New from old: Recent results and publications on Jacobi-type J-fractions

My mathematical research interests primarily fall into the realm of combinatorial number theory and generating function techniques in one form or another. The generating function of a given combinatorial sequence, $\{d_n\}_{n\geq 1}$, defined formally in q as $\sum_{n\geq 1} d_n q^n$, encodes many deep properties of the sequence. The combinatorial sequences we consider are the divisor function which counts the number of divisors of a natural number n, generalized factorial functions, and the partition function $p(n)$ which counts the number of distinct partitions of n into a non-increasing sum of positive integers. The function $p(n)$ has long been an object of study since the time of Euler, and even so, many of its basic elementary properties remain a mystery. The properties we are focused on are exact formulas for the parity of these functions, as well as their asymptotic properties. One shortcoming of traditional enumerative methods applied to such sequences is the typical divergence of their ordinary generating functions.

A useful combinatorial tool for working with the generating functions of many interesting sequences was re-addressed in the early 1980's by P. Flajolet. He pioneered a number of new results for formal infinite Jacobi-type continued fractions in this subject whose expansions have particularly "nice" properties and rational approximations [1]. More formally, an infinite Jacobi-type continued fraction, or *J-fraction* for short, has the expansion

$$J_\infty(z) = \cfrac{1}{1 - c_1 z - \cfrac{\mathrm{ab}_2\, z^2}{1 - c_2 z - \cfrac{\mathrm{ab}_3\, z^2}{\cdots}}} = \begin{array}{l} 1 + c_1 z + (\mathrm{ab}_2 + c_1^2)\, z^2 \\ \quad + (2\,\mathrm{ab}_2\, c_1 + c_1^3 + \mathrm{ab}_2\, c_2)\, z^3 + \cdots \end{array}, \qquad (1)$$

for sequence parameters, $\{c_k\}_{k\geq 1}$ and $\{\mathrm{ab}_k\}_{k\geq 2}$ which are unique to a given sequence in applications. Given any sequence, $\{d_n\}_{n\geq 1}$, we can always solve for appropriate unique choices of these sequences that (1) depends on such that the coefficients of $J_\infty(z)$ satisfy $[z^n]J_\infty(z) = d_n$ for all $n \geq 1$. In such cases, we say that $J_\infty(z)$ *generates*, or *enumerates*, the sequence over z. I have recently made use of these constructions of the Jacobi-type J-fractions generating special sequences to identify the next new generating function for the divisor function.

Theorem 1 (*The Divisor Function, Schmidt, J. Number Theor. 2017*)

For a fixed $|q| < 1$ with the sequences

$$c_i = \frac{q^i\,(q^{2i} - q^i(1 + q - q^2) + 2q - q^2 - q^3)}{(q^{2i} - 1)(q^{2i} - q^2)}, \quad \mathrm{ab}_i = \frac{q^{2i+1}(q^i - q)^2}{(q^i + q)^2(q^{2i} - q)(q^{2i} - q^3)}$$

for $i \geq 2$ with $c_1 := (1+q)^{-1}$, the expansion in (1) generates the scaled terms $q^n \cdot (1 - q^n)^{-1}$ corresponding to the generating function for the divisor function $d(n)$.

By differentiating the forms of $J_\infty(qz)$ in Theorem 1, we can obtain new exact generating functions in q and new congruences for the generalized sum-of-divisors functions, $\sigma_\alpha(n)$, for $\alpha \in \mathbb{Z}$ by letting $z \mapsto q$ as in the applications of [3]. Many properties of these special divisor functions, including asymptotics, can be derived from the continued-fraction-generated generating functions constructed above.

In [4] I have also recently employed the rationality of the h^{th} convergent functions to the infinite J-fraction defined by a special case of the expansion in (1) to expand the typically divergent generating functions for an expansive class of generalized factorial functions, denoted by $p_n(\alpha, R) = R(R+\alpha)\cdots(R+(n-1)\alpha)$ for $n \geq 1$. The multifactorial functions, $n!_{(\alpha)}$, studied coefficient-wise in my first 2010 publication in *J. Integer Seq.* (JIS) correspond to the the special cases of $n!_{(\alpha)} \equiv p_{\lfloor (n+\alpha-1)/\alpha \rfloor}(-\alpha, n)$ where the indeterminate parameter R in the generalized sequences varies linearly with n. The new results proved in the 2017 JIS article then imply new congruences for these generalized factorial functions involving often studied zeros of certain classes of special functions. These results similarly imply new exact identities for the parity of the Stirling numbers of the first kind, $\left[{n \atop k}\right] = (-1)^{n-k} s(n,k)$, which may always be used to expand these functions as polynomials in n. In particular, we have that the next remarkable formula is a direct (and as it turns out characteristic) consequence of the J-fraction expansion in (1) stated as

$$\left[{n \atop 5}\right] \equiv 2^{n-13}(27n^3 - 279n^2 + 934n - 1008)(n-1)\,[n \geq 6]_\delta + \delta_{n,5} \pmod{2},$$

where $[\mathtt{cond}]_\delta$ denotes Iverson's convention and $\delta_{n,k}$ is Kronecker's delta function.

Looking forward: Parity problems for partition functions

My advisor Professor Ernie Croot has already turned my attention to the partition function $p(n)$. I am now working on determining new significant bounds related to the parity of Euler's partition function $p(n)$, or more precisely stated, the values of $p(n) \bmod 2$. In particular, we define the following notation for our even bounded parity counts of interest: $N_e(x) := \#\{n \leq x : p(n) \equiv 0 \bmod 2\}$. It is known that for large enough $x > x_0$, we have the lower bound that $N_e(x) > C_1\sqrt{x}$ [2], though the so-termed "folklore" bound that $N_e(x) \approx \frac{x}{2}$ is the expected behavior of this function for large x. Improvements to these bounds have been established, though the problem of optimally bounding these functions remains a significant open problem of interest to many number theorists. For example, breaking the known "square root barrier" bound by proving the small power-factor improvement that $N_e(x) > C \cdot x^{0.51} + o\left(x^{0.51}\right)$ is considered significant progress on the problem.

My goal is to use the ideas on J-fractions above to develop methods to attack this well-studied parity problem. I am optimistic that the many forms of approximate generating functions for $p(n)$, such as the starting point of generating $(q;q)_n^{-1}$, among many other possibilities, that we can generate using the generalized J-fractions modulo 2 will be useful in developing breakthroughs. These expansions will eventually, through the application of more refined techniques along the way, lead to new identities and better bounds on the parity of $p(n)$ which should extend the known square-root-barrier proved by the elementary methods in [2]. Many other engaging variations are possible such as restricting parity counts to arithmetic progressions over the partition function and considering analogous bounds for other well-known restricted partition functions generated by infinite products and q-series expansions.

References

[1] P. Flajolet, Combinatorial aspects of continued fractions, *Discrete Math.* (1980). Reprinted 2006.

[2] J. L. Nicolas and I. Z. Ruzsa, On the parity of additive representation functions, *J. Number Theor.* (1998).

[3] M. D. Schmidt, Continued fractions and q-series generating functions for the generalized sum-of-divisor functions, *J. Number Theor.* (2017).

[4] M. D. Schmidt, Jacobi-type continued fractions for the ordinary generating functions of generalized factorial functions, *J. Integer Seq.* (2017).