Lower bounds on the summatory function of the Möbius function along infinite subsequences

Maxie Dion Schmidt
Georgia Institute of Technology
School of Mathematics

Last Revised: Sunday 7th June, 2020 @ 04:49:13 – Compiled with LaTeX2e

Abstract

The Mertens function, $M(x) = \sum_{n \leqslant x} \mu(n)$, is classically defined as the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture stating that $|M(x)| < C \cdot \sqrt{x}$ with come absolute $C > 0$ for all $x \geqslant 1$ has a well-known disproof due to Odlyzko and té Riele given in the early 1980's by computation of non-trivial zeta function zeros in conjunction with integral formulas expressing $M(x)$. It is conjectured that $M(x)/\sqrt{x}$ changes sign infinitely often and grows unbounded in the direction of both $\pm\infty$ along subsequences of integers $x \geqslant 1$. We prove a weaker property related to the unboundedness of $|M(x)|/\sqrt{x}$ by showing that

$$\limsup_{x \to \infty} \frac{|M(x)|(\log\log x)^{\frac{3}{4}}(\log\log\log x)^2}{\sqrt{x} \cdot (\log x)^{\frac{1}{2}}} > 0.$$

There is a distinct stylistic flavor and new element of combinatorial analysis to our proof peppered in with the standard methods from analytic, additive and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

Keywords and Phrases: *Möbius function; Mertens function; summatory function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting functions; Dirichlet generating function; asymptotic lower bounds; Mertens conjecture.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; and 11N64.*

1 Introduction

1.1 Definitions

Suppose that $n \geqslant 2$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ so that $r = \omega(n)$. We define the *Möebius function* to be the signed indicator function of the squarefree integers as follows:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leqslant i \leqslant k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many other variants and special properties of the Möebius function and its generalizations [15, *cf.* §2]. A crucial role of the classical $\mu(n)$ forms an inversion relation for arithmetic functions convolved with one by *Möbius inversion*:

$$g(n) = (f * 1)(n) \iff f(n) = (g * \mu)(n), \forall n \geqslant 1.$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined as

$$M(x) = \sum_{n \leqslant x} \mu(n), x \geqslant 1.$$

The sequence of the oscillatory values of this summatory function begins as [16, A002321]

$$\{M(x)\}_{x \geqslant 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2, \ldots\}$$

Clearly, a positive integer $n \geqslant 1$ is *squarefree*, or contains no (prime power) divisors which are squares, if and only if $\mu^2(n) = 0$. A related summatory function which counts the number of *squarefree* integers $n \leqslant x$ then satisfies [4, §18.6] [16, A013928]

$$Q(x) = \sum_{n \leqslant x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \to \infty$:

$$\mu_+(x) = \frac{\#\{1 \leqslant n \leqslant x : \mu(n) = +1\}}{Q(x)} \overset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leqslant n \leqslant x : \mu(n) = -1\}}{Q(x)} \xrightarrow{x \to \infty} \frac{3}{\pi^2}.$$

The actual local oscillations between the approximate densities of the sets $\mu_\pm(x)$ lend an unpredictable nature to the function and characterize the oscillatory sawtooth shaped plot of $M(x)$ over the positive integers.

1.2 Properties

One conventional approach to evaluating the behavior of $M(x)$ for large $x \to \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is expressed given the inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \int_1^\infty \frac{s \cdot M(x)}{x^{s+1}} dx,$$

we obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T - \imath\infty}^{T + \imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation, along with the standard Euler product representation for the reciprocal zeta function cited in the first equation above, leads us to the exact expression for $M(x)$ for any real $x > 0$ given by the next theorem due to Titchmarsh.

Theorem 1.1 (Analytic Formula for $M(x)$)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k\geqslant 1}$ satisfying $k \leqslant T_k \leqslant k+1$ for each k such that for any real $x > 0$*

$$M(x) = \lim_{k\to\infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |\operatorname{Im}(\rho)|<T_k}} \frac{x^\rho}{\rho\cdot\zeta'(\rho)} - 2 + \sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n\cdot(2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2}\left[x\in\mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x\cdot\exp\left(-C\cdot\log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan recently proved new updated estimates bounding $M(x)$ for large x in the following forms [17]:

$$\begin{aligned} M(x) &\ll \sqrt{x}\cdot\exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x}\cdot\exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right),\ \forall\epsilon>0. \end{aligned}$$

1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which posits that

$$|M(x)| < C\cdot\sqrt{x},\quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens for $C = 1$ and all $x < 10000$. Since its beginnings in 1897, the Mertens conjecture has been disproven by computation of non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele from the early 1980's [12]. Since the truth of the conjecture would have implied the RH, more recent attempts at bounding $M(x)$ consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound towards both $\pm\infty$ along infinite subsequences.

One of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ is in actuality unbounded on the natural numbers. A precise statement of this problem is to produce an affirmative answer whether $\limsup_{x\to\infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x\to\infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there are an infinite subsequences of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound towards either $\pm\infty$ along the subsequence. We cite that prior to this point it is only known by computation that [14, *cf.* §4.1] [16, *cf.* A051400; A051401]

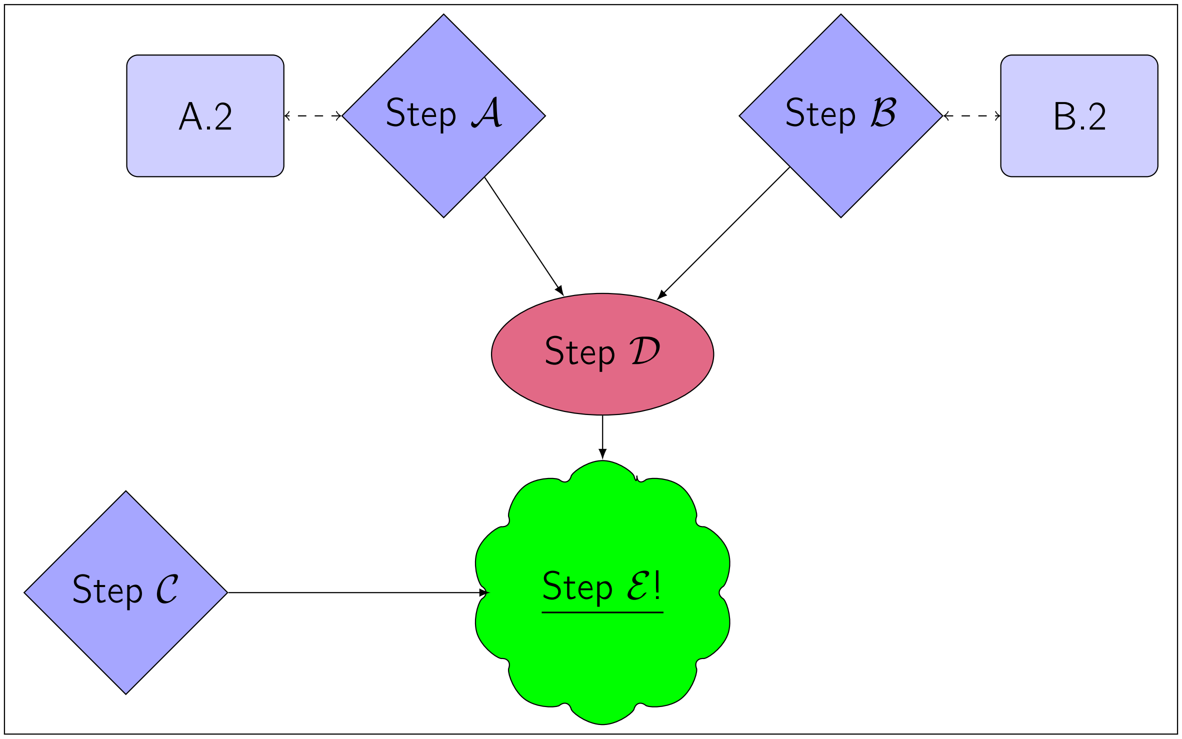
$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geqslant 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leqslant -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [12, 7, 8, 5]. Extensive computational evidence has produced a conjecture due to Gonek (among attempts on exact bounds by others) that in fact the limiting behavior of $M(x)$ satisfies [11]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}\cdot(\log\log x)^{5/4}} = O(1).$$



Legend to the diagram stages:

- **Step A:** *Citations and re-statements of existing theorems proved elsewhere.*
  - **A.A:** Key results and constructions:
    - Theorem 3.6
    - Corollary 5.5
    - The results, lemmas, and facts cited in Section 4.3
  - **A.2:** Lower bounds on the Abel summation based formula for $G^{-1}(x)$:
    - Theorem 3.7 (on page 18)
    - Proposition 5.6
    - Theorem 8.5
- **Step B:** *Constructions of an exact formula for $M(x)$.*
  - **B.B:** Key results and constructions:
    - Corollary 3.3 (follows from Theorem 3.1 proved on page 12)
    - Proposition 4.1
  - **B.2:** Asymptotics for the component functions $g^{-1}(n)$ and $G^{-1}(x)$:
    - Theorem 6.3 (on page 21)
    - Lemma 6.4
- **Step C:** *A justification for why lower bounds obtained roughly "on average" suffice.*
  - The results proved in Section 7
- **Step D:** *Interpreting the exact formula for $M(x)$.*
  - Proposition 8.1
  - Theorem 8.5
- **Step E:** *The Holy Grail.* Proving that $\frac{|M(x)|}{\sqrt{x}}$ grows without bound in the limit supremum sense.
  - Corollary 3.8 (on page 38)