2.3 Statements of useful asymptotic formulas

Facts 2.2 (The incomplete gamma function)**.** The (upper) *incomplete gamma function* is defined by [16, §8.4]

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt, \mathrm{Re}(s) > 0.$$

The following properties of $\Gamma(a,x)$ hold at positive real $a > 0$:

$$\Gamma(a,x) = (a-1)! \cdot e^{-x} \times \sum_{k=0}^{a-1} \frac{x^k}{k!}, a \in \mathbb{Z}^+, x > 0, \tag{10a}$$

$$\Gamma(a,x) \sim x^{a-1} \cdot e^{-x}, \text{ for fixed } a > 0, \text{ as } x \to \infty. \tag{10b}$$

Moreover, for real $z > 0$, as $z \to +\infty$ we have that [13]

$$\Gamma(z,z) \sim \sqrt{\frac{\pi}{2}} z^{z-\frac{1}{2}} e^{-z} + O\left(z^{z-1}e^{-z}\right), \tag{10c}$$

and if $z, a \to \infty$ with $z = \lambda a$ for some $\lambda > 0$ such that $(\lambda - 1)^{-1} = o(|a|^{1/2})$, then

$$\Gamma(a,z) \sim z^a e^{-z} \times \sum_{n \geq 0} \frac{(-a)^n b_n(\lambda)}{(z-a)^{2n+1}}, \tag{10d}$$

where the sequence $b_n(\lambda)$ satisfies the characteristic relation that $b_0(\lambda) = 1$ and[B]

$$b_n(\lambda) = \lambda(1-\lambda)b'_{n-1}(\lambda) + \lambda(2n-1)b_{n-1}(\lambda), n \geq 1.$$

Proposition 2.3. *Suppose that $z, a > 0$ are real parameters. If $z = \lambda a$ for some $\lambda > 1$, then as $z \to +\infty$ we have that*

$$\Gamma(a,z) \sim \frac{z^{a-1}e^{-z}}{1-\lambda} + O_\lambda\left(z^{a-2}e^{-z}\right).$$

Proof. We can see that for $\lambda > 1$, $b_n(\lambda) \sim \lambda^n \cdot n!$. It follows from (10d) that (*cf.* [13, §A.1])

$$\begin{aligned}\Gamma(a,z) &\sim z^{a-1}e^{-z} \times \sum_{0 \leq n < z} \frac{(-\lambda)^n \cdot n!}{z^n \cdot (1-\lambda)^{2n+1}} \\ &= z^{a-1}e^{-z} \times \sum_{0 \leq n < z} \frac{n!}{(1-\lambda)(z-a)^n}\end{aligned}$$

Since $z - a \asymp z$, or $z - a = z(1 - 1/\lambda)$ is proportional to z, as $z \to \infty$ we get that for all indices of the previous sum $1 \leq n < z$ the asymptotic order of these terms is of lesser order than that of the summand corresponding to $n := 0$. Also, we see that the n^{th} summand above is at most c^n for some bounded constant $0 < c < 1$ whenever $0 \leq n < z$. □

Lemma 2.4. *For $x > e$, we have that*

$$S_1(x) := \frac{x}{\log x} \times \sum_{1 \leq k \leq \log\log x} \frac{(-1)^k (\log\log x)^{k-1}}{(k-1)!} \sim x\left(1 - \frac{e^2}{\sqrt{2\pi}\sqrt{\log\log x}}\right), \tag{11a}$$

$$S_2(x) := \frac{x}{\log x} \times \sum_{1 \leq k \leq \log\log x} \frac{k(\log\log x)^{k-1}}{(k-1)!} \sim \frac{x}{2}, \tag{11b}$$

$$S_3(x) := x \times \sum_{1 \leq k \leq \log\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \sim \frac{x(\log x)}{2}. \tag{11c}$$

[B] An exact formula for $b_n(\lambda)$ is given in terms of the *second-order Eulerian number triangle* [22, A008517] as follows:

$$b_n(\lambda) = \sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k-1} \right\rangle\!\!\right\rangle \lambda^k.$$

Proof of (11a). Let the component sums $T_{1,1}(x)$ and $T_{1,2}(x)$ be defined by

$$T_{1,1}(x) := \sum_{\substack{k \leq \log\log x \\ k \text{ odd}}} \frac{(\log\log x)^{k-1}}{(k-1)!}$$

$$T_{1,2}(x) := \sum_{\substack{k \leq \log\log x \\ k \text{ even}}} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

Then we have that

$$S_1(x) = \frac{x}{\log x}\left(T_{1,2}(x) - T_{1,1}(x)\right).$$

We find that for all sufficiently large x, the difference of the component sums can be expressed as a sum of convergent generalized hypergeometric functions such that for $b := \log\log x$ we have the following expansions:

$$\begin{aligned}\frac{T_{1,2}(x) - T_{1,1}(x)}{\log x} &\sim 1 - \frac{2e^2}{\sqrt{2\pi}\cdot b^{1/2}} \times \left[1 + \sum_{k\geq 1} \frac{b^{2k}}{(2+b)\cdots(2k+2+b)\times(k+3+b)\cdots(2k+2+b)}\right] \\ &= 1 - \frac{2e^2}{\sqrt{2\pi}\cdot b^{1/2}} \times \sum_{k\geq 0} \frac{1}{\left(1+\frac{2}{b}\right)\cdots\left(1+\frac{k+1}{b}\right)} \cdot \frac{1}{\left(1+\frac{k+3}{b}\right)\cdots\left(1+\frac{2k+2}{b}\right)}.\end{aligned}$$

When $k < \left\lfloor \frac{\log\log x}{2} \right\rfloor$, we see that the denominator terms coincide with a convergent geometric series, e.g., since $j/b < 1$ for all $2 \leq j \leq 2k+2$, and hence the product for the k^{th} summand in these cases is approximated by $(1+o(1))^{2k} = 1 + o(1)$. Then the main term contribution of the sum over k within this range corresponds to $\left\lfloor \frac{\log\log x}{2} \right\rfloor \sim \frac{\log\log x}{2}$. For $k \geq \left\lfloor \frac{\log\log x}{2} \right\rfloor$, we have convergence of the series to an absolute constant $0 < C_0(x) < +\infty$ that varies (only slighly) with b. These cases of the previous displayed formula lead us to conclude that

$$T_{1,2}(x) - T_{1,1}(x) \sim 1 - \frac{e^2\sqrt{\log\log x}}{\sqrt{2\pi}} + O\left(\frac{1}{\sqrt{\log\log x}}\right). \qquad \square$$

Proof of (11b). We can sum exactly to see that

$$S_2(x) = \frac{x\cdot\Gamma(\log\log x, \log\log x)}{\Gamma(\log\log x)}.$$

When we apply the form of the known incomplete gamma function asymptotics from (10c) as $x \to \infty$ and apply Stirling's formula to approximate $\Gamma(N+1) = N\cdot\Gamma(N)$ for large N, we can see that $S_2(x) \sim \frac{x}{2}$. $\square$

Proof of (11c). We can sum $S_3(x)$ exactly to arrive at the following expression:

$$S_3(x) = \frac{x\cdot\Gamma(\log\log x, \log\log x)}{\Gamma(\log\log x)}.$$

When we apply (10c) with Stirlings formula to show that $\Gamma(\log\log x+1) \sim \sqrt{2\pi(\log\log x)}\left(\frac{\log\log x}{e}\right)^{\log\log x}$ for all x sufficiently large, we can then conclude the result. We have written $\Gamma(\log\log x) = \frac{1}{\log\log x}\times\Gamma(\log\log x+1)$ to apply Stirling's formula in the last step. $\square$

2.4 Results on the distribution of exceptional values of $\omega(n)$ and $\Omega(n)$

The next theorems reproduced from [12, §7.4] characterize the relative scarcity of the distributions of $\omega(n)$ and $\Omega(n)$ for $n \leq x$ such that $\omega(n), \Omega(n) > \log\log x$. Since $\mathbb{E}[\omega(n)], \mathbb{E}[\Omega(n)] = \log\log n + B$ for $B \in (0,1)$ an absolute constant in each case, these results imply a very regular, normal tendency of these arithmetic functions towards their respective average order.