SIGN SMOOTHING CONVOLUTIONS OF THE DIRICHLET INVERSES OF ARITHMETIC FUNCTIONS

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Abstract. Sign changes in sums of arithmetic functions and their inverses are a subtle topic with room to grow new results. Suppose that $S_f(x) := \sum_{n\leq x} f(n)$ is the summatory function of some arithmetic function f such that $f(1) \neq 1$. There are known lower bounds on the limiting growth of $V(S_f, Y)$ – the number of sign changes of $S_f(y)$ on the interval $y \in (0, Y]$ as $Y \to \infty$. We observe a partition theoretic smoothing discrete convolution of the local oscillatory properties of sums of the Dirichlet inverse of f, $S_{f^{-1}}(x)$, which leads to a sequence of convolution sums which are eventually constant in sign. We investigate exponential function sign smoothing convolutions and then generalize our results to prove more optimal sign smoothing weight functions for any fixed Dirichlet invertible f. We give applications a plenty of these sign smoothing convolution sums.

1. Introduction

1.1. **Dirichlet convolutions and Dirichlet inverse functions.** The sign changes of an arithmetic function f are often considered in applications where we must estimate the growth of sums depending on f. For any fixed f, we define its summatory function for all positive integers $x \geq 1$ by

$$S_f(x) := \sum_{n\leq x} f(n).$$

Given any two arithmetic functions f and g, we define their *Dirichlet convolution*, $f * g$, to be the divisor sum

$$(f * g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right), \forall n \geq 1.$$

The multiplicative inverse with respect to Dirichlet convolution is defined by $\varepsilon(n) \equiv \delta_{n,1}$ so that $f * \varepsilon = \varepsilon * f = f$ for any arithmetic f. If $f(1) \neq 1$, then it is Dirichlet invertible. That is, there is another arithmetic function $f^{-1}(n)$ such that $f * f^{-1} = f^{-1} * f = \varepsilon$. Moreover, the function f^{-1} is unique when it exists and satisfies the recursive formula

$$f^{-1}(n) = \begin{cases} \frac{1}{f(1)}, & \text{if } n = 1; \\ -\frac{1}{f(1)} \times \sum\limits_{\substack{d|n \\ d>1}} f(d) f^{-1}\left(\frac{n}{d}\right), & \text{if } n \geq 2. \end{cases}$$

We find that the signedness of f^{-1} is dictated, or prescribed, by the local sign change patterns of f. Because such locally unpredictable signage is crucial to the understanding of many classical problems and

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applications, we state the next proposition to clarify the situation for special classes of nicely behaved invertible arithmetic functions $f \geq 0$.

Proposition 1.1 (The Sequence of Signs of the Dirichlet Inverse)**.** *Suppose that $f(1) := c_f \neq 0$ and that $f(n) \geq 0$ for all $n \geq 2$. Then*

1. *If f is completely multiplicative then $\operatorname{sgn}(f^{-1}(n)) = \mu(n)$;*
2. *If f is multiplicative and $c_f \geq 1$, then $\operatorname{sgn}(f^{-1}(n)) = \lambda_*(n) = (-1)^{\omega(n)}$;*
3. *If f is additive and $c_f \geq 1$, then $\operatorname{sgn}((f+1)^{-1}(n)) = \lambda(n) = (-1)^{\Omega(n)}$.*

In the previous equations we denote by $\omega(n) := \sum_{p|n} 1$ and $\Omega(n) := \sum_{p^\alpha || n} \alpha$ the strongly and completely additive functions that count the number of distinct prime factors of n with and without counting multiplicity, respectively.

Other formulas for f^{-1} when f is any Dirichlet invertible arithmetic function provide limited insight into distributions of the signs of the inverse function over $n \geq 1$. A partition theoretic motivation for expressing the Dirichlet inverse of any f such that $f(1) \neq 0$ provides that for $n > 1$:

$$f^{-1}(n) = \sum_{k=1}^{\Omega(n)} (-1)^k \left\{ \sum_{\substack{\lambda_1 + 2\lambda_2 + \cdots + k\lambda_k = n \\ \lambda_1, \lambda_2, \ldots, \lambda_k | n}} \frac{(\lambda_1 + \lambda_2 + \cdots + \lambda_k)!}{1!2! \cdots k!} f(\lambda_1) f(\lambda_2)^2 \cdots f(\lambda_k)^k \right\}. \tag{1}$$

Let the m-fold convolution of an arithmetic function g with itself (i.e., convolve g with itself m times in a row at n) be denoted by $[g]_{*_m}$. Then Mousavi and Schmidt proved that [5]

$$f^{-1}(n) = \frac{\varepsilon(n)}{f(1)} + \sum_{j=0}^{\left\lfloor \frac{\Omega(n)}{2} \right\rfloor} \left([f - f(1)\cdot\varepsilon]_{*_{2j+1}}(n) - f(1) \times [f - f(1)\cdot\varepsilon]_{*_{2j}}(n) \right) \frac{1}{f(1)^{2j+1}}. \tag{2}$$

Since we can expand m-fold convolutions of a sum of k arithmetic functions using the multinomial (with k terms at m) theorem as

$$[f_1 + f_2 + \cdots + f_k]_{*_m} = \sum_{\substack{i_1 + i_2 + \cdots + i_k = m \\ i_1, i_2, \ldots, i_k \geq 0}} \binom{m}{i_1, i_2, \ldots, i_k} [f_1]_{*_{i_1}} [f_2]_{*_{i_2}} \cdots [f_k]_{*_{i_k}},$$

this last formula for f^{-1} leads to another set of insights we can apply in expressing the sign of $f^{-1}(n)$.

1.2. **Local sign changes of an arithmetic function.** The *Dirichlet generating function* (or DGF) of an arithmetic function f is defined as follows for all $s := \sigma + \imath t \in \mathbb{C}$ such that the following sequence converges:

$$D_f(s) := \sum_{n \geq 1} \frac{f(n)}{n^s} = \sum_{n \geq 1} \frac{f(n) \left[\cos(t \log n) + \imath \sin(t \log n)\right]}{n^\sigma}, \operatorname{Re}(s) > \sigma_{c,f}.$$

Provided that f is multiplicative, we have the *Euler product* representation of the DGF of f given by the prime-indexed product

$$D_f(s) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{r=1}^{\infty} \frac{f(p^r)}{p^{rs}} \right), \operatorname{Re}(s) > \sigma_{c,f}.$$

The DGF of f is related to the Mellin transform of its summatory function $S_f(x) := \sum_{n \leq x} f(n)$ by

$$D_f(s) = s \cdot \int_1^{\infty} \frac{S_f(x)}{x^{s+1}} dx, \operatorname{Re}(s) > \sigma_{a,f},$$

where $\sigma_{a,f}$ is the abscissa of absolute convergence of the DGF $D_f(s)$. An inversion formula for recovering the coefficients $f(n)$ of n^{-s} from a given DGF is stated as follows [1, §11.11]:

$$f(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^{\sigma + \imath t} D_f(\sigma + \imath t) dt, \forall x \in \mathbb{Z}^+; \forall \sigma > \sigma_{a,f}.$$

One heuristic that recurs in applications is that if an invertible $f \geq 0$ is non-negative for all $n \geq 1$, then the corresponding sequence of sign changes for its inverse f^{-1} is oscillatory and typically hard to predict. The same is true for invertible non-negative integer matrices: the corresponding inverse matrices as a general measure tend to display semi-random, highly oscillatory, and variably signed behavior.

We have several existing results that characterize the expected number of sign changes of well enough behaved f on increasingly large intervals of consecutive integers. Let $V(f, Y)$ denote the number of sign changes of f on the interval $(0, Y]$ for real $Y > 0$:

$$V(f, Y) := \sup \left\{N : \exists \{x_i\}_{i=1}^N, 0 < x_1 < \cdots < x_N \leq Y, f(x_i) \neq 0, \operatorname{sgn}(f(x_i)) \neq \operatorname{sgn}(f(x_{i+1})), \forall 1 \leq i < N\right\}.$$

It is nown that the analytic properties, poles, and zeros of the DGF $D_f(s)$ of f provide key insights into the sign changes of these functions [3]. For example, if the DGF of f is analytic on some half-plane, subject to certain restrictions, then Landau showed in 1905 that the summatory function of f, $S_f(x)$, changes signs infinitely often as we let x tend to infinity. We also have the next theorem that extends Landau's and which provides a more precise minimal statement concerning the frequency of the sign changes of $S_f(x)$.

Theorem 1.2 (Pólya)**.** *Suppose that $S_f(x)$ is real-valued for all $x \geq x_0$, and define the function $\hat{F}_f(s)$ by the Mellin transform at $-s$ as*

$$\hat{F}_f(s) := \int_{x_0}^{\infty} \frac{S_f(x)}{x^{s+1}} dx.$$

Suppose that $\hat{F}_f(s)$ is analytic for all $\operatorname{Re}(s) > \theta$*, but is not analytic in any half-plane* $\operatorname{Re}(s) > \theta - \varepsilon$ *for* $\varepsilon > 0$*. Furthermore, suppose that $\hat{F}_f(s)$ is meromorphic in some half-plane* $\operatorname{Re}(s) > \theta - c_0$ *for some $c_0 > 0$. Let*

$$\gamma_f := \begin{cases} \inf\{|t| : \hat{F}_f(s) \textit{ is not analytic at } s = \theta + \imath t\}, & \text{if } f \text{ is not analytic at } \operatorname{Re}(s) = \theta; \\ \infty, & \text{otherwise}. \end{cases}.$$

Then

$$\limsup_{Y \to \infty} \left\{\frac{V(S_f, Y)}{\log Y}\right\} \geq \frac{\gamma_f}{\pi}.$$

1.3. **Motivation for a new approach to determining the signs of $f^{-1}(n)$ and $S_f(x)$.** We have experimentally observed an interesting new trend of so-called *sign smoothing transformations* of certain signed integer sequences under invertible discrete convolution based encodings by special partition functions. Let the partition functions $p_1(n)$ and $p_2(n)$ be defined for integers $n \geq 0$ as the coefficients of the following generating functions:

$$\begin{aligned} p_1(n) &:= [q^n] \prod_{m \geq 1} (1 + q^m) && (3) \\ &= [q^n] \left(1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + 8q^9 + 10q^{10} + 12q^{11} + 15q^{12} + \cdots\right) \end{aligned}$$

$$\begin{aligned} p_2(n) &:= [q^n] \prod_{m\geq 1} (1+q^m)^{-1} \\ &= [q^n]\left(1 - q - q^3 + q^4 - q^5 + q^6 - q^7 + 2q^8 - 2q^9 + 2q^{10} - 2q^{11} + 3q^{12} + \cdots\right) \end{aligned}$$

By convention, we denote $\widehat{p}_2(n) = |p_2(n)| = (-1)^n p_2(n)$. We can apply the circle method to the q-series generating functions of these sequences to obtain limiting asymptotic formulas for the partition numbers of the form the following limiting asymptotics for these two functions [**?**]:

$$\begin{aligned} p_1(n) &\sim \frac{3^{3/4}}{12 \cdot n^{3/4}} \exp\left(\pi \sqrt{\frac{n}{3}}\right) \\ \widehat{p}_2(n) &\sim \frac{1}{2 \cdot 24^{1/4} n^{3/4}} \exp\left(\pi \sqrt{\frac{n}{6}}\right). \end{aligned} \tag{4}$$

We define two invertible transformations, or encodings, that are respective inverses of one another on any fixed arithmetic f as the discrete convolution sums given by

$$\begin{aligned} s_1[f](n) &:= \sum_{j=1}^{n} f(j) p_1(n-j) \\ s_2[f](n) &:= \sum_{j=1}^{n} f(j) p_2(n-j). \end{aligned} \tag{5}$$

We say that a arithmetic sequence $\{f(n)\}_{n\geq 1}$ has *property* $\mathcal{P}_1$ at N if the sign of $s_1[f](n)$ is constant for all $n \geq N$. Likewise, we say that the sequence has *property* $\mathcal{P}_2$ at N if the sign of $s_2[f](n)$ alternates for all $n \geq N$. We define

$$\begin{aligned} M_{f,1} &:= \sup\left\{n \geq 1 : f \text{ does not have property } \mathcal{P}_1 \text{ at } n\right\} \\ M_{f,2} &:= \sup\left\{n \geq 1 : f \text{ does not have property } \mathcal{P}_2 \text{ at } n\right\}. \end{aligned}$$

The characteristic limiting behavior we observe in the encoding transformations in (5) is typified by the next conjectured properties based on empirical observation and computational data sets.

Conjecture 1.3 (Sign Smoothing Convolution Operators)**.** *For any arithmetic function f which is non-vanishing on the positive integers, both $M_{f,1}$ and $M_{f,2}$ are finite. Moreover, provided that $f(n) \ll p_1(n)$, we have $\|s_1[f](n)\| \rightarrow +\infty$ as $n \rightarrow \infty$. If $f(n) \ll p_2(n)$, then*

$$\|s_2[f](n)\| \xrightarrow{n\rightarrow\infty} +\infty.$$

The truth of Conjecture 1.3 implies that if an arithmetic function f satisfies $\mathcal{P}_1$ at some finite $N \geq 1$, then for all sufficiently large $n \geq N$, the signed magnitude of the real part of the transformation $s_1[f](n)$ tends to one side of the real line or the other. A similar obervation is made for arithmetic functions f satisfying property $\mathcal{P}_2$ at some finite $N \in \mathbb{N}$.

1.4. **Generalizations.** Given the essentially exponential nature of the limiting asymptotics in (4), we can reconcile the expected behavior from Conjecture 1.3 with a more general phenomenon that characterizes a large class of more general *exponential sign smoothing operators*. Let $\zeta_m := \exp(2\pi\imath/m)$ denote the primitive m^{th} root of unity. We define the next two classes of invertible transformations in terms of the real parameters m, k for any integers $n \geq 1$:

$$s_{m,k}[f](n) := \sum_{j=1}^{n} f(j)\zeta_m^{n-j} \exp\left(\pi\sqrt{k(n-j)}\right), \forall n \geq 1; m \in \mathbb{Z}^+; k \in (0,\infty); \tag{6}$$

$$t_{m,k}[f](n) := \sum_{j|n} f(j)\zeta_m^{n-j} \exp\left(\pi\sqrt{k(n-j)}\right), \forall n \geq 1; m \in \mathbb{Z}^+; k \in (0,\infty).$$

We say that a arithmetic sequence $\{f(n)\}_{n\geq 1}$ has *property* $\mathcal{P}_{1,m,k}$ at N if the sign of $\operatorname{Re}\{s_{m,k}[f](n) \cdot \zeta_m^{-n}\}$ is constant for all $n \geq N$. Similarly, we say that f has *property* $\mathcal{P}_{2,m,k}$ at N if the sign of $\operatorname{Im}\{s_{m,k}[f](n) \cdot \zeta_m^{-n}\}$ is constant for all $n \geq N$. We define for $i := 1, 2$

$$M_{i,m,k}(f) := \sup\{n \geq 1 : f \text{ does not have property } \mathcal{P}_{i,m,k} \text{ at } n\}.$$

The above definition similarly shows that if an arithmetic function f satisfies $\mathcal{P}_{m,k}$ at some finite $N \geq 1$, then for all sufficiently large $n \geq N$, the signed magnitude of the real part of the transformation $s_{m,k}[f](n)$ tends to one side of the real line or the other. The definition does not provide the limiting signage of the transformation sequence even if the function f satisfies property $\mathcal{P}_{m,k}$. We prove the following three main theorems in the next section of the article.

Theorem 1.4 (A Sign Smoothing Convolution Operator by Exponential Function Scaling)**.** *For any Dirichlet invertible arithmetic function f which is non-vanishing on the positive integer, any $m \in \mathbb{Z}^+$, and any $k \in (0,\infty)$ such that $f(n) \ll \exp(\pi\sqrt{kn})$, $M_{1,m,k}(f^{-1})$ and $M_{2,m,k}(f^{-1})$ are finite. Moreover,*

$$\lim_{n\to\infty} \left|\operatorname{Re}\left\{\frac{s_{m,k}[f^{-1}](n)}{\zeta_m^n}\right\}\right| = +\infty, \quad \textit{and} \quad \lim_{n\to\infty} \left|\operatorname{Im}\left\{\frac{s_{m,k}[f^{-1}](n)}{\zeta_m^n}\right\}\right| = +\infty$$

Theorem 1.5. *For any Dirichlet invertible f which is non-vanishing on the positive integers, any $m \in \mathbb{Z}^+$, and any $k \in (0,\infty)$, we have that*

$$M_{1,m,k}(f^{-1}) = TODO_{1,m,k}; \tag{A}$$

$$M_{2,m,k}(f^{-1}) = TODO_{2,m,k}; \tag{B}$$

$$\limsup_{n\to\infty}\left(\operatorname{sgn}\left\{\operatorname{Re}\left[s_{m,k}[f^{-1}](n)\cdot\zeta_m^{-n}\right]\right\}\right) = TODO_{m,k}; \tag{C}$$

$$\limsup_{n\to\infty}\left(\operatorname{sgn}\left\{\operatorname{Im}\left[s_{m,k}[f^{-1}](n)\cdot\zeta_m^{-n}\right]\right\}\right) = TODO_{m,k}. \tag{D}$$

Theorem 1.6 (A Sign Smoothing Convolution Operator by Exponential Function Scaling)**.** *For any Dirichlet invertible arithmetic function f which is non-vanishing on the positive integers, any $m \in \mathbb{Z}^+$, and any $k \in (0,\infty)$ such that $f(n) \ll \exp(\pi\sqrt{kn})$, the sign of $\operatorname{Re}\left[t_{m,k}[f^{-1}](n)\cdot\zeta_m^{-n}\right]$ is eventually constant. That is, there exists a finite $N_f \geq 1$ such that for all $n > N_f$,*

$$\operatorname{sgn}\left\{\operatorname{Re}\left[t_{m,k}[f^{-1}](n)\zeta_m^{-n}\right]\right\} - \operatorname{sgn}\left\{\operatorname{Re}\left[t_{m,k}[f^{-1}](n-1)\zeta_m^{1-n}\right]\right\} = 0.$$

2. Proofs of the theorems and key consequences

2.1. **Proofs of the main theorems stated in the introduction.**

2.2. **Some immediate corollaries.** We can relate the sequence of $s_{1,k}[f](x)$ to the summatory function $S_f(x)$ of f closely within some bounded error term. In particular, it is not difficult to prove that

$$S_f(x) = \frac{s_{1,k}[f](x)}{\exp(\pi\sqrt{kx})} + O\left(\sum_{j=1}^{x-1} S_f(j) e^{-\pi\sqrt{kx}}\right).$$

So provided that the growth rates of $S_f(x)$ are sub-polynomial (significantly sub-exponential), we have a good approximation to $S_f(x)$ when $x \gg 1$ is large. This leads us to the following corollary:

Corollary 2.1 (A Sign Bias for General Summatory Functions)**.** *For limiting large $x \gg 1$, the summatory function $S_f(x)$ has a sign bias towards*

$$\limsup_{x\to\infty} \left(\operatorname{sgn}\left\{s_{1,k}[f](x)\right\}\right).$$

Proof. **TODO ...** This is easy given the eventually constant sign theorems ... □

2.3. **Local optimality of the convolution weight functions.**

Remark 2.2 (A Necessary Condition for Sign Smoothing Convolutions)**.** We call $s_f[E]$ a *sign smoothing convolution* if the sign of $s_f[E](x)$ is constant for all sufficiently large $x \gg 1$. Notice from the proof of the previous result that we have shown

$$\mathcal{M}[s_f[E]](-s) = \frac{(1-s)D_f(s)}{s}\Gamma(s)\sin(\pi s)\left[E(0) + \int_1^\infty \frac{E(x)-E(x-1)}{x^{s+1}}dx\right].$$

Suppose that this Mellin transformation is analytic for $\operatorname{Re}(s) > \theta_0$, but is not analytic in any larger half-plane $\operatorname{Re}(s) > \theta_0 - \varepsilon$ for $\varepsilon > 0$. If $s_f[E](x)$ *does not change sign* infinitely often as $x \to \infty$, then it must be the case that $\mathcal{M}[s_f[E]](-s)$ is not analytic at $s := \theta_0$ [3, *cf.* Landau's Theorem].

Proposition 2.3 (Growth of Sign Smoothing Convolutions)**.** *For any arithmetic function f and positive (strictly increasing) functions E on the non-negative reals, we define*

$$s_f[E](n) := \sum_{j=1}^{n} f(j)E(n-j), \forall n \in \mathbb{Z}^+.$$

We can assert conditions on the rate of growth of these sums given by the functions ϕ such that

$$s_f[E](x) = \sum_{k\geq 0} \phi(k)\frac{(-x)^k}{k!}.$$

In particular, we have that for any $k > \sigma_{a,f}$, these functions are given by

$$\phi(k) = \frac{(1-k)D_f(k)}{k}\Gamma(k)\sin(\pi k)\left[E(0) + \int_1^\infty \frac{E(x)-E(x-1)}{x^{k+1}}dx\right].$$

Proof. **TODO ...** Apply Ramanujan's Master Theorem for Mellin transforms ... □

3. Applications

3.1. **Improving bounds on sign changes of arithmetic functions on short intervals.**

3.2. **Partition theoretic applications (smoothness, or roundness, etc.)**

4. Generalizations of the sign smoothing convolutions

For any $\mathcal{A}_n \subseteq \{1, 2, \ldots, n\}$, and monotone increasing function $E : \mathbb{N} \to \mathbb{R}$ such that $f(n) \ll E(n)$, that

$$\begin{aligned}\sum_{j\in\mathcal{A}_n} f(j)E(n-j) &= \sum_{j\in\mathcal{A}_n} \left[1 - 2(V(f,j) - V(f,j-1))\right] |f|(j)E(n-j) \\ &= \sum_{j\in\mathcal{A}_n} |f|(j)E(n-j) - 2V(f,n)|f|(n)E(0) \\ &\quad + 2\sum_{\substack{j\in\mathcal{A}_n \\ j<n}} V(f,j)\left[|f|(j+1)E(n-1-j) - |f|(j)E(n-j)\right].\end{aligned}$$

The definitions of $\mathcal{A}_n$ corresponded to $\{1 \leq d \leq n : d|n\}$ for the sums $t_{m,k}[f](n)$ and to $\{1, 2, \ldots, n\}$ for the sums $s_{m,k}[f](n)$ we defined in (6) of the previous subsection.

We now arrive at a natural question to consider about sign smoothing convolutions with the previous eventually constant sign properties where the weight function (previously of exponentials to a square root power of the input) is in some sense "optimal" for the Dirichlet invertible function f and its inverse f^{-1}. That is, what is the most natural choice of the sign smoothing weight function we use in our convolved sums that results in

1. Optimal growth rates of the convolved sums (or otherwise nice properties to work with); and
2. The resulting sequence of convolved sums is constant in sign for the maximal number of natural numbers possible?

Consider the following questions in particular:

Question 4.1 (A Natural Sign Smoothing Function for f)**.** Given that $f(n) = O(U_f(n))$ and $V(f,Y) = O(V_f(Y))$ – and these upper bounds are *tight* in so much as $\forall\varepsilon, \delta > 0$, $\exists N, Y \geq 1$ such that $f(N)+\varepsilon > U_f(N)$ and $V(f,Y) + \delta > V_f(Y)$ – what is the optimal (minimal) choice of monotone increasing functions E_f, E_f^* such that the signs of

$$t_f[E^*](n) := \sum_{j|n} f(j)E_f^*(n/j), s_f[E](n) := \sum_{j=1}^{n} f(j)E_f(n-j),$$

are eventually constant? Can we choose the best possible functions E_f, E_f^* so that the signs of the corresponding sequences above are constant for all $n \geq N_f$ with $N_f \geq 1$ minimal over the natural numbers? That is, given some function space $\mathcal{E}_f$, what is

$$E_f := \operatorname{argmin}_{E\in\mathcal{E}_f}\left\{\min_{N\in\mathbb{Z}^+}\left[s_f[E](n+1) - s_f[E](n) = 0, \forall n \geq N\right]\right\},$$

and is this function unique?

4.1. Remarks about smoothing by partition functions with sub-exponential growth.

Why do the functions $E_k(n) := \exp(\pi\sqrt{kn})$ seem to do universally so well for all functions f we have tried (what is special about these forms)? Why do some partition function sequences work better than others (or not at all) with respect to preserving the property of sign smoothing of any arithmetic f?

5. Conclusions

5.1. Summary.

5.2. Open questions and possible generalizations.

5.3. Working topics list.
We also should consider the roles of the following properties:

- The role of local osciallations of an arithmetic function?
- Sign changes of $\nabla[f](n)$?

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