ASYMPTOTIC BOUNDS FOR THE MERTENS FUNCTION

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Abstract. The Mertens function is defined as the average order of the Möbius function, or as the summatory function $M(x) = \sum_{n \leq x} \mu(n)$, for all $x \geq 1$. There are many open problems are related to determining optimal asymptotic bounds for this function. The famous statement of Mertens' conjecture which says that $|M(x)| < \sqrt{x}$ has been disproved, though is it known that the Riemann Hypothesis is equivalent to showing that $|M(x)| \ll \sqrt{x} \exp\left(B \frac{\log x}{\log\log x}\right)$ for some constant B. Another unresolved problem related to this function is whether $\limsup_{x \to \infty} |M(x)|/\sqrt{x} = \infty$. In this article, we employ the recent construction of new formulas for the generalized sum-of-divisors functions proved by Schmidt to obtain new results which exactly sum the classical Mertens function for all finite x. We state and prove analogous results for the generalized Mertens function which we define to be $M_\alpha^*(x) = \sum_{n \leq x} n^\alpha \mu(n)$ for any fixed $\alpha \in \mathbb{C}$.

1. Introduction

1.1. Mertens summatory functions. The Mertens summatory function, or *Mertens function*, is defined as

$$M(x) = \sum_{n \leq x} \mu(n),\ x \geq 1,$$

where $\mu(n)$ denotes the Möbius function which is in some sense a signed indicator function for the squarefree integers. A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

We define the notion of a *generalized Mertens summatory function* for fixed $\alpha \in \mathbb{C}$ as

$$M_\alpha^*(x) = \sum_{n \leq x} n^\alpha \mu(n),\ x \geq 1,$$

where the special case of $M_0^*(x)$ corresponds to the definition of the classical Mertens function $M(x)$ defined above. The plots shown in Figure 1.1 illustrate the chaotic behavior of the growth of these functions for x in small intervals when $\alpha \in \{-1, 0, 1, 2\}$. In particular, there are many open problems related to bounding $M(x)$ for large x. The Riemann Hypothesis is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. It is still unresolved whether

$$\limsup_{x \to \infty} |M(x)|/\sqrt{x} = \infty,$$

although computational evidence suggests that this is a likely conjecture [5, 4]. We make a newly well-founded attempt to prove that this conjecture is true in Theorem 2.1.

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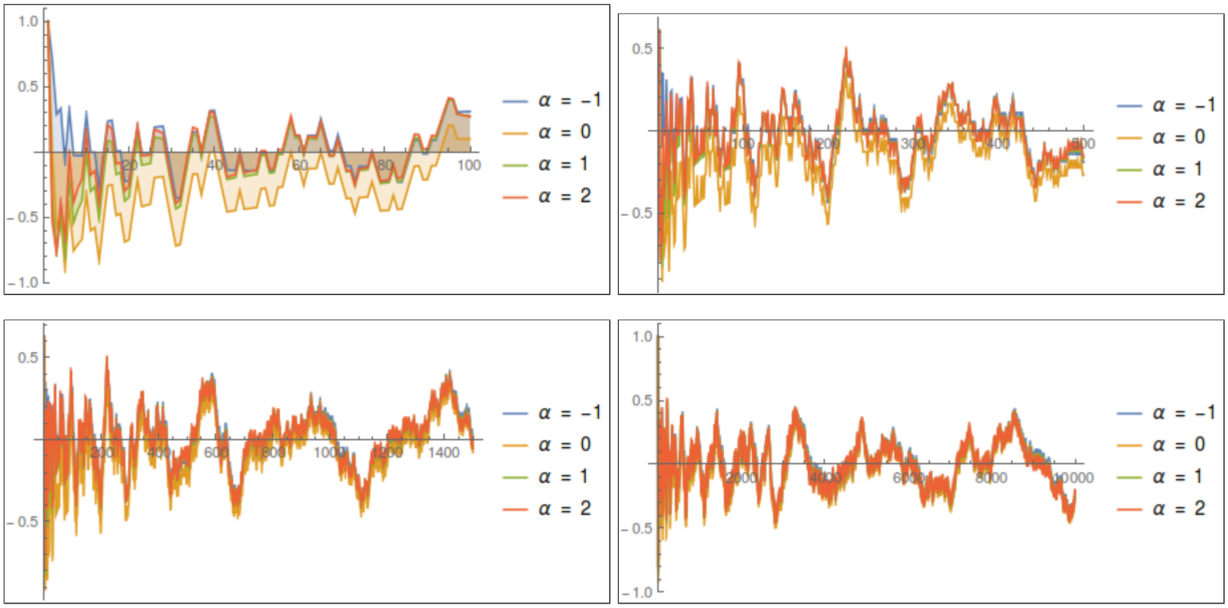


Figure 1.1. *Comparison of the Mertens Summatory Functions $M_\alpha(x)/x^{\frac{1}{2}+\alpha}$ for Small x and α*

1.2. **Exact formulas for the generalized sum-of-divisors functions.** Schmidt has recently proved (2017) several new exact formulas for the *generalized sum-of-divisors functions*, $\sigma_\alpha(x)$, defined for any $x \geq 1$ as

$$\sigma_\alpha(x) = \sum_{d|n} d^\alpha, \ \alpha \in \mathbb{C}.$$

In particular, if we let $H_n^{(r)} = \sum_{k=1}^n k^{-r}$ denote the sequence of *r-order harmonic numbers*, where [9, §2.4(iii)]

$$H_n^{(-t)} = \frac{B_{t+1}(n+1) - B_{t+1}}{(t+1)} = \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + \sum_{k=1}^{t-1} \binom{t}{k} \frac{B_{k+1} n^{t-k}}{(k+1)},$$

is a *Bernoulli polynomial* for any $n \geq 0$ when $t \in \mathbb{Z}^+$, then we can restate the next theorem from [12]. Within this article we assume that an index of summation p denotes that the sum is taken over only prime values of p. We also use the notation that the valuation function

$$\nu_p(x) = m \quad \text{if and only if} \quad p^m \| x,$$

to denote the exact exponent of the prime p dividing x.

Theorem 1.1 (Schmidt, 2017)**.** *For any fixed $\alpha \in \mathbb{C}$ and all $x \geq 1$, we have that*

$$\begin{aligned}\sigma_\alpha(x) = H_x^{(1-\alpha)} + \sum_{d|n} \tau_x^{(\alpha)}(d) + \sum_{2 \leq p \leq x} \sum_{k=1}^{\nu_p(x)+1} p^{\alpha k} H^{(1-\alpha)}_{\left\lfloor \frac{x}{p^k} \right\rfloor} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x - p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right) \\ + \sum_{3 \leq p \leq x} \sum_{k=1}^{\nu_p(x)+1} 2^{\alpha-1} p^{\alpha k} (-1)^{\lfloor x/p^{k-1} \rfloor} H^{(1-\alpha)}_{\left\lfloor \frac{x}{2p^k} \right\rfloor} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x - p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right),\end{aligned}$$

where the divisor sum over the function $\tau_x^{(\alpha)}(d)$ is defined precisely by Lemma 2.4.

Remark 1.2 (Restatement of the Theorem)**.** For $x \geq 1$ and fixed $\alpha \in \mathbb{C}$, we define the sums

$$S_1^{(\alpha)}(x) = \sum_{2 \leq p \leq x} \sum_{k=1}^{\nu_p(x)+1} p^{\alpha k} H^{(1-\alpha)}_{\left\lfloor \frac{x}{p^k} \right\rfloor} \left(\left\lfloor \frac{x}{p^k} \right\rfloor - \left\lfloor \frac{x - p^{k-1}}{p^k} \right\rfloor - \frac{1}{p} \right)$$

$$S_2^{(\alpha)}(x) = \sum_{3\leq p\leq x} \sum_{k=1}^{\nu_p(x)+1} 2^{\alpha-1} p^{\alpha k} (-1)^{\lfloor x/p^{k-1}\rfloor} H^{(1-\alpha)}_{\left\lfloor \frac{x}{2p^k}\right\rfloor} \left(\left\lfloor \frac{x}{p^k}\right\rfloor - \left\lfloor \frac{x-p^{k-1}}{p^k}\right\rfloor - \frac{1}{p}\right).$$

Then we prefer to work with the next form of Theorem 1.1 stated in terms of our new shorthand sum functions as follows:

$$\left|\sum_{d|x} \tau_x^{(\alpha)}(d)\right| = \left|H_x^{(1-\alpha)} - \sigma_\alpha(x) + S_1^{(\alpha)}(x) + S_2^{(\alpha)}(x)\right|. \tag{1}$$

The statement of the theorem given in (1) is important and significant since it implies deep connections between the sum-of-divisors functions, the generalized Mertens summatory functions, and the partial sums of the Riemann zeta function for real $\alpha < 0$, each related to one another in a convolved formula taken over sums of successive powers of the primes $p \leq x$. Thus we immediately see new relations from the restatement of the key results in [12] above. Moreover, from the previous result, we then obtain our main new results in the article given in the results in the next section as consequences of this restatement in terms of the Mertens functions.

2. New results and proofs of key lemmas

2.1. Statement of the main theorem.

Theorem 2.1 (The Limit Supremum of $M(x)$ and Its Values at Large Prime Powers)**.** *Let $x = q^r$ denote a large odd prime power for some $r \geq 1$. Then we have that*

$$\limsup_{\substack{x\to\infty \\ x=q^r}} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

Proof (Sketch). The complete proof of the theorem is given at conclusion of this section. For now, we will elaborate on the key steps in proving the theorem. We begin by noting that

$$\left|\sum_{d|x} \tau_x^{(\alpha+1)}(d)\right| \leq \left|T_v^{(\alpha)}(x)\right| + \left(2\cdot \sup_{1\leq i\leq x} |M_\alpha^*(i)| + 1\right) \times D_v^{(\alpha)}(x) - M_\alpha^*(1) d_x^{(\alpha)}(1),$$

where the upper bound is obtained by summation by parts. We then need to show that infinitely and predictably often at least (and not necessarily for all large x) that we can bound the ratio of the next sums by $x \log\log x$. We consider the cases of large x when $x := q^r$ is a large prime power for some $r \geq 1$ and employ the resulting expansions to complete our proof. The next step in the proof is to show that (1) is approximately

$$\begin{aligned}\left|\sum_{d|x} \tau_x^{(\alpha+1)}(d)\right| &= \left|H_x^{(1-\alpha)} - \sigma_\alpha(x) + S_1^{(\alpha)}(x) + S_2^{(\alpha)}(x)\right| \\ &\geq \widetilde{C}\cdot x\log\log(x-1).\end{aligned}$$

We then prove by a key (and not at all obvious) construction in Proposition 2.8 that the functions $|T_v^{(0)}(x)| \leq x\log\log\log x$ and $D_v^{(0)}(x) \leq \sqrt{x}$ when x is a sufficiently large prime. Then for all large primes x we have that

$$\frac{1}{\sqrt{x}}\left(\sup_{1\leq i\leq x} |M(i)|\right) \geq \widetilde{C}\cdot \log\log(x-1) - \log\log\log x.$$

Thus as the lower bound stated in the previous equations increases with x and tends to infinity infinitely often, i.e., whenever we input x as one of our large primes, we see that the right-hand-side supremum must tend to infinity infinitely often as well. This is the basic sketch of the argument we will employ when we give the full proof of Theorem 2.1 in the next subsections. For now, we need to develop more machinery and state several lemmas to establish this claim. □

Remark 2.2 (History of the Mertens Conjecture)**.** There is a rich history to the original statement of the *Mertens conjecture* which states that

$$|M(x)| < c \cdot x^{1/2}, \quad \text{some constant } c > 0,$$

which was first verified by Mertens for $c = 1$ and $x < 10000$, although since its beginnings in 1897 has since been disproved by computation. We cite that prior to this point it is known that [10, *cf.* §4.1]

$$\limsup_{x\rightarrow\infty} \frac{M(x)}{\sqrt{x}} > 1.06,$$

and

$$\liminf_{x\rightarrow\infty} \frac{M(x)}{\sqrt{x}} < -1.009,$$

although based on work by Odlyzyko and te Riele (1985) it seems probable that each of these limits should be $\pm\infty$, respectively [8].

2.2. **Key asymptotic bounds and formulas.**

Definition 2.3 (Prime Power Indicator Functions)**.** For $x \geq 1$, let $\chi_{\mathrm{pp}}(x)$ denote the indicator function for prime powers, i.e., the function defined precisely as

$$\chi_{\mathrm{pp}}(x) = \begin{cases} 1, & \text{if } x = p^k \text{ for some prime } p \geq 2 \text{ and } k \geq 1; \\ 0, & \text{otherwise,} \end{cases}$$

and define the composite indicator function for the prime powers $p^k, 2p^k$ as follows where $\chi_{\mathrm{pp}}(x) = 0$ if $x \in \mathbb{Q} \setminus \mathbb{Z}$[1]:

$$\begin{aligned} \widetilde{\chi}_{\mathrm{pp}}(x) &= [\chi_{\mathrm{pp}}(x) = 0]_{\delta} \left[\chi_{\mathrm{pp}}\left(\frac{x}{2}\right) = 0\right]_{\delta} \\ &= (1 - \chi_{\mathrm{pp}}(x)) \left(1 - \chi_{\mathrm{pp}}\left(\frac{x}{2}\right)\right). \end{aligned}$$

We adopt the notation $\omega(n)$ to denote the number of distinct prime factors of a natural number $n \geq 1$ where $\omega(1) = 0$.

Lemma 2.4 (Exact Formulas for the Divisor Sums $\sum_{d|x} \tau_x^{(\alpha)}(d)$)**.** *Let $v(x,d)$ be a boolean-valued function whose negation is given by $\neg v(x,d)$. For $\alpha \in \mathbb{N}$, $m \geq 1$, and $x \geq 12$, let*

$$d_{x,v}^{(\alpha)}(m) = \sum_{k=1}^{x} \sum_{\substack{d|k \\ v(x,d)}} \sum_{r|(d,x)} r^{\alpha+1} \left(\frac{k}{d}\right)^{\alpha} \widetilde{\chi}_{\mathrm{pp}}(d) \left[m = \frac{d}{r}\right]_{\delta},$$

and let

$$T_v^{(\alpha)}(x) = \sum_{k=1}^{x} \sum_{\substack{d|k \\ \neg v(x,d)}} \sum_{r|(d,x)} r \cdot \mu(d/r) \widetilde{\chi}_{\mathrm{pp}}(d) \cdot k^{\alpha}.$$

Then for any condition v on the divisors in the above sums, we can expand the divisor sums in Theorem 1.1 exactly in the following forms:

$$\sum_{d|x} \tau_x^{(\alpha+1)}(d) = T_v^{(\alpha)}(x) + \sum_{m=1}^{x} \mu(m) m^{\alpha} \cdot d_{x,v}^{(\alpha)}(m).$$

[1] Notation: We use *Iverson's convention* $[\texttt{cond} = \texttt{True}]_{\delta} \equiv \delta_{\texttt{cond},\texttt{True}}$ according to whether the condition `cond` is true or false where $\delta_{n,k}$ denotes Kronecker's delta function.

Proof. We start with the following formula for computing the divisor sum over $\tau_x^{(\alpha)}(d)$ from [12, §2]:

$$\begin{aligned}
\sum_{d|x} \tau_x^{(\alpha+1)}(d) &= [q^x]\left(\sum_{k=1}^{x}\sum_{d|k}\sum_{r|d} \frac{r\cdot\widetilde{\chi}_{\mathrm{pp}}(d)\cdot\mu(d/r)}{(1-q^r)}k^{\alpha}\right) \\
&= \sum_{k=1}^{x}\sum_{r|x}\sum_{d|k} r\cdot\widetilde{\chi}_{\mathrm{pp}}(d)\cdot\mu(d/r)\cdot[r|d]_{\delta}\cdot k^{\alpha} \\
&= \sum_{k=1}^{x}\sum_{d|k}\sum_{r|(d,x)} r\cdot\widetilde{\chi}_{\mathrm{pp}}(d)\cdot\mu(d/r)\cdot k^{\alpha} && (2)\\
&= T_v^{(\alpha)}(x) + \sum_{m=1}^{x}\mu(m)m^{\alpha}\cdot d_{x,v}^{(\alpha)}(m). && (3)
\end{aligned}$$

We notice that the left-hand-side divisor sum in the previous equations is given by the large-order case of $T_{d\leq x}^{(\alpha)}(x)$, or say $T_{d\leq\infty}^{(\alpha)}(x)$ to distinguish between our cases. The delicate balance that the role of the parameter v plays in bounding this divisor sum motivates the result in Proposition 2.8. ☐

Remark 2.5 (Connection to Ramanujan's Sum)**.** We have a deep connection between the divisor sums in Lemma 2.4 and Ramanujan's sum $c_q(n)$ given by

$$\begin{aligned}
\sum_{d|x}\tau_x^{(1)}(d) &= \sum_{k=1}^{x}\sum_{d|k}\widetilde{\chi}_{\mathrm{pp}}(d)\cdot c_d(x) \\
&= \sum_{k=1}^{x}\sum_{d|k}\widetilde{\chi}_{\mathrm{pp}}(d)\cdot\mu\left(\frac{d}{(d,x)}\right)\frac{\varphi(d)}{\varphi\left(\frac{d}{(d,x)}\right)},
\end{aligned}$$

where $\varphi(x)$ denotes Euler's totient function. These identities follow by expanding out Ramanujan's sum in the form of [9, §27.10] [7, §A.7] [3, *cf.* §5.6]

$$c_q(n) = \sum_{d|(q,n)} d\cdot\mu(q/d),$$

and then applying the formula in (2) from the proof of the lemma above.

Lemma 2.6 (A Lower Bound for the Magnitude of $M(x)$)**.** *For fixed integers $\alpha\geq 0$ and $x\geq 1$, let*

$$D_v^{(\alpha)}(x) = \sum_{m=1}^{x}\left|d_{x,v}^{(\alpha)}(m)\right| = \sum_{k=1}^{x}\sum_{\substack{d|k\\ v(x,d)}}\sum_{r|(d,x)} r^{\alpha+1}\left(\frac{k}{d}\right)^{\alpha}\widetilde{\chi}_{\mathrm{pp}}(d).$$

For all sufficiently large $x\geq 14$ and any $v\geq 0$, we have the following bound on the supremum of $|M(i)|$ taken over all $i\leq x$:

$$\frac{\left|\sum_{d|x}\tau_x^{(1)}(d)\right| - \left|T_v^{(0)}(x)\right| + d_{x,v}^{(0)}(1)}{2\cdot D_v^{(0)}(x)} - \frac{1}{2} \leq \sup_{1\leq i\leq x}|M(i)|.$$

Proof. We first observe the equivalences of the sums for (1) given in Lemma 2.4. For fixed x, we then proceed from here by summation by parts to obtain that

$$\begin{aligned}
\left|\sum_{d|x}\tau_x^{(\alpha+1)}(d)\right| &\leq \left|T_v^{(\alpha)}(x)\right| + \left|\sum_{m=1}^{x}\mu(m)m^{\alpha}\cdot d_{x,v}^{(\alpha)}(m)\right| \\
&\leq \left|T_v^{(\alpha)}(x)\right| + |M_{\alpha}^{*}(x)|\,d_{x,v}^{(\alpha)}(x) + \sum_{m=1}^{x-1}|M_{\alpha}^{*}(x)|\left|d_{x,v}^{(\alpha)}(m+1)-d_{x,v}^{(\alpha)}(m)\right|
\end{aligned}$$

$$\leq \left|T_v^{(\alpha)}(x)\right| + 2\sum_{m=1}^{x} |M_\alpha^*(x)|\, d_{x,v}^{(\alpha)}(m) + \sum_{m=1}^{x} m^\alpha |\mu(m)| \cdot |d_{x,v}^{(\alpha)}(m)| - M_\alpha^*(1) d_{x,v}^{(\alpha)}(1)$$

$$\leq \left|T_v^{(\alpha)}(x)\right| + \left(2 \sup_{1\leq i\leq x} |M_\alpha^*(i)| + 1\right) \cdot D_v^{(\alpha)}(x) - d_{x,v}^{(\alpha)}(1)$$

We then consider the special case when $\alpha := 0$ to obtain the correct bound for the Mertens function $M(x)$ stated above. □

Remark 2.7 (Appropriate Condition Functions v)**.** There is a delicate balancing act we must now play to ensure that we can bound the supremum of the $|M(i)|$ for $i \leq x$ in the lemma below by an increasingly large function of x tending to infinity. Since (2) implies that

$$\sum_{d|x} \tau_x^{(\alpha+1)}(d) = T_{d\leq\infty}^{(\alpha)}(x),$$

we can always subtract off our factor of $T_v^{(\alpha)}(x)$ and obtain a correct bound in the form of Lemma 2.6. In particular, we need to select a function $v \equiv v(x,d)$ such that the following two properties hold whenever $x = q^r$ is a power of a large odd prime for some $r \geq 1$:

- **(P1)** $T_v^{(0)}(x) \leq x \log\log\log x$, and
- **(P2)** $D_v^{(0)}(x) = O\left(x^{1/2-\delta} \log\log x\right)$ for some $\delta > 0$, *or alternately*, $D_v^{(0)}(x) = O\left(\sqrt{x}\right)$.

We will require the next Proposition 2.8 in the complete proof of Theorem 2.1 given in the next subsection. For now, we have several observations to make on the apparent choice of a condition function which depends on x. In each of the plots shown in Figure 2.1, we have selected a condition function v of the form

$$v_1(x,d) \equiv d < \frac{x^{\varepsilon(x)}}{\log\log x} \qquad \text{or} \qquad v_2(x,d) \equiv d < x^{\varepsilon(x)}$$

The plots in the figure show the resulting computations of $D_v^{(0)}(x)$ restricted to prime x. The corresponding plots of the same quantities for $D_v^{(0)}(x)$ taken over all $x \geq 1$ are much more chaotic and irregular. What we see in the prime plot cases is remarkable. Now we need only figure out how to translate this apparent behavior into a conjecture on an appropriate condition function v in the proposition below.

Proposition 2.8 (The Condition Function v)**.** *Let the condition function v defined by*

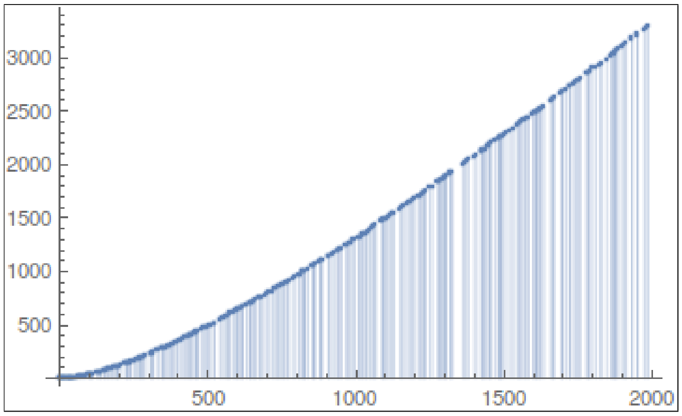
$$v(x,d) \equiv d \geq \left\lfloor (x-1)e^{-1/\sqrt{x}} \right\rfloor.$$

Then for all large primes x we have that

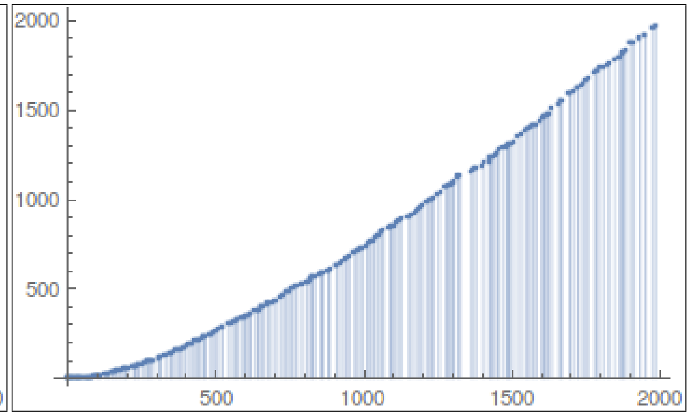
i. $D_v^{(0)}(x) \leq \sqrt{x}$, *and*
ii. $T_v^{(0)}(x) \leq x \log\log\log x$.

Proof. We begin with the first bound for $D_v^{(0)}(x)$ when x is prime using the shorthand condition function $v(x,d) \equiv d \geq x-1-m$ where m is a function of x using an asymptotic result for the first-order harmonic numbers:

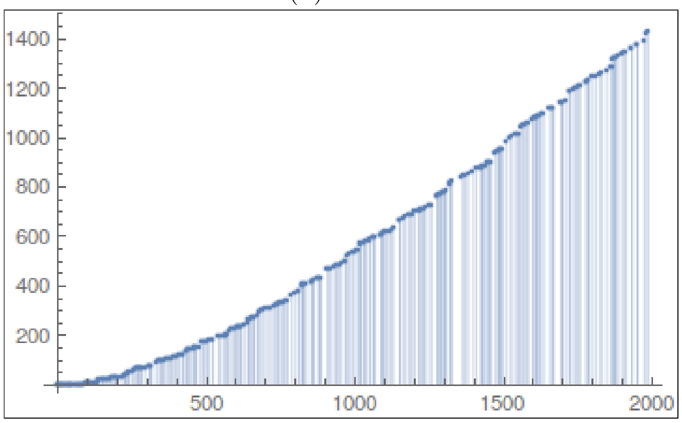
$$\begin{aligned}
D_v^{(0)}(x) &= \sum_{k=1}^{x} \sum_{\substack{d|k \\ v(x,d)}} \sum_{r|(d,x)} r\widetilde{\chi}_{\mathrm{pp}}(d) \\
&= \sum_{1\leq k<x} \sum_{\substack{d|k \\ d\geq x-1-m}} \widetilde{\chi}_{\mathrm{pp}}(d) \\
&= \sum_{d=x-1-m}^{x-1} \widetilde{\chi}_{\mathrm{pp}}(d) \cdot \#\{1 \leq k < x : d|k\}
\end{aligned}$$



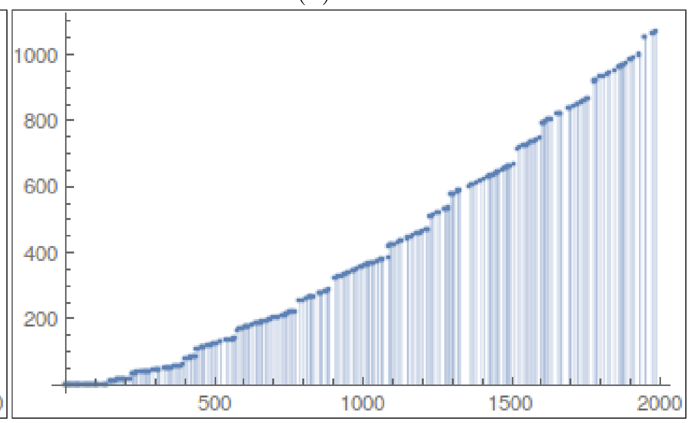
$\varepsilon(x) = 0.75$



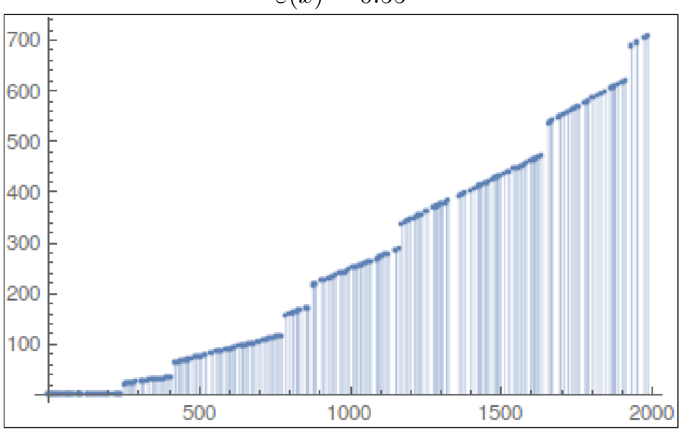
$\varepsilon(x) = 0.61$



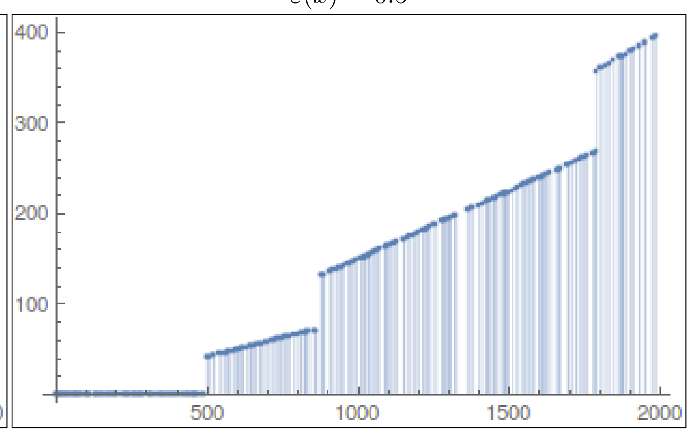
$\varepsilon(x) = 0.55$



$\varepsilon(x) = 0.5$



$\varepsilon(x) = 0.45$



$\varepsilon(x) = 0.4$

Figure 2.1. *The Difference Function $D_v^{(0)}(x)$ Restricted to Primes x for Specific Condition Function $v(x,d) \equiv d < x^{\varepsilon(x)}$. As the powers of $\varepsilon(x)$ decrease and lead to slopes of the resulting curve which are smaller (more desirable here), these plots breakdown into roughly piecewise linear, or piecewise power function, graphs when x is restricted to the primes.*

$$\leq \sum_{d=x-1-m}^{x-1} \frac{x}{d}$$

$$\sim x\left(\log(x-1)-\log(x-1-m)\right).$$

Since we require that $D_v^{(0)}(x) \leq \sqrt{x}$, we must solve for m such that

$$1+\frac{m}{x-1-m} \leq \exp(1/\sqrt{x}) \implies m \leq (x-1)\left(1-e^{-1/\sqrt{x}}\right),$$

or equivalently that

$$x-1-m \geq \left\lfloor (x-1)e^{-1/\sqrt{x}} \right\rfloor.$$

In order to bound the second function $T_v^{(0)}(x)$, we first note a standard asymptotic approximation to the sum of the reciprocals of all squarefree numbers $d \leq x$ given by

$$\sum_{d\leq x} \frac{|\mu(d)|}{d} = \prod_{p\leq x}\left(1+\frac{1}{p}\right) = \frac{\prod_{p\leq x}\left(1-\frac{1}{p^2}\right)}{\prod_{p\leq x}\left(1-\frac{1}{p}\right)} \sim \frac{e^{\gamma}}{\zeta(2)}\log x,$$

where $\gamma \approx 0.577216$ is Euler's gamma constant. Now we proceed to rewrite the expression for $T_v^{(0)}(x)$ from its definition in Lemma 2.4 when x is prime as in the expansion for the previous function:

$$\begin{aligned}
\left|T_v^{(\alpha)}(x)\right| &= \left|\sum_{k=1}^{x} \sum_{\substack{d|k \\ d<x-1-m}} \mu(d)\widetilde{\chi}_{\mathrm{pp}}(d)\right| \\
&\leq \sum_{d=1}^{x-2-m} \frac{x\cdot|\mu(d)|}{d} \\
&\sim \frac{6e^{\gamma}}{\pi^2}\log(x-2-m).
\end{aligned}$$

Since we require that $T_v^{(0)}(x) \leq x\log\log\log x$, we see that we must have

$$m \geq x-2-(\log\log x)^{\frac{\pi^2}{6e^{\gamma}}}.$$

Finally, since

$$\left\lfloor (x-1)e^{-1/\sqrt{x}} \right\rfloor \geq \left\lceil 1+(\log\log x)^{\frac{\pi^2}{6e^{\gamma}}} \right\rceil,$$

for all large x, we see that our choice of m is consistent with the bounds we have imposed on both functions. We also observe that our claimed form of v produces the desired bounds on both $D_v^{(0)}(x)$ and $T_v^{(0)}(x)$ for large primes x. ☐

Proposition 2.9 (Asymptotic Bound for the Tau Function Divisor Sum)**.** *Let $x = q^r$ denote a power of the large prime q for some $r \geq 1$. Then when the x tending to infinity of these forms is sufficiently large, we obtain*

$$\left|\sum_{d|x} \tau_x^{(\alpha+1)}(d)\right| \geq \widetilde{C}\cdot x\log\log(x-1),$$

for some absolute constant $\widetilde{C}$.

Proof. By the statement of Theorem 1.1 rephrased in (1), we see that

$$\begin{aligned}
\left|\sum_{d|x} \tau_x^{(1)}(d)\right| &= \left|x-\sigma_1(x)+S_1^{(1)}(x)+S_2^{(1)}(x)\right| \\
&= \left|S_1^{(1)}(q^r)+S_2^{(1)}(q^r)-\left(1+q+\cdots+q^{r-1}\right)\right| \\
&\geq \left|S_1^{(1)}(q^r)+S_2^{(1)}(q^r)\right| - \frac{x-1}{x^{\frac{1}{r}}-1}.
\end{aligned}$$

We next use the result of *Mertens' theorem* which implies that [7, §6.3] [1, §4.9] [3, §22.8] [9, §27.11]

$$\sum_{p \leq x} \frac{1}{p} = \log\log(x) + A + O\left(\frac{1}{\log x}\right),$$

where A is a limiting constant. In particular, when x is large we can expand the sum for $S_1^{(0)}(x)$ as

$$\begin{aligned}
S_1^{(0)}(x) &= \sum_{2 \leq p < q^r} p \cdot \left\lfloor \frac{q^r}{p} \right\rfloor \left(\left\lfloor \frac{q^r}{p} \right\rfloor - \left\lfloor \frac{q^r}{p} - \frac{1}{p} \right\rfloor - \frac{1}{p} \right) + \sum_{k=1}^{r+1} q^k \left\lfloor q^{r-k} \right\rfloor \left(\left\lfloor q^{r-k} \right\rfloor - \left\lfloor q^{r-k} - \frac{1}{q} \right\rfloor - \frac{1}{q} \right) \\
&= \sum_{2 \leq p < q^r} -\frac{p}{p} \left[\frac{q^r}{p} - \left\{ \frac{q^r}{p} \right\} \right] + \sum_{k=1}^{r} q^r \left(1 - \frac{1}{q}\right) - \frac{1}{q} \\
&= C_1 \pi(q^r - 1) - q^r \left(\log\log(q^r - 1) + A + O\left(\frac{1}{\log(q^r - 1)} \right) \right) + r \cdot q^{r-1}(q-1) - \frac{1}{q} \\
&\sim \frac{C_1(x-1)}{\log(x-1)} - x\left(\log\log(x-1) + A\right) + r \cdot x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1 \right) - \frac{1}{x^{\frac{1}{r}}},
\end{aligned}$$

and similarly, the sum $S_2^{(0)}(x)$ is expanded as

$$\begin{aligned}
S_2^{(0)}(x) &= \sum_{2 \leq p < q^r} C_2 \cdot p \left\lfloor \frac{q^r}{2p} \right\rfloor \cdot \frac{1}{p} - \sum_{k=1}^{r} q^k \left\lfloor \frac{q^{r-k}}{2} \right\rfloor \frac{(q-1)}{q} + \frac{1}{q} - 2C_3 \left\lfloor \frac{q^r}{4} \right\rfloor \cdot \frac{1}{2} \\
&= \sum_{2 \leq p < q^r} C_2 \left(\frac{q^r}{2p} - C_4 \right) - \sum_{k=1}^{r} q^k \left(\frac{q^{r-k}}{2} - C_5 \right) \frac{(q-1)}{q} + \frac{1}{q} - C_3 \left(\frac{q^r}{4} - C_6 \right) \\
&\sim \frac{C_2}{2} x \left(\log\log(x-1) + A\right) - \frac{C_2 C_4 (x-1)}{\log(x-1)} - \frac{r}{2} x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1 \right) + \frac{1}{x^{\frac{1}{r}}} + C_5(x-1) - \frac{C_3}{4} x + C_3 C_6.
\end{aligned}$$

Hence when we add these two sums cancellation of symmetric terms results in

$$\begin{aligned}
S_1^{(0)}(x) + S_2^{(0)}(x) \sim \frac{(C_1 - C_2 C_4)(x-1)}{\log(x-1)} &+ \left(\frac{C_2}{2} - 1 \right) x \left(\log\log(x-1) + A\right) \\
&+ \frac{r}{2} x^{1-\frac{1}{r}} \left(x^{\frac{1}{r}} - 1 \right) - \frac{C_3}{4} x + C_3 C_6,
\end{aligned}$$

which proves our result. □

2.3. **The complete proof of Theorem 2.1.** We are now at the point where we can assemble the complete results necessary to prove Theorem 2.1. The key idea here is that while the value of $|M(x)|$ is oscillating with x, we can bound the value of $\sup_{1 \leq i \leq x} |M(i)|$ below by something increasingly large and tending to infinity infinitely often, i.e., since there are an infinitude of large primes $q \to \infty$. Then using the lower bound in Lemma 2.6, and combining the bounds in Proposition 2.9 and Proposition 2.8, we see that when $x = q$ is large we have

$$\frac{1}{\sqrt{x}} \left(\sup_{1 \leq i \leq x} |M(i)| \right) \geq \widetilde{C} \cdot \log\log(x-1) - \log\log\log x. \tag{4}$$

Next, for $x = q$ a large odd prime q let

$$x_{0,q} = \operatorname*{argmax}_{1 \leq i \leq q} |M(i)|.$$

Then we see from (4) that

$$\frac{|M(x_{0,q})|}{\sqrt{x_{0,q}}} \geq \frac{|M(x_{0,q})|}{\sqrt{q}} \geq \widetilde{C} \cdot \log\log(q-1) - \log\log\log q.$$

Moreover, since the lower bound in (4) and in the previous equation is increasing with q, i.e., as $q \to \infty$, we see that the non-decreasing sequence of $x_{0,q}$ must gradually increase with larger and larger q. Thus we see that for any $L > 0$, there are infinitely many $x \in \mathbb{N}$ such that $|M(x)|/\sqrt{x} > L$. Hence the result is proved.

3. Conclusions

3.1. **Generalizations.** We remark that Theorem 2.1 can be effectively generalized to a result of the more general form

$$\limsup_{\substack{x\to\infty \\ x=q^r}} \frac{|M_\alpha^*(x)|}{(\sqrt{x})^{2\alpha+1}} = +\infty.$$

The only caveat here is that we need to know more precise forms of Mertens' theorem for general sums of the form $\sum_{p \leq x} p^{\alpha k}$ depending on the parameter $\alpha \geq 0$. For now, we will leave the generalizations of our main theorem as an exercise for future research on the generalized Mertens summatory functions $M_\alpha^*(x)$ defined in the introduction.

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