It will take hundreds of hours of work yet to get this argument in shape to review.

You need a manuscript that

1. Only states & proves what is needed.

   Example:

4.2 Proving the crucial signedness property from the conjecture

Proposition 4.1 (The characteristic function of the primes). *Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the incompletely additive function that counts the number of distinct prime factors of n. Then we have the convolution identity given by*

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu.$$

The summatory function of the left-hand-side of the previous equation is clearly $\widetilde{G}(x) = \pi(x) + 1$ in the notation of Corollary 3.2 for all $x \geq 1$.

You don't need this

It is contained in

Lemma 6.1 (An exact formula for $g^{-1}(n)$). *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu(n/d)\lambda(d)C_{\Omega(d)}(d).$$

2. Proofs are correct. Typo-free. And no harder than they have to be.

3. Keep editorial/side remarks to a minimum. Examples

(a)

Proof of Theorem 3.6. We showed how to compute the formulas for the base cases in the preceeding examples discussed above in Example 6.2. We can also see that $C_1(n)$ satsfies the formula we must establish when $k := 1$. Let's proceed by using induction to prove that our asymptotics hold for all $k \geq 1$ using the recurrence formula from (12) relating $C_k(n)$ to $C_{k-1}(n)$ whenever $k \geq 2$. In particular, suppose that $k \geq 2$ and let the inductive assumption for all $1 \leq m < k$ be that

$$\mathbb{E}[C_m(n)] = (\log \log n)^{2m-1}.$$

Should be: We induct on m.

The base case is ....

(b)

where we can drop the asymptotically unnecessary floored integer-valued arguments to $\pi(x)$ in place of its approximation by $\pi(x) \sim \frac{x}{\log x}$. In fact, since we can always bound

$$\frac{Ax}{\log x} \leq \pi(x) \leq \frac{Bx}{\log x},$$

for suitably defined absolute constants, $A, B > 0$, we are not losing any precision asymptotically by making this small leap in approximation from exact summation (by the first formula) to the integral formula case approximating $M(x)$ established below.

This sentence is hard to understand.

If sentence takes up >3 lines it is too long.

(c) Another confusing bit

Now we need to determine which values of u minimize the expression for the function defined in (10). For this we will use a somewhat weak elementary method from introdutory calculus in the form of the second derivative test with respect to u that immediately discards most of the dependence of (10) on x as we apply it. In particular, we can symbolically invoke the equation solver functionality in *Mathematica* to see that

$$\frac{\partial}{\partial u}\left[\widehat{\mathcal{C}}(u, x; z)\right]\Bigg|_{u \mapsto u_0} = 0 \implies u_0 \in \left\{\frac{1}{x}, \frac{1}{x} e^{-\frac{4}{3}(z-1)}\right\}.$$

When we substitute this outstanding parameter value of $u_0 =: \hat{u}_0 \mapsto \frac{1}{x} e^{-\frac{4}{3}(z-1)}$ into the next expression for the second derivative of the same function $\widehat{\mathcal{C}}(u, x; z)$ we obtain

(d)

Theorem 3.6 (Asymptotics for the functions $C_k(n)$). *For $k := 0$, we have by definition that $C_0(n) = \delta_{n,1}$. For all $k \geq 1$, we obtain that the dominant asymptotic term for $C_k(n)$ is given by*

$$\mathbb{E}[C_k(n)] = (\log\log n)^{2k-1}, \text{ as } n \to \infty.$$

Since we have that

$$(g^{-1} * 1)(n) = \lambda(n) \cdot C_{\Omega(n)}(n), \forall n \geq 1, \tag{2}$$

Möbius inversion provides us with an exact divisor sum based expression for $g^{-1}(n)$ (see Lemma 6.1). Then we can prove (see Corollary 6.5) that we can obtain lower bounds on the magnitude of $g^{-1}(n)$ by approximating it by the simpler divisor sums

$$\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

Notice that this formula is substantially easier to evaluate than the corresponding sums in (2) given directly through Möbius inversion. Hence, we prefer to work with bounds on it that we prove as new results instead rather than with results relying on the more complicated exact formula from the cited equation above.

Specifically, the last result in turn implies that

$$|G^{-1}(x)| \overset{\blacktriangle}{\gtrsim} \left| \sum_{n \leq x} \lambda(n) \cdot C_{\Omega(n)}(n) \times \sum_{d=1}^{\left\lfloor \frac{x}{n} \right\rfloor} \lambda(d) \right|. \tag{3}$$

In light of the fact that (by an integral-based interpretation of integer convolution using summation by parts, see Proposition 7.1)

$$M(x) \sim G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)},$$

the formula in (3) implies that we can establish new *lower bounds* on $M(x)$ by appropriate estimates of the summatory function $G^{-1}(x)$ where trivially we have the bounded inner sums $L_0(x) := \sum_{n \leq x} \lambda(n) \in [-x, x]$ for all $x \geq 2$. As explicit lower bounds for $M(x)$ along particular subsequences are not obvious, and are historically ellusive non-trivial features of the function to obtain as we expect sign changes of this function infinitely often, we find this approach to be an effective one.

What part of this has to be said? Just say it. Thus, and their proofs

3.4 Enumerative (or counting based) DGFs from Mongomery and Vaughan

(e)

Thm 3.1, Cor. 3.2, Cor 3.3

can't all be key!

B.B Key results and constructions:
- Theorem 3.1
- Corollary 3.2
- Corollary 3.3
- Conjecture 3.4 (to a lesser expository only extent)
- Proposition 4.1
- Proposition 4.2

B.2 Asymptotics for the component functions $g^{-1}(n)$ and $G^{-1}(x)$:

④ Only state Lemmas / Thms that have to be stated.

Any extra item is distracting.

Any extra argument is a distraction. Use standard language. Use new language very sparingly.

Things to take care with

You are trying to prove a very refined sq root law.

- Average case estimates are most likely too weak.

- Consider $C_k(n)$. You argue that

$$(*) \quad \mathbb{E}\, C_k(n) \sim (\log\log n)^{2k-1}$$

(I did not find your argument convincing.)

- But you want to use this calculation in

$$\sum_{n \le X} \lambda(n) \sum_{d \mid n} C_{\Omega(d)}(d)$$

Notice that the result $(*)$ does NOT apply, because of the $\Omega(d)$.

Namely in $(*)$ you need error estimates for the '$\sim$'.

But $C_k(n)$ can take much larger

values than avg case. So the error estimates may not help.

— The only theorem possible is a **very small** improvement over known $\sqrt{x}$, order of $\log\log x$. You can lose the gain in ANY error term along the way.

If the adjustment is not a constant multiplier, it has to be analyzed very carefully.

Any "exchange of limits" argument requires very careful justification.

Keep in mind that your claim here is exceptional. Imagine a skeptical reader. What part would they find hardest to believe?