Maxie

(A) You need a version which proves the main result in the MOST DIRECT FASHION possible. No extra arguments or editorializing.

(B) You need to use STANDARD notation. Anything exceptional needs a new symbol.

(C) Each statement of Thm or Lemma needs to be understandable on it's own terms.

Symbol	**Definition**
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d\mid n \\ d>1}} f(d)f^{-1}(n/d)$ provided that $f(1) \neq 0$. The inverse function, when it exists, is unique and satisfies the relations that $f^{-1} * f = f * f^{-1} = \varepsilon$.
$\lfloor x \rfloor$	The floor function $\lfloor x \rfloor := x - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$. ~~This function definition is the key to unraveling our new bounds proved in the article, and so, is henceforth taken as standard notation moving on.~~
$\mathrm{Id}_k(n)$	The power-scaled identity function, $\mathrm{Id}_k(n) := n^k$ for $n \geq 1$.
$\mathbb{1}_{\mathbb{S}}$, $\chi_{\mathtt{cond}(x)}$	We use the notation $\mathbb{1}, \chi : \mathbb{N} \to \{0,1\}$ to denote indicator, or characteristic functions. In paticular, $\mathbb{1}_{\mathbb{S}}(n) = 1$ if and only if $n \in \mathbb{S}$, and $\chi_{\mathtt{cond}}(n) = 1$ if and only if n satisfies the condition `cond`.
$\log_*^m(x)$	The iterated logarithm function defined recursively for integers $m \geq 0$ by $\log_*^m(x) := \begin{cases} x, & \text{if } m = 0; \\ \log x, & \text{if } m = 1; \\ \log\left[\log_*^{m-1}(x)\right], & \text{if } m \geq 2. \end{cases}$, $x >$!?
$[n = k]_\delta$	Synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and zero otherwise.
$[\mathtt{cond}]_\delta$	For a boolean-valued `cond`, $[\mathtt{cond}]_\delta$ evaluates to one precisely when `cond` is true, and zero otherwise.
$\lambda(n)$	The Liouville lambda function, $\lambda(n) := (-1)^{\Omega(n)}$, denotes the parity of $\Omega(n)$, the number of distinct prime factors of n counting multiplicity. That is, $\lambda(n) \in \{\pm 1\}$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$. Notice that if n is squarefree, then $\lambda(n) = \mu(n)$, where more generally we have that $\lambda(n) = \sum_{d^2 \mid n} \mu\left(\frac{n}{d^2}\right)$. This function is Dirichlet invertible with inverse function given by $\lambda^{-1}(n) = \mu^2(n)$, the (unsigned) characteristic function of the squarefree integers.
$\gcd(m,n); (m,n)$	The greatest common divisor of m and n. Both notations for the GCD are used interchangably within the article.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree, i.e., has no prime power divisors with exponent greater than one.
$M(x)$	The Mertens function which is the summatory function over $\mu(n)$, $M(x) := \sum\limits_{n \leq x} \mu(n)$.

Symbol	Definition
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^\alpha \|\| n$ (p^α exactly divides n) for p prime and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the distinct prime factor counting functions as $\omega(n) := \sum_{p\mid n} 1$ and $\Omega(n) := \sum_{p^\alpha \|\| n} \alpha$. Equivalently, if the factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we define that $\omega(1) = \Omega(1) = 0$.
$\sum_{p\leq x}, \prod_{p\leq x}$	Unless otherwise specified by context, we use the index variable p to denote that the summation (product) is to be taken only over prime values within the summation bounds.
$P(s)$	For complex s with $\Re(s) > 1$, we define the prime zeta function to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$. This function has an analytic continuation to $\Re(s) \in (0,1)$ with a logarithmic singularity near $s := 1$: $P(1+\varepsilon) = -\log\varepsilon + C + O(\varepsilon)$.
$\sigma_\alpha(n)$	The generalized sum-of-divisors function, $\sigma_\alpha(n) := \sum\limits_{d\mid n} d^\alpha$, for any $n \geq 1$ and $\alpha \in \mathbb{C}$.
$\left[{n \atop k}\right]$	The unsigned Stirling numbers of the first kind, $\left[{n \atop k}\right] = (-1)^{n-k} \cdot s(n,k)$.
$\sim, \approx, \gtrsim, \lesssim$	We say that two functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $$\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1.$$ We also sometimes express the *average order* of an arithmetic function $f \sim h$ that may actually oscillate, or say have value of one infinitely often, in the cases that $\frac{1}{x} \cdot \sum_{n\leq x} f(n) \sim h(x)$ (for example, we often would write that $\Omega(n) \sim \log\log n$, even though technically, $1 \leq \Omega(n) \leq \frac{\log n}{\log 2}$). We write that $f(x) \approx g(x)$ if $\|f(x) - g(x)\| = O(1)$. We say that $h(x) \gtrsim r(x)$ if $h \gg r$ as $x \to \infty$, and define the relation $\lesssim$ similarly as $h(x) \lesssim r(x)$ if $h \ll r$ as $x \to \infty$. When applying these relations we still consider leading constants to be meaningful terms that are preserved.
$\sum'_{n\leq x}$	We denote by $\sum'_{n\leq x} f(n)$ the summatory function of f at x minus $\frac{f(x)}{2}$ if $x \in \mathbb{Z}$.
$\tau_m(n)$	Let $\tau_m(n) \equiv \mathbb{1}_{*_m}(n)$ denote the m-fold Dirichlet convolution of one with itself at n. Note that $\tau_2(n)$ yields the divisor function, $d(n) \equiv \sigma_0(n)$.
$\zeta(s)$	The Riemann zeta function, defined by $\zeta(s) := \sum_{n\geq 1} n^{-s}$ when $\Re(s) > 1$, and by analytic continuation to the entire complex plane with the exception of a simple pole at $s = 1$.

I have pointed to this before.

What does $\gg$ mean?

?

Most readers would find this confusing. Not a good thing in the notation section

Insisting on keeping the constants is not expected, will cause prblms for your readers.

I am concerned that $f \lesssim g$ is typically used for positive fns, but you want to use it for SIGNED functions.

1 Introduction

1.1 Preface on notation: Unconventional notions of asymptotics

The notation of $\gtrsim, \lesssim$ defined in the prior section on notation employed in the article is convenient for expressing upper and lower bounds on functions given by asymptotically dominant main terms in the expansion of more complicated symbolic expansions. For example, suppose that exactly

$$f(x) \geq -(\log\log\log x)^2 + 3 \times 10^{1000000} \cdot (\log\log\log x)^{1.999999999} + E(x),$$

where $E(x) = o\left((\log\log\log x)^2\right)$ and the unusually complicated expression for $E(x)$ requires more than 100000 ascii characters to typeset accurately. Then naturally, we prefer to work with only the expression for the asymptotically dominant main term in the lower bounds stated above. Note that since this main term contribution does not dominate the bound until x is very large, so that replacing the right-hand-side expression with just this term yields an invalid inequality except for in limiting cases. In this instance, we prefer to write

$$f(x) \gtrsim -(\log\log\log x)^2,$$

which indicates that this substantially simplifed form of the lower bound on f holds as $x \rightarrow \infty$.

Hence, we use these new symbols, $\gtrsim, \lesssim$, as asymptotic relations defined to simplify our results by dropping expressions involving more precise, exact terms that are nonetheless asymptotically insignificant, to obtain accurate statements in limiting cases of large x that hold as $x \rightarrow \infty$ that capture the more simple essence of the bound as we choose to view it. This notation is particularly powerful and is utilized in this article when we express many lower bound estimates for functions that would otherwise require literally pages of typeset symbols to state exactly, but which have simple enough formulae when considered as bounds that hold in this type of limiting asymptotic context.

To distinguish between classical and modern usages of the notation $\sim$, we will write $A(x) \sim B(x)$ to denote that $A(x)/B(x) \xrightarrow{x\rightarrow\infty} 1$. In place of the use of $\sim$ to denote the *average order* of an arithmetic function as in [7], we will use a modern, probabilistically themed expectation symbol as follows:

$$\mathbb{E}[f(x)] = g(x) \iff \frac{1}{x}\sum_{n \leq x} f(x) \sim g(x).$$

Why isnt this in notation section.

We note that these subtle distinctions in usage of traditional notation for asymptotic relations are key to understanding our choices of upper and lower bound expressions given throughout the article.

1.2 The Mertens function – definition, properties, known results and conjectures

Suppose that $n \geq 1$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. We define the *Möebius function* to be the signed indicator function of the squarefree integers:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1,\ \forall 1 \leq i \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many known variants and special properties of the Möebius function and its generalizations [15, *cf.* §2], however, for our purposes we seek to explore the properties and asymptotics of weighted summatory functions over $\mu(n)$. The Mertens summatory function, or *Mertens function*, is defined as [17, A002321]

$$M(x) = \sum_{n \leq x} \mu(n), \ x \geq 1,$$
$$\longmapsto \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2\}$$

A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as [17, A013928]

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \to \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{Q(x)} = \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{Q(x)} \xrightarrow[n \to \infty]{} \frac{3}{\pi^2}.$$

While this limiting law suggests an even bias for the Mertens function, in practice $M(x)$ has a noted negative bias in its values, and the actual local oscillations between the approximate densities of the sets $\mu_\pm(x)$ lend an unpredictable nature to the function and its characteistic oscillatory sawtooth shaped plot.

1.2.1 Properties

The well-known approach to evaluating the behavior of $M(x)$ for large $x \to \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is easily expressed via an inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since by Perron's formula we have

$$\frac{1}{\zeta(s)} = \int_1^\infty \frac{s \cdot M(x)}{x^{s+1}} dx,$$

we then obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation along with the standard Euler product representation for the reciprocal zeta function leads us to the exact expression for $M(x)$ when $x > 0$ given by the next theorem.

Theorem 1.1 (Analytic Formula for $M(x)$)**.** *Assuming the RH, we can show that there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any $x \in \mathbb{R}_{>0}$*

$$M(x) = \lim_{k \to \infty} \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |\Im(\rho)| < T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

property established by Ng in 2008. We cite that prior to this point it is known that [14, *cf.* §4.1]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad \text{(now } 1.826054\text{)},$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad \text{(now } -1.837625\text{)},$$

although based on work by Odlyzko and te Riele it seems probable that each of these limits should be $\pm\infty$, respectively [12, 9, 10, 8]. It is also known that $M(x) = \Omega_{\pm}(\sqrt{x})$ and $M(x)/\sqrt{x} = \Omega_{\pm}(1)$.

1.3 A new approach to bounding $M(x)$ from below

1.3.1 Summing series over Dirichlet convolutions

Theorem 1.2 (Summatory functions of Dirichlet convolutions)**.** *Let $f, g : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n\leq x} f(n)$ and $G(x) := \sum_{n\leq x} g(n)$ denote the summatory functions of f, g, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse $f^{-1}(n)$ of f, i.e., the unique arithmetic function such that $f * f^{-1} = \varepsilon$ where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity with respect to Dirichlet convolution. Then, letting the counting function $\pi_{f*g}(x)$ be defined as in the first equation below, we have the following equivalent expressions for the summatory function of $f * g$ for integers $x \geq 1$:*

Again, you are not using a single notation for Summatory fns

$$\begin{aligned}
\pi_{f*g}(x) &= \sum_{n\leq x} \sum_{d|n} f(d) g(n/d) \\
&= \sum_{d\leq x} f(d) G\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} G(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].
\end{aligned}$$

Moreover, we can invert the linear system determining the coefficients of $G(k)$ for $1 \leq k \leq x$ naturally to express $G(x)$ as a linear combination of the original left-hand-side summatory function as:

I dont think you want to me G here.

$$\begin{aligned}
G(x) &= \sum_{j=1}^{x} \pi_{f*g}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\
&= \sum_{n=1}^{x} f^{-1}(n) \pi_{f*g}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).
\end{aligned}$$

Corollary 1.3 (Convolutions Arising From Möbius Inversion)**.** *Suppose that g is an arithmetic function with $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n\leq x} (g * \mu)(n)$. Then the Mertens function equals*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geq 1.$$

1.3.2 A motivating special case

Is this part of the argument? If not, doesnt belong.

Using $\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu$, where $\chi_{\mathbb{P}}$ is the characteristic function of the primes, we have that $\widetilde{G}(x) = \pi(x) + 1$ in Corollary 1.3. In particular, the corollary implies that

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

We can compute the first few terms for the Dirichlet inverse sequence of $g(n) := \omega(n) + 1$ numerically for the first few sequence values as

$$\{g^{-1}(n)\}_{n\geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is given by $\lambda(n) = \frac{g^{-1}(n)}{|g^{-1}(n)|}$ (see Proposition 2.3). Note that since the DGF of $\omega(n)$ is given by $D_\omega(s) = P(s)\zeta(s)$ where $P(s)$ is the *prime zeta function*, we do have a Dirichlet series for the inverse functions to invert coefficient-wise using more classical contour integral methods*. Our new methods do not rely on typical constructions for bounding $M(x)$ based on estimates of the non-trivial zeros of the Riemann zeta function that have so far to date been employed to bound the Mertens function from above. We will instead take a more combinatorial tack to investigating bounds on this inverse function sequence in the coming sections.

Consider the following motivating conjecture:

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Conjecture 1.4. *Suppose that $n \geq 1$ is a squarefree integer. We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n) = (\omega + 1)^{-1}(n)$ over these integers:*

(A) $g^{-1}(1) = 1$;

(B) $\operatorname{sgn}(g^{-1}(n)) = \mu(n) \equiv \lambda(n)$;

(C) We can write the inverse function at squarefree n as

$$g^{-1}(n) = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!.$$

We illustrate parts (B)–(C) of this conjecture clearly using Table T.1 given on page 41 of the appendix section. A table of the first several explicit values of $(f + 1)^{-1}(n)$ for $f(1) = 0$ and symbolic additive f are also given in Table T.2 on page 42.

The realization that the beautiful and remarkably simple form of property (C) in Conjecture 1.4 holds for all squarefree $n \geq 1$ motivates our pursuit of formulas for the inverse functions $g^{-1}(n)$ based on the configuration of the exponents in the prime factorization of any $n \geq 2$. In Section 4 we consider expansions of these inverse functions recursively, starting from a few first exact cases of an auxillary function, $C_k(n)$, that depends on the precise exponents in the prime factorization of n. We then prove limiting asymptotics for

*E.g., using [1, §11]

$$f(n) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{n^{\sigma+\imath t}}{\zeta(\sigma+\imath t)(P(\sigma+\imath t)+1)}, \sigma > 1.$$

Fröberg has also previously done some preliminary investigation as to the properties of the inversion to find the coefficients of $(1 + P(s))^{-1}$ in [4].

In light of the fact that (by an integral-based interpretation of integer convolution using summation by parts)

$$M(x) \sim G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)},$$

the formula in (3) implies that we can establish new *lower bounds* on $M(x)$ by appropriate estimates of the summatory function $G^{-1}(x)$ where trivially we have the bounded inner sums $L_0(x) := \sum_{n \leq x} \lambda(n) \in [-x, x]$ for all $x \geq 2$.

As explicit lower bounds for $M(x)$ along subsequences are not obvious, and are historically ellusive and non-trivial to obtain as we expect sign changes of this function infinitely often, we find this approach to be an effective one. Now, having motivated why we must carefully estimate the $G^{-1}(x)$ bounds using our new methods, we will require the bounds suggested in the next section to work at summing the summatory functions, $G^{-1}(x)$, for large x as $x \to \infty$.

1.3.4 Some enumerative (or counting) DGFs from Mongomery and Vaughan

Our inspiration for the new bounds found in the last sections of this article allows us to sum non-negative arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. In particular, it uses a hybrid generating function and DGF method under which we are able to recover "good enough" asymptotics about the summatory functions that encapsulate the parity of $\lambda(n)$:

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}, k \geq 1.$$

Notation section?

The precise statement of the theorem that we transform for these new bounds is re-stated as follows:

Theorem 1.6 (Montgomery and Vaughan, §7.4)**.** *Let* $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$*. For* $R < 2$ *we have that*

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

uniformly for $1 \leq k \leq R \log\log x$ *where*

$$\mathcal{G}(z) := \frac{F(1,z)}{\Gamma(z+1)} = \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z.$$

The precise formulations of the inverse function asymptotics proved in Section 4 depend on being able to form significant lower bounds on the summatory functions of an always positive arithmetic function weighted by $\lambda(n)$. The next theorem, proved carefully in Section 3, is the primary starting point for our new asymptotic lower bounds.

I cant understand this statement

Theorem 1.7 (Generating functions of symmetric functions)**.** *We obtain lower bounds of the following form for* $A_0 > 0$ *an absolute constant, and* $C_0(x)$ *a function only of* x *where we take* $z \geq 0$ *to be a real-valued parameter uniformly bounded in* $x \geq 2$*:*

$$\mathcal{G}(z) \geq A_0 \cdot C_0(x)^z$$

It suffices to take

$$A_0 = \frac{4}{3 \cdot \log 2 \cdot 2^{27/4} \cdot \Gamma\left(\frac{5}{2}\right)} \approx 0.0134439$$

$$C_0(x) = \frac{4}{3 \log 2}.$$

1.4 Cracking the classical unboundedness result, so to speak

In Section 5, we provide the culmination of what we build up to in the proofs established in prior sections of the article. Namely, we prove the form of an explicit limiting lower bound for the summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, along a specific subsequence over which the parity of both $\left\lfloor \frac{1}{2} \log\log\log\log x \right\rfloor$ and $\left\lfloor \frac{3}{2} \log\log\log\log x \right\rfloor$ are predictably signed. What we obtain is the following important summary corollary verifying the unboundedness of the scaled function $|M(x)|/\sqrt{x}$ in the limit supremum sense:

Corollary 1.8 (Bounds for the classically scaled Mertens function)**.** *Let $u_0 := e^{e^{e^e}}$ and define the infinite increasing subsequence, $\{x_n\}_{n \geq 1}$, by $x_n := e^{e^{e^{e^{6n}}}}$. We have that along the increasing subsequence x_y for large $y \geq \max\left(\left\lceil e^{e^{e^e}} \right\rceil, u_0 + 1\right)$:*

$$\frac{|M(x_y)|}{\sqrt{x_y}} \gtrsim 2C_{\ell,1} \cdot (\log\log\sqrt{x_y})(\log\log\log\sqrt{x_y})^{4 + \frac{3}{\log 2} - \frac{3}{\log 3}} + o(1),$$

as $y \to \infty$. In the previous equation, we adopt the notation for the absolute constant $C_{\ell,1} > 0$ defined more precisely by

$$C_{\ell,1} := \frac{1}{36 \cdot 2^{3/4}\sqrt{\pi} \cdot \log 2} \approx 0.000183209.$$

This is all to say that in establishing the rigorous proof of Corollary 1.8 based on our new methods, we not only show that

$$\limsup_{x \to \infty} \frac{|M(x)|}{\sqrt{x}} = +\infty,$$

but also set a minimal rate (along a large subsequence) at which the scaled Mertens function grows without bound.

1.5 Outline: Core components to the proof

We offer another brief step-by-step summary overview of the critical components to our proof outlined in the introduction above, and then which are proved piece-by-piece in the next sections of the article below. This outline is provided to help the reader see our logic and proof methodology as easily and quickly as possible.

(1) We prove ~~an apparently yet undiscovered~~ matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). ~~Namely, a careful matrix and symmetry transformation based proof of~~ Theorem 1.2 is given in Section 2. See Thm 1.2

(2) This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the seemingly unconnected prime counting function, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is already stated in (1) expanded above.

   (i) The average order, $\mathbb{E}[\omega(n)] = \log\log n$, imparts an iterated logarithmic structure to our expansions, which many have conjectured we should see in limiting bounds on $M(x)$, but which are practically elusive in most non-conjectural known formulas I have seen proved rigorously in print. Is this needed?

Only write what is needed.

(ii) The additivity of $\omega(n)$ dictates that the sign of $g^{-1}(n) = (\omega+1)^{-1}(n)$ is $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ (see Proposition 2.3). The corresponding weighted summatory functions of $\lambda(n)$ have more established predictable properties, such as known sign biases and upper bounds. These summatory functions are generally speaking more regular and easier to work with than traditional approaches to summing $M(x)$ and its complicating summand terms of the Möbius function. Note that our proof is essentially much different than what is known about sums of consecutive values of $\mu(n)$ over short intervals, both in interpretation and methodology.

Just say the minimum. No side remarks.

(3) We tighten a result from [11, §7] providing summatory functions that indicate the parity of $\lambda(n)$ using elementary arguments and more combinatorially flavored expansions of Dirichlet series in our proof of Theorem 1.7. Our motivations are different than in the reference for exploiting the unique properties of this construction. Namely, we are not after a CLT-like statement for the functions $\Omega(n)$ and $\omega(n)$. Rather, we seek to sum $\sum_{n\leq x} \lambda(n) f(n)$ for general non-negative arithmetic functions f using Abel summation when x is large.

(4) We then turn to the asymptotics if the quasi-periodic $g^{-1}(n)$, estimating this inverse function's limiting asymptotics for large n (or $n \leq x$ when x is very large) in Section 4. We eventually use these estimates to prove a substantially unique new lower bound formula for the summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$ along prescribed asymptotically large infinite subsequences that tend to $+\infty$ (see Theorem 5.2).

(5) When we return to (2) with our new lower bounds, and bootstrap, we recover "magic" in the form of showing the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. What we recover is a quick, and rigorous, proof of Corollary 1.8.

(6) We remark that while this technique and approach to the classical problem at hand is certainly new, it is not just novel, and its discovery will invariably lead to similar applications given careful study of limsup-type bounds on the summatory functions of other special signed arithmetic function sequences.

Note that in these cases, if f is multiplicative and $f(n) > 0$ for all $n \geq 1$, then $\operatorname{sgn}(f^{-1}(n)) = (-1)^{\omega(n)}$. This variation in signedness tends to complicate, but still closely parallel our argument involving the parity of $\lambda(n) = (-1)^{\Omega(n)}$ for the Mertens function case.

Need a summary of just the main logical steps. No side remarks.

3 Summing arithmetic functions weighted by $\lambda(n)$

3.1 Discussion: The enumerative DGF result in Theorem 1.6 from Montgomery and Vaughan

What this enumeratively-flavored result of Montgomery and Vaughan allows us to do is get a "good enough" lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. For comparison, we already have known, more classical bounds due to Erdös (or earlier) that state for

$$\pi_k(x) := \#\{n \leq x : \omega(n) = k\},$$

we have tightly that [2, 11]

$$\pi_k(x) = (1 + o(1)) \cdot \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

We seek to approximate the right-hand-side of $\mathcal{G}(z)$ by only taking the products of the primes $p \leq u$, e.g., $p \in \{2, 3, 5, \ldots, u\}$, of which the last element in this set has average order of $\log\left[\frac{u}{\log u}\right]$ for some minimal $u \geq 2$ as $x \to \infty$. We also state the following theorem reproduced from [11, Thm. 7.20] that handles the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \frac{3}{2}\log\log x$. This allows us later to show that taking just $k \in [1, \frac{3}{2}\log\log x]$ and summing over such k in Theorem 1.6 captures the asymptotically relevant, dominant behavior of the values of $\pi_k(x)$ for $k \leq \frac{\log x}{\log 2}$ (where $\Omega(n) \leq \frac{\log n}{\log 2}$ for all $n \geq 2$).

Theorem 3.1 (Bounds on exceptional values of $\Omega(n)$ for large n, MV 7.20)**.** *Let*

$$B(x, r) := \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\}.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x, r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

In particular, we have that for $r \in \left(\frac{3}{2}, 2\right)$,

$$\left|1 - \frac{B(x, r)}{B(x, 3/2)}\right| \xrightarrow{x \to \infty} 1.$$

This follows from the 1st part?

The proof of Theorem 3.1 is found in the cited reference as Chapter 7 of Montgomery and Vaughan.

3.2 The key new results utilizing Theorem 1.6

Corollary 3.2. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \gg 2.$$

When $s := 1$, we have the known bound in Mertens theorem. For $s > 1$, we obtain that

$$P_s(x) \approx E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1).$$

It follows that

$$\gamma_0(s, x) + o(1) \leq P_s(x) \leq \gamma_1(s, x) + o(1),$$

Both γ_0 & $\gamma_1 \longrightarrow +\infty$??

$P_s(x)$ is finite

$\gamma_{0,1}(x) = c_s \log x \pm d_s \log_2 x \pm e_s.$

where it suffices to take z ?

$$\gamma_0(z, x) = -s \log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4} s(s-1)\log(x/2) - \frac{11}{36} s(s-1)^2 \log^2(x)$$
$$\gamma_1(z, x) = -s \log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4} s(s-1)\log(x/2) + \frac{11}{36} s(s-1)^2 \log^2(2).$$

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Proof. Let $s > 1$ be real-valued. By Abel summation where our summatory function is given by $A(x) = \pi(x) \sim \frac{x}{\log x}$ and our function $f(t) = t^{-s}$ so that $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1), |x| \to \infty. \end{aligned}$$

Now using the inequalities in Facts 2.6, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\frac{P_s(x)}{s} \geq -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1)\log(x/2) - \frac{11}{36}(s-1)^2 \log^2(x)$$
$$\frac{P_s(x)}{s} \leq -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1)\log(x/2) + \frac{11}{36}(s-1)^2 \log^2(2).$$

This completes the proof of the bounds cited above in the statement of this lemma. □

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Proof of Theorem 1.7. We have that for all integers $0 \leq k \leq m$

$$[z^k] \prod_{1 \leq i \leq m} (1 - f(i)z)^{-1} = [z^k] \exp\left(\sum_{j \geq 1} \left(\sum_{i=1}^m f(i)^j\right) \frac{z^j}{j}\right).$$

In our case we have that $f(i)$ denotes the i^{th} prime. Hence, summing over all $p \leq x$ in place of $0 \leq k \leq m$ in the previous formula applied in tandem with Corollary 3.2, we obtain that the logarithm of the generating function series we are after when we sum over all $p \leq u$ for some $u \geq 2$ corresponds to

$$\begin{aligned} \log\left[\prod_{p \leq u}\left(1 - \frac{z}{p}\right)^{-1}\right] &\geq (B + \log\log u)z + \sum_{j \geq 2}\left[a(u) + b(u)(j-1) + c(u)(j-1)^2\right] z^j \\ &= (B + \log\log u)z - a(u)\left(1 + \frac{1}{z-1} + z\right) + b(u)\left(1 + \frac{2}{z-1} + \frac{1}{(z-1)^2}\right) \\ &\quad - c(u)\left(1 + \frac{4}{z-1} + \frac{5}{(z-1)^2} + \frac{2}{(z-1)^3}\right). \end{aligned}$$

In the previous equations, the lower bounds formed by the functions (a, b, c) are given by the corresponding upper bounds from Corollary 3.2 due to the leading sign on the following expansions:

$$(a_\ell, b_\ell, c_\ell) := \left(-\log\left(\frac{\log u}{\log 2}\right), \frac{3}{4}\log\left(\frac{u}{2}\right), \frac{11}{36}\log^2 2\right).$$

(10). $A_x \approx B_x$ means $|A_x - B_x| = O(1)$.

You will need a proof of (10.)

G^{-1} is signed, and you seem to be using a continuous approx of $\pi(x/k)$. Not obvious to me that (10) is true.

5 Key applications: Establishing lower bounds for $M(x)$ by cases along infinite subsequences

5.1 The culmination of what we have done so far

As noted before in the previous subsections, we cannot hope to evaluate functions weighted by $\lambda(n)$ except for on average using Abel summation. For this task, we need to know the bounds on $\widehat{\pi}_k(x)$ we developed in the proof of Corollary 1.7. A summation by parts argument shows that[*] [†]

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k)+1) \\ &\approx G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)} \qquad (10) \\ &\approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt. \end{aligned}$$

The result proved in Lemma 5.1 is key to justifying the asymptotics obtained next in Theorem 5.2.

What role does (ℓ) play??

Lemma 5.1. *Suppose that $f_k(n)$ is a sequence of arithmetic functions such that $f_k(n) > 0$ for all $n \geq 1$, $f_0(n) = \delta_{n,1}$, and $f_{\Omega(n)}(n) \gtrsim \widehat{\tau}_\ell(n)$ as $n \to \infty$ where $\widehat{\tau}_\ell(t)$ is a continuously differentiable function of t for all large enough $t \gg 1$. We define the λ-sign-scaled summatory function of f as follows:*

$$F_\lambda(x) := \sum_{\substack{n \leq x \\ \Omega(n) \leq x}} \lambda(n) \cdot f_{\Omega(n)}(n).$$

Let

$$A_\Omega^{(\ell)}(t) := \sum_{k=1}^{\left\lfloor \frac{3}{2}\log\log t \right\rfloor} (-1)^k \widehat{\pi}_k(t).$$

$\widehat{\pi}_\ell(t) = ?$

Then we have that

no ℓ here.

$$F_\lambda(\log\log x) \gtrsim A_\Omega^{(\ell)}(\log\log x)\widehat{\tau}_\ell(\log\log x) - \int_1^{\log\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_\ell'(t)dt.$$

Proof. The formula for $F_\lambda(x)$ is valid by Abel summation provided that

These are signed functions.

$$\left| \frac{\sum_{\frac{3}{2}\log\log t < k \leq \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_\Omega^{(\ell)}(t)} \right| = o(1),$$

[*]Here, we drop the unnecessary floored integer-valued arguments to $\pi(x)$ in place of its approximation by $\pi(x) \sim \frac{x}{\log x}$. In fact, since we can always bound

$$\frac{Ax}{\log x} \leq \pi(x) \leq \frac{Bx}{\log x},$$

for suitably defined absolute constants, $A, B > 0$, we are not losing any precision asymptotically by making this small leap in approximation from exact summation (in the first formula) to the integral formula representing convolution (in the second formula below).

[†]Since $\pi(1) = 0$, the actual range of summation corresponds to $k \in [1, \frac{x}{2}]$.

You want to write

$$A \lesssim B$$

when A & B are

SIGNED functions.

This is not the way readers understand this notation.

Your assumption

$$f_{\Omega(n)}(n) \gtrsim \hat{\tau}_\ell(n)$$

where $\hat{\tau}_\ell(n)$ is $C^{\underline{1}}$ $f^{\underline{n}}$

I guess that the derivative of $\hat{\tau}_\ell$ plays a role. If latter application has derivatives blowing up

That will be a problem.

Need to spell out what is required on

1st line of your proof.

Abel summation requires a C^1 f^n. Where is it?

Also, based on $(*)$, you are ~~proving more~~. why aren't you stating it?

$$\frac{3}{2} \log_2 x = \frac{\log t}{\log 2}$$

$$x = e^{e^{\frac{2}{3} \frac{\log t}{\log 2}}}$$

$$\boxed{x_t = e^{t^{2/3 \log 2}}}$$

(*) Above is

$$\left| \frac{A_{\Omega}^{(\ell)}(x_t) - A_{\Omega}^{(\ell)}(t)}{A_{\Omega}^{(\ell)}(t)} \right| = o(1)$$

I dont see why that proves your conclusion.

e.g., the asymptotically dominant terms indicating the parity of $\lambda(n)$ are encompassed by the terms summed by $A_{\Omega}^{(\ell)}(t)$ for sufficiently large t as $t \to \infty$. Using the arguments in Montgomery and Vaughan [11, §7; Thm. 7.20] (see Theorem 3.1), we can see that uniformly in x

$$\left| \frac{\sum_{k \leq x} \pi_k(x)}{B\left(x, \frac{3}{2}\right)} \right| \sim 1, \tag{11}$$

as $x \to \infty$ where $B(x, r)$ is defined as in the cited theorem re-stated on page 19 from the reference. Thus we have captured the asymptotically dominant main order terms in our formula as $x \to \infty$. $\square$

To simplify notation, for integers $m \geq 1$, let the *iterated logarithm function* (not to be confused with powers of $\log x$) be defined for $x > 0$ by

$$\log_*^m(x) := \begin{cases} x, & \text{if } m = 0; \\ \log x, & \text{if } m = 1; \\ \log\left(\log_*^{m-1}(x)\right), & \text{if } m \geq 2. \end{cases}$$

So $\log_*^2(x) = \log\log x$, $\log_*^3(x) = \log\log\log x$, $\log_*^4(x) = \log\log\log\log x$, $\log_*^5(x) = \log\log\log\log\log x$, and so on. This notation will come in handy to abbreviate the dominant asymptotic terms we find in Theorem 5.2 proved below.

We use the result of Corollary 4.6 and Corollary 1.7 to prove the following central theorem:

Theorem 5.2 (Asymptotics and bounds for the summatory functions $G^{-1}(x)$)**.** *We define the lower summatory function, $G_\ell^{-1}(x)$, to provide bounds on the magnitude of $G^{-1}(x)$:*

$$|G_\ell^{-1}(x)| \ll |G^{-1}(x)|,$$

for all sufficiently large $x \gg 1$. We have the next asymptotic approximations for the lower summatory function where $C_{\ell,1}$ is the absolute constant defined by

$$C_{\ell,1} = \frac{4A_0^2}{\sqrt{3}\pi^2 \log^4(2)} = \frac{1}{36 \cdot 2^{3/4}\sqrt{\pi} \cdot \log 2} \approx 0.000183209.$$

That is, we have

$$\left|G_\ell^{-1}(x)\right| \gtrsim \left| C_{\ell,1} \cdot (\log x)(\log\log x)(\log\log\log x)^{4 + \frac{3}{\log 2} - \frac{3}{\log 3}} \right|.$$

The exponent in the previous equation is numerically approximated as $4 + \frac{3}{\log 2} - \frac{3}{\log 3} \approx 5.59737$.

Proof Sketch: Logarithmic scaling to the accurate order of the inverse functions. For the sums given by

$$S_{g^{-1}}(x) := \sum_{n \leq x} \lambda(n) \cdot C_{\Omega(n)}(n),$$

we notice that using the asymptotic bounds (rather than the exact formulas) for the functions $C_{\Omega(n)}(n)$, we have over-summed by quite a bit. In particular, following from the intent behind the constructions in the last sections, we are really summing only over all