# Lower bounds on the summatory function of the Möbius function along infinite subsequences

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### Abstract

The Mertens function, $M(x) = \sum_{n \leqslant x} \mu(n)$, is classically defined as the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture stating that $|M(x)| < C \cdot \sqrt{x}$ with come absolute $C > 0$ for all $x \geqslant 1$ has a well-known disproof due to Odlyzko and té Riele given in the early 1980's by computation of non-trivial zeta function zeros in conjunction with integral formulas expressing $M(x)$. It is conjectured that $M(x)/\sqrt{x}$ changes sign infinitely often and grows unbounded in the direction of both $\pm\infty$ along subsequences of integers $x \geqslant 1$. We prove a weaker property related to the unboundedness of $|M(x)| \log x / \sqrt{x}$ by showing that

$$\limsup_{x \to \infty} \frac{|M(x)|(\log x)(\log\log\log x)(\log\log\log\log x)^{\frac{5}{4}}(\log\log\log\log\log x)^{\frac{3}{2}}}{\sqrt{x}(\log\log x)} > 0.$$

There is a distinct stylistic flavor and new element of combinatorial analysis to our proof peppered in with the standard methods from analytic, additive and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

**Keywords and Phrases:** *Möbius function; Mertens function; summatory function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting functions; Dirichlet generating function; asymptotic lower bounds; Mertens conjecture.*

**Math Subject Classifications (MSC 2010):** *11N37; 11A25; 11N60; and 11N64.*

# Glossary of special notation and conventions

| Symbol | Definition |
|---|---|
| $\mathbb{E}[f(x)], \overset{\mathbb{E}}{\sim}$ | We use the expectation notation $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \overset{\mathbb{E}}{\sim} h(x)$, to denote that $f$ has a so-called *average order* growth rate of $h(x)$. What this means is that $\frac{1}{x}\sum_{n \leqslant x} f(n) \sim h(x)$, or equivalently that $$\lim_{x \to \infty} \frac{\frac{1}{x}\sum_{n \leqslant x} f(n)}{h(x)} = 1.$$ |
| $B$ | The absolute constant $B \approx 0.2614972128476427837554$ from the statement of Mertens theorem. |
| $o(f), O_\alpha(g)$ | We write that $f = o(g)$ if $$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$ We sometimes adapt the standard big-$O$ notation, writing $f = O_{\alpha_1,\ldots,\alpha_k}(g)$ for some parameters $\alpha_1, \ldots, \alpha_k$ that do not depend on $x$, if $f(x) = O(g(x))$ subject only to the stated upper bound on $f$ having an implicit dependence only on $x$ (as usual) and on the $\alpha_i$ for $1 \leqslant i \leqslant k$. |
| $C_k(n)$ | These auxiliary functions are defined recursively for $n \geqslant 1$ and $1 \leqslant k \leqslant \Omega(n)$ according to the formula $$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geqslant 1. \end{cases}$$ |
| $[q^n]F(q)$ | The coefficient of $q^n$ in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n \geqslant 0}$. |
| DGF | Given a sequence $\{f(n)\}_{n \geqslant 0}$, its *Dirichlet generating function* (DGF) is defined by $D_f(s) := \sum_{n \geqslant 1} f(n)/n^s$ subject to suitable constraints on the real part of the parameter $s \in \mathbb{C}$ that guarantee convergence of $D_f(s)$. |
| $d(n)$ | The divisor function, $d(n) := \sum_{d|n} 1$, for $n \geqslant 1$. |
| $\varepsilon(n)$ | The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) = \delta_{n,1}$, defined such that for any arithmetic $f$ we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (defined below). |
| $f * g$ | The Dirichlet convolution of $f$ and $g$, $(f * g)(n) := \sum_{d|n} f(d)g(n/d)$, where the sum is taken over the divisors $d$ of $n$ for $n \geqslant 1$. |
| $f^{-1}(n)$ | The Dirichlet inverse of $f$ with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d|n \\ d>1}} f(d)f^{-1}(n/d)$ for $n \geqslant 1$ with $f^{-1}(1) = 1/f(1)$. The Dirichlet inverse of $f$ exists if and only if $f(1) \neq 0$. This inverse function, provided it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$. |
| $\lfloor x \rfloor, \lceil x \rceil$ | The floor function is defined as $\lfloor x \rfloor := x - \{x\}$ where $0 \leqslant \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$. The floor function is sometimes also written as $[x] \equiv \lfloor x \rfloor$. The corresponding ceiling function $\lceil x \rceil$ denotes the smallest integer $m \geqslant x$. |

| Symbol | Definition |
|--------|-----------|
| $g^{-1}(n), G^{-1}(x)$ | The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$. |
| $H_n$ | The *first-order harmonic numbers*, $H_n := \sum_{k=1}^{n} \frac{1}{k}$, satisfy the limiting asymptotic relation $$\lim_{n \to \infty} [H_n - \log(n)] = \gamma,$$ where $\gamma \approx 0.577216$ denotes Euler's gamma constant. |
| $\mathbb{1}_{\mathbb{S}}, \chi_{\text{cond}(x)}$ | We use the notation $\mathbb{1}, \chi : \mathbb{N} \to \{0,1\}$ to denote indicator, or characteristic functions of a set. In paticular, $\mathbb{1}_{\mathbb{S}}(n) = 1$ if and only if $n \in \mathbb{S}$, and $\chi_{\text{cond}}(n) = 1$ if and only if $n$ satisfies the boolean-valued condition `cond`. |
| $[n = k]_{\delta}, [\text{cond}]_{\delta}$ | The symbol $[n = k]_{\delta}$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise. For a boolean-valued conditions, `cond`, $[\text{cond}]_{\delta}$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention.* |
| $\lambda(n)$ | The Liouville lambda function, $\lambda(n) := (-1)^{\Omega(n)}$, denotes the signed parity of $\Omega(n)$, the number of distinct prime factors of $n$ counting their multiplicity. That is, $\lambda(n) \in \{\pm 1\}$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$. |
| $\mu(n)$ | The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever $n$ is squarefree, i.e., $n$ has no prime power divisors with exponent greater than one. |
| $M(x)$ | The Mertens function is the summatory function over $\mu(n)$ defined for all integers $x \geqslant 1$ by $M(x) := \sum_{n \leqslant x} \mu(n)$. |
| $\nu_p(n)$ | The valuation function that extracts the maximal exponent of $p$ in the prime factorization of $n$, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^{\alpha} \| n$ (or when $p^{\alpha}$ exactly divides $n$) for $p$ prime and $n \geqslant 2$. |
| $\omega(n), \Omega(n)$ | We define the strongly additive function $\omega(n) := \sum_{p|n} 1$ and the completely additive function $\Omega(n) := \sum_{p^{\alpha} \| n} \alpha$. Equivalently, if the prime factorization of $n \geqslant 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$. |
| $\pi_k(x), \widehat{\pi}_k(x)$ | The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leqslant n \leqslant x$ for $x > 1$ with exactly $k$ distinct prime factors: $\pi_k(x) := \#\{n \leqslant x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leqslant x : \Omega(n) = k\}$ for $x \geqslant 2$. |
| $\sum_{p \leqslant x}, \prod_{p \leqslant x}$ | Unless otherwise specified by context, we use the index variable $p$ to denote that the summation (product) is to be taken only over prime values within the summation bounds. |
| $P(s)$ | For complex $s$ with $\text{Re}(s) > 1$, we define the *prime zeta function* to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$. For $\text{Re}(s) > 1$, the prime zeta function is related to $\zeta(s)$ according to the formula $P(s) = \sum_{k \geqslant 1} \frac{\mu(k)}{k} \log[\zeta(ks)]$. |
| $Q(x)$ | For $x \geqslant 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leqslant x$. More precisely, this function is summed and identified with its limiting asymptotic formula as $x \to \infty$ in the following form: $Q(x) := \sum_{n \leqslant x} \mu^2(n) \sim \frac{6}{\pi^2} x + O(\sqrt{x})$. |
| $\zeta(s)$ | The Riemann zeta function, defined by $\zeta(s) := \sum_{n \geqslant 1} n^{-s}$ when $\text{Re}(s) > 1$, and by analytic continuation on the entire complex plane with the exception of a simple pole at $s = 1$. |

# 1    Introduction

## 1.1    Definitions

Suppose that $n \geqslant 2$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ so that $r = \omega(n)$. We define the *Möebius function* to be the signed indicator function of the squarefree integers as follows:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leqslant i \leqslant k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many other variants and special properties of the Möebius function and its generalizations [15, *cf.* §2]. A crucial role of the classical $\mu(n)$ forms an inversion relation for arithmetic functions convolved with one by *Möbius inversion*:

$$g(n) = (f * 1)(n) \iff f(n) = (g * \mu)(n), \forall n \geqslant 1.$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined as

$$M(x) = \sum_{n \leqslant x} \mu(n), x \geqslant 1.$$

The sequence of the oscillatory values of this summatory function begins as [16, A002321]

$$\{M(x)\}_{x \geqslant 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2, \ldots\}$$

Clearly, a positive integer $n \geqslant 1$ is *squarefree*, or contains no (prime power) divisors which are squares, if and only if $\mu^2(n) = 0$. A related summatory function which counts the number of *squarefree* integers $n \leqslant x$ then satisfies [4, §18.6] [16, A013928]

$$Q(x) = \sum_{n \leqslant x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \to \infty$:

$$\mu_+(x) = \frac{\#\{1 \leqslant n \leqslant x : \mu(n) = +1\}}{Q(x)} \overset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leqslant n \leqslant x : \mu(n) = -1\}}{Q(x)} \xrightarrow{x \to \infty} \frac{3}{\pi^2}.$$

The actual local oscillations between the approximate densities of the sets $\mu_\pm(x)$ lend an unpredictable nature to the function and characterize the oscillatory sawtooth shaped plot of $M(x)$ over the positive integers.

## 1.2    Properties

One conventional approach to evaluating the behavior of $M(x)$ for large $x \to \infty$ results from a formulation of this summatory function as a predictable exact sum involving $x$ and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is expressed given the inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \int_1^\infty \frac{s \cdot M(x)}{x^{s+1}} dx,$$

we obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation, along with the standard Euler product representation for the reciprocal zeta function cited in the first equation above, leads us to the exact expression for $M(x)$ for any real $x > 0$ given by the next theorem due to Titchmarsh.

**Theorem 1.1** (Analytic Formula for $M(x)$). *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k \geqslant 1}$ satisfying $k \leqslant T_k \leqslant k+1$ for each $k$ such that for any real $x > 0$*

$$M(x) = \lim_{k \to \infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |\operatorname{Im}(\rho)|<T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geqslant 1} \frac{(-1)^{n-1}}{n \cdot (2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp\left(-C \cdot \log^{3/5}(x)(\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundararajan recently proved new updated estimates bounding $M(x)$ for large $x$ in the following forms [17]:

$$M(x) \ll \sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{14}\right),$$

$$M(x) = O\left(\sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right), \quad \forall \epsilon > 0.$$

## 1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which posits that

$$|M(x)| < C \cdot \sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens for $C = 1$ and all $x < 10000$. Since its beginnings in 1897, the Mertens conjecture has been disproven by computation of non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele from the early 1980's [12]. Since the truth of the conjecture would have implied the RH, more recent attempts at bounding $M(x)$ consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound towards both $\pm\infty$ along infinite subsequences.

One of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ is in actuality unbounded on the natural numbers. A precise statement of this problem is to produce an affirmative answer whether $\limsup_{x\to\infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x\to\infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there are an infinite subsequences of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound towards either $\pm\infty$ along the subsequence. We cite that prior to this point it is only known by computation that [14, *cf.* §4.1] [16, *cf.* A051400; A051401]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now} \geqslant 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now} \leqslant -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [12, 7, 8, 5]. Extensive computational evidence has produced a conjecture due to Gonek (among attempts on exact bounds by others) that in fact the limiting behavior of $M(x)$ satisfies [11]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log\log x)^{5/4}} = O(1).$$

# 2 An overview of the core logical steps and components to the proof

We offer an initial step-by-step summary overview of the core components to our proof outlined in the next. As our proof methodology is new and relies on non-standard elements compared to more traditional methods of bounding $M(x)$, we hope that this sketch of the logical components to this argument makes the article easier to parse.

## 2.1 Step-by-step overview

**(1)** We prove a matrix inversion formula relating the summatory functions of an arithmetic function $f$ and its Dirichlet inverse $f^{-1}$ (for $f(1) \neq 0$). See Theorem 3.1 in Section 4.

**(2)** This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the seemingly unconnected prime counting function, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is stated in (1).

The strong additivity of $\omega(n)$ imparts the characteristic signedness of $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ for all $n \geqslant 1$, which is weighted according to the parity of $\Omega(n)$. The link relating (1) to canonical additive functions and their distributions then lends a recent distinguishing element to the success of the methods in our proof.

**(3)** We tighten an updated result from [10, §7] providing uniform asymptotic formulas for the summatory functions, $\widehat{\pi}_k(x)$, that indicate the parity of $\Omega(n)$ (sign of $\lambda(n)$) for $n \leqslant x$ and $1 \leqslant k \leqslant \log\log x$. These formulas are proved using expansions of more combinatorially motivated Dirichlet series (see Theorem 3.7). We use this result to sum $\sum_{n \leqslant x} \lambda(n) f(n)$ for particular non-negative arithmetic functions $f$ when $x$ is large.

**(4)** We then turn to bounding the asymptotics of the quasi-periodic functions, $g^{-1}(n)$, by estimating this inverse function's limiting order for large $n \leqslant x$ as $x \to \infty$ in Section 6. We eventually use these estimates to prove a substantially unique new lower bound formula for the summatory function $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$ along certain asymptotically large infinite subsequences (see Theorem 8.5).

**(5)** We spend some interim time in Section 7 carefully working out a rigorous justification for why the limiting lower bounds we obtain from average order case analysis of our arithmetic function approximations to $g^{-1}(n)$ are sufficient to prove the corollary on the unboundedness of $M(x)$ below.

**(6)** When we return to step **(2)** with our new lower bounds at hand, we have a new unconditional proof of the unboundedness of $\frac{|M(x)| \log x}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. What we recover is a quick, and rigorous, proof of Corollary 3.8 given in Section 8.2.

## 2.2 Schematic flowchart of the proof logic

The next flowchart diagramed below shows how the seemingly disparate components of the proof are organized.
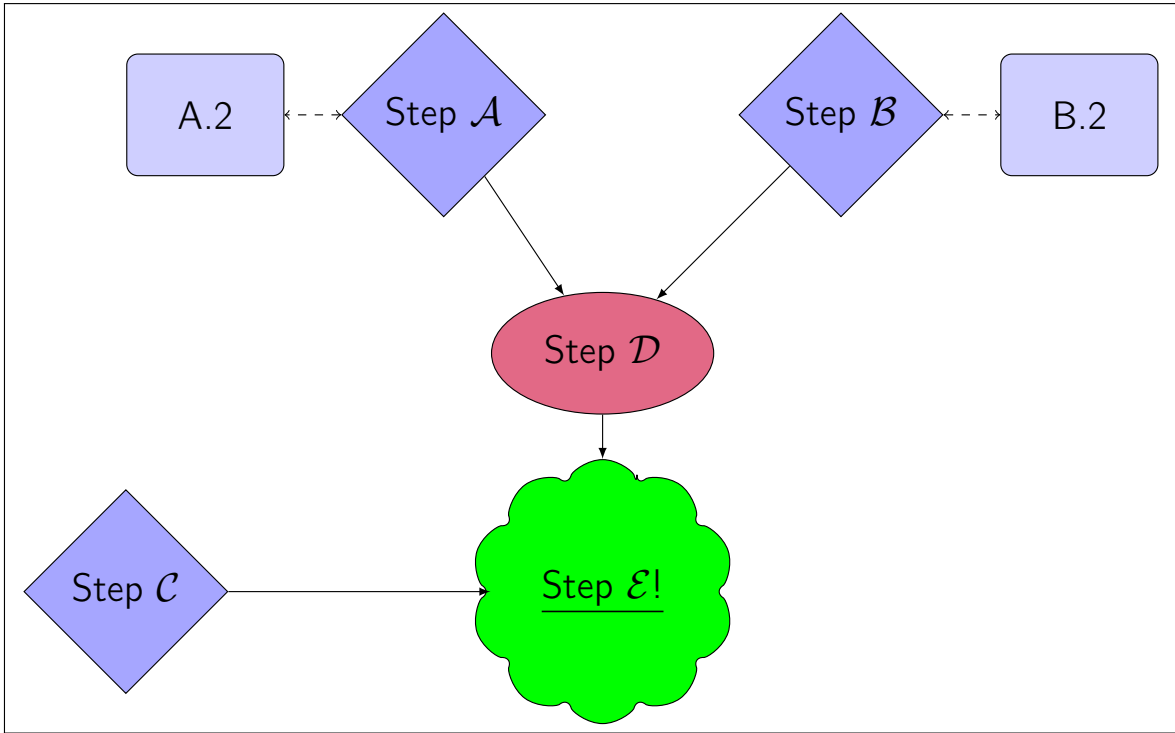
**Legend to the diagram stages:**

▶ **Step A:** *Citations and re-statements of existing theorems proved elsewhere.*

      **A.A:** Key results and constructions:
- Theorem 3.6
- Corollary 5.5
- The results, lemmas, and facts cited in Section 4.3

      **A.2:** Lower bounds on the Abel summation based formula for $G^{-1}(x)$:
- Theorem 3.7 (on page 17)
- Proposition 5.6
- Theorem 8.5

▶ **Step B:** *Constructions of an exact formula for $M(x)$.*

      **B.B:** Key results and constructions:
- Corollary 3.3 (follows from Theorem 3.1 proved on page 11)
- Proposition 4.1

      **B.2:** Asymptotics for the component functions $g^{-1}(n)$ and $G^{-1}(x)$:
- Theorem 6.3 (on page 20)
- Lemma 6.4

▶ **Step C:** *A justification for why lower bounds obtained roughly "on average" suffice.*
- The results proved in Section 7

▶ **Step D:** *Interpreting the exact formula for $M(x)$.*
- Proposition 8.1
- Theorem 8.5

▶ **Step E:** *The Holy Grail.* Proving that $\frac{|M(x)|\log x}{\sqrt{x}}$ grows without bound in the limit supremum sense.
- Corollary 3.8 (on page 37)

# 3    A concrete new approach for bounding $M(x)$ from below

## 3.1    Summatory functions of Dirichlet convolutions of arithmetic functions

**Theorem 3.1** (Summatory functions of Dirichlet convolutions). *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leqslant x} f(n)$ and $H(x) := \sum_{n \leqslant x} h(n)$ denote the summatory functions of $f, h$, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse $f^{-1}$ of $f$. Then, letting the counting function $\pi_{f*h}(x)$ be defined as in the first equation below, we have the following equivalent expressions for the summatory function of $f * h$ for all integers $x \geqslant 1$:*

$$\pi_{f*h}(x) = \sum_{n \leqslant x} \sum_{d|n} f(d) h(n/d)$$

$$= \sum_{d \leqslant x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right)$$

$$= \sum_{k=1}^{x} H(k) \left[ F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right].$$

*Moreover, we can invert the linear system determining the coefficients of $H(k)$ for $1 \leqslant k \leqslant x$ naturally to express $H(x)$ as a linear combination of the original left-hand-side summatory function as follows:*

$$H(x) = \sum_{j=1}^{x} \pi_{f*h}(j) \left[ F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \right]$$

$$= \sum_{n=1}^{x} f^{-1}(n) \pi_{f*h}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).$$

**Corollary 3.2** (Convolutions Arising From Möbius Inversion). *Suppose that $g$ is an arithmetic function on the positive integers such that $g(1) \neq 0$. Define the summatory function of the convolution of $g$ with $\mu$ by $\widetilde{G}(x) := \sum_{n \leqslant x} (g * \mu)(n)$. Then the Mertens function equals*

$$M(x) = \sum_{k=1}^{x} \left( \sum_{j=\lfloor \frac{x}{k+1} \rfloor + 1}^{\lfloor \frac{x}{k} \rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geqslant 1.$$

**Corollary 3.3** (A motivating special case). *We have exactly that for all $x \geqslant 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[ \pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1 \right]. \tag{1}$$

## 3.2    An exact expression for $M(x)$ in terms of strongly additive functions

From this point on, we fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and denote its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute the first few terms for the Dirichlet inverse of $g(n)$ exactly for the first few sequence values as (see Table T.1 of the appendix section)

$$\{g^{-1}(n)\}_{n \geqslant 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is given by $\text{sgn}(g^{-1}(n)) = \frac{g^{-1}(n)}{|g^{-1}(n)|} = \lambda(n)$ (see Proposition 4.1). This useful property is inherited from the distinctly additive nature of the component function $\omega(n)$*[F3.A].

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*[F3.A]Indeed, for any non-negative additive arithmetic function $a(n)$, $(a + 1)^{-1}(n)$ has leading sign given by $\lambda(n)$ for any $n \geqslant 1$. For multiplicative $f$, we obtain a related condition that $\text{sgn}(f(n)) = (-1)^{\omega(n)}$ for all $n \geqslant 1$.

There does not appear to be an easy, nor subtle direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences. However, the distribution of distinct sets of prime exponents is fairly regular so that $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of $g^{-1}(n)$ over $n \geqslant 2$:

**Heuristic 3.4** (Symmetry in $g^{-1}(n)$ in the exponents in the prime factorization of $n$)**.** Suppose that $n_1, n_2 \geqslant 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for some $r \geqslant 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, $g^{-1}$ has the same values on the squarefree integers with exactly two, three, and so on prime factors (compare with Table T.1 starting on page 43).

**Conjecture 3.5.** *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

> *(A)* $g^{-1}(1) = 1$;

> *(B)* *For all $n \geqslant 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*

> *(C)* *For all squarefree integers $n \geqslant 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!.$$

We illustrate parts (B)–(C) of the conjecture more clearly using Table T.1 given starting on page 43. The realization that the beautiful and remarkably simple combinatorial form of property (C) in Conjecture 3.5 holds for all squarefree $n \geqslant 1$ motivates our pursuit of formulas for the inverse functions $g^{-1}(n)$ expressed by sums of auxiliary sequences of arithmetic functions*[F3.B] (see Section 6).

For natural numbers $n \geqslant 1, k \geqslant 0$, let

$$C_k(n) := \begin{cases} \varepsilon(n) = \delta_{n,1}, & \text{if } k = 0; \\ \sum_{d \mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geqslant 1. \end{cases}$$

For any $n \geqslant 1$, we can prove that (see Lemma 6.4)

$$g^{-1}(n) = \lambda(n) \times \sum_{d \mid n} \mu^2 \left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{2}$$

In light of the fact that (see Proposition 8.1)

$$M(x) \approx G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)},$$

the formula in (2) implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by appropriate estimates of the summatory function $G^{-1}(x)$*[F3.C].

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*[F3.B]A proof of this property is not difficult to give using Lemma 6.4 stated on page 20.
*[F3.C]We can also prove that
$$M(x) = G^{-1}(x) + \sum_{p \leqslant x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right),$$
by inversion since
$$G^{-1}(x) = \sum_{d \leqslant x} (g^{-1} * 1)(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right),$$
with $(g^{-1} * 1)^{-1} = g * \mu = \chi_{\mathbb{P}} + \varepsilon$ defined such that $\chi_{\mathbb{P}}$ is the characteristic function of the primes.

## 3.3 Uniform asymptotics from enumerative counting DGFs in Mongomery and Vaughan

Our inspiration for the new bounds found in the last sections of this article allows us to sum non-negative arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. We utilize a somewhat more general hybrid generating function and enumerative DGF method under which we are able to recover "good enough" asymptotics about the summatory functions that encapsulate the parity of $\Omega(n)$ (or sign of $\lambda(n)$) through the summatory tally functions $\widehat{\pi}_k(x)$ (see Section 5.1).

**Theorem 3.6** (Montgomery and Vaughan). *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leqslant x : \Omega(n) = k\}.$$

*For $R < 2$ we have that*

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right)\frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

*uniformly for $1 \leqslant k \leqslant R\log\log x$ where*

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1}\left(1 - \frac{1}{p}\right)^z, z \geqslant 0.$$

The proof of the next result is combinatorially motivated in so much as it interprets lower bounds on a key infinite product factor of $\mathcal{G}(z)$ defined in Theorem 3.6 as corresponding to an ordinary generating function of certain homogeneous symmetric polynomials involving reciprocals of the primes.

**Theorem 3.7.** *For all large $x$ we have uniformly for $1 \leqslant k \leqslant \log\log x$ that*

$$\widehat{\pi}_k(x) \gg \frac{4}{3\sqrt{\pi}}\frac{x}{\log x}\left(\frac{\log 2}{\log x}\right)^{\frac{k-1}{\log\log x}}\frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 + O\left(\frac{k}{(\log\log x)^2}\right)\right).$$

## 3.4 Cracking the classical unboundedness barrier

In Section 8, we are able to state what forms a culmination of the results we carefully build up to in the proofs established in prior sections of the article. What we eventually obtain at the conclusion of the section is the next important summary corollary that partially resolves the classical question of the unboundedness of the scaled function Mertens function $q(x) := |M(x)|/\sqrt{x}$ scaled by a logarithmic factor in the limit supremum sense.

**Corollary 3.8** (Unboundedness of the the Mertens function, $q(x)\log x$). *We have that*

$$\limsup_{x\to\infty}\frac{|M(x)|\log x}{\sqrt{x}} = +\infty.$$

In establishing the rigorous proof of Corollary 3.8 based on our new methods, we not only show unboundedness of $q(x)\log x$, but also set a minimal rate (along a large infinite subsequence) at which this form of the scaled Mertens function grows without bound.

# 4 Preliminary proofs of new results

## 4.1 Establishing the summatory function properties and inversion identities

We will first prove Theorem 3.1 using matrix methods and similarity transforms by shift matrices. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95].

*Proof of Theorem 3.1.* Let $h, g$ be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of $h$ and $g$, respectively, by $H(x) = \sum_{n \leqslant x} h(n)$ and $G(x) = \sum_{n \leqslant x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of $g$ with $h$: $g*h$. Then we can readily see that the following initial formulas hold for all $x \geqslant 1$:

$$\pi_{g*h}(x) := \sum_{n=1}^{x} \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d)H\left(\left\lfloor \frac{x}{d} \right\rfloor\right)$$

$$= \sum_{i=1}^{x} \left[ G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] H(i).$$

We form the matrix of coefficients associated with this linear system defining $H(n)$ for all $n \leqslant x$. We then invert the system to express an exact solution for $H(x)$ at any $x \geqslant 1$. Let the matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right), \forall 1 \leqslant j \leqslant x.$$

The matrix we must invert in this problem is lower triangular, with ones on its diagonals, and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

Here, $U$ is a square matrix with finite dimensions whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$\left[(I - U^T)^{-1}\right]_{i,j} = [j \leqslant i]_\delta.$$

It is a useful fact that if we take successive differences in $x$ of the floor of certain fractions, $\left\lfloor \frac{x}{j} \right\rfloor$, we get non-zero behavior at the divisors of $x$:

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j | x; \\ 0, & \text{otherwise.} \end{cases}$$

We use this property to shift the matrix $\hat{G}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of $g$ of the following form:

$$\left[(I - U^T)\hat{G}\right]^{-1} = \left( g\left(\frac{x}{j}\right) [j|x]_\delta \right)^{-1} = \left( g^{-1}\left(\frac{x}{j}\right) [j|x]_\delta \right).$$

Now we can express the inverse of the target matrix,

$$(g_{x,j}) = (I - U^T)^{-1} \left( g\left(\frac{x}{j}\right) [j|x]_\delta \right) (I - U^T),$$

using a similarity transformation conjugated by shift operators as follows:

$$(g_{x,j})^{-1} = (I - U^T)^{-1} \left( g^{-1}\left(\frac{x}{j}\right) [j|x]_\delta \right) (I - U^T)$$

$$= \left( \sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k) \right) (I - U^T)$$

$$= \left( \sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} g^{-1}(k) \right).$$

Hence, the summatory function $H(x)$ is exactly expressed for any $x \geqslant 1$ by a vector product with the inverse matrix from the previous equation in the form of

$$H(x) = \sum_{k=1}^{x} g_{x,k}^{-1} \cdot \pi_{g*h}(k)$$

$$= \sum_{k=1}^{x} \left( \sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \cdot \pi_{g*h}(k). \qquad \square$$

## 4.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of $n$. Then we can easily prove that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu. \tag{3}$$

When combined with Corollary 3.2 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 3.3.

**Proposition 4.1** (The key signedness property of $g^{-1}(n)$). *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)| + [h(n)=0]_\delta} \in \{0, \pm 1\}$ *denote the sign of the arithmetic function $h$ at integers $n \geqslant 1$. For the Dirichlet invertible function,* $g(n) := \omega(n) + 1$, *we have that* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ *for all $n \geqslant 1$.*

*Proof.* Recall that $D_f(s) := \sum_{n \geqslant 1} f(n) n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ for $\sigma_f$ the abcissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = 1/\zeta(s)$ and $D_\omega(s) = P(s)\zeta(s)$. Then by (3) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any invertible arithmetic function $f$, for all $\operatorname{Re}(s) > 1$ we have

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s) + 1)\zeta(s)}. \tag{4}$$

It follows that $(\omega + 1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + 1$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. From this fact, it follows by inspection that $\operatorname{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make this intuition rigorous.

By the standard recurrence relation that defines the Dirichlet inverse function of any arithmetic function $h$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum_{\substack{d|n \\ d>1}} h(d) h^{-1}(n/d), & n \geqslant 2. \end{cases} \tag{5}$$

For $n \geqslant 2$, the summands in (5) can be simply indexed over the primes $p|n$ given our definition of $h$ from above. This observation yields that we can inductively expand these sums into nested divisor sums provided the depth

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of the sums does not exceed the capacity to index summations over the primes dividing $n$. Namely, notice that for $n \geqslant 2$

$$h^{-1}(n) = -\sum_{p|n} h^{-1}(n/p), \qquad\qquad\qquad \text{if } \Omega(n) \geqslant 1$$

$$= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right), \qquad\qquad \text{if } \Omega(n) \geqslant 2$$

$$= -\sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1 p_2}} h^{-1}\left(\frac{n}{p_1 p_2 p_3}\right), \qquad \text{if } \Omega(n) \geqslant 3.$$

Then by induction, again with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1, n \geqslant 2. \tag{6}$$

If for $n \geqslant 2$ we write the prime factorization of $n$ as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{\omega(n)}^{\alpha_{\omega(n)}}$ where the exponents $\alpha_i \geqslant 1$ for all $1 \leqslant i \leqslant \omega(n)$, we can see that*[F4.A]

$$|h^{-1}(n)| \geqslant (\omega(n))! \qquad\qquad\qquad =: h_\ell^{-1}(n), n \geqslant 2, \tag{7}$$
$$|h^{-1}(n)| \leqslant (\omega(n))!^{\max(\alpha_1,\alpha_2,\ldots,\alpha_{\omega(n)})} \qquad =: h_u^{-1}(n), n \geqslant 2,$$

where the bounding functions $h_\ell^{-1}(n), h_u^{-1}(n) > 0$ are positive for all $n \geqslant 1$. What these bounds show is that for all $n \geqslant 1$ (with $\lambda(1) = 1$) the following property holds:

$$\text{sgn}(h^{-1}(n)) = \lambda(n).$$

Since $\lambda$ is completely multiplicative, and since $\mu(n) = \lambda(n)$ whenever $n$ is squarefree, we obtain that

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geqslant 1.$$

Finally, since $|h^{-1}(n)| > 0$ for all $n \geqslant 1$ by the bounds we proved in (7), the previous equation implies our result. □

## 4.3 Statements of other facts and known limiting asymptotics

**Theorem 4.2** (Mertens theorem). *For all $x \geqslant 2$ we have that*

$$P_1(x) := \sum_{p \leqslant x} \frac{1}{p} = \log \log x + B + o(1),$$

*where $B \approx 0.261497212847642783755 4$ is an absolute constant*[F4.B].

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*[F4.A]In fact, we recover that

$$\lambda(n) h^{-1}(n) = \frac{(\alpha_1 + \cdots + \alpha_{\omega(n)})!}{\alpha_1! \alpha_2! \cdots \alpha_{\omega(n)}!},$$

so that since $h^{-1} = g^{-1} * 1$ by the DGF above, when $n \geqslant 1$ is squarefree, we recover property (C) stated in Conjecture 3.5.

*[F4.B]Exactly, we have that the *Mertens constant* is defined by

$$B = \gamma + \sum_{m \geqslant 2} \frac{\mu(m)}{m} \log\left[\zeta(m)\right],$$

where $\gamma \approx 0.577215664902$ is Euler's gamma constant.

**Corollary 4.3** (Product form of Mertens theorem). *We have that for all sufficiently large $x \gg 2$*

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) = \frac{e^{-B}}{\log x}(1 + o(1)),$$

*where the notation for the absolute constant $0 < B < 1$ coincides with the definition of Mertens constant from Theorem 4.2. Hence, for any real $z \geqslant 0$ we obtain that*

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right)^z = \frac{e^{-Bz}}{(\log x)^z}(1 + o(1))^z \sim \frac{e^{-Bz}}{(\log x)^z}, \quad \text{as } x \to \infty.$$

Proofs of Theorem 4.2 and Corollary 4.3 are found in [4, §22.7; §22.8].

**Facts 4.4** (Exponential integrals and the incomplete gamma function). The following two variants of the *exponential integral function* are defined by the integral representations [13, §8.19]

$$\text{Ei}(x) := \int_{-x}^{\infty} \frac{e^{-t}}{t} dt,$$

$$E_1(z) := \int_{1}^{\infty} \frac{e^{-tz}}{t} dt, \text{Re}(z) \geqslant 0.$$

These two functions are related by $\text{Ei}(-kz) = -E_1(kz)$ for real $k, z > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $\text{Ei}(\pm x)$ for real $x > 0$:

$$\gamma + \log x - x \leqslant \text{Ei}(-x) \leqslant \gamma + \log x - x + \frac{x^2}{4}, \tag{8a}$$

$$1 + \gamma + \log x - \frac{3}{4}x \leqslant \text{Ei}(x) \quad \leqslant 1 + \gamma + \log x - \frac{3}{4}x + \frac{11}{36}x^2.$$

The (upper) *incomplete gamma function* is defined by [13, §8.4]

$$\Gamma(s, x) = \int_{x}^{\infty} t^{s-1} e^{-t} dt, \text{Re}(s) > 0.$$

We have the following properties of $\Gamma(s, x)$:

$$\Gamma(s, x) = (s - 1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^{+}, \tag{8b}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, \text{ as } x \to \infty. \tag{8c}$$

# 5 Components to the asymptotic analysis of lower bounds for sums of arithmetic functions weighted by $\lambda(n)$

## 5.1 A discussion of the results proved by Montgomery and Vaughan

**Remark 5.1** (Intuition and constructions in Theorem 3.6). For $|z| < 2$ and $\mathrm{Re}(s) > 1$, let

$$F(s, z) := \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \tag{9}$$

and define the DGF coefficients, $a_z(n)$ for $n \geqslant 1$, by the relation

$$\zeta(s)^z \cdot F(s, z) := \sum_{n \geqslant 1} \frac{a_z(n)}{n^s}, \mathrm{Re}(s) > 1.$$

Suppose that $A_z(x) := \sum_{n \leqslant x} a_z(n)$ for $x \geqslant 1$. Then for the choice of the function $F(s, z)$ defined in (9), we obtain

$$A_z(x) = \sum_{n \leqslant x} z^{\Omega(n)} = \sum_{k \geqslant 0} \widehat{\pi}_k(x) z^k.$$

Thus for $r < 2$, by Cauchy's integral formula we have

$$\widehat{\pi}_k(x) = \frac{1}{2\pi \imath} \int_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz.$$

Selecting $r := \frac{k-1}{\log \log x}$ leads to the uniform asymptotic formulas for $\widehat{\pi}_k(x)$ given in Theorem 3.6.

We also require the next theorems reproduced from [10, §7.4] that handle the relative scarcity of the distribution of the $\Omega(n)$ for $n \leqslant x$ such that $\Omega(n) > \log \log x$.

**Theorem 5.2** (Upper bounds on exceptional values of $\Omega(n)$ for large $n$). *Let*

$$A(x, r) := \# \{n \leqslant x : \Omega(n) \leqslant r \cdot \log \log x\},$$
$$B(x, r) := \# \{n \leqslant x : \Omega(n) \geqslant r \cdot \log \log x\}.$$

*If $0 < r \leqslant 1$ and $x \geqslant 2$, then*

$$A(x, r) \ll x(\log x)^{r-1-r \log r}, \quad as \ x \to \infty.$$

*If $1 \leqslant r \leqslant R < 2$ and $x \geqslant 2$, then*

$$B(x, r) \ll_R x \cdot (\log x)^{r-1-r \log r}, \quad as \ x \to \infty.$$

**Theorem 5.3** (Exact bounds on exceptional values of $\Omega(n)$ for large $n$). *We have that uniformly*

$$\# \{3 \leqslant n \leqslant x : \Omega(n) - \log \log n \leqslant 0\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log \log x}}\right).$$

**Remark 5.4.** The proofs of Theorem 5.2 and Theorem 5.3 are found in §7.4 of Montgomery and Vaughan. The key interpretation we need is the result stated in the next corollary. The precise way in which the bound stated in the previous theorem depends on the indeterminate paramater $R$ can be reviewed for reference in the proof algebra and relations cited in the reference [10, §7]. The role of the parameter $R$ involved in stating the previous theorem is more notably critical as the scalar factor the upper bound on $k \leqslant R \log \log x$ in Theorem 3.6 up to which we obtain the valid uniform bounds in $x$ on the asymptotic formulas for $\widehat{\pi}_k(x)$.

We have a discrepancy to work out in so much as we can only form summatory functions over the $\widehat{\pi}_k(x)$ for $1 \leqslant k \leqslant R \log \log x$ using the asymptotic formulas guaranteed by Theorem 3.6, even though we can actually have contributions from values distributed throughout the range $1 \leqslant \Omega(n) \leqslant \log_2(n)$ infinitely often. It is then crucial that we can show that the dominant growth of the asymptotic formulas we obtain for these summatory functions is captured by summing only over $k$ in the truncated range where the uniform bounds hold.

**Corollary 5.5.** *Using the notation for $A(x,r)$ and $B(x,r)$ from Theorem 5.2, we have that for $\delta > 0$,*

$$o(1) \leqslant \left| \frac{B(x, 1+\delta)}{A(x,1)} \right| \ll 2, \text{ as } \delta \to 0^+, x \to \infty.$$

*Proof.* The lower bound stated above should be clear. To show that the asymptotic upper bound is correct, we compute using Theorem 5.2 and Theorem 5.3 that

$$\left| \frac{B(x, 1+\delta)}{A(x,1)} \right| \ll \left| \frac{x \cdot (\log x)^{\delta - \delta \log(1+\delta)}}{O(1) + \frac{x}{2} + O\left( \frac{x}{\sqrt{\log \log x}} \right)} \right| \sim \left| \frac{(\log x)^{\delta - \delta \log(1+\delta)}}{\frac{1}{2} + o(1)} \right| \xrightarrow{\delta \to 0^+} 2,$$

as $x \to \infty$. Notice that since $\mathbb{E}[\Omega(n)] = \log \log n + B$, with $0 < B < 1$ the absolute constant from Mertens theorem, when we denote the range of $k > \log \log x$ as holding in the form of $k > (1 + \delta) \log \log x$ for $\delta > 0$, we can assume that $\delta \to 0^+$ as $x \to \infty$. $\qquad \square$

## 5.2 New results based on refinements of Theorem 3.6

What the enumeratively flavored result in Theorem 3.6 allows us to do is get a sufficient lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. We seek to approximate $\mathcal{G}(z)$ defined in this theorem by only taking finite products of the primes in the factor of $\prod_p (1 - z/p)^{-1}$ for $p \leqslant ux$, e.g., indexing the component products only over those primes $p \in \{2, 3, 5, \ldots, x\}$ as $x \to \infty$. We can extend the argument behind the constructions sketched in Remark 5.1 to justify that it suffices to take only these finite products to obtain a lower bound on $\widehat{\pi}_k(x)$.

**Proposition 5.6.** *For real $s \geqslant 1$, let*

$$P_s(x) := \sum_{p \leqslant x} p^{-s}, x \geqslant 2.$$

*When $s := 1$, we have the asymptotic formula from Mertens theorem (see Theorem 4.2). For all integers $s \geqslant 2$ there is an absolutely defined bounding function $\gamma_0(s, x)$ such that*

$$\gamma_0(s, x) + o(1) \leqslant P_s(x), \text{ as } x \to \infty.$$

*It suffices to take the bound in the previous equation as*

$$\gamma_0(z, x) = s \log \left( \frac{\log x}{\log 2} \right) - s(s-1) \log \left( \frac{x}{2} \right) - \frac{1}{4} s(s-1)^2 \log^2(2).$$

*Proof.* Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$ and where our target function smooth function is $f(t) = t^{-s}$ with $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$P_s(x) = \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t}$$
$$= \operatorname{Ei}(-(s-1) \log x) - \operatorname{Ei}(-(s-1) \log 2) + o(1), \text{ as } x \to \infty.$$

Now using the inequalities in Facts 4.4, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\frac{P_s(x)}{s} \geqslant \log \left( \frac{\log x}{\log 2} \right) - (s-1) \log \left( \frac{x}{2} \right) - \frac{1}{4}(s-1)^2 \log^2(2)$$
$$\frac{P_s(x)}{s} \leqslant \log \left( \frac{\log x}{\log 2} \right) - (s-1) \log \left( \frac{x}{2} \right) + \frac{1}{4}(s-1)^2 \log^2(x).$$

This completes the proof of the bound cited above in this lemma. $\qquad \square$

*Proof of Theorem 3.7.* Notice that for real $0 \leqslant z < 2$ and any prime $p \geqslant 2$, we have that $(1 - z/p)^{-1} \geqslant 1$ with equality if and only if $z := 0$. So if for $0 \leqslant z < 2$ and integers $x \geqslant 2$, we define the function

$$\widehat{P}(z, x) := \prod_{p \leqslant x} \left( 1 - \frac{z}{p} \right)^{-1},$$

then the right-hand-side product is finite as $x \to \infty$. Moreover, for $x \geqslant 2$ the product function $\widehat{P}(z, x)$ is a non-decreasing function of $x$ when $0 \leqslant z < 2$. Thus, if we form an approximation to the function $\mathcal{G}(z)$ from Theorem 3.6 by truncating the product factor corresponding to the limiting case of $\widehat{P}(z, x)$ above by indexing only for primes $0 \leqslant p \leqslant x$, then what we obtain in the formula for $\widehat{\pi}_k(x)$ guaranteed by the theorem is indeed a *lower bound* for the summatory function.

For fixed, finite $x \geqslant 2$ let

$$\mathbb{P}_x := \{n \geqslant 1 : \text{all prime factors } p | n \text{ satisfy } p \leqslant x\}.$$

Then we can see as in the constructions from Montgomery and Vaughan sketched in Remark 5.1 that

$$\prod_{p \leqslant x} \left( 1 - \frac{z}{p^s} \right)^{-1} = \sum_{n \in \mathbb{P}_x} \frac{z^{\Omega(n)}}{n^s}, \, x \geqslant 2. \tag{10}$$

So by extending the argument we employed in the remark summarizing the proof given in [10, §7.4], we have that the formulas for

$$A_z(x) := \sum_{n \leqslant x} z^{\Omega(n)} = \sum_{k \geqslant 0} \widehat{\pi}_k(x) z^k,$$

depending on approximations (or inputs) to $\mathcal{G}(z)$ will also be valid by taking the finite products in (10). This happens since the products of all non-negative integral powers of the primes $p \leqslant x$ generate the integers $\{1 \leqslant n \leqslant x\}$ as a subset. So the bound on $\widehat{\pi}_k(x)$ that depends on $\mathcal{G}(z)$ formed by such an approximation to this product factor yields a new asymptotic formula for these summatory functions from below for each sufficiently large finite $x$ as $x \to \infty$.

We must now find effective bounds on the truncated products in (10) that are both meaningful and still simple enough to use in our new formulas. We have for all integers $0 \leqslant k < +\infty$, and any sequence $\{f(n)\}_{n \geqslant 1}$ with bounded partial sums, that [9, §2]

$$[z^k] \prod_{i \geqslant 1} (1 - f(i)z)^{-1} = [z^k] \exp \left( \sum_{j \geqslant 1} \left( \sum_{i=1}^{m} f(i)^j \right) \frac{z^j}{j} \right), |z| < 1. \tag{11}$$

In our case we have that $f(i)$ denotes the $i^{th}$ prime in the generating function expansion of (11). It follows from Proposition 5.6 that for $0 \leqslant z < 1$ we obtain

$$\log \left[ \prod_{p \leqslant x} \left( 1 - \frac{z}{p} \right)^{-1} \right] \geqslant (B + \log \log x)z + \sum_{j \geqslant 2} \left[ a(x) - b(x)(j-1) - c(x)(j-1)^2 \right] z^j$$

$$= (B + \log \log x)z - a(x) \left( 1 - \frac{1}{1-z} + z \right)$$

$$+ b(x) \left( 1 - \frac{2}{1-z} + \frac{1}{(1-z)^2} \right)$$

$$+ c(x) \left( 1 - \frac{4}{1-z} + \frac{5}{(1-z)^2} - \frac{2}{(1-z)^3} \right)$$

$$=: \widehat{\mathcal{B}}(x; z). \tag{12}$$

In the previous equations, the lower bounds formed by the functions $(a, b, c) \equiv (a_\ell, b_\ell, c_\ell)$ evaluated at $x$ are given by the corresponding lower bounds from Proposition 5.6 as

$$(a_\ell, b_\ell, c_\ell) := \left( \log \left( \frac{\log x}{\log 2} \right), \log \left( \frac{x}{2} \right), \frac{1}{4} \log^2 2 \right).$$

We make a decision to set the uniform bound parameter to a middle ground value of $1 < R < 2$ at $R := \frac{3}{2}$ so that

$$z \equiv z(k, x) = \frac{k-1}{\log\log x} \in [0, R),$$

in the notation of Theorem 3.6 for $x \gg 1$ sufficiently large. Thus $(z-1)^{-m} \in [(-1)^m, 2^m]$ for integers $m \geqslant 1$, which implies that

$$c_\ell(x)\left(1 + \frac{4}{z-1} + \frac{5}{(z-1)^2} + \frac{2}{(z-1)^3}\right) \geqslant 0.$$

Moreover, by taking the first derivative of the function

$$f(z) := 1 - \frac{2}{1-z} + \frac{1}{(1-z)^2}, z \in [0, 1),$$

we can compute that its extrema occur at $z := 0$, or as $z \to 1^-$. In the former case, we obtain that the limiting exponent is zero. Clearly, as $z \to 1$, $f(z)$ becomes unbounded, which would lead to a zero tending lower bound that is also inconsistent with the known bounds we obtain when $k := 1, 2$ in [10, §7.4, p. 228]. We thus let this term become zero-valued when we evaluate the function $\widehat{\mathcal{B}}(x; z)$ in practice. We also conclude that

$$1 + z - \frac{1}{1-z} \geqslant z, z \in [0, 1).$$

Then we obtain the next effective lower bounds on the function from (12):

$$\frac{e^{-Bz}}{(\log x)^z} \times \exp\left(\widehat{\mathcal{B}}(u, x; z)\right) \gg \left(\frac{\log 2}{\log x}\right)^z =: \widehat{\mathcal{C}}(u, x; z) \tag{13}$$

Indeed, setting $z := 0$ on these select terms in (12) does still in fact lead to lower bounds up to a constant by considering the series expansions of the product term we have estimated as a series in $z$. In particular, what we need to show to justify this step is that as $x \to \infty$

$$\mathcal{G}\left(\frac{k-1}{\log\log x}\right) \gg \left(\frac{\log 2}{\log x}\right)^{\frac{k-1}{\log\log x}}.$$

As we approach large limiting cases of $x$, we approximate (bounding $\Gamma(1 + z)$ by constants as below)

$$\mathcal{G}\left(\frac{k-1}{\log\log x}\right) \sim C \times \left(\sum_{n \geqslant 1} \frac{1}{n} \cdot \left(\frac{k-1}{\log\log x}\right)^{\Omega(n)}\right) \times \frac{\exp\left(-\frac{B(k-1)}{\log\log x}\right)}{(\log x)^{\frac{k-1}{\log\log x}}}.$$

Now taking the left-hand-side series terms over only the primes $p$, we can justify that

$$\lim_{x \to \infty}\left[\sum_{p \leqslant x}\left(\frac{k}{\log\log x}\right) \cdot \frac{1}{p} \times \left(1 + o\left(\frac{k}{\log\log x}\right)\right)\right] \sim k \gg (\log 2)^{\frac{k}{\log\log x}} \gg \log 2.$$

Finally, to finish our proof of the new form of the lower bound on $\mathcal{G}(z)$ in Theorem 3.6, we need to bound the reciprocal factor of $\Gamma(z+1)$. Since $z \equiv z(k, x) = \frac{k-1}{\log\log x}$ and $k \in [1, R\log\log x]$, or with $z \in [0, R)$, we obtain for minimal $k$ and all large enough $x \gg 1$ that $\Gamma(z+1) \approx 1$, and for $k$ towards the upper range of its interval that $\Gamma(z+1) \leqslant \Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$. □

# 6   Average case analysis of bounds on the Dirichlet inverse functions, $g^{-1}(n)$

The property in (C) of Conjecture 3.5 along squarefree $n \geqslant 1$ captures an important characteristic of $g^{-1}(n)$ that holds more globally for all $n \geqslant 1$. In particular, these functions can be expressed via more simple formulas than inspection of the first few initial values of the repetitive, quasi-periodic sequence otherwise suggests. The pages of tabular data given as Table T.1 in the appendix section starting on page 43 are intended to provide clear insight into why we arrived at the convenient approximations to $g^{-1}(n)$ proved in this section. The table offers illustrative numerical data formed by examining the approximate behavior at hand for the first several cases of $1 \leqslant n \leqslant 500$ with *Mathematica*.

## 6.1   Definitions and basic properties of component function sequences

We define the following sequence for integers $n \geqslant 1, k \geqslant 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geqslant 1. \end{cases} \tag{14}$$

The sequence of important semi-diagonals of these functions begins as [16, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geqslant 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \ -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

**Remark 6.1** (An effective range of $k$ depending on a fixed large $n$). Notice that by expanding the recursively-based definition in (14) out to its maximal depth by nested divisor sums, for fixed $n$, $C_k(n)$ is seen to only ever possibly be non-zero for $k \leqslant \Omega(n)$. Thus, the effective range of $k$ for fixed $n$ is restricted by the conditions of $C_0(n) = \delta_{n,1}$ and that $C_k(n) = 0, \forall k > \Omega(n)$ whenever $n \geqslant 2$.

**Example 6.2** (Special cases of the functions $C_k(n)$ for small $k$). We cite the following special cases which are verified by explicit computation using (14)$^{*\text{F6.A}}$:

$$C_0(n) = \delta_{n,1}$$
$$C_1(n) = \omega(n)$$
$$C_2(n) = d(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n) + 1} - \gcd(\Omega(n), \omega(n)).$$

## 6.2   Uniform asymptotics of $C_k(n)$ for large all $n$ and fixed, bounded $k$

The next theorem makes precise what these formulas already suggest about the main terms of the growth rates of $C_k(n)$ as functions of $k, n$ for limiting cases of $n$ large and fixed $k$ which is necessarily bounded in $n$, but still taken as an independent parameter.

**Theorem 6.3** (Asymptotics for the functions $C_k(n)$). *For $k := 0$, we have by definition that $C_0(n) = \delta_{n,1}$. For all sufficiently large $n > 1$ and any fixed $1 \leqslant k \leqslant \Omega(n)$ taken independently of $n$, we obtain that the dominant asymptotic term for $C_k(n)$ is given uniformly by*

$$\mathbb{E}[C_k(n)] \gg (\log \log n)^{2k-1}, \text{ as } n \to \infty.$$

---

$^{*\text{F6.A}}$For all $n, k \geqslant 2$, we have the following recurrence relation satisfied by $C_k(n)$ between successive values of $k$:

$$C_k(n) = \sum_{p|n} \sum_{d | \frac{n}{p^{\nu_p(n)}}} \sum_{i=0}^{\nu_p(n)-1} C_{k-1}\left(dp^i\right), n \geqslant 1.$$

*Proof.* We prove our bounds by induction on $k$. We can see by Example 6.2 that $C_1(n)$ satsfies the formula we must establish when $k := 1$ since $\mathbb{E}[\omega(n)] = \log\log n$. Suppose that $k \geqslant 2$ and let the inductive assumption state that for all $1 \leqslant m < k$

$$\mathbb{E}[C_m(n)] \gg (\log\log n)^{2m-1}.$$

Now using the recursive formula we used to define the sequences of $C_k(n)$ in (14), we have that as $n \to \infty$*F6.B

$$\mathbb{E}[C_k(n)] = \mathbb{E}\left[\sum_{d|n} \omega(n/d)C_{k-1}(d)\right]$$

$$= \frac{1}{n} \times \sum_{d \leqslant n} C_{k-1}(d) \times \sum_{r=1}^{\lfloor\frac{n}{d}\rfloor} \omega(r)$$

$$\sim \sum_{d \leqslant n} C_{k-1}(d) \left[\frac{\log\log(n/d)\left[d \leqslant \frac{n}{e}\right]_\delta}{d} + \frac{B}{d}\right]$$

$$\sim \sum_{d \leqslant \frac{n}{e}} \left[\sum_{m<d} \frac{\mathbb{E}[C_{k-1}(m)]}{m}\log\log\left(\frac{n}{m}\right) + B \cdot \mathbb{E}[C_{k-1}(d)] + B \cdot \sum_{m<d} \frac{\mathbb{E}[C_{k-1}(m)]}{m}\right]$$

$$\gg B\left[n\log n \cdot (\log\log n)^{2k-3} - \log n \cdot (\log\log n)^{2k-3}\right] \times \left(1 + \frac{\log n}{2}\right)$$

$$\gg (\log\log n)^{2k-1}.$$

In transitioning to the last equation from the previous step, we have used that $\frac{Bn}{2} \cdot (\log n)^2 \gg (\log\log n)^2$ as $n \to \infty$. We have also used that for large $n \to \infty$ and fixed $m$, we have by an approximation to the incomplete gamma function given by

$$\int_e^n \frac{(\log\log t)^m}{t} \sim (\log n)(\log\log n)^m, \text{ as } n \to \infty.$$

Thus the claim holds by mathematical induction whenever $n \to \infty$ is large and $1 \leqslant k \leqslant \Omega(n)$. $\qquad\square$

## 6.3   Relating the auxiliary functions $C_k(n)$ to formulas approximating $g^{-1}(n)$

**Lemma 6.4** (An exact formula for $g^{-1}(n)$). *For all $n \geqslant 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

*Proof.* We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d) + 1)g^{-1}(n/d) \implies (g^{-1} * 1)(n) = -(\omega * g^{-1})(n).$$

Now by repeatedly expanding the right-hand-side, and removing corner cases in the nested sums with $\omega(1) = 0$, we find inductively that

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n)C_{\Omega(n)}(n).$$

The statement then follows by Möbius inversion applied to each side of the last equation. $\qquad\square$

---

*F6.B For all large $x \gg 2$ the summatory function of $\omega(n)$ satisfies [4, §22.10]

$$\sum_{n \leqslant x} \omega(n) = x\log\log x + Bx + O\left(\frac{x}{\log x}\right).$$

**Corollary 6.5.** *For all squarefree integers $n \geqslant 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \tag{15}$$

*Proof.* Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geqslant 2$ and that $n$ is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where $p_i$ is prime for all $1 \leqslant i \leqslant \omega(n)$. So we can transform the exact divisor sum guaranteed for all $n$ in Lemma 6.4 as follows:

$$g^{-1}(n) = \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i}(-1)^i \cdot C_{\Omega(d)}(d)$$

$$= \lambda(n) \times \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d)$$

$$= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

The signed contributions in the first of the previous equations is justified by noting that $\lambda(n) = (-1)^{\omega(n)}$ whenever $n$ is squarefree, and that for $d$ squarefree with $\omega(d) = i$, $\Omega(d) = i$. $\qquad\square$

**Corollary 6.6.** *We have that*

$$\frac{6}{\pi^2} \log x \ll \mathbb{E}|g^{-1}(n)| \leqslant \mathbb{E}\left[\sum_{d|n} C_{\Omega(d)}(d)\right].$$

*Proof.* To prove the lower bound, first notice that by Lemma 6.4, Proposition 4.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain that

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{16}$$

In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $n \geqslant 1$ we have $\mu(n) = (-1)^{\omega(n)} = \lambda(n)$, Lemma 6.4 implies

$$|g^{-1}(n)| = \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d)$$

$$= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d)$$

$$= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d).$$

Notice in the above equation that $\lambda(n^2) = +1$ for all $n \geqslant 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is necessarily even.

Recall from the introduction that the summatory function of the squarefree integers is given by

$$Q(x) := \sum_{n \leqslant x} \mu^2(n) = \frac{6}{\pi^2} x + O(\sqrt{x}).$$

Then since $C_{\Omega(d)}(d) \geqslant 1$ for all $d \geqslant 1$, we obtain by summing over (16) that as $x \to \infty$

$$\frac{1}{x} \times \sum_{n \leqslant x} |g^{-1}(n)| = \frac{1}{x} \times \sum_{d \leqslant x} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{x}{d} \right\rfloor\right)$$

$$\sim \sum_{d \leqslant x} C_{\Omega(d)}(d) \left[ \frac{6}{d \cdot \pi^2} + O\left( \frac{1}{\sqrt{dx}} \right) \right]$$

$$\geqslant \sum_{d \leqslant x} \left[ \frac{6}{d \cdot \pi^2} + O\left( \frac{1}{\sqrt{dx}} \right) \right]$$

$$\sim \frac{6}{\pi^2} \left( \log x + \gamma + O\left( \frac{1}{x} \right) \right) + O\left( \frac{1}{\sqrt{x}} \times \int_0^x t^{-1/2} dt \right)$$

$$= \frac{6}{\pi^2} \log x + O(1).$$

To prove the upper bound, notice that by Lemma 6.4 and Corollary 6.5,

$$|g^{-1}(n)| \leqslant \sum_{d|n} C_{\Omega(d)}(d).$$

Now since both of the above quantities are positive for all $n \geqslant 1$, we must obtain the upper bound on the average order of $|g^{-1}(n)|$ stated above. $\qquad \square$

Proposition 7.2 improves on the naïve lower bound on the average order expectations of $|g^{-1}(n)|$ proved in Corollary 6.6 for large $n \to \infty$. We rely on the proof mechanisms to this less subtle corollary above in establishing the refinement (by a factor of $\log \log n$) we require in the next section.

# 7    Establishing initial lower bounds on the summatory functions $G^{-1}(x)$

**Lemma 7.1** (Asymptotic densities of exceptional values of positive arithmetic functions)**.** *Let $F \in C^1(0, \infty)$ be a monotone non-decreasing function such that $F(x) \nrightarrow 0$ as $x \to \infty$. Suppose that $f$ is an arithmetic function such that $f(n) > 0$ for all $n \geqslant 1$ and so that*

$$\sum_{n \leqslant x} f(n) = x \cdot F(x) + o(xF(x)), \text{ as } x \to \infty.$$

*Let the set defined by*

$$\mathcal{F}_- := \{n \geqslant 1 : f(n) < F(n)\},$$

*have corresponding limiting upper and lower densities given by*

$$\gamma_-^{(-)} := \liminf_{x \to \infty} \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{F}_-\},$$

$$\gamma_-^{(+)} := \limsup_{x \to \infty} \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{F}_-\}.$$

*Then the limit $\gamma_- = \gamma_-^{(-)} = \gamma_-^{(+)}$ exists and $\gamma_- = 0$. In other words, for almost every sufficiently large $n \to \infty$, $f(n) \geqslant F(n)$ and $-f(n) \leqslant -F(n)$ where $\mathbb{E}[f(n)] = F(n)$ as $n \to \infty$.*

*Proof.* First, suppose that the the following limit exists with $\gamma_- \in [0, 1)$:

$$\gamma_- := \lim_{x \to \infty} \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{F}_-\}.$$

By the positivity of $f(n)$, we know that $F(x)$ is positive for all sufficiently large $x$. So we have that as $x \to \infty$

$$
\begin{aligned}
\sum_{n \leqslant x} f(n) &\leqslant \sum_{\substack{n \leqslant x \\ n \notin \mathcal{F}_-}} f(n) + \sum_{\substack{n \leqslant x \\ n \in \mathcal{F}_-}} F(n) \\
&< \sum_{n \leqslant (1-\gamma_-)x} f(n) + \sum_{(1-\gamma_-)x \leqslant n \leqslant x} F(n) \\
&\sim xF(x) - \int_{(1-\gamma_-)x}^{x} tF'(t)dt + o(xF(x)) \\
&= (1 - \gamma_-)x \cdot F((1 - \gamma_-)x) + \gamma_- x \cdot F(c) + o(xF(x)),
\end{aligned}
\tag{17}
$$

integrating by parts, and for some $c \in [(1 - \gamma_-)x, x]$ by the mean value theorem, respectively. So by (17) and the monotonicity of $F$ on $(0, x]$, we have that

$$\sum_{n \leqslant x} f(n) < x \cdot F(x) + o(xF(x)), \text{ as } x \to \infty.$$

Unless $\gamma_- = 0$ this property contradicts our hypothesis on the limiting behavior of the summatory function of $f(n)$. Notice also that the limiting density cannot be identically one since if $\gamma_- = 1$, then

$$
\begin{aligned}
\frac{1}{x} \times \sum_{n \leqslant x} f(n) &< \frac{1}{x} \times \sum_{n \leqslant x} F(n) \\
&\leqslant \max_{1 \leqslant j \leqslant x} F(j) + o(F(x)) = F(x) + o(F(x)),
\end{aligned}
$$

by the assumption of the monotonicity of $F$ on $(0, \infty)$. Hence, we conclude that $\gamma_- = 0$ provided that the limit exists.

If a limiting value for $\gamma_-$ does not exist, i.e., we have $0 \leqslant \gamma_-^{(-)} < \gamma_-^{(+)} \leqslant 1$, then for infinitely many large $x \geqslant 2$ within a set $\mathcal{M} \subset \mathbb{Z}^+$ where the density of $\mathcal{M} \cap \{n \leqslant x\}$ is positive, we have that

$$M_x := \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{F}_-\} \in (0,1),$$

where $M_x$ non-monotonically oscillates in value along a subsequence. Using a similar method to what we argued above, in this case there are infinitely many $x$ within a set of eventually always positive limiting densities such that our assumption on the asymptotic lower bound on the summatory function of $f(n)$ also does not hold. This contradiction shows that the limit $\gamma_-$ must in fact exist, and as we have shown above, is then necessarily zero. □

**Proposition 7.2.** *For all sufficiently large $n$ on a set of asymptotic density one, we have that*

$$|g^{-1}(n)| \gg \frac{3}{\pi^2}(\log n)^2(\log \log n).$$

*Proof.* Reusing part of the argument given in the proof of Corollary 6.6, recall that we have the following holds:

$$\mathbb{E}|g^{-1}(n)| \sim \sum_{d \leqslant n} C_{\Omega(d)}(d)\left[\frac{6}{\pi^2 d} + O\left(\frac{1}{\sqrt{nd}}\right)\right].$$

Then we obtain that*[F7.A]

$$\mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(n)}] + \sum_{d<n}\frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O\left(\mathbb{E}[C_{\Omega(n)}(n)] + \frac{1}{\sqrt{n}} \times \sum_{d<n}\frac{\mathbb{E}[C_{\Omega(d)}(d)]}{\sqrt{d}}\right). \tag{18}$$

Now using Theorem 6.3, we have proved that

$$\mathbb{E}[C_{\Omega(n)}(n)] \gg \frac{(\log \log n)^{2\Omega(n)}}{\log \log n}.$$

To prove our claimed bound, it suffices to require that

$$\frac{(\log \log n)^{2\Omega(n)}}{\log \log n} > (\log n)(\log \log n) \iff \Omega(n) > \frac{\log \log n}{2(\log \log \log n)} + 1. \tag{19}$$

We have an Erdös-Kac type theorem stating that for any real $z \in (-\infty, +\infty)$ with $\Phi(z) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^z e^{-t^2/2}dt$, we have [10, §7.4] [6, *cf.* §1.7]

$$\#\left\{n \leqslant x : \frac{\Omega(n) - \log \log n}{\sqrt{\log \log n}} > z\right\} = x\left(1 - \Phi(z)\right) + o(x).$$

Then taking

$$z := -\sqrt{\log \log x}\left(1 - \frac{1}{\log \log x} - \frac{1}{2(\log \log \log x)}\right) \xrightarrow{x \to \infty} -\infty,$$

we obtain that with $\Phi(z) = o(1)$

$$\#\left\{n \leqslant x : \Omega(n) > \frac{\log \log n}{2(\log \log \log n)} + 1\right\} = x(1 + o(1)).$$

---

*[F7.A]To see why the error term holds observe that

$$\frac{1}{\sqrt{d+1}} - \frac{1}{\sqrt{d}} \sim \frac{1 - \sqrt{1 + \frac{1}{d}}}{\sqrt{d}} = \frac{1}{d} \times \sum_{s \geqslant 1}\binom{\frac{1}{2}}{s}\frac{1}{d^s} = O\left(\frac{1}{d^{3/2}}\right).$$

Thus for almost every sufficiently large $n$, we have that (19) holds.

Then by (18), we obtain that*[F7.B]

$$\mathbb{E}|g^{-1}(n)| \geqslant \frac{6}{\pi^2}(\log n)(\log \log n) + \frac{6}{\pi^2} \times \int_e^n \frac{(\log t)(\log \log t)}{t} dt$$
$$+ O\left((\log n)(\log \log n) + \frac{1}{\sqrt{n}} \times \int_e^n \frac{(\log t)(\log \log t)}{\sqrt{t}}\right)$$
$$\geqslant \frac{3}{\pi^2}(\log n)^2(\log \log n) + O(\log n \cdot \log \log n).$$

Finally, to see that our claimed property holds on a set of asymptotic density one, we recall that by Lemma 7.1, we have that since $(\log n)^2 \cdot \log \log n \nrightarrow 0$ as $n \to \infty$

$$\#\left\{n \leqslant x : |g^{-1}(n)| \geqslant \frac{6}{\pi^2}(\log n)^2(\log \log n)\right\} = x(1 + o(1)), \text{ as } x \to \infty. \tag{20}$$

In other words, the claimed asymptotic lower bound for $|g^{-1}(n)|$ holds for almost every sufficiently large $n \in \mathbb{Z}^+$. $\qquad\square$

Let the summatory function $G_E^{-1}(x)$ be defined for $x \geqslant 1$ by*[F7.C]

$$G_E^{-1}(x) := \sum_{n \leqslant (\log x)^{\frac{3}{2}}(\log \log x)} \lambda(n) \times \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d}. \tag{21}$$

The next remark serves to clarify some of the intentional details behind the construction of our proof, and in particular, why we have defined an intentional lower bound for $|G^{-1}(x)|$ through this summatory function indexed only for $n \leqslant (\log x)^{\frac{3}{2}}(\log \log x) = o(x)$.

**Remark 7.3.** It is not difficult to prove using Proposition 7.2 that for almost every sufficiently large $n$, we have that

$$\sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} - |g^{-1}(n)| \leqslant 0. \tag{22}$$

This property follows by computing the expectation

$$\mathbb{E}\left[\sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d}\right] = \frac{1}{n} \times \sum_{e < d \leqslant n} \frac{(\log d)^{\frac{1}{4}}}{\log \log d}\left\lfloor\frac{n}{d}\right\rfloor \ll \frac{25}{64}(\log \log n)^2.$$

However, when we weight the differences in (22) by the signed $\lambda(n)$, it is in general non-trivial to reconcile the observed inequalities showing $|G^{-1}(x)| \gg |G_E^{-1}(x)|$ – even when we intentionally under sum over an index range bounded by some $U(x) = o(x)$ to form the function $G_E^{-1}(x)$.

---

*[F7.B]We integrate to form the limiting error term as

$$\frac{1}{\sqrt{n}} \times \int_e^n \frac{\log \log t}{\sqrt{t}} dt = 2\log \log n + \frac{4}{\sqrt{n}} \text{Ei}\left(\frac{\log n}{2}\right) + o(1)$$
$$\leqslant 2\log \log n + \frac{4}{\sqrt{n}}\left(1 + \log \log n - \log 2 - \frac{3\log n}{8} + \frac{11}{144}(\log n)^2\right) + o(1)$$
$$= O(\log n \cdot \log \log n), \text{ as } n \to \infty.$$

*[F7.C]The subscript of $E$ (as in to be evaluated in expectation) on the function $G_E^{-1}(x)$ is purely for formality of notation and does not correspond to an actual parameter or any implicit dependence on $E$ in the formula that defines this function.

Based on our observations so far, and in the proof of Theorem 7.4 given below, the challenge posed in resolving that just such an inequality to bound $|G^{-1}(x)|$ from below holds is fairly minimal when we take $U(x)$ to be a fixed, finite-degree polynomial in $\log x$. In our case, to prove the classical order unboundedness conjecture posited by Corollary 3.8, it suffices to allow indices up to $U(x) := (\log x)^{\frac{3}{2}}(\log\log x)$ (see the restrictions in Corollary 8.4). On the other hand, suppose that we can define

$$\widehat{G}_E^{-1}(x) := \sum_{n \leqslant x^{\varpi}} \lambda(n) \times \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log\log d}$$

for some small, absolute $0 < \varpi < \frac{1}{2}$. Then asserting that for all sufficiently large $x$ taken within an infinite set $\mathcal{S}_{\varpi} \subseteq \mathbb{Z}^{+}$ of bounded asymptotic density in $(0, 1]$, we have

$$|G^{-1}(x)| \gg |\widehat{G}_E^{-1}(x)|,$$

firmly encroaches on the territory of the RH. That is to say, proving an analog to the preliminaries we need to prove our corollary on unboundedness in Section 8.2 for a variant of (21) summed up to even a small positive polynomial power is intentionally as difficult as we surmise it should be based on the grounding characterization above.

What we recover in the form of the next theorem for *all* sufficiently large integers $x \to \infty$ is however uncomplicated and allows us a good enough window into proving the classical unboundedness result of the form stated by Corollary 3.8. For the sake of clarity and exposition, we will not complicate our method of proof used to show that the corollary holds at this point by appending notation to justify that we can in fact actually take improved logarithmically polynomial upper bounds on the sums in (21).

**Theorem 7.4.** *For almost all sufficiently large integers $x \to \infty$, we have that*

$$|G^{-1}(x)| \gg |G_E^{-1}(x)|.$$

*Proof.* We bound the magnitude of each respective function respectively below and above in the worst cases of the cancellation imparted by the signage on the otherwise positive terms when weighted by $\lambda(n)$. First, consider the following upper bound on $|G_E^{-1}(x)|$:

$$
\begin{aligned}
|G_E^{-1}(x)| &= \left| \sum_{e \leqslant n \leqslant (\log x)^{\frac{3}{2}}(\log\log x)} \lambda(n) \times \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log\log d} \right| \\
&\ll \sum_{e < d \leqslant (\log x)^{\frac{3}{2}}(\log\log x)} \frac{(\log d)^{\frac{1}{4}}}{\log\log d} \cdot \left\lfloor \frac{(\log x)^{\frac{3}{2}}(\log\log x)}{d} \right\rfloor \\
&\approx (\log x)^{\frac{3}{2}}(\log\log x) \times \int_{e}^{(\log x)^{\frac{3}{2}}(\log\log x)} \frac{(\log t)^{\frac{1}{4}}}{t \cdot \log\log t} dt \\
&= (\log x)^{\frac{3}{2}}(\log\log x) \times \operatorname{Ei}\left( \frac{5}{4} \log\log \left( (\log x)^{\frac{3}{2}}(\log\log x) \right) \right) \\
&\ll \frac{25}{64} \cdot (\log x)^{\frac{3}{2}}(\log\log x)(\log\log\log x)^2.
\end{aligned}
$$

Next, we bound the summatory function $|G^{-1}(x)|$ from below. For this, notice that from Lemma 6.4, we can see that $g^{-1}(n) = o(n)$. This observation combined with (20) shows that we can form this lower bound by taking an alternating sum involving the monotone bound of $\mathbb{E}|g^{-1}(n)| \gg \frac{6}{\pi^2}(\log n)^2(\log\log n)$ from Proposition 7.2 as follows:

$$|G^{-1}(x)| \gg \frac{6}{\pi^2} \times \left| (\log x)^2(\log\log x) - \sum_{n \leqslant x} (-1)^n \cdot (\log n)^2(\log\log n) \right|$$

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$$= \frac{6}{\pi^2} \times \left| (\log x)^2 (\log \log x) - \sum_{n \leqslant \frac{x}{2}} \left[ \log(2n)^2 \log \log(2n) - \log(2n+1)^2 \log \log(2n+1) \right] \right|$$

$$\gg \frac{6}{\pi^2} \times \left| (\log x)(\log \log x) + O(1) + \sum_{n \leqslant \frac{x}{2}} \left[ \frac{\log(2n)}{2n} + \frac{\log(2n) \log \log(2n)}{2n} \right] \right|$$

$$\gg \frac{6}{\pi^2} \times \left[ (\log x)^2 (\log \log x) + \frac{(\log x)^2}{4} + \frac{1}{4}(\log x)^2(\log \log x) \right]$$

$$\gg \frac{15}{2\pi^2} (\log x)^2 (\log \log x).$$

Now since for all sufficiently large $x \to \infty$, we have that

$$\frac{15}{2\pi^2}(\log x)^2(\log \log x) \gg \frac{25}{64} \cdot (\log x)^{\frac{3}{2}}(\log \log x)(\log \log \log x)^2,$$

we have that our claimed relation between the two key summatory functions holds. $\square$

# 8 Lower bounds for $M(x)$ along infinite subsequences

## 8.1 The culmination of what we have done so far

**Proposition 8.1.** *For all sufficiently large $x$, we have that*

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt. \tag{23}$$

*Proof.* We know by applying Corollary 3.3 that

$$M(x) = \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k) + 1)$$

$$\approx G^{-1}(x) + \sum_{k=1}^{x} g^{-1}(k)\pi(x/k), \tag{24}$$

We can replace the asymptotically unnecessary floored integer-valued arguments to $\pi(x)$ in (24) using its approximation by the monotone non-decreasing asymptotic order, $\pi(x) \sim \frac{x}{\log x}$. Moreover, we can always bound

$$\frac{Ax}{\log x} \leqslant \pi(x) \leqslant \frac{Bx}{\log x},$$

for suitably defined absolute constants, $A, B > 0$ whenever $x \geqslant 2$. Therefore the approximation obtained by replacing $\pi(x)$ by the dominant term in its limting asymptotic formula is actually valid for all $x > 1$ up to at most a small constant difference.

What we require to sum and simplify the right-hand-side terms from (24) is a summation by parts argument. In particular, we argue that for sufficiently large $x \geqslant 2$*[F8.A]

$$\sum_{k=1}^{x} g^{-1}(k)\pi(x/k) = G^{-1}(x)\pi(1) - \sum_{k=1}^{x-1} G^{-1}(k) \left[ \pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right) \right]$$

$$= -\sum_{k=1}^{x/2} G^{-1}(k) \left[ \pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right) \right]$$

$$\approx -\sum_{k=1}^{x/2} G^{-1}(k) \left[ \frac{x}{k \cdot \log(x/k)} - \frac{x}{(k+1) \cdot \log(x/k)} \right]$$

$$\approx -\sum_{k=1}^{x/2} G^{-1}(k) \frac{x}{k^2 \cdot \log(x/k)}.$$

Since for $x$ large enough the summand factor $\frac{x}{k^2 \cdot \log(x/k)}$ is monotonic as $k$ ranges over $k \in [1, x/2]$ in ascending order, since this summand factor is a smooth function of $k$ (and $x$), and since $G^{-1}(x)$ is a summatory function with jumps only at the positive integers, we can approximate $M(x)$ for any finite $x \geqslant 2$ by

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt.$$

We will later only use unsigned lower bound approximations to this function in the next theorems so that the signedness of the summatory function term in the integral formula above as $x \to \infty$ is a moot point entirely. $\square$

---

*[F8.A]Since $\pi(1) = 0$, the actual range of summation corresponds to $k \in \left[1, \frac{x}{2}\right]$.

### 8.1.1 A few more necessary results

We now use the superscript and subscript notation of $(\ell)$ not to denote a formal parameter to the functions we define below, but instead to denote that these functions form *lower bound* (rather than exact) approximations to other forms of the functions without the scripted $(\ell)$.

**Lemma 8.2.** *Suppose that $\widehat{\pi}_k(x) \geqslant \widehat{\pi}_k^{(\ell)}(x) \geqslant 0$ for $\widehat{\pi}_k^{(\ell)}(x)$ a monotone real-valued function of $x$ for all integers $k \geqslant 1$ whenever $x \geqslant 2$ is sufficiently large. Let*

$$A_\Omega^{(\ell)}(x) := \sum_{k \leqslant \log\log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x)$$

$$A_\Omega(x) := \sum_{k \leqslant \log\log x} (-1)^k \widehat{\pi}_k(x).$$

*Then for all sufficiently large $x$, we have that*

$$|A_\Omega(x)| \gg |A_\Omega^{(\ell)}(x)|.$$

*Proof.* Given an explicit smooth lower bounding function, $\widehat{\pi}_k^{(\ell)}(x)$, we define the similarly smooth and monotone residual terms in approximating $\widehat{\pi}_k(x)$ using the following notation:

$$\widehat{\pi}_k(x) = \widehat{\pi}_k^{(\ell)}(x) + \widehat{E}_k(x).$$

Then we can form the ordinary exact form of the summatory function as

$$|A_\Omega(x)| \gg \left| \sum_{k \leqslant \frac{\log\log x}{2}} \left[ \widehat{\pi}_{2k}(x) - \widehat{\pi}_{2k-1}(x) \right] \right|$$

$$\geqslant \left| A_\Omega^{(\ell)}(x) - \sum_{k \leqslant \frac{\log\log x}{2}} \left[ \widehat{E}_{2k}(x) - \widehat{E}_{2k-1}(x) \right] \right|$$

$$\geqslant \left| A_\Omega^{(\ell)}(x) \right| - \left| \sum_{k \leqslant \frac{\log\log x}{2}} \left[ \widehat{E}_{2k}(x) - \widehat{E}_{2k-1}(x) \right] \right|.$$

If the latter sum, denoted

$$\mathrm{ES}(x) := \left| \sum_{k \leqslant \frac{\log\log x}{2}} \left[ \widehat{E}_{2k}(x) - \widehat{E}_{2k-1}(x) \right] \right| \to \infty,$$

as $x \to \infty$, then we can always find some absolute $C_0 > 0$ (by monotonicity) such that $\mathrm{ES}(x) \leqslant C_0 \cdot A_\Omega(x)$:

$$\mathrm{ES}(x) = \left| A_\Omega(x) - A_\Omega^{(\ell)}(x) \right| \leqslant \left| |A_\Omega(x)| + \left| A_\Omega^{(\ell)}(x) \right| \right| \ll 2 \left| A_\Omega(x) \right|.$$

If on the other hand this sum becomes constant, or is bounded as $x \to +\infty$, then we also clearly have another absolute $C_1 > 0$ such that $|A_\Omega(x)| \geqslant C_1 \cdot |A_\Omega^{(\ell)}(x)|$. In either case, the claimed result holds for all large enough $x$. $\qquad\square$

**Lemma 8.3.** *Suppose that $f(n)$ is an arithmetic functions such that $f(n) > 0$ for all $n > u_0$ where $f(n) \gg \widehat{\tau}_\ell(n)$ as $n \to \infty$. Assume that the bounding function $\widehat{\tau}_\ell(t)$ is a non-negative continuously differentiable function of $t$ for all large enough $t \gg u_0$. We define the $\lambda$-sign-scaled summatory function of $f$ as follows:*

$$F_\lambda(x) := \sum_{u_0 < n \leqslant x} \lambda(n) \cdot f(n).$$

*Let*

$$A_{\Omega}^{(\ell)}(t) := \sum_{k=1}^{\lfloor \log \log t \rfloor} (-1)^k \widehat{\pi}_k^{(\ell)}(t),$$

$$A_{\Omega}(t) := \sum_{k=1}^{\lfloor \log \log t \rfloor} (-1)^k \widehat{\pi}_k(t),$$

*where $\widehat{\pi}_k(x) \geqslant \widehat{\pi}_k^{(\ell)}(x) \geqslant 0$ for $\widehat{\pi}_k^{(\ell)}(t)$ some smooth monotone function of $t$ at all sufficiently large $t \to \infty$. Then we have that*

$$|F_{\lambda}(x)| \gg \left| A_{\Omega}^{(\ell)}(x)\widehat{\tau}_{\ell}(x) - \int_{u_0}^{x} A_{\Omega}^{(\ell)}(t)\widehat{\tau}_{\ell}'(t)dt \right|. \tag{25}$$

*Proof.* We can form an accurate $C^1(\mathbb{R})$ approximation by the smoothness of $\widehat{\pi}_k^{(\ell)}(x)$ that allows us to apply the Abel summation formula using the summatory function $A_{\Omega}^{(\ell)}(t)$ for $t$ on any bounded connected subinterval of $[1,\infty)$. The stated lower bound formula for $F_{\lambda}(x)$ in (25) above is valid by Abel summation and by applying Lemma 8.2. In particular, whenever

$$0 \leqslant \left| \frac{\displaystyle\sum_{\log\log t < k \leqslant \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_{\Omega}(t)} \right| \ll 2, \text{ as } t \to \infty.$$

the asymptotically dominant terms indicating the parity of $\lambda(n)$ are captured up to a constant factor by the terms in the range over $k$ summed by $A_{\Omega}(t)$ for sufficiently large $t \to \infty$. In other words, taking the sum over the summands that defines $A_{\Omega}(x)$ only over the truncated range of $k \in [1, \log\log x]$ does not non-trivially change the limiting asymptotically dominant terms in the lower bound obtained from using this form of the summatory function in conjunction with the claimed Abel summation formula. This property holds even when we should technically index over all $k \in [1, \log_2(x)]$ to obtain an exact formula for the summatory weight function. By Corollary 5.5, we have that the assertion above holds as $t \to \infty$.

Secondly, observe that provided sufficiently smoothness (differentiability) of close approximations to $A_{\Omega}(t)$ (to $f(t)$) on $(u_0, x)$, we have that

$$|F_{\lambda}(x)| \geqslant \left| |A_{\Omega}(x)f(x)| - \int_{u_0}^{x} |A_{\Omega}(t)f'(t)|dt \right|$$

$$\gg \left| |A_{\Omega}^{(\ell)}(x)\widehat{\tau}_{\ell}(x)| - \int_{u_0}^{x} |A_{\Omega}^{(\ell)}(t)\widehat{\tau}_{\ell}'(t)|dt \right|.$$

The previous equations follow from the ordinary Abel summation method by applying the argument in Lemma 8.2 with the triangle inequality. $\qquad\square$

**Corollary 8.4.** *We have that for almost every sufficiently large $x$, that as $x \to \infty$ (TODO)*

$$\left|G_E^{-1}(x)\right| \gg \frac{2\sqrt{2}e\log 2}{3\pi} \times \frac{(\log x)^{\frac{3}{2}}(\log\log x)}{(\log\log x)\sqrt{\log\log\log x}} \times \left| \sum_{e < d \leqslant \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d \cdot \log\log d} \right|.$$

*Proof.* Using the definition in (21), we obtain on average that*[F8.B]

$$\left| G_E^{-1}(x) \right| = \left| \sum_{n \leqslant (\log x)^{\frac{3}{2}} (\log \log x)} \lambda(n) \times \sum_{\substack{d|n \\ d > e}} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log \log d} \right|$$

$$= \left| \sum_{e < d \leqslant (\log x)^{\frac{3}{2}} (\log \log x)} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} \times \sum_{n=1}^{\left\lfloor \frac{\log x}{d} \right\rfloor} \lambda(dn) \right|.$$

We see that by complete additivity of $\Omega(n)$ (complete multiplicativity of $\lambda(n)$) that

$$\sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn) = \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(d) \times \lambda(n) = \lambda(d) \times \sum_{n \leqslant \left\lfloor \frac{x}{d} \right\rfloor} \lambda(n).$$

Now using Theorem 3.7 and Lemma 8.2, we can establish that

$$\left| \sum_{n \leqslant x} \lambda(n) \right| \gg \left| \sum_{k \leqslant \log \log x} (-1)^k \cdot \widehat{\pi}_k(x) \right| \tag{26}$$

$$\gg \frac{2\sqrt{2}e \log 2}{3\pi} \cdot \frac{x}{(\log x)\sqrt{\log \log x}} =: \widehat{L}_0(x).$$

The sign of the sum obtained by taking the right-hand-side of (26) without the absolute value operation is given by $(-1)^{\lfloor \log \log x \rfloor}$. The precise formula for the limiting lower bound stated above for $\widehat{L}_0(x)$ is computed by symbolic summation in *Mathematica* using the new bounds on $\widehat{\pi}_k(x)$ guaranteed by the theorem, and then by applying subsequent standard asymptotic estimates to the resulting formulas for large $x \to \infty$, e.g., in the form of (8c) and Stirling's formula. It follows that

$$\left| G_E^{-1}(x) \right| \gg \left| \sum_{e < d \leqslant (\log x)^{\frac{3}{2}} (\log \log x)} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log \log d} \times (-1)^{\left\lfloor \log \log \left( \frac{(\log x)^{\frac{3}{2}} (\log \log x)}{d} \right) \right\rfloor} \cdot \widehat{L}_0 \left( \frac{(\log x)^{\frac{3}{2}} (\log \log x)}{d} \right) \right|. \tag{27}$$

**Outline for the remainder of the proof.** We sketch the following core sections remaining to prove our claimed lower bound on $|G_E^{-1}(x)|$:

**(A)** We identify and extract an initial subinterval of our full bounds on the summation defined by (21). On this subinterval we prove that we can expect constant sign term contributions from the inputs to the function $\widehat{L}_0$ involving (a priori) both $d, x$.

**(B)** We then factor out easily bounded terms from the expansion of the monotone $\widehat{L}_0$ on this interval.

**(C)** We define and determine additional characteristic formulas we will refer to in later sections for the resulting lower bounds that are formed by restricting the range of $d$ in (27) to just this initial subinterval range.

**(D)** Finally, being absolutely rigorous and careful with this approach, we must argue precisely that the oscillatory, signed terms from the upper end of the deleted interval cannot generate trivial bounds by cancellation with the stated lower bounds.

---

*[F8.B]For any arithmetic functions $f, h$, we have that [1, *cf.* §3.10; §3.12]

$$\sum_{n \leqslant x} h(n) \times \sum_{d|n} f(d) = \sum_{d \leqslant x} f(d) \times \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} h(dn).$$

The arguments used to establish the form of the lower bounds stated in this corollary are longer, and are certainly more involved than the proofs of our previous results so far. Since further disassembly of these distinct parts to proving the bound into external makes the overall logic to the article harder to interpret, we will continue to label subsections of the remaining proof components corresponding to the headings itemized as above to maintain clarity.

**Part A.** We will simplify (27) using an appeal to accessible contiguous ranges of consecutive integers over which we obtain constant sign contributions from the function $\widehat{L}_0((\log x)^{\frac{3}{2}}(\log \log x)/d)$ as a function of both $x, d$. We find such simplifications by modifying the limits of summation on $d$ to be bounded by a suitable function of $x$ so that for almost every large $x$ and with the exception of at most finitely many $d$ on this new interval, we obtain the constant sign from the weight on $\widehat{L}_0\left((\log x)^{\frac{3}{2}}(\log \log x)/d\right)$, or equivalently a constant parity for $\left\lfloor \log \log \left((\log x)^{\frac{3}{2}}(\log \log x)/d\right)\right\rfloor$ at such $d$.

An initial contiguous interval is not difficult to extract for large $x$, though for general $d \in \left(e, (\log x)^{\frac{3}{2}}(\log \log x)\right)$, the sign contributions from this weight function are muddled by a dual dependence on the fluctuations of both the fractional part of $\left\{(\log x)^{\frac{3}{2}}(\log \log x)\right\}$ and on the precise location of $d$ within the interval. The idea is to identify this initial accesible interval case, and then prove that we can form a lower bound on $G_E^{-1}(x)$ by truncating and summing only over the $d$ in this range.

In particular, consider that

$$\log \log \left(\frac{(\log x)^{\frac{3}{2}}(\log \log x)}{d}\right) = \log \log \left((\log x)^{\frac{3}{2}}(\log \log x)\right)$$

$$+ \log \left(1 - \frac{\log d}{(\log x)^{\frac{3}{2}}(\log \log x)\log\left((\log x)^{\frac{3}{2}}(\log \log x)\right)}\right), \quad \text{as } x \to \infty.$$

If we take $d \in (e, \log x] =: \mathcal{R}_x$, we have that

$$\frac{\log d}{(\log x)^{\frac{3}{2}}(\log \log x)\log\left((\log x)^{\frac{3}{2}}(\log \log x)\right)} = o(1) \to 0,$$

as $x \to \infty$. So it stands to reason that for $d$ taken within $\mathcal{R}_x$, we expect that for almost every $x$ there are at most a handful of negligible cases of comparitively small order $d \leqslant d_0(x)$ such that

$$\left\lfloor \log \log \left(\frac{(\log x)^{\frac{3}{2}}(\log \log x)}{d}\right)\right\rfloor \sim \left\lfloor \log \log \left((\log x)^{\frac{3}{2}}(\log \log x)\right) + o(1)\right\rfloor,$$

changes in parity from its value modulo 2 from $d_0(x) - 1$ to $d_0(x)$. An argument making this assertion precise brings leads us to two primary cases that rely on the distribution of the fractional parts of $\left\{(\log x)^{\frac{3}{2}}(\log \log x)\right\}$ within $[0, 1)$ for integers $x \to \infty$. Consider the following points for sufficiently large $x$ and any $\log d \in \mathcal{R}_x$:

**(1)** If the fractional part $\left\{\log \log \left((\log x)^{\frac{3}{2}}(\log \log x)\right)\right\} = 0$, then (TODO)

$$\left\lfloor \log \log \left(\frac{(\log x)^{\frac{3}{2}}(\log \log x)}{d}\right)\right\rfloor = \lfloor \log \log \log \log x\rfloor + \left\lfloor -\frac{\log d}{(\log \log x)(\log \log \log x)} + o\left(\frac{1}{(\log \log x)^2}\right)\right\rfloor.$$

This implies that provided that

$$-1 \leqslant -\frac{\log d}{(\log \log x)(\log \log \log x)} + o\left(\frac{1}{(\log \log x)^2}\right) < 0,$$

we obtain a constant sign term for $\text{sgn}\left\lfloor \widehat{L}_0\left(\frac{(\log x)^{\frac{3}{2}}(\log \log x)}{d}\right)\right\rfloor$. Since this happens for all sufficiently large $x$, we obtain the constant sign term for $d$ on $\mathcal{R}_x$.

**(2)** If the fractional part $\{\log\log\log\log x\} \in (0,1)$, then

$$\left\lfloor \log\log\left(\frac{\log\log x}{d}\right)\right\rfloor = \lfloor\log\log\log\log x\rfloor + \left\lfloor \{\log\log\log\log x\} - \frac{\log d}{(\log\log x)(\log\log\log x)} + o\left(\frac{1}{(\log\log x)^2}\right)\right\rfloor.$$

Now provided that

$$0 \leqslant (\log\log x)(\log\log\log x)\{\log\log\log\log x\} - \log d + o(1) < 1,$$

or that

$$-1 \leqslant (\log\log x)(\log\log\log x)\{\log\log\log\log x\} - \log d + o(1) < 0,$$

we similarly obtain the constant signedness of $\mathrm{sgn}\left[\widehat{L}_0\left(\frac{\log\log x}{d}\right)\right]$ for $d \in \mathcal{R}_x$ whenever $x$ is sufficiently large.

We do not delve too deeply into the distribution of the fractional parts $\{\log\log\log\log x\}$ throughout $[0,1)$ for our purposes here, since a rudimentary analysis by cases argument serves our purposes well. If $\{\log\log\log\log x\} \xrightarrow{x\to\infty} 0$, then we can proceed with the analysis we used to justify case **(1)** above as $x \to \infty$. Otherwise, we can assume that for almost every large enough $x$, the conditions in case **(2)** provide us with the necessary constant signedness for $d \in \mathcal{R}_x$ that we require as $x \to \infty$. We expect a non-uniform distribution along the second case analysis for almost every large $x$.

**Part B.** Then provided that the sign term involving both $d$ and $x$ from (27) does not change for $d$ within our new interval, $\mathcal{R}_x$, we can factor out the dependence of the sign on the monotonically decreasing function $\widehat{L}_0(\log\log x/d)$ in the variable $d$ as we sum along the lower interval $\mathcal{R}_x$. We can see that the function $\widehat{L}_0(\log\log x/d)$ is decreasing for $d \in \mathcal{R}_x$, as can be viewed by computing its first derivative as

$$\frac{\partial}{\partial d}\left[\widehat{L}_0\left(\frac{\log\log x}{d}\right)\right] = \frac{2\sqrt{2}e\log 2 \cdot (\log\log x)\left[1 + 2\log\log\left(\frac{\log\log x}{d}\right) - \log\left(\frac{\log\log x}{d}\right)\log\log\left(\frac{\log\log x}{d}\right)^2\right]}{3\pi \cdot d^2 \cdot \log\left(\frac{\log\log x}{d}\right)^2 \log\log\left(\frac{\log\log x}{d}\right)^{\frac{3}{2}}},$$

and observing that the sign on this function is negative for all sufficiently large $x$ and for $d$ taken within the smaller interval. So we determine that we should select $d := \log\log\log x$ in (27) to obtain a global lower bound on $|G_E^{-1}(x)|$ if we truncate the sum defined by (21) to only indices $d \in \mathcal{R}_x$.

**Part C.** We require a couple of key arguments to ensure that we have covered all problematic corner cases that can arise in forming the lower bound based on a truncation of the interval of summation from the original definition of $G_E^{-1}(x)$ to summing only over $d \in \mathcal{R}_x$:

**(i)** Our bounds make sense as stated when we remove the signed contributions from the so-termed extraneous terms added by summing over $\bar{\mathcal{R}}_x := \left(\log\log\log x, \frac{\log\log x}{e}\right)$. To make this point we will define several shorthands for component functions to the bound below.

**(ii)** We cannot arrive at trivial lower bounds that tend to zero as $x \to \infty$ due to cancellation of the lower bounds based on $d \in \mathcal{R}_x$ with the more oscillatory signed terms corresponding to the sums over $d > \log\log\log x$ (for $d$ on $\bar{\mathcal{R}}_x$). This argument is not sophisticated in light of the intentional lack of tightness (or optimality in the largest possible order sense) of the upper bound we have chosen on the interval $\mathcal{R}_x$. In fact, we can evaluate a routine sum for the absolute contribution of the unsigned remainder terms to show that it cannot non-trivially interfere with the primary terms in the bound when $x$ is large.

Let the magnitudes of the oscillatory remainder term sums be defined for all sufficiently large $x$ by

$$R_E(x) := \left| \sum_{\log\log\log x < d < \frac{\log\log x}{e}} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\lfloor \log\log\left(\frac{\log x}{d}\right)\rfloor} \cdot \widehat{L}_0\left(\frac{\log x}{d}\right)\right|.$$

Next, let the function $T_E(x)$ be defined for all large enough $x$ as

$$T_E(x) := \frac{1}{\log\left(\frac{\log\log x}{\log\log\log x}\right)\sqrt{\log\log\left(\frac{\log\log x}{\log\log\log x}\right)}} \gg \frac{1}{(\log\log\log x)\sqrt{\log\log\log\log x}}. \tag{28}$$

Then, as we argued before, we see that as $x \to \infty$

$$
\begin{aligned}
S_{E,1}(x) &:= \left| \sum_{e < d \leqslant \log\log\log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log\log d} \times (-1)^{\lfloor \log\log\left(\frac{\log\log x}{d}\right)\rfloor} \widehat{L}_0\left(\frac{\log\log x}{d}\right) \right| \\
&\gg \frac{2\sqrt{2}e\log 2}{3\pi} \times (\log\log x) T_E(x) \times \left| \sum_{e < d \leqslant \log\log\log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d \cdot \log\log d} \right| \\
&\gg \frac{2\sqrt{2}e\log 2}{3\pi} \times (\log\log x) T_E(x) \times \left| A_\Omega^{(\ell)}(\log\log\log x)\widehat{\tau}_0(\log\log\log x) - \int_e^{\log\log\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_0'(t)dt \right|,
\end{aligned}
\tag{29}
$$

where we select the functions $\widehat{\tau}_0(t) := \frac{(\log t)^{1/4}}{t \cdot \log\log t}$ and $-\widehat{\tau}_0'(t) \gg \frac{(\log t)^{1/4}}{t^2 \cdot \log\log t}$ in the notation of Lemma 8.3.

What we then obtain from (27) and (29) is the following lower bound by the triangle inequality that holds for all sufficiently large $x$:

$$|G_E^{-1}(x)| \gg \left| S_{E,1}(x) - R_E(x) \right| \gg S_{E,1}(x), \text{ as } x \to \infty. \tag{30}$$

**Part D.** We have claimed that in fact we can drop the sum terms over upper range of $d$ and still obtain an asymptotic lower bound on $|G_E^{-1}(x)|$ as $x \to \infty$ on the right-hand-side of (30). To justify this step in the proof, we will provide asymptotic lower bounds on $R_E(x)$ that show that the contribution from these terms in absolute value exceeds the magnitude of the corresponding sums for $d \in \mathcal{R}_x$ when $x$ is large. In other words, the magnitudes of the terms in the sums defining the difference of the functions in (30) cannot match in approximate order to cause a zero-tending cancellation of the difference of terms in the middle expression from (30) when $x$ is taken sufficiently large.

In Theorem 8.5 stated below, we prove lower bounds on the sums we used to define $S_{E,1}(x)$ above of the form

$$S_{E,1}(x) \gg \frac{(\log\log x)}{(\log\log\log x)(\log\log\log\log x)^{\frac{5}{4}}(\log\log\log\log\log x)^{\frac{3}{2}}},$$

where the lower bounds on the right-hand-side of the previous equation are clearly $o(\log\log x)$, though still grows without bound as $x \to \infty$. In contrast, we can bound from below to show that the contribution from $R_E(x)$ is at least on the order of a constant times $\log\log x$. To obtain this lower bound, note that since $\frac{(\log d)^{\frac{1}{4}}}{d \cdot \log\log d}$ is monotone decreasing for all large enough $d > e$, we obtain the smallest possible magnitude on the sum by alternating signs on consecutive terms in the sum. So we have

$$
\begin{aligned}
\frac{R_E(x)}{\log\log x} &\gg \left| \frac{(\log\log\log x)^{1/4}}{(\log\log x)(\log\log\log\log x)} - \sum_{\log\log\log x < d < \frac{\log\log x}{2e}} \frac{\log(2d)^{1/4}}{2d \cdot \log\log(2d)} - \frac{\log(2d+1)^{1/4}}{(2d+1)\log\log(2d+1)} \right| \\
&\sim \left| \frac{(\log\log\log x)^{1/4}}{(\log\log x)(\log\log\log\log x)} - \sum_{\log\log\log x < d < \frac{\log\log x}{2e}} \frac{\log(2d)^{1/4}}{2d \cdot \log\log(2d)} - \frac{1}{(2d+1)} \frac{\left(\log(2d)+\frac{1}{2d}\right)^{1/4}}{\left(\log\log(2d)+\frac{1}{2d\cdot\log(2d)}\right)} \right| \\
&\approx \left| \frac{(\log\log\log x)^{1/4}}{(\log\log x)(\log\log\log\log x)} - \sum_{\log\log\log x < d < \frac{\log\log x}{2e}} \frac{\log(2d)^{1/4}}{\log\log(2d)} \left[ \frac{1}{2d} - \frac{\left(1+\frac{1}{2d\cdot\log(2d)}\right)^{1/4}}{(2d+1)\left(1+\frac{1}{2d\cdot\log(2d)\,\log\log(2d)}\right)} \right] \right|.
\end{aligned}
$$

Then by an appeal to binomial and geometric series expansions, we obtain that the significant terms in the above sum are given by*[F8.C]

$$\frac{R_E(x)}{\log\log x} \gg \left| \frac{(\log\log\log x)^{1/4}}{(\log\log x)(\log\log\log\log x)} \right.$$

$$- \sum_{\log\log\log x < d < \frac{\log\log x}{2e}} \frac{\log(2d)^{1/4}}{\log\log(2d)} \left[ \frac{1}{2d} - \frac{1}{(2d+1)} \left( 1 + \frac{1}{8d\cdot\log(2d)} - \frac{1}{2d\cdot\log(2d)\log\log(2d)} \right) \right] \Bigg|$$

$$= \left| \frac{(\log\log\log x)^{1/4}}{(\log\log x)(\log\log\log\log x)} - \sum_{\log\log\log x < d < \frac{\log\log x}{2e}} O\left( \frac{\log(2d)^{1/4}}{2d(2d+1)\cdot\log\log(2d)} \right) \right|$$

$$= O\left( 1 + \frac{(\log\log\log x)^{1/4}}{(\log\log x)(\log\log\log\log x)} \right).$$

In total, what we obtain is that the magnitude of $R_E(x)$ always exceeds that of the lower bound we establish in Theorem 8.5 for the sums over $d \in \mathcal{R}_x$. So we obtain the lower bounds on $G_E(x)$ that correspond to the smaller order terms resulting from the first summation ranges above to be bounded by the functions stated in Theorem 8.5 below. $\qquad\square$

### 8.1.2 The proof of a central lower bound on the magnitude of $G_E^{-1}(x)$

The next central theorem is the last key barrier required to prove Corollary 3.8 in the next subsection. Combined with Theorem 7.4 proved in the last section, the new lower bounds we establish below provide us with a sufficient mechanism to bound the formula from Proposition 8.1. Since these lower bounds tend to $+\infty$ as $x \to \infty$ (along an infinite subsequence of positive integers), this is a sufficient condition to guarantee the unboundedness of the scaled Mertens function of the form claimed in the corollary.

**Theorem 8.5** (Asymptotics and bounds for the summatory function $G^{-1}(x)$). *We define a lower summatory function, $G_\ell^{-1}(x)$, to provide bounds on the magnitude of $G_E^{-1}(x)$ such that*

$$|G_E^{-1}(x)| \gg |G_\ell^{-1}(x)|,$$

*for all sufficiently large $x > e$. Let $C_{\ell,1} > 0$ be the absolute constant defined by*

$$C_{\ell,1} = \frac{8e^2 \log^2(2)}{9\pi^2} \approx 0.319733.$$

*We obtain the following limiting estimate for the bounding function $G_\ell^{-1}(x)$ as $x \to \infty$:*

$$\left| G_\ell^{-1}(x) \right| \gg \frac{C_{\ell,1}\cdot(\log x)^{\frac{3}{2}}}{2\cdot(\log\log x)^{\frac{3}{4}}(\log\log\log x)^2}.$$

*Proof.* Recall from our proof of Corollary 3.7 that a lower bound on the variant prime form counting function, $\widehat{\pi}_k(x)$, is given by

$$\widehat{\pi}_k(x) \gg \frac{4}{3\sqrt{\pi}} \frac{x}{\log x} \left( \frac{\log 2}{\log x} \right)^{\frac{k-1}{\log\log x}} \frac{(\log\log x)^{k-1}}{(k-1)!} \left( 1 + O\left( \frac{k}{(\log\log x)^2} \right) \right), \quad \text{as } x \to \infty.$$

---

*[F8.C]In particular, with $|2d\log(2d)|^{-1}, |2d\log(2d)\log\log(2d)|^{-1} < 1$ we can compute that

$$\left( 1 + \frac{1}{2d\cdot\log(2d)} \right)^{\frac{1}{4}} \times \left( 1 + \frac{1}{2d\cdot\log(2d)\cdot\log\log(2d)} \right)^{-1} = 1 + \frac{1}{8d\cdot\log(2d)} - \frac{1}{2d\cdot\log(2d)\cdot\log\log(2d)} + O\left( \frac{1}{d^2} \right).$$

We can then form a lower summatory function indicating the signed contributions over the distinct parity of $\Omega(n)$ for all $n \leqslant x$ as follows by applying (8b) and Stirling's approximation as already noted in the proof of Corollary 8.4 given above:

$$
\begin{aligned}
\left| A_\Omega^{(\ell)}(t) \right| = & \left| \sum_{k \leqslant \log \log t} (-1)^k \widehat{\pi}_k(t) \right| \\
& \gg \frac{2\sqrt{2}e \log 2}{3\pi} \cdot \frac{t}{(\log t)\sqrt{\log \log t}} \left( 1 + O\left( \frac{1}{\log \log t} \right) \right) \\
& \gg \frac{2\sqrt{2}e \log 2}{3\pi} \cdot \frac{t}{(\log t)\sqrt{\log \log t}}, \text{ as } t \to \infty.
\end{aligned}
\tag{31}
$$

The actual sign on this function is given by $\text{sgn}(A_\Omega^{(\ell)}(t)) = (-1)^{\lfloor \log \log t \rfloor}$ (see Lemma 8.2). By Lemma 8.3 we know that this summatory function forms a lower bound in absolute value for the actual weight of the signed terms indicated by $\lambda(n)$.

As we determined in (29) from the proof of Corollary 8.4, we take the function $\widehat{\tau}_0(t) = \frac{(\log t)^{1/4}}{t \cdot \log \log t}$ that satisfies

$$
-\widehat{\tau}_0'(t) = -\frac{d}{dt}\left[ \frac{(\log t)^{\frac{1}{4}}}{t \cdot \log \log t} \right] \gg \frac{(\log t)^{1/4}}{t^2 \cdot \log \log t}.
$$

Moreover, we have using the notation from the proof above that we can select the initial form of the lower bound function $G_\ell^{-1}(x)$ to be defined as follows:

$$
G_\ell^{-1}(x) := \frac{2\sqrt{2}e \log 2}{3\pi} \cdot (\log x)^{\frac{3}{2}} (\log \log x) \cdot T_E(x) \times \left| A_\Omega^{(\ell)}(\log x)\widehat{\tau}_0(\log x) - \int_e^{\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_0'(t)dt \right|.
\tag{32}
$$

The inner integral term on the rightmost side of (32) is summed approximately by splitting the terms weighted by $(-1)^{\lfloor \log \log t \rfloor}$ in the form of*[F8.D]

$$
\frac{2\sqrt{2}e \log 2}{3\pi} \times \left| \int_e^{\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_0'(t)dt \right| \gg \frac{2\sqrt{2}e \log 2}{3\pi} \times \left| \sum_{k=e+1}^{\frac{1}{2}\log \log\left[ (\log x)^{\frac{3}{2}} (\log \log x) \right]} \left[ I_\ell\left( e^{e^{2k+1}} \right) - I_\ell\left( e^{e^{2k}} \right) \right] e^{e^{2k}} \right|
\tag{33}
$$

$$
\gg \frac{2\sqrt{2}e \log 2}{3\pi} \times \left| \int_{\frac{1}{2}\log \log\left[ (\log x)^{\frac{3}{2}} (\log \log x) \right] - \frac{1}{2}}^{\frac{1}{2}\log \log\left[ (\log x)^{\frac{3}{2}} (\log \log x) \right]} I_\ell\left( e^{e^{2k}} \right) e^{e^{2k}} dk \right|.
$$

We express the integrand function,

$$
I_\ell(t) := \frac{2\sqrt{2}e \log 2}{3\pi} \times \widehat{\tau}_0'(t)A_\Omega^{(\ell)}(t),
$$

defined implicitly as in (33) as the following function of $k$:

$$
I_\ell\left( e^{e^{2k}} \right) e^{e^{2k}} \gg \frac{2e^2 \log^2(2)}{9\pi^2} \cdot \frac{e^{-\frac{3k}{2}}}{k^{\frac{3}{2}}} =: \widehat{I}_\ell(k).
\tag{34}
$$

---

*[F8.D]That is, we form the disjoint union of the range of integration into subintervals along which the signedness of the integrands are constant according to

$$
\left\{ e^e \leqslant t \leqslant (\log x)^{\frac{3}{2}} (\log \log x) : (-1)^{\lfloor \log \log t \rfloor} = +1 \right\} = \left( \bigcup_{k=1}^{\frac{1}{2}(\log x)^{\frac{3}{2}} (\log \log x)} \left[ e^{e^{2k}}, e^{e^{2k+1}} \right) \right) \bigcup \mathcal{S}_{0,+},
$$

where $|\mathcal{S}_{0,+}| \leqslant \frac{1}{2}$. We can similarly split the interval of integration corresponding to the negatively biased terms on the unsigned integrand functions for $t \in \left[ e^e, (\log x)^{\frac{3}{2}} (\log \log x) \right]$.

So upon input of the upper bound on the range of integration in (33), at the point $k := \dfrac{\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]}{2}$, we find from the mean value theorem with the monotone function from (34) that

$$\frac{2\sqrt{2}e\log 2}{3\pi} \times (\log x)^{\frac{3}{2}}(\log\log x) \times T_E(x) \times \left|\int_{\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]-\frac{1}{2}}^{\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]} I_\ell\left(e^{e^{2k}}\right)e^{e^{2k}}\,dk\right|$$

$$\gg \frac{2\sqrt{2}e\log 2}{3\pi} \times (\log x)^{\frac{3}{2}}(\log\log x) \times T_E(x) \times \left|\widehat{I}_\ell\left(\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]\right)\right|$$

$$\gg \frac{C_{\ell,1}\cdot(\log x)^{\frac{3}{2}}}{2\cdot(\log\log x)^{\frac{3}{4}}(\log\log\log x)^2}. \tag{35}$$

Similarly, by evaluating $\widehat{I}_\ell(t)$ at the lower bound on the integral above with $k := \dfrac{\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]-1}{2}$, we can similarly conclude that

$$\frac{2\sqrt{2}e\log 2}{3\pi} \times (\log x)^{\frac{3}{2}}(\log\log x) \times T_E(x) \times \left|\int_{\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]-\frac{1}{2}}^{\frac{1}{2}\log\log\left[(\log x)^{\frac{3}{2}}(\log\log x)\right]} I_\ell\left(e^{e^{2k}}\right)e^{e^{2k}}\,dk\right|$$

$$\ll \frac{e^{3/4}\cdot C_{\ell,1}\cdot(\log x)^{\frac{3}{2}}}{2\cdot(\log\log x)^{\frac{3}{4}}(\log\log\log x)^2}. \tag{36}$$

To make it clear which terms in (32) the limiting lower bounds correspond to, consider the following expansion for the leading term in the Abel summation formula from (32) for comparison with (35):

$$\frac{2\sqrt{2}e\log 2}{3\pi} \times (\log x)^{\frac{3}{2}}(\log\log x) \times T_E(x) \times \left|\widehat{\tau}_0(\log x)A_\Omega^{(\ell)}(\log x)\right|$$

$$\gg \frac{C_{\ell,1}\cdot(\log x)^{\frac{3}{2}}}{(\log\log x)^{\frac{3}{4}}(\log\log\log x)^2}. \tag{37}$$

Hence, by Lemma 8.2 and the triangle inequality, we conclude that we can take $\left|G_\ell^{-1}(x)\right|$ bounded below by the term in (35). $\square$

**Remark 8.6.** What is key to observe about the distinct lower bounds obtained in the proof of the previous theorem is that each of them scaled by $(\log x)^{-1}$ monotonically increases without bound as $x \to \infty$. In particular, the remaining factor after rescaling dominates the asymptotics of the reciprocal powers of iterated logarithms. It is fortunate, and a sign of our correct calculations, that up to distinct constant factors, the asymptotic orders of each of (35), (36) and (37) match identically. We expect this correspondence to be somewhat of a rarity that still coincides in these cases even though one of these terms is formed by a product of component functions, where the other two correspond to distinct particular values of another separate product of related functions over which we perform a definite integral operation.

## 8.2   Proof of the unboundedness of the scaled Mertens function

We finally address the main conclusion of our arguments given so far with the following proof:

*Proof of Corollary 3.8.* We break up the integral term in Proposition 8.1 over $t \in [u_0, x/2]$ into two pieces: one that is easily bounded from $u_0 \leqslant t \leqslant \sqrt{x}$, and then another that will conveniently give us our slow-growing tendency towards infinity along the subsequence when evaluated using Theorem 8.5. Given a fixed large infinitely tending $x$, we have some (at least one) point $x_0 \in \left[\sqrt{x}, \frac{x}{2}\right]$ defined such that $|G^{-1}(t)|$ is minimal and non-vanishing as

$$\left|G^{-1}(x_0)\right| := \min_{\substack{\sqrt{x}\leqslant t\leqslant\frac{x}{2} \\ G^{-1}(t)\neq 0}} |G^{-1}(t)|.$$

We can then apply Proposition 8.1 to bound

$$
\begin{aligned}
\frac{|M(x)|}{\sqrt{x}} &= \frac{1}{\sqrt{x}}\left| G^{-1}(x) - x \cdot \int_{1}^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt \right| \\
&\gg \left| \sqrt{x} \times \int_{\sqrt{x}}^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt \right| \\
&\gg \left| \int_{\sqrt{x_0}}^{\frac{x}{2}} \frac{2\sqrt{x_0}}{t^2 \cdot \log(x_0)} dt \right| \times \left( \min_{\sqrt{x} \leqslant t \leqslant \frac{x}{2}} |G^{-1}(t)| \right) \\
&\gg \frac{2\left|G^{-1}(x_0)\right|}{\log(x_0)}.
\end{aligned}
\tag{38}
$$

Then by Theorem 7.4 proved in Section 7, the result in (38) implies that

$$
\frac{|M(x)|}{\sqrt{x}} \gg \frac{2\left|G_E^{-1}(x_0)\right|}{\log(x_0)}.
\tag{39}
$$

Define the infinite increasing subsequence, $\{x_{0,y}\}_{y \geqslant Y_0}$, by $x_{0,y} := e^{2e^{e^{e^{2y}}}}$ for some sufficiently large finite integer $Y_0 \gg 1$. When we assume that $x \mapsto x_{0,y}$ is taken along this subsequence, we can transform the bound in the last equation into a statement about a lower bound for $|M(x)|\log x/\sqrt{x}$ along an infinitely tending subsequence in the following form by applying Theorem 8.5 to (39):

$$
\frac{|M(x_{0,y})|\log(x_{0,y})}{\sqrt{x_{0,y}}} \gg \frac{2C_{\ell,1} \cdot (\log\log\sqrt{x_{0,y}})}{(\log\log\log\sqrt{x_{0,y}})(\log\log\log\log\sqrt{x_{0,y}})^{\frac{5}{4}}(\log\log\log\log\log\sqrt{x_{0,y}})^{\frac{3}{2}}}, \text{ as } y \to \infty.
\tag{40}
$$

Notice by a small, but insightful, technicality in stating (40), we are not actually asserting that $|M(x)|\log x/\sqrt{x}$ grows unbounded along the precise subsequence of $x \mapsto x_{0,y}$. Rather, we are asserting that the unboundedness of this function can be witnessed along some subsequence $\hat{x}_{0,y} \in \left[\sqrt{x_{0,y}}, \frac{x_{0,y}}{2}\right]$, where we conveniently choose to state the lower bound given on the right-hand-side of (40) using the monotonicity of the lower bound on $|G_E^{-1}(x)|$ we proved in Theorem 8.5. We also can verify that for sufficiently large $y \to \infty$, this infinitely tending subsequence is well defined as $\hat{x}_{0,y+1} > \hat{x}_{0,y}$ for all sufficiently large $y \geqslant Y_0$.

Finally, we can compute that

$$
\lim_{x \to \infty} \left[ \frac{(\log\log x)}{(\log\log\log x)(\log\log\log\log x)^{\frac{5}{4}}(\log\log\log\log\log x)^{\frac{3}{2}}} \right] = +\infty.
$$

The scaled Mertens function is then unbounded in the limit supremum sense, as we have claimed, since the right-hand-side of (40) tends to positive infinity as $x_{0,y} \to \infty$, or equivalently as $y \to \infty$. $\qquad \square$

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# A   Appendix: Notation to express asymptotic relations

We preface the article starting with this section detailing our precise definitions, meanings and assumptions on the uses of certain symbols, operators, and relations as $x \to \infty$ [13, *cf.* §2] [2].

## A.1   Average order, similarity and approximation of asymptotic growth rates

### A.1.1   Similarity and average order (expectation)

We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if

$$\lim_{x \to \infty} \frac{A(x)}{B(x)} = 1.$$

It is conventional to express the *average order* of an arithmetic function f as $f \sim h$, even when the values of $f(n)$ may actually non-monotonically oscillate in magnitude infinitely often. What the notation $f \sim h$ means when using this notation to express the average order of $f$ is that

$$\frac{1}{x} \cdot \sum_{n \leqslant x} f(n) \sim h(x).$$

For example, in the tradition of [4] we would normally write that $\Omega(n) \sim \log\log n$, even though technically, $1 \leqslant \Omega(n) \leqslant \frac{\log n}{\log 2}$ where $\Omega(n)$ attains values along this entire bounded range in $n$ infinitely often. To be absolutely clear about notation, we intentionally do not re-use the $\sim$ relation by instead writing $\mathbb{E}[f(x)] = h(x)$ (as in expectation of $f$), or sometimes $\overset{\mathbb{E}}{\sim}$ for convenience, to denote that $f$ has a limiting average order growing at the rate of $h$.

### A.1.2   Abel summation

The formula we prefer for the Abel summation form of summation by parts is stated as follows [1, *cf.* §4.3]*FA.A:

**Proposition A.1** (Abel Summation Integral Formula)**.** *Suppose that $t, y > 0$ are real-valued, and that $A(t) \sim \sum_{n \leqslant t} a(n)$ for some weighting arithmetic function $a(n)$ with $A(t)$ continuously differentiable on $(0, \infty) \cap (y, \infty)$. Furthermore, suppose that $b(n) \sim f(n)$ with $f$ a differentiable function of $n \geqslant 0$. That is, $f'(t)$ exists and is smooth for all $t \in (0, \infty) \cap (y, \infty)$. Then for $0 \leqslant y < x$ we have that*

$$\sum_{y < n \leqslant x} a(n)b(n) \sim A(x)b(x) - A(y)b(y) - \int_y^x A(t)f'(t)dt.$$

### A.1.3   Approximation

We adopt the convention that $f(x) \approx g(x)$ if $|f(x) - g(x)| = O(1)$ as $x \to \infty$. That is, we write $f(x) \approx g(x)$ to denote that $f$ is approximately equal to $g$ at $x$ modulo at most a small constant difference between the functions when $x$ is large.

---

*FA.ACompare to the exact formula for *summation by parts* of any arithmetic functions, $u_n, v_n$, stated as in [13, §2.10(ii)] for $U_j := u_1 + u_2 + \cdots + u_j$ when $j \geqslant 1$:

$$\sum_{j=1}^{n-1} u_j \cdot v_j = U_{n-1}v_n + \sum_{j=1}^{n-1} U_j \left( v_j - v_{j+1} \right), n \geqslant 2.$$

### A.1.4 Vinogradov's notation for asymptotics

We use the relations $f(x) \gg g(x)$ and $h(x) \ll r(x)$ to symbolically express that we should expect $f$ to be "substantially" larger than $g$, and respectively $h$ to be "significantly" smaller than $r$, in asymptotic order (e.g., rate of growth when $x$ is large). In practice, we adopt a somewhat looser definition of these symbols which allows $f \gg g$ and $h \ll r$ provided that there are constants $C, D \geqslant 1$ such that whenever $x$ is sufficiently large we have that $C \cdot f(x) \geqslant g(x)$ and $h(x) \leqslant D \cdot r(x)$. This notation is sometimes called *Vinogradov's asymptotic notation*.

## A.2 Asymptotic expansions and uniformity

We introduce the notation for asymptotic expansions of a function $f : \mathbb{R} \to \mathbb{R}$ from [13, §2.1(iii)] in the next subsections.

### A.2.1 Ordinary asymptotic expansions of a function

Let $\sum_n a_n x^{-n}$ denote a formal power series expansion in $x$ where we ignore any necessary conditions to guarantee convergence of the series. For each integer $n \geqslant 1$, suppose that

$$f(x) = \sum_{s=0}^{n-1} a_s x^{-s} + O(x^{-n}),$$

as $|x| \to \infty$ where this limiting bound holds for $x \in \mathbb{X}$ in some unbounded set $\mathbb{X} \subseteq \mathbb{C}$. When such a bound holds, we say that $\sum_s a_s x^{-s}$ is a *Poincaré asymptotic expansion*, or just an *asymptotic series expansion*, of $f(x)$ as $|x| \to \infty$ along the fixed set $\mathbb{X}$. The condition in the previous equation is equivalent to writing

$$f(x) \sim a_0 + a_1 x^{-1} + a_2 x^{-2} + \cdots \, ; x \in \mathbb{X}, \text{ as } |x| \to \infty.$$

The prior two characterizations of an asymptotic expansion for $f$ are also equivalent to the statement that

$$x^n \left( f(x) - \sum_{s=0}^{n-1} a_s x^{-s} \right) \xrightarrow{x \to \infty} a_n.$$

### A.2.2 Uniform asymptotic expansions of a function

Let the set $\mathbb{X}$ from the definition in the last subsection correspond to a closed sector of the form

$$\mathbb{X} := \{ x \in \mathbb{C} : \alpha \leqslant \arg(x) \leqslant \beta \}.$$

Then we say that the asymptotic expansion

$$f(x) = \sum_{s=0}^{n-1} a_s x^{-s} + O(x^{-n}),$$

holds *uniformly* with respect to $\arg(x) \in [\alpha, \beta]$ as $|x| \to \infty$.

Another useful notion of uniform asymptotic bounds is taken with respect to some parameter $u$ (or set of parameters, respectively) that ranges over the point set (point sets, respectively) $u \in \mathbb{U}$. In this case, if we have that the $u$-parameterized expressions

$$\left| x^n \left( f(u, x) - \sum_{s=0}^{n-1} a_s(u) x^{-s} \right) \right|,$$

are bounded for all integers $n \geqslant 1$ with $x \in \mathbb{X}$ as $|x| \to \infty$, then we say that the asymptotic expansion of $f$ holds *uniformly* for $u \in \mathbb{U}$. Now the function $f \equiv f(u, x)$ and the asymptotic series coefficients $a_s(u)$ may have an implicit dependence on the parameter $u$. If the previous boundedness condition holds for all positive integers $n$, we write that

$$f(u, x) \sim \sum_{s=0}^{\infty} a_s(u) x^{-s}; x \in \mathbb{X}, \text{ as } |x| \to \infty,$$

and say that this asymptotic expansion holds *uniformly with respect to* $u \in \mathbb{U}$. For $u$ taken outside of $\mathbb{U}$, the stated limiting bound may fail to be valid even for $x \in \mathbb{X}$ as $|x| \to \infty$.

## A.3   Limiting densities of subsets of the integers

In the proofs given in Section 7 of the article, we will require a precise notion of the *asymptotic density* of a set $\mathcal{S} \subseteq \mathbb{Z}^+$. When this limit exists, we denote the asymptotic density of $\mathcal{S}$ by $\alpha_{\mathcal{S}} \in [0, 1]$, defined as follows:

$$\alpha_{\mathcal{S}} := \lim_{x \to \infty} \frac{1}{x} \cdot \# \{n \leqslant x : n \in \mathcal{S}\}.$$

In other words, if the set $\mathcal{S}$ has asymptotic density $\alpha_{\mathcal{S}}$, then for all sufficiently large $x$

$$\alpha_{\mathcal{S}} + o(1) \leqslant \frac{1}{x} \cdot \# \{n \leqslant x : n \in \mathcal{S}\} \leqslant \alpha_{\mathcal{S}} + o(1).$$

Clearly, finite or bounded subsets of the positive integers have limiting asymptotic density of zero. If the asymptotic density of $\mathcal{S}$ is one, and some property $\mathcal{P}(n)$ holds for all $n \in \mathcal{S}$, then we say that $\mathcal{P}(n)$ is true *almost everywhere* (on the integers), also abbreviated as holding "a.e." on the positive integers as $n \to \infty$.

When the limit definition of $\alpha_{\mathcal{S}}$ does not exist, or if by some pathology of the way $\mathcal{S}$ is defined we cannot express the exact limit, we are often interested in sets of bounded asymptotic density in $(0, 1)$. We can define the following operators on any infinite subset $\mathcal{S} \subseteq \mathbb{Z}^+$:

$$\underline{d}(\mathcal{S}) := \liminf_{x \to \infty} \frac{1}{x} \cdot |\mathcal{S} \cap \{1, 2, \ldots, x\}|,$$
$$\overline{d}(\mathcal{S}) := \limsup_{x \to \infty} \frac{1}{x} \cdot |\mathcal{S} \cap \{1, 2, \ldots, x\}|.$$

Then we say that $\mathcal{S}$ has *bounded (asymptotic) density* in $(0, 1)$ if both $\underline{d}(\mathcal{S}), \overline{d}(\mathcal{S}) \in (0, 1)$.

# T.1 Table: The Dirichlet inverse function $g^{-1}(n)$ and the distribution of its summatory function

| $n$ | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$ | $\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $1^1$ | Y | N | 1 | 0 | 1.0000000 | 1.000000 | 0.000000 | 1 | 1 | 0 |
| 2 | $2^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.500000 | 0.500000 | $-1$ | 1 | $-2$ |
| 3 | $3^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.333333 | 0.666667 | $-3$ | 1 | $-4$ |
| 4 | $2^2$ | N | Y | 2 | 0 | 1.5000000 | 0.500000 | 0.500000 | $-1$ | 3 | $-4$ |
| 5 | $5^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.400000 | 0.600000 | $-3$ | 3 | $-6$ |
| 6 | $2^1 3^1$ | Y | N | 5 | 0 | 1.0000000 | 0.500000 | 0.500000 | 2 | 8 | $-6$ |
| 7 | $7^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.428571 | 0.571429 | 0 | 8 | $-8$ |
| 8 | $2^3$ | N | Y | $-2$ | 0 | 2.0000000 | 0.375000 | 0.625000 | $-2$ | 8 | $-10$ |
| 9 | $3^2$ | N | Y | 2 | 0 | 1.5000000 | 0.444444 | 0.555556 | 0 | 10 | $-10$ |
| 10 | $2^1 5^1$ | Y | N | 5 | 0 | 1.0000000 | 0.500000 | 0.500000 | 5 | 15 | $-10$ |
| 11 | $11^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.454545 | 0.545455 | 3 | 15 | $-12$ |
| 12 | $2^2 3^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.416667 | 0.583333 | $-4$ | 15 | $-19$ |
| 13 | $13^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.384615 | 0.615385 | $-6$ | 15 | $-21$ |
| 14 | $2^1 7^1$ | Y | N | 5 | 0 | 1.0000000 | 0.428571 | 0.571429 | $-1$ | 20 | $-21$ |
| 15 | $3^1 5^1$ | Y | N | 5 | 0 | 1.0000000 | 0.466667 | 0.533333 | 4 | 25 | $-21$ |
| 16 | $2^4$ | N | Y | 2 | 0 | 2.5000000 | 0.500000 | 0.500000 | 6 | 27 | $-21$ |
| 17 | $17^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.470588 | 0.529412 | 4 | 27 | $-23$ |
| 18 | $2^1 3^2$ | N | N | $-7$ | 2 | 1.2857143 | 0.444444 | 0.555556 | $-3$ | 27 | $-30$ |
| 19 | $19^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.421053 | 0.578947 | $-5$ | 27 | $-32$ |
| 20 | $2^2 5^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.400000 | 0.600000 | $-12$ | 27 | $-39$ |
| 21 | $3^1 7^1$ | Y | N | 5 | 0 | 1.0000000 | 0.428571 | 0.571429 | $-7$ | 32 | $-39$ |
| 22 | $2^1 11^1$ | Y | N | 5 | 0 | 1.0000000 | 0.454545 | 0.545455 | $-2$ | 37 | $-39$ |
| 23 | $23^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.434783 | 0.565217 | $-4$ | 37 | $-41$ |
| 24 | $2^3 3^1$ | N | N | 9 | 4 | 1.5555556 | 0.458333 | 0.541667 | 5 | 46 | $-41$ |
| 25 | $5^2$ | N | Y | 2 | 0 | 1.5000000 | 0.480000 | 0.520000 | 7 | 48 | $-41$ |
| 26 | $2^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.500000 | 0.500000 | 12 | 53 | $-41$ |
| 27 | $3^3$ | N | Y | $-2$ | 0 | 2.0000000 | 0.481481 | 0.518519 | 10 | 53 | $-43$ |
| 28 | $2^2 7^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.464286 | 0.535714 | 3 | 53 | $-50$ |
| 29 | $29^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.448276 | 0.551724 | 1 | 53 | $-52$ |
| 30 | $2^1 3^1 5^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.433333 | 0.566667 | $-15$ | 53 | $-68$ |
| 31 | $31^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.419355 | 0.580645 | $-17$ | 53 | $-70$ |
| 32 | $2^5$ | N | Y | $-2$ | 0 | 3.0000000 | 0.406250 | 0.593750 | $-19$ | 53 | $-72$ |
| 33 | $3^1 11^1$ | Y | N | 5 | 0 | 1.0000000 | 0.424242 | 0.575758 | $-14$ | 58 | $-72$ |
| 34 | $2^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.441176 | 0.558824 | $-9$ | 63 | $-72$ |
| 35 | $5^1 7^1$ | Y | N | 5 | 0 | 1.0000000 | 0.457143 | 0.542857 | $-4$ | 68 | $-72$ |
| 36 | $2^2 3^2$ | N | N | 14 | 9 | 1.3571429 | 0.472222 | 0.527778 | 10 | 82 | $-72$ |
| 37 | $37^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.459459 | 0.540541 | 8 | 82 | $-74$ |
| 38 | $2^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473684 | 0.526316 | 13 | 87 | $-74$ |
| 39 | $3^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487179 | 0.512821 | 18 | 92 | $-74$ |
| 40 | $2^3 5^1$ | N | N | 9 | 4 | 1.5555556 | 0.500000 | 0.500000 | 27 | 101 | $-74$ |
| 41 | $41^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.487805 | 0.512195 | 25 | 101 | $-76$ |
| 42 | $2^1 3^1 7^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.476190 | 0.523810 | 9 | 101 | $-92$ |
| 43 | $43^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.465116 | 0.534884 | 7 | 101 | $-94$ |
| 44 | $2^2 11^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.454545 | 0.545455 | 0 | 101 | $-101$ |
| 45 | $3^2 5^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.444444 | 0.555556 | $-7$ | 101 | $-108$ |
| 46 | $2^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.456522 | 0.543478 | $-2$ | 106 | $-108$ |
| 47 | $47^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.446809 | 0.553191 | $-4$ | 106 | $-110$ |
| 48 | $2^4 3^1$ | N | N | $-11$ | 6 | 1.8181818 | 0.437500 | 0.562500 | $-15$ | 106 | $-121$ |

**Table T.1:** *Computations with* $\mathbf{g^{-1}(n)} \equiv (\omega + 1)^{-1}(\mathbf{n})$ *for* $1 \leqslant \mathbf{n} \leqslant 500$.

▶ *The column labeled* `Primes` *provides the prime factorization of each $n$ so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled* `Sqfree` *and* `PPower`*, respectively, list inclusion of $n$ in the sets of squarefree integers and the prime powers.*

▶ *The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.*

▶ *The last several columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_\pm(x) := \frac{1}{n} \cdot \# \{n \leqslant x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geqslant 1$.*

| $n$ | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 49 | $7^2$ | N | Y | 2 | 0 | 1.5000000 | 0.448980 | 0.551020 | $-13$ | 108 | $-121$ |
| 50 | $2^1 5^2$ | N | N | $-7$ | 2 | 1.2857143 | 0.440000 | 0.560000 | $-20$ | 108 | $-128$ |
| 51 | $3^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.450980 | 0.549020 | $-15$ | 113 | $-128$ |
| 52 | $2^2 13^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.442308 | 0.557692 | $-22$ | 113 | $-135$ |
| 53 | $53^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.433962 | 0.566038 | $-24$ | 113 | $-137$ |
| 54 | $2^1 3^3$ | N | N | 9 | 4 | 1.5555556 | 0.444444 | 0.555556 | $-15$ | 122 | $-137$ |
| 55 | $5^1 11^1$ | Y | N | 5 | 0 | 1.0000000 | 0.454545 | 0.545455 | $-10$ | 127 | $-137$ |
| 56 | $2^3 7^1$ | N | N | 9 | 4 | 1.5555556 | 0.464286 | 0.535714 | $-1$ | 136 | $-137$ |
| 57 | $3^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473684 | 0.526316 | 4 | 141 | $-137$ |
| 58 | $2^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482759 | 0.517241 | 9 | 146 | $-137$ |
| 59 | $59^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.474576 | 0.525424 | 7 | 146 | $-139$ |
| 60 | $2^2 3^1 5^1$ | N | N | 30 | 14 | 1.1666667 | 0.483333 | 0.516667 | 37 | 176 | $-139$ |
| 61 | $61^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.475410 | 0.524590 | 35 | 176 | $-141$ |
| 62 | $2^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 40 | 181 | $-141$ |
| 63 | $3^2 7^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.476190 | 0.523810 | 33 | 181 | $-148$ |
| 64 | $2^6$ | N | Y | 2 | 0 | 3.5000000 | 0.484375 | 0.515625 | 35 | 183 | $-148$ |
| 65 | $5^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492308 | 0.507692 | 40 | 188 | $-148$ |
| 66 | $2^1 3^1 11^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.484848 | 0.515152 | 24 | 188 | $-164$ |
| 67 | $67^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.477612 | 0.522388 | 22 | 188 | $-166$ |
| 68 | $2^2 17^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.470588 | 0.529412 | 15 | 188 | $-173$ |
| 69 | $3^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478261 | 0.521739 | 20 | 193 | $-173$ |
| 70 | $2^1 5^1 7^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.471429 | 0.528571 | 4 | 193 | $-189$ |
| 71 | $71^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.464789 | 0.535211 | 2 | 193 | $-191$ |
| 72 | $2^3 3^2$ | N | N | $-23$ | 18 | 1.4782609 | 0.458333 | 0.541667 | $-21$ | 193 | $-214$ |
| 73 | $73^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.452055 | 0.547945 | $-23$ | 193 | $-216$ |
| 74 | $2^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.459459 | 0.540541 | $-18$ | 198 | $-216$ |
| 75 | $3^1 5^2$ | N | N | $-7$ | 2 | 1.2857143 | 0.453333 | 0.546667 | $-25$ | 198 | $-223$ |
| 76 | $2^2 19^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.447368 | 0.552632 | $-32$ | 198 | $-230$ |
| 77 | $7^1 11^1$ | Y | N | 5 | 0 | 1.0000000 | 0.454545 | 0.545455 | $-27$ | 203 | $-230$ |
| 78 | $2^1 3^1 13^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.448718 | 0.551282 | $-43$ | 203 | $-246$ |
| 79 | $79^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.443038 | 0.556962 | $-45$ | 203 | $-248$ |
| 80 | $2^4 5^1$ | N | N | $-11$ | 6 | 1.8181818 | 0.437500 | 0.562500 | $-56$ | 203 | $-259$ |
| 81 | $3^4$ | N | Y | 2 | 0 | 2.5000000 | 0.444444 | 0.555556 | $-54$ | 205 | $-259$ |
| 82 | $2^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.451220 | 0.548780 | $-49$ | 210 | $-259$ |
| 83 | $83^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.445783 | 0.554217 | $-51$ | 210 | $-261$ |
| 84 | $2^2 3^1 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.452381 | 0.547619 | $-21$ | 240 | $-261$ |
| 85 | $5^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.458824 | 0.541176 | $-16$ | 245 | $-261$ |
| 86 | $2^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.465116 | 0.534884 | $-11$ | 250 | $-261$ |
| 87 | $3^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.471264 | 0.528736 | $-6$ | 255 | $-261$ |
| 88 | $2^3 11^1$ | N | N | 9 | 4 | 1.5555556 | 0.477273 | 0.522727 | 3 | 264 | $-261$ |
| 89 | $89^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.471910 | 0.528090 | 1 | 264 | $-263$ |
| 90 | $2^1 3^2 5^1$ | N | N | 30 | 14 | 1.1666667 | 0.477778 | 0.522222 | 31 | 294 | $-263$ |
| 91 | $7^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483516 | 0.516484 | 36 | 299 | $-263$ |
| 92 | $2^2 23^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.478261 | 0.521739 | 29 | 299 | $-270$ |
| 93 | $3^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 34 | 304 | $-270$ |
| 94 | $2^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.489362 | 0.510638 | 39 | 309 | $-270$ |
| 95 | $5^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.494737 | 0.505263 | 44 | 314 | $-270$ |
| 96 | $2^5 3^1$ | N | N | 13 | 8 | 2.0769231 | 0.500000 | 0.500000 | 57 | 327 | $-270$ |
| 97 | $97^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.494845 | 0.505155 | 55 | 327 | $-272$ |
| 98 | $2^1 7^2$ | N | N | $-7$ | 2 | 1.2857143 | 0.489796 | 0.510204 | 48 | 327 | $-279$ |
| 99 | $3^2 11^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.484848 | 0.515152 | 41 | 327 | $-286$ |
| 100 | $2^2 5^2$ | N | N | 14 | 9 | 1.3571429 | 0.490000 | 0.510000 | 55 | 341 | $-286$ |
| 101 | $101^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.485149 | 0.514851 | 53 | 341 | $-288$ |
| 102 | $2^1 3^1 17^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.480392 | 0.519608 | 37 | 341 | $-304$ |
| 103 | $103^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.475728 | 0.524272 | 35 | 341 | $-306$ |
| 104 | $2^3 13^1$ | N | N | 9 | 4 | 1.5555556 | 0.480769 | 0.519231 | 44 | 350 | $-306$ |
| 105 | $3^1 5^1 7^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.476190 | 0.523810 | 28 | 350 | $-322$ |
| 106 | $2^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.481132 | 0.518868 | 33 | 355 | $-322$ |
| 107 | $107^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.476636 | 0.523364 | 31 | 355 | $-324$ |
| 108 | $2^2 3^3$ | N | N | $-23$ | 18 | 1.4782609 | 0.472222 | 0.527778 | 8 | 355 | $-347$ |
| 109 | $109^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.467890 | 0.532110 | 6 | 355 | $-349$ |
| 110 | $2^1 5^1 11^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.463636 | 0.536364 | $-10$ | 355 | $-365$ |
| 111 | $3^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.468468 | 0.531532 | $-5$ | 360 | $-365$ |
| 112 | $2^4 7^1$ | N | N | $-11$ | 6 | 1.8181818 | 0.464286 | 0.535714 | $-16$ | 360 | $-376$ |
| 113 | $113^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.460177 | 0.539823 | $-18$ | 360 | $-378$ |
| 114 | $2^1 3^1 19^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.456140 | 0.543860 | $-34$ | 360 | $-394$ |
| 115 | $5^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.460870 | 0.539130 | $-29$ | 365 | $-394$ |
| 116 | $2^2 29^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.456897 | 0.543103 | $-36$ | 365 | $-401$ |
| 117 | $3^2 13^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.452991 | 0.547009 | $-43$ | 365 | $-408$ |
| 118 | $2^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.457627 | 0.542373 | $-38$ | 370 | $-408$ |
| 119 | $7^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.462185 | 0.537815 | $-33$ | 375 | $-408$ |
| 120 | $2^3 3^1 5^1$ | N | N | $-48$ | 32 | 1.3333333 | 0.458333 | 0.541667 | $-81$ | 375 | $-456$ |
| 121 | $11^2$ | N | Y | 2 | 0 | 1.5000000 | 0.462810 | 0.537190 | $-79$ | 377 | $-456$ |
| 122 | $2^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467213 | 0.532787 | $-74$ | 382 | $-456$ |
| 123 | $3^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.471545 | 0.528455 | $-69$ | 387 | $-456$ |
| 124 | $2^2 31^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.467742 | 0.532258 | $-76$ | 387 | $-463$ |

| $n$ | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 125 | $5^3$ | N | Y | $-2$ | 0 | 2.0000000 | 0.464000 | 0.536000 | $-78$ | 387 | $-465$ |
| 126 | $2^1 3^2 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.468254 | 0.531746 | $-48$ | 417 | $-465$ |
| 127 | $127^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.464567 | 0.535433 | $-50$ | 417 | $-467$ |
| 128 | $2^7$ | N | Y | $-2$ | 0 | 4.0000000 | 0.460938 | 0.539062 | $-52$ | 417 | $-469$ |
| 129 | $3^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.465116 | 0.534884 | $-47$ | 422 | $-469$ |
| 130 | $2^1 5^1 13^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.461538 | 0.538462 | $-63$ | 422 | $-485$ |
| 131 | $131^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.458015 | 0.541985 | $-65$ | 422 | $-487$ |
| 132 | $2^2 3^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.462121 | 0.537879 | $-35$ | 452 | $-487$ |
| 133 | $7^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.466165 | 0.533835 | $-30$ | 457 | $-487$ |
| 134 | $2^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.470149 | 0.529851 | $-25$ | 462 | $-487$ |
| 135 | $3^3 5^1$ | N | N | 9 | 4 | 1.5555556 | 0.474074 | 0.525926 | $-16$ | 471 | $-487$ |
| 136 | $2^3 17^1$ | N | N | 9 | 4 | 1.5555556 | 0.477941 | 0.522059 | $-7$ | 480 | $-487$ |
| 137 | $137^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.474453 | 0.525547 | $-9$ | 480 | $-489$ |
| 138 | $2^1 3^1 23^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.471014 | 0.528986 | $-25$ | 480 | $-505$ |
| 139 | $139^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.467626 | 0.532374 | $-27$ | 480 | $-507$ |
| 140 | $2^2 5^1 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.471429 | 0.528571 | 3 | 510 | $-507$ |
| 141 | $3^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475177 | 0.524823 | 8 | 515 | $-507$ |
| 142 | $2^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478873 | 0.521127 | 13 | 520 | $-507$ |
| 143 | $11^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482517 | 0.517483 | 18 | 525 | $-507$ |
| 144 | $2^4 3^2$ | N | N | 34 | 29 | 1.6176471 | 0.486111 | 0.513889 | 52 | 559 | $-507$ |
| 145 | $5^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.489655 | 0.510345 | 57 | 564 | $-507$ |
| 146 | $2^1 73^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493151 | 0.506849 | 62 | 569 | $-507$ |
| 147 | $3^1 7^2$ | N | N | $-7$ | 2 | 1.2857143 | 0.489796 | 0.510204 | 55 | 569 | $-514$ |
| 148 | $2^2 37^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.486486 | 0.513514 | 48 | 569 | $-521$ |
| 149 | $149^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.483221 | 0.516779 | 46 | 569 | $-523$ |
| 150 | $2^1 3^1 5^2$ | N | N | 30 | 14 | 1.1666667 | 0.486667 | 0.513333 | 76 | 599 | $-523$ |
| 151 | $151^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.483444 | 0.516556 | 74 | 599 | $-525$ |
| 152 | $2^3 19^1$ | N | N | 9 | 4 | 1.5555556 | 0.486842 | 0.513158 | 83 | 608 | $-525$ |
| 153 | $3^2 17^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.483660 | 0.516340 | 76 | 608 | $-532$ |
| 154 | $2^1 7^1 11^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.480519 | 0.519481 | 60 | 608 | $-548$ |
| 155 | $5^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 65 | 613 | $-548$ |
| 156 | $2^2 3^1 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.487179 | 0.512821 | 95 | 643 | $-548$ |
| 157 | $157^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.484076 | 0.515924 | 93 | 643 | $-550$ |
| 158 | $2^1 79^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487342 | 0.512658 | 98 | 648 | $-550$ |
| 159 | $3^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490566 | 0.509434 | 103 | 653 | $-550$ |
| 160 | $2^5 5^1$ | N | N | 13 | 8 | 2.0769231 | 0.493750 | 0.506250 | 116 | 666 | $-550$ |
| 161 | $7^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.496894 | 0.503106 | 121 | 671 | $-550$ |
| 162 | $2^1 3^4$ | N | N | $-11$ | 6 | 1.8181818 | 0.493827 | 0.506173 | 110 | 671 | $-561$ |
| 163 | $163^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.490798 | 0.509202 | 108 | 671 | $-563$ |
| 164 | $2^2 41^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.487805 | 0.512195 | 101 | 671 | $-570$ |
| 165 | $3^1 5^1 11^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.484848 | 0.515152 | 85 | 671 | $-586$ |
| 166 | $2^1 83^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487952 | 0.512048 | 90 | 676 | $-586$ |
| 167 | $167^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.485030 | 0.514970 | 88 | 676 | $-588$ |
| 168 | $2^3 3^1 7^1$ | N | N | $-48$ | 32 | 1.3333333 | 0.482143 | 0.517857 | 40 | 676 | $-636$ |
| 169 | $13^2$ | N | Y | 2 | 0 | 1.5000000 | 0.485207 | 0.514793 | 42 | 678 | $-636$ |
| 170 | $2^1 5^1 17^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.482353 | 0.517647 | 26 | 678 | $-652$ |
| 171 | $3^2 19^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.479532 | 0.520468 | 19 | 678 | $-659$ |
| 172 | $2^2 43^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.476744 | 0.523256 | 12 | 678 | $-666$ |
| 173 | $173^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.473988 | 0.526012 | 10 | 678 | $-668$ |
| 174 | $2^1 3^1 29^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.471264 | 0.528736 | $-6$ | 678 | $-684$ |
| 175 | $5^2 7^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.468571 | 0.531429 | $-13$ | 678 | $-691$ |
| 176 | $2^4 11^1$ | N | N | $-11$ | 6 | 1.8181818 | 0.465909 | 0.534091 | $-24$ | 678 | $-702$ |
| 177 | $3^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.468927 | 0.531073 | $-19$ | 683 | $-702$ |
| 178 | $2^1 89^1$ | Y | N | 5 | 0 | 1.0000000 | 0.471910 | 0.528090 | $-14$ | 688 | $-702$ |
| 179 | $179^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.469274 | 0.530726 | $-16$ | 688 | $-704$ |
| 180 | $2^2 3^2 5^1$ | N | N | $-74$ | 58 | 1.2162162 | 0.466667 | 0.533333 | $-90$ | 688 | $-778$ |
| 181 | $181^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.464088 | 0.535912 | $-92$ | 688 | $-780$ |
| 182 | $2^1 7^1 13^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.461538 | 0.538462 | $-108$ | 688 | $-796$ |
| 183 | $3^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.464481 | 0.535519 | $-103$ | 693 | $-796$ |
| 184 | $2^3 23^1$ | N | N | 9 | 4 | 1.5555556 | 0.467391 | 0.532609 | $-94$ | 702 | $-796$ |
| 185 | $5^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.470270 | 0.529730 | $-89$ | 707 | $-796$ |
| 186 | $2^1 3^1 31^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.467742 | 0.532258 | $-105$ | 707 | $-812$ |
| 187 | $11^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.470588 | 0.529412 | $-100$ | 712 | $-812$ |
| 188 | $2^2 47^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.468085 | 0.531915 | $-107$ | 712 | $-819$ |
| 189 | $3^3 7^1$ | N | N | 9 | 4 | 1.5555556 | 0.470899 | 0.529101 | $-98$ | 721 | $-819$ |
| 190 | $2^1 5^1 19^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.468421 | 0.531579 | $-114$ | 721 | $-835$ |
| 191 | $191^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.465969 | 0.534031 | $-116$ | 721 | $-837$ |
| 192 | $2^6 3^1$ | N | N | $-15$ | 10 | 2.3333333 | 0.463542 | 0.536458 | $-131$ | 721 | $-852$ |
| 193 | $193^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.461140 | 0.538860 | $-133$ | 721 | $-854$ |
| 194 | $2^1 97^1$ | Y | N | 5 | 0 | 1.0000000 | 0.463918 | 0.536082 | $-128$ | 726 | $-854$ |
| 195 | $3^1 5^1 13^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.461538 | 0.538462 | $-144$ | 726 | $-870$ |
| 196 | $2^2 7^2$ | N | N | 14 | 9 | 1.3571429 | 0.464286 | 0.535714 | $-130$ | 740 | $-870$ |
| 197 | $197^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.461929 | 0.538071 | $-132$ | 740 | $-872$ |
| 198 | $2^1 3^2 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.464646 | 0.535354 | $-102$ | 770 | $-872$ |
| 199 | $199^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.462312 | 0.537688 | $-104$ | 770 | $-874$ |
| 200 | $2^3 5^2$ | N | N | $-23$ | 18 | 1.4782609 | 0.460000 | 0.540000 | $-127$ | 770 | $-897$ |

| $n$ | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\mid g^{-1}(n)\mid}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 201 | $3^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.462687 | 0.537313 | $-122$ | 775 | $-897$ |
| 202 | $2^1 101^1$ | Y | N | 5 | 0 | 1.0000000 | 0.465347 | 0.534653 | $-117$ | 780 | $-897$ |
| 203 | $7^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467980 | 0.532020 | $-112$ | 785 | $-897$ |
| 204 | $2^2 3^1 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.470588 | 0.529412 | $-82$ | 815 | $-897$ |
| 205 | $5^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473171 | 0.526829 | $-77$ | 820 | $-897$ |
| 206 | $2^1 103^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475728 | 0.524272 | $-72$ | 825 | $-897$ |
| 207 | $3^2 23^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.473430 | 0.526570 | $-79$ | 825 | $-904$ |
| 208 | $2^4 13^1$ | N | N | $-11$ | 6 | 1.8181818 | 0.471154 | 0.528846 | $-90$ | 825 | $-915$ |
| 209 | $11^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473684 | 0.526316 | $-85$ | 830 | $-915$ |
| 210 | $2^1 3^1 5^1 7^1$ | Y | N | 65 | 0 | 1.0000000 | 0.476190 | 0.523810 | $-20$ | 895 | $-915$ |
| 211 | $211^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.473934 | 0.526066 | $-22$ | 895 | $-917$ |
| 212 | $2^2 53^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.471698 | 0.528302 | $-29$ | 895 | $-924$ |
| 213 | $3^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.474178 | 0.525822 | $-24$ | 900 | $-924$ |
| 214 | $2^1 107^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476636 | 0.523364 | $-19$ | 905 | $-924$ |
| 215 | $5^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.479070 | 0.520930 | $-14$ | 910 | $-924$ |
| 216 | $2^3 3^3$ | N | N | 46 | 41 | 1.5000000 | 0.481481 | 0.518519 | 32 | 956 | $-924$ |
| 217 | $7^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 37 | 961 | $-924$ |
| 218 | $2^1 109^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486239 | 0.513761 | 42 | 966 | $-924$ |
| 219 | $3^1 73^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488584 | 0.511416 | 47 | 971 | $-924$ |
| 220 | $2^2 5^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.490909 | 0.509091 | 77 | 1001 | $-924$ |
| 221 | $13^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493213 | 0.506787 | 82 | 1006 | $-924$ |
| 222 | $2^1 3^1 37^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.490991 | 0.509009 | 66 | 1006 | $-940$ |
| 223 | $223^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.488789 | 0.511211 | 64 | 1006 | $-942$ |
| 224 | $2^5 7^1$ | N | N | 13 | 8 | 2.0769231 | 0.491071 | 0.508929 | 77 | 1019 | $-942$ |
| 225 | $3^2 5^2$ | N | N | 14 | 9 | 1.3571429 | 0.493333 | 0.506667 | 91 | 1033 | $-942$ |
| 226 | $2^1 113^1$ | Y | N | 5 | 0 | 1.0000000 | 0.495575 | 0.504425 | 96 | 1038 | $-942$ |
| 227 | $227^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.493392 | 0.506608 | 94 | 1038 | $-944$ |
| 228 | $2^2 3^1 19^1$ | N | N | 30 | 14 | 1.1666667 | 0.495614 | 0.504386 | 124 | 1068 | $-944$ |
| 229 | $229^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.493450 | 0.506550 | 122 | 1068 | $-946$ |
| 230 | $2^1 5^1 23^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.491304 | 0.508696 | 106 | 1068 | $-962$ |
| 231 | $3^1 7^1 11^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.489177 | 0.510823 | 90 | 1068 | $-978$ |
| 232 | $2^3 29^1$ | N | N | 9 | 4 | 1.5555556 | 0.491379 | 0.508621 | 99 | 1077 | $-978$ |
| 233 | $233^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.489270 | 0.510730 | 97 | 1077 | $-980$ |
| 234 | $2^1 3^2 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.491453 | 0.508547 | 127 | 1107 | $-980$ |
| 235 | $5^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493617 | 0.506383 | 132 | 1112 | $-980$ |
| 236 | $2^2 59^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.491525 | 0.508475 | 125 | 1112 | $-987$ |
| 237 | $3^1 79^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493671 | 0.506329 | 130 | 1117 | $-987$ |
| 238 | $2^1 7^1 17^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.491597 | 0.508403 | 114 | 1117 | $-1003$ |
| 239 | $239^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.489540 | 0.510460 | 112 | 1117 | $-1005$ |
| 240 | $2^4 3^1 5^1$ | N | N | 70 | 54 | 1.5000000 | 0.491667 | 0.508333 | 182 | 1187 | $-1005$ |
| 241 | $241^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.489627 | 0.510373 | 180 | 1187 | $-1007$ |
| 242 | $2^1 11^2$ | N | N | $-7$ | 2 | 1.2857143 | 0.487603 | 0.512397 | 173 | 1187 | $-1014$ |
| 243 | $3^5$ | N | Y | $-2$ | 0 | 3.0000000 | 0.485597 | 0.514403 | 171 | 1187 | $-1016$ |
| 244 | $2^2 61^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.483607 | 0.516393 | 164 | 1187 | $-1023$ |
| 245 | $5^1 7^2$ | N | N | $-7$ | 2 | 1.2857143 | 0.481633 | 0.518367 | 157 | 1187 | $-1030$ |
| 246 | $2^1 3^1 41^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.479675 | 0.520325 | 141 | 1187 | $-1046$ |
| 247 | $13^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.481781 | 0.518219 | 146 | 1192 | $-1046$ |
| 248 | $2^3 31^1$ | N | N | 9 | 4 | 1.5555556 | 0.483871 | 0.516129 | 155 | 1201 | $-1046$ |
| 249 | $3^1 83^1$ | Y | N | 5 | 0 | 1.0000000 | 0.485944 | 0.514056 | 160 | 1206 | $-1046$ |
| 250 | $2^1 5^3$ | N | N | 9 | 4 | 1.5555556 | 0.488000 | 0.512000 | 169 | 1215 | $-1046$ |
| 251 | $251^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.486056 | 0.513944 | 167 | 1215 | $-1048$ |
| 252 | $2^2 3^2 7^1$ | N | N | $-74$ | 58 | 1.2162162 | 0.484127 | 0.515873 | 93 | 1215 | $-1122$ |
| 253 | $11^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486166 | 0.513834 | 98 | 1220 | $-1122$ |
| 254 | $2^1 127^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488189 | 0.511811 | 103 | 1225 | $-1122$ |
| 255 | $3^1 5^1 17^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.486275 | 0.513725 | 87 | 1225 | $-1138$ |
| 256 | $2^8$ | N | Y | 2 | 0 | 4.5000000 | 0.488281 | 0.511719 | 89 | 1227 | $-1138$ |
| 257 | $257^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.486381 | 0.513619 | 87 | 1227 | $-1140$ |
| 258 | $2^1 3^1 43^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.484496 | 0.515504 | 71 | 1227 | $-1156$ |
| 259 | $7^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486486 | 0.513514 | 76 | 1232 | $-1156$ |
| 260 | $2^2 5^1 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.488462 | 0.511538 | 106 | 1262 | $-1156$ |
| 261 | $3^2 29^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.486590 | 0.513410 | 99 | 1262 | $-1163$ |
| 262 | $2^1 131^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488550 | 0.511450 | 104 | 1267 | $-1163$ |
| 263 | $263^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.486692 | 0.513308 | 102 | 1267 | $-1165$ |
| 264 | $2^3 3^1 11^1$ | N | N | $-48$ | 32 | 1.3333333 | 0.484848 | 0.515152 | 54 | 1267 | $-1213$ |
| 265 | $5^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486792 | 0.513208 | 59 | 1272 | $-1213$ |
| 266 | $2^1 7^1 19^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.484962 | 0.515038 | 43 | 1272 | $-1229$ |
| 267 | $3^1 89^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486891 | 0.513109 | 48 | 1277 | $-1229$ |
| 268 | $2^2 67^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.485075 | 0.514925 | 41 | 1277 | $-1236$ |
| 269 | $269^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.483271 | 0.516729 | 39 | 1277 | $-1238$ |
| 270 | $2^1 3^3 5^1$ | N | N | $-48$ | 32 | 1.3333333 | 0.481481 | 0.518519 | $-9$ | 1277 | $-1286$ |
| 271 | $271^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.479705 | 0.520295 | $-11$ | 1277 | $-1288$ |
| 272 | $2^4 17^1$ | N | N | $-11$ | 6 | 1.8181818 | 0.477941 | 0.522059 | $-22$ | 1277 | $-1299$ |
| 273 | $3^1 7^1 13^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.476190 | 0.523810 | $-38$ | 1277 | $-1315$ |
| 274 | $2^1 137^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478102 | 0.521898 | $-33$ | 1282 | $-1315$ |
| 275 | $5^2 11^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.476364 | 0.523636 | $-40$ | 1282 | $-1322$ |
| 276 | $2^2 3^1 23^1$ | N | N | 30 | 14 | 1.1666667 | 0.478261 | 0.521739 | $-10$ | 1312 | $-1322$ |
| 277 | $277^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.476534 | 0.523466 | $-12$ | 1312 | $-1324$ |

| $n$ | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum_{d\|n} C_{\Omega(d)}(d)}{\|g^{-1}(n)\|}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 278 | $2^1 139^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478417 | 0.521583 | $-7$ | 1317 | $-1324$ |
| 279 | $3^2 31^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.476703 | 0.523297 | $-14$ | 1317 | $-1331$ |
| 280 | $2^3 5^1 7^1$ | N | N | $-48$ | 32 | 1.3333333 | 0.475000 | 0.525000 | $-62$ | 1317 | $-1379$ |
| 281 | $281^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.473310 | 0.526690 | $-64$ | 1317 | $-1381$ |
| 282 | $2^1 3^1 47^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.471631 | 0.528369 | $-80$ | 1317 | $-1397$ |
| 283 | $283^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.469965 | 0.530035 | $-82$ | 1317 | $-1399$ |
| 284 | $2^2 71^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.468310 | 0.531690 | $-89$ | 1317 | $-1406$ |
| 285 | $3^1 5^1 19^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.466667 | 0.533333 | $-105$ | 1317 | $-1422$ |
| 286 | $2^1 11^1 13^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.465035 | 0.534965 | $-121$ | 1317 | $-1438$ |
| 287 | $7^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.466899 | 0.533101 | $-116$ | 1322 | $-1438$ |
| 288 | $2^5 3^2$ | N | N | $-47$ | 42 | 1.7659574 | 0.465278 | 0.534722 | $-163$ | 1322 | $-1485$ |
| 289 | $17^2$ | N | Y | 2 | 0 | 1.5000000 | 0.467128 | 0.532872 | $-161$ | 1324 | $-1485$ |
| 290 | $2^1 5^1 29^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.465517 | 0.534483 | $-177$ | 1324 | $-1501$ |
| 291 | $3^1 97^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467354 | 0.532646 | $-172$ | 1329 | $-1501$ |
| 292 | $2^2 73^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.465753 | 0.534247 | $-179$ | 1329 | $-1508$ |
| 293 | $293^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.464164 | 0.535836 | $-181$ | 1329 | $-1510$ |
| 294 | $2^1 3^1 7^2$ | N | N | 30 | 14 | 1.1666667 | 0.465986 | 0.534014 | $-151$ | 1359 | $-1510$ |
| 295 | $5^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467797 | 0.532203 | $-146$ | 1364 | $-1510$ |
| 296 | $2^3 37^1$ | N | N | 9 | 4 | 1.5555556 | 0.469595 | 0.530405 | $-137$ | 1373 | $-1510$ |
| 297 | $3^3 11^1$ | N | N | 9 | 4 | 1.5555556 | 0.471380 | 0.528620 | $-128$ | 1382 | $-1510$ |
| 298 | $2^1 149^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473154 | 0.526846 | $-123$ | 1387 | $-1510$ |
| 299 | $13^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.474916 | 0.525084 | $-118$ | 1392 | $-1510$ |
| 300 | $2^2 3^1 5^2$ | N | N | $-74$ | 58 | 1.2162162 | 0.473333 | 0.526667 | $-192$ | 1392 | $-1584$ |
| 301 | $7^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475083 | 0.524917 | $-187$ | 1397 | $-1584$ |
| 302 | $2^1 151^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476821 | 0.523179 | $-182$ | 1402 | $-1584$ |
| 303 | $3^1 101^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478548 | 0.521452 | $-177$ | 1407 | $-1584$ |
| 304 | $2^4 19^1$ | N | N | $-11$ | 6 | 1.8181818 | 0.476974 | 0.523026 | $-188$ | 1407 | $-1595$ |
| 305 | $5^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478689 | 0.521311 | $-183$ | 1412 | $-1595$ |
| 306 | $2^1 3^2 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.480392 | 0.519608 | $-153$ | 1442 | $-1595$ |
| 307 | $307^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.478827 | 0.521173 | $-155$ | 1442 | $-1597$ |
| 308 | $2^2 7^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.480519 | 0.519481 | $-125$ | 1472 | $-1597$ |
| 309 | $3^1 103^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482201 | 0.517799 | $-120$ | 1477 | $-1597$ |
| 310 | $2^1 5^1 31^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.480645 | 0.519355 | $-136$ | 1477 | $-1613$ |
| 311 | $311^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.479100 | 0.520900 | $-138$ | 1477 | $-1615$ |
| 312 | $2^3 3^1 13^1$ | N | N | $-48$ | 32 | 1.3333333 | 0.477564 | 0.522436 | $-186$ | 1477 | $-1663$ |
| 313 | $313^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.476038 | 0.523962 | $-188$ | 1477 | $-1665$ |
| 314 | $2^1 157^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477707 | 0.522293 | $-183$ | 1482 | $-1665$ |
| 315 | $3^2 5^1 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.479365 | 0.520635 | $-153$ | 1512 | $-1665$ |
| 316 | $2^2 79^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.477848 | 0.522152 | $-160$ | 1512 | $-1672$ |
| 317 | $317^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.476341 | 0.523659 | $-162$ | 1512 | $-1674$ |
| 318 | $2^1 3^1 53^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.474843 | 0.525157 | $-178$ | 1512 | $-1690$ |
| 319 | $11^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476489 | 0.523511 | $-173$ | 1517 | $-1690$ |
| 320 | $2^6 5^1$ | N | N | $-15$ | 10 | 2.3333333 | 0.475000 | 0.525000 | $-188$ | 1517 | $-1705$ |
| 321 | $3^1 107^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476636 | 0.523364 | $-183$ | 1522 | $-1705$ |
| 322 | $2^1 7^1 23^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.475155 | 0.524845 | $-199$ | 1522 | $-1721$ |
| 323 | $17^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476780 | 0.523220 | $-194$ | 1527 | $-1721$ |
| 324 | $2^2 3^4$ | N | N | 34 | 29 | 1.6176471 | 0.478395 | 0.521605 | $-160$ | 1561 | $-1721$ |
| 325 | $5^2 13^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.476923 | 0.523077 | $-167$ | 1561 | $-1728$ |
| 326 | $2^1 163^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478528 | 0.521472 | $-162$ | 1566 | $-1728$ |
| 327 | $3^1 109^1$ | Y | N | 5 | 0 | 1.0000000 | 0.480122 | 0.519878 | $-157$ | 1571 | $-1728$ |
| 328 | $2^3 41^1$ | N | N | 9 | 4 | 1.5555556 | 0.481707 | 0.518293 | $-148$ | 1580 | $-1728$ |
| 329 | $7^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483283 | 0.516717 | $-143$ | 1585 | $-1728$ |
| 330 | $2^1 3^1 5^1 11^1$ | Y | N | 65 | 0 | 1.0000000 | 0.484848 | 0.515152 | $-78$ | 1650 | $-1728$ |
| 331 | $331^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.483384 | 0.516616 | $-80$ | 1650 | $-1730$ |
| 332 | $2^2 83^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.481928 | 0.518072 | $-87$ | 1650 | $-1737$ |
| 333 | $3^2 37^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.480480 | 0.519520 | $-94$ | 1650 | $-1744$ |
| 334 | $2^1 167^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482036 | 0.517964 | $-89$ | 1655 | $-1744$ |
| 335 | $5^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483582 | 0.516418 | $-84$ | 1660 | $-1744$ |
| 336 | $2^4 3^1 7^1$ | N | N | 70 | 54 | 1.5000000 | 0.485119 | 0.514881 | $-14$ | 1730 | $-1744$ |
| 337 | $337^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.483680 | 0.516320 | $-16$ | 1730 | $-1746$ |
| 338 | $2^1 13^2$ | N | N | $-7$ | 2 | 1.2857143 | 0.482249 | 0.517751 | $-23$ | 1730 | $-1753$ |
| 339 | $3^1 113^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483776 | 0.516224 | $-18$ | 1735 | $-1753$ |
| 340 | $2^2 5^1 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.485294 | 0.514706 | 12 | 1765 | $-1753$ |
| 341 | $11^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486804 | 0.513196 | 17 | 1770 | $-1753$ |
| 342 | $2^1 3^2 19^1$ | N | N | 30 | 14 | 1.1666667 | 0.488304 | 0.511696 | 47 | 1800 | $-1753$ |
| 343 | $7^3$ | N | Y | $-2$ | 0 | 2.0000000 | 0.486880 | 0.513120 | 45 | 1800 | $-1755$ |
| 344 | $2^3 43^1$ | N | N | 9 | 4 | 1.5555556 | 0.488372 | 0.511628 | 54 | 1809 | $-1755$ |
| 345 | $3^1 5^1 23^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.486957 | 0.513043 | 38 | 1809 | $-1771$ |
| 346 | $2^1 173^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488439 | 0.511561 | 43 | 1814 | $-1771$ |
| 347 | $347^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.487032 | 0.512968 | 41 | 1814 | $-1773$ |
| 348 | $2^2 3^1 29^1$ | N | N | 30 | 14 | 1.1666667 | 0.488506 | 0.511494 | 71 | 1844 | $-1773$ |
| 349 | $349^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.487106 | 0.512894 | 69 | 1844 | $-1775$ |
| 350 | $2^1 5^2 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.488571 | 0.511429 | 99 | 1874 | $-1775$ |

| $n$ | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 351 | $3^3 13^1$ | N | N | 9 | 4 | 1.5555556 | 0.490028 | 0.509972 | 108 | 1883 | −1775 |
| 352 | $2^5 11^1$ | N | N | 13 | 8 | 2.0769231 | 0.491477 | 0.508523 | 121 | 1896 | −1775 |
| 353 | $353^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.490085 | 0.509915 | 119 | 1896 | −1777 |
| 354 | $2^1 3^1 59^1$ | Y | N | −16 | 0 | 1.0000000 | 0.488701 | 0.511299 | 103 | 1896 | −1793 |
| 355 | $5^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490141 | 0.509859 | 108 | 1901 | −1793 |
| 356 | $2^2 89^1$ | N | N | −7 | 2 | 1.2857143 | 0.488764 | 0.511236 | 101 | 1901 | −1800 |
| 357 | $3^1 7^1 17^1$ | Y | N | −16 | 0 | 1.0000000 | 0.487395 | 0.512605 | 85 | 1901 | −1816 |
| 358 | $2^1 179^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488827 | 0.511173 | 90 | 1906 | −1816 |
| 359 | $359^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.487465 | 0.512535 | 88 | 1906 | −1818 |
| 360 | $2^3 3^2 5^1$ | N | N | 145 | 129 | 1.3034483 | 0.488889 | 0.511111 | 233 | 2051 | −1818 |
| 361 | $19^2$ | N | Y | 2 | 0 | 1.5000000 | 0.490305 | 0.509695 | 235 | 2053 | −1818 |
| 362 | $2^1 181^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491713 | 0.508287 | 240 | 2058 | −1818 |
| 363 | $3^1 11^2$ | N | N | −7 | 2 | 1.2857143 | 0.490358 | 0.509642 | 233 | 2058 | −1825 |
| 364 | $2^2 7^1 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.491758 | 0.508242 | 263 | 2088 | −1825 |
| 365 | $5^1 73^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493151 | 0.506849 | 268 | 2093 | −1825 |
| 366 | $2^1 3^1 61^1$ | Y | N | −16 | 0 | 1.0000000 | 0.491803 | 0.508197 | 252 | 2093 | −1841 |
| 367 | $367^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.490463 | 0.509537 | 250 | 2093 | −1843 |
| 368 | $2^4 23^1$ | N | N | −11 | 6 | 1.8181818 | 0.489130 | 0.510870 | 239 | 2093 | −1854 |
| 369 | $3^2 41^1$ | N | N | −7 | 2 | 1.2857143 | 0.487805 | 0.512195 | 232 | 2093 | −1861 |
| 370 | $2^1 5^1 37^1$ | Y | N | −16 | 0 | 1.0000000 | 0.486486 | 0.513514 | 216 | 2093 | −1877 |
| 371 | $7^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487871 | 0.512129 | 221 | 2098 | −1877 |
| 372 | $2^2 3^1 31^1$ | N | N | 30 | 14 | 1.1666667 | 0.489247 | 0.510753 | 251 | 2128 | −1877 |
| 373 | $373^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.487936 | 0.512064 | 249 | 2128 | −1879 |
| 374 | $2^1 11^1 17^1$ | Y | N | −16 | 0 | 1.0000000 | 0.486631 | 0.513369 | 233 | 2128 | −1895 |
| 375 | $3^1 5^3$ | N | N | 9 | 4 | 1.5555556 | 0.488000 | 0.512000 | 242 | 2137 | −1895 |
| 376 | $2^3 47^1$ | N | N | 9 | 4 | 1.5555556 | 0.489362 | 0.510638 | 251 | 2146 | −1895 |
| 377 | $13^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490716 | 0.509284 | 256 | 2151 | −1895 |
| 378 | $2^1 3^3 7^1$ | N | N | −48 | 32 | 1.3333333 | 0.489418 | 0.510582 | 208 | 2151 | −1943 |
| 379 | $379^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.488127 | 0.511873 | 206 | 2151 | −1945 |
| 380 | $2^2 5^1 19^1$ | N | N | 30 | 14 | 1.1666667 | 0.489474 | 0.510526 | 236 | 2181 | −1945 |
| 381 | $3^1 127^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490814 | 0.509186 | 241 | 2186 | −1945 |
| 382 | $2^1 191^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492147 | 0.507853 | 246 | 2191 | −1945 |
| 383 | $383^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.490862 | 0.509138 | 244 | 2191 | −1947 |
| 384 | $2^7 3^1$ | N | N | 17 | 12 | 2.5882353 | 0.492188 | 0.507812 | 261 | 2208 | −1947 |
| 385 | $5^1 7^1 11^1$ | Y | N | −16 | 0 | 1.0000000 | 0.490909 | 0.509091 | 245 | 2208 | −1963 |
| 386 | $2^1 193^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492228 | 0.507772 | 250 | 2213 | −1963 |
| 387 | $3^2 43^1$ | N | N | −7 | 2 | 1.2857143 | 0.490956 | 0.509044 | 243 | 2213 | −1970 |
| 388 | $2^2 97^1$ | N | N | −7 | 2 | 1.2857143 | 0.489691 | 0.510309 | 236 | 2213 | −1977 |
| 389 | $389^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.488432 | 0.511568 | 234 | 2213 | −1979 |
| 390 | $2^1 3^1 5^1 13^1$ | Y | N | 65 | 0 | 1.0000000 | 0.489744 | 0.510256 | 299 | 2278 | −1979 |
| 391 | $17^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491049 | 0.508951 | 304 | 2283 | −1979 |
| 392 | $2^3 7^2$ | N | N | −23 | 18 | 1.4782609 | 0.489796 | 0.510204 | 281 | 2283 | −2002 |
| 393 | $3^1 131^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491094 | 0.508906 | 286 | 2288 | −2002 |
| 394 | $2^1 197^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492386 | 0.507614 | 291 | 2293 | −2002 |
| 395 | $5^1 79^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493671 | 0.506329 | 296 | 2298 | −2002 |
| 396 | $2^2 3^2 11^1$ | N | N | −74 | 58 | 1.2162162 | 0.492424 | 0.507576 | 222 | 2298 | −2076 |
| 397 | $397^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.491184 | 0.508816 | 220 | 2298 | −2078 |
| 398 | $2^1 199^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492462 | 0.507538 | 225 | 2303 | −2078 |
| 399 | $3^1 7^1 19^1$ | Y | N | −16 | 0 | 1.0000000 | 0.491228 | 0.508772 | 209 | 2303 | −2094 |
| 400 | $2^4 5^2$ | N | N | 34 | 29 | 1.6176471 | 0.492500 | 0.507500 | 243 | 2337 | −2094 |
| 401 | $401^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.491272 | 0.508728 | 241 | 2337 | −2096 |
| 402 | $2^1 3^1 67^1$ | Y | N | −16 | 0 | 1.0000000 | 0.490050 | 0.509950 | 225 | 2337 | −2112 |
| 403 | $13^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491315 | 0.508685 | 230 | 2342 | −2112 |
| 404 | $2^2 101^1$ | N | N | −7 | 2 | 1.2857143 | 0.490099 | 0.509901 | 223 | 2342 | −2119 |
| 405 | $3^4 5^1$ | N | N | −11 | 6 | 1.8181818 | 0.488889 | 0.511111 | 212 | 2342 | −2130 |
| 406 | $2^1 7^1 29^1$ | Y | N | −16 | 0 | 1.0000000 | 0.487685 | 0.512315 | 196 | 2342 | −2146 |
| 407 | $11^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488943 | 0.511057 | 201 | 2347 | −2146 |
| 408 | $2^3 3^1 17^1$ | N | N | −48 | 32 | 1.3333333 | 0.487553 | 0.512255 | 153 | 2347 | −2194 |
| 409 | $409^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.486553 | 0.513447 | 151 | 2347 | −2196 |
| 410 | $2^1 5^1 41^1$ | Y | N | −16 | 0 | 1.0000000 | 0.485366 | 0.514634 | 135 | 2347 | −2212 |
| 411 | $3^1 137^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486618 | 0.513382 | 140 | 2352 | −2212 |
| 412 | $2^2 103^1$ | N | N | −7 | 2 | 1.2857143 | 0.485437 | 0.514563 | 133 | 2352 | −2219 |
| 413 | $7^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486683 | 0.513317 | 138 | 2357 | −2219 |
| 414 | $2^1 3^2 23^1$ | N | N | 30 | 14 | 1.1666667 | 0.487923 | 0.512077 | 168 | 2387 | −2219 |
| 415 | $5^1 83^1$ | Y | N | 5 | 0 | 1.0000000 | 0.489157 | 0.510843 | 173 | 2392 | −2219 |
| 416 | $2^5 13^1$ | N | N | 13 | 8 | 2.0769231 | 0.490385 | 0.509615 | 186 | 2405 | −2219 |
| 417 | $3^1 139^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491607 | 0.508393 | 191 | 2410 | −2219 |
| 418 | $2^1 11^1 19^1$ | Y | N | −16 | 0 | 1.0000000 | 0.490431 | 0.509569 | 175 | 2410 | −2235 |
| 419 | $419^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.489260 | 0.510740 | 173 | 2410 | −2237 |
| 420 | $2^2 3^1 5^1 7^1$ | N | N | −155 | 90 | 1.1032258 | 0.488095 | 0.511905 | 18 | 2410 | −2392 |
| 421 | $421^1$ | Y | Y | −2 | 0 | 1.0000000 | 0.486936 | 0.513064 | 16 | 2410 | −2394 |
| 422 | $2^1 211^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488152 | 0.511848 | 21 | 2415 | −2394 |
| 423 | $3^2 47^1$ | N | N | −7 | 2 | 1.2857143 | 0.486998 | 0.513002 | 14 | 2415 | −2401 |
| 424 | $2^3 53^1$ | N | N | 9 | 4 | 1.5555556 | 0.488208 | 0.511792 | 23 | 2424 | −2401 |
| 425 | $5^2 17^1$ | N | N | −7 | 2 | 1.2857143 | 0.487059 | 0.512941 | 16 | 2424 | −2408 |

| $n$ | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum_{d\|n} C_{\Omega(d)}(d)}{\|g^{-1}(n)\|}$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 426 | $2^1 3^1 71^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.485915 | 0.514085 | 0 | 2424 | $-2424$ |
| 427 | $7^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487119 | 0.512881 | 5 | 2429 | $-2424$ |
| 428 | $2^2 107^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.485981 | 0.514019 | $-2$ | 2429 | $-2431$ |
| 429 | $3^1 11^1 13^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.484848 | 0.515152 | $-18$ | 2429 | $-2447$ |
| 430 | $2^1 5^1 43^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.483721 | 0.516279 | $-34$ | 2429 | $-2463$ |
| 431 | $431^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.482599 | 0.517401 | $-36$ | 2429 | $-2465$ |
| 432 | $2^4 3^3$ | N | N | $-80$ | 75 | 1.5625000 | 0.481481 | 0.518519 | $-116$ | 2429 | $-2545$ |
| 433 | $433^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.480370 | 0.519630 | $-118$ | 2429 | $-2547$ |
| 434 | $2^1 7^1 31^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.479263 | 0.520737 | $-134$ | 2429 | $-2563$ |
| 435 | $3^1 5^1 29^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.478161 | 0.521839 | $-150$ | 2429 | $-2579$ |
| 436 | $2^2 109^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.477064 | 0.522936 | $-157$ | 2429 | $-2586$ |
| 437 | $19^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478261 | 0.521739 | $-152$ | 2434 | $-2586$ |
| 438 | $2^1 3^1 73^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.477169 | 0.522831 | $-168$ | 2434 | $-2602$ |
| 439 | $439^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.476082 | 0.523918 | $-170$ | 2434 | $-2604$ |
| 440 | $2^3 5^1 11^1$ | N | N | $-48$ | 32 | 1.3333333 | 0.475000 | 0.525000 | $-218$ | 2434 | $-2652$ |
| 441 | $3^2 7^2$ | N | N | 14 | 9 | 1.3571429 | 0.476190 | 0.523810 | $-204$ | 2448 | $-2652$ |
| 442 | $2^1 13^1 17^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.475113 | 0.524887 | $-220$ | 2448 | $-2668$ |
| 443 | $443^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.474041 | 0.525959 | $-222$ | 2448 | $-2670$ |
| 444 | $2^2 3^1 37^1$ | N | N | 30 | 14 | 1.1666667 | 0.475225 | 0.524775 | $-192$ | 2478 | $-2670$ |
| 445 | $5^1 89^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476404 | 0.523596 | $-187$ | 2483 | $-2670$ |
| 446 | $2^1 223^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477578 | 0.522422 | $-182$ | 2488 | $-2670$ |
| 447 | $3^1 149^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478747 | 0.521253 | $-177$ | 2493 | $-2670$ |
| 448 | $2^6 7^1$ | N | N | $-15$ | 10 | 2.3333333 | 0.477679 | 0.522321 | $-192$ | 2493 | $-2685$ |
| 449 | $449^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.476615 | 0.523385 | $-194$ | 2493 | $-2687$ |
| 450 | $2^1 3^2 5^2$ | N | N | $-74$ | 58 | 1.2162162 | 0.475556 | 0.524444 | $-268$ | 2493 | $-2761$ |
| 451 | $11^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476718 | 0.523282 | $-263$ | 2498 | $-2761$ |
| 452 | $2^2 113^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.475664 | 0.524336 | $-270$ | 2498 | $-2768$ |
| 453 | $3^1 151^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476821 | 0.523179 | $-265$ | 2503 | $-2768$ |
| 454 | $2^1 227^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477974 | 0.522026 | $-260$ | 2508 | $-2768$ |
| 455 | $5^1 7^1 13^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.476923 | 0.523077 | $-276$ | 2508 | $-2784$ |
| 456 | $2^3 3^1 19^1$ | N | N | $-48$ | 32 | 1.3333333 | 0.475877 | 0.524123 | $-324$ | 2508 | $-2832$ |
| 457 | $457^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.474836 | 0.525164 | $-326$ | 2508 | $-2834$ |
| 458 | $2^1 229^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475983 | 0.524017 | $-321$ | 2513 | $-2834$ |
| 459 | $3^3 17^1$ | N | N | 9 | 4 | 1.5555556 | 0.477124 | 0.522876 | $-312$ | 2522 | $-2834$ |
| 460 | $2^2 5^1 23^1$ | N | N | 30 | 14 | 1.1666667 | 0.478261 | 0.521739 | $-282$ | 2552 | $-2834$ |
| 461 | $461^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.477223 | 0.522777 | $-284$ | 2552 | $-2836$ |
| 462 | $2^1 3^1 7^1 11^1$ | Y | N | 65 | 0 | 1.0000000 | 0.478355 | 0.521645 | $-219$ | 2617 | $-2836$ |
| 463 | $463^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.477322 | 0.522678 | $-221$ | 2617 | $-2838$ |
| 464 | $2^4 29^1$ | N | N | $-11$ | 6 | 1.8181818 | 0.476293 | 0.523707 | $-232$ | 2617 | $-2849$ |
| 465 | $3^1 5^1 31^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.475269 | 0.524731 | $-248$ | 2617 | $-2865$ |
| 466 | $2^1 233^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476395 | 0.523605 | $-243$ | 2622 | $-2865$ |
| 467 | $467^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.475375 | 0.524625 | $-245$ | 2622 | $-2867$ |
| 468 | $2^2 3^2 13^1$ | N | N | $-74$ | 58 | 1.2162162 | 0.474359 | 0.525641 | $-319$ | 2622 | $-2941$ |
| 469 | $7^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475480 | 0.524520 | $-314$ | 2627 | $-2941$ |
| 470 | $2^1 5^1 47^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.474468 | 0.525532 | $-330$ | 2627 | $-2957$ |
| 471 | $3^1 157^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475584 | 0.524416 | $-325$ | 2632 | $-2957$ |
| 472 | $2^3 59^1$ | N | N | 9 | 4 | 1.5555556 | 0.476695 | 0.523305 | $-316$ | 2641 | $-2957$ |
| 473 | $11^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477801 | 0.522199 | $-311$ | 2646 | $-2957$ |
| 474 | $2^1 3^1 79^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.476793 | 0.523207 | $-327$ | 2646 | $-2973$ |
| 475 | $5^2 19^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.475789 | 0.524211 | $-334$ | 2646 | $-2980$ |
| 476 | $2^2 7^1 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.476891 | 0.523109 | $-304$ | 2676 | $-2980$ |
| 477 | $3^2 53^1$ | N | N | $-7$ | 2 | 1.2857143 | 0.475891 | 0.524109 | $-311$ | 2676 | $-2987$ |
| 478 | $2^1 239^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476987 | 0.523013 | $-306$ | 2681 | $-2987$ |
| 479 | $479^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.475992 | 0.524008 | $-308$ | 2681 | $-2989$ |
| 480 | $2^5 3^1 5^1$ | N | N | $-96$ | 80 | 1.6666667 | 0.475000 | 0.525000 | $-404$ | 2681 | $-3085$ |
| 481 | $13^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476091 | 0.523909 | $-399$ | 2686 | $-3085$ |
| 482 | $2^1 241^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477178 | 0.522822 | $-394$ | 2691 | $-3085$ |
| 483 | $3^1 7^1 23^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.476190 | 0.523810 | $-410$ | 2691 | $-3101$ |
| 484 | $2^2 11^2$ | N | N | 14 | 9 | 1.3571429 | 0.477273 | 0.522727 | $-396$ | 2705 | $-3101$ |
| 485 | $5^1 97^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478351 | 0.521649 | $-391$ | 2710 | $-3101$ |
| 486 | $2^1 3^5$ | N | N | 13 | 8 | 2.0769231 | 0.479424 | 0.520576 | $-378$ | 2723 | $-3101$ |
| 487 | $487^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.478439 | 0.521561 | $-380$ | 2723 | $-3103$ |
| 488 | $2^3 61^1$ | N | N | 9 | 4 | 1.5555556 | 0.479508 | 0.520492 | $-371$ | 2732 | $-3103$ |
| 489 | $3^1 163^1$ | Y | N | 5 | 0 | 1.0000000 | 0.480573 | 0.519427 | $-366$ | 2737 | $-3103$ |
| 490 | $2^1 5^1 7^2$ | N | N | 30 | 14 | 1.1666667 | 0.481633 | 0.518367 | $-336$ | 2767 | $-3103$ |
| 491 | $491^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.480652 | 0.519348 | $-338$ | 2767 | $-3105$ |
| 492 | $2^2 3^1 41^1$ | N | N | 30 | 14 | 1.1666667 | 0.481707 | 0.518293 | $-308$ | 2797 | $-3105$ |
| 493 | $17^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482759 | 0.517241 | $-303$ | 2802 | $-3105$ |
| 494 | $2^1 13^1 19^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.481781 | 0.518219 | $-319$ | 2802 | $-3121$ |
| 495 | $3^2 5^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.482828 | 0.517172 | $-289$ | 2832 | $-3121$ |
| 496 | $2^4 31^1$ | N | N | $-11$ | 6 | 1.8181818 | 0.481855 | 0.518145 | $-300$ | 2832 | $-3132$ |
| 497 | $7^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482897 | 0.517103 | $-295$ | 2837 | $-3132$ |
| 498 | $2^1 3^1 83^1$ | Y | N | $-16$ | 0 | 1.0000000 | 0.481928 | 0.518072 | $-311$ | 2837 | $-3148$ |
| 499 | $499^1$ | Y | Y | $-2$ | 0 | 1.0000000 | 0.480962 | 0.519038 | $-313$ | 2837 | $-3150$ |
| 500 | $2^2 5^3$ | N | N | $-23$ | 18 | 1.4782609 | 0.480000 | 0.520000 | $-336$ | 2837 | $-3173$ |