Lower bounds on the summatory function of the Möbius function along infinite subsequences

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Abstract

The Mertens function, $M(x) := \sum_{n \leq x} \mu(n)$, is defined as the summatory function of the Möbius function. The Mertens conjecture states that $|M(x)| < C \cdot \sqrt{x}$ for some absolute $C > 0$ for all $x \geq 1$. This classical conjecture has a well-known disproof due to Odlyzko and té Riele. We prove the unboundedness of $|M(x)|/\sqrt{x}$ using new methods by showing that

$$\limsup_{x \rightarrow \infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log\log x)^{\frac{1}{2}}} > 0.$$

The new methods we draw upon connect formulas and recent Dirichlet generating function (or DGF) series expansions related to the canonically additive functions $\Omega(n)$ and $\omega(n)$. The connection between $M(x)$ and the distribution of these core additive functions we prove at the start of the article in the form of

$$M(x) = \sum_{k=1}^{x} (\omega+1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right],$$

is an indispensible component to the proof. It also leads to regular properties of component sequences in the new formula for $M(x)$ that include generalizations of Erdös-Kac like theorems satisfied by the distributions of these special auxiliary sequences.

Keywords and Phrases: *Möbius function; Mertens function; Dirichlet inverse function; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function; Erdös-Kac theorem; strongly additive functions.*

Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; 11N64; and 11-04.*

- Only include what you need.
- Use "simple" language. Use it consistently.
- Every proof needs to be a simple as possible.
- If something is important, and don't have a pinpoint reference, prove it.
- I have difficulty verifying the DGF proofs. I assume those are right.
- Many analysis proofs have concerns. I have not checked all of them.

- I point out conceptual problems to Thm 6.4, as well as some calculations at the end of the proof.

Expectation calcs do <u>not</u> imply distributional statements

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Some of your distributional claims are very strange.

Glossary of special notation and conventions

Symbol	Definition
$\approx$	We write that $f(x) \approx g(x)$ if $\lvert f(x) - g(x) \rvert = O(1)$ as $x \to \infty$.
$\mathbb{E}[f(x)], \overset{\mathbb{E}}{\sim}$	We use the expectation notation of $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \overset{\mathbb{E}}{\sim} h(x)$, to denote that f has an *average order* growth rate of $h(x)$. This means that $\frac{1}{x}\sum_{n \leq x} f(n) \sim h(x)$, or equivalently that $$\lim_{x\to\infty} \frac{\frac{1}{x}\sum\limits_{n \leq x} f(n)}{h(x)} = 1.$$
B	The absolute constant $B \approx 0.2614972$ from the statement of Mertens theorem.
$\chi_{\mathbb{P}}(n)$	The characteristic (indicator) function of the primes equals one if and only if $n \in \mathbb{Z}^+$ is prime, and is zero-valued otherwise.
$C_k(n)$	The sequence is defined recursively for $n \geq 1$ as follows: $$C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d \mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$ It represents the multiple, k-fold convolution of the function $\omega(n)$ with itself.
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n\geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n\geq 0} f_n q^n$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (see definition below).
$f * g$	The Dirichlet convolution of f and g, $(f * g)(n) := \sum\limits_{d \mid n} f(d) g(n/d)$, where the sum is taken over the divisors d of any $n \geq 1$.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d \mid n \\ d > 1}} f(d) f^{-1}(n/d)$ for $n \geq 2$ with $f^{-1}(1) = 1/f(1)$. The Dirichlet inverse of f exists if and only if $f(1) \neq 0$. This inverse function, denoted by f^{-1} when it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$.
γ	The Euler gamma constant defined by $\gamma := \lim\limits_{n\to\infty} \left(\sum\limits_{k=1}^{n} \frac{1}{k} - \log n\right) \approx 0.5772157$.
$\gg, \ll, \asymp$	For functions A, B in x, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A \ll B$ and $B \gg A$, we write $A \asymp B$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$.

Symbol	Definition
$[n=k]_{\delta}$,$[\texttt{cond}]_{\delta}$	The symbol $[n=k]_{\delta}$ is a synonym for $\delta_{n,k}$ which is one if and only if $n=k$, and is zero otherwise. For boolean-valued conditions, `cond`, the symbol $[\texttt{cond}]_{\delta}$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention*.
$\lambda_*(n)$	For positive integers $n \geq 2$, we define the next variant of the Liouville lambda function, $\lambda(n)$, as follows: $\lambda_*(n) := (-1)^{\omega(n)}$. We have the initial condition that $\lambda_*(1) = 1$.
$\lambda(n), L(x)$	The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. That is, $\lambda(n) \in \{\pm 1\}$ for all integers $n \geq 1$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$. Its summatory function is defined by $L(x) := \sum_{n \leq x} \lambda(n)$.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree.
$\mu_x(C), \sigma_x(C)$	We define these analogs to the approximate mean and variance of the function $C_{\Omega(n)}(n)$ in the context of our new Erdös-Kac like theorems as $\mu_x(C) := \log\log x + \hat{a} - \frac{1}{2}\log\log\log x$ and $\sigma_x(C) := \sqrt{\mu_x(C)}$ where $\hat{a} := \log\left(\frac{1}{\sqrt{2\pi}}\right) \approx -0.918939$ is an absolute constant.
$M(x)$	The Mertens function is the summatory function of $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum\limits_{n \leq x} \mu(n)$.
$\Phi(z)$	For $x \in \mathbb{R}$, we define the function giving the normal distribution CDF by $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{z} e^{-t^2/2} dt$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^{\alpha} \vert\vert n$ (or when p^{α} exactly divides n) for p prime, $\alpha \geq 1$ and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p \mid n} 1$ and the completely additive function $\Omega(n) := \sum_{p^{\alpha} \vert\vert n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leq n \leq x$ for $x \geq 2$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$ for $x \geq 2$.
$P(s)$	For complex s with $\mathrm{Re}(s) > 1$, we define the prime zeta function to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s} = \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s}$.
$Q(x)$	For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. More precisely, this function is summed and identified with its limiting asymptotic formula as $x \to \infty$ in the following form: $Q(x) := \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O(\sqrt{x})$.
$\sim$	We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x \to \infty} \frac{A(x)}{B(x)} = 1$.
$\zeta(s)$	The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation on the rest of the complex plane with the exception of a simple pole at $s = 1$ of residue one.

1 Introduction

1.1 Definitions

We define the *Möbius function* to be the signed indicator function of the squarefree integers in the form of [20, A008683]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \text{ and } n \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined on the positive integers as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as follows [20, A002321]:

$$\{M(x)\}_{x\geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \ldots\}.$$

The Mertens function satisfies that $\sum_{n \leq x} M\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = 1$, and is related to the summatory function $L(x) := \sum_{n \leq x} \lambda(n)$ via the relation [10]

$$L(x) = \sum_{d \leq \sqrt{x}} M\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), x \geq 1.$$

Clearly, a positive integer $n \geq 1$ is *squarefree*, or contains no divisors (other than one) which are squares, if and only if $\mu^2(n) = 1$. A related summatory function which counts the number of *squarefree* integers $n \leq x$ satisfies [5, §18.6] [20, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers characterized by the sign of the Möbius function are in fact equal as $x \to \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{x} \overset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{x} \xrightarrow{x \to \infty} \frac{3}{\pi^2}.$$

1.2 Properties

A conventional approach to evaluating the limiting asymptotic behavior of $M(x)$ for large $x \to \infty$ considers an inverse Mellin transformation of the reciprocal of the Riemann zeta function. In particular, since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \cdot \int_1^\infty \frac{M(x)}{x^{s+1}} dx, \operatorname{Re}(s) > 1,$$

we obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

The previous two representations lead us to the exact expression of $M(x)$ for any real $x > 0$ given by the next theorem.

Theorem 1.1 (Analytic Formula for $M(x)$, Titchmarsh)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k\geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any real $x > 0$*

$$M(x) = \lim_{k \to \infty} \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |\operatorname{Im}(\rho)| < T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp\left(-C \cdot \log^{\frac{3}{5}}(x)(\log\log x)^{-\frac{3}{5}}\right).$$

Under the assumption of the RH, Soundararajan more recently proved new updated estimates bounding $M(x)$ from above for large x in the following forms [21]:

$$M(x) \ll \sqrt{x} \cdot \exp\left((\log x)^{\frac{1}{2}}(\log\log x)^{14}\right),$$
$$M(x) = O\left(\sqrt{x} \cdot \exp\left((\log x)^{\frac{1}{2}}(\log\log x)^{\frac{5}{2}+\epsilon}\right)\right), \ \forall \epsilon > 0.$$

1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$ for any $0 < \epsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C \cdot \sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens for $C = 1$ and all $x < 10000$. Since its beginnings in 1897, the Mertens conjecture has been disproven by computation of non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele [14]. Since the truth of the conjecture would have implied the RH, more recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound along infinite subsequences, e.g., considering the asymptotics of the function in the limit supremum and limit infimum senses.

We cite that it is only known by computation that [17, *cf.* §4.1] [20, *cf.* A051400; A051401]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leq -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [14, 8, 9, 6]. Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies [13]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log\log\log x)^{\frac{5}{4}}} = O(1).$$

2 A concrete new approach to bounding $M(x)$ from below

2.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 2.1 (Summatory functions of Dirichlet convolutions)**.** *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of f and h, respectively, and that $F^{-1}(x) := \sum_{n \leq x} f^{-1}(n)$ denotes the summatory function of the Dirichlet inverse of f for any $x \geq 1$. We have the following exact expressions for the summatory function of $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned}\pi_{f*h}(x) &:= \sum_{n \leq x} \sum_{d|n} f(d) h(n/d) \\ &= \sum_{d \leq x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].\end{aligned}$$

Moreover, for all $x \geq 1$

$$\begin{aligned}H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\ &= \sum_{k=1}^{x} f^{-1}(k) \cdot \pi_{f*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right).\end{aligned}$$

Corollary 2.2 (Convolutions arising from Möbius inversion)**.** *Suppose that h is an arithmetic function such that $h(1) \neq 0$. Define the summatory function of the convolution of h with μ by $\widetilde{H}(x) := \sum_{n \leq x} (h * \mu)(n)$. Then the Mertens function is expressed by the sum*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} h^{-1}(j) \right) \widetilde{H}(k), \forall x \geq 1.$$

Corollary 2.3 (A motivating special case)**.** *We have that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

2.2 An exact expression for $M(x)$ in terms of strongly additive functions

Fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute exactly that (see Table T.1 starting on page 39)

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

There is not a simple meaningful direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences. The distribution of distinct sets of prime exponents is still regular since $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of this inverse function over $n \geq 2$:

Heuristic 2.4 (Symmetry in $g^{-1}(n)$ in the prime factorizations of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for $= \omega(n_i) \geq 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly one, two, three, and so on prime factors.

Conjecture 2.5 (Characteristic properties of the inverse sequence)**.** *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) *For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*

(B) *For all squarefree integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!;$$

(C) *If $n \geq 2$ and $\Omega(n) = k$, then*

$$2 \leq |g^{-1}(n)| \leq \sum_{j=0}^{k} \binom{k}{j} \cdot j!.$$

We illustrate the conjecture clearly using the computation of initial values of this inverse sequence in Table T.1. The signedness property in (A) is proved exactly in Proposition 3.1. A proof of (B) in fact follows from Lemma 5.1 stated on page 21. The realization that the beautiful and remarkably simple combinatorial form of property (B) in Conjecture 2.5 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through sums of auxiliary subsequences of arithmetic functions denoted by $C_k(n)$ (see Section 5). An exact expression for $g^{-1}(n)$ through a key semi-diagonal of these subsequences is given by

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d), n \geq 1,$$

where the sequence $\lambda(n) C_{\Omega(n)}(n)$ has DGF $(P(s)+1)^{-1}$ for $\operatorname{Re}(s) > 1$.

In Corollary 6.5, we prove that

$$\mathbb{E}|g^{-1}(n)| \asymp (\log n)^2 \sqrt{\log\log n}, \text{ as } n \to \infty.$$

The regularity and quasi-periodicity we have alluded to in the remarks above are actually quantifiable in so much as $|g^{-1}(n)|$ for $n \leq x$ tends to its average order with a non-central normal tendency depending on x as $x \to \infty$. In Section 6, we prove the next variant of an Erdös-Kac theorem like analog for a component sequence $C_{\Omega(n)}(n)$. We have the following statement for $\mu_x(C) := \log\log x + \hat{a} - \frac{1}{2}\log\log\log x$, $\sigma_x(C) := \sqrt{\mu_x(C)}$, $\hat{a}$ an absolute constant, and any $y \in \mathbb{R}$ (see Corollary 6.7):

$$\frac{1}{x} \cdot \#\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq y\} = \Phi\left(\frac{\frac{\pi^2}{6} y - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

We also prove that (see Proposition 7.4)

$$M(x) = G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right]. \tag{2}$$

This formula implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by bounding appropriate estimates of the summatory function $G^{-1}(x)$. This take on the regularity of $|g^{-1}(n)|$ is imperative to our argument formally bounding the growth $G^{-1}(x)$ from below as $|G^{-1}(x)| \gg (\log x)\sqrt{\log\log x}$ as $x \to \infty$ (see Theorem 7.3).

2.3 Uniform asymptotics from certain bivariate counting DGFs

Theorem 2.6 (Montgomery and Vaughan)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $R < 2$ we have that uniformly for all $1 \leq k \leq R \cdot \log\log x$

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right],$$

where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, 0 \leq |z| < R.$$

The proof of the next result is combinatorially motivated in so much as it interprets lower bounds on a key infinite product factor of $\mathcal{G}(z)$ defined in Theorem 2.6 as corresponding to an ordinary generating function of certain homogeneous symmetric polynomials involving the primes (see (14) in the proof of Theorem 2.7). This interpretation allows us to recover the following uniform lower bounds on $\widehat{\pi}_k(x)$ as $x \to \infty$:

Theorem 2.7 (Schmidt, 2020)**.** *For all sufficiently large x we have uniformly for $1 \leq k \leq \log\log x$ that*

$$\widehat{\pi}_k(x) \gg \frac{x^{3/4}}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \times \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

? dont know this word.

Remark 2.8. We emphasize the relevant recency of the method demonstrated by Montgomery and Vaughan in constructing a proof of Theorem 2.6. To the best of our knowledge, this textbook reference is one of the first clear-cut applications documenting something of a hybrid DGF-and-OGF approach to enumerating sequences of arithmetic functions and their summatory functions. This interpretation of certain bivariate DGFs offers a window into the best of both generating function series worlds: It combines the additivity implicit to the coefficients indexed by a formal power series variable formed by multiplication of these structures, while coordinating the distinct DGF-best property of the multiplicativity of distinct prime powers invoked by taking powers of a reciprocal Euler product. For example, for any additive arithmetic function $a(n)$, characterized by the property that $a(n) = \sum_{p^\alpha || n} a(p^\alpha)$ for all $n \geq 2$, we have that [7, *cf.* §1.7]

$$\prod_p \left(1 - \sum_{m\geq 1} \frac{z^{a(p^m)}}{p^{ms}}\right)^{-1} = \sum_{n\geq 1} \frac{z^{a(n)}}{n^s}, \mathrm{Re}(s) > 1.$$

do you need this remark?

Another set of proofs constructed based on this type of hybrid power series enabling DGF is key in Section 6 when we prove an Erdös-Kac theorem like analog that holds for the component sequence $C_{\Omega(n)}(n)$, crucially related to $g^{-1}(n)$ by the results in Section 5.

2.4 Cracking the classical unboundedness barrier

In Section 7, we are able to state what forms a bridge between the results we carefully prove up to that point the article. What we obtain at the conclusion of this last section is the next summary theorem that unconditionally resolves the classical question of the unboundedness of the scaled function Mertens function $q(x) := |M(x)|/\sqrt{x}$ in the limit supremum sense.

Theorem 2.9 (Unboundedness of the the Mertens function, $q(x)$)**.** *We have that*

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

The proof of Theorem 2.9 is the main result we build up to in the article. It motivates all of our new constructions behind the additive function based sequences we employ to expand $M(x)$ via (1) and (2). This link relating our new formula for $M(x)$ to canonical additive functions and their distributions lends a recent distinguishing element to the success and characterization of the methods in our proof.

2.5 An overview of the core components to the proof

We offer the following initial step-by-step summary overview of the core components to our proof with the intention of making this new argument easier to parse:

(1) We prove a matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). See Theorem 2.1 in Section 3.

(2) This crucial step provides us with an exact formula for $M(x)$ in terms of the prime counting function $\pi(x)$, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n)+1$. This formula is stated in (1) (see Proposition 7.4).

(3) We tighten bounds from a less classical result proved in [12, §7] providing uniform asymptotic formulas for lower bounds on the summatory functions, $\widehat{\pi}_k(x)$, large $x \gg e$ and $1 \leq k \leq \log\log x$ (see Theorem 2.7). This allows us to eventually approximate the magnitude of the summatory function

$$L(x) := \sum_{n \leq x} \lambda(n) \asymp \sum_{k=1}^{\log\log x} (-1)^k \widehat{\pi}_k(x), \text{ as } x \to \infty,$$

well from below (see the proof of Theorem 7.3; and Table T.2 starting on page 46).

(4) In Section 5. we relate $g^{-1}(n)$ to a subsequence of recursively-defined auxiliary functions, $C_k(n)$, that respectively express multiple k-convolutions of $\omega(n)$ with itself for $1 \leq k \leq \Omega(n)$ (see Lemma 5.1 and Lemma 5.3).

(5) In Section 6, we prove new expectation formulas for $|g^{-1}(n)|$ and the related component sequence $C_{\Omega(n)}(n)$ by first proving an Erdös-Kac like theorem satisfied by $C_{\Omega(n)}(n)$. This allows us to prove asymptotic lower bounds on $|G^{-1}(x)| \gg (\log x)\sqrt{\log\log x}$ when x is large and such that $G^{-1}(x) \neq 0$ in Section 7.

(6) When we return to step **(2)** with our new lower bounds at hand, we are led to a new unconditional proof of the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along an exponentially very large increasing infinite subsequence of positive natural numbers (see Section 7.2).

3 Preliminary proofs of new results

3.1 Establishing the summatory function properties and inversion identities

We will offer a proof of Theorem 2.1 suggested by an intuitive construction through matrix methods. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95].

Proof of Theorem 2.1. Let h, g be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h. We have that the following formulas hold for all $x \geq 1$:

$$\begin{aligned}\pi_{g*h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n) h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] H(i). \end{aligned} \tag{3}$$

The first formula above is well known. The second formula is justified directly using summation by parts as [15, §2.10(ii)]

$$\begin{aligned}\pi_{g*h}(x) &= \sum_{d=1}^{x} h(d) G\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i \leq x} \left(\sum_{j \leq i} h(j) \right) \times \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right]. \end{aligned}$$

We next form the invertible matrix of coefficients associated with this linear system defining $H(j)$ for all $1 \leq j \leq x$ in (3) by setting

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right), 1 \leq j \leq x.$$

Since $g_{x,x} = G(1) = g(1)$ and $g_{x,j} = 0$ for all $j > x$, the matrix we must invert in this problem is lower triangular with a non-zero constant on its diagonals, and is hence invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressed by applying an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

Here, U is a square matrix with sufficiently large finite dimensions whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$\left[(I - U^T)^{-1}\right]_{i,j} = [j \leq i]_{\delta}.$$

Observe that

$$\left\lfloor \frac{x}{j} \right\rfloor - \left\lfloor \frac{x-1}{j} \right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

The previous property implies that

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

We use the last property in (4) to shift the matrix $\hat{G}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of g in the following form:

$$\left[(I-U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Our target matrix in the inversion problem is defined by

$$(g_{x,j}) = (I-U^T)\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators as follows:

$$\begin{aligned}(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T)\\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k)\right)(I-U^T)\\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1}\right\rfloor} g^{-1}(k)\right).\end{aligned}$$

Hence, the summatory function $H(x)$ is given exactly for any $x \geq 1$ by a vector product with the inverse matrix from the previous equation in the form of

$$H(x) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right)\cdot \pi_{g*h}(k).$$

We can prove an inversion formula providing the coefficients of the summatory function $G^{-1}(i)$ for $1 \leq i \leq x$ given by the last equation by adapting our argument to prove (3) above. This leads to the following identity:

$$H(x) = \sum_{k=1}^{x} g^{-1}(x)\cdot \pi_{g*h}\left(\left\lfloor \frac{x}{k}\right\rfloor\right). \qquad \square$$

3.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, let $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n. Then we can easily prove using DGFs that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega+1) * \mu. \tag{5}$$

When combined with Corollary 2.2 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 2.3.

Proposition 3.1 (The signedness property of $g^{-1}(n)$)**.** *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_\delta} \in \{0,\pm 1\}$ *denote the sign of the arithmetic function h at integers $n \geq 1$. For the Dirichlet invertible function $g(n) := \omega(n)+1$, we have that* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ *for all $n \geq 1$.*

Proof. The function $D_f(s) := \sum_{n\geq 1} f(n)n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ for σ_f the abcissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = 1/\zeta(s)$ and $D_\omega(s) = P(s)\zeta(s)$ for $\operatorname{Re}(s) > 1$. Then by (5) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any arithmetic function f such that $f(1) \neq 0$ (e.g., this relation between the DGFs of these two functions holds whenever f^{-1} exists), we have for all $\operatorname{Re}(s) > 1$ that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{6}$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. This observation implies that $\operatorname{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make our claims and intuition rigorous.

By the recurrence relation that defines the Dirichlet inverse function of any arithmetic function h such that $h(1) = 1$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d|n \\ d>1}} h(d)h^{-1}(n/d), & n \geq 2. \end{cases} \tag{7}$$

For $n \geq 2$, the summands in (7) can be simply indexed over the primes $p|n$ given our definition of h from above. This observation yields that we can inductively unfold these sums into nested divisor sums provided the depth of the expanded divisor sums does not exceed the capacity to index summations over the primes dividing n. Namely, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p|n} h^{-1}\left(\frac{n}{p}\right), && \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1p_2}\right), && \text{if } \Omega(n) \geq 2 \\ &= -\sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1p_2}} h^{-1}\left(\frac{n}{p_1p_2p_3}\right), && \text{if } \Omega(n) \geq 3. \end{aligned}$$

Then by induction with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1p_2\cdots p_{\Omega(n)-1}}} 1, n \geq 2. \tag{8}$$

In fact, by a combinatorial argument we recover exactly that

$$h^{-1}(n) = \lambda(n)(\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}. \tag{9}$$

The last two expansions imply that the following property holds for all $n \geq 1$:

$$\operatorname{sgn}(h^{-1}(n)) = \lambda(n).$$

Since λ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$ for all divisors $d|n$ when $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever n is squarefree, so that we obtain the following result:

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1. \qquad \square$$

3.3 Statements of known limiting asymptotics

Theorem 3.2 (Mertens theorem)**.** *For all $x \geq 2$ we have that*

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log\log x + B + o(1), \text{ as } x \to \infty,$$

where $B \approx 0.2614972128476427837554$ *is an absolute constant.*

Corollary 3.3 (Product form of Mertens theorem)**.** *We have that for all sufficiently large* $x \gg 2$

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} \left(1 + o(1)\right), \text{ as } x \to \infty.$$

Hence, for any real z we obtain that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^z \sim \frac{e^{-\gamma z}}{(\log x)^z}, \text{ as } x \to \infty.$$

Proofs of Theorem 3.2 and Corollary 3.3 are given in [5, §22.7; §22.8].

Facts 3.4 (Exponential integrals and the incomplete gamma function)**.** Two variants of the *exponential integral function* are defined by the integral next representations [15, §8.19] [3, §3.3].

$$\begin{aligned}
\mathrm{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, x \in \mathbb{R} \\
E_1(z) &:= \int_{1}^{\infty} \frac{e^{-tz}}{t} dt, \mathrm{Re}(z) \geq 0
\end{aligned}$$

These functions are related by $\mathrm{Ei}(-kz) = -E_1(kz)$ for real $k, z > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $\mathrm{Ei}(\pm x)$ for all real $x > 0$:

$$\begin{aligned}
\gamma + \log x - x \leq \mathrm{Ei}(-x) &\leq \gamma + \log x - x + \frac{x^2}{4}, \qquad (10a) \\
1 + \gamma + \log x - \frac{3}{4}x \leq \mathrm{Ei}(x) \quad &\leq 1 + \gamma + \log x - \frac{3}{4}x + \frac{11}{36}x^2.
\end{aligned}$$

The (upper) *incomplete gamma function* is defined by [15, §8.4]

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt, \mathrm{Re}(s) > 0.$$

The following properties of $\Gamma(s, x)$ hold:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, x > 0, \qquad (10b)$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, s > 0, \text{ as } x \to \infty. \qquad (10c)$$

4 Components to the asymptotic analysis of lower bounds for sums of arithmetic functions weighted by $\lambda(n)$

4.1 A discussion of the results proved by Montgomery and Vaughan

Remark 4.1 (Intuition and constructions behind the proof of Theorem 2.6)**.** For $|z| < 2$ and $\mathrm{Re}(s) > 1$, let

$$F(s,z) := \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \tag{11}$$

and define the DGF coefficients, $a_z(n)$ for $n \geq 1$, by the product

$$\zeta(s)^z \cdot F(s,z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s}, \mathrm{Re}(s) > 1.$$

Suppose that $A_z(x) := \sum_{n \leq x} a_z(n)$ for $x \geq 1$. We obtain the next generating function like identity in z enumerating the $\widehat{\pi}_k(x)$ for $1 \leq k < 2 \log\log x$.

$$A_z(x) = \sum_{n \leq x} z^{\Omega(n)} = \sum_{0 \leq k \leq \log_2(x)} \widehat{\pi}_k(x) z^k \tag{12}$$

Thus for $r < 2$, by Cauchy's integral formula we have

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|v|=r} \frac{A_v(x)}{v^{k+1}} dv.$$

Selecting $r := \frac{k-1}{\log\log x}$ for $1 \leq k < 2\log\log x$ leads to the uniform asymptotic formulas for $\widehat{\pi}_k(x)$ given in Theorem 2.6. Montgomery and Vaughan then consider individual analysis of the main and error terms for $A_z(x)$ to prove that

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

We will require estimates of $A_{-z}(x)$ from below to form summatory functions that weight the terms of $\lambda(n)$ in our new formulas derived in the next sections.

4.2 New uniform asymptotics based on refinements of Theorem 2.6

Proposition 4.2. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \geq 2.$$

When $s := 1$, we have the asymptotic formula from Mertens theorem (see Theorem 3.2). For all integers $s \geq 2$ there are absolutely defined quasi-polynomial bounding functions $\gamma_0(s,x)$ and $\gamma_1(s,x)$ in s and x such that

$$\gamma_0(s,x) + o(1) \leq P_s(x) \leq \gamma_1(s,x) + o(1), \text{ as } x \to \infty.$$

It suffices to define the bounds in the previous equation by the functions

$$\gamma_0(s,x) = s \log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}s(s-1)^2 \log^2(2),$$
$$\gamma_1(s,x) = s \log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) + \underline{\frac{1}{4}s(s-1)^2 \log^2(x)}.$$

$\longrightarrow +\infty$??

$$P_2(x) \longrightarrow \sum_p \frac{1}{p^2}$$

$$\gamma_0(2,x) = 2\log\frac{\log x}{\log 2} - 2\log\frac{x}{2} - \frac{2}{4}\log^2(2)$$

$$\longrightarrow -\infty \qquad x \to \infty$$

So that does not inspire confidence.

Likewise $\gamma_1(2,x) \longrightarrow +\infty$.

The prop. 4.2 is not wrong, but it is a strange way to approximate.

Why not just take $\gamma_0(s,x) \equiv 0$?

Seems to me that I have complained about this before.

And you have not changed it.

Not good.

Proof. Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$, and where our target smooth function is $f(t) = t^{-s}$ with $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

I dont follow.

$$\begin{aligned} P_s(x) &\sim \frac{1}{x^s \cdot \log x} + s \times \int_2^x \frac{dt}{t^s \log t} \\ &= \mathrm{Ei}(-(s-1)\log x) - \mathrm{Ei}(-(s-1)\log 2) + o(1), \text{ as } x \to \infty. \end{aligned}$$

Now using the inequalities in Facts 3.4, we obtain that the difference of the exponential integral functions in the previous equation is respectively bounded below and above by

$$\begin{aligned} \frac{P_s(x)}{s} &\geq \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}(s-1)^2\log^2(2) + o(1) \\ \frac{P_s(x)}{s} &\leq \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) + \frac{1}{4}(s-1)^2\log^2(x) + o(1). \end{aligned}$$ □

The utility to the quasi-logarithmic bounds tending to infinity as $x \to \infty$ stated in Proposition 4.2 will become apparent when we take the exponential of sums of the functions $P_j(x)$ for $j \geq 2$ in order to form a lower bound on $\mathcal{G}(-z)$ for $z := \frac{k-1}{\log\log x}$ in the next subsection.

4.2.1 The proof of Theorem 2.7

We will first prove the stated form of the lower bound on $\mathcal{G}(-z)$ for $z := \frac{k-1}{\log\log x}$. Then we will discuss the technical adaptations to Montgomery and Vaughan's proof of Theorem 2.6 in Section 4.2.2 below to rigorously prove that the new asymptotic lower bounds on $\widehat{\pi}_k(x)$ that hold uniformly for all $1 \leq k \leq \log\log x$.

If it is essential It needs a full proof.

Lemma 4.3. *For sufficiently large $x > e$ and $1 \leq k \leq \log\log x$, we have that*

$$\left|\mathcal{G}\left(\frac{1-k}{\log\log x}\right)\right| \gg x^{-\frac{1}{4}}.$$

Proof. For $-2 < z < 2$ and integers $x \geq 2$, the right-hand-side of the following product is finite:

$$\widehat{P}(z,x) := \prod_{p \leq x}\left(1 - \frac{z}{p}\right)^{-1}.$$

For fixed $x \geq 2$ let

$$\mathbb{P}_x := \left\{n \in \mathbb{Z}^+ : \text{all prime divisors } p|n \text{ satisfy } p \leq x\right\}.$$

Then we can see that for $x \geq 2$

$$\prod_{p \leq x}\left(1 - \frac{z}{p^s}\right)^{-1} = \sum_{n \in \mathbb{P}_x} \frac{z^{\Omega(n)}}{n^s}. \tag{13}$$

By extending the argument in the proof given in [12, §7.4], we have that

$$A_{-z}(x) := \sum_{n \leq x} \lambda(n) z^{\Omega(n)} = \sum_{0 \leq k \leq \log_2(x)} \widehat{\pi}_k(x)(-z)^k,$$

Let $a_n(z,x)$ be defined as the coefficients of the DGF

$$\widehat{P}(z,x) =: \sum_{n \geq 1} \frac{a_n(z,x)}{n^s}.$$

We have argued that

$$\sum_{n \leq x} a_n(-z,x) = \sum_{k=0}^{\log_2(x)} \widehat{\pi}_k(x)(-z)^k + \sum_{k > \log_2(x)} e_k(x)(-z)^k.$$

Lemma 4.3 seems odd.

g is a nice function,

and $-1 < \frac{1-k}{\log\log x} < 0$, so

the bound should be

$$\left| g\left(\frac{1-k}{\log\log x}\right)\right| \gg 1.$$

$$g\left(\frac{1-k}{\log\log x}\right) \gg x^{-1/4}$$

$$g(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1-\frac{z}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{z}$$

$$-1 < \underset{\overset{\|}{z}}{\frac{1-k}{\log\log x}} \leq 0$$

• $\Gamma(z+1)$ is irrelevant

Take log of the remaing product

$$-\sum_p \log 1-\frac{z}{p} + z \log 1-\frac{1}{p}$$

$$\log 1-x = -x + \frac{x^2}{2} + O(x^3)$$

$$\sim -\sum_p \cancel{-\frac{z}{p}} + \frac{z^2}{p^2} - z\left(\cancel{-\frac{1}{p}} + \frac{1}{p^2}\right)$$

$$= -\sum \frac{z^2}{p^2} - \frac{z}{p^2}$$

$$= +\sum \frac{z-z^2}{p^2}$$

z is negative

This assertion is correct since the products of all non-negative integral powers of the primes $p \leq x$ (counting multiplicity) generate the integers $\{1 \leq n \leq x\}$ as a subset. Thus we capture all of the relevant terms needed to express $(-1)^k \cdot \widehat{\pi}_k(x)$ via the Cauchy integral formula representation over $A_{-z}(x)$ by replacing the corresponding infinite product terms with $\widehat{P}(-z, x)$ in the definition of $\mathcal{G}(-z)$.

Now we argue that

$$\mathcal{G}(-z) \gg \prod_{p \leq x} \left(1 + \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{-z}, 0 \leq z < 1, x \geq 2.$$

For $0 \leq z < 1$ and $x \geq 2$, we see that

$$\begin{aligned} \mathcal{G}(-z) &= \exp\left(-\sum_p \left[\log\left(1 + \frac{z}{p}\right) + z \cdot \log\left(1 - \frac{1}{p}\right)\right]\right) \\ &\gg \exp\left(-z \times \sum_{p > x} \left[\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right] - \sum_{p \leq x} \left[\log\left(1 + \frac{z}{p}\right) + z \cdot \log\left(1 - \frac{1}{p}\right)\right]\right) \\ &\gg_z \widehat{P}(-z, x), \text{ as } x \to \infty, \end{aligned}$$

where the *Mertens constant* B is defined exactly by the prime sum [5, §22.8]

$$B := \gamma + \sum_p \left[\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right].$$

Next, we have for all integers $0 \leq k \leq m < \infty$, and any sequence $\{f(n)\}_{n \geq 1}$ with sufficiently bounded partial power sums, that [11, §2]

$$[z^k] \prod_{1 \leq i \leq m} (1 - f(i)z)^{-1} = [z^k] \exp\left(\sum_{j \geq 1} \left(\sum_{i=1}^m f(i)^j\right) \frac{z^j}{j}\right), |z| < 1. \tag{14}$$

In our case, $f(i)$ denotes the reciprocal of the i^{th} prime in the generating function expansion of (14). It follows from Proposition 4.2 that for any real $0 \leq z < 1$ we obtain

$$\begin{aligned} \log\left[\prod_{p \leq x} \left(1 + \frac{z}{p}\right)^{-1}\right] &\geq -(\log\log x + B)z + \sum_{j \geq 0} \left[\log\left(\frac{\log x}{\log 2}\right) - (2j+1)\log\left(\frac{x}{2}\right) - (2j+1)^2 \frac{\log^2 2}{4}\right] z^{2j+2} \\ &\quad - \sum_{j \geq 0} \left[\log\left(\frac{\log x}{\log 2}\right) - (2j+2)\log\left(\frac{x}{2}\right) + (2j+2)^2 \frac{\log^2 x}{4}\right] z^{2j+3} \\ &= -(\log\log x + B)z + \sum_{j \geq 0} \left[\log\left(\frac{\log x}{\log 2}\right) - (j+1)\log\left(\frac{x}{2}\right)\right] (-z)^{j+2} \\ &\quad - \frac{1}{4} \times \sum_{j \geq 0} \left[(\log 2)^2 (2j+1)^2 z^{2j+2} + (\log x)^2 (2j+2)^2 z^{2j+3}\right] \\ &= -(\log\log x + B)z + \log\left(\frac{\log x}{\log 2}\right)\left[z - 1 + \frac{1}{z+1}\right] + \log\left(\frac{x}{2}\right)\left[\frac{2}{1+z} - 1 - \frac{1}{(1+z)^2}\right] \\ &\quad - (\log x)^2 \times \frac{(z^3 + z^5)}{(1-z^2)^3} - (\log 2)^2 \times \frac{(z^2 + 6z^4 + z^6)}{4(1-z^2)^3} \\ &=: \widehat{\mathcal{B}}(x; z). \end{aligned} \tag{15}$$

We adjust the uniform bound parameter R so that

$$z \equiv z(k, x) = \frac{k-1}{\log\log x} \in [0, 1),$$

e.g., whenever $1 \leq k \leq \log\log x$ in the notation of Theorem 2.6. We have that

$$\min_{0 \leq z \leq 1} \left[z - 1 + \frac{1}{z+1}\right] = 0$$
$$\min_{0 \leq z \leq 1} \left[\frac{2}{1+z} - 1 - \frac{1}{(1+z)^2}\right] = -\frac{1}{4}.$$

Moreover, when we expand out the coefficients of $(\log 2)^2$ and $(\log x)^2$ in (15) by partial fractions of z, we see that all of the terms with an infinitely tending singularity as $z \to 1^-$ are positive. This means to obtain the lower bound, we can drop these contributions. What we are left to minimize is the following terms:

$$(\log 2)^2 \times \min_{0 \leq z \leq 1} \left[\frac{1}{4} - \frac{1}{4(1+z)^3} + \frac{5}{8(1+z)^2} - \frac{1}{2(1+z)}\right] = \frac{13}{108}(\log 2)^2$$
$$(\log x)^2 \times \min_{0 \leq z \leq 1} \left[\frac{1}{4(1+z)^3} - \frac{5}{8(1+z)^2} + \frac{1}{2(1+z)}\right] = \frac{7}{54}(\log x)^2.$$

So we have from (15) that

$$\widehat{\mathcal{B}}(x;z) \gg \left(\frac{2}{x}\right)^{\frac{1}{4}} \times \exp\left(\frac{13}{108}(\log 2)^2\right) \times \exp\left(\frac{7}{54}(\log x)^2\right) \gg x^{-\frac{1}{4}}.$$

In summary, we have arrived at a proof that as $x \to \infty$

$$\frac{e^{\gamma z}}{(\log x)^{-z}} \times \exp\left(\widehat{\mathcal{B}}(u,x;z)\right) \gg x^{-\frac{1}{4}}. \tag{16}$$

Finally, to finish our proof of the new lower bound on $\mathcal{G}(-z)$, we need only bound the reciprocal factor of $\Gamma(1-z) = -z \cdot \Gamma(-z)$. Since $z \equiv z(k,x) = \frac{k-1}{\log\log x}$ for $k \in [1, \log\log x]$, or again with $z \in [0,1)$, we obtain for minimal k and all large enough $x \gg 1$ that $\Gamma(1-z) = \Gamma(1) = 1$, and for k towards the upper range of its interval that

$$\Gamma(1-z) \approx \Gamma\left(\frac{1}{\log\log x}\right) = \frac{1}{\log\log x} \times \Gamma\left(1 + \frac{1}{\log\log x}\right) \approx \frac{1}{\log\log x}.$$

Therefore, our assertion that the claimed lower bound holds is correct. □

If important then prove it.

4.2.2 Technical adjustments in the proof of Theorem 2.7

We now discuss the differences between our construction and that in the technical proof of Theorem 2.6 in the reference when we bound $\mathcal{G}(-z)$ from below as in the previous lemma. The reference proves that for $0 \leq z < 2$ [12, Thm. 7.18]

$$A_{-z}(x) = -\frac{zF(1,-z)}{\Gamma(1-z)} \cdot x(\log x)^{-(z+1)} + O\left(x(\log x)^{-\operatorname{Re}(z)-2}\right). \tag{17}$$

Recall that for $r < 2$ we have by Cauchy's integral formula that

$$(-1)^k \widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|v|=r} \frac{A_{-v}(x)}{v^{k+1}} dv. \tag{18}$$

We first claim that uniformly for large x and $1 \leq k \leq \log\log x$ we have

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{1-k}{\log\log x}\right) \times \frac{x(\log\log x)^{k-1}}{(\log x)(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right]. \tag{19}$$

Then since we have proved in Lemma 4.3 that

$$\left|\mathcal{G}\left(\frac{1-k}{\log\log x}\right)\right| \gg \frac{1}{x^{\frac{1}{4}}},$$

the result in (19) implies our stated uniform asymptotic bound. Namely, we obtain that

$$\widehat{\pi}_k(x) \gg \frac{x^{\frac{3}{4}}}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

We must provide analogs to the proofs of the two separate bounds from the reference corresponding to the error and main terms of our estimate according to (17) and (18).

Step I: Error Term Bound. To prove that the error term bound holds, we estimate the following bounds for $r := \frac{k-1}{\log\log x}$ with $r < 1$ whenever $2 \leq k \leq \log\log x$:

$$\begin{aligned} \left|\frac{1}{2\pi\imath}\int_{|v|=r} \frac{x}{(\log x)^2} \frac{(\log x)^{-\operatorname{Re}(v)}}{v^{k+1}} dv\right| &\ll x(\log x)^{-(r+2)} r^{-(k+1)} \ll \frac{x}{(\log x)^2} \frac{(\log\log x)^{k+1}}{e^{k-1}(k-1)^{k+1}} \\ &\ll \frac{x}{(\log x)^2} \frac{(\log\log x)^{k+1}}{e^{2(k-1)}(k-1)!(k-1)^{\frac{3}{2}}} \ll \frac{x}{(\log x)^2} \frac{(\log\log x)^{k+1}}{(k-1)!} \\ &\ll \frac{x}{\log x} \frac{(\log\log x)^{k-4}}{(k-1)!}. \end{aligned} \tag{20}$$

By the Cauchy integral formula, we can verify that

$$\left|\frac{1}{2\pi\imath}\int_{|v|=r} \frac{x}{(\log x)^2} \frac{(\log x)^{-\operatorname{Re}(v)}}{v^2} dv\right| = \frac{x}{(\log x)^2} \cdot (\log\log x)^2 \ll \frac{x}{(\log x)(\log\log x)^2},$$

so that the formula for the error term in (20) also matches when $k := 1$.

We can calculate that for $0 \leq z < 1$

$$\begin{aligned} \prod_p \left(1 + \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{-z} &= \exp\left(-\sum_p \left[\log\left(1 + \frac{z}{p}\right) + z\log\left(1 - \frac{1}{p}\right)\right]\right) \\ &\sim \exp\left(-o(z) \times \sum_p \frac{1}{p^2}\right) \\ &\gg \exp\left(-o(z) \cdot P(2)\right) \gg_z 1. \end{aligned}$$

In other words, we have that $\mathcal{G}\left(\frac{1-k}{\log\log x}\right) \gg 1$ whenever $1 \leq k \leq \log\log x$. So the error term in (20) is majorized by taking $O\left(\frac{k}{(\log\log x)^3}\right)$ as our upper bound.

Step II: Main Term Bound. By (17) the main term estimate for (18) is given by $\frac{x}{\log x} \cdot I_x$, where

$$I_x := \frac{(-1)^{k-1}}{2\pi\imath} \int_{|v|=r} G(-z)(\log x)^{-v} v^{-k} dv.$$

In particular, we can write $I_x = I_{1,x} + I_{2,x}$ where we define

$$\begin{aligned} I_{1,x} &:= \frac{G(-r)}{2\pi\imath} \int_{|v|=r} (\log x)^{-v} v^{-k} dv \\ &= \frac{(-1)^{k-1} G(-r)(\log\log x)^{k-1}}{(k-1)!} \\ I_{2,x} &:= \frac{1}{2\pi\imath} \int_{|v|=r} (G(-v) - G(-r))(\log x)^{-v} v^{-k} dv \\ &= \frac{1}{2\pi\imath} \int_{|v|=r} (G(-v) - G(-r) + G'(-r)(v-r))(\log x)^{-v} v^{-k} dv. \end{aligned}$$

The second integral formula for $I_{2,x}$ results from integration by parts.

We have by taking a power series expansion of $G''(-w)$ about $-r$ and integrating the resulting series termwise with respect to w that when $|v| = r$

$$\left|G(-v) - G(-r) + G'(-r)(v-r)\right| = \left|\int_r^v (v-w)G''(-w)dw\right| \ll |v-r|^2.$$

Now we parameterize the curve in the contour for $I_{2,x}$ by writing $v = re^{2\pi\imath t}$ for $t \in [-1/2, 1/2]$. This leads us to the bounds

$$\begin{aligned}|I_{2,x}| &\ll r^{3-k} \times \int_{-\frac{1}{2}}^{\frac{1}{2}} |e^{2\pi\imath t} - 1|^2 \cdot (\log x)^{re^{2\pi\imath t}} \cdot e^{2\pi\imath t} dt \\ &\ll r^{3-k} \times \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin^2(\pi t) \cdot e^{(k-1)\cos(2\pi t)} dt.\end{aligned}$$

Whenever $|x| \leq 1$, we know that $|\sin x| \leq |x|$. Also, $\cos(2\pi t) \leq 1 - 8t^2$ whenever $|t| \leq \frac{1}{2}$. Thus the last bound for $|I_{2,x}|$ becomes

$$\begin{aligned}|I_{2,x}| &\ll r^{3-k}e^{k-1} \times \int_0^\infty t^2 \cdot e^{-8(k-1)t^2} dt \\ &\ll \frac{r^{3-k}e^{k-1}}{(k-1)^{3/2}} = \frac{(\log\log x)^{k-3}e^{k-1}}{(k-1)^{k-3/2}} \\ &\ll \frac{k \cdot (\log\log x)^{k-3}}{(k-1)!}.\end{aligned}$$

Thus the contribution from the term $|I_{2,x}|$ can then be absorbed into the error term bound in (19).

4.3 The distribution of exceptional values of $\Omega(n)$

The next theorems reproduced from [12, §7.4] characterize the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \log\log x$.

Theorem 4.4 (Upper bounds on exceptional values of $\Omega(n)$ for large n)**.** *Let*

$$\begin{aligned}A(x,r) &:= \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\}, \\ B(x,r) &:= \#\left\{n \leq x : \Omega(n) \geq r \cdot \log\log x\right\}.\end{aligned}$$

If $0 < r \leq 1$ and $x \geq 2$, then

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x,r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

Theorem 4.5 is a special case analog to the celebrated Erdös-Kac theorem typically stated for the normally distributed values of the scaled-shifted function $\omega(n)$ over $n \leq x$ as $x \to \infty$ [12, *cf.* Thm. 7.21].

Theorem 4.5 (Exact limiting bounds on exceptional values of $\Omega(n)$ for large n)**.** *We have that as $x \to \infty$*

$$\#\left\{3 \leq n \leq x : \Omega(n) - \log\log n \leq 0\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

The key interpretation we need to take away from the statements of Theorem 4.4 and Theorem 4.5 is the result proved in the next corollary. The role of the parameter R involved in stating the previous theorem is a critical bound as the scalar factor in the upper bound on $k \leq \log\log x$ in Theorem 2.6 up to which our uniform bounds given by Theorem 2.7 hold. In contrast, for $n \geq 2$ we can actually have contributions from values distributed throughout the range $1 \leq \Omega(n) \leq \log_2(n)$ infinitely often. It is then crucial that we can show that the main term in the asymptotic formulas we obtain for the summatory function over $\widehat{\pi}_k(x)$ is captured by summing only over the truncated range of $k \in [1, \log\log x]$ where the uniform bounds guaranteed by Theorem 2.6 and Theorem 2.7 hold.

Corollary 4.6. *Using the notation for $A(x,r)$ and $B(x,r)$ from Theorem 4.4, we have that for $x \geq 2$ and $\delta > 0$,*

$$\frac{B(x, 1+\delta)}{A(x,1)} = o_\delta(1), \text{ as } x \to \infty.$$

Proof. To show that the asymptotic bound is correct, we compute using Theorem 4.4 and Theorem 4.5 that

confusing

$$\frac{B(x,1+\delta)}{A(x,1)} \ll \frac{x \cdot (\log x)^{\delta-(1+\delta)\log(1+\delta)}}{O(1) + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right)} \sim o_\delta(1),$$

as $x \to \infty$. Notice that since $\mathbb{E}[\Omega(n)] = \log\log n + B$, with $0 < B < 1$ the absolute constant from Mertens theorem, when we denote the range of $k > \log\log x$ as holding in the form of $k > (1+\delta)\log\log x$ for $\delta > 0$ at large x, we can assume that $\delta \to 0^+$ as $x \to \infty$. In particular, this holds since $k > \log\log x$ implies that

δ is fixed in the statement.

$$\lfloor \log\log x \rfloor + 1 \geq (1+\delta)\log\log x \quad \implies \quad \delta \leq \frac{1 + \{\log\log x\}}{\log\log x} = o(1), \text{ as } x \to \infty.$$

The key consequence is that the ratio

$$\left| \frac{\sum\limits_{k > \log\log x} (-1)^k \widehat{\pi}_k(x)}{\sum\limits_{k \leq \log\log x} (-1)^k \widehat{\pi}_k(x)} \right| \ll \frac{\sqrt{\log\log x} \cdot B(x, 1+\delta)}{x} = o_\delta(1),$$

is bounded above by at most a small constant for any $\delta > 0$ when x is large. The second term in the last bound is obtained by summing over the uniform estimates guaranteed by Theorem 2.6 and applying (10c) to the resulting expression involving the incomplete gamma function. $\square$

$$\log 1+\delta = \delta - \delta^2/2 + O(\delta^3)$$

$$\delta - (1+\delta)(\delta - \delta^2/2)$$

$$= \delta - (\delta - \delta^2/2 + \delta^2 + O(\delta^3))$$

$$= -\delta^2/2 + o(\delta^3)$$

$$\log 1+\delta \Big|_{\delta=0} = 1$$

$$\frac{d}{d\delta} \log 1+\delta \Big|_{\delta=0} = \frac{1}{1+\delta}\Big|_{\delta=0} = 1$$

$$\frac{d^2}{d\delta^2} \log 1+\delta \Big|_{\delta=0} = -\frac{1}{(1+\delta)^2}\Big|_{\delta=0} = -1$$

5 Auxiliary sequences to express the Dirichlet inverse function, $g^{-1}(n)$

The pages of tabular data given as Table T.1 in the appendix section (refer to page 39) are intended to provide clear insight into why we eventually arrived at the approximations to $g^{-1}(n)$ proved in this section. The table provides illustrative numerical data by examining the approximate behavior at hand for the cases of $1 \leq n \leq 500$ with *Mathematica* [19].

5.1 Definitions and basic properties of component function sequences

We define the following auxiliary coefficient sequence for integers $n \geq 1$ and $k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{21}$$

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (21) inductively. By the same argument, we see that at fixed n, the function $C_k(n)$ is seen to be non-zero only for positive integers $k \leq \Omega(n)$ whenever $n \geq 2$. A sequence of relevant signed semi-diagonals of the functions $C_k(n)$ begins as follows [20, A008480]:

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \; -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

We can see that $C_{\Omega(n)}(n) \leq (\Omega(n))!$ for all $n \geq 1$. In fact, $h^{-1}(n) \equiv \lambda(n) C_{\Omega(n)}(n)$ is the same function given by the formula in (9) from Proposition 3.1. This sequence of semi-diagonals of (21) is precisely related to $g^{-1}(n)$ in the next subsection. In Section 6 we prove exact probabilistic distributions for the values of $C_{\Omega(n)}(n)$.

5.2 Relating the auxiliary functions $C_{\Omega(n)}(n)$ to formulas approximating $g^{-1}(n)$

Lemma 5.1 (An exact initial formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) g^{-1}(n/d) \quad \implies \quad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \tag{22}$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (22) in the form of $(g^{-1} * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m \cdot (C_m(-) * g^{-1})(n)$, or so that

$$F_m(n) = -\begin{cases} \sum\limits_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum\limits_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & m \geq 2, \\ (\omega * g^{-1})(n), & m = 1. \end{cases}$$

By repeatedly expanding the right-hand-side of the previous equation, we find that for $m := \Omega(n)$ (i.e., with the expansions at a maximal depth in the previous equation)

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \tag{23}$$

The formula then follows from (23) by Möbius inversion applied to each side of the last equation. $\square$

Corollary 5.2. *For all squarefree integers $n \geq 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \tag{24}$$

Proof. Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geq 2$ and that n is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where p_i is prime for all $1 \leq i \leq \omega(n)$. Since all divisors of any squarefree n are necessarily also squarefree, we can transform the exact divisor sum guaranteed for all n in Lemma 5.1 into a sum that partitions the divisors according to the number of distinct prime factors as follows:

$$\begin{aligned} g^{-1}(n) &= \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i}(-1)^i \cdot C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \end{aligned}$$

The signed contributions in the first of the previous equations is justified by noting that $\lambda(n) = \mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree, and that for $d \geq 1$ squarefree we have the correspondence $\omega(d) = k \implies \Omega(d) = k$. ☐

Since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ using the notation defined in the the proof of Proposition 3.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for squarefree $n \geq 1$. A proof of part (C) of Conjecture 2.5 follows as an immediate consequence.

Lemma 5.3. *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{25}$$

Proof. By applying Lemma 5.1, Proposition 3.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$, Lemma 5.1 implies

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \end{aligned}$$

In the last equation, we see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. ☐

Combined with the signedness property of $g^{-1}(n)$ guaranteed by Proposition 3.1, Lemma 5.3 shows that its summatory function is expressed as

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_{\Omega(d)}(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Additionally, since (5) implies that

$$\lambda(d)C_{\Omega(d)}(d) = (g^{-1} * 1)(d) = (\chi_{\mathbb{P}} + \varepsilon)^{-1}(d),$$

where $\chi_{\mathbb{P}}$ denotes the characteristic function of the primes, we also clearly recover by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1.$$

5.3 A connection to the distribution of the primes

too long

The combinatorial complexity of $g^{-1}(n)$ is deeply tied to the distribution of the primes $p \leq n$ as $n \to \infty$. While the magnitudes and dispersion of the primes $p \leq x$ certainly restricts the repeating of these distinct sequence values, we can see that the following is still clear about the relation of the weight functions $|g^{-1}(n)|$ to the distribution of the primes: The value of $|g^{-1}(n)|$ is entirely dependent on the pattern of the *exponents* (viewed as multisets) of the distinct prime factors of $n \geq 2$ (*cf.* Heuristic 2.4). The relation of the repitition of the distinct values of $|g^{-1}(n)|$ in forming bounds on $G^{-1}(x)$ makes another clear tie to $M(x)$ through Proposition 7.4.

Is this needed?

Example 5.4 (Combinatorial significance to the distribution of $g^{-1}(n)$)**.** We have a natural extremal behavior with respect to distinct values of $\Omega(n)$ corresponding to squarefree integers and prime powers. Namely, if for $k \geq 1$ we define the infinite sets M_k and m_k to correspond to the maximal (minimal) sets of positive integers such that

$$M_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j) = k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

$$m_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j) = k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

By defn of M_k, it is a subset of $\mathbb{Z}^+$.

then any element of M_k is squarefree and any element of m_k is a prime power. In particular, we have that for any $N_k \in M_k$ and $n_k \in m_k$

$$N_k = \sum_{j=0}^{k} \binom{k}{j} \cdot j!, \quad \text{and} \quad n_k = 2 \cdot (-1)^k.$$

The formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 3.1 implies that we can express an exact formula for $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n. Namely, for $n \geq 2$ and $0 \leq k \leq \omega(n)$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z \cdot \nu_p(n)) = [z^k] \prod_{p^\alpha || n} (1 + \alpha z).$$

Then we have essentially shown using (9) and (25) that we can expand formulas for these inverse functions in the following form:

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, n \geq 2.$$

Too long.

The combinatorial formula for $h^{-1}(n) = \lambda(n) \cdot (\Omega(n))! \times \prod_{p^\alpha || n} (\alpha!)^{-1}$ we derived in the proof of the key signedness proposition in Section 3 suggests further patterns and more regularity in the contributions of the distinct weighted terms for $G^{-1}(x)$ when we sum over all of the distinct prime exponent patterns that factorize $n \leq x$. Our interpretations leading to the proof of the bounds on $|G^{-1}(x)|$ from below via Theorem 7.3 is less combinatorially motivated.

6 The precise limiting distributions of $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$

We have remarked already in the introduction that the relation of the component functions, $g^{-1}(n)$ and $C_{\Omega(n)}(n)$, to the canonical additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties of these functions witnessed (á priori) in Table T.1. In particular, each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdös-Kac theorem that shows that the density of a shifted and scaled variant of each of the sets of these function values for $n \leq x$ can be expressed through a limiting normal distribution as $x \to \infty$ [4, 2, 16]. In the remainder of this section we establish more analytically motivated proofs of related properties of these key sequences used to express $G^{-1}(x)$, again in the spirit of Montgomery and Vaughan's reference manual (*cf.* Remark 2.8).

Proposition 6.1. *Let the function $\widehat{F}(s,z)$ is defined for $\operatorname{Re}(s) \geq 2$ and $|z| < |P(s)|^{-1}$ in terms of the prime zeta function by*

be

$$\widehat{F}(s,z) := \frac{1}{1-P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

For $|z| < P(2)^{-1}$, let the summatory function of the coefficients of the DGF expansion of $\widehat{F}(s,z)$ be defined as follows:

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

We have that for all sufficiently large x

$$\widehat{A}_z(x) = \frac{x}{\Gamma(z)} \cdot \widehat{F}(2,z) \cdot (\log x)^{z-1} + O_z\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right), |z| < P(2)^{-1}.$$

Proof. We know from the proof of Proposition 3.1 that for $n \geq 2$

This does NOT match (9), page 12.

Pinpoint ref. needed.

$$C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}.$$

We can generate scaled forms of these terms through the Dirichlet series identity

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! \cdot p^{rs}}\right)^{-1} = \exp\left(z \cdot P(s)\right), \operatorname{Re}(s) \geq 2, z \in \mathbb{C}.$$

By computing a Laplace transform on the right-hand-side of the above, we obtain

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n) \cdot (-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp\left(tz \cdot P(s)\right) dt = \frac{1}{1-P(s)z}, \operatorname{Re}(s) \geq 2, |z| < |P(s)|^{-1}.$$

It follows that

$$\sum_{n \geq 1} \frac{\lambda_*(n) C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times \widehat{F}(s,z), \operatorname{Re}(s) \geq 2, |z| < |P(s)|^{-1}.$$

Since $\widehat{F}(s,z)$ is convergent as an analytic function of s for all $\operatorname{Re}(s) > 1$ whenever $|z| < |P(s)|^{-1}$, if $b_z(n)$ are the coefficients in the DGF expansion of $\widehat{F}(s,z)$, then

$$\left| \sum_{n \geq 1} \frac{b_z(n) (\log n)^{2R+1}}{n^s} \right| < +\infty,$$

is uniformly bounded for $|z| \leq R$. This fact follows by repeated termwise differentiation with respect to s.

We must adapt the details to the case where the next proof method arises in the first application instance from the reference [12, §7.4; Thm. 7.18]. Let the function $d_z(n)$ be generated as the coefficients of the DGF $\zeta(s)^z$ for

$\mathrm{Re}(s) > 1$, with corresponding summatory function $D_z(x) := \sum_{n \leq x} d_z(n)$. The theorem in [12, Thm. 7.17; §7.4] implies that for any $z \in \mathbb{C}$ and $x \geq 2$

$$D_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\left(x \cdot (\log x)^{\mathrm{Re}(z)-2}\right).$$

Taking the notation from the reference, we set $b_z(n) \equiv (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$, set the convolution $a_z(n) := \sum_{d|n} b_z(d) d_z(n/d)$, and define its summatory function $A_z(x) := \sum_{n \leq x} a_z(n)$. Then we have that

$$\begin{aligned} A_z(x) &= \sum_{m \leq x/2} b_z(m) D_z(x/m) + \sum_{x/2 < m \leq x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \times \sum_{m \leq x/2} \frac{b_z(m)}{m^2} \times m \cdot \log\left(\frac{x}{m}\right)^{z-1} + O\left(\sum_{m \leq x} \frac{x \cdot |b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2}\right). \qquad (26) \end{aligned}$$

We can sum the coefficients for $u > e$ large as

$$\sum_{m \leq u} \frac{b_z(m)}{m} = (\widehat{F}(2,z) + O(u^{-2}))u - \int_1^u (\widehat{F}(2,z) + O(t^{-2}))dt = \widehat{F}(2,z) + O(1 + u^{-1}). \qquad (27)$$

Suppose that $|z| \leq R < P(2)^{-1}$. The error term in (26) satisfies

$$\begin{aligned} \sum_{m \leq x} \frac{x \cdot |b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\mathrm{Re}(z)-2} &\ll x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m} \\ &\quad + x(\log x)^{-(R+2)} \times \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m} (\log m)^{2R} \\ &\ll x(\log x)^{\mathrm{Re}(z)-2} \cdot \widehat{F}(2,z) = O_z\left(x \cdot (\log x)^{\mathrm{Re}(z)-2}\right), |z| \leq R. \end{aligned}$$

not a pos. functions.

In the main term estimate for $A_z(x)$ from (26), when $m \leq \sqrt{x}$ we have

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\mathrm{Re}(z)-2}\right).$$

The total sum over the interval $m \leq x/2$ then corresponds to bounding

$$\begin{aligned} \sum_{m \leq x/2} b_z(m) D_z(x/m) &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \times \sum_{m \leq x/2} \frac{b_z(m)}{m} \\ &\quad + O_z\left(x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m} + x(\log x)^{R-1} \times \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \widehat{F}(2,z) + O_z\left(x(\log x)^{\mathrm{Re}(z)-2} \times \sum_{m \geq 1} \frac{b_z(m)(\log m)^{2R+1}}{m^2}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \widehat{F}(2,z) + O_z\left(x(\log x)^{\mathrm{Re}(z)-2}\right). \end{aligned}$$

??

□

Theorem 6.2. *We have uniformly for $1 \leq k < \log\log x$ that as $x \to \infty$*

$$\widehat{C}_k(x) := \sum_{\substack{n \leq x \\ \Omega(n) = k}} (-1)^{\omega(n)} C_k(n) \asymp -\frac{x}{\log x} \cdot \frac{(\log\log x - \log\zeta(2))^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

This is a confusing approx.

Take $m = \sqrt{x}$

$$\left(\log x/m\right)^2 = \left(\frac{1}{2}\log x\right)^2$$

The two terms on the right are of same order of magnitude.

↓ I dont like this proof

Proof. The proof is a similar adaptation of the method of Montgomery and Vaughan we cited in Remark 4.1 and Remark **??** to prove our variant of Theorem 2.7. We begin by bounding a contour integral over the error term for fixed large x when $r := \frac{k-1}{\log\log x}$ with $r < 2$:

$$\left|\int_{|v|=r} \frac{x \cdot (\log x)^{-(\mathrm{Re}(v)+2)}}{v^{k+1}} dv\right| \ll x(\log x)^{-(r+2)} r^{-(k+1)} \ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{k+1}} \cdot \frac{1}{e^{k-1}}$$
$$\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{3/2}} \cdot \frac{1}{e^{2k}(k-1)!}$$
$$\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \ll \frac{x}{\log x} \cdot \frac{k \cdot (\log\log x)^{k-5}}{(k-1)!}.$$

We must find an asymptotically accurate main term approximation to the coefficients of the following contour integral for $r \in [0, z_{\max}]$ where $z_{\max} < P(2)^{-1}$:

$$\widetilde{A}_r(x) := -\int_{|v|=r} \frac{x \cdot (\log x)^{-v} \zeta(2)^v}{(\log x)\Gamma(1+v) \cdot v^k (1+P(2)v)} dv. \tag{28}$$

Sounds fishy

Finding an exact formula for the derivatives of the function that is implicit to the Cauchy integral formula (CIF) for (28) is complicated significantly by the need to differentiate $\Gamma(1+v)^{-1}$ up to any integer order k in the formula. We can show that provided a restriction on the uniform bound parameter to $1 \leq r < 1$, ? we can approximate the contour integral in (28) where the resulting main term is accurate up to a bounded constant factor. This procedure removes the gamma function term in the denominator of the integrand by essentially applying a mean value theorem type analog for contours.

We observe that for $r := 1$, the function $|\Gamma(1 + re^{2\pi\imath t})|$ has a singularity (pole) when $t := \frac{1}{2}$. Thus we restrict the range of $|v| = r$ so that $0 \leq r < 1$ to necessarily avoid this problematic value of t when we parameterize $v = re^{2\pi\imath t}$ by a real-line integral over $t \in [0,1]$. We can compute the finite extremal values of this function as

$$\min_{\substack{0 \leq r < 1 \\ 0 \leq t \leq 1}} |\Gamma(1 + re^{2\pi\imath t})| = |\Gamma(1 + re^{2\pi\imath t})|\Big|_{(r,t)\approx(1,0.740592)} \approx 0.520089$$
$$\max_{\substack{0 \leq r < 1 \\ 0 \leq t \leq 1}} |\Gamma(1 + re^{2\pi\imath t})| = |\Gamma(1 + re^{2\pi\imath t})|\Big|_{(r,t)\approx(1,0.999887)} \approx 1.$$

This shows that

$$\widetilde{A}_r(x) \asymp -\int_{|v|=r} \frac{x \cdot (\log x)^{-v} \zeta(2)^v}{(\log x) \cdot v^k (1+P(2)v)} dv, \tag{29}$$

where as $x \to \infty$

$$\frac{\widetilde{A}_r(x)}{-\int_{|v|=r} \frac{x(\log x)^{-v}\zeta(2)^v}{(\log x)\cdot v^k(1+P(2)v)} dv} \in [1, 1.92275].$$

By induction we can compute the remaining coefficients $[z^k]\Gamma(1+z) \times \widehat{A}_z(x)$ with respect to x for fixed $k \leq \log\log x$ using the CIF. Namely, it is not difficult to see that for any integer $m \geq 0$, we have the m^{th} partial derivative of the integrand with respect to z has the following limiting expansion by applying (10c):

$$\frac{1}{m!} \times \frac{\partial^{(m)}}{\partial v^{(m)}} \left[\frac{(\log x)^{-v}\zeta(2)^v}{1+P(2)v}\right]\Bigg|_{v=0} = \sum_{j=0}^{m} \frac{(-1)^m P(2)^j (\log\log x - \log\zeta(2))^{m-j}}{(m-j)!}$$
$$= \frac{(-P(2))^m (\log x)^{\frac{1}{P(2)}} \zeta(2)^{-\frac{1}{P(2)}}}{m!} \times \Gamma\left(m+1, \frac{\log\log x - \log\zeta(2)}{P(2)}\right)$$
$$\sim \frac{(-1)^m (\log\log x - \log\zeta(2))^m}{m!}.$$

Now by parameterizing the countour around $|z| = r := \frac{k-1}{\log\log x} < 1$ we deduce that the the main term of our approximation corresponds to

$$-\int_{|z|=r} \frac{x \cdot (\log x)^{-z}\zeta(2)^z}{(\log x) z^k (1+P(2)z)} dz \asymp -\frac{x}{\log x} \cdot \frac{(-1)^{k-1}(\log\log x - \log\zeta(2))^{k-1}}{(k-1)!}. \qquad \square$$

An exact DGF expression for $\lambda(n) C_{\Omega(n)}(n)$ is in fact very much complicated by the need to estimate the asymptotics of the coefficients of the more difficult right-hand-side product forms of

$$\sum_{n\geq 1} \frac{\lambda(n) C_{\Omega(n)}(n) z^{\Omega(n)}}{(\Omega(n))! \cdot n^s} = \prod_p \left(2 - \exp\left(-z \cdot p^{-s}\right)\right)^{-1}, \operatorname{Re}(s) > 1, |z| < \log 2.$$

It is unclear how to exactly, and effectively, bound the coefficients of powers of z in the DGF expansion defined by the last equation. We use an alternate intermediate method in Corollary 6.3 to obtain the asymptotics for the summatory functions on which we require average case bounds.

Corollary 6.3. *We have that for large $x \geq 2$ that uniformly for $1 \leq k \leq \log\log x$*

$$\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp 2\sqrt{2\pi} \cdot x \times \frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!}.$$

Proof. We have an integral formula involving the non-sign-weighted sequence that results by applying ordinary Abel summation (and integrating by parts) in the form of

Maybe you need a better notation than λ_*.

$$\sum_{n\leq x} \lambda_*(n) h(n) = \left(\sum_{n\leq x} \lambda_*(n)\right) h(x) - \int_1^x \left(\sum_{n\leq t} \lambda_*(n)\right) h'(t) dt \tag{30}$$

$$\left\{\begin{matrix} u_t = L_*(t) & v_t' = h'(t)dt \\ u_t' = L_*'(t)dt & v_t = h(t) \end{matrix}\right\}$$

$$\asymp \int_1^x \frac{d}{dt}\left[\sum_{n\leq t} \lambda_*(n)\right] h(t) dt. \tag{31}$$

Since uniformity in k and constants is important.

Let the signed left-hand-side summatory function for our function corresponding to (30) be defined by

$$\begin{aligned}
\widehat{C}_{k,*}(x) &:= \sum_{\substack{n\leq x\\ \Omega(n)=k}} (-1)^{\omega(n)} C_{\Omega(n)}(n) \\
&\asymp -\frac{x}{\log x} \cdot \frac{(\log\log x - \log\zeta(2))^{k-1}}{(k-1)!}\left[1 + O\left(\frac{1}{\log\log x}\right)\right] \\
&\asymp -\frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O\left(\frac{1}{\log\log x}\right)\right]
\end{aligned}$$

where the second equation above follows from the proof of Theorem 6.2.

We handle transforming our previous results for the partial sums over the unsigned sequence $C_{\Omega(n)}(n)$ such that $\Omega(n) = k$. The argument is based on approximating the smooth summatory function of $\lambda_*(n) := (-1)^{\omega(n)}$ using the following uniform approximation of $\pi_k(x)$ when $1 \leq k \leq \log\log x$ as $x \to \infty$:

$$\pi_k(x) \asymp \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}(1 + o(1)).$$

In particular, we have that (compare to Table T.2 starting on page 46)

$$L_*(t) := \left|\sum_{n\leq t}(-1)^{\omega(n)}\right| = \left|\sum_{k=1}^{\log\log x}(-1)^k\pi_k(x)\right| \sim \frac{t}{\sqrt{2\pi}\sqrt{\log\log t}}, \text{ as } t\to\infty.$$

The main term for the reciprocal of the derivative of this summatory function is given by

$$\frac{1}{L_*'(t)} \asymp \sqrt{2\pi}\cdot(\log\log t)^{\frac{1}{2}}.$$

After applying the formula from (30), we deduce that the unsigned summatory function variant satisfies

$$\begin{aligned}
\widehat{C}_{k,*}(x) = \int_1^x L_*'(t)C_{\Omega(t)}(t)dt \qquad &\implies C_{\Omega(x)}(x) \asymp \frac{\widehat{C}'_{k,*}(x)}{L_*'(x)} \qquad \implies \\
C_{\Omega(x)}(x) \asymp \sqrt{2\pi}\cdot\frac{(\log\log x)^{\frac{1}{2}}}{\log x}&\cdot\left[\frac{(\log\log x)^{k-1}}{(k-1)!}\left(1-\frac{1}{\log x}\right)+\frac{(\log\log x)^{k-2}}{(\log x)(k-2)!}\right] \\
\asymp \sqrt{2\pi}\cdot\frac{(\log\log x)^{k-\frac{1}{2}}}{(\log x)(k-1)!} &=: \widehat{C}_{k,**}(x).
\end{aligned}$$

So applying to the ordinary Abel summation formula, and integrating by parts, we obtain that the main term for this function is given by

$$\begin{aligned}
\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n) &\asymp \int \widehat{C}_{k,**}(x)dx \\
&\asymp 2\sqrt{2\pi}\cdot x\times\frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!}.
\end{aligned}$$

□

Lemma 6.4. *We have that as $x\to\infty$*

$$\mathbb{E}\left[C_{\Omega(n)}(n)\right] \asymp 2\sqrt{2\pi}\cdot(\log x)\sqrt{\log\log x}.$$

Proof. We first compute the absolute value of the following summatory function by applying Corollary 6.3:

$$\sum_{k=1}^{\log\log x}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp 2\sqrt{2\pi}\cdot x\cdot(\log x)\sqrt{\log\log x}. \tag{32}$$

We claim that

$$\sum_{n\leq x} C_{\Omega(n)}(n) = \sum_{k=1}^{\log_2(x)}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp \sum_{k=1}^{\log\log x}\sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n). \tag{33}$$

To prove (33), it suffices to show that

$$\frac{\sum\limits_{\log\log x<k\leq\log_2(x)}\ \sum\limits_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n)}{\sum\limits_{k=1}^{\log\log x}\ \sum\limits_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n)} = o(1), \text{ as } x\to\infty. \tag{34}$$

Next, define the following component sums for large x and $0<\varepsilon<1$ so that $(\log\log x)^{\frac{\varepsilon\log\log x}{\log\log\log x}} = o(\log x)$:

$$S_{2,\varepsilon}(x) := \sum_{\log\log x<k\leq\log x}\ \sum_{\substack{n\leq x\\ \Omega(n)=k}} C_{\Omega(n)}(n).$$

I have not checked Cor. 6.3.

Using it w/ the constants uniform in k,

$$\frac{1}{x}\sum_{k=1}^{\log\log x}\sum_{\substack{n\le x\\ \Omega(n)=k}} C_{\Omega(n)}(n)$$

$$\asymp \sum_{k=1}^{\log\log x}\frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)\ (k-1)!}$$

How do you get a $\log x\,(\log\log x)^{1/2}$ out of this??

Your use of $\asymp$ here does $\boxed{\text{not}}$ match your definition.

Then

$$\sum_{k=1}^{\log_2(x)} \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \gg S_{2,\varepsilon}(x),$$

$\log_2 x = ??$

↑ is this proved?

with equality as $\varepsilon \to 1$ so that the upper bound of summation tends to $\log x$. To show that (34) holds, observe that whenever $\Omega(n) = k$, we have that $C_{\Omega(n)}(n) \leq k!$. We can then bound the sum defined above using Theorem 2.7 and Theorem 4.4 for large $x \to \infty$ as

$\sim (\log x)^{\varepsilon}$

$$\begin{aligned} S_{2,\varepsilon}(x) &\leq \sum_{\log\log x \leq k \leq \log x} \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \ll \sum_{k=\log\log x}^{(\log\log x)^{\frac{\varepsilon \log\log x}{\log\log\log x}}} \frac{\widehat{\pi}_k(x)}{x} \cdot k! \\ &\ll \sum_{k=\log\log x}^{(\log\log x)^{\frac{\varepsilon \log\log x}{\log\log\log x}}} (\log x)^{\frac{k}{\log\log x} - 1 - \frac{k}{\log\log x}(\log k - \log\log\log x)} \cdot \left(\frac{k}{e}\right)^k \sqrt{2\pi k} \\ &\ll \sum_{k=\log\log x}^{\frac{\varepsilon \log\log x}{\log\log\log x}} (\log x)^{\frac{2k \cdot \log\log\log x}{\log\log x} - 1} \sqrt{k} \\ &\ll \frac{1}{(\log x)} \times \int_{\log\log x}^{\frac{\varepsilon \log\log x}{\log\log\log x}} (\log\log x)^{2t} \sqrt{t} \cdot dt \\ &\ll \frac{1}{(\log x)} \sqrt{\frac{\varepsilon \cdot \log\log x}{\log\log\log x}} (\log\log x)^{\frac{2\varepsilon \cdot \log\log x}{\log\log\log x}} = o(x), \end{aligned}$$

??

where does this come from?

where we have a simplification by noticing that $\lim_{x\to\infty} (\log x)^{\frac{1}{\log\log x}} = e$. So by (32) this form of the ratio in (34) clearly tends to zero. If we have a contribution from the terms as $\varepsilon \to 1$, e.g., if x is a power of two, then $C_{\Omega(x)}(x) = 1$ by the formula in (9), so that the contribution from this upper-most indexed term is negligible:

$$x = 2^k \implies \Omega(x) = k \implies C_{\Omega(x)}(x) = \frac{(\Omega(x))!}{k!} = 1.$$

The formula for the expectation claimed in the statement of this lemma above then follows from (32) by scaling by $\frac{1}{x}$ and dropping the asymptotically lesser error terms in the bound. □

I dont understand this part either

Corollary 6.5 (Expectation formulas). *We have that as $n \to \infty$*

$$\mathbb{E}|g^{-1}(n)| \asymp (\log n)^2 \sqrt{\log\log n}.$$

Proof. We use the formula from Lemma 6.4 to find $\mathbb{E}[C_{\Omega(n)}(n)]$ up to a small bounded multiplicative constant factor as $n \to \infty$. This implies that for large x

$$\begin{aligned} \int \frac{\mathbb{E}[C_{\Omega(t)}(t)]}{t} dt &\asymp \sqrt{2\pi} \cdot (\log t)^2 \sqrt{\log\log t} - \frac{\pi}{2} \operatorname{erfi}\left(\sqrt{2\log\log t}\right) \\ &\asymp \sqrt{2\pi} \cdot (\log t)^2 \sqrt{\log\log t}. \end{aligned}$$

Recall from the introduction that the summatory function of the squarefree integers is given by

$$Q(x) := \sum_{n\leq x} \mu^2(n) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Therefore summing over (25) we find that

$$\mathbb{E}|g^{-1}(n)| = \frac{1}{n} \times \sum_{d\leq n} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{n}{d} \right\rfloor\right)$$

$$
\begin{aligned}
&\sim \sum_{d \leq n} C_{\Omega(d)}(d) \left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dn}} \right) \right] \\
&= \frac{6}{\pi^2} \left[\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d<n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d} \right] + O\left(\frac{1}{\sqrt{n}} \times \int_0^n t^{-1/2} dt \right) \\
&= \frac{6}{\pi^2} \left[\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d<n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d} \right] + O(1) \qquad (35) \\
&\asymp \frac{6\sqrt{2}}{\pi^{\frac{3}{2}}} (\log n)^2 \sqrt{\log\log n}. \qquad \square
\end{aligned}
$$

Theorem 6.6. *Let the mean and variance analogs be denoted by*

$$
\mu_x(C) := \log\log x + \hat{a} - \frac{1}{2} \cdot \log\log\log x, \qquad \text{and} \qquad \sigma_x(C) := \sqrt{\mu_x(C)},
$$

where the absolute constant $\hat{a} := \log\left(\frac{1}{\sqrt{2\pi}}\right) \approx -0.918939$. *Set* $Y > 0$ *and suppose that* $z \in [-Y, Y]$. *Then we have uniformly for all* $-Y \leq z \leq Y$ *that*

$$
\frac{1}{x} \cdot \#\left\{ 2 \leq n \leq x : \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)} \leq z \right\} = \Phi(z) + O\left(\frac{1}{\sqrt{\log\log x}} \right), \quad \text{as } x \to \infty.
$$

Proof. For large x and $n \leq x$, define the following auxiliary variables:

$\beta_{n,x}$ appears in statement

$$
\alpha_n := \frac{C_{\Omega(n)}(n) - \mu_n(C)}{\sigma_n(C)}, \quad \text{and} \quad \beta_{n,x} := \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)}.
$$

Let the corresponding densities (whose limiting distributions we must verify) be defined by the functions

$$
\Phi_1(x, z) := \frac{1}{x} \cdot \#\{n \leq x : \alpha_n \leq z\},
$$

and

$$
\Phi_2(x, z) := \frac{1}{x} \cdot \#\{n \leq x : \beta_{n,x} \leq z\}.
$$

We first argue that it suffices to consider the distribution of $\Phi_2(x, z)$ as $x \to \infty$ in place of $\Phi_1(x, z)$ to obtain our desired result. The difference of the two auxiliary variables is neglibible as $x \to \infty$ for (n, x) taken over the ranges that contribute the non-trivial weight to the main term of each density function. In particular, we have for $\sqrt{x} \leq n \leq x$ and $C_{\Omega(n)}(n) \leq 2 \cdot \mu_x(C)$ that the following is true:

$$
|\alpha_n - \beta_{n,x}| \ll \frac{1}{\sigma_x(C)} \xrightarrow{x \to \infty} 0.
$$

Thus we can replace α_n by $\beta_{n,x}$ and estimate the limiting densities corresponding to the alternate terms. The rest of our argument follows the method in the proof of the related theorem in [12, Thm. 7.21; §7.4] closely.

We use the formula proved in Corollary 6.3 to estimate the densities claimed within the ranges bounded by z as $x \to \infty$. Let $k \geq 1$ be a natural number defined by $k := t + \mu_x(C)$. We write the small parameter $\delta_{t,x} := \frac{t}{\mu_x(C)}$. When $|t| \leq \frac{1}{2}\mu_x(C)$, we have by Stirling's formula that

$$
\begin{aligned}
2\sqrt{2\pi} \cdot x \times \frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!} &\sim \frac{e^{\hat{a}+t} (\log\log x)^{\mu_x(C)(1+\delta_{t,x})}}{\sigma_x(C) \cdot \mu_x(C)^{\mu_x(C)(1+\delta_{t,x})} (1+\delta_{t,x})^{\mu_x(C)(1+\delta_{t,x})+\frac{3}{2}}} \\
&\sim \frac{e^t}{\sqrt{2\pi} \cdot \sigma_x(C)} (1+\delta_{t,x})^{-\left(\mu_x(C)(1+\delta_{t,x})+\frac{3}{2}\right)},
\end{aligned}
$$

since $\frac{\mu_x(C)}{\log\log x} = 1 + o(1)$ as $x \to \infty$.

This is confusing wording.
Thm 6.6 does not mention
anything about α_n

You do not say where this expression comes from.
It appears to come from Cor. 6.3.
That Corollary is a statement about
expected value in $\{1, \dots, x\}$

$$\underset{\Omega(n)=k}{\mathbb{E}} C_k(n).$$

Statements about expectation do not
imply distributional statements. There
appears to be a conceptual problem w/ the approach.

We have the uniform estimate $\log(1+\delta_{t,x}) = \delta_{t,x} - \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)$ whenever $|\delta_{t,x}| \leq \frac{1}{2}$. Then we can expand the factor involving $\delta_{t,x}$ in the previous equation as follows:

$$\begin{aligned}(1+\delta_{t,x})^{-\mu_x(C)(1+\delta_{t,x})-\frac{1}{2}} &= \exp\left(\left(\frac{1}{2} + \mu_x(C)(1+\delta_{t,x})\right) \times \left(-\delta_{t,x} + \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)\right)\right) \\ &= \exp\left(-t - \frac{3t+t^2}{2\mu_x(C)} + \frac{3t^2}{4\mu_x(C)^2} + O\left(\frac{|t|^3}{\mu_x(C)^2}\right)\right).\end{aligned}$$

For both $|t| \leq \mu_x(C)^{1/2}$ and $\mu_x(C)^{1/2} < |t| \leq \mu_x(C)^{2/3}$, we see that

$$\frac{t}{\mu_x(C)} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Similarly, for $|t| \leq 1$ and $|t| > 1$, we see that both

$$\frac{t^2}{\mu_x(C)^2} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Let the corresponding error terms in (x,t) be denoted by

$$\widetilde{E}(x,t) := O\left(\frac{1}{\sigma_x(C)}\right) + O\left(\frac{|t|^3}{\mu_x(C)^2}\right).$$

Combining these estimates with the previous computations, we can deduce that uniformly for $|t| \leq \mu_x(C)^{2/3}$

$$2\sqrt{2\pi}\cdot x \times \frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!} \sim \frac{1}{\sqrt{2\pi}\cdot\sigma_x(C)} \cdot \exp\left(-\frac{t^2}{2\sigma_x(C)^2}\right) \times \left[1+\widetilde{E}(x,t)\right].$$

By the same argument utilized in the proof of Lemma 6.4, we see that the contributions of these summatory functions for $k \leq \mu_x(C) - \mu_x(C)^{2/3}$ is negligible. We also require that $k \leq \log\log x$ as we have worked out in Theorem 6.2. So we sum over a corresponding range of

$$\mu_x(C) - \mu_x(C)^{2/3} \leq k \leq R_{z,x}\cdot\mu_x(C) + z\cdot\sigma_x(C),$$

for $R_{z,x} := 1 - \frac{z}{\sigma_x(C)}$ to approximate the stated normalized densities. Then finally as $x \to \infty$, the three terms that result (one main term, two error terms, respectively) can be considered to each correspond to a Riemann sum for an associated integral. □

Corollary 6.7. *Let $Y > 0$. Uniformly for all $-Y \leq y \leq Y$ we have that*

This is

$$\frac{1}{x}\cdot\#\left\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq y\right\} = \Phi\left(\frac{\frac{\pi^2}{6}y - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

problematic

Proof. We claim that

$$|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2}C_{\Omega(n)}(n).$$

From (35) we obtain that

$$\frac{1}{x} \times \sum_{n\leq x}|g^{-1}(n)| = \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(x)}(x)] + \sum_{d<x}\frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O(1).$$

Let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) := f(x) - f(x-1)$. Then from the proof of Corollary 6.5, we see that for large n

$$\begin{aligned}
|g^{-1}(n)| = \Delta_n(n \cdot \mathbb{E}|g^{-1}(n)|) &\sim \Delta_n\left(\sum_{d \leq n} \frac{6}{\pi^2} \cdot C_{\Omega(d)}(d) \cdot \frac{n}{d}\right) \\
&= \frac{6}{\pi^2}\left[C_{\Omega(n)}(n) + \sum_{d<n} C_{\Omega(d)}(d)\frac{n}{d} - \sum_{d<n} C_{\Omega(d)}(d)\frac{(n-1)}{d}\right] \\
&= \frac{6}{\pi^2} C_{\Omega(n)}(n) + \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n-1)|, \text{ as } n \rightarrow \infty.
\end{aligned}$$

The result finally follows from Theorem 6.6. $\square$

7 Lower bounds for $M(x)$ along infinite subsequences

7.1 Establishing initial lower bounds on the summatory function $G^{-1}(x)$

Lemma 7.1. *If x is sufficiently large and we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds:*

$$\mathbb{P}\left(|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq 0\right) = o(1) \tag{A}$$

$$\mathbb{P}\left(|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\mu_x(C)\right) = \frac{1}{2} + o(1). \tag{B}$$

Moreover, for any positive real $\delta > 0$ we have that

$$\mathbb{P}\left(|g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\mu_x(C)^{1+\delta}\right) = 1 + o_\delta(1), \text{ as } x \rightarrow \infty. \tag{C}$$

Proof. Each of these results is a consequence of Corollary 6.7. Let the densities $\gamma_z(x)$ be defined for $z \in \mathbb{R}$ and large $x > e$ as follows:

$$\gamma_z(x) := \frac{1}{x} \cdot \#\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq z\}.$$

To prove (A), observe that by Corollary 6.7 for $z := 0$ we have that

$$\gamma_0(x) = \Phi\left(-\sigma_x(C)\right) + o(1), \text{ as } x \rightarrow \infty.$$

We can see that $\sigma_x(C) \xrightarrow{x\rightarrow\infty} +\infty$ where for $z \geq 0$ we have the reflection identity $\Phi(z) = 1 - \Phi(-z)$. Combined we have by an asymptotic approximation to the error function expanded by

$$\begin{aligned}\Phi(z) &= \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right) \\ &= 1 - \frac{2e^{-z^2/2}}{\sqrt{2\pi}}\left[z^{-1} - z^{-3} + 3z^{-5} - 15z^{-7} + \cdots\right], \text{ as } |z| \rightarrow \infty,\end{aligned}$$

that

$$\gamma_0(x) = \Phi\left(-\sigma_x(C)\right) \asymp \frac{1}{\sigma_x(C)\exp(\mu_x(C)/2)} = o(1).$$

To prove (B), observe that setting $z_1 := \frac{6}{\pi^2}\mu_x(C)$ yields

$$\gamma_{z_1}(x) = \Phi(0) + o(1) = \frac{1}{2} + o(1), \text{ as } x \rightarrow \infty.$$

To prove (C), we require that $\mu_x(C)^{\frac{1}{2}+\delta} - \sigma_x(C) \xrightarrow{x\rightarrow\infty} +\infty$. Since this happens as $x \rightarrow \infty$ for any fixed $\delta > 0$, we have that with $z(\delta) := \frac{6}{\pi^2}\mu_x(C)^{1+\delta}$

$$\begin{aligned}\gamma_{z(\delta)} &= \Phi\left(\mu_x(C)^{\frac{1}{2}+\delta} - \sigma_x(C)\right) + o(1) \\ &\sim 1 - \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\left(\mu_x(C)^{\frac{1}{2}+\delta} - \sigma_x(C)\right)} \times \exp\left(-\frac{\mu_x(C)}{4} \cdot \left(\mu_x(C)^\delta - 1\right)^2\right) \\ &= 1 + o_\delta(1), \text{ as } x \rightarrow \infty.\end{aligned}$$

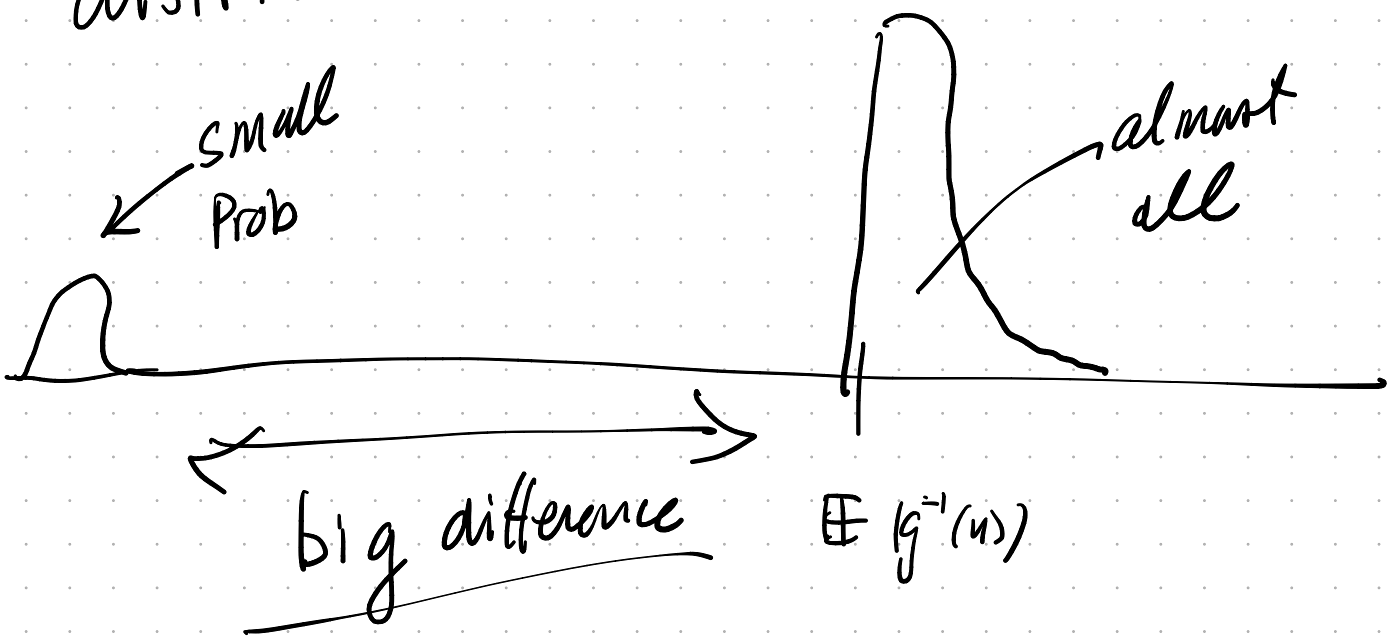
□

Remark 7.2 (Interpretations for constructing bounds on $G^{-1}(x)$)**.** A consequence of (A) and (C) in Lemma 7.1 is that for any fixed $\delta > 0$ and $n \in \mathcal{S}_1(\delta)$ taken within a set of asymptotic density one we have that

$$\mathbb{E}|g^{-1}(n)| \leq |g^{-1}(n)| \leq \mathbb{E}|g^{-1}(n)| + \frac{6}{\pi^2}\mu_x(C)^{\frac{1}{2}+\delta}. \tag{36}$$

(A) says that $|g^{-1}(u)|$ has a very <u>strange</u> distribution.

It MUST take values less than $\mathbb{E}|g^{-1}(u)|$, but those must be unlikely. So the distribution has to look like



So it hard to believe that it is true.

Thus when we integrate over a sufficiently spaced set of disjoint consecutive intervals containing large enough integer values, we can assume that an asymptotic lower bound on the contribution of $|g^{-1}(n)|$ is given by its average order, and an upper bound is given by the upper limit above for any fixed $\delta > 0$. In particular, observe that by Corollary 6.7 and Corollary 6.5 we can see that

$$\frac{\pi^2}{6 \cdot \sigma_x(C)} \times \int_{-\infty}^{\infty} z \cdot \Phi'\left(\frac{\frac{\pi^2}{6} z - \mu_x(C)}{\sigma_x(C)}\right) dz = \frac{6}{\pi^2} \cdot \sigma_x(C) = o\left(\mathbb{E}|g^{-1}(x)|\right).$$

Emphasizing the point above, we can thus again interpret the previous calculation as implying that for n on a large interval, the contribution from $|g^{-1}(n)|$ can be approximated above and below accurately as in the bounds from (36).

Theorem 7.3. *For all sufficiently large integers x, whenever $G^{-1}(x) \neq 0$ we have that*

$$|G^{-1}(x)| \gg (\log x)\sqrt{\log\log x}, \text{ as } x \to \infty.$$

Proof. We will require a couple of observations to sum $G^{-1}(x)$ in absolute value and bound it from below. We will use a lower bound approximating the summatory function of $\lambda(n)$ for $n \leq t$ and t large by summing over the uniform asymptotic lower bounds proved in Theorem 2.7. To be careful about the expected sign of this summatory function, we first appeal to the original uniform approximations to the functions $\widehat{\pi}_k(x)$ given by Theorem 2.6. As noted in [12, §7.4], the function $\mathcal{G}(z)$ satisfies

$$\mathcal{G}\left(\frac{k-1}{\log\log x}\right) = O(1), 1 \leq k \leq \log\log x,$$

so that uniformly for $1 \leq k \leq \log\log x$ we can write

How is this relevant?

$$\widehat{\pi}_k(x) \asymp \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O\left(\frac{1}{\log\log x}\right)\right].$$

By Corollary 4.6, the following summatory function represents the asymptotic main term in the summation $L(x) := \sum_{n \leq x} \lambda(n)$ as $x \to \infty$ (see Table T.2 on page 46):

I do not understand.

$$\widehat{L}_2(x) = \sum_{k=1}^{\log\log x} (-1)^k \widehat{\pi}_k(x) = -\frac{x}{(\log x)^2} \times \Gamma(\log\log x, -\log\log x) \sim \frac{(-1)^{1+\lceil \log\log x\rceil} \cdot x}{\sqrt{2\pi}\sqrt{\log\log x}}$$

So we expect the sign of our summatory function approximation to be approximately given by $(-1)^{1+\lceil \log\log x\rceil}$ for sufficiently large x.

We now find a lower bound on the unsigned magnitude of these summatory functions. In particular, using Theorem 2.7, we have that $\widehat{\pi}_k(x) \gg \widehat{\pi}_k^{(\ell)}(x)$ where

$$\widehat{\pi}_k^{(\ell)}(x) := \frac{x^{\frac{3}{4}}}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

Thus we define our lower bound by

$$\widehat{L}_0(x) := \left|\sum_{k=1}^{\log\log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x)\right| \asymp \frac{x^{\frac{3}{4}}}{\sqrt{\log\log x}}.$$

Requires proof

The derivative of this summatory function satisfies

$$\widehat{L}_0'(x) \asymp \frac{1}{x^{\frac{1}{4}} \cdot \sqrt{\log\log x}}.$$

What is the connection between G^{-1} and what you have written for the proof of Thm 7.3.

You dont point to any formula for $G^{-1}(a)$ that you are using.

We observe that we can break the interval $t \in (e, x]$ into disjoint subintervals according to which we have the expected sign contributions from the summatory function $\widehat{L}_0(x)$. Namely, we expect that for $1 \leq k \leq \frac{\log\log x}{2}$ we expect that (compare to Table T.2)

$$\begin{aligned}\operatorname{sgn}\left(\widehat{L}_0(x)\right) &= -1 \text{ on } \left[e^{e^{2k}}, e^{e^{2k+1}}\right) \\ \operatorname{sgn}\left(\widehat{L}_0(x)\right) &= +1 \text{ on } \left[e^{e^{2k+1}}, e^{e^{2k+2}}\right).\end{aligned}$$

Moreover, since the derivative $\widehat{L}_0'(x)$ is monotone decreasing in x, we can construct our lower bounds by placing the input points to this function in the Abel summation formula from (30) over these signed intervals at the extremal endpoints depending on the leading sign terms. As we have argued in Lemma 7.1 and observed in the preceeding remark, we have the bounds in (36) upon which we can similarly construct the lower bound on $|G^{-1}(x)|$ based on the sign term of the subinterval (as above) and the extremal points within the interval. The idea used to conclude this proof below is to underestimate the resulting integral formula beyond all reasonable doubt.

Let $I_k := \left[e^{e^k}, e^{e^{k+1}}\right)$. By Lemma 7.1 and its consequence stated in (36), we know that for sufficiently large k for approximately one half of the integers n in the interval I_k satisfy $|g^{-1}(n)| \leq k$. Thus by applying the inequalities from (10a) in the last step below we have the following asymptotic lower bounds:

$$\begin{aligned}|G^{-1}(x)| &\gg \left|\int_2^x \widehat{L}_0'(t)|g^{-1}(t)|dt\right| \\ &\gg \sum_{k=1}^{\frac{\log\log x}{2}} (-1)^k \widehat{L}_0'\left(e^{e^k}\right)\left|g^{-1}\left(e^{e^k}\right)\right| \\ &\gg \left|\sum_{k=\frac{\log\log x}{4}}^{\frac{\log\log x}{2}} \left[\widehat{L}_0'\left(e^{e^{2k}}\right)\left|g^{-1}\left(e^{e^{2k}}\right)\right| - \widehat{L}_0'\left(e^{e^{2k-1}}\right)\left|g^{-1}\left(e^{e^{2k-1}}\right)\right|\right]\right| \\ &\gg \left|\sum_{k=\frac{\log\log x}{4}}^{\frac{\log\log x}{2}} \left[\sqrt{2k}\exp\left(-\frac{1}{4}e^{2k}\right) - \sqrt{2k-1}\exp\left(-\frac{1}{4}e^{2k-1}\right)\right]\right| \\ &\gg \left|\int_{k=\frac{\log\log x}{4}}^{\frac{\log\log x}{2}} \sqrt{t}\cdot\exp\left(-\frac{1}{4}e^{2t}\right)\right| \\ &\gg \left|\sqrt{\log\log x}\cdot \operatorname{Ei}\left(-\frac{\log x}{4}\right)\right| \\ &\gg (\log x)\cdot\sqrt{\log\log x}.\end{aligned}$$

□

7.2 Proof of the unboundedness of the scaled Mertens function

Proposition 7.4. *For all sufficiently large x, we have that*

$$M(x) = G^{-1}(x) + G^{-1}\left(\left\lfloor\frac{x}{2}\right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left[\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor\frac{x}{k+1}\right\rfloor\right)\right]. \tag{37}$$

Proof. We know by applying Corollary 2.3 that

$$M(x) = \sum_{k=1}^{x} g^{-1}(k)\left[\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) + 1\right]$$

Strategic Thinking You want to base a proof on (37) above. Now $M(x)$ is basically $\sqrt{x}$

For most values of $1 \leq k \leq x/2$

$$\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)$$

$$\approx \frac{x}{k^2 \log x}$$

This will be much large than $\sqrt{x}$ for $1 < k \ll \sqrt{x}$.

But you are claiming that $G^{-1}(k)$ does not change signs very much.

(Yet dont seek to prove this?)

$$= G^{-1}(x) + \sum_{k=1}^{x/2} g^{-1}(k)\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right)$$

$$= G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) + \sum_{k=1}^{x/2-1} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1}\right\rfloor\right)\right],$$

where the upper bound on the sum is truncated in the second equation by the fact that $\pi(1) = 0$. □

Lemma 7.5. *For sufficiently large x, $k \in \left[\sqrt{x}, \frac{x}{2}\right]$ and integers $m \geq 0$, we have that*

$$\frac{x}{k \cdot \log^m\left(\frac{x}{k}\right)} - \frac{x}{(k+1)\cdot \log^m\left(\frac{x}{k+1}\right)} \gg \frac{x}{(\log x)^m \cdot k(k+1)}, \tag{A}$$

Need to restrict M

and

$$\sum_{k=\sqrt{x}}^{\frac{x}{2}} \frac{x}{k(k+1)} = \sum_{k=\sqrt{x}}^{\frac{x}{2}} \frac{x}{k^2} + O(1). \tag{B}$$

Proof. The proof of (A) is obvious since for $k_0 \in \left[\sqrt{x}, \frac{x}{2}\right]$ we have that

$$\log(2)(1+o(1)) \leq \log\left(\frac{x}{k_0}\right) \leq \log(x).$$

To prove (B), notice that

$$\frac{x}{k(k+1)} - \frac{x}{k^2} = -\frac{x}{k^2(k+1)}.$$

Then we see that

$$\left|\int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^2(t+1)}dt\right| \leq \left|\int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^3}dt\right| \asymp 1.$$ □

We finally address the main conclusion of our arguments given so far with the following proof:

Proof of Theorem 2.9. Define the infinite increasing subsequence, $\{x_{0,y}\}_{y \geq Y_0}$, by $x_{0,y} := e^{2e^{2y+1}}$ for the sequence indices y starting at some sufficiently large finite integer Y_0. We can verify that for sufficiently large $y \to \infty$, this infinitely tending subsequence is well defined as $x_{0,y+1} > x_{0,y}$, and also importantly $\log\log(x_{0,y+1}) > \log\log(x_{0,y})$ whenever $y \geq Y_0$ (see concluding argument below). Given a fixed large infinitely tending y, we have some (at least one) point $\widehat{x}_0(y) \in \mathbb{X}_y$ defined such that $|G^{-1}(t)|$ is minimal and non-vanishing on the interval $\mathbb{X}_y := \left[\sqrt{x_{0,y+1}}, \frac{x_{0,y+1}}{2}\right)$ in the form of ?

$$\left|G^{-1}(\widehat{x}_0(y))\right| := \min_{\substack{\sqrt{x_{0,y+1}} \leq t < \frac{x_{0,y+1}}{2} \\ G^{-1}(t) \neq 0}} |G^{-1}(t)|.$$

Thm 7.3 says this doesn't happen.

In the last step, we observe that $G^{-1}(x) = 0$ for x on a set of asymptotic density *at least* bounded below by $\frac{1}{2}$, so that our claim is accurate as the summand's lower bound on this interval does not trivially vanish at large y. This happens since the sequence $g^{-1}(n)$ is non-zero for all $n \geq 1$, so that if we do encounter a zero of the summatory function at x, we find a non-zero summatory function value at $x+1$. Let the shorthand notation $|G^{-1}_{\min}(x_y)| := |G^{-1}(\hat{x}_0(y))|$.

We need to bound the prime counting function differences in the formula given by Proposition 7.4. We will require the following known bounds on the prime counting function due to Rosser and Schoenfeld for large $x \gg 59$ [18, Thm. 1]:

$$\frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right). \tag{38}$$

Let the component function $U_M(y)$ be defined for all large y as

$$U_M(y) := \sum_{k=1}^{\sqrt{\hat{x}_{0,y+1}}} |G^{-1}(k)| \left[\pi\left(\frac{\hat{x}_{0,y+1}}{k}\right) - \pi\left(\frac{\hat{x}_{0,y+1}}{k+1}\right)\right].$$

(*)

Combined with Lemma 7.5, these estimates on $\pi(x)$ lead to the following approximations that hold on the increasing sequences taken within the subintervals defined by $\widehat{x}_0(y)$:

??

I have no idea why this ≫ is true.

$$\begin{aligned}
|U_M(y)| &\gg \sum_{k=1}^{\frac{\hat{x}_{0,y+1}}{2}-1} |G^{-1}(k)| \left[\frac{\hat{x}_{0,y+1}}{k \cdot \log\left(\frac{\hat{x}_{0,y+1}}{k}\right)} \left(1 + \frac{1}{2 \cdot \log\left(\frac{\hat{x}_{0,y+1}}{k}\right)}\right) \right. \\
&\qquad \left. - \frac{\hat{x}_{0,y+1}}{(k+1) \cdot \log\left(\frac{\hat{x}_{0,y+1}}{k+1}\right)} \left(1 + \frac{3}{2 \cdot \log\left(\frac{\hat{x}_{0,y+1}}{k+1}\right)}\right)\right] \\
&\gg \sum_{k=\sqrt{\hat{x}_{0,y+1}}}^{\frac{\hat{x}_{0,y+1}}{2}-1} \frac{\hat{x}_{0,y+1} \cdot |G^{-1}_{\min}(x_y)|}{k^2} \left[\frac{1}{\log(\hat{x}_{0,y+1})} + \frac{1}{2\log^2(\hat{x}_{0,y+1})}\right] \\
&\gg \hat{x}_{0,y+1} \times |G^{-1}_{\min}(x_y)| \left(\frac{1}{\log(\hat{x}_{0,y+1})} + \frac{1}{2\log^2(\hat{x}_{0,y+1})}\right) \times \left|\int_{\sqrt{\hat{x}_{0,y+1}}}^{\frac{\hat{x}_{0,y+1}}{2}} \frac{dt}{t^2}\right| \\
&\gg \sqrt{\hat{x}_{0,y+1}} \times \frac{|G^{-1}_{\min}(x_y)|}{\log(\hat{x}_{0,y+1})} + o(1), \text{ as } y \to \infty.
\end{aligned}$$

− ??

We clearly see from Theorem 7.3 and Proposition 7.4 that

$$\begin{aligned}
\frac{|M(\hat{x}_{0,y+1})|}{\sqrt{\hat{x}_{0,y+1}}} &\gg \frac{1}{\sqrt{\hat{x}_{0,y+1}}} \times \left|\left|G^{-1}(\hat{x}_{0,y+1}) + G^{-1}\left(\frac{\hat{x}_{0,y+1}}{2}\right)\right| + |U_M(y)|\right| \\
&\gg \frac{1}{\sqrt{\hat{x}_{0,y+1}}} \times |U_M(y)| \\
&\gg \log\log\left(\sqrt{\hat{x}_{0,y+1}}\right)^{\frac{1}{2}}. \qquad (39)
\end{aligned}$$

(**)

everything is the same sign?

There is a small, but nonetheless insightful point in question to explain about a technicality in stating (39). Namely, we are not asserting that $|M(x)|/\sqrt{x}$ grows unbounded along the precise subsequence of $x \mapsto \hat{x}_{0,y+1}$ itself as $y \to \infty$. Rather, we are asserting that the unboundedness of this function can be witnessed along some subsequence whose points are taken within a large interval window of $x \in \mathbb{X}_y$ as $y \to \infty$. We choose to state the lower bound given on the right-hand-side of (39) using the lower bound on $|G^{-1}(x)|$ we proved in Theorem 7.3 minimally with $\hat{x}_0(y) \geq \sqrt{\hat{x}_{0,y+1}}$ on the interval for all $y \geq Y_0$. It is also necessary that $\log\log(x_{0,y+1}) > \log\log(x_{0,y})$ for all sufficiently large y so that we indeed to obtain an increasing infinite subsequence along which to show the unboundedness of (39). $\square$

Why is this true.

I dont understand.

In order for your defn of $\mathcal{U}_M(y)$ to be useful, you need a statement that G^{-1} doesn't change signs.

You clearly believe that, but where is it proved.

I dont understand these lines. And I am concerned that there is some deep problem. In place of the masked $\gg$, I could just as easily argue that

$$\gg \sum_{k=(\hat{x}_{0,y+1})^{1/4}}^{\frac{\hat{x}_{0,y+1}}{2}-1} \ldots \text{(all the same stuff)}$$

$$\gg x_{0,y+1}^{3/4} \times \text{logs}$$

But this would lead to a false conclusion.

What you should do:

After Prop 7.4 state a Lemma:

If x_0 is an integer which meets these conditions

(A)
(B)
(C)

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ whatever you need about $\hat{x}$ & G^{-1}

Then $M(x_0)/\sqrt{x_0} > \log x_0$

That will make the strategy clear. And a guide to readers