New characterizations of partial sums of the Möbius function

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Abstract

The Mertens function, $M(x) \coloneqq \sum_{n \leq x} \mu(n)$, is defined as the summatory function of the classical Möbius function for $x \geq 1$. The inverse function $g^{-1}(n) \coloneqq (\omega + 1)^{-1}(n)$ taken with respect to Dirichlet convolution is defined in terms of the strongly additive function $\omega(n)$ that counts the number of distinct prime factors of the integers $n \geq 2$ without multiplicity. For large x and $n \leq x$, we associate a natural combinatorial significance to the magnitude of the distinct values of $g^{-1}(n)$ that depends directly on the exponent patterns in the prime factorizations of the integers in $\{2, 3, \ldots, x\}$ viewed as multisets. We have an Erdős-Kac theorem analog for the distribution of the unsigned sequence $|g^{-1}(n)|$ over $n \leq x$ as $x \to \infty$. The key connection of the partial sums of the auxiliary function $C_{\Omega(n)}(n) \coloneqq (\Omega(n))! \times \prod_{p^{\alpha} \| n} (\alpha!)^{-1}$ to $|g^{-1}(n)|$ is proved using assumptions on the independence of the completely additive function $\Omega(n)$ and the distribution of the exponents of the distinct prime factors of $2 \leq n \leq x$ when x is large. Discrete convolutions of the summatory function $G^{-1}(x) \coloneqq \sum_{n \leq x} \lambda(n) |g^{-1}(n)|$ with the prime counting function $\pi(x)$ determine exact formulas and new characterizations of asymptotic approaches to $M(x)$. In this way, we prove another characteristic link of the Mertens function to the distribution of the partial sums $L(x) \coloneqq \sum_{n \leq x} \lambda(n)$ and connect these two classical summatory functions with an explicit probability distribution at large x.

Keywords and Phrases: *Möbius function; Mertens function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function; Erdős-Kac theorem; strongly additive function.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; 11N64; and 11-04.*

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Notation and conventions

The next listing provides a glossary of common notation, conventions and abbreviations employed throughout the article.

Symbols	**Definition**
$\gg, \ll, \asymp$	For functions A, B, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A, B \geq 0$, $A \ll B$ and $B \ll A$, we write $A \asymp B$.
$\approx, \sim$	We write that $f(x) \approx g(x)$ if $\lvert f(x) - g(x)\rvert \ll 1$ as $x \to \infty$. Two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1$.
$\chi_{\mathbb{P}}(n), P(s)$	The indicator function of the primes equals one if and only if $n \in \mathbb{Z}^+$ is prime, and is zero-valued otherwise. For any $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > 1$, we define the prime zeta function to be the Dirichlet generating function (DGF) defined by $P(s) = \sum_{n\geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s}$. The function $P(s)$ has an analytic continuation to the half-plane $\operatorname{Re}(s) > 0$ through the formula $P(s) = \sum_{k\geq 1} \frac{\mu(k)}{k} \log \zeta(ks)$ with poles at the reciprocal of each positive integer and a natural boundary at the line $\operatorname{Re}(s) = 0$.
$C_k(n), C_{\Omega(n)}(n)$	The sequence is defined recursively for integers $n \geq 1$ and $k \geq 0$ as follows: $$C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum_{d\mid n} \omega(d) C_{k-1}\left(\frac{n}{d}\right), & \text{if } k \geq 1. \end{cases}$$ It represents the multiple (k-fold) convolution of the function $\omega(n)$ with itself. The function $C_{\Omega(n)}(n)$ has the DGF $(1 - P(s))^{-1}$ for $\operatorname{Re}(s) > 1$.
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function (OGF) of some sequence, $\{f_n\}_{n\geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n\geq 0} f_n q^n$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic function f we have that $f * \varepsilon = \varepsilon * f = f$ where the operation $*$ denotes Dirichlet convolution (see definition below).
$f * g$	The Dirichlet convolution of any two arithmetic functions f and g is denoted by the divisor sum $(f * g)(n) := \sum_{d\mid n} f(d) g\left(\frac{n}{d}\right)$ for $n \geq 1$.
$f^{-1}(n)$	The Dirichlet inverse f^{-1} of an arithmetic function f exists if and only if $f(1) \neq 0$. The Dirichlet inverse of any f such that $f(1) \neq 0$ is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \times \sum_{\substack{d\mid n \\ d>1}} f(d) f^{-1}\left(\frac{n}{d}\right)$ for $n \geq 2$ with $f^{-1}(1) = f(1)^{-1}$. When it exists, this inverse function is unique and satisfies $f^{-1} * f = f * f^{-1} = \varepsilon$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$ for $x \geq 1$.

Symbols	**Definition**
$[n=k]_{\delta}$,$[\mathtt{cond}]_{\delta}$	The symbol $[n=k]_{\delta}$ is a synonym for $\delta_{n,k}$ which is one if and only if $n=k$, and is zero otherwise. For boolean-valued conditions, `cond`, the symbol $[\mathtt{cond}]_{\delta}$ evaluates to one precisely when `cond` is true, and to zero otherwise.
$\lambda(n), L(x)$	The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. Its summatory function is defined by the partial sums $L(x) := \sum_{n\leq x} \lambda(n)$ for $x \geq 1$.
$\mu(n), M(x)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers $n \geq 1$ where $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree. The Mertens function is the summatory function defined for all integers $x \geq 1$ by $M(x) := \sum_{n\leq x} \mu(n)$.
$\Phi(z)$	For $z \in \mathbb{R}$, we take the cumulative density function of the standard normal distribution to be denoted by $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^{\alpha} \Vert n$ for $p \geq 2$ prime, $\alpha \geq 1$ and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p\mid n} 1$ and the completely additive function $\Omega(n) := \sum_{p^{\alpha}\Vert n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \times \cdots \times p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. We set $\omega(1) = \Omega(1) = 0$ by convention.
$\pi_k(x), \widehat{\pi}_k(x)$	For integers $k \geq 1$, the function $\pi_k(x)$ denotes the number of $2 \leq n \leq x$ with exactly k distinct prime factors: $\pi_k(x) := \#\{2 \leq n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{2 \leq n \leq x : \Omega(n) = k\}$ for $x \geq 2$ and fixed $k \geq 1$.
$Q(x)$	For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. That is, $Q(x) := \sum_{n\leq x} \mu^2(n)$ where $Q(x) = \frac{6x}{\pi^2} + O(\sqrt{x})$.
$W(x)$	For $x, y \in \mathbb{R}_{\geq 0}$, we write that $x = W(y)$ if and only if $xe^x = y$. This function denotes the principal branch of the multi-valued Lambert W function defined on the non-negative reals.
$\zeta(s)$	The Riemann zeta function is defined by $\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation to any $s \in \mathbb{C}$ with the exception of a simple pole at $s = 1$ of residue one.

1 Introduction

The *Mertens function*, or the summatory function of $\mu(n)$, is defined on the positive integers by the partial sums

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The first several values of this summatory function are calculated as follows [27, A008683; A002321]:

$$\{M(x)\}_{x \geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \ldots\}.$$

The Mertens function is related to the partial sums of the Liouville lambda function, denoted by $L(x) := \sum\limits_{n \leq x} \lambda(n)$, via the relation [10, 16] [27, A008836; A002819]

$$L(x) = \sum_{d \leq \sqrt{x}} M\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), x \geq 1.$$

The main interpretation to take away from the article is the new characterization of $M(x)$ through two primary auxiliary unsigned sequences and their summatory functions, namely, the functions $C_{\Omega(n)}(n)$, $g^{-1}(n)$ and their partial sums. This characterization is formed by constructing the combinatorially motivated sequences related to the distribution of the primes by convolutions of the strongly additive function $\omega(n)$. The methods in this article initially stem from a curiosity about an elementary identity from the list of exercises in [1, §2; *cf.* §11]. In particular, the indicator function of the primes is given by Möbius inversion as the Dirichlet convolution $\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu$. We form partial sums of $(\omega + 1) * \mu(n)$ over $n \leq x$ for any $x \geq 1$ and then apply classical inversion theorems to relate $M(x)$ to the partial sums of $g^{-1}(n) := (\omega + 1)^{-1}(n)$ (*cf.* Theorem 1.2; Corollary 1.3; and Corollary 1.4).

1.1 Motivation

There is a natural relationship of $g^{-1}(n)$ with the auxiliary function $C_{\Omega(n)}(n)$, or the $\Omega(n)$-fold Dirichlet convolution of $\omega(n)$ with itself at n, which we prove by elementary methods in Section 3. These identities inspire the deep connection between the unsigned inverse function and additive prime counting combinatorics we find in Section 3.3. In this sense, the new results stated within this article diverge from the proofs typified by previous anayltic and combinatorial methods to bound $M(x)$ cited in the references. The function $C_{\Omega(n)}(n)$ was considered under alternate notation by Fröberg (circa 1968) in his work on the series expansions of the *prime zeta function*, $P(s)$, e.g., the prime sums defined as the Dirichlet generating function (DGF) of $\chi_{\mathbb{P}}(n)$. The clear interpretation of the function $C_{\Omega(n)}(n)$ in connection with $M(x)$ is unique to our work to establish the properties of this auxiliary sequence. References to uniform asymptotics for restricted partial sums of $C_{\Omega(n)}(n)$ and the features of the limiting distribution of this function are missing in surrounding literature (*cf.* Corollary 4.4; Proposition 4.5; and Theorem 4.8).

We utilize the results in [17, §7.4; §2.4] that apply traditional analytic methods to formulate limiting asymptotics and to prove an Erdős-Kac theorem analog characterizing key properties of the distribution of the completely additive function $\Omega(n)$. Adaptations of the key ideas from the exposition in the reference provide a foundation for analytic proofs of several limiting properties of, asymptotic formulae for restricted partial sums involving, and in part the Erdős-Kac type theorem for both $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$. Our Erdős-Kac type theorem variants characterizing the distributions of both $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$ are established under reasonable limiting assumptions on the random variables $X_{n,k} := \frac{C_{\Omega(n)}(n)}{(\log n)(\log\log n)}$ when $\Omega(n) = k$ for $k \geq 1$ and $n \leq x$ as $x \to \infty$. The sequence $g^{-1}(n)$ and its partial sums defined by $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ are linked to canonical examples of strongly and completely additive functions, e.g., in relation to $\omega(n)$ and $\Omega(n)$, respectively. The definitions of the sequences we define, and the proof methods given in the spirit of

Montgomery and Vaughan's work, allow us to reconcile the property of strong additivity with the signed partial sums of a multiplicative function. We leverage the connection of $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$ with the canonical number theoretic additive functions to obtain the results proved primarily in Section 4.

We also formulate a probabilistic perspective from which to express our intuition about features of the distribution of $G^{-1}(x)$ via the properties of its summands. Since we prove that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ for all $n \geq 1$ in Proposition 2.1, the partial sums defined by $G^{-1}(x)$ are precisely related to the properties of $|g^{-1}(n)|$ and asymptotics for $L(x)$. Our new results then relate the distribution of $L(x)$, an explicitly identified probability distribution, and $M(x)$ as $x \to \infty$. Formalizing the properties of the distribution of $L(x)$ is still viewed as a problem that is equally as difficult as understanding the properties of $M(x)$ well at large x or along infinite subsequences.

Our characterizations of $M(x)$ by the summatory function of the signed inverse sequence, $G^{-1}(x)$, is suggestive of new approaches to bounding the Mertens function. These results motivate future work to state upper (and possibly lower) bounds on $M(x)$ in terms of the additive combinatorial properties of the repeated distinct values of the sign weighted summands of $G^{-1}(x)$. We also expect that an outline of the method behind the collective proofs we provide with respect to the Mertens function case can be generalized to identify associated additive functions with the same role of $\omega(n)$ in this paper to express asymptotics for partial sums of other signed multiplicative functions.

1.2 Preliminaries on the Mertens function

An approach to evaluating the limiting asymptotic behavior of $M(x)$ for large $x \to \infty$ considers an inverse Mellin transform of the reciprocal of the Riemann zeta function given by

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \times \int_1^{\infty} \frac{M(x)}{x^{s+1}} dx, \operatorname{Re}(s) > 1.$$

In particular, we obtain that

$$M(x) = \lim_{T\to\infty} \frac{1}{2\pi\imath} \times \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s\zeta(s)} ds.$$

The previous formulas lead to the exact expression of $M(x)$ for any $x > 0$ given by the next theorem.

Theorem 1.1 (Titchmarsh)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k\geq 1}$ satisfying $k \leq T_k \leq k+1$ for each integer $k \geq 1$ such that for any real $x > 0$*

$$M(x) = \lim_{k\to\infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ 0<|\operatorname{Im}(\rho)|<T_k}} \frac{x^\rho}{\rho\zeta'(\rho)} - 2 + \sum_{n\geq 1} \frac{(-1)^{n-1}}{n(2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

An unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C_1 > 0$ such that

$$M(x) \ll x \times \exp\left(-C_1 \log^{\frac{3}{5}}(x)(\log\log x)^{-\frac{1}{5}}\right).$$

Under the assumption of the RH, Soundararajan and Humphries, respectively, improved estimates bounding $M(x)$ from above for large x in the following forms [28, 10]:

$$\begin{aligned} M(x) &\ll \sqrt{x} \times \exp\left(\sqrt{\log x}(\log\log x)^{14}\right), \\ M(x) &\ll \sqrt{x} \times \exp\left(\sqrt{\log x}(\log\log x)^{\frac{5}{2}+\epsilon}\right), \text{ for all } \epsilon > 0. \end{aligned}$$

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$ for any $0 < \epsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that $|M(x)| < C_2\sqrt{x}$ for some absolute constant $C_2 > 0$. The conjecture was first verified by F. Mertens himself for $C_2 = 1$ and all $x < 10000$ without the benefit of modern computation. Since its beginnings in 1897, the Mertens conjecture was disproved by computational methods involving non-trivial simple zeta function zeros with comparatively small imaginary parts in the famous paper from the mid 1980's by Odlyzko and te Riele [22].

More recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $M(x)x^{-\frac{1}{2}}$ grows with or without bound along infinite subsequences, i.e., considering the asymptotics of the function in the limit supremum and limit infimum senses.

It is verified by computation that [25, *cf.* §4.1] [27, *cf.* A051400; A051401]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad \text{(more recently } \geq 1.826054\text{)},$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad \text{(more recently } \leq -1.837625\text{)}.$$

Based on the work by Odlyzko and te Riele, it is likely that each of these limiting bounds evaluates to $\pm\infty$, respectively [22, 14, 15, 11]. A conjecture due to Gonek asserts that in fact $M(x)$ satisfies [21]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}(\log\log\log x)^{\frac{5}{4}}} = C_3,$$

for C_3 an absolute constant.

1.3 A concrete new approach to characterizing $M(x)$

1.3.1 Summatory functions of Dirichlet convolutions of arithmetic functions

We prove the formulas in the next inversion theorem by matrix methods in Section 2.1.

Theorem 1.2 (Partial sums of Dirichlet convolutions and their inversions)**.** *Let $r, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $r(1) \neq 0$. Suppose that $R(x) := \sum_{n\leq x} r(n)$ and $H(x) := \sum_{n\leq x} h(n)$ denote the summatory functions of r and h, respectively, and that $R^{-1}(x) := \sum_{n\leq x} r^{-1}(n)$ denotes the summatory function of the Dirichlet inverse of r for any $x \geq 1$. We have the following exact expressions that hold for all integers $x \geq 1$:*

$$\begin{aligned}
\pi_{r*h}(x) &:= \sum_{n\leq x}\sum_{d|n} r(d) h\left(\frac{n}{d}\right) \\
&= \sum_{d\leq x} r(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} H(k)\left(R\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - R\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right).
\end{aligned}$$

Moreover, for any $x \geq 1$ we have

$$\begin{aligned}
H(x) &= \sum_{j=1}^{x} \pi_{r*h}(j)\left(R^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - R^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right) \\
&= \sum_{k=1}^{x} r^{-1}(k)\pi_{r*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right).
\end{aligned}$$

Key consequences of Theorem 1.2 as it applies to $M(x)$ in the special case of $h(n) := \mu(n)$ for all $n \geq 1$ are stated as the next two corollaries.

Corollary 1.3 (Applications of Möbius inversion)**.** *Suppose that r is an arithmetic function such that $r(1) \neq 0$. Define the summatory function of the convolution of r with μ by $\widetilde{R}(x) \coloneqq \sum_{n \leq x} (r * \mu)(n)$. Then the Mertens function is expressed by the partial sums*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} r^{-1}(j) \right) \widetilde{R}(k), \forall x \geq 1.$$

Corollary 1.4 (Key Identity)**.** *We have that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left(\pi \left(\left\lfloor \frac{x}{k} \right\rfloor \right) + 1 \right). \tag{1}$$

1.3.2 An exact expression for $M(x)$ via strongly additive functions

We fix the notation for the Dirichlet invertible function $g(n) \coloneqq \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n)$ [27, A341444]. We compute the first several values of this sequence as follows:

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

There is not a simple direct recursion between the distinct values of $g^{-1}(n)$ that holds for all $n \geq 1$. However, the next observation is suggestive of the quasi-periodicity of the distribution of distinct values of this inverse function over $n \geq 2$.

Observation 1.5 (Additive symmetry in $g^{-1}(n)$ from the prime factorizations of $n \leq x$)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \times \cdots \times p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \times \cdots \times q_s^{\beta_s}$. If $r = s$ and $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of the prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly one, two, three (and so on) prime factors. Hence, there is an essentially additive structure underneath the sequence $\{g^{-1}(n)\}_{n \geq 2}$.

Proposition 1.6. *We have the following properties of the Dirichlet inverse function $g^{-1}(n)$:*

(A) For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;

(B) For all squarefree integers $n \geq 1$, we have that

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \times m!;$$

(C) If $n \geq 2$ and $\Omega(n) = k$ for some $k \geq 1$, then

$$2 \leq |g^{-1}(n)| \leq \sum_{j=0}^{k} \binom{k}{j} \times j!.$$

The signedness property in (A) is proved precisely in Proposition 2.1. A proof of (B) follows from Lemma 3.1. The realization that the beautiful and remarkably simple combinatorial form of property (B) in Proposition 1.6 holds for all squarefree integers motivates our pursuit of simpler formulas for the inverse function $g^{-1}(n)$ through the sums of auxiliary subsequences $C_k(n)$ when $k \coloneqq \Omega(n)$ defined in Section 3. That is, we observe a familiar formula for $g^{-1}(n)$ on an asymptotically dense infinite subset of integers (with density $\frac{6}{\pi^2}$), e.g., that holds for all squarefree $n \geq 2$, and then seek to extrapolate by proving there are in fact regular properties of the distribution of this sequence when viewed more generally over the positive integers.

An exact expression for $g^{-1}(n)$ is given by

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d), n \geq 1,$$

where the sequence $\lambda(n)C_{\Omega(n)}(n)$ has the DGF $(1+P(s))^{-1}$ and $C_{\Omega(n)}(n)$ has DGF $(1-P(s))^{-1}$ for $\mathrm{Re}(s) > 1$ (see Proposition 2.1). The function $C_{\Omega(n)}(n)$ was considered in [8] with its exact formula given by [12, *cf.* §3]

$$C_{\Omega(n)}(n) = \begin{cases} 1, & \text{if } n = 1; \\ (\Omega(n))! \times \prod\limits_{p^\alpha || n} \frac{1}{\alpha!}, & \text{if } n \geq 2. \end{cases}$$

In Corollary 4.4, we use the result proved in Theorem 4.2 to show that uniformly for $1 \leq k \leq 2 \log\log x$ there is an absolute constant $A_0 > 0$ such that

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) = \frac{A_0\sqrt{2\pi}x}{\log x} \times \widehat{G}\left(\frac{k-1}{\log\log x}\right) \frac{(\log\log x)^{k-\frac{1}{2}}}{(k-1)!}\left(1 + O\left(\frac{1}{\log\log x}\right)\right), \text{ as } x \to \infty,$$

where $\widehat{G}(z) := \frac{\zeta(2)^{-z}}{\Gamma(1+z)(1+P(2)z)}$ for $0 \leq |z| < P(2)^{-1}$.

In Proposition 4.5, we use an adaptation of the asymptotic formulas for the summations proved in the appendix section of this article combined with the form of *Rankin's method* from [17, Thm. 7.20] to show that there is another absolute constant $B_0 > 0$ such that

$$\frac{1}{n} \times \sum_{k \leq n} C_{\Omega(k)}(k) = B_0(\log n)\sqrt{\log\log n}\left(1 + O\left(\frac{1}{\log\log n}\right)\right), \text{ as } n \to \infty.$$

In Corollary 4.6, we prove that the average order of $|g^{-1}(n)|$ is

$$\frac{1}{n} \times \sum_{k \leq n} |g^{-1}(k)| = \frac{6B_0(\log n)^2\sqrt{\log\log n}}{\pi^2}\left(1 + O\left(\frac{1}{\log\log n}\right)\right), \text{ as } n \to \infty.$$

In Section 4.3, we prove a variant of the Erdős-Kac theorem that characterizes the distribution of $C_{\Omega(n)}(n)$ which holds under reasonable assumptions on independence (see Theorem 4.8; *cf.* Ansatz 4.7). The theorem leads the conclusion of the following statement for any fixed $Y > 0$, with $\mu_x(C) := \log\log x - \log\left(\frac{\sqrt{2\pi}A_0}{\zeta(2)(1+P(2))}\right)$ and $\sigma_x(C) := \sqrt{\log\log x}$, and holds uniformly for any $-Y \leq y \leq Y$ (see Corollary 4.9):

$$\begin{aligned} &\frac{1}{x} \times \#\left\{3 \leq n \leq x : \frac{|g^{-1}(n)|}{(\log n)\sqrt{\log\log n}} - \frac{6}{\pi^2 n(\log n)\sqrt{\log\log n}} \times \sum_{k \leq n} |g^{-1}(k)| \leq y\right\} \\ &= \Phi\left(\frac{\frac{\pi^2 y}{6} - \mu_x(C)}{\sigma_x(C)}\right) + o(1), \text{ as } x \to \infty. \end{aligned}$$

The regularity and quasi-periodicity we alluded to in the previous few remarks are then quantifiable insomuch as $|g^{-1}(n)|$ tends to a scaled multiple of its average order with a non-centrally normal tendency. If x is sufficiently large and if we pick any integer $n \in [2, x]$ uniformly at random, then the following statement also holds as $x \to \infty$:

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2 n} \times \sum_{k \leq n} |g^{-1}(k)| \leq \frac{6}{\pi^2}(\log n)\sqrt{\log\log n}\,(\alpha\sigma_x(C) + \mu_x(C))\right) = \Phi(\alpha) + o(1), \alpha \in \mathbb{R}.$$

1.3.3 Formulas illustrating the new characterizations of $M(x)$

Let the partial sums $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$ for integers $x \geq 1$ [27, A341472]. We prove that (see Proposition 5.1)

$$M(x) = G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} G^{-1}(k) \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right), x \geq 1, \tag{2}$$

and that (*cf.* Section 3.2)

$$M(x) = G^{-1}(x) + \sum_{p\leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1.$$

These formulas imply that we can establish asymptotic bounds on $M(x)$ along infinite subsequences by sharply bounding the summatory function $G^{-1}(x)$ along those points. We also have an identification of $G^{-1}(x)$ with $L(x)$ given by

$$G^{-1}(x) = L(x)|g^{-1}(x)| - \sum_{n<x} L(n)\left(\left|g^{-1}(n+1)\right| - \left|g^{-1}(n)\right|\right),$$

where the distribution of $|g^{-1}(n)|$ is characterized by Corollary 4.9. In Section 5.2, we use the anaytic methods due to H. Davenport and H. Heilbronn suggested by R. C. Vaughan to prove that for $\sigma_1 \approx 1.39943$ the unique solution to $P(\sigma) = 1$ on $(1, \infty)$ we have

$$\limsup_{x\to\infty} \frac{\log\left|G^{-1}(x)\right|}{\log x} \geq \sigma_1.$$

Hence, for any $\epsilon > 0$, Corollary 5.3 proves that there are arbitrarily large x such that

$$|G^{-1}(x)| > x^{\sigma_1-\epsilon}.$$

Nonetheless, we still expect substantial local cancellation in the terms involving $G^{-1}(x)$ in our new formulas for $M(x)$ at almost every large x (see Section 5.3).

2 Initial elementary proofs of new results

2.1 Establishing the summatory function properties and inversion identities

We give a proof of the inversion type results in Theorem 1.2 by matrix methods in this section. Related results on summations of Dirichlet convolutions and their inversion appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95]. It is similarly not difficult to establish the identity

$$\sum_{n\leq x} h(n)(q*r)(n) = \sum_{n\leq x} q(n) \times \sum_{k\leq \left\lfloor \frac{x}{n}\right\rfloor} r(k)h(kn).$$

Proof of Theorem 1.2. Let h, r be arithmetic functions such that $r(1)\neq 0$. Denote the summatory functions of h, r and r^{-1}, respectively, by $H(x) = \sum_{n\leq x} h(n)$, $R(x) = \sum_{n\leq x} r(n)$, and $R^{-1}(x) = \sum_{n\leq x} r^{-1}(n)$. We define $\pi_{r*h}(x)$ to be the summatory function of the Dirichlet convolution of r with h. We have that the following formulas hold for all $x\geq 1$:

$$\begin{aligned}\pi_{r*h}(x) &:= \sum_{n=1}^{x}\sum_{d\mid n} r(n)h\left(\frac{n}{d}\right) = \sum_{d=1}^{x} r(d)H\left(\left\lfloor \frac{x}{d}\right\rfloor\right) \\ &= \sum_{i=1}^{x}\left(R\left(\left\lfloor \frac{x}{i}\right\rfloor\right) - R\left(\left\lfloor \frac{x}{i+1}\right\rfloor\right)\right)H(i). \qquad (3)\end{aligned}$$

The first formula above is well known from the references cited above. The second formula is justified directly using summation by parts as [23, §2.10(ii)]

$$\begin{aligned}\pi_{r*h}(x) &= \sum_{d=1}^{x} h(d)R\left(\left\lfloor \frac{x}{d}\right\rfloor\right) \\ &= \sum_{i\leq x}\left(\sum_{j\leq i} h(j)\right)\times\left(R\left(\left\lfloor \frac{x}{i}\right\rfloor\right) - R\left(\left\lfloor \frac{x}{i+1}\right\rfloor\right)\right).\end{aligned}$$

We form the invertible matrix of coefficients $\hat{R}$ associated with the linear system defining $H(j)$ for all $1\leq j\leq x$ in (3) by setting

$$r_{x,j} := R\left(\left\lfloor \frac{x}{j}\right\rfloor\right) - R\left(\left\lfloor \frac{x}{j+1}\right\rfloor\right) \equiv R_{x,j} - R_{x,j+1},$$

where

$$R_{x,j} := R\left(\left\lfloor \frac{x}{j}\right\rfloor\right), \text{ for } 1\leq j\leq x.$$

Since $r_{x,x} = R(1) = r(1)\neq 0$ for all $x\geq 1$ and $r_{x,j} = 0$ for all $j > x$, the matrix we have defined in this problem is lower triangular with a non-zero constant on its diagonals, and is hence invertible. If we let $\hat{R} := (R_{x,j})$, then the next matrix is expressed by applying an invertible shift operation as

$$(r_{x,j}) = \hat{R}\left(I - U^T\right).$$

Note that the square matrix U of sufficiently large finite dimensions $N\times N$ has $(i,j)^{th}$ entries for all $1\leq i,j\leq N$ that are defined by $(U)_{i,j} = \delta_{i+1,j}$ so that

$$\left[(I-U^T)^{-1}\right]_{i,j} = [j\leq i]_{\delta}.$$

We also observe that

$$\left\lfloor \frac{x}{j}\right\rfloor - \left\lfloor \frac{x-1}{j}\right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.}\end{cases}$$

The previous equation implies that

$$R\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - R\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} r\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

We use the property in (4) to shift the matrix $\hat{R}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of r as

$$\left(\left(I - U^T\right)\hat{R}\right)^{-1} = \left(r\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(r^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

In particular, our target matrix in the inversion problem is defined by

$$(r_{x,j}) = \left(I - U^T\right)\left(r\left(\frac{x}{j}\right)[j|x]_\delta\right)\left(I - U^T\right)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators in the form of

$$\begin{aligned} (r_{x,j})^{-1} &= \left(I - U^T\right)^{-1}\left(r^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)\left(I - U^T\right) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} r^{-1}(k)\right)\left(I - U^T\right) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} r^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} r^{-1}(k)\right). \end{aligned}$$

Hence, the summatory function $H(x)$ is given exactly for any integers $x \geq 1$ by a vector product with the inverse matrix from the previous equation by

$$H(x) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} r^{-1}(j)\right) \times \pi_{r*h}(k).$$

We can prove a second inversion formula providing the coefficients of the summatory function $R^{-1}(j)$ for $1 \leq j \leq x$ from the last equation by adapting our argument to prove (3) above. This leads to the following alternate identity expressing $H(x)$:

$$H(x) = \sum_{k=1}^{x} r^{-1}(k) \times \pi_{r*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right). \qquad \square$$

2.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}(n)$ denote the characteristic function of the primes, let $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n (without multiplicity). We can see using elementary methods that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu. \tag{5}$$

Namely, since $\mu * 1 = \varepsilon$ and

$$\omega(n) = \sum_{p|n} 1 = \sum_{d|n} \chi_{\mathbb{P}}(d), \text{ for } n \geq 1,$$

the result in (5) follows by Möbius inversion. When combined with Corollary 1.3, this convolution identity yields the key exact formula for $M(x)$ stated in (1) of Corollary 1.4.

Proposition 2.1 (The signedness of $g^{-1}(n)$)**.** *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0, \pm 1\}$ *denote the signedness of the arithmetic function* h *at any* $n \geq 1$*. For the Dirichlet invertible function* $g(n) := \omega(n) + 1$*, we have that* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ *for all* $n \geq 1$*.*

Proof. The function $D_f(s) := \sum_{n\geq 1} f(n) n^{-s}$ defines the Dirichlet generating function (DGF) of any arithmetic function f which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ where σ_f is the abscissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = \zeta(s)^{-1}$ and $D_\omega(s) = P(s)\zeta(s)$ for $\operatorname{Re}(s) > 1$. Then by (5) and the fact that whenever $f(1) \neq 0$, the DGF of $f^{-1}(n)$ is $D_f(s)^{-1}$, we have that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{\zeta(s)(1+P(s))}, \operatorname{Re}(s) > 1. \tag{6}$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. We see that this observation implies $\operatorname{sgn}(h^{-1} * \mu) = \lambda$.

First, by a combinatorial argument related to multinomial coefficient expansions of these sums, we recover exactly that [8, *cf.* §2]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ \lambda(n)(\Omega(n))! \times \prod\limits_{p^\alpha || n} \frac{1}{\alpha!}, & n \geq 2. \end{cases} \tag{7}$$

In particular, by expanding the DGF of h^{-1} in powers of $P(s)$ we count that

$$\begin{aligned} \frac{1}{1+P(s)} &= \sum_{n\geq 1} \frac{h^{-1}(n)}{n^s} = \sum_{k\geq 0} (-1)^k P(s)^k \\ &= \sum_{\substack{n\geq 1 \\ n=p_1^{\alpha_1} p_2^{\alpha_2} \times \cdots \times p_k^{\alpha_k}}} \frac{(-1)^{\alpha_1+\alpha_2+\cdots+\alpha_k}}{n^s} \times \binom{\alpha_1+\alpha_2+\cdots+\alpha_k}{\alpha_1, \alpha_2, \ldots, \alpha_k} = \sum_{\substack{n\geq 1 \\ n=p_1^{\alpha_1} p_2^{\alpha_2} \times \cdots \times p_k^{\alpha_k}}} \frac{\lambda(n)}{n^s} \times \binom{\Omega(n)}{\alpha_1, \alpha_2, \ldots, \alpha_k}. \end{aligned}$$

Since λ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$ for all divisors $d|n$ when $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever n is squarefree, so that we obtain the following results:

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1. \qquad \square$$

The conclusion of the proof of Proposition 2.1 implies the stronger result that

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d).$$

We have adopted the notation that for $n \geq 2$, $C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha||n} (\alpha!)^{-1}$, where the same function, $C_0(n)$, is taken to be one for $n := 1$ (see Section 3). We see that the scaled functions $f_1(n) := \frac{C_{\Omega(n)}(n)}{(\Omega(n))!}$ and $f_2(n) := \frac{\lambda(n) C_{\Omega(n)}(n)}{(\Omega(n))!}$ are multiplicative.

2.3 The distributions of $\omega(n)$ and $\Omega(n)$

The next theorems reproduced from [17, §7.4] characterize the relative scarcity of the distributions of $\omega(n)$ and $\Omega(n)$ for $n \leq x$ such that $\omega(n), \Omega(n) > \log\log x$. Since $\frac{1}{n} \times \sum_{k\leq n} \omega(k) = \log\log n + B_1$ and $\frac{1}{n} \times \sum_{k\leq n} \Omega(k) = \log\log n + B_2$ for $B_1 \approx 0.261497$ and $B_2 \approx 1.03465$ absolute constants in each case, these results imply a distinctively regular tendency of these additive arithmetic functions towards their respective average orders.

Theorem 2.2 (Upper bounds on exceptional values of $\Omega(n)$ for large n)**.** *For $x \geq 2$ and $r > 0$, let*

$$\begin{aligned} A(x,r) &:= \#\left\{n \leq x : \Omega(n) \leq r \log\log x\right\}, \\ B(x,r) &:= \#\left\{n \leq x : \Omega(n) \geq r \log\log x\right\}. \end{aligned}$$

If $0 < r \leq 1$ and $x \geq 2$, then

$$A(x,r) \ll \quad x(\log x)^{r-1-r\log r}, \text{ as } x \to \infty.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x,r) \ll_R x(\log x)^{r-1-r\log r}, \text{ as } x \to \infty.$$

Theorem 2.3 is a special case analog to the Erdős-Kac theorem stated for the normally distributed values of $\frac{\omega(n)-\log\log n}{\sqrt{\log\log n}}$ over $n \leq x$ as $x \to \infty$ [17, *cf.* Thm. 7.21] [13, *cf.* §1.7].

Theorem 2.3. *We have that as $x \to \infty$*

$$\#\left\{3 \leq n \leq x : \Omega(n) \leq \log\log n\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Theorem 2.4 (Montgomery and Vaughan)**.** *Recall that for integers $k \geq 1$ and $x \geq 2$ we have defined*

$$\widehat{\pi}_k(x) := \#\{2 \leq n \leq x : \Omega(n) = k\}.$$

For $0 < R < 2$ we have uniformly for all $1 \leq k \leq R\log\log x$ that

$$\widehat{\pi}_k(x) = \frac{x}{\log x} \times \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

where we define

$$\mathcal{G}(z) := \frac{1}{\Gamma(1+z)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1}\left(1 - \frac{1}{p}\right)^z, 0 \leq |z| < R.$$

Remark 2.5. We can extend the work in [17] on the distribution of $\Omega(n)$ to obtain corresponding analogs for the distribution of $\omega(n)$. For $0 < R < 2$ we have that as $x \to \infty$

$$\pi_k(x) = \frac{x}{\log x} \times \widetilde{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) \frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right), \tag{8}$$

uniformly for any $1 \leq k \leq R\log\log x$. The analogous function to express these bounds for $\omega(n)$ is defined by $\widetilde{\mathcal{G}}(z) := \widetilde{F}(1,z) \times \Gamma(1+z)^{-1}$ where we define

$$\widetilde{F}(s,z) := \prod_p \left(1 + \frac{z}{p^s - 1}\right)\left(1 - \frac{1}{p^s}\right)^z, \mathrm{Re}(s) > \frac{1}{2}; |z| \leq R < 2.$$

Let the functions

$$\begin{aligned} C(x,r) &:= \#\{n \leq x : \omega(n) \leq r\log\log x\}, \\ D(x,r) &:= \#\{n \leq x : \omega(n) \geq r\log\log x\}. \end{aligned}$$

Then we have upper bounds given by the following asymptotics as $x \to \infty$:

$$\begin{aligned} C(x,r) &\ll \quad x(\log x)^{r-1-r\log r}, \text{ uniformly for } 0 < r \leq 1, \\ D(x,r) &\ll_R x(\log x)^{r-1-r\log r}, \text{ uniformly for } 1 \leq r \leq R < 2. \end{aligned}$$

3 Auxiliary sequences related to the inverse function $g^{-1}(n)$

The computational data given as Table B in the appendix section is intended to provide clear insight into the significance of the few characteristic formulas for $g^{-1}(n)$ proved in this section. The table provides illustrative numerical data by examining the first cases of $1 \leq n \leq 500$ with *Mathematica* and *Sage* [26].

3.1 Definitions and properties of triangular component function sequences

We define the following sequence for integers $n \geq 1$ and $k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}\left(\frac{n}{d}\right), & \text{if } k \geq 1. \end{cases} \tag{9}$$

The Dirichlet inverse $f^{-1}(n)$ of any arithmetic function f such that $f(1) \neq 0$ is computed exactly by an $\Omega(n)$-fold convolution of f with itself. The motivation for considering the auxiliary sequence representing the k-fold Dirichlet convolution of $\omega(n)$ with itself follows from our definition of $g^{-1}(n) := (\omega + 1)^{-1}(n)$. We prove a few precise relations of the function $C_{\Omega(n)}(n)$ to the inverse sequence $g^{-1}(n)$ that result in the next subsections. Indeed, $h^{-1}(n) \equiv \lambda(n) C_{\Omega(n)}(n)$ is the same function given by (7) from Proposition 2.1.

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (9) inductively. By the same argument, we see that at fixed n, the function $C_k(n)$ is non-zero only possibly when $1 \leq k \leq \Omega(n)$ whenever $n \geq 2$. A sequence of signed semi-diagonals of the functions $C_k(n)$ begins as follows [27, A008480]:

$$\{\lambda(n) C_{\Omega(n)}(n)\}_{n \geq 1} = \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

We see by (7) that $C_{\Omega(n)}(n) \leq (\Omega(n))!$ for all $n \geq 1$ with equality precisely at the squarefree integers so that $(\Omega(n))! = (\omega(n))!$.

3.2 Formulas relating $C_{\Omega(n)}(n)$ and $g^{-1}(n)$

Lemma 3.1. *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first expand the recurrence relation for the Dirichlet inverse when $g^{-1}(1) = g(1)^{-1} = 1$ as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d) + 1) g^{-1}\left(\frac{n}{d}\right) \quad \Longrightarrow \quad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \tag{10}$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (10) in the form of $(g^{-1} * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m (C_m(-) * g^{-1})(n)$, so that

$$F_m(n) = -\begin{cases} (\omega * g^{-1})(n), & m = 1; \\ \sum\limits_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum\limits_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & 2 \leq m \leq \Omega(n); \\ 0, & \text{otherwise.} \end{cases}$$

When $m := \Omega(n)$, e.g., with the expansions in the previous equation taken to a maximal depth, we obtain the relation

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \tag{11}$$

The formula for $g^{-1}(n)$ then follows from (11) by Möbius inversion. $\square$

Corollary 3.2. *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{12}$$

Proof. By applying Lemma 3.1, Proposition 2.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and since at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$, Lemma 3.1 and Proposition 2.1 imply that

$$\begin{aligned}|g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d).\end{aligned}$$

We see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. □

Remark 3.3. Since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ in the notation from the proof of Proposition 2.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for all squarefree $n \geq 1$. We also have that whenever $n \geq 1$ is squarefree

$$|g^{-1}(n)| = \sum_{d|n} C_{\Omega(d)}(d).$$

Since all divisors of a squarefree integer are squarefree, a proof of part (B) of Proposition 1.6 follows by an elementary counting argument as an immediate consequence of the previous equation.

Remark 3.4. Lemma 3.1 shows that the summatory function of this sequence satisfies

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_{\Omega(d)}(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Equation (5) implies that

$$\lambda(d) C_{\Omega(d)}(d) = (g^{-1} * 1)(d) = (\chi_{\mathbb{P}} + \varepsilon)^{-1}(d).$$

We recover by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1. \tag{13}$$

The proof of Corollary 4.6 shows that

$$\sum_{n \leq x} |g^{-1}(n)| = \sum_{d \leq x} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{x}{d} \right\rfloor\right), x \geq 1,$$

where $Q(x) := \sum_{n \leq x} \mu^2(n)$ counts the number of squarefree $n \leq x$.

3.3 Combinatorial connections to the distribution of the primes

The combinatorial properties of $g^{-1}(n)$ are deeply tied to the distribution of the primes $p \leq n$ as $n \to \infty$. The magnitudes of and spacings between the primes $p \leq n$ certainly restricts the repeating of these distinct sequence values. We can see that the following is still clear about the relation of the weight functions $|g^{-1}(n)|$ to the distribution of the primes: The value of $|g^{-1}(n)|$ is entirely dependent only on the pattern of the exponents (viewed as multisets) of the distinct prime factors of $n \geq 2$, rather than on the prime factor weights themselves (*cf.* Observation 1.5). This property implies that $|g^{-1}(n)|$ has an inherently additive, rather than multiplicative, structure underneath the distribution of its distinct values over $n \leq x$.

Example 3.5. There is a natural extremal behavior of $|g^{-1}(n)|$ with respect to the distinct values of $\Omega(n)$ at squarefree integers and prime powers. For integers $k \geq 1$ we define the infinite sets $\overline{M}_k$ and $\underline{m}_k$ to correspond to the maximal (minimal) sets of positive integers such that

$$\overline{M}_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

$$\underline{m}_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+.$$

Any element of $\overline{M}_k$ is squarefree and any element of $\underline{m}_k$ is a prime power. Moreover, for any fixed $k \geq 1$ we have that for any $N_k \in \overline{M}_k$ and $n_k \in \underline{m}_k$

$$(-1)^k g^{-1}(N_k) = \sum_{j=0}^{k} \binom{k}{j} \times j!, \qquad \text{and} \qquad (-1)^k g^{-1}(n_k) = 2.,$$

where $\lambda(N_k) = \lambda(n_k) = (-1)^k$.

Remark 3.6. The formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 2.1 shows that we can express $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n. For $n \geq 2$ and $0 \leq k \leq \omega(n)$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z\nu_p(n)) = [z^k] \prod_{p^\alpha || n} (1 + \alpha z).$$

Then we can prove using (7) and (12) that the following formula holds:

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, n \geq 2.$$

The combinatorial formula for $h^{-1}(n) = \lambda(n)(\Omega(n))! \times \prod_{p^\alpha || n} (\alpha!)^{-1}$ suggests additional patterns and regularity in the contributions of the distinct sign weighted terms in the summands of $G^{-1}(x)$[1]. Sections 5.2 and 5.3 discuss limiting asymptotic properties and local cancellation in the formula for $M(x)$ from (13) that is expanded exactly through the auxiliary sums $G^{-1}(x)$.

[1] This sequence is also considered using a different motivation based on the DGFs $(1 \pm P(s))^{-1}$ in [8, §2].

4 The distributions of $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$ and their partial sums

We observed an intuition in the introduction that the relation of the unsigned auxiliary functions, $g^{-1}(n)$ and $C_{\Omega(n)}(n)$, to the canonically additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties illustrated in Table B. Each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdős-Kac theorem that provides a central limiting distribution for each of these functions over $n \leq x$ as $x \to \infty$ [7, 2, 24] (*cf.* [12]). In the remainder of this section, we use analytic methods in the spirit of [17, §7.4] to prove new properties that characterize the distributions of the auxiliary functions in analogous ways. The probabilistic ansatz given at the start of Section 4.3 is reminiscent of preliminaries behind the first proofs of the Erdős-Kac theorem. It is thus suggestive of deeper connections of $C_{\Omega(n)}(n)$, $|g^{-1}(n)|$, and classes of functions constructed (and enumerated) through similar procedures to strong additivity.

4.1 Analytic proofs extending bivariate DGF methods for additive functions

Theorem 4.1. *Let the bivariate DGF $\widehat{F}(s,z)$ be defined in terms of the prime zeta function, $P(s)$, for $\operatorname{Re}(s) > 1$ and $|z| < |P(s)|^{-1}$ by*

$$\widehat{F}(s,z) := \frac{1}{1+P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

The partial sums of the coefficients of $\widehat{F}(s,z)\zeta(s)^z$ are given by

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

We have for all sufficiently large x and any $|z| < P(2)^{-1} \approx 2.21118$ that

$$\widehat{A}_z(x) = \frac{x\widehat{F}(2,z)}{\Gamma(z)} (\log x)^{z-1} + O_z\left(x(\log x)^{\operatorname{Re}(z)-2}\right).$$

Proof. It follows from (7) that we can generate exponentially scaled forms of the function $C_{\Omega(n)}(n)$ by product identity of the following form:

$$\sum_{n\geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! p^{rs}}\right)^{-1} = \exp\left(-zP(s)\right), \text{ for } \operatorname{Re}(s) > 1 \wedge \operatorname{Re}(P(s)z) > -1.$$

This Euler type product expansion is similar in construction to the parameterized bivariate DGFs in [17, §7.4]. By computing a termwise Laplace transform applied to the right-hand-side of the above equation, we obtain that

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n)(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp\left(-tzP(s)\right) dt = \frac{1}{1+P(s)z}, \text{ for } \operatorname{Re}(s) > 1 \wedge \operatorname{Re}(P(s)z) > -1.$$

It follows from the Euler product representation of $\zeta(s)$ which holds for any $\operatorname{Re}(s) > 1$ that

$$\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \widehat{F}(s,z)\zeta(s)^z, \text{ for } \operatorname{Re}(s) > 1 \wedge |z| < |P(s)|^{-1}.$$

The bivariate DGF $\widehat{F}(s,z)$ is an analytic function of s for all $\operatorname{Re}(s) > 1$ whenever the parameter $|z| < |P(s)|^{-1}$. If the sequence $\{b_z(n)\}_{n\geq 1}$ indexes the coefficients in the DGF expansion of $\widehat{F}(s,z)\zeta(s)^z$, then the series

$$\left|\sum_{n \geq 1} \frac{b_z(n)(\log n)^{2R+1}}{n^s}\right| < +\infty.$$

Moreover, the series in the last equation is uniformly bounded for all $\mathrm{Re}(s) \geq 2$ and $|z| \leq R < |P(s)|^{-1}$. This fact follows by repeated termwise differentiation of the series for the original function $\lceil 2R+1 \rceil$ times with respect to s.

For fixed $0 < |z| < 2$, let the sequence $\{d_z(n)\}_{n\geq 1}$ be generated as the coefficients of the DGF

$$\zeta(s)^z = \sum_{n\geq 1} \frac{d_z(n)}{n^s}, \text{ for } \mathrm{Re}(s) > 1.$$

The corresponding summatory function of $d_z(n)$ is defined by $D_z(x) := \sum_{n\leq x} d_z(n)$. The theorem proved in [17, Thm. 7.17; §7.4] shows that for any $0 < |z| < 2$ and all integers $x \geq 2$ we have

$$D_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O_z\left(x(\log x)^{\mathrm{Re}(z)-2}\right).$$

Set $b_z(n) := (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$, define the convolution $a_z(n) := \sum_{d|n} b_z(d) d_z\left(\frac{n}{d}\right)$, and take its partial sums to be $A_z(x) := \sum_{n\leq x} a_z(n)$. Then we have that

$$\begin{aligned} A_z(x) &= \sum_{m\leq \frac{x}{2}} b_z(m) D_z\left(\frac{x}{m}\right) + \sum_{\frac{x}{2}<m\leq x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \times \sum_{m\leq\frac{x}{2}} \frac{b_z(m)}{m} \log\left(\frac{x}{m}\right)^{z-1} + O\left(\sum_{m\leq x} \frac{x|b_z(m)|}{m} \times \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2}\right). \end{aligned} \tag{14}$$

We can sum the coefficients $\frac{b_z(m)}{m}$ for integers $m \leq u$ when u is taken sufficiently large as

$$\sum_{m\leq u} \frac{b_z(m)}{m^2} \times m = \left(\widehat{F}(2,z) + O_z\left(u^{-2}\right)\right) u - \int_1^u \left(\widehat{F}(2,z) + O_z\left(t^{-2}\right)\right) dt = \widehat{F}(2,z) + O_z\left(u^{-1}\right).$$

Suppose that $0 < |z| \leq R < P(2)^{-1}$. For large x, the error term in (14) satisfies

$$\begin{aligned} \sum_{m\leq x} \frac{x|b_z(m)|}{m} \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2} &\ll x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m\leq\sqrt{x}} \frac{|b_z(m)|}{m} \\ &\quad + x(\log x)^{-(R+2)} \times \sum_{m>\sqrt{x}} \frac{|b_z(m)|}{m} (\log m)^{2R} \\ &= O_z\left(x(\log x)^{\mathrm{Re}(z)-2}\right), \end{aligned}$$

whenever $0 < |z| \leq R$. When $m \leq \sqrt{x}$ we have

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\mathrm{Re}(z)-2}\right).$$

A related upper bound is obtained for the left-hand-side of the previous equation when $\sqrt{x} < m < x$ and $0 < |z| < R$. The combined sum over the interval $m \leq \frac{x}{2}$ corresponds to bounding the sum components when we take $0 < |z| \leq R$ by

$$\begin{aligned} \sum_{m\leq\frac{x}{2}} b_z(m) D_z\left(\frac{x}{m}\right) &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \times \sum_{m\leq\frac{x}{2}} \frac{b_z(m)}{m} \\ &\quad + O_R\left(x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m\leq\sqrt{x}} \frac{|b_z(m)|\log m}{m} + x(\log x)^{R-1} \times \sum_{m>\sqrt{x}} \frac{|b_z(m)|}{m}\right) \end{aligned}$$

$$= \frac{x\widehat{F}(2,z)}{\Gamma(z)}(\log x)^{z-1} + O_R\left(x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m\geq 1} \frac{b_z(m)(\log m)^{2R+1}}{m^2}\right)$$
$$= \frac{x\widehat{F}(2,z)}{\Gamma(z)}(\log x)^{z-1} + O_R\left(x(\log x)^{\mathrm{Re}(z)-2}\right). \qquad \square$$

Theorem 4.2. *For all large $x \geq 3$ and integers $k \geq 1$, let*

$$\widehat{C}_{k,*}(x) := \sum_{\substack{n\leq x\\ \Omega(n)=k}} (-1)^{\omega(n)} C_k(n)$$

Let $\widehat{G}(z) := \widehat{F}(2,z) \times \Gamma(1+z)^{-1}$ when $0 \leq |z| < P(2)^{-1}$ where $\widehat{F}(s,z)$ is defined as in Theorem 4.1. As $x \to \infty$, we have uniformly for any $1 \leq k \leq 2\log\log x$ that

$$\widehat{C}_{k,*}(x) = -\widehat{G}\left(\frac{k-1}{\log\log x}\right)\frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 + O\left(\frac{k}{(\log\log x)^2}\right)\right).$$

Proof. When $k = 1$, we have that $\Omega(n) = \omega(n)$ for all $n \leq x$ such that $\Omega(n) = k$. The positive integers n that satisfy this requirement are precisely the primes $p \leq x$. Hence, the formula is satisfied as

$$\sum_{p\leq x} (-1)^{\omega(p)} C_1(p) = -\sum_{p\leq x} 1 = -\frac{x}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Since $O\left((\log x)^{-1}\right) = O\left((\log\log x)^{-2}\right)$ as $x \to \infty$, we obtain the required error term bound at $k = 1$. For $2 \leq k \leq 2\log\log x$, we will apply the error estimate from Theorem 4.1 with $r := \frac{k-1}{\log\log x}$ in the formula

$$\widehat{C}_{k,*}(x) = \frac{(-1)^{k+1}}{2\pi\imath} \times \int_{|v|=r} \frac{\widehat{A}_{-v}(x)}{v^{k+1}} dv.$$

Since $(\log x)^{\frac{1}{\log\log x}} = e$, the error in the formula contributes terms that are bounded by

$$\left|x(\log x)^{-(\mathrm{Re}(v)+2)} v^{-(k+1)}\right| \ll \left|x(\log x)^{-(r+2)} r^{-(k+1)}\right| \ll \frac{x}{(\log x)^{2-\frac{k-1}{\log\log x}}} \cdot \frac{(\log\log x)^k}{(k-1)^k}$$
$$\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^k}{(k-1)^{\frac{1}{2}}(k-1)!} \ll \frac{x}{\log x} \cdot \frac{k(\log\log x)^{k-5}}{(k-1)!}, \text{ as } x \to \infty.$$

We next find the main term for the coefficients of the following contour integral when $r \in [0, z_{\max}] \subseteq \left[0, P(2)^{-1}\right)$:

$$\widehat{C}_{k,*}(x) \sim \frac{(-1)^k x}{\log x} \times \int_{|v|=r} \frac{(\log x)^{-v}\zeta(2)^v}{\Gamma(1-v)v^k(1-P(2)v)} dv. \tag{15}$$

The main term of $\widehat{C}_{k,*}(x)$ is given by $-\frac{x}{\log x} \times I_k(r,x)$, where we define

$$I_k(r,x) = \frac{1}{2\pi\imath} \times \int_{|v|=r} \frac{\widehat{G}(v)(\log x)^v}{v^k} dv$$
$$=: I_{1,k}(r,x) + I_{2,k}(r,x).$$

Taking $r = \frac{k-1}{\log\log x}$, the first of the component integrals is defined to be

$$I_{1,k}(r,x) := \frac{\widehat{G}(r)}{2\pi\imath} \times \int_{|v|=r} \frac{(\log x)^v}{v^k} dv = \widehat{G}(r) \times \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

The second integral, $I_{2,k}(r,x)$, corresponds to an error term in our approximation. This component function is defined by

$$I_{2,k}(r,x) := \frac{1}{2\pi\imath} \times \int_{|v|=r} \left(\widehat{G}(v) - \widehat{G}(r)\right) \frac{(\log x)^v}{v^k} dv.$$

Integrating by parts shows that [17, *cf.* Thm. 7.19; §7.4]

$$\frac{(r-v)}{2\pi\imath} \times \int_{|v|=r} (\log x)^v v^{-k} dv = 0,$$

so that integrating by parts once again we have

$$I_{2,k}(r,x) := \frac{1}{2\pi\imath} \times \int_{|v|=r} \left(\widehat{G}(v) - \widehat{G}(r) - \widehat{G}'(r)(v-r)\right) (\log x)^v v^{-k} dv.$$

We find that

$$\widehat{G}(v) - \widehat{G}(r) - \widehat{G}'(r)(v-r) = \int_r^v (v-w)\widehat{G}''(w)dw \ll |v-r|^2.$$

With the parameterization $v = re^{2\pi\imath\theta}$ for $\theta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and selecting $r := \frac{k-1}{\log\log x}$, we obtain

$$|I_{2,k}(r,x)| \ll r^{3-k} \times \int_{-\frac{1}{2}}^{\frac{1}{2}} (\sin \pi\theta)^2 e^{(k-1)\cos(2\pi\theta)} d\theta.$$

Since $|\sin x| \leq |x|$ for all $|x| < 1$ and $\cos(2\pi\theta) \leq 1 - 8\theta^2$ if $-\frac{1}{2} \leq \theta \leq \frac{1}{2}$, we arrive at the next bounds by again taking setting $r = \frac{k-1}{\log\log x}$ at any $1 \leq k \leq 2\log\log x$.

$$\begin{aligned} |I_{2,k}(r,x)| &\ll r^{3-k} e^{k-1} \times \int_0^\infty \theta^2 e^{-8(k-1)\theta^2} d\theta \\ &\ll \frac{r^{3-k} e^{k-1}}{(k-1)^{\frac{3}{2}}} = \frac{(\log\log x)^{k-3} e^{k-1}}{(k-1)^{k-\frac{3}{2}}} \ll \frac{k(\log\log x)^{k-3}}{(k-1)!}. \end{aligned}$$

Finally, whenever $1 \leq k \leq 2\log\log x$ we have

$$1 = \widehat{G}(0) \geq \widehat{G}\left(\frac{k-1}{\log\log x}\right) = \frac{1}{\Gamma\left(1 + \frac{k-1}{\log\log x}\right)} \times \frac{\zeta(2)^{\frac{1-k}{\log\log x}}}{\left(1 + \frac{P(2)(k-1)}{\log\log x}\right)} \geq \widehat{G}(2) \approx 0.097027.$$

In particular, the function $\widehat{G}\left(\frac{k-1}{\log\log x}\right) \gg 1$ for all $1 \leq k \leq 2\log\log x$. This implies the result of the theorem. □

Lemma 4.3. *As $x \to \infty$, there is an absolute constant $A_0 > 0$ such that*

$$\sum_{n \leq x} (-1)^{\omega(n)} = \frac{(-1)^{\lfloor \log\log x \rfloor} x}{A_0 \sqrt{2\pi \log\log x}} + O\left(\frac{x}{\log\log x}\right).$$

Proof. An adaptation of the proof of Lemma A.3 from the appendix provides that for any $a \in (1, 1.76322)$

$$\begin{aligned} S_a(x) &:= \frac{x}{\log x} \times \left| \sum_{k=1}^{\lfloor a\log\log x \rfloor} \frac{(-1)^k (\log\log x)^{k-1}}{(k-1)!} \right| \\ &= \frac{\sqrt{a}x}{\sqrt{2\pi}(a+1)a^{\{a\log\log x\}}} \times \frac{(\log x)^{a-1-a\log a}}{\sqrt{\log\log x}} \left(1 + O\left(\frac{1}{\log\log x}\right)\right). \end{aligned} \tag{16}$$

Here, we define $\{x\} = x - \lfloor x \rfloor \in [0,1)$ to be the *fractional part* of x. Suppose that we take $a := \frac{3}{2}$ so that $a - 1 - a\log a \approx -0.108198$. We define and expand the next partial sums as

$$L_{**}(x) := \sum_{n \leq x} (-1)^{\omega(n)} = \sum_{k \leq \log\log x} 2(-1)^k \pi_k(x) + S_{\frac{3}{2}}(x) + O\left(\#\left\{n \leq x : \omega(n) \geq \frac{3}{2}\log\log x\right\}\right).$$

We can show that for any $1 < k \leq \log\log x$, the function $\widetilde{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right)$ from Remark 2.5 is decreasing in k with $\widetilde{\mathcal{G}}(0) = 1$ and satisfies

$$\widetilde{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) \geq \widetilde{\mathcal{G}}\left(1 - \frac{1}{\log\log x}\right) \geq \widetilde{\mathcal{G}}(1) = 1.$$

We apply the uniform asymptotics for $\pi_k(x)$ that hold as $x \to \infty$ when $1 \leq k \leq R\log\log x$ for $1 \leq R < 2$. We then see by Lemma A.3 and (16) that at sufficiently large x there is some absolute constant $A_0 > 0$ such that

$$L_{**}(x) = \frac{(-1)^{\lfloor \log\log x \rfloor} x}{A_0\sqrt{2\pi\log\log x}} + O\left(E_\omega(x) + \frac{x}{(\log x)^{0.108198}\sqrt{\log\log x}} + \#\left\{n \leq x : \omega(x) \geq \frac{3}{2}\log\log x\right\}\right).$$

The error term in the previous equation is bounded by the next sum as $x \to \infty$. In particular, the following estimate is obtained from Stirling's formula, (27a) and (27c) from the appendix:

$$\begin{aligned} E_\omega(x) &\ll \frac{x}{\log x} \times \sum_{1 \leq k \leq \log\log x} \frac{(\log\log x)^{k-2}}{(k-1)!} \\ &= \frac{x\Gamma(\log\log x, \log\log x)}{\Gamma(\log\log x + 1)} \sim \frac{x}{2\log\log x}\left(1 + O\left(\frac{1}{\sqrt{\log\log x}}\right)\right). \end{aligned}$$

By an application of the second set of results in Remark 2.5, we see that

$$\#\left\{n \leq x : \omega(x) \geq \frac{3}{2}\log\log x\right\} \ll \frac{x}{(\log x)^{0.108198}}.$$

□

Corollary 4.4. *We have at all sufficiently large x uniformly for $1 \leq k \leq \frac{3}{2}\log\log x$ that*

$$\widehat{C}_k(x) := \sum_{\substack{n \leq x \\ \Omega(n) = k}} C_{\Omega(n)}(n) = A_0\sqrt{2\pi}x\widehat{G}\left(\frac{k-1}{\log\log x}\right)\frac{(\log\log x)^{k-\frac{1}{2}}}{(k-1)!}\left(1 + O\left(\frac{1}{\log\log x}\right)\right).$$

Proof. Suppose that $h(t)$ and $\sum_{n \leq t} \lambda_*(n)$ are piecewise smooth and differentiable functions on $\mathbb{R}^+$. The next integral formulas result by Abel summation and integration by parts.

$$\sum_{n \leq x} \lambda_*(n)h(n) = \left(\sum_{n \leq x} \lambda_*(n)\right)h(x) - \int_1^x \left(\sum_{n \leq t} \lambda_*(n)\right)h'(t)dt \tag{17a}$$

$$\sim \int_1^x \frac{d}{dt}\left[\sum_{n \leq t} \lambda_*(n)\right]h(t)dt \tag{17b}$$

We transform our previous results for the partial sums of $(-1)^{\omega(n)}C_{\Omega(n)}(n)$ such that $\Omega(n) = k$ to approximate the corresponding partial sums of only $C_{\Omega(n)}(n)$. In particular, since $1 \leq k \leq \frac{3}{2}\log\log x$, we have that

$$\widehat{C}_{k,*}(x) = \sum_{\substack{n \leq x \\ \Omega(n) = k}} (-1)^{\omega(n)}C_{\Omega(n)}(n) = \sum_{n \leq x} (-1)^{\omega(n)}\left[\omega(n) \leq \frac{3}{2}\log\log x\right]_\delta \times C_{\Omega(n)}(n)\left[\Omega(n) = k\right]_\delta.$$

We have by the proof of Lemma 4.3 that as $t \to \infty$

$$L_*(t) := \sum_{\substack{n \leq t \\ \omega(n) \leq \frac{3}{2}\log\log t}} (-1)^{\omega(n)} = \frac{(-1)^{\lfloor \log\log t \rfloor} t}{A_0\sqrt{2\pi \log\log t}} \left(1 + O\left(\frac{1}{\sqrt{\log\log t}}\right)\right). \tag{18}$$

Except for t within a subset of $(0, \infty)$ of measure zero on which $L_*(t)$ changes sign, the main term of the derivative of this summatory function is given almost everywhere by

$$L_*'(t) \sim \frac{(-1)^{\lfloor \log\log t \rfloor}}{A_0\sqrt{2\pi \log\log t}}.$$

We apply the formula from (17b), to deduce that as $x \to \infty$ with $1 \leq k \leq \frac{3}{2}\log\log x$

$$\begin{aligned}
\widehat{C}_{k,*}(x) &\sim \sum_{j=1}^{\log\log x - 1} \frac{2 \cdot (-1)^{j+1}}{A_0\sqrt{2\pi}} \times \int_{e^{e^j}}^{e^{e^{j+1}}} \frac{C_{\Omega(t)}(t)\left[\Omega(t) = k\right]_{\delta}}{\sqrt{\log\log t}} dt \\
&\sim -\int_1^{\frac{\log\log x}{2}} \int_{e^{e^{2s-1}}}^{e^{e^{2s}}} \frac{2C_{\Omega(t)}(t)\left[\Omega(t) = k\right]_{\delta}}{A_0\sqrt{2\pi \log\log t}} dt ds + \frac{1}{A_0\sqrt{2\pi}} \times \int_{e^e}^{x} \frac{C_{\Omega(t)}(t)\left[\Omega(t) = k\right]_{\delta}}{\sqrt{\log\log t}} dt.
\end{aligned}$$

For large x, $(\log\log t)^{-\frac{1}{2}}$ is continuous and monotone decreasing on $\left[x^{e^{-1}}, x\right]$ with

$$\frac{1}{\sqrt{\log\log x}} - \frac{1}{\sqrt{\log\log\left(x^{e^{-1}}\right)}} = O\left(\frac{1}{(\log x)\sqrt{\log\log x}}\right),$$

Hence, we have that

$$-A_0\sqrt{2\pi}x(\log x)\sqrt{\log\log x}\widehat{C}_{k,*}'(x) = \left(\widehat{C}_k(x) - \widehat{C}_k\left(x^{e^{-1}}\right)\right)(1 + o(1)) - x(\log x)\widehat{C}_k'(x). \tag{19}$$

For $1 \leq k < \frac{3}{2}\log\log x$, we expect contributions from the squarefree integers $n \leq x$ such that $\omega(n) = \Omega(n) = k$ to be on the order of

$$\widehat{C}_k'(x) \asymp \frac{6}{\pi^2} \times k! \times \widehat{\pi}_k(x) \sim \frac{6xk}{\pi^2} \times \frac{(\log\log x)^{k-1}}{\log x}.$$

We conclude that $\widehat{C}_k\left(x^{e^{-1}}\right) = o\left(\widehat{C}_k(x)\right)$. Then equation (19) becomes an ordinary differential equation for $\widehat{C}_k(x)$ under this observation. Its solution has the form

$$\widehat{C}_k(x) = A_0\sqrt{2\pi}(\log x) \times \int_3^x \frac{\sqrt{\log\log t}}{\log t}\widehat{C}_{k,*}'(t)dt + O(\log x).$$

When we integrate by parts and apply the result from Theorem 4.2, we find that

$$\begin{aligned}
\widehat{C}_k(x) &= \frac{\sqrt{\log\log x}}{\log x}\widehat{C}_{k,*}(x) + O\left(x \times \int_3^x \frac{\sqrt{\log\log t}\widehat{C}_{k,*}(t)}{t^2(\log t)^2} dt\right) \\
&= \frac{\sqrt{\log\log x}}{\log x}\widehat{C}_{k,*}(x) + O\left(\frac{x}{2^k} \times \Gamma\left(k + \frac{1}{2}, 2\log\log x\right)\right).
\end{aligned}$$

Finally, whenever we assume that $1 \leq k \leq \frac{3}{2}\log\log x$ so that $\lambda > 1$ in Proposition A.2 (*cf.* Facts A.1 for k of substantially lesser order than this upper bound), Theorem 4.2 implies the conclusion of our corollary. □

4.2 Average orders of the unsigned sequences

Proposition 4.5. *There is an absolute constant $B_0 > 0$ such that as $n \to \infty$*

$$\frac{1}{n} \times \sum_{k \leq n} C_{\Omega(k)}(k) = B_0 (\log n) \sqrt{\log\log n} \left(1 + O\left(\frac{1}{\log\log n}\right)\right).$$

Proof. By Corollary 4.4 and Proposition A.2 with $\lambda = \frac{1}{2}$, we have that

$$\begin{aligned}
\sum_{k=1}^{\frac{3}{2}\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) &\asymp \sum_{k=1}^{\frac{3}{2}\log\log x} \frac{x(\log\log x)^{k-\frac{1}{2}}}{(k-1)!} \left(1 + O\left(\frac{1}{\log\log x}\right)\right) \\
&= \frac{x(\log x)\sqrt{\log\log x}\,\Gamma\left(\frac{3}{2}\log\log x, \log\log x\right)}{\Gamma\left(\frac{3}{2}\log\log x\right)} \left(1 + O\left(\frac{1}{\log\log x}\right)\right) \\
&= \frac{4x(\log x)}{\sqrt{2\pi \log\log x}} \left(1 + O\left(\frac{1}{\log\log x}\right)\right).
\end{aligned}$$

For real $0 \leq z \leq 2$, the function $\widehat{G}(z)$ is piecewise monotone in z with $\widehat{G}(0) = 1$ and $\widehat{G}(2) \approx 0.303964$. Then we see that there is an absolute constant $B_0 > 0$ such that

$$\frac{1}{x} \times \sum_{k=1}^{\frac{3}{2}\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) = B_0 (\log x)\sqrt{\log\log x} \left(1 + O\left(\frac{1}{\log\log x}\right)\right).$$

We claim that

$$\begin{aligned}
\frac{1}{x} \times \sum_{n \leq x} C_{\Omega(n)}(n) &= \frac{1}{x} \times \sum_{k \geq 1} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \\
&= \frac{1}{x} \times \sum_{k=1}^{\frac{3}{2}\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n)(1 + o(1)), \text{ as } x \to \infty.
\end{aligned}$$

To prove the claim it suffices to show that

$$\frac{1}{x} \times \sum_{\substack{n \leq x \\ \Omega(n) \geq \frac{3}{2}\log\log x}} C_{\Omega(n)}(n) = o\left(\frac{\log x}{\log\log x}\right). \tag{20}$$

We proved in Theorem 4.1 that for all sufficiently large x and $|z| < P(2)^{-1}$

$$\sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)} = \frac{x\widehat{F}(2,z)}{\Gamma(z)} (\log x)^{z-1} + O\left(x(\log x)^{\operatorname{Re}(z)-2}\right).$$

By Lemma 4.3, we have that the summatory function

$$\sum_{n \leq x} (-1)^{\omega(n)} = \frac{(-1)^{\lfloor \log\log x \rfloor} x}{A_0\sqrt{2\pi \log\log x}} \left(1 + O\left(\frac{1}{\sqrt{\log\log x}}\right)\right),$$

where $\frac{d}{dx}\left[\frac{x}{\sqrt{\log\log x}}\right] = \frac{1}{\sqrt{\log\log x}} + o(1)$. We can argue as in the proof of Corollary 4.4 that whenever $0 < |z| < P(2)^{-1}$ and x is sufficiently large we have

$$\sum_{n \leq x} C_{\Omega(n)}(n) z^{\Omega(n)} \ll \frac{\widehat{F}(2,z) x(\log x)\sqrt{\log\log x}}{\Gamma(z)} \times \frac{\partial}{\partial x}\left[x(\log x)^{z-1}\right]$$

$$\ll \frac{\widehat{F}(2,z)x\sqrt{\log\log x}}{\Gamma(z)}(\log x)^z. \tag{21}$$

For large x and any fixed $0 < r < P(2)^{-1}$, we define

$$\widehat{B}(x,r) := \sum_{\substack{n \le x \\ \Omega(n) \ge r\log\log x}} C_{\Omega(n)}(n).$$

We adapt the proof from the reference [17, *cf.* Thm. 7.20; §7.4] by applying (21) when $1 \le r < P(2)^{-1}$. Since $r\widehat{F}(2,r) = \frac{r\zeta(2)^{-r}}{1+P(2)r} \ll 1$ for $r \in [1, P(2)^{-1})$, and similarly since we have that $\frac{1}{\Gamma(1+r)} \gg 1$ for r within the same range, we find that

$$x\sqrt{\log\log x}(\log x)^r \gg \sum_{\substack{n \le x \\ \Omega(n) \ge r\log\log x}} C_{\Omega(n)}(n) r^{\Omega(n)} \gg \sum_{\substack{n \le x \\ \Omega(n) \ge r\log\log x}} C_{\Omega(n)}(n) r^{r\log\log x}.$$

This implies that for $r := \frac{3}{2}$ we have

$$\widehat{B}(x,r) \ll x(\log x)^{r - r\log r}\sqrt{\log\log x} = O\left(x(\log x)^{0.891802}\sqrt{\log\log x}\right) \tag{22}$$

We evaluate the limiting asymptotics of the sum

$$S_2(x) := \frac{1}{x} \times \sum_{k \ge \frac{3}{2}\log\log x} \sum_{\substack{n \le x \\ \Omega(n) = k}} C_{\Omega(n)}(n) \ll \frac{1}{x} \times \widehat{B}(x,2) = O\left((\log x)^{0.891802}\sqrt{\log\log x}\right), \text{ as } x \to \infty.$$

This implies that (20) holds. ☐

Corollary 4.6. *We have that as $n \to \infty$*

$$\frac{1}{n} \times \sum_{k \le n} |g^{-1}(k)| = \frac{6B_0(\log n)^2\sqrt{\log\log n}}{\pi^2}\left(1 + O\left(\frac{1}{\log\log n}\right)\right).$$

Proof. As $|z| \to \infty$, the *imaginary error function*, $\operatorname{erfi}(z)$, has the following asymptotic expansion [23, §7.12]:

$$\operatorname{erfi}(z) := \frac{2}{\sqrt{\pi}\imath} \times \int_0^{\imath z} e^{t^2} dt = \frac{e^{z^2}}{\sqrt{\pi}}\left(\frac{1}{z} + \frac{1}{2z^3} + \frac{3}{4z^5} + \frac{15}{8z^7} + O\left(\frac{1}{z^9}\right)\right). \tag{23}$$

We use the formula from Proposition 4.5 to sum the average order of $C_{\Omega(n)}(n)$. The proposition and error terms obtained from (23) imply that for all sufficiently large $t \to \infty$

$$\begin{aligned}\int \frac{\sum_{n \le t} C_{\Omega(n)}(n)}{t^2} dt &= B_0(\log t)^2\sqrt{\log\log t} - \frac{1}{4}\sqrt{\frac{\pi}{2}}\operatorname{erfi}\left(\sqrt{2\log\log t}\right) \\ &= B_0(\log t)^2\sqrt{\log\log t}\left(1 + O\left(\frac{1}{\log\log t}\right)\right).\end{aligned}$$

The summatory function that counts the number of *squarefree* integers $n \le x$ satisfies [9, §18.6] [27, A013928]

$$Q(x) = \sum_{n \le x} \mu^2(n) = \frac{6x}{\pi^2} + O\left(\sqrt{x}\right), \text{ as } x \to \infty.$$

Therefore, summing over the formula from (12) in Section 3.2, we find that

$$\begin{aligned}\frac{1}{n} \times \sum_{k \le n} |g^{-1}(k)| &= \frac{1}{n} \times \sum_{d \le n} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{n}{d} \right\rfloor\right) \\ &\sim \sum_{d \le n} C_{\Omega(d)}(d)\left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dn}}\right)\right] \\ &= \frac{6}{\pi^2}\left[\frac{1}{n} \times \sum_{k \le n} C_{\Omega(k)}(k) + \sum_{d < n}\sum_{k \le d} \frac{C_{\Omega(k)}(k)}{d^2}\right] + O(1).\end{aligned}$$

☐

4.3 Erdős-Kac theorem analogs for the distributions of the unsigned functions

We show in the proof of Theorem 4.8 that for $1 \leq k \leq \frac{3}{2}\log\log x$

$$\frac{1}{x} \times \sum_{\substack{n \leq x \\ \Omega(n)=k}} \frac{C_{\Omega(n)}(n)}{(\log n)\sqrt{\log\log n}} \asymp \frac{(\log\log x)^{k-1}}{(\log x)(k-1)!}\left(1 + O\left(\frac{1}{\log\log x}\right)\right). \tag{24}$$

The non-centrally normal tending densities resulting from the right-hand-side of the previous equation summed over k follow from the analytic arguments given in [17, Thm. 7.21; §7.4]. Nonetheless, showing that the distribution of $\frac{C_{\Omega(n)}(n)}{(\log n)\sqrt{\log\log n}}$ over $n \leq x$ as $x \to \infty$ has the same non-centrally normal CDF does not follow from (24) in the same manner as the corresponding distribution obtained in the reference. We need a deeper assumption to prove this result. Namely, we need the next probabilistic ansatz to prove Theorem 4.8.

Ansatz 4.7. We require the assumption that the functions

$$X_{n,k} := \frac{C_{\Omega(n)}(n)}{(\log n)\sqrt{\log\log n}},$$

defined for distinct $n \leq x$ such that $\Omega(n) = k$ when $1 \leq k \leq \frac{3}{2}\log\log x$ can be viewed as independent random variables (*cf.* [2]). The reasoning for this assumption on the independence of $X_{n_1,k}, X_{n_2,k}$ whenever $\Omega(n_1) = \Omega(n_2) = k$ is explained using the notation for the asymptotic densities defined in [17, §2.4] as

$$N_m(x) := \#\{n \leq x : \Omega(n) - \omega(n) = m\} = d_m x + O\left(\left(\frac{3}{4}\right)^k \sqrt{x}(\log x)^{\frac{4}{3}}\right), m \geq 0,$$

where

$$\sum_{k\geq 0} d_k z^k = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p-z}\right),$$

and such that the possible d_m sum to $\sum_{m\geq 0} d_m \sim 1$ as $x \to \infty$. For $1 \leq k \leq \frac{3}{2}\log\log x$ the total sum of $n \leq x$ such that $\Omega(n) = k$ over all possible exponent patterns that contribute to the distinct values of $C_{\Omega(n)}(n)$ has main term $(d_0 + d_1 + \cdots + d_{k-1}) \times \widehat{\pi}_k(x)$ for large x. Since $\frac{(p\widehat{\pi}_k(x)-1)}{\widehat{\pi}_k(x)} = p + o(1)$, when there is overlap in the values r_1, r_2 assumed by $X_{n_1,k}, X_{n_2,k}$ for fixed k, we still see that these variables are (approximately) independent as $x \to \infty$ by computing

$$\mathbb{P}(X_{n_1,k} = r_1 \mid X_{n_2,k} = r_2) = \begin{cases} \mathbb{P}(X_{n_1,k} = r_1), & r_1 \neq r_2; \\ \mathbb{P}(X_{n_1,k} = r_1) + O\left(\frac{(\log x)\sqrt{\log\log x}}{x}\right), & r_1 = r_2, \end{cases}$$

and vice versa.

Theorem 4.8. *For sufficiently large x, let the mean and variance parameters be defined by*

$$\mu_x(C) := \log\log x - \log\left(\sqrt{2\pi}A_0\widehat{G}(1)\right), \quad \text{and} \quad \sigma_x(C) := \sqrt{\log\log x}.$$

We have that

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : \frac{\frac{C_{\Omega(n)}(n)}{(\log n)\sqrt{\log\log n}} - \mu_x(C)}{\sigma_x(C)} \leq z\right\} = \Phi(z) + o(1), \text{ as } x \to \infty.$$

Proof. We will provide a rigorous outline to prove the theorem under the assumption of the ansatz. The complete remaining details behind the rest of the proof are left to the reader to verify. For $1 \leq k \leq \frac{3}{2}\log\log x$, let

$$\widehat{\mu}_k(x) := \frac{1}{x} \times \sum_{\substack{n \leq x \\ \Omega(n)=k}} \frac{C_{\Omega(n)}(n)}{(\log n)\sqrt{\log\log n}}.$$

Using integration by parts applied to Corollary 4.4, we have uniformly for any $1 \leq k \leq \frac{3}{2}\log\log x$ that

$$\begin{aligned} x \cdot \widehat{\mu}_k(x) &= \frac{\widehat{C}_k(x)}{(\log x)\sqrt{\log\log x}} + O\left(\int_3^x \frac{dt}{(\log t)(\log\log t)}\right) \\ &= \frac{\widehat{C}_k(x)}{(\log x)\sqrt{\log\log x}} + O\left(\frac{x}{(\log x)^2\sqrt{\log\log x}}\right) \\ &= \frac{A_0\sqrt{2\pi}x}{\log x} \times \widehat{G}\left(\frac{k-1}{\log\log x}\right)\frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 + O\left(\frac{1}{\log\log x}\right)\right), \text{ as } x \to \infty. \end{aligned} \tag{25}$$

For $1 \leq k \leq \frac{3}{2}\log\log x$, let

$$\sigma_k^2(x) := \frac{1}{x} \times \sum_{\substack{n \leq x \\ \Omega(n)=k}} \frac{C_{\Omega(n)}(n)^2}{(\log n)^2(\log\log n)}.$$

We then define the following variance parameters for large x:

$$s_x^2 := \sum_{n \leq x} \sigma_{\Omega(n)}^2(n).$$

We can show that the sequence of random variables $\{X_{n,\Omega(n)}\}_{n\geq 1}$ satisfies *Lindeberg's condition*, i.e., for all fixed $\epsilon > 0$

$$\lim_{x\to\infty} \frac{1}{s_x^2} \times \sum_{n \leq x}\left(\frac{1}{n} \times \sum_{m \leq n}(X_{m,\Omega(m)} - \widehat{\mu}_{\Omega(m)}(m))^2 \mathbb{1}_{\{|X_{m,\Omega(m)} - \widehat{\mu}_{\Omega(m)}(m)| > \epsilon s_n\}}(m)\right) = 0.$$

Then we have convergence in distribution to standard normal in the form of

$$\frac{1}{x \cdot s_x} \times \sum_{1 \leq k \leq 2\log\log x}\left(\sum_{\substack{n \leq x \\ \Omega(n)=k}} \frac{C_{\Omega(n)}(n)}{(\log n)\sqrt{\log\log n}} - x \cdot \widehat{\mu}_k(x)\right) \overset{d}{\Longrightarrow} \mathcal{N}(0,1), \text{ as } x \to \infty.$$

We find that $s_x^2 = o(1)$ so that both

$$\frac{1}{x} \times \sum_{1 \leq k \leq 2\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} \frac{C_{\Omega(n)}(n)}{(\log n)\sqrt{\log\log n}}, \qquad \text{and} \qquad \sum_{1 \leq k \leq 2\log\log x} \widehat{\mu}_k(x),$$

have identical distributions as $x \to \infty$. A straightforward extension of the arguments given in [17, Thm. 7.21; §7.4] shows for any $Y > 0$ uniformly for $-Y \leq z \leq Y$ that

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : \frac{\widehat{\mu}_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)} \leq z\right\} = \Phi(z) + O\left(\frac{1}{\sqrt{\log\log x}}\right).$$

In fact we see that as $x \to \infty$

$$\sum_k \widehat{\mu}_k(x) \overset{d}{\Longrightarrow} \mathcal{N}\left(\mu_x(C), \sigma_x^2(C)\right).$$

Hence, we also have that

$$\frac{1}{x} \times \sum_{n \leq x} \frac{C_{\Omega(n)}(n)}{(\log n)\sqrt{\log\log n}} \overset{d}{\Longrightarrow} \mathcal{N}\left(\mu_x(C), \sigma_x^2(C)\right),$$

with maximally the same error term. □

Corollary 4.9. *Suppose that $\mu_x(C)$ and $\sigma_x(C)$ are defined as in Theorem 4.8 for large x. Let $Y > 0$. We have uniformly for all $-Y \leq y \leq Y$ that as $x \to \infty$*

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : \frac{|g^{-1}(n)|}{(\log n)\sqrt{\log\log n}} - \frac{6}{\pi^2 n(\log n)\sqrt{\log\log n}} \times \sum_{k\leq n} |g^{-1}(k)| \leq y\right\} = \Phi\left(\frac{\frac{\pi^2 y}{6} - \mu_x(C)}{\sigma_x(C)}\right) + o(1).$$

Proof. We claim that

$$|g^{-1}(n)| - \frac{6}{\pi^2 n} \times \sum_{k\leq n} |g^{-1}(k)| \sim \frac{6}{\pi^2} C_{\Omega(n)}(n), \text{ as } n \to \infty.$$

As in the proof of Corollary 4.6, we obtain that

$$\frac{1}{x} \times \sum_{n\leq x} |g^{-1}(n)| = \frac{6}{\pi^2}\left(\frac{1}{x} \times \sum_{n\leq x} C_{\Omega(n)}(n) + \sum_{d<x}\sum_{k\leq d} \frac{C_{\Omega(k)}(k)}{d^2}\right) + O(1).$$

Let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) := f(x) - f(x-1)$. We see that for large n

$$\begin{aligned}
|g^{-1}(n)| &= \Delta_n\left(\sum_{k\leq n} g^{-1}(k)\right) \sim \frac{6}{\pi^2} \times \Delta_n\left(\sum_{d\leq n} C_{\Omega(d)}(d) \cdot \frac{n}{d}\right) \\
&= \frac{6}{\pi^2}\left(C_{\Omega(n)}(n) + \sum_{d<n} C_{\Omega(d)}(d)\frac{n}{d} - \sum_{d<n} C_{\Omega(d)}(d)\frac{(n-1)}{d}\right) \\
&\sim \frac{6}{\pi^2}\left(C_{\Omega(n)}(n) + \frac{1}{n-1} \times \sum_{k<n} |g^{-1}(k)|\right), \text{ as } n \to \infty.
\end{aligned}$$

Since $\frac{1}{n-1} \times \sum_{k<n} |g^{-1}(k)| \sim \frac{1}{n} \times \sum_{k\leq n} |g^{-1}(k)|$ for all sufficiently large n, the result follows by a re-normalization of Theorem 4.8. □

Lemma 4.10. *Let $\mu_x(C)$ and $\sigma_x(C)$ be defined as in Theorem 4.8. For all sufficiently large x, if we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds as $x \to \infty$:*

$$\mathbb{P}\left(\frac{|g^{-1}(n)|}{(\log n)\sqrt{\log\log n}} - \frac{6}{\pi^2 n(\log n)\sqrt{\log\log n}} \times \sum_{k\leq n} |g^{-1}(k)| \leq \frac{6}{\pi^2}\mu_x(C)\right) = \frac{1}{2} + o(1) \tag{A}$$

$$\mathbb{P}\left(\frac{|g^{-1}(n)|}{(\log n)\sqrt{\log\log n}} - \frac{6}{\pi^2 n(\log n)\sqrt{\log\log n}} \times \sum_{k\leq n} |g^{-1}(k)| \leq \frac{6}{\pi^2}\left(\alpha\sigma_x(C) + \mu_x(C)\right)\right) = \Phi(\alpha) + o(1), \alpha \in \mathbb{R}. \tag{B}$$

Proof. Each of these results is a consequence of Corollary 4.9. The result in (A) follows since $\Phi(0) = \frac{1}{2}$ by taking

$$y = \frac{6}{\pi^2}\left(\alpha\sigma_x(C) + \mu_x(C)\right),$$

in Corollary 4.9 for $\alpha = 0$. □

5 New formulas and limiting relations characterizing $M(x)$

5.1 Formulas relating $M(x)$ to the summatory function $G^{-1}(x)$

Proposition 5.1. *For all sufficiently large x, we have that*

$$M(x) = G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} G^{-1}(k) \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right). \tag{26}$$

Proof. We know by applying Corollary 1.4 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k) \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right) \\ &= G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} g^{-1}(k) \pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) \\ &= G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k) \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right). \end{aligned}$$

The upper bound on the sum is truncated to $k \in \left[1, \frac{x}{2}\right]$ in the second equation above due to the fact that $\pi(1) = 0$. The third formula above follows directly by (ordinary) summation by parts. □

By the result from (13) proved in Section 3.2, we recall that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), \text{ for } x \geq 1.$$

Summation by parts implies that we can also express $G^{-1}(x)$ in terms of the summatory function $L(x)$ and differences of the unsigned sequence whose distribution is given by Corollary 4.9. That is, we have

$$G^{-1}(x) = \sum_{n \leq x} \lambda(n) |g^{-1}(n)| = L(x) |g^{-1}(x)| - \sum_{n < x} L(n) \left(|g^{-1}(n+1)| - |g^{-1}(n)|\right), \text{ for } x \geq 1.$$

5.2 Asymptotics of $G^{-1}(x)$

The following proofs are credited to Professor R. C. Vaughan and his suggestions about approaches to upper bounds on $|G^{-1}(x)|$ that are attained along infinite subsequences as $x \to \infty$. The ideas at the crux of the proof of the next theorem are found in the references by Davenport and Heilbronn [3, 4] and are known to date back to the work of Hans Bohr [29, *cf.* §11].

Theorem 5.2. *Let σ_1 denote the unique solution to the equation $P(\sigma) = 1$ for $\sigma > 1$. There are complex s with $\mathrm{Re}(s)$ arbitrarily close to σ_1 such that $1 + P(s) = 0$.*

Proof. The function $P(\sigma)$ is decreasing on $(1, \infty)$, tends to $+\infty$ as $\sigma \to 1^+$, and tends to zero as $\sigma \to \infty$. Thus we find that the equation $P(\sigma) = 1$ has a unique solution for $\sigma > 1$, which we denote by $\sigma = \sigma_1 \approx 1.39943$. Let $\delta > 0$ be chosen small enough that $|1 - P(z)| > 0$ for all z such that $|z - \sigma_1| = \delta$. Set

$$\eta = \min_{\substack{z \in \mathbb{C} \\ |z - \sigma_1| = \delta}} |1 - P(z)|.$$

Since $P(z)$ is continuous whenever $\mathrm{Re}(z) > 1$, we have that $\eta > 0$. Let $X \geq 2$ be a sufficiently large integer so that

$$\sum_{p > X} p^{\delta - \sigma_1} < \frac{\eta}{4}.$$

Kronecker's theorem provides a fixed t such that the following inequality holds [9, §XXIII]:

$$\max_{2<p\leq X} \min_{n\in\mathbb{Z}} \left|\frac{t\log p}{2\pi} - n - \frac{1}{2}\right| < \delta\eta.$$

Thus we have that

$$\sum_{p>2} p^{\delta-\sigma_1} \left|p^{\imath t}+1\right| < \frac{\eta}{2}.$$

Hence, for all z such that $|z-\sigma_1| = \delta$, we have

$$|P(z+\imath t)+P(z)| < \frac{\eta}{2}.$$

We apply Rouché's theorem to see that the functions $1-P(z)$ and $1-P(z)+P(z+\imath t)+P(z)$ have the same number of zeros in the disk $\mathcal{D}_\delta = \{z\in\mathbb{C} : |z-\sigma_1| < \delta\}$. Since $1-P(z)$ has at least one zero within $\mathcal{D}_\delta$, we must have that $1+P(w)$ has at least one zero with $|w-\sigma_1-\imath t| < \delta$. Since we can take δ as small as necessary, there are zeros of the function $1+P(s)$ that are arbitrarily close to the line $s=\sigma_1$. □

Corollary 5.3. *Let $\sigma_1 > 1$ be defined as in Theorem 5.2. For any $\epsilon > 0$, there are arbitrarily large x such that*

$$|G^{-1}(x)| > x^{\sigma_1-\epsilon}.$$

Proof. We have by (6) that

$$D_{g^{-1}}(s) := \sum_{n\geq 1} \frac{g^{-1}(n)}{n^s} = \frac{1}{\zeta(s)(1+P(s))}, \text{ for } \operatorname{Re}(s) > 1.$$

Theorem 5.2 implies that $D_{g^{-1}}(s)$ has singularities $s\in\mathbb{C}$ such that the $\operatorname{Re}(s)$ are arbitrarily close to σ_1. By applying [17, Cor. 1.2; §1.2], we have that any Dirichlet series is locally uniformly convergent in its half-plane of convergence, e.g., for $\operatorname{Re}(s) > \sigma_c$, and is hence analytic in this half-plane. It follows that the abscissa of convergence of $D_{g^{-1}}(s)$ is given by $\sigma_c \geq \sigma_1 > 1$. In particular, the abscissa of convergence of this DGF cannot be smaller than σ_1. The result proved in [17, Thm. 1.3; §1.2] then shows that

$$\limsup_{x\to\infty} \frac{\log|G^{-1}(x)|}{\log x} = \sigma_c \geq \sigma_1. \qquad \square$$

5.3 Local cancellation of $G^{-1}(x)$ in the new formulas for $M(x)$

Lemma 5.4. *Suppose that p_n denotes the n^{th} prime for $n\geq 1$ [27, A000040]. Let $\mathcal{P}_\#$ denote the set of positive primorial integers as [27, A002110]*

$$\mathcal{P}_\# = \{n\#\}_{n\geq 1} = \left\{\prod_{k=1}^{n} p_k : n\geq 1\right\} = \{2, 6, 30, 210, 2310, 30030, \ldots\}.$$

As $m\to\infty$ we have

$$\begin{aligned} -G^{-1}((4m+1)\#) &= (4m+1)!\left(1+O\left(\frac{1}{m^2}\right)\right), \\ G^{-1}\left(\frac{(4m+1)\#}{p_k}\right) &= (4m)!\left(1+O\left(\frac{1}{m^2}\right)\right), \text{ for all } 1\leq k\leq 4m+1. \end{aligned}$$

Proof. We have by part (B) of Proposition 1.6 that for all squarefree integers $n\geq 1$

$$|g^{-1}(n)| = \sum_{j=0}^{\omega(n)} \binom{\omega(n)}{j} \times j! = (\omega(n))! \times \sum_{j=0}^{\omega(n)} \frac{1}{j!}$$

$$= (\omega(n))! \times \left(e + O\left(\frac{1}{(\omega(n)+1)!}\right)\right).$$

Let m be a large positive integer. We obtain main terms of the form

$$\begin{aligned}
G_U^{-1}((4m+1)\#) &:= \sum_{\substack{n \leq (4m+1)\# \\ \omega(n)=\Omega(n)}} \lambda(n)|g^{-1}(n)| \\
&= \sum_{0 \leq k \leq 4m+1} \binom{4m+1}{k} (-1)^k k! \left(e + O\left(\frac{1}{(k+1)!}\right)\right) \\
&= -(4m+1)! + O(1).
\end{aligned}$$

We argue that the analogous sums over the non-squarefree $n \leq (4m+1)\#$ contribute strictly less than the order of $G_U^{-1}((4m+1)\#)$ to the main term of $G^{-1}((4m+1)\#)$. Suppose that $2 \leq n \leq (4m+1)\#$ is not squarefree. We have the next largest order of growth of the sequence along those n with $|g^{-1}(n)| \leq |g^{-1}(p_s^2 t)|$ for some $1 \leq s \leq 4m+1$ and where t is squarefree. If $s = 1$ so that $p_s = 2$, we have that the largest possible squarefree part t satisfies $t \leq p_3 p_4 \cdots p_{4m+1}$. A corresponding t with $\omega(t) = 4m-1$ that attains the same bound on $|g^{-1}(n)|$ corresponds to taking any (unordered) rearrangement of the distinct prime factors bounding t from above by the previous product. By Corollary 3.2, we have that

$$\left|g^{-1}(p_1^k t)\right| = \sum_{\substack{d = p_1^k d_0, p_1^{k-1} d_0 \\ d_0 | t}} C_{\Omega(d)}(d) = \sum_{d_0 | t} \left(\binom{k+\omega(d_0)}{k} + \binom{k-1+\omega(d_0)}{k-1}\right)(\omega(d_0))!.$$

Then we see that

$$\begin{aligned}
\left| \sum_{k=2}^{\log_2((4m+1)\#)} \sum_{\substack{1 \leq t \leq \frac{(4m+1)\#}{p_1^k} \\ \omega(t)=\Omega(t)=4m-1}} g^{-1}(p_1^k t) \right| &\leq \sum_{k \geq 2} \sum_{i=0}^{4m-1} \binom{4m-1}{i} (-1)^{k+i} i! \left(\binom{k+i}{k} + \binom{k-1+i}{k-1}\right) \\
&= \frac{(4m-1)!(4m+1)}{4em} + O(1).
\end{aligned}$$

We consider the contributions from subsequent leading powers of the other $p_s \leq (4m+1)\#$ when $2 \leq s \leq 4m+1$. When we have that $|g^{-1}(n)| \leq |g^{-1}(p_s^2 t)|$ for $p_s \geq 3$ and $t \leq p_{r+1} p_{r+2} \cdots p_{4m+1}$ squarefree, we obtain

$$\begin{aligned}
\left| \sum_{k=2}^{\log_{p_s}((4m+1)\#)} \sum_{\substack{1 \leq t \leq \frac{(4m+1)\#}{p_1^k} \\ \omega(t)=\Omega(t)=4m+1-r}} g^{-1}(p_s^k t) \right| &\leq \frac{(4m-r)!(4m+1-r)}{e} + O(1) \\
&\ll \frac{(4m-1)!(4m+1-r)}{r!}.
\end{aligned}$$

For any fixed p_s with $2 \leq s \leq 4m+1$, we bound the lower index r according to $p_s^2(1+o(1)) \leq r \log r$ using the prime number theorem. The inequality requires that

$$r \geq e^{W_0(p_s^2(1+o(1)))} = e^{2\log p_s - \log\log(p_s^2) + o(1)} \sim p_s^2 - 2\log p_s.$$

The lower order term sums $G_L^{-1}((4m+1)\#)$ are then bounded from above by

$$G_L^{-1}((4m+1)\#) := \left| \sum_{\substack{n \leq (4m+1)\# \\ \omega(n) < \Omega(n)}} g^{-1}(n) \right|$$

$$\begin{aligned} &\leq \sum_{r=2}^{4m} \frac{(4m-1)!(4m+1-r)}{er!} \\ &\asymp -(4m)!\left(1+O\left(\frac{1}{m}\right)\right), \text{ as } m \to \infty. \end{aligned}$$

Hence, we find that $-G^{-1}((4m+1)\#) \sim (4m+1)!$. We can similarly derive for any $1 \leq k \leq 4m+1$ that

$$G^{-1}\left(\frac{(4m+1)\#}{p_k}\right) \sim \sum_{0\leq k\leq 4m} \binom{4m}{k} (-1)^k k! \left(e + O\left(\frac{1}{(k+1)!}\right)\right) \sim (4m)!. \qquad \square$$

Remark 5.5. The analysis of the maximal limiting bounds on $G^{-1}(x)$ from below as $x \to \infty$ guaranteed by Corollary 5.3 complicate the interpretation of Proposition 5.1 to form new asymptotics for $M(x)$. Even though we get comparitively large order growth of $G^{-1}(x)$ infinitely often, we expect that there is usually (nearly almost always) a large cancellation between the successive values of this summatory function in (13). Lemma 5.4 demonstrates the phenomenon well along the asymptotically large infinite subsequence of x taken along the primorials, or the integers $x = (4m+1)\#$ that each precisely the product of the first $4m+1$ primes.

Since we have for sufficiently large n that [5, 6]

$$n\# \sim e^{\vartheta(p_n)} \asymp n^n (\log n)^n e^{-n(1+o(1))}, \text{ as } n \to \infty,$$

the RH requires that the leading constants with opposing signs on the asymptotics for the functions from the last lemma match. This observation follows from the fact that if we obtain a contrary result, equation (13) would imply that

$$\frac{M((4m+1)\#}{\sqrt{(4m+1)\#}} \gg [(4m+1)\#]^{\delta_0}, \text{ as } m \to \infty,$$

for some fixed $\delta_0 > 0$. The formula in (13) implies that under the RH we witness the expected substantial cancellation from the summatory function terms involving $G^{-1}(x)$ in the formula for $M(x)$ along this notable subsequence. In fact, for sufficiently large m, we have that the following properties holds:

(i) $\operatorname{sgn}\left(G^{-1}((4m+1)\#)\right) = -\operatorname{sgn}\left(\sum\limits_{p\leq(4m+1)\#} G^{-1}\left(\frac{(4m+1)\#}{p}\right)\right)$;

(ii) $\lim\limits_{m\to\infty} \frac{G^{-1}((4m+1)\#)}{\sum\limits_{p\leq(4m+1)\#} G^{-1}\left(\frac{(4m+1)\#}{p}\right)} = -1$;

(iii) $M((4m+1)\#) \gg \sum\limits_{\substack{n\leq(4m+1)\# \\ \omega(n)=\Omega(n)}} g^{-1}(n)\left(1+\pi\left(\frac{(4m+1)\#}{n}\right)\right)$.

That is, along this primorial subsequence, the contributions of the local maxima for the absolute values of $|g^{-1}(n)|$ at the squarefree integers cancel considerably and do not contribute the main term for the limiting asymptotic expansion of $M(x)$ along $x = (4m+1)\#$ as $m \to \infty$.

6 Conclusions

We have identified a new sequence, $\{g^{-1}(n)\}_{n\geq 1}$, that is the Dirichlet inverse of the shifted strongly additive function $\omega(n)$. Section 3.3, shows that there is a natural combinatorial interpretation to the distribution of distinct values of $|g^{-1}(n)|$ for $n \leq x$ involving the distribution of the primes $p \leq x$ at large x. In particular, the magnitude of $g^{-1}(n)$ depends only on the pattern of the exponents of the prime factorization of n. The signedness of $g^{-1}(n)$ is given by $\lambda(n)$ for all $n \geq 1$. This leads to a new relations of the summatory function $G^{-1}(x)$ that characterize the distribution of $M(x)$ to the distribution of the summatory function $L(x)$.

We emphasize that our new work on the Mertens function proved within this article is significant in providing a new window through which we can view bounding $M(x)$ through asymptotics of auxiliary sequences and partial sums. The computational data generated in Table B of the appendix section suggests numerically that the distribution of $G^{-1}(x)$ may be easier to work with than that of $M(x)$ or $L(x)$. The additively combinatorial relation of the distinct (and repetition of) values of $|g^{-1}(n)|$ for $n \leq x$ are suggestive towards bounding main terms for $G^{-1}(x)$ along infinite subsequences in future work.

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A Appendix: Asymptotic formulas for partial sums

We appreciate the kind online correspondence with Gergő Nemes from the Alfréd Rényi Institute of Mathematics and his careful notes on the limiting asymptotics for the sums identified in this section. We have adapted the communication of his proofs to establish the next few lemmas based on his recent work in the references [18, 19, 20].

Facts A.1 (The incomplete gamma function)**.** The (upper) *incomplete gamma function* is defined by [23, §8.4]

$$\Gamma(a,z) = \int_z^{\infty} t^{a-1} e^{-t} dt, a \in \mathbb{R}, |\arg z| < \pi.$$

The function $\Gamma(a,z)$ can be continued to an analytic function of z on the universal covering of $\mathbb{C} \setminus \{0\}$. For $a \in \mathbb{Z}^+$, the function $\Gamma(a,z)$ is an entire function of z. The following properties of $\Gamma(a,z)$ hold [23, §8.4; §8.11(i)]:

$$\Gamma(a,z) = (a-1)! e^{-z} \times \sum_{k=0}^{a-1} \frac{z^k}{k!}, \text{ for } a \in \mathbb{Z}^+, z \in \mathbb{C}, \tag{27a}$$

$$\Gamma(a,z) \sim z^{a-1} e^{-z}, \text{ for fixed } a \in \mathbb{C}, \text{ as } z \to +\infty. \tag{27b}$$

Moreover, for real $z > 0$, as $z \to +\infty$ we have that [18]

$$\Gamma(z,z) = \sqrt{\frac{\pi}{2}} z^{z-\frac{1}{2}} e^{-z} + O\left(z^{z-1} e^{-z}\right), \tag{27c}$$

If $z, a \to \infty$ with $z = \lambda a$ for some $\lambda > 1$ such that $(\lambda - 1)^{-1} = o\left(\sqrt{|a|}\right)$, then [18]

$$\Gamma(a,z) = z^a e^{-z} \times \sum_{n \geq 0} \frac{(-a)^n b_n(\lambda)}{(z-a)^{2n+1}}. \tag{27d}$$

The sequence $b_n(\lambda)$ satisfies the characteristic recurrence relation that $b_0(\lambda) = 1$ and[2]

$$b_n(\lambda) = \lambda(1-\lambda) b'_{n-1}(\lambda) + \lambda(2n-1) b_{n-1}(\lambda), n \geq 1.$$

Proposition A.2. *Let a, z, λ be positive real parameters such that $z = \lambda a$. If $\lambda \in (0,1)$, then as $z \to \infty$*

$$\Gamma(a,z) = \Gamma(a) + O_{\lambda}\left(z^{a-1} e^{-z}\right).$$

If $\lambda > 1$, then as $z \to \infty$

$$\Gamma(a,z) = \frac{z^{a-1} e^{-z}}{1 - \lambda^{-1}} + O_{\lambda}\left(z^{a-2} e^{-z}\right).$$

If $\lambda > 0.567142 > W(1)$ where $W(x)$ denotes the principal branch of the Lambert W-function for $x \geq 0$, then as $z \to \infty$

$$\Gamma(a, z e^{\pm \pi \imath}) = -e^{\pm \pi \imath a} \frac{z^{a-1} e^{z}}{1 + \lambda^{-1}} + O_{\lambda}\left(z^{a-2} e^{z}\right).$$

[2] An exact formula for $b_n(\lambda)$ is given in terms of the *second-order Eulerian number triangle* [27, A008517] as follows:

$$b_n(\lambda) = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \lambda^{k+1}.$$

Note that the first two asymptotic estmates are only useful when λ is bounded away from the transition point at 1. We cannot write the last expansion above as $\Gamma(a,-z)$ directly unless $a \in \mathbb{Z}^+$ as the incomplete gamma function has a branch point at the origin with respect to its second variable. This function becomes a single-valued analytic function of its second input by continuation on the universal covering of $\mathbb{C} \setminus \{0\}$.

Proof. The first asymptotic estimate follows directly from the following asymptotic series expansion that holds as $z \to +\infty$ [20, Eq. (2.1)]:

$$\Gamma(a,z) \sim \Gamma(a) + z^a e^{-z} \times \sum_{k \geq 0} \frac{(-a)^k b_k(\lambda)}{(z-a)^{2k+1}}.$$

Using the notation from (27d) and [19], we have that

$$\Gamma(a,z) = \frac{z^{a-1}e^{-z}}{1-\lambda^{-1}} + z^a e^{-z} R_1(a,\lambda).$$

From the bounds in [19, §3.1], we have that

$$|z^a e^{-z} R_1(a,\lambda)| \leq z^a e^{-z} \times \frac{a \cdot b_1(\lambda)}{(z-a)^3} = \frac{z^{a-2}e^{-z}}{(1-\lambda^{-1})^3}$$

The main and error terms in the previous equation can also be seen by applying the asymptotic series in (27d) directly.

The proof of the third equation above follows from the following asymptotics [18, Eq. (1.1)]

$$\Gamma(-a,z) \sim z^{-a} e^{-z} \times \sum_{n \geq 0} \frac{a^n b_n(-\lambda)}{(z+a)^{2n+1}},$$

by setting $(a,z) \mapsto (ae^{\pm \pi \imath}, ze^{\pm \pi \imath})$ so that $\lambda = \frac{z}{a} > 0.567142 > W(1)$. The restriction on the range of λ over which the third formula holds is made to ensure that the last formula from the reference is valid at negative real a. $\square$

Lemma A.3. *For $x \to +\infty$, we have that*

$$S_1(x) := \frac{x}{\log x} \times \left| \sum_{1 \leq k \leq \lfloor \log\log x \rfloor} \frac{(-1)^k (\log\log x)^{k-1}}{(k-1)!} \right| = \frac{x}{2\sqrt{2\pi \log\log x}} + O\left(\frac{x}{(\log\log x)^{\frac{3}{2}}} \right).$$

Proof. We have for $n \geq 1$ and any $t > 0$ by (27a) that

$$\sum_{1 \leq k \leq n} \frac{(-1)^k t^{k-1}}{(k-1)!} = -e^{-t} \times \frac{\Gamma(n,-t)}{(n-1)!}.$$

Suppose that $t = n + \xi$ with $\xi = O(1)$, e.g., so we can formally take the floor of the input n to truncate the last sum. By the third formula in Proposition A.2 with the parameters $(a,z,\lambda) \mapsto \left(n, t, 1 + \frac{\xi}{n}\right)$, we deduce that as $n, t \to +\infty$.

$$\Gamma(n,-t) = (-1)^{n+1} \times \frac{t^n e^t}{t+n} + O\left(\frac{nt^n e^t}{(t+n)^3} \right) = (-1)^{n+1} \frac{t^n e^t}{2n} + O\left(\frac{t^{n-1} e^t}{n} \right). \tag{28}$$

Accordingly, we see that

$$\sum_{1 \leq k \leq n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{t^n}{2n!} + O\left(\frac{t^{n-1}}{n!} \right).$$

By the variant of Stirling's formula in [23, *cf.* Eq. (5.11.8)], we have

$$n! = \Gamma(1+t-\xi) = \sqrt{2\pi} t^{t-\xi+\frac{1}{2}} e^{-t} \left(1 + O\left(t^{-1}\right)\right) = \sqrt{2\pi} t^{n+\frac{1}{2}} e^{-t} \left(1 + O\left(t^{-1}\right)\right).$$

Hence, as $n \to +\infty$ with $t \coloneqq n+\xi$ and $\xi = O(1)$, we obtain that

$$\sum_{k=1}^{n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{e^t}{2\sqrt{2\pi t}} + O\left(e^t t^{-\frac{3}{2}}\right).$$

The conclusion follows by taking $n \coloneqq \lfloor \log\log x \rfloor$, $t \coloneqq \log\log x$ and applying the triangle inequality to obtain the result. □

B Table: Compuations involving $g^{-1}(n)$ and $G^{-1}(n)$ for $1 \leq n \leq 500$

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1^1	Y	N	1	0	1.0000000	1.000000	0.000000	1	1	0
2	2^1	Y	Y	−2	0	1.0000000	0.500000	0.500000	−1	1	−2
3	3^1	Y	Y	−2	0	1.0000000	0.333333	0.666667	−3	1	−4
4	2^2	N	Y	2	0	1.5000000	0.500000	0.500000	−1	3	−4
5	5^1	Y	Y	−2	0	1.0000000	0.400000	0.600000	−3	3	−6
6	$2^1 3^1$	Y	N	5	0	1.0000000	0.500000	0.500000	2	8	−6
7	7^1	Y	Y	−2	0	1.0000000	0.428571	0.571429	0	8	−8
8	2^3	N	Y	−2	0	2.0000000	0.375000	0.625000	−2	8	−10
9	3^2	N	Y	2	0	1.5000000	0.444444	0.555556	0	10	−10
10	$2^1 5^1$	Y	N	5	0	1.0000000	0.500000	0.500000	5	15	−10
11	11^1	Y	Y	−2	0	1.0000000	0.454545	0.545455	3	15	−12
12	$2^2 3^1$	N	N	−7	2	1.2857143	0.416667	0.583333	−4	15	−19
13	13^1	Y	Y	−2	0	1.0000000	0.384615	0.615385	−6	15	−21
14	$2^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	−1	20	−21
15	$3^1 5^1$	Y	N	5	0	1.0000000	0.466667	0.533333	4	25	−21
16	2^4	N	Y	2	0	2.5000000	0.500000	0.500000	6	27	−21
17	17^1	Y	Y	−2	0	1.0000000	0.470588	0.529412	4	27	−23
18	$2^1 3^2$	N	N	−7	2	1.2857143	0.444444	0.555556	−3	27	−30
19	19^1	Y	Y	−2	0	1.0000000	0.421053	0.578947	−5	27	−32
20	$2^2 5^1$	N	N	−7	2	1.2857143	0.400000	0.600000	−12	27	−39
21	$3^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	−7	32	−39
22	$2^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−2	37	−39
23	23^1	Y	Y	−2	0	1.0000000	0.434783	0.565217	−4	37	−41
24	$2^3 3^1$	N	N	9	4	1.5555556	0.458333	0.541667	5	46	−41
25	5^2	N	Y	2	0	1.5000000	0.480000	0.520000	7	48	−41
26	$2^1 13^1$	Y	N	5	0	1.0000000	0.500000	0.500000	12	53	−41
27	3^3	N	Y	−2	0	2.0000000	0.481481	0.518519	10	53	−43
28	$2^2 7^1$	N	N	−7	2	1.2857143	0.464286	0.535714	3	53	−50
29	29^1	Y	Y	−2	0	1.0000000	0.448276	0.551724	1	53	−52
30	$2^1 3^1 5^1$	Y	N	−16	0	1.0000000	0.433333	0.566667	−15	53	−68
31	31^1	Y	Y	−2	0	1.0000000	0.419355	0.580645	−17	53	−70
32	2^5	N	Y	−2	0	3.0000000	0.406250	0.593750	−19	53	−72
33	$3^1 11^1$	Y	N	5	0	1.0000000	0.424242	0.575758	−14	58	−72
34	$2^1 17^1$	Y	N	5	0	1.0000000	0.441176	0.558824	−9	63	−72
35	$5^1 7^1$	Y	N	5	0	1.0000000	0.457143	0.542857	−4	68	−72
36	$2^2 3^2$	N	N	14	9	1.3571429	0.472222	0.527778	10	82	−72
37	37^1	Y	Y	−2	0	1.0000000	0.459459	0.540541	8	82	−74
38	$2^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	13	87	−74
39	$3^1 13^1$	Y	N	5	0	1.0000000	0.487179	0.512821	18	92	−74
40	$2^3 5^1$	N	N	9	4	1.5555556	0.500000	0.500000	27	101	−74
41	41^1	Y	Y	−2	0	1.0000000	0.487805	0.512195	25	101	−76
42	$2^1 3^1 7^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	9	101	−92
43	43^1	Y	Y	−2	0	1.0000000	0.465116	0.534884	7	101	−94
44	$2^2 11^1$	N	N	−7	2	1.2857143	0.454545	0.545455	0	101	−101
45	$3^2 5^1$	N	N	−7	2	1.2857143	0.444444	0.555556	−7	101	−108
46	$2^1 23^1$	Y	N	5	0	1.0000000	0.456522	0.543478	−2	106	−108
47	47^1	Y	Y	−2	0	1.0000000	0.446809	0.553191	−4	106	−110
48	$2^4 3^1$	N	N	−11	6	1.8181818	0.437500	0.562500	−15	106	−121

Table B: Computations involving $g^{-1}(n) \equiv (\omega+1)^{-1}(n)$ and $G^{-1}(x)$ for $1 \leq n \leq 500$.

- *The column labeled* `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled* `Sqfree` *and* `PPower`*, respectively, list inclusion of n in the sets of squarefree integers and the prime powers.*
- *The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \times k!$.*
- *The last columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{n} \times \#\{n \leq x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geq 1$. That is, the component functions $G_{\pm}^{-1}(x)$ displayed in the last two columns of the table correspond to the summatory function $G^{-1}(x)$ with summands that are positive and negative, respectively.*

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
49	7^2	N	Y	2	0	1.5000000	0.448980	0.551020	−13	108	−121
50	$2^1 5^2$	N	N	−7	2	1.2857143	0.440000	0.560000	−20	108	−128
51	$3^1 17^1$	Y	N	5	0	1.0000000	0.450980	0.549020	−15	113	−128
52	$2^2 13^1$	N	N	−7	2	1.2857143	0.442308	0.557692	−22	113	−135
53	53^1	Y	Y	−2	0	1.0000000	0.433962	0.566038	−24	113	−137
54	$2^1 3^3$	N	N	9	4	1.5555556	0.444444	0.555556	−15	122	−137
55	$5^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−10	127	−137
56	$2^3 7^1$	N	N	9	4	1.5555556	0.464286	0.535714	−1	136	−137
57	$3^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	4	141	−137
58	$2^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	9	146	−137
59	59^1	Y	Y	−2	0	1.0000000	0.474576	0.525424	7	146	−139
60	$2^2 3^1 5^1$	N	N	30	14	1.1666667	0.483333	0.516667	37	176	−139
61	61^1	Y	Y	−2	0	1.0000000	0.475410	0.524590	35	176	−141
62	$2^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	40	181	−141
63	$3^2 7^1$	N	N	−7	2	1.2857143	0.476190	0.523810	33	181	−148
64	2^6	N	Y	2	0	3.5000000	0.484375	0.515625	35	183	−148
65	$5^1 13^1$	Y	N	5	0	1.0000000	0.492308	0.507692	40	188	−148
66	$2^1 3^1 11^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	24	188	−164
67	67^1	Y	Y	−2	0	1.0000000	0.477612	0.522388	22	188	−166
68	$2^2 17^1$	N	N	−7	2	1.2857143	0.470588	0.529412	15	188	−173
69	$3^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	20	193	−173
70	$2^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.471429	0.528571	4	193	−189
71	71^1	Y	Y	−2	0	1.0000000	0.464789	0.535211	2	193	−191
72	$2^3 3^2$	N	N	−23	18	1.4782609	0.458333	0.541667	−21	193	−214
73	73^1	Y	Y	−2	0	1.0000000	0.452055	0.547945	−23	193	−216
74	$2^1 37^1$	Y	N	5	0	1.0000000	0.459459	0.540541	−18	198	−216
75	$3^1 5^2$	N	N	−7	2	1.2857143	0.453333	0.546667	−25	198	−223
76	$2^2 19^1$	N	N	−7	2	1.2857143	0.447368	0.552632	−32	198	−230
77	$7^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−27	203	−230
78	$2^1 3^1 13^1$	Y	N	−16	0	1.0000000	0.448718	0.551282	−43	203	−246
79	79^1	Y	Y	−2	0	1.0000000	0.443038	0.556962	−45	203	−248
80	$2^4 5^1$	N	N	−11	6	1.8181818	0.437500	0.562500	−56	203	−259
81	3^4	N	Y	2	0	2.5000000	0.444444	0.555556	−54	205	−259
82	$2^1 41^1$	Y	N	5	0	1.0000000	0.451220	0.548780	−49	210	−259
83	83^1	Y	Y	−2	0	1.0000000	0.445783	0.554217	−51	210	−261
84	$2^2 3^1 7^1$	N	N	30	14	1.1666667	0.452381	0.547619	−21	240	−261
85	$5^1 17^1$	Y	N	5	0	1.0000000	0.458824	0.541176	−16	245	−261
86	$2^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	−11	250	−261
87	$3^1 29^1$	Y	N	5	0	1.0000000	0.471264	0.528736	−6	255	−261
88	$2^3 11^1$	N	N	9	4	1.5555556	0.477273	0.522727	3	264	−261
89	89^1	Y	Y	−2	0	1.0000000	0.471910	0.528090	1	264	−263
90	$2^1 3^2 5^1$	N	N	30	14	1.1666667	0.477778	0.522222	31	294	−263
91	$7^1 13^1$	Y	N	5	0	1.0000000	0.483516	0.516484	36	299	−263
92	$2^2 23^1$	N	N	−7	2	1.2857143	0.478261	0.521739	29	299	−270
93	$3^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	34	304	−270
94	$2^1 47^1$	Y	N	5	0	1.0000000	0.489362	0.510638	39	309	−270
95	$5^1 19^1$	Y	N	5	0	1.0000000	0.494737	0.505263	44	314	−270
96	$2^5 3^1$	N	N	13	8	2.0769231	0.500000	0.500000	57	327	−270
97	97^1	Y	Y	−2	0	1.0000000	0.494845	0.505155	55	327	−272
98	$2^1 7^2$	N	N	−7	2	1.2857143	0.489796	0.510204	48	327	−279
99	$3^2 11^1$	N	N	−7	2	1.2857143	0.484848	0.515152	41	327	−286
100	$2^2 5^2$	N	N	14	9	1.3571429	0.490000	0.510000	55	341	−286
101	101^1	Y	Y	−2	0	1.0000000	0.485149	0.514851	53	341	−288
102	$2^1 3^1 17^1$	Y	N	−16	0	1.0000000	0.480392	0.519608	37	341	−304
103	103^1	Y	Y	−2	0	1.0000000	0.475728	0.524272	35	341	−306
104	$2^3 13^1$	N	N	9	4	1.5555556	0.480769	0.519231	44	350	−306
105	$3^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	28	350	−322
106	$2^1 53^1$	Y	N	5	0	1.0000000	0.481132	0.518868	33	355	−322
107	107^1	Y	Y	−2	0	1.0000000	0.476636	0.523364	31	355	−324
108	$2^2 3^3$	N	N	−23	18	1.4782609	0.472222	0.527778	8	355	−347
109	109^1	Y	Y	−2	0	1.0000000	0.467890	0.532110	6	355	−349
110	$2^1 5^1 11^1$	Y	N	−16	0	1.0000000	0.463636	0.536364	−10	355	−365
111	$3^1 37^1$	Y	N	5	0	1.0000000	0.468468	0.531532	−5	360	−365
112	$2^4 7^1$	N	N	−11	6	1.8181818	0.464286	0.535714	−16	360	−376
113	113^1	Y	Y	−2	0	1.0000000	0.460177	0.539823	−18	360	−378
114	$2^1 3^1 19^1$	Y	N	−16	0	1.0000000	0.456140	0.543860	−34	360	−394
115	$5^1 23^1$	Y	N	5	0	1.0000000	0.460870	0.539130	−29	365	−394
116	$2^2 29^1$	N	N	−7	2	1.2857143	0.456897	0.543103	−36	365	−401
117	$3^2 13^1$	N	N	−7	2	1.2857143	0.452991	0.547009	−43	365	−408
118	$2^1 59^1$	Y	N	5	0	1.0000000	0.457627	0.542373	−38	370	−408
119	$7^1 17^1$	Y	N	5	0	1.0000000	0.462185	0.537815	−33	375	−408
120	$2^3 3^1 5^1$	N	N	−48	32	1.3333333	0.458333	0.541667	−81	375	−456
121	11^2	N	Y	2	0	1.5000000	0.462810	0.537190	−79	377	−456
122	$2^1 61^1$	Y	N	5	0	1.0000000	0.467213	0.532787	−74	382	−456
123	$3^1 41^1$	Y	N	5	0	1.0000000	0.471545	0.528455	−69	387	−456
124	$2^2 31^1$	N	N	−7	2	1.2857143	0.467742	0.532258	−76	387	−463

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
125	5^3	N	Y	−2	0	2.0000000	0.464000	0.536000	−78	387	−465
126	$2^1 3^2 7^1$	N	N	30	14	1.1666667	0.468254	0.531746	−48	417	−465
127	127^1	Y	Y	−2	0	1.0000000	0.464567	0.535433	−50	417	−467
128	2^7	N	Y	−2	0	4.0000000	0.460938	0.539062	−52	417	−469
129	$3^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	−47	422	−469
130	$2^1 5^1 13^1$	Y	N	−16	0	1.0000000	0.461538	0.538462	−63	422	−485
131	131^1	Y	Y	−2	0	1.0000000	0.458015	0.541985	−65	422	−487
132	$2^2 3^1 11^1$	N	N	30	14	1.1666667	0.462121	0.537879	−35	452	−487
133	$7^1 19^1$	Y	N	5	0	1.0000000	0.466165	0.533835	−30	457	−487
134	$2^1 67^1$	Y	N	5	0	1.0000000	0.470149	0.529851	−25	462	−487
135	$3^3 5^1$	N	N	9	4	1.5555556	0.474074	0.525926	−16	471	−487
136	$2^3 17^1$	N	N	9	4	1.5555556	0.477941	0.522059	−7	480	−487
137	137^1	Y	Y	−2	0	1.0000000	0.474453	0.525547	−9	480	−489
138	$2^1 3^1 23^1$	Y	N	−16	0	1.0000000	0.471014	0.528986	−25	480	−505
139	139^1	Y	Y	−2	0	1.0000000	0.467626	0.532374	−27	480	−507
140	$2^2 5^1 7^1$	N	N	30	14	1.1666667	0.471429	0.528571	3	510	−507
141	$3^1 47^1$	Y	N	5	0	1.0000000	0.475177	0.524823	8	515	−507
142	$2^1 71^1$	Y	N	5	0	1.0000000	0.478873	0.521127	13	520	−507
143	$11^1 13^1$	Y	N	5	0	1.0000000	0.482517	0.517483	18	525	−507
144	$2^4 3^2$	N	N	34	29	1.6176471	0.486111	0.513889	52	559	−507
145	$5^1 29^1$	Y	N	5	0	1.0000000	0.489655	0.510345	57	564	−507
146	$2^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	62	569	−507
147	$3^1 7^2$	N	N	−7	2	1.2857143	0.489796	0.510204	55	569	−514
148	$2^2 37^1$	N	N	−7	2	1.2857143	0.486486	0.513514	48	569	−521
149	149^1	Y	Y	−2	0	1.0000000	0.483221	0.516779	46	569	−523
150	$2^1 3^1 5^2$	N	N	30	14	1.1666667	0.486667	0.513333	76	599	−523
151	151^1	Y	Y	−2	0	1.0000000	0.483444	0.516556	74	599	−525
152	$2^3 19^1$	N	N	9	4	1.5555556	0.486842	0.513158	83	608	−525
153	$3^2 17^1$	N	N	−7	2	1.2857143	0.483660	0.516340	76	608	−532
154	$2^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.480519	0.519481	60	608	−548
155	$5^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	65	613	−548
156	$2^2 3^1 13^1$	N	N	30	14	1.1666667	0.487179	0.512821	95	643	−548
157	157^1	Y	Y	−2	0	1.0000000	0.484076	0.515924	93	643	−550
158	$2^1 79^1$	Y	N	5	0	1.0000000	0.487342	0.512658	98	648	−550
159	$3^1 53^1$	Y	N	5	0	1.0000000	0.490566	0.509434	103	653	−550
160	$2^5 5^1$	N	N	13	8	2.0769231	0.493750	0.506250	116	666	−550
161	$7^1 23^1$	Y	N	5	0	1.0000000	0.496894	0.503106	121	671	−550
162	$2^1 3^4$	N	N	−11	6	1.8181818	0.493827	0.506173	110	671	−561
163	163^1	Y	Y	−2	0	1.0000000	0.490798	0.509202	108	671	−563
164	$2^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	101	671	−570
165	$3^1 5^1 11^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	85	671	−586
166	$2^1 83^1$	Y	N	5	0	1.0000000	0.487952	0.512048	90	676	−586
167	167^1	Y	Y	−2	0	1.0000000	0.485030	0.514970	88	676	−588
168	$2^3 3^1 7^1$	N	N	−48	32	1.3333333	0.482143	0.517857	40	676	−636
169	13^2	N	Y	2	0	1.5000000	0.485207	0.514793	42	678	−636
170	$2^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.482353	0.517647	26	678	−652
171	$3^2 19^1$	N	N	−7	2	1.2857143	0.479532	0.520468	19	678	−659
172	$2^2 43^1$	N	N	−7	2	1.2857143	0.476744	0.523256	12	678	−666
173	173^1	Y	Y	−2	0	1.0000000	0.473988	0.526012	10	678	−668
174	$2^1 3^1 29^1$	Y	N	−16	0	1.0000000	0.471264	0.528736	−6	678	−684
175	$5^2 7^1$	N	N	−7	2	1.2857143	0.468571	0.531429	−13	678	−691
176	$2^4 11^1$	N	N	−11	6	1.8181818	0.465909	0.534091	−24	678	−702
177	$3^1 59^1$	Y	N	5	0	1.0000000	0.468927	0.531073	−19	683	−702
178	$2^1 89^1$	Y	N	5	0	1.0000000	0.471910	0.528090	−14	688	−702
179	179^1	Y	Y	−2	0	1.0000000	0.469274	0.530726	−16	688	−704
180	$2^2 3^2 5^1$	N	N	−74	58	1.2162162	0.466667	0.533333	−90	688	−778
181	181^1	Y	Y	−2	0	1.0000000	0.464088	0.535912	−92	688	−780
182	$2^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.461538	0.538462	−108	688	−796
183	$3^1 61^1$	Y	N	5	0	1.0000000	0.464481	0.535519	−103	693	−796
184	$2^3 23^1$	N	N	9	4	1.5555556	0.467391	0.532609	−94	702	−796
185	$5^1 37^1$	Y	N	5	0	1.0000000	0.470270	0.529730	−89	707	−796
186	$2^1 3^1 31^1$	Y	N	−16	0	1.0000000	0.467742	0.532258	−105	707	−812
187	$11^1 17^1$	Y	N	5	0	1.0000000	0.470588	0.529412	−100	712	−812
188	$2^2 47^1$	N	N	−7	2	1.2857143	0.468085	0.531915	−107	712	−819
189	$3^3 7^1$	N	N	9	4	1.5555556	0.470899	0.529101	−98	721	−819
190	$2^1 5^1 19^1$	Y	N	−16	0	1.0000000	0.468421	0.531579	−114	721	−835
191	191^1	Y	Y	−2	0	1.0000000	0.465969	0.534031	−116	721	−837
192	$2^6 3^1$	N	N	−15	10	2.3333333	0.463542	0.536458	−131	721	−852
193	193^1	Y	Y	−2	0	1.0000000	0.461140	0.538860	−133	721	−854
194	$2^1 97^1$	Y	N	5	0	1.0000000	0.463918	0.536082	−128	726	−854
195	$3^1 5^1 13^1$	Y	N	−16	0	1.0000000	0.461538	0.538462	−144	726	−870
196	$2^2 7^2$	N	N	14	9	1.3571429	0.464286	0.535714	−130	740	−870
197	197^1	Y	Y	−2	0	1.0000000	0.461929	0.538071	−132	740	−872
198	$2^1 3^2 11^1$	N	N	30	14	1.1666667	0.464646	0.535354	−102	770	−872
199	199^1	Y	Y	−2	0	1.0000000	0.462312	0.537688	−104	770	−874
200	$2^3 5^2$	N	N	−23	18	1.4782609	0.460000	0.540000	−127	770	−897

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
201	$3^1 67^1$	Y	N	5	0	1.0000000	0.462687	0.537313	−122	775	−897
202	$2^1 101^1$	Y	N	5	0	1.0000000	0.465347	0.534653	−117	780	−897
203	$7^1 29^1$	Y	N	5	0	1.0000000	0.467980	0.532020	−112	785	−897
204	$2^2 3^1 17^1$	N	N	30	14	1.1666667	0.470588	0.529412	−82	815	−897
205	$5^1 41^1$	Y	N	5	0	1.0000000	0.473171	0.526829	−77	820	−897
206	$2^1 103^1$	Y	N	5	0	1.0000000	0.475728	0.524272	−72	825	−897
207	$3^2 23^1$	N	N	−7	2	1.2857143	0.473430	0.526570	−79	825	−904
208	$2^4 13^1$	N	N	−11	6	1.8181818	0.471154	0.528846	−90	825	−915
209	$11^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	−85	830	−915
210	$2^1 3^1 5^1 7^1$	Y	N	65	0	1.0000000	0.476190	0.523810	−20	895	−915
211	211^1	Y	Y	−2	0	1.0000000	0.473934	0.526066	−22	895	−917
212	$2^2 53^1$	N	N	−7	2	1.2857143	0.471698	0.528302	−29	895	−924
213	$3^1 71^1$	Y	N	5	0	1.0000000	0.474178	0.525822	−24	900	−924
214	$2^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−19	905	−924
215	$5^1 43^1$	Y	N	5	0	1.0000000	0.479070	0.520930	−14	910	−924
216	$2^3 3^3$	N	N	46	41	1.5000000	0.481481	0.518519	32	956	−924
217	$7^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	37	961	−924
218	$2^1 109^1$	Y	N	5	0	1.0000000	0.486239	0.513761	42	966	−924
219	$3^1 73^1$	Y	N	5	0	1.0000000	0.488584	0.511416	47	971	−924
220	$2^2 5^1 11^1$	N	N	30	14	1.1666667	0.490909	0.509091	77	1001	−924
221	$13^1 17^1$	Y	N	5	0	1.0000000	0.493213	0.506787	82	1006	−924
222	$2^1 3^1 37^1$	Y	N	−16	0	1.0000000	0.490991	0.509009	66	1006	−940
223	223^1	Y	Y	−2	0	1.0000000	0.488789	0.511211	64	1006	−942
224	$2^5 7^1$	N	N	13	8	2.0769231	0.491071	0.508929	77	1019	−942
225	$3^2 5^2$	N	N	14	9	1.3571429	0.493333	0.506667	91	1033	−942
226	$2^1 113^1$	Y	N	5	0	1.0000000	0.495575	0.504425	96	1038	−942
227	227^1	Y	Y	−2	0	1.0000000	0.493392	0.506608	94	1038	−944
228	$2^2 3^1 19^1$	N	N	30	14	1.1666667	0.495614	0.504386	124	1068	−944
229	229^1	Y	Y	−2	0	1.0000000	0.493450	0.506550	122	1068	−946
230	$2^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.491304	0.508696	106	1068	−962
231	$3^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.489177	0.510823	90	1068	−978
232	$2^3 29^1$	N	N	9	4	1.5555556	0.491379	0.508621	99	1077	−978
233	233^1	Y	Y	−2	0	1.0000000	0.489270	0.510730	97	1077	−980
234	$2^1 3^2 13^1$	N	N	30	14	1.1666667	0.491453	0.508547	127	1107	−980
235	$5^1 47^1$	Y	N	5	0	1.0000000	0.493617	0.506383	132	1112	−980
236	$2^2 59^1$	N	N	−7	2	1.2857143	0.491525	0.508475	125	1112	−987
237	$3^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	130	1117	−987
238	$2^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.491597	0.508403	114	1117	−1003
239	239^1	Y	Y	−2	0	1.0000000	0.489540	0.510460	112	1117	−1005
240	$2^4 3^1 5^1$	N	N	70	54	1.5000000	0.491667	0.508333	182	1187	−1005
241	241^1	Y	Y	−2	0	1.0000000	0.489627	0.510373	180	1187	−1007
242	$2^1 11^2$	N	N	−7	2	1.2857143	0.487603	0.512397	173	1187	−1014
243	3^5	N	Y	−2	0	3.0000000	0.485597	0.514403	171	1187	−1016
244	$2^2 61^1$	N	N	−7	2	1.2857143	0.483607	0.516393	164	1187	−1023
245	$5^1 7^2$	N	N	−7	2	1.2857143	0.481633	0.518367	157	1187	−1030
246	$2^1 3^1 41^1$	Y	N	−16	0	1.0000000	0.479675	0.520325	141	1187	−1046
247	$13^1 19^1$	Y	N	5	0	1.0000000	0.481781	0.518219	146	1192	−1046
248	$2^3 31^1$	N	N	9	4	1.5555556	0.483871	0.516129	155	1201	−1046
249	$3^1 83^1$	Y	N	5	0	1.0000000	0.485944	0.514056	160	1206	−1046
250	$2^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	169	1215	−1046
251	251^1	Y	Y	−2	0	1.0000000	0.486056	0.513944	167	1215	−1048
252	$2^2 3^2 7^1$	N	N	−74	58	1.2162162	0.484127	0.515873	93	1215	−1122
253	$11^1 23^1$	Y	N	5	0	1.0000000	0.486166	0.513834	98	1220	−1122
254	$2^1 127^1$	Y	N	5	0	1.0000000	0.488189	0.511811	103	1225	−1122
255	$3^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.486275	0.513725	87	1225	−1138
256	2^8	N	Y	2	0	4.5000000	0.488281	0.511719	89	1227	−1138
257	257^1	Y	Y	−2	0	1.0000000	0.486381	0.513619	87	1227	−1140
258	$2^1 3^1 43^1$	Y	N	−16	0	1.0000000	0.484496	0.515504	71	1227	−1156
259	$7^1 37^1$	Y	N	5	0	1.0000000	0.486486	0.513514	76	1232	−1156
260	$2^2 5^1 13^1$	N	N	30	14	1.1666667	0.488462	0.511538	106	1262	−1156
261	$3^2 29^1$	N	N	−7	2	1.2857143	0.486590	0.513410	99	1262	−1163
262	$2^1 131^1$	Y	N	5	0	1.0000000	0.488550	0.511450	104	1267	−1163
263	263^1	Y	Y	−2	0	1.0000000	0.486692	0.513308	102	1267	−1165
264	$2^3 3^1 11^1$	N	N	−48	32	1.3333333	0.484848	0.515152	54	1267	−1213
265	$5^1 53^1$	Y	N	5	0	1.0000000	0.486792	0.513208	59	1272	−1213
266	$2^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.484962	0.515038	43	1272	−1229
267	$3^1 89^1$	Y	N	5	0	1.0000000	0.486891	0.513109	48	1277	−1229
268	$2^2 67^1$	N	N	−7	2	1.2857143	0.485075	0.514925	41	1277	−1236
269	269^1	Y	Y	−2	0	1.0000000	0.483271	0.516729	39	1277	−1238
270	$2^1 3^3 5^1$	N	N	−48	32	1.3333333	0.481481	0.518519	−9	1277	−1286
271	271^1	Y	Y	−2	0	1.0000000	0.479705	0.520295	−11	1277	−1288
272	$2^4 17^1$	N	N	−11	6	1.8181818	0.477941	0.522059	−22	1277	−1299
273	$3^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−38	1277	−1315
274	$2^1 137^1$	Y	N	5	0	1.0000000	0.478102	0.521898	−33	1282	−1315
275	$5^2 11^1$	N	N	−7	2	1.2857143	0.476364	0.523636	−40	1282	−1322
276	$2^2 3^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−10	1312	−1322
277	277^1	Y	Y	−2	0	1.0000000	0.476534	0.523466	−12	1312	−1324

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
278	$2^1 139^1$	Y	N	5	0	1.0000000	0.478417	0.521583	−7	1317	−1324
279	$3^2 31^1$	N	N	−7	2	1.2857143	0.476703	0.523297	−14	1317	−1331
280	$2^3 5^1 7^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−62	1317	−1379
281	281^1	Y	Y	−2	0	1.0000000	0.473310	0.526690	−64	1317	−1381
282	$2^1 3^1 47^1$	Y	N	−16	0	1.0000000	0.471631	0.528369	−80	1317	−1397
283	283^1	Y	Y	−2	0	1.0000000	0.469965	0.530035	−82	1317	−1399
284	$2^2 71^1$	N	N	−7	2	1.2857143	0.468310	0.531690	−89	1317	−1406
285	$3^1 5^1 19^1$	Y	N	−16	0	1.0000000	0.466667	0.533333	−105	1317	−1422
286	$2^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.465035	0.534965	−121	1317	−1438
287	$7^1 41^1$	Y	N	5	0	1.0000000	0.466899	0.533101	−116	1322	−1438
288	$2^5 3^2$	N	N	−47	42	1.7659574	0.465278	0.534722	−163	1322	−1485
289	17^2	N	Y	2	0	1.5000000	0.467128	0.532872	−161	1324	−1485
290	$2^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.465517	0.534483	−177	1324	−1501
291	$3^1 97^1$	Y	N	5	0	1.0000000	0.467354	0.532646	−172	1329	−1501
292	$2^2 73^1$	N	N	−7	2	1.2857143	0.465753	0.534247	−179	1329	−1508
293	293^1	Y	Y	−2	0	1.0000000	0.464164	0.535836	−181	1329	−1510
294	$2^1 3^1 7^2$	N	N	30	14	1.1666667	0.465986	0.534014	−151	1359	−1510
295	$5^1 59^1$	Y	N	5	0	1.0000000	0.467797	0.532203	−146	1364	−1510
296	$2^3 37^1$	N	N	9	4	1.5555556	0.469595	0.530405	−137	1373	−1510
297	$3^3 11^1$	N	N	9	4	1.5555556	0.471380	0.528620	−128	1382	−1510
298	$2^1 149^1$	Y	N	5	0	1.0000000	0.473154	0.526846	−123	1387	−1510
299	$13^1 23^1$	Y	N	5	0	1.0000000	0.474916	0.525084	−118	1392	−1510
300	$2^2 3^1 5^2$	N	N	−74	58	1.2162162	0.473333	0.526667	−192	1392	−1584
301	$7^1 43^1$	Y	N	5	0	1.0000000	0.475083	0.524917	−187	1397	−1584
302	$2^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−182	1402	−1584
303	$3^1 101^1$	Y	N	5	0	1.0000000	0.478548	0.521452	−177	1407	−1584
304	$2^4 19^1$	N	N	−11	6	1.8181818	0.476974	0.523026	−188	1407	−1595
305	$5^1 61^1$	Y	N	5	0	1.0000000	0.478689	0.521311	−183	1412	−1595
306	$2^1 3^2 17^1$	N	N	30	14	1.1666667	0.480392	0.519608	−153	1442	−1595
307	307^1	Y	Y	−2	0	1.0000000	0.478827	0.521173	−155	1442	−1597
308	$2^2 7^1 11^1$	N	N	30	14	1.1666667	0.480519	0.519481	−125	1472	−1597
309	$3^1 103^1$	Y	N	5	0	1.0000000	0.482201	0.517799	−120	1477	−1597
310	$2^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.480645	0.519355	−136	1477	−1613
311	311^1	Y	Y	−2	0	1.0000000	0.479100	0.520900	−138	1477	−1615
312	$2^3 3^1 13^1$	N	N	−48	32	1.3333333	0.477564	0.522436	−186	1477	−1663
313	313^1	Y	Y	−2	0	1.0000000	0.476038	0.523962	−188	1477	−1665
314	$2^1 157^1$	Y	N	5	0	1.0000000	0.477707	0.522293	−183	1482	−1665
315	$3^2 5^1 7^1$	N	N	30	14	1.1666667	0.479365	0.520635	−153	1512	−1665
316	$2^2 79^1$	N	N	−7	2	1.2857143	0.477848	0.522152	−160	1512	−1672
317	317^1	Y	Y	−2	0	1.0000000	0.476341	0.523659	−162	1512	−1674
318	$2^1 3^1 53^1$	Y	N	−16	0	1.0000000	0.474843	0.525157	−178	1512	−1690
319	$11^1 29^1$	Y	N	5	0	1.0000000	0.476489	0.523511	−173	1517	−1690
320	$2^6 5^1$	N	N	−15	10	2.3333333	0.475000	0.525000	−188	1517	−1705
321	$3^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−183	1522	−1705
322	$2^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.475155	0.524845	−199	1522	−1721
323	$17^1 19^1$	Y	N	5	0	1.0000000	0.476780	0.523220	−194	1527	−1721
324	$2^2 3^4$	N	N	34	29	1.6176471	0.478395	0.521605	−160	1561	−1721
325	$5^2 13^1$	N	N	−7	2	1.2857143	0.476923	0.523077	−167	1561	−1728
326	$2^1 163^1$	Y	N	5	0	1.0000000	0.478528	0.521472	−162	1566	−1728
327	$3^1 109^1$	Y	N	5	0	1.0000000	0.480122	0.519878	−157	1571	−1728
328	$2^3 41^1$	N	N	9	4	1.5555556	0.481707	0.518293	−148	1580	−1728
329	$7^1 47^1$	Y	N	5	0	1.0000000	0.483283	0.516717	−143	1585	−1728
330	$2^1 3^1 5^1 11^1$	Y	N	65	0	1.0000000	0.484848	0.515152	−78	1650	−1728
331	331^1	Y	Y	−2	0	1.0000000	0.483384	0.516616	−80	1650	−1730
332	$2^2 83^1$	N	N	−7	2	1.2857143	0.481928	0.518072	−87	1650	−1737
333	$3^2 37^1$	N	N	−7	2	1.2857143	0.480480	0.519520	−94	1650	−1744
334	$2^1 167^1$	Y	N	5	0	1.0000000	0.482036	0.517964	−89	1655	−1744
335	$5^1 67^1$	Y	N	5	0	1.0000000	0.483582	0.516418	−84	1660	−1744
336	$2^4 3^1 7^1$	N	N	70	54	1.5000000	0.485119	0.514881	−14	1730	−1744
337	337^1	Y	Y	−2	0	1.0000000	0.483680	0.516320	−16	1730	−1746
338	$2^1 13^2$	N	N	−7	2	1.2857143	0.482249	0.517751	−23	1730	−1753
339	$3^1 113^1$	Y	N	5	0	1.0000000	0.483776	0.516224	−18	1735	−1753
340	$2^2 5^1 17^1$	N	N	30	14	1.1666667	0.485294	0.514706	12	1765	−1753
341	$11^1 31^1$	Y	N	5	0	1.0000000	0.486804	0.513196	17	1770	−1753
342	$2^1 3^2 19^1$	N	N	30	14	1.1666667	0.488304	0.511696	47	1800	−1753
343	7^3	N	Y	−2	0	2.0000000	0.486880	0.513120	45	1800	−1755
344	$2^3 43^1$	N	N	9	4	1.5555556	0.488372	0.511628	54	1809	−1755
345	$3^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.486957	0.513043	38	1809	−1771
346	$2^1 173^1$	Y	N	5	0	1.0000000	0.488439	0.511561	43	1814	−1771
347	347^1	Y	Y	−2	0	1.0000000	0.487032	0.512968	41	1814	−1773
348	$2^2 3^1 29^1$	N	N	30	14	1.1666667	0.488506	0.511494	71	1844	−1773
349	349^1	Y	Y	−2	0	1.0000000	0.487106	0.512894	69	1844	−1775
350	$2^1 5^2 7^1$	N	N	30	14	1.1666667	0.488571	0.511429	99	1874	−1775

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
351	$3^3 13^1$	N	N	9	4	1.5555556	0.490028	0.509972	108	1883	−1775
352	$2^5 11^1$	N	N	13	8	2.0769231	0.491477	0.508523	121	1896	−1775
353	353^1	Y	Y	−2	0	1.0000000	0.490085	0.509915	119	1896	−1777
354	$2^1 3^1 59^1$	Y	N	−16	0	1.0000000	0.488701	0.511299	103	1896	−1793
355	$5^1 71^1$	Y	N	5	0	1.0000000	0.490141	0.509859	108	1901	−1793
356	$2^2 89^1$	N	N	−7	2	1.2857143	0.488764	0.511236	101	1901	−1800
357	$3^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.487395	0.512605	85	1901	−1816
358	$2^1 179^1$	Y	N	5	0	1.0000000	0.488827	0.511173	90	1906	−1816
359	359^1	Y	Y	−2	0	1.0000000	0.487465	0.512535	88	1906	−1818
360	$2^3 3^2 5^1$	N	N	145	129	1.3034483	0.488889	0.511111	233	2051	−1818
361	19^2	N	Y	2	0	1.5000000	0.490305	0.509695	235	2053	−1818
362	$2^1 181^1$	Y	N	5	0	1.0000000	0.491713	0.508287	240	2058	−1818
363	$3^1 11^2$	N	N	−7	2	1.2857143	0.490358	0.509642	233	2058	−1825
364	$2^2 7^1 13^1$	N	N	30	14	1.1666667	0.491758	0.508242	263	2088	−1825
365	$5^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	268	2093	−1825
366	$2^1 3^1 61^1$	Y	N	−16	0	1.0000000	0.491803	0.508197	252	2093	−1841
367	367^1	Y	Y	−2	0	1.0000000	0.490463	0.509537	250	2093	−1843
368	$2^4 23^1$	N	N	−11	6	1.8181818	0.489130	0.510870	239	2093	−1854
369	$3^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	232	2093	−1861
370	$2^1 5^1 37^1$	Y	N	−16	0	1.0000000	0.486486	0.513514	216	2093	−1877
371	$7^1 53^1$	Y	N	5	0	1.0000000	0.487871	0.512129	221	2098	−1877
372	$2^2 3^1 31^1$	N	N	30	14	1.1666667	0.489247	0.510753	251	2128	−1877
373	373^1	Y	Y	−2	0	1.0000000	0.487936	0.512064	249	2128	−1879
374	$2^1 11^1 17^1$	Y	N	−16	0	1.0000000	0.486631	0.513369	233	2128	−1895
375	$3^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	242	2137	−1895
376	$2^3 47^1$	N	N	9	4	1.5555556	0.489362	0.510638	251	2146	−1895
377	$13^1 29^1$	Y	N	5	0	1.0000000	0.490716	0.509284	256	2151	−1895
378	$2^1 3^3 7^1$	N	N	−48	32	1.3333333	0.489418	0.510582	208	2151	−1943
379	379^1	Y	Y	−2	0	1.0000000	0.488127	0.511873	206	2151	−1945
380	$2^2 5^1 19^1$	N	N	30	14	1.1666667	0.489474	0.510526	236	2181	−1945
381	$3^1 127^1$	Y	N	5	0	1.0000000	0.490814	0.509186	241	2186	−1945
382	$2^1 191^1$	Y	N	5	0	1.0000000	0.492147	0.507853	246	2191	−1945
383	383^1	Y	Y	−2	0	1.0000000	0.490862	0.509138	244	2191	−1947
384	$2^7 3^1$	N	N	17	12	2.5882353	0.492188	0.507812	261	2208	−1947
385	$5^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.490909	0.509091	245	2208	−1963
386	$2^1 193^1$	Y	N	5	0	1.0000000	0.492228	0.507772	250	2213	−1963
387	$3^2 43^1$	N	N	−7	2	1.2857143	0.490956	0.509044	243	2213	−1970
388	$2^2 97^1$	N	N	−7	2	1.2857143	0.489691	0.510309	236	2213	−1977
389	389^1	Y	Y	−2	0	1.0000000	0.488432	0.511568	234	2213	−1979
390	$2^1 3^1 5^1 13^1$	Y	N	65	0	1.0000000	0.489744	0.510256	299	2278	−1979
391	$17^1 23^1$	Y	N	5	0	1.0000000	0.491049	0.508951	304	2283	−1979
392	$2^3 7^2$	N	N	−23	18	1.4782609	0.489796	0.510204	281	2283	−2002
393	$3^1 131^1$	Y	N	5	0	1.0000000	0.491094	0.508906	286	2288	−2002
394	$2^1 197^1$	Y	N	5	0	1.0000000	0.492386	0.507614	291	2293	−2002
395	$5^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	296	2298	−2002
396	$2^2 3^2 11^1$	N	N	−74	58	1.2162162	0.492424	0.507576	222	2298	−2076
397	397^1	Y	Y	−2	0	1.0000000	0.491184	0.508816	220	2298	−2078
398	$2^1 199^1$	Y	N	5	0	1.0000000	0.492462	0.507538	225	2303	−2078
399	$3^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.491228	0.508772	209	2303	−2094
400	$2^4 5^2$	N	N	34	29	1.6176471	0.492500	0.507500	243	2337	−2094
401	401^1	Y	Y	−2	0	1.0000000	0.491272	0.508728	241	2337	−2096
402	$2^1 3^1 67^1$	Y	N	−16	0	1.0000000	0.490050	0.509950	225	2337	−2112
403	$13^1 31^1$	Y	N	5	0	1.0000000	0.491315	0.508685	230	2342	−2112
404	$2^2 101^1$	N	N	−7	2	1.2857143	0.490099	0.509901	223	2342	−2119
405	$3^4 5^1$	N	N	−11	6	1.8181818	0.488889	0.511111	212	2342	−2130
406	$2^1 7^1 29^1$	Y	N	−16	0	1.0000000	0.487685	0.512315	196	2342	−2146
407	$11^1 37^1$	Y	N	5	0	1.0000000	0.488943	0.511057	201	2347	−2146
408	$2^3 3^1 17^1$	N	N	−48	32	1.3333333	0.487745	0.512255	153	2347	−2194
409	409^1	Y	Y	−2	0	1.0000000	0.486553	0.513447	151	2347	−2196
410	$2^1 5^1 41^1$	Y	N	−16	0	1.0000000	0.485366	0.514634	135	2347	−2212
411	$3^1 137^1$	Y	N	5	0	1.0000000	0.486618	0.513382	140	2352	−2212
412	$2^2 103^1$	N	N	−7	2	1.2857143	0.485437	0.514563	133	2352	−2219
413	$7^1 59^1$	Y	N	5	0	1.0000000	0.486683	0.513317	138	2357	−2219
414	$2^1 3^2 23^1$	N	N	30	14	1.1666667	0.487923	0.512077	168	2387	−2219
415	$5^1 83^1$	Y	N	5	0	1.0000000	0.489157	0.510843	173	2392	−2219
416	$2^5 13^1$	N	N	13	8	2.0769231	0.490385	0.509615	186	2405	−2219
417	$3^1 139^1$	Y	N	5	0	1.0000000	0.491607	0.508393	191	2410	−2219
418	$2^1 11^1 19^1$	Y	N	−16	0	1.0000000	0.490431	0.509569	175	2410	−2235
419	419^1	Y	Y	−2	0	1.0000000	0.489260	0.510740	173	2410	−2237
420	$2^2 3^1 5^1 7^1$	N	N	−155	90	1.1032258	0.488095	0.511905	18	2410	−2392
421	421^1	Y	Y	−2	0	1.0000000	0.486936	0.513064	16	2410	−2394
422	$2^1 211^1$	Y	N	5	0	1.0000000	0.488152	0.511848	21	2415	−2394
423	$3^2 47^1$	N	N	−7	2	1.2857143	0.486998	0.513002	14	2415	−2401
424	$2^3 53^1$	N	N	9	4	1.5555556	0.488208	0.511792	23	2424	−2401
425	$5^2 17^1$	N	N	−7	2	1.2857143	0.487059	0.512941	16	2424	−2408

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\|n} C_{\Omega(d)}(d)}{\|g^{-1}(n)\|}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
426	$2^1 3^1 71^1$	Y	N	−16	0	1.0000000	0.485915	0.514085	0	2424	−2424
427	$7^1 61^1$	Y	N	5	0	1.0000000	0.487119	0.512881	5	2429	−2424
428	$2^2 107^1$	N	N	−7	2	1.2857143	0.485981	0.514019	−2	2429	−2431
429	$3^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	−18	2429	−2447
430	$2^1 5^1 43^1$	Y	N	−16	0	1.0000000	0.483721	0.516279	−34	2429	−2463
431	431^1	Y	Y	−2	0	1.0000000	0.482599	0.517401	−36	2429	−2465
432	$2^4 3^3$	N	N	−80	75	1.5625000	0.481481	0.518519	−116	2429	−2545
433	433^1	Y	Y	−2	0	1.0000000	0.480370	0.519630	−118	2429	−2547
434	$2^1 7^1 31^1$	Y	N	−16	0	1.0000000	0.479263	0.520737	−134	2429	−2563
435	$3^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.478161	0.521839	−150	2429	−2579
436	$2^2 109^1$	N	N	−7	2	1.2857143	0.477064	0.522936	−157	2429	−2586
437	$19^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	−152	2434	−2586
438	$2^1 3^1 73^1$	Y	N	−16	0	1.0000000	0.477169	0.522831	−168	2434	−2602
439	439^1	Y	Y	−2	0	1.0000000	0.476082	0.523918	−170	2434	−2604
440	$2^3 5^1 11^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−218	2434	−2652
441	$3^2 7^2$	N	N	14	9	1.3571429	0.476190	0.523810	−204	2448	−2652
442	$2^1 13^1 17^1$	Y	N	−16	0	1.0000000	0.475113	0.524887	−220	2448	−2668
443	443^1	Y	Y	−2	0	1.0000000	0.474041	0.525959	−222	2448	−2670
444	$2^2 3^1 37^1$	N	N	30	14	1.1666667	0.475225	0.524775	−192	2478	−2670
445	$5^1 89^1$	Y	N	5	0	1.0000000	0.476404	0.523596	−187	2483	−2670
446	$2^1 223^1$	Y	N	5	0	1.0000000	0.477578	0.522422	−182	2488	−2670
447	$3^1 149^1$	Y	N	5	0	1.0000000	0.478747	0.521253	−177	2493	−2670
448	$2^6 7^1$	N	N	−15	10	2.3333333	0.477679	0.522321	−192	2493	−2685
449	449^1	Y	Y	−2	0	1.0000000	0.476615	0.523385	−194	2493	−2687
450	$2^1 3^2 5^2$	N	N	−74	58	1.2162162	0.475556	0.524444	−268	2493	−2761
451	$11^1 41^1$	Y	N	5	0	1.0000000	0.476718	0.523282	−263	2498	−2761
452	$2^2 113^1$	N	N	−7	2	1.2857143	0.475664	0.524336	−270	2498	−2768
453	$3^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−265	2503	−2768
454	$2^1 227^1$	Y	N	5	0	1.0000000	0.477974	0.522026	−260	2508	−2768
455	$5^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476923	0.523077	−276	2508	−2784
456	$2^3 3^1 19^1$	N	N	−48	32	1.3333333	0.475877	0.524123	−324	2508	−2832
457	457^1	Y	Y	−2	0	1.0000000	0.474836	0.525164	−326	2508	−2834
458	$2^1 229^1$	Y	N	5	0	1.0000000	0.475983	0.524017	−321	2513	−2834
459	$3^3 17^1$	N	N	9	4	1.5555556	0.477124	0.522876	−312	2522	−2834
460	$2^2 5^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−282	2552	−2834
461	461^1	Y	Y	−2	0	1.0000000	0.477223	0.522777	−284	2552	−2836
462	$2^1 3^1 7^1 11^1$	Y	N	65	0	1.0000000	0.478355	0.521645	−219	2617	−2836
463	463^1	Y	Y	−2	0	1.0000000	0.477322	0.522678	−221	2617	−2838
464	$2^4 29^1$	N	N	−11	6	1.8181818	0.476293	0.523707	−232	2617	−2849
465	$3^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.475269	0.524731	−248	2617	−2865
466	$2^1 233^1$	Y	N	5	0	1.0000000	0.476395	0.523605	−243	2622	−2865
467	467^1	Y	Y	−2	0	1.0000000	0.475375	0.524625	−245	2622	−2867
468	$2^2 3^2 13^1$	N	N	−74	58	1.2162162	0.474359	0.525641	−319	2622	−2941
469	$7^1 67^1$	Y	N	5	0	1.0000000	0.475480	0.524520	−314	2627	−2941
470	$2^1 5^1 47^1$	Y	N	−16	0	1.0000000	0.474468	0.525532	−330	2627	−2957
471	$3^1 157^1$	Y	N	5	0	1.0000000	0.475584	0.524416	−325	2632	−2957
472	$2^3 59^1$	N	N	9	4	1.5555556	0.476695	0.523305	−316	2641	−2957
473	$11^1 43^1$	Y	N	5	0	1.0000000	0.477801	0.522199	−311	2646	−2957
474	$2^1 3^1 79^1$	Y	N	−16	0	1.0000000	0.476793	0.523207	−327	2646	−2973
475	$5^2 19^1$	N	N	−7	2	1.2857143	0.475789	0.524211	−334	2646	−2980
476	$2^2 7^1 17^1$	N	N	30	14	1.1666667	0.476891	0.523109	−304	2676	−2980
477	$3^2 53^1$	N	N	−7	2	1.2857143	0.475891	0.524109	−311	2676	−2987
478	$2^1 239^1$	Y	N	5	0	1.0000000	0.476987	0.523013	−306	2681	−2987
479	479^1	Y	Y	−2	0	1.0000000	0.475992	0.524008	−308	2681	−2989
480	$2^5 3^1 5^1$	N	N	−96	80	1.6666667	0.475000	0.525000	−404	2681	−3085
481	$13^1 37^1$	Y	N	5	0	1.0000000	0.476091	0.523909	−399	2686	−3085
482	$2^1 241^1$	Y	N	5	0	1.0000000	0.477178	0.522822	−394	2691	−3085
483	$3^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−410	2691	−3101
484	$2^2 11^2$	N	N	14	9	1.3571429	0.477273	0.522727	−396	2705	−3101
485	$5^1 97^1$	Y	N	5	0	1.0000000	0.478351	0.521649	−391	2710	−3101
486	$2^1 3^5$	N	N	13	8	2.0769231	0.479424	0.520576	−378	2723	−3101
487	487^1	Y	Y	−2	0	1.0000000	0.478439	0.521561	−380	2723	−3103
488	$2^3 61^1$	N	N	9	4	1.5555556	0.479508	0.520492	−371	2732	−3103
489	$3^1 163^1$	Y	N	5	0	1.0000000	0.480573	0.519427	−366	2737	−3103
490	$2^1 5^1 7^2$	N	N	30	14	1.1666667	0.481633	0.518367	−336	2767	−3103
491	491^1	Y	Y	−2	0	1.0000000	0.480652	0.519348	−338	2767	−3105
492	$2^2 3^1 41^1$	N	N	30	14	1.1666667	0.481707	0.518293	−308	2797	−3105
493	$17^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	−303	2802	−3105
494	$2^1 13^1 19^1$	Y	N	−16	0	1.0000000	0.481781	0.518219	−319	2802	−3121
495	$3^2 5^1 11^1$	N	N	30	14	1.1666667	0.482828	0.517172	−289	2832	−3121
496	$2^4 31^1$	N	N	−11	6	1.8181818	0.481855	0.518145	−300	2832	−3132
497	$7^1 71^1$	Y	N	5	0	1.0000000	0.482897	0.517103	−295	2837	−3132
498	$2^1 3^1 83^1$	Y	N	−16	0	1.0000000	0.481928	0.518072	−311	2837	−3148
499	499^1	Y	Y	−2	0	1.0000000	0.480962	0.519038	−313	2837	−3150
500	$2^2 5^3$	N	N	−23	18	1.4782609	0.480000	0.520000	−336	2837	−3173