Lower bounds on the Mertens function $M(x)$ for $x \gg 2.3315 \times 10^{1656520}$

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Last Revised: Thursday 16^{th} April, 2020 – Compiled with LaTeX2e

Abstract

The Mertens function, $M(x) = \sum_{n \leq x} \mu(n)$, is classically defined to be the summatory function of the Möbius function $\mu(n)$. In some sense, the Móbius function can be viewed as a signed indicator function of the squarefree integers which have asymptotic density of $6/\pi^2 \approx 0.607927$ and a corresponding well-known asymptotic average order formula. The signed terms in the sums in the definition of the Mertens function introduce complications in the form of semi-randomness and cancellation inherent to the distribution of the Möbius function over the natural numbers. The Mertens conjecture which states that $|M(x)| < C \cdot \sqrt{x}$ for all $x \geq 1$ has a well-known disproof due to Odlyzko et. al. It is widely believed that $M(x)/\sqrt{x}$ is an unbounded function which changes sign infinitely often and exhibits a negative bias over all natural numbers $x \geq 1$.

We focus on obtaining new lower bounds for $M(x)$ by methods that generalize to handle other related cases of special number theoretic summatory functions. The key to our proofs calls upon a known result from the standardized summatory function enumeration by Dirichlet generating functions (DGFs) found in Chapter 7 of Montogomery and Vaughan. There is also a distinct flavor of combinatorial analysis peppered in with the standard methods from analytic number theory which distinguishes our methods. We make surprising new claims about a classical conjecture on the boundedness of $|M(x)|/\sqrt{x}$ along an asymptotically large growing infinite subsequence of reals as a concluding corollary to the lower bounds on components of our updated formulas for $M(x)$ proved within the article.

Keywords and Phrases: *Möbius function sums; Mertens function; summatory function; arithmetic functions; Dirichlet inverse; Liouville lambda function; prime omega functions; prime counting functions; Dirichlet series and DGFs; asymptotic lower bounds; Mertens conjecture; asymptotic methods from the Montgomery and Vaughan textbook.*

Primary Math Subject Classifications (2010): *11N37; 11A25; 11N60; 11N64; and 11-04.*

Draft prepared for Professors Kevin Ford, Michael Lacey and Bruce Reznick.

I have checked these results by careful reading, and re-reading at least twice myself with input from others, before emailing this document. I will eventually also have organized `Mathematica` *notebook references to help with verfiying the claims and performing some of the more involved integration and summation procedures. Please let me know if there is any other information or clarifications I can provide to help you with understanding my argument and the new constructions.*

Reference on abbreviations, special notation and other conventions

Symbol	Definition
$o(g), O_\alpha(h)$	Using standard notation, we write that $f = o(g)$ if $$\lim_{x\rightarrow\infty} \frac{f(x)}{g(x)} = 0.$$ We adapt the stock big-Oh notation, writing $f = O_{\alpha_1,\ldots,\alpha_k}(g)$ for some parameters $\alpha_1, \ldots, \alpha_k$ if $f = O(g)$ subject only to some potentially fluctuating parameters that depend on the fixed α_i.
$\lceil x \rceil$	The ceiling function $\lceil x \rceil := x + 1 - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$.
$C_k(n)$	Auxillary component functions in obtaining asymptotic bounds on $g^{-1}(n)$. These functions are defined for $k \geq 0$ and $n \geq 1$ by the formula $$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum_{d\mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero.
DGF	*Dirichlet generating function (or DGF).* Given a sequence $\{f(n)\}_{n\geq 0}$, its DGF enumerates the sequence in a different way than formal generating functions in an auxiliary variable. Namely, for $\lvert s \rvert < \sigma_a$, the abcissa of absolute convergence of the series, the DGF $D_f(s)$ constitutes an analytic function of s given by: $D_f(s) := \sum_{n\geq 1} f(n)/n^s$. The DGF is alternately called the *Dirichlet series* of an arithmetic function f. The DGF of f can be inverted via a contour-based integral formula representation to solve for $f(n)$. It is also closely related to the Mellin transform of the summatory function of f at $-s$. type
$\sigma_0(n), d(n)$	The ordinary divisor function, $d(n) := \sum_{d\mid n} 1$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) = \delta_{n,1}$, defined such that for any arithmetic f we have that $f \ast \varepsilon = \varepsilon \ast f = f$.
$f \ast g$	The Dirichlet convolution of f and g, $f \ast g(n) := \sum\limits_{d\mid n} f(d)g(n/d)$, for $n \geq 1$. This symbol for the discrete convolution of two arithmetic functions is the only notion of convolution of functions we employ within the article that is not explicitly expanded by the definition of another relvant convolution operation, e.g., integral formula or summation with exactly specified indices as input to the functions at hand.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ provided that $f(1) \neq 0$. The inverse function, when it exists, is unique and satisfies the relations that $f^{-1} \ast f = f \ast f^{-1} = \varepsilon$.
$\lfloor x \rfloor$	The floor function $\lfloor x \rfloor := x - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$.

Symbol	Definition
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega+1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$. This function definition is the key to unraveling our new bounds proved in the article, and so, is henceforth taken as standard notation moving on.
$\mathrm{Id}_k(n)$	The power-scaled identity function, $\mathrm{Id}_k(n) := n^k$ for $n \geq 1$.
$\mathbb{1}_{\mathbb{S}}$, $\chi_{\mathtt{cond}(x)}$	We use the notation $\mathbb{1}, \chi : \mathbb{N} \rightarrow \{0,1\}$ to denote indicator, or characteristic functions. In paticular, $\mathbb{1}_{\mathbb{S}}(n) = 1$ if and only if $n \in \mathbb{S}$, and $\chi_{\mathtt{cond}}(n) = 1$ if and only if n satisfies the condition `cond`.
$\log_*^m(x)$	The iterated logarithm function defined recursively for integers $m \geq 0$ by $$\log_*^m(x) := \begin{cases} x, & \text{if } m = 0; \\ \log x, & \text{if } m = 1; \\ \log\left[\log_*^{m-1}(x)\right], & \text{if } m \geq 2. \end{cases}$$
$[n = k]_{\delta}$	Synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and zero otherwise.
$[\mathtt{cond}]_{\delta}$	For a boolean-valued `cond`, $[\mathtt{cond}]_{\delta}$ evaluates to one precisely when `cond` is true, and zero otherwise.
$\lambda(n)$	The Liouville lambda function, $\lambda(n) := (-1)^{\Omega(n)}$, denotes the parity of $\Omega(n)$, the number of distinct prime factors of n counting multiplicity. That is, $\lambda(n) \in \{\pm 1\}$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$. Notice that if n is squarefree, then $\lambda(n) = \mu(n)$, where more generally we have that $\lambda(n) = \sum_{d^2 \mid n} \mu\left(\frac{n}{d^2}\right)$. This function is Dirichlet invertible with inverse function given by $\lambda^{-1}(n) = \mu^2(n)$, the (unsigned) characteristic function of the squarefree integers.
$\gcd(m,n); (m,n)$	The greatest common divisor of m and n. Both notations for the GCD are used interchangably within the article.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree, i.e., has no prime power divisors with exponent greater than one.
$M(x)$	The Mertens function which is the summatory function over $\mu(n)$, $M(x) := \sum\limits_{n \leq x} \mu(n)$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^{\alpha} \vert\vert n$ (p^{α} exactly divides n) for p prime and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the distinct prime factor counting functions as $\omega(n) := \sum_{p \mid n} 1$ and $\Omega(n) := \sum_{p^{\alpha} \vert\vert n} \alpha$. Equivalently, if the factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we define that $\omega(1) = \Omega(1) = 0$.
$\sum_{p \leq x}, \prod_{p \leq x}$	Unless otherwise specified by context, we use the index variable p to denote that the summation (product) is to be taken only over prime values within the summation bounds.
$P(s)$	For complex s with $\Re(s) > 1$, we define the prime zeta function to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$. This function has an analytic continuation to $\Re(s) \in (0,1)$ with a logarithmic singularity near $s := 1$: $P(1+\varepsilon) = -\log \varepsilon + C + O(\varepsilon)$.

Symbol	**Definition**
$\sigma_\alpha(n)$	The generalized sum-of-divisors function, $\sigma_\alpha(n) := \sum_{d\mid n} d^\alpha$, for any $n \geq 1$ and $\alpha \in \mathbb{C}$.
$\left[{n \atop k}\right]$	The unsigned Stirling numbers of the first kind, $\left[{n \atop k}\right] = (-1)^{n-k} \cdot s(n,k)$.
$\sim, \approx, \gtrsim, \lesssim$	We say that two functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $$\lim_{x\rightarrow\infty} \frac{A(x)}{B(x)} = 1.$$ We also sometimes express the *average order* of an arithmetic function $f \sim h$ that may actually oscillate, or say have value of one infinitely often, in the cases that $\frac{1}{x} \cdot \sum_{n\leq x} f(n) \sim h(x)$ (for example, we often would write that $\Omega(n) \sim \log\log n$, even though technically, $1 \leq \Omega(n) \leq \frac{\log n}{\log 2}$). We write that $f(x) \approx g(x)$ if $\lvert f(x) - g(x)\rvert = O(1)$. We say that $h(x) \gtrsim r(x)$ if $h \gg r$ as $x \rightarrow \infty$, and define the relation $\lesssim$ similarly as $h(x) \lesssim r(x)$ if $h \ll r$ as $x \rightarrow \infty$. When applying these relations we still consider leading constants to be meaningful terms that are preserved. *Observation on usage.* The notation of $\gtrsim, \lesssim$ is convenient for expressing upper and lower bounds on functions given by asymptotically dominant main terms in the expansion of more complicated symbolic expansions. Hence, we use these relations to simplify our results by dropping expressions involving more precise, exact terms that are nonetheless asymptotically insignificant, to obtain accurate statements in limiting cases of large x that hold as $x \rightarrow \infty$. This notation is particularly powerful and is utilized in this article when we express many lower bound estimates for functions that would otherwise require literally pages of typeset symbols to state exactly, but which have simple enough formulae when considered as bounds that hold in this type of limiting asymptotic context.
$\sum_{n\leq x}'$	We denote by $\sum_{n\leq x}' f(n)$ the summatory function of f at x minus $\frac{f(x)}{2}$ if $x \in \mathbb{Z}$.
$\zeta(s)$	The Riemann zeta function, defined by $\zeta(s) := \sum_{n\geq 1} n^{-s}$ when $\Re(s) > 1$, and by analytic continuation to the entire complex plane with the exception of a simple pole at $s = 1$.

Listing of major sections and topics

1 Key notes: Core components to the proof

We offer a brief overview of the critical components to our proof outlined in the introduction, and then piece-by-piece in the next sections of the article:

(1) We prove an apparently yet undiscovered matrix inversion formula relating the summatory o functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$).

(2) This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$ and the Dirichlet inverse of the shifted additive function $\omega(n)+1$.

   (i) The average order, $\omega(n) \sim \log\log n$, imparts an iterated logarithmic structure to our expansions, which many have conjectured we should see in limiting bounds on $M(x)$, but which are practically elusive in most non-conjectural known formulas I have seen.

   (ii) The additivity of $\omega(n)$ dictates that the sign of $g^{-1}(n) = (\omega+1)^{-1}(n)$ is $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$. The corresponding weighted summatory functions of $\lambda(n)$ have more established predictable properties, such as known sign biases and upper bounds. These summatory functions are generally speaking more regular and easier to work with that $M(x)$ and its summand of the Möbius function.

(3) We tighten a result from [9, §7] providing summatory functions that indicate the parity of $\lambda(n)$ using elementary arguments and combinatorially motivated expansions of Dirichlet series. Our motivations are different than in the reference for exploiting its unique properties. Namely, we are not after a CLT-like statement for the functions $\Omega(n)$ and $\omega(n)$. Rather, we seek to sum $\sum_{n\leq x} \lambda(n) f(n)$ for general non-negative arithmetic functions f.

(4) We then turn to the asymptotics if the quasi-periodic $g^{-1}(n)$, estimating their limiting asymptotics for large n. We eventually use these estimates to prove a substantially unique new lower bound formula for the summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$.

(5) When we return to (2) with our new lower bounds, and bootstrap, we recover "magic" in the form of showing the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers.

(6) We remark that while this technique and approach to the classical problem at hand is certainly new, it is not just novel, and its discovery will invariably lead to similar applications given careful study of limsup-type bounds on the summatory functions of other special signed arithmetic function sequences.

   Note that in these cases, if f is multiplicative and $f(n) > 0$ for all $n \geq 1$, then $\operatorname{sgn}(f^{-1}(n)) = (-1)^{\omega(n)}$. This signedness tends to complicate, but still closely parallel our argument involving the parity of $\lambda(n) = (-1)^{\Omega(n)}$ for the Mertens function case.

2 Introduction

2.1 The Mertens function – definition, properties, known results and conjectures

Suppose that $n \geq 1$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2 \alpha_2 \cdots p_k^{\alpha_k}$. We define the *Möebius function* to be the signed indicator function of the squarefree integers:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leq i \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many known variants and special properties of the Möebius function and its generalizations [13, *cf.* §2], however, for our purposes we seek to explore the properties and asymptotics of weighted summatory functions over $\mu(n)$. The Mertens summatory function, or *Mertens function*, is defined as [15, A002321]

$$\begin{aligned} M(x) &= \sum_{n \leq x} \mu(n), \ x \geq 1, \\ &\longmapsto \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2\} \end{aligned}$$

A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as [15, A013928]

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \rightarrow \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{Q(x)} = \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{Q(x)} \xrightarrow[n \rightarrow \infty]{} \frac{3}{\pi^2}.$$

While this limiting law suggests an even bias for the Mertens function, in practice $M(x)$ has a noted negative bias in its values, and the actual local oscillations between the approximate densities of the sets $\mu_{\pm}(x)$ lend an unpredictable nature to the function and its characteistic oscillatory sawtooth shaped plot.

2.1.1 Properties

The well-known approach to evaluating the behavior of $M(x)$ for large $x \rightarrow \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is easily expressed via an inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since by Perron's formula we have

$$\frac{1}{\zeta(s)} = \int_1^{\infty} \frac{s \cdot M(x)}{x^{s+1}} dx,$$

we then obtain that

$$M(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation along with the standard Euler product representation for the reciprocal zeta function leads us to the exact expression for $M(x)$ when $x > 0$ given by the next theorem.

Theorem 2.1 (Analytic Formula for $M(x)$)**.** *Assuming the RH, we can show that there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any $x \in \mathbb{R}_{>0}$*

$$M(x) = \lim_{k \rightarrow \infty} \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |\Im(\rho)| < T_k}} \frac{x^{\rho}}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_{\delta}.$$

An unconditional bound on the Mertens function due to Walfisz (1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \exp\left(-C \cdot \log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan in 2009 proved new updated estimates bounding $M(x)$ for large x of the following forms:

$$\begin{aligned} M(x) &\ll \sqrt{x} \exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x} \exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right), \ \forall \epsilon > 0. \end{aligned}$$

Other explicit bounds due to the article by Kotnik include the following simpler estimates for the Mertens function when x is sufficiently large:

$$\begin{aligned} |M(x)| &< \frac{x}{4345}, \ \forall x > 2160535, \\ |M(x)| &< \frac{0.58782 \cdot x}{\log^{11/9}(x)}, \ \forall x > 685. \end{aligned}$$

2.1.2 Conjectures

The Riemann Hypothesis (RH) is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. It is still unresolved whether

$$\limsup_{x\to\infty} |M(x)|/\sqrt{x} = \infty,$$

although computational evidence suggests that this is a likely conjecture [8, 6]. There is a rich history to the original statement of the *Mertens conjecture* which states that

$$|M(x)| < c \cdot x^{1/2}, \quad \text{some constant } c > 0,$$

which was first verified by Mertens for $c = 1$ and $x < 10000$, although since its beginnings in 1897 has since been disproved by computation by Odlyzko and té Riele in the early 1980's.

There are a number of other interesting unsolved and at least somewhat accessible open problems related to the asymptotic behavior of $M(x)$ at large x. It is believed that the sign of $M(x)$ changes infinitely often. That is to say that it is widely believed that $M(x)$ is ocsillatory and exhibits a negative bias insomuch as $M(x) < 0$ more frequently than $M(x) > 0$ over all $x \in \mathbb{N}$. One of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ is unbounded on the natural numbers. In particular, the precise statement of this problem is to produce an affirmative answer whether $\limsup_{x\to\infty} |M(x)|/\sqrt{x} = +\infty$, or equivalently whether there is an infinite sequence of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that $M(x_i)x_i^{-1/2}$ grows without bound along this subsequence.

Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies that

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}(\log\log x)^{5/4}},$$

corresponds to some bounded constant. To date an exact rigorous proof that $M(x)/\sqrt{x}$ is unbounded still remains elusive, though there is suggestive probabilistic evidence of this property established by Ng in 2008. We cite that prior to this point it is known that [12, *cf.* §4.1]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } -1.837625),$$

although based on work by Odlyzyko and te Riele it seems probable that each of these limits should be $\pm\infty$, respectively [10, 7, 8, 6]. It is also known that $M(x) = \Omega_{\pm}(\sqrt{x})$ and $M(x)/\sqrt{x} = \Omega_{\pm}(1)$.

2.2 A new approach to bounding $M(x)$ from below

2.2.1 Summing series over Dirichlet convolutions

Theorem 2.2 (Summatory functions of Dirichlet convolutions)**.** *Let $f, g : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $G(x) := \sum_{n \leq x} g(n)$ denote the summatory functions of f, g, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse $f^{-1}(n)$ of f, i.e., the unique arithmetic function such that $f * f^{-1} = \varepsilon$ where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity with respect to Dirichlet convolution. Then, letting the counting function $\pi_{f*g}(x)$ be defined as in the first equation below, we have the following equivalent expressions for the summatory function of $f * g$ for integers $x \geq 1$:*

$$\begin{aligned}
\pi_{f*g}(x) &= \sum_{n \leq x} \sum_{d|n} f(d) g(n/d) \\
&= \sum_{d \leq x} f(d) G\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} G(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].
\end{aligned}$$

Moreover, we can invert the linear system determining the coefficients of $G(k)$ for $1 \leq k \leq x$ naturally to express $G(x)$ as a linear combination of the original left-hand-side summatory function as:

$$\begin{aligned}
G(x) &= \sum_{j=1}^{x} \pi_{f*g}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\
&= \sum_{n=1}^{x} f^{-1}(n) \pi_{f*g}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).
\end{aligned}$$

Corollary 2.3 (Convolutions Arising From Möbius Inversion)**.** *Suppose that g is an arithmetic function with $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n \leq x} (g * \mu)(n)$. Then the Mertens function equals*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geq 1.$$

2.2.2 A motivating special case

Using $\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu$, where $\chi_{\mathbb{P}}$ is the characteristic function of the primes, we have that $\widetilde{G}(x) = \pi(x) + 1$ in Corollary 2.3. In particular, the corollary implies that

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

We can compute the first few terms for the Dirichlet inverse sequence of $g(n) := \omega(n) + 1$ numerically for the first few sequence values as

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is given by $\lambda(n) = \frac{g^{-1}(n)}{|g^{-1}(n)|}$ (see Proposition 3.3). Note that since the DGF of $\omega(n)$ is given by $D_\omega(s) = P(s)\zeta(s)$ where $P(s)$ is the *prime zeta function*, we do have a Dirichlet series for the inverse functions to invert coefficient-wise using more classical contour integral methods[1]. Our new methods do not

[1] E.g., using [1, §11]

$$f(n) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \frac{n^{\sigma + \imath t}}{\zeta(\sigma + \imath t)(P(\sigma + \imath t) + 1)}, \sigma > 1.$$

rely on typical constructions for bounding $M(x)$ based on estimates of the non-trivial zeros of the Riemann zeta function that have so far to date been employed to bound the Mertens function from above. We will instead take a more combinatorial tack to investigating bounds on this inverse function sequence in the coming sections. Consider the following motivating conjecture:

Conjecture 2.4. *Suppose that $n \geq 1$ is a squarefree integer. We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n) = (\omega+1)^{-1}(n)$ over these integers:*

(A) $g^{-1}(1) = 1$*;*

(B) $\operatorname{sgn}(g^{-1}(n)) = \mu(n) \equiv \lambda(n)$*;*

(C) We can write the inverse function at squarefree n as

$$g^{-1}(n) = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!.$$

We illustrate parts (B)–(C) of this conjecture clearly using Table T.1 given on page 40 of the appendix section. A table of the first several explicit values of $(f+1)^{-1}(n)$ for $f(1) = 0$ and symbolic additive f are also given in Table T.2 on page 41.

The realization that the beautiful, and simplistic, e.g., not terribly complicated considering the subject matter, form of property (C) in Conjecture 2.4 holds for all squarefree $n \geq 1$ motivates our pursuit of formulas for the inverse functions $g^{-1}(n)$ based on the configuration of the exponents in the prime factorization of any $n \geq 2$. In Section 5 we consider expansions of these inverse functions recursively, starting from a few first exact cases of an auxillary function, $C_k(n)$, that depends on the precise exponents in the prime factorization of n. We then prove limiting asymptotics for these functions and assemble the main terms in the expansion of $g^{-1}(n)$ using artifacts from combinatorial analysis.

Combined with the DGF-based generating function for certain summatory functions indicating the parity of $\Omega(n)$ introduced in the next subsection of this introduction, this take on the identity in (1) provides us with a powerful new method to bound $M(x)$ from below. We will sketch the key results and formulation to the construction we actually use to prove the new lower bounds on $M(x)$ next.

From this point on, we fix the Dirichlet invertible function $g(n) := \omega(n) + 1$ and denote its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega+1)^{-1}(n)$. For natural numbers $n \geq 1, k \geq 0$, let

$$C_k(n) := \begin{cases} \varepsilon(n) = \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$

By Möbius inversion (see Lemma 5.2), we have that

$$(g^{-1} * 1)(n) = \lambda(n) \cdot C_{\Omega(n)}(n), \forall n \geq 1. \tag{2}$$

We have limiting asymptotics on these functions given by the following theorem:

Theorem 2.5 (Asymptotics for the functions $C_k(n)$)**.** *Let $\mathbb{1}_{*_m}(n)$ denote the m-fold Dirichlet convolution of one with itself at n. The function $\sigma_0 * 1_{*_m}$ is multiplicative with values at prime powers given by*

$$(\sigma_0 * \mathbb{1}_{*_m})(p^\alpha) = \binom{\alpha+m+1}{m+1}.$$

Fröberg has also previously done some preliminary investigation as to the properties of the inversion to find the coefficients of $(1+P(s))^{-1}$ [3].

We have the following asymptotic base cases for the functions $C_k(n)$:

$$\begin{aligned}
C_1(n) &\sim \log\log n \\
C_2(n) &\sim \frac{\sigma_0(n)n}{\log n} + O(\log\log n) \\
C_3(n) &\sim -\frac{(\sigma_0 * 1)(n)n^2}{\log n} + O\left(n \cdot \log\log n\right).
\end{aligned}$$

For all $k \geq 4$, we obtain that the dominant asymptotic term and the error bound terms for $C_k(n)$ are given by

$$C_k(n) \sim (\sigma_0 * \mathbb{1}_{*_{k-2}})(n) \times \frac{(-1)^k n^{k-1}}{(\log n)^{k-1}(k-1)!} + O_k\left(\frac{n^{k-2}}{(k-2)!} \cdot \frac{(\log\log n)^{k-2}}{(\log n)^{k-2}}\right), \text{ as } n \to \infty.$$

Then we can prove (see Corollary 5.7) that

$$g^{-1}(n) \sim \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

Notice that this formula is substantially easier to evaluate than the corresponding sums in (2) given directly by Möbius inversion – and hence, we prefer to work with bounds on it implied by our new results than the exact formula from the cited equation above. The last result in turn implies that

$$G^{-1}(x) \gtrsim \sum_{n \leq x} \lambda(n) \cdot C_{\Omega(n)}(n) \times \sum_{d=1}^{\left\lfloor \frac{x}{n} \right\rfloor} \lambda(d). \tag{3}$$

In light of the fact that (by an integral-based interpretation of integer convolution using summation by parts)

$$M(x) \sim G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)},$$

the formula in (3) implies that we can establish new *lower bounds* on $M(x)$ by appropriate estimates of the summatory function $G^{-1}(x)$ where trivially we have the bounded inner sums $L_0(x) := \sum_{n \leq x} \lambda(n) \in [-x, x]$ for all $x \geq 2$. As lower bounds for $M(x)$ along subsequences are not obvious, and historically non-trivial to obtain as we expect sign changes of this function infinitely often, we find this approach to be an effective one. Now, having motivated why we must carefully estimate the $G^{-1}(x)$ bounds using our new methods, we will require the bounds suggested in the next section to work at summing the summatory functions, $G^{-1}(x)$, for large x as $x \to \infty$.

2.2.3 Some enumerative (e.g., counting) DGFs from Mongomery and Vaughan

Our inspiration for the new bounds found in the last sections of this article allows us to sum non-negative arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. In particular, it uses a hybrid generating function and DGF method under which we are able to recover "good enough" asymptotics about the summatory functions that encapsulate the parity of $\lambda(n)$:

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}, k \geq 1.$$

The precise statement of the theorem that we transform for these new bounds is re-stated as follows:

Theorem 2.6 (Montgomery and Vaughan, §7.4)**.** *Let $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$. For $R < 2$ we have that*

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

uniformly for $1 \leq k \leq R \log\log x$ *where*

$$\mathcal{G}(z) := \frac{F(1,z)}{\Gamma(z+1)} = \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z.$$

The precise formulations of the inverse function asymptotics proved in Section 5 depend on being able to form significant lower bounds on the summatory functions of an always positive arithmetic function weighted by $\lambda(n)$. The next theorem, proved in Section 4, is the primary starting point for our new asymptotic lower bounds.

Theorem 2.7 (Generating functions of symmetric functions)**.** *We obtain upper and lower bounds of the form*

$$\alpha_0(z,x) \leq \prod_{p \leq x} \left(1 - \frac{z}{p}\right)^{-1} \leq \alpha_1(z,x),$$

where it suffices to take

$$\begin{aligned}
\alpha_0(z,x) &= \frac{\exp\left(\frac{55}{4}\log^2 2\right)}{\log^3 2} (\log x)^3 \left(\frac{e^B \log^2 x}{\log 2}\right)^z \\
\alpha_1(z,x) &= \exp\left(\frac{11}{3}\log^2 x\right) \left(e^B \log 2\right)^z,
\end{aligned}$$

where we take $z \geq 0$ *to be a real-valued parameter uniformly bounded in* $x \geq 2$*.*

The argument providing new lower bounds for $\mathcal{G}(z)$ is completed by the proof given in Corollary 4.3. This leads to a structure involving the incomplete gamma function, $\Gamma(s,z)$ defined as in the statements given in Section 3.3, inherited from Theorem 2.6. In Lemma 4.4, we justify that this construction, which holds uniformly for $k \leq \frac{3}{2}\log\log x$ (taking $R := \frac{3}{2}$), allows us to asymptotically enumerate the main terms in the expansions of $\widehat{\pi}_k(x)$ when we sum over just k in this range (as opposed to $k \leq \frac{\log x}{\log 2}$).

3 Preliminary proofs of lemmas and new results

3.1 Establishing the summatory function inversion identities

Given the interpretation of the summatory functions over an arbitrary Dirichlet convolution (and the vast number of such identities for special number theoretic functions, it is not surprising that this formulation of the first theorem from the introduction will provide many fruitful applications, indeed. For example, in addition to those cited in the compendia of the catalog references in [4, 14], we have notable identities of the form:

$$(f*1)(n) = [q^n]\sum_{m\geq 1} f(m)q^m/(1-q^m), \sigma_k = \mathrm{Id}_k *1, \mathrm{Id}_1 = \phi*\sigma_0, \mathrm{Id}_k = J_k*1, \log = \Lambda*1, 2^{\omega} = \mu^2*1.$$

Thus by the matrix inversion result in Theorem 2.2, we can express most generally new formulas for the summatory function bounds (e.g., average order sums) of many special (and arbitrary) arithmetic functions f in terms of sums of signed Dirichlet inverse functions. We will go ahead and prove this useful theorem, a crucial component to our new more combinatorial formulations used to bound $M(x)$ in later sections, up front at this point before moving on.

Remark 3.1. The proof given below follows from a straightforward matrix inversion procedure involving standard *shift matrix* operations, and similarity transformations of these operators. Notice that this proof provides a natural analog (and alternate expansion) to the already well known, established inversion relations for general arithmetic-function-weighted sums of summatory functions as typically cited in [1, §2.14] as follows for Dirichlet invertible functions $\alpha(1)\neq 0$ and any function F convolved with such an arithmetic function:

$$G(x) := \sum_{n\leq x}\alpha(n)\cdot F\left(\frac{x}{n}\right) \iff F(x) = \sum_{n\leq x}\alpha^{-1}(n)\cdot G\left(\frac{x}{n}\right).$$

Our theorem statements, proof, and motivation for constructing these sums are, however, somewhat different and worth expanding in their alternate forms from Theorem 2.2 compared to the somewhat less informative classical inversion formulae given in the last equation. Related results for summations of Dirichlet convolutions appear in [1, §3.10; §3.12; *cf.* §4.9, p. 95].

Proof of Theorem 2.2. Let h, g be arithmetic functions where $g(1)\neq 1$ has a Dirichlet inverse. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n\leq x} h(n)$ and $G(x) = \sum_{n\leq x} g(n)$. We define $S_{g,h}(x)$ to be the summatory function of the Dirichlet convolution of g with h: $g*h$. Then we can easily see that the following expansions hold:

$$\begin{aligned}\pi_{g*h}(x) := \sum_{n=1}^{x}\sum_{d|n} g(n)h(n/d) &= \sum_{d=1}^{x} g(d)H\left(\left\lfloor\frac{x}{d}\right\rfloor\right)\\ &= \sum_{i=1}^{x}\left[G\left(\left\lfloor\frac{x}{i}\right\rfloor\right) - G\left(\left\lfloor\frac{x}{i+1}\right\rfloor\right)\right]H(i).\end{aligned}$$

We form the matrix of coefficients associated with this system for $H(x)$, and proceed to invert it to express an exact solution for this function over all $x\geq 1$. Let the ordinary (initial, non-inverse) matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor\frac{x}{j}\right\rfloor\right) - G\left(\left\lfloor\frac{x}{j+1}\right\rfloor\right) \equiv G_{x,j} - G_{x,j+1}.$$

Then the matrix we must invert in this problem is lower triangular, with ones on its diagonals – and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I-U^T); \qquad U = (\delta_{i,j+1}), (I-U^T)^{-1} = ([j\leq i]_{\delta}).$$

Here, U is the $N\times N$ matrix whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$.

It is a useful fact that if we take successive differences of floor functions, we get non-zero behavior at divisors:

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

We use this property to shift the matrix $\hat{G}$, and then invert the result, to obtain a matrix involving the Dirichlet inverse of g:

$$\left[(I-U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Now we can express the inverse of the target matrix $(g_{x,j})$ in terms of these Dirichlet inverse functions as follows:

$$\begin{aligned}
(g_{x,j}) &= (I-U^T)^{-1}\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\
(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k)\right)(I-U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} g^{-1}(k)\right).
\end{aligned}$$

Thus the summatory function H is exactly expressed by the inverse vector product of the form

$$\begin{aligned}
H(x) &= \sum_{k=1}^{x} g_{x,k}^{-1} \cdot \pi_{g*h}(k) \\
&= \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor+1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j)\right) \cdot \pi_{g*h}(k). \square
\end{aligned}$$

3.2 Proving the crucial property from the conjecture over the squarefree integers

Proposition 3.2 (The characteristic function of the primes)**.** *Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the additive function that counts the number of distinct prime factors of n. Then*

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu.$$

The summatory function of the LHS is $\widetilde{G}(x) = \pi(x) + 1$. The corresponding characteristic function for the prime powers is similarly given by $\chi_{\mathrm{PP}} = \Omega * \mu$.

Proof. The core is to prove that for all $n \geq 1$, $\chi_{\mathbb{P}}(n) = (\mu * \omega)(n)$ – our claim. We notice that the Mellin transform of $\pi(x)$ – the summatory function of $\chi_{\mathbb{P}}(n)$ – at $-s$ is given by

$$\begin{aligned}
s \cdot \int_1^\infty \frac{\pi(x)}{x^{s+1}} dx &= \sum_{n \geq 1} \frac{\nabla[\pi](n-1)}{n^s} \\
&= \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s} = P(s),
\end{aligned}$$

where $\nabla[f](n) := f(n+1) - f(n)$ denotes the standard *forward difference operator* used to express a discrete derivative type operation on arithmetic functions. This is typical construction which more generally relates the Mellin transform $s \cdot \mathcal{M}[S_f](-s)$ to the DGF of the sequence $f(n)$ as cited, for example, in [1, §11]. Now we consider the DGF of the right-hand-side function, $f(n) := (\mu * \omega)(n)$, as

$$D_f(s) = \frac{1}{\zeta(s)} \times \sum_{n\geq 1} \frac{\omega(n)}{n^s} = P(s).$$

Thus for any $\Re(s) > 1$, the DGFs of each side of the claimed equation coincide. So by uniqueness of Dirichlet series, we see that in fact the claim holds. To obtain the full result, we add to each side of this equation a term of $\varepsilon(n) \equiv (\mu * 1)(n)$, and then factor the resulting convolution identity. □

Proposition 3.3 (The sign of $g^{-1}(n)$)**.** *For the Dirichlet invertible function, $g(n) := \omega(n) + 1$ defined such that $g(1) = 1$, at any $n \geq 1$, we have that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$. Here, the notation for the operation given by $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|} \in \{0, \pm 1\}$ denotes the sign, or signed parity, of the arithmetic function h at n.*

Proof. Let $D_f(s) := \sum_{n\geq 1} f(n) n^{-s}$ denote the Dirichlet generating function (DGF) of $f(n)$. Then we have that

$$D_{(\omega+1)^{-1}}(s) = \frac{D_\lambda(s)}{(P(s)+1)\zeta(2s)}.$$

Let $h^{-1}(n) := (\omega * \mu + \varepsilon)^{-1}(n) = [n^{-s}](P(s)+1)^{-1}$. Then we have that

$$\begin{aligned}
(h^{-1} * 1)(n) &= -\sum_{p_1 | n} h^{-1}\left(\frac{n}{p_1}\right) = \lambda(n) \times \sum_{p_1 | n} \sum_{p_2 | \frac{n}{p_1}} \cdots \sum_{p_{\Omega(n)} | \frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1 \\
&= \begin{cases} \lambda(n) \times (\Omega(n) - 1)!, & n \geq 2; \\ \lambda(n), & n = 1. \end{cases}
\end{aligned}$$

So by Möbius inversion

$$h^{-1}(n) = \lambda(n) \left[\sum_{\substack{d|n \\ d<n}} \lambda(d)\mu(d)(\Omega(n/d) - 1)! + 1\right] = \lambda(n) \left[\sum_{\substack{d|n \\ d<n}} \mu^2(d)(\Omega(n/d) - 1)! + 1\right].$$

Then we finally have that

$$(\omega+1)^{-1}(n) = \lambda(n) \times \sum_{d|n} \lambda(d) \left[\sum_{\substack{r | \frac{n}{d} \\ r < \frac{n}{d}}} \mu^2(r)(\Omega\left(\frac{n}{dr}\right) - 1)! + 1\right] \chi_{\mathrm{sq}}(d)\mu(\sqrt{d}),$$

where χ_{sq} is the characteristic function of the squares. In either case of $\lambda(n) = \pm 1$, there are positive constants $C_{1,n}, C_{2,n} > 0$ such that

$$\lambda(n) C_{1,n} \times \sum_{d^2|n} \lambda(d^2)\mu(d) \leq g^{-1}(n) \leq \lambda(n) C_{1,n} \times \sum_{d^2|n} \lambda(d^2)\mu(d),$$

where $\sum_{d^2|n} \lambda(d^2)\mu(d) = \sum_{d^2|n} \mu^2(n) > 0$. This proves the result. □

More generally, we have that for f a non-negative additive arithmetic function that vanishes at one, $\operatorname{sgn}((f+1)^{-1}) = \lambda(n) = (-1)^{\Omega(n)}$. We can state similar properties for the common case of multiplicative f in the form of the following result: If $f(n) > 0$ for all $n \geq 1$ and f is multiplicative, then $\operatorname{sgn}(f^{-1}(n)) = (-1)^{\omega(n)}$.

3.3 Other facts and listings of results we will need in our proofs

Theorem 3.4 (Mertens theorem)**.**

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log\log x + B + O\left(e^{-(\log x)^{\frac{1}{14}}}\right),$$

where $B \approx 0.2614972128476427837554$ *is an absolute constant.*

Corollary 3.5. *We have that for sufficiently large* $x \gg 1$

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right) = \frac{e^{-B}}{\log x}\left[1 - \frac{(\log x)^{1/14}}{B} + o\left((\log x)^{1/14}\right)\right].$$

Hence, for $1 < |z| < R < 2$ *we obtain that*

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right)^z = \frac{e^{-Bz}}{(\log x)^z}\left[1 - \frac{z}{B}(\log x)^{\frac{1}{14}} + o_{z,R}\left(z^2 \cdot (\log x)^{\frac{1}{14}}\right)\right].$$

Proof. By taking logarithms and using Mertens theorem above, we obtain that

$$\begin{aligned}\log \prod_{p \leq x}\left(1 - \frac{1}{p}\right) &= \sum_{p \leq x} \log\left(1 - \frac{1}{p}\right) \\ &\approx -\log\log x - B + O\left(e^{-(\log x)^{1/14}}\right).\end{aligned}$$

Hence, the first formula follows by expanding out an alternating series for the exponential function. The second formula follows for $z \notin \mathbb{Z}$ by an application of the generalized binomial series given by

$$\log \prod_{p \leq x}\left(1 - \frac{1}{p}\right)^z \approx \frac{e^{-Bz}}{(\log x)^z} \times \sum_{r \geq 0}\binom{z}{r}\frac{(-1)^r}{B^r}(\log x)^{\frac{r}{14}},$$

where for $1 < |z| < 2$, we obtain the next result stated above with $\binom{z}{1} = z$ and $\binom{z}{2} = z(z-1)/2$. □

Facts 3.6 (Exponential Integrals and Incomplete Gamma Functions)**.** The following two variants of the *exponential integral function* are defined by

$$\begin{aligned}\operatorname{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, \\ E_1(z) &:= \int_1^{\infty} \frac{e^{-tz}}{t} dt, \Re(z) \geq 0,\end{aligned}$$

where $\operatorname{Ei}(-kz) = -E_1(kz)$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $E_1(z)$:

$$1 - \frac{3}{4}z \leq E_1(z) - \gamma - \log z \leq 1 - \frac{3}{4}z + \frac{11}{36}z^2. \tag{4a}$$

A related function is the (upper) *incomplete gamma function* defined by

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt, \Re(s) > 0.$$

We have the following properties of $\Gamma(s, x)$:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, \tag{4b}$$

$$\Gamma(s+1, 1) = e^{-1}\left\lfloor \frac{s!}{e} \right\rfloor, s \in \mathbb{Z}^+, \tag{4c}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, |x| \to +\infty. \tag{4d}$$

4 Summing arithmetic functions weighted by the function $\lambda(n)$

4.1 Discussion: The enumerative DGF result in Theorem 2.6 from Montgomery and Vaughan

In the reference we have defined $F(s,z)$ for complex-valued $z \neq 0$ and $\Re(s) > 1$ taken such that the Dirichlet series coefficient functions, $a_z(n)$, are defined by

$$\zeta(s)^z F(s,z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s}, \Re(s) > 1.$$

Then for the function F defined by

$$F(s,z) := \prod_p \left(1 - \frac{z}{p^s}\right)\left(1 - \frac{1}{p^s}\right)^z,$$

we obtain in the notation above that $a_z(n) \equiv z^{\Omega(n)}$, and that the summatory function of $a_z(n)$ satisfies

$$A_z(x) := \sum_{n \leq x} z^{\Omega(n)} = \sum_{k \geq 0} \widehat{\pi}_k(x) z^k.$$

Hence, by the Cauchy integral formula, for $r < 2$ we get that

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \oint_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz,$$

from which we obtain the formula re-stated (and reproduced from the reference text) in the theorem.

What this enumeratively-flavored result of Montgomery and Vaughan allows us to do is get a "good enough" lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. For comparison, we already have known, more classical bounds due to Erdös (or earlier) that state for

$$\pi_k(x) := \#\{n \leq x : \omega(n) = k\},$$

we have tightly that [2, 9]

$$\pi_k(x) = (1 + o(1)) \cdot \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

We seek to approximate the right-hand-side of $\mathcal{G}(z)$ by only taking the products of the primes $p \leq x$, e.g., $p \in \{2, 3, 5, \ldots, x\}$, of which the last element in this set has average order of $\log\left[\frac{x}{\log x}\right]$ as $x \to \infty$. To refine and adjust the prime products that define $F(s,z)$ to only include the primes bounded up to the $p \leq x$ marker we seek, we will require some fairly elementary estimates of products of primes, the classical Mertens theorem on the rate of divergence of the sum of the reciprocals of the primes, and of some generating function techniques involving elementary symmetric functions. The statements in Section 3.3 provide the basis for proving most of the lemmas we require to build our refined estimates and appropriately bound $F(s,z)$.

We also state the following theorem reproduced from [9, Thm. 7.20] that handles the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \frac{3}{2}\log\log x$. This allows us later to show that taking just $k \in [1, \frac{3}{2}\log\log x]$ and summing over such k in Theorem 2.6 captures the asymptotically relevant, dominant behavior of the values of $\pi_k(x)$ for $k \leq \frac{\log x}{\log 2}$ (where $\Omega(n) \leq \frac{\log n}{\log 2}$ for all $n \geq 2$). The proof of the main statement of the next result is found in the main text of the cited reference in Chapter 7 of Montgomery and Vaughan.

Theorem 4.1 (Bounds on exceptional values of $\Omega(n)$ for large n, MV 7.20)**.** *Let*

$$B(x,r) := \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\}.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x, r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \textit{as } x \to \infty.$$

In particular, we have that for $r \in \left(\frac{3}{2}, 2\right)$,

$$\left|1 - \frac{B(x,r)}{B(x,3/2)}\right| \xrightarrow{x\to\infty} 1.$$

4.2 The key new results utilizing Theorem 2.6

Corollary 4.2. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \gg 2.$$

When $s := 1$, we have the known bound in Mertens theorem. For $s > 1$, we obtain that

$$P_s(x) \approx E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1).$$

It follows that

$$\gamma_0(s,x) + o(1) \leq P_s(x) \leq \gamma_1(s,x) + o(1),$$

where it suffices to take

$$\begin{aligned} \gamma_0(z,x) &= -s\log\left(\frac{\log x}{\log 2}\right) - \frac{3}{4}s(s-1)\log(x/2) - \frac{11}{36}s(s-1)^2\log^2(2) \\ \gamma_1(z,x) &= \quad s\log\left(\frac{\log x}{\log 2}\right) - \frac{3}{4}s(s-1)\log(x/2) + \frac{11}{36}s(s-1)^2\log^2(x). \end{aligned}$$

Proof. Let $s > 1$ be real-valued. By Abel summation where our summatory function is given by $A(x) = \pi(x) \sim \frac{x}{\log x}$ and our function $f(t) = t^{-s}$ so that $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= E_1((s-1)\log x) - E_1((s-1)\log 2) + o(1), |x| \to \infty. \end{aligned}$$

Now using the inequalities in Facts 3.6, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned} \frac{P_s(x)}{s} &\geq -\log\left(\frac{\log x}{\log 2}\right) - \frac{3}{4}(s-1)\log(x/2) - \frac{11}{36}(s-1)^2\log^2(2) \\ \frac{P_s(x)}{s} &\leq \quad \log\left(\frac{\log x}{\log 2}\right) - \frac{3}{4}(s-1)\log(x/2) + \frac{11}{36}(s-1)^2\log^2(x). \end{aligned}$$

This completes the proof of the bounds cited above in the statement of this lemma. □

Proof of Theorem 2.7. We have that for all integers $0 \leq k \leq m$

$$[z^k] \prod_{1 \leq i \leq m} (1 - f(i)z)^{-1} = [z^k] \exp\left(\sum_{j \geq 1} \left(\sum_{i=1}^m f(i)^j\right) \frac{z^j}{j}\right).$$

In our case we have that $f(i)$ denotes the i^{th} prime. Hence, summing over all $p \leq x$ in place of $0 \leq k \leq m$ in the previous formula applied in tandem with Corollary 4.2, we obtain that the logarithm of the generating function series we are after corresponds to

$$\log\left[\prod_{p \leq x}\left(1 - \frac{z}{p}\right)^{-1}\right] = (B + \log\log x)z + \sum_{j \geq 2}\left[a(x) + b(x)(j-1) + c(x)(j-1)^2\right] z^j$$

$$\begin{aligned}&= (B+\log\log x)z - a(x)\left(1+\frac{1}{z-1}+z\right) + b(x)\left(1+\frac{2}{z-1}+\frac{1}{(z-1)^2}\right)\\ &\quad - c(x)\left(1+\frac{4}{z-1}+\frac{5}{(z-1)^2}+\frac{2}{(z-1)^3}\right).\end{aligned}$$

In the previous equations, the upper and lower bounds formed by the functions (a,b,c) are given by

$$\begin{aligned}(a_\ell,b_\ell,c_\ell) &:= \left(-\log\left(\frac{\log x}{\log 2}\right), \frac{3}{4}\log\left(\frac{x}{2}\right), -\frac{11}{36}\log^2 2\right)\\ (a_u,b_u,c_u) &:= \left(\log\left(\frac{\log x}{\log 2}\right), -\frac{3}{4}\log\left(\frac{x}{2}\right), \frac{11}{36}\log^2 x\right).\end{aligned}$$

Now we make a prudent decision to set the uniform bound parameter to a middle ground value of $1<R<2$ as $R:=\frac{3}{2}$ so that

$$z \equiv z(k,x) = \frac{k-1}{\log\log x} \in [0,R),$$

for $x \gg 1$ very large. Thus $(z-1)^{-m} \in [(-1)^m, 2^m]$ for integers $m \geq 1$, and we can then form the upper and lower bounds from above. What we get out of these formulas is stated as in the theorem bounds. □

Corollary 4.3 (Bounds on $\mathcal{G}(z)$ from MV)**.** *We have that for the function $\mathcal{G}(z) := F(1,z)/\Gamma(z+1)$ from Montgomery and Vaughan, there is a constant A_0 and functions of x only, $B_0(x), C_0(x)$, so that*

$$A_0 \cdot B_0(x) \cdot C_0(x)^z \left(1 - \frac{z}{B}(\log x)^{\frac{1}{14}}\right) \leq \mathcal{G}(z).$$

It suffices to take

$$\begin{aligned}A_0 &= \frac{\exp\left(\frac{55}{4}\log^2 2\right)}{\log^3(2)\cdot\Gamma(5/2)} \approx 1670.84511225\\ B_0(x) &= \log^3 x\\ C_0(x) &= \frac{\log x}{\log 2}.\end{aligned}$$

Proof. This result is a consequence of applying both Corollary 3.5 and Theorem 2.7 to the definition of $G(z)$. In particular, we obtain bounds of the following form from the theorem:

$$\frac{A_0 \cdot B_0(x)\cdot C_0(x)^z}{\Gamma(z+1)} \leq \frac{\mathcal{G}(z)}{\prod_p \left(1-\frac{1}{p}\right)^z}.$$

Since $z \equiv z(k,x) = \frac{k-1}{\log\log x}$ and $k \in [1, R\log\log x]$, we obtain that for small k and $x \gg 1$ large $\Gamma(z+1) \approx 1$, and for k towards the upper bound of its interval that $\Gamma(z+1) \approx \Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$ (recall that we set $R := 3/2$ in the preceeding proof of Theorem 2.7). Thus when we expand out the formula given by the corollary in conjunction with these bounds on the gamma function, we obtain the claimed results. □

Lemma 4.4. *Suppose that $f_k(n)$ is a sequence of arithmetic functions such that $f_k(n) > 0$ for all $n \geq 1$, $f_0(n) = \delta_{n,1}$, and $f_{\Omega(n)}(n) \gtrsim \widehat{\tau}_\ell(n)$ as $n \to \infty$ where $\widehat{\tau}_\ell(t)$ is a continuously differentiable function of t for all large enough $t \gg 1$. We define the λ-sign-scaled summatory function of f as follows:*

$$F_\lambda(x) := \sum_{\substack{n\leq x\\ \Omega(n)\leq x}} \lambda(n)\cdot f_{\Omega(n)}(n).$$

Let

$$A_\Omega^{(\ell)}(t) := \sum_{k=1}^{\left\lfloor \frac{3}{2}\log\log t\right\rfloor} (-1)^k \widehat{\pi}_k(t).$$

Then we have that

$$F_\lambda(\log\log x) \gtrsim A_\Omega^{(\ell)}(x)\widehat{\tau}_\ell(\log\log x) - \int_1^{\log\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_\ell'(t)dt.$$

Proof. The formula for $F_\lambda(x)$ is valid by Abel summation provided that

$$\left|\frac{\sum\limits_{\frac{3}{2}\log\log t < k \leq \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_\Omega^{(\ell)}(t)}\right| = o(1),$$

e.g., the asymptotically dominant terms indicating the parity of $\lambda(n)$ are encompassed by the terms summed by $A_\Omega^{(\ell)}(t)$ for sufficiently large t as $t \to \infty$. Using the arguments in Montgomery and Vaughan [9, §7; Thm. 7.20] (see Theorem 4.1), we can see that uniformly in x

$$\left|\frac{\sum_{k \leq x} \pi_k(x)}{B\left(x, \frac{3}{2}\right)}\right| \sim 1, \tag{5}$$

as $x \to \infty$ where $B(x, r)$ is defined as in the cited theorem re-stated on page 17 from the reference. Thus we have captured the asymptotically dominant main order terms in our formula as $x \to \infty$. □

5 Precisely enumerating and bounding the Dirichlet inverse functions, $g^{-1}(n) := (\omega+1)^{-1}(n)$

5.1 Developing an improved conjecture: Proving precise bounds on the inverse functions

Conjecture 2.4 is not the most accurate way to express the limiting behavior of the Dirichlet inverse functions $g^{-1}(n)$, though it does capture an important characteristic – namely, that these functions can be expressed via more simple formulas than inspection of the initial sequence properties might otherwise suggest. This is to say that, though an exact expression for $g^{-1}(n)$ that holds for all $n \geq 1$ is difficult to write down, the conjecture suggests simple formulas that hold in most important, non-corner cases of $n \geq 1$ (i.e., that should hold for sets $n \leq x$ with limiting asymptotic density approaching one, or in most cases to give this property easily). We still need to actually come up with better bounds to plug back into the asymptotic analysis we obtain in the next sections.

It turns out that these results are related to symmetric functions of the exponents in the prime factorizations of each $n \leq x$. The idea is that by having information about $g^{-1}(n)$ in terms of its prime factorization exponents for $n \leq x$, we should be able to extrapolate what we need which is information about the average behavior of the summatory functions, $G^{-1}(x)$, from the proofs above. Moreover, we notice the following observation that is suggestive of the semi-periodicity at play with the distinct values of $g^{-1}(n)$ distributed over $n \geq 2$.

Heuristic 5.1 (Symmetry in $g^{-1}(n)$ in the exponents in the prime factorization of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly two, three, and so on prime factors. There does not appear to be an easy, nor subtle direct recursion between the distinct g^{-1} values, except through auxiliary function sequences. We will settle for an asymptotically accurate main term approximation to $g^{-1}(n)$ for large n as $n \to \infty$ in the average case.

With all of this in mind, we define the following sequence for integers $n \geq 1, k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{6}$$

We will illustrate by example the first few cases of these functions for small k after we prove the next lemma. The sequence of important semi-diagonals of these functions begins as [15, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n\geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

Lemma 5.2 (An exact formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu(n/d) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$\begin{aligned} g^{-1}(n) &= -\sum_{\substack{d|n \\ d>1}} (\omega(d) + 1) f^{-1}(n/d) && \implies \\ (g^{-1} * 1)(n) &= -(\omega * g^{-1})(n). \end{aligned}$$

Now by repeatedly expanding the right-hand-side, and removing corner cases in the nested sums since $\omega(1) = 0$ by convention, we find that

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n).$$

The statement follows by Möbius inversion applied to each side of the last equation. $\square$

Example 5.3 (Special cases of the functions $C_k(n)$ for small k)**.** We cite the following special cases which should be easy enough to see on paper:

$$\begin{aligned}
C_0(n) &= \delta_{n,1} \\
C_1(n) &= \omega(n) \\
C_2(n) &= \sigma_0(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} - \gcd\left(\Omega(n), \omega(n)\right).
\end{aligned}$$

We also can see a recurrence relation between successive $C_k(n)$ values over k of the form

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} C_{k-1}\left(d \cdot p^i\right). \tag{7}$$

Thus we can work out further cases of the $C_k(n)$ for a while until we are able to understand the general trends of its asymptotic behaviors. In particular, we can compute the main term of $C_3(n)$ as follows where we use the notation that p, q are prime indices:

$$\begin{aligned}
C_3(n) &\sim \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \sum_{q|dp^i} \frac{\nu_q(dp^i)}{\nu_q(dp^i)+1} \sigma_0(d)(i+1) \\
&= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \left[\sum_{q|d} \frac{\nu_q(d)}{\nu_q(d)+1} \sigma_0(d)(i+1) + \sum_{j=1}^{i} \frac{j}{(j+1)} \sigma_0(d)(i+1)\right] \\
&= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{q|d} \sigma_0(d) \left[\frac{\nu_p(n)(\nu_p(n)+3)}{2} \frac{\nu_q(d)}{\nu_q(d)+1} + \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2)\left(4\nu_p(n)+9-6H_{\nu_p(n)+2}^{(1)}\right)\right].
\end{aligned}$$

We will break the two key component sums into separate calculations. First, we compute that[2]

$$\begin{aligned}
C_{3,1}(n) &= \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \frac{\nu_p(n)(\nu_p(n)+3)}{2} \times \sum_{q|d} \frac{\nu_q(d)}{\nu_q(d)+1} \sigma_0(d) \\
&= \sum_{\substack{p,q|n \\ p \neq q}} \sum_{d|\frac{n}{p^{\nu_p(n)} q^{\nu_q(n)}}} \frac{\nu_p(n)(\nu_p(n)+3)}{2} \times \sum_{i=1}^{\nu_q(n)} \frac{\nu_q(dq^i)}{\nu_q(dq^i)+1} \sigma_0(dq^i) \\
&= \sum_{\substack{p,q|n \\ p \neq q}} \sum_{d|\frac{n}{p^{\nu_p(n)} q^{\nu_q(n)}}} \frac{\nu_p(n)(\nu_p(n)+3)\nu_q(n)(\nu_q(n)+3)}{4} \sigma_0(d) \\
&= (\sigma_0 * 1)(n) \times \sum_{\substack{p,q|n \\ p \neq q}} \frac{\nu_p(n)(\nu_p(n)+3)\nu_q(n)(\nu_q(n)+3)}{(\nu_p(n)+1)(\nu_p(n)+2)(\nu_q(n)+1)(\nu_q(n)+2)}.
\end{aligned}$$

Next, we have that

$$C_{3,2}(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{q|d} \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2)\left(4\nu_p(n)+9-6H_{\nu_p(n)+2}^{(1)}\right)\sigma_0(d)$$

[2]Here, the arithmetic function $\sigma_0 * 1$ is multiplicative. Its value at prime powers can be computed as

$$(\sigma_0 * 1)(p^\alpha) = \sum_{i=0}^{\alpha} (i+1) = \frac{(\alpha+1)(\alpha+2)}{2},$$

where $\sigma_0(p^\beta) = \beta + 1$.

$$\begin{aligned}
&= \sum_{\substack{p,q|n \\ p \neq q}} \sum_{d | \frac{n}{p^{\nu_p(n)} q^{\nu_q(n)}}} \sum_{i=1}^{\nu_q(n)} \frac{1}{12} (\nu_p(n)+1)(\nu_p(n)+2) \left(4\nu_p(n) + 9 - 6H_{\nu_p(n)+2}^{(1)}\right) \sigma_0(d)(i+1) \\
&= (\sigma_0 * 1)(n) \times \sum_{\substack{p,q|n \\ p \neq q}} \frac{1}{6} \frac{\nu_q(n)(\nu_q(n)+3)}{(\nu_q(n)+1)(\nu_q(n)+2)} \left(4\nu_p(n) + 9 - 6H_{\nu_p(n)+2}^{(1)}\right).
\end{aligned}$$

Now to roughly bound the error term, e.g., the GCD of prime omega functions from the exact formula for $C_3(n)$, we observe that the divisor function has average order of the form:

$$d(n) \sim \log n + (2\gamma - 1) + O\left(\frac{1}{\sqrt{n}}\right).$$

Then using that $\omega(n), \Omega(n) \sim \log\log n$ (except in rare cases when n is primorial, a power of 2, etc. [3]), as discussed in the next remarks, we bound the error as

$$\begin{aligned}
C_{3,3}(n) &= -\sum_{p|n} \sum_{d | \frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \gcd\left(\Omega(d) + i, \omega(d) + 1\right) \\
&= \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} O\left(\sigma_0(n) \cdot \log\log n\right) \\
&= O\left(\pi(n) \cdot \log n \cdot \log\log n\right) \\
&= O\left(n \cdot \log\log n\right).
\end{aligned}$$

In total, we obtain that

$$\begin{aligned}
C_3(n) = (\sigma_0 * 1)(n) \times & \sum_{\substack{p,q|n \\ p \neq q}} \frac{1}{6} \frac{\nu_q(n)(\nu_q(n)+3)}{(\nu_q(n)+1)(\nu_q(n)+2)} \left(4\nu_p(n) + 9 - 6H_{\nu_p(n)+2}^{(1)}\right) \\
& + \sigma_0(n) \times \sum_{\substack{p,q|n \\ q \neq p}} \frac{2^{\nu_q(n)} \nu_p(n)(\nu_p(n)+3)}{4(\nu_p(n)+1)(\nu_q(n)+1)} \\
& + O\left(n \cdot \log\log n\right).
\end{aligned} \tag{8}$$

For the next cases, we would use similar techniques. The key is to compute enough small cases that we can see the dominant asymptotic terms in these expansions. We will expand more on this below.

Remark 5.4 (Recursive growth of the functions $C_k(n)$ in the average case)**.** We note that the average order of $\Omega(n) \sim \log\log n$, so that for large $x \gg 1$ tending to infinity, we can expect (on average) that for $p|n$, $1 \leq \nu_p(n)$ (for large $p|x, p \sim \frac{x}{\log x}$) and $\nu_p(n) \approx \log\log n$. However, if x is primorial, we can have $\Omega(x) \sim \frac{\log x}{\log\log x}$. There is, however, a duality with the size of $\Omega(x)$ and the rate of growth of the $\nu_p(x)$ exponents. That is to say that on average, even though $\nu_p(x) \sim \log\log n$ for most $p|x$, if $\Omega(x) = m \approx O(1)$ is small, then

$$\nu_p(x) \approx \log_{\sqrt[m]{\frac{x}{\log x}}}(x) = \frac{m \log x}{\log\left(\frac{x}{\log x}\right)}.$$

Since we will be essentially averaging the inverse functions, $g^{-1}(n)$, via their summatory functions over the range $n \leq x$ for x large, we tend not to worry about bounding anything but by the average order case, which

[3] In this context, we write $\Omega(n) \sim \log\log n$ to denote the *average order* of this arithmetic function – even though its actual values may fluctuate non-uniformly infinitely often, e.g., $\Omega(2^m) = m$ and for primes p we have that $\Omega(p) = 1$, but most of the time the asymptotic relation holds when we sum, or average over all possible $n \leq x$. What this notation corresponds to, or means in practice, is that the average of the summatory function satisfies: $\frac{1}{x} \cdot \sum_{n \leq x} \Omega(n) \sim \log\log x$.

wins out when we sum (i.e., average) and tend to infinity. Given these observations, we can use the function $C_3(n)$ we just computed exactly as an asymptotic benchmark to build further approximations. In particular, the dominant order terms in $C_3(n)$ are given by

$$C_3(n) \sim \frac{(\sigma_0 * 1)(n)n^2}{\log^2 n} - \frac{(\sigma_0 * 1)(n)n^2}{\log n} + O\left(n \cdot \log\log n\right).$$

We will leave the terms involving the divisor function $\sigma_0(n)$ and convolutions involving it unevaluated because of how much their growth can fluctuate depending on prime factorizations for now.

Summary 5.5 (Asymptotics of the $C_k(n)$)**.** We have the following asymptotic relations relations for the growth of small cases of the functions $C_k(n)$:

$$\begin{aligned}
C_1(n) &\sim \log\log n \\
C_2(n) &\sim \frac{\sigma_0(n)n}{\log n} + O(\log\log n) \\
C_3(n) &\sim \frac{(\sigma_0 * 1)(n)n^2}{\log^2 n} - \frac{(\sigma_0 * 1)(n)n^2}{\log n} + O\left(n \cdot \log\log n\right).
\end{aligned}$$

Theorem 2.5 proved next makes precise what these formulas suggest about the growth rates of $C_k(n)$.

Proof of Theorem 2.5. We showed how to compute the formulas for the base cases in the preceeding examples discussed above in Example 5.3. We can also see that $C_3(n)$ satsfies the formula we must establish when $k := 3$. Let's proceed by using induction with the recurrence formula from (7) relating $C_k(n)$ to $C_{k-1}(n)$ for all $k \geq 1$. The strategy is to precisely evaluate the sums recursively. The strategy is to drop asymptotically insignificant terms that, indeed make the formulas most precise, but which will along the way contribute negligible weight to our target bounds. What results after performing this procedure is a main term formula that is precise for sufficiently large n as we let $n \to \infty$. We will compute the main term formula first, then complete the proof by bounding the easier big-Oh error term calculations to wrap up our induction.

Main term formula inductive proof. Suppose that $k \geq 4$. By the recurrence formula for $C_k(n)$, we have that

$$C_k(n) \sum_{p|n} \sum_{d|np^{-\nu_p(n)}} \sum_{i=1}^{\nu_p(n)} -\frac{(dp^i)^{k-1}}{(\log(dp^i))^{k-1}} \binom{i+k-1}{k-1} (\sigma_0 * \mathbb{1}_{*_{k-2}})(d).$$

Now to handle the inner sum, we bound by setting $\alpha \equiv \nu_p(n)$ and invoking *Mathematica* in the form of

$$\begin{aligned}
\mathrm{IC}_k(n) &= \sum_{i=1}^{\alpha} -\frac{(dp^i)^{k-1}}{(\log(dp^i))^{k-1}} \binom{i+k-1}{k-1} \\
&\approx \int -\frac{(dp^\alpha)^{k-1}}{(\log(dp^\alpha))^{k-1}} \binom{\alpha+k-1}{k-1} \\
&\sim \frac{1}{(k-1)! \log^k p} \left(\mathrm{Ei}((k-2)\log(dp^\alpha)) \left[\log^{k-1}(d) - (k-1)! \log^{k-1}(p)\right]\right) \\
&\qquad - \frac{1}{(k-2)(k-1)! \log^k p} \left(\log^{k-2}(d) + \alpha^{k-2} \log^{k-2}(p)\right).
\end{aligned}$$

We now simplify somewhat again by setting

$$p \mapsto \left(\frac{n}{e}\right)^{\frac{1}{\log\log n}}, \alpha \mapsto \log\log n, \log p \mapsto \frac{\log n}{\log\log n}.$$

Also, since $p \gg_n d$, we obtain the dominant asymptotic growth terms of

$$\mathrm{IC}_k(n) \sim \frac{\alpha^{k-2}}{(k-2)(k-1)! \log^2 p}$$

$$\approx \frac{(\log\log n)^k}{(k-2)(k-1)!\log^2 n}.$$

Now, as we did in the previous example work, we handle the sums by pulling out a factor of the inner divisor sum depending only on n (and k):

$$\begin{aligned} C_k(n) &= \sum_{p|n} (\sigma_0 * \mathbb{1}_{*_{k-1}})(n) \binom{p^{\nu_p(n)}+k}{k}^{-1} \times \mathrm{IC}_k(n) \\ &= (\sigma_0 * \mathbb{1}_{*_{k-1}})(n) \binom{p^{\nu_p(n)}+k}{k}^{-1} \cdot \pi(n) \times \mathrm{IC}_k(n) \end{aligned}$$

Combining with the remaining terms we get by induction a proof of our target bounds for $C_k(n)$.

Establishing the error term bound inductively. To bound the error terms, again suppose inductively that $k \geq 4$. We compute the big-O bounds as follows letting $\alpha \equiv \nu_p(n)$:

$$\begin{aligned} \mathrm{ET}_k(n) &= \sum_{i=1}^{\nu_p(n)} n^{k-2} \cdot \frac{(\log\log n)^{k-2}}{(\log n)^{k-2}} \\ &\approx \int (dp^\alpha)^{k-2} \log\log(dp^\alpha) d\alpha \\ &= -\frac{\mathrm{Ei}((k-2)\log(dp^\alpha))}{(k-2)\log p} + \frac{d^{k-2}p^{(k-2)\alpha}}{(k-2)\log p}\log(dp^\alpha) \\ &\sim \frac{d^{k-2}p^{(k-2)\alpha}}{(k-2)\log p}\log(dp^\alpha). \end{aligned}$$

In the last expansion, we have dropped the exponential integral terms since they provide at most polynomial powers of the logarithm of their inputs.

To evaluate the outer divisor sum from the recurrence relation for $C_k(n)$, we will require the following bound providing an average order on the *generalized sum-of-divisors functions*, $\sigma_\alpha(n) := \sum_{d|n} d^\alpha$. In particular, we have that for integers $\alpha \geq 2$ [11, §27.11]:

$$\sigma_\alpha(n) \sim \frac{\zeta(\alpha+1)}{\alpha+1} x^\alpha + O(x^{\alpha-1}).$$

Approximating the number of terms in the prime divisor sum by $\pi(x) \sim \frac{x}{\log x}$, we thus obtain

$$\mathrm{ET}_k(n) \approx \frac{(\log\log n)^{k-1} e^{k-2}}{(k-1)(k-2)} x^{(k-2)\left(1-\frac{1}{\log\log x}\right)+1+\log\log x} \zeta(k-1).$$

So up to what is effectively constant in k, and dropping lower order terms for a slightly suboptimal, but still sufficient for our purposes, error bound formula, we have completed the proof by induction. $\square$

Corollary 5.6 (Computing the inverse functions)**.** *In contrast to the complicated formulation given by Lemma 5.2, we have that*

$$g^{-1}(n) \sim \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

This is to say that for all $n \geq 2$

$$\left| 1 - \frac{\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)}{g^{-1}(n)} \right| = o\left(\sum_{d|n} C_{\Omega(d)}(d) \right).$$

Moreover, we can bound the error terms as

$$\left| \frac{\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)}{g^{-1}(n)} \right| = O\left(\frac{(\log\log n)^2}{\log n} \cdot \frac{\Gamma(\log\log n)}{n^{\log\log n} \cdot (\log n)^{\log\log n}} \right) \xrightarrow{n\rightarrow\infty} 0.$$

Proof. Using Lemma 5.2, it suffices to show that the squarefree divisors $d|n$ such that $\operatorname{sgn}(\mu(d)\lambda(n/d)) = -1$ have an order of magnitude smaller magnitude (in the little-o notation sense) than the corresponding cases of positive sign on the terms in the divisor sum from the lemma. This is because the sign of the terms in the Möbius inversion sum from the lemma will have already matched exactly that of the terms in $\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)$ except possibly in these comparitively rare cases when $\operatorname{sgn}(\mu(d)\lambda(n/d)) = -1$. Thus, we need only compute a reasonable bound of when such innacuracies between the exact formula from Lemma 5.2 differs from the approximation we claim works when we take the divisor sum over the unsigned $C_{\Omega(d)}(d)$ terms and weight by an overall factor of $\lambda(n)$. It is obvious that if we can show that the difference in these two formulas is asymptotically negligible as we let $n \to \infty$, then we can use the substantially easier to evaluate approximation to calculate accurate new bounds on the summatory function of $g^{-1}(n)$ later on in the results found below (following the conclusion of this proof).

Let n have m_1 prime factors p_1 such that $v_{p_1}(n) = 1$, m_2 such that $v_{p_2}(n) = 2$, and the remaining $m_3 := \Omega(n) - m_1 - 2m_2$ prime factors of higher-order exponentation. We have a few cases to consider after re-writing the sum from the lemma in the following form:

$$g^{-1}(n) = \lambda(n)C_{\Omega(n)}(n) + \sum_{i=1}^{\omega(n)} \left\{ \sum_{\substack{d|n \\ \omega(d)=\Omega(d)=i \\ \#\{p|d:\nu_p(d)=1\}=k_1 \\ \#\{p|d:\nu_p(d)=2\}=k_2 \\ \#\{p|d:\nu_p(d)\geq 3\}=k_3}} \mu(d)\lambda(n/d)C_{\Omega(n/d)}(n/d) \right\}.$$

We obtain the following cases of the squarefree divisors contributing to the signage on the terms in the above sum:

- The sign of $\mu(d)$ is $(-1)^i = (-1)^{k_1+k_2+k_3}$;
- If $m_3 < \#\{p|n : \nu_p(n) \geq 3\}$, then $\lambda(n/d) = 1$ (since $\mu(n/d) = 0$);
- Given (k_1, k_2, k_3) as above, since $\lambda(n) = (-1)^{\Omega(n)}$, we have that $\mu(d) \cdot \lambda(n/d) = (-1)^{i-k_1-k_2}\lambda(n)$.

Thus we define the following sums, parameterized in the $(m_1, m_2, m_3; n)$, which corresponds to a change in expected parity transitioning from the Möbius inversion sum from Lemma 5.2 to the sum approximating $g^{-1}(n)$ defined at the start of this result:

$$\widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) := \sum_{i=1}^{\omega(n)/2} \sum_{k_1=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{i}{2} \rfloor - k_1} \left[\binom{m_1}{2k_1+1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1}\binom{2m_2}{2k_2}\right] [i - k_1 - k_2 = k_3 \equiv m_3]_\delta$$

$$\widetilde{S}_{\mathrm{even}}(m_1, m_2, m_3; n) := \sum_{i=1}^{\omega(n)/2} \sum_{k_1=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{i}{2} \rfloor - k_1} \left[\binom{m_1}{2k_1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1+1}\binom{2m_2}{2k_2}\right] [i - k_1 - k_2 = k_3 \equiv m_3]_\delta.$$

Part I (Lower bounds on the inner sums of the count functions). We claim that

$$\begin{aligned} \widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) &\gg \binom{m_1}{i+1} + \binom{m_1}{\frac{i}{2}}\binom{2m_2-1}{\frac{i}{2}+1} \\ \widetilde{S}_{\mathrm{even}}(m_1, m_2, m_3; n) &\gg \binom{m_1}{i+1} + \binom{m_1}{\frac{i}{2}-1}\binom{2m_2}{\frac{i}{2}+1}. \end{aligned} \tag{9}$$

To prove (9) we have to provide a straightforward bound that represents the maximums of the terms in m_1, m_2. In particular, observe that for

$$\widetilde{S}_{\mathrm{odd}}(m_1, m_2; u) = \sum_{k_1=0}^{u} \sum_{k_2=0}^{u-k_1} \left[\binom{m_1}{2k_1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1+1}\binom{2m_2}{2k_2}\right]$$

$$\widetilde{S}_{\text{even}}(m_1,m_2;u) = \sum_{k_1=0}^{u}\sum_{k_2=0}^{u-k_1}\left[\binom{m_1}{2k_1+1}\binom{2m_2}{2k_2+1}+\binom{m_1}{2k_1}\binom{2m_2}{2k_2}\right],$$

we have that

$$\begin{aligned}
\widetilde{S}_{\text{odd}}(m_1,m_2;u) &\gtrsim \binom{m_1}{2u+1} + \max_{1\leq k_1\leq u}\binom{m_1}{2k_1+1}\binom{2m_2}{2u+1-2k_1} \\
&= \binom{m_1}{2u+1} + \binom{m_1}{2k_1+1}\binom{2m_2}{2u+1-2k_1}\Bigg|_{k_1=\frac{u}{2}} \\
&= \binom{m_1}{2u+1} + \binom{m_1}{u+1}\binom{2m_2}{u+1} \\
\widetilde{S}_{\text{even}}(m_1,m_2;u) &\gtrsim \binom{m_1}{2u+1} + \max_{1\leq k_1\leq u}\binom{m_1}{2k_1}\binom{2m_2}{2u+1-2k_1} \\
&= \binom{m_1}{2u+1} + \binom{m_1}{u-1}\binom{2m_2}{u+1}.
\end{aligned}$$

The lower bounds in (9) then follow by setting $u \equiv \left\lfloor \frac{i}{2} \right\rfloor$.

Part II (Bounding m_1, m_2, m_3 and effective (i, k_1, k_2) contributing to the count). We thus have to determine the asymptotic growth rate of $\widetilde{S}_{\text{odd}}(m_1,m_2,m_3;n)+\widetilde{S}_{\text{even}}(m_1,m_2,m_3;n)$, and show that it is of comparatively small order. First, we bound the count of non-zero m_3 for $n \leq x$ from below. For the cases where we expect differences in signage, it's the last Iverson convention term that kills the order of growth, e.g., we expect differences when the parameter m_3 is larger than the usual configuration. We know that

$$\pi_k(x) \sim \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}.$$

Using the formula for $\pi_k(x)$, we can count the average orders of m_1, m_2 as

$$\begin{aligned}
N_{m_1}(x) &\approx \frac{1}{x}\#\{n\leq x:\omega(n)=1\} \sim \frac{\log\log x}{\log x} \\
N_{m_2}(x) &\approx \frac{1}{x}\#\{n\leq x:\omega(n)=2\} \sim \frac{(\log\log x)^2}{\log x}.
\end{aligned}$$

Additionally, in Corollary 4.3 on page 19 we will prove a lower bound on $\widehat{\pi}_k(x)$. We use this result immediately below without proof.

When we have parameters with respect to some $n \geq 1$ such that $m_3 > 0$, it must be the case that

$$\Omega(n)-\omega(n) > \begin{cases} 0, & \text{if } \omega(n)\geq 2; \\ 1, & \text{if } \omega(n)=1. \end{cases}$$

To count the number of cases $n \leq x$ where this happens, we form the sums

$$\begin{aligned}
N_{m_3}(x) &\gg \pi_1(x)\times\sum_{k=3}^{\frac{3}{2}\log\log x}\widehat{\pi}_k(x) + \sum_{k=2}^{\frac{3}{2}\log\log x}\sum_{j=k+1}^{\frac{3}{2}\log\log x}\pi_k(x)\widehat{\pi}_j(x) \\
&\gtrsim \frac{Ae}{B}\frac{x}{\log^{\frac{13}{14}}(x)} + \sum_{k=2}^{\log\log x}\pi_k(x)\left[\frac{Ae}{B}\log^{\frac{1}{14}}(x)\right] \\
&\gtrsim \frac{Ae}{B}\frac{x}{\log^{\frac{13}{14}}(x)} + \frac{Ae\sqrt{2}}{2\sqrt{\pi}B}\frac{x}{\log^{\frac{13}{14}}(x)\sqrt{\log\log x}}.
\end{aligned}$$

Now in practice, we are not summing up $n \leq x$, but rather $n \leq \log\log x$. So the above function evaluates to

$$N_{m_3}(\log\log x) \gg \frac{\log\log x}{(\log\log\log x)^{13/14}} \gg \frac{\log\log x}{\log\log\log x}.$$

Next, we go about solving the subproblem of finding when $i - k_1 - k_2 = m_3$. First, we find a lower solution index on i using asymptotics for the *Lambert W-function*, $W_0(x) = \log x - \log\log x + o(1)$:

$$\begin{aligned} \frac{i}{2} = \frac{\log\log x}{\log\log\log x} &\iff \log\log x \gtrsim \frac{i}{2}(\log i + \log\log i) \\ &\iff \frac{i}{2} \sim \frac{\log\log x}{\log\log\log x}. \end{aligned}$$

Now since $2 \leq k_1 + k_2 \leq i/2$, when x is large, we actually obtain a number of soultions on the order of

$$\frac{\log\log x}{2} - \frac{\log\log x}{\log\log\log x} = \frac{\log\log x}{2}(1 + o(1)).$$

Part III (Putting it all together). Using the binomial coefficient inequality

$$\binom{n}{k} \geq \frac{n^k}{k^k},$$

we can work out carefully on paper using (9) that

$$\begin{aligned} \widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) &\gtrsim \frac{\log\log x}{2}\left(\frac{\log\log\log x}{2\log x}\right)^{\frac{2\log\log x}{\log\log\log x}+1}\left[1 + \frac{(\log\log\log x)^2}{\log^2 x}(4\log\log x \cdot \log\log\log x)^{\frac{\log\log x}{2\log\log\log x}+1}\right] \\ \widetilde{S}_{\mathrm{odd}}(m_1, m_2, m_3; n) &\gtrsim \frac{\log\log x}{2}\left(\frac{\log\log\log x}{2\log x}\right)^{\frac{2\log\log x}{\log\log\log x}+1}\left[1 + \left(\frac{\log x}{2\log\log\log x}\right)(8\log\log x)^{\frac{\log\log x}{\log\log\log x}+1}\right]. \end{aligned}$$

Part IV (Obtaining the rate at which the ratio goes to zero). Let

$$S_{\mathrm{diff}}(m_1, m_2, m_3; n) := S_{\mathrm{even}}(m_1, m_2, m_3; n) + S_{\mathrm{odd}}(m_1, m_2, m_3; n).$$

Then we have that

$$\begin{aligned} \left|\frac{\lambda(x) \times \sum_{d|x} C_{\Omega(d)}(d)}{g^{-1}(x)}\right| &= \frac{\sum\limits_{\substack{d|x \\ d \leq \log\log x}} C_{\Omega(d)}(d)}{(\log\log d)}|g^{-1}(x)| = \\ &= O\left(\frac{S_{\mathrm{diff}}\left(\frac{\log\log x}{\log x}, \frac{(\log\log x)^2}{\log x}, \frac{(\log\log x)^2}{\log\log\log x}; x\right)}{C_{\Omega(x)}(x)}\right) \\ &= O\left(\frac{\log x \cdot \log\log x}{\log\log\log x \cdot C_{\Omega(x)}(x)}\left(\frac{\sqrt{2\log\log x} \cdot \log\log\log x}{\log x}\right)^{\frac{2\log\log x}{\log\log\log x}}\right) \\ &= O\left(\frac{(\log\log x)^2 \cdot \log x}{\log\log\log x \cdot C_{\Omega(x)}(x)}\right). \end{aligned}$$

We borrow from Corollary 5.8 proved below in this section to get a lower bound on $C_{\Omega(x)}(x)$. This result implies the stated bound, which tends to zero as $x \to \infty$. Thus the divisor sum in the corollary statement accuaretely approximates the main term and sign of $g^{-1}(n)$ as $n \to \infty$. □

Corollary 5.7. *We have that for sufficiently large x, as $x \to \infty$ that*

$$G^{-1}(x) \gtrsim \widehat{L}_0(\log\log x) \times \sum_{n \leq \log\log x} \lambda(n) \cdot C_{\Omega(n)}(n),$$

where the function

$$\widehat{L}_0(\log\log x) := (-1)^{\left\lfloor \frac{3}{2}\log\log\log\log x \right\rfloor + 1} \left\{ \sqrt{\frac{3}{\pi}} \frac{A(2e+3)}{4B \log^{\frac{3}{2}}(2)} \right\} \cdot \frac{(\log\log\log x)^{\frac{43}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3}}}{\sqrt{\log\log\log\log x}},$$

with the exponent $\frac{43}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3} \approx 3.87011$.

Proof. Using Corollary 5.6, we have that

$$\begin{aligned} G^{-1}(x) &\approx \sum_{n \leq x} \lambda(n) \cdot (g^{-1} * 1)(n) \\ &= \sum_{d \leq \log\log x} C_{\Omega(d)}(d) \times \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn). \end{aligned}$$

Now we see that by complete additivity (multiplicativity) of $\Omega(n)$ (as indicated by the sign of $\lambda(n)$) that

$$\sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn) = \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(d)\lambda(n) = \lambda(d) \sum_{n \leq \left\lfloor \frac{x}{d} \right\rfloor} \lambda(n).$$

Borrowing a result from the next sections (proved in Section 4), we can establish that

$$\begin{aligned} \sum_{n \leq x} \lambda(n) &\gg \sum_{n \leq \frac{3}{2}\log\log x} (-1)^k \cdot \widehat{\pi}_k(x) \\ &\gtrsim (-1)^{\left\lfloor \frac{3}{2}\log\log x \right\rfloor + 1} \left(\sqrt{\frac{3}{\pi}} \frac{A(2e+3)}{4B \log^{\frac{3}{2}}(2)} \right) \cdot \frac{(\log x)^{\frac{43}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3}}}{\sqrt{\log\log x}} \\ &=: \widehat{L}_0(x). \end{aligned}$$

Then since for large enough x and $d \leq x$,

$$\log(x/d) \sim \log x, \log\log(x/d) \sim \log\log x,$$

we can obtain the stated result, e.g., so that $\widehat{L}_0(x) \sim \widehat{L}_0(x/d)$ for large $x \to \infty$. $\square$

The previous corollary is employed to prove the exact lower bounds on $G^{-1}(x)$ given in Theorem 6.1 in the next section. The parity of $\lfloor 2\log\log\log\log x \rfloor$ determines subsequences of real $x \gg 1$ along which we break these bounds into cases. The next result provides complete asymptotic upper and lower bound information on the functions $C_k(n)$ when $k \equiv \Omega(n)$.

Corollary 5.8 (Asymptotics for very special case of the functions $C_k(n)$)**.** *For $k \gg 1$ sufficiently large, we have that*

$$C_{\Omega(n)}(n) \sim (\sigma_0 * \mathbb{1}_{*_{\log\log n - 2}})(n) \times \lambda(n) \frac{n^{\log\log n - 1}}{(\log n)^{\log\log - 1} \Gamma(\log\log n)}.$$

Moreover, by considering the average orders of the function $\nu_p(n)$ for p large and tending to infinity, we have bounds on the aysmptotic behavior of these functions of the form

$$\lambda(n)\widehat{\tau}_0(n) \precsim C_{\Omega(n)}(n) \precsim \lambda(n)\widehat{\tau}_1(n).$$

It suffices to take the functions

$$\widehat{\tau}_0(n) := \frac{1}{\log 2} \cdot \frac{\log n}{(\log n)^{\log\log n}} \cdot \frac{n^{\log\log n - 1}}{\Gamma(\log\log n)}$$
$$\widehat{\tau}_1(n) := \frac{1}{2e\log 2} \cdot \frac{(\log n)^2}{(\log n)^{\log\log n}} \cdot \frac{n^{\log\log n}}{\Gamma(\log\log n)}.$$

Proof. The first stated formula follows from Theorem 2.5 by setting $k := \Omega(n) \sim \log\log n$ and simplifying. We evaluate the Dirichlet convolution functions and approximate as follows:

$$\begin{aligned}
(\sigma_0 * \mathbb{1}_{\log\log n - 2})(n) &= \sum_{p|n} \binom{\nu_p(n) + \log\log n - 1}{\log\log n - 1} \\
&\geq \sum_{p|n} \frac{(\nu_p(n) + \log\log n - 1)^{\log\log n - 1}}{(\log\log n)^{\log\log n - 1}} \\
&\sim \frac{n}{\log 2} \\
(\sigma_0 * \mathbb{1}_{\log\log n - 2})(n) &\leq \left(\frac{(\nu_p(n) + \log\log n - 1)e}{\log\log n - 1}\right)^{\log\log n - 1} \\
&\sim (2e)^{\log\log n - 1} \\
&= \frac{n \cdot \log n}{2e\log 2}.
\end{aligned}$$

The upper and lower bounds are obtained from the next well known binomial coefficient approximations using Stirling's formula.

$$\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{n^k}{k!} < \left(\frac{ne}{k}\right)^k \qquad \square$$

Now that we have accurate asymptotic bounds on $|g^{-1}(n)|$ as $n \to \infty$, we must form the summatory functions $G^{-1}(x)$ of g^{-1} whose terms vary widely when including the parity of $\Omega(n)$ (sign of $\lambda(n)$). The natural mechanism for this is to employ Abel summation. However, we do not yet have a sufficient grasp on the summatory functions, $A_{\Omega}(x)$, that indicate the sign shifts of $\lambda(n)$ for $n \leq x$. To effectively bound these functions for large x, we will require asymptotic lower bounds on $\widehat{\pi}_k(x)$ for $k \geq 1$ and k bounded high enough above (with respect to x) so that the resulting functions $A_{\Omega}(x)$ are asymptotically accurate.

6 Key applications: Establishing lower bounds for $M(x)$ by cases along infinite subsequences

6.1 The culmination of what we have done so far

As noted before in the previous subsections, we cannot hope to evaluate functions weighted by $\lambda(n)$ except for on average using Abel summation. For this task, we need to know the bounds on $\widehat{\pi}_k(x)$ we developed in the proof of Corollary 4.3. A summation by parts argument shows that[4] [5]

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k)+1) \\ &\approx G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)}. \end{aligned} \tag{10}$$

The result proved in Lemma 4.4 is key to justifying the asymptotics obtained next in Theorem 6.1.

To simplify notation, for integers $m \geq 1$, let the *iterated logarithm function* (not to be confused with powers of $\log x$) be defined for $x > 0$ by

$$\log_*^m(x) := \begin{cases} x, & \text{if } m = 0; \\ \log x, & \text{if } m = 1; \\ \log\left(\log_*^{m-1}(x)\right), & \text{if } m \geq 2. \end{cases}$$

So $\log_*^2(x) = \log\log x$, $\log_*^3(x) = \log\log\log x$, $\log_*^4(x) = \log\log\log\log x$, $\log_*^5(x) = \log\log\log\log\log x$, and so on. This notation will come in handy to abbreviate the dominant asymptotic terms we find next in Theorem 6.1.

We use the result of Corollary 5.8 and Corollary 4.3 to prove the following central theorem:

Theorem 6.1 (Asymptotics and bounds for the summatory functions $G^{-1}(x)$)**.** *We define the lower summatory function, $G_u^{-1}(x)$, to provide bounds on the magnitude of $G^{-1}(x)$:*

$$|G_\ell^{-1}(x)| \ll |G^{-1}(x)|,$$

for all sufficiently large $x \gg 1$. We have the following asymptotic approximations for the lower summatory function where $C_{\ell,1}, C_{\ell,2}$ are absolute constants defined by

$$C_{\ell,1} = \frac{3}{16}\sqrt{\frac{3}{2}}\frac{A_0^2(2e+3)^2}{\pi e B^2 (\log 2)^3}, C_{\ell,2} = \frac{27 A_0^2 (2e+3)^3}{128\pi^{3/2} B^2 (\log 2)^3},$$

and $\widehat{L}_0(x)$ is the multiplier function from Corollary 5.7:

$$\begin{aligned} &\left|G_\ell^{-1}(x)\right| \gtrsim \\ &\quad\Bigg|(-1)^{\left\lfloor \frac{3}{2}\log_*^4(x)\right\rfloor} C_{\ell,1} \cdot (\log x)^{\frac{11}{7}} (\log\log x)^{\frac{71}{14}+\frac{3}{2\log 2}-\frac{3}{2\log 3}-\log_*^4(x)} \log_*^3(x)^{1+\frac{3}{2}\log\log x+\log_*^4(x)} \log_*^4(x)^{\log_*^4(x)-\frac{1}{2}} \\ &\quad\quad - (-1)^{\left\lfloor 2\log_*^4(x)\right\rfloor} C_{\ell,2} \cdot \frac{\log_*^3(x)^{\frac{9}{2}+\frac{25}{6}\log 2+\frac{3}{2\log 2}-\frac{4}{3}\log 3-\frac{3}{2\log 3}}}{\sqrt{\log_*^4(x)}} \log_*^5(x)^{\frac{11}{7}+\frac{3}{2}\log_*^7(x)}\Bigg|. \end{aligned}$$

---

[4] Here, we drop the unnecessary floored integer-valued arguments to $\pi(x)$ in place of its approximation by $\pi(x) \sim \frac{x}{\log x}$. In fact, since we can always bound

$$\frac{Ax}{\log x} \leq \pi(x) \leq \frac{Bx}{\log x},$$

for suitably defined absolute constants, $A, B > 0$, we are not losing any precision asymptotically by making this small leap in approximation from exact summation (in the first formula) to the integral formula representing convolution (in the second formula below).

[5] Since $\pi(1) = 0$, the actual range of summation corresponds to $k \in \left[1, \frac{x}{2}\right]$.

Proof Sketch: Logarithmic scaling to the accurate order of the inverse functions. For the sums given by

$$S_{g^{-1}}(x) := \sum_{n\leq x} \lambda(n) \cdot C_{\Omega(n)}(n),$$

we notice that using the asymptotic bounds (rather than the exact formulas) for the functions $C_{\Omega(n)}(n)$, we have over-summed by quite a bit. In particular, following from the intent behind the constructions in the last sections, we are really summing only over all $n \leq x$ with $\Omega(n) \leq x$. Since $\Omega(n) \leq \lfloor \log_2 n \rfloor = \left\lfloor \frac{\log n}{\log 2} \right\rfloor$, many of the terms in the previous equation are actually zero (recall that $C_0(n) = \delta_{n,1}$). So we are actually only going to sum up to the average order of $\Omega(n) \sim \log\log n$ in practice, or to the slightly larger bound if the leading sign term on $G_\ell^{-1}(x)$ is negative. Hence, the sum (in general) that we are really interested in bounding is bounded below in magnitude by $S_{g^{-1}}(\log\log x)$ or $S_{g^{-1}}(\log_2(x))$, where we can now safely apply the asymptotic formulas for the $C_k(n)$ functions from Corollary 5.8 that hold once we have verified these constraints. □

Proof. Recall from our proof of Corollary 4.3 that a lower bound on the function $\widehat{\pi}_k(x)$ is given by $G\left(\frac{k-1}{\log\log x}\right)$ where the function $G(z)$ is bounded below by

$$G(z) \gtrsim A_0 x \frac{(\log\log x)^{k-1}}{(k-1)!} \left(\frac{\log x}{\log 2}\right)^z \log^2 x \left(1 - \frac{z}{B} \log^{\frac{1}{14}}(x)\right).$$

Thus we can form a lower summatory function indicating the parity of all $\Omega(n)$ for $n \leq x$ as

$$\begin{aligned} A_\Omega^{(\ell)}(t) &= \sum_{k \leq \frac{3}{2}\log\log t} (-1)^k G\left(\frac{k-1}{\log\log t}\right) \\ &\sim (-1)^{1+\left\lfloor \frac{3\log\log t}{2} \right\rfloor} \cdot \frac{3A_0}{4eB \log^{\frac{3}{2}}(2) \Gamma\left(1 + \frac{3}{2}\log\log t\right)} \left((2e+3)\log^{\frac{1}{14}}(t) - 2B\right) \log^{\frac{3}{2}}(t) (\log\log t)^{\frac{3}{2}\log\log t}. \end{aligned} \tag{11}$$

Next, as in Lemma 4.4, we apply Abel summation to obtain that

$$G_\ell^{-1}(x) = \widehat{\tau}_0(\log\log x) A_\Omega^{(\ell)}(x) - \widehat{\tau}_0(u_0) A_\Omega^{(\ell)}(u_0) - \int_{u_0}^{\log\log x} \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t) dt, \tag{12}$$

where we define the integrand function, $I_\ell(t) := \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t)$, with some limiting simplifications as

$$\begin{aligned} I_\ell\left(e^{e^{\frac{4k}{3}}}\right) e^{e^{\frac{4k}{3}}} &= \frac{4A_0 4^{2k-1} 9^{-k} k^{2k} \left((3+2e)e^{2k/21} - 2B)\right] \exp\left(-\frac{16k^2}{9} + 2k + e^{4k/3}\left(\frac{4k}{3} - 1\right) - 1\right)}{3B \log^{\frac{5}{2}}(2)} \times \\ &\quad \times \left(4e^{4k/3}k - 8k - 3\log k - 3\gamma + 6 + 3\log 3 - 6\log 2\right). \end{aligned}$$

The integration term in (12) is summed approximately as follows:

$$\begin{aligned} \int_{u_0-1}^{\log\log x} \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t) dt &\sim \sum_{k=u_0+1}^{\frac{1}{2}\log\log\log\log x} \left(\frac{I_\ell\left(e^{e^{\frac{4k+2}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!} - \frac{I_\ell\left(e^{e^{\frac{4k}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!} \right) e^{e^{\frac{4k}{3}}} \\ &\approx C_0(u_0) + (-1)^{\left\lfloor \frac{\log\log\log\log x}{2} \right\rfloor} \times \int_{\frac{\log\log\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log\log\log x}{2}} \frac{I_\ell\left(e^{e^{\frac{4k}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!} e^{e^{\frac{4k}{3}}} dk. \end{aligned}$$

The differences on the upper and lower bounds on each integral in the last equation is small, and in particular $\frac{1}{2} \lll \log\log x$. So we can use a small perturbation of $+1$ in the power terms of $I_\ell(t)$ combined with an appeal to the binomial series, the expansion of binomial coefficients by the Stirling numbers of the first kind, and the following exact indefinite integral for $x, z \in \mathbb{R}$ moving forward:

$$\int t^p e^{ct} dt = \frac{(-1)^p}{c^{p+1}} \Gamma(p+1, -ct) \sim \frac{e^{ct} t^p}{c}.$$

Define the following function of t and note the change of variable $t \mapsto \frac{k-1}{2}$:

$$I_\ell\left(e^{e^{\frac{4k}{3}}}\right) e^{e^{\frac{4k}{3}}} = (1+k)^{2k} \exp\left(-\frac{16k^2}{9}\left(\frac{4k}{3}-1\right) e^{\frac{4k}{3}}\right) e^{2k-1} \widehat{f}(t_0).$$

So we take one reciprocal factor in the next integrand, and set the remaining powers of t^p to be t_0^p for t_0 a bound of integration which results in a lower bound on our target integrand from Abel summation.

From this perspective, we obtain using the exponential generating functions for the Stirling numbers of the first kind that [5, §7.4][6]

$$\begin{aligned}
\widehat{T}_\ell(t_0; t) &= \int \widehat{I}_\ell(t) dt \\
&\gg \sum_{m \geq 0} \sum_{n \geq 0} \sum_{q \geq 0} \sum_{j \geq 0} \sum_{r \geq 0} \frac{(-1)^{m+q+j+r}}{m! n! q! j!} \left(\frac{4}{3}\right)^{2m+n} \begin{bmatrix} j \\ r \end{bmatrix} \left\{\int t^{2m+n+j+r} \exp\left(\left(2 + \frac{4}{3}(n+q)\right) t\right) dt\right\} \frac{\widehat{f}(t_0)}{e} \\
&\gtrsim -\frac{3\widehat{f}(t_0)}{4e} e^{2t} e^{-\frac{16k^2}{9}} \left(\gamma + \frac{e^{te^{\frac{4t}{3}}}}{te^{\frac{4t}{3}}} + \frac{4t}{3}\right) \left(\gamma + \frac{e^{te^{\frac{4t}{3}}}}{ke^{\frac{4t}{3}}} - \frac{4t}{3}\right) t^{2t}
\end{aligned}$$

In the previous equation, we have used that $(n+q+12)^{-1} \gtrsim \frac{1}{nq}$ and that for large $x \ggg 1$ tending to infinity

$$\sum_{m \geq 1} \frac{(-x)^m}{m \cdot m!} = -\left(\gamma + \Gamma(0, x) + \log x\right) \sim -\left(\gamma + \frac{e^{-x}}{x} + \log x\right).$$

Now we can define the coefficient functions, which as multipliers above would have otherwise complicated our integrals, in the form of $\widehat{f}(t_0) = \mathrm{cf}_+(t_0) - \mathrm{cf}_-(t_0)$ as

$$\begin{aligned}
\mathrm{cf}_+(t) &:= \left(\frac{16}{9}\right)^t \left(2B(8t + 3\gamma + 6\log 2) + 6B \log t + 12e^{10t/7} t + 8e^{\frac{10t}{7}+1} t + 6e^{\frac{2t}{21}+1}(2 + \log 3) + 9e^{2t/21}(2 + \log 3)\right) \\
\mathrm{cf}_-(t) &:= \left(\frac{16}{9}\right)^t \left(2B\left(4e^{4t/3} t + 6 + 3\log 3\right) + (3+2e)e^{2t/21}(8t + 3\gamma + 6\log 2) + 3(3+2e)e^{2t/21} \log t\right).
\end{aligned}$$

Let

$$\widehat{h}(t) := 3 \cdot 4^{-t-1} \left(\frac{3}{4}\right)^{\frac{4t}{3}} \frac{\sqrt{3}}{16\pi t^{\frac{10t}{3}+1}}.$$

Applying Stirling's formula again when x is large, we have that

$$\begin{aligned}
\widehat{R}_\ell(x) &= (-1)^{\left\lfloor \frac{\log\log\log\log x}{2} \right\rfloor} \times \int_{\frac{\log\log\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log\log\log x}{2}} \frac{I_\ell\left(e^{e^{\frac{4k}{3}}}\right)}{(2k)! \left(\frac{4k}{3}\right)!} e^{e^{\frac{4k}{3}}} dk \qquad (13) \\
&\gtrsim (-1)^{\left\lfloor \frac{x}{2} \right\rfloor} \times \widehat{h}\left(\frac{\log\log\log\log x}{2}\right) \Bigg[
\end{aligned}$$

[6]Namely, that for natural numbers $j \geq 0$

$$\sum_{k \geq 0} \begin{bmatrix} k \\ j \end{bmatrix} \frac{z^k}{k!} = \frac{(-1)^j}{j!} \operatorname{Log}(1-z)^j.$$

$$\widehat{T}_\ell\left(\frac{\log\log\log\log x}{2};\frac{\log\log\log\log x}{2}\right)\left(\mathrm{cf}_+\left(\frac{\log\log\log\log x-1}{2}\right)-\mathrm{cf}_-\left(\frac{\log\log\log\log x}{2}\right)\right)$$
$$-\widehat{T}_\ell\left(\frac{\log\log\log\log x-1}{2};\frac{\log\log\log\log x-1}{2}\right)\left(\mathrm{cf}_+\left(\frac{\log\log\log\log x}{2}\right)-\mathrm{cf}_-\left(\frac{\log\log\log\log x-1}{2}\right)\right)\Bigg].$$

Since for real $0<s<1$ such that $s\to 0$, we have that $\log(1+s)\sim s$ and $(1+s)^{-1}\sim 1-s$, we can approximate the differences implied by the last estimate using that for t large tending to infinity we have

$$\log_*^m\left(t-\frac{1}{2}\right)\sim\log_*^m(t)-\frac{1}{2\log^{m-1}t}, m\geq 1.$$

Then applying these simplifications to (13) above and removing lower-order terms that do not contribute to the dominant asymptotic terms, we find that

$$\int_{u_0}^{\log\log x}\widehat{\tau}_0'(t)A_\Omega^{(\ell)}(t) \tag{14}$$
$$\gtrsim C_0(u_0)+\frac{(-1)^{\left\lfloor\frac{\log\log\log\log x}{2}\right\rfloor}\cdot 9\sqrt{3}A_0(2e+3)^2}{32B\pi e\log^{3/2}(2)}(\log\log\log x)^{\frac{10}{7}+\frac{25}{6}\log 2-\frac{4}{3}\log 3-\frac{5}{2}\log_*^5(x)}\log_*^5(x)^{\frac{11}{7}+\frac{3}{2}\log_*^7(x)}.$$

Finally, using Stirling's formula for very large x and (11), we can see that

$$\widehat{\tau}_0(x)\sim\frac{\log^2 x\cdot\log\log x}{\sqrt{2\pi}\cdot x}\left(\frac{x}{\log x\cdot\log\log x}\right)^{\log\log x}$$
$$A_\Omega^{(\ell)}(x)\sim\frac{3A_0(2e+3)}{4eB\log^{\frac{3}{2}}(2)}\log^{\frac{11}{7}}(x)(\log\log x)^{\frac{3}{2}\log\log x}.$$

So we have that the first terms in (12) are given by

$$\widehat{\tau}_0(\log\log x)A_\Omega^{(\ell)}(x)\gtrsim\frac{3A_0(2e+3)}{4\sqrt{2\pi}\cdot eB\log^{\frac{3}{2}}(2)}\cdot\log^{\frac{11}{7}}(x)\frac{\log\log\log x\cdot(\log\log x)^{1+\frac{3\log\log x}{2}+\log\log\log\log x}}{(\log\log\log x\cdot\log\log\log\log x)^{\log\log\log\log x-1}}.$$

These last formulas imply the forms of the stated bounds when we drop the lower-order constant term and multiply through by the bounds for the function $\widehat{L}_0(\log\log x)$ proved in Corollary 5.7. $\square$

6.2 Lower bounds on the scaled Mertens function along an infinite subsequence

Corollary 6.2 (Bounds for the classically scaled Mertens function)**.** *Let $u_0:=e^{e^{e^e}}$ and define the infinite increasing subsequence, $\{x_n\}_{n\geq 1}$, by $x_n:=e^{e^{e^{e^{6n}}}}$. We have that along the increasing subsequence x_y for large $y\geq\max\left(\left\lceil e^{e^{e^e}}\right\rceil,u_0+1\right)$:*

$$\frac{|M(x_y)|}{\sqrt{x_y}}\gtrsim 6C_{\ell,1}\cdot(\log\sqrt{x_y})^{\frac{3}{2}\log_*^4(\sqrt{x_y})-\frac{10}{7}}(\log\log\sqrt{x_y})^{\frac{71}{14}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}(\log\log\log\sqrt{x_y})\sqrt{\log\log\log\log\sqrt{x_y}}+o(1),$$

as $y\to\infty$.

Proof of the Asymptotic Lower Bound. It suffices to take $u_0=e^{e^{e^e}}$, and a sufficient requirement on x is that the parity of $\lfloor\log\log\log\log x\rfloor\equiv 0 \pmod 2$ using the formula for $-G_\ell^{-1}(t)$ proved in Theorem 6.1. Since on $x/2\geq t\gg u_0$, we have that[7]

$$\frac{d}{dt}\left[G_\ell^{-1}(t)\right]\gtrsim\frac{3C_{\ell,1}}{t}(\log t)^{\frac{3}{2}\log_*^4(t)-\frac{3}{7}}(\log\log t)^{\frac{71}{14}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}(\log\log\log t)\sqrt{\log\log\log\log t}. \tag{15}$$

[7] The derivative main term, as conveyed by the $\gtrsim$ relation, is computed from the asymptotic formula we proved in Theorem 6.1 above. This calculation is tedious and messy to perform exactly on paper. We have used *Mathematica* to compute the derivative, and then manually separate the distinguishing main term it represents. Otherwise, an exact formula would be necessary – and that complication actually distorts the presentation of the asymptotically significant behavior we are looking at in bounding these derivatives along our subsequence.

The input to the derivative operator in the last equation is justified by establishing the limit

$$\lim_{x\to\infty} (\log\log x)^{\frac{(\log\log\log\log x)^2}{\log\log x}} = 1.$$

Now, we break up the integral over $t \in [u_0, x/2]$ into two pieces: one that is easily bounded from $u_0 \leq t \leq \sqrt{x}$, and then another that will conveniently give us our logarithmically slow-growing tendency towards infinity along the subsequence.

First, since $\pi(j) = \pi(\sqrt{x})$ for all $\sqrt{x} \leq j < x$, we can take the first chunk of the interval of integration and bound it as

$$\begin{aligned}\int_{u_0}^{\sqrt{x}} \frac{2\sqrt{x}}{t^2\log(x)} \frac{d}{dt}\left[G_\ell^{-1}(t)\right] dt &\gtrsim \frac{2\sqrt{x}}{\log(x)} \cdot \left(\max_{u_0 \leq t \leq \sqrt{x}} \widehat{d}_\ell(t)\right) \times \int_{u_0}^{\sqrt{x}} \frac{dt}{t^3} \\ &= o(\sqrt{x}),\end{aligned}$$

where the function $\widehat{d}_\ell(t)$ corresponds to the terms in (15) with the reciprocal multiple of t removed. The maximum in the previous equation is clearly attained by taking $t := \sqrt{x}$. The bound follows, and will be good enough to dispense with this term when we scale the function as $|M(x)|/\sqrt{x}$.

Next, we have to prove a related bound on the second portion of the interval from $\sqrt{x} \leq t \leq x/2$:

$$\begin{aligned}-\int_{\sqrt{x}}^{x/2} \frac{2x}{t^3\log(x)} \cdot \widehat{d}_\ell(t)dt &\gtrsim -\frac{2}{\log x} \cdot \left(\max_{\sqrt{x} \leq t \leq x/2} \widehat{d}_\ell(t)\right) \\ &= 6C_{\ell,1} \cdot (\log\sqrt{x})^{\frac{3}{2}\log_*^4(\sqrt{x}) - \frac{10}{7}} (\log\log\sqrt{x})^{\frac{71}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3}} (\log\log\log\sqrt{x}) \times \\ &\quad \times \sqrt{\log\log\log\log\sqrt{x}}.\end{aligned}$$

Finally, since $G_\ell^{-1}(x) = o(\sqrt{x})$, we obtain in total that as $x \to \infty$ along this infinite subsequence:

$$\frac{|M(x)|}{\sqrt{x}} \gtrsim 6C_{\ell,1} \cdot (\log\sqrt{x})^{\frac{3}{2}\log_*^4(\sqrt{x}) - \frac{10}{7}} (\log\log\sqrt{x})^{\frac{71}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3}} (\log\log\log\sqrt{x}) \sqrt{\log\log\log\log\sqrt{x}} + o(1),$$

Note that the constant power is approximately evaluated as $\frac{71}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3} \approx 5.87011$. The above expression tends to $+\infty$ as $x \to \infty$, however, only extremly slowly and along the defined infinite subsequence of asymptotically very large x. Remark 6.4 discusses the limitations by computational infeasibility of coming anywhere close to verifying this result numerically with current computational tools and CPU power. $\square$

6.3 Remarks

Remark 6.3 (Tightness of the lower bounds)**.** One remaining question for scaling $|M(x)|/f(x)$ is exactly how tight can the function $f \in \mathcal{F}$ be made so that (for $\mathcal{F}$ some reasonable function space with bases in polynomials of x and powers of iterated logarithms)

$$f(x) := \mathrm{argmax}_{h \in \mathcal{F}} \left\{\limsup_{x\to\infty} \frac{|M(x)|}{h(x)} = C_h\right\},$$

for some absolute constants $C_h > 0$? What we have proved is that we can take

$$f(x) = \sqrt{x} \cdot TODO,$$

to obtain the above where the limiting constant is

$$C_f \mapsto \frac{9}{8}\sqrt{\frac{3}{2}\frac{A_0^2(2e+3)^2}{\pi e B^2(\log 2)^3}}.$$

But is this the tightest possible f provably?

There is also, of course, the possibility of tightening the bound from above using the upper bounds proved in Theorem 2.7 along an infinite subsequence tending to infinity. We do not approach this problem here due to length constraints and that our lower bounds seem to have done much better than was previously known before about the (un)boundedness of the scaled Mertens function – our initial so-called "pipe dream", or "impossible", result.

Remark 6.4 (Computational limitations on numerically verifying the new lower bounds)**.** To the best of my knowledge, the most efficient method for computing $M(x)$ uses a time complexity of $O\left(x^{2/3}(\log\log x)^{2/3}\right)$ operations and space complexity of $O\left(x^{1/3}(\log\log x)^{2/3}\right)$. Practically speaking, this restriction on computing values of $M(x)$ makes numerically verifying the results claimed in Corollary 6.2 completely infeasible by modern computational standards. It does, however, seem to "gel" with the difficulty of even proving that the liminf (limsup) of $M(x)/\sqrt{x}$ is lesser (greater) than a small constant that we should only expect to observe the unboundedness of this function along such an asymptotically large subsequence. Or else, it stands to reason, that current algorithms for bounding the limit infimum and limit supremum values would have led to improved known markers prior to this point.

7 Conclusions (TODO)

7.1 Summary

- Using average order bounds, summatory functions, and the $\precsim$-type relations for lower bounds.
- Somewhat oddly, we did not need substantially improved bounds on $L_0(x) := \sum_{n \leq x} \lambda(n)$ than what is already known in upper bound form to obtain our new bounds on the Mertens function, aka, summatory function of the "testier" Möbius function.
- The upper bounds proved in Theorem 2.7 remain untouched. It may be possible to refine the best known upper bounds on $M(x)$ using these methods.

7.2 Future research and work that still needs to be done

- Refinements of these bounds to find the tightest possible lower (limit supremum) bounds, e.g., proofs of an optimal version of Gonek's original conjecture.
- Generalizations to weighted Mertens functions of the form $M_\alpha(x) := \sum_{n \leq x} \mu(n) n^{-\alpha}$.
- Indications of sign changes and exceptionally small, or zero values of $M(x)$.
- What our more combinatorial approach to bounding $M(x)$ effectively suggests about necessary, but unproved, zeta zero bounds that have historically formed the basis for arguments bounding $M(x)$ using Mellin inversion.
- Evaluate alternate strategies and approaches using different Dirichlet convolution functions besides g and $g^{-1}(n)$ (corresponding to $\pi(x)$) with Theorem 2.2.

7.3 Motivating a general technique towards bounding the summatory functions of arbitrary arithmetic f

7.4 The general construction using Theorem 2.2

7.4.1 A proposed generalization

For each $n \geq 1$, let $A(n) \subseteq \{d : 1 \leq d \leq n, d|n\}$ be a subset of the divisors of n. We say that a natural number $n \geq 1$ is *A-primitive* if $A(n) = \{1, n\}$. Under a list of assumptions so that the resulting A-convolutions are *regular convolutions*, we get a generalized multiplicative Möbius function [13, §2.2]:

$$\mu_A(p^\alpha) = \begin{cases} 1, & \alpha = 0; \\ -1, & p^\alpha > 1 \text{ is } A\text{-primitive}; \\ 0, & \text{otherwise.} \end{cases}$$

We also define the functions $\omega_A(n) := \#\{d|n : \mathrm{d\ is\ an\ A-primitive\ factor\ of\ n}\}$ and $\Omega_A(n) := \#\{p^\alpha || n : \mathrm{p\ is\ an\ A-primitive\ factor\ of\ n}\}$. Then the characteristic function of the set $A := \cup_{n \geq 1} A(n)$ is given by $\chi_A(n) = [n \in A]_\delta$. By Möbius inversion, we have that $\chi_A = \omega_A *_A \mu_A$. Moreover, for the A-counting function, $\pi_A(x)$, defined by

$$\pi_A(x) := \#\{n \leq x : n \in A\},$$

we can define a corresponding notion of a generalized A-Mertens function, $M_A(x) := \sum_{n \leq x} \mu_A(n)$. This function then satisfies (by Theorem 2.2) the relation that

$$M_A(x) = \sum_{k=1}^{x} (\omega_A + 1)^{-1}(k) \cdot \pi_A(x/k),$$

where the inverse function, $(\omega_A + 1)^{-1}(n)$, is defined with respect to A-convolution. We conjecture, but do not prove here, that $\operatorname{sgn}((\omega_A + 1)^{-1}(n)) = \lambda_A(n) =: (-1)^{\Omega_A(n)}$.

7.4.2 Ideas towards why the proposed generalization could be useful

Consider the following construction in the direction of proving that the set of *twin primes*

$$\mathbb{TP} := \left\{n \in \mathbb{Z}^+ : (n, n+2) \in \mathbb{P}^2\right\},$$

contains infinitely many elements. In particular, if we can use the setup from the previous subsection using characteristic functions, and write the *twin prime counting function* as

$$\pi_{\mathrm{TP}}(x) := \#\{n \leq x : n \in \mathbb{TP}\},$$

then for the Mertens-style function variant, $M_{\mathrm{TP}}(x) := \sum_{n \leq x} \mu_{\mathrm{TP}}(n)$, we obtain bounds of the form

$$-\frac{d}{dx}\left[M_{\mathrm{TP}}(x) + G_{\mathrm{TP}}^{-1}(x)\right] \geq \left(\min_{1 \leq t \leq \frac{x}{2}} \left|\frac{G_{\mathrm{TP}}^{-1}(t)}{t^3}\right|\right) \cdot \pi_{\mathrm{TP}}(x).$$

Provided the generalized Mertens set function $M_{\mathrm{TP}}(x)$ can be shown to satisfy certain limiting bounds along a nicely defined convenient infinite subsequence, this trick is an obvious way to attack the problem of infinitude (or lack thereof) of the twin primes $\mathbb{TP}$. We suggest that proving the existence of sufficient lower bounds on $M_{\mathrm{TP}}(x)$ along an infinite subsequence ought be a substantially difficult problem, though our new more combinatorial analytic approaches to the classical $M(x)$ bounds certainly suggest new ways to challenge what is known on this problem type.

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A Appendix: Supplementary tables and data

T.1 Table: Computations with a highly signed Dirichlet inverse function

n	**Primes**		**Sqfree**	**PPower**	$\bar{\mathbb{S}}$		$g^{-1}(n)$	$\lambda(n)\,\mathrm{sgn}(g^{-1}(n))$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_2(n)$		$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1^1	–	Y	N	N	–	1	1	0	0	–	1	1	0
2	2^1	–	Y	Y	N	–	−2	1	0	0	–	−1	1	−2
3	3^1	–	Y	Y	N	–	−2	1	0	0	–	−3	1	−4
4	2^2	–	N	Y	N	–	2	1	0	−1	–	−1	3	−4
5	5^1	–	Y	Y	N	–	−2	1	0	0	–	−3	3	−6
6	$2^1 3^1$	–	Y	N	N	–	5	1	0	−1	–	2	8	−6
7	7^1	–	Y	Y	N	–	−2	1	0	0	–	0	8	−8
8	2^3	–	N	Y	N	–	−2	1	0	−2	–	−2	8	−10
9	3^2	–	N	Y	N	–	2	1	0	−1	–	0	10	−10
10	$2^1 5^1$	–	Y	N	N	–	5	1	0	−1	–	5	15	−10
11	11^1	–	Y	Y	N	–	−2	1	0	0	–	3	15	−12
12	$2^2 3^1$	–	N	N	Y	–	−7	1	2	−2	–	−4	15	−19
13	13^1	–	Y	Y	N	–	−2	1	0	0	–	−6	15	−21
14	$2^1 7^1$	–	Y	N	N	–	5	1	0	−1	–	−1	20	−21
15	$3^1 5^1$	–	Y	N	N	–	5	1	0	−1	–	4	25	−21
16	2^4	–	N	Y	N	–	2	1	0	−3	–	6	27	−21
17	17^1	–	Y	Y	N	–	−2	1	0	0	–	4	27	−23
18	$2^1 3^2$	–	N	N	Y	–	−7	1	2	−2	–	−3	27	−30
19	19^1	–	Y	Y	N	–	−2	1	0	0	–	−5	27	−32
20	$2^2 5^1$	–	N	N	Y	–	−7	1	2	−2	–	−12	27	−39
21	$3^1 7^1$	–	Y	N	N	–	5	1	0	−1	–	−7	32	−39
22	$2^1 11^1$	–	Y	N	N	–	5	1	0	−1	–	−2	37	−39
23	23^1	–	Y	Y	N	–	−2	1	0	0	–	−4	37	−41
24	$2^3 3^1$	–	N	N	Y	–	9	1	4	−3	–	5	46	−41
25	5^2	–	N	Y	N	–	2	1	0	−1	–	7	48	−41
26	$2^1 13^1$	–	Y	N	N	–	5	1	0	−1	–	12	53	−41
27	3^3	–	N	Y	N	–	−2	1	0	−2	–	10	53	−43
28	$2^2 7^1$	–	N	N	Y	–	−7	1	2	−2	–	3	53	−50
29	29^1	–	Y	Y	N	–	−2	1	0	0	–	1	53	−52
30	$2^1 3^1 5^1$	–	Y	N	N	–	−16	1	0	−4	–	−15	53	−68
31	31^1	–	Y	Y	N	–	−2	1	0	0	–	−17	53	−70
32	2^5	–	N	Y	N	–	−2	1	0	−4	–	−19	53	−72
33	$3^1 11^1$	–	Y	N	N	–	5	1	0	−1	–	−14	58	−72
34	$2^1 17^1$	–	Y	N	N	–	5	1	0	−1	–	−9	63	−72
35	$5^1 7^1$	–	Y	N	N	–	5	1	0	−1	–	−4	68	−72
36	$2^2 3^2$	–	N	N	Y	–	14	1	9	1	–	10	82	−72
37	37^1	–	Y	Y	N	–	−2	1	0	0	–	8	82	−74
38	$2^1 19^1$	–	Y	N	N	–	5	1	0	−1	–	13	87	−74
39	$3^1 13^1$	–	Y	N	N	–	5	1	0	−1	–	18	92	−74
40	$2^3 5^1$	–	N	N	Y	–	9	1	4	−3	–	27	101	−74
41	41^1	–	Y	Y	N	–	−2	1	0	0	–	25	101	−76
42	$2^1 3^1 7^1$	–	Y	N	N	–	−16	1	0	−4	–	9	101	−92
43	43^1	–	Y	Y	N	–	−2	1	0	0	–	7	101	−94
44	$2^2 11^1$	–	N	N	Y	–	−7	1	2	−2	–	0	101	−101
45	$3^2 5^1$	–	N	N	Y	–	−7	1	2	−2	–	−7	101	−108
46	$2^1 23^1$	–	Y	N	N	–	5	1	0	−1	–	−2	106	−108
47	47^1	–	Y	Y	N	–	−2	1	0	0	–	−4	106	−110
48	$2^4 3^1$	–	N	N	Y	–	−11	1	6	−4	–	−15	106	−121

Table T.1: ***Computations of the first several cases of*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for*** $\mathbf{1 \leq n \leq 56}$***.***
The column labeled `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled, respectively,* `Sqfree`, `PPower` *and $\bar{\mathbb{S}}$ list inclusion of n in the sets of squarefree integers, prime powers, and the set $\bar{\mathbb{S}}$ that denotes the positive integers n which are neither squarefree nor prime powers. The next two columns provide the explicit values of the inverse function $g^{-1}(n)$ and indicate that the sign of this function at n is given by $\lambda(n) = (-1)^{\Omega(n)}$.*

Then the next two columns show the small-ish magnitude differences between the unsigned magnitude of $g^{-1}(n)$ and the summations $\widehat{f}_1(n) := \sum_{k\geq 0} \binom{\omega(n)}{k} \cdot k!$ and $\widehat{f}_2(n) := \sum_{k \geq 0} \binom{\omega(n)}{k} \cdot \#\{d|n : \omega(d) = k\}$. Finally, the last three columns show the summatory function of $g^{-1}(n)$, $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$, deconvolved into its respective positive and negative components: $G_+^{-1}(x) := \sum_{n \leq x} g^{-1}(n) \left[g^{-1}(n) > 0\right]_\delta$ and $G_-^{-1}(x) := \sum_{n\leq x} g^{-1}(n) \left[g^{-1}(n) < 0\right]_\delta$.

T.2 Table: Dirichlet inverse functions of $(f+1)(n)$ for f additive

n	$\lambda(n)$	$(f+1)^{-1}(n)$
1	1	1
2	-1	$-f(2)-1$
3	-1	$-f(3)-1$
4	1	$f(2)^2+2f(2)-f(4)$
5	-1	$-f(5)-1$
6	1	$2f(3)f(2)+f(2)+f(3)+1$
7	-1	$-f(7)-1$
8	-1	$-f(2)^3-3f(2)^2+2f(4)f(2)-f(2)+2f(4)-f(8)$
9	1	$f(3)^2+2f(3)-f(9)$
10	1	$2f(5)f(2)+f(2)+f(5)+1$
11	-1	$-f(11)-1$
12	-1	$-3f(3)f(2)^2-f(2)^2-4f(3)f(2)-2f(2)+2f(3)f(4)+f(4)$
13	-1	$-f(13)-1$
14	1	$2f(7)f(2)+f(2)+f(7)+1$
15	1	$2f(5)f(3)+f(3)+f(5)+1$
16	1	$f(2)^4+4f(2)^3-3f(4)f(2)^2+3f(2)^2-6f(4)f(2)+2f(8)f(2)+f(4)^2-f(4)+2f(8)-f(16)$
17	-1	$-f(17)-1$
18	-1	$-3f(2)f(3)^2-f(3)^2-4f(2)f(3)-2f(3)+2f(2)f(9)+f(9)$
19	-1	$-f(19)-1$
20	-1	$-3f(5)f(2)^2-f(2)^2-4f(5)f(2)-2f(2)+f(4)+2f(4)f(5)$
21	1	$2f(7)f(3)+f(3)+f(7)+1$
22	1	$2f(11)f(2)+f(2)+f(11)+1$
23	-1	$-f(23)-1$
24	1	$4f(3)f(2)^3+f(2)^3+9f(3)f(2)^2+3f(2)^2+2f(3)f(2)-6f(3)f(4)f(2)-2f(4)f(2)+f(2)-4f(3)f(4)-2f(4)+2f(3)f(8)+f(8)$
25	1	$f(5)^2+2f(5)-f(25)$
26	1	$2f(13)f(2)+f(2)+f(13)+1$
27	-1	$-f(3)^3-3f(3)^2+2f(9)f(3)-f(3)+2f(9)-f(27)$
28	-1	$-3f(7)f(2)^2-f(2)^2-4f(7)f(2)-2f(2)+f(4)+2f(4)f(7)$
29	-1	$-f(29)-1$
30	-1	$-2f(3)f(2)-6f(3)f(5)f(2)-2f(5)f(2)-f(2)-f(3)-2f(3)f(5)-f(5)-1$
31	-1	$-f(31)-1$

Table T.2: ***Dirichlet inverse functions of additive arithmetic functions.*** *The table provides a list of the Dirichlet inverse functions of $(f+1)(n)$ for f additive such that $f(1)=0$.*

T.3 Table: Dirichlet inverse functions of $g(n)$ for g multiplicative

n	$(-1)^{\omega(n)}$	$f^{-1}(n)$
2	-1	$-g(2)$
3	-1	$-g(3)$
4	-1	$g(2)^2 - g(4)$
5	-1	$-g(5)$
6	1	$g(2)g(3)$
7	-1	$-g(7)$
8	-1	$-g(2)^3 + 2g(4)g(2) - g(8)$
9	-1	$g(3)^2 - g(9)$
10	1	$g(2)g(5)$
11	-1	$-g(11)$
12	1	$g(3)g(4) - g(2)^2g(3)$
13	-1	$-g(13)$
14	1	$g(2)g(7)$
15	1	$g(3)g(5)$
16	-1	$g(2)^4 - 3g(4)g(2)^2 + 2g(8)g(2) + g(4)^2 - g(16)$
17	-1	$-g(17)$
18	1	$g(2)g(9) - g(2)g(3)^2$
19	-1	$-g(19)$
20	1	$g(4)g(5) - g(2)^2g(5)$
21	1	$g(3)g(7)$
22	1	$g(2)g(11)$
23	-1	$-g(23)$
24	1	$g(3)g(2)^3 - 2g(3)g(4)g(2) + g(3)g(8)$
25	-1	$g(5)^2 - g(25)$
26	1	$g(2)g(13)$
27	-1	$-g(3)^3 + 2g(9)g(3) - g(27)$
28	1	$g(4)g(7) - g(2)^2g(7)$
29	-1	$-g(29)$
30	-1	$-g(2)g(3)g(5)$
31	-1	$-g(31)$
32	-1	$-g(2)^5 + 4g(4)g(2)^3 - 3g(8)g(2)^2 - 3g(4)^2g(2) + 2g(16)g(2) + 2g(4)g(8) - g(32)$
33	1	$g(3)g(11)$
34	1	$g(2)g(17)$

Table T.3: *Dirichlet inverse functions of multiplicative arithmetic functions.* *The table provides a list of the Dirichlet inverse functions of $g(n)$ for g multiplicative such that $g(1) = 1$.*