Lower bounds on the Mertens function $M(x)$ for $x \gg 2.3315 \times 10^{1656520}$

Maxie Dion Schmidt
maxieds@gmail.com
Georgia Institute of Technology

Last Revised: Tuesday 31st March, 2020 – Compiled with LaTeX2e

1 Introduction

1.1 The Mertens function – definition, properties, known results and conjectures

Suppose that $n \geq 1$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2 \alpha_2 \cdots p_k^{\alpha_k}$. We define the *Möebius function* to be the signed indicator function of the squarefree integers:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leq i \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many known variants and special properties of the Möebius function and its generalizations [8, *cf.* §2], however, for our purposes we seek to explore the properties and asymptotics of weighted summatory functions

$$\frac{\cdots}{\zeta(s)} = \int_1 \frac{\cdots}{x^{s+1}} dx,$$

we then obtain that

$$M(x) = \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation along with the standard Euler product representation for the reciprocal zeta function leads us to the exact expression for $M(x)$ when $x > 0$ given by the next theorem.

Theorem 1.1 (Analytic Formula for $M(x)$)**.** *If the RH is true, the there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any $x \in \mathbb{R}_{>0}$*

$$M(x) = \lim_{k \to \infty} \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |\Im(\rho)| < T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

An unconditional bound on the Mertens function due to Walfisz [?] states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \exp\left(-C \cdot \log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan proved new updated estimates bounding $M(x)$ for large x in 2009 of the following forms:

$$M(x) \ll \sqrt{x} \exp\left(\log^{1/2}(x)(\log\log x)^{14}\right),$$
$$M(x) = O\left(\sqrt{x} \exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right), \ \forall \epsilon > 0.$$

Other explicit bounds due to the article by Kotnik include the following simpler estimates for the Mertens function when x is sufficiently large:

Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies that

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}(\log\log x)^{5/4}},$$

corresponds to some bounded constant. A probabilistic proof along these lines has been given by Ng in 2008. To date an exact rigorous proof that $M(x)/\sqrt{x}$ is unbounded still remains elusive. We cite that prior to this point it is known that [?, *cf.* §4.1]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad \text{(now 1.826054)},$$

[1] See, for example, the discussion in the following thread:
https://mathoverflow.net/questions/98174/is-mertens-function-negatively-biased.

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } -1.837625),$$

although based on work by Odlyzyko and te Riele it seems probable that each of these limits should be $\pm\infty$, respectively [?, ?, ?, ?]. It is also known that $M(x) = \Omega_{\pm}(\sqrt{x})$ and $M(x)/\sqrt{x} = \Omega_{\pm}(1)$.

1.2 A new approach to bounding $M(x)$ from below

1.2.1 Summing series over Dirichlet convolutions

Theorem 1.2 (Summatory functions of Dirichlet convolutions). *Let $f, g : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n\leq x} f(n)$ and $G(x) := \sum_{n\leq x} g(n)$ denote the summatory functions*

1.2.2 A motivating special case

Using $\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu$, where $\chi_{\mathbb{P}}$ is the characteristic function of the primes, we have that $\widetilde{G}(x) = \pi(x) + 1$ in Corollary 1.3. In particular, the corollary implies that

$$M(x) = \sum_{k=1}^{x} (\omega+1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

We can compute the first few terms for the Dirichlet inverse sequence of $g(n) := \omega(n) + 1$ numerically for the first few sequence values as

$$\{g^{-1}(n)\}_{n\geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is given by $\lambda(n) = (-1)^{\Omega(n)}$ (see Proposition 2.2). Note that since the DGF of $\omega(n)$ is given by $D_\omega(s) = P(s)\zeta(s)$ where $P(s)$ is the *prime zeta function*, we do have a Dirichlet series for the inverse functions to invert coefficient-wise using more classical contour integral methods, e.g., using [1, §11]

$$f(n) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{n^{\sigma+\imath t}}{\zeta(\sigma+\imath t)(P(\sigma+\imath t)+1)}, \sigma > 1.$$

Fröberg has previously done some preliminary investigation as to the properties of the inversion to find the coefficients of $(1+P(s))^{-1}$ [2].

We will instead take a more combinatorial tact to investigating bounds on this inverse function sequence in the coming sections. Consider the following motivating conjecture:

Conjecture 1.4. *Suppose that $n \geq 1$ is a squarefree integer. We have the following properties characterizing*

inverse functions $g^{-1}(n)$ based on the configuration of the exponents in the prime factorization of any $n \geq 2$. In Section 3 we consider expansions of these inverse functions recursively, starting from a few first exact cases of an auxillary function, $C_k(n)$, that depends on the precise exponents in the prime factorization of n. We then prove limiting asymptotics for these functions and assemble the main terms in the expansion of $g^{-1}(n)$ using artifacts from combinatorial analysis. Combined with the DGF-based generating function for certain summatory functions indicating the parity of $\Omega(n)$ introduced in the next subsection of this introduction, this take on the identity in (1) provides us with a powerful new method to bound $M(x)$ from below. We will sketch the key results and formulation to the construction we actually use to prove the new lower bounds on $M(x)$ next.

For natural numbers $n \geq 1, k \geq 0$, let

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$

By Möbius inversion (see Lemma 3.2), we have that

$$(g^{-1} * 1)(n) = \lambda(n) \cdot C_{\Omega(n)}(n), \forall n \geq 1.$$

We have limiting asymptotics on these functions given by the following theorem:

Theorem 1.5 (Asymptotics for the functions $C_k(n)$)**.** *Let $\mathbb{1}_{*m}(n)$ denote the m-fold Dirichlet convolution of one with itself at n. The function $\sigma_0 * 1_{*m}$ is multiplicative with values at prime powers given by*

$$(\sigma_0 * \mathbb{1}_{*m})(p^{\alpha}) = \binom{\alpha+m+1}{m+1}.$$

We have the following asymptotic bases cases for the functions $C_k(n)$:

$$C_1(n) \sim \log\log n$$

summatory functions that encapsulate the parity of $\lambda(n)$:

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}, k \geq 1.$$

The precise statement of the theorem that we transform for these new bounds is re-stated as follows:

Theorem 1.6 (Montgomery and Vaughan, §7.4)**.** *Let $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$. For $R < 2$ we have that*

$$\widehat{\pi}_k(x) = G\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

uniformly for $1 \leq k \leq R\log\log x$ where

$$G(z) := \frac{F(1,z)}{\Gamma(z+1)} = \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{z}.$$

The precise formulations of the inverse function asymptotics proved in Section 3 depend on being able to form significant lower bounds on the summatory functions of an always positive arithmetic function weighted by $\lambda(n)$. The next theorem, proved in Section 4, is the crux of the starting point for our new asymptotic lower bounds.

Theorem 1.7 (Generating functions of symmetric functions)**.** *We obtain upper and lower bounds of the partial prime products of the form*

$$\alpha_0(z,x) \leq \prod_{p \leq x} \left(1 - \frac{z}{p}\right)^{-1} \leq \alpha_1(z,x),$$

where it suffices to take

$$\exp\left(\tfrac{55}{4}\log^2 2\right) \quad \left(e^B \log^2 x\right)^z$$

We will make much reference to these results in Section 5.

- **Section 4 (Refining a result from MV)**: In this section, we cite a more enumerative DGF construction for finding uniform bounds on the summatory functions, $\widehat{\pi}_k(x)$, that indicate the parity of $\lambda(n)$ on average (e.g., when using Abel summation). It turns out that it is not necessary to form the infinite products over the primes involved in the original statement of the component DGFs from Theorem 1.6.

  More precisely, to obtain "good enough" bounds on $\widehat{\pi}_k(x)$, it is only necessary to take the primes $p \leq x$ in these products. We use generating functions for elementary symmetric polynomials, along with variants of *Mertens theorem* bounding finite sums of the reciprocals of the primes, to establish upper and lower bounds on the function $G(z)$ from Theorem 1.6.

- **Section 5 (Assembling the proof of lower bounds on $M(x)$)**: This section is the culmination in all of the work in the pages that preceed it. Namely, we prove effective asymptotic lower bounds on the summatory function of g^{-1}. The bounds are stated case-wise in Theorem 5.1 corresponding to the leading sign of $G^{-1}(x)$ and the parity of $\lfloor \log\log\log\log x \rfloor$.

  It turns out that along one particular infinite subsequence of (asymptotically huge) reals, $x_n := \exp\left\{e^{e^{4n}}\right\}$, we "win" largely with x and can obtain new, significant scaled lower bounds for the Mertens function $|M(x)|/\sqrt{x}$. This last statement provides a solution, and partial answer, to the classical question of boundedness of $M(x)$, at which rate along subsequences it is unbounded (very slowly), and addresses an logarithmic power improvement to *Gonek's conjecture*. The key result of Corolllary 5.2 to cite here is found on page 31.

- **Section 6 (Generalizations)**: This section introduces a class of weighted Mertens functions, $M_\alpha(x)$,

2 Preliminary proofs and configuration

2.1 Establishing the summatory function inversion identities

Given the interpretation of the summatory functions over an arbitrary Dirichlet convolution (and the vast number of such identities for special number theoretic functions – *cf.* [3, **?**]), it is not surprising that this formulation of the first theorem may well provide many fruitful applications, indeed. In addition to those cited in the compendia of the catalog reference, we have notable identities of the form: $(f * 1)(n) = [q^n] \sum_{m \geq 1} f(m) q^m / (1 - q^m)$, $\sigma_k = \mathrm{Id}_k * 1$, $\mathrm{Id}_1 = \phi * \sigma_0$, $\chi_{\mathrm{sq}} = \lambda * 1$ (see sections below), $\mathrm{Id}_k = J_k * 1$, $\log = \Lambda * 1$, and of course $2^{\omega} = \mu^2 * 1$. The result in Theorem 1.2 is natural and displays a quite beautiful form of symmetry between the initial matrix terms,

form the matrix of coefficients associated with this system for $H(x)$, and proceed to invert it to express an exact solution for this function over all $x \geq 1$. Let the ordinary (initial, non-inverse) matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1}.$$

Then the matrix we must invert in this problem is lower triangular, with ones on its diagonals – and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by an invertible shift operation as[2]

$$(g_{x,j}) = \hat{G}(I - U^T); U = (\delta_{i,j+1}), (I - U^T)^{-1} = ([j \leq i]_{\delta}).$$

[2]Here, U is the $N \times N$ matrix whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$.

It is a nice round fact that if we take successive differences of floor functions, we get non-zero behavior at divisors:

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

We use this property to invertibly shift the matrix $\hat{G}$, and then invert the result, to obtain a matrix involving the Dirichlet inverse of g:

$$\left[(I - U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Now we can express the inverse of the target matrix $(g_{x,j})$ in terms of these Dirichlet inverse functions as follows:

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu.$$

The summatory function of the LHS is $\widetilde{G}(x) = \pi(x) + 1$.

Proof. The the core of the stated identity is to prove that for all $n \geq 1$, $\chi_{\mathbb{P}}(n) = (\mu * \omega)(n)$ – our claim. We notice that the Mellin transform of $\pi(x)$ – the summatory function of $\chi_{\mathbb{P}}(n)$ – at $-s$ is given by

$$\begin{aligned} s \cdot \int_1^\infty \frac{\pi(x)}{x^{s+1}} dx &= \sum_{n \geq 1} \frac{\nabla[\pi](n-1)}{n^s} \\ &= \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s} = P(s). \end{aligned}$$

This is typical fodder which more generally relates the Mellin transform $\mathcal{M}[S_f](-s)$ to the DGF of the sequence $f(n)$ as cited, for example, in [1, §11]. Now we consider the DGF of the right-hand-side function, $f(n) := (\mu * \omega)(n)$, as

$$D_f(s) = \frac{1}{\zeta(s)} \times \sum_{n\geq 1} \frac{\omega(n)}{n^s} = P(s).$$

Thus for any $\Re(s) > 1$, the DGFs of each side of the claimed equation coincide. So by uniqueness of Dirichlet series, we see that in fact the claim holds. To obtain the full result, we add to each side of this equation a term of $\varepsilon(n) \equiv (\mu * 1)(n)$, and then factor the resulting convolution identity. $\square$

Proposition 2.2 (Sign of Dirichlet inverses of positive and bounded arithmetic functions)**.** *Suppose that f is an arithmetic function with $f(1) = 1$ and such that $f(n) > 0$ for all $n \geq 2$. Then for all $n \geq 1$,*

$$\sum_{j=1}^{\frac{\Omega(n)}{2}} TODO.$$

- Suppose that $\Omega(n)$ is odd. Then

$$TODO.$$

In either case, we obtain the signedness on the Dirichlet inverse functions that we have stated in the proposition. $\square$

2.3 Other facts and listings of results we will need in our proofs

Theorem 2.3 (Mertens theorem)**.**

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log\log x + B + o\left(\frac{1}{\log x}\right),$$

where $B \approx 0.2614972128476427837554$ *is an absolute constant. We actually can bound the left-hand-side more explicitly by*

$$P_1(x) = \log\log x + B + O\left(e^{-(\log x)^{\frac{1}{14}}}\right).$$

Corollary 2.4. *We have that for sufficiently large* $x \gg 1$

$$\mathrm{Ei}(x) := \int_{-x} \frac{}{t} dt,$$

$$E_1(z) := \int_1^{\infty} \frac{e^{-tz}}{t} dt, \Re(z) \geq 0,$$

where $\mathrm{Ei}(-kz) = -E_1(kz)$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $E_1(z)$:

$$1 - \frac{3}{4}z \leq E_1(z) - \gamma - \log z \leq 1 - \frac{3}{4}z + \frac{11}{36}z^2. \tag{3a}$$

A related function is the (upper) *incomplete gamma function* defined by

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt, \Re(s) > 0.$$

We have the following properties of $\Gamma(s,x)$:

$$\Gamma(s,x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, \tag{3b}$$

$$\Gamma(s+1,1) = e^{-1} \left\lfloor \frac{s!}{e} \right\rfloor, s \in \mathbb{Z}^+, \tag{3c}$$

$$\Gamma(s,x) \sim x^{s-1} \cdot e^{-x}, |x| \to +\infty. \tag{3d}$$

3 Precisely enumerating and bounding the Dirichlet inverse functions, $g^{-1}(n) := (\omega + 1)^{-1}(n)$

3.1 Developing an improved conjecture: Proving precise bounds on the inverse functions $g^{-1}(n)$ for all n

Conjecture 1.4 is not the most accurate fomulation of the limiting behavior of the Dirichlet inverse functions $g^{-1}(n)$ that we can see and prove. We need to come up with better bounds to plug back into the asymptotic analysis we obtain in the next sections. It turns out that these results are related to symmetric functions of the exponents in the prime factorizations of each $n \leq x$. The idea is that by having information about $g^{-1}(n)$ in terms of its prime factorization exponents for $n \leq x$, we should be able to extrapolate what we need which is information about the average behavior of the summatory functions, $G^{-1}(x)$, from the proofs above. Moreover,

$$(g^{-1} * 1)(n) = -(\omega * g^{-1})(n).$$

Now by repeatedly expanding the right-hand-side, and removing corner cases in the nested sums since $\omega(1) = 0$ by convention, we find that

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n).$$

The statement follows by Möbius inversion applied to each side of the last equation. $\square$

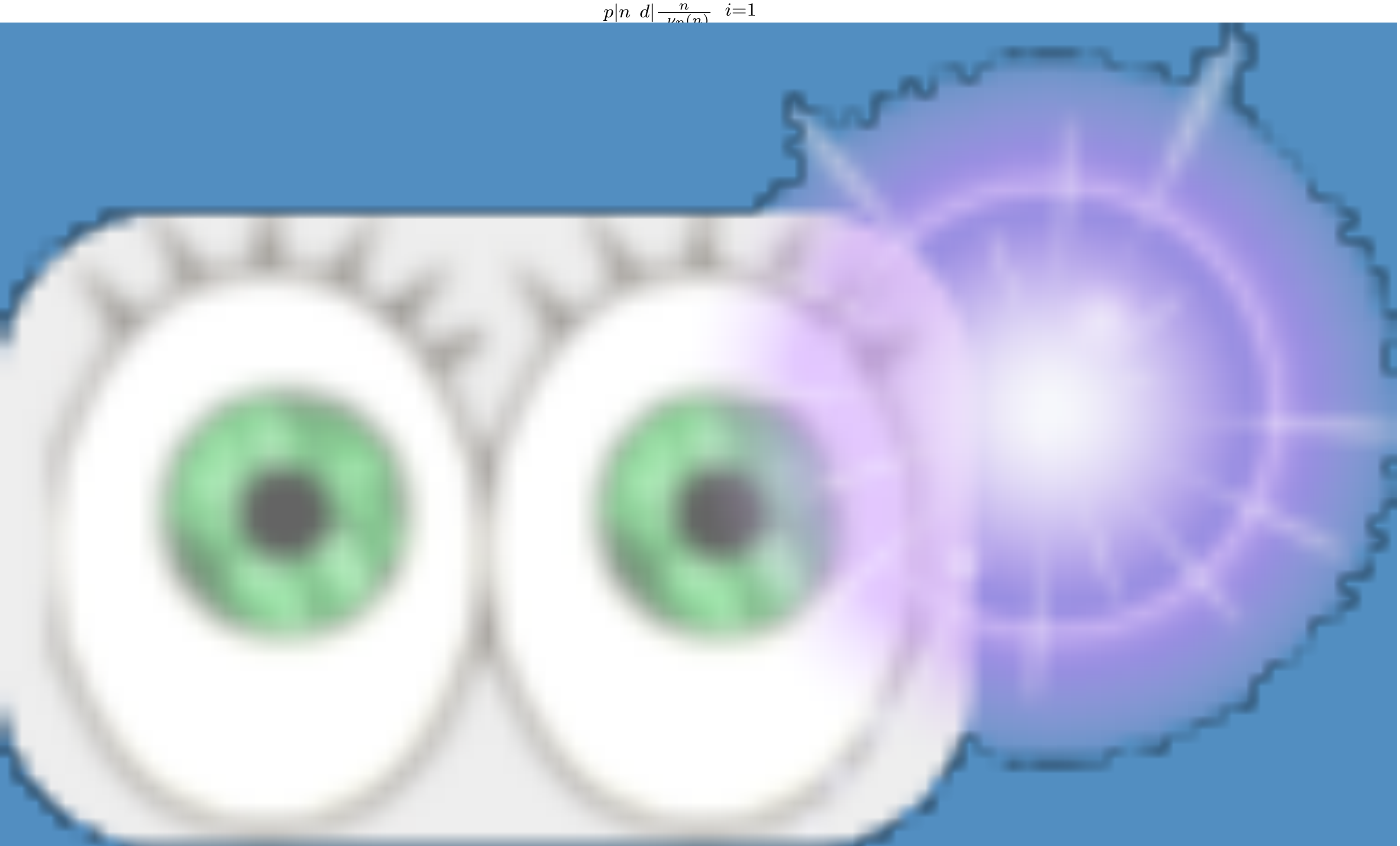
Notice that this approach, while it definitely has its complications due to the necessary step of Möbius inversion, is somewhat simpler that trying to form the Dirichlet inverse of the sum of $\omega + 1$ directly, though this is also a possible approach.

Example 3.3 (Special cases of the functions $C_k(n)$ for small k)**.** We cite the following special cases which should be easy enough to see on paper:

$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= \sigma_0(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} - \gcd\left(\Omega(n), \omega(n)\right). \end{aligned}$$

We also can see a recurrence relation between successive $C_k(n)$ values over k of the form

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} C_{k-1}\left(d \cdot p^i\right). \tag{5}$$



$$= (\sigma_0 * 1)(n) \times \sum_{\substack{p,q|n \\ p \neq q}} \frac{\nu_p(n)(\nu_p(n)+3)\nu_q(n)(\nu_q(n)+3)}{(\nu_p(n)+1)(\nu_p(n)+2)(\nu_q(n)+1)(\nu_q(n)+2)}.$$

Next, we have that

$$C_{3,2}(n) = \sum_{p|n} \sum_{d|\frac{n}{p^{\nu_p(n)}}} \sum_{q|d} \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2)\left(4\nu_p(n) + 9 - 6H^{(1)}_{\nu_p(n)+2}\right)\sigma_0(d)$$

[3]Here, the arithmetic function $\sigma_0 * 1$ is multiplicative. It's value at prime powers can be computed as

$$(\sigma_0 * 1)(p^\alpha) = \sum_{i=0}^{\alpha}(i+1) = \frac{(\alpha+1)(\alpha+2)}{2},$$

where $\sigma_0(p^\beta) = \beta + 1$.

$$= \sum_{\substack{p,q|n \\ p\neq q}} \sum_{d|\frac{n}{p^{\nu_p(n)}q^{\nu_q(n)}}} \sum_{i=1}^{\nu_q(n)} \frac{1}{12}(\nu_p(n)+1)(\nu_p(n)+2)\left(4\nu_p(n)+9-6H^{(1)}_{\nu_p(n)+2}\right)\sigma_0(d)(i+1)$$
$$= (\sigma_0 * 1)(n) \times \sum_{\substack{p,q|n \\ p\neq q}} \frac{1}{6}\frac{\nu_q(n)(\nu_q(n)+3)}{(\nu_q(n)+1)(\nu_q(n)+2)}\left(4\nu_p(n)+9-6H^{(1)}_{\nu_p(n)+2}\right).$$

Now to roughly bound the error term, e.g., the GCD of prime omega functions from the exact formula for $C_3(n)$, we observe that the divisor function has average order of the form:

$$d(n) \sim \log n + (2\gamma - 1) + O\left(\frac{1}{\sqrt{n}}\right).$$

$\Omega(n) \sim \log\log n$, so that for large $x \gg 1$ tending to infinity, we can expect (on average) that for $p|n$, $1 \leq \nu_p(n)$ (for large $p|x, p \sim \frac{x}{\log x}$) and $\nu_p(n) \approx \log\log n$. However, if x is primorial, we can have $\Omega(x) \sim \frac{\log x}{\log\log x}$. There is, however, a duality with the size of $\Omega(x)$ and the rate of growth of the $\nu_p(x)$ exponents. That is to say that on average, even though $\nu_p(x) \sim \log\log n$ for most $p|x$, if $\Omega(x) = m \approx O(1)$ is small, then

$$\nu_p(x) \approx \log_{\sqrt[m]{\frac{x}{\log x}}}(x) = \frac{m\log x}{\log\left(\frac{x}{\log x}\right)}.$$

Since we will be essentially averaging the inverse functions, $g^{-1}(n)$, via their summatory functions over the range $n \leq x$ for x large, we tend not to worry about bounding anything but by the average case, which wins when we sum (i.e., average) and tend to infinity. Given these observations, we can use the function $C_3(n)$ we just painstakingly computed exactly as an asymptotic benchmark to build further approximations. In particular, the dominant order terms in $C_3(n)$ are given by

$$C_3(n) \sim \frac{(\sigma_0 * 1)(n)n^2}{\log^2 n} - \frac{(\sigma_0 * 1)(n)n^2}{\log n} + O\left(n \cdot \log\log n\right).$$

We will leave the terms involving the divisor function $\sigma_0(n)$ and convolutions involving it unevaluated because of how much their growth can fluctuate depending on prime factorizations for now.

Summary 3.5 (Asymptotics of the $C_k(n)$)**.** We have the following asymptotic relations relations for the growth of small cases of the functions $C_k(n)$:

$$C_1(n) \sim \log\log n$$

$$\sim \frac{1}{(k-1)!\log^k p}\left(\mathrm{Ei}((k-2)\log(dp^\alpha))\left[\log^{k-1}(d) - (k-1)!\log^{k-1}(p)\right]\right)$$
$$- \frac{1}{(k-2)(k-1)!\log^k p}\left(\log^{k-2}(d) + \alpha^{k-2}\log^{k-2}(p)\right).$$

We now simplify somewhat again by setting

$$p \mapsto \left(\frac{n}{e}\right)^{\frac{1}{\log\log n}}, \alpha \mapsto \log\log n, \log p \mapsto \frac{\log n}{\log\log n}.$$

Also, since $p \gg_n d$, we obtain the dominant asymptotic growth terms of

$$\mathrm{IC}_k(n) \sim \frac{\alpha^{k-2}}{(k-2)(k-1)!\log^2 p}$$

$$\approx \frac{(\log\log n)^k}{(k-2)(k-1)!\log^2 n}.$$

Now, as we did in the previous example work, we handle the sums by pulling out a factor of the inner divisor sum depending only on n (and k):

$$\begin{aligned}
C_k(n) &= \sum_{p|n} (\sigma_0 * \mathbb{1}_{*_{k-1}})(n) \binom{p^{\nu_p(n)}+k}{k}^{-1} \times \mathrm{IC}_k(n) \\
&= (\sigma_0 * \mathbb{1}_{*_{k-1}})(n) \binom{p^{\nu_p(n)}+k}{k}^{-1} \cdot \pi(n) \times \mathrm{IC}_k(n)
\end{aligned}$$

Combining with the remaining terms we get by induction, we have proved our target bound holds for $C_k(n)$.

To bound the error terms, again suppose inductively that $k \geq 4$. We compute the big-O bounds as follows

Corollary 3.6 (Computing the inverse functions)**.** *In contrast to the complicated formulation given by Lemma 3.2, we have that*

$$g^{-1}(n) \sim \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

This is to say that for all $n \geq 2$

$$\left|1 - \frac{\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)}{g^{-1}(n)}\right| = o\left(\sum_{d|n} C_{\Omega(d)}(d)\right).$$

Moreover, we can bound the error terms as

$$\left|\frac{\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d)}{g^{-1}(n)}\right| = O\left(\frac{(\log\log n)^2}{\log n} \cdot \frac{\Gamma(\log\log n)}{n^{\log\log n} \cdot (\log n)^{\log\log n}}\right) \xrightarrow{n\to\infty} 0.$$

Proof. Using Lemma 3.2, it suffices to show that the squarefree divisors $d|n$ such that $\operatorname{sgn}(\mu(d)\lambda(n/d)) = -1$ have an order of magnitude less adbundancy than the corresponding cases of positive sign on the terms in the divisor sum from the lemma. Let n have m_1 prime factors p_1 such that $v_{p_1}(n) = 1$, m_2 such that $v_{p_2}(n) = 2$, and the remaining $m_3 := \Omega(n) - m_1 - 2m_2$ prime factors of higher-order exponentation. We have a few cases to consider after re-writing the sum from the lemma in the following form:

$$g^{-1}(n) = \lambda(n)C_{\Omega(n)}(n) + \sum_{i=1}^{\omega(n)} \left\{ \sum_{\substack{d|n \\ \omega(d)=\Omega(d)=i \\ \#\{p|d:\nu_p(d)=1\}=k_1}} \mu(d)\lambda(n/d)C_{\Omega(n/d)}(n/d) \right\}.$$

To prove (7) we have to provide a straightforward bound that represents the maximums of the terms in m_1, m_2. In particular, observe that for

$$\widetilde{S}_{\mathrm{odd}}(m_1, m_2; u) = \sum_{k_1=0}^{u} \sum_{k_2=0}^{u-k_1} \left[\binom{m_1}{2k_1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1+1}\binom{2m_2}{2k_2} \right]$$

$$\widetilde{S}_{\mathrm{even}}(m_1, m_2; u) = \sum_{k_1=0}^{u} \sum_{k_2=0}^{u-k_1} \left[\binom{m_1}{2k_1+1}\binom{2m_2}{2k_2+1} + \binom{m_1}{2k_1}\binom{2m_2}{2k_2} \right],$$

we have that

$$\widetilde{S}_{\mathrm{odd}}(m_1, m_2; u) \gtrsim \binom{m_1}{2u+1} + \max_{1 \leq k_1 \leq u} \binom{m_1}{2k_1+1}\binom{2m_2}{2u+1-2k_1}$$

$$= \binom{m_1}{2u+1} + \binom{m_1}{2k_1+1}\binom{2m_2}{2u+1-2k_1}\Bigg|_{k_1=\frac{u}{2}}$$
$$= \binom{m_1}{2u+1} + \binom{m_1}{u+1}\binom{2m_2}{u+1}$$
$$\widetilde{S}_{\mathrm{even}}(m_1, m_2; u) \gtrsim \binom{m_1}{2u+1} + \max_{1\leq k_1\leq u}\binom{m_1}{2k_1}\binom{2m_2}{2u+1-2k_1}$$
$$= \binom{m_1}{2u+1} + \binom{m_1}{u-1}\binom{2m_2}{u+1}.$$

The lower bounds in (7) then follow by setting $u \equiv \left\lfloor \frac{i}{2} \right\rfloor$.
Part II (Bounding m_1, m_2, m_3 and effective (i, k_1, k_2) contributing to the count). We thus have to determine the

$$\gtrsim \frac{Ae}{B}\frac{x}{\log^{\frac{13}{14}}(x)} + \frac{Ae\sqrt{2}}{2\sqrt{\pi}B}\frac{x}{\log^{\frac{13}{14}}(x)\sqrt{\log\log x}}.$$

Now in practice, we are not summing up $n \leq x$, but rather $n \leq \log\log x$. So the above function evaluates to

$$N_{m_3}(\log\log x) \gg \frac{\log\log x}{(\log\log\log x)^{13/14}} \gg \frac{\log\log x}{\log\log\log x}.$$

Next, we go about solving the subproblem of finding when $i - k_1 - k_2 = m_3$. First, we find a lower solution index on i using asymptotics for the *Lambert W-function*, $W_0(x) = \log x - \log\log x + o(1)$:

$$\frac{i}{2} = \frac{\log\log x}{\log\log\log x} \iff \log\log x \gtrsim \frac{i}{2}\left(\log i + \log\log i\right)$$

$$\iff \frac{i}{2} \sim \frac{\log\log x}{\log\log\log x}.$$

Now since $2 \leq k_1 + k_2 \leq i/2$, when x is large, we actually obtain a number of soultions on the order of

$$\frac{\log\log x}{2} - \frac{\log\log x}{\log\log\log x} = \frac{\log\log x}{2}(1 + o(1)).$$

Part III (Putting it all together). Using the binomial coefficient inequality

$$\binom{n}{k} \geq \frac{n^k}{k^k},$$

we can work out carefully on paper using (7) that

the stated bound, which tends to zero as $x \to \infty$. Thus the divisor sum in the corollary statement accurately approximates the main term and sign of $g^{-1}(n)$ as $n \to \infty$. □

Corollary 3.7. *We have that for sufficiently large x, as $x \to \infty$ that*

$$G^{-1}(x) \gtrsim \widehat{L}_0(\log\log x) \times \sum_{n \leq \log\log x} \lambda(n) \cdot C_{\Omega(n)}(n),$$

where the function

$$\widehat{L}_0(x) := (-1)^{\left\lfloor \frac{3}{2}\log\log\log\log x \right\rfloor + 1} \left\{ \sqrt{\frac{3}{\pi}} \frac{A(2e+3)}{4B \log^{\frac{3}{2}}(2)} \right\} \cdot \frac{(\log\log\log x)^{\frac{43}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3}}}{\sqrt{\log\log\log\log x}},$$

with the exponent $\frac{43}{14} + \frac{3}{2\log 2} - \frac{3}{2\log 3} \approx 3.87011$.

Proof. Using Corollary 3.6, we have that

$$G^{-1}(x) \approx \sum_{n \leq x} \lambda(n) \cdot (g^{-1} * 1)(n)$$
$$= \sum_{d \leq \log\log x} C_{\Omega(d)}(d) \times \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn).$$

Now we see that by complete additivity (multiplicativity) of Ω (λ) that

$$\sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn) = \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(d)\lambda(n) = \lambda(d) \sum_{n \leq \left\lfloor \frac{x}{d} \right\rfloor} \lambda(n).$$

$$\lambda(n)\widehat{\tau}_0(n) \lesssim C_{\Omega(n)}(n) \lesssim \lambda(n)\widehat{\tau}_1(n).$$

It suffices to take the functions

$$\widehat{\tau}_0(n) := \frac{1}{\log 2} \cdot \frac{\log n}{(\log n)^{\log\log n}} \cdot \frac{n^{\log\log n - 1}}{\Gamma(\log\log n)}$$
$$\widehat{\tau}_1(n) := \frac{1}{2e\log 2} \cdot \frac{(\log n)^2}{(\log n)^{\log\log n}} \cdot \frac{n^{\log\log n}}{\Gamma(\log\log n)}.$$

Proof. The first stated formula follows from Theorem 1.5 by setting $k := \Omega(n) \sim \log\log n$ and simplifying. We evaluate the Dirichlet convolution functions and approximate as follows:

$$(\sigma_0 * \mathbb{1}_{\log\log n - 2})(n) = \sum_{p|n} \binom{\nu_p(n) + \log\log n - 1}{\log\log n - 1}$$

$$
\begin{aligned}
&\geq \sum_{p|n} \frac{(\nu_p(n) + \log\log n - 1)^{\log\log n - 1}}{(\log\log n)^{\log\log n - 1}} \\
&\sim \frac{n}{\log 2} \\
(\sigma_0 * \mathbb{1}_{\log\log n - 2})(n) &\leq \left(\frac{(\nu_p(n) + \log\log n - 1)e}{\log\log n - 1} \right)^{\log\log n - 1} \\
&\sim (2e)^{\log\log n - 1} \\
&= \frac{n \cdot \log n}{2e \log 2}.
\end{aligned}
$$

The upper and lower bounds are obtained from the next well known binomial coefficient approximations using Stirling's formula

4 Summing functions weighted by the Liouville lambda function, $\lambda(n) := (-1)^{\Omega(n)}$: *Borrowing a method of enumeration of summatory functions by Dirichlet series and Euler products from Montgomery and Vaughan, Chapter 7*

4.1 Discussion: The enumerative DGF result in Theorem 1.6 from Montgomery and Vaughan

In the reference we have defined $F(s, z)$ for $\Re(s) > 1$ such that the Dirichlet series coefficients, $a_z(n)$, are defined by

$$\zeta(s)^z F(s, z) := \sum \frac{a_z(n)}{n^s}, \Re(s) > 1.$$

the rate of divergence of the sum of the reciprocals of the primes, and on some generating function techniques involving elementary symmetric functions. The statements in Section 2.3 provide the basis for proving most of the lemmas we require.

4.2 The key new results utilizing Theorem 1.6

Corollary 4.1. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \gg 1.$$

When $s := 1$, we have the known bound in Mertens theorem. For $s > 1$, we obtain that

$$P_s(x) \approx E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1).$$

It follows that

$$\gamma_0(s,x)+o(1)\leq P_s(x)\leq \gamma_1(s,x)+o(1),$$

where it suffices to take

$$\begin{aligned}\gamma_0(z,x) &= -s\log\left(\frac{\log x}{\log 2}\right)-\frac{3}{4}s(s-1)\log(x/2)-\frac{11}{36}s(s-1)^2\log^2(2)\\ \gamma_1(z,x) &= \quad s\log\left(\frac{\log x}{\log 2}\right)-\frac{3}{4}s(s-1)\log(x/2)+\frac{11}{36}s(s-1)^2\log^2(x).\end{aligned}$$

Proof. Let $s>1$ be real-valued. By Abel summation where our summatory function is given by $A(x)=\pi(x)\sim\frac{x}{\log x}$ and our function $f(t)=t^{-s}$ so that $f'(t)=-s\cdot t^{-(s+1)}$, we obtain that

$$-c(x)\left(1+\frac{4}{z-1}+\frac{5}{(z-1)^2}+\frac{2}{(z-1)^3}\right).$$

In the previous equations, the upper and lower bounds formed by the functions (a,b,c) are given by

$$\begin{aligned}(a_\ell,b_\ell,c_\ell) &:= \left(-\log\left(\frac{\log x}{\log 2}\right),\frac{3}{4}\log\left(\frac{x}{2}\right),-\frac{11}{36}\log^2 2\right)\\ (a_u,b_u,c_u) &:= \left(\log\left(\frac{\log x}{\log 2}\right),-\frac{3}{4}\log\left(\frac{x}{2}\right),\frac{11}{36}\log^2 x\right).\end{aligned}$$

Now we make a prudent decision to set the uniform bound parameter to a middle ground value of $1<R<2$ as $R:=\frac{3}{2}$ so that

$$z\equiv z(k,x)=\frac{k-1}{\log\log x}\in[0,R),$$

for $x \gg 1$ very large. Thus $(z-1)^{-m} \in [(-1)^m, 2^m]$ for integers $m \geq 1$, and we can then form the upper and lower bounds from above. What we get out of these formulas is stated as in the theorem bounds. ☐

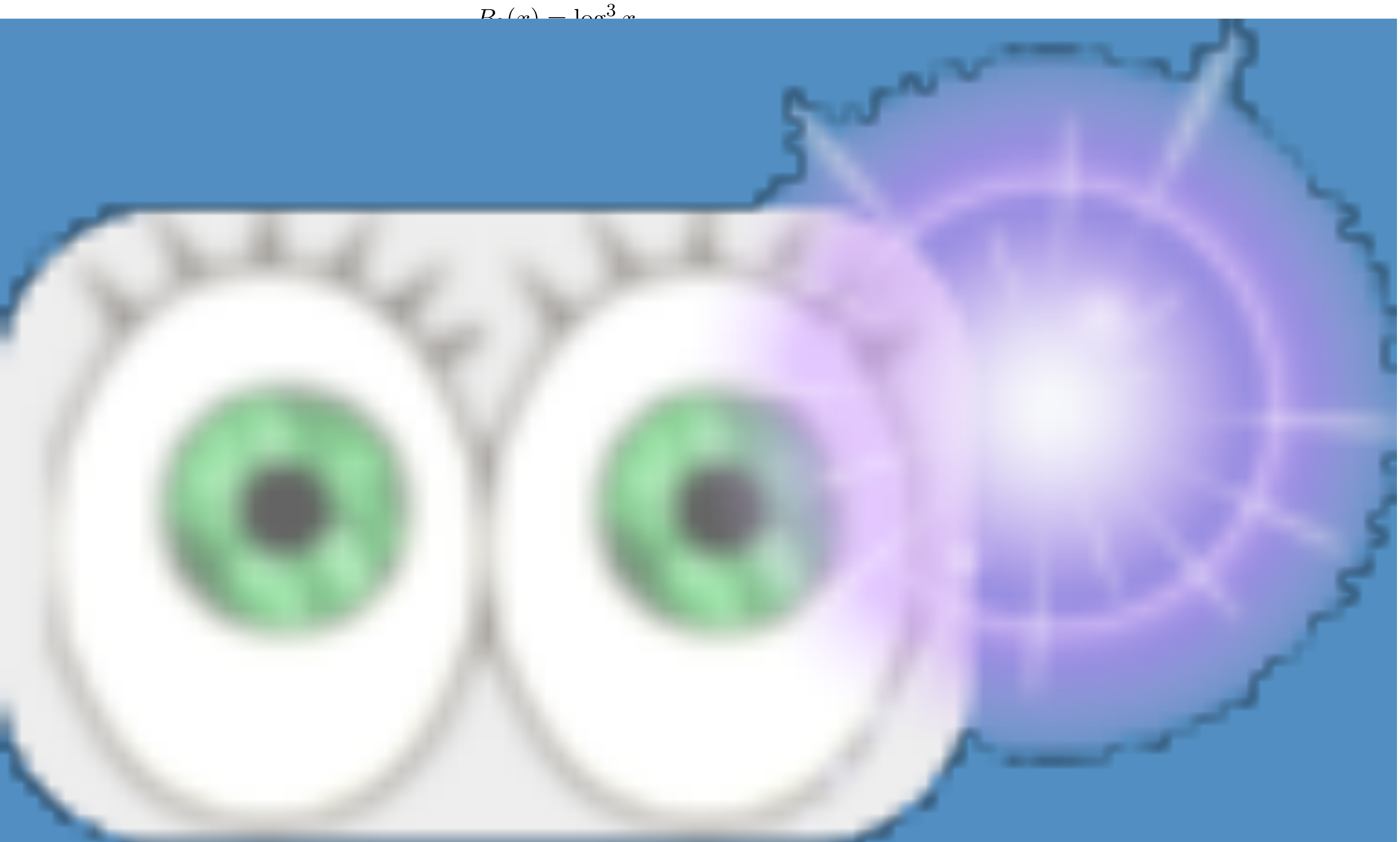
Corollary 4.2 (Bounds on $G(z)$ from MV)**.** *We have that for the function $G(z) := F(1,z)/\Gamma(z+1)$ from Montgomery and Vaughan, there is a constant A_0 and functions of x only, $B_0(x), C_0(x)$, so that*

$$A_0 \cdot B_0(x) \cdot C_0(x)^z \left(1 - \frac{z}{B}(\log x)^{\frac{1}{14}}\right) \leq G(z).$$

It suffices to take

$$A_0 = \frac{\exp\left(\frac{55}{4}\log^2 2\right)}{\log^3(2) \cdot \Gamma(5/2)} \approx 1670.84511225$$

$$B_0(x) = \log^3 x$$



$$F_\lambda(\log\log x) \gtrsim A_\Omega^{(\ell)}(x)\widehat{\tau}_\ell(\log\log x) - \int_1^{\log\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_\ell'(t)dt.$$

Proof. The formula for $F_\lambda(x)$ is valid by Abel summation provided that

$$\left| \frac{\sum\limits_{\frac{3}{2}\log\log t < k \leq \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_\Omega^{(\ell)}(t)} \right| = o(1),$$

e.g., the asymptotically dominant terms indicating the parity of $\lambda(n)$ are encompassed by the terms summed by $A_{\Omega}^{(\ell)}(t)$ for sufficiently large t as $t \to \infty$. Using the arguments in Montgomery and Vaughan [?, §7; Thm. 7.21], we can see that uniformly in x

$$\begin{aligned}\#\left\{n \leq x : \frac{\Omega(n) - \frac{3}{2}\log\log n}{\sqrt{\log\log n}} > 0\right\} &\sim x\left(1 - \Phi(\sqrt{\log\log x})\right) \\ &= x \cdot \Phi(-\sqrt{\log\log x}) \xrightarrow{x \to \infty} 0,\end{aligned} \tag{8}$$

where $\Phi(z)$ is the CDF of a standard normal random variable. Thus we have captured the asymptotically dominant main order terms in our formula as $x \to \infty$. $\square$

5 Key applications: Establishing lower bounds for $M(x)$ by cases along infinite subsequences

5.1 The culmination of what we have done so far

As noted before in the previous subsections, we cannot hope to evaluate functions weighted by $\lambda(n)$ except for on average using Abel summation. For this task, we need to know the bounds on $\widehat{\pi}_k(x)$ we developed in the proof of Corollary 4.2. A summation by parts argument shows that

$$M(x) = \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k) + 1)$$

$x/2$

$$C_{\ell,1} = \frac{\text{[illegible]}}{16}\sqrt{\frac{\text{[illegible]}}{2}}\,\frac{\text{[illegible]}}{\pi e B^2(\log 2)^3}, \quad C_{\ell,2} = \frac{\text{[illegible]}}{128\pi^{3/2}B^2(\log 2)^3},$$

and $\widehat{L}_0(x)$ is the multiplier function from Corollary 3.7:

$$\left|G_\ell^{-1}(x)\right| \gtrsim$$
$$\Bigg|(-1)^{\left\lfloor \frac{3}{2}\log_*^4(x)\right\rfloor} C_{\ell,1} \cdot (\log x)^{\frac{11}{7}} (\log\log x)^{\frac{71}{14}+\frac{3}{2\log 2}-\frac{3}{2\log 3}-\log_*^4(x)} \log_*^3(x)^{1+\frac{3}{2}\log\log x+\log_*^4(x)} \log_*^4(x)^{\log_*^4(x)-\frac{1}{2}}$$
$$- (-1)^{\left\lfloor 2\log_*^4(x)\right\rfloor} C_{\ell,2} \cdot \frac{\log_*^3(x)^{\frac{9}{2}+\frac{25}{6}\log 2+\frac{3}{2\log 2}-\frac{4}{3}\log 3-\frac{3}{2\log 3}}}{\sqrt{\log_*^4(x)}} \log_*^5(x)^{\frac{11}{7}+\frac{3}{2}\log_*^7(x)}\Bigg|.$$

Proof Sketch: Logarithmic scaling to the accurate order of the inverse functions. For the sums given by

$$S_{g^{-1}}(x) := \sum_{n \leq x} \lambda(n) \cdot C_{\Omega(n)}(n),$$

we notice that using the asymptotic bounds (rather than the exact formulas) for the functions $C_{\Omega(n)}(n)$, we have over-summed by quite a bit. In particular, following from the intent behind the constructions in the last sections, we are really summing only over all $n \leq x$ with $\Omega(n) \leq x$. Since $\Omega(n) \leq \lfloor \log_2 n \rfloor = \left\lfloor \frac{\log n}{\log 2} \right\rfloor$, many of the terms in the previous equation are actually zero (recall that $C_0(n) = \delta_{n,1}$). So we are actually only going to sum up to the average order of $\Omega(n) \sim \log\log n$ in practice, or to the slightly larger bound if the leading sign term on $G_\ell^{-1}(x)$ is negative. Hence, the sum (in general) that we are really interested in bounding is bounded below in magnitude by $S_{g^{-1}}(\log\log x)$ or $S_{g^{-1}}(\log_2(x))$, where we can now safely apply the asymptotic formulas for the $C_l(n)$ functions from Corollary 3.8 that hold once we have verified these constraints. □

The integration term in (11) is summed approximately as follows:

$$\int_{u_0-1}^{\log\log x} \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t) dt \sim \sum_{k=u_0+1}^{\frac{1}{2}\log\log\log\log x} \left(\frac{I_\ell\left(e^{e^{\frac{4k+2}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!} - \frac{I_\ell\left(e^{e^{\frac{4k}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!} \right) e^{e^{\frac{4k}{3}}}$$

$$\approx C_0(u_0) + (-1)^{\left\lfloor \frac{\log\log\log\log x}{2} \right\rfloor} \times \int_{\frac{\log\log\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log\log\log x}{2}} \frac{I_\ell\left(e^{e^{\frac{4k}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!} e^{e^{\frac{4k}{3}}} dk.$$

The differences on the upper and lower bounds on each integral in the last equation is small, and in particular $\frac{1}{2} \lll \log\log x$. So we can use a small perturbation of $+1$ in the power terms of $I_\ell(t)$ combined with an appeal to the binomial series, the expansion of binomial coefficients by the Stirling numbers of the first kind, and the following exact indefinite integral for $x, z \in \mathbb{R}$ moving forward:

$$\int t^p e^{ct} dt = \frac{(-1)^p}{c^{p+1}} \Gamma(p+1, -ct) \sim \frac{e^{ct} t^p}{c}.$$

Define the following function of t and note the change of variable $t \mapsto \frac{k-1}{2}$:

$$I_\ell\left(e^{e^{\frac{4k}{3}}}\right) e^{e^{\frac{4k}{3}}} = (1+k)^{2k} \exp\left(-\frac{16k^2}{9}\left(\frac{4k}{3}-1\right) e^{\frac{4k}{3}}\right) e^{2k-1} \widehat{f}(t_0).$$

So we take one reciprocal factor in the next integrand, and set the remaining powers of t^p to be t_0^p for t_0 a bound of integration which results in a lower bound on our target integrand from Abel summation.

Applying Stirling's formula again when x is large, we have that

$$\widehat{R}_\ell(x) = (-1)^{\left\lfloor \frac{\log\log\log\log x}{2} \right\rfloor} \times \int_{\frac{\log\log\log\log x}{2} - \frac{1}{2}}^{\frac{\log\log\log\log x}{2}} \frac{I_\ell\left(e^{e^{\frac{4k}{3}}}\right)}{(2k)!\left(\frac{4k}{3}\right)!} e^{e^{\frac{4k}{3}}} dk \tag{12}$$

$$\gtrsim (-1)^{\left\lfloor \frac{x}{2} \right\rfloor} \times \widehat{h}\left(\frac{\log\log\log\log x}{2}\right)\Bigg[$$

[4]Namely, that for natural numbers $j \geq 0$

$$\sum_{k \geq 0} \begin{bmatrix} k \\ j \end{bmatrix} \frac{z^k}{k!} = \frac{(-1)^j}{j!} \operatorname{Log}^j(1-z).$$

$$\widehat{T}\left(\frac{\log\log\log\log x}{2};\frac{\log\log\log\log x}{2}\right)\left(\mathrm{cf}_+\left(\frac{\log\log\log\log x-1}{2}\right)-\mathrm{cf}_-\left(\frac{\log\log\log\log x}{2}\right)\right)$$
$$-\widehat{T}\left(\frac{\log\log\log\log x-1}{2};\frac{\log\log\log\log x-1}{2}\right)\left(\mathrm{cf}_+\left(\frac{\log\log\log\log x}{2}\right)-\mathrm{cf}_-\left(\frac{\log\log\log\log x-1}{2}\right)\right)\Bigg].$$

Since for real $0 < s < 1$ such that $s \to 0$, we have that $\log(1+s) \sim s$ and $(1+s)^{-1} \sim 1-s$, we can approximate the differences implied by the last estimate using that for t large tending to infinity we have

$$\log_*^m\left(t-\frac{1}{2}\right) \sim \log_*^m(t) - \frac{1}{2\log^{m-1} t}, m \geq 1.$$

Then applying these simplifications to (12) above and removing lower-order terms that do not contribute to the dominant asymptotic terms, we find that

$$\frac{|M(x_y)|}{\sqrt{x_y}} \gtrsim \frac{1}{2}\sqrt{\frac{3}{2}}\frac{A^2(2e+3)^2}{\pi e B^2(\log 2)^3}\left[\left(\frac{\log x_y}{4}\right)^{\frac{3}{2}\log\log\log\log x_y-\frac{3}{7}}\log\log\log x_y\cdot\sqrt{\log\log\log\log x_y}\cdot(\log\log x_y)^{\frac{71}{14}+\frac{3}{2\log 2}-\frac{3}{2\log 3}}\right],$$

as $y \to \infty$.

Proof of the Asymptotic Lower Bound. Since $\pi(1) = 0$ and $\pi(j) = \pi(\sqrt{x})$ for all $\sqrt{x} \leq j < x$, we can write (9) in the following form using Abel summation:

$$M(x) \gtrsim G_\ell^{-1}(x) + G_\ell^{-1}(u_0)A_\Omega^{(\ell)}(u_0) - G_\ell^{-1}(x)A_\Omega^{(\ell)}(x) \tag{14}$$
$$+\int_{u_0}^{\sqrt{x}}\frac{2\sqrt{x}}{t^2\log(x)}\frac{d}{dt}\left[G_\ell^{-1}(t)\right]dt + \int_{\sqrt{x}}^{x/2}\frac{x}{t^2\log(x/t)}\frac{d}{dt}\left[G_\ell^{-1}(t)\right]dt.$$

We have that one of these integrals contributes a term of $o(\sqrt{x})$ in the form of

$$\left| \int_{u_0}^{\sqrt{x}} \frac{2\sqrt{x}}{t^2 \log x} \frac{d}{dt} \left[G_\ell^{-1}(t) \right] dt \right| \leq \left| \frac{2\sqrt{x}}{\log x} \left(\frac{1}{\sqrt{x}} - \frac{1}{u_0} \right) \frac{d}{dx} \left[|G_\ell^{-1}(t)| \right] \Big|_{t=\sqrt{x}} \right| = o\left(\sqrt{x}\right).$$

We select x and u_0 so that $G_\ell^{-1}(u_0) = 0$, as is it's derivative, and $\min(-G_\ell^{-1}(u_0), -G_\ell^{-1}(\sqrt{x})) = -G_\ell^{-1}(\sqrt{x})$. Then (14) becomes

$$\begin{aligned} M(x) &\gg G_\ell^{-1}(x) - G_\ell^{-1}(x) A_\Omega^{(\ell)}(x) + \int_{u_0}^{\sqrt{x}} \frac{x}{t^2 \log(x/t)} \frac{d}{dt} \left[G_\ell^{-1}(t) \right] dt + o(1) \\ &\gg G_\ell^{-1}(x) - G_\ell^{-1}(x) A_\Omega^{(\ell)}(x) + \int_{u_0}^{\sqrt{x}} \frac{x}{t^2 \log(x/t)} \frac{d}{dt} \left[G_\ell^{-1}(t) \right] dt + o(1) \end{aligned}$$

5.3 Remarks

Remark 5.3 (Tightness of the lower bounds)**.** One remaining question for scaling $|M(x)|/f(x)$ is exactly how tight can the function $f \in \mathcal{F}$ be made so that (for $\mathcal{F}$ some reasonable function space with bases in polynomials of x and powers of iterated logarithms)

$$f(x) := \operatorname{argmax}_{h \in \mathcal{F}} \left\{ \limsup_{x \to \infty} \frac{|M(x)|}{h(x)} = C_h \right\},$$

for some absolute constants $C_h > 0$? What we have proved is that we can take

$$f(x) = \sqrt{x} \cdot TODO,$$

to obtain the above where the limiting constant is

$$C_f \mapsto TODO.$$

But is this the tightest possible f provably? (*Note that Gonek's famous conjecture states that we ought have* $f(x) := \sqrt{x} \cdot (\log\log x)^{5/4}$. *We apparently are able to asymptotically win here by a logarithmic landslide over this estimate, but at a cost of our witness infinite subsequence of positive reals being gigantic in asymptotic order, and hence numerically infeasible to compute* $M(x)$ *along with modern methods* .)

There is also, of course, the possibility of tightening the bound from above using the upper bounds proved in Theorem 1.7 along an infinite subsequence tending to infinity. We do not approach this problem here due to length constraints and that our lower bounds seem to have done much better than was previously known before about the (un)boundedness of the scaled Mertens function – our initial so-called "pipe dream", or "impossible", result.

6 Generalizations to weighted Mertens functions (TODO)

6.1 A standard notion of generalizing the Mertens function

We define the notion of a *generalized, or weighted, Mertens summatory function* for fixed $\alpha \in \mathbb{C}$ as

$$M_\alpha(x) = \sum_{n \leq x} \frac{\mu(n)}{n^\alpha},\ x \geq 1, \tag{15}$$

where the special case of $M_0^*(x)$ coincides with the definition of the classical Mertens function $M(x)$. We have in these cases that

$$M_\alpha(x) = \sum_{n \leq x} g^{-1}(x) \left[\pi\left(\frac{x}{n}\right) + \left(\frac{n}{x}\right)^\alpha\right]. \tag{16}$$

$$\lambda(n) = (-1)^{\Omega(n)} = \sum_{d^2 | n} \mu(n/d^2).$$

For example, it is known that [?]

- For $\alpha = 0, 1$ and $0 < \alpha < \frac{1}{2}$, a counting argument shows that $L_\alpha(x)$ changes sign infinitely often, though these functions exhibit a known negative bias that is numerically apparent for large x.
- We have the trivial *upper bounds* providing that

$$|L_\alpha(x)| \leq \sum_{n \leq x} n^{-\alpha} \ll x^{1-\alpha}.$$

- For some absolute constant $C > 0$,

$$|L_\alpha(x)| = O\left(x^{1-\alpha} \exp\left\{-C\frac{(\log x)^3 5}{(\log\log x)^{1/5}}\right\}\right), 0 \leq \alpha \leq \frac{1}{2}.$$

- Since

$$\frac{\zeta(2\alpha+2s)}{\zeta(\alpha+s)} = s\int_1^\infty \frac{L_\alpha(x)}{x^s}\frac{dx}{x},$$

  we have exact formulas for $L_\alpha(x)$ involving the non-trivial zeros of the Riemann zeta function by Mellin inversion (*cf.* Theorem 1.1).

- The RH is equivalent to $L_\alpha(x) = O(x^{\frac{1}{2}-\alpha+\varepsilon})$ for all $\varepsilon > 0$ when $0 \leq \alpha \leq \frac{1}{2}$.

7 Conclusions (TODO)

References

[1] T. M. Apostol. *Introduction to Analytic Number Theory.* Springer–Verlag, 1976.

[2] C. E. Fröberg. On the prime zeta function. *BIT Numerical Mathematics*, 8:87–202, 1968.

[3] H. W. Gould and T. Shonhiwa. A catalog of interesting Dirichlet series. *Missouri J. Math. Sci.*, 20(1):2–18, 2008.

[4] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science.* Addison-Wesley, 1994.

[5] M. Merca and M. D. Schmidt. Factorization theorems for generalized Lambert series and applications. *Ramanujan Journal*, 2019.

T.1 Table: Computations with a highly signed Dirichlet inverse function

Appendix A Reference on common abbreviations, special notation and other conventions

Symbol | **Definition**

$\lceil x \rceil$ — The ceiling function $\lceil x \rceil := x + 1 - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$.

$C_k(n)$ — Auxillary component functions in obtaining asymptotic bounds on $g^{-1}(n)$. These functions are defined for $k \geq 0$ and $n \geq 1$ by the formula

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \ldots \end{cases}$$

$\mathbb{1}_{\mathbb{S}}, \chi_{\mathtt{cond}}(x)$ — We use the notation $\mathbb{1}, \chi : \mathbb{N} \to \{0, 1\}$ to denote indicator, or characteristic functions. In paticular, $\mathbb{1}_{\mathbb{S}}(n) = 1$ if and only if $n \in \mathbb{S}$, and $\chi_{\mathtt{cond}}(n) = 1$ if and only if n satisfies the condition $\mathtt{cond}$.

$\log_*^m(x)$ — The iterated logarithm function defined recursively for integers $m \geq 0$ by

$$\log_*^m(x) := \begin{cases} x, & \text{if } m = 0; \\ \log x, & \text{if } m = 1; \\ \log\left[\log_*^{m-1}(x)\right], & \text{if } m \geq 2. \end{cases}$$

$[n = k]_\delta$ — Synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and zero otherwise.

$[\mathtt{cond}]_\delta$ — For a boolean-valued $\mathtt{cond}$, $[\mathtt{cond}]_\delta$ evaluates to one precisely when $\mathtt{cond}$ is true, and zero otherwise.

Symbol	**Definition**
$\gcd(m,n); (m,n)$	The greatest common divisor of m and n. Both notations for the GCD are used interchangably within the article.
$\mu(n)$	The Möbius function.
$M(x)$	The Mertens function which is the summatory function over $\mu(n)$, $M(x) := \sum\limits_{n \leq x} \mu(n)$.
$\nu_p(n)$	The function that extracts the prime exponent of p from the prime factorization of n.
$\sum_{p \leq x}, \prod_{p \leq x}$	Unless otherwise specified by context, we use the index variable p to denote that the summation (product) is to be taken only over prime values within the summation bounds.

Appendix B Accompanying software references and examples (TODO)

n	**Primes**		**Sqfree**	**PPower**	$\bar{\mathbb{S}}$		$g^{-1}(n)$	$\lambda(n)\operatorname{sgn}(g^{-1}(n))$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_2(n)$		$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1^1	–	Y	N	N	–	1	1	0	0	–	1	1	0
2	2^1	–	Y	Y	N	–	-2	1	0	0	–	-1	1	-2
3	3^1	–	Y	Y	N	–	-2	1	0	0	–	-3	1	-4
4	2^2	–	N	Y	N	–	2	1	0	-1	–	-1	3	-4
5	5^1	–	Y	Y	N	–	-2	1	0	0	–	-3	3	-6
6	$2^1 3^1$	–	Y	N	N	–	5	1	0	-1	–	2	8	-6
7	7^1	–	Y	Y	N	–	-2	1	0	0	–	0	8	-8
8	2^3	–	N	Y	N	–	-2	1	0	-2	–	-2	8	-10
9	3^2	–	N	Y	N	–	2	1	0	-1	–	0	10	-10
10	$2^1 5^1$	–	Y	N	N	–	5	1	0	-1	–	5	15	-10
11	11^1	–	Y	Y	N	–	-2	1	0	0	–	3	15	-12
12	$2^2 3^1$	–	N	N	Y	–	-7	1	2	-2	–	-4	15	-19
13	13^1	–	Y	Y	N	–	-2	1	0	0	–	-6	15	-21
14	$2^1 7^1$	–	Y	N	N	–	5	1	0	-1	–	-1	20	-21
15	$3^1 5^1$	–	Y	N	N	–	5	1	0	-1	–	4	25	-21
16	2^4	–	N	Y	N	–	2	1	0	-3	–	6	27	-21

Table T.1: ***Computations of the first several cases of*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for*** $\mathbf{1 \leq n \leq 56}$.
The column labeled `Primes` *provides the prime factorization of each* n *so that the values of* $\omega(n)$ *and* $\Omega(n)$ *are easily extracted. The columns labeled, respectively,* `Sqfree`, `PPower` *and* $\bar{\mathbb{S}}$ *list inclusion of* n *in the sets of squarefree integers, prime powers, and the set* $\bar{\mathbb{S}}$ *that denotes the positive integers* n *which are neither squarefree nor prime powers. The next two columns provide the explicit values of the inverse function* $g^{-1}(n)$ *and indicate that the sign of this function at* n *is given by* $\lambda(n) = (-1)^{\Omega(n)}$.

Then the next two columns show the small-ish magnitude differences between the unsigned magnitude of $g^{-1}(n)$ *and the summations* $\widehat{f}_1(n) := \sum_{k \geq 0} \binom{\omega(n)}{k} \cdot k!$ *and* $\widehat{f}_2(n) := \sum_{k \geq 0} \binom{\omega(n)}{k} \cdot \#\{d|n : \omega(d) = k\}$. *Finally, the last three columns show the summatory function of* $g^{-1}(n)$, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, *borken down into its respective positive and negative components:* $G_+^{-1}(x) := \sum_{n \leq x} g^{-1}(n) \left[g^{-1}(n) > 0\right]_\delta$ *and* $G_-^{-1}(x) := \sum_{n \leq x} g^{-1}(n) \left[g^{-1}(n) < 0\right]_\delta$.