Fall 2013 GRFP Application

Personal Statement, Relevant Background and Future Goals

Essay Prompt: (3 Pages Including Figures)

Please outline your educational and professional development plans and career goals. How do you envision graduate school preparing you for a career that allows you to contribute to expanding scientific understanding as well as broadly benefit society?

Describe your personal, educational and/or professional experiences that motivate your decision to pursue advanced study in science, technology, engineering or mathematics (STEM). Include specific examples of any research and/or professional activities in which you have participated. Present a concise description of the activities, highlight the results and discuss how these activities have prepared you to seek a graduate degree. Specify your role in the activity including the extent to which you worked independently and/or as part of a team. Describe the contributions of your activity to advancing knowledge in STEM fields as well as the potential for broader societal impacts (See Solicitation, Section VI, for more information about Broader Impacts).

NSF Fellows are expected to become globally engaged knowledge experts and leaders who can contribute significantly to research, education, and innovations in science and engineering. The purpose of this statement is to demonstrate your potential to satisfy this requirement. Your ideas and examples do not have to be confined necessarily to the discipline that you have chosen to pursue.

Important questions to ask yourself before starting the essay:

1) Why are you fascinated by your research area?
2) What examples of leadership skills and unique characteristics do you bring to your chosen field?
3) What personal and individual strengths do you have that make you a qualified applicant?
4) How will receiving the fellowship contribute to your career goals?
5) What are all of your applicable experiences?
6) For each experience, what were the key questions, methodology, findings, and conclusions?
7) Did you work in a team and/or independently?
8) How did you assist in the analysis of results?
9) How did your activities address the Intellectual Merit and Broader Impacts criteria?

Overview

In this essay I will begin by discussing how my personal research interests have evolved since I first enrolled for study at the University of Illinois and how these interests have influenced my motivations for both continued graduate study and for a career as a professional researcher. I will also address how the support offered by the GRFP will assist me in pursuing my research, graduate studies, and future career goals.

Motivations for Research, Graduate Studies, and Future Career Goals

I want a career working on topics that are both important and interesting. I have always been drawn to the forms and properties of recurrence relations. I was introduced to the subject through study of the recursively–defined sequence of the Fibonacci numbers, F_n. A surprising and well-known result for the seemingly simple Fibonacci numbers is that the numbers can be generated in order through the closed–form of Binet's formula. In the summer before my first semester at the University of Illinois I became interested in finding forms related to the set of prime numbers.

Inspired by the unusual result of Binet's formula for the Fibonacci numbers, I started challenging myself to find such a non–intuitive result for the fascinating sequence of primes. The explorations of this topic first sparked my interest in developing research of my own and has led me to consider many other topics such as by providing the original motivation for considering the topics of my 2010 publication in the *Journal of Integer Sequences* related to expansions of generalized factorial functions. This research experience has also gradually led me to consider many other topics in my research, including those described in the next sections of this essay and the topics considered in my *Graduate Research Statement* essay.

In my sabbatical from the University of Illinois (2005–2009) I had medical issues that prevented me from continuing in formal academic studies for an extended period of time. I knew that my interests would eventually lead me to graduate studies and I pursued research offers from professors to work on their projects. Prior to my time away from a campus environment, I had never had so much free time to devote to my own software projects and personal interests in Mathematics research. During this time I was able to focus my efforts almost exclusively on my own work. It was through this experience that I found a strong passion for my own independent and original research. I have decided that I can look forward to giving the same full–time intensity to my studies that is required by a rigorous Ph.D. program in my fields of study. This realization has strengthened my dedication to a career as a professional researcher.

Since my Honorable Mention last year for the 2012 GRFP application, I have decided to change my field of graduate study from Computer Science to pursue a Ph.D. in Mathematics. I plan to complete a thesis–based masters program in Computer Science in the Spring of 2014 and am currently applying to other graduate programs in Mathematics. This preparation will allow me to contribute to my fields by providing me with experience and interdisciplinary skills that I can apply to aspects of my future studies – both in graduate school and in my future professional research career. For example, I have recently become involved with several projects in the *Illinois Geometry Lab* (IGL) facility in the Mathematics department at the University of Illinois by assisting with programming tasks in various projects within the lab. The projects in the IGL focus on mathematical visualization and community engagement. Within the lab, undergraduate students work closely with graduate students and postdocs on visualization projects set forth by University of Illinois faculty members, as well as to bring mathematics to the community through school visits and other activities. I feel that have personally benefited from undergraduate research experiences such as the Math 496 course and found being taken seriously as an undergraduate pursuing original research of my own particularly encouraging. I plan to pass on this experience to other motivated researchers as my professional career progresses.

Intellectual Merit: Prior Studies, Publications, and Research Experience

Since returning to my undergraduate studies at the University of Illinois in 2009 I have completed a dual degree program in both Computer Science in the College of Engineering and in Mathematics in the College of Liberal Arts and Sciences. In May of 2012 I was awarded both a Bachelor of Science degree in Computer Science with institutional honors of *Highest Honors* and a B.S. degree for Mathematics with institutional honors of *Cum Laude* and departmental honors of *Highest Distinction MATH*. In 2010, I received the Barry M. Goldwater scholarship. The scholarship is awarded to current undergraduate students who demonstrate strong research potential and who are committed to a Ph.D. degree in fields of Mathematics, Engineering, and other scientific disciplines.

I published an article in the *Journal of Integer Sequences* (JIS) in 2010 based on a significant subset of my independent Mathematics research conducted in my time away from formal academic studies over 2005–2009. The original research contained in the article primarily considers the polynomial expansions of generalized factorial functions through sets of coefficients defined by triangular recurrence relations that generalize the form of the Stirling numbers of the first kind. A review of the 54–page *JIS* publication appeared on *MathSciNet* in 2011 as article `MR2659223 (2011h:05009)`. The review states that the research presented in the article is excellent and is fit as a starting point for further study in enumerative combinatorics.

Other applications of my *JIS* article research have led me to consider additional topics in my original research including (i) the continued fraction expansions of the generating functions of generalized factorial functions that formed the topic of my honors project for the *Introduction to Mathematical Research* (Math 496) course taught by Bruce Reznick in 2010, (ii) the research for my *Senior Thesis* (CS 499) course in 2011 considering further generalizations of the triangular recurrences given in the *JIS* article in the context of the partial sums of infinite zeta–function–related series of the form $\sum_n t^n/f(n)^p$, and then (iii) to eventually explore the topics of my Summer 2012 REGS research that forms the foundation for my *Graduate Research Statement* contained in this application. Besides their significance in number theory and appeal as objects in pure mathematics, the study of the odd–valued zeta function constants considered in the *Research Statement* essay also have applications in broader contexts of other sciences such as quantum physics.

I also plan on submitting several other articles I have authored for publication in peer–reviewed journals in the near future. I have updated and added several new results to the continued fractions research from the 2010 Math496 course and also plan to submit the article to the *Journal of Integer Sequences* once I receive feedback on the article from an expert in continued fractions research that I recently contacted about the research. The research that has grown out of my Summer 2012 REGS (*Research Experience for Graduate Students*) research in the Department of Mathematics at the University of Illinois is closely–

related to the forms of Euler sums and Dirichlet series and was sponsored by NSF grant DMS–0838434. I intend on submitting this research for publication in the *Online Journal of Analytic Combinatorics* (OJAC) suggested by Bruce Berndt later on this Fall. I will also submit another article containing my original research related to so–termed "*square series*" to the OJAC, an outlet which was also suggested as a good fit for the research by Professor Berndt. This article presents integral–based transformations of the generating functions for an arbitrary sequence, $\langle g_n \rangle$, into the form of a series enumerating sequence terms of $\langle q^{n^2} g_n \rangle$ for some fixed $q \in \mathbb{C}$. The particular cases of the geometric and exponential series where $g_n \equiv 1$ and $g_n \equiv 1/n!$, respectively, have a number of applications to series for theta functions and for generating functions that arise in enumerative combinatorics and graph theory.

Research Interests, Broader Impacts, and Participation in Conferences

My particular areas of interest combine Mathematics and Computer Science with emphasis on number theory, enumerative combinatorics, and software development. I intend to conduct research and publish in my fields of study, both working on my personal research in my free time and as a professional in industry. I will continue to publish significant findings in professional journals and other peer–reviewed scholarly outlets. As a female in my fields of study, I will continue to pursue challenging research problems with cross-disciplinary applications in Mathematics, Computer Science, and other engineering sciences. I intend to publish the results of my research on these topics in peer–reviewed journals, present the results through talks at professional conferences, and make the research broadly available for educational, teaching, and other purposes in venues such as the web.

I presented a report talk based on a subset of the research from my *JIS* article at the *Young Mathematicians Conference* (YMC) held at The Ohio State University in the Summer of 2012 with support of NSF grant DMS–0841054. My presentation at the YMC conference also included a summary of results in some of my more recent work with harmonic number sequences and more general zeta function series that were originally considered as applications in the original article. I was recently awarded a scholarship by the Computer Science department at the University of Illinois to attend the *Grace Hopper Celebration of Women in Computing* conference held in Minneapolis, MN from October 2–5 of this year.

I have also independently developed two separate open–source–software projects where the program source code is freely available in the project webpages for purposes including educational study.

Motivation for Applying for GRFP Support of My Graduate Studies

I am currently a graduate student in Computer Science at the University of Illinois at Urbana–Champaign where I have held a teaching assistant position for the CS 173 *Discrete Structures* course in the past three semesters. I chose to attend the University of Illinois for my graduate studies for its strong academic programs in both of my primary undergraduate areas of study and for the university's strengths in other engineering sciences. In addition to the University of Illinois, I am applying to Mathematics Ph.D. programs at several other universities that also have world-class research faculty in areas of Mathematics including Number Theory and Combinatorics, Computer Science, and other scientific disciplines.

I feel that the support from the NSF GRFP will allow me to focus more fully on my research, publications, and graduate studies at the University of Illinois. The fellowship award will also allow me the flexibility to pursue more research–based graduate coursework more quickly in the timeline of my program of study for the Ph.D. program. The fellowship support will also allow me the freedom to get involved in more projects like those in the IGL in place of my current TA position which mostly involves the grading of large quantities of online homework assignments. The fellowship award and tuition support will then allow me to focus more completely on continuing along both of these lines of research and mentoring work.

Software Projects and Other Research Experience

Classnotes Optical Character Recognition Application (2006–Present)

Classnotes is a GUI–based (Graphical User Interface) Optical Character Recognition (OCR) application that is designed to generate plaintext output from scanned images of non–standard font printouts. The algorithm employed by the program is similar to trainable Bayesian e–mail spam filters, except that instead of finding patterns in text, the program operates on gridded image data. The C++ source code for the program is available on the project's webpage at `http://classnotes.sourceforge.net`. I developed the application as an extension of my Computer Science studies using at least two original algorithms created by myself for the project.

Math Pattern Hunter Project (2006–Present)

The project is a math–related program written in C++ that attempts to find closed–form formulas and other identities for input integer and rational sequences by means of programmed and configurable brute force. The source code for the program can be downloaded from the project's webpage at `http://pattern-hunter.sourceforge.net`. I developed the software in my free time as an extension of my mathematics research and am the only developer for the project.

Intel / Lockheed Martin Scholars Program (2005–2006)

The program is a salaried research experience at the University of Illinois for the Laser Charger project. My responsibilities involved development of software to process and recognize input video and microcontroller programming to control electronic peripherals. I learned quite a bit about C and low–level Linux programming through the project.

Graduate Research Statement

Essay Prompt: (2 Pages)

Present an original research topic that you would like to pursue in graduate school. Describe the research idea, your general approach, as well as any unique resources that may be needed for accomplishing the research goal (i.e., access to national facilities or collections, collaborations, overseas work, etc.) You may choose to include important literature citations. Address the potential of the research to advance knowledge and understanding within science as well as the potential for broader impacts on society. The research discussed must be in a field listed in the Solicitation (Section X, Fields of Study).

Important questions to ask yourself before starting the essay:

1) What issues in the scientific community are you most passionate about?
2) Do you possess the technical knowledge and skills necessary for conducting this work, or will you have sufficient mentoring and training to complete the study?
3) Is this plan feasible for the allotted time and institutional resources?
4) How will your research contribute to the "big picture" outside the academic context?
5) How can you draft a plan using the guidelines presented in the essay instructions?
6) How does your proposed research address the Intellectual Merit and Broader Impacts criteria?

Introduction: Evaluation of the Riemann Zeta Function at Odd Positive Integers

Historical Background and Broader Significance of the Problem

The treatment in Havil's *Gamma* provides an engaging and storied account of the history related to the series that defines the Riemann zeta function $\zeta(p) := \sum_{n\geq 1} n^{-p}$ at real $p > 1$. The particular task of summing the first series for $\zeta(2)$ is commonly known as the *Basel Problem* after the famous mathematician Jacob Bernoulli referred to the problem in his 1689 tract published in that namesake (TODO) location. In 1735 Leonhard Euler first expressed this distinguished constant in closed–form as $\zeta(2) = \pi^2/6$ using his original method of proof based on equating the coefficients of the Taylor series and an infinite product for $\sin(z)$. Euler later extended his result for $\zeta(2)$ to a formula for all $k \in \mathbb{Z}^+$ in terms of the rational sequence of Bernoulli numbers, B_n, by obtaining the formula in (1).

$$\zeta(2k) = \frac{(-1)^{k-1} (2\pi)^{2k} B_{2k}}{2 \cdot (2k)!} \tag{1}$$

The even–indexed cases of $\zeta(2k)$ then form a sequence of transcendental constants over $k \in \mathbb{Z}^+$. The question of the rationality of $\zeta(2k+1)$ taken over $k \in \mathbb{Z}^+$ remains a mystery that has fascinated both famous mathematicians and modern researchers alike for over 250 years. Surprisingly, even though Euler's result for the even cases was established in 1750, still no such analogous or obvious closed–form representations exist for the odd–indexed $\zeta(2k+1)$ and considerably much less is known about the properties of these constants – even at small special case values such as $\zeta(3)$ or $\zeta(5)$.

Existing Results, Current Research, and Open Problems

It is known that infinitely–many of the $\zeta(2k+1)$ are irrational. In 1978, mathematician Roger Apéry produced the first proof that $\zeta(3)$ is irrational, though no attempts at a generalization of Apéry's original method have been successful so far. More recently, W. Zudilin proved that at least one of the constants $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, to $\zeta(11)$ is irrational. Other similar interval bounds for the irrationality of consecutive odd–indexed $\zeta(p)$ are known, including those summarized in the survey given in [3], though to date only the concrete case of Apéry's constant, $\zeta(3)$, has been shown to be irrational.

My Interest in the Problem, Motivations for Research, and Prior Work

My research on *square series* generating function transformations starting in 2011 led me to consider the following identity given in terms of the j^{th} derivatives, $F^{(j)}(z)$, of the ordinary generating function (OGF) of the sequence $\langle f_n \rangle$ and the Stirling numbers of the second kind, $\left\{ {n \atop k} \right\}$, for $k \in \mathbb{N}$.

$$\sum_{n=0}^{\infty} n^k f_n z^n = \sum_{j=0}^{k} \left\{ {k \atop j} \right\} z^j F^{(j)}(z) \tag{2}$$

In my REGS research at the University of Illinois over the Summer of 2012, I considered an analog to the formula in (2) for series of the form $\sum_n n^{-k} f_n z^n$. The special case of $f_n \equiv 1$ corresponds to the series for the polylogarithm function, $\mathrm{Li}_k(z) := \sum_{n\geq 1} n^{-k} z^n$, in (3).

$$\mathrm{Li}_k(z) = \sum_{j=1}^{\infty} \left\{ {k+2 \atop j} \right\}_* \frac{z^j j!}{(1-z)^{j+1}} \tag{3}$$

With this motivation in mind, I was able to obtain a non–triangular recurrence for the terms of the right–hand–side of (3) and then to arrive at the following pair of formulas for these coefficients where the r–order harmonic numbers are defined as $H_n^{(r)} := \sum_{k=1}^{n} k^{-r}$.

$$\left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_* = \sum_{m=1}^{j} \binom{j}{m} \frac{(-1)^{j-m}}{j! \cdot m^k} \tag{4}$$

$$\left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_* = \sum_{i=0}^{k-1} \frac{H_j^{(i+1)}}{k} \left\{ \begin{matrix} k+1-i \\ j \end{matrix} \right\}_* + \left(\frac{(-1)^{j-1}}{j!} \right) \cdot [i=0]_\delta . \tag{5}$$

The harmonic–number–based formulas obtained from (5) are similar to those for the Stirling numbers of the first kind given in [1] that I first considered as an application in my 2010 *JIS* article. For example, consider the following special case expansion of (5) when $k=4$:

$$\begin{aligned} \left\{ \begin{matrix} 6 \\ j \end{matrix} \right\}_* &= \frac{(-1)^{j-1}}{24 j!} \left(H_j^4 + 6 H_j^2 H_j^{(2)} + 3 \left(H_j^{(2)} \right)^2 + 8 H_j H_j^{(3)} + 6 H_j^{(4)} \right) \\ &= (TODO). \end{aligned} \tag{6}$$

A Recursive Approach to the Zeta Functions Involving Harmonic Numbers

Motivation and Formulation of the Harmonic–Number–Based Recurrence Relations

Let $\mathcal{C} := \{c_0, c_1, \ldots, c_{r-1}\}$ denote a fixed set of constants. The motivation for this section is to construct recurrence relations of the form in (7) for $p \geq 3$ and where the coefficients $\widetilde{c}_r(\mathcal{C}), \widetilde{b}_i(\mathcal{C})$ are rational linear combinations of the fixed terms in $\mathcal{C}$.

$$\begin{aligned} c_0 H_n^{(p)} &= c_1 H_n^{(p-1)} + \cdots + c_{r-1} H_n^{(p+1-r)} + \widetilde{c}_r\left(\mathcal{C}\right) H_n^{(p-r)} \\ &\quad + \sum_{j=1}^{n} \binom{n}{j} \left\{ \begin{matrix} p+2 \\ j \end{matrix} \right\}_* \left[\frac{\widetilde{b}_2(\mathcal{C})}{(j+2)} \cdot \frac{(n+2)!}{n!} + \cdots + \frac{\widetilde{b}_r(\mathcal{C})}{(j+r)} \cdot \frac{(n+r)!}{n!} \right] \end{aligned} \tag{7}$$

Provided that $p-r \geq 2$ for integers $p \geq 2$, the limiting cases of (7) formed by letting $n \to \infty$ provide new relations between the zeta function constants, $\zeta(p)$, where the even–indexed cases are known in closed–form as the formula in (1).

Since the sequences of harmonic numbers are generated by $H_n^{(r)} = [z^n] \operatorname{Li}_r(z)/(1-z)$, the approach to establishing the recurrences of the form in (7) follows from repeatedly differentiating the series in (3). The intuition behind this series–based approach follows by observing that differentiation of a power series is expanded through polynomial multiples of the terms of the original power series, where in this case $\operatorname{Li}_{k-m}(z) = \sum_{n \geq 1} n^m n^{-k} z^n$ for $k, m \in \mathbb{N}$.

Examples: Recurrences and New Relations Between the Zeta Function Constants

Suppose that the function $M_{k+1}^{(d)}(n)$ is defined as (8) for $k, d, n \in \mathbb{N}$ with $k \geq 1$ and $d \geq 2$.

$$M_{k+1}^{(d)}(n) := \sum_{j=1}^{n} \binom{n}{j} \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_* \frac{(-1)^{j-1}}{(j+d)} \cdot \frac{(n+d)!}{n!} \tag{8}$$

The recurrences given in the form of (7) can be obtained in terms of (8) for any desired p, r, and degree of difference $p-r$. The following results provide several specific examples of the recurrences I have been able to obtain using this method and where the last recurrence holds for any fixed $m \in \mathbb{R}$.

$$H_n^{(p)} = H_n^{(p-2)} + 3 L_{p+1}^{(2)}(n) + L_{p+1}^{(3)}(n) \tag{9}$$

$$2 H_n^{(p)} = -3 H_n^{(p-1)} - H_n^{(p-2)} - L_{p+1}^{(3)}(n) \tag{10}$$

$$7 H_n^{(p)} = -12 H_n^{(p-1)} - 6 H_n^{(p-2)} - H_n^{(p-3)} + L_{p+1}^{(2)}(n) + L_{p+1}^{(4)}(n) \tag{11}$$

$$5 H_n^{(p)} = -9 H_n^{(p-1)} - 5 H_n^{(p-2)} - H_n^{(p-3)} + L_{p+1}^{(2)}(n) + L_{p+1}^{(3)}(n) + L_{p+1}^{(4)}(n) \tag{12}$$

$$H_n^{(p)} = 2H_n^{(p-2)} + H_n^{(p-3)} - L_{p+1}^{(2)}(n) - 4L_{p+1}^{(3)}(n) - L_{p+1}^{(4)}(n) \tag{13}$$

$$\begin{aligned} H_n^{(p)} = (1-m)H_n^{(p-2)} + mH_n^{(p-4)} + (12m+3)L_{p+1}^{(2)}(n) + (24m+1)L_{p+1}^{(3)}(n) \\ + 10mL_{p+1}^{(4)}(n) + mL_{p+1}^{(5)}(n) \end{aligned} \tag{14}$$

By letting $p = 5, 7, 9, 11$ in these formulas, the limit of the left–hand–side of these equations corresponds to one of the particular odd–indexed values of $\zeta(p)$ mentioned in the introduction.

My Current Progress, Future Research Directions, Generalizations, and Other Impacts of the Research

For $k, d, n \in \mathbb{N}$ with $k \geq 1$ and $d \geq 2$, let the function $S_{k+1}^{(d)}(n)$ be defined as (15) where the (unsigned) Stirling numbers of the first kind are denoted in the bracket notation of $\left[{n \atop k}\right]$.

$$S_{k+1}^{(d)}(n) := \sum_{j=1}^{n} \binom{n}{j} \left[{j+1 \atop k+1}\right] \frac{(-1)^{j-1}}{j! \cdot (j+d)} \cdot \frac{(n+d)!}{n!} \tag{15}$$

As already pointed out in the previous section, the coefficients defined by (5) can be expressed in terms of the Stirling numbers with remainder terms involving products of harmonic numbers. In particular, this observation leads to the definition of (8) through (15) in the form of $M_{k+1}^{(d)}(n) := S_{k+1}^{(d)}(n) + T_{k+1}^{(d)}(n)$. So far, I have been able to find the following generating function for (15) as

$$\sum_{n=0}^{\infty} S_{k+1}^{(d)}(n) z^n = \frac{\operatorname{Log}(1-z)^{k+1-d}}{(1-z)^d} \times \sum_{m=0}^{d-1} \frac{\operatorname{Log}(1-z)^m}{(k+1-d+m)!} \cdot g_m^{(d)}(z) \tag{16}$$

$$g_m^{(d)}(z) = \sum_{i=0}^{m} \left[{d-i \atop d-m}\right] \binom{d-1}{i}^2 (-1)^i \cdot z^{m-i} \cdot i!.$$

I have also been able to obtain the recurrence relation for the full terms of (8) in (17).

$$\begin{aligned} \left(2n^2 + (2d+5)n + 3d + 3\right) M_{k+1}^{(d)}(n) - \left(3n^2 + (2d+9)n + 3d + 7\right) M_{k+1}^{(d)}(n) \\ + (n+2)^2 M_{k+1}^{(d)}(n) = \frac{(n+d+1)!}{(n+1)!} \left(\frac{(2n+3)}{(n+1)^k} - \frac{1}{(n+2)^{k-1}}\right) \end{aligned} \tag{17}$$

The generating function in (16), as well as the recurrence relation in (17), suggest a number of possible approaches to studying the limiting behavior of recurrences of the form in (7) by applying various asymptotic methods.

This approach to recurrence relations in (7) can also be extended to cases of $p \in \mathbb{R}^+ \setminus \mathbb{N}$ using the formula for these cases given in (4). Specifically, the method outlined in this essay can also be extended to studying the properties and the irrationality of $\zeta(p)$ for real, non–integral $p > 1$.

References and Literature Citations

[1] V. Adamchik. On Stirling numbers and Euler sums. *J. Comput. Appl. Math.*, 79(1):119–130, 1997.

[2] J. Havil. *Gamma: Exploring Euler's Constant.* Princeton University Press, 2003.

[3] W. Zudilin. An elementary proof of Apéry's theorem. *math.NT*, 2002.