Exact formulas for partial sums of the Möbius function expressed by partial sums weighted by the Liouville lambda function

Maxie Dion Schmidt

maxieds@gmail.com

mschmidt34@gatech.edu

Georgia Institute of Technology

School of Mathematics

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Abstract

The Mertens function, $M(x) := \sum_{n \leq x} \mu(n)$, is defined as the summatory function of the classical Möbius function for $x \geq 1$. The Dirichlet inverse function $g(n) := (\omega + \mathbb{1})^{-1}(n)$ is defined in terms of the shifted strongly additive function $\omega(n)$ that counts the number of distinct prime factors of n without multiplicity. Discrete convolutions of the partial sums of $g(n)$ with the prime counting function provide new exact formulas for $M(x)$ that are weighted sums of the Liouville function multiplied by the unsigned summands $|g(n)|$ for $n \leq x$. We study the distribution of the unsigned function $|g(n)|$ whose Dirichlet generating function (DGF) is $\zeta(2s)^{-1}(1-P(s))^{-1}$ through the auxiliary unsigned sequence $C_{\Omega}(n)$ whose DGF is given by $(1-P(s))^{-1}$ for $\mathrm{Re}(s) > 1$ where $P(s) = \sum_p p^{-s}$ is the prime zeta function. We prove formulas for the average order and variance of both $\log C_{\Omega}(n)$ and $\log |g(n)|$ and conjecture a central limit theorem for the distribution of their values over $n \leq x$ as $x \to \infty$.

Keywords and Phrases: *Möbius function; Mertens function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function; prime zeta function.*

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1 Introduction

1.1 Definitions

For integers $n \geq 2$, we define the strongly and completely additive functions, respectively, that count the number of prime divisors of n by

$$\omega(n) = \sum_{p|n} 1, \text{ and } \Omega(n) = \sum_{p^{\alpha}||n} \alpha.$$

We use the convention that when $n = 1$, the functions $\omega(1) = \Omega(1) = 0$. The Möbius function is defined as the signed indicator function of the squarefree integers by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } n \geq 2 \text{ and } \omega(n) = \Omega(n); \\ 0, & \text{otherwise.} \end{cases}$$

The Mertens function is the summatory function defined by the partial sums [21, A008683; A002321]

$$M(x) = \sum_{n \leq x} \mu(n), \text{ for } x \geq 1. \tag{1.1}$$

The Liouville lamda function is defined for all $n \geq 1$ by $\lambda(n) := (-1)^{\Omega(n)}$. The partial sums of this function are defined by [21, A008836; A002819]

$$L(x) := \sum_{n \leq x} \lambda(n), \text{ for } x \geq 1. \tag{1.2}$$

For any arithmetic functions f and h, we define their Dirichlet convolution at $n \geq 1$ by

$$(f * h)(n) := \sum_{d|n} f(d) h\left(\frac{n}{d}\right).$$

The arithmetic function f has a unique inverse with respect to Dirichlet convolution, denoted by f^{-1}, that satisfies $(f * f^{-1})(n) = (f^{-1} * f)(n) = \delta_{n,1}$ if and only if $f(1) \neq 0$. We fix the notation for the Dirichlet inverse function [21, A341444]

$$g(n) := (\omega + \mathbb{1})^{-1}(n), \text{ for } n \geq 1. \tag{1.3}$$

We use the notation $|g(n)|$ to denote the absolute value of $g(n)$. The sign of $g(n)$ is given by $\lambda(n)$ for all $n \geq 1$ (see Proposition 3.3). We define the partial sums $G(x)$ for integers $x \geq 1$ as follows [21, A341472]:

$$G(x) := \sum_{n \leq x} g(n) = \sum_{n \leq x} \lambda(n)|g(n)|. \tag{1.4}$$

1.2 Main results

For any $x \geq 1$, the function $\pi(x) := \sum_{p \leq x} 1$ in the next theorem denotes the classical prime counting function.

Theorem 1.1. *For all* $x \geq 1$

$$M(x) = G(x) + \sum_{1 \leq k \leq x} |g(k)| \pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) \lambda(k), \tag{1.5a}$$

$$M(x) = G(x) + \sum_{1 \leq k \leq \frac{x}{2}} \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right) G(k), \tag{1.5b}$$

$$M(x) = G(x) + \sum_{p \leq x} G\left(\left\lfloor \frac{x}{p} \right\rfloor\right). \tag{1.5c}$$

An exact expression for $g(n)$ is given by (see Lemma 3.4 and Corollary 3.5)

$$\lambda(n)g(n) = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_\Omega(d), n \geq 1. \tag{1.6}$$

The sequence $\lambda(n)C_\Omega(n)$ has the Dirichlet generating function (DGF) of $(1+P(s))^{-1}$ and $C_\Omega(n)$ has the DGF $(1-P(s))^{-1}$ for $\mathrm{Re}(s) > 1$ where $P(s) := \sum_p p^{-s}$ is the prime zeta function. The function $C_\Omega(n)$ was considered in [8] with an exact formula given by (*cf.* equation (3.4) in the proof of Proposition 3.3)

$$C_\Omega(n) = \begin{cases} 1, & \text{if } n = 1; \\ (\Omega(n))! \times \prod\limits_{p^\alpha \| n} \frac{1}{\alpha!}, & \text{if } n \geq 2. \end{cases} \tag{1.7}$$

Theorem 1.2. *As $n \to \infty$*

$$\frac{1}{n} \times \sum_{k \leq n} \log C_\Omega(k) = B_0 (\log\log n)(\log\log\log n)\left(1 + o(1)\right).$$

Conjecture. *For any fixed $z > 0$, there is an absolute constant $B_0 > 0$ so that as $x \to \infty$*

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : -z \leq |g(n)| - \frac{1}{n} \times \sum_{k \leq n} |g(k)| \leq z\right\} = \Phi\left(\frac{\log\left(\frac{\pi^2 |z|}{6}\right) - B_0(\log\log x)(\log\log\log x)}{B_0(\log\log x)(\log\log\log x)}\right) + o(1).$$

We can show that the limiting absolute constant B_0 in the conjecture is actually identically one assuming that as $x \to \infty$ the following result is true for any fixed, finite $y > 0$:

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : \frac{C_\Omega(n)}{(\log\log x)(\log\log\log x)} \leq y\right\} = \Phi\left(y - 1\right) + o(1).$$

The motivation for why the last equation is expected to hold is discussed in Section 4.

1.3 Discussion of the new results

For $n \geq 2$, let the function $\mathcal{E}[n] := (\alpha_1, \alpha_2, \ldots, \alpha_r)$ denote the unordered partition of exponents for which $n = p_1^{\alpha_1} \times \cdots \times p_r^{\alpha_r}$ is the factorization of n into powers of distinct primes. For any $n_1, n_2 \geq 2$

$$\mathcal{E}[n_1] = \mathcal{E}[n_2] \implies g(n_1) = g(n_2). \tag{1.8}$$

The Mertens function is related to the partial sums in (1.2) via the relation [11, 13]

$$M(x) = \sum_{d \leq \sqrt{x}} \mu(d) L\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), \text{ for } x \geq 1. \tag{1.9}$$

The relation in (1.9) gives an exact expression for $M(x)$ with summands involving $L(x)$ that are oscillatory. In contrast, the exact expansions for the Mertens function given in Theorem 1.1 express $M(x)$ as finite sums over $\lambda(n)$ with weight coefficients that are unsigned. The property of the symmetry of the distinct values of $|g(n)|$ with respect to the prime factorizations of $n \geq 2$ in (1.8) suggests that the unsigned weights on $\lambda(n)$ in the new formulas from the theorem should be comparatively easier to work with than the known exact expressions for $M(x)$ in terms of $L(x)$ that have the less predictably signed summands from equation (1.9) above.

1.4 Organization of the manuscript

The focus of the article is on studying the unsigned functions $C_\Omega(n)$ and $|g(n)|$. The new formulas for $M(x)$ given in Theorem 1.1 provide a window from which we can view classically difficult problems about asymptotics for this function partially in terms of the properties of the auxiliary unsigned functions and their distributions. We first prove the new results for statistics and properties of the functions $C_\Omega(n)$ and $g(n)$. We then establish the proofs of the exact formulas for $M(x)$ stated in Theorem 1.1. The appendix sections provide a glossary of notation and supplementary material on topics that can be separated from the organization of the main sections of the article.

2 Properties of the function $C_\Omega(n)$

The function $C_\Omega(n)$ is key to understanding the unsigned inverse sequence $|g(n)|$ through equation (1.6). In this section, we define the unsigned function $C_\Omega(n)$ precisely and explore its properties.

2.1 Definitions

Definition 2.1. We define the following bivariate sequence for integers $n \geq 1$ and $k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum_{d|n} \omega(d) C_{k-1}\left(\frac{n}{d}\right), & \text{if } k \geq 1. \end{cases} \tag{2.1}$$

Using the notation for iterated convolution in Bateman and Diamond [2, Def. 2.3; §2], we have $C_0(n) \equiv \omega^{*0}(n)$ and $C_k(n) \equiv \omega^{*k}(n)$ for integers $k \geq 1$ and $n \geq 1$. The special case of (2.1) where $k := \Omega(n)$ occurs frequently in the next sections of the article. To avoid cumbersome notation when referring to this common function variant, we suppress the duplicate index n by writing $C_\Omega(n) := C_{\Omega(n)}(n)$ [21, A008480].

Remark 2.2. By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (2.1) inductively. We also see that at fixed n, the function $C_k(n)$ is non-zero only possibly for $1 \leq k \leq \Omega(n)$ when $n \geq 2$. By (1.7) we have that $C_\Omega(n) \leq (\Omega(n))!$ for all $n \geq 1$ with equality precisely at the squarefree integers so that $(\Omega(n))! = (\omega(n))!$ if and only if $\mu^2(n) = 1$.

2.2 Average order and variance

Proof of Theorem 1.2. We first use (1.7) to see that there is an absolute constant $P_0 \geq \frac{6}{\pi^2}$ such that

$$\sum_{k\geq 1} \sum_{\substack{n\leq x \\ \Omega(n)=k}} \log C_\Omega(n) = \sum_{k\geq 1} P_0 \times \#\{n \leq x : \Omega(n) = k\} \times \log(k!)(1 + o(1)). \tag{2.2}$$

A complete justification of why equation (2.2) is correct is given after completing this proof below. We will split the full sum on the left-hand-side of (2.2) into two sums over disjoint indices and show that one of the sums contributes the main term to the average order formula, and that the other may be regarded as an error term. For $x \geq 3$, consider the following partial sums:

$$L_\Omega(x) := \sum_{1\leq k\leq \frac{3}{2}\log\log x} \sum_{\substack{n\leq x \\ \Omega(n)=k}} \log C_\Omega(n).$$

For any $x \geq 0$, we cite the following known form of Binet's formula for the log-gamma function [19, §5.9(i)]:

$$\log x! = \left(x + \frac{1}{2}\right)\log(1 + x) - x + O(1).$$

Provided that (2.2) holds, we apply Theorem B.2 (see appendix) to show that there is an absolute constant $B_0 > 0$ such that

$$L_{\Omega}(x) = \sum_{1 \leq k \leq \frac{3}{2}\log\log x} \frac{B_0 x (\log\log x)^{k-1}}{(\log x)(k-1)!} \times \left(\left(k + \frac{1}{2}\right)\log(1+k) - k\right)(1 + o(1)). \tag{2.3}$$

The right-hand-side of (2.3) can be approximated by Abel summation using the functions

$$A_x(u) := \frac{B_0 x \Gamma(u, \log\log x)}{\Gamma(u)}; f(u) := \frac{(2u+1)}{2}\log(1+u) - \frac{(2u+1)}{2}, f'(u) = \log(1+u) - \frac{1}{2(1+u)}.$$

Then we have by Proposition C.3 that

$$\begin{aligned} L_{\Omega}(x) &= A_x\left(\frac{3}{2}\log\log x\right) f\left(\frac{3}{2}\log\log x\right) - \int_0^{\frac{3}{2}} A_x(\alpha \log\log x) f'(\alpha\log\log x) d\alpha \\ &= B_0 x (\log\log x)(\log\log\log x)(1 + o(1)). \end{aligned}$$

It suffices to show

$$\sum_{\substack{n \leq x \\ \Omega(n) \geq \frac{3}{2}\log\log x}} \log C_{\Omega}(n) = o\left(x (\log\log x)(\log\log\log x)\right), \text{ as } x \to \infty. \tag{2.4}$$

Because $r - 1 - r\log r \approx -0.108198$ when $r := \frac{3}{2}$ and

$$\log C_{\Omega}(n) \ll \Omega(n)\log\Omega(n), \text{ for } n \leq x, \tag{2.5}$$

we can argue using Rankin's method as in [15, Thm. 7.20; §7.4] that (2.4) holds. In particular, the bounds provided in Theorem B.1 together with applications of the Cauchy-Schwarz and the (logarithmic) AGM inequalities fill in the complete details to a proof verifying that the bound in (2.4) is attained at all sufficiently large x. The assertion on the upper bound for $\log C_{\Omega}(n)$ in (2.5) holds for all n even though the right-hand-side terms involving $\Omega(n)$ oscillate in magnitude for $1 \leq n \leq x$. This is justified by maximizing (minimizing) the ratio of the right-hand-side above to Binet's log-gamma formula cited above numerically to find explicit bounded real $x \equiv \Omega(n) \in [1, 11)$ that yield the extrema of the function. $\square$

Proof of equation (2.2). The key to this argument is in understanding that the main term of the sum on the left-hand-side of the equation is obtained by summing over only those $n \leq x$ which are squarefree. That is, we will show that

$$\sum_{k \geq 1} \sum_{\substack{n \leq x \\ \Omega(n) = k}} \log C_{\Omega}(n) \sim \sum_{k \geq 1} \sum_{\substack{n \leq x \\ \mu^2(n) = 1 \\ \Omega(n) = k}} \log C_{\Omega}(n).$$

For integers $x, k \geq 1$ with $k \leq \log_2(x)$, we define the sets

$$\mathcal{S}_k\left(\{\varpi_j\}_{j=1}^k; x\right) := \left\{2 \leq n \leq x : \mu(n) = 0, \omega(n) = k, \frac{n}{\operatorname{rad}(n)} = p_1^{\varpi_1} \times \cdots \times p_k^{\varpi_k}, \ p_i \neq p_j \text{ prime if } 1 \leq i < j \leq k\right\}.$$

The notation $\operatorname{rad}(n)$ is the radix of n which evaluates to the largest squarefree factor of any $n \geq 2$. Let

$$\mathcal{N}_k(\varpi_1, \ldots, \varpi_k; x) := \frac{\left|\mathcal{S}_k\left(\{\varpi_j\}_{j=1}^k; x\right)\right|}{x},$$

and let notation for the special case where $\{\varpi_j\}_{1 \leq j \leq k} \equiv \{1\}$ (with multiplicity of exactly one) be given by

$$\widehat{T}_k(x) := \mathcal{N}_k(1, 0, \ldots, 0; x).$$

If $2 \leq n \leq x$ is not squarefree and $n \in \mathcal{S}_k\left(\{\varpi_j\}_{j=1}^k; x\right)$, then we must have that $\varpi_j \geq 1$ for at least one $1 \leq j \leq k$. We then clearly conclude that for any $k \geq 1$ and non-trivial $\{\varpi_j\}_{1 \leq j \leq k} \neq \{0\}$, we have

$$\lim_{x \to \infty} \mathcal{N}_k(\varpi_1, \varpi_2, \ldots, \varpi_k; x) \leq \lim_{x \to \infty} \binom{k}{1} \times \widehat{T}_k(x).$$

It suffices to establish bounds on the $\widehat{T}_k(x)$ that show

$$\widehat{T}_k(x) \ll \#\{n \leq x : \Omega(n) = k\}, \text{ for all } k \geq 1, \text{ as } x \to \infty.$$

We can obtain intuition on the quality of the upper bounds we will require for this task by explicitly evaluating asymptotic formulae for the first cases of $k \in \{1, 2\}$ explicitly as follows:

$$\begin{aligned}
\widehat{T}_1(x) &= \sum_{p \leq \sqrt{x}} 1 = \frac{2\sqrt{x}}{\log x}(1 + o(1)), \\
\widehat{T}_2(x) &= \sum_{p \leq \sqrt{x}} \widehat{T}_1\left(\frac{x}{p}\right) \ll \frac{\sqrt{x}}{\log x} \times \sum_{p \leq \sqrt{x}} \frac{1}{\sqrt{p}} \times \left(1 + \frac{\log p}{\log x}\right) \ll \frac{\sqrt{x}(\log\log x)}{\log x}.
\end{aligned}$$

We have applied a famous theorem of Mertens to reach the last equation. This result proves that the partial sums of the reciprocals of the primes are given by $\sum_{p \leq x} p^{-1} = (\log\log x)(1 + o(1))$. We can argue by induction that for any $k \geq 1$

$$\widehat{T}_k(x) \ll x^{0.905466} \times (\log x)^{k-2}, \text{ as } x \to \infty.$$

We must have that $k \ll \log x$ and that $(\log x)^{\log x} = o(x)$ by L'Hopital's rule as $x \to \infty$. It follows that for large x

$$\sum_{k \geq 1} \#\{n \leq x : \Omega(n) = k, \mu(n) = 0\} \times \log(k!) \ll \sum_{k \geq 1} \#\{n \leq x : \Omega(n) = k, \mu^2(n) = 1\} \times \log(k!).$$

By an extension of the previous argument applied to the formula for $C_\Omega(n)$ from equation (1.7), we similarly conclude that as $x \to \infty$

$$\sum_{\substack{n \leq x \\ \mu(n) = 0}} \log C_\Omega(n) \ll \sum_{\substack{n \leq x \\ \mu^2(n) = 1}} \log C_\Omega(n).$$

That is, we have that the main term of the sums defined on the left of equation (2.2) is

$$\sum_{n \leq x} \log C_\Omega(n) \sim \sum_{k \geq 1} \sum_{\substack{n \leq x \\ \Omega(n) = k}} \log(k!).$$

The corresponding denominator differences from (1.7) that we subtract of from the main term identified in the last equation is asymptotically insubstantial compared to the right-hand-side of the previous equation in light of the argument given above to establish the upper bounds on the functions $\widehat{T}_k(x)$ for any $1 \leq k \leq \log_2(x)$. Finally, since $C_\Omega(n) = (\Omega(n))!$ for all squarefree $n \geq 1$ and the limiting proportion of positive integers that are squarefree is $\frac{6}{\pi^2}$, the limiting constant P_0 has the (sharp) bound stated before equation (2.2) in the main proof of the theorem outlined above. $\square$

Definition 2.3. For any integers $x \geq 1$, we define the alternate notation for the *average order*, or expected (averaged) value, of the function $\log C_\Omega(n)$ on the integers $1 \leq n \leq x$ by

$$\mathbb{E}\left[\log C_\Omega(x)\right] := \frac{1}{x} \times \sum_{n \leq x} \log C_\Omega(n).$$

The *variance* of the logarithm of this function corresponds to the centralized second moments

$$\operatorname{Var}\left(\log C_\Omega(x)\right) := \sum_{n \leq x} \left(\log C_\Omega(n) - \mathbb{E}\left[\log C_\Omega(x)\right]\right)^2.$$

Proposition 2.4. *For $n > e^e$, the variance of the function $\log C_\Omega(n)$ is given by*

$$\sqrt{\operatorname{Var}\left(\log C_\Omega(x)\right)} = B_0(\log\log n)(\log\log\log n)\left(1+o(1)\right).$$

Proof. Suppose that $n \geq 16$. We have a standard rearrangement of the terms in the sample variance of the values $\{\log C_\Omega(n)\}_{1\leq n\leq x}$ in the form of

$$S_{2,\Omega}(n) := \sum_{k\leq n} \log^2 C_\Omega(k) - \left(\sum_{k\leq n} \log C_\Omega(k)\right)^2 = 2 \times \sum_{1\leq j<k\leq n} \log C_\Omega(j) \log C_\Omega(k). \tag{2.6}$$

Let the first and second moment sums for the function be denoted in respective order by

$$E_\Omega(n) := \frac{1}{n} \times \sum_{k\leq n} \log C_\Omega(k), \text{ and } V_\Omega(n) := \sqrt{\frac{1}{n} \times \sum_{k\leq n} \log^2 C_\Omega(k)}, \text{ for } n \geq 1.$$

The expansion on the right-hand-side of (2.6) is rewritten as

$$S_{2,\Omega}(n) = V_\Omega^2(n) - E_\Omega^2(n) = 2 \times \sum_{1\leq j<n} \log C_\Omega(j)\left(E_\Omega(n) - E_\Omega(j)\right). \tag{2.7}$$

Equation (2.7) implies that as $n \to \infty$

$$\begin{aligned} V_\Omega^2(n) &\sim B_0^2\left(3E_\Omega^2(n) - 2(\log\log n)^2(\log\log\log n)^2 + I_A(n)\right) \\ &= B_0^2\left((\log\log n)^2(\log\log\log n)^2 + I_A(n)\right)(1+o(1)). \end{aligned} \tag{2.8}$$

We define the integral term in the last equations by

$$I_A(x) := 2 \times \int_{e^e}^x t(\log\log t)^2(\log\log\log t)^2 dt.$$

For $x > e^e$, we can exactly integrate

$$\int_{e^e}^x \frac{(\log\log t)^2(\log\log\log t)^2}{\log t} \cdot \frac{dt}{t} = \frac{1}{3}(\log\log x)^3(\log\log\log x)^3(1+o(1)), \text{ as } x \to \infty.$$

The mean value theorem shows that for all sufficiently large x there is a $c \equiv c(x) \in [e^e, x]$ (i.e., a bounded constant depending on x) such that we have exactly that

$$I_A(x) = \frac{2}{3}c(x)\log c(x)(\log\log x)^3(\log\log\log x)^3(1+o(1)).$$

We can differentiate the previous equation, discarding lower order terms, to solve for the main term of $c(x)$ as $x \to \infty$:

$$c(x) \ll \frac{\log\log\log\log x}{W\left(\log\log\log\log x\right)} \ll \frac{\log\log\log\log x}{\log\log\log\log\log x}.$$

This implies that $I_A(x) = o\left(E_\Omega(x)\right)$ for all large x. The conclusion follows from this observation input into the formula we derived in equation (2.8). □

3 Properties of the function $g(n)$

In this section, we explore and enumerate several key properties of the inverse function $g(n)$. The partial sums of this sequence yield the new formulas for $M(x)$ stated in Theorem 1.1 proved in Section 5 below.

Definition 3.1. For integers $n \geq 1$, we define the Dirichlet inverse function taken with respect to the operation of Dirichlet convolution to be

$$g(n) = (\omega + \mathbb{1})^{-1}(n), \text{ for } n \geq 1.$$

The function $|g(n)|$ denotes the unsigned inverse function.

We briefly motivate the definition of $g(n)$ given in Definition 3.1 using the next argument.

Remark 3.2. Let $\chi_{\mathbb{P}}(n)$ denote the characteristic function of the primes, let $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n (without multiplicity). We can see using elementary methods that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + \mathbb{1}) * \mu. \tag{3.1}$$

Namely, the result in (3.1) follows by Möbius inversion since $\mu * 1 = \varepsilon$ and

$$\omega(n) = \sum_{p|n} 1 = \sum_{d|n} \chi_{\mathbb{P}}(d), \text{ for } n \geq 1.$$

We recall the classic inversion theorem of summatory functions (of generalized convolutions) proved in [1, §2.14] for any Dirichlet invertible arithmetic function $\alpha(n)$ as follows:

$$G(x) = \sum_{n \leq x} \alpha(n) F\left(\frac{x}{n}\right) \Longrightarrow F(x) = \sum_{n \leq x} \alpha^{-1}(n) G\left(\frac{x}{n}\right), \text{ for } x \geq 1. \tag{3.2}$$

Hence, to express the new formulas for $M(x)$ we may consider the inversion of the right-hand-side of the partial sums

$$\pi(x) + 1 = \sum_{n \leq x} \left(\chi_{\mathbb{P}} + \varepsilon\right)(n) = \sum_{n \leq x} (\omega + \mathbb{1}) * \mu(n), \text{ for } x \geq 1.$$

3.1 Signedness

Proposition 3.3. *The sign of the function $g(n)$ is $\lambda(n)$ for all $n \geq 1$.*

Proof. The series $D_f(s) := \sum_{n \geq 1} f(n) n^{-s}$ defines the Dirichlet generating function (DGF) of any arithmetic function f which is convergent for all $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) > \sigma_f$ where σ_f is the abscissa of convergence of the series. Recall that $D_{\mathbb{1}}(s) = \zeta(s)$, $D_\mu(s) = \zeta(s)^{-1}$ and $D_\omega(s) = P(s)\zeta(s)$ for $\mathrm{Re}(s) > 1$. By (3.1) and the fact that whenever $f(1) \neq 0$, the DGF of $f^{-1}(n)$ is $D_f(s)^{-1}$, we have

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{\zeta(s)(1 + P(s))}, \text{ for } \mathrm{Re}(s) > 1. \tag{3.3}$$

It follows that $(\omega + 1)^{-1}(n) = (h^{-1} * \mu)(n)$ for $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\mathrm{sgn}(h^{-1}) = \lambda$. This observation then implies that $\mathrm{sgn}(h^{-1} * \mu) = \lambda$.

We recover exactly that [8, *cf.* §2]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ \lambda(n)(\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}, & n \geq 2. \end{cases}$$

In particular, by expanding the DGF of h^{-1} formally in powers of $P(s)$ (where $|P(s)| < 1$ whenever $\mathrm{Re}(s) \geq 2$), we count that

$$\frac{1}{1 + P(s)} = \sum_{n \geq 1} \frac{h^{-1}(n)}{n^s} = \sum_{k \geq 0} (-1)^k P(s)^k,$$

$$= 1 + \sum_{\substack{n \geq 2 \\ n = p_1^{\alpha_1} p_2^{\alpha_2} \times \cdots \times p_k^{\alpha_k}}} \frac{(-1)^{\alpha_1+\alpha_2+\cdots+\alpha_k}}{n^s} \times \binom{\alpha_1+\alpha_2+\cdots+\alpha_k}{\alpha_1, \alpha_2, \ldots, \alpha_k},$$

$$= 1 + \sum_{\substack{n \geq 2 \\ n = p_1^{\alpha_1} p_2^{\alpha_2} \times \cdots \times p_k^{\alpha_k}}} \frac{\lambda(n)}{n^s} \times \binom{\Omega(n)}{\alpha_1, \alpha_2, \ldots, \alpha_k}. \tag{3.4}$$

Since λ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$ for all divisors $d|n$ when $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever n is squarefree so that

$$g(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, \text{ for } n \geq 1. \qquad \square$$

The notation $|h^{-1}(n)|$ from the last proof is the same as the function $C_\Omega(n)$ for all $n \geq 1$.

3.2 Precise relations to $C_\Omega(n)$

Lemma 3.4. *For all $n \geq 1$*

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_\Omega(d).$$

Proof. We first expand the recurrence relation for the Dirichlet inverse when $g(1) = g(1)^{-1} = 1$ as

$$g(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d) + 1) g\left(\frac{n}{d}\right) \quad \Longrightarrow \quad (g * 1)(n) = -(\omega * g)(n). \tag{3.5}$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (3.5) in the form of $(g * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m (C_m(-) * g)(n)$ as

$$F_m(n) = -\begin{cases} (\omega * g)(n), & m = 1; \\ \sum\limits_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum\limits_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g\left(\frac{n}{dr}\right), & 2 \leq m \leq \Omega(n); \\ 0, & \text{otherwise.} \end{cases}$$

When $n \geq 2$ and $m := \Omega(n)$, i.e., with the expansions in the previous equation taken to a maximal depth, we obtain the relation

$$(g * 1)(n) = (-1)^{\Omega(n)} C_\Omega(n) = \lambda(n) C_\Omega(n). \tag{3.6}$$

The formula for $g(n)$ follows from (3.6) by Möbius inversion. $\square$

Corollary 3.5. *For all $n \geq 1$*

$$|g(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_\Omega(d). \tag{3.7}$$

Proof. The result follows by applying Lemma 3.4, Proposition 3.3 and the complete multiplicativity of $\lambda(n)$. Since $\mu(n)$ is non-zero only at squarefree integers and since at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$, we have

$$|g(n)| = \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_\Omega(d)$$

$$= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_\Omega(d).$$

The leading term $\lambda(n^2) = 1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. $\square$

Remark 3.6. We have the following remarks on consequences of Corollary 3.5:

- Whenever $n \geq 1$ is squarefree
$$|g(n)| = \sum_{d|n} C_{\Omega}(d). \tag{3.8a}$$
Since all divisors of a squarefree integer are squarefree, for all squarefree integers $n \geq 1$, we have that
$$|g(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \times m!. \tag{3.8b}$$
- The formula in (3.7) shows that the DGF of the unsigned inverse function $|g(n)|$ is given by the meromorphic function $\zeta(2s)^{-1}(1 - P(s))^{-1}$ for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. This DGF has a pole to the right of the line at $\mathrm{Re}(s) = 1$ which occurs for the unique real $\sigma \approx 1.39943$ such that $P(\sigma) = 1$ on $(1, \infty)$.

3.3 Average order and variance

Theorem 3.7. *As $n \to \infty$*
$$\frac{1}{n} \times \sum_{k \leq n} \log|g(k)| = \frac{B_0}{2}\left((\log\log n)(\log\log\log n) - \log\left(\frac{\pi^2}{6}\right)\right)(1 + o(1)).$$

Proof. A classical formula for the number of squarefree integers $n \leq x$ shows that [10, §18.6] [21, A013928]
$$Q(x) = \sum_{n \leq x} \mu^2(n) = \frac{6x}{\pi^2} + O\left(\sqrt{x}\right), \text{ as } x \to \infty.$$
Therefore, summing over the formula from (3.7), we find that for large n
$$\begin{aligned}
\frac{1}{n} \times \sum_{k \leq n} |g(k)| &= \frac{1}{n} \times \sum_{d \leq n} C_{\Omega}(d) Q\left(\left\lfloor \frac{n}{d} \right\rfloor\right) \\
&\sim \sum_{d \leq n} C_{\Omega}(d)\left(\frac{6}{\pi^2 d} + O\left(\frac{1}{\sqrt{dn}}\right)\right) \\
&= \frac{6}{\pi^2}\left(\frac{1}{n} \times \sum_{k \leq n} C_{\Omega}(k) + \sum_{d < n}\sum_{k \leq d} \frac{C_{\Omega}(k)}{d^2}\right) + O(1).
\end{aligned} \tag{3.9}$$
We claim that
$$|g(n)| - \frac{1}{n} \times \sum_{k \leq n} |g(k)| \sim \frac{6}{\pi^2} C_{\Omega}(n), \text{ as } n \to \infty. \tag{3.10}$$
Let the backwards difference operator with respect to x be defined for $x \geq 2$ and any arithmetic function f by $\Delta_x[f] := f(x) - f(x-1)$. Using this notation, we see from (3.9) that
$$\begin{aligned}
|g(n)| = \Delta_n\left[\sum_{k \leq n} g(k)\right] &\sim \frac{6}{\pi^2} \times \Delta_n\left[\sum_{d \leq n} C_{\Omega}(d) \frac{n}{d}\right] \\
&= \frac{6}{\pi^2}\left(C_{\Omega}(n) + \sum_{d < n} C_{\Omega}(d)\frac{n}{d} - \sum_{d < n} C_{\Omega}(d) \frac{(n-1)}{d}\right) \\
&\sim \frac{6}{\pi^2} C_{\Omega}(n) + \frac{1}{n-1} \times \sum_{k < n} |g(k)|, \text{ as } n \to \infty.
\end{aligned}$$
By taking the logarithm of (3.10), we find that
$$\frac{1}{n} \times \sum_{k \leq n} \log|g(k)| = \frac{B_0}{2}(\log\log n)(\log\log\log n) - \frac{B_0}{2}\log\left(\frac{\pi^2}{6}\right) + O\left(\frac{1}{n^2} \times \sum_{k \leq n} \log|g(k)|\right). \qquad \square$$

A similar argument to that given in the proof of Proposition 2.4 shows that the variance of $\log|g(n)|$ is given by

$$\sqrt{\operatorname{Var}\left(\log|g(x)|\right)} = \frac{\sqrt{2}B_0}{2}(\log\log n)(\log\log\log n)(1+o(1)), \text{ as } n \to \infty.$$

4 Conjectures on limiting distributions

In this section, we motivate a conjecture that provides a limiting central limit type distribution for the function $\log C_\Omega(n)$. The relations between $C_\Omega(n)$ and $g(n)$ we proved in Section 3.2 then allow us to formulate a limiting central limit theorem for the distribution of the unsigned inverse sequence $|g(n)|$ under the assumption that the conjecture holds. For any $z \in (-\infty, \infty)$, the cumulative density function of any standard normal distributed random variable is denoted by

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{z} e^{\frac{-t^2}{2}}\,dt.$$

Conjecture 4.1. *For any real z, as $x \to \infty$*

$$\frac{1}{x} \times \#\left\{2 \le n \le x : \frac{\log C_\Omega(n) - B_0(\log\log x)(\log\log\log x)}{B_0(\log\log x)(\log\log\log x)} \le z\right\} = \Phi(z) + o(1).$$

A rigorous proof of the conjecture is outside of the scope of this manuscript. There are distributions of the probability weights on the log-multinomial distributions associated with the distinct values of $C_\Omega(n)$ on $n \le x$ in [20, *cf.* §1.2]. These limiting distributions may yield a useful probability model under which we can prove the conjectured convergence in distribution.

Proposition 4.2. *Suppose that Conjecture 4.1 is true. For any $z > 0$ as $x \to \infty$*

$$\frac{1}{x} \times \#\left\{2 \le n \le x : -z \le |g(n)| - \frac{1}{n} \times \sum_{k \le n} |g(k)| \le z\right\} = \Phi\left(\frac{\log\left(\frac{\pi^2|z|}{6}\right) - B_0(\log\log x)(\log\log\log x)}{B_0(\log\log x)(\log\log\log x)}\right) + o(1).$$

Proof. The result follows from (3.10) as a re-normalization of Conjecture 4.1. □

Remark 4.3 (Applications)**.** An obvious application of the proposition is to apply the limiting distribution of $|g(n)|$ from the last conjectured proposition to estimate best and worst case growth of the $\lambda(n)$-signed summands of the partial sums of $g(n)$. The final result stated in Theorem 1.1 yields bounds on $M(x)$ given estimates of this type. We observe that to cover the spread at the center of the right-hand-side distribution as $\Phi(w) \le M \in (0,1]$, the relevant values of $\pm z$ in Proposition 4.2 are bounded by

$$|z| \ll \left(\frac{\Gamma(\log\log x + 1)(\log x)}{\sqrt{2\pi\log\log x}}\right)^{B_0\left(1+\sqrt{2}\operatorname{erf}^{-1}(|2M-1|)\right)}.$$

We can consider $1 \le M_x \ll \sqrt{\log\log x}$ so that for large x we have $\Phi(M_x) = 1 + O\left(\frac{1}{\log x}\right)$. We may then apply the bound on z in the previous equation to evaluate the cases of z that contribute only non-negligible weight to sums over the function of n in Proposition 4.2. That is, those differences where $|g(n)|$ diverges from its average order with substantial weight. Evaluating the distribution of $|g(n)|$ predicted by the proposition to evaluate the new formulas for $M(x)$ in Theorem 1.1 still requires more information about the sign weights by $\lambda(n)$ on the summands of the summatory function $G(x)$ (*cf.* [12]).

The large order growth of the average order of $|g(n)|$ is problematic in predicting the likelihood (on average) that $|\sum_{n\le x} g(n)| \le T$ for fixed $T > 0$. We still should expect enormous and miraculous cancellation almost everywhere in the summatory function terms involving discrete convolutions of one with $G(t)$ from the exact expression for $M(x)$ proved as (1.5c) of Theorem 1.1. A future extension of the work in this article we suggest is to find new ways to exploit the cancellation in this formula to extract hidden information about the frequency of the sign changes of $\lambda(n)$ on bounded subintervals of $[1,x]$ given how large of a spread of the inner difference from Proposition 4.2 we anticipate (at least on average) as $x \to \infty$.

5 Proofs of the new exact formulas for $M(x)$

In this section, we prove the formulas for $M(x)$ involving the partial sums of the function $g(n)$ stated in Theorem 1.1. These new formulas exactly identify the Mertens function with partial sums of positive unsigned arithmetic functions whose summands are weighted by the sign of $\lambda(n)$. The formulas in equations (1.5b) and (1.5c) suggest that a more complete understanding of the asymptotics of the summatory function of $g(n)$ may yield new insights into the behavior of $M(x)$. We take the time to explore the properties of these partial sums in this section as well.

5.1 Formulas relating $M(x)$ to the partial sums of $g(n)$

Definition 5.1. For any $x \geq 1$, let the partial sums of the Dirichlet convolution $r * h$ be defined by

$$S_{r*h}(x) := \sum_{n \leq x} \sum_{d|n} r(d) h\left(\frac{n}{d}\right).$$

Theorem 5.2. *Let $r, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $r(1) \neq 0$. Suppose that $R(x) := \sum_{n \leq x} r(n)$, $H(x) := \sum_{n \leq x} h(n)$, and that $R^{-1}(x) := \sum_{n \leq x} r^{-1}(n)$ for $x \geq 1$. The following formulas hold for all integers $x \geq 1$:*

$$\begin{aligned} S_{r*h}(x) &= \sum_{d=1}^{x} r(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ S_{r*h}(x) &= \sum_{k=1}^{x} H(k) \left(R\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - R\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right). \end{aligned}$$

Moreover, for any $x \geq 1$

$$\begin{aligned} H(x) &= \sum_{j=1}^{x} S_{r*h}(j) \left(R^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - R^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right) \\ &= \sum_{k=1}^{x} r^{-1}(k) S_{r*h}(x). \end{aligned}$$

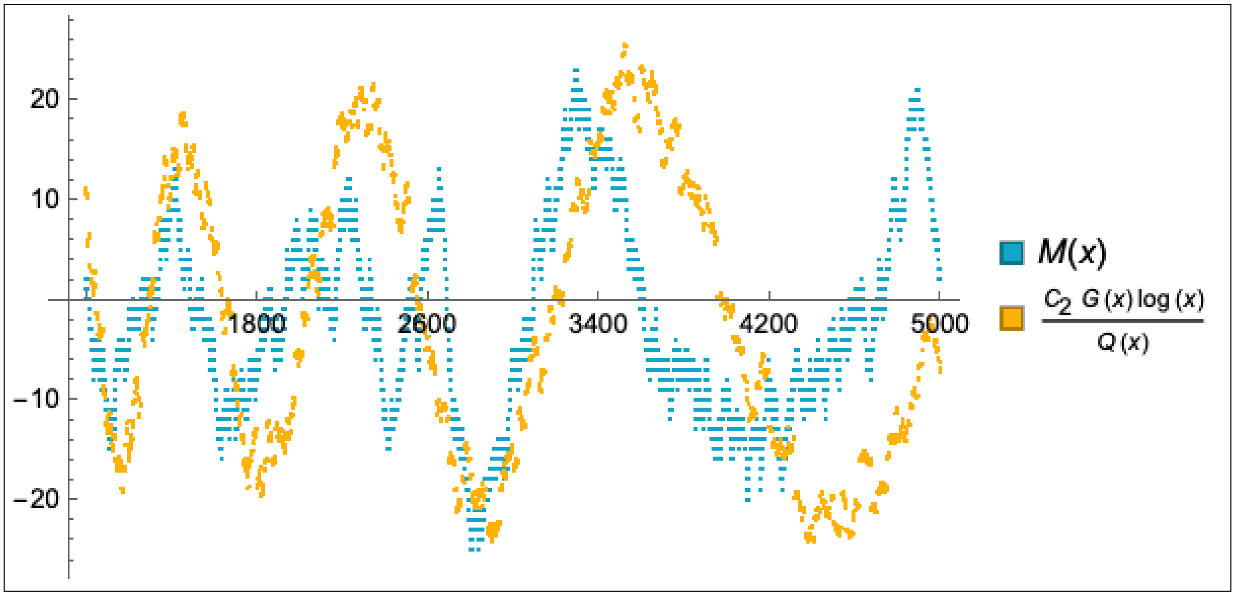
Theorem 5.2 is proved in Appendix D.

Corollary 5.3. *Suppose that r is an arithmetic function such that $r(1) \neq 0$. Let the summatory function $\widetilde{R}(x) := \sum_{n \leq x} (r * \mu)(n)$. The Mertens function is expressed by the following partial sums for any $x \geq 1$:*

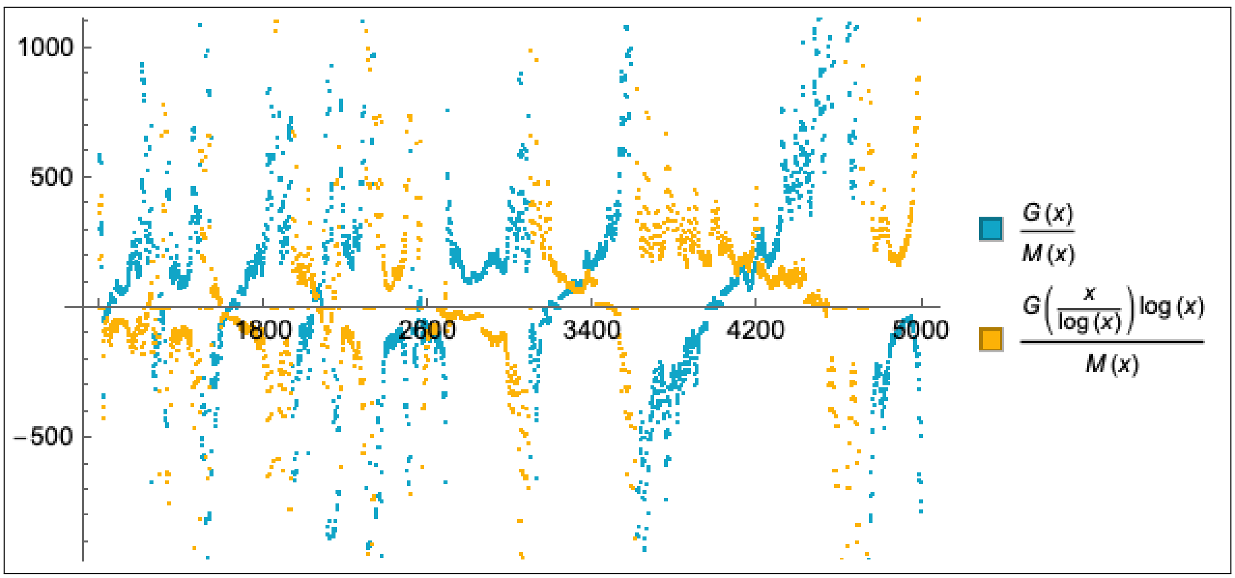
$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} r^{-1}(j) \right) \widetilde{R}(k).$$

Definition 5.4. The summatory functions of $g(n)$ and $|g(n)|$, respectively, are defined for all $x \geq 1$ by the partial sums

$$G(x) := \sum_{n \leq x} g(n) = \sum_{n \leq x} \lambda(n) |g(n)|, \text{ and } |G|(x) := \sum_{n \leq x} |g(n)|.$$



(a)



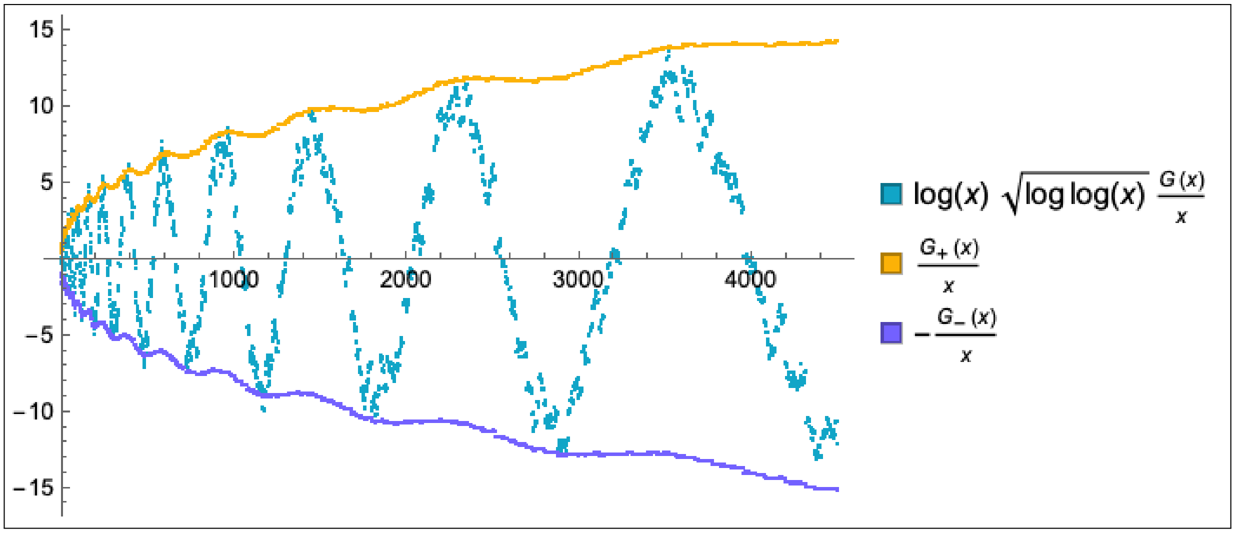
(b)

Figure 5.1

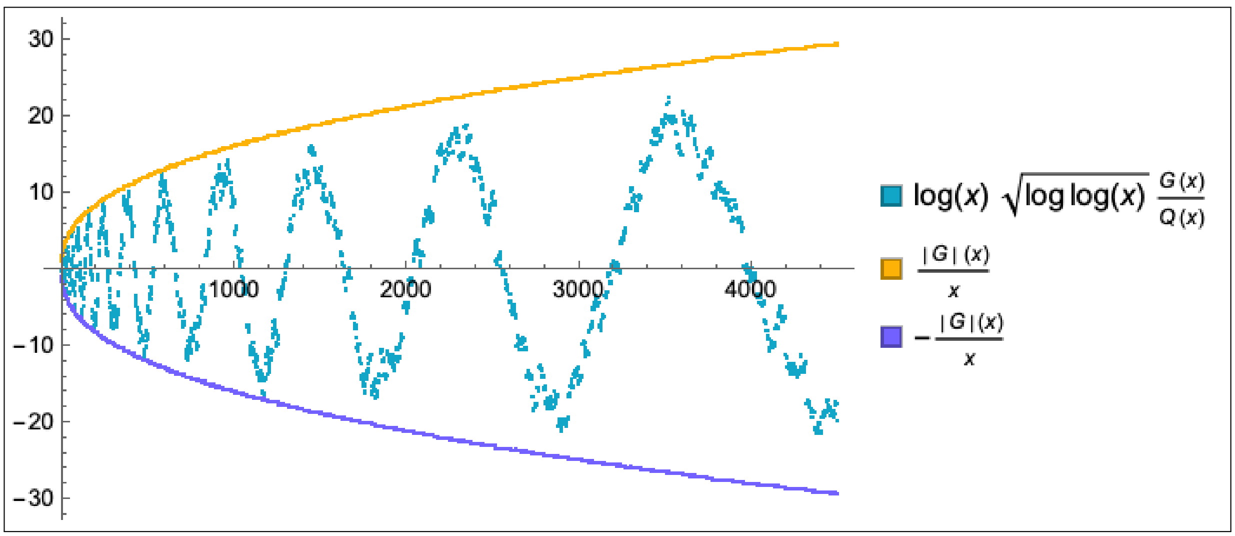
Based on the convolution identity in (3.1), we prove the formulas in Theorem 1.1 as special cases of Corollary 5.3 below.

Proof of (1.5a) *and* (1.5b) *of Theorem 1.1.* By applying Theorem 5.2 to equation (3.1) we have that

$$\begin{aligned}
M(x) &= \sum_{k=1}^{x} \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor \right) + 1 \right) g(k) \\
&= G(x) + \sum_{k=1}^{\frac{x}{2}} \pi\left(\left\lfloor \frac{x}{k} \right\rfloor \right) g(k) \\
&= G(x) + G\left(\left\lfloor \frac{x}{2} \right\rfloor \right) + \sum_{k=1}^{\frac{x}{2}-1} \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor \right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor \right) \right) G(k).
\end{aligned}$$



(a)



(b)

Figure 5.2

The upper bound on the sum is truncated to $k \in \left[1, \frac{x}{2}\right]$ in the second equation above because $\pi(1) = 0$. The third formula above follows directly by summation by parts. $\square$

Proof of (1.5c) *of Theorem 1.1.* Lemma 3.4 shows that

$$G(x) = \sum_{d \leq x} \lambda(d) C_\Omega(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

The identity in (3.1) implies

$$\lambda(d) C_\Omega(d) = (g * 1)(d) = (\chi_{\mathbb{P}} + \varepsilon)^{-1}(d).$$

We recover the stated result from the classical inversion of summatory functions in equation (3.2). $\square$

5.2 Plots and numerical experiments

The plots shown in the figures in this section compare the values of $M(x)$ and $G(x)$ with scaled forms of related auxiliary partial sums:

- In Figure 5.1, we plot comparisons of $M(x)$ to scaled forms of $G(x)$ for $x \leq 5000$. The absolute constant $C_2 := \zeta(2)$ and where the function $Q(x) := \sum_{n \leq x} \mu^2(n)$ counts the number of squarefree integers $n \leq x$ for any $x \geq 1$. In (a) the shift to the left on the x-axis of the former function is compared and seen to be similar in shape to the magnitude of $M(x)$ on this initial subinterval. It is unknown whether the similar shape and magnitude of these two functions persists for much larger x. In (b) we have observed unusual reflections and symmetry between the two ratios plotted in the figure. We have numerically modified the plot values to shift the denominators of $M(x)$ by one at each $x \leq 5000$ for which $M(x) = 0$.
- In Figure 5.2, we compare envelopes on the logarithmically scaled values of $G(x)x^{-1}$ to other variants of the partial sums of $g(n)$ for $x \leq 4500$. In (a) we define $G(x) := G_+(x) - G_-(x)$ where the functions $G_+(x) > 0$ and $G_-(x) > 0$ for all $x \geq 1$. That is, the signed component functions $G_\pm(x)$ denote the unsigned contributions of only those summands $|g(n)|$ over $n \leq x$ where $\lambda(n) = \pm 1$, respectively. The summatory function $Q(x) = \frac{6x}{\pi^2}\left(1 + O\left(\frac{1}{\sqrt{x}}\right)\right)$ in (b) has the same definition as in Figure 5.1 above. The second plot suggests that for large x there is enough cancellation in the signed summatory function so that
$$|G(x)| \ll \frac{|G|(x)}{(\log x)\sqrt{\log\log x}} = \frac{1}{(\log x)\sqrt{\log\log x}} \times \sum_{n \leq x} |g(n)|.$$

5.3 Local cancellation in the new formulas for the Mertens function

Definition 5.5. Let p_n denote the n^{th} prime for $n \geq 1$ [21, A000040]. The set of primorial integers is defined by [21, A002110]
$$\{n\#\}_{n \geq 1} = \left\{\prod_{k=1}^{n} p_k\right\}_{n \geq 1}.$$

Proposition 5.6. *As $m \to \infty$, each of the following holds:*
$$-G((4m+1)\#) \asymp (4m+1)!, \tag{A}$$
$$G\left(\frac{(4m+1)\#}{p_k}\right) \asymp (4m)!, \text{ for any } 1 \leq k \leq 4m+1. \tag{B}$$

Proof. We have by (3.8b) that for all squarefree integers $n \geq 1$
$$\begin{aligned} |g(n)| &= \sum_{j=0}^{\omega(n)} \binom{\omega(n)}{j} \times j! = (\omega(n))! \times \sum_{j=0}^{\omega(n)} \frac{1}{j!} \\ &= (\omega(n))! \times \left(e + O\left(\frac{1}{(\omega(n)+1)!}\right)\right). \end{aligned}$$

Let m be a large positive integer. We obtain main terms of the form
$$\begin{aligned} \sum_{\substack{n \leq (4m+1)\# \\ \omega(n) = \Omega(n)}} \lambda(n)|g(n)| &= \sum_{0 \leq k \leq 4m+1} \binom{4m+1}{k} (-1)^k k! \left(e + O\left(\frac{1}{(k+1)!}\right)\right) \\ &= -(4m+1)! + O\left(\frac{1}{4m+1}\right). \end{aligned} \tag{5.2}$$

The formula for $C_\Omega(n)$ stated in (1.7) then implies the result in (A). This follows since the contributions from the summands of the inner summation on the right-hand-side of (5.2) off of the squarefree integers are at most a bounded multiple of $(-1)^k k!$ when $\Omega(n) = k$. We can similarly derive that for any $1 \leq k \leq 4m+1$
$$G\left(\frac{(4m+1)\#}{p_k}\right) \asymp \sum_{0 \leq k \leq 4m} \binom{4m}{k} (-1)^k k! \left(e + O\left(\frac{1}{(k+1)!}\right)\right) = (4m)! + O\left(\frac{1}{4m+1}\right). \qquad \square$$

Remark 5.7. The Riemann hypothesis (RH) is equivalent to showing that

$$M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right), \text{ for all } 0 < \epsilon < \frac{1}{2}. \tag{5.3}$$

We expect that there is usually (almost always) a large amount cancellation between the successive values of the summatory function in (1.5c). Proposition 5.6 demonstrates the phenomenon well along the infinite subsequence of the primorials $\{(4m+1)\#\}_{m\geq 1}$. If the RH is true, the sums of the leading constants with opposing signs on the asymptotic bounds for the functions from the last proposition are necessarily required to match. Namely, we have that [4, 5]

$$n\# \sim e^{\vartheta(p_n)} \asymp n^n (\log n)^n e^{-n(1+o(1))}, \text{ as } n \to \infty.$$

The observation on the necessary cancellation in (1.5c) then follows from the fact that if we obtain a contrary result

$$\frac{M((4m+1)\#)}{\sqrt{(4m+1)\#}} \gg [(4m+1)\#]^{\delta_0}, \text{ as } m \to \infty,$$

for some fixed $\delta_0 > 0$. If the last equation were to hold, we would find a contradiction to the condition required by equation (5.3). Assuming the RH, we can state a stronger bound for the Mertens function along this subsequence by considering the error terms given in the proof of Proposition 5.6.

6 Conclusions

6.1 Summary

We have identified a sequence, $\{g(n)\}_{n\geq 1}$, that is the Dirichlet inverse of the shifted strongly additive function $\omega(n)$. There is a natural combinatorial interpretation to the repetition of distinct values of $|g(n)|$ in terms of the configuration of the exponents in the prime factorization of any $n \geq 2$. The sign of $g(n)$ is given by $\lambda(n)$ for all $n \geq 1$. This leads to a new exact relations of the summatory function $G(x)$ to $M(x)$ and the classical partial sums $L(x)$. We have formalized a new perspective from which we might express our intuition about features of the distribution of $G(x)$ via the properties of its $\lambda(n)$-sign-weighted summands. The new results proved within this article are significant in providing a new window through which we can view bounding $M(x)$ through asymptotics of the unsigned sequences and their partial sums.

6.2 Discussion of the new results

Probabilistic models of the Möbius function lead us to consider the behavior of $M(x)$ as a sum of independent and identically distributed (i.i.d.) random variables. Suppose that $\{X_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables such that for all $n \geq 1$, $\mathbb{P}[X_n = 1] = \frac{3}{\pi^2}$, $\mathbb{P}[X_n = 0] = 1 - \frac{6}{\pi^2}$ and $\mathbb{P}[X_n = -1] = \frac{3}{\pi^2}$, e.g., as providing a randomized model of the values of $\mu(n)$ on the average, so that we can model its partial sums by $M(x) \cong \sum_{n\leq x} X_n$. This viewpoint is used to model and predict certain limiting asymptotic behavior of the Mertens function. We can show that

$$\mathbb{E}\left[\sum_{1\leq n\leq x} X_n\right] = 0, \mathrm{Var}\left(\sum_{1\leq n\leq x} X_n\right) = \sqrt{\frac{6x}{\pi^2}}, \text{ and } \limsup_{x\to\infty} \frac{\left|\sum\limits_{1\leq n\leq x} X_n\right|}{\sqrt{n\log\log n}} = \frac{2\sqrt{3}}{\pi} \text{ (almost surely)}.$$

The property of the symmetry of the distinct values of $|g(n)|$ with respect to the prime factorizations of $n \geq 2$ in (1.8) shows that the unsigned weights on $\lambda(n)$ in the new formulas Theorem 1.1 are comparatively easier to work with than the known exact expressions for $M(x)$ like equation (1.9).

Stating tight bounds on the distribution of $L(x)$ is a problem that is equally as difficult as understanding the properties of $M(x)$ well at large x or along infinite subsequences (*cf.* [9, 7, 23]). Indeed, $\lambda(n) = \mu(n)$ for all squarefree $n \geq 1$ so that $\lambda(n)$ agrees with $\mu(n)$ at most large n as the asymptotic density of the squarefree integers is $\frac{6}{\pi^2}$. We infer that $\lambda(n)$ must then inherit the pseudo-randomized quirks of $\mu(n)$ predicted by models of this function in Sarnak's conjecture. On the other hand, arguments for why the formulas in Theorem 1.1 are more desirable to explore than classical formulae for $M(x)$ yield three counter points:

(1) Breakthrough work in recent years due to Matomäki, Radziwiłł and Soundararajan to bound multiplicative functions in short intervals has proven fruitful when applied to $\lambda(n)$ [22, 14]. The analogs of results of this type corresponding to the Möbius function are not clearly attained;

(2) The squarefree $n \geq 1$ on which $\lambda(n)$ and $\mu(n)$ must identically agree are in some senses easier integer cases to handle insomuch as we can prove very regular properties that govern the distributions of the distinct values of $\omega(n)$, $\Omega(n)$ and their difference over $n \leq x$ as $x \to \infty$ [15, *cf.* §2.4; §7.4];

(3) The function $\lambda(n)$ is completely multiplicative. Hence, sign weighting by the function $\lambda(n)$ may eventually reflect a nicer cousin to the multiplicative $\mu(n)$ along the integers $n \geq 4$ for which $\mu(n) = 0$. This notion, and applications to bounding $M(x)$ via $G(x)$ in equation (1.5c), are intentionally stated imprecisely at the time of writing this manuscript.

References

[1] T. M. Apostol. *Introduction to Analytic Number Theory.* Springer–Verlag, 1976.

[2] P. T. Bateman and H. G. Diamond. *Analytic Number Theory.* World Scientific Publishing, 2004.

[3] P. Billingsley. On the central limit theorem for the prime divisor function. *Amer. Math. Monthly*, 76(2):132–139, 1969.

[4] P. Dusart. The k^{th} prime is greater than $k(\log k + \log\log k - 1)$ for $k \geq 2$. *Math. Comp.*, 68(225):411–415, 1999.

[5] P. Dusart. Estimates of some functions over primes without R.H., 2010.

[6] P. Erdős and M. Kac. The Gaussian errors in the theory of additive arithmetic functions. *American Journal of Mathematics*, 62(1):738–742, 1940.

[7] N. Frantzikinakis and B. Host. The logarithmic Sarnak conjecture for ergodic weights. *Ann. of Math. (2)*, 187(3):869–931, 2018.

[8] C. E. Fröberg. On the prime zeta function. *BIT Numerical Mathematics*, 8:87–202, 1968.

[9] B. Green and T. Tao. The Möbius function is strongly orthogonal to nilsequences. *Ann. of Math. (2)*, 175(2):541–566, 2012.

[10] G. H. Hardy and E. M. Wright, editors. *An Introduction to the Theory of Numbers.* Oxford University Press, 2008 (Sixth Edition).

[11] P. Humphries. The distribution of weighted sums of the Liouville function and Pólya's conjecture. *J. Number Theory*, 133:545–582, 2013.

[12] J. Kaczorowski and J. Pintz. Oscillatory properties of arithmetical functions, i. *Acta Math. Hungar.*, 48(1–2):173–185, 1986.

[13] R. S. Lehman. On Liouville's function. *Math. Comput.*, 14:311–320, 1960.

[14] K. Matomäki and M. Radziwiłł. Multiplicative functions in short intervals. *Ann. of Math.*, 183:1015–1056, 2016.

[15] H. L. Montgomery and R. C. Vaughan. *Multiplicative Number Theory: I. Classical Theory*. Cambridge, 2006.

[16] G. Nemes. The resurgence properties of the incomplete gamma function II. *Stud. Appl. Math.*, 135(1):86–116, 2015.

[17] G. Nemes. The resurgence properties of the incomplete gamma function I. *Anal. Appl. (Singap.)*, 14(5):631–677, 2016.

[18] G. Nemes and A. B. Olde Daalhuis. Asymptotic expansions for the incomplete gamma function in the transition regions. *Math. Comp.*, 88(318):1805–1827, 2019.

[19] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.

[20] A. D. Barbour R. Arratia and Simon Tavaré. *Logarithmic Combinatorial Structures: A Probabilistic Approach*. Preprint draft, 2002.

[21] N. J. A. Sloane. The Online Encyclopedia of Integer Sequences, 2021. http://oeis.org.

[22] K. Soundararajan. The Liouville function in short intervals (after Matomaki and Radziwill). *arXiv:1606.08021*, 2016.

[23] T. Tao. The logarithmically averaged Chowla and Elliott conjectures for two-point correlations. *Forum of Mathematics*, 4:e8, 2016.

A Glossary of notation and conventions

Symbols	Definition
$\gg, \ll, \asymp, \sim$	For functions A, B, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A, B \geq 0$, $A \ll B$ and $B \ll A$, we write $A \asymp B$. Two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1$.
$\chi_{\mathbb{P}}(n), P(s)$	The indicator function of the primes equals one if and only if $n \in \mathbb{Z}^+$ is prime and is defined to be zero-valued otherwise. For any $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$, we define the prime zeta function to be the Dirichlet generating function (DGF) defined by $P(s) = \sum_{n\geq 1} \chi_{\mathbb{P}}(n) n^{-s}$. The function $P(s)$ has an analytic continuation to the half-plane $\mathrm{Re}(s) > 0$ with the exception of $s = 1$ through the formula $P(s) = \sum_{k\geq 1} \frac{\mu(k)}{k} \log \zeta(ks)$. The DGF $P(s)$ poles at the reciprocal of each positive integer and a natural boundary at the line $\mathrm{Re}(s) = 0$.
$C_k(n), C_\Omega(n)$	The first sequence is defined recursively for integers $n \geq 1$ and $k \geq 0$ as follows: $$C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum_{d\mid n} \omega(d) C_{k-1}\left(\frac{n}{d}\right), & \text{if } k \geq 1. \end{cases}$$ It represents the multiple (k-fold) convolution of the function $\omega(n)$ with itself. The function $C_\Omega(n) := C_{\Omega(n)}(n)$ has the DGF $(1 - P(s))^{-1}$ for $\mathrm{Re}(s) > 1$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic function f we have that $f * \varepsilon = \varepsilon * f = f$ where the operation $*$ denotes Dirichlet convolution.
$f * g$	The Dirichlet convolution of any two arithmetic functions f and g at n is defined to be the divisor sum $(f * g)(n) := \sum_{d\mid n} f(d) g\left(\frac{n}{d}\right)$ for $n \geq 1$.
$f^{-1}(n)$	The Dirichlet inverse f^{-1} of an arithmetic function f exists if and only if $f(1) \neq 0$. The Dirichlet inverse of any f such that $f(1) \neq 0$ is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \times \sum_{\substack{d\mid n \\ d>1}} f(d) f^{-1}\left(\frac{n}{d}\right)$ for $n \geq 2$ with $f^{-1}(1) = f(1)^{-1}$. When it exists, this inverse function is unique and satisfies $f^{-1} * f = f * f^{-1} = \varepsilon$.
$\Gamma(a, z)$	The incomplete gamma function is defined as $\Gamma(a, z) := \int_z^\infty t^{a-1} e^{-t} dt$ by continuation for $a \in \mathbb{R}$ and $\lvert\arg(z)\rvert < \pi$.
$g(n), G(x), \lvert G\rvert(x)$	The Dirichlet inverse function, $g(n) = (\omega + 1)^{-1}(n)$, has the summatory function $G(x) := \sum_{n\leq x} g(n)$ for $x \geq 1$. We define the partial sums of the unsigned inverse function to be $\lvert G\rvert(x) := \sum_{n\leq x} \lvert g(n)\rvert$ for $x \geq 1$.
$[n = k]_\delta$, $[\texttt{cond}]_\delta$	The symbol $[n = k]_\delta$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise. For Boolean-valued conditions, `cond`, the symbol $[\texttt{cond}]_\delta$ evaluates to one precisely when `cond` is true or to zero otherwise.

Symbols	Definition
$\lambda(n), L(x)$	The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. Its summatory function is defined by the partial sums $L(x) := \sum_{n\leq x} \lambda(n)$ for $x \geq 1$.
$\mu(n), M(x)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers $n \geq 1$ where $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree. The Mertens function is the summatory function defined for all integers $x \geq 1$ by the partial sums $M(x) := \sum_{n\leq x} \mu(n)$.
$\Phi(z)$	For $z \in \mathbb{R}$, we take the cumulative density function of the standard normal distribution to be denoted by $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p\mid n} 1$ and the completely additive function $\Omega(n) := \sum_{p^\alpha \| n} \alpha$. This means that if the prime factorization of any $n \geq 2$ is given by $n := p_1^{\alpha_1} \times \cdots \times p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. We set $\omega(1) = \Omega(1) = 0$ by convention.
$\pi_k(x), \widehat{\pi}_k(x)$	For integers $k \geq 1$, the function $\pi_k(x)$ denotes the number of $2 \leq n \leq x$ with exactly k distinct prime factors: $\pi_k(x) := \#\{2 \leq n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{2 \leq n \leq x : \Omega(n) = k\}$ for $x \geq 2$ and fixed $k \geq 1$.
$Q(x)$	For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. That is, $Q(x) = \sum_{n\leq x} \mu^2(n)$ for $x \geq 1$.
$W(x)$	For $x, y \in [0, \infty)$, we write that $x = W(y)$ if and only if $xe^x = y$. This function denotes the principal branch of the multi-valued Lambert W function taken over the non-negative reals.
$\zeta(s)$	The Riemann zeta function is defined by $\zeta(s) := \sum_{n\geq 1} n^{-s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation to any $s \in \mathbb{C}$ with the exception of a simple pole at $s = 1$ of residue one.

B The distributions of $\omega(n)$ and $\Omega(n)$

As $n \to \infty$, we have that

$$\frac{1}{n} \times \sum_{k\leq n} \omega(k) = \log\log n + B_1 + o(1),$$

and

$$\frac{1}{n} \times \sum_{k\leq n} \Omega(k) = \log\log n + B_2 + o(1),$$

for $B_1 \approx 0.261497$ and $B_2 \approx 1.03465$ absolute constants [10, §22.10]. The next theorems reproduced from [15, §7.4] bound the frequency of the number of $\omega(n)$ and $\Omega(n)$ over $n \leq x$ such that these functions diverge substantially from their average order (*cf.* [6, 3] [15, §7.4]).

Theorem B.1. *For $x \geq 2$ and $r > 0$, let*

$$A(x, r) := \#\{n \leq x : \Omega(n) \leq r \log\log x\},$$
$$B(x, r) := \#\{n \leq x : \Omega(n) \geq r \log\log x\}.$$

If $0 < r \leq 1$, then

$$A(x, r) \ll \quad x(\log x)^{r-1-r\log r}, \ \textit{as } x \to \infty.$$

If $1 \leq r \leq R < 2$, *then*

$$B(x, r) \ll_R x(\log x)^{r-1-r\log r}, \quad \textit{as } x \to \infty.$$

Theorem B.2. *For integers* $k \geq 1$ *and* $x \geq 2$

$$\widehat{\pi}_k(x) := \#\{2 \leq n \leq x : \Omega(n) = k\}.$$

For $0 < R < 2$, *uniformly for* $1 \leq k \leq R \log\log x$

$$\widehat{\pi}_k(x) = \frac{x}{\log x} \times \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right), \quad \textit{as } x \to \infty,$$

where

$$\mathcal{G}(z) := \frac{1}{\Gamma(1+z)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1}\left(1 - \frac{1}{p}\right)^z, \quad \textit{for } 0 \leq |z| < R.$$

We can extend the work in [15] on the distribution of $\Omega(n)$ to obtain corresponding analogous results for the distribution of $\omega(n)$.

Remark B.3. For integers $k \geq 1$ and $x \geq 2$, we define

$$\pi_k(x) := \#\{2 \leq n \leq x : \omega(n) = k\}.$$

For $0 < R < 2$ and as $x \to \infty$

$$\pi_k(x) = \frac{x}{\log x} \times \widetilde{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) \frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right), \tag{B.1}$$

uniformly for $1 \leq k \leq R \log\log x$. The factor involving the function $\widetilde{\mathcal{G}}(z)$ is defined by $\widetilde{\mathcal{G}}(z) := \widetilde{F}(1, z) \times \Gamma(1+z)^{-1}$ where

$$\widetilde{F}(s, z) := \prod_p \left(1 + \frac{z}{p^s - 1}\right)\left(1 - \frac{1}{p^s}\right)^z, \text{ for } \operatorname{Re}(s) > \frac{1}{2} \text{ and } |z| \leq R < 2.$$

Let the functions

$$\begin{aligned} C(x, r) &:= \#\{n \leq x : \omega(n) \leq r \log\log x\}, \\ D(x, r) &:= \#\{n \leq x : \omega(n) \geq r \log\log x\}. \end{aligned}$$

The following upper bounds hold as $x \to \infty$:

$$\begin{aligned} C(x, r) &\ll \ \ x(\log x)^{r-1-r\log r}, \text{ uniformly for } 0 < r \leq 1, \\ D(x, r) &\ll_R x(\log x)^{r-1-r\log r}, \text{ uniformly for } 1 \leq r \leq R < 2. \end{aligned}$$

C Asymptotics of the incomplete gamma function

We cite the correspondence with Gergő Nemes from the Alfréd Rényi Institute of Mathematics and his careful notes on the limiting asymptotics for the sums identified in this section. The communication of his proofs are adapted to establish the next lemmas based on [16, 17, 18].

Definition C.1. The (upper) incomplete gamma function is defined by [19, §8.4]

$$\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt, \text{ for } a \in \mathbb{R} \text{ and } |\arg z| < \pi.$$

The function $\Gamma(a,z)$ can be continued to an analytic function of z on the universal covering of $\mathbb{C}\setminus\{0\}$. For $a\in\mathbb{Z}^+$, the function $\Gamma(a,z)$ is an entire function of z.

Facts C.2. The following properties hold [19, §8.4; §8.11(i)]:

$$\Gamma(a,z) = (a-1)!e^{-z}\times\sum_{k=0}^{a-1}\frac{z^k}{k!},\ \text{for } a\in\mathbb{Z}^+ \text{ and } z\in\mathbb{C}, \tag{C.1a}$$

$$\Gamma(a,z) \sim z^{a-1}e^{-z},\ \text{for fixed } a\in\mathbb{R} \text{ and } z>0 \text{ as } z\to\infty. \tag{C.1b}$$

For $z>0$, as $z\to\infty$ we have that [16]

$$\Gamma(z,z) = \sqrt{\frac{\pi}{2}}z^{z-\frac{1}{2}}e^{-z} + O\left(z^{z-1}e^{-z}\right), \tag{C.1c}$$

For fixed, finite real $|\rho|>0$, we define the sequence $\{b_n(\rho)\}_{n\geq 0}$ by the following recurrence relation for $n\geq 0$:

$$b_n(\rho) = \rho(1-\rho)b_{n-1}'(\rho) + \rho(2n-1)b_{n-1}(\rho) + \delta_{n,0}.$$

If $z,a\to\infty$ with $z=\rho a$ for some $\rho>1$ such that $(\rho-1)^{-1} = o\left(\sqrt{|a|}\right)$, then [16]

$$\Gamma(a,z) \sim z^a e^{-z}\times\sum_{n\geq 0}\frac{(-a)^n b_n(\rho)}{(z-a)^{2n+1}}. \tag{C.1d}$$

Proposition C.3. *Let a,z,ρ be positive real parameters such that $z=\rho a$. If $\rho\in(0,1)$, then as $z\to\infty$*

$$\Gamma(a,z) = \Gamma(a) + O_\rho\left(z^{a-1}e^{-z}\right). \tag{C.2a}$$

If $\rho>1$, then as $z\to\infty$

$$\Gamma(a,z) = \frac{z^{a-1}e^{-z}}{1-\rho^{-1}} + O_\rho\left(z^{a-2}e^{-z}\right). \tag{C.2b}$$

If $\rho>W(1)$, then as $z\to\infty$

$$\Gamma(a,ze^{\pm\pi\imath}) = -e^{\pm\pi\imath a}\frac{z^{a-1}e^{z}}{1+\rho^{-1}} + O_\rho\left(z^{a-2}e^{z}\right). \tag{C.2c}$$

Remark C.4. The first two estimates in the proposition are only useful when ρ is bounded away from the transition point at one. We cannot write the last expansion above as $\Gamma(a,-z)$ directly unless $a\in\mathbb{Z}^+$ as the incomplete gamma function has a branch point at the origin with respect to its second variable. This function becomes a single-valued analytic function of its second input by continuation on the universal covering of $\mathbb{C}\setminus\{0\}$.

Proof of Proposition C.3. The first asymptotic estimate follows directly from the following asymptotic series expansion that holds as $z\to\infty$ [18, Eq. (2.1)]:

$$\Gamma(a,z) \sim \Gamma(a) + z^a e^{-z}\times\sum_{k\geq 0}\frac{(-a)^k b_k(\rho)}{(z-a)^{2k+1}}.$$

Using the notation from (C.1d) and [17]

$$\Gamma(a,z) = \frac{z^{a-1}e^{-z}}{1-\rho^{-1}} + z^a e^{-z}R_1(a,\rho).$$

From the bounds in [17, §3.1], we have

$$|z^a e^{-z}R_1(a,\rho)| \leq z^a e^{-z}\times\frac{a\cdot b_1(\rho)}{(z-a)^3} = \frac{z^{a-2}e^{-z}}{(1-\rho^{-1})^3}$$

The main and error terms in the previous equation can also be seen by applying the asymptotic series in (C.1d) directly.

The proof of the third equation above follows from the asymptotics [16, Eq. (1.1)]

$$\Gamma(-a,z) \sim z^{-a}e^{-z} \times \sum_{n\geq 0} \frac{a^n b_n(-\rho)}{(z+a)^{2n+1}},$$

by setting $(a,z) \mapsto (ae^{\pm\pi\imath}, ze^{\pm\pi\imath})$ so that $\rho = \frac{z}{a} > W(1) \approx 0.56714$. The restriction on the range of ρ over which the third formula holds is made to ensure that the formula from the reference is valid at negative real a. □

D Inversion theorems for partial sums of Dirichlet convolutions

Proof of Theorem 5.2. Suppose that h, r are arithmetic functions such that $r(1) \neq 0$. The following formulas hold for all $x \geq 1$:

$$\begin{aligned} S_{r*h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} r(n) h\left(\frac{n}{d}\right) = \sum_{d=1}^{x} r(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left(R\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - R\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right) H(i). \end{aligned} \tag{D.1}$$

The first formula on the right-hand-side above is well known from the references. The second formula is justified directly using summation by parts as [19, §2.10(ii)]

$$\begin{aligned} S_{r*h}(x) &= \sum_{d=1}^{x} h(d) R\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i\leq x} \left(\sum_{j\leq i} h(j)\right) \times \left(R\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - R\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right). \end{aligned}$$

We form the invertible matrix of coefficients, denoted by $\hat{R}$ below, associated with the linear system defining $H(j)$ for $1 \leq j \leq x$ in (D.1) by defining

$$R_{x,j} := R\left(\left\lfloor \frac{x}{j} \right\rfloor\right) [j \leq x]_{\delta},$$

and

$$r_{x,j} := R_{x,j} - R_{x,j+1}, \text{ for } j \geq 1.$$

Since $r_{x,x} = R(1) = r(1) \neq 0$ for all $x \geq 1$ and $r_{x,j} = 0$ for all $j > x$, the matrix we have defined in this problem is lower triangular with a non-zero constant on its diagonals, and so is invertible. If we let $\hat{R} := (R_{x,j})$, then the next matrix is expressed by applying an invertible shift operation as

$$(r_{x,j}) = \hat{R}\left(I - U^T\right).$$

The $N \times N$ square matrix U has $(i,j)^{th}$ entries for all $1 \leq i, j \leq N$ when $N \geq x$ that are defined by $(U)_{i,j} = \delta_{i+1,j}$ so that

$$\left[\left(I - U^T\right)^{-1}\right]_{i,j} = [j \leq i]_{\delta}.$$

We observe the identity

$$\left\lfloor \frac{x}{j} \right\rfloor - \left\lfloor \frac{x-1}{j} \right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

The previous equation implies that

$$R\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - R\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} r\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \tag{D.2}$$

We use the property in (D.2) to shift the matrix $\hat{R}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of r as

$$\left(\left(I - U^T\right)\hat{R}\right)^{-1} = \left(r\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(r^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Our target matrix in the inversion problem is

$$(r_{x,j}) = \left(I - U^T\right)\left(r\left(\frac{x}{j}\right)[j|x]_\delta\right)\left(I - U^T\right)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators by expanding

$$\begin{aligned} (r_{x,j})^{-1} &= \left(I - U^T\right)^{-1}\left(r^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)\left(I - U^T\right) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} r^{-1}(k)\right)\left(I - U^T\right) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} r^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} r^{-1}(k)\right). \end{aligned}$$

The summatory function $H(x)$ is given exactly for any integers $x \geq 1$ by a vector product with the inverse matrix from the previous equation as

$$H(x) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor+1}^{\left\lfloor \frac{x}{k} \right\rfloor} r^{-1}(j)\right) \times S_{r*h}(k).$$

We can prove a second inversion formula providing the coefficients of the summatory function $R^{-1}(j)$ for $1 \leq j \leq x$ from the last equation by adapting our argument to prove (D.1) above. This leads to the following alternate identity expressing $H(x)$:

$$H(x) = \sum_{k=1}^{x} r^{-1}(k) \times S_{r*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right). \qquad \square$$