5 New formulas and limiting relations characterizing $M(x)$

5.1 Formulas relating $M(x)$ to the summatory function $G^{-1}(x)$

Proposition 5.1. *For all sufficiently large x, we have that*

$$M(x) = G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} G^{-1}(k)\left(\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1}\right\rfloor\right)\right). \tag{26}$$

Proof. We know by applying Corollary 1.4 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)\left(\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right) + 1\right) \\ &= G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} g^{-1}(k)\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right) \\ &= G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2}\right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left(\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1}\right\rfloor\right)\right). \end{aligned}$$

The upper bound on the sum is truncated to $k \in \left[1, \frac{x}{2}\right]$ in the second equation above due to the fact that $\pi(1) = 0$. The third formula above follows directly by (ordinary) summation by parts. ☐

By the result from (13) proved in Section 3.2, we recall that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p}\right\rfloor\right), \text{ for } x \geq 1.$$

Summation by parts implies that we can also express $G^{-1}(x)$ in terms of the summatory function $L(x)$ and differences of the unsigned sequence whose distribution is given by Corollary 4.9. That is, we have

$$G^{-1}(x) = \sum_{n \leq x} \lambda(n)|g^{-1}(n)| = L(x)|g^{-1}(x)| - \sum_{n<x} L(n)\left(|g^{-1}(n+1)| - |g^{-1}(n)|\right), \text{ for } x \geq 1.$$

5.2 Asymptotics of $G^{-1}(x)$

The following proofs are credited to Professor R. C. Vaughan and his suggestions about approaches to upper bounds on $|G^{-1}(x)|$ that are attained along infinite subsequences as $x \to \infty$. The ideas at the crux of the proof of the next theorem are found in the references by Davenport and Heilbronn [3, 4] and are known to date back to the work of Hans Bohr [29, *cf.* §11].

Theorem 5.2. *Let σ_1 denote the unique solution to the equation $P(\sigma) = 1$ for $\sigma > 1$. There are complex s with $\mathrm{Re}(s)$ arbitrarily close to σ_1 such that $1 + P(s) = 0$.*

Proof. The function $P(\sigma)$ is decreasing on $(1, \infty)$, tends to $+\infty$ as $\sigma \to 1^+$, and tends to zero as $\sigma \to \infty$. Thus we find that the equation $P(\sigma) = 1$ has a unique solution for $\sigma > 1$, which we denote by $\sigma = \sigma_1 \approx 1.39943$. Let $\delta > 0$ be chosen small enough that $|1 - P(z)| > 0$ for all z such that $|z - \sigma_1| = \delta$. Set

$$\eta = \min_{\substack{z \in \mathbb{C} \\ |z-\sigma_1| = \delta}} |1 - P(z)|.$$

Since $P(z)$ is continuous whenever $\mathrm{Re}(z) > 1$, we have that $\eta > 0$. Let $X \geq 2$ be a sufficiently large integer so that

$$\sum_{p > X} p^{\delta - \sigma_1} < \frac{\eta}{4}.$$

Kronecker's theorem provides a fixed t such that the following inequality holds [9, §XXIII]:

$$\max_{2<p\leq X}\min_{n\in\mathbb{Z}}\left|\frac{t\log p}{2\pi}-n-\frac{1}{2}\right|<\delta\eta.$$

Thus we have that

$$\sum_{p>2}p^{\delta-\sigma_1}\left|p^{\imath t}+1\right|<\frac{\eta}{2}.$$

Hence, for all z such that $|z-\sigma_1|=\delta$, we have

$$|P(z+\imath t)+P(z)|<\frac{\eta}{2}.$$

We apply Rouché's theorem to see that the functions $1-P(z)$ and $1-P(z)+P(z+\imath t)+P(z)$ have the same number of zeros in the disk $\mathcal{D}_\delta=\{z\in\mathbb{C}:|z-\sigma_1|<\delta\}$. Since $1-P(z)$ has at least one zero within $\mathcal{D}_\delta$, we must have that $1+P(w)$ has at least one zero with $|w-\sigma_1-\imath t|<\delta$. Since we can take δ as small as necessary, there are zeros of the function $1+P(s)$ that are arbitrarily close to the line $s=\sigma_1$. ☐

Corollary 5.3. *Let $\sigma_1>1$ be defined as in Theorem 5.2. For any $\epsilon>0$, there are arbitrarily large x such that*

$$\boxed{|G^{-1}(x)|>x^{\sigma_1-\epsilon}.}$$

? for $G>1$

Proof. We have by (6) that

$$D_{g^{-1}}(s):=\sum_{n\geq 1}\frac{g^{-1}(n)}{n^s}=\frac{1}{\zeta(s)(1+P(s))},\quad\text{for }\operatorname{Re}(s)>1.$$

Theorem 5.2 implies that $D_{g^{-1}}(s)$ has singularities $s\in\mathbb{C}$ such that the $\operatorname{Re}(s)$ are arbitrarily close to σ_1. By applying [17, Cor. 1.2; §1.2], we have that any Dirichlet series is locally uniformly convergent in its half-plane of convergence, e.g., for $\operatorname{Re}(s)>\sigma_c$, and is hence analytic in this half-plane. It follows that the abscissa of convergence of $D_{g^{-1}}(s)$ is given by $\sigma_c\geq\sigma_1>1$. In particular, the abscissa of convergence of this DGF cannot be smaller than σ_1. The result proved in [17, Thm. 1.3; §1.2] then shows that

$$\limsup_{x\to\infty}\frac{\log|G^{-1}(x)|}{\log x}=\sigma_c\geq\sigma_1.$$ ☐

5.3 Local cancellation of $G^{-1}(x)$ in the new formulas for $M(x)$

Lemma 5.4. *Suppose that p_n denotes the n^{th} prime for $n\geq 1$ [27, A000040]. Let $\mathcal{P}_\#$ denote the set of positive primorial integers as [27, A002110]*

$$\mathcal{P}_\#=\{n\#\}_{n\geq 1}=\left\{\prod_{k=1}^n p_k:n\geq 1\right\}=\{2,6,30,210,2310,30030,\ldots\}.$$

As $m\to\infty$ we have

$$-G^{-1}((4m+1)\#)=(4m+1)!\left(1+O\left(\frac{1}{m^2}\right)\right),$$
$$G^{-1}\left(\frac{(4m+1)\#}{p_k}\right)=(4m)!\left(1+O\left(\frac{1}{m^2}\right)\right),\quad\text{for all }1\leq k\leq 4m+1.$$

Proof. We have by part (B) of Proposition 1.6 that for all squarefree integers $n\geq 1$

$$|g^{-1}(n)|=\sum_{j=0}^{\omega(n)}\binom{\omega(n)}{j}\times j!=(\omega(n))!\times\sum_{j=0}^{\omega(n)}\frac{1}{j!}$$

My adviser brought up the next issue with the result in Cor 5.3:

- we claim that for ∞-many large x, $x^{G_1-\varepsilon} < |G^{-1}(x)|$, for any $\varepsilon > 0$ and $G_1 \simeq 1.39943$.
- Yet I proved that

$$\frac{1}{x} \cdot \sum_{n \leq x} |g^{-1}(n)|$$

$$= \frac{6B_0}{\pi^2} (\log x)^2 \sqrt{\log\log x}\,(1+o(1)),$$

as $x \rightarrow \infty$, where $B_0 > 0$ is an absolute constant.

- However, he points out the following contrary argument:

$$x^{G_1-\varepsilon} < |G^{-1}(x)|$$

$$\le \sum_{n\le x} |g^{-1}(n)|$$

$$\ll x(\log x)^2\sqrt{\log\log x}$$

(with $G_1 > 1$).

I do not see the error looking back through the proofs.

Can you please offer some suggestions for where we went wrong?

Thank you for kind correspondence. :)

THE FUNCTION $G^{-1}(x)$

1. Notation

As usual $\omega(n)$ is the number of distinct prime divisors on n. Then $g(n)$ is defined by

$$g(n) = 1 + \omega(n).$$

In connection with this define for $\sigma > 1$

$$P(s) = \sum_p p^{-s}.$$

If I recall correctly Landau showed (I think in Rendiconti di Palermo very roughly about 1920) that this has an analytic continuation to the half-plane $\sigma > 0$ given by

$$P(s) = -\sum_{m=1}^{\infty} \frac{\mu(m)}{m} \log \zeta(sm)$$

and has the line $\operatorname{Re} s = 0$ as a natural boundary (every point of the imaginary axis is a limit point of singularities of the function). This is not really relevant to our discussion but is interesting background.

Now define for $\sigma > 1$

$$\mathcal{P}(s) = 1 + P(s).$$

Then $\mathcal{G}(s)$ defined by

$$\mathcal{G}(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

satisfies

$$\mathcal{G}(s) = \zeta(s)\mathcal{P}(s).$$

Thus, if we write $\mathbf{1}(n)$ for the function which is 1 for every n and $p(n)$ for the function which is 1 when $n = 1$ or n is prime and 0 for all other n, then we have

$$g = \mathbf{1} * p.$$

Now, as long as $\sigma > 1$ and the factors on the right are non-zero one has

$$\mathcal{G}(s)^{-1} = \zeta(s)^{-1}\mathcal{P}(s)^{-1}.$$

In other words

$$g^{-1} = \mu * p^{-1}$$

and

$$\mu = g^{-1} * p.$$

Hence

$$M(x) = G^{-1}(x) + \sum_p G^{-1}(x/p)$$

so that

$$M(x) = G^{-1}(x) + G^{-1}(x/2) + \sum_{p>2} G^{-1}(x/p) \tag{1.1}$$

which is essentially Proposition 5.2 since

$$\begin{aligned}\sum_{p>2} G^{-1}(x/p) &= \sum_{k\le x/2} \sum_{k\le x/p<k+1} G^{-1}(x/p)\\ &= \sum_{k\le x/2} G^{-1}(k)\big(\pi(x/k) - \pi(x/(k+1))\big).\end{aligned}$$

The function $P(\sigma)$ is strictly decreasing on $(1,\infty)$, tends to ∞ as $\sigma \to 1$ and tends to 0 as $\sigma \to \infty$. Thus the equation

$$P(\sigma) = 1$$

has a unique solution $\sigma = \sigma_1$ with $\sigma_1 > 1$. Now we can prove the following theorem.

Theorem 1.1. *There are s with σ arbitrarily close to σ_1 such that*

$$\mathcal{P}(s) = 0.$$

Corollary 1.2. *The functions*

$$\mathcal{G}(s)^{-1},$$

$$\mathcal{G}(s)^{-1}2^{-s},$$

and

$$\zeta(s)^{-1}\big(1 - (1+2^{-s})\mathcal{P}(s)^{-1}\big)$$

have singularities at points s with σ arbitrarily close to σ_1, and so in each case the corresponding Dirichlet series has abscissa of convergence $\sigma_c \ge \sigma_1$.

Note that, by Corollary 1.2 in Chapter 1 of M&V, a Dirichlet series converges locally uniformly in its half-plane of convergence and so is analytic there. Thus in each case the abscissa of convergence cannot be smaller than σ_1. Hence, by Theorem 1.3 in Chapter 1 of M&V

$$\limsup_{x\to\infty} \frac{\log|G^{-1}(x)|}{\log x} \ge \sigma_1,$$

$$\limsup_{x\to\infty} \frac{\log|G^{-1}(x/2)|}{\log x} \geq \sigma_1,$$

$$\limsup_{x\to\infty} \frac{\log|\sum_{2<p\leq x} G^{-1}(x/p)|}{\log x} \geq \sigma_1.$$

Thus, for example, given any $\varepsilon > 0$, there are arbitrarily large x such that

$$|G^{-1}(x)| > x^{\sigma_1-\varepsilon}.$$

This contradicts Theorem 5.1.

I am guessing that these three terms are all this large most of the time, even though they largely cancel each other out. The coefficients of the three sums in (1.1) are precisely the coefficients of the three Dirichlet series above. Thus each of the three sums could be very large. I doubt that the third term can be ignored even in special cases.

2. Proof of Theorem 1.1

The ideas of this proof go back certainly to Davenport and Heilbronn [1936a] and [1936b], and perhaps even to Hans Bohr (see Chapter 11 of Titchmarsh [1986]). Choose δ arbitrarily small so that

$$|1 - P(z)| > 0$$

for all z with $|z - \sigma_1| = \delta$. Let

$$\eta = \min_{\substack{z \\ |z-\sigma_1|=\delta}} |1 - P(z)|.$$

Since $P(z)$ is continuous we have

$$\eta > 0.$$

Choose X so that

$$\sum_{p>X} p^{\delta-\sigma_1} < \frac{\eta}{4}.$$

Then we can use Kronecker's theorem to provide us with a t such that

$$\max_{2<p\leq X} \min_{n\in\mathbb{Z}} \left| \frac{t\log p}{2\pi} - n - \frac{1}{2} \right| < \delta\eta.$$

Thus

$$\sum_{p>2} p^{\delta-\sigma_1}|p^{it}+1| < \frac{\eta}{2}.$$

Hence, whenever $|z - \sigma_1| = \delta$ we have

$$|P(z+it) + P(z)| < \frac{\eta}{2}.$$

Then, by Rouché's theorem, $1 - P(z)$ and $1 - P(z) + P(z+it) + P(z)$ have the same number of zeros with $|z - \sigma_1| < \delta$. Since $1 - P(z)$

has at least one zero there, $1 + P(w)$ will have at least one zero with $|w - \sigma_1 - it| < \delta$. Since δ can be made arbitrarily small there are zeros arbitrarily close to the line $s = \sigma_1$.

References

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