Lower bounds on the summatory function of the Möbius function along infinite subsequences

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Abstract

The Mertens function, $M(x) = \sum_{n \leq x} \mu(n)$, is classically defined as the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture states that $|M(x)| < C \cdot \sqrt{x}$ for some absolute $C > 0$ for all $x \geq 1$. This classical conjecture has a well-known disproof due to Odlyzko and té Riele. We prove the unboundedness of $|M(x)|/\sqrt{x}$ using new methods by showing that

$$\limsup_{x \to \infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log \log x)^{\frac{1}{2}}} > 0.$$

The new methods we draw upon connect formulas and recent DGF series expansions involving the canonically additive functions $\Omega(n)$ and $\omega(n)$. The relation of $M(x)$ to the distribution of these core additive functions we prove at the start of the article is an indispensible component to the proof.

Keywords and Phrases: *Möbius function; Mertens function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function; Erdös-Kac theorem; additive functions.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; and 11N64.*

Glossary of special notation and conventions

Symbol	**Definition**
$\approx$	We write that $f(x) \approx g(x)$ if $\lvert f(x) - g(x)\rvert = O(1)$ as $x \rightarrow \infty$.
$\mathbb{E}[f(x)], \overset{\mathbb{E}}{\sim}$	We adapt the expectation notation $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \overset{\mathbb{E}}{\sim} h(x)$, to denote that f has an *average order* growth rate of $h(x)$. This means that $\frac{1}{x}\sum_{n \leq x} f(n) \sim h(x)$, or equivalently that $$\lim_{x\rightarrow\infty} \frac{\frac{1}{x}\sum_{n\leq x} f(n)}{h(x)} = 1.$$
B	The absolute constant $B \approx 0.2614972$ from the statement of Mertens theorem.
$C_k(n)$	The sequence is defined recursively for $n \geq 1$ as follows where we assume that $1 \leq k \leq \Omega(n)$: $$C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d\mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n\geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n\geq 0} f_n q^n$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (see below).
$f * g$	The Dirichlet convolution of f and g, $(f * g)(n) := \sum\limits_{d\mid n} f(d) g(n/d)$, where the sum is taken over the divisors d of n for $n \geq 1$.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ for $n \geq 2$ with $f^{-1}(1) = 1/f(1)$. The Dirichlet inverse of f exists if and only if $f(1) \neq 0$. This inverse function, denoted by f^{-1} when it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$.
γ	The Euler gamma constant defined by $\gamma := \lim\limits_{n\rightarrow\infty}\left(\sum\limits_{k=1}^{n} \frac{1}{k} - \log n\right) \approx 0.5772157$.
$\gg, \ll, \asymp$	For functions A, B in x, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A \ll B$ and $B \gg A$, we write $A \asymp B$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$.
$[n = k]_\delta$,$[\mathtt{cond}]_\delta$	The symbol $[n = k]_\delta$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise. For boolean-valued conditions, `cond`, the symbol $[\mathtt{cond}]_\delta$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention*.

Symbol	Definition
$\lambda_*(n)$	For positive integers $n \geq 2$, we define the next variant of the Liouville lambda function, $\lambda(n)$, as follows: $\lambda_*(n) := (-1)^{\omega(n)}$. We have the initial condition that $\lambda_*(1) = 1$.
$\lambda(n)$	The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. That is, $\lambda(n) \in \{\pm 1\}$ for all integers $n \geq 1$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree.
$\mu_x(C), \sigma_x(C)$	We define these analogs to the approximate mean and variance of the function $C_{\Omega(n)}(n)$ in the context of our new Erdös-Kac like theorems as $\mu_x(C) := \log\log x + \hat{a} - \frac{3}{2}\log\log\log x$ and $\sigma_x(C) := \sqrt{\mu_x(C)}$ where $\hat{a} := \log\left(\frac{1}{2\sqrt{2\pi}}\right) \approx -1.61209$ is an absolute constant.
$M(x)$	The Mertens function is the summatory function of $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum\limits_{n \leq x} \mu(n)$.
$\Phi(z)$	For $x \in \mathbb{R}$, we define the function giving the normal distribution CDF by $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{z} e^{-t^2/2} dt$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^\alpha \vert\vert n$ (or when p^α exactly divides n) for p prime, $\alpha \geq 1$ and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p \vert n} 1$ and the completely additive function $\Omega(n) := \sum_{p^\alpha \vert\vert n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leq n \leq x$ for $x > 1$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$ for $x \geq 2$.
$P(s)$	For complex s with $\mathrm{Re}(s) > 1$, we define the *prime zeta function* to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$. For $\mathrm{Re}(s) > 1$, the prime zeta function is related to $\zeta(s)$ according to the formula $P(s) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log[\zeta(ks)]$.
$Q(x)$	For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. More precisely, this function is summed and identified with its limiting asymptotic formula as $x \to \infty$ in the following form: $Q(x) := \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O(\sqrt{x})$.
$\sim$	We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x \to \infty} \frac{A(x)}{B(x)} = 1$.
$\zeta(s)$	The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation on the rest of the complex plane with the exception of a simple pole at $s = 1$ of residue one.

1 Introduction

1.1 Definitions

We define the *Möbius function* to be the signed indicator function of the squarefree integers in the form of [20, A008683]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \text{ and } n \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

There are many variants and special properties of the Möbius function and its generalizations [18, *cf.* §2]. One crucial role of the classical $\mu(n)$ is that the function forms an inversion relation for the divisor sums formed by arithmetic functions convolved with one through *Möbius inversion*:

$$g(n) = (f * 1)(n) \iff f(n) = (g * \mu)(n), \forall n \geq 1.$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined on the positive integers as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as follows [20, A002321]:

$$\{M(x)\}_{x \geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \ldots\}.$$

Clearly, a positive integer $n \geq 1$ is *squarefree*, or contains no (prime power) divisors which are squares, if and only if $\mu^2(n) = 1$. A related summatory function which counts the number of *squarefree* integers $n \leq x$ satisfies [5, §18.6] [20, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers characterized by the sign of the Möbius function are in fact equal as $x \to \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{x} \overset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{x} \xrightarrow{x \to \infty} \frac{3}{\pi^2}.$$

1.2 Properties

A conventional approach to evaluating the limiting asymptotic behavior of $M(x)$ for large $x \to \infty$ results by considering an inverse Mellin transformation of the reciprocal of the Riemann zeta function. In particular, since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \cdot \int_1^\infty \frac{M(x)}{x^{s+1}} dx, \operatorname{Re}(s) > 1,$$

we obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T - \imath\infty}^{T + \imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

The previous two representations lead us to the exact expression of $M(x)$ for any real $x > 0$ given by the next theorem due to Titchmarsh.

Theorem 1.1 (Analytic Formula for $M(x)$)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any real $x > 0$*

$$M(x) = \lim_{k \to \infty} \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |\operatorname{Im}(\rho)| < T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp\left(-C \cdot \log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan more recently proved new updated estimates bounding $M(x)$ from above for large x in the following forms [21]:

$$\begin{aligned} M(x) &\ll \sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right), \ \forall \epsilon > 0. \end{aligned}$$

1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C \cdot \sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens for $C = 1$ and all $x < 10000$. Since its beginnings in 1897, the Mertens conjecture has been disproven by computation of non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele [13]. Since the truth of the conjecture would have implied the RH, more recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound along infinite subsequences, e.g., considering the asymptotics of the function in the limit supremum and limit infimum senses.

A precise statement of this problem is to produce an unconditional proof of whether $\limsup_{x\to\infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x\to\infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there are infinite subsequences of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound towards either $\pm\infty$ along the subsequence. We cite that it is only known by computation that [16, *cf.* §4.1] [20, *cf.* A051400; A051401]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leq -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [13, 8, 9, 6]. Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies [12]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log\log\log x)^{5/4}} = O(1).$$

2 A concrete new approach to bounding $M(x)$ from below

2.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 2.1 (Summatory functions of Dirichlet convolutions)**.** *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of f and h, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse of f. We have the following exact expressions for the summatory function of $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned} \pi_{f*h}(x) &:= \sum_{n \leq x} \sum_{d|n} f(d) h(n/d) \\ &= \sum_{d \leq x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right]. \end{aligned}$$

Moreover, for all $x \geq 1$

$$\begin{aligned} H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\ &= \sum_{n=1}^{x} f^{-1}(n) \pi_{f*h}\left(\left\lfloor \frac{x}{n} \right\rfloor\right). \end{aligned}$$

Corollary 2.2 (Convolutions arising from Möbius inversion)**.** *Suppose that g is an arithmetic function such that $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n \leq x} (g * \mu)(n)$. The Mertens function is expressed by the sum*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geq 1.$$

Corollary 2.3 (A motivating special case)**.** *We have exactly that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

2.2 An exact expression for $M(x)$ in terms of strongly additive functions

Fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute exactly that (see Table T.1 starting on page 40)

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these positive terms is given by $\operatorname{sgn}(g^{-1}(n)) = \frac{g^{-1}(n)}{|g^{-1}(n)|} = \lambda(n)$ for all $n \geq 1$ (see Proposition 4.1).

There is not an easy, nor subtle direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences. The distribution of distinct sets of prime exponents is still fairly regular so that $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of $g^{-1}(n)$ over $n \geq 2$:

Heuristic 2.4 (Symmetry in $g^{-1}(n)$ in the prime factorizations of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for $= \omega(n_i) \geq 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly one, two, three, and so on prime factors.

Conjecture 2.5. *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) $g^{-1}(1) = 1$;

(B) *For all $n \geq 1$,* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;

(C) *For all squarefree integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!;$$

(D) *If $n \geq 2$ and $\Omega(n) = k$, then*

$$2 \leq |g^{-1}(n)| \leq \sum_{m=0}^{k} \binom{k}{m} \cdot m!.$$

We illustrate parts (B)–(D) of the conjecture clearly using the computation of initial values of this inverse sequence in Table T.1. A proof of (C) in fact follows from Lemma 6.2 stated on page 21. The realization that the beautiful and remarkably simple combinatorial form of property (C) in Conjecture 2.5 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through sums of auxiliary sequences of arithmetic functions (see Section 6).

We prove that (see Proposition 8.4)

$$M(x) = G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{\sqrt{x}} G^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].$$

This formula implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by bounding appropriate estimates of the summatory function $G^{-1}(x)$. The regularity of $|g^{-1}(n)|$ is useful to our argument in formally bounding $G^{-1}(x)$ from below.

The regularity and quasi-periodicity we alluded to in the previous remarks are actually quantifiable in so much as $|g^{-1}(n)|$ for $n \leq x$ tends to its average order with a skew normal tendency depending on x as $x \to \infty$. In Section 7, we prove the next variant of an Erdös-Kac theorem like analog for a component sequence closely related to $g^{-1}(n) = \lambda(n) \cdot |g^{-1}(n)|$. What results is the following statement for $\mu_x(C) := \log\log x + \hat{a}$, $\sigma_x(C) := \sqrt{\mu_x(C)}$, $\hat{a} \approx -1.37662$ an absolute constant, and any $y \in \mathbb{R}$ (see Corollary 7.9):

$$\#\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq y\} = x \cdot \Phi\left(\frac{\frac{\pi^2}{6} y - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{x}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

These clear probabilistic statements on the distribution of $|g^{-1}(n)|$ allow us to bound $|G^{-1}(x)| \gg (\log x)\sqrt{\log\log x}$ as $x \to \infty$ (see Theorem 8.3).

2.3 Uniform asymptotics from enumerative bivariate DGFs from Mongomery and Vaughan

Theorem 2.6 (Montgomery and Vaughan)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $R < 2$ we have that uniformly for all $1 \leq k \leq R \log\log x$

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right],$$

where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, 0 \leq |z| \leq R.$$

The proof of the next result is combinatorially motivated in so much as it interprets lower bounds on a key infinite product factor of $\mathcal{G}(z)$ defined in Theorem 2.6 as corresponding to an ordinary generating function of certain homogeneous symmetric polynomials involving the primes. This interpretation allows us to recover the following uniform lower bounds on $\widehat{\pi}_k(x)$ as $x \to \infty$:

Theorem 2.7. *For all sufficiently large x we have uniformly for $1 \leq k \leq \log\log x$ that*

$$\widehat{\pi}_k(x) \gg \frac{x^{3/4}}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^3}\right)\right].$$

2.3.1 Remarks

We emphasize the relevant recency of the method demonstrated by Montgomery and Vaughan in constructing a proof of Theorem 2.6. To the best of our knowledge, this textbook reference is one of the first clear-cut applications documenting something of a hybrid DGF-and-OGF approach to enumerating sequences of arithmetic functions and their summatory functions. The hybrid method is motivated by the fact that it does not require a direct appeal to traditional highly oscillatory DGF-only inversions and integral formulas involving the Riemmann zeta function. This newer interpretaion of certain bivariate DGFs offers a window into the best of both generating function series worlds: It combines the additivity implicit to the coefficients indexed by a formal power series variable formed by multiplication of these structures, while coordinating the distinct DGF-best property of the multiplicativity of distinct prime powers invoked by taking powers of an Euler product. Another set of proofs constructed based on this type of hybrid power series enabling DGF is given in Section 7 when we prove an Erdös-Kac theorem like analog that holds for a component sequence related to $g^{-1}(n)$.

2.4 Cracking the classical unboundedness barrier

In Section 8, we are able to state what forms a bridge between the results we carefully prove up to that point the article. What we obtain at the conclusion of the section is the next summary theorem that unconditionally resolves the classical question of the unboundedness of the scaled function Mertens function $q(x) := |M(x)|/\sqrt{x}$ in the limit supremum sense.

Theorem 2.8 (Unboundedness of the the Mertens function, $q(x)$)**.** *We have that*

$$\limsup_{x \to \infty} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

In establishing the rigorous proof of Theorem 2.8 based on our new methods, we not only show unboundedness of $q(x)$, but also set a minimal rate (along a large infinite subsequence) at which this form of the scaled Mertens function grows without bound.

3 An overview of the core components to the proof

We offer an initial step-by-step summary overview of the core components to our proof outlined in the next points. We hope that this sketch of the logical components to this argument makes the article easier to parse.

(1) We prove a matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). See Theorem 2.1 in Section 4.

(2) This crucial step provides us with an exact formula for $M(x)$ in terms of the prime counting function $\pi(x)$, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is stated in (1). The link relating our new formula in (1) to canonical additive functions and their distributions lends a recent distinguishing element to the success of the methods in our proof.

(3) We tighten bounds from a less classical result proved in [11, §7] providing uniform asymptotic formulas for the summatory functions, $\widehat{\pi}_k(x)$, large $x \gg e$ and $1 \leq k \leq \log\log x$ (see Theorem 2.7).

(4) We then turn to estimating the limiting asymptotics of the quasi-periodic function, $|g^{-1}(n)|$, by proving several formulas bounding its average order as $x \to \infty$ in Section 6.

(5) In Section 7, we prove new expectation formulas for $|g^{-1}(n)|$ and the related component sequence $C_{\Omega(n)}(n)$ by first proving an Erdös-Kac like theorem satisfied by $C_{\Omega(n)}(n)$. This allows us to prove asymptotic lower bounds on $|G^{-1}(x)|$ when x is large.

(6) When we return to step **(2)** with our new lower bounds at hand, we are led to a new unconditional proof of the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers.

4 Preliminary proofs of new results

4.1 Establishing the summatory function properties and inversion identities

We will offer a proof of Theorem 2.1 suggested by an intuitive construction through matrix methods. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95].

Proof of Theorem 2.1. Let h, g be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h. We have that the following formulas hold for all $x \geq 1$:

$$\begin{aligned}\pi_{g*h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] H(i). \end{aligned} \tag{2}$$

The first formula above is well known. The second formula is justified directly using summation by parts as[A]

$$\begin{aligned}\pi_{g*h}(x) &= \sum_{d=1}^{x} h(d) G\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i \leq x} \left(\sum_{j \leq i} h(j) \right) \times \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right]. \end{aligned}$$

We next form the invertible matrix of coefficients associated with this linear system defining $H(j)$ for all $1 \leq j \leq x$ in (2) by setting

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right), 1 \leq j \leq x.$$

Since $g_{x,x} = G(1) = g(1)$ and $g_{x,j} = 0$ for all $j > x$, the matrix we must invert in this problem is lower triangular with a non-zero constant on its diagonals, and is hence invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressed by applying an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

Here, U is a square matrix with sufficiently large finite dimensions whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$\left[(I - U^T)^{-1}\right]_{i,j} = [j \leq i]_{\delta}.$$

Observe that

$$\left\lfloor \frac{x}{j} \right\rfloor - \left\lfloor \frac{x-1}{j} \right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

The previous property implies that

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

[A] For any arithmetic functions, u_n, v_n, with $U_j := u_1 + u_2 + \cdots + u_j$ for $j \geq 1$, we have that [14, §2.10(ii)]

$$\sum_{j=1}^{n-1} u_j \cdot v_j = U_{n-1} v_n + \sum_{j=1}^{n-1} U_j \left(v_j - v_{j+1}\right), n \geq 2.$$

We use the last property in (3) to shift the matrix $\hat{G}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of g in the following form:

$$\left[(I-U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Our target matrix in the inversion problem is defined by

$$(g_{x,j}) = (I-U^T)\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators as follows:

$$\begin{aligned}(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k)\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1}\right\rfloor} g^{-1}(k)\right).\end{aligned}$$

Hence, the summatory function $H(x)$ is given exactly for any $x \geq 1$ by a vector product with the inverse matrix from the previous equation as

$$H(x) = \sum_{k=1}^{x} g_{x,k}^{-1} \cdot \pi_{g*h}(k) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right)\cdot \pi_{g*h}(k).$$

We can prove an inversion formula providing the coefficients of $G^{-1}(i)$ for $1 \leq i \leq x$ given by the last equation by adapting our argument to prove (2) above. This leads to the identity that

$$H(x) = \sum_{k=1}^{x} g^{-1}(x)\pi_{g*h}\left(\left\lfloor \frac{x}{k}\right\rfloor\right). \qquad \square$$

4.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, let $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n. Then we can easily prove using DGFs that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega+1) * \mu. \tag{4}$$

When combined with Corollary 2.2 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 2.3.

Proposition 4.1 (The signedness property of $g^{-1}(n)$). *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0, \pm 1\}$ *denote the sign of the arithmetic function h at integers $n \geq 1$. For the Dirichlet invertible function,* $g(n) := \omega(n)+1$*, we have that* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ *for all $n \geq 1$.*

Proof. The function $D_f(s) := \sum_{n\geq 1} f(n)n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ for σ_f the abcissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = 1/\zeta(s)$ and $D_\omega(s) = P(s)\zeta(s)$ for $\operatorname{Re}(s) > 1$. Then by (4) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any arithmetic function f such that $f(1) \neq 0$, we have for all $\operatorname{Re}(s) > 1$ that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{5}$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. This observation implies that $\operatorname{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make our claims rigorous.

By the recurrence relation that defines the Dirichlet inverse function of any arithmetic function h such that $h(1) = 1$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d\mid n \\ d>1}} h(d) h^{-1}(n/d), & n \geq 2. \end{cases} \tag{6}$$

For $n \geq 2$, the summands in (6) can be simply indexed over the primes $p|n$ given our definition of h from above. This observation yields that we can inductively unfold these sums into nested divisor sums provided the depth of the expanded divisor sums does not exceed the capacity to index summations over the primes dividing n. Namely, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p|n} h^{-1}\left(\frac{n}{p}\right), && \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right), && \text{if } \Omega(n) \geq 2 \\ &= -\sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1 p_2}} h^{-1}\left(\frac{n}{p_1 p_2 p_3}\right), && \text{if } \Omega(n) \geq 3. \end{aligned}$$

Then by induction with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1, n \geq 2. \tag{7}$$

In fact, by a combinatorial argument we recover exactly that

$$h^{-1}(n) = \lambda(n)(\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}. \tag{8}$$

These expansions imply that the following property holds for all $n \geq 1$:

$$\operatorname{sgn}(h^{-1}(n)) = \lambda(n).$$

Since λ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$ for all $d|n$ and $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever n is squarefree, so that we obtain

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1. \qquad \square$$

4.3 Statements of known limiting asymptotics

Theorem 4.2 (Mertens theorem)**.** *For all $x \geq 2$ we have that*

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log\log x + B + o(1), \text{ as } x \rightarrow \infty,$$

where $B \approx 0.2614972128476427837554$ is an absolute constant[B].

Corollary 4.3 (Product form of Mertens theorem)**.** *We have that for all sufficiently large $x \gg 2$*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\left(1 + o(1)\right), \text{ as } x \rightarrow \infty,$$

where the notation for the absolute constant $0 < B < 1$ coincides with the definition of Mertens constant from Theorem 4.2. Hence, for any real z we obtain that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^z \sim \frac{e^{-\gamma z}}{(\log x)^z}, \text{ as } x \rightarrow \infty.$$

Proofs of Theorem 4.2 and Corollary 4.3 are given in [5, §22.7; §22.8]. We have a related analog of Corollary 4.3 that is justified using the Euler product representation for the Riemann zeta function:

$$\prod_{p \leq x} \left(1 + \frac{1}{p}\right) = \prod_{p \leq x} \frac{\left(1 - p^{-2}\right)}{\left(1 - p^{-1}\right)} = \zeta(2) e^{\gamma} (\log x)(1 + o(1)), \text{ as } x \rightarrow \infty.$$

Facts 4.4 (Exponential integrals and the incomplete gamma function)**.** Two variants of the *exponential integral function* are defined by the integral next representations [14, §8.19] [3, §3.3].

$$\begin{aligned}\operatorname{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, x \in \mathbb{R} \\ E_1(z) &:= \int_{1}^{\infty} \frac{e^{-tz}}{t} dt, \operatorname{Re}(z) \geq 0\end{aligned}$$

These functions are related by $\operatorname{Ei}(-kz) = -E_1(kz)$ for real $k, z > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $\operatorname{Ei}(\pm x)$ for all real $x > 0$:

$$\begin{aligned}\gamma + \log x - x \leq \operatorname{Ei}(-x) &\leq \gamma + \log x - x + \frac{x^2}{4}, \qquad (9a) \\ 1 + \gamma + \log x - \frac{3}{4}x \leq \operatorname{Ei}(x) \quad &\leq 1 + \gamma + \log x - \frac{3}{4}x + \frac{11}{36}x^2.\end{aligned}$$

The (upper) *incomplete gamma function* is defined by [14, §8.4]

$$\Gamma(s, x) = \int_{x}^{\infty} t^{s-1} e^{-t} dt, \operatorname{Re}(s) > 0.$$

The following properties of $\Gamma(s, x)$ hold:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, x > 0, \qquad (9b)$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, s > 0, \text{ as } x \rightarrow \infty. \qquad (9c)$$

[B]Precisely, we have that the *Mertens constant* is defined by [20, A077761]

$$B = \gamma + \sum_{m \geq 2} \frac{\mu(m)}{m} \log\left[\zeta(m)\right].$$

5 Components to the asymptotic analysis of lower bounds for sums of arithmetic functions weighted by $\lambda(n)$

5.1 A discussion of the results proved by Montgomery and Vaughan

Remark 5.1 (Intuition and constructions in Theorem 2.6)**.** For $|z| < 2$ and $\mathrm{Re}(s) > 1$, let

$$F(s, z) := \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \tag{10}$$

and define the DGF coefficients, $a_z(n)$ for $n \geq 1$, by the product

$$\zeta(s)^z \cdot F(s, z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s}, \mathrm{Re}(s) > 1.$$

Suppose that $A_z(x) := \sum_{n \leq x} a_z(n)$ for $x \geq 1$. Then we obtain the next generating function like identity in z enumerating the $\widehat{\pi}_k(x)$ for $1 \leq k \leq \log\log x$ [A]

$$A_z(x) = \sum_{n \leq x} z^{\Omega(n)} = \sum_{k \geq 0} \widehat{\pi}_k(x) z^k \tag{11}$$

Thus for $r < 2$, by Cauchy's integral formula we have

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz.$$

Selecting $r := \frac{k-1}{\log\log x}$ for $1 \leq k < 2\log\log x$ leads to the uniform asymptotic formulas for $\widehat{\pi}_k(x)$ given in Theorem 2.6. Montgomery and Vaughan then consider individual analysis of the main and error terms for $A_z(x)$ to prove that

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

We will require estimates of $A_{-z}(x)$ from below to form summatory functions that weight the terms of $\lambda(n)$ in our new formulas derived in the next sections.

5.2 New uniform asymptotics based on refinements of Theorem 2.6

Proposition 5.2. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \geq 2.$$

When $s := 1$, we have the asymptotic formula from Mertens theorem (see Theorem 4.2). For all integers $s \geq 2$ there is are absolutely defined quasi-polynomial bounding functions $\gamma_0(s, x)$ and $\gamma_1(s, x)$ in s, x such that

$$\gamma_0(s, x) + o(1) \leq P_s(x) \leq \gamma_1(s, x) + o(1), \text{ as } x \to \infty.$$

It suffices to define the bounds in the previous equation by the functions

$$\begin{aligned} \gamma_0(s, x) &= s \log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}s(s-1)^2\log^2(2) \\ \gamma_1(s, x) &= s \log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) + \frac{1}{4}s(s-1)^2\log^2(x). \end{aligned}$$

[A] In fact, for any additive arithmetic function $a(n)$, characterized by the property that $a(n) = \sum_{p^\alpha || n} a(p^\alpha)$ for all $n \geq 2$, we have that [7, *cf.* §1.7]

$$\prod_p \left(1 - \sum_{m \geq 1} \frac{z^{a(p^m)}}{p^{ms}}\right)^{-1} = \sum_{n \geq 1} \frac{z^{a(n)}}{n^s}, \mathrm{Re}(s) > 1.$$

Proof. Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$, and where our target function smooth function is $f(t) = t^{-s}$ so that $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= \operatorname{Ei}(-(s-1)\log x) - \operatorname{Ei}(-(s-1)\log 2) + o(1), \text{ as } x \to \infty. \end{aligned}$$

Now using the inequalities in Facts 4.4, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned} \frac{P_s(x)}{s} &\geq \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}(s-1)^2\log^2(2) + o(1) \\ \frac{P_s(x)}{s} &\leq \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) + \frac{1}{4}(s-1)^2\log^2(x) + o(1). \end{aligned}$$

□

We will first prove the stated form of the lower bound on $\mathcal{G}(-z)$ for $z := \frac{k-1}{\log\log x}$. Then we will discuss the technical adaptations to Montgomery and Vaughan's proof of Theorem 2.6 in Remark 5.3 to justify the new asymptotic lower bounds on $\widehat{\pi}_k(x)$ that hold uniformly for all $1 \leq k \leq \log\log x$.

Proof of Theorem 2.7. For $0 \leq z < 2$ and integers $x \geq 2$, the right-hand-side of the following product is finite.

$$\widehat{P}(z,x) := \prod_{p \leq x} \left(1 - \frac{z}{p}\right)^{-1}.$$

For fixed, finite $x \geq 2$ let

$$\mathbb{P}_x := \{n \geq 1 : \text{all prime divisors } p|n \text{ satisfy } p \leq x\}.$$

Then we can see that

$$\prod_{p \leq x} \left(1 - \frac{z}{p^s}\right)^{-1} = \sum_{n \in \mathbb{P}_x} \frac{z^{\Omega(n)}}{n^s}, x \geq 2. \tag{12}$$

By extending the argument in the proof given in [11, §7.4], we have that

$$A_{-z}(x) := \sum_{n \leq x} \lambda(n) z^{\Omega(n)} = \sum_{k \geq 0} \widehat{\pi}_k(x)(-z)^k,$$

If we let $a_n(z,x)$ be defined by the DGF

$$\widehat{P}(z,x) := \sum_{n \geq 1} \frac{a_n(z,x)}{n^s},$$

then we show that

$$\sum_{n \leq x} a_n(-z,x) = \sum_{k=0}^{\log_2(x)} \widehat{\pi}_k(x)(-z)^k + \sum_{k > \log_2(x)} e_k(x)(-z)^k.$$

This assertion if correct since the products of all non-negative integral powers of the primes $p \leq x$ generate the integers $\{1 \leq n \leq x\}$ as a subset. Thus we capture all of the relevant terms needed to express $(-1)^k \cdot \widehat{\pi}_k(x)$ via the Cauchy integral formula representation over $A_{-z}(x)$ by replacing the corresponding infinite product terms with $\widehat{P}(-z,x)$ in the definition of $\mathcal{G}(-z)$.

Now we must argue that

$$\mathcal{G}(-z) \gg \prod_{p \leq x} \left(1 + \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{-z}, 0 \leq z < 1, x \geq 2.$$

For $0 \leq z < 1$ and $x \geq 2$, we see that

$$\begin{aligned}
\mathcal{G}(-z) &= \exp\left(-\sum_p \left[\log\left(1+\frac{z}{p}\right) + \log\left(1-\frac{1}{p}\right)\right]\right) \\
&\gg \exp\left(-z \times \sum_{p>x}\left[\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right] - \sum_{p \leq x}\left[\log\left(1+\frac{z}{p}\right)+\log\left(1-\frac{1}{p}\right)\right]\right) \\
&= \widehat{P}(-z,x) \times \exp\left(-z(B+o(1))\right) \gg_z \widehat{P}(-z,x), \text{ as } x \rightarrow \infty.
\end{aligned}$$

Next, we have for all integers $0 \leq k \leq m < \infty$, and any sequence $\{f(n)\}_{n \geq 1}$ with sufficiently bounded partial power sums, that [10, §2]

$$[z^k] \prod_{1 \leq i \leq m} (1-f(i)z)^{-1} = [z^k] \exp\left(\sum_{j \geq 1}\left(\sum_{i=1}^m f(i)^j\right)\frac{z^j}{j}\right), |z| < 1. \tag{13}$$

In our case we have that $f(i)$ denotes the reciprocal of the i^{th} prime in the generating function expansion of (13). It follows from Proposition 5.2 that for any real $0 \leq z < 1$ we obtain

$$\begin{aligned}
\log\left[\prod_{p \leq x}\left(1+\frac{z}{p}\right)^{-1}\right] &\geq -(B+\log\log x)z + \sum_{j \geq 0}\left[\log\left(\frac{\log x}{\log 2}\right) - (2j+1)\log\left(\frac{x}{2}\right) - (2j+1)^2\frac{\log^2 2}{4}\right] z^{2j+2} \\
&\quad - \sum_{j \geq 0}\left[\log\left(\frac{\log x}{\log 2}\right) - (2j+2)\log\left(\frac{x}{2}\right) + (2j+2)^2\frac{\log^2 x}{4}\right] z^{2j+3} \\
&= -(B+\log\log x)z + \sum_{j \geq 0}\left[\log\left(\frac{\log x}{\log 2}\right) - (j+1)\log\left(\frac{x}{2}\right)\right](-z)^{j+2} \\
&\quad - \frac{1}{4} \times \sum_{j \geq 0}\left[(\log 2)^2(2j+1)^2 z^{2j+2} + (\log x)^2(2j+2)^2 z^{2j+3}\right] \\
&= -(B+\log\log x)z + \log\left(\frac{\log x}{\log 2}\right)\left[z-1+\frac{1}{z+1}\right] + \log\left(\frac{x}{2}\right)\left[\frac{2}{1+z}-1-\frac{1}{(1+z)^2}\right] \\
&\quad + (\log 2)^2 \cdot \frac{z^2+z^4}{(z^2-1)^3} + (\log x)^2 \cdot \frac{z^2+6z^4+z^6}{4(z^2-1)^3} \\
&=: \widehat{\mathcal{B}}(x;z).
\end{aligned} \tag{14}$$

We adjust the uniform bound parameter R so that

$$z \equiv z(k,x) = \frac{k-1}{\log\log x} \in [0,1),$$

whenever $1 \leq k \leq \log\log x$ in the notation of Theorem 2.6. This implies that $(1+z)^{-1} \in \left(\frac{1}{2}, 1\right]$, and so

$$\begin{aligned}
\min_{0 \leq z \leq 1}\left[z-1+\frac{1}{z+1}\right] &= 0 \\
\min_{0 \leq z \leq 1}\left[\frac{2}{1+z}-1-\frac{1}{(1+z)^2}\right] &= -\frac{1}{4}.
\end{aligned}$$

When we expand out the coefficients of $(\log 2)^2$ and $(\log x)^2$ in partial fractions of z, we see that all of the terms with a singularity as $z \rightarrow 1^-$ are positive. This means to obtain the lower bound, we can drop these contributions. What we are left to minimize are the following terms:

$$(\log 2)^2 \times \min_{0 \leq z \leq 1}\left[\frac{1}{4} - \frac{1}{4(1+z)^3} + \frac{5}{8(1+z)^2} - \frac{1}{2(1+z)}\right] = \frac{13}{108}(\log 2)^2$$

$$(\log x)^2 \times \min_{0\leq z\leq 1}\left[-\frac{1}{4(1+z)^3}+\frac{3}{8(1+z)^2}-\frac{1}{8(1+z)}\right]=0.$$

In total, we have from (14) that

$$\widehat{\mathcal{B}}(x;z) \gg \left(\frac{2}{x}\right)^{\frac{1}{4}} \cdot \exp\left(\frac{13}{108}(\log 2)^2\right) \asymp x^{-\frac{1}{4}}.$$

In summary, we have arrived at a proof that as $x\to\infty$

$$\frac{e^{\gamma z}}{(\log x)^{-z}} \times \exp\left(\widehat{\mathcal{B}}(u,x;z)\right) \gg x^{-\frac{1}{4}}. \tag{15}$$

Finally, to finish our proof of the new form of the lower bound on $\mathcal{G}(-z)$, we need to bound the reciprocal factor of $\Gamma(1-z)$. Since $z \equiv z(k,x) = \frac{k-1}{\log\log x}$ and $k \in [1, \log\log x]$, or again with $z \in [0,1)$, we obtain for minimal k and all large enough $x \gg 1$ that $\Gamma(1-z) = \Gamma(1) = 1$, and for k towards the upper range of its interval that

$$\Gamma(1-z) \approx \Gamma\left(\frac{1}{\log\log x}\right) = \frac{1}{\log\log x}\Gamma\left(1+\frac{1}{\log\log x}\right) \approx \frac{1}{\log\log x}. \qquad \square$$

Remark 5.3 (Technical adjustments in the proof of Theorem 2.7)**.** We now discuss the differences between our construction and that in the technical proof of Theorem 2.6 in the reference when we bound $\mathcal{G}(-z)$ from below as in Theorem 2.7. The reference proves that for real $0 \leq z < 2$

$$A_{-z}(x) = -\frac{zF(1,-z)}{\Gamma(1-z)} \cdot x(\log x)^{-(z+1)} + O\left(x(\log x)^{-\operatorname{Re}(z)-2}\right). \tag{16}$$

Recall that for $r < 2$ we have by Cauchy's integral formula that

$$(-1)^k \widehat{\pi}_k(x) = \frac{1}{2\pi\imath}\int_{|z|=r} \frac{A_{-z}(x)}{z^{k+1}} dz. \tag{17}$$

We first claim that uniformly for large x and $1 \leq k \leq \log\log x$ we have

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{1-k}{\log\log x}\right) \times \frac{x(\log\log x)^{k-1}}{(\log x)(k-1)!}\left[1+O\left(\frac{k}{(\log\log x)^3}\right)\right]. \tag{18}$$

Then since we have proved in Theorem 2.6 above that

$$\mathcal{G}\left(\frac{1-k}{\log\log x}\right) \gg \frac{1}{x^{1/4}} \cdot \frac{(k-1)}{\log\log x},$$

the result in (18) implies our stated uniform asymptotic bound. Namely, we obtain that

$$\widehat{\pi}_k(x) \gg \frac{x^{3/4}}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1+O\left(\frac{k}{(\log\log x)^3}\right)\right].$$

We have to provide analogs to the two separate bounds corresponding to the error and main terms of our estimate according to (16) and (17). The error term estimate is simpler, so we tackle it first in the next argument. The second part of our proof establishing the main term in (18) requires us to duplicate and adjust significant parts of the fine-tuned reasoning given in the reference.

Error Term Bound. To prove that the error term bound holds, we estimate that

$$\begin{aligned}\left|\frac{1}{2\pi\imath}\int_{|z|=r}\frac{x}{(\log x)^2}\frac{(\log x)^{-\operatorname{Re}(z)}}{z^{k+1}}\right| &\ll x(\log x)^{-(r+2)}r^{-(k+1)} \ll \frac{x}{(\log x)^2}\frac{(\log\log x)^{k+1}}{e^{k-1}(k-1)^{k+1}} \\ &\ll \frac{x}{(\log x)^2}\frac{(\log\log x)^{k+1}}{e^{2(k-1)}(k-1)!(k-1)} \ll \frac{x}{(\log x)^2}\frac{(\log\log x)^{k+1}}{(k-1)!}\end{aligned}$$

$$\ll \frac{x}{\log x} \frac{(\log\log x)^{k-4}}{(k-1)!}. \tag{19}$$

We can calculate that for $0 \leq z < 1$

$$\begin{aligned}
\prod_p \left(1 + \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{-z} &= \exp\left(-\sum_p \left[\log\left(1 + \frac{z}{p}\right) + z\log\left(1 - \frac{1}{p}\right)\right]\right) \\
&\sim \exp\left(-o(z) \times \sum_p \frac{1}{p^2}\right) \\
&\gg \exp\left(-o(z)\frac{\pi^2}{6}\right) \gg_z 1.
\end{aligned}$$

In other words, we have that $\mathcal{G}\left(\frac{1-k}{\log\log x}\right) \gg 1$. So the error term in (19) is majorized by taking $O\left(\frac{k}{(\log\log x)^3}\right)$ as our upper bound.

Main Term Bounds. Notice that the main term estimate corresponfing to (16) and (17) is given by $\frac{x}{\log x} I$, where

$$I := \frac{(-1)^{k-1}}{2\pi\imath} \int_{|z|=r} G(-z)(\log x)^{-z} z^{-k} dz.$$

In particular, we can write $I = I_1 + I_2$ where we define

$$\begin{aligned}
I_1 &:= \frac{(-1)^{k-1} G(-r)}{2\pi\imath} \int_{|z|=r} (\log x)^{-z} z^{-k} dz \\
&= \frac{G(-r)(\log\log x)^{k-1}}{(k-1)!} \\
I_2 &:= \frac{(-1)^{k-1}}{2\pi\imath} \int_{|z|=r} (G(-z) - G(-r))(\log x)^{-z} z^{-k} dz \\
&= \frac{(-1)^{k-1}}{2\pi\imath} \int_{|z|=r} (G(-z) - G(-r) + G'(-r)(z+r))(\log x)^{-z} z^{-k} dz.
\end{aligned}$$

We have by a power series expansion of $G''(-w)$ about $-z$ and integrating the resulting series termwise with respect to w that

$$\left|G(-z) - G(-r) + G'(-r)(z+r)\right| = \left|\int_{-r}^{z} (z+w)G''(-w)dw\right| \ll G''(-r) \times |z+r|^2 \ll |z+r|^2.$$

Now we parameterize the curve in the contour for I_2 by writing $z = re^{2\pi\imath t}$ for $t \in [-1/2, 1/2]$. This leads us to the bounds

$$\begin{aligned}
|I_2| &= r^{3-k} \times \int_{-1/2}^{1/2} |e^{2\pi\imath t} + 1|^2 \cdot (\log x)^{re^{2\pi\imath t}} \cdot e^{2\pi\imath t} dt \\
&\ll r^{3-k} \times \int_{-1/2}^{1/2} \sin^2(\pi t) \cdot e^{(1-k)\cos(2\pi t)} dt.
\end{aligned}$$

Whenever $|x| \leq 1$, we know that $|\sin x| \leq |x|$. We can construct bounds on $-\cos(2\pi t)$ for $t \in [-1/2, 1/2]$ by writing $\cos(2x) = 1 - 2\sin^2 x$ for $|x| < 1/2$. Then by the alternating Taylor series expansions of the sine function

$$\begin{aligned}
1 - 2\sin^2(2\pi t) &\geq 1 - 2\left(1 - \frac{\pi t}{3}\right)^2 \geq -1 - \frac{2\pi^2 t^2}{9} \qquad \implies \\
-\cos(2\pi t) &\leq 1 + \frac{2\pi^2 t^2}{9} \leq \left(4 + \frac{2\pi^2}{9}\right) t^2 \leq 1 + 3t^2.
\end{aligned}$$

So it follows that

$$\begin{aligned}
|I_2| &\ll r^{3-k}e^{k-1} \times \left|\int_0^\infty t^2 e^{3(k-1)t^2}\,dt\right| \\
&\ll \frac{r^{3-k}e^{k-1}}{(k-1)^{3/2}} = \frac{(\log\log x)^{k-3}e^{k-1}}{(k-1)^{k-3/2}} \\
&\ll \frac{k\cdot(\log\log x)^{k-3}}{(k-1)!}.
\end{aligned}$$

Thus the contribution from the term $|I_2|$ can then be asborbed into the error term bound in (18).

5.3 The distribution of exceptional values of $\Omega(n)$

The next theorems reproduced from [11, §7.4] characterize the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \log\log x$. The tendency of this canonical completely additive function to not deviate substantially from its average order is an extraordinary property that allows us to prove asymptotic relations on summatory functions that are weighted by its parity without having to account for significant local oscillations when we average over a large interval.

Theorem 5.4 (Upper bounds on exceptional values of $\Omega(n)$ for large n)**.** *Let*

$$\begin{aligned}
A(x,r) &:= \#\left\{n \leq x : \Omega(n) \leq r\cdot\log\log x\right\}, \\
B(x,r) &:= \#\left\{n \leq x : \Omega(n) \geq r\cdot\log\log x\right\}.
\end{aligned}$$

If $0 < r \leq 1$ and $x \geq 2$, then

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x\to\infty.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x,r) \ll_R x\cdot(\log x)^{r-1-r\log r}, \quad \text{as } x\to\infty.$$

Theorem 5.5 is an analog to the celebrated Erdös-Kac theorem typically stated for the normally distributed values of the scaled-shifted $\omega(n)$ function over $n \leq x$ as $x \to \infty$.

Theorem 5.5 (Exact bounds on exceptional values of $\Omega(n)$ for large n)**.** *We have that as $x \to \infty$*

$$\#\left\{3 \leq n \leq x : \Omega(n) - \log\log n \leq 0\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Remark 5.6. The key interpretation we need to take away from the statements of Theorem 5.4 and Theorem 5.5 is the result proved in the next corollary. The role of the parameter R involved in stating the previous theorem is a critical bound as the scalar factor in the upper bound on $k \leq R\log\log x$ in Theorem 2.6 up to which our uniform bounds given by Theorem 2.7 hold. In contrast, for $n \geq 2$ we can actually have contributions from values distributed throughout the range $1 \leq \Omega(n) \leq \log_2(n)$ infinitely often. It is then crucial that we can show that the main term in the asymptotic formulas we obtain for these summatory functions is captured by summing only over the truncated range of $k \in [1, \log\log x]$ where the uniform bounds hold.

Corollary 5.7. *Using the notation for $A(x,r)$ and $B(x,r)$ from Theorem 5.4, we have that for $x \geq 2$ and $\delta > 0$,*

$$o(1) \leq \frac{B(x,1+\delta)}{A(x,1)} \ll 2, \text{ as } \delta\to 0^+, x\to\infty.$$

Proof. The lower bound stated above is clear. To show that the asymptotic upper bound is correct, we compute using Theorem 5.4 and Theorem 5.5 that

$$\frac{B(x,1+\delta)}{A(x,1)} \ll \frac{x\cdot(\log x)^{\delta-\delta\log(1+\delta)}}{O(1)+\frac{x}{2}+O\left(\frac{x}{\sqrt{\log\log x}}\right)} \sim o_\delta(1),$$

as $x \to \infty$. Notice that since $\mathbb{E}[\Omega(n)] = \log\log n + B$, with $0 < B < 1$ the absolute constant from Mertens theorem, when we denote the range of $k > \log\log x$ as holding in the form of $k > (1+\delta)\log\log x$ for $\delta > 0$ at large x, we can assume that $\delta \to 0^+$ as $x \to \infty$. In particular, this holds since $k > \log\log x$ implies that

$$\lfloor \log\log x \rfloor + 1 \geq (1+\delta)\log\log x \qquad \implies \qquad \delta \leq \frac{1 + \{\log\log x\}}{\log\log x} = o(1), \text{ as } x \to \infty.$$

The key consequence is that $B(x, 1+\delta)$ is at most a bounded constant multiple of $A(x,1)$ for all large x. $\square$

6 Component sequences expressing the Dirichlet inverse functions, $g^{-1}(n)$

The pages of tabular data given as Table T.1 in the appendix section (refer to page 40) are intended to provide clear insight into why we arrived at the approximations to $g^{-1}(n)$ proved in this section. The table provides illustrative numerical data by examining the approximate behavior at hand for the cases of $1 \leq n \leq 500$ with *Mathematica*.

6.1 Definitions and basic properties of component function sequences

We define the following auxiliary coefficient sequence for integers $n \geq 1, k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{20}$$

Example 6.1 (Special cases of the functions $C_k(n)$ for small k)**.** We cite the following special cases which are verified by explicit computation using (20) [20, A066922][A]:

$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= d(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} - \gcd\left(\Omega(n), \omega(n)\right). \end{aligned}$$

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (20) inductively. By the same argument, we see that at fixed n, the function $C_k(n)$ is seen to be non-zero only for positive integers $k \leq \Omega(n)$ whenever $n \geq 2$. A sequence of relevant signed semi-diagonals of the functions $C_k(n)$ begins as [20, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \ -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

6.2 Relating the auxiliary functions $C_k(n)$ to formulas approximating $g^{-1}(n)$

Lemma 6.2 (An exact formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) g^{-1}(n/d) \quad \implies \quad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \tag{21}$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (21) in the form of $(g^{-1} * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m \cdot (C_m(-) * g^{-1})(n)$, or so that

$$F_m(n) = -\begin{cases} \sum\limits_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum\limits_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & m \geq 2, \\ (\omega * g^{-1})(n), & m = 1. \end{cases}$$

[A] For all $n, k \geq 2$, we have the following recurrence relation satisfied by $C_k(n)$ between successive values of k:

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=0}^{\nu_p(n)-1} C_{k-1}\left(dp^i\right), n \geq 1.$$

By repeatedly expanding the right-hand-side of the previous equation, we find that for $m := \Omega(n)$

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \tag{22}$$

The formula then follows from (22) by Möbius inversion applied to each side of the last equation. □

Corollary 6.3. *For all squarefree integers $n \geq 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \tag{23}$$

Proof. Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geq 2$ and that n is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where p_i is prime for all $1 \leq i \leq \omega(n)$. Since all divisors of any squarefree n are necessarily also squarefree, we can transform the exact divisor sum guaranteed for all n in Lemma 6.2 into a sum that partitions the divisors according to the number of distinct prime factors:

$$\begin{aligned} g^{-1}(n) &= \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i} (-1)^i \cdot C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \end{aligned}$$

The signed contributions in the first of the previous equations is justified by noting that $\lambda(n) = (-1)^{\omega(n)}$ whenever n is squarefree, and that for $d \geq 1$ squarefree we have the correspondence $\omega(d) = k \implies \Omega(d) = k$ for $1 \leq k \leq \log_2(d)$. □

Since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ using the notation defined in the the proof of Proposition 4.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for squarefree $n \geq 1$. A proof of part (C) of Conjecture 2.5 follows as an immediate consequence.

Lemma 6.4. *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{24}$$

Proof. By applying Lemma 6.2, Proposition 4.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$. Lemma 6.2 implies

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \end{aligned}$$

In the last equation, we see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. □

Combined with the signedness property of $g^{-1}(n)$ guaranteed by Proposition 4.1, Lemma 6.4 shows that the summatory function is expressed as

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_{\Omega(d)}(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Since $\lambda(d) C_{\Omega(d)}(d) = (g^{-1} * 1)^{-1}(d) = (\chi_{\mathbb{P}} + \varepsilon)(d)$ where $\chi_{\mathbb{P}}$ denotes the characteristic function of the primes, we also clearly recover by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1.$$

6.2.1 A connection to the distribution of the primes

Remark 6.5. The combinatorial complexity of $g^{-1}(n)$ is deeply tied to the distribution of the primes $p \leq n$ as $n \to \infty$. While the magnitudes and dispersion of the primes $p \leq x$ certainly restricts the repeating of these distinct sequence values we can see in the contributions to $G^{-1}(x)$, the following statement is still clear about the relation of the weight functions $|g^{-1}(n)|$ to the distribution of the primes: The value of $|g^{-1}(n)|$ is entirely dependent on the pattern of the *exponents* (viewed as multisets) of the distinct prime factors of $n \geq 2$. The relation of the repitition of the distinct values of $|g^{-1}(n)|$ in forming bounds on $G^{-1}(x)$ makes another clear tie to $M(x)$ through Proposition 8.4 in the next section.

Example 6.6 (Combinatorial significance to the distribution of $g^{-1}(n)$)**.** We have a natural extremal behavior with respect to distinct values of $\Omega(n)$ corresponding to squarefree integers, and prime powers. Namely, if for $k \geq 1$ we define the infinite sets M_k and m_k to correspond to the maximal (minimal) positive integers such that

$$M_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\},$$

$$m_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\},$$

then any element of M_k is squarefree and any element of m_k is a prime power. In particular, we have that for any $N_k \in M_k$ and $n_k \in m_k$

$$N_k = \sum_{j=0}^{k} \binom{k}{j} \cdot j!, \quad \text{and} \quad n_k = 2 \cdot (-1)^k.$$

The formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 4.1 implies that we can express an exact formula for $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n. Namely, for $n \geq 2$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z \cdot \nu_p(n)) = [z^k] \prod_{p^\alpha || n} (1 + \alpha z), 0 \leq k \leq \omega(n).$$

Then we have essentially shown using (8) and (24) that we can expand

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, n \geq 2.$$

The combinatorial formula for $h^{-1}(n) = \lambda(n) \cdot (\Omega(n))! \times \prod_{p^\alpha || n} (\alpha!)^{-1}$ we derived in the proof of the key signedness proposition in Section 4 suggests further patterns and more regularity in the contributions of the distinct weighted terms for $G^{-1}(x)$ when we sum over all of the distinct prime exponent patterns that factorize $n \leq x$.

7 The distribution of $C_{\Omega(n)}(n)$ and $g^{-1}(n)$

We have remarked already in the introduction that the relation of the component functions, $g^{-1}(n)$ and $C_k(n)$, to the canonical additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties of these functions witnessed in Table T.1. In particular, each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdös-Kac theorem that shows that a shifted and scaled variant of each of the sets of these function values can be expressed through a limiting normal distribution as $n \to \infty$. This extremely regular tendency of these functions towards their average order is inherited by the component function sequences we are summing in the approximation of $M(x)$ stated by Proposition 8.4. In the remainder of this section we establish more technical analytic proofs of related properties of our key sequences, again in the spirit of Montgomery and Vaughan's reference.

Proposition 7.1. *For $|z| < P(2)^{-1}$, let the summatory function be defined as*

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

Let the function $F(s, z)$ is defined for $\operatorname{Re}(s) > 1$ and $|z| < 2$ in terms of the prime zeta function by

$$F(s, z) := \frac{1}{1 - P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

Then we have that for large x

$$\widehat{A}_z(x) = \frac{x}{\Gamma(z)} \cdot F(2, z) \cdot (\log x)^{z-1} + O_z\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right), |z| < P(2)^{-1}.$$

Proof. We know from the proof of Proposition 4.1 that for $n \geq 2$

$$C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}.$$

Then we can generate the denominator terms by the Dirichlet series

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! \cdot p^{rs}}\right)^{-1} = \exp\left(z \cdot P(s)\right), \operatorname{Re}(s) > 1, z \in \mathbb{C}.$$

By computing a Laplace transform on the right-hand-side of the above with respect to the variable z, we obtain

$$\sum_{n \geq 1} C_{\Omega(n)}(n) \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp\left(tz \cdot P(s)\right) dt = \frac{1}{1 - P(s)z}, \operatorname{Re}(s) > 1, |z| < |P(s)|^{-1}.$$

It follows that

$$\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times F(s, z),$$

where

$$F(s, z) := \frac{1}{1 - P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z, \operatorname{Re}(s) > 1, |z| < |P(s)|^{-1}.$$

Since $F(s, z)$ is convergent as an analytic function of s for all $\operatorname{Re}(s) > 1$ whenever $|z| < 2$, if $b_z(n)$ are the coefficients of the DGF $F(s, z)$, then

$$\left| \sum_{n \geq 1} \frac{b_z(n) (\log n)^{2R+1}}{n^s} \right| < +\infty,$$

is uniformly bounded for $|z| \leq R$. We must adapt the details to the case where the next proof method arises in the first application from [11, §7.4; Thm. 7.18] so that we can sum over our modified function depending on $\Omega(n)$. In particular, we cannot guarantee convergence of $F(s,z)$ by setting $s := 1$, so we modify the proof to show that we can in fact set $s := 2$ in this function to obtain a related result.

Let the function $d_z(n)$ be generated as the coefficients of the DGF $\zeta(s)^z$ for $\mathrm{Re}(s) > 1$, with corresponding summatory function $D_z(x) := \sum_{n \leq x} d_z(n)$. Taking the notation from the reference, we set $b_z(n) := (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$, let the convolution $a_z(n) := \sum_{d|n} b_z(d) d_z(n/d)$, and define the summatory function $A_z(x) := \sum_{n \leq x} a_z(n)$. The theorem in [11, Thm. 7.17; §7.4] implies that for any $z \in \mathbb{C}$ and $x \geq 2$

$$D_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\left(x \cdot (\log x)^{\mathrm{Re}(z)-2}\right).$$

Then we have that

$$\begin{aligned}
A_z(x) &= \sum_{m \leq x/2} b_z(m) D_z(x/m) + \sum_{x/2 < m \leq x} b_z(m) \\
&= \frac{x}{\Gamma(z)} \times \sum_{m \leq x/2} \frac{b_z(m)}{m^2} \times m \cdot \log\left(\frac{x}{m}\right)^{z-1} + O\left(x \sum_{m \leq x} \frac{|b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2}\right).
\end{aligned} \tag{25}$$

We can sum the coefficients for $u \geq e$ large as

$$\sum_{m \leq u} \frac{b_z(m)}{m} = (F(2,z) + O(u^{-2}))u - \int_1^u (F(2,z) + O(t^{-2}))dt = F(2,z) + O(u^{-1}).$$

The error term in (25) satisfies

$$\begin{aligned}
x \sum_{m \leq x} \frac{|b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2} &\ll x(\log x)^{\mathrm{Re}(z)-2} \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m} + x(\log x)^{-(R+2)} \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m} (\log m)^{2R} \\
&\ll x(\log x)^{\mathrm{Re}(z)-2} \cdot F(2,z) = O_z\left(x \cdot (\log x)^{\mathrm{Re}(z)-2}\right), |z| \leq R.
\end{aligned}$$

In the main term estimate for $A_z(x)$ from (25), when $m \leq \sqrt{x}$ we have

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\mathrm{Re}(z)-2}\right).$$

The total main term sum over the interval $m \leq x/2$ then corresponds to bounding

$$\begin{aligned}
\sum_{m \leq x/2} b_z(m) D_z(x/m) &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \sum_{m \leq x/2} \frac{b_z(m)}{m} \\
&\quad + O_z\left(x(\log x)^{\mathrm{Re}(z)-2} \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m} + x(\log x)^{R-1} \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m}\right) \\
&= \frac{x}{\Gamma(z)} (\log x)^{z-1} F(2,z) + O_z\left(x(\log x)^{\mathrm{Re}(z)-2} \sum_{m \geq 1} \frac{b_z(m)(\log m)^{2R+1}}{m^2}\right) \\
&= \frac{x}{\Gamma(z)} (\log x)^{z-1} F(2,z) + O_z\left(x(\log x)^{\mathrm{Re}(z)-2}\right).
\end{aligned}$$

□

Theorem 7.2. *We have uniformly for $1 \leq k < \log\log x$ that as $x \to \infty$*

$$\widehat{C}_k(x) := \sum_{\substack{n \leq x \\ \Omega(n) = k}} \lambda(n)(-1)^{\omega(n)} C_k(n) \asymp \frac{x}{\log x} \cdot \frac{(-1)^k (\log\log x - \log\zeta(2))^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^3}\right)\right].$$

Proof. The proof is a similar adaptation of the method of Montgomery and Vaughan we cited in Remark 5.3 to prove our variant of Theorem 2.7. We begin by bounding a contour integral over the error term for fixed large x for $r := \frac{k-1}{\log\log x}$ with $r < 2$:

$$\left|\int_{|z|=r} \frac{x \cdot (\log x)^{-(\mathrm{Re}(z)+2)}}{z^{k+1}} dz\right| \ll x(\log x)^{-(r+2)} r^{-(k+1)} \ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{k+1}} \cdot \frac{1}{e^{k-1}}$$
$$\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{3/2}} \cdot \frac{1}{e^{2k}(k-1)!}$$
$$\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \ll \frac{x}{\log x} \cdot \frac{k \cdot (\log\log x)^{k-5}}{(k-1)!}.$$

We must find an asymptotically accurate main term approximation to the coefficients of the following contour integral for $r \in [0, z_{\max}]$ where $z_{\max} < 2$:

$$\widetilde{A}_r(x) := -\int_{|z|=r} \frac{x \cdot (\log x)^{-z} \zeta(2)^z}{(\log x)\Gamma(1+z) \cdot z^k (1+P(2)z)} dz. \tag{26}$$

Finding an exact formula for the derivatives of the function that is implicit to the Cauchy integral formula (CIF) for (26) is complicated significantly by the need to differentiate $\Gamma(1+z)^{-1}$ up to integer order k in the formula. We can show that provided a restriction on the uniform bound parameter to $1 \leq r < 1$, we can approximate the contour integral in (26) using a sane bounding procedure where the resulting main term is accurate up to a bounded constant factor.

We observe that for $r := 1$, the function $|\Gamma(1 + re^{2\pi\imath t})|$ has a singularity (pole) when $t := \frac{1}{2}$. Thus we restrict the range of $|z| = r$ so that $0 \leq r < 1$ to necessarily avoid this problematic value of t when we parameterize $z = re^{2\pi\imath t}$ as a real integral over $t \in [0, 1]$. Then we can compute the finite extremal values as

$$\min_{\substack{0 \leq r < 1 \\ 0 \leq t \leq 1}} |\Gamma(1 + re^{2\pi\imath t})| = |\Gamma(1 + re^{2\pi\imath t})|\Big|_{(r,t)\approx(1,0.740592)} \approx 0.520089$$
$$\max_{\substack{0 \leq r < 1 \\ 0 \leq t \leq 1}} |\Gamma(1 + re^{2\pi\imath t})| = |\Gamma(1 + re^{2\pi\imath t})|\Big|_{(r,t)\approx(1,0.999887)} \approx 1.$$

This shows that

$$\widetilde{A}_r(x) \asymp -\int_{|z|=r} \frac{x \cdot \exp(-P(2)z)(\log x)^{-z}}{(\log x) \cdot z^k (1+P(2)z)} dz, \tag{27}$$

where as $x \to \infty$

$$\frac{\widetilde{A}_r(x)}{-\int_{|z|=r} \frac{x(\log x)^{-z}\zeta(2)^z}{(\log x)\cdot z^k(1+P(2)z)} dz} \in [1, 1.92275].$$

In particular, this argument holds by an analog to the mean value theorem for real integrals based on sufficient continuity conditions on the parameterized path and the smoothness of the integrand viewed as a function of z.

By induction we can compute the remaining coefficients $[z^k]\Gamma(1+z) \times \widehat{A}_z(x)$ with respect to x for fixed $k \leq \log\log x$ using the CIF. Namely, it is not difficult to see that for any integer $m \geq 0$, we have the m^{th} partial derivative of the integrand with respect to z has the following expansion:

$$\frac{1}{m!} \times \frac{\partial^{(m)}}{\partial z^{(m)}}\left[\frac{(\log x)^{-z}\zeta(2)^z}{1+P(2)z}\right]\Bigg|_{z=0} = \sum_{j=0}^{m} \frac{(-1)^m P(2)^j (\log\log x - \log\zeta(2))^{m-j}}{(m-j)!}$$
$$= \frac{(-P(2))^m (\log x)^{\frac{1}{P(2)}} \zeta(2)^{-\frac{1}{P(2)}}}{m!} \times \Gamma\left(m+1, \frac{\log\log x - \log\zeta(2)}{P(2)}\right)$$

$$\sim \frac{(-1)^m(\log\log x - \log\zeta(2))^m}{m!}.$$

Now by parameterizing the countour around $|z| = r := \frac{k-1}{\log\log x} < 1$ we deduce that the the main term of our approximation corresponds to

$$-\int_{|z|=r} \frac{x \cdot (\log x)^{-z}\zeta(2)^z}{(\log x)z^k(1+P(2)z)}dz \asymp \frac{x}{\log x} \cdot \frac{(-1)^k(\log\log x - \log\zeta(2))^{k-1}}{(k-1)!}. \qquad \square$$

Remark 7.3. An exact DGF expression for $\lambda(n)C_{\Omega(n)}(n)$ is in fact very much complicated by the need to estimate the asymptotics of the coefficients of the right-hand-side products

$$\begin{aligned}\sum_{n\geq 1} \frac{\lambda(n)C_{\Omega(n)}(n)z^{\Omega(n)}}{(\Omega(n))! \cdot n^s} &= \prod_p \left(2 - \exp\left(-z \cdot p^{-s}\right)\right)^{-1}, \mathrm{Re}(s) > 1, |z| < \log 2 \\ &= \exp\left(\sum_{j\geq 1}\sum_p \left(e^{-zp^{-s}} - 1\right)^j \frac{1}{j}\right).\end{aligned}$$

It is unclear how to exactly, and effectively, bound the coefficients of powers of z in the DGF expansion defined by the last equation. We use an alternate method in Corollary 7.5 to obtain the asymptotics for the actual summatory functions on which we require tight average case bounds.

Remark 7.4 (A standard simplifying assumption)**.** For $m \leq \omega_{\max}$ and $k \leq \Omega_{\max}$, as $n \to \infty$ we expect

$$\mathbb{P}\left(\omega(n) = m | \Omega(n) = k\right) \approx \frac{\omega_{\max} + 1 - k}{\omega_{\max}},$$

so that the conditional distribution of $\omega(n), \Omega(n)$ is not uniform over its bounded range. However, we do as is standard fare in proofs of the more traditional Erdös-Kac theorems require the simplifying assumption that as $n \to \infty$, we expect independently that $\omega(n), \Omega(n)$ are approximately equally likely to assume any values in some bounded $[1, M]$. This means we can treat the difference $\Omega(n) - \omega(n)$ as being approximately randomly distributed over some bounded range of its possible values. For a more rigorous treatment of this underlying principle see [4, 2, 15].

Corollary 7.5 (Summatory functions of the unsigned component sequences)**.** *We have that for large $x \geq 2$ that uniformly for $1 \leq k \leq \log\log x$*

$$\sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp 2\sqrt{2\pi} \cdot x \cdot (\log\log x)^{\frac{3}{2}} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

Proof. We have an integral formula involving the non-sign-weighted sequence that results by again applying ordinary Abel summation (and integrating by parts) in the form of

$$\begin{aligned}\sum_{n\leq x} \lambda_*(n)h(n) &= \left(\sum_{n\leq x}\lambda_*(n)\right)h(x) - \int_1^x \left(\sum_{n\leq t}\lambda_*(n)\right)h'(t)dt \qquad (28) \\ &\asymp \left\{\begin{array}{ll} u_t = L_*(t) & v_t' = h'(t)dt \\ u_t' = L_*'(t)dt & v_t = h(t)\end{array}\right\} \int_1^x \frac{d}{dt}\left[\sum_{n\leq t}\lambda_*(n)\right]h(t)dt.\end{aligned}$$

Let the signed left-hand-side summatory function in (28) for our function be defined by

$$\widehat{C}_{k,*}(x) := \sum_{\substack{n\leq x \\ \Omega(n)=k}} (-1)^{\omega(n)}C_{\Omega(n)}(n)$$

$$= \frac{x}{\log x} \cdot \frac{(\log\log x - \log\zeta(2))^{k-1}}{(k-1)!}\left[1 + O\left(\frac{1}{(\log\log x)^2}\right)\right]$$
$$= \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O\left(\frac{1}{(\log\log x)^2}\right)\right]$$

where the second equation follows from the proof of Theorem 7.2.

We handle transforming our previous results for the sum over the unsigned sequence $C_{\Omega(n)}(n)$ such that $\Omega(n) = k$. The argument is based on approximating the smooth summatory function of $\lambda_*(n) := (-1)^{\omega(n)}$ using the following uniform approximation of $\pi_k(x)$ for $1 \leq k \leq \log\log x$ as $x \to \infty$:

$$\pi_k(x) \asymp \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}(1 + o(1)).$$

In particular, we have that (*cf.* Table T.2 starting on page 47)

$$L_*(t) := \left|\sum_{n \leq t} (-1)^{\omega(n)}\right| = \left|\sum_{k=1}^{\log\log x} (-1)^k \pi_k(x)\right| \sim \frac{t}{\sqrt{2\pi}\sqrt{\log\log t}}, \text{ as } t \to \infty.$$

The derivative of this summatory function is given by

$$\frac{1}{L_*'(t)} \asymp -2\sqrt{2\pi}(\log t)(\log\log t)^{\frac{3}{2}}.$$

After applying the formula from (28), we deduce that the unsigned summatory function variant satisfies

$$\widehat{C}_{k,*}(x) = \int_1^x L_*'(t) C_{\Omega(t)}(t) dt \qquad \implies C_{\Omega(x)}(x) \asymp \frac{\widehat{C}_{k,*}'(x)}{L_*'(x)}$$
$$C_{\Omega(x)}(x) \asymp -2\sqrt{2\pi}(\log t)(\log\log t)^{\frac{3}{2}}\left[\frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 - \frac{1}{\log x}\right) + \frac{(\log\log x)^{k-2}}{(\log x)(k-2)!}\right] =: \widehat{C}_{k,**}(x).$$

So applying to the ordinary Abel summation formula, and integrating by parts, we obtain that the main term for this function is given by

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp \int \frac{d}{dx}\left[\widehat{C}_{k,**}(x)\right] dx$$
$$\asymp 2\sqrt{2\pi} \cdot x \cdot (\log\log x)^{\frac{3}{2}} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}. \qquad \square$$

Lemma 7.6. *We have that as* $x \to \infty$

$$\mathbb{E}\left[\sum_{n \leq x} C_{\Omega(n)}(n)\right] \asymp (\log x)(\log\log x).$$

Proof. We claim that

$$\sum_{n \leq x} C_{\Omega(n)}(n) = \sum_{k=1}^{\log_2(x)} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp \sum_{k=1}^{\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n). \tag{29}$$

To prove (29), it suffices to show that

$$\frac{\displaystyle\sum_{\log\log x < k \leq \log_2(x)} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n)}{\displaystyle\sum_{k=1}^{\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n)} = o(1), \text{ as } x \to \infty. \tag{30}$$

We first compute the absolute value of the following summatory function by applying Corollary 7.5:

$$\sum_{k=1}^{\log\log x} \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp \sum_{k=1}^{\log\log x} 2\sqrt{2\pi}\cdot x\cdot(\log\log x)^{\frac{3}{2}}\cdot\frac{(\log\log x)^{k-1}}{(k-1)!} \asymp x\cdot(\log x)(\log\log x). \tag{31}$$

We define the following component sums for large x and $0<\varepsilon<1$ so that $(\log\log x)^{\varepsilon\frac{\log\log x}{\log\log\log x}} = o(\log x)$:

$$S_{2,\varepsilon}(x) := \sum_{\log\log x<k\leq\log\log x} \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n).$$

Then

$$\sum_{k=1}^{\log_2(x)} \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \gg S_{2,\varepsilon}(x),$$

with equality as $\varepsilon\to 1$ so that the upper bound of summation tends to $\log x$. To show that (30) holds, observe that whenever $\Omega(n)=k$, we have that $C_{\Omega(n)}(n)\leq k!$. We can bound the sum defined above using Theorem 5.4 for large $x\to\infty$ as

$$\begin{aligned} S_{2,\varepsilon}(x) &\leq \sum_{\log\log x\leq k\leq\log\log x} \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \ll \sum_{k=\log\log x}^{(\log\log x)^{\varepsilon\frac{\log\log x}{\log\log\log x}}} \frac{\widehat{\pi}_k(x)}{x}\cdot k! \\ &\ll \sum_{k=\log\log x}^{(\log\log x)^{\varepsilon\frac{\log\log x}{\log\log\log x}}} (\log x)^{\frac{k}{\log\log x}-1-\frac{k}{\log\log x}(\log k-\log\log\log x)}\cdot\left(\frac{k}{e}\right)^k\sqrt{2\pi k} \\ &\ll \sum_{k=\log\log x}^{\varepsilon\frac{\log\log x}{\log\log\log x}} (\log x)^{k\frac{\log\log\log x}{\log\log x}-1}\sqrt{k} \ll \frac{1}{(\log x)}\times\int_{\log\log x}^{\varepsilon\frac{\log\log x}{\log\log\log x}} (\log\log x)^t\sqrt{t}\cdot dt \\ &\ll \frac{1}{(\log x)}\sqrt{\frac{\varepsilon\cdot\log\log x}{\log\log\log x}}(\log\log x)^{\frac{\varepsilon\cdot\log\log x}{\log\log\log x}} = o(x), \end{aligned}$$

where $\lim_{x\to\infty}(\log x)^{\frac{1}{\log\log x}} = e$. By (31) this form of the ratio in (30) clearly tends to zero. If we have a contribution from the terms $\widehat{\pi}_k(x)$ as $\varepsilon\to 1$, e.g., if x is a power of two, then $C_{\Omega(x)}(x)=1$ by the formula in (8), so that the contribution from this upper-most indexed term is negligible:

$$x=2^k \implies \Omega(x)=k \implies C_{\Omega(x)}(x)=\frac{(\Omega(x))!}{k!}=1.$$

The formula for the expectation claimed in the statement of this lemma above then follows from (31) by scaling by $\frac{1}{x}$ and dropping the asymptotically lesser error terms in the bound. □

Corollary 7.7 (Expectation formulas). *We have that as $n\to\infty$*

$$\mathbb{E}|g^{-1}(n)| \asymp \frac{3}{\pi^2}(\log x)^2(\log\log x).$$

Proof. We use the formula from Lemma 7.6 to find $\mathbb{E}[C_{\Omega(n)}(n)]$ up to a small bounded multiplicative constant factor as $n\to\infty$. This implies that for large x

$$\int\frac{\mathbb{E}[C_{\Omega(x)}(x)]}{x}dx = \frac{1}{2}(\log x)^2(\log\log x)-\frac{1}{4}(\log x)^2.$$

Therefore we find that

$$\mathbb{E}|g^{-1}(n)| = \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d<n}\frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O(1)$$
$$\sim \frac{3}{\pi^2}(\log x)^2(\log\log x). \qquad \square$$

Theorem 7.8. *Let the mean and variance analogs be denoted by*

$$\mu_x(C) := \log\log x + \hat{a} - \frac{3}{2}\cdot\log\log\log x, \qquad \text{and} \qquad \sigma_x(C) := \sqrt{\mu_x(C)},$$

where the absolute constant $\hat{a} := \log\left(\frac{1}{2\sqrt{2\pi}}\right) \approx -1.61209$*. Set* $Y > 0$ *and suppose that* $z \in [-Y, Y]$*. Then we have uniformly for all* $-Y \leq z \leq Y$ *that*

$$\frac{1}{x}\cdot\#\left\{2 \leq n \leq x : \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)} \leq z\right\} = \Phi(z) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

Proof. For large x and $n \leq x$, define the following auxiliary variables:

$$\alpha_n := \frac{C_{\Omega(n)}(n) - \mu_n(C)}{\sigma_n(C)}, \quad \beta_{n,x} := \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)}.$$

Let the corresponding densities (whose limiting distributions we must verify) be defined by the functions

$$\Phi_1(x, z) := \frac{1}{x}\cdot\#\{n \leq x : \alpha_n \leq z\},$$

and

$$\Phi_2(x, z) := \frac{1}{x}\cdot\#\{n \leq x : \beta_{n,x} \leq z\}.$$

We first argue that it suffices to consider the distribution of $\Phi_2(x, z)$ as $x \to \infty$ in place of $\Phi_1(x, z)$ to obtain our desired result statement. In particular, the difference of the two auxiliary variables is neglibible as $x \to \infty$ for n, x taken over the ranges that contribute the non-trivial weight to the main term of each density function. We have for $\sqrt{x} \leq n \leq x$ and $C_{\Omega(n)}(n) \leq 2\cdot\mu_x(C)$ that

$$|\alpha_n - \beta_{n,x}| \ll \frac{1}{\sigma_x(C)} \xrightarrow{x\to\infty} 0.$$

Then we can replace α_n by $\beta_{n,x}$ and estimate the limiting densities corresponding to these terms. The rest of our argument follows the method in the proof of the related theorem in [11, Thm. 7.21; §7.4] closely.

We use the formula proved in Corollary 7.5 to estimate the densities claimed within the ranges bounded by z as $x \to \infty$. Let $k \geq 1$ be a natural number defined by $k := t + \mu_x(C)$. We write the small parameter $\delta_{t,x} := \frac{t}{\mu_x(C)}$. When $|t| \leq \frac{1}{2}\mu_x(C)$, we have by Stirling's formula that

$$2\sqrt{2\pi}\cdot(\log\log x)^{\frac{3}{2}}\cdot\frac{(\log\log x)^{k-1}}{(k-1)!} \sim \frac{2\cdot e^{\hat{a}+t}(\log\log x)^{\mu_x(C)(1+\delta_{t,x})}}{\sigma_x(C)\cdot\mu_x(C)^{\mu_x(C)(1+\delta_{t,x})}(1+\delta_{t,x})^{\mu_x(C)(1+\delta_{t,x})+\frac{1}{2}}}$$
$$\sim \frac{e^t}{\sqrt{2\pi}\cdot\sigma_x(C)}(1+\delta_{t,x})^{-\left(\mu_x(C)(1+\delta_{t,x})+\frac{1}{2}\right)},$$

since $\frac{\mu_x(C)}{\log\log x} = 1 + o(1)$ as $x \to \infty$.

We have the uniform estimate $\log(1+\delta_{t,x}) = \delta_{t,x} - \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)$ whenever $|\delta_{t,x}| \leq \frac{1}{2}$. Then we can expand the factor involving $\delta_{t,x}$ in the previous equation as follows:

$$(1+\delta_{t,x})^{-\mu_x(C)(1+\delta_{t,x})-\frac{1}{2}} = \exp\left(\left(\frac{1}{2} + \mu_x(C)(1+\delta_{t,x})\right)\times\left(-\delta_{t,x} + \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)\right)\right)$$

$$= \exp\left(-t + \frac{t-t^2}{2\mu_x(C)} - \frac{t^2}{4\mu_x(C)^2} + O\left(\frac{|t|^3}{\mu_x(C)^2}\right)\right).$$

For both $|t| \leq \mu_x(C)^{1/2}$ and $\mu_x(C)^{1/2} < |t| \leq \mu_x(C)^{2/3}$, we see that

$$\frac{t}{\mu_x(C)} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Similarly, for $|t| \leq 1$ and $|t| > 1$, we see that both

$$\frac{t^2}{\mu_x(C)^2} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Let the error terms in (x, t) be denoted by

$$\widetilde{E}(x,t) := O\left(\frac{1}{\sigma_x(C)}\right) + O\left(\frac{|t|^3}{\mu_x(C)^2}\right).$$

Combining these estimates with the previous computations, we can deduce that uniformly for $|t| \leq \mu_x(C)^{2/3}$

$$2\sqrt{2\pi} \cdot (\log\log x)^{\frac{3}{2}} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \sim \frac{1}{\sqrt{2\pi} \cdot \sigma_x(C)} \cdot \exp\left(-\frac{t^2}{2\sigma_x(C)^2}\right) \times \left[1 + \widetilde{E}(x,t)\right].$$

By the argument in the proof of Lemma 7.6, we see that the contributions of these summatory functions for $k \leq \mu_x(C) - \mu_x(C)^{2/3}$ is negligible. We also require that $k \leq \log\log x$ as we have worked out in Theorem 7.2. So we sum over a corresponding range of

$$\mu_x(C) - \mu_x(C)^{2/3} \leq k \leq R_{z,x} \cdot \mu_x(C) + z \cdot \sigma_x(C),$$

for $R_{z,x} := 1 - \frac{z}{\sigma_x(C)}$ to approximate the stated normalized densities. Then finally as $x \to \infty$, the three terms that result (one main term, two error terms) can be considered to correspond to a Riemann sum for an associated integral. □

Corollary 7.9. *Let $Y > 0$. Then uniformly for all $-Y \leq y \leq Y$ we have that*

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq y\right\} = \Phi\left(\frac{\frac{\pi^2}{6}y - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

Proof. We claim that

$$|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2} C_{\Omega(n)}(n).$$

Recall from the introduction that the summatory function of the squarefree integers is given by

$$Q(x) := \sum_{n \leq x} \mu^2(n) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Then summing over (24) we obtain that

$$\begin{aligned} \frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| &= \frac{1}{x} \times \sum_{d \leq x} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &\sim \sum_{d \leq x} C_{\Omega(d)}(d) \left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dx}}\right)\right] \end{aligned}$$

$$= \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(x)}(x)] + \sum_{d<x} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O\left(\frac{1}{\sqrt{x}} \times \int_0^x t^{-1/2} dt\right).$$

Let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) := f(x) - f(x-1)$. Then from the proof of the initial corollary, we see that for large n

$$\begin{aligned}
|g^{-1}(n)| = \Delta_n(n \cdot \mathbb{E}|g^{-1}(n)|) &\sim \Delta_n\left(\sum_{d \leq n} \frac{6}{\pi^2} \cdot C_{\Omega(d)}(d) \cdot \frac{x}{d}\right) \\
&= \frac{6}{\pi^2}\left[C_{\Omega(n)}(n) + \sum_{d<n} C_{\Omega(d)}(d) \frac{n}{d} - \sum_{d<n} C_{\Omega(d)}(d) \frac{(n-1)}{d}\right] \\
&= \frac{6}{\pi^2} C_{\Omega(n)}(n) + \frac{6}{\pi^2} \mathbb{E}[C_{\Omega(n)}(n)] \\
&= \frac{6}{\pi^2} C_{\Omega(n)}(n) + o(1), \text{ as } n \to \infty,
\end{aligned}$$

where the last step is a consequence of Lemma 7.6. The result finally follows from Theorem 7.8. $\square$

8 Lower bounds for $M(x)$ along infinite subsequences

8.1 Establishing initial lower bounds on the summatory function $G^{-1}(x)$

Lemma 8.1 (Effective ranges of $|g^{-1}(n)|$ for large n)**.** *If x is sufficiently large and we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds:*

$$\mathbb{P}\left(|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq 0\right) = o(1) \tag{A}$$

$$\mathbb{P}\left(|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\mu_x(C)\right) = \frac{1}{2} + o(1). \tag{B}$$

Moreover, for any real $\delta > 0$ we have that

$$\mathbb{P}\left(|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\mu_x(C)^{1+\delta}\right) = 1 + o_\delta(1), \text{ as } x \rightarrow \infty. \tag{C}$$

Proof. Each of these results is a consequence of Corollary 7.9. Let the densities $\gamma_z(x)$ be defined for $z \in \mathbb{R}$ and large $x > e$ as follows:

$$\gamma_z(x) := \frac{1}{x} \cdot \#\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq z\}.$$

To prove (A), observe that for $z := 0$ we have that

$$\gamma_0(x) = \Phi\left(-\sigma_x(C)\right) + o(1), \text{ as } x \rightarrow \infty.$$

Then since $\sigma_x(C) \xrightarrow{x\rightarrow\infty} +\infty$, we have by an asymptotic approximation to the error function as

$$\begin{aligned} \Phi(z) &= \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right) \\ &= 1 - \frac{e^{-z^2/2}}{\sqrt{2\pi}}\left[z^{-1} - z^{-3} + 3z^{-5} - 15z^{-7} + \cdots\right], \text{ for } |z| \rightarrow \infty, \end{aligned}$$

that

$$\Phi\left(-\sigma_x(C)\right) \sim \frac{1}{\sigma_x(C)\exp(\mu_x(C))} = o(1).$$

To prove (B), observe that setting $z := \frac{6}{\pi^2}\mu_x(C)$ yields

$$\gamma_z(x) = \Phi(0) + o(1) = \frac{1}{2} + o(1), \text{ as } x \rightarrow \infty.$$

The point in (C), and transition from the implies range of values from (B) to (C), is more subtle. We require that $\mu_x(C)^{\frac{1}{2}+\delta} - \sigma_x(C) \xrightarrow{x\rightarrow\infty} +\infty$. Since this happens as $x \rightarrow \infty$ for any fixed $\delta > 0$, we have that for $z \equiv z(\delta) := \frac{6}{\pi^2}\mu_x(C)^{1+\delta}$

$$\begin{aligned} \gamma_{z(\delta)} &= \Phi\left(\frac{6}{\pi^2}\left(\mu_x(C)^{\frac{1}{2}+\delta} - \sigma_x(C)\right)\right) + o(1) \\ &= 1 - \Phi\left(-\frac{6}{\pi^2}\left(\mu_x(C)^{\frac{1}{2}+\delta} - \sigma_x(C)\right)\right) \\ &\sim 1 - \frac{\pi^2}{6} \cdot \frac{1}{\mu_x(C)^{\frac{1}{2}+\delta}} \cdot \exp\left(-\frac{36}{\pi^4}(\log\log x)^{1+2\delta}\right) \\ &= 1 + o_\delta(1), \text{ as } x \rightarrow \infty. \end{aligned}$$

□

Remark 8.2 (Interpretations for constructing bounds on $G^{-1}(x)$)**.** Note that we technically cannot allow $\delta := 0$ to obtain the stated probablility of almost one in Lemma 8.1, but for any increasingly small $\delta > 0$, this property does hold when x is sufficiently large. A consequence of (A) and (C) is that for any fixed $\delta > 0$ and $n \in \mathcal{S}_1(\delta)$ taken within a set of asymptotic density one

$$\mathbb{E}|g^{-1}(n)| \leq |g^{-1}(n)| \leq \mathbb{E}|g^{-1}(n)| + \frac{6}{\pi^2}\mu_x(C)^{\frac{1}{2}+\delta}. \tag{32}$$

Thus when we integrate over a sufficiently spaced set of disjoint consecutive intervals, we can assume that a lower bound on the contribution of $|g^{-1}(n)|$ is given by its average order, and an upper bound is given by the upper limit above for some fixed $\delta > 0$. In particular, observe that

$$\frac{\pi^2}{6 \cdot \sigma_x(C)} \times \int_{-\infty}^{\infty} z \cdot \Phi'\left(\frac{\frac{\pi^2}{6}x - \mu_x(C)}{\sigma_x(C)}\right) dz = \frac{6}{\pi^2} \cdot \sigma_x(C) = o\left(\mathbb{E}|g^{-1}(x)|\right).$$

We can interpret the previous calculation as implying that for n on a large interval, the contribution from $|g^{-1}(n)|$ can be approximated above and below accurately as in the bounds from (32).

Theorem 8.3. *For all sufficiently large integers x, we have that*

$$|G^{-1}(x)| \gg (\log x)\sqrt{\log\log x}, \text{ as } x \to \infty.$$

Proof. We need a couple of observations to sum $G^{-1}(x)$ in absolute value and bound it from below. We will use a lower bound approximating the summatory function of $\lambda(n)$ for $n \leq t$ and t large by summing over the uniform asymptotic bounds proved in Theorem 2.7. To be careful about the expected sign of this summatory function, we first appeal to the original approximation to the functions $\widehat{\pi}_k(x)$ given by Theorem 2.6. As noted in [11, §7.4], the function $\mathcal{G}(z)$ from Theorem 2.6 satisfies

$$\mathcal{G}\left(\frac{k-1}{\log\log x}\right) = 1 + O(1), k \leq \log\log x,$$

so that uniformly for $1 \leq k \leq \log\log x$ we can write

$$\widehat{\pi}_k(x) \asymp \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O\left(\frac{1}{\log\log x}\right)\right].$$

By Corollary 5.7, the following summatory function represents the asymptotic main term in the summation $L(x) := \sum_{n \leq x} \lambda(n)$ as $x \to \infty$:

$$\widehat{L}_2(x) = \sum_{k=1}^{\log\log x} (-1)^k \widehat{\pi}_k(x) = -\frac{x}{(\log x)^2} \cdot \Gamma(\log\log x, -\log\log x) \sim \frac{(-1)^{\lceil \log\log x \rceil} \cdot x}{\sqrt{2\pi}\sqrt{\log\log x}}$$

So we expect the sign of our summatory function approximation to be approximately given by $(-1)^{\lceil \log\log x \rceil}$ for large x. We now find a lower bound on the unsigned magnitude of these summatory functions. In particular, using Theorem 2.7, we have that $\widehat{\pi}_k(x) \gg \widehat{\pi}_k^{(\ell)}(x)$ where (see Table T.2 on page 47)

$$\widehat{\pi}_k^{(\ell)}(x) := \frac{x^{3/4}}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O\left(\frac{k}{(\log\log x)^3}\right)\right].$$

So we define our lower bound by

$$\widehat{L}_0(x) := \left|\sum_{k=1}^{\log\log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x)\right| \asymp \frac{x^{\frac{3}{4}}}{\sqrt{\log\log x}},$$

where the derivative of this summatory function satisfies

$$\widehat{L}_0'(x) \asymp \frac{1}{x^{1/4} \cdot \sqrt{\log\log x}}.$$

We observe that we can break the interval $t \in (e, x]$ into disjoint subintervals according to which we have the expected sign contributions from the summatory function $\widehat{L}_0(x)$. Namely, we expect that for $1 \leq k \leq \frac{\log\log x}{2}$ we expect that

$$\begin{aligned}
\operatorname{sgn}\left(\widehat{L}_0(x)\right) &= +1 \text{ on } \left[e^{e^{2k}}, e^{e^{2k+1}}\right) \\
\operatorname{sgn}\left(\widehat{L}_0(x)\right) &= -1 \text{ on } \left[e^{e^{2k+1}}, e^{e^{2k+2}}\right).
\end{aligned}$$

Moreover, since the derivative $\widehat{L}_0'(x)$ is monotone decreasing in x, we can construct our lower bounds by placing the input points to this function in the Abel summation formula from (28) over these signed intervals at te extremal endpoints depending on the leading sign terms. As we have argued in Lemma 8.1 and observed in the preceeding remark, we have the bounds in (32) on which we can similarly construct the lower bound on $|G^{-1}(x)|$ based on the sign term of the subinterval and the extremal points within the interval.

For any $\delta > 0$ we have the following bounds on the summatory function:

$$\begin{aligned}
|G^{-1}(x)| &\gg \left|\int_2^x \widehat{L}_0'(t)|g^{-1}(t)|dt\right| \\
&\gg \left|\sum_{k=1}^{\frac{\log\log x}{2}} \widehat{L}_0'\left(e^{e^{2k}}\right)\left[\mathbb{E}\left|g^{-1}\left(e^{e^{2k-1}}\right)\right| - \mathbb{E}\left|g^{-1}\left(e^{e^{2k+1}}\right)\right| - \frac{6}{\pi^2}\log\log\left(e^{e^{2k+1}}\right)^{1+\delta}\right]\right|.
\end{aligned}$$

Now we will separate the two inner component integrals to see that one is asymptotically dominant, and hence forms the main term of the lower bound we seek. First, we compute that for any $p > \frac{1}{(1+2\delta)}$

$$\begin{aligned}
I_1(x) &:= \int_e^{\frac{\log\log x}{2}} \widehat{L}_0'\left(e^{e^{2t}}\right)(2t+1)^{1+\delta} = dt \\
&\gg \left(t^{\frac{1}{2}+\delta}\right)\Bigg|_{t=(\log\log x)^p} \times \int_{(\log\log x)^p}^{\frac{\log\log x}{2}} \exp\left(-\frac{e^{2t}}{4}\right) dt \\
&\gg (\log\log x)^{\frac{1}{2}} \times \operatorname{Ei}\left(-\frac{\log x}{4}\right) \\
&\gg (\log x)(\log\log x)^{\frac{1}{2}}.
\end{aligned}$$

Next, we compute the contribution from the remaining integral terms for the difference of expectations as follows:

$$\begin{aligned}
I_2(x) &:= \int_e^{\frac{\log\log x}{2}} \widehat{L}_0'\left(e^{e^{2t}}\right)\left[\mathbb{E}\left|g^{-1}\left(e^{e^{2t-1}}\right)\right| - \mathbb{E}\left|g^{-1}\left(e^{e^{2t+1}}\right)\right|\right] dt \\
&\gg \int_e^{\frac{\log\log x}{2}} \exp\left(-\frac{e^{2t}}{4} + 4t\right) dt \gg \frac{(\log x)}{x^{\frac{1}{4}}}.
\end{aligned}$$

Combining the difference of these two estimates and then taking the main term, we clearly obtain that stated result follows. □

Remark 8.4 (On the approximation to the signedness of $L(x)$).

8.2 Proof of the unboundedness of the scaled Mertens function

Proposition 8.5. *For all sufficiently large x, we have that*

$$M(x) = G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{\sqrt{x}} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right]. \tag{33}$$

Proof. We know by applying Corollary 2.3 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)\left(\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right) + 1\right) \\ &= G^{-1}(x) + \sum_{k=1}^{x/2} g^{-1}(k)\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right) \qquad (34) \\ &= G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{x/2-1} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1}\right\rfloor\right)\right] \qquad (35) \end{aligned}$$

where the upper bound on the sum is truncated by the fact that $\pi(1) = 0$. We see that

$$\frac{x}{k} - \frac{x}{k+1} = \frac{x}{k(k+1)} \sim \frac{x}{k^2},$$

so that $\frac{x}{k^2} \geq 1 \implies k \leq \sqrt{x}$. Thus we can re-write the latter sum to obtain

$$M(x) = G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{\sqrt{x}} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].$$

We will require more assumptions and information about the behavior of the summatory functions, $G^{-1}(x)$, before we can further bound and simplify this expression for $M(x)$. □

Lemma 8.6. *For sufficiently large x, $k \in [1, \sqrt{x}]$ and integers $m \geq 0$, we have that*

$$\frac{x}{k \cdot \log^m\left(\frac{x}{k}\right)} - \frac{x}{(k+1)\cdot \log^m\left(\frac{x}{k+1}\right)} \asymp \frac{x}{(\log x)^m \cdot k(k+1)}, \tag{A}$$

and

$$\sum_{k=1}^{\sqrt{x}} \frac{x}{k(k+1)} = \sum_{k=1}^{\sqrt{x}} \frac{x}{k^2} + O(1). \tag{B}$$

Proof. The proof of (A) is obvious since $\log(x/k_0) \asymp \log(x)$ for all $k_0 \in [1, \sqrt{x}+1]$ when x is large. In particular, for $k_0 \in [1, \sqrt{x}+1]$ we have that

$$\frac{1}{2}\log(x)(1+o(1)) \leq \log(x/k_0) \leq \log(x).$$

To prove (B), notice that

$$\frac{x}{k(k+1)} - \frac{x}{k^2} = -\frac{x}{k^2(k+1)}.$$

Then we see that

$$\left|\int_e^{\sqrt{x}} \frac{x}{t^2(t+1)}dt\right| \leq \left|\int_e^{\sqrt{x}} \frac{x}{t^3}dt\right| \asymp \left|\frac{x}{2(\sqrt{x})^2)}\right| = \frac{1}{2}.$$ □

We finally address the main conclusion of our arguments given so far with the following proof:

Proof of Theorem 2.8. Define the infinite increasing subsequence, $\{x_{0,y}\}_{y \geq Y_0}$, by $x_{0,y} := e^{2e^{2y+1}}$ for the sequence indices y starting at some sufficiently large finite integer $Y_0 \gg 1$. We can verify that for sufficiently large $y \to \infty$, this infinitely tending subsequence is well defined as $x_{0,y+1} > x_{0,y}$ whenever $y \geq Y_0$. Given a fixed large infinitely tending y, we have some (at least one) point $\widehat{x}_0 \in \left[\sqrt{x}, \frac{x}{2}\right]$ defined such that $|G^{-1}(t)|$ is minimal and non-vanishing on the interval $\mathbb{X}_y := (\sqrt{x_{0,y}}, \sqrt{x_{0,y+1}}]$ in the form of

$$\left|G^{-1}(\widehat{x}_0)\right| := \min_{\substack{\sqrt{x_{0,y}} < t \leq \sqrt{x_{0,y+1}} \\ G^{-1}(t) \neq 0}} |G^{-1}(t)|.$$

Let the shorthand notation $|G^{-1}_{\min}(x_y)| := |G^{-1}(\widehat{x_0})|$. In the last step, we observe that $G^{-1}(x) = 0$ for x on a set of asymptotic density *at least* bounded below by $\frac{1}{2}$, so that our claim is accurate as the integrand lower bound on this interval does not trivially vanish at large y. This happens since the sequence $g^{-1}(n)$ is non-zero for all $n \geq 1$, so that if we do encounter a zero of the summatory function at x, we find a non-zero function value at $x+1$.

We need to bound the prime counting function differences in the formula given by Proposition 8.4 in tandem with enforcing minimal values of the absolute value of $G^{-1}(k)$ for $k \in \mathbb{X}_y$. We will require the following known bounds on the prime counting function due to Rosser and Schoenfeld [17, Thm. 1] for large $x \gg 59$:

$$\frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right). \tag{36}$$

Let the component function $U_M(y)$ be defined for all large y as

$$U_M(y) := -\sum_{k=1}^{\sqrt{\hat{x}_{0,y+1}}} |G^{-1}(k)| \left[\pi\left(\frac{\hat{x}_{0,y+1}}{k}\right) - \pi\left(\frac{\hat{x}_{0,y+1}}{k+1}\right)\right].$$

Combined with Lemma 8.5, these estimates on $\pi(x)$ lead to the following approximations that hold on the increasing sequences taken within the subintervals defined by $\widehat{x}_0$:

$$\begin{aligned}
U_M(y) &\gg -\sum_{k=1}^{\sqrt{\hat{x}_{0,y+1}}} |G^{-1}(k)| \left[\frac{\hat{x}_{0,y+1}}{k \cdot \log\left(\frac{\hat{x}_{0,y+1}}{k}\right)} + \frac{\hat{x}_{0,y+1}}{2k \cdot \log^2\left(\frac{\hat{x}_{0,y+1}}{k}\right)} - \frac{\hat{x}_{0,y+1}}{(k+1) \cdot \log\left(\frac{\hat{x}_{0,y+1}}{k+1}\right)} - \frac{3\hat{x}_{0,y+1}}{2(k+1) \cdot \log^2\left(\frac{\hat{x}_{0,y+1}}{k+1}\right)}\right] \\
&\gg -\sum_{k=\sqrt{\hat{x}_{0,y}}}^{\sqrt{\hat{x}_{0,y+1}}} \frac{\hat{x}_{0,y+1} \cdot |G^{-1}_{\min}(\hat{x_0})|}{k^2} \left[\frac{1}{\log(\hat{x}_{0,y+1})} + \frac{1}{2\log^2(\hat{x}_{0,y+1})}\right] \\
&\gg -\hat{x}_{0,y+1} |G^{-1}_{\min}(\hat{x_0})| \left(\frac{1}{\log(\hat{x}_{0,y+1})} + \frac{1}{2\log^2(\hat{x}_{0,y+1})}\right) \times \int_{\sqrt{\hat{x}_{0,y}}}^{\sqrt{\hat{x}_{0,y+1}}} \frac{dt}{t^2} \\
&\gg \sqrt{\hat{x}_{0,y+1}} \times \frac{|G^{-1}_{\min}(\hat{x_0})|}{\log(\hat{x}_{0,y+1})} \times \left(1 + \frac{1}{\log(\hat{x}_{0,y+1})}\right).
\end{aligned}$$

Now by applying the lower bounds proved in Theorem 8.3, we can see that in fact the following is true:

$$U_M(y) \gg \sqrt{\hat{x}_{0,y+1}} \times \frac{|G^{-1}_{\min}(\hat{x_0})|}{\log(\hat{x}_{0,y+1})} + o(1), \text{ as } y \to \infty.$$

Now we need to assemble this bound on the summation term in the formula for $M(x)$ from Proposition 8.4 with the leading terms involving the summatory function G^{-1}. In particular, we need to argue that we can effectively drop these leading terms to obtain a lower bound. Then we succeed by applying Theorem 8.3 since the remaining terms given by the function $U_M(y)$ are infinitely tending as $y \to \infty$.

Namely, we clearly see from Theorem 8.3 and the proposition that

$$\frac{|M(\hat{x}_{0,y+1})|}{\sqrt{\hat{x}_{0,y+1}}} \gg \frac{1}{\sqrt{\hat{x}_{0,y+1}}} \times \left|\left|G^{-1}(\hat{x}_{0,y+1}) + G^{-1}\left(\frac{\hat{x}_{0,y+1}}{2}\right)\right| + U_M(y)\right|$$

$$\gg \frac{1}{\sqrt{\hat{x}_{0,y+1}}} \times |U_M(y)|$$
$$\gg \log\log\left(\sqrt{\hat{x}_{0,y+1}}\right)^{\frac{1}{2}}. \tag{37}$$

There is a small, but nonetheless insightful point in question to explain about a technicality in stating (37). Namely, we are not asserting that $|M(x)|/\sqrt{x}$ grows unbounded along the precise subsequence of $x \mapsto \hat{x}_{0,y+1}$ itself as $y \to \infty$. Rather, we are asserting that the unboundedness of this function can be witnessed along some subsequence whose points are taken within a large interval window of $x \in \left(\sqrt{\hat{x}_{0,y}}, \sqrt{\hat{x}_{0,y+1}}\right]$ as $y \to \infty$. We choose to state the lower bound given on the right-hand-side of (37) using the nicely formulated monotone lower bound on $|G^{-1}(x)|$ we proved in Theorem 8.3 with $\hat{x}_0 \geq \sqrt{\hat{x}_{0,y}}$ for all $y \geq Y_0$. $\square$

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T.1 Table: The Dirichlet inverse function $g^{-1}(n)$ and the distribution of its summatory function

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1^1	Y	N	1	0	1.0000000	1.000000	0.000000	1	1	0
2	2^1	Y	Y	-2	0	1.0000000	0.500000	0.500000	-1	1	-2
3	3^1	Y	Y	-2	0	1.0000000	0.333333	0.666667	-3	1	-4
4	2^2	N	Y	2	0	1.5000000	0.500000	0.500000	-1	3	-4
5	5^1	Y	Y	-2	0	1.0000000	0.400000	0.600000	-3	3	-6
6	$2^1 3^1$	Y	N	5	0	1.0000000	0.500000	0.500000	2	8	-6
7	7^1	Y	Y	-2	0	1.0000000	0.428571	0.571429	0	8	-8
8	2^3	N	Y	-2	0	2.0000000	0.375000	0.625000	-2	8	-10
9	3^2	N	Y	2	0	1.5000000	0.444444	0.555556	0	10	-10
10	$2^1 5^1$	Y	N	5	0	1.0000000	0.500000	0.500000	5	15	-10
11	11^1	Y	Y	-2	0	1.0000000	0.454545	0.545455	3	15	-12
12	$2^2 3^1$	N	N	-7	2	1.2857143	0.416667	0.583333	-4	15	-19
13	13^1	Y	Y	-2	0	1.0000000	0.384615	0.615385	-6	15	-21
14	$2^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-1	20	-21
15	$3^1 5^1$	Y	N	5	0	1.0000000	0.466667	0.533333	4	25	-21
16	2^4	N	Y	2	0	2.5000000	0.500000	0.500000	6	27	-21
17	17^1	Y	Y	-2	0	1.0000000	0.470588	0.529412	4	27	-23
18	$2^1 3^2$	N	N	-7	2	1.2857143	0.444444	0.555556	-3	27	-30
19	19^1	Y	Y	-2	0	1.0000000	0.421053	0.578947	-5	27	-32
20	$2^2 5^1$	N	N	-7	2	1.2857143	0.400000	0.600000	-12	27	-39
21	$3^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-7	32	-39
22	$2^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-2	37	-39
23	23^1	Y	Y	-2	0	1.0000000	0.434783	0.565217	-4	37	-41
24	$2^3 3^1$	N	N	9	4	1.5555556	0.458333	0.541667	5	46	-41
25	5^2	N	Y	2	0	1.5000000	0.480000	0.520000	7	48	-41
26	$2^1 13^1$	Y	N	5	0	1.0000000	0.500000	0.500000	12	53	-41
27	3^3	N	Y	-2	0	2.0000000	0.481481	0.518519	10	53	-43
28	$2^2 7^1$	N	N	-7	2	1.2857143	0.464286	0.535714	3	53	-50
29	29^1	Y	Y	-2	0	1.0000000	0.448276	0.551724	1	53	-52
30	$2^1 3^1 5^1$	Y	N	-16	0	1.0000000	0.433333	0.566667	-15	53	-68
31	31^1	Y	Y	-2	0	1.0000000	0.419355	0.580645	-17	53	-70
32	2^5	N	Y	-2	0	3.0000000	0.406250	0.593750	-19	53	-72
33	$3^1 11^1$	Y	N	5	0	1.0000000	0.424242	0.575758	-14	58	-72
34	$2^1 17^1$	Y	N	5	0	1.0000000	0.441176	0.558824	-9	63	-72
35	$5^1 7^1$	Y	N	5	0	1.0000000	0.457143	0.542857	-4	68	-72
36	$2^2 3^2$	N	N	14	9	1.3571429	0.472222	0.527778	10	82	-72
37	37^1	Y	Y	-2	0	1.0000000	0.459459	0.540541	8	82	-74
38	$2^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	13	87	-74
39	$3^1 13^1$	Y	N	5	0	1.0000000	0.487179	0.512821	18	92	-74
40	$2^3 5^1$	N	N	9	4	1.5555556	0.500000	0.500000	27	101	-74
41	41^1	Y	Y	-2	0	1.0000000	0.487805	0.512195	25	101	-76
42	$2^1 3^1 7^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	9	101	-92
43	43^1	Y	Y	-2	0	1.0000000	0.465116	0.534884	7	101	-94
44	$2^2 11^1$	N	N	-7	2	1.2857143	0.454545	0.545455	0	101	-101
45	$3^2 5^1$	N	N	-7	2	1.2857143	0.444444	0.555556	-7	101	-108
46	$2^1 23^1$	Y	N	5	0	1.0000000	0.456522	0.543478	-2	106	-108
47	47^1	Y	Y	-2	0	1.0000000	0.446809	0.553191	-4	106	-110
48	$2^4 3^1$	N	N	-11	6	1.8181818	0.437500	0.562500	-15	106	-121

Table T.1: ***Computations with*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for*** $\mathbf{1 \leq n \leq 500}$**.**

- ► *The column labeled* `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled* `Sqfree` *and* `PPower`*, respectively, list inclusion of n in the sets of squarefree integers and the prime powers.*
- ► *The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.*
- ► *The last several columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{n} \cdot \#\{n \leq x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geq 1$.*

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
49	7^2	N	Y	2	0	1.5000000	0.448980	0.551020	-13	108	-121
50	$2^1 5^2$	N	N	-7	2	1.2857143	0.440000	0.560000	-20	108	-128
51	$3^1 17^1$	Y	N	5	0	1.0000000	0.450980	0.549020	-15	113	-128
52	$2^2 13^1$	N	N	-7	2	1.2857143	0.442308	0.557692	-22	113	-135
53	53^1	Y	Y	-2	0	1.0000000	0.433962	0.566038	-24	113	-137
54	$2^1 3^3$	N	N	9	4	1.5555556	0.444444	0.555556	-15	122	-137
55	$5^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-10	127	-137
56	$2^3 7^1$	N	N	9	4	1.5555556	0.464286	0.535714	-1	136	-137
57	$3^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	4	141	-137
58	$2^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	9	146	-137
59	59^1	Y	Y	-2	0	1.0000000	0.474576	0.525424	7	146	-139
60	$2^2 3^1 5^1$	N	N	30	14	1.1666667	0.483333	0.516667	37	176	-139
61	61^1	Y	Y	-2	0	1.0000000	0.475410	0.524590	35	176	-141
62	$2^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	40	181	-141
63	$3^2 7^1$	N	N	-7	2	1.2857143	0.476190	0.523810	33	181	-148
64	2^6	N	Y	2	0	3.5000000	0.484375	0.515625	35	183	-148
65	$5^1 13^1$	Y	N	5	0	1.0000000	0.492308	0.507692	40	188	-148
66	$2^1 3^1 11^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	24	188	-164
67	67^1	Y	Y	-2	0	1.0000000	0.477612	0.522388	22	188	-166
68	$2^2 17^1$	N	N	-7	2	1.2857143	0.470588	0.529412	15	188	-173
69	$3^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	20	193	-173
70	$2^1 5^1 7^1$	Y	N	-16	0	1.0000000	0.471429	0.528571	4	193	-189
71	71^1	Y	Y	-2	0	1.0000000	0.464789	0.535211	2	193	-191
72	$2^3 3^2$	N	N	-23	18	1.4782609	0.458333	0.541667	-21	193	-214
73	73^1	Y	Y	-2	0	1.0000000	0.452055	0.547945	-23	193	-216
74	$2^1 37^1$	Y	N	5	0	1.0000000	0.459459	0.540541	-18	198	-216
75	$3^1 5^2$	N	N	-7	2	1.2857143	0.453333	0.546667	-25	198	-223
76	$2^2 19^1$	N	N	-7	2	1.2857143	0.447368	0.552632	-32	198	-230
77	$7^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-27	203	-230
78	$2^1 3^1 13^1$	Y	N	-16	0	1.0000000	0.448718	0.551282	-43	203	-246
79	79^1	Y	Y	-2	0	1.0000000	0.443038	0.556962	-45	203	-248
80	$2^4 5^1$	N	N	-11	6	1.8181818	0.437500	0.562500	-56	203	-259
81	3^4	N	Y	2	0	2.5000000	0.444444	0.555556	-54	205	-259
82	$2^1 41^1$	Y	N	5	0	1.0000000	0.451220	0.548780	-49	210	-259
83	83^1	Y	Y	-2	0	1.0000000	0.445783	0.554217	-51	210	-261
84	$2^2 3^1 7^1$	N	N	30	14	1.1666667	0.452381	0.547619	-21	240	-261
85	$5^1 17^1$	Y	N	5	0	1.0000000	0.458824	0.541176	-16	245	-261
86	$2^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	-11	250	-261
87	$3^1 29^1$	Y	N	5	0	1.0000000	0.471264	0.528736	-6	255	-261
88	$2^3 11^1$	N	N	9	4	1.5555556	0.477273	0.522727	3	264	-261
89	89^1	Y	Y	-2	0	1.0000000	0.471910	0.528090	1	264	-263
90	$2^1 3^2 5^1$	N	N	30	14	1.1666667	0.477778	0.522222	31	294	-263
91	$7^1 13^1$	Y	N	5	0	1.0000000	0.483516	0.516484	36	299	-263
92	$2^2 23^1$	N	N	-7	2	1.2857143	0.478261	0.521739	29	299	-270
93	$3^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	34	304	-270
94	$2^1 47^1$	Y	N	5	0	1.0000000	0.489362	0.510638	39	309	-270
95	$5^1 19^1$	Y	N	5	0	1.0000000	0.494737	0.505263	44	314	-270
96	$2^5 3^1$	N	N	13	8	2.0769231	0.500000	0.500000	57	327	-270
97	97^1	Y	Y	-2	0	1.0000000	0.494845	0.505155	55	327	-272
98	$2^1 7^2$	N	N	-7	2	1.2857143	0.489796	0.510204	48	327	-279
99	$3^2 11^1$	N	N	-7	2	1.2857143	0.484848	0.515152	41	327	-286
100	$2^2 5^2$	N	N	14	9	1.3571429	0.490000	0.510000	55	341	-286
101	101^1	Y	Y	-2	0	1.0000000	0.485149	0.514851	53	341	-288
102	$2^1 3^1 17^1$	Y	N	-16	0	1.0000000	0.480392	0.519608	37	341	-304
103	103^1	Y	Y	-2	0	1.0000000	0.475728	0.524272	35	341	-306
104	$2^3 13^1$	N	N	9	4	1.5555556	0.480769	0.519231	44	350	-306
105	$3^1 5^1 7^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	28	350	-322
106	$2^1 53^1$	Y	N	5	0	1.0000000	0.481132	0.518868	33	355	-322
107	107^1	Y	Y	-2	0	1.0000000	0.476636	0.523364	31	355	-324
108	$2^2 3^3$	N	N	-23	18	1.4782609	0.472222	0.527778	8	355	-347
109	109^1	Y	Y	-2	0	1.0000000	0.467890	0.532110	6	355	-349
110	$2^1 5^1 11^1$	Y	N	-16	0	1.0000000	0.463636	0.536364	-10	355	-365
111	$3^1 37^1$	Y	N	5	0	1.0000000	0.468468	0.531532	-5	360	-365
112	$2^4 7^1$	N	N	-11	6	1.8181818	0.464286	0.535714	-16	360	-376
113	113^1	Y	Y	-2	0	1.0000000	0.460177	0.539823	-18	360	-378
114	$2^1 3^1 19^1$	Y	N	-16	0	1.0000000	0.456140	0.543860	-34	360	-394
115	$5^1 23^1$	Y	N	5	0	1.0000000	0.460870	0.539130	-29	365	-394
116	$2^2 29^1$	N	N	-7	2	1.2857143	0.456897	0.543103	-36	365	-401
117	$3^2 13^1$	N	N	-7	2	1.2857143	0.452991	0.547009	-43	365	-408
118	$2^1 59^1$	Y	N	5	0	1.0000000	0.457627	0.542373	-38	370	-408
119	$7^1 17^1$	Y	N	5	0	1.0000000	0.462185	0.537815	-33	375	-408
120	$2^3 3^1 5^1$	N	N	-48	32	1.3333333	0.458333	0.541667	-81	375	-456
121	11^2	N	Y	2	0	1.5000000	0.462810	0.537190	-79	377	-456
122	$2^1 61^1$	Y	N	5	0	1.0000000	0.467213	0.532787	-74	382	-456
123	$3^1 41^1$	Y	N	5	0	1.0000000	0.471545	0.528455	-69	387	-456
124	$2^2 31^1$	N	N	-7	2	1.2857143	0.467742	0.532258	-76	387	-463

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
125	5^3	N	Y	-2	0	2.0000000	0.464000	0.536000	-78	387	-465
126	$2^1 3^2 7^1$	N	N	30	14	1.1666667	0.468254	0.531746	-48	417	-465
127	127^1	Y	Y	-2	0	1.0000000	0.464567	0.535433	-50	417	-467
128	2^7	N	Y	-2	0	4.0000000	0.460938	0.539062	-52	417	-469
129	$3^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	-47	422	-469
130	$2^1 5^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-63	422	-485
131	131^1	Y	Y	-2	0	1.0000000	0.458015	0.541985	-65	422	-487
132	$2^2 3^1 11^1$	N	N	30	14	1.1666667	0.462121	0.537879	-35	452	-487
133	$7^1 19^1$	Y	N	5	0	1.0000000	0.466165	0.533835	-30	457	-487
134	$2^1 67^1$	Y	N	5	0	1.0000000	0.470149	0.529851	-25	462	-487
135	$3^3 5^1$	N	N	9	4	1.5555556	0.474074	0.525926	-16	471	-487
136	$2^3 17^1$	N	N	9	4	1.5555556	0.477941	0.522059	-7	480	-487
137	137^1	Y	Y	-2	0	1.0000000	0.474453	0.525547	-9	480	-489
138	$2^1 3^1 23^1$	Y	N	-16	0	1.0000000	0.471014	0.528986	-25	480	-505
139	139^1	Y	Y	-2	0	1.0000000	0.467626	0.532374	-27	480	-507
140	$2^2 5^1 7^1$	N	N	30	14	1.1666667	0.471429	0.528571	3	510	-507
141	$3^1 47^1$	Y	N	5	0	1.0000000	0.475177	0.524823	8	515	-507
142	$2^1 71^1$	Y	N	5	0	1.0000000	0.478873	0.521127	13	520	-507
143	$11^1 13^1$	Y	N	5	0	1.0000000	0.482517	0.517483	18	525	-507
144	$2^4 3^2$	N	N	34	29	1.6176471	0.486111	0.513889	52	559	-507
145	$5^1 29^1$	Y	N	5	0	1.0000000	0.489655	0.510345	57	564	-507
146	$2^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	62	569	-507
147	$3^1 7^2$	N	N	-7	2	1.2857143	0.489796	0.510204	55	569	-514
148	$2^2 37^1$	N	N	-7	2	1.2857143	0.486486	0.513514	48	569	-521
149	149^1	Y	Y	-2	0	1.0000000	0.483221	0.516779	46	569	-523
150	$2^1 3^1 5^2$	N	N	30	14	1.1666667	0.486667	0.513333	76	599	-523
151	151^1	Y	Y	-2	0	1.0000000	0.483444	0.516556	74	599	-525
152	$2^3 19^1$	N	N	9	4	1.5555556	0.486842	0.513158	83	608	-525
153	$3^2 17^1$	N	N	-7	2	1.2857143	0.483660	0.516340	76	608	-532
154	$2^1 7^1 11^1$	Y	N	-16	0	1.0000000	0.480519	0.519481	60	608	-548
155	$5^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	65	613	-548
156	$2^2 3^1 13^1$	N	N	30	14	1.1666667	0.487179	0.512821	95	643	-548
157	157^1	Y	Y	-2	0	1.0000000	0.484076	0.515924	93	643	-550
158	$2^1 79^1$	Y	N	5	0	1.0000000	0.487342	0.512658	98	648	-550
159	$3^1 53^1$	Y	N	5	0	1.0000000	0.490566	0.509434	103	653	-550
160	$2^5 5^1$	N	N	13	8	2.0769231	0.493750	0.506250	116	666	-550
161	$7^1 23^1$	Y	N	5	0	1.0000000	0.496894	0.503106	121	671	-550
162	$2^1 3^4$	N	N	-11	6	1.8181818	0.493827	0.506173	110	671	-561
163	163^1	Y	Y	-2	0	1.0000000	0.490798	0.509202	108	671	-563
164	$2^2 41^1$	N	N	-7	2	1.2857143	0.487805	0.512195	101	671	-570
165	$3^1 5^1 11^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	85	671	-586
166	$2^1 83^1$	Y	N	5	0	1.0000000	0.487952	0.512048	90	676	-586
167	167^1	Y	Y	-2	0	1.0000000	0.485030	0.514970	88	676	-588
168	$2^3 3^1 7^1$	N	N	-48	32	1.3333333	0.482143	0.517857	40	676	-636
169	13^2	N	Y	2	0	1.5000000	0.485207	0.514793	42	678	-636
170	$2^1 5^1 17^1$	Y	N	-16	0	1.0000000	0.482353	0.517647	26	678	-652
171	$3^2 19^1$	N	N	-7	2	1.2857143	0.479532	0.520468	19	678	-659
172	$2^2 43^1$	N	N	-7	2	1.2857143	0.476744	0.523256	12	678	-666
173	173^1	Y	Y	-2	0	1.0000000	0.473988	0.526012	10	678	-668
174	$2^1 3^1 29^1$	Y	N	-16	0	1.0000000	0.471264	0.528736	-6	678	-684
175	$5^2 7^1$	N	N	-7	2	1.2857143	0.468571	0.531429	-13	678	-691
176	$2^4 11^1$	N	N	-11	6	1.8181818	0.465909	0.534091	-24	678	-702
177	$3^1 59^1$	Y	N	5	0	1.0000000	0.468927	0.531073	-19	683	-702
178	$2^1 89^1$	Y	N	5	0	1.0000000	0.471910	0.528090	-14	688	-702
179	179^1	Y	Y	-2	0	1.0000000	0.469274	0.530726	-16	688	-704
180	$2^2 3^2 5^1$	N	N	-74	58	1.2162162	0.466667	0.533333	-90	688	-778
181	181^1	Y	Y	-2	0	1.0000000	0.464088	0.535912	-92	688	-780
182	$2^1 7^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-108	688	-796
183	$3^1 61^1$	Y	N	5	0	1.0000000	0.464481	0.535519	-103	693	-796
184	$2^3 23^1$	N	N	9	4	1.5555556	0.467391	0.532609	-94	702	-796
185	$5^1 37^1$	Y	N	5	0	1.0000000	0.470270	0.529730	-89	707	-796
186	$2^1 3^1 31^1$	Y	N	-16	0	1.0000000	0.467742	0.532258	-105	707	-812
187	$11^1 17^1$	Y	N	5	0	1.0000000	0.470588	0.529412	-100	712	-812
188	$2^2 47^1$	N	N	-7	2	1.2857143	0.468085	0.531915	-107	712	-819
189	$3^3 7^1$	N	N	9	4	1.5555556	0.470899	0.529101	-98	721	-819
190	$2^1 5^1 19^1$	Y	N	-16	0	1.0000000	0.468421	0.531579	-114	721	-835
191	191^1	Y	Y	-2	0	1.0000000	0.465969	0.534031	-116	721	-837
192	$2^6 3^1$	N	N	-15	10	2.3333333	0.463542	0.536458	-131	721	-852
193	193^1	Y	Y	-2	0	1.0000000	0.461140	0.538860	-133	721	-854
194	$2^1 97^1$	Y	N	5	0	1.0000000	0.463918	0.536082	-128	726	-854
195	$3^1 5^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-144	726	-870
196	$2^2 7^2$	N	N	14	9	1.3571429	0.464286	0.535714	-130	740	-870
197	197^1	Y	Y	-2	0	1.0000000	0.461929	0.538071	-132	740	-872
198	$2^1 3^2 11^1$	N	N	30	14	1.1666667	0.464646	0.535354	-102	770	-872
199	199^1	Y	Y	-2	0	1.0000000	0.462312	0.537688	-104	770	-874
200	$2^3 5^2$	N	N	-23	18	1.4782609	0.460000	0.540000	-127	770	-897

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
201	$3^1 67^1$	Y	N	5	0	1.0000000	0.462687	0.537313	−122	775	−897
202	$2^1 101^1$	Y	N	5	0	1.0000000	0.465347	0.534653	−117	780	−897
203	$7^1 29^1$	Y	N	5	0	1.0000000	0.467980	0.532020	−112	785	−897
204	$2^2 3^1 17^1$	N	N	30	14	1.1666667	0.470588	0.529412	−82	815	−897
205	$5^1 41^1$	Y	N	5	0	1.0000000	0.473171	0.526829	−77	820	−897
206	$2^1 103^1$	Y	N	5	0	1.0000000	0.475728	0.524272	−72	825	−897
207	$3^2 23^1$	N	N	−7	2	1.2857143	0.473430	0.526570	−79	825	−904
208	$2^4 13^1$	N	N	−11	6	1.8181818	0.471154	0.528846	−90	825	−915
209	$11^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	−85	830	−915
210	$2^1 3^1 5^1 7^1$	Y	N	65	0	1.0000000	0.476190	0.523810	−20	895	−915
211	211^1	Y	Y	−2	0	1.0000000	0.473934	0.526066	−22	895	−917
212	$2^2 53^1$	N	N	−7	2	1.2857143	0.471698	0.528302	−29	895	−924
213	$3^1 71^1$	Y	N	5	0	1.0000000	0.474178	0.525822	−24	900	−924
214	$2^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−19	905	−924
215	$5^1 43^1$	Y	N	5	0	1.0000000	0.479070	0.520930	−14	910	−924
216	$2^3 3^3$	N	N	46	41	1.5000000	0.481481	0.518519	32	956	−924
217	$7^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	37	961	−924
218	$2^1 109^1$	Y	N	5	0	1.0000000	0.486239	0.513761	42	966	−924
219	$3^1 73^1$	Y	N	5	0	1.0000000	0.488584	0.511416	47	971	−924
220	$2^2 5^1 11^1$	N	N	30	14	1.1666667	0.490909	0.509091	77	1001	−924
221	$13^1 17^1$	Y	N	5	0	1.0000000	0.493213	0.506787	82	1006	−924
222	$2^1 3^1 37^1$	Y	N	−16	0	1.0000000	0.490991	0.509009	66	1006	−940
223	223^1	Y	Y	−2	0	1.0000000	0.488789	0.511211	64	1006	−942
224	$2^5 7^1$	N	N	13	8	2.0769231	0.491071	0.508929	77	1019	−942
225	$3^2 5^2$	N	N	14	9	1.3571429	0.493333	0.506667	91	1033	−942
226	$2^1 113^1$	Y	N	5	0	1.0000000	0.495575	0.504425	96	1038	−942
227	227^1	Y	Y	−2	0	1.0000000	0.493392	0.506608	94	1038	−944
228	$2^2 3^1 19^1$	N	N	30	14	1.1666667	0.495614	0.504386	124	1068	−944
229	229^1	Y	Y	−2	0	1.0000000	0.493450	0.506550	122	1068	−946
230	$2^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.491304	0.508696	106	1068	−962
231	$3^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.489177	0.510823	90	1068	−978
232	$2^3 29^1$	N	N	9	4	1.5555556	0.491379	0.508621	99	1077	−978
233	233^1	Y	Y	−2	0	1.0000000	0.489270	0.510730	97	1077	−980
234	$2^1 3^2 13^1$	N	N	30	14	1.1666667	0.491453	0.508547	127	1107	−980
235	$5^1 47^1$	Y	N	5	0	1.0000000	0.493617	0.506383	132	1112	−980
236	$2^2 59^1$	N	N	−7	2	1.2857143	0.491525	0.508475	125	1112	−987
237	$3^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	130	1117	−987
238	$2^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.491597	0.508403	114	1117	−1003
239	239^1	Y	Y	−2	0	1.0000000	0.489540	0.510460	112	1117	−1005
240	$2^4 3^1 5^1$	N	N	70	54	1.5000000	0.491667	0.508333	182	1187	−1005
241	241^1	Y	Y	−2	0	1.0000000	0.489627	0.510373	180	1187	−1007
242	$2^1 11^2$	N	N	−7	2	1.2857143	0.487603	0.512397	173	1187	−1014
243	3^5	N	Y	−2	0	3.0000000	0.485597	0.514403	171	1187	−1016
244	$2^2 61^1$	N	N	−7	2	1.2857143	0.483607	0.516393	164	1187	−1023
245	$5^1 7^2$	N	N	−7	2	1.2857143	0.481633	0.518367	157	1187	−1030
246	$2^1 3^1 41^1$	Y	N	−16	0	1.0000000	0.479675	0.520325	141	1187	−1046
247	$13^1 19^1$	Y	N	5	0	1.0000000	0.481781	0.518219	146	1192	−1046
248	$2^3 31^1$	N	N	9	4	1.5555556	0.483871	0.516129	155	1201	−1046
249	$3^1 83^1$	Y	N	5	0	1.0000000	0.485944	0.514056	160	1206	−1046
250	$2^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	169	1215	−1046
251	251^1	Y	Y	−2	0	1.0000000	0.486056	0.513944	167	1215	−1048
252	$2^2 3^2 7^1$	N	N	−74	58	1.2162162	0.484127	0.515873	93	1215	−1122
253	$11^1 23^1$	Y	N	5	0	1.0000000	0.486166	0.513834	98	1220	−1122
254	$2^1 127^1$	Y	N	5	0	1.0000000	0.488189	0.511811	103	1225	−1122
255	$3^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.486275	0.513725	87	1225	−1138
256	2^8	N	Y	2	0	4.5000000	0.488281	0.511719	89	1227	−1138
257	257^1	Y	Y	−2	0	1.0000000	0.486381	0.513619	87	1227	−1140
258	$2^1 3^1 43^1$	Y	N	−16	0	1.0000000	0.484496	0.515504	71	1227	−1156
259	$7^1 37^1$	Y	N	5	0	1.0000000	0.486486	0.513514	76	1232	−1156
260	$2^2 5^1 13^1$	N	N	30	14	1.1666667	0.488462	0.511538	106	1262	−1156
261	$3^2 29^1$	N	N	−7	2	1.2857143	0.486590	0.513410	99	1262	−1163
262	$2^1 131^1$	Y	N	5	0	1.0000000	0.488550	0.511450	104	1267	−1163
263	263^1	Y	Y	−2	0	1.0000000	0.486692	0.513308	102	1267	−1165
264	$2^3 3^1 11^1$	N	N	−48	32	1.3333333	0.484848	0.515152	54	1267	−1213
265	$5^1 53^1$	Y	N	5	0	1.0000000	0.486792	0.513208	59	1272	−1213
266	$2^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.484962	0.515038	43	1272	−1229
267	$3^1 89^1$	Y	N	5	0	1.0000000	0.486891	0.513109	48	1277	−1229
268	$2^2 67^1$	N	N	−7	2	1.2857143	0.485075	0.514925	41	1277	−1236
269	269^1	Y	Y	−2	0	1.0000000	0.483271	0.516729	39	1277	−1238
270	$2^1 3^3 5^1$	N	N	−48	32	1.3333333	0.481481	0.518519	−9	1277	−1286
271	271^1	Y	Y	−2	0	1.0000000	0.479705	0.520295	−11	1277	−1288
272	$2^4 17^1$	N	N	−11	6	1.8181818	0.477941	0.522059	−22	1277	−1299
273	$3^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−38	1277	−1315
274	$2^1 137^1$	Y	N	5	0	1.0000000	0.478102	0.521898	−33	1282	−1315
275	$5^2 11^1$	N	N	−7	2	1.2857143	0.476364	0.523636	−40	1282	−1322
276	$2^2 3^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−10	1312	−1322
277	277^1	Y	Y	−2	0	1.0000000	0.476534	0.523466	−12	1312	−1324

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
278	$2^1 139^1$	Y	N	5	0	1.0000000	0.478417	0.521583	−7	1317	−1324
279	$3^2 31^1$	N	N	−7	2	1.2857143	0.476703	0.523297	−14	1317	−1331
280	$2^3 5^1 7^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−62	1317	−1379
281	281^1	Y	Y	−2	0	1.0000000	0.473310	0.526690	−64	1317	−1381
282	$2^1 3^1 47^1$	Y	N	−16	0	1.0000000	0.471631	0.528369	−80	1317	−1397
283	283^1	Y	Y	−2	0	1.0000000	0.469965	0.530035	−82	1317	−1399
284	$2^2 71^1$	N	N	−7	2	1.2857143	0.468310	0.531690	−89	1317	−1406
285	$3^1 5^1 19^1$	Y	N	−16	0	1.0000000	0.466667	0.533333	−105	1317	−1422
286	$2^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.465035	0.534965	−121	1317	−1438
287	$7^1 41^1$	Y	N	5	0	1.0000000	0.466899	0.533101	−116	1322	−1438
288	$2^5 3^2$	N	N	−47	42	1.7659574	0.465278	0.534722	−163	1322	−1485
289	17^2	N	Y	2	0	1.5000000	0.467128	0.532872	−161	1324	−1485
290	$2^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.465517	0.534483	−177	1324	−1501
291	$3^1 97^1$	Y	N	5	0	1.0000000	0.467354	0.532646	−172	1329	−1501
292	$2^2 73^1$	N	N	−7	2	1.2857143	0.465753	0.534247	−179	1329	−1508
293	293^1	Y	Y	−2	0	1.0000000	0.464164	0.535836	−181	1329	−1510
294	$2^1 3^1 7^2$	N	N	30	14	1.1666667	0.465986	0.534014	−151	1359	−1510
295	$5^1 59^1$	Y	N	5	0	1.0000000	0.467797	0.532203	−146	1364	−1510
296	$2^3 37^1$	N	N	9	4	1.5555556	0.469595	0.530405	−137	1373	−1510
297	$3^3 11^1$	N	N	9	4	1.5555556	0.471380	0.528620	−128	1382	−1510
298	$2^1 149^1$	Y	N	5	0	1.0000000	0.473154	0.526846	−123	1387	−1510
299	$13^1 23^1$	Y	N	5	0	1.0000000	0.474916	0.525084	−118	1392	−1510
300	$2^2 3^1 5^2$	N	N	−74	58	1.2162162	0.473333	0.526667	−192	1392	−1584
301	$7^1 43^1$	Y	N	5	0	1.0000000	0.475083	0.524917	−187	1397	−1584
302	$2^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−182	1402	−1584
303	$3^1 101^1$	Y	N	5	0	1.0000000	0.478548	0.521452	−177	1407	−1584
304	$2^4 19^1$	N	N	−11	6	1.8181818	0.476974	0.523026	−188	1407	−1595
305	$5^1 61^1$	Y	N	5	0	1.0000000	0.478689	0.521311	−183	1412	−1595
306	$2^1 3^2 17^1$	N	N	30	14	1.1666667	0.480392	0.519608	−153	1442	−1595
307	307^1	Y	Y	−2	0	1.0000000	0.478827	0.521173	−155	1442	−1597
308	$2^2 7^1 11^1$	N	N	30	14	1.1666667	0.480519	0.519481	−125	1472	−1597
309	$3^1 103^1$	Y	N	5	0	1.0000000	0.482201	0.517799	−120	1477	−1597
310	$2^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.480645	0.519355	−136	1477	−1613
311	311^1	Y	Y	−2	0	1.0000000	0.479100	0.520900	−138	1477	−1615
312	$2^3 3^1 13^1$	N	N	−48	32	1.3333333	0.477564	0.522436	−186	1477	−1663
313	313^1	Y	Y	−2	0	1.0000000	0.476038	0.523962	−188	1477	−1665
314	$2^1 157^1$	Y	N	5	0	1.0000000	0.477707	0.522293	−183	1482	−1665
315	$3^2 5^1 7^1$	N	N	30	14	1.1666667	0.479365	0.520635	−153	1512	−1665
316	$2^2 79^1$	N	N	−7	2	1.2857143	0.477848	0.522152	−160	1512	−1672
317	317^1	Y	Y	−2	0	1.0000000	0.476341	0.523659	−162	1512	−1674
318	$2^1 3^1 53^1$	Y	N	−16	0	1.0000000	0.474843	0.525157	−178	1512	−1690
319	$11^1 29^1$	Y	N	5	0	1.0000000	0.476489	0.523511	−173	1517	−1690
320	$2^6 5^1$	N	N	−15	10	2.3333333	0.475000	0.525000	−188	1517	−1705
321	$3^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−183	1522	−1705
322	$2^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.475155	0.524845	−199	1522	−1721
323	$17^1 19^1$	Y	N	5	0	1.0000000	0.476780	0.523220	−194	1527	−1721
324	$2^2 3^4$	N	N	34	29	1.6176471	0.478395	0.521605	−160	1561	−1721
325	$5^2 13^1$	N	N	−7	2	1.2857143	0.476923	0.523077	−167	1561	−1728
326	$2^1 163^1$	Y	N	5	0	1.0000000	0.478528	0.521472	−162	1566	−1728
327	$3^1 109^1$	Y	N	5	0	1.0000000	0.480122	0.519878	−157	1571	−1728
328	$2^3 41^1$	N	N	9	4	1.5555556	0.481707	0.518293	−148	1580	−1728
329	$7^1 47^1$	Y	N	5	0	1.0000000	0.483283	0.516717	−143	1585	−1728
330	$2^1 3^1 5^1 11^1$	Y	N	65	0	1.0000000	0.484848	0.515152	−78	1650	−1728
331	331^1	Y	Y	−2	0	1.0000000	0.483384	0.516616	−80	1650	−1730
332	$2^2 83^1$	N	N	−7	2	1.2857143	0.481928	0.518072	−87	1650	−1737
333	$3^2 37^1$	N	N	−7	2	1.2857143	0.480480	0.519520	−94	1650	−1744
334	$2^1 167^1$	Y	N	5	0	1.0000000	0.482036	0.517964	−89	1655	−1744
335	$5^1 67^1$	Y	N	5	0	1.0000000	0.483582	0.516418	−84	1660	−1744
336	$2^4 3^1 7^1$	N	N	70	54	1.5000000	0.485119	0.514881	−14	1730	−1744
337	337^1	Y	Y	−2	0	1.0000000	0.483680	0.516320	−16	1730	−1746
338	$2^1 13^2$	N	N	−7	2	1.2857143	0.482249	0.517751	−23	1730	−1753
339	$3^1 113^1$	Y	N	5	0	1.0000000	0.483776	0.516224	−18	1735	−1753
340	$2^2 5^1 17^1$	N	N	30	14	1.1666667	0.485294	0.514706	12	1765	−1753
341	$11^1 31^1$	Y	N	5	0	1.0000000	0.486804	0.513196	17	1770	−1753
342	$2^1 3^2 19^1$	N	N	30	14	1.1666667	0.488304	0.511696	47	1800	−1753
343	7^3	N	Y	−2	0	2.0000000	0.486880	0.513120	45	1800	−1755
344	$2^3 43^1$	N	N	9	4	1.5555556	0.488372	0.511628	54	1809	−1755
345	$3^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.486957	0.513043	38	1809	−1771
346	$2^1 173^1$	Y	N	5	0	1.0000000	0.488439	0.511561	43	1814	−1771
347	347^1	Y	Y	−2	0	1.0000000	0.487032	0.512968	41	1814	−1773
348	$2^2 3^1 29^1$	N	N	30	14	1.1666667	0.488506	0.511494	71	1844	−1773
349	349^1	Y	Y	−2	0	1.0000000	0.487106	0.512894	69	1844	−1775
350	$2^1 5^2 7^1$	N	N	30	14	1.1666667	0.488571	0.511429	99	1874	−1775

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
351	$3^3 13^1$	N	N	9	4	1.5555556	0.490028	0.509972	108	1883	−1775
352	$2^5 11^1$	N	N	13	8	2.0769231	0.491477	0.508523	121	1896	−1775
353	353^1	Y	Y	−2	0	1.0000000	0.490085	0.509915	119	1896	−1777
354	$2^1 3^1 59^1$	Y	N	−16	0	1.0000000	0.488701	0.511299	103	1896	−1793
355	$5^1 71^1$	Y	N	5	0	1.0000000	0.490141	0.509859	108	1901	−1793
356	$2^2 89^1$	N	N	−7	2	1.2857143	0.488764	0.511236	101	1901	−1800
357	$3^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.487395	0.512605	85	1901	−1816
358	$2^1 179^1$	Y	N	5	0	1.0000000	0.488827	0.511173	90	1906	−1816
359	359^1	Y	Y	−2	0	1.0000000	0.487465	0.512535	88	1906	−1818
360	$2^3 3^2 5^1$	N	N	145	129	1.3034483	0.488889	0.511111	233	2051	−1818
361	19^2	N	Y	2	0	1.5000000	0.490305	0.509695	235	2053	−1818
362	$2^1 181^1$	Y	N	5	0	1.0000000	0.491713	0.508287	240	2058	−1818
363	$3^1 11^2$	N	N	−7	2	1.2857143	0.490358	0.509642	233	2058	−1825
364	$2^2 7^1 13^1$	N	N	30	14	1.1666667	0.491758	0.508242	263	2088	−1825
365	$5^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	268	2093	−1825
366	$2^1 3^1 61^1$	Y	N	−16	0	1.0000000	0.491803	0.508197	252	2093	−1841
367	367^1	Y	Y	−2	0	1.0000000	0.490463	0.509537	250	2093	−1843
368	$2^4 23^1$	N	N	−11	6	1.8181818	0.489130	0.510870	239	2093	−1854
369	$3^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	232	2093	−1861
370	$2^1 5^1 37^1$	Y	N	−16	0	1.0000000	0.486486	0.513514	216	2093	−1877
371	$7^1 53^1$	Y	N	5	0	1.0000000	0.487871	0.512129	221	2098	−1877
372	$2^2 3^1 31^1$	N	N	30	14	1.1666667	0.489247	0.510753	251	2128	−1877
373	373^1	Y	Y	−2	0	1.0000000	0.487936	0.512064	249	2128	−1879
374	$2^1 11^1 17^1$	Y	N	−16	0	1.0000000	0.486631	0.513369	233	2128	−1895
375	$3^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	242	2137	−1895
376	$2^3 47^1$	N	N	9	4	1.5555556	0.489362	0.510638	251	2146	−1895
377	$13^1 29^1$	Y	N	5	0	1.0000000	0.490716	0.509284	256	2151	−1895
378	$2^1 3^3 7^1$	N	N	−48	32	1.3333333	0.489418	0.510582	208	2151	−1943
379	379^1	Y	Y	−2	0	1.0000000	0.488127	0.511873	206	2151	−1945
380	$2^2 5^1 19^1$	N	N	30	14	1.1666667	0.489474	0.510526	236	2181	−1945
381	$3^1 127^1$	Y	N	5	0	1.0000000	0.490814	0.509186	241	2186	−1945
382	$2^1 191^1$	Y	N	5	0	1.0000000	0.492147	0.507853	246	2191	−1945
383	383^1	Y	Y	−2	0	1.0000000	0.490862	0.509138	244	2191	−1947
384	$2^7 3^1$	N	N	17	12	2.5882353	0.492188	0.507812	261	2208	−1947
385	$5^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.490909	0.509091	245	2208	−1963
386	$2^1 193^1$	Y	N	5	0	1.0000000	0.492228	0.507772	250	2213	−1963
387	$3^2 43^1$	N	N	−7	2	1.2857143	0.490956	0.509044	243	2213	−1970
388	$2^2 97^1$	N	N	−7	2	1.2857143	0.489691	0.510309	236	2213	−1977
389	389^1	Y	Y	−2	0	1.0000000	0.488432	0.511568	234	2213	−1979
390	$2^1 3^1 5^1 13^1$	Y	N	65	0	1.0000000	0.489744	0.510256	299	2278	−1979
391	$17^1 23^1$	Y	N	5	0	1.0000000	0.491049	0.508951	304	2283	−1979
392	$2^3 7^2$	N	N	−23	18	1.4782609	0.489796	0.510204	281	2283	−2002
393	$3^1 131^1$	Y	N	5	0	1.0000000	0.491094	0.508906	286	2288	−2002
394	$2^1 197^1$	Y	N	5	0	1.0000000	0.492386	0.507614	291	2293	−2002
395	$5^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	296	2298	−2002
396	$2^2 3^2 11^1$	N	N	−74	58	1.2162162	0.492424	0.507576	222	2298	−2076
397	397^1	Y	Y	−2	0	1.0000000	0.491184	0.508816	220	2298	−2078
398	$2^1 199^1$	Y	N	5	0	1.0000000	0.492462	0.507538	225	2303	−2078
399	$3^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.491228	0.508772	209	2303	−2094
400	$2^4 5^2$	N	N	34	29	1.6176471	0.492500	0.507500	243	2337	−2094
401	401^1	Y	Y	−2	0	1.0000000	0.491272	0.508728	241	2337	−2096
402	$2^1 3^1 67^1$	Y	N	−16	0	1.0000000	0.490050	0.509950	225	2337	−2112
403	$13^1 31^1$	Y	N	5	0	1.0000000	0.491315	0.508685	230	2342	−2112
404	$2^2 101^1$	N	N	−7	2	1.2857143	0.490099	0.509901	223	2342	−2119
405	$3^4 5^1$	N	N	−11	6	1.8181818	0.488889	0.511111	212	2342	−2130
406	$2^1 7^1 29^1$	Y	N	−16	0	1.0000000	0.487685	0.512315	196	2342	−2146
407	$11^1 37^1$	Y	N	5	0	1.0000000	0.488943	0.511057	201	2347	−2146
408	$2^3 3^1 17^1$	N	N	−48	32	1.3333333	0.487745	0.512255	153	2347	−2194
409	409^1	Y	Y	−2	0	1.0000000	0.486553	0.513447	151	2347	−2196
410	$2^1 5^1 41^1$	Y	N	−16	0	1.0000000	0.485366	0.514634	135	2347	−2212
411	$3^1 137^1$	Y	N	5	0	1.0000000	0.486618	0.513382	140	2352	−2212
412	$2^2 103^1$	N	N	−7	2	1.2857143	0.485437	0.514563	133	2352	−2219
413	$7^1 59^1$	Y	N	5	0	1.0000000	0.486683	0.513317	138	2357	−2219
414	$2^1 3^2 23^1$	N	N	30	14	1.1666667	0.487923	0.512077	168	2387	−2219
415	$5^1 83^1$	Y	N	5	0	1.0000000	0.489157	0.510843	173	2392	−2219
416	$2^5 13^1$	N	N	13	8	2.0769231	0.490385	0.509615	186	2405	−2219
417	$3^1 139^1$	Y	N	5	0	1.0000000	0.491607	0.508393	191	2410	−2219
418	$2^1 11^1 19^1$	Y	N	−16	0	1.0000000	0.490431	0.509569	175	2410	−2235
419	419^1	Y	Y	−2	0	1.0000000	0.489260	0.510740	173	2410	−2237
420	$2^2 3^1 5^1 7^1$	N	N	−155	90	1.1032258	0.488095	0.511905	18	2410	−2392
421	421^1	Y	Y	−2	0	1.0000000	0.486936	0.513064	16	2410	−2394
422	$2^1 211^1$	Y	N	5	0	1.0000000	0.488152	0.511848	21	2415	−2394
423	$3^2 47^1$	N	N	−7	2	1.2857143	0.486998	0.513002	14	2415	−2401
424	$2^3 53^1$	N	N	9	4	1.5555556	0.488208	0.511792	23	2424	−2401
425	$5^2 17^1$	N	N	−7	2	1.2857143	0.487059	0.512941	16	2424	−2408

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
426	$2^1 3^1 71^1$	Y	N	-16	0	1.0000000	0.485915	0.514085	0	2424	-2424
427	$7^1 61^1$	Y	N	5	0	1.0000000	0.487119	0.512881	5	2429	-2424
428	$2^2 107^1$	N	N	-7	2	1.2857143	0.485981	0.514019	-2	2429	-2431
429	$3^1 11^1 13^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	-18	2429	-2447
430	$2^1 5^1 43^1$	Y	N	-16	0	1.0000000	0.483721	0.516279	-34	2429	-2463
431	431^1	Y	Y	-2	0	1.0000000	0.482599	0.517401	-36	2429	-2465
432	$2^4 3^3$	N	N	-80	75	1.5625000	0.481481	0.518519	-116	2429	-2545
433	433^1	Y	Y	-2	0	1.0000000	0.480370	0.519630	-118	2429	-2547
434	$2^1 7^1 31^1$	Y	N	-16	0	1.0000000	0.479263	0.520737	-134	2429	-2563
435	$3^1 5^1 29^1$	Y	N	-16	0	1.0000000	0.478161	0.521839	-150	2429	-2579
436	$2^2 109^1$	N	N	-7	2	1.2857143	0.477064	0.522936	-157	2429	-2586
437	$19^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	-152	2434	-2586
438	$2^1 3^1 73^1$	Y	N	-16	0	1.0000000	0.477169	0.522831	-168	2434	-2602
439	439^1	Y	Y	-2	0	1.0000000	0.476082	0.523918	-170	2434	-2604
440	$2^3 5^1 11^1$	N	N	-48	32	1.3333333	0.475000	0.525000	-218	2434	-2652
441	$3^2 7^2$	N	N	14	9	1.3571429	0.476190	0.523810	-204	2448	-2652
442	$2^1 13^1 17^1$	Y	N	-16	0	1.0000000	0.475113	0.524887	-220	2448	-2668
443	443^1	Y	Y	-2	0	1.0000000	0.474041	0.525959	-222	2448	-2670
444	$2^2 3^1 37^1$	N	N	30	14	1.1666667	0.475225	0.524775	-192	2478	-2670
445	$5^1 89^1$	Y	N	5	0	1.0000000	0.476404	0.523596	-187	2483	-2670
446	$2^1 223^1$	Y	N	5	0	1.0000000	0.477578	0.522422	-182	2488	-2670
447	$3^1 149^1$	Y	N	5	0	1.0000000	0.478747	0.521253	-177	2493	-2670
448	$2^6 7^1$	N	N	-15	10	2.3333333	0.477679	0.522321	-192	2493	-2685
449	449^1	Y	Y	-2	0	1.0000000	0.476615	0.523385	-194	2493	-2687
450	$2^1 3^2 5^2$	N	N	-74	58	1.2162162	0.475556	0.524444	-268	2493	-2761
451	$11^1 41^1$	Y	N	5	0	1.0000000	0.476718	0.523282	-263	2498	-2761
452	$2^2 113^1$	N	N	-7	2	1.2857143	0.475664	0.524336	-270	2498	-2768
453	$3^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	-265	2503	-2768
454	$2^1 227^1$	Y	N	5	0	1.0000000	0.477974	0.522026	-260	2508	-2768
455	$5^1 7^1 13^1$	Y	N	-16	0	1.0000000	0.476923	0.523077	-276	2508	-2784
456	$2^3 3^1 19^1$	N	N	-48	32	1.3333333	0.475877	0.524123	-324	2508	-2832
457	457^1	Y	Y	-2	0	1.0000000	0.474836	0.525164	-326	2508	-2834
458	$2^1 229^1$	Y	N	5	0	1.0000000	0.475983	0.524017	-321	2513	-2834
459	$3^3 17^1$	N	N	9	4	1.5555556	0.477124	0.522876	-312	2522	-2834
460	$2^2 5^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	-282	2552	-2834
461	461^1	Y	Y	-2	0	1.0000000	0.477223	0.522777	-284	2552	-2836
462	$2^1 3^1 7^1 11^1$	Y	N	65	0	1.0000000	0.478355	0.521645	-219	2617	-2836
463	463^1	Y	Y	-2	0	1.0000000	0.477322	0.522678	-221	2617	-2838
464	$2^4 29^1$	N	N	-11	6	1.8181818	0.476293	0.523707	-232	2617	-2849
465	$3^1 5^1 31^1$	Y	N	-16	0	1.0000000	0.475269	0.524731	-248	2617	-2865
466	$2^1 233^1$	Y	N	5	0	1.0000000	0.476395	0.523605	-243	2622	-2865
467	467^1	Y	Y	-2	0	1.0000000	0.475375	0.524625	-245	2622	-2867
468	$2^2 3^2 13^1$	N	N	-74	58	1.2162162	0.474359	0.525641	-319	2622	-2941
469	$7^1 67^1$	Y	N	5	0	1.0000000	0.475480	0.524520	-314	2627	-2941
470	$2^1 5^1 47^1$	Y	N	-16	0	1.0000000	0.474468	0.525532	-330	2627	-2957
471	$3^1 157^1$	Y	N	5	0	1.0000000	0.475584	0.524416	-325	2632	-2957
472	$2^3 59^1$	N	N	9	4	1.5555556	0.476695	0.523305	-316	2641	-2957
473	$11^1 43^1$	Y	N	5	0	1.0000000	0.477801	0.522199	-311	2646	-2957
474	$2^1 3^1 79^1$	Y	N	-16	0	1.0000000	0.476793	0.523207	-327	2646	-2973
475	$5^2 19^1$	N	N	-7	2	1.2857143	0.475789	0.524211	-334	2646	-2980
476	$2^2 7^1 17^1$	N	N	30	14	1.1666667	0.476891	0.523109	-304	2676	-2980
477	$3^2 53^1$	N	N	-7	2	1.2857143	0.475891	0.524109	-311	2676	-2987
478	$2^1 239^1$	Y	N	5	0	1.0000000	0.476987	0.523013	-306	2681	-2987
479	479^1	Y	Y	-2	0	1.0000000	0.475992	0.524008	-308	2681	-2989
480	$2^5 3^1 5^1$	N	N	-96	80	1.6666667	0.475000	0.525000	-404	2681	-3085
481	$13^1 37^1$	Y	N	5	0	1.0000000	0.476091	0.523909	-399	2686	-3085
482	$2^1 241^1$	Y	N	5	0	1.0000000	0.477178	0.522822	-394	2691	-3085
483	$3^1 7^1 23^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	-410	2691	-3101
484	$2^2 11^2$	N	N	14	9	1.3571429	0.477273	0.522727	-396	2705	-3101
485	$5^1 97^1$	Y	N	5	0	1.0000000	0.478351	0.521649	-391	2710	-3101
486	$2^1 3^5$	N	N	13	8	2.0769231	0.479424	0.520576	-378	2723	-3101
487	487^1	Y	Y	-2	0	1.0000000	0.478439	0.521561	-380	2723	-3103
488	$2^3 61^1$	N	N	9	4	1.5555556	0.479508	0.520492	-371	2732	-3103
489	$3^1 163^1$	Y	N	5	0	1.0000000	0.480573	0.519427	-366	2737	-3103
490	$2^1 5^1 7^2$	N	N	30	14	1.1666667	0.481633	0.518367	-336	2767	-3103
491	491^1	Y	Y	-2	0	1.0000000	0.480652	0.519348	-338	2767	-3105
492	$2^2 3^1 41^1$	N	N	30	14	1.1666667	0.481707	0.518293	-308	2797	-3105
493	$17^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	-303	2802	-3105
494	$2^1 13^1 19^1$	Y	N	-16	0	1.0000000	0.481781	0.518219	-319	2802	-3121
495	$3^2 5^1 11^1$	N	N	30	14	1.1666667	0.482828	0.517172	-289	2832	-3121
496	$2^4 31^1$	N	N	-11	6	1.8181818	0.481855	0.518145	-300	2832	-3132
497	$7^1 71^1$	Y	N	5	0	1.0000000	0.482897	0.517103	-295	2837	-3132
498	$2^1 3^1 83^1$	Y	N	-16	0	1.0000000	0.481928	0.518072	-311	2837	-3148
499	499^1	Y	Y	-2	0	1.0000000	0.480962	0.519038	-313	2837	-3150
500	$2^2 5^3$	N	N	-23	18	1.4782609	0.480000	0.520000	-336	2837	-3173

T.2 Table: Approximations of the summatory functions of $\lambda(n)$ and $\lambda_*(n)$

x	$L(x)$	$R_\pm(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$	x	$L(x)$	$R_\pm(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$

Table T.2: ***Approximations to the summatory functions of*** $\lambda(n)$ ***and*** $\lambda_*(n)$**.**

- *We define the exact summatory functions over these sequences by* $L(x) := \sum_{n \leq x} \lambda(n)$ *and* $L_*(n) := \sum_{n \leq x} \lambda_*(n)$.
- *Let the expected sign ratio function be defined by* $R_\pm(x) := \frac{\operatorname{sgn}(L(x))}{(-1)^{\lfloor \log\log x \rfloor}}$.
- *We compare the ratios of the following two functions with* $L(x)$*:* $L_{\approx,1}(x) := \sum_{k=1}^{\log\log x} \frac{x}{\log x} \cdot \frac{(-\log\log x)^{k-1}}{(k-1)!}$ *and* $L_{\approx,2}(x) := \frac{x^{1/4}}{\sqrt{\log x}\sqrt{\log\log x}}$.
- *Finally, we compare the approximations (very accurate) to* $L_*(x)$ *by the summatory function* $\sum_{k \leq x} \widehat{c}(-1)^k \cdot 2^{-k}$ *using the approximation* $L_{\approx}^*(x) := \frac{2\widehat{c}}{3} x$.

We are expecting to see and verify numerically that for sufficiently large x *the following properties:*

- *Almost always we have that* $R_\pm(x) = 1$.
- *The ratio* $\frac{L(x)}{L_{\approx,1}(x)}$ *should be bounded by a constant approximately equal to one, and the ratio* $\frac{L(x)}{L_{\approx,2}(x)}$ *should be at least one.*
- *The ratio* $\frac{L_*(x)}{L_{\approx}^*(x)}$ *tends towards an absolute constant.*

The summatory functions $L(x)$ *and* $L_*(x)$ *are numerically taxing to compute directly for large* x*. We have written a software package in [19] in* `Python3` *for use with the* `SageMath` *platform that employs known algorithms for efficiently computing these functions.*