Definition 3.8 (Precise Forms of the Generating Function Factorizations). With this in mind, let's make an ansatz that we can tease nice results out of the following definition of the product for $C_\infty(q)$ and the corresponding partition functions we get as the coefficients of it and its reciprocal. Let $\omega^*(n) := \omega(n) + \varepsilon(n)$ where $\varepsilon(n) = \delta_{n,1}$ is the identity for Dirichlet inversion and where $\omega(n) = \sum_{p|n}$. We define

$$\begin{aligned} C_\infty(q) &:= \prod_{m\geq 1}\left(1+q^{m\cdot(\omega(m)+\varepsilon(m))}\right) = \sum_{n\geq 0}\hat{p}_\infty(n)q^n \\ &= 1+q+q^2+2q^3+2q^4+3q^5+3q^6+4q^7+5q^8+6q^9+7q^{10}+8q^{11}+11q^{12}+12q^{13}+14q^{14}+\cdots \\ p_\infty(n) &:= [q^n]\frac{1}{C_\infty(q)} \\ &= 1-q-q^3+q^4-q^5+2q^6-2q^7+2q^8-3q^9+4q^{10}-4q^{11}+4q^{12}-5q^{13}+7q^{14}-7q^{15}+8q^{16}-\cdots. \end{aligned}$$

In our case, we have defined

$$\begin{aligned} \{p_\infty(n)\}_{n\geq 0} &= \{1,1,1,2,2,3,3,4,5,6,7,8,11,12,14,16,19,22,24,28,33,37,41,47,\ldots\} \\ \{\hat{p}_\infty(n)\}_{n\geq 1} &= \{1,-1,0,-1,1,-1,2,-2,2,-3,4,-4,4,-5,7,-7,8,-10,11,-13,14,-15,18\ldots\}. \end{aligned}$$

and the corresponding inversion coefficients $\mu_{x,j}$ are shown in Table A.3.

Now we should focus on finding sharp and more sophisticated techniques for bounding sequence-weighted partition functions, of which our $p_\infty(n)$, is a motivating special case to consider. This is obviously the next task forward. We can use more classical methods first to bound our partition functions of interest and hope that these methods can be sharpened later.

Remark 3.9 (Setup of the Method). We briefly sketch the details to the probabilistic proof of the asymptotic results we will need to estimate the first partition function sequence $p_\infty(n) := [q^n]F_\infty(q)$. The presentation here follows along that given in [5]. See that reference for complete details and citations which we only skim for content here:

A. For $S_k(q) := (1+q^{k\omega^*(k)})^{-1}$, we have to estimate the coefficients of the generating functions

$$p_\infty(n) = [q^n]\prod_{k\geq 1} S_k(q) = [q^n]\prod_{k=1}^n S_k(q).$$

Let $F_n(q) := \prod_{k=1}^n S_k(q)$.

B. As in the setup from §4 of the reference, we have that

$$p_\infty(n) = e^{\delta n}\int_0^1 F_n\left(e^{-\delta+2\pi\imath\alpha}\right)d\alpha,\ n\geq 1,$$

where $\delta \equiv \delta_n$ is a free parameter which we can suitably estimate later.

C. Assign the (weighted, signed) probabilities to this interpretation by defining the independent, integer-valued random variables Y_k for $k\geq 1$ by

$$\mathbb{P}(Y_k = jk\omega^*(k)) := \frac{[q^j]S_k(q)\cdot e^{-\delta k\omega^*(k)}}{S_k(e^{-\delta})}.$$

D. Since

$$\phi_n(\alpha) = \frac{F_n(e^{-\delta+2\pi\imath\alpha})}{F_n(e^{-\delta})} = \mathbb{E}[e^{2\pi\imath\alpha Z_n}],$$

is the characteristic function of the random variable $Z_n := \sum_{k=1}^n Y_k$, we then have that

$$p_\infty(n) = (-1)^n e^{n\delta}F_n(e^{-\delta})\mathbb{P}(Z_n = n),\ n\geq 1.$$

E. Now to assign a "good" value of $\delta\equiv\delta_n$, we will require that $\mathbb{E}(Z_n)(\delta_n) = n$, where

$$\mathbb{E}(Z_n)(\delta_n) = -\left(\log F_n(e^{-\delta})\right)'.$$

Lemma 3.10 (Solving for the Parameters γ_n From Expectations). *For $n \geq 1$, we have that the expectation of the random variables $Z_n := \sum_{k=1}^{n} Y_k$ is given by*

$$n \sim \mathbb{E}[Z_n](\delta) = \sum_{k=1}^{n} \frac{k\omega^*(k) e^{-\delta k \omega^*(k)}}{1 + e^{-\delta k\omega^*(k)}}. \tag{10}$$

By strategically summing the previous equations, we solve for $\delta \equiv \delta_n$ of the form

$$\delta_n = \frac{\log n + \log\log\log n}{n \cdot \log\log n}.$$

Proof. The first result follows from the constructions in the reference [5] where we can write

$$\mathbb{E}[Z_n] = \left(\log F_n(e^{-\delta})\right)'.$$

The new way in which we sum this representation for the expectation is to first interpret the summands as generating functions for the *Euler polynomials*:

$$\frac{k\omega^*(k) e^{-\delta k\omega^*(k)}}{1 + e^{-\delta k \omega^*(k)}} = \sum_{m \geq 0} \frac{k\omega^*(k)}{2} \frac{(-\delta k \omega^*(k))^m}{m!} E_m(1).$$

Then we can obtain asymptotics for the sums of the form $\sum_{k=1}^{n} k^r \omega^*(k)^r$, when $r \geq 1$ is integer-valued by induction. In particular, from the $r := 1, 2$ cases proved in Hardy and Wright, we can generalize to see that

$$\sum_{k=1}^{n} \omega(k)^r \sim n(\log\log n)^r.$$

An appeal to Abel summation provides that

$$\sum_{k=1}^{n} k^r \omega^*(k)^r \sim n^{r+1} (\log\log n)^r.$$

So we can transform the finite sums over the Euler polynomial generating functions as

$$\begin{aligned} n &= \frac{n^2}{2} \log\log n \sum_{m \geq 0} (-\delta n \log\log n)^m \frac{E_m(1)}{m!} \\ &= \frac{n^2 \log\log n \cdot e^{-\delta n \log\log n}}{1 + e^{-\delta n \log\log n}}. \end{aligned}$$

This equation is straightforward enough to solve by hand as

$$\delta = \frac{\log(n \log\log n)}{n \cdot \log\log n}.$$ □

Lemma 3.11 (Estimating the Expectations of Z_n to Find the Variance and Form of F_n). *For $n \geq 1$, we have that*

$$\log F_n(e^{-\delta}) \sim n(\log(2) + 2) + \frac{\tanh^{-1}\left((\log n)^{\delta n}\right)}{\delta \cdot \log\log n}.$$

Proof. We will use contour integration and inverse Mellin transforms to transform the series for the logarithm of our function into a problem about residue calculus. In particular, we have that

$$\begin{aligned} \log F_n(e^{-\delta}) &= \sum_{k=1}^{n} \sum_{j \geq 1} \frac{(-1)^j}{j} e^{-\delta j k \omega^*(k)} \\ &= \sum_{k=1}^{n} \sum_{j \geq 1} \frac{1}{2\pi \imath} \int_{c - \imath\infty}^{c + \imath\infty} \frac{(-1)^j e^{-\imath \pi s} \delta^{-s}}{(k\omega^*(k))^s j^{s+1}} ds \end{aligned}$$

$$= \sum_{k=1}^{n} \frac{1}{2\pi\imath} \int_{c-\imath\infty}^{c+\imath\infty} \frac{e^{\imath\pi s}\Gamma(s)(1-2^{-s})\zeta(s+1)}{\delta^s \cdot (k\omega^*(k))^s}.$$

Let the integrand of the last equation be denoted by $I_s(k,\delta)$. We see that $I_s(k,\delta)$ has simple poles at $s := 0, -1, -2, -4, -6, -8, \ldots$, so that by the residue theorem we can write

$$\begin{aligned}\log F_n(e^{-\delta}) &= \sum_{k=1}^{n}\sum_{s_0=0}^{\infty} \operatorname{Res}_{s=s_0}[I_s(k,\delta)] \\ &= n\log(2) + 2n + \sum_{k=1}^{n}\sum_{m\geq 1} \frac{B_m k^m \omega^*(k)^m \delta^m (2^m-1)}{m\cdot m!} \\ &\sim n(\log(2)+2) + \sum_{m\geq 1} \frac{B_m n^{m+1} (\log\log n)^m \delta^m (2^m-1)}{m\cdot m!}.\end{aligned}$$

Now using the fact that

$$\begin{aligned}\sum_{n\geq 1} \frac{B_n (cz)^n (2^n-1)}{n\cdot n!} &= \int_0^z \left(\frac{1}{e^{2cz}-1} - \frac{1}{e^z-1}\right) dt \\ &= \frac{n}{c}\cdot \tanh^{-1}(e^{cz}),\end{aligned}$$

we have that

$$\log F_n(e^{-\delta}) \sim n(\log 2 + 2) + \frac{\tanh^{-1}\left((\log n)^{\delta n}\right)}{\delta\cdot\log\log n}.$$

Hence, this result allows us to estimate both the term $F_n(e^{-\delta})$ which we need to approximate (10), and also gives us a starting point for estimating the variance of Z_n, which as we will see, we will use in the final step of constructing this approximation to $p_\infty(n)$. □

Now all we need to do to induce our asymptotic formula for (10) is to accurately compute a formula for $\mathbb{P}(Z_n = n)$. Using an extension of the local limit theorem argument given in §6 (proof of Theorem 4) of the reference, we claim that

$$\mathbb{P}(Z_n = n) \sim \frac{1}{\sqrt{2\pi}\cdot\sigma_n},$$

where $\sigma_n^2 = \operatorname{Var}(Z_n)$. So we have the next result to relieve this problem.

Lemma 3.12 (An Asymptotic Formula for the Variance of Z_n). *We have that*

$$\begin{aligned}\operatorname{Var}(Z_n) \sim{}& \frac{2n\tanh^{-1}\left((\log n)^{\delta}\right)}{\delta^3\cdot\log\log n} - \frac{2n(\log n)^{\delta}}{\delta^2(1-(\log n)^{2\delta})} \\ &+ \frac{n(\log\log n)^2}{\delta(1-(\log n)^{2\delta})\log\log n}\left(\frac{2(\log n)^{3\delta}}{1-(\log n)^{2\delta}} + (\log n)^{\delta}\right).\end{aligned}$$

Proof. Using an adaptation of the interpretation given in §6 of the reference, we have that

$$\begin{aligned}\operatorname{Var}(Z_n) &= \frac{d^2}{d\delta^2}\left[\log F_n(e^{-\delta})\right] \\ &= \frac{d}{d\delta}\left[\frac{n(\log n)^{\delta}}{\delta(1-(\log n)^{2\delta}} - \frac{n\tanh^{-1}\left((\log n)^{\delta}\right)}{\delta^2\cdot\log\log n}]\right],\end{aligned}$$

where the last line follows by inputing the result of Lemma 3.11 into the derivative-based formula from the reference. The claimed result then follows by an otherwise tedious exercise using *Mathematica*. □

Why the bloody !&?*#!#*&! doesn't this work like I think it should? In particular, why are the numerics I get from this approximation method still so far off and subject to an apparent exponential blow up?