# THE MATRIX

## 1. Set up

Let $0 < b_1 < \cdots < b_r$ and $0 < c_1 < \cdots < c_r$ be real numbers. Assume that

$$A = \begin{bmatrix} e^{b_1c_1} & \cdots & e^{b_1c_r} \\ \vdots & \ddots & \vdots \\ e^{b_rc_1} & \cdots & e^{b_rc_r} \end{bmatrix}$$

We want to find a lower bound for the smallest eigenvalue $\lambda_1$ of the $r \times r$ matrix $A$. We have the result from [1, Chapter 4] that $A$ is a strictly positive matrix, meaning that all of its eigenvalues are positive. We know from [2, Remark Page 4] that the smallest singular value $\sigma_1$ is larger than

$$\sigma_1 > \frac{|\det(A)|}{2^{\frac{r}{2}-1} \parallel A \parallel_2} > 0 \tag{1.1}$$

Let $\sigma_1$ and $\lambda_1$ denote the smallest singular value and smallest eigenvalue of A, respectively. We first show that $|\sigma_1| \leq \lambda_1$. Let $v$ be a unit eigenvector of $A$ for the eigenvalue $\lambda_1$ with $\parallel v \parallel_2 = 1$. Since $Av = \lambda_1 v$, we have that

$$v^T A^T A v = \parallel Av \parallel_2^2 = \lambda_1^2 \parallel v \parallel_2^2 = \lambda_1^2.$$

It is not difficult to verify that $A$ and $A^TA$ are a positive definite matrices. Thus, we can write $A^TA = U^TDU$ for $U$ unitary and some diagonal matrix $D$ which has nonnegative diagonal entries. By definition, $\sigma_1^2$ corresponds to the minimum value of the eigenvalues of $v^TA^TAv$. Hence, we get that

$$\lambda_1^2 = v^TA^TAv \geq \min_{\|x\|=1} x^TA^TAx = \min_{\|x\|=1} (Ux)^TD(Ux) = \min_{\|y\|=1} y^TDy = \sigma_1^2.$$

The bound in (1.1) is then also a lower bound for $\lambda_1$. Since $\parallel A \parallel_2 \leq re^{b_rc_r}$ by the bound of the 2-norm from above by $\parallel A \parallel_F$, we need only to find a lower bound for $\det(A)$ to effectively bound $\lambda_1$ using (1.1).

**Definition 1.1.** Let $B, C \in \mathbb{M}_r(\mathbb{R}^+)$ be the respective Vandermonde matrices in our constants $\{b_1, \ldots, b_r\}$ and $\{c_1, \ldots, c_r\}$ defined as follows:

$$B = \begin{bmatrix} 1 & b_1 & b_1^2 & \cdots & b_1^{r-1} \\ 1 & b_2 & b_2^2 & \cdots & b_2^{r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & b_r & b_r^2 & \cdots & b_r^{r-1} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ c_1 & c_2 & c_3 & \cdots & c_r \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1^{r-1} & c_2^{r-1} & c_3^{r-1} & \cdots & c_r^{r-1} \end{bmatrix}.$$

Since $B$ is a Vandemonde matrix and $C$ is the transpose of a Vandermonde matrix, each of $B$ and $C$ are invertible. Let $m$ be a natural number such that

$$m > 3 + \max\left\{ r, \max_{\substack{1\leq i,j \leq r \\ i \neq j}} \frac{r! e^{b_r}}{(b_i - b_j)}, \max_{\substack{1\leq i,j \leq r \\ i \neq j}} \frac{r! e^{c_r}}{(c_i - c_j)}, \right\} \tag{1.2}$$

Assume that the matrix $H \in \mathbb{M}_r(\mathbb{R})$ is defined such that its $(i,j)^{th}$ entries are given by

$$H_{ij} = \sum_{\ell=m}^{\infty} \frac{b_i^\ell c_j^\ell}{\ell!}.$$

Let the matrix $E \in \mathbb{M}_r(\mathbb{R}^+)$ be defined by

$$E = [\epsilon_{ij}] := B^{-1} H C^{-1}.$$

Suppose that $D \in \mathbb{M}_r(\mathbb{R}^+)$ is the diagonal matrix defined by

$$D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 \cdots & 0 & \frac{1}{(r-1)!} \end{bmatrix}$$

We define the $r \times r$ real matrix $T$ as follows:

$$T = B(D+E)C.$$

## 2. Proofs

**Lemma 2.1.** *For every $0 < a < \log\left(\frac{m}{r!}\right)$ (tightened) and $x < \frac{\pi^{\frac{1}{4}}}{ea}\sqrt{e - \frac{1}{2}} \times (m-1)m^{\frac{1}{m-1}}$ (being precise is good – does this help?) we have*

$$e^{ax} - 2\sum_{\ell=r}^{m-1} \frac{a^\ell x^\ell}{\ell!} > \frac{1}{2}.$$

*Proof.* We prove the lemma inductively. For $a > 0$, let

$$f(x) = e^{ax} - 2\sum_{\ell=r}^{m-1} \frac{a^\ell x^\ell}{\ell!} - \frac{1}{2}.$$

For $B_m > 0$ we have that

$$f\left(\frac{B_m}{a}\right) > e^{B_m} - \frac{2mB_m^{m-1}}{(m-1)!} - \frac{1}{2}.$$

Then $f(0) = \frac{1}{2} > 0$ and by arithmetic we can verify that for all sufficiently large $m$

$$f\left(\frac{\pi^{\frac{1}{4}}}{ea}\sqrt{e - \frac{1}{2}} \times (m-1)m^{\frac{1}{m-1}}\right) > 0.$$

We conclude that if for some $x_0 \in \mathbb{R}$ that $f(x_0) = 0$, then $f$ also has a local minimum at some $x_1 > 0$. Hence, if $f(x_0) = 0$ then $f'(x_1) = 0$ as well. But one can see by direct computation that

$$f'(x) = ae^{ax} - 2a\sum_{\ell=r-1}^{m-2} \frac{a^\ell x^\ell}{\ell!}.$$

By similar reasoning, if $f'(x_0) = 0$ for some $x_0 > 0$, then we must have that $f''(x_2) = 0$ for some $x_2 > 0$. That is

$$f''(x) = a^2 e^{ax} - 2a^2 \sum_{\ell=r-2}^{m-3} \frac{a^\ell x^\ell}{\ell!} = 0, \text{ for some } x > 0.$$

Inductively applying this argument, we see that $f(x_0) = 0$ for some $x_0 > 0$ if and only if

$$e^{ax_r} - 2 \sum_{\ell=0}^{m-r-1} \frac{a^\ell x_r^\ell}{\ell!} = 0, \text{ for some } x_r \geq 0.$$

But we see that this condition can never be attained because with an appropriate choice of $m$ we always have that the tail of the exponential series satisfies

$$\sum_{\ell=0}^{m-r-1} \frac{a^\ell x^\ell}{\ell!} > \sum_{\ell=m-r}^{\infty} \frac{a^\ell x^\ell}{\ell!}.$$

We conclude that $f(x) \neq 0$ for all $x > 0$. □

**Theorem 2.2.** *We have*

$$\det(A) > \frac{2^{-r} e^{r(b_1 c_1 - b_r c_r)}}{r!^{r-1}} \times \prod_{i<j} (b_j - b_i)(c_j - c_i).$$

*Proof.* Recall that we have defined $T = B(D+E)C$ in terms of the matrices from Definition 1.1. Straightforward expansion shows that

$$T = A - H'$$

where the $(i,j)^{th}$ entries of the $r \times r$ matrix $H'$ correspond to

$$H'_{ij} = \sum_{\ell=r}^{m-1} \frac{b_i^\ell c_j^\ell}{\ell!}.$$

A simple algebraic manipulation of the formula for $A$ in terms of $T$ given above shows that

$$\det(A) = \det(T)\det(I + T^{-1}(A-T)) = \det(T)\det\left(I + T^{-1}H'\right). \tag{2.1}$$

We argue that $\| T^{-1}H' \|_2$ is small. Hence, with the identity that for any positive matrices $M_1, M_2$ we have that $\det(M_1 + M_2) \geq \det(M_1) + \det(M_2)$, we find that we can bound $\det(A)$ from below well by approximating $\det(T)$ (we do not know the matrix $H'$ is positive. We can use the continuity of det with respect to $L^2-$norm metric here). By the known determinant formula for Vandermonde matrices, we see that

$$\det(T) = \det(D+E) \times \prod_{i<j} (b_j - b_i)(c_j - c_i). \tag{2.2}$$

We have

$$\| T^{-1}H' \|_2^2 \leq \frac{\| H' \|_2^2}{\| T \|_2^2} = \frac{\operatorname{Tr}\left((A-T)(A-T)^T\right)}{\operatorname{Tr}(TT^T)}$$

$$= \frac{\mathrm{Tr}(AA^T) + Tr(TT^T) - 2Tr(AT^T)}{\mathrm{Tr}(TT^T)}$$
$$= 1 - \frac{\mathrm{Tr}\left((2T - A)A^T\right)}{\mathrm{Tr}(TT^T)}.$$

An upper bound for $\mathrm{Tr}(TT^T)$ is

$$\mathrm{Tr}(TT^T) = \sum_{j=1}^{r}\sum_{i=1}^{r}\left(e^{b_i c_j} - \sum_{\ell=r}^{m-1}\frac{b_i^\ell c_j^\ell}{\ell!}\right)^2 \le r^2 e^{2b_r c_r}.$$

We next find a lower bound for $\mathrm{Tr}\left((2T - A)A^T\right)$ as follows:

$$\mathrm{Tr}\left((2T - A)A^T\right) = \sum_{1\le i,j\le r}(2T - A)_{ij}\, A_{ij}$$
$$= \sum_{1\le i,j\le r}\left(e^{b_i c_j} - 2\sum_{\ell=r}^{m-1}\frac{b_i^\ell c_j^\ell}{\ell!}\right)e^{b_i c_j}.$$

By lemma 2.1 we conclude that

$$\mathrm{Tr}\left((2T - A)A^T\right) > \frac{r^2}{2}e^{b_1 c_1}.$$

In total, when we combine the bounds we get that

$$\| T^{-1}H' \|_2^2 \le 1 - \frac{1}{2}e^{b_1 c_1 - 2b_r c_r}.$$

If $\rho_1$ is the the largest eigenvalue of $T^{-1}H'$, then $\rho_1^2 < 1 - \frac{1}{2}e^{b_1 c_1 - 2b_r c_r}$. This implies that

$$\det\left(I + T^{-1}H'\right) > \prod_{j=1}^{r}(1 - \rho_1) > 2^{-r}e^{r(b_1 c_1 - 2b_r c_r)}.$$

Using (2.1), we combine our bounds to see that

$$\det(A) > 2^{-r}e^{r(b_1 c_1 - 2b_r c_r)} \times \det(T).$$

It remains to compute a lower bound for $\det(D + E)$ in the expression for $\det(T)$ from (2.2). Notice that

$$\det(D + E) = \det(D)\det(I + D^{-1}E) = \det(I + D^{-1}E) \times \prod_{\ell=0}^{r-1}\frac{1}{\ell!}.$$

We have that

$$\| E \|_2 = \| B^{-1}HC^{-1} \|_2$$

Also, the entries of $B^{-1}$ and $C^{-1}$ respectively are at most

$$b_r^r \times \prod_{i<j}(b_i - b_j)^{-1}, c_r^r \times \prod_{i<j}(c_i - c_j)^{-1}.$$

On the other hand, all entires of $H$ are at most $\frac{1}{(m/2)!}$. Together, these observations imply that

$$\| D^{-1}E \|_2 \ll \frac{(b_r c_r)^r}{(m/2)!} \times \prod_{\ell=0}^{r} \ell! \times \prod (c_i - c_j)^{-1}(b_i - b_j)^{-1}.$$

By the definition of $m$ from (1.2), the right-hand-side of the previous equation is very small, and hence, $\| D^{-1}E \|_2$ is also negligible. This implies that

$$\det(D+E) \gg \prod_{\ell=0}^{r-1} \frac{1}{\ell!}.$$

Hence, we see that

$$\det(T) \gg \prod_{i<j} (b_j - b_i)(c_j - c_i) \times \prod_{\ell=1}^{r-1} \frac{1}{\ell!} \qquad \square$$

## References

[1] Pinkus, A. "Totally Positive Matrices", Cambridge University Press, 2010.

[2] G.Piazza, T. Politi, "An upper bound for the condition number of a matrix in spectral norm", Journal of Computational and Applied Mathematics (143) 141-144, 2002.