New characterizations of the summatory function of the Möbius function

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Abstract

The Mertens function, $M(x) \coloneqq \sum_{n \leq x} \mu(n)$, is defined as the summatory function of the classical Möbius function for $x \geq 1$. The inverse function sequence $\{g^{-1}(n)\}_{n \geq 1}$ taken with respect to Dirichlet convolution is defined in terms of the strongly additive function $\omega(n)$ that counts the number of distinct prime factors of any integer $n \geq 2$. For large x and $n \leq x$, we associate a natural combinatorial significance to the magnitude of the distinct values of the function $g^{-1}(n)$ that depends directly on the exponent patterns in the prime factorizations of the integers in $\{2, 3, \ldots, x\}$ viewed as multisets.

We prove an Erdős-Kac theorem analog for the distribution of the unsigned sequence $|g^{-1}(n)|$ over $n \leq x$ with a limiting central limit theorem type tendency towards normal as $x \to \infty$. For all $x \geq 1$, discrete convolutions of $G^{-1}(x) \coloneqq \sum_{n \leq x} \lambda(n)|g^{-1}(n)|$ with the prime counting function $\pi(x)$ determine exact formulas and new characterizations of asymptotic bounds for $M(x)$. In this way, we prove another concrete link of the distribution of $L(x) \coloneqq \sum_{n \leq x} \lambda(n)$ with the Mertens function and connect these classical summatory functions with an explicit normal tending probability distribution at large x. The proofs of these resulting combinatorially motivated new characterizations of $M(x)$ are rigorous and unconditional.

Keywords and Phrases: *Möbius function; Mertens function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function; Erdős-Kac theorem; strongly additive function.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; 11N64; and 11-04.*

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The conclusion of the proof of Proposition 2.1 in fact implies the stronger result that

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d),$$

where we adopt the notation that for $n \geq 2$, $C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha \| n} \frac{1}{\alpha!}$, where the sequence is taken to be one at $n := 1$.

2.3 Results on the distribution of exceptional values of $\omega(n)$ and $\Omega(n)$

The next theorems reproduced from [14, §7.4] characterize the relative scarcity of the distributions of $\omega(n)$ and $\Omega(n)$ for $n \leq x$ such that $\omega(n), \Omega(n) > \log\log x$. Since $\mathbb{E}[\omega(n)], \mathbb{E}[\Omega(n)] = \log\log n + B$ for $B \in (0,1)$ an absolute constant in each case, these results imply a regular, normal tendency of these additive arithmetic functions towards their respective average orders.

Theorem 2.2 (Upper bounds on exceptional values of $\Omega(n)$ for large n). *Let*

$$A(x,r) := \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\},$$
$$B(x,r) := \#\left\{n \leq x : \Omega(n) \geq r \cdot \log\log x\right\}.$$

If $0 < r \leq 1$ *and* $x \geq 2$*, then*

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

If $1 \leq r \leq R < 2$ *and* $x \geq 2$*, then*

$$B(x,r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

Theorem 2.3 is a special case analog to the celebrated Erdős-Kac theorem typically stated for the normally distributed values of the scaled-shifted function $\omega(n)$ over $n \leq x$ as $x \to \infty$ [14, *cf.* Thm. 7.21] [10, *cf.* §1.7].

Theorem 2.3 (Exact limiting bounds on exceptional values of $\Omega(n)$ for large n). *We have that as* $x \to \infty$

$$\#\left\{3 \leq n \leq x : \Omega(n) \leq \log\log n\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Theorem 2.4 (Montgomery and Vaughan). *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $0 < R < 2$ *we have that uniformly for all* $1 \leq k \leq R \cdot \log\log x$

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right],$$

where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1}\left(1 - \frac{1}{p}\right)^z, 0 \leq |z| < R.$$

Remark 2.5. We can extend the work in [14] on the distribution of $\Omega(n)$ to find analogous results bounding the distribution of $\omega(n)$. We have that for $0 < R < 2$

$$\pi_k(x) = \widehat{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right], \text{ unif. for } 1 \leq k \leq R\log\log x. \tag{10}$$

The analogous function to express these bounds for $\omega(n)$ is defined by $\widehat{\mathcal{G}}(z) := \widehat{F}(1,z)/\Gamma(1+z)$ where we take

$$\widehat{F}(s,z) := \prod_p \left(1+\frac{z}{p^s-1}\right)^{-1}\left(1-\frac{1}{p^s}\right)^z, \operatorname{Re}(s) > \frac{1}{2}; |z| \leq R < 2.$$

Let the functions

$$\begin{aligned} C(x,r) &:= \#\{n \leq x : \omega(n) \leq r \log\log x\} \\ D(x,r) &:= \#\{n \leq x : \omega(n) \geq r \log\log x\}. \end{aligned}$$

Then we have the next uniform upper bounds given by

$$\begin{aligned} C(x,r) &\ll x(\log x)^{r-1-r\log r}, \text{ uniformly for } 0 < r \leq 1, \\ D(x,r) &\ll x(\log x)^{r-1-r\log r}, \text{ uniformly for } 1 \leq r \leq R < 2. \end{aligned}$$

With the next corollary, we can accurately approximate asymptotic order of the sums $\mathcal{A}_\omega(x)$ (defined below) for large x by only considering the truncated sums $\mathcal{D}_\omega(x)$ where we have the known uniform bounds on the summands for $1 \leq k \leq \log\log x$. This result is cited in the proof of our new result stated in Corollary 4.4 of Section 4 (see Appendix A).

Corollary 2.6. *Suppose that for $x > e$ we define the following functions:*

$$\begin{aligned} \mathcal{N}_\omega(x) &:= \left|\sum_{k > \log\log x} (-1)^k \pi_k(x)\right| \\ \mathcal{D}_\omega(x) &:= \left|\sum_{k \leq \log\log x} (-1)^k \pi_k(x)\right| \\ \mathcal{A}_\omega(x) &:= \left|\sum_{k \geq 1} (-1)^k \pi_k(x)\right|. \end{aligned}$$

As $x \to \infty$, we have that $\mathcal{D}_\omega(x)/\mathcal{N}_\omega(x) = o(1)$ and $\mathcal{A}_\omega(x) \sim \mathcal{D}_\omega(x)$.

Proof. First, we sum the main term for the function $\mathcal{D}_\omega(x)$ by applying the limiting asymptotics for the incomplete gamma function derived in Lemma A.3 to obtain that

$$\begin{aligned} \mathcal{D}_\omega(x) &= \left|\sum_{1 \leq k \leq \log\log x} \frac{(-1)^k \cdot x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\right| + O(E_\omega(x)) \\ &= \frac{x}{\sqrt{2\pi \log\log x}} + O(E_\omega(x)), \end{aligned}$$

The error term from the bound in the previous equation is defined according to (10) with $\widehat{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) \gg 1$ for all $1 \leq k \leq \log\log x$ as

$$\begin{aligned} E_\omega(x) &:= \sum_{k \leq \log\log x} \frac{x}{\log x} \cdot \frac{k(\log\log x)^{k-3}}{(k-1)!} \leq \frac{x}{\log x} \times \sum_{1 \leq k \leq \log\log x} \frac{(\log\log x)^{k-2}}{(k-1)!} \\ &\leq \frac{x}{(\log x)(\log\log x)} e^{\log\log x} \leq \frac{x}{\log\log x}. \end{aligned}$$

The right-hand-side expression in the previous equation follows by applying Lemma A.3.

Next, we utilize the notation for and bounds on the function $D(x,r)$ from Remark 2.5 to bound the function $\mathcal{N}_\omega(x)$ as follows:

$$\frac{1}{x} \times |\mathcal{N}_\omega(x)| \leq \sum_{k \geq \log\log x} \frac{\pi_k(x)}{x} = \frac{1}{x} \times \sum_{k \geq \log\log x} \#\{2 \leq n \leq x : \omega(n) = k\} \ll 1.$$

Then we see that

$$\left|\frac{\mathcal{D}_\omega(x)}{\mathcal{N}_\omega(x)}\right| = O\left(\frac{1}{\sqrt{\log\log x}}\right) = o(1), \text{ as } x \to \infty.$$

Equivalently, we have shown that $\mathcal{D}_\omega(x) = o\left(\mathcal{N}_\omega(x)\right)$. The following results from the triangle inequality when x is large:

$$1 + o(1) = \left(\frac{\mathcal{D}_\omega(x) - \mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)}\right)^{-1} \ll \frac{\mathcal{D}_\omega(x)}{\mathcal{A}_\omega(x)} \ll \left(\frac{\mathcal{D}_\omega(x) + \mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)}\right)^{-1} = 1 + o(1).$$

The last equation implies that $\mathcal{A}_\omega(x) \sim \mathcal{D}_\omega(x)$ as $x \to \infty$. $\square$

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A Appendix: Asymptotic formulas

Facts A.1 (The incomplete gamma function)**.** The (upper) *incomplete gamma function* is defined by [18, §8.4]

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt, \mathrm{Re}(s) > 0.$$

The following properties of $\Gamma(a,x)$ hold at $z,a > 0$:

$$\Gamma(a,z) = (a-1)! \cdot e^{-z} \times \sum_{k=0}^{a-1} \frac{z^k}{k!}, a \in \mathbb{Z}^+, z > 0, \tag{26a}$$

$$\Gamma(a,z) \sim x^{a-1} \cdot e^{-z}, \text{ for fixed } a > 0, \text{ as } z \to +\infty. \tag{26b}$$

Moreover, for real $z > 0$, as $z \to +\infty$ we have that [15]

$$\Gamma(z,z) = \sqrt{\frac{\pi}{2}} z^{z-\frac{1}{2}}e^{-z} + O\left(z^{z-1}e^{-z}\right), \tag{26c}$$

If $z, a \to \infty$ with $z = \lambda a$ for some $\lambda > 0$ such that $(\lambda-1)^{-1} = o(|a|^{1/2})$, then [15]

$$\Gamma(a,z) \sim z^a e^{-z} \times \sum_{n\geq 0} \frac{(-a)^n b_n(\lambda)}{(z-a)^{2n+1}}, \tag{26d}$$

where the sequence $b_n(\lambda)$ satisfies the characteristic relation that $b_0(\lambda) = 1$ and[D]

$$b_n(\lambda) = \lambda(1-\lambda)b'_{n-1}(\lambda) + \lambda(2n-1)b_{n-1}(\lambda), n \geq 1.$$

Proposition A.2. *Suppose that z and $a > 0$ are real parameters. If $|z| = \lambda a$ for some $\lambda > 1$, then as $z \to +\infty$ we have that*

$$\Gamma(a,z) = \frac{z^{a-1}e^{-z}}{1-\lambda^{-1}} + O_\lambda\left(z^{a-2}e^{-z}\right),$$

and if $a \geq 1$ is integer-valued, then as $z \to +\infty$ we have

$$\Gamma(a,-z) \sim \frac{z^{a-1}e^{z}}{1+\lambda^{-1}} + O_\lambda\left(z^{a-2}e^{-z}\right).$$

Proof. We can see that for $\lambda > 1$ and $n \geq 1$ [5, §6.2]

$$\sum_{k=1}^{n} [\lambda^k] b_n(\lambda) = \frac{(2n)!}{2^n n!} \sim \sqrt{2}\left(\frac{2n}{e}\right)^n.$$

Thus we conclude that for all large enough n

$$b_n(\lambda) \ll (2n)! \cdot \lambda^n.$$

It follows from (26d) that for $N := z\left(1 - \frac{\log\lambda}{\lambda} - \frac{1}{\lambda}\right)$ (*cf.* [15, §A.1])

$$\Gamma(a,z) \sim z^{a-1}e^{-z} \times \sum_{0\leq n<N} \frac{(-1)^n b_n(\lambda)}{z^n \cdot (1-\lambda^{-1})^{2n+1}} = z^{a-1}e^{-z}(1 + E(a,z;\lambda)),$$

[D] An exact formula for $b_n(\lambda)$ is given in terms of the *second-order Eulerian number triangle* [24, A008517] as follows:

$$b_n(\lambda) = \sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k-1} \right\rangle\!\!\right\rangle \lambda^k.$$

where

$$|E(a,z;\lambda)| \ll \sum_{1\leq n<N} \frac{(2\lambda^2 n)^n}{((\lambda-1)^2 ez)^n}.$$

For all $\lambda > 1$, we have that the n^{th} summand in the previous bound is given by c_n^n where $0 < c_n < 1$. We conclude that since $z \to \infty$, we get $E(a,z;\lambda) = o(1)$. This argument justifies the main term for $\Gamma(a,z)$ stated as our proposition.

We obtain the following expansions using a similar justification on which terms in the sum are main terms as above for $a \in \mathbb{Z}^+$ and for $N := z\left(1 + \frac{\log\lambda}{\lambda} + \frac{1}{\lambda}\right)$ as $z \to -\infty$:

$$\begin{aligned}\Gamma(a,z) &= z^a e^z \times \sum_{n\geq 0} \frac{(-a)^n b_n(\lambda)}{(z-a)^{2n+1}} \sim z^{a-1} e^z \times \sum_{n\geq 0} \frac{(-1)^n b_n(\lambda)}{z^n \left(1+\frac{1}{\lambda}\right)^{2n+1}} \\ &= \frac{z^{a-1}e^z}{\left(1+\frac{1}{\lambda}\right)} + O\left(\sum_{1\leq n<N} \frac{\lambda^{2n}(2ne^{-1})^n}{(1+\lambda)^{2n+1}}\right).\end{aligned}$$

□

Lemma A.3. *For $x > e$, we have that*

$$S_1(x) := \frac{x}{\log x} \times \left| \sum_{1\leq k\leq \log\log x} \frac{(-1)^k (\log\log x)^{k-1}}{(k-1)!} \right| \sim \frac{x}{\sqrt{2\pi \log\log x}}. \tag{27a}$$

Proof of (27a). We set $t := \log\log x$ and let $t \to +\infty$. We can write

$$\frac{\log x}{x} \cdot S_1(x) = \left| \sum_{o\leq k<t} \frac{(-t)^k}{k!} \right|.$$

By Taylor's theorem for the exponential function with remainder terms, we have that for some $0 < s < t$

$$e^{-t} = \sum_{0\leq k<t} \frac{(-t)^k}{k!} + \frac{(-s)^t}{t!} e^{-s}. \tag{27b}$$

Clearly, we have that the left-hand-side of (27b) corresponds to an $O\left(\frac{1}{\log x}\right)$ error term. We can also compute that as $t \to \infty$ the remainder term in the previous equation satisfies

$$\left| \frac{(-s)^t}{t!} e^{-s} \right| \leq \frac{t^t}{t!} \sim \frac{\log x}{\sqrt{2\pi \log\log x}},$$

by applying Stirlings formula to approximate $t!$ when t is sufficiently large. A tight bound on the main term of our sum is argued in cases on the sign of the first sum on the right-hand-side of (27b). Let $s_t := \operatorname{sgn}\left(\sum_{0\leq k<t} \frac{(-t)^k}{k!}\right)$. If $s_t = -1$, then we get that

$$\left| \sum_{0\leq k<t} \frac{(-t)^k}{k!} \right| \leq \frac{\log x}{\sqrt{2\pi \log\log x}} + O\left(\frac{1}{\log x}\right).$$

When $s_t = +1$, then for all large x we must have that the sign on the remainder term from Taylor's theorem in (27b) is negative. Reversing the corresponding inequality yields the symmetric bound that

$$\left| \sum_{0\leq k<t} \frac{(-t)^k}{k!} \right| \geq \frac{\log x}{\sqrt{2\pi \log\log x}} + O\left(\frac{1}{\log x}\right).$$

□

Lemma A.4. *For $x > e$, we have that*

$$S_3(x) := \sum_{1\leq k\leq \log\log x} \frac{(\log\log x)^{k+1/2}}{(2k+1)(k-1)!} = \frac{1}{2}(\log x)\sqrt{\log\log x} + O\left(\frac{\log x}{\sqrt{\log\log x}}\right). \tag{27c}$$

Proof. We can sum this series symbolically with *Mathematica* to find that

$$\begin{aligned}S_3(x) &= \frac{1}{2}(\log x)\sqrt{\log\log x} - \frac{\sqrt{\pi}}{4}\operatorname{erfi}\left(\sqrt{\log\log x}\right) \\ &\quad - \frac{{}_2F_2\left(1, \frac{3}{2}+\log\log x; 1+\log\log x, \frac{5}{2}+\log\log x; \log\log x\right)(\log x)(\log\log x)}{2\sqrt{2\pi}(2\log\log x+3)}.\end{aligned} \tag{27d}$$

We will bound each component term in the above expansion of $S_3(x)$ to see that the dominant asymptotic order of this function is given by the leading term.

As $|z| \to \infty$, the *imaginary error function*, denoted by $\operatorname{erfi}(z)$, has the following asymptotic expansion [18, §7.12]:

$$\operatorname{erfi}(z) = -\imath + \frac{e^{x^2}}{\sqrt{\pi}}\left(z^{-1} + \frac{z^{-3}}{2} + \frac{3z^{-5}}{4} + \frac{15z^{-7}}{8} + O(z^{-9})\right).$$

It follows that

$$\frac{\sqrt{\pi}}{4}\operatorname{erfi}\left(\sqrt{\log\log x}\right) = \frac{(\log x)}{4}\left(\frac{1}{\sqrt{\log\log x}} + O\left(\frac{1}{(\log\log x)^{3/2}}\right)\right).$$

By bounding the remaining hypergeometric series term in the expansion of $S_3(x)$, we see that

$$\begin{aligned}&\frac{{}_2F_2\left(1, \frac{3}{2}+\log\log x; 1+\log\log x, \frac{5}{2}+\log\log x; \log\log x\right)}{(2\log\log x+3)} \\ &= \frac{1}{2\log\log x} \times \sum_{k\geq 0} \frac{1}{\left(1+\frac{2k+3}{2\log\log x}\right)} \prod_{i=1}^{k}\left(1+\frac{i}{\log\log x}\right)^{-1} \\ &= \frac{1}{2\log\log x} \times \sum_{\substack{k\geq 0 \\ 2k+3<\log\log x}} \left(1+\frac{2k+3}{\log\log x}\right)^{-1} \times \prod_{i=1}^{k}\left(1+\frac{i}{\log\log x}\right)^{-1} + O\left(\frac{1}{\log\log x} \times \prod_{i=1}^{\log\log x-1}\left(1+\frac{i}{\log\log x}\right)^{-1}\right).\end{aligned} \tag{27e}$$

The rightmost sum corresponds to another error term. Indeed, we see that the inner sum for the error term in the second to last line over the bounds $2k+3 \geq \log\log x$ is convergent to some constant. The leading product factors remaining in the last equation satisfy

$$\prod_{i=1}^{\log\log x-1}\left(1+\frac{i}{\log\log x}\right) \geq \frac{1}{2} + \frac{\log\log x}{2},$$

by appealing to a polynomial expansion of the factorial product by the Stirling numbers of the first kind where each component term $\frac{j}{\log\log x} < 1$ whenever $1 \leq j < \log\log x$. It follows that the error term in (27e) is $O\left((\log\log x)^{-1}\right)$. The main term in (27e) is expanded as

$$\begin{aligned}&\sum_{\substack{k\geq 0 \\ 2k+3<\log\log x}} \left(1+\frac{2k+3}{\log\log x}\right)^{-1} \times \prod_{i=1}^{k}\left(1+\frac{i}{\log\log x}\right)^{-1} \\ &\qquad \ll \sum_{\substack{k\geq 0 \\ 2k+3<\log\log x}} \prod_{i=1}^{k}\left(1+\frac{i}{\log\log x} + O\left(\frac{i^2}{(\log\log x)^2}\right)\right) \times \left(1+\frac{2k+3}{\log\log x} + O\left(\frac{(2k+3)^2}{(\log\log x)^2}\right)\right) \\ &\qquad = \sum_{\substack{k\geq 0 \\ 2k\leq\log\log x-3}} \left[1+\frac{(2k+1)(2k+2)}{2\log\log x} + O\left(\frac{k^2}{(\log\log x)^2}\right)\right] \\ &\qquad \sim \frac{(\log\log x)^2}{12} + \frac{3(\log\log x)}{8} - \frac{19}{12} + \frac{1}{8\log\log x}.\end{aligned}$$

Hence, the main term is the first leading term in the expansion of $S_3(x)$ from (27d). □