Michael,

- I wonder if the pf. of Thm 5.2 is off in so much as we should consider $1+P(z)$ in place of the orig. $1-P(z)$?
- Note that numerically $P(s_1) = -1$ for $s_1 = \hat{\sigma}_1 + i\hat{t}_1$, where $\hat{\sigma}_1 \simeq 0.0709259$ and $\hat{t}_1 \simeq -3.72015$
- Then $\mathrm{Re}(s_1) < 1$ (cont. below)

**Theorem 5.2.** *Let $\sigma_1$ denote the unique solution to the equation $P(\sigma) = 1$ for $\sigma > 1$. There are complex $s$ with $\mathrm{Re}(s)$ arbitrarily close to $\sigma_1$ such that $1 + P(s) = 0$.*

*Proof.* The function $P(\sigma)$ is decreasing on $(1, \infty)$, tends to $+\infty$ as $\sigma \to 1^+$, and tends to zero as $\sigma \to \infty$. Thus we find that the equation $P(\sigma) = 1$ has a unique solution for $\sigma > 1$, which we denote by $\sigma = \sigma_1 \approx 1.39943$. Let $\delta > 0$ be chosen small enough that $|1 - P(z)| > 0$ for all $z$ such that $|z - \sigma_1| = \delta$. Set

$$\eta = \min_{\substack{z \in \mathbb{C} \\ |z-\sigma_1|=\delta}} |1 - P(z)|.$$

Since $P(z)$ is continuous whenever $\mathrm{Re}(z) > 1$, we have that $\eta > 0$. Let $X \geq 2$ be a sufficiently large integer so that

$$\sum_{p>X} p^{\delta-\sigma_1} < \frac{\eta}{4}.$$

Kronecker's theorem provides a fixed $t$ such that the following inequality holds [9, §XXIII]:

$$\max_{2<p\leq X}\min_{n\in\mathbb{Z}}\left|\frac{t\log p}{2\pi}-n-\frac{1}{2}\right|<\delta\eta.$$

Thus we have that

$$\sum_{p>2}p^{\delta-\sigma_1}\left|p^{\imath t}+1\right|<\frac{\eta}{2}. \quad (*)$$

Hence, for all $z$ such that $|z-\sigma_1|=\delta$, we have

$$|P(z+\imath t)+P(z)|<\frac{\eta}{2}. \quad (**)$$

We apply Rouché's theorem to see that the functions $1-P(z)$ and $1-P(z)+P(z+\imath t)+P(z)$ have the same number of zeros in the disk $\mathcal{D}_\delta=\{z\in\mathbb{C}:|z-\sigma_1|<\delta\}$. Since $1-P(z)$ has at least one zero within $\mathcal{D}_\delta$, we must have that $1+P(w)$ has at least one zero with $|w-\sigma_1-\imath t|<\delta$. Since we can take $\delta$ as small as necessary, there are zeros of the function $1+P(s)$ that are arbitrarily close to the line $s=\sigma_1$. □

**Corollary 5.3.** *Let $\sigma_1>1$ be defined as in Theorem 5.2. For any $\epsilon>0$, there are arbitrarily large $x$ such that*

$$|G^{-1}(x)|>x^{\sigma_1-\epsilon}.$$

*Proof.* We have by (6) that

$$D_{g^{-1}}(s):=\sum_{n\geq 1}\frac{g^{-1}(n)}{n^s}=\frac{1}{\zeta(s)(1+P(s))},\text{ for }\operatorname{Re}(s)>1.$$

Theorem 5.2 implies that $D_{g^{-1}}(s)$ has singularities $s\in\mathbb{C}$ such that the $\operatorname{Re}(s)$ are arbitrarily close to $\sigma_1$. By applying [17, Cor. 1.2; §1.2], we have that any Dirichlet series is locally uniformly convergent in its half-plane of convergence, e.g., for $\operatorname{Re}(s)>\sigma_c$, and is hence analytic in this half-plane. It follows that the abscissa of convergence of $D_{g^{-1}}(s)$ is given by $\sigma_c\geq\sigma_1>1$. In particular, the abscissa of convergence of this DGF cannot be smaller than $\sigma_1$. The result proved in [17, Thm. 1.3; §1.2] then shows that

$$\limsup_{x\to\infty}\frac{\log|G^{-1}(x)|}{\log x}=\sigma_c\geq\sigma_1. \quad (***)$$ □

to the point that your argument does not give a counterexample to $|G^{-1}(x_n)|>x_n^{\sigma_1-\varepsilon}$, $\varepsilon>0$, for inf. many $\{x_n\}_{n\geq 1}$.

- Any thoughts?
- I am not still 100% about deriving the ineqs. in (*) and (**).

The idea in applying (***), however, is solid with proof given in §1.2 (Thm 1.3) of [MV].

- Surely the error is in the selection of the zero/eqn. involving $P(s)$ at $Re(s) > 1$...