Report: “Exact Formulas For the Generalized Sum-of-Divisors Functions” (Maxie D. Schmidt)

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1 Summary of Results

Let $x \in \mathbb{N}$, and $\alpha \in \mathbb{C}$. The author has developed new formulas for the sum of divisors function

$$\sigma_\alpha(x) = \sum_{d|x} d^\alpha.$$

They note that $\sigma_\alpha(x)$ is the coefficient of q^x for

$$\sum_{n=1}^{x} \frac{n^\alpha q^n}{1-q^n},$$

and so devise a method to express each term $\frac{q^n}{1-q^n}$ in terms of sums of lograithmic derivatives of cyclotomic polynomials. More specifically,

$$\frac{q^n}{1-q^n} = -1 + \frac{1}{n(1-q)} + \frac{1}{n} \sum_{\substack{d|n,\\ d>1}} \tilde{\Phi}_d(q),$$

with

$$\Phi_n(q) = \prod_{\substack{1\le k\le n,\\ (k,n)=1}} \left(q - e^{2\pi i k/n}\right),$$

$$\tilde{\Phi}_d(q) = \frac{1}{q} \cdot \frac{d}{dw} \left[\log \Phi_n(w)\right]_{w=1/q}.$$

The author expresses

$$\sum_{\substack{d|n,\\ d>1}} \tilde{\Phi}_d(q) = \tilde{S}_{0,n}(q) + \tilde{S}_{1,n}(q) + \tilde{S}_{2,n}(q),$$

with each $\tilde{S}_{j,n}(q)$ summing over a class of divisors $d|n$.

The coefficient of q^x is then extracted from each sum $\tilde{S}_{j,n}(q)$, which allows an explicit formula for $\sigma_\alpha(x)$.

Once the formula for $\sigma_\alpha(x)$ is proved, applications are discussed for asymptotics involving the average order of the divisor function, and for the study of perfect numbers. Comparisons are also made to other exact formulas, including the Hardy–Ramanujan–Rademacher integer partition formula, and similar more restrictive partition functions.

2 Importance

Given the importance of $\sigma_\alpha(x)$ for elementary number theory, it is interesting to note that few exact formulas for $\sigma_\alpha(x)$ have been developed. The existence of such a formula is therefore potentially valuable.

The main formula contains the sum

$$\sum_{d|x} \tau_x^{\alpha+1}(d),$$

in which the divisors of x must be known to begin with. Given the comparable complexity of computing τ_x^α, this raises the question of whether it would be easier to simply compute $\sigma_\alpha(x)$ directly, without resorting to the computed formula.

Nevertheless, this derived formula is of considerable interest. As in the study of the primes, or even of the integer partition function $p(n)$ discussed by the author toward the end of the paper, whether or not a given formula is of optimal efficiency is not necessarily the sole criterion for importance.

The author spends Section 3 of the paper discussing applications of their results to the study of perfect numbers, plane partitions, and the general relationship between multiplicative and additive number theory. These applications ensure that there is considerable interest and importance in the author's formulæ.

3 Key Difficulties For the Paper

There are still some key errors that need to be corrected, but they are comparably mild tyops.

3.1 Section 1

In Section 1.2: *Factoring partial sums into irreducibles*, Equation (4), $\Phi_n(x)$ is incorrectly written as

$$\Phi_n(x) = \prod_{d|n} (1 - q^d)^{\mu(n/d)}.$$

This is given as the Möbius inversion to

$$q^n - 1 = \prod_{d|n} \Phi_d(q).$$

The correct Möbius inversion should produce

$$\Phi_n(q) = \prod_{d|n}(q^d - 1)^{\mu(n/d)}.$$

Note also the typo of writing $\Phi_n(x)$ rather than $\Phi_n(q)$.

The most important error is likely in Definition 1.1, in which the author provides the following identity for $n \geq 2$:

$$\Pi_n(q) = \sum_{j=0}^{n-2} \frac{(n-1-j)q^j(1-q)}{(1-q^n)} = \frac{(n-1)+nq-q^n}{1-q}.$$

This formula is incorrect. Most likely, the author meant for the right hand side to be $\frac{(n-1)-nq+q^n}{1-q}$.

3.2 Section 2

In the beginning of Section 2: *Proofs of Our New Results*, Equation (4) is erroneously referred to as Equation (3).

In the proof of Proposition 2.3, $\tau_\alpha(x)$ is written as a triple sum; the first sum is indexed over k, but the second sum excludes divisors $d = p^k$, where p is prime, and the k here is presumably an arbitrary power. This representation of d as a prime power ought to be made with another index, e.g. $d = p^j$.

In the proof of Proposition 1.3, the author writes that

$$\Pi_n(q) = \tilde{\Phi}_n(1/q).$$

Presumably, the author intended that

$$\Pi_n(q) = \sum_{\substack{d|n,\\ d>1}} \tilde{\Phi}_d(q).$$

In the proof of Proposition 1.4, $\epsilon_p(n)$ is used, when the author presumably meant $v_p(n)$.

The key error in Section 1 for $\Pi_n(q)$ is repeated in the proof of Proposition 1.4: in this case, the author writes

$$\sum_{j=0}^{p-2}(n-1-j)t^j = \frac{(p-1)+pt-t^p}{(1-t)^2}.$$

Here, $n = p$, and q is replaced with $t = q^{p^{k-1}}$. This is incorrect; again, the formula ought to be

$$\sum_{j=0}^{p-2}(n-1-j)t^j = \frac{(p-1)-pt+t^p}{(1-t)^2}.$$

This error is not present in subsequent steps of the proof. The proof of Proposition 1.4 is still valid—indeed, it is elegant. However, this error should still be corrected before publication.

3.3 Section 3

Most of Section 3 is well-written. The only error found was in Section 3.3:

$$\frac{nq}{1-q^n} = nq - \sum_{d|n} \tilde{\Phi}_d(q^{-1}).$$

Perhaps the author intended

$$\frac{nq}{1-q^n} = nq - q\sum_{d|n} \tilde{\Phi}_d(q^{-1}).$$

4 Recommendation

The referee finds that the author's result is interesting and significant. The errors listed are all mild, and the Mathematica demonstration of the formulæ was carefully written. Publication is recommended once the listed errors are corrected.