Lemma 4.3. *Suppose that for $x > e$ we define the following functions:*

$$\mathcal{N}_\omega(x) := \left| \sum_{k > \log\log x} (-1)^k \pi_k(x) \right|$$
$$\mathcal{D}_\omega(x) := \left| \sum_{k \leq \log\log x} (-1)^k \pi_k(x) \right|$$
$$\mathcal{A}_\omega(x) := \left| \sum_{k \geq 1} (-1)^k \pi_k(x) \right|.$$

As $x \to \infty$, we have that $\mathcal{D}_\omega(x)/\mathcal{N}_\omega(x) = o(1)$ and $\mathcal{A}_\omega(x) \sim \mathcal{D}_\omega(x)$.

With this lemma, we can accurately approximate asymptotic order of the sums $\mathcal{A}_\omega(x)$ for large x by only considering the truncated sums $\mathcal{D}_\omega(x)$ where we have the known uniform bounds on the summands for $1 \leq k \leq \log\log x$ by the results in Remark 2.5.

= ?? . Eq (10) ??

Proof. First, we sum the main term for the function $\mathcal{D}_\omega(x)$ by applying the limiting asymptotics for the incomplete gamma function derived in Lemma A.3 to obtain that

$$\mathcal{D}_\omega(x) = \left| \sum_{1 \leq k \leq \log\log x} \frac{(-1)^k \cdot x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \right| + O(E_\omega(x))$$
$$= \frac{x}{2\sqrt{2\pi \log\log x}} + O\left(\frac{x}{(\log\log x)^{3/2}} + E_\omega(x) \right),$$

$\tilde{\mathcal{G}}$??

The error term from the bound in the previous equation is defined according to (10) with $\widehat{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) \gg 1$ for all $1 \leq k \leq \log\log x$ as

$$E_\omega(x) := \sum_{k \leq \log\log x} \frac{x}{\log x} \cdot \frac{k(\log\log x)^{k-3}}{(k-1)!} \leq \frac{x}{\log x} \times \sum_{1 \leq k \leq \log\log x} \frac{(\log\log x)^{k-2}}{(k-1)!}$$
$$\leq \frac{x}{(\log x)(\log\log x)} e^{\log\log x} \leq \frac{x}{\log\log x}.$$

Next, we utilize the notation for and bounds on the function $D(x, r)$ from Remark 2.5 to bound the function $\mathcal{N}_\omega(x)$ as follows:

$$\frac{1}{x} \times |\mathcal{N}_\omega(x)| \leq \sum_{k \geq \log\log x} \frac{\pi_k(x)}{x} = \frac{1}{x} \times \sum_{k \geq \log\log x} \#\{2 \leq n \leq x : \omega(n) = k\} \ll 1.$$

Then we see that

$$\left| \frac{\mathcal{D}_\omega(x)}{\mathcal{N}_\omega(x)} \right| = O\left(\frac{1}{\sqrt{\log\log x}} \right) = o(1), \text{ as } x \to \infty.$$

This an upper bound on $\mathcal{N}_\omega$. You need a lower bound.

Equivalently, we have shown that $\mathcal{D}_\omega(x) = o(\mathcal{N}_\omega(x))$. The following results from the triangle inequality when x is large:

$$1 + o(1) = \left(\frac{\mathcal{D}_\omega(x) - \mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)} \right)^{-1} \ll \frac{\mathcal{D}_\omega(x)}{\mathcal{A}_\omega(x)} \ll \left(\frac{\mathcal{D}_\omega(x) + \mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)} \right)^{-1} = 1 + o(1).$$

The last equation implies that $\mathcal{A}_\omega(x) \sim \mathcal{D}_\omega(x)$ as $x \to \infty$. □

Corollary 4.4. *We have for large $x > e$ and $1 \leq k \leq \log\log x$ that*

$$\widehat{C}_k(x) := \sum_{\substack{n \leq x \\ \Omega(n) = k}} C_{\Omega(n)}(n) \sim \frac{4\sqrt{2\pi} \cdot x}{(2k-1)} \cdot \frac{(\log\log x)^{k-1/2}}{(k-1)!}.$$