Lower bounds on the Mertens function $M(x)$ for $x \gg 2.3315 \times 10^{1656520}$

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Abstract

The Mertens function, $M(x) = \sum_{n \leq x} \mu(n)$, is classically defined to be the summatory function of the Möbius function $\mu(n)$. In some sense, the Móbius function can be viewed as a signed indicator function of the squarefree integers which have asymptotic density of $6/\pi^2 \approx 0.607927$ and a corresponding well-known asymptotic average order formula. The signed terms in the sums in the definition of the Mertens function introduce complications in the form of semi-randomness and cancellation inherent to the distribution of the Möbius function over the natural numbers. The Mertens conjecture which states that $|M(x)| < C \cdot \sqrt{x}$ for all $x \geq 1$ has a well-known disproof due to Odlyzko et. al. It is widely believed that $M(x)/\sqrt{x}$ is an unbounded function which changes sign infinitely often and exhibits a negative bias over all natural numbers $x \geq 1$.

We focus on obtaining new lower bounds for $M(x)$ by methods that generalize to handle other related cases of special number theoretic summatory functions. The key to our proofs calls upon a known result from the standardized summatory function enumeration by Dirichlet generating functions (DGFs) found in Chapter 7 of Montogomery and Vaughan. There is also a distinct flavor of combinatorial analysis peppered in with the standard methods from analytic number theory which distinguishes our methods.

Keywords and Phrases: *Möbius function sums; Mertens function; summatory function; arithmetic functions; Dirichlet inverse; Liouville lambda function; prime omega functions; prime counting functions; Dirichlet series and DGFs; asymptotic lower bounds; Mertens conjecture; asymptotic methods from the Montgomery and Vaughan textbook.*
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TODO:

- references;

1 Reference on common abbreviations, special notation and other conventions

Symbol	**Definition**
$\lceil x \rceil$	The ceiling function $\lceil x \rceil := x + 1 - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$.
$C_k(n)$	Auxillary component functions in obtaining asymptotic bounds on $g^{-1}(n)$. These functions are defined for $k \geq 0$ and $n \geq 1$ by the formula $$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d\mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero.
DGF	*Dirichlet generating function.* Given a sequence $\{f(n)\}_{n\geq 0}$, its DGF enumerates the sequence in a different way than formal generating functions in an auxiliary variable. Namely, for $\lvert s \rvert < \sigma_a$, the abcissa of absolute convergence of the series, the DGF $D_f(s)$ constitutes an analytic function of s given by: $D_f(s) := \sum_{n\geq 1} f(n)/n^s$. The DGF is alternately called the *Dirichlet series* of an arithmetic function f. type
$\sigma_0(n), d(n)$	The ordinary divisor function, $d(n) := \sum_{d\mid n} 1$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) = \delta_{n,1}$.
$*; f * g$	The Dirichlet convolution of f and g, $f * g(n) := \sum\limits_{d\mid n} f(d) g(n/d)$, for $n \geq 1$. This symbol for the discrete convolution of two arithmetic functions is the only notion of convolution of functions we employ within the article.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ provided that $f(1) \neq 0$.
$\lfloor x \rfloor$	The floor function $\lfloor x \rfloor := x - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega+1)^{-1}(n)$ with summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$.
$\mathrm{Id}_k(n)$	The power-scaled identity function, $\mathrm{Id}_k(n) := n^k$ for $n \geq 1$.
$\mathbb{1}_{\mathbb{S}}, \chi_{\mathtt{cond}(x)}$	We use the notation $\mathbb{1}, \chi : \mathbb{N} \to \{0,1\}$ to denote indicator, or characteristic functions. In paticular, $\mathbb{1}_{\mathbb{S}}(n) = 1$ if and only if $n \in \mathbb{S}$, and $\chi_{\mathtt{cond}}(n) = 1$ if and only if n satisfies the condition `cond`.
$\log_*^m(x)$	The iterated logarithm function defined recursively for integers $m \geq 0$ by $$\log_*^m(x) := \begin{cases} x, & \text{if } m = 0; \\ \log x, & \text{if } m = 1; \\ \log\left[\log_*^{m-1}(x)\right], & \text{if } m \geq 2. \end{cases}$$
$[n = k]_\delta$	Synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and zero otherwise.
$[\mathtt{cond}]_\delta$	For a boolean-valued `cond`, $[\mathtt{cond}]_\delta$ evaluates to one precisely when `cond` is true, and zero otherwise.

Symbol	**Definition**
$\gcd(m,n); (m,n)$	The greatest common divisor of m and n. Both notations for the GCD are used interchangably within the article.
$\mu(n)$	The Möbius function.
$M(x)$	The Mertens function which is the summatory function over $\mu(n)$, $M(x) := \sum\limits_{n\leq x} \mu(n)$.
$\nu_p(n)$	The function that extracts the prime exponent of p from the prime factorization of n.
$\sum_{p\leq x}, \prod_{p\leq x}$	Unless otherwise specified by context, we use the index variable p to denote that the summation (product) is to be taken only over prime values within the summation bounds.
$P(s)$	For complex s with $\Re(s) > 1$, we define $P(s) = \sum_{p \text{ prime}} p^{-s}$.
$\sigma_\alpha(n)$	The generalized sum-of-divisors function, $\sigma_\alpha(n) := \sum\limits_{d\mid n} d^\alpha$, for any $n \geq 1$ and $\alpha \in \mathbb{C}$.
$\left[{n \atop k}\right]$	The unsigned Stirling numbers of the first kind, $\left[{n \atop k}\right] = (-1)^{n-k} \cdot s(n,k)$.
$\gtrsim, \lesssim$	We say that $h(x) \gtrsim r(x)$ if $h \gg r$ as $x \rightarrow \infty$, and define the relation $\lesssim$ similarly. When applying these relations we still consider leading constants to be meaningful terms that are preserved.
$\sum_{n\leq x}'$	We denote by $\sum_{n\leq x}' f(n)$ the summatory function of f at x minus $\frac{f(x)}{2}$ if $x \in \mathbb{Z}$.
$\zeta(s)$	The Riemann zeta function, defined by $\zeta(s) := \sum_{n\geq 1} n^{-s}$ when $\Re(s) > 1$, and by analytic continuation to the entire complex plane with the exception of a simple pole at $s = 1$.

2 Introduction

2.1 The Mertens function – definition, properties, known results and conjectures

Suppose that $n \geq 1$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2 \alpha_2 \cdots p_k^{\alpha_k}$. We define the *Möebius function* to be the signed indicator function of the squarefree integers:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leq i \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many known variants and special properties of the Möebius function and its generalizations [8, *cf.* §2], however, for our purposes we seek to explore the properties and asymptotics of weighted summatory functions over $\mu(n)$. The Mertens summatory function, or *Mertens function*, is defined as [9, A002321]

$$\begin{aligned} M(x) &= \sum_{n \leq x} \mu(n), \ x \geq 1, \\ &\longmapsto \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2\} \end{aligned}$$

A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as [9, A013928]

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \rightarrow \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{Q(x)} = \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{Q(x)} \xrightarrow[n \rightarrow \infty]{} \frac{3}{\pi^2}.$$

While this limiting law suggests an even bias for the Mertens function, in practice $M(x)$ has a noted negative bias in its values, and the actual local oscillations between the approximate densities of the sets $\mu_\pm(x)$ lend an unpredictable nature to the function and its characteistic oscillatory sawtooth shaped plot – even over asymptotically large and variable intervals.

2.1.1 Properties

The well-known approach to evaluating the behavior of $M(x)$ for large $x \rightarrow \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is easily expressed via an inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since by Perron's formula we have

$$\frac{1}{\zeta(s)} = \int_1^\infty \frac{s \cdot M(x)}{x^{s+1}} dx,$$

we then obtain that

$$M(x) = \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation along with the standard Euler product representation for the reciprocal zeta function leads us to the exact expression for $M(x)$ when $x > 0$ given by the next theorem.

Theorem 2.1 (Analytic Formula for $M(x)$)**.** *If the RH is true, then there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any $x \in \mathbb{R}_{>0}$*

$$M(x) = \lim_{k \rightarrow \infty} \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |\Im(\rho)| < T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

An unconditional bound on the Mertens function due to Walfisz [?] states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \exp\left(-C \cdot \log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan proved new updated estimates bounding $M(x)$ for large x in 2009 of the following forms:

$$\begin{aligned} M(x) &\ll \sqrt{x} \exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x}\exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right), \ \forall \epsilon > 0. \end{aligned}$$

Other explicit bounds due to the article by Kotnik include the following simpler estimates for the Mertens function when x is sufficiently large:

$$\begin{aligned} |M(x)| &< \frac{x}{4345}, \ \forall x > 2160535, \\ |M(x)| &< \frac{0.58782 \cdot x}{\log^{11/9}(x)}, \ \forall x > 685. \end{aligned}$$

2.1.2 Conjectures

The Riemann Hypothesis (RH) is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. It is still unresolved whether

$$\limsup_{x\rightarrow\infty} |M(x)|/\sqrt{x} = \infty,$$

although computational evidence suggests that this is a likely conjecture [?, ?]. There is a rich history to the original statement of the *Mertens conjecture* which states that

$$|M(x)| < c \cdot x^{1/2}, \quad \text{some constant } c > 0,$$

which was first verified by Mertens for $c = 1$ and $x < 10000$, although since its beginnings in 1897 has since been disproved by computation by Odlyzko and té Riele in the early 1980's.

There are a number of other interesting unsolved and at least somewhat accessible open problems related to the asymptotic behavior of $M(x)$ at large x. It is believed that the sign of $M(x)$ changes infinitely often. That is to say that it is widely believed that $M(x)$ is ocsillatory and exhibits a negative bias insomuch as $M(x) < 0$ more frequently than $M(x) > 0$ over all $x \in \mathbb{N}$. One of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ is unbounded on the natural numbers. In particular, the precise statement of this problem is to produce an affirmative answer whether $\limsup_{x\rightarrow\infty} |M(x)|/\sqrt{x} = +\infty$, or equivalently whether there is an infinite sequence of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that $M(x_i)x_i^{-1/2}$ grows without bound along this subsequence.

Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies that

$$\limsup_{x\rightarrow\infty} \frac{|M(x)|}{\sqrt{x}(\log\log x)^{5/4}},$$

corresponds to some bounded constant. A probabilistic proof along these lines has been given by Ng in 2008. To date an exact rigorous proof that $M(x)/\sqrt{x}$ is unbounded still remains elusive. We cite that prior to this point it is known that [?, *cf.* §4.1]

$$\limsup_{x\rightarrow\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad \text{(now 1.826054)},$$

and

$$\liminf_{x\rightarrow\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad \text{(now } -1.837625\text{)},$$

although based on work by Odlyzyko and te Riele it seems probable that each of these limits should be $\pm\infty$, respectively [?, ?, ?, ?]. It is also known that $M(x) = \Omega_{\pm}(\sqrt{x})$ and $M(x)/\sqrt{x} = \Omega_{\pm}(1)$.

2.2 A new approach to bounding $M(x)$ from below

2.2.1 Summing series over Dirichlet convolutions

Theorem 2.2 (Summatory functions of Dirichlet convolutions)**.** *Let $f, g : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $G(x) := \sum_{n \leq x} g(n)$ denote the summatory functions of f, g, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse $f^{-1}(n)$ of f, i.e., the unique arithmetic function such that $f * f^{-1} = \varepsilon$ where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity with respect to Dirichlet convolution. Then we have the following equivalent expressions for the summatory function of $f * g$ for integers $x \geq 1$:*

$$\begin{aligned}\pi_{f*g}(x) &= \sum_{n \leq x} \sum_{d|n} f(d) g(n/d) \\ &= \sum_{d \leq x} f(d) G\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{k=1}^{x} G(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].\end{aligned}$$

Moreover, we can invert the linear system determining the coefficients of $G(k)$ for $1 \leq k \leq x$ naturally to express $G(x)$ as a linear combination of the original left-hand-side summatory function as:

$$\begin{aligned}G(x) &= \sum_{j=1}^{x} \pi_{f*g}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\ &= \sum_{n=1}^{x} f^{-1}(n) \pi_{f*g}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).\end{aligned}$$

Corollary 2.3 (Convolutions Arising From Möbius Inversion)**.** *Suppose that g is an arithmetic function with $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n \leq x} (g * \mu)(n)$. Then the Mertens function equals*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geq 1.$$

2.2.2 A motivating special case

Using $\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu$, where $\chi_{\mathbb{P}}$ is the characteristic function of the primes, we have that $\widetilde{G}(x) = \pi(x) + 1$ in Corollary 2.3. In particular, the corollary implies that

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

We can compute the first few terms for the Dirichlet inverse sequence of $g(n) := \omega(n) + 1$ numerically for the first few sequence values as

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is given by $\lambda(n) = (-1)^{\Omega(n)}$ (see Proposition 3.2). Note that since the DGF of $\omega(n)$ is given by $D_\omega(s) = P(s)\zeta(s)$ where $P(s)$ is the *prime zeta function*, we do have a Dirichlet series for the inverse functions to invert coefficient-wise using more classical contour integral methods, e.g., using [1, §11]

$$f(n) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \frac{n^{\sigma + \imath t}}{\zeta(\sigma + \imath t)(P(\sigma + \imath t) + 1)}, \sigma > 1.$$

Fröberg has previously done some preliminary investigation as to the properties of the inversion to find the coefficients of $(1+P(s))^{-1}$ [2].

We will instead take a more combinatorial tack to investigating bounds on this inverse function sequence in the coming sections. Consider the following motivating conjecture:

Conjecture 2.4. *Suppose that $n \geq 1$ is a squarefree integer. We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n) = (\omega+1)^{-1}(n)$ over these integers:*

(A) $g^{-1}(1) = 1$, *which follows immediately by computation;*

(B) $\operatorname{sgn}(g^{-1}(n)) = \mu(n) \equiv \lambda(n)$*;*

(C) If $w(n) = k$, we can write the inverse function at k as

$$g^{-1}(n) = \sum_{m=0}^{k} \binom{k}{m} \cdot m!.$$

We illustrate parts (B)–(C) of this conjecture clearly using Table T.1 given on page 35 of the appendix section.

Why exactly is the Dirichlet inverse function, $g^{-1}(n)$, difficult to evaluate? There are several apparent reasons for this. The first is that the Dirichlet inverse function not only depends on the prime factorization of n in the typical way, involving weighted sums of $\Omega(n)$ terms of the function $\omega(n)+1$, but also in the additive nature of how we build up and assemble these terms in an essentially non-multiplicative, but instead very additive, way. Note that for distinct primes a, b and positive integers $m, n \geq 1$, the (incomplete) additivity of $\omega(n)$ implies that $\omega(a^m b^n) = \omega(a) + \omega(b)$ with $\omega(1) = 0$. Secondly, the extra additive factor of $+1$ (that was added to make the function Dirichlet invertible) also does not depend on n in the corresponding expansions of the Dirichlet inverse terms. A table of the first several explicit values of $(f+1)^{-1}(n)$ for $f(1) = 0$ and f additive are given in Table T.2 on page 36. Note that the additivity of f in forming the Dirichlet inverse of $(f+1)^{-1}(n)$ significantly influences the sign of the inverse function, given by $\lambda(n)$.

The realization that the beautiful, and simplistic, e.g., not terribly complicated considering the subject matter, form of property (C) in Conjecture 2.4 holds for all squarefree $n \geq 1$ motivates our pursuit of formulas for the inverse functions $g^{-1}(n)$ based on the configuration of the exponents in the prime factorization of any $n \geq 2$. In Section 5 we consider expansions of these inverse functions recursively, starting from a few first exact cases of an auxillary function, $C_k(n)$, that depends on the precise exponents in the prime factorization of n. We then prove limiting asymptotics for these functions and assemble the main terms in the expansion of $g^{-1}(n)$ using artifacts from combinatorial analysis. Combined with the DGF-based generating function for certain summatory functions indicating the parity of $\Omega(n)$ introduced in the next subsection of this introduction, this take on the identity in (1) provides us with a powerful new method to bound $M(x)$ from below. We will sketch the key results and formulation to the construction we actually use to prove the new lower bounds on $M(x)$ next.

From this point on, we fix the Dirichlet invertible function $g(n) := \omega(n)+1$ and denote its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega+1)^{-1}(n)$. For natural numbers $n \geq 1, k \geq 0$, let

$$C_k(n) := \begin{cases} \varepsilon(n) = \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$

By Möbius inversion (see Lemma 5.2), we have that

$$(g^{-1} * 1)(n) = \lambda(n) \cdot C_{\Omega(n)}(n), \forall n \geq 1.$$

We have limiting asymptotics on these functions given by the following theorem:

Theorem 2.5 (Asymptotics for the functions $C_k(n)$)**.** *Let $\mathbb{1}_{*_m}(n)$ denote the m-fold Dirichlet convolution of one with itself at n. The function $\sigma_0 * 1_{*_m}$ is multiplicative with values at prime powers given by*

$$(\sigma_0 * \mathbb{1}_{*_m})(p^\alpha) = \binom{\alpha+m+1}{m+1}.$$

We have the following asymptotic bases cases for the functions $C_k(n)$:

$$\begin{aligned} C_1(n) &\sim \log\log n \\ C_2(n) &\sim \frac{\sigma_0(n)n}{\log n} + O(\log\log n) \\ C_3(n) &\sim -\frac{(\sigma_0 * 1)(n)n^2}{\log n} + O\left(n \cdot \log\log n\right). \end{aligned}$$

For all $k \geq 4$, we obtain that the dominant asymptotic term and the error bound terms for $C_k(n)$ are given by

$$C_k(n) \sim (\sigma_0 * \mathbb{1}_{*_{k-2}})(n) \times \frac{(-1)^k n^{k-1}}{(\log n)^{k-1}(k-1)!} + O_k\left(\frac{n^{k-2}}{(k-2)!} \cdot \frac{(\log\log n)^{k-2}}{(\log n)^{k-2}}\right), \text{ as } n \to \infty.$$

Then we can prove (see Corollary 5.7) that

$$g^{-1}(n) \sim \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

This in turn implies that

$$G^{-1}(x) \gtrsim \sum_{n \leq x} \lambda(n) \cdot C_{\Omega(n)}(n) \times \sum_{d=1}^{\left\lfloor \frac{x}{n} \right\rfloor} \lambda(d).$$

Now we require the bounds suggested in the next section to work at summing the summatory functions, $G^{-1}(x)$, for large x as $x \to \infty$.

2.2.3 DGFs from Mongomery and Vaughan

Our inspiration for the new bounds found in the last sections of this article allows us to sum non-negative arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. In particular, it uses a hybrid generating function and DGF method under which we are able to recover "good enough" asymptotics about the summatory functions that encapsulate the parity of $\lambda(n)$:

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}, k \geq 1.$$

The precise statement of the theorem that we transform for these new bounds is re-stated as follows:

Theorem 2.6 (Montgomery and Vaughan, §7.4)**.** *Let $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$. For $R < 2$ we have that*

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

uniformly for $1 \leq k \leq R\log\log x$ where

$$\mathcal{G}(z) := \frac{F(1,z)}{\Gamma(z+1)} = \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1}\left(1 - \frac{1}{p}\right)^z.$$

The precise formulations of the inverse function asymptotics proved in Section 5 depend on being able to form significant lower bounds on the summatory functions of an always positive arithmetic function weighted by $\lambda(n)$. The next theorem, proved in Section 4, is the crux of the starting point for our new asymptotic lower bounds.

Theorem 2.7 (Generating functions of symmetric functions)**.** *We obtain upper and lower bounds of the form*

$$\alpha_0(z,x) \leq \prod_{p \leq x} \left(1 - \frac{z}{p}\right)^{-1} \leq \alpha_1(z,x),$$

where it suffices to take

$$\begin{aligned} \alpha_0(z,x) &= \frac{\exp\left(\frac{55}{4}\log^2 2\right)}{\log^3 2} (\log x)^3 \left(\frac{e^B \log^2 x}{\log 2}\right)^z \\ \alpha_1(z,x) &= \exp\left(\frac{11}{3}\log^2 x\right) \left(e^B \log 2\right)^z. \end{aligned}$$

The argument providing new lower bounds for $G(z)$ is completed by the proof given in Corollary 4.2. This leads to a structure involving the incomplete gamma function inherited from Theorem 2.6. In Lemma 4.3, we justify that this construction, which holds uniformly for $k \leq \frac{3}{2}\log\log x$ (taking $R := \frac{3}{2}$), allows us to asymptotically enumerate the main terms in the expansions of $\widehat{\pi}_k(x)$ when we sum over just k in this range (as opposed to $k \leq \frac{\log x}{\log 2}$).

3 Preliminary proofs and configuration

3.1 Establishing the summatory function inversion identities

Given the interpretation of the summatory functions over an arbitrary Dirichlet convolution (and the vast number of such identities for special number theoretic functions – *cf.* [3, ?]), it is not surprising that this formulation of the first theorem may well provide many fruitful applications, indeed. In addition to those cited in the compendia of the catalog reference, we have notable identities of the form: $(f * 1)(n) = [q^n] \sum_{m \geq 1} f(m) q^m / (1-q^m)$, $\sigma_k = \mathrm{Id}_k * 1$, $\mathrm{Id}_1 = \phi * \sigma_0$, $\chi_{\mathrm{sq}} = \lambda * 1$ (see sections below), $\mathrm{Id}_k = J_k * 1$, $\log = \Lambda * 1$, and of course $2^{\omega} = \mu^2 * 1$. The result in Theorem 2.2 is natural and displays a quite beautiful form of symmetry between the initial matrix terms,

$$t_{x,j}(f) = \sum_{k = \left\lfloor \frac{x}{j+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{j} \right\rfloor} f(k),$$

and the corresponding inverse matrix,

$$t_{x,j}^{-1}(f) = \sum_{k = \left\lfloor \frac{x}{j+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{j} \right\rfloor} f^{-1}(k),$$

as expressed by the duality of f and its Dirichlet inverse function f^{-1}. Since the recurrence relations for the summatory functions $G(x)$ arise naturally in applications where we have established bounds on sums of Dirichlet convolutions of arithmetic functions, we will go ahead and prove it here before moving along to the motivating examples of the use of this theorem.

Proof of Theorem 2.2. Let h, g be arithmetic functions where $g(1) \neq 1$ has a Dirichlet inverse. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $S_{g,h}(x)$ to be the summatory function of the Dirichlet convolution of g with h: $g * h$. Then we can easily see that the following expansions hold:

$$\begin{aligned} S_{g,h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n) h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right] H(i). \end{aligned}$$

Thus we have an implicit statement of a recurrence relation for the summatory function H, weighted by g and G, whose non-homogeneous term is the summatory function, $S_{g,h}(x)$, of the Dirichlet convolutions $g * h$. We form the matrix of coefficients associated with this system for $H(x)$, and proceed to invert it to express an exact solution for this function over all $x \geq 1$. Let the ordinary (initial, non-inverse) matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1}.$$

Then the matrix we must invert in this problem is lower triangular, with ones on its diagonals – and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T); \qquad U = (\delta_{i,j+1}), (I - U^T)^{-1} = ([j \leq i]_{\delta}).$$

Here, U is the $N \times N$ matrix whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$.

It is a useful fact that if we take successive differences of floor functions, we get non-zero behavior at divisors:

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

We use this property to shift the matrix $\hat{G}$, and then invert the result, to obtain a matrix involving the Dirichlet inverse of g:

$$\left[(I - U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Now we can express the inverse of the target matrix $(g_{x,j})$ in terms of these Dirichlet inverse functions as follows:

$$\begin{aligned}
(g_{x,j}) &= (I - U^T)^{-1}\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I - U^T) \\
(g_{x,j})^{-1} &= (I - U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I - U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k)\right)(I - U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} g^{-1}(k)\right).
\end{aligned}$$

Thus the summatory function H is exactly expressed by the inverse vector product of the form

$$\begin{aligned}
H(x) &= \sum_{k=1}^{x} g_{x,k}^{-1} \cdot S_{g,h}(k) \\
&= \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j)\right) \cdot S_{g,h}(k). \square
\end{aligned}$$

3.2 Proving the crucial property from the conjecture over the squarefree integers

Proposition 3.1 (The characteristic function of the primes)**.** *Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the additive function that counts the number of distinct prime factors of n. Then*

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu.$$

The summatory function of the LHS is $\widetilde{G}(x) = \pi(x) + 1$. The corresponding characteristic function for the prime powers is similarly given by $\chi_{\mathrm{PP}} = \Omega * \mu$.

Proof. The core is to prove that for all $n \geq 1$, $\chi_{\mathbb{P}}(n) = (\mu * \omega)(n)$ – our claim. We notice that the Mellin transform of $\pi(x)$ – the summatory function of $\chi_{\mathbb{P}}(n)$ – at $-s$ is given by

$$\begin{aligned}
s \cdot \int_1^\infty \frac{\pi(x)}{x^{s+1}} dx &= \sum_{n \geq 1} \frac{\nabla[\pi](n-1)}{n^s} \\
&= \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s} = P(s).
\end{aligned}$$

This is typical construction which more generally relates the Mellin transform $\mathcal{M}[S_f](-s)$ to the DGF of the sequence $f(n)$ as cited, for example, in [1, §11]. Now we consider the DGF of the right-hand-side function, $f(n) := (\mu * \omega)(n)$, as

$$D_f(s) = \frac{1}{\zeta(s)} \times \sum_{n\geq 1} \frac{\omega(n)}{n^s} = P(s).$$

Thus for any $\Re(s) > 1$, the DGFs of each side of the claimed equation coincide. So by uniqueness of Dirichlet series, we see that in fact the claim holds. To obtain the full result, we add to each side of this equation a term of $\varepsilon(n) \equiv (\mu * 1)(n)$, and then factor the resulting convolution identity. □

Proposition 3.2 (The sign of $g^{-1}(n)$)**.** *For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$.*

Proof. Let $D_f(s) := \sum_{n\geq 1} f(n)n^{-s}$ denote the Dirichlet generating function (DGF) of $f(n)$. Then we have that

$$D_{(\omega+1)^{-1}}(s) = \frac{D_\lambda(s)}{(P(s)+1)\zeta(2s)}.$$

Let $h^{-1}(n) := (\omega * \mu + \varepsilon)^{-1}(n) = [n^{-s}](P(s)+1)^{-1}$. Then we have that

$$\begin{aligned}(h^{-1} * 1)(n) &= -\sum_{p_1|n} h^{-1}\left(\frac{n}{p_1}\right) = \lambda(n) \times \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \cdots \sum_{p_{\Omega(n)}|\frac{n}{p_1p_2\cdots p_{\Omega(n)-1}}} 1 \\ &= \begin{cases} \lambda(n) \times (\Omega(n)-1)!, & n \geq 2; \\ \lambda(n), & n = 1.\end{cases}\end{aligned}$$

So by Möbius inversion

$$h^{-1}(n) - \lambda(n)\left[\sum_{\substack{d|n \\ d<n}} \lambda(d)\mu(d)(\Omega(n/d)-1)! + 1\right] = \lambda(n)\left[\sum_{\substack{d|n \\ d<n}} \mu^2(d)(\Omega(n/d)-1)! + 1\right].$$

Then we finally have that

$$(\omega+1)^{-1}(n) = \lambda(n) \times \sum_{d|n} \lambda(d)\left[\sum_{\substack{r|\frac{n}{d} \\ r<\frac{n}{d}}} \mu^2(r)(\Omega\left(\frac{n}{dr}\right) - 1)! + 1\right] \chi_{\mathrm{sq}}(d)\mu(\sqrt{d}),$$

where χ_{sq} is the characteristic function of the squares. In either case of $\lambda(n) = \pm 1$, there are positive constants $C_{1,n}, C_{2,n} > 0$ such that

$$\lambda(n)C_{1,n} \times \sum_{d^2|n} \lambda(d^2)\mu(d) \leq g^{-1}(n) \leq \lambda(n)C_{1,n} \times \sum_{d^2|n} \lambda(d^2)\mu(d),$$

where $\sum_{d^2|n} \lambda(d^2)\mu(d) = \sum_{d^2|n} \mu^2(n) > 0$. This proves the result. □

3.3 Other facts and listings of results we will need in our proofs

Theorem 3.3 (Mertens theorem)**.**

$$P_1(x) := \sum_{p\leq x} \frac{1}{p} = \log\log x + B + O\left(e^{-(\log x)^{\frac{1}{14}}}\right),$$

where $B \approx 0.2614972128476427837554$ is an absolute constant.

Corollary 3.4. *We have that for sufficiently large* $x \gg 1$

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right) = \frac{e^{-B}}{\log x}\left[1 - \frac{(\log x)^{1/14}}{B} + o\left((\log x)^{1/14}\right)\right].$$

Hence, for $1 < |z| < R < 2$ *we obtain that*

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right)^z = \frac{e^{-Bz}}{(\log x)^z}\left[1 - \frac{z}{B}(\log x)^{\frac{1}{14}} + o_z\left(z^2 \cdot (\log x)^{\frac{1}{14}}\right)\right].$$

Proof. By taking logarithms and using Mertens theorem above, we obtain that

$$\begin{aligned}\log \prod_{p \leq x}\left(1 - \frac{1}{p}\right) & = \sum_{p \leq x} \log\left(1 - \frac{1}{p}\right) \\ & \approx -\log\log x - B + O\left(e^{-(\log x)^{1/14}}\right).\end{aligned}$$

Hence, the first formula follows by expanding out an alternating series for the exponential function. The second formula follows for $z \notin \mathbb{Z}$ by an application of the generalized binomial series given by

$$\log \prod_{p \leq x}\left(1 - \frac{1}{p}\right)^z \approx \frac{e^{-Bz}}{(\log x)^z} \times \sum_{r \geq 0}\binom{z}{r}\frac{(-1)^r}{B^r}(\log x)^{\frac{r}{14}},$$

where for $1 < |z| < 2$, we obtain the next result stated above with $\binom{z}{1} = z$ and $\binom{z}{2} = z(z-1)/2$. ☐

Facts 3.5 (Exponential Integrals and Incomplete Gamma Functions)**.** The following two variants of the *exponential integral function* are defined by

$$\begin{aligned}\operatorname{Ei}(x) & := \int_{-x}^{\infty}\frac{e^{-t}}{t}dt, \\ E_1(z) & := \int_1^{\infty}\frac{e^{-tz}}{t}dt, \Re(z) \geq 0,\end{aligned}$$

where $\operatorname{Ei}(-kz) = -E_1(kz)$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $E_1(z)$:

$$1 - \frac{3}{4}z \leq E_1(z) - \gamma - \log z \leq 1 - \frac{3}{4}z + \frac{11}{36}z^2. \tag{2a}$$

A related function is the (upper) *incomplete gamma function* defined by

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1}e^{-t}dt, \Re(s) > 0.$$

We have the following properties of $\Gamma(s, x)$:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1}\frac{x^k}{k!}, s \in \mathbb{Z}^+, \tag{2b}$$

$$\Gamma(s+1, 1) = e^{-1}\left\lfloor\frac{s!}{e}\right\rfloor, s \in \mathbb{Z}^+, \tag{2c}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, |x| \to +\infty. \tag{2d}$$

4 Summing functions weighted by the Liouville lambda function, $\lambda(n) := (-1)^{\Omega(n)}$: *Borrowing a method of enumeration of summatory functions by Dirichlet series and Euler products from Montgomery and Vaughan, Chapter 7*

4.1 Discussion: The enumerative DGF result in Theorem 2.6 from Montgomery and Vaughan

In the reference we have defined $F(s,z)$ for $\Re(s)>1$ such that the Dirichlet series coefficients, $a_z(n)$, are defined by

$$\zeta(s)^z F(s,z) := \sum_{n\geq 1} \frac{a_z(n)}{n^s}, \Re(s) > 1.$$

For the function

$$F(s,z) := \prod_p \left(1-\frac{z}{p^s}\right)\left(1-\frac{1}{p^s}\right)^z,$$

we obtain in the notation above that $a_z(n) \equiv z^{\Omega(n)}$, and that the summatory function satisfies

$$A_z(x) := \sum_{n\leq x} z^{\Omega(n)} = \sum_{k\geq 0} \widehat{\pi}_k(x) z^k.$$

Hence, by the Cauchy integral formula, for $r<2$ we get that

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \oint_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz,$$

from which we obtain the stated formula in the theorem.

What this enumeratively-flavored result of Montgomery and Vaughan allows us to do is get a "good enough" lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Lioville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. For comparison, we have known, more classical bounds due to Erdös (or earlier) that state for

$$\pi_k(x) := \#\{n \leq x : \omega(n) = k\},$$

we have tightly that [?, ?]

$$\pi_k(x) = (1+o(1)) \cdot \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

We seek to approximate the right-hand-side of $G(z)$ by only taking the products of the primes $p \leq x$, e.g., $p \in \{2,3,5,\ldots,x\}$. We will require some fairly elementary estimates of products of primes, Mertens theorem on the rate of divergence of the sum of the reciprocals of the primes, and on some generating function techniques involving elementary symmetric functions. The statements in Section 3.3 provide the basis for proving most of the lemmas we require.

4.2 The key new results utilizing Theorem 2.6

Corollary 4.1. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p\leq x} p^{-s}, x \gg 1.$$

When $s := 1$, we have the known bound in Mertens theorem. For $s > 1$, we obtain that

$$P_s(x) \approx E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1).$$

It follows that

$$\gamma_0(s,x)+o(1)\leq P_s(x)\leq \gamma_1(s,x)+o(1),$$

where it suffices to take

$$\begin{aligned}\gamma_0(z,x) &= -s\log\left(\frac{\log x}{\log 2}\right)-\frac{3}{4}s(s-1)\log(x/2)-\frac{11}{36}s(s-1)^2\log^2(2)\\ \gamma_1(z,x) &= \quad s\log\left(\frac{\log x}{\log 2}\right)-\frac{3}{4}s(s-1)\log(x/2)+\frac{11}{36}s(s-1)^2\log^2(x).\end{aligned}$$

Proof. Let $s>1$ be real-valued. By Abel summation where our summatory function is given by $A(x)=\pi(x)\sim\frac{x}{\log x}$ and our function $f(t)=t^{-s}$ so that $f'(t)=-s\cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned}P_s(x) &= \frac{1}{x^s\cdot\log x}+s\cdot\int_2^x\frac{dt}{t^s\log t}\\ &= E_1((s-1)\log x)-E_1((s-1)\log 2)+o(1), |x|\to\infty.\end{aligned}$$

Now using the inequalities in Facts 3.5, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned}\frac{P_s(x)}{s} &\geq -\log\left(\frac{\log x}{\log 2}\right)-\frac{3}{4}(s-1)\log(x/2)-\frac{11}{36}(s-1)^2\log^2(2)\\ \frac{P_s(x)}{s} &\leq \quad \log\left(\frac{\log x}{\log 2}\right)-\frac{3}{4}(s-1)\log(x/2)+\frac{11}{36}(s-1)^2\log^2(x).\end{aligned}$$

This completes the proof of the bounds cited above in the statement of this lemma. □

Proof of Theorem 2.7. We have that for all integers $0\leq k\leq m$

$$[z^k]\prod_{1\leq i\leq m}(1-f(i)z)^{-1}=[z^k]\exp\left(\sum_{j\geq 1}\left(\sum_{i=1}^m f(i)^j\right)\frac{z^j}{j}\right).$$

In our case we have that $f(i)$ denotes the i^{th} prime. Hence, summing over all $p\leq x$ in place of $0\leq k\leq m$ in the previous formula applied in tandem with Corollary 4.1, we obtain that the logarithm of the generating function series we are after corresponds to

$$\begin{aligned}\log\left[\prod_{p\leq x}\left(1-\frac{z}{p}\right)^{-1}\right] &= (B+\log\log x)z+\sum_{j\geq 2}\left[a(x)+b(x)(j-1)+c(x)(j-1)^2\right]z^j\\ &= (B+\log\log x)z-a(x)\left(1+\frac{1}{z-1}+z\right)+b(x)\left(1+\frac{2}{z-1}+\frac{1}{(z-1)^2}\right)\\ &\qquad -c(x)\left(1+\frac{4}{z-1}+\frac{5}{(z-1)^2}+\frac{2}{(z-1)^3}\right).\end{aligned}$$

In the previous equations, the upper and lower bounds formed by the functions (a,b,c) are given by

$$\begin{aligned}(a_\ell,b_\ell,c_\ell) &:= \left(-\log\left(\frac{\log x}{\log 2}\right),\frac{3}{4}\log\left(\frac{x}{2}\right),-\frac{11}{36}\log^2 2\right)\\ (a_u,b_u,c_u) &:= \left(\log\left(\frac{\log x}{\log 2}\right),-\frac{3}{4}\log\left(\frac{x}{2}\right),\frac{11}{36}\log^2 x\right).\end{aligned}$$

Now we make a prudent decision to set the uniform bound parameter to a middle ground value of $1<R<2$ as $R:=\frac{3}{2}$ so that

$$z\equiv z(k,x)=\frac{k-1}{\log\log x}\in[0,R),$$

for $x \gg 1$ very large. Thus $(z-1)^{-m} \in [(-1)^m, 2^m]$ for integers $m \geq 1$, and we can then form the upper and lower bounds from above. What we get out of these formulas is stated as in the theorem bounds. $\square$

Corollary 4.2 (Bounds on $G(z)$ from MV)**.** *We have that for the function $G(z) := F(1,z)/\Gamma(z+1)$ from Montgomery and Vaughan, there is a constant A_0 and functions of x only, $B_0(x), C_0(x)$, so that*

$$A_0 \cdot B_0(x) \cdot C_0(x)^z \left(1 - \frac{z}{B}(\log x)^{\frac{1}{14}}\right) \leq G(z).$$

It suffices to take

$$\begin{aligned} A_0 &= \frac{\exp\left(\frac{55}{4}\log^2 2\right)}{\log^3(2) \cdot \Gamma(5/2)} \approx 1670.84511225 \\ B_0(x) &= \log^3 x \\ C_0(x) &= \frac{\log x}{\log 2}. \end{aligned}$$

Proof. This result is a consequence of applying both Corollary 3.4 and Theorem 2.7 to the definition of $G(z)$. In particular, we obtain bounds of the following form from the theorem:

$$\frac{A_0 \cdot B_0(x) \cdot C_0(x)^z}{\Gamma(z+1)} \leq \frac{G(z)}{\prod_p \left(1 - \frac{1}{p}\right)^z}.$$

Since $z \equiv z(k,x) = \frac{k-1}{\log\log x}$ and $k \in [1, R\log\log x]$, we obtain that for small k and $x \gg 1$ large $\Gamma(z+1) \approx 1$, and for k towards the upper bound of its interval that $\Gamma(z+1) \approx \Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$ (recall that we set $R := 3/2$ in the preceeding proof of Theorem 2.7). Thus when we expand out the formula given by the corollary in conjunction with these bounds on the gamma function, we obtain the claimed results. $\square$

Lemma 4.3. *Suppose that $f_k(n)$ is a sequence of arithmetic functions such that $f_k(n) > 0$ for all $n \geq 1$, $f_0(n) = \delta_{n,1}$, and $f_{\Omega(n)}(n) \gtrsim \widehat{\tau}_\ell(n)$ as $n \to \infty$ where $\widehat{\tau}_\ell(t)$ is a continuously differentiable function of t for all large enough $t \gg 1$. We define the λ-sign-scaled summatory function of f as follows:*

$$F_\lambda(x) := \sum_{\substack{n \leq x \\ \Omega(n) \leq x}} \lambda(n) \cdot f_{\Omega(n)}(n).$$

Let

$$A_\Omega^{(\ell)}(t) := \sum_{k=1}^{\left\lfloor \frac{3}{2}\log\log t \right\rfloor} (-1)^k \widehat{\pi}_k(t).$$

Then we have that

$$F_\lambda(\log\log x) \gtrsim A_\Omega^{(\ell)}(x)\widehat{\tau}_\ell(\log\log x) - \int_1^{\log\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_\ell'(t)dt.$$

Proof. The formula for $F_\lambda(x)$ is valid by Abel summation provided that

$$\left| \frac{\sum\limits_{\frac{3}{2}\log\log t < k \leq \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_\Omega^{(\ell)}(t)} \right| = o(1),$$

e.g., the asymptotically dominant terms indicating the parity of $\lambda(n)$ are encompassed by the terms summed by $A_{\Omega}^{(\ell)}(t)$ for sufficiently large t as $t \to \infty$. Using the arguments in Montgomery and Vaughan [?, §7; Thm. 7.21], we can see that uniformly in x

$$\begin{aligned} \#\left\{n \leq x : \frac{\Omega(n) - \frac{3}{2}\log\log n}{\sqrt{\log\log n}} > 0\right\} &\sim x\left(1 - \Phi(\sqrt{\log\log x})\right) \\ &= x \cdot \Phi(-\sqrt{\log\log x}) \xrightarrow{x\to\infty} 0, \end{aligned} \tag{3}$$

where $\Phi(z)$ is the CDF of a standard normal random variable. Thus we have captured the asymptotically dominant main order terms in our formula as $x \to \infty$. $\square$

5 Precisely enumerating and bounding the Dirichlet inverse functions, $g^{-1}(n) := (\omega+1)^{-1}(n)$

5.1 Developing an improved conjecture: Proving precise bounds on the inverse functions $g^{-1}(n)$ for all n

Conjecture 2.4 is not the most accurate fomulation of the limiting behavior of the Dirichlet inverse functions $g^{-1}(n)$ that we can see and prove. We need to come up with better bounds to plug back into the asymptotic analysis we obtain in the next sections. It turns out that these results are related to symmetric functions of the exponents in the prime factorizations of each $n \leq x$. The idea is that by having information about $g^{-1}(n)$ in terms of its prime factorization exponents for $n \leq x$, we should be able to extrapolate what we need which is information about the average behavior of the summatory functions, $G^{-1}(x)$, from the proofs above. Moreover, we notice the following observation that is suggestive of the semi-periodicity at play with the distinct values of $g^{-1}(n)$ distributed over $n \geq 2$.

Heuristic 5.1 (Symmetry in $g^{-1}(n)$ in the exponents in the prime factorization of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly two, three, and so on prime factors. There does not appear to be an easy, nor subtle direct recursion between the distinct g^{-1} values, except through auxiliary function sequences. We will settle for an asymptotically accurate main term approximation to $g^{-1}(n)$ for large n as $n \to \infty$ in the average case.

With all of this in mind, we define the following sequence for integers $n \geq 1, k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{4}$$

We will illustrate by example the first few cases of these functions for small k after we prove the next lemma. The sequence of important semi-diagonals of these functions begins as [9, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \ -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

Lemma 5.2 (An exact formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu(n/d) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$\begin{aligned} g^{-1}(n) &= -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) f^{-1}(n/d) \qquad \implies \\ (g^{-1} * 1)(n) &= -(\omega * g^{-1})(n). \end{aligned}$$

Now by repeatedly expanding the right-hand-side, and removing corner cases in the nested sums since $\omega(1) = 0$ by convention, we find that

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n).$$

The statement follows by Möbius inversion applied to each side of the last equation. ∎

Notice that this approach, while it definitely has its complications due to the necessary step of Möbius inversion, is somewhat simpler that trying to form the Dirichlet inverse of the sum of $\omega + 1$ directly, though this is also a possible approach.

Example 5.3 (Special cases of the functions $C_k(n)$ for small k)**.** We cite the following special cases which should be easy enough to see on paper:

$$\begin{aligned}
C_0(n) &= \delta_{n,1} \\
C_1(n) &= \omega(n) \\
C_2(n) &= \sigma_0(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} - \gcd\left(\Omega(n), \omega(n)\right).
\end{aligned}$$

We also can see a recurrence relation between successive $C_k(n)$ values over k of the form

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} C_{k-1}\left(d \cdot p^i\right). \tag{5}$$

Thus we can work out further cases of the $C_k(n)$ for a while until we are able to understand the general trends of its asymptotic behaviors. In particular, we can compute the main term of $C_3(n)$ as follows where we use the notation that p, q are prime indices:

$$\begin{aligned}
C_3(n) &\sim \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \sum_{q|dp^i} \frac{\nu_q(dp^i)}{\nu_q(dp^i)+1} \sigma_0(d)(i+1) \\
&= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \left[\sum_{q|d} \frac{\nu_q(d)}{\nu_q(d)+1} \sigma_0(d)(i+1) + \sum_{j=1}^{i} \frac{j}{(j+1)} \sigma_0(d)(i+1)\right] \\
&= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{q|d} \sigma_0(d) \left[\frac{\nu_p(n)(\nu_p(n)+3)}{2} \frac{\nu_q(d)}{\nu_q(d)+1} + \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2)\left(4\nu_p(n)+9-6H_{\nu_p(n)+2}^{(1)}\right)\right].
\end{aligned}$$

We will break the two key component sums into separate calculations. First, we compute that[1]

$$\begin{aligned}
C_{3,1}(n) &= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \frac{\nu_p(n)(\nu_p(n)+3)}{2} \times \sum_{q|d} \frac{\nu_q(d)}{\nu_q(d)+1} \sigma_0(d) \\
&= \sum_{\substack{p,q|n \\ p \neq q}} \sum_{d|\frac{n}{p^{\nu_p(n)} q^{\nu_q(n)}}} \frac{\nu_p(n)(\nu_p(n)+3)}{2} \times \sum_{i=1}^{\nu_q(n)} \frac{\nu_q(dq^i)}{\nu_q(dq^i)+1} \sigma_0(dq^i) \\
&= \sum_{\substack{p,q|n \\ p \neq q}} \sum_{d|\frac{n}{p^{\nu_p(n)} q^{\nu_q(n)}}} \frac{\nu_p(n)(\nu_p(n)+3)\nu_q(n)(\nu_q(n)+3)}{4} \sigma_0(d) \\
&= (\sigma_0 * 1)(n) \times \sum_{\substack{p,q|n \\ p \neq q}} \frac{\nu_p(n)(\nu_p(n)+3)\nu_q(n)(\nu_q(n)+3)}{(\nu_p(n)+1)(\nu_p(n)+2)(\nu_q(n)+1)(\nu_q(n)+2)}.
\end{aligned}$$

Next, we have that

$$C_{3,2}(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{q|d} \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2)\left(4\nu_p(n)+9-6H_{\nu_p(n)+2}^{(1)}\right) \sigma_0(d)$$

[1]Here, the arithmetic function $\sigma_0 * 1$ is multiplicative. Its value at prime powers can be computed as

$$(\sigma_0 * 1)(p^\alpha) = \sum_{i=0}^{\alpha} (i+1) = \frac{(\alpha+1)(\alpha+2)}{2},$$

where $\sigma_0(p^\beta) = \beta + 1$.

$$
\begin{aligned}
&= \sum_{\substack{p,q|n \\ p\neq q}} \sum_{d|\frac{n}{p^{\nu_p(n)}q^{\nu_q(n)}}} \sum_{i=1}^{\nu_q(n)} \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2)\left(4\nu_p(n)+9-6H_{\nu_p(n)+2}^{(1)}\right)\sigma_0(d)(i+1) \\
&= (\sigma_0 * 1)(n) \times \sum_{\substack{p,q|n \\ p\neq q}} \frac{1}{6}\frac{\nu_q(n)(\nu_q(n)+3)}{(\nu_q(n)+1)(\nu_q(n)+2)}\left(4\nu_p(n)+9-6H_{\nu_p(n)+2}^{(1)}\right).
\end{aligned}
$$

Now to roughly bound the error term, e.g., the GCD of prime omega functions from the exact formula for $C_3(n)$, we observe that the divisor function has average order of the form:

$$
d(n) \sim \log n + (2\gamma - 1) + O\left(\frac{1}{\sqrt{n}}\right).
$$

Then using that $\omega(n), \Omega(n) \sim \log\log n$ (except in rare cases when n is primorial, a power of 2, etc.), as discussed in the next remarks, we bound the error as

$$
\begin{aligned}
C_{3,3}(n) &= -\sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \gcd\left(\Omega(d)+i, \omega(d)+1\right) \\
&= \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} O\left(\sigma_0(n)\cdot\log\log n\right) \\
&= O\left(\pi(n)\cdot\log n\cdot\log\log n\right) \\
&= O\left(n\cdot\log\log n\right).
\end{aligned}
$$

In total, we obtain that

$$
\begin{aligned}
C_3(n) = (\sigma_0 * 1)(n) &\times \sum_{\substack{p,q|n \\ p\neq q}} \frac{1}{6}\frac{\nu_q(n)(\nu_q(n)+3)}{(\nu_q(n)+1)(\nu_q(n)+2)}\left(4\nu_p(n)+9-6H_{\nu_p(n)+2}^{(1)}\right)) \\
&+ \sigma_0(n) \times \sum_{\substack{p,q|n \\ q\neq p}} \frac{2^{\nu_q(n)}\nu_p(n)(\nu_p(n)+3)}{4(\nu_p(n)+1)(\nu_q(n)+1)} \\
&+ O\left(n\cdot\log\log n\right).
\end{aligned} \tag{6}
$$

For the next cases, we would use similar techniques. The key is to compute enough small cases that we can see the dominant asymptotic terms in these expansions. We will expand more on this below.

Remark 5.4 (Recursive growth of the functions $C_k(n)$ in the average case)**.** We note that the average order of $\Omega(n) \sim \log\log n$, so that for large $x \gg 1$ tending to infinity, we can expect (on average) that for $p|n$, $1 \leq \nu_p(n)$ (for large $p|x, p \sim \frac{x}{\log x}$) and $\nu_p(n) \approx \log\log n$. However, if x is primorial, we can have $\Omega(x) \sim \frac{\log x}{\log\log x}$. There is, however, a duality with the size of $\Omega(x)$ and the rate of growth of the $\nu_p(x)$ exponents. That is to say that on average, even though $\nu_p(x) \sim \log\log n$ for most $p|x$, if $\Omega(x) = m \approx O(1)$ is small, then

$$
\nu_p(x) \approx \log_{\sqrt[m]{\frac{x}{\log x}}}(x) = \frac{m\log x}{\log\left(\frac{x}{\log x}\right)}.
$$

Since we will be essentially averaging the inverse functions, $g^{-1}(n)$, via their summatory functions over the range $n \leq x$ for x large, we tend not to worry about bounding anything but by the average case, which wins when we sum (i.e., average) and tend to infinity. Given these observations, we can use the function $C_3(n)$ we just painstakingly computed exactly as an asymptotic benchmark to build further approximations. In particular, the dominant order terms in $C_3(n)$ are given by

$$C_3(n) \sim \frac{(\sigma_0 * 1)(n)n^2}{\log^2 n} - \frac{(\sigma_0 * 1)(n)n^2}{\log n} + O\left(n \cdot \log\log n\right).$$

We will leave the terms involving the divisor function $\sigma_0(n)$ and convolutions involving it unevaluated because of how much their growth can fluctuate depending on prime factorizations for now.

Summary 5.5 (Asymptotics of the $C_k(n)$)**.** We have the following asymptotic relations relations for the growth of small cases of the functions $C_k(n)$:

$$\begin{aligned} C_1(n) &\sim \log\log n \\ C_2(n) &\sim \frac{\sigma_0(n)n}{\log n} + O(\log\log n) \\ C_3(n) &\sim \frac{(\sigma_0 * 1)(n)n^2}{\log^2 n} - \frac{(\sigma_0 * 1)(n)n^2}{\log n} + O\left(n \cdot \log\log n\right). \end{aligned}$$

Theorem 2.5 proved in the next section makes precise what these formulas suggest about the growth rates of $C_k(n)$.

Proof of Theorem 2.5. We showed how to compute the formulas for the base cases in the preceeding examples discussed above. We can also see that $C_3(n)$ satsfies the target formula specification. Let's proceed by using induction with the recurrence formula from (5) relating $C_k(n)$ to $C_{k-1}(n)$ for all $k \geq 1$. The strategy is to precisely evaluate the sums recursively, and drop the messy troublesome lower order terms that contribute to the nuances of the full formulas. What results is precise for sufficiently large $n \gg 1$ as $n \to \infty$. We will compute the main term formula first, then complete the proof by bounding the easier error term calculations.

Suppose that $k \geq 4$. By the recurrence formula for $C_k(n)$, we have that

$$C_k(n) \sum_{p|n} \sum_{d|np^{-\nu_p(n)}} \sum_{i=1}^{\nu_p(n)} -\frac{(dp^i)^{k-1}}{(\log(dp^i))^{k-1}} \binom{i+k-1}{k-1} (\sigma_0 * \mathbb{1}_{*_{k-2}})(d).$$

Now to handle the inner sum, we bound by setting $\alpha \equiv \nu_p(n)$ and invoking *Mathematica* in the form of

$$\begin{aligned} \mathrm{IC}_k(n) &= \sum_{i=1}^{\alpha} -\frac{(dp^i)^{k-1}}{(\log(dp^i))^{k-1}} \binom{i+k-1}{k-1} \\ &\approx \int -\frac{(dp^\alpha)^{k-1}}{(\log(dp^\alpha))^{k-1}} \binom{\alpha+k-1}{k-1} \\ &\sim \frac{1}{(k-1)!\log^k p} \left(\mathrm{Ei}((k-2)\log(dp^\alpha)) \left[\log^{k-1}(d) - (k-1)!\log^{k-1}(p)\right]\right) \\ &\qquad - \frac{1}{(k-2)(k-1)!\log^k p} \left(\log^{k-2}(d) + \alpha^{k-2}\log^{k-2}(p)\right). \end{aligned}$$

We now simplify somewhat again by setting

$$p \mapsto \left(\frac{n}{e}\right)^{\frac{1}{\log\log n}}, \alpha \mapsto \log\log n, \log p \mapsto \frac{\log n}{\log\log n}.$$

Also, since $p \gg_n d$, we obtain the dominant asymptotic growth terms of

$$\mathrm{IC}_k(n) \sim \frac{\alpha^{k-2}}{(k-2)(k-1)!\log^2 p}$$

$$\approx \frac{(\log\log n)^k}{(k-2)(k-1)!\log^2 n}.$$

Now, as we did in the previous example work, we handle the sums by pulling out a factor of the inner divisor sum depending only on n (and k):

$$\begin{aligned} C_k(n) &= \sum_{p|n} (\sigma_0 * \mathbb{1}_{*_{k-1}})(n) \binom{p^{\nu_p(n)}+k}{k}^{-1} \times \mathrm{IC}_k(n) \\ &= (\sigma_0 * \mathbb{1}_{*_{k-1}})(n) \binom{p^{\nu_p(n)}+k}{k}^{-1} \cdot \pi(n) \times \mathrm{IC}_k(n) \end{aligned}$$

Combining with the remaining terms we get by induction, we have proved our target bound holds for $C_k(n)$.

To bound the error terms, again suppose inductively that $k \geq 4$. We compute the big-O bounds as follows letting $\alpha \equiv \nu_p(n)$:

$$\begin{aligned} \mathrm{ET}_k(n) &= \sum_{i=1}^{\nu_p(n)} n^{k-2} \cdot \frac{(\log\log n)^{k-2}}{(\log n)^{k-2}} \\ &\approx \int (dp^\alpha)^{k-2} \log\log(dp^\alpha) d\alpha \\ &= -\frac{\mathrm{Ei}((k-2)\log(dp^\alpha))}{(k-2)\log p} + \frac{d^{k-2}p^{(k-2)\alpha}}{(k-2)\log p} \log(dp^\alpha) \\ &\sim \frac{d^{k-2}p^{(k-2)\alpha}}{(k-2)\log p} \log(dp^\alpha). \end{aligned}$$

In the last expansion, we have dropped the exponential integral terms since they provide at most polynomial powers of the logarithm of their inputs.

To evaluate the outer divisor sum from the recurrence relation for $C_k(n)$, we will require the following bound providing an average order on the *generalied sum-of-divisors functions*, $\sigma_\alpha(n) := \sum_{d|n} d^\alpha$. In particular, we have that for integers $\alpha \geq 2$ [7, §27.11]:

$$\sigma_\alpha(n) \sim \frac{\zeta(\alpha+1)}{\alpha+1} x^\alpha + O(x^{\alpha-1}).$$

Approximating the number of terms in the prime divisor sum by $\pi(x) \sim \frac{x}{\log x}$, we thus obtain

$$\mathrm{ET}_k(n) \approx \frac{(\log\log n)^{k-1} e^{k-2}}{(k-1)(k-2)} x^{(k-2)\left(1-\frac{1}{\log\log x}\right)+1+\log\log x} \zeta(k-1).$$

So up to what is effectively constant in k, and dropping lower order terms for a slightly suboptimal, but still sufficient for our purposes, error bound formula, we have completed the proof by induction. □

Corollary 5.6 (Computing the inverse functions). *In contrast to the complicated formulation given by Lemma 5.2, we have that*

$$g^{-1}(n) \sim \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

This is to say that for all $n \geq 2$

$$\left| 1 - \frac{\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)}{g^{-1}(n)} \right| = o\left(\sum_{d|n} C_{\Omega(d)}(d) \right).$$

Moreover, we can bound the error terms as

$$\left| \frac{\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)}{g^{-1}(n)} \right| = O\left(\frac{(\log\log n)^2}{\log n} \cdot \frac{\Gamma(\log\log n)}{n^{\log\log n} \cdot (\log n)^{\log\log n}} \right) \xrightarrow{n\to\infty} 0.$$

Proof. Using Lemma 5.2, it suffices to show that the squarefree divisors $d|n$ such that $\operatorname{sgn}(\mu(d)\lambda(n/d)) = -1$ have an order of magnitude less adbundancy than the corresponding cases of positive sign on the terms in the divisor sum from the lemma. Let n have m_1 prime factors p_1 such that $v_{p_1}(n) = 1$, m_2 such that $v_{p_2}(n) = 2$, and the remaining $m_3 := \Omega(n) - m_1 - 2m_2$ prime factors of higher-order exponentation. We have a few cases to consider after re-writing the sum from the lemma in the following form:

$$g^{-1}(n) = \lambda(n)C_{\Omega(n)}(n) + \sum_{i=1}^{\omega(n)} \left\{ \sum_{\substack{d|n \\ \omega(d)=\Omega(d)=i \\ \#\{p|d:\nu_p(d)=1\}=k_1 \\ \#\{p|d:\nu_p(d)=2\}=k_2 \\ \#\{p|d:\nu_p(d)\geq 3\}=k_3}} \mu(d)\lambda(n/d)C_{\Omega(n/d)}(n/d) \right\}.$$

We obtain the following cases of the squarefree divisors contributing to the signage on the terms in the above sum:

- The sign of $\mu(d)$ is $(-1)^i = (-1)^{k_1+k_2+k_3}$;
- If $m_3 < \#\{p|n : \nu_p(n) \geq 3\}$, then $\lambda(n/d) = 1$ (since $\mu(n/d) = 0$);
- Given (k_1, k_2, k_3) as above, since $\lambda(n) = (-1)^{\Omega(n)}$, we have that $\mu(d) \cdot \lambda(n/d) = (-1)^{i-k_1-k_2}\lambda(n)$.

Thus we define the following sums, parameterized in the $(m_1, m_2, m_3; n)$, which corresponds to a change in expected parity transitioning from the Möbius inversion sum from Lemma 5.2 to the sum approximating $g^{-1}(n)$ defined at the start of this result:

$$\begin{aligned} \widetilde{S}_{\text{odd}}(m_1, m_2, m_3; n) &:= \sum_{i=1}^{\omega(n)/2} \sum_{k_1=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{i}{2} \rfloor - k_1} \left[\binom{m_1}{2k_1+1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1}\binom{2m_2}{2k_2}\right] [i - k_1 - k_2 = k_3 \equiv m_3]_\delta \\ \widetilde{S}_{\text{even}}(m_1, m_2, m_3; n) &:= \sum_{i=1}^{\omega(n)/2} \sum_{k_1=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{i}{2} \rfloor - k_1} \left[\binom{m_1}{2k_1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1+1}\binom{2m_2}{2k_2}\right] [i - k_1 - k_2 = k_3 \equiv m_3]_\delta. \end{aligned}$$

Part I (Lower bounds on the inner sums of the count functions). We claim that

$$\begin{aligned} \widetilde{S}_{\text{odd}}(m_1, m_2, m_3; n) &\gg \binom{m_1}{i+1} + \binom{m_1}{\frac{i}{2}}\binom{2m_2-1}{\frac{i}{2}+1} \\ \widetilde{S}_{\text{even}}(m_1, m_2, m_3; n) &\gg \binom{m_1}{i+1} + \binom{m_1}{\frac{i}{2}-1}\binom{2m_2}{\frac{i}{2}+1}. \end{aligned} \tag{7}$$

To prove (7) we have to provide a straightforward bound that represents the maximums of the terms in m_1, m_2. In particular, observe that for

$$\begin{aligned} \widetilde{S}_{\text{odd}}(m_1, m_2; u) &= \sum_{k_1=0}^{u} \sum_{k_2=0}^{u-k_1} \left[\binom{m_1}{2k_1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1+1}\binom{2m_2}{2k_2}\right] \\ \widetilde{S}_{\text{even}}(m_1, m_2; u) &= \sum_{k_1=0}^{u} \sum_{k_2=0}^{u-k_1} \left[\binom{m_1}{2k_1+1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1}\binom{2m_2}{2k_2}\right], \end{aligned}$$

we have that

$$\widetilde{S}_{\text{odd}}(m_1, m_2; u) \gtrsim \binom{m_1}{2u+1} + \max_{1 \leq k_1 \leq u} \binom{m_1}{2k_1+1}\binom{2m_2}{2u+1-2k_1}$$

$$\begin{aligned}
&= \binom{m_1}{2u+1} + \binom{m_1}{2k_1+1}\binom{2m_2}{2u+1-2k_1}\Bigg|_{k_1=\frac{u}{2}} \\
&= \binom{m_1}{2u+1} + \binom{m_1}{u+1}\binom{2m_2}{u+1} \\
\widetilde{S}_{\mathrm{even}}(m_1,m_2;u) &\gtrsim \binom{m_1}{2u+1} + \max_{1\le k_1\le u}\binom{m_1}{2k_1}\binom{2m_2}{2u+1-2k_1} \\
&= \binom{m_1}{2u+1} + \binom{m_1}{u-1}\binom{2m_2}{u+1}.
\end{aligned}$$

The lower bounds in (7) then follow by setting $u \equiv \left\lfloor \frac{i}{2} \right\rfloor$.

Part II (Bounding m_1, m_2, m_3 and effective (i, k_1, k_2) contributing to the count). We thus have to determine the asymptotic growth rate of $\widetilde{S}_{\mathrm{odd}}(m_1,m_2,m_3;n)+\widetilde{S}_{\mathrm{even}}(m_1,m_2,m_3;n)$, and show that it is of comparatively small order. First, we bound the count of non-zero m_3 for $n \le x$ from below. For the cases where we expect differences in signage, it's the last Iverson convention term that kills the order of growth, e.g., we expect differences when the parameter m_3 is larger than the usual configuration. We know that

$$\pi_k(x) \sim \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}.$$

Using the formula for $\pi_k(x)$, we can count the average orders of m_1, m_2 as

$$\begin{aligned}
N_{m_1}(x) &\approx \frac{1}{x}\#\{n \le x : \omega(n) = 1\} \sim \frac{\log\log x}{\log x} \\
N_{m_2}(x) &\approx \frac{1}{x}\#\{n \le x : \omega(n) = 2\} \sim \frac{(\log\log x)^2}{\log x}.
\end{aligned}$$

Additionally, in Corollary 4.2 on page 16 we will prove a lower bound on $\widehat{\pi}_k(x)$. We use this result immediately below without proof.

When we have parameters with respect to some $n \ge 1$ such that $m_3 > 0$, it must be the case that

$$\Omega(n) - \omega(n) > \begin{cases} 0, & \text{if } \omega(n) \ge 2; \\ 1, & \text{if } \omega(n) = 1. \end{cases}$$

To count the number of cases $n \le x$ where this happens, we form the sums

$$\begin{aligned}
N_{m_3}(x) &\gg \pi_1(x) \times \sum_{k=3}^{\frac{3}{2}\log\log x} \widehat{\pi}_k(x) + \sum_{k=2}^{\frac{3}{2}\log\log x}\sum_{j=k+1}^{\frac{3}{2}\log\log x} \pi_k(x)\widehat{\pi}_j(x) \\
&\gtrsim \frac{Ae}{B}\frac{x}{\log^{\frac{13}{14}}(x)} + \sum_{k=2}^{\log\log x} \pi_k(x)\left[\frac{Ae}{B}\log^{\frac{1}{14}}(x)\right] \\
&\gtrsim \frac{Ae}{B}\frac{x}{\log^{\frac{13}{14}}(x)} + \frac{Ae\sqrt{2}}{2\sqrt{\pi}B}\frac{x}{\log^{\frac{13}{14}}(x)\sqrt{\log\log x}}.
\end{aligned}$$

Now in practice, we are not summing up $n \le x$, but rather $n \le \log\log x$. So the above function evaluates to

$$N_{m_3}(\log\log x) \gg \frac{\log\log x}{(\log\log\log x)^{13/14}} \gg \frac{\log\log x}{\log\log\log x}.$$

Next, we go about solving the subproblem of finding when $i - k_1 - k_2 = m_3$. First, we find a lower solution index on i using asymptotics for the *Lambert W-function*, $W_0(x) = \log x - \log\log x + o(1)$:

$$\frac{i}{2} = \frac{\log\log x}{\log\log\log x} \iff \log\log x \gtrsim \frac{i}{2}\left(\log i + \log\log i\right)$$

$$\iff \frac{i}{2} \sim \frac{\log\log x}{\log\log\log x}.$$

Now since $2 \leq k_1 + k_2 \leq i/2$, when x is large, we actually obtain a number of soultions on the order of

$$\frac{\log\log x}{2} - \frac{\log\log x}{\log\log\log x} = \frac{\log\log x}{2}(1+o(1)).$$

Part III (Putting it all together). Using the binomial coefficient inequality

$$\binom{n}{k} \geq \frac{n^k}{k^k},$$

we can work out carefully on paper using (7) that

$$\widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) \gtrsim \frac{\log\log x}{2}\left(\frac{\log\log\log x}{2\log x}\right)^{\frac{2\log\log x}{\log\log\log x}+1}\left[1 + \frac{(\log\log\log x)^2}{\log^2 x}(4\log\log x \cdot \log\log\log x)^{\frac{\log\log x}{2\log\log\log x}+1}\right]$$
$$\widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) \gtrsim \frac{\log\log x}{2}\left(\frac{\log\log\log x}{2\log x}\right)^{\frac{2\log\log x}{\log\log\log x}+1}\left[1 + \left(\frac{\log x}{2\log\log\log x}\right)(8\log\log x)^{\frac{\log\log x}{\log\log\log x}+1}\right].$$

Part IV (Obtaining the rate at which the ratio goes to zero). Let

$$S_{\mathrm{diff}}(m_1, m_2, m_3; n) := S_{\mathrm{even}}(m_1, m_2, m_3; n) + S_{\mathrm{odd}}(m_1, m_2, m_3; n).$$

Then we have that

$$\begin{aligned}
\left|\frac{\lambda(x) \times \sum_{d|x} C_{\Omega(d)}(d)}{g^{-1}(x)}\right| &= \frac{\sum\limits_{\substack{d|x \\ d \leq \log\log x}} C_{\Omega(d)}(d)}{(\log\log d)}|g^{-1}(x)| = \\
&= O\left(\frac{S_{\mathrm{diff}}\left(\frac{\log\log x}{\log x}, \frac{(\log\log x)^2}{\log x}, \frac{(\log\log x)^2}{\log\log\log x}; x\right)}{C_{\Omega(x)}(x)}\right) \\
&= O\left(\frac{\log x \cdot \log\log x}{\log\log\log x \cdot C_{\Omega(x)}(x)}\left(\frac{\sqrt{2\log\log x} \cdot \log\log\log x}{\log x}\right)^{\frac{2\log\log x}{\log\log\log x}}\right) \\
&= O\left(\frac{(\log\log x)^2 \cdot \log x}{\log\log\log x \cdot C_{\Omega(x)}(x)}\right).
\end{aligned}$$

We borrow from Corollary 5.8 proved below in this section to get a lower bound on $C_{\Omega(x)}(x)$. This result implies the stated bound, which tends to zero as $x \to \infty$. Thus the divisor sum in the corollary statement accuarately approximates the main term and sign of $g^{-1}(n)$ as $n \to \infty$. $\square$

Corollary 5.7. *We have that for sufficiently large x, as $x \to \infty$ that*

$$G^{-1}(x) \gtrsim \widehat{L}_0(\log\log x) \times \sum_{n \leq \log\log x} \lambda(n) \cdot C_{\Omega(n)}(n),$$

where the function

$$\widehat{L}_0(\log\log x) := (-1)^{\left\lfloor \frac{3}{2}\log\log\log\log x \right\rfloor + 1}\left\{\sqrt{\frac{3}{\pi}}\frac{A(2e+3)}{4B\log^{\frac{3}{2}}(2)}\right\} \cdot \frac{(\log\log\log x)^{\frac{43}{14}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}}{\sqrt{\log\log\log\log x}},$$

with the exponent $\frac{43}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3} \approx 3.87011$.

Proof. Using Corollary 5.6, we have that

$$\begin{aligned} G^{-1}(x) &\approx \sum_{n\leq x} \lambda(n)\cdot (g^{-1}*1)(n) \\ &= \sum_{d\leq \log\log x} C_{\Omega(d)}(d) \times \sum_{n=1}^{\lfloor \frac{x}{d}\rfloor} \lambda(dn). \end{aligned}$$

Now we see that by complete additivity (multiplicativity) of Ω (λ) that

$$\sum_{n=1}^{\lfloor \frac{x}{d}\rfloor} \lambda(dn) = \sum_{n=1}^{\lfloor \frac{x}{d}\rfloor} \lambda(d)\lambda(n) = \lambda(d) \sum_{n\leq \lfloor \frac{x}{d}\rfloor} \lambda(n).$$

Borrowing a result from the next sections (proved in Section 4), we can establish that

$$\begin{aligned} \sum_{n\leq x} \lambda(n) \gg & \sum_{n\leq \frac{3}{2}\log\log x} (-1)^k \cdot \widehat{\pi}_k(x) \\ & \gtrsim (-1)^{\lfloor \frac{3}{2}\log\log x\rfloor +1} \left(\sqrt{\frac{3}{\pi}} \frac{A(2e+3)}{4B\log^{\frac{3}{2}}(2)}\right) \cdot \frac{(\log x)^{\frac{43}{14}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}}{\sqrt{\log\log x}} \\ & =: \widehat{L}_0(x). \end{aligned}$$

Then since for large enough x and $d \leq x$,

$$\log(x/d) \sim \log x, \log\log(x/d) \sim \log\log x,$$

we can obtain the stated result, e.g., so that $\widehat{L}_0(x) \sim \widehat{L}_0(x/d)$ for large $x \to \infty$. $\square$

The previous corollary is employed to prove the exact lower bounds on $G^{-1}(x)$ given in Theorem 6.1 in the next section. The parity of $\lfloor 2\log\log\log\log x\rfloor$ determines subsequences of real $x \gg 1$ along which we break these bounds into cases. The next result provides complete asymptotic upper and lower bound information on the functions $C_k(n)$ when $k \equiv \Omega(n)$.

Corollary 5.8 (Asymptotics for very special case of the functions $C_k(n)$)**.** *For $k \gg 1$ sufficiently large, we have that*

$$C_{\Omega(n)}(n) \sim (\sigma_0 * \mathbb{1}_{*\log\log n-2})(n) \times \lambda(n) \frac{n^{\log\log n-1}}{(\log n)^{\log\log -1}\Gamma(\log\log n)}.$$

Moreover, by considering the average orders of the function $\nu_p(n)$ for p large and tending to infinity, we have bounds on the aysmptotic behavior of these functions of the form

$$\lambda(n)\widehat{\tau}_0(n) \precsim C_{\Omega(n)}(n) \precsim \lambda(n)\widehat{\tau}_1(n).$$

It suffices to take the functions

$$\begin{aligned} \widehat{\tau}_0(n) &:= \frac{1}{\log 2} \cdot \frac{\log n}{(\log n)^{\log\log n}} \cdot \frac{n^{\log\log n-1}}{\Gamma(\log\log n)} \\ \widehat{\tau}_1(n) &:= \frac{1}{2e\log 2} \cdot \frac{(\log n)^2}{(\log n)^{\log\log n}} \cdot \frac{n^{\log\log n}}{\Gamma(\log\log n)}. \end{aligned}$$

Proof. The first stated formula follows from Theorem 2.5 by setting $k := \Omega(n) \sim \log\log n$ and simplifying. We evaluate the Dirichlet convolution functions and approximate as follows:

$$(\sigma_0 * \mathbb{1}_{\log\log n-2})(n) = \sum_{p|n} \binom{\nu_p(n)+\log\log n-1}{\log\log n-1}$$

$$\begin{aligned}
&\geq \sum_{p|n} \frac{(\nu_p(n) + \log\log n - 1)^{\log\log n - 1}}{(\log\log n)^{\log\log n-1}} \\
&\sim \frac{n}{\log 2} \\
(\sigma_0 * \mathbb{1}_{\log\log n - 2})(n) &\leq \left(\frac{(\nu_p(n) + \log\log n - 1)e}{\log\log n - 1}\right)^{\log\log n - 1} \\
&\sim (2e)^{\log\log n - 1} \\
&= \frac{n \cdot \log n}{2e \log 2}.
\end{aligned}$$

The upper and lower bounds are obtained from the next well known binomial coefficient approximations using Stirling's formula.

$$\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{n^k}{k!} < \left(\frac{ne}{k}\right)^k \qquad \square$$

Now that we have accurate asymptotic bounds on $|g^{-1}(n)|$ as $n \to \infty$, we must form the summatory functions $G^{-1}(x)$ of g^{-1} whose terms vary widely when including the parity of $\Omega(n)$ (sign of $\lambda(n)$). The natural mechanism for this is to employ Abel summation. However, we do not yet have a sufficient grasp on the summatory functions, $A_\Omega(x)$, that indicate the sign shifts of $\lambda(n)$ for $n \leq x$. To effectively bound these functions for large x, we will require asymptotic lower bounds on $\widehat{\pi}_k(x)$ for $k \geq 1$ and k bounded high enough above (with respect to x) so that the resulting functions $A_\Omega(x)$ are asymptotically accurate.

6 Key applications: Establishing lower bounds for $M(x)$ by cases along infinite subsequences

6.1 The culmination of what we have done so far

As noted before in the previous subsections, we cannot hope to evaluate functions weighted by $\lambda(n)$ except for on average using Abel summation. For this task, we need to know the bounds on $\widehat{\pi}_k(x)$ we developed in the proof of Corollary 4.2. A summation by parts argument shows that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k)+1) \\ &\approx G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)}. \end{aligned} \tag{8}$$

Thus it suffices for us to compute the effective *average order* of $g^{-1}(n)$ by summing its summatory function, $G^{-1}(n)$, including absorbing the parity of the $\lambda(n)$ terms into the parity of the $\Omega(n)$ terms we are summing over. The result in Lemma 4.3 is key to justifying the asymptotics obtained next in Theorem 6.1.

To simplify notation, for integers $m \geq 1$, let the *iterated logarithm function* (not to be confused with powers of $\log x$) be defined for $x > 0$ by

$$\log_*^m(x) := \begin{cases} x, & \text{if } m = 0; \\ \log x, & \text{if } m = 1; \\ \log\left(\log_*^{m-1}(x)\right), & \text{if } m \geq 2. \end{cases}$$

So $\log_*^2(x) = \log\log x$, $\log_*^3(x) = \log\log\log x$, $\log_*^4(x) = \log\log\log\log x$, $\log_*^5(x) = \log\log\log\log\log x$, and so on. This notation will come in handy to abbreviate the dominant asymptotic terms we find next in Theorem 6.1.

We use the result of Corollary 5.8 and Corollary 4.2 to prove the following central theorem:

Theorem 6.1 (Asymptotics and bounds for the summatory functions $G^{-1}(x)$)**.** *We define the lower summatory function, $G_u^{-1}(x)$, to provide bounds on the magnitude of $G^{-1}(x)$:*

$$|G_\ell^{-1}(x)| \ll |G^{-1}(x)|,$$

for all sufficiently large $x \gg 1$. We have the following asymptotic approximations for the lower summatory function where $C_{\ell,1}, C_{\ell,2}$ are absolute constants defined by

$$C_{\ell,1} = \frac{3}{16}\sqrt{\frac{3}{2}}\frac{A_0^2(2e+3)^2}{\pi e B^2 (\log 2)^3}, C_{\ell,2} = \frac{27 A_0^2 (2e+3)^3}{128\pi^{3/2} B^2 (\log 2)^3},$$

and $\widehat{L}_0(x)$ is the multiplier function from Corollary 5.7:

$$\begin{aligned} &\left|G_\ell^{-1}(x)\right| \gtrsim \\ &\left|(-1)^{\left\lfloor \frac{3}{2}\log_*^4(x)\right\rfloor} C_{\ell,1} \cdot (\log x)^{\frac{11}{7}} (\log\log x)^{\frac{71}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3} - \log_*^4(x)} \log_*^3(x)^{1+\frac{3}{2}\log\log x + \log_*^4(x)} \log_*^4(x)^{\log_*^4(x) - \frac{1}{2}}\right. \\ &\quad \left. - (-1)^{\left\lfloor 2\log_*^4(x)\right\rfloor} C_{\ell,2} \cdot \frac{\log_*^3(x)^{\frac{9}{2} + \frac{25}{6}\log 2 + \frac{3}{2\log 2} - \frac{4}{3}\log 3 - \frac{3}{2\log 3}}}{\sqrt{\log_*^4(x)}} \log_*^5(x)^{\frac{11}{7} + \frac{3}{2}\log_*^7(x)}\right|. \end{aligned}$$

Proof Sketch: Logarithmic scaling to the accurate order of the inverse functions. For the sums given by

$$S_{g^{-1}}(x) := \sum_{n \leq x} \lambda(n) \cdot C_{\Omega(n)}(n),$$

we notice that using the asymptotic bounds (rather than the exact formulas) for the functions $C_{\Omega(n)}(n)$, we have over-summed by quite a bit. In particular, following from the intent behind the constructions in the last sections, we are really summing only over all $n \leq x$ with $\Omega(n) \leq x$. Since $\Omega(n) \leq \lfloor \log_2 n \rfloor = \left\lfloor \frac{\log n}{\log 2} \right\rfloor$, many of the terms in the previous equation are actually zero (recall that $C_0(n) = \delta_{n,1}$). So we are actually only going to sum up to the average order of $\Omega(n) \sim \log\log n$ in practice, or to the slightly larger bound if the leading sign term on $G_\ell^{-1}(x)$ is negative. Hence, the sum (in general) that we are really interested in bounding is bounded below in magnitude by $S_{g^{-1}}(\log\log x)$ or $S_{g^{-1}}(\log_2(x))$, where we can now safely apply the asymptotic formulas for the $C_k(n)$ functions from Corollary 5.8 that hold once we have verified these constraints. □

Proof (Lower Bounds). Recall from our proof of Corollary 4.2 that a lower bound on the function $\widehat{\pi}_k(x)$ is given by $G\left(\frac{k-1}{\log\log x}\right)$ where the function $G(z)$ is bounded below by

$$G(z) \gg A_0 x \frac{(\log\log x)^{k-1}}{(k-1)!} \left(\frac{\log x}{\log 2}\right)^z \log^2 x \left(1 - \frac{z}{B} \log^{\frac{1}{14}}(x)\right).$$

Thus we can form a lower summatory function indicating the parity of all $\Omega(n)$ for $n \leq x$ as

$$\begin{aligned} A_\Omega^{(\ell)}(t) &= \sum_{k \leq \frac{3}{2}\log\log t} (-1)^k G\left(\frac{k-1}{\log\log t}\right) \\ &\sim (-1)^{1+\left\lfloor \frac{3\log\log t}{2} \right\rfloor} \cdot \frac{3A_0}{4eB \log^{\frac{3}{2}}(2) \Gamma\left(1 + \frac{3}{2}\log\log t\right)} \Big((2e+3)\log^{\frac{1}{14}}(t) - 2B\Big) \log^{\frac{3}{2}}(t) (\log\log t)^{\frac{3}{2}\log\log t}. \end{aligned} \tag{9}$$

Next, as in Lemma 4.3, we apply Abel summation to obtain that

$$G_\ell^{-1}(x) = \widehat{\tau}_0(\log\log x) A_\Omega^{(\ell)}(x) - \widehat{\tau}_0(u_0) A_\Omega^{(\ell)}(u_0) - \int_{u_0}^{\log\log x} \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t) dt, \tag{10}$$

where we define the integrand function, $I_\ell(t) := \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t)$, with some limiting simplifications as

$$\begin{aligned} I_\ell\left(e^{e^{\frac{4k}{3}}}\right) e^{e^{\frac{4k}{3}}} &= \frac{4A_0 4^{2k-1} 9^{-k} k^{2k} \left((3+2e)e^{2k/21} - 2B)\right] \exp\left(-\frac{16k^2}{9} + 2k + e^{4k/3}\left(\frac{4k}{3} - 1\right) - 1\right)}{3B \log^{\frac{5}{2}}(2)} \times \\ &\quad \times \Big(4e^{4k/3}k - 8k - 3\log k - 3\gamma + 6 + 3\log 3 - 6\log 2\Big). \end{aligned}$$

The integration term in (10) is summed approximately as follows:

$$\begin{aligned} \int_{u_0-1}^{\log\log x} \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t) dt &\sim \sum_{k=u_0+1}^{\frac{1}{2}\log\log\log\log x} \left(\frac{I_\ell\left(e^{e^{\frac{4k+2}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!} - \frac{I_\ell\left(e^{e^{\frac{4k}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!} \right) e^{e^{\frac{4k}{3}}} \\ &\approx C_0(u_0) + (-1)^{\left\lfloor \frac{\log\log\log\log x}{2} \right\rfloor} \times \int_{\frac{\log\log\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log\log\log x}{2}} \frac{I_\ell\left(e^{e^{\frac{4k}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!} e^{e^{\frac{4k}{3}}} dk. \end{aligned}$$

The differences on the upper and lower bounds on each integral in the last equation is small, and in particular $\frac{1}{2} \lll \log\log x$. So we can use a small perturbation of $+1$ in the power terms of $I_\ell(t)$ combined with an appeal to the binomial series, the expansion of binomial coefficients by the Stirling numbers of the first kind, and the following exact indefinite integral for $x, z \in \mathbb{R}$ moving forward:

$$\int t^p e^{ct} dt = \frac{(-1)^p}{c^{p+1}} \Gamma(p+1, -ct) \sim \frac{e^{ct} t^p}{c}.$$

Define the following function of t and note the change of variable $t \mapsto \frac{k-1}{2}$:

$$I_\ell\left(e^{e^{\frac{4k}{3}}}\right) e^{e^{\frac{4k}{3}}} = (1+k)^{2k} \exp\left(-\frac{16k^2}{9}\left(\frac{4k}{3}-1\right) e^{\frac{4k}{3}}\right) e^{2k-1} \widehat{f}(t_0).$$

So we take one reciprocal factor in the next integrand, and set the remaining powers of t^p to be t_0^p for t_0 a bound of integration which results in a lower bound on our target integrand from Abel summation.

From this perspective, we obtain using the exponential generating functions for the Stirling numbers of the first kind that [4, §7.4][2]

$$\begin{aligned}
\widehat{T}_\ell(t_0; t) &= \int \widehat{I}_\ell(t) dt \\
&\gg \sum_{m\geq 0}\sum_{n\geq 0}\sum_{q\geq 0}\sum_{j\geq 0}\sum_{r\geq 0} \frac{(-1)^{m+q+j+r}}{m!n!q!j!}\left(\frac{4}{3}\right)^{2m+n} \begin{bmatrix} j \\ r \end{bmatrix} \left\{\int t^{2m+n+j+r} \exp\left(\left(2+\frac{4}{3}(n+q)\right)t\right) dt\right\} \frac{\widehat{f}(t_0)}{e} \\
&\gtrsim -\frac{3\widehat{f}(t_0)}{4e} e^{2t} e^{-\frac{16k^2}{9}} \left(\gamma + \frac{e^{te^{\frac{4t}{3}}}}{te^{\frac{4t}{3}}} + \frac{4t}{3}\right)\left(\gamma + \frac{e^{te^{\frac{4t}{3}}}}{ke^{\frac{4t}{3}}} - \frac{4t}{3}\right) t^{2t}
\end{aligned}$$

In the previous equation, we have used that $(n+q+12)^{-1} \gtrsim \frac{1}{nq}$ and that for large $x \ggg 1$ tending to infinity

$$\sum_{m\geq 1} \frac{(-x)^m}{m \cdot m!} = -(\gamma + \Gamma(0,x) + \log x) \sim -\left(\gamma + \frac{e^{-x}}{x} + \log x\right).$$

Now we can define the coefficient functions, which as multipliers above would have otherwise complicated our integrals, in the form of $\widehat{f}(t_0) = \mathrm{cf}_+(t_0) - \mathrm{cf}_-(t_0)$ as

$$\begin{aligned}
\mathrm{cf}_+(t) &:= \left(\frac{16}{9}\right)^t \left(2B(8t+3\gamma+6\log 2) + 6B\log t + 12e^{10t/7}t + 8e^{\frac{10t}{7}+1}t + 6e^{\frac{2t}{21}+1}(2+\log 3) + 9e^{2t/21}(2+\log 3)\right) \\
\mathrm{cf}_-(t) &:= \left(\frac{16}{9}\right)^t \left(2B\left(4e^{4t/3}t + 6 + 3\log 3\right) + (3+2e)e^{2t/21}(8t+3\gamma+6\log 2) + 3(3+2e)e^{2t/21}\log t\right).
\end{aligned}$$

Let

$$\widehat{h}(t) := 3 \cdot 4^{-t-1}\left(\frac{3}{4}\right)^{\frac{4t}{3}} \frac{\sqrt{3}}{16\pi t^{\frac{10t}{3}+1}}.$$

Applying Stirling's formula again when x is large, we have that

$$\begin{aligned}
\widehat{R}_\ell(x) &= (-1)^{\left\lfloor \frac{\log\log\log\log x}{2} \right\rfloor} \times \int_{\frac{\log\log\log\log x}{2}-\frac{1}{2}}^{\frac{\log\log\log\log x}{2}} \frac{I_\ell\left(e^{e^{\frac{4k}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!} e^{e^{\frac{4k}{3}}} dk \qquad (11) \\
&\gtrsim (-1)^{\left\lfloor \frac{x}{2} \right\rfloor} \times \widehat{h}\left(\frac{\log\log\log\log x}{2}\right) \Bigg[
\end{aligned}$$

[2] Namely, that for natural numbers $j \geq 0$

$$\sum_{k\geq 0} \begin{bmatrix} k \\ j \end{bmatrix} \frac{z^k}{k!} = \frac{(-1)^j}{j!} \operatorname{Log}(1-z)^j.$$

$$\widehat{T}_\ell\left(\frac{\log\log\log\log x}{2};\frac{\log\log\log\log x}{2}\right)\left(\mathrm{cf}_+\left(\frac{\log\log\log\log x-1}{2}\right)-\mathrm{cf}_-\left(\frac{\log\log\log\log x}{2}\right)\right)$$
$$-\widehat{T}_\ell\left(\frac{\log\log\log\log x-1}{2};\frac{\log\log\log\log x-1}{2}\right)\left(\mathrm{cf}_+\left(\frac{\log\log\log\log x}{2}\right)-\mathrm{cf}_-\left(\frac{\log\log\log\log x-1}{2}\right)\right)\Bigg].$$

Since for real $0<s<1$ such that $s\to 0$, we have that $\log(1+s)\sim s$ and $(1+s)^{-1}\sim 1-s$, we can approximate the differences implied by the last estimate using that for t large tending to infinity we have

$$\log_*^m\left(t-\frac{1}{2}\right)\sim\log_*^m(t)-\frac{1}{2\log^{m-1}t}, m\geq 1.$$

Then applying these simplifications to (11) above and removing lower-order terms that do not contribute to the dominant asymptotic terms, we find that

$$\int_{u_0}^{\log\log x}\widehat{\tau}_0'(t)A_\Omega^{(\ell)}(t) \tag{12}$$
$$\gtrsim C_0(u_0)+\frac{(-1)^{\left\lfloor\frac{\log\log\log\log x}{2}\right\rfloor}\cdot 9\sqrt{3}A_0(2e+3)^2}{32B\pi e\log^{3/2}(2)}(\log\log\log x)^{\frac{10}{7}+\frac{25}{6}\log 2-\frac{4}{3}\log 3-\frac{5}{2}\log_*^5(x)}\log_*^5(x)^{\frac{11}{7}+\frac{3}{2}\log_*^7(x)}.$$

Finally, using Stirling's formula for very large x and (9), we can see that

$$\widehat{\tau}_0(x)\sim\frac{\log^2 x\cdot\log\log x}{\sqrt{2\pi}\cdot x}\left(\frac{x}{\log x\cdot\log\log x}\right)^{\log\log x}$$
$$A_\Omega^{(\ell)}(x)\sim\frac{3A_0(2e+3)}{4eB\log^{\frac{3}{2}}(2)}\log^{\frac{11}{7}}(x)(\log\log x)^{\frac{3}{2}\log\log x}.$$

So we have that the first terms in (10) are given by

$$\widehat{\tau}_0(\log\log x)A_\Omega^{(\ell)}(x)\gtrsim\frac{3A_0(2e+3)}{4\sqrt{2\pi}\cdot eB\log^{\frac{3}{2}}(2)}\cdot\log^{\frac{11}{7}}(x)\frac{\log\log\log x\cdot(\log\log x)^{1+\frac{3\log\log x}{2}+\log\log\log\log x}}{(\log\log\log x\cdot\log\log\log\log x)^{\log\log\log\log x-1}}.$$

These last formulas imply the forms of the stated bounds when we drop the lower-order constant term and multiply through by the bounds for the function $\widehat{L}_0(\log\log x)$ proved in Corollary 5.7. □

6.2 Lower bounds on the scaled Mertens function along an infinite subsequence

Corollary 6.2 (Bounds for the classically scaled Mertens function)**.** *Let $u_0:=e^{e^{e^e}}$ and define the infinite increasing subsequence, $\{x_n\}_{n\geq 1}$, by $x_n:=e^{e^{e^{e^{6n}}}}$. We have that along the increasing subsequence x_y for large $y\geq\max\left(\left\lceil e^{e^{e^e}}\right\rceil, u_0+1\right), y\gg 3\times 10^{1656520}$:*

$$\frac{|M(x_y)|}{\sqrt{x_y}}\gtrsim\frac{48C_{\ell,1}x_y^{1/4}}{17}(\log x_y^{15/32})^{\frac{3}{2}\log_*^4(x_y^{15/32})-\frac{10}{7}}(\log\log x_y^{15/32})^{\frac{71}{14}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}(\log\log\log x_y^{15/32})\times$$
$$\times\sqrt{\log\log\log\log x_y^{15/32}}+o(1),$$

as $y\to\infty$.

Proof of the Asymptotic Lower Bound. It suffices to take $u_0=e^{e^{e^e}}$, and a sufficient requirement on x is that the parity of $\lfloor\log\log\log\log x\rfloor\equiv 0 \pmod 2$ using the formula for $-G_\ell^{-1}(t)$ proved in Theorem 6.1. Since on $x/2\geq t\gg u_0$, we have that

$$\frac{d}{dt}\left[G_\ell^{-1}(t)\right]\gtrsim\frac{3C_{\ell,1}}{t}(\log t)^{\frac{3}{2}\log_*^4(t)-\frac{3}{7}}(\log\log t)^{\frac{71}{14}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}(\log\log\log t)\sqrt{\log\log\log\log t}.$$

The input to the derivative operator in the last equation is justified by establishing the limit

$$\lim_{x\to\infty} (\log\log x)^{\frac{(\log\log\log\log x)^2}{\log\log x}} = 1.$$

We can then write (8) in the following form using Abel summation:

$$\begin{aligned} |M(x)| &\gtrsim \Bigg| G_\ell^{-1}(x) + G_\ell^{-1}(u_0) A_\Omega^{(\ell)}(u_0) - G_\ell^{-1}(x) A_\Omega^{(\ell)}(x) \\ &\qquad + \left\{\int_{u_0}^{x^{1/6}} + \int_{x^{1/6}}^{x^{15/32}} + \int_{x^{15/32}}^{\sqrt{x}} + \int_{\sqrt{x}}^{x/2}\right\} \frac{x}{t^2\log(x/u_0)} \frac{d}{dt}\left[G_\ell^{-1}(t)\right] dt \Bigg| \\ &\gtrsim \Bigg| G_\ell^{-1}(x) - G_\ell^{-1}(x) A_\Omega^{(\ell)}(x) + o(\sqrt{x}) \\ &\quad + \frac{96 C_{\ell,1} x}{17} (\log x^{15/32})^{\frac{3}{2}\log_*^4(x^{15/32}) - \frac{10}{7}} (\log\log x^{15/32})^{\frac{71}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3}} (\log\log\log x^{15/32}) \times \\ &\quad \times \sqrt{\log\log\log\log x^{15/32}} \int_{x^{15/32}}^{\sqrt{x}} \frac{dt}{t^3} \Bigg| \end{aligned} \tag{13}$$

Now when we scale $M(x)$ by a reciprocal of $\sqrt{x}$ and let $x \to \infty$ along this infinite subsequence, we obtain that

$$\begin{aligned} \frac{|M(x)|}{\sqrt{x}} &\gtrsim \frac{48 C_{\ell,1} x^{1/4}}{17} (\log x^{15/32})^{\frac{3}{2}\log_*^4(x^{15/32}) - \frac{10}{7}} (\log\log x^{15/32})^{\frac{71}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3}} (\log\log\log x^{15/32}) \times \\ &\quad \times \sqrt{\log\log\log\log x^{15/32}} + o(1), \end{aligned}$$

where the constant power $\frac{71}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3} \approx 5.87011$. Notice that the above expression tends to $+\infty$ as $x \to \infty$, however, only extremly slowly and along a subsequence of asymptotically very large x. □

6.3 Remarks

Remark 6.3 (Tightness of the lower bounds)**.** One remaining question for scaling $|M(x)|/f(x)$ is exactly how tight can the function $f \in \mathcal{F}$ be made so that (for $\mathcal{F}$ some reasonable function space with bases in polynomials of x and powers of iterated logarithms)

$$f(x) := \operatorname{argmax}_{h\in\mathcal{F}} \left\{\limsup_{x\to\infty} \frac{|M(x)|}{h(x)} = C_h\right\},$$

for some absolute constants $C_h > 0$? What we have proved is that we can take

$$f(x) = \sqrt{x} \cdot TODO,$$

to obtain the above where the limiting constant is

$$C_f \mapsto \frac{9}{17}\sqrt{\frac{3}{2}\frac{A_0^2(2e+3)^2}{\pi e B^2 (\log 2)^3}}.$$

But is this the tightest possible f provably?

There is also, of course, the possibility of tightening the bound from above using the upper bounds proved in Theorem 2.7 along an infinite subsequence tending to infinity. We do not approach this problem here due to length constraints and that our lower bounds seem to have done much better than was previously known before about the (un)boundedness of the scaled Mertens function – our initial so-called “pipe dream”, or ”impossible”, result.

Remark 6.4 (Computational limitations on numerically verifying the new lower bounds)**.**

7 Conclusions (TODO)

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T.1 Table: Computations with a highly signed Dirichlet inverse function

n	**Primes**		**Sqfree**	**PPower**	$\bar{\mathbb{S}}$		$g^{-1}(n)$	$\lambda(n)\,\mathrm{sgn}(g^{-1}(n))$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_2(n)$		$G^{-1}(n)$	$G_{+}^{-1}(n)$	$G_{-}^{-1}(n)$
1	1^1	–	Y	N	N	–	1	1	0	0	–	1	1	0
2	2^1	–	Y	Y	N	–	-2	1	0	0	–	-1	1	-2
3	3^1	–	Y	Y	N	–	-2	1	0	0	–	-3	1	-4
4	2^2	–	N	Y	N	–	2	1	0	-1	–	-1	3	-4
5	5^1	–	Y	Y	N	–	-2	1	0	0	–	-3	3	-6
6	$2^1 3^1$	–	Y	N	N	–	5	1	0	-1	–	2	8	-6
7	7^1	–	Y	Y	N	–	-2	1	0	0	–	0	8	-8
8	2^3	–	N	Y	N	–	-2	1	0	-2	–	-2	8	-10
9	3^2	–	N	Y	N	–	2	1	0	-1	–	0	10	-10
10	$2^1 5^1$	–	Y	N	N	–	5	1	0	-1	–	5	15	-10
11	11^1	–	Y	Y	N	–	-2	1	0	0	–	3	15	-12
12	$2^2 3^1$	–	N	N	Y	–	-7	1	2	-2	–	-4	15	-19
13	13^1	–	Y	Y	N	–	-2	1	0	0	–	-6	15	-21
14	$2^1 7^1$	–	Y	N	N	–	5	1	0	-1	–	-1	20	-21
15	$3^1 5^1$	–	Y	N	N	–	5	1	0	-1	–	4	25	-21
16	2^4	–	N	Y	N	–	2	1	0	-3	–	6	27	-21
17	17^1	–	Y	Y	N	–	-2	1	0	0	–	4	27	-23
18	$2^1 3^2$	–	N	N	Y	–	-7	1	2	-2	–	-3	27	-30
19	19^1	–	Y	Y	N	–	-2	1	0	0	–	-5	27	-32
20	$2^2 5^1$	–	N	N	Y	–	-7	1	2	-2	–	-12	27	-39
21	$3^1 7^1$	–	Y	N	N	–	5	1	0	-1	–	-7	32	-39
22	$2^1 11^1$	–	Y	N	N	–	5	1	0	-1	–	-2	37	-39
23	23^1	–	Y	Y	N	–	-2	1	0	0	–	-4	37	-41
24	$2^3 3^1$	–	N	N	Y	–	9	1	4	-3	–	5	46	-41
25	5^2	–	N	Y	N	–	2	1	0	-1	–	7	48	-41
26	$2^1 13^1$	–	Y	N	N	–	5	1	0	-1	–	12	53	-41
27	3^3	–	N	Y	N	–	-2	1	0	-2	–	10	53	-43
28	$2^2 7^1$	–	N	N	Y	–	-7	1	2	-2	–	3	53	-50
29	29^1	–	Y	Y	N	–	-2	1	0	0	–	1	53	-52
30	$2^1 3^1 5^1$	–	Y	N	N	–	-16	1	0	-4	–	-15	53	-68
31	31^1	–	Y	Y	N	–	-2	1	0	0	–	-17	53	-70
32	2^5	–	N	Y	N	–	-2	1	0	-4	–	-19	53	-72
33	$3^1 11^1$	–	Y	N	N	–	5	1	0	-1	–	-14	58	-72
34	$2^1 17^1$	–	Y	N	N	–	5	1	0	-1	–	-9	63	-72
35	$5^1 7^1$	–	Y	N	N	–	5	1	0	-1	–	-4	68	-72
36	$2^2 3^2$	–	N	N	Y	–	14	1	9	1	–	10	82	-72
37	37^1	–	Y	Y	N	–	-2	1	0	0	–	8	82	-74
38	$2^1 19^1$	–	Y	N	N	–	5	1	0	-1	–	13	87	-74
39	$3^1 13^1$	–	Y	N	N	–	5	1	0	-1	–	18	92	-74
40	$2^3 5^1$	–	N	N	Y	–	9	1	4	-3	–	27	101	-74
41	41^1	–	Y	Y	N	–	-2	1	0	0	–	25	101	-76
42	$2^1 3^1 7^1$	–	Y	N	N	–	-16	1	0	-4	–	9	101	-92
43	43^1	–	Y	Y	N	–	-2	1	0	0	–	7	101	-94
44	$2^2 11^1$	–	N	N	Y	–	-7	1	2	-2	–	0	101	-101
45	$3^2 5^1$	–	N	N	Y	–	-7	1	2	-2	–	-7	101	-108
46	$2^1 23^1$	–	Y	N	N	–	5	1	0	-1	–	-2	106	-108
47	47^1	–	Y	Y	N	–	-2	1	0	0	–	-4	106	-110
48	$2^4 3^1$	–	N	N	Y	–	-11	1	6	-4	–	-15	106	-121

Table T.1: ***Computations of the first several cases of*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for*** $\mathbf{1 \leq n \leq 56}$.
The column labeled `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled, respectively,* `Sqfree`, `PPower` *and $\bar{\mathbb{S}}$ list inclusion of n in the sets of squarefree integers, prime powers, and the set $\bar{\mathbb{S}}$ that denotes the positive integers n which are neither squarefree nor prime powers. The next two columns provide the explicit values of the inverse function $g^{-1}(n)$ and indicate that the sign of this function at n is given by $\lambda(n) = (-1)^{\Omega(n)}$.*

Then the next two columns show the small-ish magnitude differences between the unsigned magnitude of $g^{-1}(n)$ and the summations $\widehat{f}_1(n) := \sum_{k \geq 0} \binom{\omega(n)}{k} \cdot k!$ and $\widehat{f}_2(n) := \sum_{k \geq 0} \binom{\omega(n)}{k} \cdot \#\{d|n : \omega(d) = k\}$. Finally, the last three columns show the summatory function of $g^{-1}(n)$, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, deconvolved into its respective positive and negative components: $G_{+}^{-1}(x) := \sum_{n \leq x} g^{-1}(n) \left[g^{-1}(n) > 0\right]_{\delta}$ and $G_{-}^{-1}(x) := \sum_{n \leq x} g^{-1}(n) \left[g^{-1}(n) < 0\right]_{\delta}$.

T.2 Table: Dirichlet inverse functions of $(f+1)(n)$ for f additive

n	$\lambda(n)$	$(f+1)^{-1}(n)$
1	1	1
2	-1	$-f(2)-1$
3	-1	$-f(3)-1$
4	1	$f(2)^2+2f(2)-f(4)$
5	-1	$-f(5)-1$
6	1	$2f(3)f(2)+f(2)+f(3)+1$
7	-1	$-f(7)-1$
8	-1	$-f(2)^3-3f(2)^2+2f(4)f(2)-f(2)+2f(4)-f(8)$
9	1	$f(3)^2+2f(3)-f(9)$
10	1	$2f(5)f(2)+f(2)+f(5)+1$
11	-1	$-f(11)-1$
12	-1	$-3f(3)f(2)^2-f(2)^2-4f(3)f(2)-2f(2)+2f(3)f(4)+f(4)$
13	-1	$-f(13)-1$
14	1	$2f(7)f(2)+f(2)+f(7)+1$
15	1	$2f(5)f(3)+f(3)+f(5)+1$
16	1	$f(2)^4+4f(2)^3-3f(4)f(2)^2+3f(2)^2-6f(4)f(2)+2f(8)f(2)+f(4)^2-f(4)+2f(8)-f(16)$
17	-1	$-f(17)-1$
18	-1	$-3f(2)f(3)^2-f(3)^2-4f(2)f(3)-2f(3)+2f(2)f(9)+f(9)$
19	-1	$-f(19)-1$
20	-1	$-3f(5)f(2)^2-f(2)^2-4f(5)f(2)-2f(2)+f(4)+2f(4)f(5)$
21	1	$2f(7)f(3)+f(3)+f(7)+1$
22	1	$2f(11)f(2)+f(2)+f(11)+1$
23	-1	$-f(23)-1$
24	1	$4f(3)f(2)^3+f(2)^3+9f(3)f(2)^2+3f(2)^2+2f(3)f(2)-6f(3)f(4)f(2)-2f(4)f(2)+f(2)-4f(3)f(4)-2f(4)+2f(3)f(8)+f(8)$
25	1	$f(5)^2+2f(5)-f(25)$
26	1	$2f(13)f(2)+f(2)+f(13)+1$
27	-1	$-f(3)^3-3f(3)^2+2f(9)f(3)-f(3)+2f(9)-f(27)$
28	-1	$-3f(7)f(2)^2-f(2)^2-4f(7)f(2)-2f(2)+f(4)+2f(4)f(7)$
29	-1	$-f(29)-1$
30	-1	$-2f(3)f(2)-6f(3)f(5)f(2)-2f(5)f(2)-f(2)-f(3)-2f(3)f(5)-f(5)-1$
31	-1	$-f(31)-1$

Table T.2: ***Dirichlet inverse functions of additive arithmetic functions.*** *The table provides a list of the Dirichlet inverse functions of $(f+1)(n)$ for f additive such that $f(1)=0$.*