4 The distributions of $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$

We have already suggested in the introduction that the relation of the component functions, $g^{-1}(n)$ and $C_{\Omega(n)}(n)$, to the canonical additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties of these functions cited in Table B. Each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdős-Kac theorem that provides a central limit type theorem for the distributions of these functions over $n \leq x$ as $x \to \infty$ [3, 2, 18] (*cf.* [9]). In the remainder of this section we establish more analytical proofs of related properties of these key sequences used to express $G^{-1}(x)$.

4.1 Analytic proofs and adaptations of DGF methods for summing additive functions

Theorem 4.1. *Let the function $\widehat{F}(s,z)$ be defined in terms of the prime zeta function, $P(s)$, for* $\mathrm{Re}(s) \geq 2$ *and $|z| < |P(s)|^{-1}$ by*

$$\widehat{F}(s,z) := \frac{1}{1+P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

For $|z| < P(2)^{-1}$, the summatory function of the DGF coefficients of $\widehat{F}(s,z) \cdot \zeta(s)^z$ correspond to

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

We have that for all sufficiently large $x \geq 2$ and any $|z| < P(2)^{-1}$

$$\widehat{A}_z(x) = \frac{x}{\Gamma(z)} \cdot \widehat{F}(2,z) \cdot (\log x)^{z-1} + O_z\left(x \cdot (\log x)^{\mathrm{Re}(z)-2}\right).$$

actually eqn (9)

Proof. We can see from the proof of Proposition 2.1 that

What is (N)! ??

$$C_{\Omega(n)}(n) = \begin{cases} 1, & n = 1; \\ (\Omega(n))! \times \prod_{p^\alpha \| n} \frac{1}{\alpha!}, & n \geq 2. \end{cases}$$

We can then generate exponentially scaled forms of these terms through a product identity of the following form:

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! \cdot p^{rs}}\right)^{-1} = \exp\left(-z \cdot P(s)\right), \mathrm{Re}(s) \geq 2 \wedge \mathrm{Re}(P(s)z) > -1.$$

This product based expansion is similar in construction to the parameterized bivariate DGF used in the reference [13, §7.4]. By computing a Laplace transform on the right-hand-side of the above equation, we obtain

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n) \cdot (-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp\left(-tz \cdot P(s)\right) dt = \frac{1}{1+P(s)z}, \mathrm{Re}(s) > 1 \wedge \mathrm{Re}(P(s)z) > -1.$$

It follows that

$$\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times \widehat{F}(s,z), \mathrm{Re}(s) > 1 \wedge |z| < |P(s)|^{-1}.$$

Since $\widehat{F}(s,z)$ is an analytic function of s for all $\mathrm{Re}(s) \geq 2$ whenever the parameter $|z| < |P(s)|^{-1}$, if the sequence $\{b_z(n)\}_{n \geq 1}$ indexes the coefficients in the DGF expansion of $\widehat{F}(s,z) \cdot \zeta(s)^z$, then

$$\left|\sum_{n \geq 1} \frac{b_z(n)(\log n)^{2R+1}}{n^s}\right| < +\infty, \mathrm{Re}(s) \geq 2$$

$P(z)^{-1} \geq 2.2.$ So if I set

$z = 2,$ Thm 4.1 implies

$$\hat{A}_2(x) = \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n)\, 2^{\Omega(n)}$$

$$= c_2 x (\log x) + O(x)$$

So in particular

$$\left| \hat{A}_2(x+1) - \hat{A}_2(x) \right| = O(x)$$

But this leads to a contradiction.

Take $x+1 = 2^k 3^k$ where $k \to \infty$

$$C_{\Omega(x+1)}(x+1)\, 2^{\Omega(x+1)}$$

$$= (-1)^2 \binom{2k}{k} 2^{2k}$$

$$\gtrsim \frac{2^{2k} \cdot 2^{2k}}{\sqrt{k}} = \frac{16^k}{\sqrt{k}}$$

$$\gtrsim (x+1)^{\delta} \quad \text{where } \delta > 1.$$

That is a contradiction.

Remark: any choice of $\sqrt{3/2} < |z| < \frac{1}{P(z)}$ leads to a contradiction.