where it is not difficult to prove that the DGF of $\omega(n)$ is $P(s) \cdot \zeta(s)$.

Thus for any $\Re(s) > 1$, the DGFs of each side of the claimed equation coincide. So by uniqueness of Dirichlet series, we see that in fact the claim holds. To obtain the full result, we add to each side of this equation a term of $\varepsilon(n) \equiv (\mu * 1)(n)$, and then factor the resulting convolution identity. □

Needs a crystal clear proof.

When combined with Corollary 3.2, the proof of Proposition 4.1 yields the crucial starting point providing an exact formula for $M(x)$ stated in (1) of Corollary 3.3. Thus, while the formula in (1) is a key component utilized in our proof moving forward, we do not need to explicitly show that it holds for all $x \geq 1$ from this point.

Proposition 4.2 (The key signedness property of $g^{-1}(n)$)**.** *For the Dirichlet invertible function, $g(n) := \omega(n) + 1$ defined such that $g(1) = 1$, at any $n \geq 1$, we have that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$. The notation for the operation given by $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0, \pm 1\}$ denotes the sign of the arithmetic function h at n.*

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Proof. Let $D_f(s) := \sum_{n \geq 1} f(n) n^{-s}$ denote the Dirichlet generating function (DGF) of any arithmetic function $f(n)$ convergent for $\Re(s) > \sigma_f$. Using Proposition 4.1 and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of the original arithmetic function f, we can express the DGF of our particular $g^{-1}(n)$ explicitly as an analytic function of s for $\Re(s) > 1$. For all $\Re(s) > 1$, expanding the DGF for the function $g^{-1}(n)$ yields

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{4}$$

reference?

Let $h^{-1}(n) := (\omega * \mu + \varepsilon)^{-1}(n) = [n^{-s}](P(s)+1)^{-1}$. Then we have using the standard recurrence relation for the Dirichlet inverse function h^{-1} with $\chi_{\mathbb{P}} = \omega * \mu$ that

What property of h^{-1} are you using? State it!

$$\begin{aligned}(h^{-1} * 1)(n) &= \sum_{p_1 \mid n} h^{-1}\left(\frac{n}{p_1}\right) = \lambda(n) \times \sum_{p_1 \mid n} \sum_{p_2 \mid \frac{n}{p_1}} \cdots \sum_{p_{\Omega(n)} \mid \frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1 \\ &= \begin{cases} \lambda(n) \times (\Omega(n)-1)!, & n \geq 2; \\ \cancel{\lambda(n)},\ 1 & n = 1. \end{cases}\end{aligned} \tag{5}$$

We need to compute the sign of the function $h^{-1} * \mu$, where $g^{-1} = h^{-1} * \mu$ by the DGF in (4) since the DGF of $\mu(n)$ is well-known to be $1/\zeta(s)$ for $\Re(s) > 1$ (and the DGF of a convolution is a product of the component DGFs).

First, by Möbius inversion and the formula for $h^{-1} * 1$ we proved above, for each $n \geq 2$, we have that there exist constants $C_{1,n}, C_{2,n} > 0$ so that

I do not understand what you are doing here.

see next page (**)

$$C_{1,n} \cdot (\lambda * \mu)(n) \leq h^{-1}(n) \leq C_{2,n} \cdot (\lambda * \mu)(n).$$

This observation follows from the non-negativity of the factorial function in the formula we just proved in (5). Since both λ, μ are multiplicative, $\lambda * \mu$ is multiplicative, where we know that the values of any multiplicative function are uniquely determined by its action at prime powers. So we can compute that for any prime p and integer exponents $\alpha \geq 1$,

$\alpha > 1$

$$(\lambda * \mu)(p^\alpha) = \lambda(p^\alpha) - \lambda(p^{\alpha-1}) = 2\lambda(p^\alpha).$$

?

$\lambda(p^\alpha) - \lambda(p^{\alpha-1}) = \alpha - \alpha + 1 = 1$

Then by the multiplicativity of $\lambda(n)$, the previous inequalities are re-stated in the form of

$$2C_{1,n} \cdot \lambda(n) \leq h^{-1}(n) \leq 2C_{2,n} \cdot \lambda(n).$$

$$h^{-1}(n) = (h^{-1} * 1) * \mu\,(n)$$

$$= \sum_{d|n} h^{-1} * 1\left(\tfrac{n}{d}\right) \mu(d)$$

$$\lambda\left(\tfrac{n}{d}\right) \cdot \mu(d) = (-1)^{\Omega\left(\frac{n}{d}\right) + \omega(d)}$$

which is a signed function.

(**) $$h^{-1}(n) = \sum_{d|n} \underbrace{\lambda\left(\tfrac{n}{d}\right) \mu(d)}_{\text{signed}}\ \underbrace{\gamma(n/d)}_{\text{positive}}$$

You are suggesting in (**) above

$$\sum_{d|n} \lambda\left(\tfrac{n}{d}\right) \mu(d)\ \gamma(n/d)$$

$$\leq \lambda * \mu(n) \cdot \max_{d|n} \gamma(n/d)$$

which is certainly not true.

There must be some other property you are using in (**).

You need to state it

5 Summing arithmetic functions weighted by $\lambda(n)$

5.1 Discussion: The enumerative DGF result in Theorem 3.7 from Montgomery and Vaughan

What the enumeratively-flavored result of Montgomery and Vaughan in Theorem 3.7 allows us to do is get a "good enough" lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. For comparison, we already have known, more classical bounds due to Erdös (and earlier) that we can tightly bound [2, 11]

$$\pi_k(x) = (1 + o(1)) \cdot \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!}.$$

We seek to approximate the right-hand-side of $\mathcal{G}(z)$ by only taking the products of the primes $p \leq u$, e.g., indexing the component products only over those primes $p \in \{2, 3, 5, \ldots, u\}$ for some minimal upper bound u (depending on x) as $x \to \infty$ (see Remark 5.4). The results proved in Section 5.2 identify a minimal parameter u.

We also state the following theorem reproduced from [11, Thm. 7.20] that handles the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \log \log x$.

Theorem 5.1 (Bounds on exceptional values of $\Omega(n)$ for large n, MV 7.20)**.** *Let*

$$B(x, r) := \# \{n \leq x : \Omega(n) \leq r \cdot \log \log x\}.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x, r) \ll_R x \cdot (\log x)^{r-1-r \log r}, \quad \text{as } x \to \infty.$$

what is role of R?

The proof of Theorem 5.2 is found in the cited reference as Chapter 7 of Montgomery and Vaughan. The key interpretation we need is the result stated in the next corollary.

Corollary 5.2. *Using the notation for $B(x, r)$ from Theorem 5.1, we have in particular that for $r \in (1, 2)$,*

$$\left|1 - \frac{B(x, r)}{B(x, 1)}\right| \xrightarrow{x \to \infty} 1.$$

? This does NOT follow from Thm 5.1

We emphasize that Corollary 5.2 implies that for sums involving $\widehat{\pi}_k(x)$ indexed by k, we can capture the dominant asymptotic behavior of these sums by taking k in the truncated range $1 \leq k \leq \log \log x$, e.g., $0 \leq z \leq 1$ in Theorem 3.7. This fact will be important when we prove Theorem 7.4 in Section 7 using a sign-weighted summatory function in Abel summation that depends on these functions (see Lemma 7.2).

5.2 The key new results utilizing Theorem 3.7

We will require a handle on partial sums of integer powers of the reciprocal primes as functions of the integral exponent and the upper summation index x. The next corollary is not a triviality as it comes in handy when we take to the task of proving Theorem 3.8 below. The next statement of Corollary 5.3 effectively generalizes Mertens theorem stated previously as Theorem 4.3 by providing a coarse rate in x below which the reciprocal prime sums tend to absolute constants given by the prime zeta function, $P(s)$.

Thm 5.1 suggests

$$\frac{B(x,r)}{B(x,1)} \leq \frac{x\,(\log x)^{\varrho(r)}}{x}$$

$$= (\log x)^{\varrho(r)} \nearrow \infty$$

when $\varrho(r) = r - 1 - r\log r > 0$

Maybe you mean something else.