(*)

Corollary 4.6. *We have that as $n \to \infty$*

$$\frac{1}{n} \times \sum_{k \leq n} |g^{-1}(k)| = \frac{6B_0(\log n)^2\sqrt{\log\log n}}{\pi^2}\left(1 + O\left(\frac{1}{\log\log n}\right)\right).$$

Proof. As $|z| \to \infty$, the *imaginary error function*, $\operatorname{erfi}(z)$, has the following asymptotic expansion [24, §7.12]:

$$\operatorname{erfi}(z) := \frac{2}{\sqrt{\pi}\imath} \times \int_0^{\imath z} e^{t^2} dt = \frac{e^{z^2}}{\sqrt{\pi}}\left(\frac{1}{z} + \frac{1}{2z^3} + \frac{3}{4z^5} + \frac{15}{8z^7} + O\left(\frac{1}{z^9}\right)\right). \tag{23}$$

So $\boxed{|G^{-1}|(x) =_{\log} x}$

Corollary 5.3. *Suppose that the partial sums of the unsigned inverse sequence are defined as follows:*

$$|G^{-1}|(x) := \sum_{n \leq x} |g^{-1}(n)|, x \geq 1.$$

Let $\sigma_1 > 1$ be defined as in Theorem 5.2. For any $\epsilon > 0$, there are arbitrarily large x such that

$$|G^{-1}|(x) > x^{\sigma_1 - \epsilon}.$$

So $\boxed{\overline{\lim} \frac{|G^{-1}|(x)}{x^{1+\delta}} = +\infty.}$

for a $\delta > 0$.

Michael,

Thank you for pointing out this inconsistency. You are correct that the consequence that

$$|G^{-1}|(x) > x^{G_1-\varepsilon}, \ \forall \varepsilon > 0, \text{ holds for infinitely many large } x \ (\text{with } G_1 \simeq 1.39943)$$

does not make sense in light of the average order formula in (*) (Cor. 4.6, as stated). I am confident that the expectation formula is correct here. The problem seems to be in the translation of the proof suggested by Prof. Vaughan to the context here. Allow me to explain what is going on, and then perhaps you can suggest workarounds? (I may just remove this subsection from the article before pushing the revised manuscipt to JNT tomorrow, otherwise ...)

The construction in terms of "my" unsigned functions, $|g^{-1}(n)|$ and $C_\Omega(n)$, is as follows:

$$|g^{-1}(n)| = \lambda(n) g^{-1}(n) = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_\Omega(d), \forall n \geq 1$$

For $s \in \mathbb{C}$, let the DGF

$$\underline{\overline{\mathcal{K}}}(s) := DGF|g^{-1}|(s) \quad (Re(s) > 1)$$

$$= \sum_{n \geq 1} \frac{|g^{-1}(n)|}{n^s} = \frac{1}{\zeta(2s)(1-P(s))} \quad (**),$$

Since the DGF of $f * 1$ is $\zeta(s) DGF[f](s)$, and $DGF[\mu^2] = \frac{\zeta(s)}{\zeta(2s)}$, $Re(s) > 1$.

Now, we easily find that $\underline{\overline{\mathcal{K}}}(s)$ converges $\forall Re(s) \in (1, \sigma_1 \simeq 1.39943) \cup (\sigma_1, +\infty)$, where $\zeta(2s)$ is convergent $\forall Re(s) > \frac{1}{2}$ and

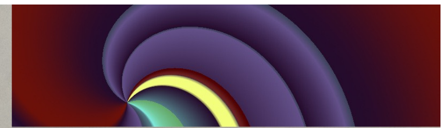
$(1-P(s))^{-1}$ has a (simple?) pole at $s=\sigma_1$.
(Recall that σ_1 is the unique real $\sigma>1$ s.t. $P(\sigma)=1$.)

So $\Xi(s)$ is a meromorphic function of s in the half-plane $Re(s)>1$. We also can obtain an analytic continuation of the DGF $\Xi(s)$ to any $s\in\mathbb{C}$ with $Re(s)>0$, with the exception of the aforementioned poles at $s=1/2$ and $s=\sigma_1$.

The problem at hand, which you noticed on the first two pages of this note, seems to be in reconciling definitions from [MV] used to define the σ_c (abscissa of convergence for $\Xi(s)$). E.g., observe the next starts of theorems from §1 of that reference reproduced below:

① Characterization of σ_c for any DGF, $\alpha(s)$:

1.2 Analytic properties of Dirichlet series



Corollary 1.2 *Any Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ has an abscissa of convergence σ_c with the property that $\alpha(s)$ converges for all s with $\sigma > \sigma_c$, and for no s with $\sigma < \sigma_c$. Moreover, if s_0 is a point with $\sigma_0 > \sigma_c$, then there is a neighbourhood of s_0 in which $\alpha(s)$ converges uniformly.*

② What the value of σ_c implies about asymptotics of the summatory functs!

Theorem 1.3 *Let $A(x) = \sum_{n \le x} a_n$. If $\sigma_c < 0$, then $A(x)$ is a bounded function, and*

$$\sum_{n=1}^{\infty} a_n n^{-s} = s \int_1^{\infty} A(x) x^{-s-1}\, dx \tag{1.10}$$

for $\sigma > 0$. If $\sigma_c \ge 0$, then

$$\limsup_{x \to \infty} \frac{\log |A(x)|}{\log x} = \sigma_c, \tag{1.11}$$

and (1.10) holds for $\sigma > \sigma_c$.

So here in lies the problem!

We have, for example, that $\underline{E}(1.25)$ is convergent (at $s=5/4$), but we also inherit the pole of $\underline{E}(s)$ at $s=\sigma_1 > 5/4$. The definition of the abscissa of convergence, σ_c, in (1) from [MV] above requires that we get convergence of $\underline{K}(s)$ $\forall Re(s) > \sigma_c$, but namely also, for NO s with $Re(s) < \sigma_c$. This obviously doesn't happen for the key DGF, $\underline{K}(s)$, defined by (**).

Pardon my "fucking" language, but WTF can we then conclude here? Does Thm 1.3 from [MV] in (2) above simply not apply, whence, we should conclude that while Prof. Vaughan's proof is slick, it contains no extra information? (Request for detailed information hence submitted, pending your approval...)

;)

FOR COMMENT

PRELIMINARY RESULTS

Happy New Years, 2022.
May this year suck substant-
-ially less than the last few
that have come before it.

— MDS