THE FUNCTION $G^{-1}(x)$

1. Notation

As usual $\omega(n)$ is the number of distinct prime divisors on n. Then $g(n)$ is defined by

$$g(n) = 1 + \omega(n).$$

In connection with this define for $\sigma > 1$

$$P(s) = \sum_p p^{-s}.$$

If I recall correctly Landau showed (I think in Rendiconti di Palermo very roughly about 1920) that this has an analytic continuation to the half-plane $\sigma > 0$ given by

$$P(s) = -\sum_{m=1}^{\infty} \frac{\mu(m)}{m} \log \zeta(sm)$$

and has the line $\operatorname{Re} s = 0$ as a natural boundary (every point of the imaginary axis is a limit point of singularities of the function). This is not really relevant to our discussion but is interesting background.

Now define for $\sigma > 1$

$$\mathcal{P}(s) = 1 + P(s).$$

Then $\mathcal{G}(s)$ defined by

$$\mathcal{G}(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

satisfies

$$\mathcal{G}(s) = \zeta(s)\mathcal{P}(s).$$

Thus, if we write $\mathbf{1}(n)$ for the function which is 1 for every n and $p(n)$ for the function which is 1 when $n = 1$ or n is prime and 0 for all other n, then we have

$$g = \mathbf{1} * p.$$

Now, as long as $\sigma > 1$ and the factors on the right are non-zero one has

$$\mathcal{G}(s)^{-1} = \zeta(s)^{-1}\mathcal{P}(s)^{-1}.$$

In other words

$$g^{-1} = \mu * p^{-1}$$

and

$$\mu = g^{-1} * p.$$

Hence

$$M(x) = G^{-1}(x) + \sum_p G^{-1}(x/p)$$

so that

$$M(x) = G^{-1}(x) + G^{-1}(x/2) + \sum_{p>2} G^{-1}(x/p) \tag{1.1}$$

which is essentially Proposition 5.2 since

$$\begin{aligned}\sum_{p>2} G^{-1}(x/p) &= \sum_{k\le x/2} \sum_{k\le x/p<k+1} G^{-1}(x/p)\\ &= \sum_{k\le x/2} G^{-1}(k)\big(\pi(x/k) - \pi(x/(k+1))\big).\end{aligned}$$

The function $P(\sigma)$ is strictly decreasing on $(1,\infty)$, tends to ∞ as $\sigma \to 1$ and tends to 0 as $\sigma \to \infty$. Thus the equation

$$P(\sigma) = 1$$

has a unique solution $\sigma = \sigma_1$ with $\sigma_1 > 1$. Now we can prove the following theorem.

Theorem 1.1. *There are s with σ arbitrarily close to σ_1 such that*

$$\mathcal{P}(s) = 0.$$

Corollary 1.2. *The functions*

$$\mathcal{G}(s)^{-1},$$

$$\mathcal{G}(s)^{-1}2^{-s},$$

and

$$\zeta(s)^{-1}\big(1 - (1+2^{-s})\mathcal{P}(s)^{-1}\big)$$

have singularities at points s with σ arbitrarily close to σ_1, and so in each case the corresponding Dirichlet series has abscissa of convergence $\sigma_c \ge \sigma_1$.

Note that, by Corollary 1.2 in Chapter 1 of M&V, a Dirichlet series converges locally uniformly in its half-plane of convergence and so is analytic there. Thus in each case the abscissa of convergence cannot be smaller than σ_1. Hence, by Theorem 1.3 in Chapter 1 of M&V

$$\limsup_{x\to\infty} \frac{\log|G^{-1}(x)|}{\log x} \ge \sigma_1,$$

$$\limsup_{x\to\infty} \frac{\log |G^{-1}(x/2)|}{\log x} \geq \sigma_1,$$
$$\limsup_{x\to\infty} \frac{\log |\sum_{2<p\leq x} G^{-1}(x/p)|}{\log x} \geq \sigma_1.$$

Thus, for example, given any $\varepsilon > 0$, there are arbitrarily large x such that
$$|G^{-1}(x)| > x^{\sigma_1 - \varepsilon}.$$
This contradicts Theorem 5.1.

I am guessing that these three terms are all this large most of the time, even though they largely cancel each other out. The coefficients of the three sums in (1.1) are precisely the coefficients of the three Dirichlet series above. Thus each of the three sums could be very large. I doubt that the third term can be ignored even in special cases.

2. Proof of Theorem 1.1

The ideas of this proof go back certainly to Davenport and Heilbronn [1936a] and [1936b], and perhaps even to Hans Bohr (see Chapter 11 of Titchmarsh [1986]). Choose δ arbitrarily small so that
$$|1 - P(z)| > 0$$
for all z with $|z - \sigma_1| = \delta$. Let
$$\eta = \min_{\substack{z \\ |z-\sigma_1|=\delta}} |1 - P(z)|.$$
Since $P(z)$ is continuous we have
$$\eta > 0.$$
Choose X so that
$$\sum_{p>X} p^{\delta-\sigma_1} < \frac{\eta}{4}.$$
Then we can use Kronecker's theorem to provide us with a t such that
$$\max_{2<p\leq X} \min_{n\in\mathbb{Z}} \left| \frac{t\log p}{2\pi} - n - \frac{1}{2} \right| < \delta\eta.$$
Thus
$$\sum_{p>2} p^{\delta-\sigma_1} |p^{it} + 1| < \frac{\eta}{2}.$$
Hence, whenever $|z - \sigma_1| = \delta$ we have
$$|P(z + it) + P(z)| < \frac{\eta}{2}.$$
Then, by Rouché's theorem, $1 - P(z)$ and $1 - P(z) + P(z + it) + P(z)$ have the same number of zeros with $|z - \sigma_1| < \delta$. Since $1 - P(z)$

has at least one zero there, $1 + P(w)$ will have at least one zero with $|w - \sigma_1 - it| < \delta$. Since δ can be made arbitrarily small there are zeros arbitrarily close to the line $s = \sigma_1$.

References

[1936a] H. Davenport & H. Heilbronn, On the zeros of certain Dirichlet series I, J. London Math. Soc. 11(1936), 181–185.

[1936b] H. Davenport & H. Heilbronn, On the zeros of certain Dirichlet series II, J. London Math. Soc. 11(1936), 307–312.

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