A Appendix: Asymptotic formulas

We thank Gergő Nemes from the Alfréd Rényi Institute of Mathematics for his careful notes on the limiting asymptotics for the sums identified in this section. We have adapted the communication of his proofs to establish the next few lemmas.

Facts A.1 (The incomplete gamma function)**.** The (upper) *incomplete gamma function* is defined by [19, §8.4]

$$\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt, a \in \mathbb{R}, |\arg z| < \pi.$$

The function $\Gamma(a,z)$ can be continued to an analytic function of z on $\mathbb{C}\setminus\{0\}$. For $a \in \mathbb{Z}^+$, $\Gamma(a,z)$ is an entire function of z. The following properties of $\Gamma(a,z)$ hold [19, §8.4; §8.11(i)]:

$$\Gamma(a,z) = (a-1)! \cdot e^{-z} \times \sum_{k=0}^{a-1} \frac{z^k}{k!}, \text{ for } a \in \mathbb{Z}^+, z \in \mathbb{C}; \tag{30a}$$

$$\Gamma(a,z) \sim z^{a-1}e^{-z}, \text{ for fixed } a \in \mathbb{C}, \text{ as } z \to +\infty. \tag{30b}$$

Moreover, for real $z > 0$, as $z \to +\infty$ we have that [14]

$$\Gamma(z,z) = \sqrt{\frac{\pi}{2}} z^{z-\frac{1}{2}} e^{-z} + O\left(z^{z-1}e^{-z}\right), \tag{30c}$$

If $z, a \to \infty$ with $z = \lambda a$ for some $\lambda > 1$ such that $(\lambda-1)^{-1} = o(|a|^{1/2})$, then [14]

$$\Gamma(a,z) = z^a e^{-z} \times \sum_{n\geq 0} \frac{(-a)^n b_n(\lambda)}{(z-a)^{2n+1}}, \tag{30d}$$

where the sequence $b_n(\lambda)$ satisfies the characteristic relation that $b_0(\lambda) = 1$ and[D]

$$b_n(\lambda) = \lambda(1-\lambda)b'_{n-1}(\lambda) + \lambda(2n-1)b_{n-1}(\lambda), n \geq 1.$$

Proposition A.2. *Suppose that $a, z > 0$ are such that $z = \lambda a$. If $\lambda > 1$, then as $z \to +\infty$*

$$\Gamma(a,z) = \frac{z^{a-1}e^{-z}}{1-\lambda^{-1}} + O_\lambda\left(z^{a-2}e^{-z}\right).$$

If $\lambda > 0.567142 > W(1)$ where $W(x)$ denotes the principal branch of the Lambert W-function, then as $z \to +\infty$

$$\Gamma(a, ze^{\pm\pi\imath}) = -e^{\pm\pi\imath a}\frac{z^{a-1}e^{z}}{1+\lambda^{-1}} + O_\lambda\left(z^{a-2}e^{z}\right).$$

Note that we cannot write this expansion as $\Gamma(a,-z)$ directly unless $a \in \mathbb{Z}^+$ as the incomplete gamma function has a branch point at the origin with respect to its second variable. This function becomes a single-valued analytic function of its second input by continuation on the universal covering of $\mathbb{C}\setminus\{0\}$.

Proof. Using the notation from (30d) and [15], we have that

$$\Gamma(a,z) = \frac{z^{a-1}e^{-z}}{1-\lambda^{-1}} + z^a e^{-z} R_1(a,\lambda).$$

[D] An exact formula for $b_n(\lambda)$ is given in terms of the *second-order Eulerian number triangle* [25, A008517] as follows:

$$b_n(\lambda) = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \lambda^{k+1}.$$

From the bounds in [15, §3.1], we get that

$$|z^a e^{-z} R_1(a,\lambda)| \leq z^a e^{-z} \times \frac{a \cdot b_1(\lambda)}{(z-a)^3} = \frac{z^{a-2} e^{-z}}{(1-\lambda^{-1})^3}$$

The main and error terms in the previous equation can also be seen by applying the asymptotic series in (30d) directly.

The proof of the second equation above follows from the following asymptotics [14, Eq. (1.1)]

$$\Gamma(-a,z) \sim z^{-a} e^{-z} \times \sum_{n\geq 0} \frac{a^n b_n(-\lambda)}{(z+a)^{2n+1}},$$

by setting $a \mapsto a e^{\pm \pi \imath}$ and $z \mapsto z e^{\pm \pi \imath}$ with $\lambda = z/a > 0.567142 > W(1)$. The restriction on the range of λ over which the second formula holds is made to ensure that the last formula from the reference is valid at negative real a. $\square$

Lemma A.3. *For $x \to +\infty$, we have that*

$$S_1(x) := \frac{x}{\log x} \times \left| \sum_{1\leq k\leq \log\log x} \frac{(-1)^k (\log\log x)^{k-1}}{(k-1)!} \right| = \frac{x}{2\sqrt{2\pi}\log\log x} + O\left(\frac{x}{(\log\log x)^{3/2}}\right).$$

Proof. We have for $n \geq 1$ and any $t > 0$ by (30a) that

$$\sum_{1\leq k\leq n} \frac{(-1)^k t^{k-1}}{(k-1)!} = -e^{-t} \times \frac{\Gamma(n,-t)}{(n-1)!}.$$

Suppose that $t = n + \xi$ with $\xi = O(1)$ (e.g., so we can take the floor of the input n to truncate the last sum). By the second formula in Proposition A.2 with the parameters $(a,z,\lambda) \mapsto \left(n, t, 1+\frac{\xi}{n}\right)$, we deduce that as $n, t \to +\infty$.

$$\Gamma(n,-t) = (-1)^{n+1} \times \frac{t^n e^t}{t+n} + O\left(\frac{n t^n e^t}{(t+n)^3}\right) = (-1)^{n+1} \frac{t^n e^t}{2n} + O\left(\frac{t^{n-1} e^t}{n}\right). \tag{31}$$

Accordingly, we see that

$$\sum_{1\leq k\leq n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{t^n}{2n!} + O\left(\frac{t^{n-1}}{n!}\right).$$

By the variant of Stirling's formula in [19, *cf.* Eq. (5.11.8)]

$$n! = \Gamma(1+t-\xi) = \sqrt{2\pi} \cdot t^{t-\xi+1/2} e^{-t} \left(1 + O(t^{-1})\right) = \sqrt{2\pi} \cdot t^{n+1/2} e^{-t} \left(1 + O(t^{-1})\right).$$

Hence, as $n \to +\infty$ with $t := n + \xi$ and $\xi = O(1)$, we obtain

$$\sum_{k=1}^{n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{e^t}{2\sqrt{2\pi t}} + O\left(\frac{e^t}{t^{3/2}}\right).$$

The conclusion follows by taking $n := \lfloor \log\log x \rfloor$, $t := \log\log x$ and applying the triangle inequality to obtain the desired result. $\square$

Remark A.4. Gergő, things have changed slightly where I need to use this lemma in my proof in the main body of the argument. I need that the upper bound of summation is $c \log\log x$ (for the same summands) where $2 \geq c > 1$ is an absolute constant (It really doesn't matter which constant, as long as it's in that range). With this, it seems that the argument using your reference at the transition point is no longer relevant. Can you please help me to adapt this proof using the references to your own work that you know so well?

Lemma A.5. *For $x \to +\infty$, we have that*

$$S_3(x) := \sum_{1 \leq k \leq 2 \log\log x} \frac{(\log\log x)^{k-1/2}}{(2k-1)(k-1)!} = \frac{\log x}{4\sqrt{\log\log x}} + O\left(\frac{\log x}{(\log\log x)^{3/2}}\right).$$

Proof. We will give an exact proof of the result directly using the bounds for the incomplete gamma function established in the recent reference [16]. The reference takes into account the behavior of the incomplete gamma function $\Gamma(a,z)$ near the transition point $\lambda = z/a = 1$ as $a, z \to +\infty$. (**TODO: Is this part of the introduction to the proof still relevant???**)

For $n \geq 1$ and any $t > 0$, let

$$\widetilde{S}_n(t) := \sum_{1 \leq k \leq n} \frac{t^{k-1}}{(2k-1)(k-1)!}.$$

By the formula in (30a) and a change of variable, we get that

$$\begin{aligned} \widetilde{S}_n(t) &= \int_0^1 \left(\sum_{k=1}^n \frac{(s^2 t)^{k-1}}{(k-1)!}\right) ds \\ &= \frac{1}{(n-1)!} \times \int_0^1 e^{s^2 t} \Gamma(n, s^2 t) ds \\ &= \frac{1}{2(n-1)!} \times \int_0^1 \frac{e^{xt}}{\sqrt{x}} \Gamma(n, xt) dx. \end{aligned}$$

Integration by parts performed one time with

$$\left\{\begin{array}{ll} u_x = \Gamma(n, xt) & v_x' = \frac{e^{xt}}{\sqrt{x}} dx \\ u_x' = -t^n x^{n-1} e^{-xt} dx & v_x = \sqrt{\frac{\pi}{t}} \operatorname{erfi}\left(\sqrt{tx}\right) \end{array}\right\},$$

implies that

$$\begin{aligned} \widetilde{S}_n(t) &= \frac{1}{2(n-1)!} \sqrt{\frac{\pi}{t}} \Gamma(n, xt) \operatorname{erfi}\left(\sqrt{xt}\right)\Big|_0^1 + \frac{\sqrt{\pi} \cdot t^{n-1/2}}{2(n-1)!} \times \int_0^1 x^{n-1} e^{-xt} \operatorname{erfi}\left(\sqrt{tx}\right) dx \\ &= \frac{1}{2(n-1)!} \sqrt{\frac{\pi}{t}} \Gamma(n, t) \operatorname{erfi}\left(\sqrt{t}\right) + \frac{\sqrt{\pi}}{2t^{3/2}(n-1)!} \times \int_0^t s^{n-1} e^{-s} \operatorname{erfi}\left(\sqrt{s}\right) ds. \end{aligned} \tag{32}$$

Using the asymptotic series for $\operatorname{erfi}(z)$, we can see that as $t \to +\infty$

$$\frac{\sqrt{\pi}}{2} \cdot \frac{e^{-t}}{\sqrt{t}} \operatorname{erfi}(\sqrt{t}) = O(t^{-1}).$$

(**TODO: Things change from here on ...**) Suppose that $t = n + \xi$ where $\xi = O(1)$. According to the reference [16, Eq. (2.4)], we have that as $t \to +\infty$

$$\Gamma(n, t) = t^n e^{-t} \sqrt{\frac{\pi}{2t}} \left(1 + O(t^{-1/2})\right).$$

By the variant of Stirling's formula expressed in [19, §5.11(i)], we again have that

$$(n-1)! = \sqrt{2\pi} \cdot t^{n-1/2} e^{-t} \left(1 + O(t^{-1})\right).$$

Whence, with $t = n + O(1)$ and as $n \to +\infty$, we obtain

$$\widetilde{S}_n(t) = \frac{e^t}{4t} + O\left(\frac{e^t}{nt^{3/2}}\right).$$

The conclusion follows taking $n = \lfloor 2 \log\log x \rfloor$, $t = \log\log x$ and mulitplying $\widetilde{S}_n(t)$ by $(\log\log x)^{1/2}$. $\square$