We have limiting asymptotics on these functions given by the following theorem.

Theorem 4.6 (Asymptotics for the functions $C_k(n)$)**.** *For $k := 0$, we have by definition that $C_0(n) = \delta_n$ all $k \geq 1$, we obtain that the dominant asymptotic term for $C_k(n)$ is given by*

$$\mathbb{E}[C_k(n)] = (\log \log n)^{2k-1}, \text{ as } n \to \infty.$$

Asymptotics claimed for each fixed k

*E.g., using contour integration or the following integral formula for Dirichlet series inversion [1, §11]:

$$f(n) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{n^{\sigma+\imath t}}{\zeta(\sigma+\imath t)(P(\sigma+\imath t)+1)}, \sigma > 1.$$

Fröberg has also previously done some preliminary investigation as to the properties of the inversion to find the coeffic $(1+P(s))^{-1}$ in [5].

Since we have that

$$(g^{-1} * 1)(n) = \lambda(n) \cdot C_{\Omega(n)}(n), \forall n \geq 1,$$

Möbius inversion provides us with an exact divisor sum based expression for $g^{-1}(n)$ (see Lemma 7.1). T can prove (see Corollary 7.5) that we can obtain lower bounds on the magnitude of $g^{-1}(n)$ by approxima by the simpler divisor sums

$$\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

Notice that this formula is substantially easier to evaluate than the corresponding sums in (2) given through Möbius inversion. Hence, we prefer to work with bounds on it that we prove as new results rather than with results relying on the more complicated exact formula from the cited equation above.

Specifically, the last result in turn implies that

$$|G^{-1}(x)| \overset{\blacktriangle}{\gtrsim} \left| \sum_{n \leq x} \lambda(n) \cdot C_{\Omega(n)}(n) \times \sum_{d=1}^{\left\lfloor \frac{x}{n} \right\rfloor} \lambda(d) \right|.$$

In light of the fact that (by an integral-based interpretation of integer convolution using summation by see Proposition 8.1)

For this to be correct, need previous asymptotics to be good for $K \lesssim \Omega(n)$.