Theorem 1.2

June 21, 2022

1 The estimate

Here I give a proof of the following estimate (Theorem 1.2 from paper draft):

$$\sum_{n \le x} \log C_\Omega(n) \;=\; x(\log\log x)(\log\log\log x)(1 + o(1)), \tag{1}$$

where

$$C_\Omega(n) \;=\; \begin{cases} 1, & \text{if } n = 1; \\ \Omega(n)! \prod_{p^a || n} \frac{1}{a!}, & \text{if } n \ge 2. \end{cases}$$

2 Proof

We have that

$$\sum_{n \le x} \log C_\Omega(n) \;=\; \sum_{k \ge 1} \sum_{\substack{n \le x \\ \Omega(n) = k}} \log C_\Omega(n) \;=\; \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where Σ_1 is the contribution of those n where

$$|\Omega(n) \;-\; \log\log x| \;\le\; (\log\log x)^{2/3}, \tag{2}$$

while Σ_2 is the contribution of the n's with

$$\Omega(n) \;<\; \log\log x - (\log\log x)^{2/3}, \tag{3}$$

and where Σ_3 is the contribution of the n's with

$$\Omega(n) \;>\; \log\log x + (\log\log x)^{2/3}. \tag{4}$$

Furthermore, we let $\mathcal{S}_1$ denote the set of $n \le x$ that satisfy (2); we let $\mathcal{S}_2$ denote those that satisfy (3); and we let $\mathcal{S}_3$ denote those that satisfy (4).

We will show that

$$\Sigma_1 \;\sim\; x(\log\log x)(\log\log\log x),$$

while

$$\Sigma_2, \Sigma_3 \;=\; o(x \log\log x \log\log\log x),$$

which altogether will imply (1).

2.1 Estimating Σ_1

We further subdivide $\Sigma_1 = \Sigma_1' + \Sigma_1''$, where Σ_1' is the contribution to Σ_1 of all those $n \in \mathcal{S}_1$ that additionally satisfy

$$\prod_{p^a || n} a! \ > \ R,$$

where R will be determined later. Taking logs, we are saying here that

$$\log R \ < \ \sum_{p^a || n} \log(a!) \ = \ \sum_{\substack{p^a || n \\ a \geq 2}} \log(a!). \qquad (5)$$

The sum Σ_1'' gives the contribution of the remaining $n \in \mathcal{S}_1$.

We will show that for an appropriate choice of $R = R(x)$, we will have that

$$\Sigma_1' \ = \ o(x \log\log x \log\log\log x), \ \Sigma_1'' \ \sim \ x(\log\log x)(\log\log\log x),$$

from which it would follow that

$$\Sigma_1 \ = \ \Sigma_1' + \Sigma_1'' \ \sim \ x(\log\log x)(\log\log\log x).$$

It will be convenient to let $\mathcal{S}_1'$ denote the $n \in \mathcal{S}_1$ contributing to Σ_1', and then letting $\mathcal{S}_1''$ denote the remaining $n \in \mathcal{S}_1$.

2.1.1 Bounding $\mathcal{S}_1'$ and Σ_1' from above

We now bound $\mathcal{S}_1'$ from above. We will do this by writing

$$\mathcal{S}_1' \ = \ \mathcal{S}_{1,1}' \ \cup \ \mathcal{S}_{1,2}',$$

where $\mathcal{S}_{1,1}'$ is the set of all those n satisfying (5) with the property that at least K (K determined later) of the prime divisors p have the property that p^2 also divides n. The set $\mathcal{S}_{1,2}'$ is the set of all n satisfying (5) with the property that fewer than K of the prime divisors p have the property that p^2 also divides n.

An $n \in \mathcal{S}_{1,1}'$ has a square divisor of size at least

$$p_1^2 p_2^2 \cdots p_K^2,$$

where p_j denotes the jth prime number. By the Prime Number Theorem we have that

$$p_1 p_2 \cdots p_K \ > \ e^{p_K(1-o(1))} \ > \ e^{K(\log K)(1-o(1))}.$$

So, in this case we would have that n has a square divisor

$$d^2 \ > \ K^{(2-o(1))K}.$$

And so,

$$
\begin{aligned}
|\mathcal{S}'_{1,1}| \ \le \ \sum_{\substack{n\in\mathcal{S}_1 \\ \exists d^2|n,\ d>K^{(1-o(1))K}}} 1 \ &\le \ \sum_{d>K^{(1-o(1))K}} \#\{n \le x \ : \ d^2|n\} \\
&\ll \ x \sum_{d>K^{(1-o(1))K}} \frac{1}{d^2} \\
&\ll \ \frac{x}{K^{(1-o(1))K}}.
\end{aligned}
$$

If we take $K \to \infty$ with x, then

$$|\mathcal{S}'_{1,1}| \ = \ o(x).$$

Next, we consider the contribution of those $n \in \mathcal{S}'_{1,2}$. Since we are assuming that n satisfies (5), we have that

$$\max_{p^a\|n,\ a\ge 2} \log(a!) \ \ge \ \frac{1}{\#\{p^a\|n,\ a\ge 2\}} \sum_{\substack{p^a\|n \\ a\ge 2}} \log(a!) \ \ge \ \frac{\log R}{K}.$$

Since $a! \le a^a$, one can see that this implies

$$\max_{p^a\|n,\ a\ge 2} a \ \ge \ \frac{\log R}{K\log\log R}.$$

Since every prime $p \ge 2$, this implies that n is divisible by a prime power p^a, $a \ge 2$, satisfying

$$p^a \ \ge \ 2^{(\log R)/(K\log\log R)}.$$

Thus, as before, we get that

$$|\mathcal{S}'_{1,2}| \ < \ \sum_{\substack{n\in\mathcal{S}_1 \\ \exists d^2|n,\ d>2^{(\log R)/(2K\log\log R)}}} 1 \ \ll \ \frac{x}{2^{(\log R)/(2K\log\log R)}}$$

Since we get to choose R and K, we choose both of them to tend to infinity slowly with x, but also choose them so that $(\log R)/(K \log\log R) \to \infty$, as well. Thus,

$$|\mathcal{S}'_{1,2}| \ = \ o(x). \tag{6}$$

We thus have that

$$|\mathcal{S}'_1| \ = \ |\mathcal{S}'_{1,1}| + |\mathcal{S}'_{1,2}| \ = \ o(x) + o(x) \ = \ o(x).$$

For this we deduce that

$$\Sigma'_1 \ < \ \sum_{n\in\mathcal{S}'_1} \log C_\Omega(n) \ \le \ |\mathcal{S}'_1| \log([\log\log x + (\log\log x)^{2/3}]!) \ = \ o(x \log\log x \log\log\log x).$$

2.1.2 Estimating $\mathcal{S}_1''$ and Σ_1''

We have that

$$|\mathcal{S}_1''| \;=\; |\mathcal{S}_1| - |\mathcal{S}_1'| \;=\; x(1 - o(1)).$$

Now, each $n \in \mathcal{S}_1''$ will fail to satisfy (5), and thus

$$\log C_\Omega(n) \;=\; \log(\Omega(n)!) - E(n) \;=\; (\log\log n)(\log\log\log n)(1 + \delta(n)) - E(n),$$

where $|\delta(n)| = o(1)$ and where $|E(n)| \leq \log R$.

We can allow $K \to \infty$ slowly enough, so that we may choose an $R \to \infty$ fast enough, so that (6) holds, while also having $E(n) = o((\log\log n)(\log\log\log n))$; and thus,

$$\log C_\Omega(n) \;\sim\; (\log\log n)(\log\log\log n).$$

It then follows that

$$\Sigma_1'' \;\sim\; |S_1''|(\log\log x)(\log\log\log x) \;\sim\; x(\log\log x)(\log\log\log x).$$

2.2 Estimating Σ_2

Using the Erdős-Kac Theorem we know that for a randomly chosen $n \leq x$,

$$\mathrm{Prob}(|\Omega(n) - \log\log x| \geq (\log\log x)^{2/3}) \;=\; o(1).$$

From this it follows that

$$|\mathcal{S}_2| \;=\; o(x),$$

and therefore

$$\Sigma_2 \;\leq\; |\mathcal{S}_2| \log([\log\log x + (\log\log x)^{2/3}]!) \;=\; o(x \log\log x \log\log\log x).$$

2.3 Estimating Σ_3

We will split the sum as follows

$$\Sigma_3 \;=\; \Sigma_3' + \Sigma_3'',$$

where Σ_3' is the sum over those n with

$$\log\log x + (\log\log x)^{2/3} \;\leq\; \Omega(n) \;\leq\; 10\log\log x, \tag{7}$$

and where Σ_3'' is the sum over those n with

$$\Omega(n) \;>\; 10\log\log x. \tag{8}$$

Using similar naming convention we used before, we let $\mathcal{S}_3'$ denote the set of n's satisfying (7), and we let $\mathcal{S}_3''$ denote the set of n's satisfying (8).

2.3.1 Upper bound for Σ_3'

From Erdős-Kac we know that

$$|\mathcal{S}_3'| \;=\; o(x).$$

Thus, one quickly sees

$$\Sigma_3' \;\leq\; |\mathcal{S}_3'| \log([10 \log\log x]!) \;=\; o(x \log\log x \log\log\log x).$$

2.3.2 Upper bound for Σ_3''

To bound Σ_3'' from above we will need the following simple lemma.

Lemma. We have that

$$\#\{n \leq x \;:\; \Omega(n) \geq k \log\log x\} \;<\; x(\log x)^{k(1-\log k)+o(1)}.$$

Proof. We could prove this by quoting a high-powered theorem from the literature; however it is easy to prove just using Rankin's method. To this end, we note that

$$\begin{aligned}
\sum_{\substack{n \leq x \\ \Omega(n) \geq k \log\log x}} 1 \quad &\leq \quad \sum_{\ell \geq k \log\log x} \; \sum_{\substack{p_1 < p_2 < \cdots < p_\ell \leq x \\ p_i' s \text{ prime}}} \; \sum_{\substack{a_1, \ldots, a_\ell \geq 1 \\ a_1 + \cdots + a_\ell = \ell}} \frac{x}{p_1^{a_1} \cdots p_\ell^{a_\ell}} \\
&< \quad x \sum_{\ell \geq k \log\log x} \frac{1}{\ell!} \left(\sum_{p^a, \text{ p prime}} \frac{1}{p^a} \right)^{\ell} \\
&< \quad x \sum_{\ell \geq k \log\log x} \left(\frac{e}{\ell} \right)^{\ell} (\log\log x + O(1))^{\ell} \\
&\ll \quad x \left(\frac{e + o(1)}{k} \right)^{k \log\log x} \quad = \quad x(\log x)^{k(1-\log k)+o(1)}.
\end{aligned}$$

■

From this it easily follows that

$$|\mathcal{S}_3''| \;<\; \frac{x}{(\log x)^2}.$$

So,

$$|\Sigma_3''| \;\leq\; |\mathcal{S}_3''| \log(x!) \;\ll\; \frac{x}{\log x}.$$

Thus, Σ_3'' contributes very little to our sum Σ, as we claimed; and so the theorem is proved.