^{k-1}}{(k-1)!} = \frac{(-1)^n}{2}\frac{t^n}{n!} + \mathcal{O}\left(\frac{t^n}{n!t}\right).$$

By [4, Eq. 5.11.8],

$$n! = \Gamma(t-\xi+1) = t^{t-\xi+1/2}e^{-t}\sqrt{2\pi}\left(1+\mathcal{O}\left(\frac{1}{t}\right)\right) = t^{n+1/2}e^{-t}\sqrt{2\pi}\left(1+\mathcal{O}\left(\frac{1}{t}\right)\right).$$

Hence,

$$\sum_{k=1}^{n}(-1)^k \frac{t^{k-1}}{(k-1)!} = \frac{(-1)^n}{2}\frac{e^t}{\sqrt{2\pi t}} + \mathcal{O}\left(\frac{e^t}{t^{3/2}}\right)$$

as $n \to +\infty$ with $t = n + \mathcal{O}(1)$. Substituting $n = \lfloor \log\log x \rfloor$, $t = \log\log x$ and taking absolute values, we obtain the desired result. $\square$

Proposition 3. *As $x \to +\infty$,*

$$\sum_{k=1}^{\lfloor \log\log x \rfloor} \frac{(\log\log x)^{k+1/2}}{(2k+1)(k-1)!} = \frac{1}{4}\log x\sqrt{\log\log x} + \mathcal{O}(\log x).$$

Proof. We have for $t > 0$

$$\begin{aligned}\sum_{k=1}^{n} \frac{t^{k-1}}{(2k+1)(k-1)!} &= \int_0^1 s^2 \sum_{k=1}^{n}\frac{(s^2t)^{k-1}}{(k-1)!}ds = \frac{1}{(n-1)!}\int_0^1 s^2 e^{s^2t}\Gamma(n,s^2t)ds \\ &= \frac{1}{2(n-1)!}\int_0^1 \sqrt{x}e^{xt}\Gamma(n,xt)dx\end{aligned}$$

(cf. [4, Eq. 8.4.8]). Integrating once by parts shows that this is further equal to

$$\begin{aligned}&\frac{\Gamma(n,t)}{2(n-1)!}\frac{e^t}{t}\left(1-\frac{\sqrt{\pi}}{2}\frac{e^{-t}}{\sqrt{t}}\operatorname{erfi}\left(\sqrt{t}\right)\right) + \frac{t^{n-1}}{(n-1)!}\frac{1}{2n+1} \\ &\quad - \frac{\sqrt{\pi}}{4}\frac{t^{n-3/2}}{(n-1)!}\int_0^1 x^{n-1}e^{-xt}\operatorname{erfi}\left(\sqrt{tx}\right)dx \\ &= \frac{\Gamma(n,t)}{2(n-1)!}\frac{e^t}{t}\left(1-\frac{\sqrt{\pi}}{2}\frac{e^{-t}}{\sqrt{t}}\operatorname{erfi}\left(\sqrt{t}\right)\right) + \frac{t^{n-1}}{(n-1)!}\frac{1}{2n+1} \\ &\quad - \frac{\sqrt{\pi}}{4}\frac{t^{-3/2}}{(n-1)!}\int_0^t s^{n-1}e^{-s}\operatorname{erfi}\left(\sqrt{s}\right)ds.\end{aligned}$$

Now, by [4, Eq. 7.12.1]) and the definition of erfi,

$$\frac{\sqrt{\pi}}{2}\frac{e^{-t}}{\sqrt{t}}\operatorname{erfi}\left(\sqrt{t}\right) = \mathcal{O}\left(\frac{1}{t}\right)$$

as $t \to +\infty$. From now on assume that $t = n + \xi$, $\xi = \mathcal{O}(1)$. According to [3, Eq. (2.4)],

$$\Gamma(n,t) = t^n e^{-t}\sqrt{\frac{\pi}{2t}}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{t}}\right)\right)$$

as $t \to +\infty$. With the same assumptions,

$$\frac{t^{n-1}}{(n-1)!} \frac{1}{2n+1} = \frac{1}{2} \frac{t^{n-2}}{(n-1)!} \left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right)$$

as $t \to +\infty$. Finally, by [4, Eq. 7.12.1]) and the definition of erfi,

$$\begin{aligned}\int_0^t s^{n-1} e^{-s} \operatorname{erfi}\left(\sqrt{s}\right) ds &= \int_0^1 s^{n-1} e^{-s} \operatorname{erfi}\left(\sqrt{s}\right) ds + \int_1^t s^{n-1} e^{-s} \operatorname{erfi}\left(\sqrt{s}\right) ds \\ &= \mathcal{O}(1) + \mathcal{O}(1) \int_1^t s^{n-3/2} ds = \mathcal{O}\left(t^{n-1/2}\right)\end{aligned}$$

as $t \to +\infty$. Collecting all the partial results, we see that

$$\sum_{k=1}^{n} \frac{t^{k-1}}{(2k+1)(k-1)!} = \frac{1}{2} \sqrt{\frac{\pi}{2}} \frac{t^{n-3/2}}{(n-1)!} \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right)\right)$$

as $t \to +\infty$. By [4, Eq. 5.11.8],

$$(n-1)! = \Gamma(t - \xi) = t^{t-\xi-1/2} e^{-t} \sqrt{2\pi} \left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right) = t^{n-1/2} e^{-t} \sqrt{2\pi} \left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right),$$

whence,

$$\sum_{k=1}^{n} \frac{t^{k-1}}{(2k+1)(k-1)!} = \frac{1}{4} \frac{e^t}{t} + \mathcal{O}\left(\frac{e^t}{t^{3/2}}\right)$$

as $n \to +\infty$ with $t = n + \mathcal{O}(1)$. Substituting $n = \lfloor \log \log x \rfloor$, $t = \log \log x$ and doing some algebra, we obtain the desired result. $\square$

References

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