Revised Manuscript and Responses to Referee Feedback

New characterizations of partial sums of the Möbius function

Revision 3 – v2022-01-03

Responses to the feedback from the referee upon return of the first revised manuscript:

Referee Point:
It takes a long time to see exactly what theorems the author is proving, and when you see 'em, it's unclear why anyone would do this.

Author Response:
In making modifications to the article, the exposition is now clearer and more accessible from a high level starting in the introduction. I have received feedback from the JNT editor and from other sources in making these changes. The main updated material with respect to this point you will find is in the first section of the article. I have added several paragraphs explaining motivation for the methods in the article, compared my techniques to previous methods exploring properties and bounds of M(x), highlighted that which is new about my perspective in characterizing these partial sums through the new sequences studied, and given more breadth and focus on the higher-level insights behind the new results. The prior draft (revisions 1 and 2) of the article contained several results that were compressed between longer technical proofs in the middle of the article. The new presentation should help clarify the results on the distributions of the unsigned functions, e.g., the average order formulas and the conjectures on the limiting distributions behind the two key auxiliary unsigned functions within the article. I have tried to illustrate the new prime related combinatorics that underly these functions in the introduction to motivate and give readers a feel for the flavor of the article at the start. The plots in Appendix B also now supplement the intuition suggested by the table of computations of the first five-hundred values related to these new auxiliary functions which is still located in the last appendix section (now Appendix C).

Referee Point:
Sound's paper is in Crelle, not Annals.

Author Response:
The point is noted and now correctly cited in the bibliography section of the article. Thank you for catching this missed citation from my old BibTeX file. Some of the results I originally had attributed to Sound's article with the reference are contained in the bibliography item by Humphries. In other words, there was a mistaken reference to which of the articles proved which bounds on M(x). Both upper bounds that were proved under the assumption of the RH are now accurately stated and cited inline in respective order.

Referee Point:
Sound's result, quoted before on page 4, just before \S 1.1.3 should have "14" as the exponent of \log\log x, and not 5/2 + \epsilon.
Author Response:
The credit to Sound's Crelle article with the correct exponent corresponding to that reference is updated in Section 1.3 (on preliminary known results). I also have added a new citation to the result for an upper bound on M(x) stated by Humphries as one of the bounds given in the discussion of preliminaries in this section.

Referee Point:
Above this, Walfisz's result should have, as the exponent of \log\log x "-3/5" instead of "-1/5".
Author Response:
Thank you again for pointing this typo out. It has been corrected in the revised article. I do have a question back for the referee to answer, which is whether a citation where Walfisz's result can be found that is translated well in English? Unfortunately, as I do not read fluent German, I was only able to scan the citation I could find to this result online and verify it exists by examining the mathematical typesetting. It seems to be a famous bound for the Mertens function that may be just as easily attributed, as I have done in this version of the manuscript, by citing the author and date of the result.

Referee Point:
"te Riele" and "Odlyzko" have interesting variations on their spelling throughout the article.
Author Response:
I removed the accent mark on the first author throughout the article and in the bibliography. By my inspection, there was a single typographical misspelling of the second author that was a single letter transposition type error. An online search of the spellings of both author names based on their prior publications suggests that the other instances in my article, and now the misspelled author name variants, are now correct. Thank you for pointing out that I needed to check the spellings more carefully.