Proposition 1. *Let a, z and λ be positive real parameters such that $z = \lambda a$. If $0 < \lambda < 1$, then*

$$\Gamma(a,z) = \Gamma(a) + \mathcal{O}_\lambda(z^{a-1}e^{-z})$$

as $z \to +\infty$.

Remark. This asymptotics is useful only when λ is bounded away from 1. The same is true for the first estimate in your Proposition A.2.

Proof. This asymptotic estimate follows directly from the asymptotic expansion

$$\Gamma(a,z) \sim \Gamma(a) + z^a e^{-z} \sum_{k=0}^{\infty} \frac{(-a)^k b_k(\lambda)}{(z-a)^{2k+1}}$$

as $z \to +\infty$ (see, e.g., [1, Eq. (2.1)]). $\square$

Proposition 2. *As $x \to +\infty$,*

$$\sum_{k=1}^{\lfloor 2\log\log x\rfloor} \frac{(\log\log x)^{k-1/2}}{(2k-1)(k-1)!} = \frac{1}{2}\frac{\log x}{\sqrt{\log\log x}} + \mathcal{O}\left(\frac{\log x}{(\log\log x)^{3/2}}\right).$$

Proof. We have for $t > 0$

$$\begin{aligned}\sum_{k=1}^{n} \frac{t^{k-1}}{(2k-1)(k-1)!} &= \int_0^1 \sum_{k=1}^{n} \frac{(s^2t)^{k-1}}{(k-1)!} ds = \frac{1}{(n-1)!}\int_0^1 e^{s^2t}\Gamma(n,s^2t)ds \\ &= \frac{1}{(n-1)!}\int_0^1 e^{s^2t}\Gamma(n,s^2t)ds = \frac{t^{-1/2}}{2(n-1)!}\int_0^t u^{-1/2}e^u\Gamma(n,u)du\end{aligned}$$

(cf. (30a)). Integrating once by parts shows that this is further equal to

$$\frac{1}{2(n-1)!}\sqrt{\frac{\pi}{t}}\Gamma(n,t)\operatorname{erfi}(\sqrt{t}) + \frac{1}{2(n-1)!}\sqrt{\frac{\pi}{t}}\int_0^t u^{n-1}e^{-u}\operatorname{erfi}(\sqrt{u})du.$$

From now on assume that $t = \frac{1}{2}n + \xi$, $\xi = \mathcal{O}(1)$. By [2, Eq. 7.12.1]) and the definition of erfi,

$$e^{-t}\operatorname{erfi}(\sqrt{t}) = \frac{1}{\sqrt{\pi t}} + \mathcal{O}\left(\frac{1}{t^{3/2}}\right) = \mathcal{O}\left(\frac{1}{t^{1/2}}\right)$$

as $t \to +\infty$. Consequently,

$$\frac{1}{2(n-1)!}\sqrt{\frac{\pi}{t}}\int_0^t u^{n-1}e^{-u}\operatorname{erfi}(\sqrt{u})du = \frac{1}{(n-1)!}\mathcal{O}(t^{n-2})$$

as $t \to +\infty$. Applying Proposition 1 with $a = n$, $z = t$ and $\lambda = \frac{1}{2} + \frac{\xi}{n}$, we find

$$\Gamma(n,t) = \Gamma(n) + \mathcal{O}\big(t^{n-1}e^{-t}\big)$$

as $t \to +\infty$. Thus,

$$\sum_{k=1}^{n} \frac{t^{k-1}}{(2k-1)(k-1)!} = \frac{1}{2}\frac{e^t}{t} + \mathcal{O}\left(\frac{e^t}{t^2}\right) + \frac{1}{(n-1)!}\mathcal{O}(t^{n-2})$$

as $t \to +\infty$. By [2, Eq. 5.11.8],

$$(n-1)! = \Gamma(2t-2\xi) = (2t)^{2t-2\xi-1/2}e^{-2t}\mathcal{O}(1) = e^{-t}\left(\frac{4}{e}\right)^t t^{n-1/2}\mathcal{O}(1),$$

whence,

$$\sum_{k=1}^{n} \frac{t^{k-1}}{(2k-1)(k-1)!} = \frac{1}{2}\frac{e^t}{t} + \mathcal{O}\left(\frac{e^t}{t^2}\right) + \left(\frac{e}{4}\right)^t \sqrt{t}\mathcal{O}\left(\frac{e^t}{t^2}\right) = \frac{1}{2}\frac{e^t}{t} + \mathcal{O}\left(\frac{e^t}{t^2}\right)$$

as $n \to +\infty$ with $t = \frac{1}{2}n + \mathcal{O}(1)$. Substituting $n = \lfloor 2\log\log x \rfloor$, $t = \log\log x$ and doing some algebra, we obtain the desired result. □

References

[1] G. Nemes, A. B. Olde Daalhuis, Asymptotic expansions for the incomplete gamma function in the transition regions, *Math. Comp.* **88** (2019), no. 318, pp. 1805–1827.

[2] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.1 of 2021-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.