Lower bounds on the summatory function of the Möbius function along infinite subsequences

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Abstract

The Mertens function, $M(x) = \sum_{n \leqslant x} \mu(n)$, is classically defined as the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture stating that $|M(x)| < C \cdot \sqrt{x}$ with come absolute $C > 0$ for all $x \geqslant 1$ has a well-known disproof due to Odlyzko and té Riele given in the early 1980's by computation of non-trivial zeta function zeros in conjunction with integral formulas expressing $M(x)$. It is conjectured that $M(x)/\sqrt{x}$ changes sign infinitely often and grows unbounded in the direction of both $\pm\infty$ along subsequences of integers $x \geqslant 1$. We prove a weaker property related to the unboundedness of $|M(x)| \log x/\sqrt{x}$ by showing that

$$\limsup_{x \rightarrow \infty} \frac{|M(x)|(\log x)(\log\log\log x)(\log\log\log\log x)^{\frac{5}{4}}(\log\log\log\log\log x)^{\frac{3}{2}}}{\sqrt{x}(\log\log x)} > 0.$$

There is a distinct stylistic flavor and new element of combinatorial analysis to our proof peppered in with the standard methods from analytic, additive and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

Keywords and Phrases: *Möbius function; Mertens function; summatory function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting functions; Dirichlet generating function; asymptotic lower bounds; Mertens conjecture.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; and 11N64.*

2 An introduction to the Mertens function

2.1 Definitions

Suppose that $n \geqslant 2$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ so that $r = \omega(n)$. We define the *Möebius function* to be the signed indicator function of the squarefree integers as follows:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leqslant i \leqslant k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many other variants and special properties of the Möebius function and its generalizations [15, *cf.* §2]. A crucial role of the classical $\mu(n)$ forms an inversion relation for arithmetic functions convolved with one by *Möbius inversion*:

$$g(n) = (f * 1)(n) \iff f(n) = (g * \mu)(n), \forall n \geqslant 1.$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined as

$$M(x) = \sum_{n \leqslant x} \mu(n), x \geqslant 1.$$

The sequence of the oscillatory values of this summatory function begins as [16, A002321]

$$\{M(x)\}_{x \geqslant 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2, \ldots\}$$

Clearly, a positive integer $n \geqslant 1$ is *squarefree*, or contains no (prime power) divisors which are squares, if and only if $\mu^2(n) = 0$. A related summatory function which counts the number of *squarefree* integers $n \leqslant x$ then satisfies [4, §18.6] [16, A013928]

$$Q(x) = \sum_{n \leqslant x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \to \infty$:

$$\mu_+(x) = \frac{\#\{1 \leqslant n \leqslant x : \mu(n) = +1\}}{Q(x)} \overset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leqslant n \leqslant x : \mu(n) = -1\}}{Q(x)} \xrightarrow{x \to \infty} \frac{3}{\pi^2}.$$

The actual local oscillations between the approximate densities of the sets $\mu_\pm(x)$ lend an unpredictable nature to the function and characterize the oscillatory sawtooth shaped plot of $M(x)$ over the positive integers.

2.2 Properties

One conventional approach to evaluating the behavior of $M(x)$ for large $x \to \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is expressed given the inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \int_1^\infty \frac{s \cdot M(x)}{x^{s+1}} dx,$$

we obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation, along with the standard Euler product representation for the reciprocal zeta function cited in the first equation above, leads us to the exact expression for $M(x)$ for any real $x > 0$ given by the next theorem due to Titchmarsh.

Theorem 2.1 (Analytic Formula for $M(x)$)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence* $\{T_k\}_{k\geqslant 1}$ *satisfying* $k \leqslant T_k \leqslant k+1$ *for each* k *such that for any real* $x > 0$

$$M(x) = \lim_{k\to\infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |\operatorname{Im}(\rho)|<T_k}} \frac{x^\rho}{\rho\cdot\zeta'(\rho)} - 2 + \sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n\cdot(2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2}\left[x\in\mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x\cdot\exp\left(-C\cdot\log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan recently proved new updated estimates bounding $M(x)$ for large x in the following forms [17]:

$$M(x) \ll \sqrt{x}\cdot\exp\left(\log^{1/2}(x)(\log\log x)^{14}\right),$$
$$M(x) = O\left(\sqrt{x}\cdot\exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right),\ \forall\epsilon > 0.$$

2.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which posits that

$$|M(x)| < C\cdot\sqrt{x},\quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens for $C = 1$ and all $x < 10000$. Since its beginnings in 1897, the Mertens conjecture has been disproven by computation of non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele from the early 1980's [12]. Since the truth of the conjecture would have implied the RH, more recent attempts at bounding $M(x)$ consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound towards both $\pm\infty$ along infinite subsequences.

One of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ is in actuality unbounded on the natural numbers. A precise statement of this problem is to produce an affirmative answer whether $\limsup_{x\to\infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x\to\infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there are an infinite subsequences of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound towards either $\pm\infty$ along the subsequence. We cite that prior to this point it is only known by computation that [14, *cf.* §4.1] [16, *cf.* A051400; A051401]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geqslant 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leqslant -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [12, 7, 8, 5]. Extensive computational evidence has produced a conjecture due to Gonek (among attempts on limiting bounds by others) that in fact the limiting behavior of $M(x)$ satisfies [11]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}\cdot(\log\log x)^{5/4}} = O(1).$$

3 An overview of the core logical steps and components to the proof

We offer an initial step-by-step summary overview of the core components to our proof outlined in the next. As our proof methodology is new and relies on non-standard elements compared to more traditional methods of bounding $M(x)$, we hope that this sketch of the logical components to this argument makes the article easier to parse.

3.1 Step-by-step overview

(1) We prove a matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). See Theorem 4.1 in Section 5.

(2) This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the seemingly unconnected prime counting function, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is stated in (1).

The strong additivity of $\omega(n)$ imparts the characteristic signedness of $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ for all $n \geqslant 1$, which is weighted according to the parity of $\Omega(n)$. The link relating (1) to canonical additive functions and their distributions then lends a recent distinguishing element to the success of the methods in our proof.

(3) We tighten an updated result from [10, §7] providing uniform asymptotic formulas for the summatory functions, $\widehat{\pi}_k(x)$, that indicate the parity of $\Omega(n)$ (sign of $\lambda(n)$) for $n \leqslant x$ and $1 \leqslant k \leqslant \log\log x$. These formulas are proved using expansions of more combinatorially motivated Dirichlet series (see Theorem 4.7). We use this result to sum $\sum_{n \leqslant x} \lambda(n) f(n)$ for particular non-negative arithmetic functions f when x is large.

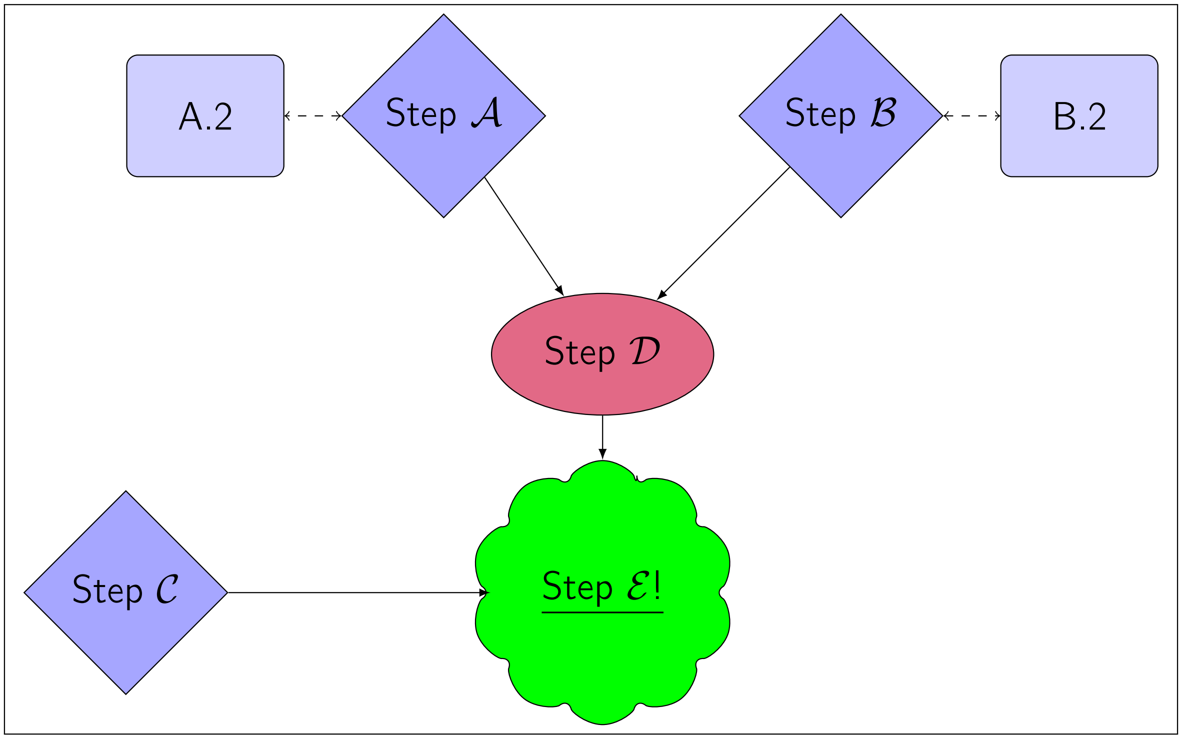
(4) We then turn to bounding the asymptotics of the quasi-periodic functions, $g^{-1}(n)$, by estimating this inverse function's limiting order for large $n \leqslant x$ as $x \to \infty$ in Section 7. We eventually use these estimates to prove a substantially unique new lower bound formula for the summatory function $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$ along certain asymptotically large infinite subsequences (see Theorem 9.5).

(5) We spend some interim time in Section 8 carefully working out a rigorous justification for why the limiting lower bounds we obtain from average order case analysis of our arithmetic function approximations to $g^{-1}(n)$ are sufficient to prove the corollary on the unboundedness of $M(x)$ below.

(6) When we return to step **(2)** with our new lower bounds at hand, we have a new unconditional proof of the unboundedness of $\frac{|M(x)| \log x}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. What we recover is a quick, and rigorous, proof of Corollary 4.8 given in Section 9.2.

3.2 Schematic flowchart of the proof logic

The next flowchart diagramed below shows how the seemingly disparate components of the proof are organized.



Legend to the diagram stages:

- **Step A:** *Citations and re-statements of existing theorems proved elsewhere.*
  - **A.A:** Key results and constructions:
    - Theorem 4.6
    - Corollary 6.5
    - The results, lemmas, and facts cited in Section 5.3
  - **A.2:** Lower bounds on the Abel summation based formula for $G^{-1}(x)$:
    - Theorem 4.7 (on page 20)
    - Proposition 6.6
    - Theorem 9.5
- **Step B:** *Constructions of an exact formula for $M(x)$.*
  - **B.B:** Key results and constructions:
    - Corollary 4.3 (follows from Theorem 4.1 proved on page 14)
    - Proposition 5.1
  - **B.2:** Asymptotics for the component functions $g^{-1}(n)$ and $G^{-1}(x)$:
    - Theorem 7.3 (on page 23)
    - Lemma 7.4
- **Step C:** *A justification for why lower bounds obtained roughly "on average" suffice.*
  - The results proved in Section 8
- **Step D:** *Interpreting the exact formula for $M(x)$.*
  - Proposition 9.1
  - Theorem 9.5
- **Step E:** *The Holy Grail.* Proving that $\frac{|M(x)| \log x}{\sqrt{x}}$ grows without bound in the limit supremum sense.
  - Corollary 4.8 (on page 39)