4 Expansions of summatory functions over special sums using generalized Dirichlet inverse functions

4.1 Exploring special cases of the theorem for summatory functions of Dirichlet convolutions

Given the interpretation of the summatory functions over an arbitrary Dirichlet convolution (and the vast number of such identities for special number theoretic functions – *cf.* [6]), it is not surprising that this formulation of the first theorem may well provide many fruitful applications, indeed. In addition to those cited in the compendia of the catalog reference, we have notable identities of the form: $(f * 1)(n) = [q^n] \sum_{m \geq 1} f(m) q^m / (1 - q^m)$, $\sigma_k = \mathrm{Id}_k * 1$, $\mathrm{Id}_1 = \phi * \sigma_0$, $\chi_{\mathrm{sq}} = \lambda * 1$ (see sections below), $\mathrm{Id}_k = J_k * 1$, $\log = \Lambda * 1$, and of course $2^{\omega} = \mu^2 * 1$. The result in Theorem 2.4.1 is natural and displays a quite beautiful form of symmetry between the initial matrix terms,

$$t_{x,j}(f) = \sum_{k = \left\lfloor \frac{x}{j+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{j} \right\rfloor} f(k),$$

and the corresponding inverse matrix,

$$t_{x,j}^{-1}(f) = \sum_{k = \left\lfloor \frac{x}{j+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{j} \right\rfloor} f^{-1}(k),$$

as expressed by the duality of f and its Dirichlet inverse function f^{-1}. Since the recurrence relations for the summatory functions $G(x)$ arise naturally in applications where we have established bounds on sums of Dirichlet convolutions of arithmetic functions, we will go ahead and prove it here before moving along to some motivating examples of the use of this theorem.

Proof of Theorem 2.4.1. Let h, g be arithmetic functions where $g(1) \neq 1$ has a Dirichlet inverse. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $S_{g,h}(x)$ to be the summatory function of the Dirichlet convolution of g with h: $g * h$. Then we can easily see that the following expansions hold:

$$\begin{aligned} S_{g,h}(x) &:= \sum_{n=1}^{x} \sum_{d \mid n} g(n) h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor \right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor \right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor \right) \right] H(i). \end{aligned}$$

Thus we have an implicit statement of a recurrence relation for the summatory function H, weighted by g and G, whose non-homogeneous term is the summatory function, $S_{g,h}(x)$, of the Dirichlet convolutions $g * h$. We form the matrix of coefficients associated with this system for $H(x)$, and proceed to invert it to express an exact solution for this function over all $x \geq 1$. Let the ordinary (initial, non-inverse) matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor \right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor \right) \equiv G_{x,j} - G_{x,j+1}.$$

Then the matrix we must invert in this problem is lower triangular, with ones on its diagonals – and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T); U = (\delta_{i,j+1}), (I - U^T)^{-1} = ([j \leq i]_{\delta}).$$

It is a nice round fact that if we take successive differences of floor functions, we get non-zero behavior at divisors:

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor \right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor \right) = \begin{cases} g\left(\frac{x}{j} \right), & \text{if } j \mid x; \\ 0, & \text{otherwise.} \end{cases}$$

We use this property to invertibly shift the matrix $\hat{G}$, and then invert the result, to obtain a matrix involving the Dirichlet inverse of g:

$$\left[(I-U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Now we can express the inverse of the target matrix $(g_{x,j})$ in terms of these Dirichlet inverse functions as follows:

$$\begin{aligned}
(g_{x,j}) &= (I-U^T)^{-1}\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\
(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k)\right)(I-U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} g^{-1}(k)\right).
\end{aligned}$$

Thus the summatory function H is exactly expressed by the inverse vector product of the form

$$\begin{aligned}
H(x) &= \sum_{k=1}^{x} g_{x,k}^{-1} \cdot S_{g,h}(k) \\
&= \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor+1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j)\right) \cdot S_{g,h}(k). \square
\end{aligned}$$

4.2 Case study by example: Proving a new upper bound for the Mertens function

4.2.1 An introduction to the classical Mertens function

Suppose that $n \geq 1$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2 \alpha_2 \cdots p_k^{\alpha_k}$. We define the *Möebius function* to be the signed indicator function of the squarefree integers. There are many known variants and special properties of the Möebius function and its generalizations [18, *cf.* §2], however, for our purposes we seek to explore the properties and asymptotics of weighted summatory functions over $\mu(n)$. The Mertens summatory function, or *Mertens function*, is defined as

$$\begin{aligned}
M(x) &= \sum_{n \leq x} \mu(n), \ x \geq 1, \\
&\longmapsto \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2\}
\end{aligned}$$

A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \to \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{Q(x)} = \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{Q(x)} \xrightarrow[n \to \infty]{} \frac{3}{\pi^2}.$$

While this limiting law suggests an even bias for the Mertens function, in practice $M(x)$ has a noted negative bias in its values, and the actual local oscillations between the approximate densities of the sets $\mu_{\pm}(x)$ lend an unpredictable nature to the function and its characteistic oscillatory sawtooth shaped plot – even over asymptotically large and variable intervals.

The well-known approach to evaluating the behavior of $M(x)$ for large $x \to \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is easily expressed via an inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since by Perron's formula we have

$$\frac{1}{\zeta(s)} = \int_1^{\infty} \frac{s \cdot M(x)}{x^{s+1}} dx,$$

we then obtain that

$$M(x) = \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation along with the standard Euler product representation for the reciprocal zeta function leads us to the exact expression for $M(x)$ when $x > 0$ given by the next theorem.

Theorem 4.2.1 (Analytic Formula for $M(x)$). *If the RH is true, the there exists an infinite sequence $\{T_k\}_{k\geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any $x \in \mathbb{R}_{>0}$*

$$M(x) = \lim_{k\to\infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |\Im(\rho)|<T_k}} \frac{x^{\rho}}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n\geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_{\delta}.$$

An unconditional bound on the Mertens function due to Walfisz [**?**] states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \exp\left(-C \cdot \log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan proved new updated estimates bounding $M(x)$ for large x in 2009 of the following forms:

$$\begin{aligned} M(x) &\ll \sqrt{x} \exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x} \exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right), \ \forall \epsilon > 0. \end{aligned}$$

Other explicit bounds due to the article by Kotnik include the following simpler estimates for the Mertens function when x is sufficiently large:

$$\begin{aligned} |M(x)| &< \frac{x}{4345}, \ \forall x > 2160535, \\ |M(x)| &< \frac{0.58782 \cdot x}{\log^{11/9}(x)}, \ \forall x > 685. \end{aligned}$$

4.2.2 An alternate more combinatorial phrasing for summing $M(x)$ in general

Corollary 4.2.2 (Convolutions Arising From Möbius Inversion). *Suppose that g is an arithmetic functions with $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n\leq x}(g * \mu)(n)$. Then*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geq 1.$$

Proposition 4.2.3 (An Antique Divisor Sum Identity). *Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the additive function that counts the number of distinct prime factors of n. Then*

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu.$$

The summatory function of the LHS is $\widetilde{G}(x) = \pi(x) + 1$.

Proof. The the crux of the stated identity is to prove that for all $n \geq 1$, $\chi_{\mathbb{P}}(n) = (\mu * \omega)(n)$ – our claim. We notice that the Mellin transform of $\pi(x)$ – the summatory function of $\chi_{\mathbb{P}}(n)$ – at $-s$ is given by

$$\begin{aligned} s \cdot \int_1^\infty \frac{\pi(x)}{x^{s+1}} dx &= \sum_{n \geq 1} \frac{\nabla[\pi](n-1)}{n^s} \\ &= \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s} = P(s). \end{aligned}$$

This is typical fodder which more generally relates the Mellin transform $\mathcal{M}[S_f](-s)$ to the DGF of the sequence $f(n)$ as cited, for example, in [2, §11]. Now we consider the DGF of the right-hand-side function, $f(n) := (\mu * \omega)(n)$, as

$$D_f(s) = \frac{1}{\zeta(s)} \times \sum_{n \geq 1} \frac{\omega(n)}{n^s} = P(s),$$

by Lemma **??**. Thus for any $\Re(s) > 1$, the DGFs of each side of the claimed equation coincide. So by uniqueness of Dirichlet series, we see that in fact the claim holds. To obtain the full result, we add to each side of this equation a term of $\varepsilon(n) \equiv (\mu * 1)(n)$, and then factor the resulting convolution identity. □

Example 4.2.4 (Inverting a recurrence relation for the Mertens function). Using $\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu$ in Corollary 4.2.2 where $\chi_{\mathbb{P}}$ is the characteristic function of the primes, we have that $\widetilde{G}(x) = \pi(x) + 1$, and can compute the Dirichlet inverse sequence of $g(n) := \omega(n) + 1$ numerically for the first few sequence values as

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is apparently dictated by $\lambda(n) = (-1)^{\Omega(n)}$, though no formula for the unsigned magnitudes is immediately obvious. Note that since the DGF of $\omega(n)$ is given by $D_\omega(s) = P(s)\zeta(s)$ where $P(s)$ is the *prime zeta function*, we do have a Dirichlet series for the inverse functions to invert coefficient-wise (see B.2). In fact, Fröberg has previously done some preliminary investigation as to the properties of the inversion to find the coefficients of $(1 + P(s))^{-1}$ [5]. We will employ a different tact to try to obtain new upper and lower bounds on $|M(x)|$ using Corollary 4.2.2. In particular, the corollary together with Proposition 4.2.3 implies that

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{19}$$

Discussion 4.2.5 (The expansions of the Dirichlet inverse functions). First, we sketch the general procedure which was computationally discovered by applying our handbook of experimental mathematics techniques [21]. We begin by using an identity for the Dirichlet inverse of a general arithmetic function developed in [12, 14]:

$$\begin{aligned} g^{-1}(n) &= \sum_{j=1}^{\Omega(n)} \left[(\omega + 1 - \varepsilon)_{*_{2j}}(n) - (\omega + 1 - \varepsilon)_{*_{2j-1}}(n)\right] \\ &= \sum_{j=1}^{\Omega(n)} \sum_{r=0}^{j} \binom{j}{r} (-1)^r (\omega + 1)_{*_r}(n) \\ &= \sum_{j=1}^{\Omega(n)} \sum_{r=0}^{j} \sum_{m=0}^{r} \binom{j}{r} \binom{r}{m} (-1)^r \omega_{*_m}(n) \end{aligned}$$

$$= TODO.$$

In the previous equations we have suppressed the long form of the multiple convolution expansions and intend f_{*_k} to denote the k-fold convolution of f with itself. Now to see the pattern we are after, note that for primes a, b and positive integers $m, n \geq 1$, the (incomplete) additivity of $\omega(n)$ implies that $\omega(a^m b^n) = \omega(a) + \omega(b)$ where by convention $\omega(1) = 0$.

Examples: We should see what happens when we evaluate special cases of the convolution functions $\omega_{*_k}(n)$ for $k \geq 1$ and varying prime factorizations of n. Since these terms contribute to the signed behavior of (19), this is a useful digression. First, when $k := 1$, we have that

$$\omega_{*_1}(n) = \sum_{\substack{d|n \\ d>1}} \omega(d) = \Omega(n).$$

Moving forward slowly to the case where $k := 2$, we see slowly that all hell breaks loose. Let's first consider the double convolution for the case where $n := p_1^{\alpha_1} p_2^{\alpha_2}$ for distinct primes p_1, p_2. Unfortunately, simply expanding out the double divisor sum in the following form leads to complications:

$$\omega_{*_2}(n) = \sum_{\substack{d|n \\ d>1}} \sum_{\substack{r|\frac{n}{d} \\ r>1}} \omega(d)\omega(r).$$

Now it might be convenient to break up these sums into the divisors so that we obtain the patterns $\omega(d)\omega(r) \in \{1^2, 2^2; 1 \cdot 2\}$, but this method is troublesome because, for example, the 2^2 pattern may not be possible depending on the integral values of $\alpha_1, \alpha_2 \geq 1$. If we define a bivariate characteristic function

$$\mathbb{1}_{x,y} = (1 - \delta_{x,0}) + (1 - \delta_{y,0}) + 2(1 - \delta_{x,0})(1 - \delta_{y,0}),$$

then `Mathematica` can quickly sum and simplify the sums when $\omega(n) = 2$:

$$\begin{aligned}
\omega_{*_2}(n) &= \sum_{i_1=0}^{\alpha_1} \sum_{i_2=0}^{\alpha_2} \sum_{j_1=0}^{\alpha_1 - i_1} \sum_{j_2=0}^{\alpha_2 - i_2} \mathbb{1}_{i_1,i_2} \cdot \mathbb{1}_{j_1,j_2} \\
&= \begin{cases} \frac{1}{2}\alpha_1(9\alpha_1 - 5), & \text{if } \alpha_2 = 1; \\ \frac{1}{2}\left(\left(8\alpha_2^2 + 1\right)\alpha_1^2 - (4\alpha_2 + 1)\alpha_1\right) + (\alpha_2 - 1)\alpha_2\big), & \text{otherwise,} \end{cases}
\end{aligned}$$

where we have assumed that $\alpha_1, \alpha_2 \geq 1$. This is a reasonable assumption since $\omega_{*_k}(n) = 0$ if $k > \Omega(n)$. Let's make a more general characteristic function definition and hope for the best in performing similar sums over products of these multivariate indicators:

$$\mathbb{1}_{i_1,i_2,\ldots,i_r} := \sum_{m=1}^{r} \sum_{1 \leq k_1 < k_2 < \cdots < k_m \leq r} m \times \prod_{j=1}^{m} (1 - \delta_{i_{k_j},0}).$$

Then with this definition, if $n := p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the prime factorization of n into distinct primes with $\alpha_1, \ldots, \alpha_r \geq 1$, i.e., so that $\omega(n) = r$, then we have that

$$\omega_{*_2}(n) = \sum_{\substack{0 \leq i_1 \leq \alpha_1 \\ 0 \leq i_2 \leq \alpha_2 \\ \cdots \\ 0 \leq i_r \leq \alpha_r}} \sum_{\substack{0 \leq j_1 \leq \alpha_1 - i_1 \\ 0 \leq j_2 \leq \alpha_2 - i_2 \\ \cdots \\ 0 \leq j_r \leq \alpha_r - i_r}} \mathbb{1}_{i_1,i_2,\ldots,i_r} \times \mathbb{1}_{j_1,j_2,\ldots,j_r}. \tag{20}$$

Thus the forms of our functions depend on a mess of cases of $2r$ iterated sums depending on whether the index of summation is zero-valued or not. This is cumbersome, but can be done. In general, the result will be a fixed degree-2 polynomial in the variables $\{\alpha_1, \ldots, \alpha_r\}$, let's say

$$\omega_{*_2}\left(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}\right) = \sum_{(p_1,\ldots,p_r) \in \{0,2\}^r} \widehat{C}_{p_1,\ldots,p_r}(r) \times \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_r^{p_r}.$$

Since we can, and might end up needing it, we can evaluate the indicator product sums in (20), by simplifying the indictor sums using *Newton's identities* for elementary symmetric polynomials, which as it happen, simplify nicely in cases. The analysis by cases boils down to which of the $2r$ summation indices are zero at once. Namely, we will extract a "main term" sum where the indices of summation do not include zero, and evaluate the somewhat messy resulting formulae that happens when we get degenerate nested sum cases which correspond to the prior sums having zero-indexed values.

Expanding the general solution in cases: First, notice that we can obviously expand out the indicator functions in terms of a sum of scaled *elementary symmetric polynomials*. For example, let $\chi_x(\cdot) \equiv 1 - \delta_{x,\cdot}$ and observe that in the trivariate case we get

$$\mathbb{1}_{x_1,x_2,x_3} = x_1 + x_2 + x_3 + 2(x_1x_2 + x_1x_3 + x_2x_3) + 3x_1x_2x_3.$$

Thus, in general if we have s zero-indexed cases (which we need to keep track of and index carefully due to the other parameters involved), the indicator sum terms simplify as follows. Note that the *power sum symmetric polynomials* are convenient here because non-unit powers of an indicator function are irrelevant in calculations:

$$\begin{aligned}
\mathbb{1}_{x_1,\ldots,x_r} &= \sum_{k=1}^{r} k \cdot e_k(\chi_{x_1},\ldots,\chi_{x_r}) \\
&= \sum_{k=1}^{r}\sum_{i=1}^{k-1}(-1)^i e_{k-i}(\chi_{x_1},\ldots,\chi_{x_r}) \cdot p_i(\chi_{x_1},\ldots,\chi_{x_r}) \\
&= \sum_{k=1}^{r}\sum_{i=1}^{k-1}(-1)^i e_{k-i}(\chi_{x_1},\ldots,\chi_{x_r}) \cdot p_1(\chi_{x_1},\ldots,\chi_{x_r}) \\
&= \sum_{k=1}^{r}\sum_{i=1}^{k-1}(-1)^i [z^{k-i}] \prod_{j=1}^{r}(1+\chi_{x_j} z) \times (\chi_{x_1} + \cdots + \chi_{x_r}).
\end{aligned}$$

Now we set any $2r \geq s \geq 1$ indices in the $2r$ sums from (20) above to zero, and while the terms involving the α_i's change depending on which indices we zero out, the indicator function terms simplify nicely to a consistent factor depending only on s, r:

$$\begin{aligned}
\mathbb{1}_{x_1,\ldots,x_r}\big|_{x_{i_1},\ldots,x_{i_s}=0} &= \sum_{k=1}^{r}\sum_{i=1}^{k-1}\binom{r}{s}(r-s)(-1)^i[z^i](1+z)^{r-s} \\
&= \sum_{k=1}^{r}\sum_{i=1}^{k-1}\binom{r}{s}\binom{r-s}{i}(r-s)(-1)^i \\
&= r \cdot \binom{r}{s}\left(s - r + \frac{(-1)^r(r+1)\binom{r-s}{r+1}}{s+1-r}\right).
\end{aligned}$$

Next, we need to handle the fomulas resulting from the $2r - s + 1$ summations (starting from one) that we have remaining as the coefficient terms of what we just derived in the previous few equations. Notice that the resulting formula *will be messy*, but is nonetheless exact given a prime factorization of $n \geq 2$ to plug into the inverse functions we will need to study more carefully below.

Let our s selected indices be partitioned as

$$\{(i_{k_1}, j_{k_1}), \ldots, (i_{k_q}, j_{k_q}); i_{k_\ell}, \ldots, i_{k_r}; j_{k_m}, \ldots, j_{k_w}\},$$

so that $2q + (r - \ell + 1) + (w - m + 1) = s$. We select the notation

$$\mathcal{N}_J(i_1, \ldots, i_s) := \#\{i_1, \ldots, i_s : i_j > r, 1 \leq j \leq s\}.$$

This notation will be used in the resulting formula we will have proven below. Now for each paired index tuple, (i, j), we have selected, we have

$$\sum_{i=1}^{\alpha_i} \sum_{j=1}^{\alpha_i - i} 1 = \sum_{i=1}^{\alpha_i} (\alpha_i - i) = \frac{\alpha_i}{2}(\alpha_i - 1).$$

In the remaining unpaired singleton index cases (setting $i_j = 0$ in the upper bounds on the j-indexed sums), we have both that

$$\sum_{i=1}^{\alpha_i} 1 = \alpha_i, \sum_{j=1}^{\alpha_i - i} 1 = \alpha_i.$$

Now in conclusion, what we have shown is that

$$\begin{aligned} \omega_{*_2}(n) = \sum_{s=0}^{r} \Bigg[& \sum_{1 \leq i_1 < \cdots < i_s \leq 2r} r \cdot \binom{r}{s+1-\mathcal{N}_J(i_1,\ldots,i_s)} \times \\ & \times \left(s+1-\mathcal{N}_J(i_1,\ldots,i_s) - r + \frac{(-1)^r (r+1)\binom{r-1-s+\mathcal{N}_J(i_1,\ldots,i_s)}{r+1}}{s+2-\mathcal{N}_J(i_1,\ldots,i_s) - r} \right) \times \\ & \times r \cdot \binom{r}{\mathcal{N}_J(i_1,\ldots,i_s)} \left(\mathcal{N}_J(i_1,\ldots,i_s) - r + \frac{(-1)^r (r+1)\binom{r-\mathcal{N}_J(i_1,\ldots,i_s)}{r+1}}{\mathcal{N}_J(i_1,\ldots,i_s) + 1 - r} \right) \times \\ & \times \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_s} \times \prod_{\ell=1}^{s} \left(\frac{(\alpha_{i_\ell} - 1)}{2} \chi_{i_j \leq r}(i_j) + 1 - \chi_{i_j \leq r}(i_j) \right) \Bigg]. \end{aligned}$$

Generalizing the formula to $k \geq 3$: At this point, a generalization to expand ω_{*_k} for subsequent cases of $k > 2$ is just a formality of notation. That is, given a known prime factorization for n (and this is important, because we shant typically be given such a nicety), we just repeat the procedure above with more (kr) nested summations and a product of k similarly expandable indicator functions to iterate over. We have already shown the internals to constructing such a formula, so let's just go ahead and state the general case in all of its symbolic formality. Suppose that $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and that $k \geq 2$ is integer-valued. Then we have that

$$\omega_{*_k}(n) = TODO. \tag{21}$$

Next plan of attack: Fortunately, we can move along to a few nicer-to-state, single summation formulas that tend to characterize the Dirichlet inverse functions, $g^{-1}(n)$, in "most cases" when n is either squarefree or a prime power. In the remaining cases, we conjecture that we can still approximate the values of the inverse sequence very well (with asymptotically small enough error). We may refer to formula (21) if we need it – and if we can, for example, make some simplifying assumptions about the distribution of the prime factorizations over which we will need to sum $g^{-1}(n)$.

Conjecture 4.2.6. *Suppose that $n \geq 1$ is a squarefree integer. We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n) = (\omega + 1)^{-1}(n)$ over these integers:*

(A) $g^{-1}(1) = 1$, which follows immediately by computation;

(B) $\operatorname{sgn}(g^{-1}(n)) = \mu(n) \equiv \lambda(n)$;

(C) If $w(n) = k$, we can write the inverse function at k as

$$g^{-1}(n) = \sum_{m=0}^{k} \binom{k}{m} \cdot m!.$$

(D) When n is not squarefree nor a prime power, we can still come close to the approximate value of $g^{-1}(n)$ using a construction similar to (C). Namely, let

$$\widehat{f}_2(n) := \sum_{m=0}^{k} \binom{k}{m} \#\{d | n : \omega(d) = k\}$$

$$= \sum_{m=0}^{k} \binom{k}{m} \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq r} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k},$$

where here we denote $n \equiv p_1^{\alpha_1} \cdots p_r^{\alpha_r}$*. We claim that for* $n \in \bar{\mathbb{S}} = \{12, 18, 20, 24, 28, 36, 40, 44, 45, 48, 50, \ldots\}$*, by which we denote the positive integers* $n \geq 2$ *which are not squarefree and are not an exact prime power [23, ATODO], that*

$$|\lambda(n) g^{-1}(n) - \widehat{f}_2(n)| \leq \left| \lambda(n) g^{-1}(n) - \sum_{m=0}^{k} \binom{k}{m} \cdot m! \right|.$$

The "obvious" interpretation to these sums is that we are counting up the factorizations of n according to the number of prime factors in each term, selecting each k of the primes, and then scaling by a "count" of how many ways there are to arrange k such factors[2]*.*

We illustrate parts (B)–(D) of Conjecture 4.2.6 using Table T.2 given on page 55 of the appendix section.

4.2.3 Rudimentary asymptotic estimates to round out our example

Remark 4.2.7 (Some Commentary)**.** I will stress, and re-stress for emphasis, that what follows is not intended to be a conclusive bit of ether to whether the Mertens function $M(x)/\sqrt{x}$ is unbounded. Rather, the author believes that it is important in the exposition here to at least follow up on what is possible given the observations from Conjecture 4.2.6 in terms of estimating new bounds on the Mertens function. The current best *upper bound* for the Mertens function is cited by Soundarajan in his 2009 *Annals* paper [24]. These bounds are recounted in the introduction to $M(x)$ given in Section 4.2.1.

In some sense, while astonishing in its construction, these bounds are still unsatisfying for a couple of reasons. First, they do not predict lower bounds on any particular intervals of $x \in \mathbb{R}$. And secondly, the bounds sidestep predictions of the signedness of $M(x)$, which is expected to oscillate in both directions of signed infinity infinitely often as $x \to \infty$. In contrast, the formulas we have seen in the previous subsection allow us to pinpoint cancellation – even if the signedness observed corresponds to taking the difference of two oppositely signed asymptotic formulae.

Theorem 4.2.8 (Montgomery and Vaughan, §7.4)**.** *Let* $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$*. For* $R < 2$ *we have that*

$$\widehat{\pi}_k(x) = G\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

uniformly for $1 \leq k \leq R \log\log x$ *where*

$$G(z) := \frac{F(1,z)}{\Gamma(z+1)} = \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z.$$

Remarks on the Proof. In particular, in the reference we have defined $F(s,z)$ for $\Re(s) > 1$ such that the Dirichlet series coefficients, $a_z(n)$, are defined by

$$\zeta(s)^z F(s,z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s}, \Re(s) > 1.$$

For the function

$$F(s,z) := \prod_p \left(1 - \frac{z}{p^s}\right) \left(1 - \frac{1}{p^s}\right)^z,$$

we obtain in the notation above that $a_z(n) \equiv z^{\Omega(n)}$, and that the summatory function satisfies

$$A_z(x) := \sum_{n \leq x} z^{\Omega(n)} = \sum_{k \geq 0} \widehat{\pi}_k(x) z^k.$$

[2] *It's a nice thought in so much as these formulas *DO* seem to accuratly approximate the inverse sequence enough of the time to be useful.*

Hence, by the Cauchy integral formula, for $r < 2$ we get that

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \oint_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz,$$

from which we obtain the stated formula in the theorem. $\square$

Strategy. We seek to approximate the right-hand-side of $G(z)$ by only taking the products of the primes $p \leq x$, e.g., $p \in \{2, 3, 5, \ldots, x\}$. We will require some fairly elementary estimates of products of primes, Mertens theorem on the rate of divergence of the sum of the reciprocals of the primes, and on some generating function techniques involving elementary symmetric functions. $\square$

Theorem 4.2.9 (Mertens theorem)**.**

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log\log x + B + o\left(\frac{1}{\log x}\right),$$

where $B \approx 0.2614972128476427837554$ is an absolute constant. We actually can bound the left-hand-side more explicitly by

$$P_1(x) = \log\log x + B + O\left(e^{-(\log x)^{\frac{1}{14}}}\right).$$

Corollary 4.2.10. *We have that for sufficiently large $x \gg 1$*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-B}}{\log x} \left[1 - \frac{(\log x)^{1/14}}{B} + o\left((\log x)^{1/14}\right)\right].$$

Hence, for $1 < |z| < R < 2$ we obtain that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^z = \frac{e^{-Bz}}{(\log x)^z} \left[1 - \frac{z}{B} (\log x)^{\frac{1}{14}} + o_z\left(z^2 \cdot (\log x)^{\frac{1}{14}}\right)\right].$$

Proof. By taking logarithms and using Mertens theorem above, we obtain that

$$\begin{aligned}
\log \prod_{p \leq x} \left(1 - \frac{1}{p}\right) &= \sum_{p \leq x} \log\left(1 - \frac{1}{p}\right) \\
&\approx -\log\log x - B + O\left(e^{-(\log x)^{1/14}}\right).
\end{aligned}$$

Hence, the first formula follows by expanding out an alternating series for the exponential function. The second formula follows for $z \notin \mathbb{Z}$ by an application of the generalized binomial series given by

$$\log \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^z \approx \frac{e^{-Bz}}{(\log x)^z} \times \sum_{r \geq 0} \binom{z}{r} \frac{(-1)^r}{B^r} (\log x)^{\frac{r}{14}},$$

where for $1 < |z| < 2$, we obtain the next result stated above with $\binom{z}{1} = z$ and $\binom{z}{2} = z(z-1)/2$. $\square$

Facts 4.2.11 (Exponential Integrals and Incomplete Gamma Functions)**.** The following two variants of the *exponential integral function* are defined by

$$\begin{aligned}
\mathrm{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, \\
E_1(z) &:= \int_1^{\infty} \frac{e^{-tz}}{t} dt, \Re(z) \geq 0,
\end{aligned}$$

where $\mathrm{Ei}(-kz) = -E_1(kz)$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $E_1(z)$:

$$1 - \frac{3}{4}z \leq E_1(z) - \gamma - \log z \leq 1 - \frac{3}{4}z + \frac{11}{36}z^2. \tag{22a}$$

A related function is the (uppper) *incomplete gamma function* defined by

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt, \Re(s) > 0.$$

We have the following properties of $\Gamma(s, x)$:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, \tag{22b}$$

$$\Gamma(s+1, 1) = e^{-1} \left\lfloor \frac{s!}{e} \right\rfloor, s \in \mathbb{Z}^+, \tag{22c}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, |x| \to +\infty. \tag{22d}$$

Corollary 4.2.12. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \gg 1.$$

When $s := 1$, we have the known bound in Mertens theorem. For $s > 1$, we obtain that

$$P_s(x) \approx E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1).$$

It follows that

$$\gamma_0(s, x) + o(1) \leq P_s(x) \leq \gamma_1(s, x) + o(1),$$

where it suffices to take

$$\begin{aligned}
\gamma_0(z, x) &= -s \log\left[\frac{\log x}{\log 2}\right] - \frac{3}{4}s(s-1)\log(x/2) - \frac{11}{36}s(s-1)^2\log^2(2) \\
\gamma_1(z, x) &= s \log\left[\frac{\log x}{\log 2}\right] - \frac{3}{4}s(s-1)\log(x/2) + \frac{11}{36}s(s-1)^2\log^2(x).
\end{aligned}$$

Proof. Let $s > 1$ be real-valued. By Abel summation where our summatory function is given by $A(x) = \pi(x) \sim \frac{x}{\log x}$ and our function $f(t) = t^{-s}$ so that $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned}
P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\
&= E_1((s-1)\log x) - E_1((s-1)\log 2) + o(1), |x| \to \infty.
\end{aligned}$$

Now using the inequalities in Facts 4.2.11, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned}
\frac{P_s(x)}{s} &\geq -\log\left[\frac{\log x}{\log 2}\right] - \frac{3}{4}(s-1)\log(x/2) - \frac{11}{36}(s-1)^2\log^2(2) \\
\frac{P_s(x)}{s} &\leq \log\left[\frac{\log x}{\log 2}\right] - \frac{3}{4}(s-1)\log(x/2) + \frac{11}{36}(s-1)^2\log^2(x).
\end{aligned}$$

This completes the proof of the bounds cited above in the statement of this lemma. $\square$

Theorem 4.2.13 (Generating Functions of Symmetric Functions)**.** *We have that for all integers $0 \leq k \leq m$*

$$[z^k] \prod_{1 \leq i \leq m} (1 - f(i)z)^{-1} = [z^k] \exp\left(\sum_{j \geq 1} \left(\sum_{i=1}^{m} f(i)^j\right) \frac{z^j}{j}\right).$$

Thus we obtain upper and lower bounds of the partial prime products of the form

$$\alpha_0(z, x) \leq \prod_{p \leq x} \left(1 - \frac{z}{p}\right)^{-1} \leq \alpha_1(z, x),$$

where it suffices to take

$$\begin{aligned}
\alpha_0(z, x) &= \frac{\exp\left(\frac{55}{4} \log^2 2\right)}{\log^3 2} (\log x)^3 \left(\frac{e^B \log^2 x}{\log 2}\right)^z \\
\alpha_1(z, x) &= \exp\left(\frac{11}{3} \log^2 x\right) \left(e^B \log 2\right)^z.
\end{aligned}$$

Proof. In our case we have that $f(i)$ denotes the i^{th} prime. Hence, summing over all $p \leq x$ in place of $0 \leq k \leq m$ in the previous formula applied in tandem with Corollary 4.2.12, we obtain that the logarithm of the generating function series we are after corresponds to

$$\begin{aligned}
\log\left[\prod_{p \leq x} \left(1 - \frac{z}{p}\right)^{-1}\right] &= (B + \log\log x)z + \sum_{j \geq 2} \left[a(x) + b(x)(j-1) + c(x)(j-1)^2\right] z^j \\
&= (B + \log\log x)z - a(x)\left(1 + \frac{1}{z-1} + z\right) + b(x)\left(1 + \frac{2}{z-1} + \frac{1}{(z-1)^2}\right) \\
&\quad - c(x)\left(1 + \frac{4}{z-1} + \frac{5}{(z-1)^2} + \frac{2}{(z-1)^3}\right).
\end{aligned}$$

In the previous equations, the upper and lower bounds formed by the functions (a, b, c) are given by

$$\begin{aligned}
(a_\ell, b_\ell, c_\ell) &:= \left(-\log\left[\frac{\log x}{\log 2}\right], \frac{3}{4} \log\left(\frac{x}{2}\right), -\frac{11}{36} \log^2 2\right) \\
(a_u, b_u, c_u) &:= \left(\log\left[\frac{\log x}{\log 2}\right], -\frac{3}{4} \log\left(\frac{x}{2}\right), \frac{11}{36} \log^2 x\right).
\end{aligned}$$

Now we make a prudent decision to set $R := \frac{3}{2}$ so that

$$z \equiv z(k, x) = \frac{k-1}{\log\log x} \in [0, R),$$

for $x \gg 1$ very large. Thus $(z-1)^{-m} \in [(-1)^m, 2^m]$ for integers $m \geq 1$, and we can then form the upper and lower bounds from above. What we get out of these formulas is stated as in the theorem bounds. $\square$

Corollary 4.2.14 (Bounds on $G(z)$ from MV)**.** *We have that for the function $G(z) := F(1, z)/\Gamma(z+1)$ from Montgomery and Vaughan, there are constants A_0, A_1 and functions of x only, $B_0(x), B_1(x), C_0(x), C_1(x)$, so that*

$$A_0 \cdot B_0(x) \cdot C_0(x)^z \left(1 - \frac{z}{B} (\log x)^{\frac{1}{14}}\right) \leq G(z) \leq A_1 \cdot B_1(x) \cdot C_1(x)^z \left(1 - \frac{z}{B} (\log x)^{\frac{1}{14}}\right).$$

It suffices to take

$$\begin{aligned}
A_0 &= \frac{\exp\left(\frac{55}{4} \log^2 2\right)}{\log^3(2) \cdot \Gamma(5/2)} \approx 1670.84511225 \\
B_0(x) &= \log^3 x
\end{aligned}$$

$$
\begin{aligned}
C_0(x) &= \frac{\log x}{\log 2} \\
A_1 &= 1 \\
B_1(x) &= \exp\left(\frac{11}{3}\log^2 x\right) \\
C_1(x) &= \frac{\log 2}{\log x}.
\end{aligned}
$$

Proof. This result is a consequence of applying both Corollary 4.2.10 and Theorem 4.2.13 to the definition of $G(z)$. In particular, we obtain bounds of the following form from the theorem:

$$\frac{A_0 \cdot B_0(x) \cdot C_0(x)^z}{\Gamma(z+1)} \leq \frac{G(z)}{\prod_p \left(1-\frac{1}{p}\right)^z} \leq \frac{A_1 \cdot B_1(x) \cdot C_1(x)^z}{\Gamma(z+1)}.$$

Since $z \equiv z(k,x) = \frac{k-1}{\log\log x}$ and $k \in [1, R\log\log x]$, we obtain that for small k and $x \gg 1$ large $\Gamma(z+1) \approx 1$, and for k towards the upper bound of its interval that $\Gamma(z+1) \approx \Gamma(5/2)$ (recall that we set $R := 3/2$ in the preceeding proof of Theorem 4.2.13). Thus when we expand out the formula given by the corollary in conjunction with these bounds on the gamma function, we obtain the claimed results. $\square$

Lemma 4.2.15. *Suppose that $\Omega(n) = m$ where $n \geq 2$ and $m \geq 1$. Then*

$$g^{-1}(n) \approx e^{-1} \times m!.$$

Proof. We form the Laplace transform of the binomial theorem

$$(1+t)^m = \sum_{j=0}^{m} \binom{m}{j} t^j.$$

In particular, we see that

$$\int_0^\infty e^{-t}(1+t)^m dt = e \cdot \Gamma(m+1, 1).$$

So since $m \geq 1$ is an integer, we can use the facts list for the incomplete gamma function provided above to see that the result is correct. $\square$

Theorem 4.2.16. *For very large $x \gg 1$, we find that*

$$\frac{|M(x)|}{\sqrt{x}} \gg \left| o(1) + \int_{\frac{\log\log\log\log x}{2}}^{\frac{\log\log\log\log x}{2}+\frac{1}{2}} \frac{2\widetilde{A}_0\sqrt{x}}{(m+1)} (2t+1)^{\frac{3}{2}(2t+1)} e^{(2t+1)(m+1)} dt \right|.$$

A corresponding upper bound is possible to formulate, but more difficult to express exactly. It is given by

$$\frac{|M(x)|}{\sqrt{x}} \ll \sum_{m \geq 0} \left[\left(\frac{11}{3}\right)^m \frac{14\widetilde{A}_1\sqrt{x}}{(14m-3)m!}\right] \left(O(1) + \int_{\frac{\log\log\log\log x}{2}}^{\frac{\log\log\log\log x}{2}+\frac{1}{2}} e^{(2t+1)(2m-10/7)} (2t+1)^{\frac{3}{2}(2t+1)} dt\right).$$

Proof. We first see easily that the summatory functions satisfy

$$G^{-1}(x) \approx \sum_{k=1}^{R\log\log x} \left[\sum_{j=0}^{k} \binom{k}{j} j!\right] (-1)^k \widehat{\pi}_k(x),$$

by a straightforward counting argument. Thus, using Lemma 4.2.15, our lower bound is of the form

$$G^{-1}(x) \approx \frac{A_0 \cdot x \cdot B_0(x)}{e\log x} \times \sum_{k=1}^{R\log\log x} (-1)^k k(\log\log x)^{k-1} \left(1 - \frac{k-1}{\log\log x}(\log x)^{1/14}\right) C_0(x)^{\frac{k-1}{\log\log x}}$$

$$
\begin{aligned}
&\approx \frac{A_0 x B_0(x)}{e\log x}\left(\frac{B+2e(\log x)^{1/14}}{Be^3(\log\log x)^3}+\frac{1}{e^2(\log\log x)^2}\right)\\
&\quad+\frac{A_0 x B_0(x)}{e\log x}\left(\frac{3(-1)^{\lfloor R\log\log x\rfloor}}{2Be^3}(2B-3\log^{1/14}(x))(\log\log x)^{\frac{3}{2}\log\log x}\right)\left[1+\sum_{i=1}^{3}O\left(\frac{1}{(\log\log x)^i}\right)\right]
\end{aligned}
$$

Notice that we have used that for any fixed constant C_0,

$$
\lim_{x\to\infty}\left(C_0(\log x)^{\pm 1}\right)^{\frac{1}{\log\log x}}=e^{\pm 1}.
$$

For x sufficiently large, the dominant order terms in the previous expansion correspond to

$$
\begin{aligned}
G^{-1}(x) &\gg -\frac{3(-1)^{\lfloor R\log\log x\rfloor}\cdot A_0 x B_0(x) C_0(x)^{3/2}}{2Be^4\log x}\left(2B-3\log^{\frac{1}{14}}x\right)(\log\log x)^{\frac{3}{2}\log\log x}\\
&\sim \widetilde{A}_0 x(\log x)^{\frac{25}{7}}(1+\log\log x)^{\frac{3}{2}\log\log x},
\end{aligned}
\tag{23}
$$

for some absolute constant $\widetilde{A}_0$. Now by an approximate summation by parts argument, we have that

$$
\begin{aligned}
M(x) &= \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k)+1)\\
&\approx G^{-1}(x)-\sum_{k=1}^{x/2}G^{-1}(k)\cdot\frac{x}{k^2\log(x/k)}.
\end{aligned}
\tag{24}
$$

There are two primary problems left to tackle here. The first is that the expression we obtained as the dominant lower bound terms for $G^{-1}(x)$ in (23) are signed as the floor of $\frac{3}{2}\log\log x$. So we need to break up our intervals of integration piecewise into chunks of the sizes

$$
\left[e^{e^{\frac{2k}{3}}}, e^{e^{\frac{2(k+1)}{3}}}\right],\ \text{for } k\in[15,\log\log[R\log\log x]].
$$

This part is not so bad once we get a handle on the construction.

The second problem is that we only really have a good, reasonably simple closed-form expression for the integrals resulting from the right-hand-side of (24) if we bound some terms. In particular, we have the following exact antiderivative, which is similar, but not identical to the integral we are trying to evaluate:

$$
\begin{aligned}
\int \log^m(t)(\log\log t)^p\frac{x}{t}dt &= \frac{x(-1)^p}{(m+1)^{p+1}}\Gamma\left(p+1,-(m+1)\log\log t\right)\\
&\sim \frac{x}{m+1}(\log t)^{m+1}(\log\log t)^p, t\to\infty.
\end{aligned}
$$

We will just disregard the missing reciprocal term of $\log(x/t)$ in the integral since it will not be asymptotically significant – even if we were to include it.

Let $z\equiv z(t):=\log\log t$. In this case, with $m:=\frac{25}{7}\approx 3.57143$, we can expand (23) and (24) using the binomial series and the Stirling numbers of the first kind as

$$
\begin{aligned}
\frac{(m+1)I(t)}{\widetilde{A}_0 x\cdot\log^{m+1}(t)} &= \sum_{p\geq 0}\sum_{k\geq 0}\begin{bmatrix}p\\k\end{bmatrix}\frac{(-z)^{p+k}}{p!}\left(\frac{3}{2}\right)^k\\
&= \sum_{k\geq 0}\frac{(3/2)^k z^k}{k!}\log(1+z)^k\\
&= (1+z)^{\frac{3}{2}z}.
\end{aligned}
$$

Now by summing over the intervals depending on k, and separating the parity of the corresponding terms, we obtain that for some small $u_0 \approx 15$, we need to perform the sums

$$\widehat{S}_2(x) := \sum_{k=u_0}^{\frac{\log\log\log\log x}{2}} \frac{2\widetilde{A}_0 x}{(m+1)} \left[(2k)^{\frac{3}{2}(2k)} e^{(2k)(m+1)} - (2k)^{\frac{3}{2}(2k+1)} e^{(2k+1)(m+1)}\right].$$

When $k \gg 1$ is very large, the tails of the sum shrink as the terms are approximately zero in difference. The question in how much the index shift contributes to the asymptotics of this sum can be made precise by a change of variable:

$$\frac{1}{\sqrt{x}}\widehat{S}_2(x) \approx o(1) + \int_{\frac{\log\log\log\log x}{2}}^{\frac{\log\log\log\log x}{2}+\frac{1}{2}} \frac{2\widetilde{A}_0\sqrt{x}}{(m+1)}(2t+1)^{\frac{3}{2}(2t+1)} e^{(2t+1)(m+1)} dt.$$

Thus, in absolute value, a factor of $\gg \sqrt{x}(\log\log x)$ survives. This would appear to be a good thing! A similar argument gives the upper bound. $\square$

Lemma 4.2.17. *Let* $\widehat{g}^{-1}(x)$ *denote the* average value *of* $g^{-1}(n)$ *over the integers* $n \leq x$*:*

$$\widehat{g}^{-1}(x) = \frac{1}{x} \sum_{n \leq x} g^{-1}(n).$$

Let the function $\widehat{M}(x)$ *denote the Mertens-like summatory function formed by replacing* $g^{-1}(k)$ *by* $\widehat{g}^{-1}(k)$ *in* (19)*. Then*

$$M(x) \approx \widehat{M}(x) + G^{-1}(x) - \int_2^x \frac{G^{-1}(t)}{t} dt,$$

where $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ *is the summatory function of* g^{-1}*.*

Proof. We have that

$$\begin{aligned}
\widehat{M}(x) &= \sum_{k=1}^{x} \widehat{g}^{-1}(k) \left(\pi(x/k) + 1\right) \\
&= \sum_{k=1}^{x} \frac{G^{-1}(k)}{k} \left(\pi(x/k) + 1\right) \\
&= \sum_{k=1}^{x} g^{-1}(k) \times \sum_{j=k}^{x} \frac{1}{j} \left(\pi(x/j) + 1\right) \\
&\approx \sum_{k=1}^{x} g^{-1}(k) \left[\log x - \log k + \mathrm{li}(x/k)\right] \\
&\approx \sum_{k=1}^{x} g^{-1}(k) \left[\log\left(\frac{x}{ek}\right) + \pi(x/k) + 1\right].
\end{aligned}$$

Now we apply (19) to extract one additive term of $M(x)$, and apply summation by parts (Abel summation) to formulate the remaining terms in the statement above. □

Note that á priori, we expect (for a number of reasons), the sums $\widehat{M}(x)$ to be simpler to evaluate than the corresponding exact sum for $M(x)$ from (19). What Lemma 4.2.17 then allows us to do is relate the computationaly easier sums $\widehat{M}(x)$ and $G^{-1}(x)$ to $M(x)$ and proceed by evaluating the average order and summatory function forms of $g^{-1}(n)$.