Revised Manuscript and Responses to Referee Feedback

New characterizations of partial sums of the Möbius function

Responses to the feedback from the referee on the first revised manuscript:

Referee Point: It takes a long time to see exactly what theorems the author is proving, and when you see 'em, it's unclear why anyone would do this.
Author Response: I have reviewed the article with multiple drafts to address this concern of the referee in full. In making modifications to the article to make the exposition clearer and more accessible from a high level in the introduction, I have received feedback from the JNT editor and from other sources. The main changes you will notice are in the first section of the article. I have added several paragraphs explaining motivation for the methods in the article, compared my techniques to previous methods exploring properties and bounds of M(x), highlighted that which is new about my perspective in characterizing these partial sums through the new sequences studied, and given more breadth and focus on the higher-level insights behind the new results. In Section 1.3.2, I state more results proved in Section 4 of the article to summarize and give some perspective on the direction and flow within this important later section. The prior draft of the article contained several results that were compressed between longer technical proofs in the middle of the article. The new presentation should help clarify the results on the distributions of the unsigned functions, e.g., the average order formulas and limiting distributions behind the two key auxiliary functions within the article. I have also tried to illustrate the new prime related combinatorics that underly these functions in the introduction to motivate and give readers a feel for the flavor of the article at the start.

Referee Point: Sound's paper is in Crelle, not Annals.
Author Response: Point noted, and now correctly cited in the bibliography section of the article. Thank you for catching this missed citation. Some of the results I attributed to Sound's article were actually contained in the reference by Humphries. In other words, there was a mistaken reference to which article proved which bounds on M(x), which I believe are now accurately stated and cited inline.

Referee Point: Sound's result, quoted before on page 4, just before \S 1.1.3 should have "14" as the exponent of \log\log x, and not 5/2 + \epsilon.
Author Response: The credit to Sound's Crelle article with the correct exponent corresponding to that reference is updated in Section 1.2. I added a new citation to the result for an upper bound on M(x) stated by Humphries as one of the bounds given in the discussion of preliminaries in this section.

Referee Point: Above this, Walfisz's result should have, as the exponent of \log\log x "-3/5" instead of "-1/5".

Author Response: Thank you again for pointing this typo out. It has been corrected in the revised article. I have a question for the referee, which is whether a citation where Walfisz's result can be found that is translated well in English? Unfortunately, as I do not read fluent German, I was only able to scan the citation I could find to this result online and verify it exists by examining the mathematical typesetting. It seems to be a famous bound for the Mertens function that may be just as easily attributed as I have done in this version by citing the author and date of the result.

Referee Point: "te Riele" and "Odlyzko" have interesting variations on their spelling throughout the article.
Author Response: I removed the accent mark on the first author throughout the article and in the bibliography. By my inspection, there was a single typographical misspelling of the second author that was a single letter transposition type typographical error. An online search of the spellings of both author names based on their prior publications suggests that the other instances in my article are correct. Thank you for pointing out that I needed to check the spellings carefully.

<u>Other changes and modifications to note in this revised manuscript</u>:

Technical changes are made where necessary for rigor and correctness of the arguments used to justify the results in the article. In spirit, the results and key properties and methods underneath the results in Section 4 are the same. For example, I realized that a couple of the results I had originally stated in Section 5 were either incorrect upper bounds on G^{-1}(x) or were trivialities in comparison with the best known bounds for M(x) under the RH. The following listing is intentionally detailed and should allow the referee and editors to understand where and why the technical modifications were made throughout the article including the changes elaborated on above to ease exposition and motivate the work. Please do not judge the article unfairly because these changes were necessary. I am now confident that the proofs are correct after multiple drafts and edit sessions with pen, paper and a LaTeX text editor these last few months.

The next points organized by article section are intended to be a comprehensive guide to important modifications to annotate in the new revised manuscript:

<u>Section 2: Initial elementary proofs of new results:</u>

- The proof of the \lambda(n) sign-weighted-ness of the inverse sequence is updated in Section 2.2. It now includes a short explicit formula of how the multinomial coefficient-based formula involving factorial products is obtained from the DGF expansions. This is an asset to article because it quickly demonstrates a combinatorial argument for how to prove the formula stated in Froberg's reference on the prime zeta function where this sequence was first considered (to the best of my knowledge). Since the factorial ratios formula for $C_{\Omega(n)}(n)$ is utilized several times in proofs of later results, I explained how to derive it more carefully in this version of the article.

- The note about the scaled multiplicative function variants based on this function given at the end of Section 2.2 was pointed out to me in conversation by Professor Bob Vaughan. I will talk more about his feedback in addressing the points and changes to the content in Section 5 below related to limiting bounds for $G^{-1}(x)$.

Section 3: Auxiliary sequences related to the inverse function $g^{-1}(n)$:

- In Section 3.1, I elaborated more on motivation as to why the recursively defined sequences $C_k(n)$ are considered in the context of the topics we study here. These points help to address the referee's comment about requiring a better explanation to motivate the use and definition of these sequences. Note that for squarefree n, these convolutions correspond to the divisor sums

$$|g^{-1}(n)| = \sum_{d|n} C_{\Omega(d)}(d).$$

The remarkable combinatorial formulas obtained at the squarefree integers were a topic of numerical exploration from the early stages of the article. The connection proved in Lemma 3.1 is in fact difficult to initially identify in the general case without this type of insight.

Section 4: The distributions of the unsigned functions and their partial sums:

- Note that the relation of the two unsigned auxiliary sequences, $C_{\Omega(n)}(n)$ and $g^{-1}(n)$, to strongly additive functions hints at (a provably quantifiable) regular and orderly structure underneath these functions that is, at least in passing, analogous in many respects to the limiting probability measures we find for the distributions of other strongly additive functions (cf. [13]). I added a few brief sentences to the introductory paragraph in Section 4 to lead into and motivate the technical analytic methods that are used to rigorously show that these types of properties hold here for the special case functions.

- The bounds on the bivariate DGF $\widehat{G}(z)$ in the last typeset equations in the proof of Theorem 4.2 are now accurately stated when $z = (k-1)/\log\log x$ within the uniform bounds when $1 \leq k \leq 2\log\log x$. The core outline and methodology behind this proof involves minor adaptations of the argument given in the proof of Theorem 7.19 in Montgomery and Vaughan (MV).

- Lemma 4.3 is extended and provided with an explicit sign on the main term. It now appears before the statement and proof of Corollary 4.4 and Proposition 4.5. The logic employed in obtaining the asymptotic expansions of these partial sums is needed for reference within those two proofs. Note that I have been in correspondence with Gergő Nemes, an expert in special functions and asymptotic expansions of the incomplete gamma function, to add the appendix section results needed to establish main and error term bounds on the partial sums considered in Section 4. I have also added three citations to his work on the asymptotics of $\Gamma(a, z)$ from 2015, 2016, and most recently from 2019 that are in the *NIST Handbook of Mathematical Functions* (DLMF) as a primary source and reference point.

- The proof of Corollary 4.4, which is an important result for establishing the bounds and theorems in the rest of this section, has changed substantially from the original version. The key ideas are still in place, and it happens that the resulting asymptotics obtained after the corrections are still of accurate order (up to the leading constant factor), but I had to side-step a few obstacles to formulate a rigorous proof due to signed summands (integrands in the approximation). The key error in the last version of the article to note here is that the summatory function, $L_{\ast}(t)$, is variably signed on the interval $t \leq x$. The problem was that I originally attempted to take the derivative of an integral whose integrands are variably signed within intervals of the $t \leq x$. It turns out that the signed terms on this function are "nicely" enough behaved at large x so that the main term contributions I obtained before are still semi-accurate in order up to the leading sign factor. The resulting proof is cleaner and easier to read as well in my view.

- A related error to that originally present in the proof of Corollary 4.4 was also made for the same set of reasons (with the analogous partial sums over difference ranges) in the proof of Proposition 4.5. It is now corrected using the same method I used in the original proof of the corollary. The key idea to adapt the form of Rankin's method from the MV reference is similarly applied. The difference to note is that the formula I obtained before was inaccurate by a factor of $\log x$. The main idea is still to sum the restricted functions from Corollary 4.4 over the range where we are guaranteed uniform asymptotics, and then use Rankin's method to show that the extremal values of the function when $k \geq 3/2\log\log x$, contribute negligible weight to the sums that define the average order.

- The changes at the start of Section 4.3 form the most significant new material in the article. That is, the updated manuscript brings in new ideas that are required to justify the original results for the normal tending distribution of $C_{\Omega(n)}(n)$. I believe the introduction to this section clearly spells out the problem from before. In particular, that we have the matching asymptotic expansions of the summatory function for the sequence in question compared to the well-known proof in the MV reference, does not imply in and of itself that the distribution of the summands is identical as $x \rightarrow \infty$. Nonetheless, as I pointed out in the previous bullet points, since the function is apparently so regular, it is not a stretch that it has such a beautiful natural relation to the standard normal distribution.

What has changed is much more probabilistic in spirit than the analytic proof I cited before. In the original heuristics made to justify the Erdos-Kac theorem for \omega(n) last century, a similar justification was required based on the view of the arithmetic function as a random variable under independence assumptions. Please reference this first part of Section 4.3 for more information about the method and the more recent assumptions on independence as it has been carefully prepared. I am grateful for feedback and corroboration from the referees and editor on these points. ☺

Section 5: New formulas and limiting relations characterizing M(x):

- Professor Bob (R. C.) Vaughan is a coauthor with Montgomery of a primary reference and source of inspiration for the analytic methods employed in much of the proofs in the article [17]. I have been in conversation with him electronically recently since receiving the referee's feedback on the first revised version of the article. He spotted a few issues with the proofs of the bounds on the summatory function G^{-1}(x) in Section 5 of the first revision of the article. I thank and credit him ***very, very graciously*** for taking the time to read through the article and give me his feedback! After reading the notes on the errors he found in that section, I have removed proofs of the two results offering bounds on G^{-1}(x) from the previous version. One was flawed for technical reasons and the other, after some review, does not come close enough to the best-known bounds for M(x) under the RH to be worth highlighting as a significant application of this new work. However, I am optimistic that future work studying M(x) through these new functions and their partial sums will offer insight into more significant bounds on the Mertens function.

- Professor Vaughan communicated the proofs now found in Section 5.2 to me carefully in writing. I have obtained his implicit permission to reproduce what he says he considers to be standardized analytic methods found in the work of Davenport and Heilbronn (circa 1936 in the two papers added to the bibliography) and which date back to the ideas of Hans Bohr as documented in the second edition of the reference by Titchmarsh. Using his arguments, the article now contains a careful (and slick) analytic proof that there arbitrarily large integers x such that

$$|G^{-1}(x)| > x^{\sigma_1 - \epsilon}.$$

  where \sigma_1 \approx 1.39943 is the unique real \sigma > 1 that solves the equation P(\sigma) = 1 on (1, \infty). Here, P(s) denotes the prime zeta function in our usual notation for it within the article.

- Despite the existence of comparatively large growing values of G^{-1}(x) infinitely often, we still witness substantial dampening of the magnitude expressed by the formulas for M(x) involving the successive values of this summatory function, e.g., as summed by

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), \text{ for all } x \geq 1.$$

- The result in Lemma 5.4, and the remarks that follow its proof, in Section 5.3 concretely illustrate the inherent complications in applying limiting asymptotics for G^{-1}(x) to sum M(x) with precision. That is, the special case example identified in the lemma clearly shows how much signed cancellation can occur between the summands in the previous equation along an infinite subsequence of the positive integers.
