When combined with Corollary 3.2, the proof of Proposition 4.1 yields the crucial starting point providing an exact formula for $M(x)$ stated in (1) of Corollary 3.3.

Proposition 4.2 (The key signedness property of $g^{-1}(n)$)**.** *For the Dirichlet invertible function, $g(n) := \omega(n) + 1$ defined such that $g(1) = 1$, at any $n \geq 1$, we have that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$. The notation for the operation given by $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0, \pm 1\}$ denotes the sign of the arithmetic function h at n.*

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Proof. Let $D_f(s) := \sum_{n\geq 1} f(n) n^{-s}$ denote the Dirichlet generating function (DGF) of an aeithmetic function $f(n)$ convergent for $\Re(s) > \sigma_f$. For all $\Re(s) > 1$, expanding the DGF for the function $g^{-1}(n)$ yields

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}.$$

Let $h^{-1}(n) := (\omega * \mu + \varepsilon)^{-1}(n) = [n^{-s}](P(s)+1)^{-1}$. Then we have using the recurrence relation for h^{-1} with $\chi_{\mathbb{P}} = \omega * \mu$ that

I have no idea where this comes from.

$$\begin{aligned}(h^{-1} * 1)(n) &= \sum_{p_1 | n} h^{-1}\left(\frac{n}{p_1}\right) = \lambda(n) \times \sum_{p_1|n} \sum_{p_2 | \frac{n}{p_1}} \cdots \sum_{p_{\Omega(n)} | \frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1 \\ &= \begin{cases} \lambda(n) \times (\Omega(n)-1)!, & n \geq 2; \\ \lambda(n), & n = 1. \end{cases}\end{aligned}$$

$h^{-1} * \mu = g^{-1}$??

We need to compute the sign of the function $h^{-1} * \mu$. First, by Möbius inversion and the formula for $h^{-1} * 1$ we proved above, for each $n \geq 2$, there exist constants $C_{1,n}, C_{2,n} > 0$ so that

$$C_{1,n} \cdot (\lambda * \mu)(n) \leq h^{-1}(n) \leq C_{2,n} \cdot (\lambda * \mu)(n).$$

?

Since both λ, μ are multiplicative, we can compute that for any prime p and integers $\alpha \geq 1$,

$$(\lambda * \mu)(p^\alpha) = \lambda(p^\alpha) - \lambda(p^{\alpha-1}) = 2\lambda(p^\alpha).$$

I do not understand.

Thus the previous inequalities are re-stated in the form of

$$2C_{1,n} \cdot \lambda(n) \leq h^{-1}(n) \leq 2C_{2,n} \cdot \lambda(n).$$

Now to bound $h^{-1} * \mu$, we similarly can see by multiplicativity that

$$4C_{1,n} \cdot \lambda(n) \leq (h^{-1} * \mu)(n) \leq 4C_{2,n} \cdot \lambda(n).$$

Since the absolute constants (for each n) are positive, we recover the signedness of $g^{-1}(n)$ as $\lambda(n)$. $\square$

4.3 Other facts and listings of results we will need in our proofs

Theorem 4.3 (Mertens theorem)**.**

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log\log x + B + o(1),$$

where $B \approx 0.2614972128476427837554$ *is an absolute constant.*

Prop 4.2 needs a rock solid proof. Everyone must be able to read & understand it.

Summary 6.3 (Asymptotics of the $C_k(n)$)**.** We have the following asymptotic relations relations for the growth of small cases of the functions $C_k(n)$:

$$C_1(n) \sim \log\log n$$
$$C_2(n) \sim (\log\log n)^3.$$

I thought you were using expectation.

The previous limiting asymptotics are computed from the explicit formulas for small k in Example 6.2 using the average order arguments such that $\mathbb{E}[\nu_p(n)] = \log\log n$ and for $p|n$, $\mathbb{E}[p] = \frac{n}{\log n}$.

Theorem 3.6 is proved next. The theorem makes precise what these formulas already suggest about the main terms of the growth rates of $C_k(n)$ as functions of k, n for limiting cases of n large for fixed k. Since we will be essentially averaging the inverse functions, $g^{-1}(n)$, via their summatory functions over the range $n \leq x$ for x large, we tend not to bound any relevant components to obtaining these results but by the average order case, which evens out when we sum (i.e., average) and tend to infinity.

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Proof of Theorem 3.6. We showed how to compute the formulas for the base cases in the preceeding examples discussed above in Example 6.2. We can also see that $C_1(n)$ satsfies the formula we must establish when $k := 1$. Let's proceed by using induction to prove that our asymptotics hold for all $k \geq 1$ using the recurrence formula from (8) relating $C_k(n)$ to $C_{k-1}(n)$ whenever $k \geq 2$. In particular, suppose that $k \geq 2$ and let the inductive assumption for all $1 \leq m < k$ be that

$$C_m(n) \sim (\log\log n)^{2m-1}.$$

Now we have by the recursive formula that

$$\begin{aligned} C_k(n) &= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} (\log\log(dp^i))^{2k-3} \\ &\sim \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \left[\int (\log\log(dp^\alpha))^{2k-3} d\alpha\right]\Bigg|_{\alpha=\nu_p(n)}. \end{aligned} \tag{9}$$

The inner integral in the previous equation can be evaluated using the limiting asymptotic expansions for the incomplete gamma function stated in Section 4.3. In particular, for $p|n$ and $n \geq 2$ large, we let the parameters assume average order values of

$$\mathbb{E}[\nu_p(n)] = \log\log n, \mathbb{E}[p] = \frac{n}{\log n}.$$

Now you are using expectation??

Then we evaluate the integral from above as

$$\begin{aligned} \int (\log\log(dp^\alpha))^{2k-3} d\alpha &\sim \alpha\left(\log d + \alpha \cdot \log p\right)^{2k-3} \\ &\sim \alpha\left(\log\alpha + \log\log p + \frac{d}{\alpha\log p}\right)^{2k-3}. \end{aligned}$$

We know that the average order of the number of primes $p|n$ is given by $\mathbb{E}[\omega(n)] = \log\log n$, so approximating p as the cited function of n initially allows us to take a factor of $\log\log n$

and remove the outer divisor sum in (9). So we obtain that *

$$\begin{aligned}
C_k(n) &\sim (\log\log n)^2 \left[\log\log\log n + \log\log n + \frac{\pi^2}{12}\frac{n}{\log n}\frac{1}{\left(\frac{n}{\log n}\right)^{\log\log n}}\right]^{2k-3} \\
&\sim (\log\log n)^{2k-1}.
\end{aligned}$$

In the previous equation, we have used that the average order of the sum-of-divisors function, $\sigma_1(n)$, is given by $\mathbb{E}[\sigma_1(n)] = \frac{\pi^2 \cdot n}{12}$ [13, §27.11]. Thus by mathematical induction, we have proved that the claimed limiting asymptotic behavior holds for $C_k(n)$ whenever $k \geq 1$ as $n \to \infty$. □

Using Lemma 6.1 directly is problematic since forming the summatory function of the exact $g^{-1}(n)$ that obey this formula leads to a nested recurrence relation involving $M(x)$, e.g., more in-order sums of consecutive Möbius function terms appear yet again. Some suggestive numerical experiments illustrate that this implicit recursive dependence of our new formulas for $M(x)$ can be avoided simply by using an inexact, but still provably asymptotically sufficient in form expression approximating $g^{-1}(n)$. The next corollary provides the specific inexact, asymptotically accurate formula for these inverse functions we have in mind.

What Corollary 6.4, allows us to do is provide a substantially simpler formula and limiting bound on the summatory functions $G^{-1}(x)$ of $g^{-1}(n)$. The form of this new formula for $G^{-1}(x)$ is established in Corollary 6.5, which is subsequently stated and easily given a short proof immediately after the next result is proved. This is an important leap in expressing a workable formula that we can use to bound these summatory functions from below when x is large as rigorously justified in Theorem 7.4.

Corollary 6.4 (Computing the inverse functions)**.** *For $n \geq 2$ as $n \to \infty$ we have that*

$$g^{-1}(n) \sim \left(\frac{\pi^2}{3} - \frac{1}{2}\right) \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

In particular, we can bound the error terms in the approximation of Lemma 6.1 by the previous formula to ensure that

$$\left|\frac{\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)}{g^{-1}(n)}\right| \xrightarrow{n\to\infty} \frac{\pi^2}{3} - \frac{1}{2} \approx 2.78987.$$

Proof. Let

$$S_R(n) := \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

*Here, we simplify the iterated logarithm expansions as $n \to \infty$ by writing

$$\begin{aligned}
\log\log\left(\frac{n}{\log n}\right) &= \log\left[\log n + \log\left(1 + \frac{1}{n\log n}\right)\right] \\
&\sim \log\log n + \frac{1}{n(\log n)^2} \\
&\sim \log\log n.
\end{aligned}$$

USE eqref, to recall facts that you are referring to.

The argument for Cor 6.4 looks like it does not account for an exchange of limits.

$$g^{-1}(n) = \sum_{d|n} \mu(n/d)\,\lambda(d)\,\tilde{C}_d\,, \quad \text{where } \tilde{C}_d > 0$$

$$\sim c\,\lambda(n) \sum_{d|n} \tilde{C}_d$$

$n = P_1^2 P_2^2$, $\lambda(n) = 1$

d	$\mu(n/d)$	$\lambda(d)$	$\mu(n/d)\,\lambda(d)$
P_1	0		0
P_1^2	0		0
P_2	0		0
P_2^2	0		0
P_1P_2	1	1	1

For Corollary 6.4 to be true as $p_2 \longrightarrow \infty$ must be that

$$\frac{C_{p_1} + C_{p_1^2} + C_{p_2} + C_{p_1 p_2} + C_{p_2^2}}{C_{p_1 p_2}} \xrightarrow[p_2 \to \infty]{} \alpha$$

where $0 < \alpha < 1$. This doesn't make sense.