SUPER SQUARE ROOT LOWER BOUNDS ON THE PARITY OF THE PARTITION FUNCTION

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Abstract. We define the function $N_e(x)$ to be the number of times the parity partition function $p(n)$ is even for $n \leq x$. We expect that on average the value of $N_e(x)$ is approximately $x/2$. However, currently the best known lower bound for the function is given by $N_e(x) \geq C \cdot \sqrt{x} \log\log(x)$ for sufficiently large x. We provide several new plausible and numerically supported conjectures which suggest that we can in fact prove significant increases to the known lower bounds for $N_e(x)$ by factors of small powers of x – effectively "*squashing*" (*cf.* language of Croot) the existing square root barrier encountered in known methods for bounding the problem.

1. Introduction

1.1. **Euler's partition function.**

1.1.1. *Definitions.* The *partition function* $p(n)$ counts the number of distinct partitions of a natural number $n \geq 1$ into parts of size at least one [5, A000041]. For example, the partition number $p(5) = 7$ as the natural number $n := 5$ is decomposed into the following distinct sums of the form $n = \lambda_1 + \cdots + \lambda_k$ for $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ such that each $\lambda_i \in \{1, 2, \ldots, n\}$:

$$5 = 5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

Alternately, we can define $p(n)$ to be the number of distinct non-negative solutions to the equation

$$n = x_1 + 2x_2 + 3x_3 + \cdots + nx_n,$$

for integers $n \geq 1$. The partition function $p(n)$ is generated by the reciprocal of the infinite q–*Pochhammer symbol*, $(q;q)_\infty$, which is expanded via *Euler's pentagonal theorem* as

$$\begin{aligned}(q;q)_\infty &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \\ &= 1 + \sum_{n\geq 1} (-1)^n \left(q^{n(3n-1)/2} + q^{n(3n+1)/2}\right) \\ &= \sum_{j\geq 0} (-1)^{\lceil \frac{j}{2} \rceil} q^{G_j}\end{aligned}$$

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$$= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots,$$

where the exponents of q in the power series expansion in the previous equations are the *pentagonal numbers*, $\{\omega(0), \omega(1), \omega(-1), \omega(2), \omega(-2), \ldots\}$, also denoted by G_j where for $j \geq 0$ [5, A001318]

$$G_j := \frac{1}{2}\left\lceil \frac{j}{2} \right\rceil \left\lceil \frac{3j+1}{2} \right\rceil = \begin{cases} \frac{j(3j+1)}{2}, & \text{if } j \text{ is even;} \\ \frac{j(3j-1)}{2}, & \text{if } j \text{ is odd.} \end{cases}$$

1.1.2. *Properties of the partition function.* The partition function satisfies a number of known basic properties such as the fundamental recurrence relation

$$\sum_{k \geq 0} (-1)^{\lceil k/2 \rceil} p(n - 1 - G_k) = \delta_{n,1}, \tag{1}$$

which holds for all integers $n \geq 1$, as well as properties such as Rademacher's exact formula formula for $p(n)$ which can be proved by more advanced methods. The identity in (1) is the only known *exact* result relating sums of the partition function minus the values of a quadratic function, namely the pentagonal numbers, and moreover, appears to be the only formula which relates sums of the partition function at a sequence of *polynomial* values identically to zero[1]. While this function is well-studied in number theory and of general interest to many mathematicians who have been studying its properties since the time of Euler, many elementary properties of this function remain a mystery to this day. Our particular motivation for studying this function is to determine new lower bounds for the tally of its values $p(n)$ for $n \leq x$ modulo 2 when $x \gg 1$ is large.

1.2. **The parity problem.** One property of the partition function which is an open unresolved problem concerns the distribution of the parity of $p(n)$, or its reduced values in $\{0, 1\}$ modulo 2, for arbitrary n. More precisely, we are concerned with determining the best possible lower bounds for the function[2]

$$\begin{aligned} N_e(x) &:= \#\{n \leq x : p(n) \equiv 0 \bmod 2\} \\ &= \sum_{n \leq x} [p(n) \equiv 0 \bmod 2]_\delta \\ &= x - N_o(x), \end{aligned}$$

for all $x \geq x_0$ sufficiently large where $N_o(x)$ is the corresponding count of odd parity partition numbers $p(n)$ for $n \leq x$ [5, *cf.* A040051]. It is well-known and can be proved by even elementary methods that this tally function satisfies the so-termed *square root barrier* of

$$N_e(x) \geq C \cdot \sqrt{x}, \quad \text{for } x \geq x_0 \text{ sufficiently large,}$$

for some constant $C > 0$, for example as shown in [4] circa 1998. Using more advanced methods involving modular forms, the first bound for $N_e(x)$ can improved slightly to [1]

$$N_e(x) \geq 0.069 \cdot \sqrt{x} \log\log(x),$$

[1] Though we point out that a cubic, or even higher-order, polynomial index relation is desirable in that it would likely lead to more elementary proofs of better parity bounds for the partition function as in the methods of Nicolas and Rusza [4].

[2] Notation: *Iverson's convention* compactly specifies boolean-valued conditions and is equivalent to the *Kronecker delta function*, $\delta_{i,j}$, as $[n = k]_\delta \equiv \delta_{n,k}$. Similarly, $[\mathtt{cond} = \mathtt{True}]_\delta \equiv \delta_{\mathtt{cond},\mathtt{True}}$ in the remainder of the article.

for all $x > 1$. Computation and intuition on the problem suggest that $N_e(x) \sim \frac{x}{2}$ for all large x. However, showing even a relatively small power-of-x factor improvement relative to the order of the expected bound in the form of

$$N_e(x) \geq C_3 \cdot x^{\frac{1}{2}+\varepsilon} + o\left(x^{\frac{1}{2}+\varepsilon}\right), \quad \text{when } x \gg 1,$$

for a particular concrete value of $\varepsilon > 0$ would constitute a significant breakthrough on progress to the parity problem.

1.3. **Our aims in the article.** We aim to make just such progress by showing that, or at least presenting a plausible and extensively numerically supported approach to showing that,

$$N_e(x) \geq C \cdot x^{0.51} + o\left(x^{0.51}\right).$$

The construction we suggest in the next sections is also easily generalized to showing significant power-of-x improvements to known (if any) lower bounds on the even parity of other special partition functions whose reciprocal q-series-related generating function satisfies a well-known, or at least identifiable closed-form, expansion as a power series in q. In particular, if our conjectures made within this article do in fact lead to significant progress on the parity problem for $p(n)$, it is likely that we will be able to prove analogous bounds for the partition function $q(n)$, among others, as a straightforward direct extension of our reasoning here (see [2] for comparable results).

2. Constructing a new approach to the parity problem

2.1. **New identities for the Möbius function.** In previous drafts of this article, I proved the following proposition using the theory of so-termed "*Lambert series factorizations*" which I worked on with Mircea Merca over the summer of 2017 as summarized in [3]. It turns out that the recurrence formula stated in (1) combined with a simple application of Möbius inversion is enough to prove this simple variant of a more general result which can be applied in the context of bounds on the parity of other partition functions. As Ernie has pointed out, the result in Proposition 2.1 itself is somewhat trivial on its own. However, it is useful in the constructions below (in particular, given its application in the form of Corollary 2.2) so we prove it first using the straightforward method immediately below.

Proposition 2.1. *For natural numbers $n \geq 1$, we have the identity that*

$$\mu(n) = \sum_{d|n} \left[p(d-1) + \sum_{k=1}^{d} (-1)^{\lceil k/2 \rceil} p(d-1-G_k)\right] \mu(n/d),$$

where $\mu(n)$ is the Möbius function.

Proof (Möbius Inversion). The well-known identity provided in (1) is first restated in the following form:

$$\begin{aligned}\delta_{n,1} &= p(n-1) + \sum_{k=1}^{n} (-1)^{\lceil k/2 \rceil} p(n-1-G_k) \\ &= \sum_{d|n} \mu(d).\end{aligned}$$

By Möbius inversion, the last equation holds if and only if the statement of the proposition is correct for all $n \geq 1$. □

Corollary 2.2 (An Exact Identity for the Möbius Function)**.** *For $q \geq 2$ prime, we have that*

$$0 \equiv p(q-1) + \sum_{k=1}^{2\mu_q} p(q-1-G_k) \pmod{2},$$

where

$$\mu_q := \left\lfloor \frac{\sqrt{24(q-1)+1}+1}{6} \right\rfloor.$$

In particular, when q is prime, we can count that at least one of the elements in the set

$$\{p(q-1-G_k) : 1 \leq k \leq 2\mu_q\} \cup \{p(q-1)\},$$

is of even parity.

Proof. We can prove this theorem using the known identities related to sums of multiplicative functions and the partition function $p(n)$ stated in Proposition 2.1. In particular, if the integer n is prime then it has only the divisors $d \in \{1, n\}$ and $\mu(n) = -1$ in the expansion guaranteed by the proposition. Then plugging in these explicit and simple forms of the divisors $d|n$ to the formula implies the stated result above in the form of

$$-1 = p(q-1) - p(0) + \sum_{k=1}^{\mu(n)} p(n-1-G_k) \pmod{2}. \qquad \square$$

Remark 2.3 (Applications to Other Special Partition Functions)**.** As hinted at in the beginning of this section, the identity proved by Möbius inversion in Proposition 2.1, we can state this result in a more general form derived from the so-termed *Lambert series factorization theorems* studied in [3]. In particular, suppose that $P(n)$ denotes any special partition function of interest whose reciprocal generating function, denoted $(q;q)_{P,\infty}$ say, has a power series expansion of the following form:

$$(q;q)_{P,\infty} := 1 + \sum_{j \geq 1} C_{P,G_{P,j}} q^{G_{P,j}}.$$

We can then modify the result in the proposition to the more general statement that

$$\mu(n) = \sum_{d|n} \sum_{k=1}^{n} P(k-1)\mu(n/d) \left([G_{P,d-k} > 0]_\delta \, C_{P,G_{P,d-k}} + [d-k=0]_\delta \right), \tag{2}$$

which as in the corollary above then implies that

$$0 \equiv P(q-1) + \sum_{\substack{k \geq 1 \\ G_{P,k} < q}} C_{P,G_{P,k}} P(q-1-G_{P,k}) \pmod{2}.$$

In other words, if we wish to study the corresponding parity problem for the partition function $q(n) := [q^n](q;q^2)_\infty^{-1}$ counting the number of partitions of n whose parts are *distinct*, or equivalently the number of partitions of n into odd parts, [5, A000009], we can define a pentagonal number analog by the series coefficients of $(q;q^2)_\infty$ [5, A081362] and proceed to consider the parity problem on subsets of the primes such that the total number of terms involving $P(\cdot)$ are odd on the right-hand-side of the last equation. Thus we have an effective generalization of our method described below in the article to form analogous lower bounds on the even-ness tallies of other special partition functions.

2.2. **A Counting Heuristic for Bounding $N_e(x)$.** We can make extensive use – perhaps even of more utility than the proof of Proposition 2.1 itself warrants – of the identity in Corollary 2.2 to finding improved lower bounds on $N_e(x)$ counting the even-ness of the partition function on a subset of primes $q \leq x$ rather than on the entire subset of the integers $n \geq x$ implicit to the definition of this parity counting function. In particular, if we denote the set of primes by $\mathbb{P}$, then we have the following heuristic for bounding $N_e(x)$ from below:

Proposition 2.4. *Suppose that $Q \subseteq \mathbb{P} \cap \{1, 2, \ldots, x\}$ for some $x > 1$ denotes an appropriately chosen bounded subset of the primes $q \leq x$. Then we have the bound*

$$N_e(x) \geq |Q| - I_{e,Q}, \tag{3}$$

where the indicator function $I_{e,Q}$ counts the maximum possible overlap of indices $q_1 - G_j = q_2 - G_k$ for $1 \leq k < j \leq 2\mu_{q_1}$ and where $q_1 > q_2$ for $q_1, q_2 \in Q$. More precisely, we have that the stated bound holds where the indicator function $I_{e,Q}$ with respect to the set Q is defined to be

$$\begin{aligned} I_{e,Q} = \sum_{\substack{q_1,q_2 \in Q \\ q_1 > q_2}} \sum_{t=1}^{2\mu_{q_1}} \sum_{k=1}^{\min(t-1, 2\mu_{q_2})} & \Big(\big[2q_1 - 2q_2 - (3t^2 + (6k+1)t) = 0\big]_\delta \\ & + \big[2q_1 - 2q_2 - (3t^2 + (6k-1)t) = 0\big]_\delta \\ & + \big[2q_1 - 2q_2 - (3t^2 + (6k+1)t + 2k) = 0\big]_\delta \\ & + \big[2q_1 - 2q_2 - (3t^2 + (6k-1)t - 2k) = 0\big]_\delta \Big). \end{aligned}$$

Proof. The inequality for a lower bound on $N_e(x)$ follows from a straightforward counting argument based on Corollary 2.2. In particular, if we order the subset $Q := \{q_1, q_2, \ldots, q_r\}$ of primes $q_j \leq x$, we see from the corollary that we should have at least one even value of $p(n)$ corresponding to the formula at each prime $q_j \in Q$. However, it is possible that we are over counting the same partition number index across the multiple prime values in the set Q, i.e., for $(q_1, q_2, j, k) := TODO$ we have that $q_1 - G_j = q_2 - G_k$, so by our náive counting above where we assume nothing more about the parity of $p(n)$ than what is implied by the corollary, if $q_1, q_2 \in Q$ it is possible that we have over counted the same even partition number twice in the main term of $|Q|$! Therefore, if we subtract the count (strictly speaking a náive over count of the duplicate indices) of times where the indices in the formula at q_j is duplicated in the formula corresponding to another $q_{j'}$ then we have arrived at our stated lower bound.

What remains is to prove the second indicator sum formula for the occurrences of these duplicate indices denoted by $I_{e,Q}$. In particular, for distinct $q_1 > q_2$ and

$j > k$ where $q_1 - G_j, q_2 - G_k \geq 0$, then we have the four separate cases below corresponding to $G_j = \frac{j(3j+1)}{2}, \frac{j(3j-1)}{2}$ and similarly $G_k = \frac{k(3k+1)}{2}, \frac{k(3k-1)}{2}$ where we define $t := j - k$ to simplify notation:

$$(G_j, G_k) := \left(\frac{j(3j+1)}{2}, \frac{k(3k+1)}{2}\right): \quad G_j - G_k = \frac{1}{2}(3t^2 + (6k+1)t) \qquad \text{(I)}$$

$$(G_j, G_k) := \left(\frac{j(3j-1)}{2}, \frac{k(3k-1)}{2}\right): \quad G_j - G_k = \frac{1}{2}(3t^2 + (6k-1)t) \qquad \text{(II)}$$

$$(G_j, G_k) := \left(\frac{j(3j+1)}{2}, \frac{k(3k-1)}{2}\right): \quad G_j - G_k = \frac{1}{2}(3t^2 + (6k+1)t + 2k) \qquad \text{(III)}$$

$$(G_j, G_k) := \left(\frac{j(3j-1)}{2}, \frac{k(3k+1)}{2}\right): \quad G_j - G_k = \frac{1}{2}(3t^2 + (6k-1)t - 2k). \qquad \text{(IV)}$$

We give more refined versions of the indicator sums for $I_{e,Q}$ which are defined for specific subsets $Q_x \subseteq \mathbb{P} \cap \{1, 2, \ldots, x\}$ when we define them in the context below. □

2.3. **Formally defining our construction.** We notice that if we choose our subsets Q in the previous proposition to contain all of the primes $q \leq x$, then $|Q| = \pi(x) \sim x/\log(x)$ whose main term approximation has a fantastically simple improvement to the known bounds for $N_e(x)$ – namely, $x/\log(x) >> C \cdot \sqrt{x} \log\log(x)$. However, with the laws of physics currently in existence on this planet available for us to manipulate at this time, we are not so lucky to obtain our desired bound improvement by this method as $I_{e,Q} \gg \pi(x)$ for all large enough x which we have computed. We must then modify our construction to find prime subsets Q_x such that $|Q_x|$ represents the main term in the lower bound that we aim to obtain and *more importantly* such that the corresponding over count I_{e,Q_x} in our heuristic for the bound satisfies $I_{e,Q_x} = o(|Q_x|)$ for all large x.

We claim based on intuition and an extensive body of computational evidence that we can construct such sets Q_x by taking *subsets* of appropriately spaced "*fortunate*" primes, or rather fortunately chosen primes, from the set $\mathbb{P} \cap \{1, 2, \ldots, x\}$ when $x \gg 1$ is sufficiently large. In our case, this amounts to being able to chose some optimally spaced subset Q_x of the primes $q \leq x$ such that $|Q_x| \geq \lfloor x^{0.51} \rfloor$ and such that the overlap in the distribution of the primes in our subset Q_x minus all possible pentagonal numbers are minimized to be at most $o(x^{0.51})$. We note here that the subproblem this phrasing seems to invoke – namely, considering the distribution of subsets of primes minus quadratic functions over some preset range – is highly non-trivial! After some numerical trial and error and thought on the particular construction we require, we instead make the problem simpler, by choosing suitable primes in logarithmically spaced intervals with respect to powers of our $p := 1.96 = (0.51)^{-1}$. In particular, we make the following subproblem-specific definitions:

Definition 2.5. For a fixed $t \geq 1$ we define the intervals

$$I_p(t) := [t^p, (t+1)^p],$$

for some prescribed $p \in (1, 2)$ (i.e., our chosen $p := 1.96 = (0.51)^{-1}$). For each $t \geq 1$, we define the corresponding "*fortunate*" prime $q_{p,t}$ to be the prime in the interval $I_p(t)$ of *minimal weight* (more on this below) and then our more specific special case prime subsequence of $q_t := q_{1.96,t}$. Finally, for sufficiently large $x > 1$

we define the prime subset $Q_x \subseteq \mathbb{P} \cap \{1, 2, \ldots, x\}$ by selecting the $\lceil x^{0.51}/\log(x) \rceil$ terms of minimal weight from the next subset Q_x'.

$$Q_x' := \left\{q_t : 1 \leq t \leq \lfloor x^{0.51} \rfloor \right\}.$$

We say that a prime has *minimal weight* with respect to another prime subset S if (TODO: More details on the prime selection algorithm).

Claim 2.6 (Existence of Super Square Root Power Bounds)**.** *For all $x > 1$, we have that*

$$|Q_x| = \lceil x^{0.51}/\log(x) \rceil = C_{0.51} \cdot \frac{x^{0.51}}{\log(x)},$$

for an absolute limiting constant $C_{0.51} > 0$. Moreover, we can bound the indicator function sums in our corresponding heuristic for $N_e(x)$ as $I_{e,Q_x} = o(x^{0.51}/\log(x))$, where we can expand this duplicate index counting function as follows:

$$\begin{aligned} I_{e,Q} \geq \sum_{s=1}^{\lfloor x^{0.51}/\log(x) \rfloor} \sum_{i=1}^{s-1} \sum_{t=1}^{2\mu_{q_s}} \sum_{k=1}^{2\mu_{q_{s-i}}} \Big(& \left[2q_s - 2q_{s-i} - (3t^2 + (6k+1)t) = 0\right]_\delta \\ & + \left[2q_s - 2q_{s-i} - (3t^2 + (6k-1)t) = 0\right]_\delta \\ & + \left[2q_s - 2q_{s-i} - (3t^2 + (6k+1)t + 2k) = 0\right]_\delta \\ & + \left[2q_s - 2q_{s-i} - (3t^2 + (6k-1)t - 2k) = 0\right]_\delta \Big). \end{aligned}$$

2.4. **Evaluating and bounding the indicator sums in the error terms.** Let's first observe that all positive solutions to the equation

$$m = 3t^2 + ct, \;\; m, c, t \in \mathbb{Z}^+$$

require that

$$6t = -c + \sqrt{c^2 + 12m},$$

i.e., that in order for us to have integer solutions t to the above exact equation, we have the necessary condition that $c^2 + 24m = s^2$ is square where $s \equiv 1 \pmod{6}$. We need to evaluate the asymptotic behavior of the sums

$$\begin{aligned} I_{e,Q_x} = & \sum_{s=1}^{|Q_x|} \sum_{i=1}^{s-1} \#\left\{(u,k) : 12(q_s - q_{s-i}) = (6u+1)^2 - (6k+1)^2\right\} \\ & + \sum_{s=1}^{|Q_x|} \sum_{i=1}^{s-1} \#\left\{(u,k) : 12(q_s - q_{s-i}) = (6u+1)^2 - (6k-1)^2\right\} \\ & + \sum_{s=1}^{|Q_x|} \sum_{i=1}^{s-1} \#\left\{(u,k) : 12(q_s - q_{s-i}) = (6u+1)^2 - (6k+1)^2 - 2k\right\} \\ & + \sum_{s=1}^{|Q_x|} \sum_{i=1}^{s-1} \#\left\{(u,k) : 12(q_s - q_{s-i}) = (6u+1)^2 - (6k-1)^2 + 2k\right\}. \end{aligned} \tag{4}$$

It is clear that we will need some more detailed knowledge of the properties of the prime differences, $q_s - q_{s-i}$, in order to bound these sums. The question remains as to whether we really need deep knowledge of the distribution of these primes, or whether it will suffice to place them only sufficiently near the end of the intervals $I_p(t)$ to appropriately bound the indicator sums we need by $o(x^{0.51}/\log(x))$.

In this formulation it becomes necessary to know more information about the positive *differences* between primes defined by the *multiset*

$$D_x := \left\{q_s - q_{s-i} : 1 \leq s \leq \left\lfloor x^{0.51} \right\rfloor, 1 \leq i < s\right\},$$

where for convenience, we take $d_{\max,x} := \max(D_x)$. As an example, if we have our prime subset Q_x defined to be

$$Q_x = \{3, 7, 13, 23, 31, 43, 53, 73\},$$

then our corresponding difference multiset of elements of the form (d, count) is given by

$$\begin{aligned} D_x = \big\{&(4,1),(6,1),(8,1),(10,3),(12,1),(16,1),(18,1),(20,3),(22,1),(24,1), \\ &(28,1),(30,3),(36,1),(40,2),(42,1),(46,1),(50,2),(60,1),(66,1),(70,1)\big\}. \end{aligned}$$

Now with this in mind, we have our criteria for evaluating, and hence bounding, the error term I_{e,Q_x} through the formula proved in the next theorem (refer to Table 2.1):

#	**Diophantine Eqn. for $24\mathbf{n_0}$**	**Formula for $24\mathbf{n_0}$**	**Extra Assumptions**
I	$(6u+1)^2 - (6k+1)^2$	$12[6jk + j(3j+1)]$	*None.*
II	$(6u+1)^2 - (6k-1)^2$	$12(3j+1)(j+2k)$	$u \neq k$.
III	$(6u+1)^2 - (6k+1)^2 - 2k$	$2[(36j-1)k + 6j(3j+1)]$	*None.*
IV	$(6u+1)^2 - (6k-1)^2 + 2k$	$2[(36j+13)k + 6j(3j+1)]$	$u \neq k$; $k \equiv 0 \bmod 12$.

Table 2.1. *Formulas for the Prime Gap Indices $24n_0$ satisfying the Diophantine Equations in (4). Common assumptons in solving each equation are that (i) $u, k \geq 1$; (ii) $(6u+1)^2 > 6k+1$; and (iii) $u - k \equiv 0 \bmod 2$. In each formula, the difference $u-k$ corresponds to the solution parameter $j \geq 1$ where we require the $k \geq 1$ to denote the other integer parameter.*

Theorem 2.7 (An Exact Formula for I_{e,Q_x} Involving Congruences for Prime Gaps)**.** *For sufficiently large $x \geq x_0$, we have the following formula for the indicator sum error terms in (3):*

$$\begin{aligned} I_{e,Q_x} = &\sum_{j=1}^{d_{\max,x}} \#\{d \in D_x : d \equiv 12j(3j+1) \pmod{72j}\} && \text{(Indicator I)} \\ &+ \sum_{j=1}^{d_{\max,x}} \#\{d \in D_x : d \equiv 0 \pmod{36j+12}\} && \text{(Indicator II)} \\ &+ \sum_{j=1}^{d_{\max,x}} \#\{d \in D_x : d \equiv 6j(3j+1) \pmod{72j-2}\} && \text{(Indicator III)} \end{aligned}$$

$$+\sum_{j=1}^{d_{\max,x}} \#\left\{d \in D_x : d \equiv 6j(3j+1) \pmod{864j+312}\right\}. \quad \text{(Indicator IV)}$$

Proof. Based on some computational observations and noticing that $24 \mid 12(q_s - q_{s-i})$, we have the four cases for our indicator sums corresponding to the distinct sums in (4). Table 2.1 summarizes the criteria and the resulting formulas for the integers (prime gaps) $d \equiv 0 \bmod 24$ in each case, each of which we will prove below. We first notice that the statment of the theorem implicitly depends on summing over the indicators for prime gaps of a particular form depending on two parameters: some implicit $m \geq 1$ and the sum index $j \geq 1$. Then we see that our formula is also equivalent to showing that

$$\begin{aligned}
I_{e,Q_x} = \quad & \sum_{j=1}^{d_{\max,x}} \#\left\{d \in D_x : d = 12[6jk + j(3j+1)],\ \text{for some } k \geq 1\right\} \\
& + \sum_{j=1}^{d_{\max,x}} \#\left\{d \in D_x : d = 12(3j+1)(j+2k),\ \text{for some } k \geq 1\right\} \\
& + \sum_{j=1}^{d_{\max,x}} \#\left\{d \in D_x : d = 2[(36j-1)k + 6j(3j+1)],\ \text{some } k \geq 1\right\} \\
& + \sum_{j=1}^{d_{\max,x}} \#\left\{d \in D_x : d = 2[(36j+13)k + 6j(3j+1)],\ \text{some } k \geq 1,\ k \equiv 0 \bmod 12\right\}.
\end{aligned} \tag{5}$$

<u>*Case I*</u>*:* In the first case from the table, we write $j := u-k$ and expand the equation as

$$\begin{aligned}
& 36j(j+2k) + 12j = 24n_0 \\
\iff\ & 3j(j+2k) + j = 2n_0 \\
\iff\ & j(3j+1) + 6jk = 2n_0,
\end{aligned}$$

which implies the formula given in the table as

$$(6u+1)^2 + (6k+1)^2 = 24n_0 \iff 24n_0 = 24(6jm + j(3j+1)), \quad \text{for some } m, j \geq 1.$$

<u>*Case II*</u>*:* Similarly, in the second case in the table we have that

$$\begin{aligned}
& (6u+1)^2 + (6k-1)^2 = 24n_0 \\
\iff\ & 36j(j+2k) + 12(j+2k) = 24n_0 \\
\iff\ & 3j(j+2k) + j + 2k = 2n_0 \\
\iff\ & j(3j+1) + 6jk + 2k = 2n_0 \\
\iff\ & 12(3j+1)(j+2k) = 24n_0.
\end{aligned}$$

<u>*Case III*</u>*:* In this case we have that

$$\begin{aligned}
& (6u+1)^2 + (6k+1)^2 - 2k = 24n_0 \\
\iff\ & 36j(j+2k) + 12jk - 2k = 24n_0 \\
\iff\ & 6j(3j+1) + (36j-1)k = 12n_0 \\
\iff\ & 2[(36j-1)k + 6j(3j+1)] = 24n_0.
\end{aligned}$$

Case IV: Finally, we can expand our Diophantine equation solutions in this last case in the form of

$$\begin{aligned}
&(6u+1)^2+(6k-1)^2+2k = 24n_0 \\
&\iff 36j(j+2k)+12(j+2k)+2k = 24n_0 \\
&\iff 6j(3j+1)+(36j+13)k = 12n_0 \\
&\iff 2[(36j+13)k+6j(3j+1)] = 24n_0.
\end{aligned}$$

Thus it is clear that the identity in (5) is correct. Then to obtain the stated formula, we consider the forms of the prime gaps in the equation respectively modulo $\{72j, 36j+12, 72j-2, 72j+26\}$ where we recall that we have specified that $k \equiv 0 \bmod 12$ in the last indicator sum case. $\square$

Remark 2.8 (Connections to Bounded Differences Between Primes in Intervals)**.** We now remark that Theorem 2.7 effectively ties the parity problem for the partition function to the hard and deep question of understanding gaps between primes in some specified subset (in intervals in our case). This is a new and significant connection in the parity problem for $p(n)$. [Discuss known references on gaps between primes].

3. Computational Data

Given that *Mathematica* is not especially well-suited for performing computationally and memory intensive tasks on a home desktop machine, the current plan of attack towards verifying the conjecture that

$$C_{0.51} \cdot \frac{x^{0.51}}{\log(x)} + o(x^{0.51}/\log(x)),$$

holds for large $x \in [10, 10^8]$ (should be sufficient to make the point) is to rewrite the *Mathematica* code in C++ and apply for supercomputer time at Georgia Tech for this project. For up to 136 intervals, $I_{1.96}(t)$ corresponding to an approximate x value of $x \approx 10^{4.188}$ as input to $N_e(x)$, we have verified the claim.

Without dividing through by the $\log(x)$ factor, we unfortunately do not obtain the corresponding power factor better bound. At the moment, with the computational data I am able to obtain without extended computing power, it is unclear whether the modified "*squash–by–log*" approach will indeed do the trick, or whether instead we will eventually have that I_{e,Q_x} eventually overtakes $|Q_x|$ according to the formulas we have carefully developed in the previous sections – and that the division by a logarithmic number of terms in x only slows down this eventuality (see Figure 3.1). In short, we just don't have enough data to be sure either way at this point. However, the ratio image featured in the figure does have some encouraging properties that suggest the bound is attainable in this form, but we again need significantly more data points to support the claim.

4. Conclusions

4.1. **Summary.**

4.2. **Topics for future research and investigation.**

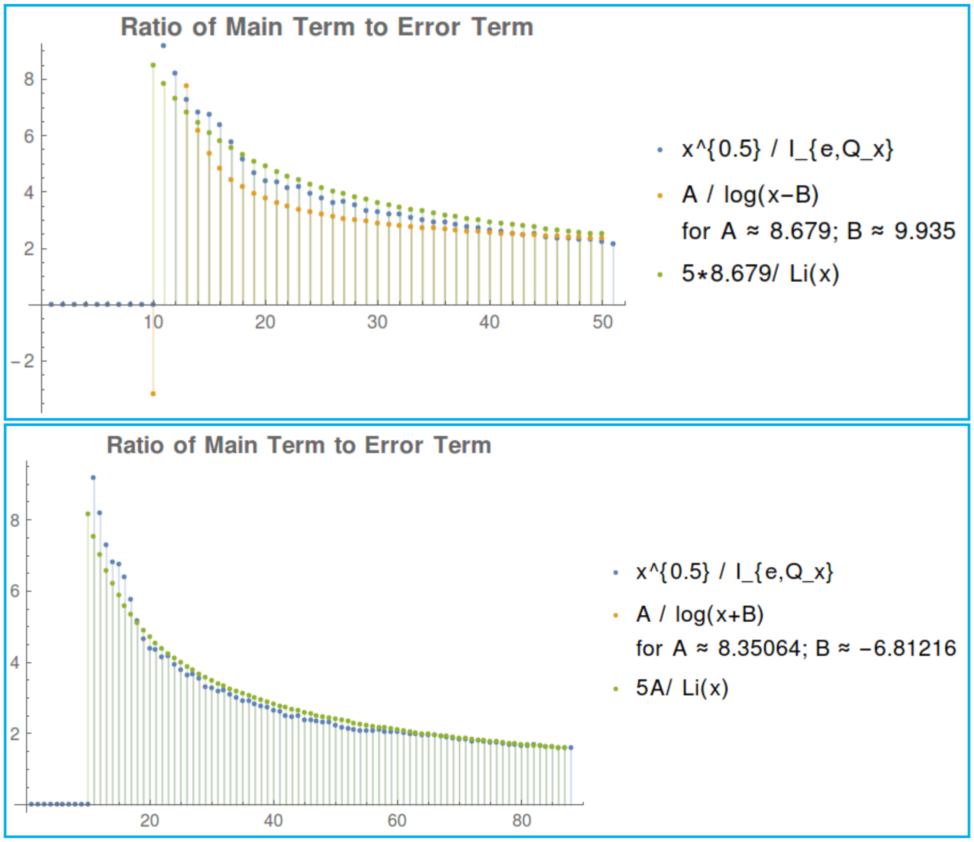

Figure 3.1. *The Ratio of the Main Term* $|Q_x| \approx x^{0.51}$ *to the Error Term* I_{e,Q_x} *Obtained Computationally for* $10 \leq t \leq 10$ *Intervals With* `Mathematica`*. Our goal is to have this ratio flatten out as* $t \to \infty$*, i.e., as* $x \to \infty$*, and converge to a limiting ratio strictly greater than unity. The ratio of a constant (presumably depending on* $t_{\max} = 50$*) over the logarithmic integral,* $\mathrm{Li}(x) = \int_2^x dt/\log(t)$*, appears to give an accurate fit for the decay of the ratio of main term to error featured in the plot.*

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