Lower bounds on the summatory function of the Möbius function along infinite subsequences

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Abstract

The Mertens function, $M(x) := \sum_{n \leq x} \mu(n)$, is defined as the summatory function of the Möbius function. The Mertens conjecture states that $|M(x)| < C \cdot \sqrt{x}$ for some absolute $C > 0$ for all $x \geq 1$. This classical conjecture has a well-known disproof due to Odlyzko and té Riele. We prove the unboundedness of $|M(x)|/\sqrt{x}$ using new methods by showing that

$$\limsup_{x \to \infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log\log x)^{\frac{1}{2}}} > 0.$$

The new methods we draw upon connect formulas and recent Dirichlet generating function (or DGF) series expansions related to the canonically additive functions $\Omega(n)$ and $\omega(n)$. The connection between $M(x)$ and the distribution of these core additive functions we prove at the start of the article in the form of

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right],$$

is an indispensible component to the proof. It also leads to regular properties of component sequences in the new formula for $M(x)$ that include generalizations of Erdös-Kac like theorems satisfied by the distributions of these special auxiliary sequences.

Keywords and Phrases: *Möbius function; Mertens function; Dirichlet inverse function; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function; Erdös-Kac theorem; strongly additive functions.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; 11N64; and 11-04.*

Glossary of notation and conventions

Symbol	Definition
$\approx$	We write that $f(x) \approx g(x)$ if $\lvert f(x) - g(x) \rvert = O(1)$ as $x \rightarrow \infty$.
$\mathbb{E}[f(x)], \overset{\mathbb{E}}{\sim}$	We use the expectation notation of $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \overset{\mathbb{E}}{\sim} h(x)$, to denote that f has an *average order* growth rate of $h(x)$. This means that $\frac{1}{x}\sum_{n \leq x} f(n) \sim h(x)$, or equivalently that $$\lim_{x \rightarrow \infty} \frac{\frac{1}{x}\sum\limits_{n \leq x} f(n)}{h(x)} = 1.$$
B	The absolute constant $B \approx 0.2614972$ from the statement of Mertens theorem.
$\chi_{\mathbb{P}}(n)$	The characteristic (or indicator) function of the primes equals one if and only if $n \in \mathbb{Z}^{+}$ is prime, and is zero-valued otherwise.
$C_k(n)$	The sequence is defined recursively for $n \geq 1$ as follows: $$C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d \mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$ It represents the multiple, k-fold convolution of the function $\omega(n)$ with itself.
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n \geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n \geq 0} f_n q^n$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic f we have that $f \ast \varepsilon = \varepsilon \ast f = f$ where $\ast$ denotes Dirichlet convolution (see definition below).
$f \ast g$	The Dirichlet convolution of f and g, $(f \ast g)(n) := \sum\limits_{d \mid n} f(d) g(n/d)$, where the sum is taken over the divisors of any $n \geq 1$.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d \mid n \\ d > 1}} f(d) f^{-1}(n/d)$ for $n \geq 2$ with $f^{-1}(1) = 1/f(1)$. The Dirichlet inverse of f exists if and only if $f(1) \neq 0$. This inverse function, denoted by f^{-1} when it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} \ast f = f \ast f^{-1} = \varepsilon$.
γ	The Euler gamma constant defined by $\gamma := \lim\limits_{n \rightarrow \infty} \left(\sum\limits_{k=1}^{n} \frac{1}{k} - \log n\right) \approx 0.5772157$.
$\gg, \ll, \asymp$	For functions A, B, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A \ll B$ and $B \ll A$, we write $A \asymp B$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$.

Symbol	**Definition**
$[n=k]_\delta$,$[\texttt{cond}]_\delta$	The symbol $[n=k]_\delta$ is a synonym for $\delta_{n,k}$ which is one if and only if $n=k$, and is zero otherwise. For boolean-valued conditions, `cond`, the symbol $[\texttt{cond}]_\delta$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention*.
$\lambda_*(n)$	For positive integers $n \geq 2$, we define the next variant of the Liouville lambda function, $\lambda(n)$, as follows: $\lambda_*(n) := (-1)^{\omega(n)}$. We have the initial condition that $\lambda_*(1)=1$.
$\lambda(n), L(x)$	The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. Its summatory function is defined by $L(x) := \sum_{n \leq x} \lambda(n)$.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree.
$\mu_x(C), \sigma_x(C)$	We define these analogs to the mean and variance of the function $C_{\Omega(n)}(n)$ in the context of our new Erdös-Kac like theorems as $\mu_x(C) := \log\log x + \hat{a} - \frac{1}{2}\log\log\log x$ and $\sigma_x(C) := \sqrt{\mu_x(C)}$ where $\hat{a} := \log\left(\frac{1}{\sqrt{2\pi}}\right) \approx -0.918939$ is an absolute constant.
$M(x)$	The Mertens function is the summatory function of $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum\limits_{n \leq x} \mu(n)$.
$\Phi(z)$	For $x \in \mathbb{R}$, we define the function giving the normal distribution CDF by $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{z} e^{-t^2/2} dt$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^\alpha \vert\vert n$ (or when p^α exactly divides n) for p prime, $\alpha \geq 1$ and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p \vert n} 1$ and the completely additive function $\Omega(n) := \sum_{p^\alpha \vert\vert n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leq n \leq x$ for $x \geq 2$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$ for $x \geq 2$.
$P(s)$	For complex s with $\mathrm{Re}(s) > 1$, we define the prime zeta function to be the Dirichlet generating function $P(s) = \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s}$.
$Q(x)$	For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. More precisely, this function is summed and identified with its limiting asymptotic formula as $x \rightarrow \infty$ in the following form: $Q(x) := \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O(\sqrt{x})$.
$\sim$	We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} = 1$.
$\zeta(s)$	The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation on the rest of the complex plane with the exception of a simple pole at $s=1$ of residue one.

1 Introduction

1.1 Definitions

We define the *Möbius function* to be the signed indicator function of the squarefree integers in the form of [20, A008683]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \text{ and } n \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined on the positive integers as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as follows [20, A002321]:

$$\{M(x)\}_{x \geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \ldots\}.$$

The Mertens function satisfies that $\sum_{n \leq x} M\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = 1$, and is related to the summatory function $L(x) := \sum_{n \leq x} \lambda(n)$ via the relation [10]

$$L(x) = \sum_{d \leq \sqrt{x}} M\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), x \geq 1.$$

Clearly, a positive integer $n \geq 1$ is *squarefree*, or contains no divisors (other than one) which are squares, if and only if $\mu^2(n) = 1$. A related summatory function which counts the number of *squarefree* integers $n \leq x$ satisfies [5, §18.6] [20, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers characterized by the sign of the Möbius function are in fact equal as $x \to \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{x} \overset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{x} \xrightarrow{x \to \infty} \frac{3}{\pi^2}.$$

1.2 Properties

An approach to evaluating the limiting asymptotic behavior of $M(x)$ for large $x \to \infty$ considers an inverse Mellin transformation of the reciprocal of the Riemann zeta function. In particular, since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \cdot \int_1^\infty \frac{M(x)}{x^{s+1}} dx, \operatorname{Re}(s) > 1,$$

we obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

The previous two representations lead us to the exact expression of $M(x)$ for any real $x > 0$ given by the next theorem.

Theorem 1.1 (Analytic Formula for $M(x)$, Titchmarsh)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any real $x > 0$*

$$M(x) = \lim_{k \to \infty} \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |\operatorname{Im}(\rho)| < T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp\left(-C \cdot \log^{\frac{3}{5}}(x)(\log\log x)^{-\frac{3}{5}}\right).$$

Under the assumption of the RH, Soundararajan proved new updated estimates bounding $M(x)$ from above for large x in the following forms [21]:

$$\begin{aligned} M(x) &\ll \sqrt{x} \cdot \exp\left((\log x)^{\frac{1}{2}}(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x} \cdot \exp\left((\log x)^{\frac{1}{2}}(\log\log x)^{\frac{5}{2}+\epsilon}\right)\right), \ \forall \epsilon > 0. \end{aligned}$$

1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$ for any $0 < \epsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C \cdot \sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens himself for $C = 1$ and all $x < 10000$ without the benefit of modern computation. Since its beginnings in 1897, the Mertens conjecture has been disproven by computational methods with non-trivial simple zeta function zeros with comparitively small imaginary parts in a famous paper by Odlyzko and té Riele [14]. More recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound along infinite subsequences, e.g., considering the asymptotics of the function in the limit supremum and limit infimum senses.

We cite that it is only known by computation that [17, *cf.* §4.1] [20, *cf.* A051400; A051401]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leq -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [14, 8, 9, 6]. Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies [13]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log\log\log x)^{\frac{5}{4}}} = O(1).$$

2 A concrete new approach to bounding $M(x)$ from below

2.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 2.1 (Summatory functions of Dirichlet convolutions). *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of f and h, respectively, and that $F^{-1}(x) := \sum_{n \leq x} f^{-1}(n)$ denotes the summatory function of the Dirichlet inverse of f for any $x \geq 1$. We have the following exact expressions for the summatory function of the convolution $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned}
\pi_{f*h}(x) &:= \sum_{n \leq x} \sum_{d \mid n} f(d) h(n/d) \\
&= \sum_{d \leq x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].
\end{aligned}$$

Moreover, for all $x \geq 1$

$$\begin{aligned}
H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\
&= \sum_{k=1}^{x} f^{-1}(k) \cdot \pi_{f*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right).
\end{aligned}$$

Corollary 2.2 (Convolutions arising from Möbius inversion). *Suppose that h is an arithmetic function such that $h(1) \neq 0$. Define the summatory function of the convolution of h with μ by $\widetilde{H}(x) := \sum_{n \leq x} (h * \mu)(n)$. Then the Mertens function is expressed by the sum*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} h^{-1}(j) \right) \widetilde{H}(k), \forall x \geq 1.$$

Corollary 2.3 (A motivating special case). *We have that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

2.2 An exact expression for $M(x)$ in terms of strongly additive functions

Fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute exactly that (see Table T.1 starting on page 38)

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

There is not a simple meaningful direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences whose properties we will discuss in detail. The distribution of distinct sets of prime exponents is still clearly quite regular since $\omega(n)$ and $\Omega(n)$ play a crucial role in the repitition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of this inverse function we notice below over $n \geq 2$:

Heuristic 2.4 (Symmetry in $g^{-1}(n)$ in the prime factorizations of n)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for $= \omega(n_i) \geq 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly one, two, three, and so on prime factors.

Conjecture 2.5 (Characteristic properties of the inverse sequence)**.** *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

(A) *For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*

(B) *For all squarefree integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!;$$

(C) *If $n \geq 2$ and $\Omega(n) = k$, then*

$$2 \leq |g^{-1}(n)| \leq \sum_{j=0}^{k} \binom{k}{j} \cdot j!.$$

We illustrate the conjecture clearly using the computation of initial values of this inverse sequence in Table T.1. The signedness property in (A) is proved precisely in Proposition 3.1. A proof of (B) in fact follows from Lemma 5.1 stated on page 21. The realization that the beautiful and remarkably simple combinatorial form of property (B) in Conjecture 2.5 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through sums of auxiliary subsequences of arithmetic functions denoted by $C_k(n)$ (see Section 5). That is, we observe a familiar formula for $g^{-1}(n)$ at many integers and then seek to extrapolate and prove more regular tendencies of this sequence more generally at any $n \geq 2$.

An exact expression for $g^{-1}(n)$ through a key semi-diagonal of these subsequences is given by

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d), n \geq 1,$$

where the sequence $\lambda(n) C_{\Omega(n)}(n)$ has DGF $(P(s)+1)^{-1}$ for $\operatorname{Re}(s) > 1$ (see Proposition 3.1). In Corollary 6.5, we prove that the approximate mean of the unsigned sequence satisfies

$$\mathbb{E}|g^{-1}(n)| \asymp (\log n)^2 \sqrt{\log\log n}, \text{ as } n \to \infty.$$

The regularity and quasi-periodicity we have alluded to in the remarks above are actually quantifiable in so much as $|g^{-1}(n)|$ for $n \leq x$ tends to its average order with a non-central normal tendency depending on x as $x \to \infty$. In Section 6, we prove the next variant of an Erdös-Kac theorem like analog for a component sequence $C_{\Omega(n)}(n)$. Namely, we prove the following statement for $\mu_x(C) := \log\log x + \hat{a} - \frac{1}{2}\log\log\log x$, $\sigma_x(C) := \sqrt{\mu_x(C)}$, $\hat{a}$ an absolute constant, and any $y \in \mathbb{R}$ (see Corollary 6.7):

$$\frac{1}{x} \cdot \#\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq y\} = \Phi\left(\frac{\frac{\pi^2}{6}y - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

We also prove that (see Proposition 7.4)

$$M(x) = G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right]. \tag{2}$$

This formula implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by bounding appropriate estimates of the summatory function $G^{-1}(x)$. This take on the regularity of $|g^{-1}(n)|$ is imperative to our argument formally bounding the growth $G^{-1}(x)$ from below as $|G^{-1}(x)| \gg (\log x)\sqrt{\log\log x}$ as $x \to \infty$ (see Theorem 7.3). A more combinatorial approach to summing $G^{-1}(x)$ for large x based on the distribution of the primes is outlined in our remarks in Section 5.3.

2.3 Uniform asymptotics from certain bivariate counting DGFs

Theorem 2.6 (Montgomery and Vaughan)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $R < 2$ we have that uniformly for all $1 \leq k \leq R \cdot \log\log x$

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right],$$

where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, 0 \leq |z| < R.$$

The proof of the next result is also combinatorially motivated in so much as it interprets lower bounds on a key infinite product factor of $\mathcal{G}(z)$ defined in Theorem 2.6 as corresponding to an ordinary generating function of certain homogeneous symmetric polynomials involving the primes (see (14) in the proof of Theorem 2.7). This interpretation allows us to recover the following uniform lower bounds on $\widehat{\pi}_k(x)$ as $x \to \infty$:

Theorem 2.7 (Schmidt, 2020)**.** *For all sufficiently large x we have uniformly for $1 \leq k \leq \log\log x$ that*

$$\widehat{\pi}_k(x) \gg \frac{x^{3/4}}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \times \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

Remark 2.8. We emphasize the recency of the method demonstrated by Montgomery and Vaughan in constructing their original proof of Theorem 2.6. To the best of our knowledge, this textbook reference is one of the first clear-cut applications documenting something of a hybrid DGF-and-OGF approach to enumerating sequences of arithmetic functions and their summatory functions. This interpretation of certain bivariate DGFs offers a window into the best of both generating function series worlds. It combines the additivity implicit to the coefficients indexed by a formal power series variable formed by multiplication of these structures, while coordinating the distinct DGF-best property of the multiplicativity of distinct prime powers invoked by taking powers of a reciprocal Euler product. Another set of proofs constructed based on this type of hybrid power series DGF is utilized in Section 6 when we prove an Erdös-Kac theorem like analog that holds for the component sequence $C_{\Omega(n)}(n)$, which is crucially related to $g^{-1}(n)$ by the results in Section 5.

2.4 Cracking the classical unboundedness barrier

What we obtain at the conclusion of Section 7 is the next summary theorem that unconditionally resolves the classical question of the unboundedness of the scaled function Mertens function $q(x) := |M(x)|/\sqrt{x}$ in the limit supremum sense.

Theorem 2.9 (Unboundedness of the the Mertens function, $q(x)$)**.** *We have that*

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}} = +\infty.$$

The proof of Theorem 2.9 is the main result we build up to in the article. It motivates all of our new constructions behind the additive function based sequences we employ to expand $M(x)$ via (1) and (2). This link relating our new formula for $M(x)$ to canonical additive functions and their distributions lends a recent distinguishing element to the success and characterization of the methods in our proof.

2.5 An overview of the core components to the proof

We offer the following initial step-by-step summary overview of the core components to our proof with the intention of making this new argument easier to parse in stages:

(1) We directly prove a matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). See Theorem 2.1 in Section 3.

(2) This crucial step provides us with an exact formula for $M(x)$ in terms of the prime counting function $\pi(x)$, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n)+1$. This formula is stated in (1) (see also Proposition 7.4).

(3) We tighten bounds given by a newer result proved in [12, §7] providing uniform asymptotic formulas for lower bounds on the summatory functions, $\widehat{\pi}_k(x)$, large $x > e$ when $1 \leq k \leq \log\log x$ (see Theorem 2.7). This allows us to eventually approximate the magnitude of the summatory functions

$$L(x) := \sum_{n \leq x} \lambda(n) \asymp \sum_{k=1}^{\log\log x} (-1)^k \widehat{\pi}_k(x), \text{ as } x \to \infty,$$

well from below (see the proof of Theorem 7.3; and Table T.2 starting on page 45 for numerical data).

(4) In Section 5. we relate $g^{-1}(n)$ to a subsequence of recursively-defined auxiliary functions, $C_k(n)$, that respectively express multiple k-convolutions of $\omega(n)$ with itself for $1 \leq k \leq \Omega(n)$ (see Lemma 5.1 and Lemma 5.3).

(5) In Section 6, we prove new expectation formulas for $|g^{-1}(n)|$ and the related component sequence $C_{\Omega(n)}(n)$ by first proving an Erdös-Kac like theorem satisfied by $C_{\Omega(n)}(n)$. This allows us to prove asymptotic lower bounds on $|G^{-1}(x)| \gg (\log x)\sqrt{\log\log x}$ when x is large and such that $G^{-1}(x) \neq 0$ in Section 7.

(6) When we return to step **(2)** with our new lower bounds at hand, we are led to a new unconditional proof of the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along an exponentially very large increasing infinite subsequence of positive natural numbers (see Section 7.2).

3 Preliminary proofs of new results

3.1 Establishing the summatory function properties and inversion identities

We will offer a proof of Theorem 2.1 suggested by an intuitive construction through matrix based methods. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95].

Proof of Theorem 2.1. Let h, g be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h. We have that the following formulas hold for all $x \geq 1$:

$$\begin{aligned}\pi_{g*h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d)H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right] H(i). \qquad (3)\end{aligned}$$

The first formula above is well known. The second formula is justified directly using summation by parts as [15, §2.10(ii)]

$$\begin{aligned}\pi_{g*h}(x) &= \sum_{d=1}^{x} h(d)G\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i \leq x} \left(\sum_{j \leq i} h(j)\right) \times \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right].\end{aligned}$$

We next form the invertible matrix of coefficients associated with this linear system defining $H(j)$ for all $1 \leq j \leq x$ in (3) by setting

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right), 1 \leq j \leq x.$$

Since $g_{x,x} = G(1) = g(1)$ and $g_{x,j} = 0$ for all $j > x$, the matrix we must work with in this problem is lower triangular with a non-zero constant on its diagonals, and is hence invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressed by applying an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

Here, U is a square matrix with sufficiently large finite dimensions whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$\left[(I - U^T)^{-1}\right]_{i,j} = [j \leq i]_{\delta}.$$

Observe that

$$\left\lfloor \frac{x}{j} \right\rfloor - \left\lfloor \frac{x-1}{j} \right\rfloor = \begin{cases} 1, & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

The previous property implies that

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \qquad (4)$$

We use the last property in (4) to shift the matrix $\hat{G}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of g in the following forms:

$$\left[(I-U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_{\delta}\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_{\delta}\right).$$

In particular, our target matrix in the inversion problem is defined by

$$(g_{x,j}) = (I-U^T)\left(g\left(\frac{x}{j}\right)[j|x]_{\delta}\right)(I-U^T)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators as

$$\begin{aligned}(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_{\delta}\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k)\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1}\right\rfloor} g^{-1}(k)\right).\end{aligned}$$

Hence, the summatory function $H(x)$ is given exactly for any $x \geq 1$ by a vector product with the inverse matrix from the previous equation in the form of

$$H(x) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right)\cdot \pi_{g*h}(k).$$

We can prove an inversion formula providing the coefficients of the summatory function $G^{-1}(i)$ for $1 \leq i \leq x$ given by the last equation by adapting our argument to prove (3) above. This leads to the following identity:

$$H(x) = \sum_{k=1}^{x} g^{-1}(x)\cdot \pi_{g*h}\left(\left\lfloor \frac{x}{k}\right\rfloor\right). \qquad \square$$

3.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, let $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n. Then we can easily prove using DGFs that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega+1) * \mu. \tag{5}$$

When combined with Corollary 2.2 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 2.3.

Proposition 3.1 (The signedness property of $g^{-1}(n)$)**.** *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_{\delta}} \in \{0,\pm 1\}$ *denote the sign of the arithmetic function h at integers $n \geq 1$. For the Dirichlet invertible function $g(n) := \omega(n)+1$, we have that* $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ *for all $n \geq 1$.*

Proof. The function $D_f(s) := \sum_{n\geq 1} f(n)n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ for σ_f the abcissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_{\mu}(s) = \zeta(s)^{-1}$ and $D_{\omega}(s) = P(s)\zeta(s)$ for $\operatorname{Re}(s) > 1$. Then by (5) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any arithmetic function f such that $f(1) \neq 0$ (e.g., this relation between the DGFs of these two functions holds whenever f^{-1} exists), we have for all $\mathrm{Re}(s) > 1$ that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{6}$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. This observation implies that $\operatorname{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make our claims and intuition rigorous.

By the recurrence relation that defines the Dirichlet inverse function of any arithmetic function h such that $h(1) = 1$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d|n \\ d>1}} h(d) h^{-1}(n/d), & n \geq 2. \end{cases} \tag{7}$$

For $n \geq 2$, the summands in (7) can be simply indexed over the primes $p|n$ given our definition of h from above. This observation yields that we can inductively unfold these sums into nested divisor sums provided the depth of the expanded divisor sums does not exceed the capacity to index summations over the primes dividing n. Namely, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p|n} h^{-1}\left(\frac{n}{p}\right), && \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right), && \text{if } \Omega(n) \geq 2 \\ &= -\sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1 p_2}} h^{-1}\left(\frac{n}{p_1 p_2 p_3}\right), && \text{if } \Omega(n) \geq 3. \end{aligned}$$

Then by induction with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1, n \geq 2. \tag{8}$$

In fact, by a combinatorial argument related to multinomial coefficient expansions of the sums in (8), we recover exactly that

$$h^{-1}(n) = \lambda(n)(\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}. \tag{9}$$

The last two expansions imply that the following property holds for all $n \geq 1$:

$$\operatorname{sgn}(h^{-1}(n)) = \lambda(n).$$

Since λ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$ for all divisors $d|n$ when $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever n is squarefree, so that we obtain the following result:

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1. \qquad \square$$

3.3 Statements of known limiting asymptotics

Theorem 3.2 (Mertens theorem). *For all $x \geq 2$ we have that*

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log\log x + B + o(1), \text{ as } x \to \infty,$$

where $B \approx 0.2614972128476427837554$ is an absolute constant.

Corollary 3.3 (Product form of Mertens theorem). *We have that for all sufficiently large $x \geq 2$*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} \left(1 + o(1)\right), \text{ as } x \to \infty.$$

Hence, for any real z we obtain that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^z \sim \frac{e^{-\gamma z}}{(\log x)^z}, \text{ as } x \to \infty.$$

Proofs of Theorem 3.2 and Corollary 3.3 are given in [5, §22.7; §22.8].

Facts 3.4 (Exponential integrals and the incomplete gamma function). Two variants of the *exponential integral function* are defined by the integral next representations [15, §8.19] [3, §3.3].

$$\begin{aligned} \mathrm{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, x \in \mathbb{R} \\ E_1(z) &:= \int_{1}^{\infty} \frac{e^{-tz}}{t} dt, \mathrm{Re}(z) \geq 0 \end{aligned}$$

These functions are related by $\mathrm{Ei}(-kz) = -E_1(kz)$ for real $k, z > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $\mathrm{Ei}(\pm x)$ for all real $x > 0$:

$$\begin{aligned} \gamma + \log x - x \leq \mathrm{Ei}(-x) &\leq \gamma + \log x - x + \frac{x^2}{4}, \\ 1 + \gamma + \log x - \frac{3}{4}x \leq \mathrm{Ei}(x) \quad &\leq 1 + \gamma + \log x - \frac{3}{4}x + \frac{11}{36}x^2. \end{aligned} \tag{10a}$$

The (upper) *incomplete gamma function* is defined by [15, §8.4]

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt, \mathrm{Re}(s) > 0.$$

The following properties of $\Gamma(s, x)$ hold:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, x > 0, \tag{10b}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, s > 0, \text{ as } x \to \infty. \tag{10c}$$

4 Components to the asymptotic analysis of lower bounds for sums of arithmetic functions weighted by $\lambda(n)$

4.1 A discussion of the results proved by Montgomery and Vaughan

Remark 4.1 (Intuition and constructions behind the proof of Theorem 2.6)**.** For $|z| < 2$ and $\mathrm{Re}(s) > 1$, let

$$F(s,z) := \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \tag{11}$$

and define the DGF coefficients, $a_z(n)$ for $n \geq 1$, by the product

$$\zeta(s)^z \cdot F(s,z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s}, \mathrm{Re}(s) > 1.$$

Suppose that $A_z(x) := \sum_{n \leq x} a_z(n)$ for $x \geq 1$. We obtain the next generating function like identity in z enumerating the $\widehat{\pi}_k(x)$ for $1 \leq k < 2\log\log x$.

$$A_z(x) = \sum_{n \leq x} z^{\Omega(n)} = \sum_{0 \leq k \leq \log_2(x)} \widehat{\pi}_k(x) z^k \tag{12}$$

Thus for $r < 2$, by Cauchy's integral formula we have

$$\widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|v|=r} \frac{A_v(x)}{v^{k+1}} dv.$$

Selecting $r := \frac{k-1}{\log\log x}$ for $1 \leq k < 2\log\log x$ leads to the uniform asymptotic formulas for $\widehat{\pi}_k(x)$ given in Theorem 2.6. Montgomery and Vaughan then consider individual analysis of the main and error terms for $A_z(x)$ to prove that

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

The estimate in the previous equation agrees with non-uniform estimates for $\widehat{\pi}_k(x)$ obtained roughly by an inductive type heuristic in [12, §7.4]. The multiple of $\zeta(s)^{-z}$ in the form of $F(s,z)$ is suggested as a smoothing factor to approach proving the theorem. We will require estimates of $A_{-z}(x)$ from below to form summatory functions that weight the terms of $\lambda(n)$ in our new formulas derived in the next sections.

4.2 New uniform asymptotics based on refinements of Theorem 2.6

Proposition 4.2. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \geq 2.$$

When $s := 1$, we have the asymptotic formula from Mertens theorem (see Theorem 3.2). For all integers $s \geq 2$ there are absolutely defined quasi-polynomial bounding functions $\gamma_0(s,x)$ and $\gamma_1(s,x)$ in s and x such that

$$\gamma_0(s,x) + o(1) \leq P_s(x) \leq \gamma_1(s,x) + o(1), \text{ as } x \to \infty.$$

It suffices to define the bounds in the previous equation by the functions

$$\begin{aligned}\gamma_0(s,x) &= s\log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}s(s-1)^2\log^2(2),\\ \gamma_1(s,x) &= s\log\left(\frac{\log x}{\log 2}\right) - s(s-1)\log\left(\frac{x}{2}\right) + \frac{1}{4}s(s-1)^2\log^2(x).\end{aligned}$$

Proof. Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$, and where our target smooth function is $f(t) = t^{-s}$ with $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &\sim \frac{1}{x^s \cdot \log x} + s \times \int_2^x \frac{dt}{t^s \log t} \\ &= \mathrm{Ei}(-(s-1)\log x) - \mathrm{Ei}(-(s-1)\log 2) + o(1), \text{ as } x \to \infty. \end{aligned}$$

Now using the inequalities in Facts 3.4, we obtain that the difference of the exponential integral functions in the previous equation is respectively bounded below and above by

$$\begin{aligned} \frac{P_s(x)}{s} &\geq \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) - \frac{1}{4}(s-1)^2\log^2(2) + o(1) \\ \frac{P_s(x)}{s} &\leq \log\left(\frac{\log x}{\log 2}\right) - (s-1)\log\left(\frac{x}{2}\right) + \frac{1}{4}(s-1)^2\log^2(x) + o(1). \end{aligned}$$ □

The utility to the quasi-logarithmic bounds tending to infinity as $x \to \infty$ stated in Proposition 4.2 will become apparent when we take the exponential of sums of the functions $P_j(x)$ for $j \geq 2$ in order to form a lower bound on $\mathcal{G}(-z)$ for $z := \frac{k-1}{\log\log x}$ in the next subsection. We will use the lower bound obtained in Theorem 2.7 to prove Theorem 7.3 where we show that as $x \to \infty$, $|G^{-1}(x)| \gg (\log x)\sqrt{\log\log x}$.

4.2.1 The proof of Theorem 2.7

We will first prove the stated form of the lower bound on $\mathcal{G}(-z)$ for $z := \frac{k-1}{\log\log x}$. Then we will discuss the technical adaptations to Montgomery and Vaughan's proof of Theorem 2.6 below to rigorously prove that the new asymptotic lower bounds on $\widehat{\pi}_k(x)$ that hold uniformly for all $1 \leq k \leq \log\log x$.

Lemma 4.3. *For sufficiently large $x > e$ and $1 \leq k \leq \log\log x$, we have that*

$$\left|\mathcal{G}\left(\frac{1-k}{\log\log x}\right)\right| \gg x^{-\frac{1}{4}}.$$

Proof. For $-2 < z < 2$ and integers $x \geq 2$, the right-hand-side of the following product is finite:

$$\widehat{P}(z,x) := \prod_{p \leq x}\left(1 - \frac{z}{p}\right)^{-1}.$$

For fixed $x \geq 2$ let

$$\mathbb{P}_x := \left\{n \in \mathbb{Z}^+ : \text{all prime divisors } p|n \text{ satisfy } p \leq x\right\}.$$

Then we can see that for $x \geq 2$

$$\prod_{p \leq x}\left(1 - \frac{z}{p^s}\right)^{-1} = \sum_{n \in \mathbb{P}_x} \frac{z^{\Omega(n)}}{n^s}. \tag{13}$$

By extending the argument in the proof given in [12, §7.4], we have that

$$A_{-z}(x) := \sum_{n \leq x} \lambda(n) z^{\Omega(n)} = \sum_{0 \leq k \leq \log_2(x)} \widehat{\pi}_k(x)(-z)^k,$$

Let $a_n(z,x)$ be defined as the coefficients of the DGF

$$\widehat{P}(z,x) =: \sum_{n \geq 1} \frac{a_n(z,x)}{n^s}.$$

We have argued that

$$\sum_{n\leq x} a_n(-z,x) = \sum_{k=0}^{\log_2(x)} \widehat{\pi}_k(x)(-z)^k + \sum_{k>\log_2(x)} e_k(x)(-z)^k.$$

This assertion is correct since the products of all non-negative integral powers of the primes $p \leq x$ (counting multiplicity) generate the integers $\{1 \leq n \leq x\}$ as a subset. Thus we capture all of the relevant terms needed to express $(-1)^k \cdot \widehat{\pi}_k(x)$ via the Cauchy integral formula representation over $A_{-z}(x)$ by replacing the corresponding infinite product terms with $\widehat{P}(-z,x)$ in the definition of $\mathcal{G}(-z)$.

Now we argue that

$$\mathcal{G}(-z) \gg \prod_{p\leq x}\left(1+\frac{z}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{-z}, 0 \leq z < 1, x \geq 2.$$

For $0 \leq z < 1$ and $x \geq 2$, we see that

$$\begin{aligned}
\mathcal{G}(-z) &= \exp\left(-\sum_p\left[\log\left(1+\frac{z}{p}\right)+z\cdot\log\left(1-\frac{1}{p}\right)\right]\right) \\
&\gg \exp\left(-z\times\sum_{p>x}\left[\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right]-\sum_{p\leq x}\left[\log\left(1+\frac{z}{p}\right)+z\cdot\log\left(1-\frac{1}{p}\right)\right]\right) \\
&\gg_z \widehat{P}(-z,x), \text{ as } x\to\infty,
\end{aligned}$$

where the *Mertens constant* B is defined exactly by the prime sum [5, §22.8]

$$B := \gamma + \sum_p\left[\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right].$$

Next, we have for all integers $0 \leq k \leq m < \infty$, and any sequence $\{f(n)\}_{n\geq 1}$ with sufficiently bounded partial power sums, that [11, §2]

$$[z^k]\prod_{1\leq i\leq m}(1-f(i)z)^{-1} = [z^k]\exp\left(\sum_{j\geq 1}\left(\sum_{i=1}^m f(i)^j\right)\frac{z^j}{j}\right), |z|<1. \tag{14}$$

In our case, $f(i)$ denotes the reciprocal of the i^{th} prime in the generating function expansion of (14). It follows from Proposition 4.2 that for any real $0 \leq z < 1$ we obtain

$$\begin{aligned}
\log\left[\prod_{p\leq x}\left(1+\frac{z}{p}\right)^{-1}\right] &\geq -(\log\log x + B)z + \sum_{j\geq 0}\left[\log\left(\frac{\log x}{\log 2}\right)-(2j+1)\log\left(\frac{x}{2}\right)-(2j+1)^2\frac{\log^2 2}{4}\right]z^{2j+2} \\
&\quad - \sum_{j\geq 0}\left[\log\left(\frac{\log x}{\log 2}\right)-(2j+2)\log\left(\frac{x}{2}\right)+(2j+2)^2\frac{\log^2 x}{4}\right]z^{2j+3} \\
&= -(\log\log x + B)z + \sum_{j\geq 0}\left[\log\left(\frac{\log x}{\log 2}\right)-(j+1)\log\left(\frac{x}{2}\right)\right](-z)^{j+2} \\
&\quad -\frac{1}{4}\times\sum_{j\geq 0}\left[(\log 2)^2(2j+1)^2z^{2j+2}+(\log x)^2(2j+2)^2z^{2j+3}\right] \\
&= -(\log\log x + B)z + \log\left(\frac{\log x}{\log 2}\right)\left[z-1+\frac{1}{z+1}\right]+\log\left(\frac{x}{2}\right)\left[\frac{2}{1+z}-1-\frac{1}{(1+z)^2}\right] \\
&\quad -(\log x)^2\times\frac{(z^3+z^5)}{(1-z^2)^3}-(\log 2)^2\times\frac{(z^2+6z^4+z^6)}{4(1-z^2)^3} \\
&=: \widehat{\mathcal{B}}(x;z).
\end{aligned} \tag{15}$$

We adjust the uniform bound parameter R so that

$$z \equiv z(k, x) = \frac{k-1}{\log\log x} \in [0, 1),$$

e.g., whenever $1 \leq k \leq \log\log x$ in the notation of Theorem 2.6. We have that

$$\min_{0 \leq z \leq 1} \left[z - 1 + \frac{1}{z+1}\right] = 0$$
$$\min_{0 \leq z \leq 1} \left[\frac{2}{1+z} - 1 - \frac{1}{(1+z)^2}\right] = -\frac{1}{4}.$$

Moreover, when we expand out the coefficients of $(\log 2)^2$ and $(\log x)^2$ in (15) by partial fractions of z, we see that all of the terms with an infinitely tending singularity as $z \to 1^-$ are positive. This means to obtain the lower bound, we can drop these contributions. What we are left to minimize is the following terms:

$$(\log 2)^2 \times \min_{0 \leq z \leq 1} \left[\frac{1}{4} - \frac{1}{4(1+z)^3} + \frac{5}{8(1+z)^2} - \frac{1}{2(1+z)}\right] = \frac{13}{108}(\log 2)^2$$
$$(\log x)^2 \times \min_{0 \leq z \leq 1} \left[\frac{1}{4(1+z)^3} - \frac{5}{8(1+z)^2} + \frac{1}{2(1+z)}\right] = \frac{7}{54}(\log x)^2.$$

So we have from (15) that

$$\widehat{\mathcal{B}}(x; z) \gg \left(\frac{2}{x}\right)^{\frac{1}{4}} \times \exp\left(\frac{13}{108}(\log 2)^2\right) \times \exp\left(\frac{7}{54}(\log x)^2\right) \gg x^{-\frac{1}{4}}.$$

In summary, we have arrived at a proof that as $x \to \infty$

$$\frac{e^{\gamma z}}{(\log x)^{-z}} \times \exp\left(\widehat{\mathcal{B}}(u, x; z)\right) \gg x^{-\frac{1}{4}}. \tag{16}$$

Finally, to finish our proof of the new lower bound on $\mathcal{G}(-z)$, we need only bound the reciprocal factor of $\Gamma(1-z) = -z \cdot \Gamma(-z)$. Since $z \equiv z(k, x) = \frac{k-1}{\log\log x}$ for $k \in [1, \log\log x]$, or again with $z \in [0, 1)$, we obtain for minimal k and all large enough $x \gg 1$ that $\Gamma(1-z) = \Gamma(1) = 1$, and for k towards the upper range of its interval that

$$\Gamma(1-z) \approx \Gamma\left(\frac{1}{\log\log x}\right) = \frac{1}{\log\log x} \times \Gamma\left(1 + \frac{1}{\log\log x}\right) \approx \frac{1}{\log\log x}.$$

Therefore, our assertion that the claimed lower bound holds is correct. □

Proof of Theorem 2.7. We now discuss the differences between our construction and that in the technical proof of Theorem 2.6 in the reference when we bound $\mathcal{G}(-z)$ from below as in the previous lemma. We will use v in place of the parameter z as our variable of complex contour integration. The reference proves that for $0 \leq z < 2$ [12, Thm. 7.18]

$$A_{-z}(x) = -\frac{zF(1, -z)}{\Gamma(1-z)} \cdot x(\log x)^{-(z+1)} + O\left(x(\log x)^{-\operatorname{Re}(z)-2}\right). \tag{17}$$

Recall that for $r < 2$ we have by Cauchy's integral formula that

$$(-1)^k \widehat{\pi}_k(x) = \frac{1}{2\pi\imath} \int_{|v|=r} \frac{A_{-v}(x)}{v^{k+1}} dv. \tag{18}$$

We first claim that uniformly for large x and $1 \leq k \leq \log\log x$ we have

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{1-k}{\log\log x}\right) \times \frac{x(\log\log x)^{k-1}}{(\log x)(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right]. \tag{19}$$

Then since we have proved in Lemma 4.3 that

$$\left|\mathcal{G}\left(\frac{1-k}{\log\log x}\right)\right| \gg x^{-\frac{1}{4}},$$

the result in (19) implies our stated uniform asymptotic bound. Namely, we obtain that

$$\widehat{\pi}_k(x) \gg \frac{x^{\frac{3}{4}}}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

We must provide analogs to the proofs of the two separate bounds from the reference corresponding to the error and main terms of our estimate according to (17) and (18).

Step I: Error Term Bound. To prove that the error term bound holds, we estimate the following bounds for $r := \frac{k-1}{\log\log x}$ with $r < 1$ whenever $2 \leq k \leq \log\log x$:

$$\begin{aligned}
\left|\frac{1}{2\pi\imath}\int_{|v|=r} \frac{x}{(\log x)^2}\frac{(\log x)^{-\operatorname{Re}(v)}}{v^{k+1}}dv\right| &\ll x(\log x)^{-(r+2)}r^{-(k+1)} \ll \frac{x}{(\log x)^2}\frac{(\log\log x)^{k+1}}{e^{k-1}(k-1)^{k+1}} \\
&\ll \frac{x}{(\log x)^2}\frac{(\log\log x)^{k+1}}{e^{2(k-1)}(k-1)!(k-1)^{\frac{3}{2}}} \ll \frac{x}{(\log x)^2}\frac{(\log\log x)^{k+1}}{(k-1)!} \\
&\ll \frac{x}{\log x}\frac{(\log\log x)^{k-4}}{(k-1)!}.
\end{aligned} \tag{20}$$

By the Cauchy integral formula, we can similarly verify that

$$\left|\frac{1}{2\pi\imath}\int_{|v|=r} \frac{x}{(\log x)^2}\frac{(\log x)^{-\operatorname{Re}(v)}}{v^{2}}dv\right| = \frac{x}{(\log x)^2}\cdot(\log\log x)^2 \ll \frac{x}{(\log x)(\log\log x)^2},$$

so that the formula for the error term in (20) also matches when $k := 1$.

We can calculate that for $0 \leq z < 1$

$$\begin{aligned}
\prod_p \left(1+\frac{z}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{-z} &= \exp\left(-\sum_p\left[\log\left(1+\frac{z}{p}\right) + z\log\left(1-\frac{1}{p}\right)\right]\right) \\
&\sim \exp\left(-o(z)\times\sum_p \frac{1}{p^2}\right) \\
&\gg \exp\left(-o(z)\cdot P(2)\right) \gg_z 1.
\end{aligned}$$

In other words, we have that $\mathcal{G}\left(\frac{1-k}{\log\log x}\right) \gg 1$ whenever $1 \leq k \leq \log\log x$. So the error term in (20) is majorized by taking $O\left(\frac{k}{(\log\log x)^3}\right)$ as our upper bound.

Step II: Main Term Bound. By (17) the main term estimate for (18) is given by $\frac{x}{\log x}\cdot I_x$, where

$$I_x := \frac{(-1)^{k-1}}{2\pi\imath}\int_{|v|=r} G(-z)(\log x)^{-v}v^{-k}dv.$$

In particular, we can write $I_x = I_{1,x} + I_{2,x}$ where we define

$$\begin{aligned}
I_{1,x} &:= \frac{G(-r)}{2\pi\imath}\int_{|v|=r}(\log x)^{-v}v^{-k}dv \\
&= \frac{(-1)^{k-1}G(-r)(\log\log x)^{k-1}}{(k-1)!} \\
I_{2,x} &:= \frac{1}{2\pi\imath}\int_{|v|=r}(G(-v)-G(-r))(\log x)^{-v}v^{-k}dv
\end{aligned}$$

$$= \frac{1}{2\pi\imath} \int_{|v|=r} (G(-v) - G(-r) + G'(-r)(v-r))(\log x)^{-v} v^{-k} dv.$$

The second integral formula for $I_{2,x}$ results from integration by parts.

We have by taking a power series expansion of $G''(-w)$ about $-r$ and integrating the resulting series termwise with respect to w that when $|v| = r$

$$\left|G(-v) - G(-r) + G'(-r)(v-r)\right| = \left| \int_r^v (v-w) G''(-w) dw \right| \ll |v-r|^2.$$

Now we parameterize the curve in the contour for $I_{2,x}$ by writing $v = re^{2\pi\imath t}$ for $t \in [-1/2, 1/2]$. This leads us to the bounds

$$\begin{aligned} |I_{2,x}| &\ll r^{3-k} \times \int_{-\frac{1}{2}}^{\frac{1}{2}} |e^{2\pi\imath t} - 1|^2 \cdot (\log x)^{re^{2\pi\imath t}} \cdot e^{2\pi\imath t} dt \\ &\ll r^{3-k} \times \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin^2(\pi t) \cdot e^{(k-1)\cos(2\pi t)} dt. \end{aligned}$$

Whenever $|x| \leq 1$, we know that $|\sin x| \leq |x|$. Also, $\cos(2\pi t) \leq 1 - 8t^2$ whenever $|t| \leq \frac{1}{2}$. Thus the last bound for $|I_{2,x}|$ becomes

$$\begin{aligned} |I_{2,x}| &\ll r^{3-k} e^{k-1} \times \int_0^\infty t^2 \cdot e^{-8(k-1)t^2} dt \\ &\ll \frac{r^{3-k} e^{k-1}}{(k-1)^{3/2}} = \frac{(\log\log x)^{k-3} e^{k-1}}{(k-1)^{k-\frac{3}{2}}} \\ &\ll \frac{k \cdot (\log\log x)^{k-3}}{(k-1)!}. \end{aligned}$$

Thus the contribution from the term $|I_{2,x}|$ can then be absorbed into the error term bound in (19). □

4.3 The distribution of exceptional values of $\Omega(n)$

The next theorems reproduced from [12, §7.4] characterize the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \log\log x$. Since $\mathbb{E}[\Omega(n)] = \log\log n + B$, these results imply a very regular, normal tendency of this arithmetic function towards its average order.

Theorem 4.4 (Upper bounds on exceptional values of $\Omega(n)$ for large n)**.** *Let*

$$\begin{aligned} A(x,r) &:= \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\}, \\ B(x,r) &:= \#\left\{n \leq x : \Omega(n) \geq r \cdot \log\log x\right\}. \end{aligned}$$

If $0 < r \leq 1$ and $x \geq 2$, then

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x,r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

Theorem 4.5 is a special case analog to the celebrated Erdös-Kac theorem typically stated for the normally distributed values of the scaled-shifted function $\omega(n)$ over $n \leq x$ as $x \to \infty$ [12, *cf.* Thm. 7.21].

Theorem 4.5 (Exact limiting bounds on exceptional values of $\Omega(n)$ for large n)**.** *We have that as $x \to \infty$*

$$\#\left\{3 \leq n \leq x : \Omega(n) - \log\log n \leq 0\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

The key interpretation we need to take away from the statements of Theorem 4.4 and Theorem 4.5 is the result proved in the next corollary. The role of the parameter R involved in stating the previous theorem is a critical bound as the scalar factor in the upper bound on $k \leq \log\log x$ in Theorem 2.6 up to which our uniform bounds given by Theorem 2.7 hold. In contrast, for $n \geq 2$ we can actually have contributions from values distributed throughout the range $1 \leq \Omega(n) \leq \log_2(n)$ infinitely often. It is then crucial that we can show that the main term in the asymptotic formulas we obtain for the summatory function over $\widehat{\pi}_k(x)$ is captured by summing only over the truncated range of $k \in [1, \log\log x]$ where the uniform bounds guaranteed by Theorem 2.6 and Theorem 2.7 are valid.

Corollary 4.6. *Using the notation for $A(x,r)$ and $B(x,r)$ from Theorem 4.4, we have that for $x \geq 2$ and $\delta > 0$,*

$$\frac{B(x,1+\delta)}{A(x,1)} = o_\delta(1), \text{ as } x \to \infty.$$

Proof. To show that the asymptotic bound is correct, we compute using Theorem 4.4 and Theorem 4.5 that

$$\frac{B(x,1+\delta)}{A(x,1)} \ll \frac{x \cdot (\log x)^{\delta-(1+\delta)\log(1+\delta)}}{O(1) + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right)} \sim o_\delta(1),$$

as $x \to \infty$. □

Remark 4.7 (Applications and key consequences)**.** Since $\mathbb{E}[\Omega(n)] = \log\log n + B$, again with $0 < B < 1$ the absolute constant from Mertens theorem, when we denote the range of $k > \log\log x$ as holding in the form of $k > (1+\delta)\log\log x$ for $\delta > 0$ at large x, we can assume that $\delta \to 0^+$ as $x \to \infty$ when we apply Corollary 4.6 in practice. In particular, this type of bound holds since $k > \log\log x$ implies that

$$\lfloor \log\log x \rfloor + 1 \geq (1+\delta)\log\log x \qquad \implies \qquad \delta \leq \frac{1 + \{\log\log x\}}{\log\log x} = o(1), \text{ as } x \to \infty.$$

The key consequence of this observation is that the ratio

$$\left| \frac{\sum\limits_{k > \log\log x} (-1)^k \widehat{\pi}_k(x)}{\sum\limits_{k \leq \log\log x} (-1)^k \widehat{\pi}_k(x)} \right| \ll \frac{\sqrt{\log\log x} \cdot B(x,1+\delta)}{x} = o_\delta(1),$$

is bounded above by at most a small constant for any $\delta > 0$ when x is large. The second term in the last bound is obtained by summing over the uniform estimates guaranteed by Theorem 2.6 and applying (10c) to the resulting expression involving the incomplete gamma function (see [19] for symbolic simplifications in *Mathematica*).

5 Auxiliary sequences to express the Dirichlet inverse function, $g^{-1}(n)$

The pages of tabular data given as Table T.1 in the appendix section (refer to page 38) are intended to provide clear insight into why we eventually arrived at the approximations to $g^{-1}(n)$ initially proved in this section. The table provides illustrative numerical data by examining the approximate behavior at hand for the cases of $1 \leq n \leq 500$ with *Mathematica* [19]. In Section 6, we will use these relations between $g^{-1}(n)$ and $C_{\Omega(n)}(n)$ to prove an Erdös-Kac like analog for the distribution of this function.

5.1 Definitions and basic properties of component function sequences

We define the following auxiliary coefficient sequence for integers $n \geq 1$ and $k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{21}$$

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (21) inductively. By the same argument, we see that at fixed n, the function $C_k(n)$ is seen to be non-zero only for positive integers $k \leq \Omega(n)$ whenever $n \geq 2$. A sequence of relevant signed semi-diagonals of the functions $C_k(n)$ begins as follows [20, A008480]:

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, \ -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

We can see that $C_{\Omega(n)}(n) \leq (\Omega(n))!$ for all $n \geq 1$. In fact, $h^{-1}(n) \equiv \lambda(n) C_{\Omega(n)}(n)$ is the same function given by the formula in (9) from Proposition 3.1. This sequence of semi-diagonals of (21) is precisely related to $g^{-1}(n)$ in the next subsection.

5.2 Relating the auxiliary functions $C_{\Omega(n)}(n)$ to formulas approximating $g^{-1}(n)$

Lemma 5.1 (An exact initial formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) g^{-1}(n/d) \quad \Longrightarrow \quad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \tag{22}$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (22) in the form of $(g^{-1} * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m \cdot (C_m(-) * g^{-1})(n)$, or so that

$$F_m(n) = -\begin{cases} \sum\limits_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum\limits_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & m \geq 2, \\ (\omega * g^{-1})(n), & m = 1. \end{cases}$$

By repeatedly expanding the right-hand-side of the previous equation, we find that for $m := \Omega(n)$ (i.e., with the expansions at a maximal depth in the previous equation)

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \tag{23}$$

The formula then follows from (23) by Möbius inversion applied to each side of the last equation. $\square$

Corollary 5.2. *For all squarefree integers $n \geq 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \tag{24}$$

Proof. Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geq 2$ and that n is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where p_i is prime for all $1 \leq i \leq \omega(n)$. Since all divisors of any squarefree n are necessarily also squarefree, we can transform the exact divisor sum guaranteed for all n in Lemma 5.1 into a sum that partitions the divisors according to the number of distinct prime factors as follows:

$$\begin{aligned} g^{-1}(n) &= \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i}(-1)^i \cdot C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \end{aligned}$$

The signed contributions in the first of the previous equations is justified by noting that $\lambda(n) = \mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree, and that for $d \geq 1$ squarefree we have the correspondence $\omega(d) = k \implies \Omega(d) = k$. □

Since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ using the notation defined in the the proof of Proposition 3.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for squarefree $n \geq 1$. A proof of part (B) of Conjecture 2.5 follows as an immediate consequence.

Lemma 5.3. *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{25}$$

Proof. By applying Lemma 5.1, Proposition 3.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$, Lemma 5.1 implies

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \end{aligned}$$

In the last equation, we see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. □

Combined with the signedness property of $g^{-1}(n)$ guaranteed by Proposition 3.1, Lemma 5.3 shows that its summatory function is expressed as

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_{\Omega(d)}(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Additionally, since (5) implies that

$$\lambda(d)C_{\Omega(d)}(d) = (g^{-1} * 1)(d) = (\chi_{\mathbb{P}} + \varepsilon)^{-1}(d),$$

where $\chi_{\mathbb{P}}$ denotes the characteristic function of the primes, we also clearly recover by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1.$$

This connection between the summatory function of $g^{-1}(n)$ and the primes is also relayed by the form of the identity we prove for $M(x)$ in Proposition 7.4 involving the prime counting function, $\pi(x)$.

5.3 A connection to the distribution of the primes

The combinatorial complexity of $g^{-1}(n)$ is deeply tied to the distribution of the primes $p \leq n$ as $n \to \infty$. The magnitudes and dispersion of the primes $p \leq x$ certainly restricts the repeating of these distinct sequence values. Nonetheless, we can see that the following is still clear about the relation of the weight functions $|g^{-1}(n)|$ to the distribution of the primes: The value of $|g^{-1}(n)|$ is entirely dependent on the pattern of the *exponents* (viewed as multisets) of the distinct prime factors of $n \geq 2$ (*cf.* Heuristic 2.4).

Example 5.4 (Combinatorial significance to the distribution of $g^{-1}(n)$)**.** We have a natural extremal behavior with respect to distinct values of $\Omega(n)$ corresponding to squarefree integers and prime powers. Namely, if for $k \geq 1$ we define the infinite sets M_k and m_k to correspond to the maximal (minimal) sets of positive integers such that

$$M_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

$$m_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

then any element of M_k is squarefree and any element of m_k is a prime power. In particular, we have that for any $N_k \in M_k$ and $n_k \in m_k$

$$N_k = \sum_{j=0}^{k} \binom{k}{j} \cdot j!, \qquad \text{and} \qquad n_k = 2 \cdot (-1)^k.$$

The formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 3.1 implies that we can express an exact formula for $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n. Namely, for $n \geq 2$ and $0 \leq k \leq \omega(n)$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z \cdot \nu_p(n)) = [z^k] \prod_{p^\alpha || n} (1 + \alpha z).$$

Then we have essentially shown using (9) and (25) that we can expand formulas for these inverse functions in the following form:

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, n \geq 2.$$

The combinatorial formula for $h^{-1}(n) = \lambda(n) \cdot (\Omega(n))! \times \prod_{p^\alpha || n} (\alpha!)^{-1}$ we derived in the proof of the key signedness proposition in Section 3 suggests further patterns and more regularity in the contributions of the distinct weighted terms for $G^{-1}(x)$. Our interpretations leading to the proof of the bounds on $|G^{-1}(x)|$ from below via Theorem 7.3 are more analytically motivated.

6 The distributions of $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$

We have remarked already in the introduction that the relation of the component functions, $g^{-1}(n)$ and $C_{\Omega(n)}(n)$, to the canonical additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties of these functions witnessed á priori in Table T.1. In particular, each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdös-Kac theorem that shows that the density of a shifted and scaled variant of each of the sets of these function values for $n \leq x$ can be expressed through a limiting normal distribution as $x \to \infty$ [4, 2, 16]. In the remainder of this section we establish more analytical proofs of related properties of these key sequences used to express $G^{-1}(x)$, again in the spirit of Montgomery and Vaughan's reference manual (*cf.* Remark 2.8).

Proposition 6.1. *Let the function $\widehat{F}(s,z)$ is defined for $\operatorname{Re}(s) \geq 2$ and $|z| < |P(s)|^{-1}$ in terms of the prime zeta function by*

$$\widehat{F}(s,z) := \frac{1}{1-P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

For $|z| < P(2)^{-1}$, the summatory function of the coefficients of the DGF expansion of $\widehat{F}(s,z)$ are defined as follows:

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

We have that for all sufficiently large x

$$\widehat{A}_z(x) = \frac{x}{\Gamma(z)} \cdot \widehat{F}(2,z) \cdot (\log x)^{z-1} + O_z\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right), |z| < P(2)^{-1}.$$

Proof. We can see by adapting the notation from the proof of Proposition 3.1 that for $n \geq 2$

$$C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}.$$

We can generate scaled forms of these terms through a product identity of the form

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! \cdot p^{rs}}\right)^{-1} = \exp\left(z \cdot P(s)\right), \operatorname{Re}(s) \geq 2, z \in \mathbb{C}.$$

By computing a Laplace transform on the right-hand-side of the above, we obtain

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n) \cdot (-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp\left(tz \cdot P(s)\right) dt = \frac{1}{1-P(s)z}, \operatorname{Re}(s) \geq 2, |z| < |P(s)|^{-1}.$$

It follows that

$$\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times \widehat{F}(s,z), \operatorname{Re}(s) \geq 2, |z| < |P(s)|^{-1}.$$

Since $\widehat{F}(s,z)$ is an analytic function of s for all $\operatorname{Re}(s) > 1$ whenever the parameter $|z| < |P(s)|^{-1}$, if $b_z(n)$ are the coefficients in the DGF expansion of $\widehat{F}(s,z)$ (as above), then

$$\left| \sum_{n \geq 1} \frac{b_z(n) (\log n)^{2R+1}}{n^s} \right| < +\infty,$$

is uniformly bounded for $|z| \leq R$. This fact follows by repeated termwise differentiation with respect to s.

Let the function $d_z(n)$ be generated as the coefficients of the DGF $\zeta(s)^z = \sum_{n\geq 1} \frac{d_z(n)}{n^s}$ for $\operatorname{Re}(s) > 1$, and with corresponding summatory function $D_z(x) := \sum_{n\leq x} d_z(n)$. The theorem in [12, Thm. 7.17; §7.4] implies that for any $z \in \mathbb{C}$ and integers $x \geq 2$

$$D_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right).$$

We set $b_z(n) \equiv (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$, set the convolution $a_z(n) := \sum_{d|n} b_z(d) d_z(n/d)$, and define its summatory function $A_z(x) := \sum_{n\leq x} a_z(n)$. Then we have that

$$\begin{aligned} A_z(x) &= \sum_{m\leq x/2} b_z(m) D_z(x/m) + \sum_{x/2<m\leq x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \times \sum_{m\leq x/2} \frac{b_z(m)}{m^2} \times m \cdot \log\left(\frac{x}{m}\right)^{z-1} + O\left(\sum_{m\leq x} \frac{x \cdot |b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\operatorname{Re}(z)-2}\right). \end{aligned} \tag{26}$$

We can sum the coefficients for $u > e$ large as follows:

$$\sum_{m\leq u} \frac{b_z(m)}{m} = \left(\widehat{F}(2,z) + O(u^{-2})\right) u - \int_1^u \left(\widehat{F}(2,z) + O(t^{-2})\right) dt = \widehat{F}(2,z) + O(1 + u^{-1}).$$

Suppose that $|z| \leq R < P(2)^{-1}$. The error term in (26) satisfies

$$\begin{aligned} \sum_{m\leq x} \frac{x \cdot |b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\operatorname{Re}(z)-2} &\ll x(\log x)^{\operatorname{Re}(z)-2} \times \sum_{m\leq\sqrt{x}} \frac{|b_z(m)|}{m} \\ &\quad + x(\log x)^{-(R+2)} \times \sum_{m>\sqrt{x}} \frac{|b_z(m)|}{m} (\log m)^{2R} \\ &\ll x(\log x)^{\operatorname{Re}(z)-2} \cdot \widehat{F}(2,z) = O_z\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right), |z| \leq R. \end{aligned}$$

In the main term estimate for $A_z(x)$ from (26), when $m \leq \sqrt{x}$ we have

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\operatorname{Re}(z)-2}\right).$$

The total sum over the interval $m \leq x/2$ corresponds to bounding the following sum components when we take $|z| \leq R$:

$$\begin{aligned} \sum_{m\leq x/2} b_z(m) D_z(x/m) &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \times \sum_{m\leq x/2} \frac{b_z(m)}{m} \\ &\quad + O_z\left(x(\log x)^{\operatorname{Re}(z)-2} \times \sum_{m\leq\sqrt{x}} \frac{|b_z(m)|}{m} + x(\log x)^{R-1} \times \sum_{m>\sqrt{x}} \frac{|b_z(m)|}{m}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \widehat{F}(2,z) + O_z\left(x(\log x)^{\operatorname{Re}(z)-2} \times \sum_{m\geq 1} \frac{b_z(m)(\log m)^{2R+1}}{m^2}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \widehat{F}(2,z) + O_z\left(x(\log x)^{\operatorname{Re}(z)-2}\right). \end{aligned}$$

□

Theorem 6.2. *We have uniformly for $1 \leq k < \log\log x$ that as $x \to \infty$*

$$\widehat{C}_k(x) := \sum_{\substack{n\leq x \\ \Omega(n)=k}} (-1)^{\omega(n)} C_k(n) \asymp \frac{x}{\log x} \cdot \frac{(\log\log x - \log\zeta(2))^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

Proof. The proof is an adaptation of the method of Montgomery and Vaughan we cited in Remark 4.1 to prove our variant of Theorem 2.7 in that section. We begin by bounding a contour integral over the error term for fixed large x when $r := \frac{k-1}{\log\log x}$ with $r < 2$:

$$\begin{aligned}\left|\int_{|v|=r} \frac{x \cdot (\log x)^{-(\operatorname{Re}(v)+2)}}{v^{k+1}} dv\right| &\ll x(\log x)^{-(r+2)} r^{-(k+1)} \ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{k+1}} \cdot \frac{1}{e^{k-1}} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{3/2}} \cdot \frac{1}{e^{2k}(k-1)!} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \ll \frac{x}{\log x} \cdot \frac{k \cdot (\log\log x)^{k-5}}{(k-1)!}.\end{aligned}$$

We must find an asymptotically accurate main term approximation to the coefficients of the following contour integral for $r \in [0, z_{\max}]$ where $z_{\max} < P(2)^{-1}$ according to Proposition 6.1:

$$\widetilde{A}_r(x) := \int_{|v|=r} \frac{x \cdot (\log x)^{-v} \zeta(2)^{-v}}{(\log x)\Gamma(1+v) \cdot v^k (1+P(2)v)} dv. \tag{27}$$

We can show that provided a restriction to $1 \leq r < 1$, we can approximate the contour integral in (27) where the resulting main term is accurate up to a bounded constant factor. This procedure removes the gamma function term in the denominator of the integrand by essentially applying a mean value theorem type analog for smoothly parameterized contours. The logic used to justify this type of simplification of form argument is discussed next. We observe that for $r := 1$, the function $|\Gamma(1+re^{2\pi\imath t})|$ has a singularity (pole) when $t := \frac{1}{2}$. We restrict the range of $|v| = r$ so that $0 \leq r < 1$ to necessarily avoid this problematic value of t when we parameterize $v = re^{2\pi\imath t}$ by a real-line integral over $t \in [0, 1]$. We can compute finite extremal values of this function as

$$\min_{\substack{0 \leq r < 1 \\ 0 \leq t \leq 1}} |\Gamma(1+re^{2\pi\imath t})| = |\Gamma(1+re^{2\pi\imath t})|\Bigg|_{(r,t)\approx(1,0.740592)} \approx 0.520089$$

$$\max_{\substack{0 \leq r < 1 \\ 0 \leq t \leq 1}} |\Gamma(1+re^{2\pi\imath t})| = |\Gamma(1+re^{2\pi\imath t})|\Bigg|_{(r,t)\approx(1,0.999887)} \approx 1.$$

This shows that

$$\widetilde{A}_r(x) \asymp \int_{|v|=r} \frac{x \cdot (\log x)^{-v} \zeta(2)^{-v}}{(\log x) \cdot v^k (1+P(2)v)} dv, \tag{28}$$

where as $x \to \infty$

$$\frac{\widetilde{A}_r(x)}{\int_{|v|=r} \frac{x(\log x)^{-v}\zeta(2)^{-v}}{(\log x)\cdot v^k(1+P(2)v)} dv} \in [1, 1.92275].$$

By induction we can compute the remaining coefficients $[z^k]\Gamma(1+z) \times \widehat{A}_z(x)$ with respect to x for fixed $k \leq \log\log x$ using the Cauchy integral formula. Namely, it is not difficult to see that for any integer $m \geq 0$, we have the m^{th} partial derivative of the integrand with respect to z has the following limiting expansion by applying (10c):

$$\begin{aligned}\frac{1}{m!} \times \frac{\partial^{(m)}}{\partial v^{(m)}} \left[\frac{(\log x)^{-v}\zeta(2)^{-v}}{1+P(2)v}\right]\Bigg|_{v=0} &= \sum_{j=0}^{m} \frac{(-1)^m P(2)^j (\log\log x + \log\zeta(2))^{m-j}}{(m-j)!} \\ &= \frac{(-P(2))^m (\log x)^{\frac{1}{P(2)}} \zeta(2)^{\frac{1}{P(2)}}}{m!} \times \Gamma\left(m+1, \frac{\log\log x + \log\zeta(2)}{P(2)}\right) \\ &\sim \frac{(-1)^m (\log\log x + \log\zeta(2))^m}{m!}.\end{aligned}$$

Now by parameterizing the countour around $|z| = r := \frac{k-1}{\log\log x} < 1$ we deduce that the the main term of our approximation corresponds to

$$\int_{|v|=r} \frac{x \cdot (\log x)^{-v} \zeta(2)^{-v}}{(\log x) v^k (1+P(2)v)} dv \asymp \frac{x}{\log x} \cdot \frac{(-1)^{k-1} (\log\log x + \log \zeta(2))^{k-1}}{(k-1)!}. \qquad \square$$

Corollary 6.3. *We have that for large $x \geq 2$ that uniformly for $1 \leq k \leq \log\log x$*

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp 2\sqrt{2\pi} \cdot x \times \frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!}.$$

Proof. We have an integral formula involving the non-sign-weighted sequence that results by applying ordinary Abel summation (and integrating by parts) in the form of

$$\sum_{n \leq x} \lambda_*(n) h(n) = \left(\sum_{n \leq x} \lambda_*(n)\right) h(x) - \int_1^x \left(\sum_{n \leq t} \lambda_*(n)\right) h'(t) dt \tag{29}$$

$$\left\{ \begin{array}{ll} u_t = L_*(t) & v_t' = h'(t)dt \\ u_t' = L_*'(t)dt & v_t = h(t) \end{array} \right\}$$

$$\asymp \int_1^x \frac{d}{dt}\left[\sum_{n \leq t} \lambda_*(n)\right] h(t) dt. \tag{30}$$

Let the signed left-hand-side summatory function for our function corresponding to (29) be defined by

$$\begin{aligned} \widehat{C}_{k,*}(x) &:= \sum_{\substack{n \leq x \\ \Omega(n)=k}} (-1)^{\omega(n)} C_{\Omega(n)}(n) \\ &\asymp \frac{x}{\log x} \cdot \frac{(\log\log x + \log\zeta(2))^{k-1}}{(k-1)!} \left[1 + O\left(\frac{1}{\log\log x}\right)\right] \\ &\asymp \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{1}{\log\log x}\right)\right] \end{aligned}$$

where the second equation above follows from the proof of Theorem 6.2. We adopt the notation that $\lambda_*(n) = (-1)^{\omega(n)}$ for $n \geq 1$.

We next transform our previous results for the partial sums over the signed sequences $\lambda_*(n) \cdot C_{\Omega(n)}(n)$ such that $\Omega(n) = k$. The argument is based on approximating the smooth summatory function of $\lambda_*(n) := (-1)^{\omega(n)}$ using the following known uniform approximation of $\pi_k(x)$ when $1 \leq k \leq \log\log x$ as $x \to \infty$:

$$\pi_k(x) \asymp \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}(1 + o(1)).$$

In particular, we have by an asymptotic approximation to the incomplete gamma function that (compare to Table T.2 starting on page 45)

$$L_*(t) := \left|\sum_{n \leq t} (-1)^{\omega(n)}\right| = \left|\sum_{k=1}^{\log\log x} (-1)^k \pi_k(x)\right| \sim \frac{t}{\sqrt{2\pi}\sqrt{\log\log t}}, \text{ as } t \to \infty.$$

The main term for the reciprocal of the derivative of this summatory function is asymptotic to

$$\frac{1}{L_*'(t)} \asymp \sqrt{2\pi} \cdot (\log\log t)^{\frac{1}{2}}.$$

After applying the formula from (29), we thus deduce that the unsigned summatory function variant satisfies

$$\begin{aligned}
\widehat{C}_{k,*}(x) &= \int_1^x L_*'(t) C_{\Omega(t)}(t) dt \qquad \implies C_{\Omega(x)}(x) \asymp \frac{\widehat{C}_{k,*}'(x)}{L_*'(x)} \qquad \implies \\
C_{\Omega(x)}(x) &\asymp \sqrt{2\pi} \cdot \frac{(\log\log x)^{\frac{1}{2}}}{\log x} \cdot \left[\frac{(\log\log x)^{k-1}}{(k-1)!}\left(1 - \frac{1}{\log x}\right) + \frac{(\log\log x)^{k-2}}{(\log x)(k-2)!}\right] \\
&\asymp \sqrt{2\pi} \cdot \frac{(\log\log x)^{k-\frac{1}{2}}}{(\log x)(k-1)!} =: \widehat{C}_{k,**}(x).
\end{aligned}$$

The ordinary Abel summation formula, and integration by parts, implies that we obtain a main term is given by

$$\begin{aligned}
\sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) &\asymp \int \widehat{C}_{k,**}(x) dx \\
&\asymp 2\sqrt{2\pi} \cdot x \times \frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!}.
\end{aligned}$$

□

Lemma 6.4. *We have that as $x \to \infty$*

$$\mathbb{E}\left[C_{\Omega(n)}(n)\right] \asymp 2\sqrt{2\pi} \cdot (\log n)\sqrt{\log\log n}.$$

Proof. We first compute the following summatory function by applying Corollary 6.3:

$$\sum_{k=1}^{\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp 2\sqrt{2\pi} \cdot x \cdot (\log x)\sqrt{\log\log x}. \tag{31}$$

We claim that

$$\sum_{n \leq x} C_{\Omega(n)}(n) = \sum_{k=1}^{\log_2(x)} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \asymp \sum_{k=1}^{\log\log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n). \tag{32}$$

Then (31) clearly implies our result. To prove (32) it suffices to show that

$$\frac{\sum\limits_{\log\log x < k \leq \log_2(x)} \sum\limits_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n)}{\sum\limits_{k=1}^{\log\log x} \sum\limits_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n)} = o(1), \text{ as } x \to \infty. \tag{33}$$

We define the following component sums for large x and $0 < \varepsilon < 1$ so that $(\log\log x)^{\frac{\varepsilon \log\log x}{\log\log\log x}} = o(\log x)$:

$$S_{2,\varepsilon}(x) := \sum_{\log\log x < k \leq (\log\log x)^{\frac{\varepsilon \log\log x}{\log\log\log x}}} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n).$$

Then

$$\sum_{k=\log\log x}^{\log_2(x)} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \gg S_{2,\varepsilon}(x),$$

with equality as $\varepsilon \to 1$ when the upper bound of summation tends to $\log x$. Observe that whenever $\Omega(n) = k$, we have that $C_{\Omega(n)}(n) \leq k!$. We can then bound the sums defined above using Theorem 2.7 and Theorem 4.4 for large $x \to \infty$ as

$$\begin{aligned} S_{2,\varepsilon}(x) &\leq \sum_{\log\log x \leq k \leq \log x} \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \ll \sum_{k=\log\log x}^{(\log\log x)^{\frac{\varepsilon \log\log x}{\log\log\log x}}} \frac{\widehat{\pi}_k(x)}{x} \cdot k! \\ &\ll \sum_{k=\log\log x}^{(\log\log x)^{\frac{\varepsilon \log\log x}{\log\log\log x}}} (\log x)^{\frac{k}{\log\log x} - 1 - \frac{k}{\log\log x}(\log k - \log\log\log x)} \cdot \left(\frac{k}{e}\right)^k \sqrt{2\pi k} \\ &\ll \sum_{k=\log\log x}^{\frac{\varepsilon \log\log x}{\log\log\log x}} (\log x)^{\frac{2k \cdot \log\log\log x}{\log\log x} - 1} \sqrt{k} \\ &\ll \frac{1}{(\log x)} \times \int_{\log\log x}^{\frac{\varepsilon \log\log x}{\log\log\log x}} (\log\log x)^{2t} \sqrt{t} \cdot dt \\ &\ll \frac{1}{(\log x)} \sqrt{\frac{\varepsilon \cdot \log\log x}{\log\log\log x}} (\log\log x)^{\frac{2\varepsilon \cdot \log\log x}{\log\log\log x}} = o(x), \end{aligned}$$

where we have a simplification for large x by noticing that $\lim_{x\to\infty} (\log x)^{\frac{1}{\log\log x}} = e$. So by (31) this form of the ratio in (33) clearly tends to zero. $\square$

Corollary 6.5. *We have that as $n \to \infty$, the unsigned sequence mean satisfies*

$$\mathbb{E}|g^{-1}(n)| \asymp (\log n)^2 \sqrt{\log\log n}.$$

Proof. We use the formula from Lemma 6.4 to find $\mathbb{E}[C_{\Omega(n)}(n)]$ up to a small bounded multiplicative constant factor as $n \to \infty$. This implies that for large t

$$\begin{aligned} \int \frac{\mathbb{E}[C_{\Omega(t)}(t)]}{t} dt &\asymp \sqrt{2\pi} \cdot (\log t)^2 \sqrt{\log\log t} - \frac{\pi}{2} \operatorname{erfi}\left(\sqrt{2\log\log t}\right) \\ &\asymp \sqrt{2\pi} \cdot (\log t)^2 \sqrt{\log\log t}. \end{aligned}$$

Recall from the introduction that the summatory function of the squarefree integers is approximated by

$$Q(x) := \sum_{n \leq x} \mu^2(n) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Therefore summing over (25) we find that

$$\begin{aligned} \mathbb{E}|g^{-1}(n)| &= \frac{1}{n} \times \sum_{d \leq n} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{n}{d} \right\rfloor\right) \\ &\sim \sum_{d \leq n} C_{\Omega(d)}(d) \left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dn}}\right)\right] \\ &= \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d<n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O\left(\frac{1}{\sqrt{n}} \times \int_0^n t^{-1/2} dt\right) \\ &= \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d<n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O(1) \qquad (34) \\ &\asymp \frac{6\sqrt{2}}{\pi^{\frac{3}{2}}} (\log n)^2 \sqrt{\log\log n}. \end{aligned}$$

$\square$

Theorem 6.6. *Let the mean and variance analogs be denoted by*

$$\mu_x(C) := \log\log x + \hat{a} - \frac{1}{2} \cdot \log\log\log x, \qquad \text{and} \qquad \sigma_x(C) := \sqrt{\mu_x(C)},$$

where the absolute constant $\hat{a} := \log\left(\frac{1}{\sqrt{2\pi}}\right) \approx -0.918939$. *Set* $Y > 0$ *and suppose that* $z \in [-Y, Y]$. *Then we have uniformly for all* $-Y \leq z \leq Y$ *that*

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)} \leq z\right\} = \Phi(z) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \rightarrow \infty.$$

Proof. For large x and $n \leq x$, define the following auxiliary variables:

$$\alpha_n := \frac{C_{\Omega(n)}(n) - \mu_n(C)}{\sigma_n(C)}, \quad \text{and} \quad \beta_{n,x} := \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)}.$$

Let the corresponding densities (whose limiting distributions we must verify) be defined by the functions

$$\Phi_1(x, z) := \frac{1}{x} \cdot \#\{n \leq x : \alpha_n \leq z\},$$

and

$$\Phi_2(x, z) := \frac{1}{x} \cdot \#\{n \leq x : \beta_{n,x} \leq z\}.$$

We first argue that it suffices to consider the distribution of $\Phi_2(x, z)$ as $x \rightarrow \infty$ in place of $\Phi_1(x, z)$ to obtain our desired result. The difference of the two auxiliary variables is neglibible as $x \rightarrow \infty$ for (n, x) taken over the ranges that contribute the non-trivial weight to the main term of each density function. In particular, we have for $\sqrt{x} \leq n \leq x$ and $C_{\Omega(n)}(n) \leq 2 \cdot \mu_x(C)$ that the following is true:

$$|\alpha_n - \beta_{n,x}| \ll \frac{1}{\sigma_x(C)} \xrightarrow{x \rightarrow \infty} 0.$$

Thus we can replace α_n by $\beta_{n,x}$ and estimate the limiting densities corresponding to the alternate terms. The rest of our argument follows the method in the proof of the related theorem in [12, Thm. 7.21; §7.4] closely. Readers familiar with the methods in the reference will see many parallels to that construction.

We use the formula proved in Corollary 6.3 to estimate the densities claimed within the ranges bounded by z as $x \rightarrow \infty$. Let $k \geq 1$ be a natural number defined by $k := t + \mu_x(C)$. We write the small parameter $\delta_{t,x} := \frac{t}{\mu_x(C)}$. When $|t| \leq \frac{1}{2}\mu_x(C)$, we have by Stirling's formula that

$$\begin{aligned} 2\sqrt{2\pi} \cdot x \times \frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!} &\sim \frac{e^{\hat{a}+t}(\log\log x)^{\mu_x(C)(1+\delta_{t,x})}}{\sigma_x(C) \cdot \mu_x(C)^{\mu_x(C)(1+\delta_{t,x})}(1+\delta_{t,x})^{\mu_x(C)(1+\delta_{t,x})+\frac{3}{2}}} \\ &\sim \frac{e^t}{\sqrt{2\pi} \cdot \sigma_x(C)}(1+\delta_{t,x})^{-\left(\mu_x(C)(1+\delta_{t,x})+\frac{3}{2}\right)}, \end{aligned}$$

since $\frac{\mu_x(C)}{\log\log x} = 1 + o(1)$ as $x \rightarrow \infty$.

We have the uniform estimate $\log(1+\delta_{t,x}) = \delta_{t,x} - \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)$ whenever $|\delta_{t,x}| \leq \frac{1}{2}$. Then we can expand the factor involving $\delta_{t,x}$ in the previous equation as follows:

$$\begin{aligned} (1+\delta_{t,x})^{-\mu_x(C)(1+\delta_{t,x})-\frac{1}{2}} &= \exp\left(\left(\frac{1}{2} + \mu_x(C)(1+\delta_{t,x})\right) \times \left(-\delta_{t,x} + \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)\right)\right) \\ &= \exp\left(-t - \frac{3t+t^2}{2\mu_x(C)} + \frac{3t^2}{4\mu_x(C)^2} + O\left(\frac{|t|^3}{\mu_x(C)^2}\right)\right). \end{aligned}$$

For both $|t| \leq \mu_x(C)^{1/2}$ and $\mu_x(C)^{1/2} < |t| \leq \mu_x(C)^{2/3}$, we see that

$$\frac{t}{\mu_x(C)} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Similarly, for $|t| \leq 1$ and $|t| > 1$, we see that both

$$\frac{t^2}{\mu_x(C)^2} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Let the corresponding error terms in (x,t) be denoted by

$$\widetilde{E}(x,t) := O\left(\frac{1}{\sigma_x(C)}\right) + O\left(\frac{|t|^3}{\mu_x(C)^2}\right).$$

Combining these estimates with the previous computations, we can deduce that uniformly for $|t| \leq \mu_x(C)^{2/3}$

$$2\sqrt{2\pi} \cdot x \times \frac{(\log\log x)^{k+\frac{1}{2}}}{(2k+1)(k-1)!} \sim \frac{1}{\sqrt{2\pi} \cdot \sigma_x(C)} \cdot \exp\left(-\frac{t^2}{2\sigma_x(C)^2}\right) \times \left[1 + \widetilde{E}(x,t)\right].$$

By the same argument utilized in the proof of Lemma 6.4, we see that the contributions of these summatory functions for $k \leq \mu_x(C) - \mu_x(C)^{2/3}$ is negligible. We also require that $k \leq \log\log x$ as we have worked out in Theorem 6.2. So we sum over a corresponding range of

$$\mu_x(C) - \mu_x(C)^{2/3} \leq k \leq R_{z,x} \cdot \mu_x(C) + z \cdot \sigma_x(C),$$

for $R_{z,x} := 1 - \frac{z}{\sigma_x(C)}$ to approximate the stated normalized densities. Then finally as $x \to \infty$, the three terms that result (one main term, two error terms, respectively) can be considered to each correspond to a Riemann sum for an associated integral. $\square$

Corollary 6.7. *Let $Y > 0$. Suppose that $\mu_x(C)$ and $\sigma_x(C)$ are defined as in Theorem 6.6. Uniformly for all $-Y \leq y \leq Y$ we have that*

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : |g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq y\right\} = \Phi\left(\frac{\frac{\pi^2}{6}y - \mu_x(C)}{\sigma_x(C)}\right) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

Proof. We claim that

$$|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2} C_{\Omega(n)}(n).$$

From (34) we obtain that

$$\frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| = \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(x)}(x)] + \sum_{d<x} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O(1).$$

Let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) := f(x) - f(x-1)$. Then from the proof of Corollary 6.5, we see that for large n

$$\begin{aligned}
|g^{-1}(n)| = \Delta_n(n \cdot \mathbb{E}|g^{-1}(n)|) &\sim \Delta_n\left(\sum_{d \leq n} \frac{6}{\pi^2} \cdot C_{\Omega(d)}(d) \cdot \frac{n}{d}\right) \\
&= \frac{6}{\pi^2}\left[C_{\Omega(n)}(n) + \sum_{d<n} C_{\Omega(d)}(d)\frac{n}{d} - \sum_{d<n} C_{\Omega(d)}(d)\frac{(n-1)}{d}\right] \\
&= \frac{6}{\pi^2} C_{\Omega(n)}(n) + \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n-1)|, \text{ as } n \to \infty.
\end{aligned}$$

The result finally follows from Theorem 6.6. $\square$

7 Lower bounds for $M(x)$ along infinite subsequences

7.1 Establishing initial lower bounds on the summatory function $G^{-1}(x)$

Lemma 7.1. *Suppose that $\mu_x(C)$ and $\sigma_x(C)$ are defined as in Theorem 6.6. If x is sufficiently large and we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds:*

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq 0\right) = o(1) \tag{A}$$

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\mu_x(C)\right) = \frac{1}{2} + o(1). \tag{B}$$

Moreover, for any positive real $\delta > 0$ we have that

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\mu_x(C)^{1+\delta}\right) = 1 + o_\delta(1), \text{ as } x \to \infty. \tag{C}$$

Proof. Each of these results is a consequence of Corollary 6.7. Let the densities $\gamma_z(x)$ be defined for $z \in \mathbb{R}$ and sufficiently large $x > e$ as follows:

$$\gamma_z(x) := \frac{1}{x} \cdot \#\{2 \leq n \leq x : |g^{-1}(n)| - \mathbb{E}|g^{-1}(n)| \leq z\}.$$

To prove (A), observe that by Corollary 6.7 for $z := 0$ we have that

$$\gamma_0(x) = \Phi\left(-\sigma_x(C)\right) + o(1), \text{ as } x \to \infty.$$

We can see that $\sigma_x(C) \xrightarrow{x\to\infty} +\infty$ where for $z \geq 0$ we have the reflection identity for the normal distribution CDF $\Phi(z) = 1 - \Phi(-z)$. Since we have by an asymptotic approximation to the error function expanded by

$$\begin{aligned}
\Phi(z) &= \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right) \\
&= 1 - \frac{2e^{-z^2/2}}{\sqrt{2\pi}}\left[z^{-1} - z^{-3} + 3z^{-5} - 15z^{-7} + \cdots\right], \text{ as } |z| \to \infty,
\end{aligned}$$

we can see that

$$\gamma_0(x) = \Phi\left(-\sigma_x(C)\right) \asymp \frac{1}{\sigma_x(C)\exp(\mu_x(C)/2)} = o(1).$$

To prove (B), observe setting $z_1 := \frac{6}{\pi^2}\mu_x(C)$ yields that

$$\gamma_{z_1}(x) = \Phi(0) + o(1) = \frac{1}{2} + o(1), \text{ as } x \to \infty.$$

To prove (C), we require that $\mu_x(C)^{\frac{1}{2}+\delta} - \sigma_x(C) \xrightarrow{x\to\infty} +\infty$. Since this happens as $x \to \infty$ for any fixed $\delta > 0$, we have that with $z(\delta) := \frac{6}{\pi^2}\mu_x(C)^{1+\delta}$

$$\begin{aligned}
\gamma_{z(\delta)} &= \Phi\left(\mu_x(C)^{\frac{1}{2}+\delta} - \sigma_x(C)\right) + o(1) \\
&\sim 1 - \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\left(\mu_x(C)^{\frac{1}{2}+\delta} - \sigma_x(C)\right)} \times \exp\left(-\frac{\mu_x(C)}{4} \cdot \left(\mu_x(C)^{\delta} - 1\right)^2\right) \\
&= 1 + o_\delta(1), \text{ as } x \to \infty.
\end{aligned}$$

□

Remark 7.2. A consequence of (A) and (C) in Lemma 7.1 is that for any fixed $\delta > 0$ and $n \in \mathcal{S}_1(\delta)$ taken within a set of asymptotic density one we have that

$$\frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq |g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| + \frac{6}{\pi^2}\mu_x(C)^{\frac{1}{2}+\delta}. \tag{35}$$

Thus when we integrate over a sufficiently spaced set of (e.g., set of wide enough) disjoint consecutive intervals containing large enough integer values, we can assume that an asymptotic lower bound on the contribution of $|g^{-1}(n)|$ is given by the function's average order, and an upper bound is given by the related upper limit above for any fixed $\delta > 0$. In particular, observe that by Corollary 6.7 and Corollary 6.5 we can see that

$$\frac{\pi^2}{6 \cdot \sigma_x(C)} \times \int_{-\infty}^{\infty} z \cdot \Phi' \left(\frac{\frac{\pi^2}{6} z - \mu_x(C)}{\sigma_x(C)} \right) dz = \frac{6}{\pi^2} \cdot \sigma_x(C) = o\left(\mathbb{E}|g^{-1}(x)|\right).$$

Emphasizing the point above, we can thus again interpret the previous calculation as implying that for n on a large interval, the contribution from $|g^{-1}(n)|$ can be approximated above and below accurately as in the bounds from (35).

Theorem 7.3. *For all sufficiently large integers x, whenever $G^{-1}(x) \neq 0$ we have that*

$$|G^{-1}(x)| \gg (\log x)\sqrt{\log\log x}, \text{ as } x \to \infty.$$

Proof. We will use a lower bound approximating the summatory function $L(t)$ of $\lambda(n)$ for $n \leq t$ and large $t \to \infty$ by summing over the uniform asymptotic lower bounds proved in Theorem 2.7. To be careful about the expected sign of this summatory function, we first appeal to the first uniform approximations to the functions $\widehat{\pi}_k(x)$ given by Theorem 2.6. As noted in [12, §7.4], the function $\mathcal{G}(z)$ satisfies

$$\mathcal{G}\left(\frac{k-1}{\log\log x}\right) = O(1), 1 \leq k \leq \log\log x,$$

so that uniformly for all integers $1 \leq k \leq \log\log x$ we can write

$$\widehat{\pi}_k(x) \asymp \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{1}{\log\log x}\right)\right].$$

By the argument leading up to Corollary 4.6, the following summatory function represents the asymptotic main term in the summation $L(x) := \sum_{n \leq x} \lambda(n)$ as $x \to \infty$ (see Table T.2 on page 45):

$$\widehat{L}_2(x) = \sum_{k=1}^{\log\log x} (-1)^k \widehat{\pi}_k(x) = -\frac{x}{(\log x)^2} \times \Gamma(\log\log x, -\log\log x) \sim \frac{(-1)^{1+\lceil \log\log x \rceil} \cdot x}{\sqrt{2\pi}\sqrt{\log\log x}}$$

Then we expect the sign of our summatory function approximation to be approximately given by $(-1)^{1+\lceil \log\log x \rceil}$ for sufficiently large x. This observation is empirically verified in the second table for pages of large x values.

We will now find a lower bound on the unsigned magnitude of these summatory functions. In particular, using Theorem 2.7, we have that $\widehat{\pi}_k(x) \gg \widehat{\pi}_k^{(\ell)}(x)$ where

$$\widehat{\pi}_k^{(\ell)}(x) := \frac{x^{\frac{3}{4}}}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O\left(\frac{k}{(\log\log x)^2}\right)\right].$$

Thus we can define our lower bound in absolute value by

$$\widehat{L}_0(x) := \left| \sum_{k=1}^{\log\log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x) \right| \asymp \frac{x^{\frac{3}{4}}}{\sqrt{\log\log x}},$$

where limiting asymptotics for the incomplete gamma function expression for the inner summation dominates the main term on the right-hand-side of the previous equation. The derivative of the lower summatory function satisfies

$$\widehat{L}_0'(x) \asymp \frac{1}{x^{\frac{1}{4}} \cdot \sqrt{\log\log x}}.$$

We observe that we can break the interval $t \in (e, x]$ into disjoint subintervals according to which we have expected sign contributions from the summatory function $\widehat{L}_0(x)$ as $t \to x$ grows without bound. Namely, we expect that for $1 \leq k \leq \frac{\log\log x}{2}$ we have (compare to Table T.2)

$$\begin{aligned}
\operatorname{sgn}\left(\widehat{L}_0(x)\right) &= -1 \text{ on } \left[e^{e^{2k}}, e^{e^{2k+1}}\right) \\
\operatorname{sgn}\left(\widehat{L}_0(x)\right) &= +1 \text{ on } \left[e^{e^{2k+1}}, e^{e^{2k+2}}\right).
\end{aligned}$$

Since the derivative $\widehat{L}_0'(x)$ is monotone decreasing in x, we can construct our lower bounds by placing the input points to this function in the Abel summation formula from (29) over these signed subintervals at extremal endpoints. As we have argued in Lemma 7.1 and observed in Remark 7.2, we have the bounds in (35) upon which we can similarly construct the lower bound on $|G^{-1}(x)|$ based on the sign term of the subinterval (as above) and the extremal points within the interval. The idea used to conclude this proof below underestimatesm the resulting integral formulas approximating $|G^{-1}(x)|$ as $x \to \infty$.

Let $I_k := \left[e^{e^k}, e^{e^{k+1}}\right)$ for $k \geq 1$. We have argued that for sufficiently large k approximately one half of the integers n in the interval I_k satisfy $|g^{-1}(n)| \leq k$. Thus by applying the inequalities from (10a) in the last step below we obtain the following asymptotic lower bounds:

$$\begin{aligned}
|G^{-1}(x)| &\gg \left|\int_2^x \widehat{L}_0'(t)|g^{-1}(t)|dt\right| \\
&\gg \sum_{k=1}^{\frac{\log\log x}{2}} (-1)^k \widehat{L}_0'\left(e^{e^k}\right)\left|g^{-1}\left(e^{e^k}\right)\right| \\
&\gg \left|\sum_{k=\frac{\log\log x}{4}}^{\frac{\log\log x}{2}} \left[\widehat{L}_0'\left(e^{e^{2k}}\right)\left|g^{-1}\left(e^{e^{2k}}\right)\right| - \widehat{L}_0'\left(e^{e^{2k-1}}\right)\left|g^{-1}\left(e^{e^{2k-1}}\right)\right|\right]\right| \\
&\gg \left|\sum_{k=\frac{\log\log x}{4}}^{\frac{\log\log x}{2}} \left[\sqrt{2k}\exp\left(-\frac{1}{4}e^{2k}\right) - \sqrt{2k-1}\exp\left(-\frac{1}{4}e^{2k-1}\right)\right]\right| \\
&\gg \left|\int_{k=\frac{\log\log x}{4}}^{\frac{\log\log x}{2}} \sqrt{t}\cdot\exp\left(-\frac{1}{4}e^{2t}\right)\right| \\
&\gg \left|\sqrt{\log\log x}\cdot \operatorname{Ei}\left(-\frac{\log x}{4}\right)\right| \\
&\gg (\log x)\cdot\sqrt{\log\log x}.
\end{aligned}$$

□

7.2 Proof of the unboundedness of the scaled Mertens function

Proposition 7.4. *For all sufficiently large x, we have that the Mertens function satisfies*

$$M(x) = G^{-1}(x) + G^{-1}\left(\left\lfloor\frac{x}{2}\right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left[\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor\frac{x}{k+1}\right\rfloor\right)\right]. \tag{36}$$

Proof. We know by applying Corollary 2.3 that

$$\begin{aligned}
M(x) &= \sum_{k=1}^{x} g^{-1}(k)\left[\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) + 1\right] \\
&= G^{-1}(x) + \sum_{k=1}^{x/2} g^{-1}(k)\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right)
\end{aligned}$$

$$= G^{-1}(x) + G^{-1}\left(\left\lfloor\frac{x}{2}\right\rfloor\right) + \sum_{k=1}^{x/2-1} G^{-1}(k)\left[\pi\left(\left\lfloor\frac{x}{k}\right\rfloor\right) - \pi\left(\left\lfloor\frac{x}{k+1}\right\rfloor\right)\right].$$

The upper bound on the sum is truncated in the second equation above due to the fact that $\pi(1) = 0$. □

Lemma 7.5. *For sufficiently large x, integers $k \in \left[\sqrt{x}, \frac{x}{2}\right]$ and $m \geq 0$, we have that*

$$\frac{x}{k \cdot \log^m\left(\frac{x}{k}\right)} - \frac{x}{(k+1)\cdot \log^m\left(\frac{x}{k+1}\right)} \gg \frac{x}{(\log x)^m \cdot k(k+1)}, \tag{A}$$

and

$$\sum_{k=\sqrt{x}}^{\frac{x}{2}} \frac{x}{k(k+1)} = \sum_{k=\sqrt{x}}^{\frac{x}{2}} \frac{x}{k^2} + O(1). \tag{B}$$

Proof. The proof of (A) is obvious since for $k_0 \in \left[\sqrt{x}, \frac{x}{2}\right]$ we have that

$$\log(2)(1+o(1)) \leq \log\left(\frac{x}{k_0}\right) \leq \log(x).$$

To prove (B), notice that

$$\frac{x}{k(k+1)} - \frac{x}{k^2} = -\frac{x}{k^2(k+1)}.$$

Then we see that

$$\left|\int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^2(t+1)} dt\right| \leq \left|\int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^3} dt\right| \asymp 1.$$ □

We finally address the main conclusion of our arguments given so far with the following proof:

Proof of Theorem 2.9. Define the infinite increasing subsequence, $\{x_{0,y}\}_{y \geq Y_0}$, by $x_{0,y} := e^{2e^{2y+1}}$ for the sequence indices y starting at some sufficiently large finite integer Y_0. We can verify that for sufficiently large $y \to \infty$, this infinitely tending subsequence is well defined as $x_{0,y+1} > x_{0,y}$, and also importantly $\log\log(x_{0,y+1}) > \log\log(x_{0,y})$ whenever $y \geq Y_0$ (see concluding argument below). Given a fixed large infinitely tending y, we have some (at least one) point $\widehat{x}_0(y) \in \mathbb{X}_y$ defined such that $|G^{-1}(t)|$ is minimal and non-vanishing on the interval $\mathbb{X}_y := \left[\sqrt{x_{0,y+1}}, \frac{x_{0,y+1}}{2}\right)$ in the form of

$$\left|G^{-1}(\widehat{x}_0(y))\right| := \min_{\substack{\sqrt{x_{0,y+1}} \leq t < \frac{x_{0,y+1}}{2} \\ G^{-1}(t) \neq 0}} |G^{-1}(t)|.$$

In the last step, we observe that $G^{-1}(x) \neq 0$ for x on a set of asymptotic density *at least* bounded below by $\frac{1}{2}$, so that our claim is accurate as the summand's lower bound on this interval does not trivially vanish at large y. This happens since the sequence $g^{-1}(n)$ is non-zero for all $n \geq 1$, so that if we do encounter a zero of the summatory function at x, we find a non-zero summatory function value at $x+1$. Let the shorthand notation $|G^{-1}_{\min}(x_y)| := |G^{-1}(\hat{x}_0(y))|$.

We need to bound the prime counting function differences in the formula given by Proposition 7.4. We will require the following known bounds on the prime counting function due to Rosser and Schoenfeld for large $x > 59$ [18, Thm. 1]:

$$\frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right). \tag{37}$$

Let the component function $U_M(y)$ be defined for all large y as follows:

$$U_M(y) := \sum_{k=1}^{\sqrt{\hat{x}_{0,y+1}}} |G^{-1}(k)|\left[\pi\left(\frac{\hat{x}_{0,y+1}}{k}\right) - \pi\left(\frac{\hat{x}_{0,y+1}}{k+1}\right)\right].$$

Combined with Lemma 7.5, the estimates on $\pi(x)$ in (37) lead to the following approximations that hold on the increasing sequences of y taken within the subintervals defined by $\widehat{x}_0(y)$:

$$\begin{aligned}
|U_M(y)| &\gg \sum_{k=1}^{\frac{\hat{x}_{0,y+1}}{2}-1} |G^{-1}(k)| \Bigg[\frac{\hat{x}_{0,y+1}}{k \cdot \log\left(\frac{\hat{x}_{0,y+1}}{k}\right)} \left(1 + \frac{1}{2 \cdot \log\left(\frac{\hat{x}_{0,y+1}}{k}\right)}\right) \\
&\qquad - \frac{\hat{x}_{0,y+1}}{(k+1) \cdot \log\left(\frac{\hat{x}_{0,y+1}}{k+1}\right)} \left(1 + \frac{3}{2 \cdot \log\left(\frac{\hat{x}_{0,y+1}}{k+1}\right)}\right)\Bigg] \\
&\gg \sum_{k=\sqrt{\hat{x}_{0,y+1}}}^{\frac{\hat{x}_{0,y+1}}{2}-1} \frac{\hat{x}_{0,y+1} \cdot |G_{\min}^{-1}(x_y)|}{k^2} \left[\frac{1}{\log(\hat{x}_{0,y+1})} + \frac{1}{2\log^2(\hat{x}_{0,y+1})}\right] \\
&\gg \hat{x}_{0,y+1} \times |G_{\min}^{-1}(x_y)| \left(\frac{1}{\log(\hat{x}_{0,y+1})} + \frac{1}{2\log^2(\hat{x}_{0,y+1})}\right) \times \left| \int_{\sqrt{\hat{x}_{0,y+1}}}^{\frac{\hat{x}_{0,y+1}}{2}} \frac{dt}{t^2} \right| \\
&\gg \sqrt{\hat{x}_{0,y+1}} \times \frac{|G_{\min}^{-1}(x_y)|}{\log(\hat{x}_{0,y+1})} + o(1), \text{ as } y \to \infty.
\end{aligned}$$

We clearly see from Theorem 7.3 and Proposition 7.4 that

$$\begin{aligned}
\frac{|M(\hat{x}_{0,y+1})|}{\sqrt{\hat{x}_{0,y+1}}} &\gg \frac{1}{\sqrt{\hat{x}_{0,y+1}}} \times \left| \left| G^{-1}(\hat{x}_{0,y+1}) + G^{-1}\left(\frac{\hat{x}_{0,y+1}}{2}\right)\right| + |U_M(y)| \right| \\
&\gg \frac{1}{\sqrt{\hat{x}_{0,y+1}}} \times |U_M(y)| \\
&\gg \log\log\left(\sqrt{\hat{x}_{0,y+1}}\right)^{\frac{1}{2}}. \qquad (38)
\end{aligned}$$

There is a small, but nonetheless necessary point in question to explain about a technicality in stating (38). Namely, we are not asserting that $|M(x)|/\sqrt{x}$ grows unbounded along the precise subsequence of $x \mapsto \hat{x}_{0,y+1}$ itself we have defined as $y \to \infty$. Rather, we are asserting that the unboundedness of this function can be witnessed along some subsequence whose points are taken within a large interval window of $x \in \mathbb{X}_y$ as $y \to \infty$. We choose to state the lower bound given on the right-hand-side of (38) using the lower bound on $|G^{-1}(x)|$ we proved in Theorem 7.3 minimally with $\hat{x}_0(y) \geq \sqrt{\hat{x}_{0,y+1}}$ on the interval for all $y \geq Y_0$. It is also necessary that $\log\log(x_{0,y+1}) > \log\log(x_{0,y})$ for all sufficiently large y so that we indeed to obtain an increasing infinite subsequence along which to show the unboundedness of (38). $\square$

References

[1] T. M. Apostol. *Introduction to Analytic Number Theory.* Springer–Verlag, 1976.

[2] P. Billingsly. On the central limit theorem for the prime divisor function. *Amer. Math. Monthly*, 76(2):132–139, 1969.

[3] M. A. Chaudhury and S. M. Zubair. *On a class of incomplete gamma functions with applications.* Chapman and Hall / CRC, 2000.

[4] P. Erdös and M. Kac. The guassian errors in the theory of additive arithmetic functions. *American Journal of Mathematics*, 62(1):738–742, 1940.

[5] G. H. Hardy and E. M. Wright, editors. *An Introduction to the Theory of Numbers.* Oxford University Press, 2008 (Sixth Edition).

[6] G. Hurst. Computations of the Mertens function and improved bounds on the Mertens conjecture. *Math. Comp.*, 87:1013–1028, 2018.

[7] H. Iwaniec and E. Kowalski. *Analytic Number Theory*, volume 53. AMS Colloquium Publications, 2004.

[8] T. Kotnik and H. té Riele. The Mertens conjecture revisited. *Algorithmic Number Theory*, 7^{th} International Symposium, 2006.

[9] T. Kotnik and J. van de Lune. On the order of the Mertens function. *Exp. Math.*, 2004.

[10] R. S. Lehman. On Liouville's function. *Math. Comput.*, 14:311–320, 1960.

[11] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford: The Clarendon Press, 1995.

[12] H. L. Montgomery and R. C. Vaughan. *Multiplicative Number Theory: I. Classical Theory.* Cambridge, 2006.

[13] N. Ng. The distribution of the summatory function of the Móbius function. *Proc. London Math. Soc.*, 89(3):361–389, 2004.

[14] A. M. Odlyzko and H. J. J. té Riele. Disproof of the Mertens conjecture. *J. Reine Angew. Math*, 1985.

[15] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. *NIST Handbook of Mathematical Functions.* Cambridge University Press, 2010.

[16] A. Renyi and P. Turan. On a theorem of Erdös-Kac. *Acta Arithmetica*, 4(1):71–84, 1958.

[17] P. Ribenboim. *The new book of prime number records.* Springer, 1996.

[18] J. B. Rosser and L. Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.*, 6:64–94, 1962.

[19] M. D. Schmidt. SageMath and Mathematica software for computations with the Mertens function, 2020. https://github.com/maxieds/MertensFunctionComputations.

[20] N. J. A. Sloane. The Online Encyclopedia of Integer Sequences, 2020. http://oeis.org.

[21] K. Soundararajan. Partial sums of the Möbius function. *Annals of Mathematics*, 2009.

[22] E. C. Titchmarsh. *The theory of the Riemann zeta function.* Clarendon Press, 1951.

T.1 Table: The Dirichlet inverse function $g^{-1}(n)$ and the distribution of its summatory function

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1^1	Y	N	1	0	1.0000000	1.000000	0.000000	1	1	0
2	2^1	Y	Y	-2	0	1.0000000	0.500000	0.500000	-1	1	-2
3	3^1	Y	Y	-2	0	1.0000000	0.333333	0.666667	-3	1	-4
4	2^2	N	Y	2	0	1.5000000	0.500000	0.500000	-1	3	-4
5	5^1	Y	Y	-2	0	1.0000000	0.400000	0.600000	-3	3	-6
6	$2^1 3^1$	Y	N	5	0	1.0000000	0.500000	0.500000	2	8	-6
7	7^1	Y	Y	-2	0	1.0000000	0.428571	0.571429	0	8	-8
8	2^3	N	Y	-2	0	2.0000000	0.375000	0.625000	-2	8	-10
9	3^2	N	Y	2	0	1.5000000	0.444444	0.555556	0	10	-10
10	$2^1 5^1$	Y	N	5	0	1.0000000	0.500000	0.500000	5	15	-10
11	11^1	Y	Y	-2	0	1.0000000	0.454545	0.545455	3	15	-12
12	$2^2 3^1$	N	N	-7	2	1.2857143	0.416667	0.583333	-4	15	-19
13	13^1	Y	Y	-2	0	1.0000000	0.384615	0.615385	-6	15	-21
14	$2^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-1	20	-21
15	$3^1 5^1$	Y	N	5	0	1.0000000	0.466667	0.533333	4	25	-21
16	2^4	N	Y	2	0	2.5000000	0.500000	0.500000	6	27	-21
17	17^1	Y	Y	-2	0	1.0000000	0.470588	0.529412	4	27	-23
18	$2^1 3^2$	N	N	-7	2	1.2857143	0.444444	0.555556	-3	27	-30
19	19^1	Y	Y	-2	0	1.0000000	0.421053	0.578947	-5	27	-32
20	$2^2 5^1$	N	N	-7	2	1.2857143	0.400000	0.600000	-12	27	-39
21	$3^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	-7	32	-39
22	$2^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-2	37	-39
23	23^1	Y	Y	-2	0	1.0000000	0.434783	0.565217	-4	37	-41
24	$2^3 3^1$	N	N	9	4	1.5555556	0.458333	0.541667	5	46	-41
25	5^2	N	Y	2	0	1.5000000	0.480000	0.520000	7	48	-41
26	$2^1 13^1$	Y	N	5	0	1.0000000	0.500000	0.500000	12	53	-41
27	3^3	N	Y	-2	0	2.0000000	0.481481	0.518519	10	53	-43
28	$2^2 7^1$	N	N	-7	2	1.2857143	0.464286	0.535714	3	53	-50
29	29^1	Y	Y	-2	0	1.0000000	0.448276	0.551724	1	53	-52
30	$2^1 3^1 5^1$	Y	N	-16	0	1.0000000	0.433333	0.566667	-15	53	-68
31	31^1	Y	Y	-2	0	1.0000000	0.419355	0.580645	-17	53	-70
32	2^5	N	Y	-2	0	3.0000000	0.406250	0.593750	-19	53	-72
33	$3^1 11^1$	Y	N	5	0	1.0000000	0.424242	0.575758	-14	58	-72
34	$2^1 17^1$	Y	N	5	0	1.0000000	0.441176	0.558824	-9	63	-72
35	$5^1 7^1$	Y	N	5	0	1.0000000	0.457143	0.542857	-4	68	-72
36	$2^2 3^2$	N	N	14	9	1.3571429	0.472222	0.527778	10	82	-72
37	37^1	Y	Y	-2	0	1.0000000	0.459459	0.540541	8	82	-74
38	$2^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	13	87	-74
39	$3^1 13^1$	Y	N	5	0	1.0000000	0.487179	0.512821	18	92	-74
40	$2^3 5^1$	N	N	9	4	1.5555556	0.500000	0.500000	27	101	-74
41	41^1	Y	Y	-2	0	1.0000000	0.487805	0.512195	25	101	-76
42	$2^1 3^1 7^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	9	101	-92
43	43^1	Y	Y	-2	0	1.0000000	0.465116	0.534884	7	101	-94
44	$2^2 11^1$	N	N	-7	2	1.2857143	0.454545	0.545455	0	101	-101
45	$3^2 5^1$	N	N	-7	2	1.2857143	0.444444	0.555556	-7	101	-108
46	$2^1 23^1$	Y	N	5	0	1.0000000	0.456522	0.543478	-2	106	-108
47	47^1	Y	Y	-2	0	1.0000000	0.446809	0.553191	-4	106	-110
48	$2^4 3^1$	N	N	-11	6	1.8181818	0.437500	0.562500	-15	106	-121

Table T.1: ***Computations with*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for*** $\mathbf{1 \leq n \leq 500}$**.**

- ► *The column labeled* `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled* `Sqfree` *and* `PPower`*, respectively, list inclusion of n in the sets of squarefree integers and the prime powers.*
- ► *The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.*
- ► *The last several columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{n} \cdot \#\{n \leq x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geq 1$.*

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
49	7^2	N	Y	2	0	1.5000000	0.448980	0.551020	-13	108	-121
50	$2^1 5^2$	N	N	-7	2	1.2857143	0.440000	0.560000	-20	108	-128
51	$3^1 17^1$	Y	N	5	0	1.0000000	0.450980	0.549020	-15	113	-128
52	$2^2 13^1$	N	N	-7	2	1.2857143	0.442308	0.557692	-22	113	-135
53	53^1	Y	Y	-2	0	1.0000000	0.433962	0.566038	-24	113	-137
54	$2^1 3^3$	N	N	9	4	1.5555556	0.444444	0.555556	-15	122	-137
55	$5^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-10	127	-137
56	$2^3 7^1$	N	N	9	4	1.5555556	0.464286	0.535714	-1	136	-137
57	$3^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	4	141	-137
58	$2^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	9	146	-137
59	59^1	Y	Y	-2	0	1.0000000	0.474576	0.525424	7	146	-139
60	$2^2 3^1 5^1$	N	N	30	14	1.1666667	0.483333	0.516667	37	176	-139
61	61^1	Y	Y	-2	0	1.0000000	0.475410	0.524590	35	176	-141
62	$2^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	40	181	-141
63	$3^2 7^1$	N	N	-7	2	1.2857143	0.476190	0.523810	33	181	-148
64	2^6	N	Y	2	0	3.5000000	0.484375	0.515625	35	183	-148
65	$5^1 13^1$	Y	N	5	0	1.0000000	0.492308	0.507692	40	188	-148
66	$2^1 3^1 11^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	24	188	-164
67	67^1	Y	Y	-2	0	1.0000000	0.477612	0.522388	22	188	-166
68	$2^2 17^1$	N	N	-7	2	1.2857143	0.470588	0.529412	15	188	-173
69	$3^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	20	193	-173
70	$2^1 5^1 7^1$	Y	N	-16	0	1.0000000	0.471429	0.528571	4	193	-189
71	71^1	Y	Y	-2	0	1.0000000	0.464789	0.535211	2	193	-191
72	$2^3 3^2$	N	N	-23	18	1.4782609	0.458333	0.541667	-21	193	-214
73	73^1	Y	Y	-2	0	1.0000000	0.452055	0.547945	-23	193	-216
74	$2^1 37^1$	Y	N	5	0	1.0000000	0.459459	0.540541	-18	198	-216
75	$3^1 5^2$	N	N	-7	2	1.2857143	0.453333	0.546667	-25	198	-223
76	$2^2 19^1$	N	N	-7	2	1.2857143	0.447368	0.552632	-32	198	-230
77	$7^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	-27	203	-230
78	$2^1 3^1 13^1$	Y	N	-16	0	1.0000000	0.448718	0.551282	-43	203	-246
79	79^1	Y	Y	-2	0	1.0000000	0.443038	0.556962	-45	203	-248
80	$2^4 5^1$	N	N	-11	6	1.8181818	0.437500	0.562500	-56	203	-259
81	3^4	N	Y	2	0	2.5000000	0.444444	0.555556	-54	205	-259
82	$2^1 41^1$	Y	N	5	0	1.0000000	0.451220	0.548780	-49	210	-259
83	83^1	Y	Y	-2	0	1.0000000	0.445783	0.554217	-51	210	-261
84	$2^2 3^1 7^1$	N	N	30	14	1.1666667	0.452381	0.547619	-21	240	-261
85	$5^1 17^1$	Y	N	5	0	1.0000000	0.458824	0.541176	-16	245	-261
86	$2^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	-11	250	-261
87	$3^1 29^1$	Y	N	5	0	1.0000000	0.471264	0.528736	-6	255	-261
88	$2^3 11^1$	N	N	9	4	1.5555556	0.477273	0.522727	3	264	-261
89	89^1	Y	Y	-2	0	1.0000000	0.471910	0.528090	1	264	-263
90	$2^1 3^2 5^1$	N	N	30	14	1.1666667	0.477778	0.522222	31	294	-263
91	$7^1 13^1$	Y	N	5	0	1.0000000	0.483516	0.516484	36	299	-263
92	$2^2 23^1$	N	N	-7	2	1.2857143	0.478261	0.521739	29	299	-270
93	$3^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	34	304	-270
94	$2^1 47^1$	Y	N	5	0	1.0000000	0.489362	0.510638	39	309	-270
95	$5^1 19^1$	Y	N	5	0	1.0000000	0.494737	0.505263	44	314	-270
96	$2^5 3^1$	N	N	13	8	2.0769231	0.500000	0.500000	57	327	-270
97	97^1	Y	Y	-2	0	1.0000000	0.494845	0.505155	55	327	-272
98	$2^1 7^2$	N	N	-7	2	1.2857143	0.489796	0.510204	48	327	-279
99	$3^2 11^1$	N	N	-7	2	1.2857143	0.484848	0.515152	41	327	-286
100	$2^2 5^2$	N	N	14	9	1.3571429	0.490000	0.510000	55	341	-286
101	101^1	Y	Y	-2	0	1.0000000	0.485149	0.514851	53	341	-288
102	$2^1 3^1 17^1$	Y	N	-16	0	1.0000000	0.480392	0.519608	37	341	-304
103	103^1	Y	Y	-2	0	1.0000000	0.475728	0.524272	35	341	-306
104	$2^3 13^1$	N	N	9	4	1.5555556	0.480769	0.519231	44	350	-306
105	$3^1 5^1 7^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	28	350	-322
106	$2^1 53^1$	Y	N	5	0	1.0000000	0.481132	0.518868	33	355	-322
107	107^1	Y	Y	-2	0	1.0000000	0.476636	0.523364	31	355	-324
108	$2^2 3^3$	N	N	-23	18	1.4782609	0.472222	0.527778	8	355	-347
109	109^1	Y	Y	-2	0	1.0000000	0.467890	0.532110	6	355	-349
110	$2^1 5^1 11^1$	Y	N	-16	0	1.0000000	0.463636	0.536364	-10	355	-365
111	$3^1 37^1$	Y	N	5	0	1.0000000	0.468468	0.531532	-5	360	-365
112	$2^4 7^1$	N	N	-11	6	1.8181818	0.464286	0.535714	-16	360	-376
113	113^1	Y	Y	-2	0	1.0000000	0.460177	0.539823	-18	360	-378
114	$2^1 3^1 19^1$	Y	N	-16	0	1.0000000	0.456140	0.543860	-34	360	-394
115	$5^1 23^1$	Y	N	5	0	1.0000000	0.460870	0.539130	-29	365	-394
116	$2^2 29^1$	N	N	-7	2	1.2857143	0.456897	0.543103	-36	365	-401
117	$3^2 13^1$	N	N	-7	2	1.2857143	0.452991	0.547009	-43	365	-408
118	$2^1 59^1$	Y	N	5	0	1.0000000	0.457627	0.542373	-38	370	-408
119	$7^1 17^1$	Y	N	5	0	1.0000000	0.462185	0.537815	-33	375	-408
120	$2^3 3^1 5^1$	N	N	-48	32	1.3333333	0.458333	0.541667	-81	375	-456
121	11^2	N	Y	2	0	1.5000000	0.462810	0.537190	-79	377	-456
122	$2^1 61^1$	Y	N	5	0	1.0000000	0.467213	0.532787	-74	382	-456
123	$3^1 41^1$	Y	N	5	0	1.0000000	0.471545	0.528455	-69	387	-456
124	$2^2 31^1$	N	N	-7	2	1.2857143	0.467742	0.532258	-76	387	-463

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\|n} C_{\Omega(d)}(d)}{\|g^{-1}(n)\|}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
125	5^3	N	Y	-2	0	2.0000000	0.464000	0.536000	-78	387	-465
126	$2^1 3^2 7^1$	N	N	30	14	1.1666667	0.468254	0.531746	-48	417	-465
127	127^1	Y	Y	-2	0	1.0000000	0.464567	0.535433	-50	417	-467
128	2^7	N	Y	-2	0	4.0000000	0.460938	0.539062	-52	417	-469
129	$3^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	-47	422	-469
130	$2^1 5^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-63	422	-485
131	131^1	Y	Y	-2	0	1.0000000	0.458015	0.541985	-65	422	-487
132	$2^2 3^1 11^1$	N	N	30	14	1.1666667	0.462121	0.537879	-35	452	-487
133	$7^1 19^1$	Y	N	5	0	1.0000000	0.466165	0.533835	-30	457	-487
134	$2^1 67^1$	Y	N	5	0	1.0000000	0.470149	0.529851	-25	462	-487
135	$3^3 5^1$	N	N	9	4	1.5555556	0.474074	0.525926	-16	471	-487
136	$2^3 17^1$	N	N	9	4	1.5555556	0.477941	0.522059	-7	480	-487
137	137^1	Y	Y	-2	0	1.0000000	0.474453	0.525547	-9	480	-489
138	$2^1 3^1 23^1$	Y	N	-16	0	1.0000000	0.471014	0.528986	-25	480	-505
139	139^1	Y	Y	-2	0	1.0000000	0.467626	0.532374	-27	480	-507
140	$2^2 5^1 7^1$	N	N	30	14	1.1666667	0.471429	0.528571	3	510	-507
141	$3^1 47^1$	Y	N	5	0	1.0000000	0.475177	0.524823	8	515	-507
142	$2^1 71^1$	Y	N	5	0	1.0000000	0.478873	0.521127	13	520	-507
143	$11^1 13^1$	Y	N	5	0	1.0000000	0.482517	0.517483	18	525	-507
144	$2^4 3^2$	N	N	34	29	1.6176471	0.486111	0.513889	52	559	-507
145	$5^1 29^1$	Y	N	5	0	1.0000000	0.489655	0.510345	57	564	-507
146	$2^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	62	569	-507
147	$3^1 7^2$	N	N	-7	2	1.2857143	0.489796	0.510204	55	569	-514
148	$2^2 37^1$	N	N	-7	2	1.2857143	0.486486	0.513514	48	569	-521
149	149^1	Y	Y	-2	0	1.0000000	0.483221	0.516779	46	569	-523
150	$2^1 3^1 5^2$	N	N	30	14	1.1666667	0.486667	0.513333	76	599	-523
151	151^1	Y	Y	-2	0	1.0000000	0.483444	0.516556	74	599	-525
152	$2^3 19^1$	N	N	9	4	1.5555556	0.486842	0.513158	83	608	-525
153	$3^2 17^1$	N	N	-7	2	1.2857143	0.483660	0.516340	76	608	-532
154	$2^1 7^1 11^1$	Y	N	-16	0	1.0000000	0.480519	0.519481	60	608	-548
155	$5^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	65	613	-548
156	$2^2 3^1 13^1$	N	N	30	14	1.1666667	0.487179	0.512821	95	643	-548
157	157^1	Y	Y	-2	0	1.0000000	0.484076	0.515924	93	643	-550
158	$2^1 79^1$	Y	N	5	0	1.0000000	0.487342	0.512658	98	648	-550
159	$3^1 53^1$	Y	N	5	0	1.0000000	0.490566	0.509434	103	653	-550
160	$2^5 5^1$	N	N	13	8	2.0769231	0.493750	0.506250	116	666	-550
161	$7^1 23^1$	Y	N	5	0	1.0000000	0.496894	0.503106	121	671	-550
162	$2^1 3^4$	N	N	-11	6	1.8181818	0.493827	0.506173	110	671	-561
163	163^1	Y	Y	-2	0	1.0000000	0.490798	0.509202	108	671	-563
164	$2^2 41^1$	N	N	-7	2	1.2857143	0.487805	0.512195	101	671	-570
165	$3^1 5^1 11^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	85	671	-586
166	$2^1 83^1$	Y	N	5	0	1.0000000	0.487952	0.512048	90	676	-586
167	167^1	Y	Y	-2	0	1.0000000	0.485030	0.514970	88	676	-588
168	$2^3 3^1 7^1$	N	N	-48	32	1.3333333	0.482143	0.517857	40	676	-636
169	13^2	N	Y	2	0	1.5000000	0.485207	0.514793	42	678	-636
170	$2^1 5^1 17^1$	Y	N	-16	0	1.0000000	0.482353	0.517647	26	678	-652
171	$3^2 19^1$	N	N	-7	2	1.2857143	0.479532	0.520468	19	678	-659
172	$2^2 43^1$	N	N	-7	2	1.2857143	0.476744	0.523256	12	678	-666
173	173^1	Y	Y	-2	0	1.0000000	0.473988	0.526012	10	678	-668
174	$2^1 3^1 29^1$	Y	N	-16	0	1.0000000	0.471264	0.528736	-6	678	-684
175	$5^2 7^1$	N	N	-7	2	1.2857143	0.468571	0.531429	-13	678	-691
176	$2^4 11^1$	N	N	-11	6	1.8181818	0.465909	0.534091	-24	678	-702
177	$3^1 59^1$	Y	N	5	0	1.0000000	0.468927	0.531073	-19	683	-702
178	$2^1 89^1$	Y	N	5	0	1.0000000	0.471910	0.528090	-14	688	-702
179	179^1	Y	Y	-2	0	1.0000000	0.469274	0.530726	-16	688	-704
180	$2^2 3^2 5^1$	N	N	-74	58	1.2162162	0.466667	0.533333	-90	688	-778
181	181^1	Y	Y	-2	0	1.0000000	0.464088	0.535912	-92	688	-780
182	$2^1 7^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-108	688	-796
183	$3^1 61^1$	Y	N	5	0	1.0000000	0.464481	0.535519	-103	693	-796
184	$2^3 23^1$	N	N	9	4	1.5555556	0.467391	0.532609	-94	702	-796
185	$5^1 37^1$	Y	N	5	0	1.0000000	0.470270	0.529730	-89	707	-796
186	$2^1 3^1 31^1$	Y	N	-16	0	1.0000000	0.467742	0.532258	-105	707	-812
187	$11^1 17^1$	Y	N	5	0	1.0000000	0.470588	0.529412	-100	712	-812
188	$2^2 47^1$	N	N	-7	2	1.2857143	0.468085	0.531915	-107	712	-819
189	$3^3 7^1$	N	N	9	4	1.5555556	0.470899	0.529101	-98	721	-819
190	$2^1 5^1 19^1$	Y	N	-16	0	1.0000000	0.468421	0.531579	-114	721	-835
191	191^1	Y	Y	-2	0	1.0000000	0.465969	0.534031	-116	721	-837
192	$2^6 3^1$	N	N	-15	10	2.3333333	0.463542	0.536458	-131	721	-852
193	193^1	Y	Y	-2	0	1.0000000	0.461140	0.538860	-133	721	-854
194	$2^1 97^1$	Y	N	5	0	1.0000000	0.463918	0.536082	-128	726	-854
195	$3^1 5^1 13^1$	Y	N	-16	0	1.0000000	0.461538	0.538462	-144	726	-870
196	$2^2 7^2$	N	N	14	9	1.3571429	0.464286	0.535714	-130	740	-870
197	197^1	Y	Y	-2	0	1.0000000	0.461929	0.538071	-132	740	-872
198	$2^1 3^2 11^1$	N	N	30	14	1.1666667	0.464646	0.535354	-102	770	-872
199	199^1	Y	Y	-2	0	1.0000000	0.462312	0.537688	-104	770	-874
200	$2^3 5^2$	N	N	-23	18	1.4782609	0.460000	0.540000	-127	770	-897

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
201	$3^1 67^1$	Y	N	5	0	1.0000000	0.462687	0.537313	−122	775	−897
202	$2^1 101^1$	Y	N	5	0	1.0000000	0.465347	0.534653	−117	780	−897
203	$7^1 29^1$	Y	N	5	0	1.0000000	0.467980	0.532020	−112	785	−897
204	$2^2 3^1 17^1$	N	N	30	14	1.1666667	0.470588	0.529412	−82	815	−897
205	$5^1 41^1$	Y	N	5	0	1.0000000	0.473171	0.526829	−77	820	−897
206	$2^1 103^1$	Y	N	5	0	1.0000000	0.475728	0.524272	−72	825	−897
207	$3^2 23^1$	N	N	−7	2	1.2857143	0.473430	0.526570	−79	825	−904
208	$2^4 13^1$	N	N	−11	6	1.8181818	0.471154	0.528846	−90	825	−915
209	$11^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	−85	830	−915
210	$2^1 3^1 5^1 7^1$	Y	N	65	0	1.0000000	0.476190	0.523810	−20	895	−915
211	211^1	Y	Y	−2	0	1.0000000	0.473934	0.526066	−22	895	−917
212	$2^2 53^1$	N	N	−7	2	1.2857143	0.471698	0.528302	−29	895	−924
213	$3^1 71^1$	Y	N	5	0	1.0000000	0.474178	0.525822	−24	900	−924
214	$2^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−19	905	−924
215	$5^1 43^1$	Y	N	5	0	1.0000000	0.479070	0.520930	−14	910	−924
216	$2^3 3^3$	N	N	46	41	1.5000000	0.481481	0.518519	32	956	−924
217	$7^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	37	961	−924
218	$2^1 109^1$	Y	N	5	0	1.0000000	0.486239	0.513761	42	966	−924
219	$3^1 73^1$	Y	N	5	0	1.0000000	0.488584	0.511416	47	971	−924
220	$2^2 5^1 11^1$	N	N	30	14	1.1666667	0.490909	0.509091	77	1001	−924
221	$13^1 17^1$	Y	N	5	0	1.0000000	0.493213	0.506787	82	1006	−924
222	$2^1 3^1 37^1$	Y	N	−16	0	1.0000000	0.490991	0.509009	66	1006	−940
223	223^1	Y	Y	−2	0	1.0000000	0.488789	0.511211	64	1006	−942
224	$2^5 7^1$	N	N	13	8	2.0769231	0.491071	0.508929	77	1019	−942
225	$3^2 5^2$	N	N	14	9	1.3571429	0.493333	0.506667	91	1033	−942
226	$2^1 113^1$	Y	N	5	0	1.0000000	0.495575	0.504425	96	1038	−942
227	227^1	Y	Y	−2	0	1.0000000	0.493392	0.506608	94	1038	−944
228	$2^2 3^1 19^1$	N	N	30	14	1.1666667	0.495614	0.504386	124	1068	−944
229	229^1	Y	Y	−2	0	1.0000000	0.493450	0.506550	122	1068	−946
230	$2^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.491304	0.508696	106	1068	−962
231	$3^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.489177	0.510823	90	1068	−978
232	$2^3 29^1$	N	N	9	4	1.5555556	0.491379	0.508621	99	1077	−978
233	233^1	Y	Y	−2	0	1.0000000	0.489270	0.510730	97	1077	−980
234	$2^1 3^2 13^1$	N	N	30	14	1.1666667	0.491453	0.508547	127	1107	−980
235	$5^1 47^1$	Y	N	5	0	1.0000000	0.493617	0.506383	132	1112	−980
236	$2^2 59^1$	N	N	−7	2	1.2857143	0.491525	0.508475	125	1112	−987
237	$3^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	130	1117	−987
238	$2^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.491597	0.508403	114	1117	−1003
239	239^1	Y	Y	−2	0	1.0000000	0.489540	0.510460	112	1117	−1005
240	$2^4 3^1 5^1$	N	N	70	54	1.5000000	0.491667	0.508333	182	1187	−1005
241	241^1	Y	Y	−2	0	1.0000000	0.489627	0.510373	180	1187	−1007
242	$2^1 11^2$	N	N	−7	2	1.2857143	0.487603	0.512397	173	1187	−1014
243	3^5	N	Y	−2	0	3.0000000	0.485597	0.514403	171	1187	−1016
244	$2^2 61^1$	N	N	−7	2	1.2857143	0.483607	0.516393	164	1187	−1023
245	$5^1 7^2$	N	N	−7	2	1.2857143	0.481633	0.518367	157	1187	−1030
246	$2^1 3^1 41^1$	Y	N	−16	0	1.0000000	0.479675	0.520325	141	1187	−1046
247	$13^1 19^1$	Y	N	5	0	1.0000000	0.481781	0.518219	146	1192	−1046
248	$2^3 31^1$	N	N	9	4	1.5555556	0.483871	0.516129	155	1201	−1046
249	$3^1 83^1$	Y	N	5	0	1.0000000	0.485944	0.514056	160	1206	−1046
250	$2^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	169	1215	−1046
251	251^1	Y	Y	−2	0	1.0000000	0.486056	0.513944	167	1215	−1048
252	$2^2 3^2 7^1$	N	N	−74	58	1.2162162	0.484127	0.515873	93	1215	−1122
253	$11^1 23^1$	Y	N	5	0	1.0000000	0.486166	0.513834	98	1220	−1122
254	$2^1 127^1$	Y	N	5	0	1.0000000	0.488189	0.511811	103	1225	−1122
255	$3^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.486275	0.513725	87	1225	−1138
256	2^8	N	Y	2	0	4.5000000	0.488281	0.511719	89	1227	−1138
257	257^1	Y	Y	−2	0	1.0000000	0.486381	0.513619	87	1227	−1140
258	$2^1 3^1 43^1$	Y	N	−16	0	1.0000000	0.484496	0.515504	71	1227	−1156
259	$7^1 37^1$	Y	N	5	0	1.0000000	0.486486	0.513514	76	1232	−1156
260	$2^2 5^1 13^1$	N	N	30	14	1.1666667	0.488462	0.511538	106	1262	−1156
261	$3^2 29^1$	N	N	−7	2	1.2857143	0.486590	0.513410	99	1262	−1163
262	$2^1 131^1$	Y	N	5	0	1.0000000	0.488550	0.511450	104	1267	−1163
263	263^1	Y	Y	−2	0	1.0000000	0.486692	0.513308	102	1267	−1165
264	$2^3 3^1 11^1$	N	N	−48	32	1.3333333	0.484848	0.515152	54	1267	−1213
265	$5^1 53^1$	Y	N	5	0	1.0000000	0.486792	0.513208	59	1272	−1213
266	$2^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.484962	0.515038	43	1272	−1229
267	$3^1 89^1$	Y	N	5	0	1.0000000	0.486891	0.513109	48	1277	−1229
268	$2^2 67^1$	N	N	−7	2	1.2857143	0.485075	0.514925	41	1277	−1236
269	269^1	Y	Y	−2	0	1.0000000	0.483271	0.516729	39	1277	−1238
270	$2^1 3^3 5^1$	N	N	−48	32	1.3333333	0.481481	0.518519	−9	1277	−1286
271	271^1	Y	Y	−2	0	1.0000000	0.479705	0.520295	−11	1277	−1288
272	$2^4 17^1$	N	N	−11	6	1.8181818	0.477941	0.522059	−22	1277	−1299
273	$3^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−38	1277	−1315
274	$2^1 137^1$	Y	N	5	0	1.0000000	0.478102	0.521898	−33	1282	−1315
275	$5^2 11^1$	N	N	−7	2	1.2857143	0.476364	0.523636	−40	1282	−1322
276	$2^2 3^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−10	1312	−1322
277	277^1	Y	Y	−2	0	1.0000000	0.476534	0.523466	−12	1312	−1324

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
278	$2^1 139^1$	Y	N	5	0	1.0000000	0.478417	0.521583	−7	1317	−1324
279	$3^2 31^1$	N	N	−7	2	1.2857143	0.476703	0.523297	−14	1317	−1331
280	$2^3 5^1 7^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−62	1317	−1379
281	281^1	Y	Y	−2	0	1.0000000	0.473310	0.526690	−64	1317	−1381
282	$2^1 3^1 47^1$	Y	N	−16	0	1.0000000	0.471631	0.528369	−80	1317	−1397
283	283^1	Y	Y	−2	0	1.0000000	0.469965	0.530035	−82	1317	−1399
284	$2^2 71^1$	N	N	−7	2	1.2857143	0.468310	0.531690	−89	1317	−1406
285	$3^1 5^1 19^1$	Y	N	−16	0	1.0000000	0.466667	0.533333	−105	1317	−1422
286	$2^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.465035	0.534965	−121	1317	−1438
287	$7^1 41^1$	Y	N	5	0	1.0000000	0.466899	0.533101	−116	1322	−1438
288	$2^5 3^2$	N	N	−47	42	1.7659574	0.465278	0.534722	−163	1322	−1485
289	17^2	N	Y	2	0	1.5000000	0.467128	0.532872	−161	1324	−1485
290	$2^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.465517	0.534483	−177	1324	−1501
291	$3^1 97^1$	Y	N	5	0	1.0000000	0.467354	0.532646	−172	1329	−1501
292	$2^2 73^1$	N	N	−7	2	1.2857143	0.465753	0.534247	−179	1329	−1508
293	293^1	Y	Y	−2	0	1.0000000	0.464164	0.535836	−181	1329	−1510
294	$2^1 3^1 7^2$	N	N	30	14	1.1666667	0.465986	0.534014	−151	1359	−1510
295	$5^1 59^1$	Y	N	5	0	1.0000000	0.467797	0.532203	−146	1364	−1510
296	$2^3 37^1$	N	N	9	4	1.5555556	0.469595	0.530405	−137	1373	−1510
297	$3^3 11^1$	N	N	9	4	1.5555556	0.471380	0.528620	−128	1382	−1510
298	$2^1 149^1$	Y	N	5	0	1.0000000	0.473154	0.526846	−123	1387	−1510
299	$13^1 23^1$	Y	N	5	0	1.0000000	0.474916	0.525084	−118	1392	−1510
300	$2^2 3^1 5^2$	N	N	−74	58	1.2162162	0.473333	0.526667	−192	1392	−1584
301	$7^1 43^1$	Y	N	5	0	1.0000000	0.475083	0.524917	−187	1397	−1584
302	$2^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−182	1402	−1584
303	$3^1 101^1$	Y	N	5	0	1.0000000	0.478548	0.521452	−177	1407	−1584
304	$2^4 19^1$	N	N	−11	6	1.8181818	0.476974	0.523026	−188	1407	−1595
305	$5^1 61^1$	Y	N	5	0	1.0000000	0.478689	0.521311	−183	1412	−1595
306	$2^1 3^2 17^1$	N	N	30	14	1.1666667	0.480392	0.519608	−153	1442	−1595
307	307^1	Y	Y	−2	0	1.0000000	0.478827	0.521173	−155	1442	−1597
308	$2^2 7^1 11^1$	N	N	30	14	1.1666667	0.480519	0.519481	−125	1472	−1597
309	$3^1 103^1$	Y	N	5	0	1.0000000	0.482201	0.517799	−120	1477	−1597
310	$2^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.480645	0.519355	−136	1477	−1613
311	311^1	Y	Y	−2	0	1.0000000	0.479100	0.520900	−138	1477	−1615
312	$2^3 3^1 13^1$	N	N	−48	32	1.3333333	0.477564	0.522436	−186	1477	−1663
313	313^1	Y	Y	−2	0	1.0000000	0.476038	0.523962	−188	1477	−1665
314	$2^1 157^1$	Y	N	5	0	1.0000000	0.477707	0.522293	−183	1482	−1665
315	$3^2 5^1 7^1$	N	N	30	14	1.1666667	0.479365	0.520635	−153	1512	−1665
316	$2^2 79^1$	N	N	−7	2	1.2857143	0.477848	0.522152	−160	1512	−1672
317	317^1	Y	Y	−2	0	1.0000000	0.476341	0.523659	−162	1512	−1674
318	$2^1 3^1 53^1$	Y	N	−16	0	1.0000000	0.474843	0.525157	−178	1512	−1690
319	$11^1 29^1$	Y	N	5	0	1.0000000	0.476489	0.523511	−173	1517	−1690
320	$2^6 5^1$	N	N	−15	10	2.3333333	0.475000	0.525000	−188	1517	−1705
321	$3^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−183	1522	−1705
322	$2^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.475155	0.524845	−199	1522	−1721
323	$17^1 19^1$	Y	N	5	0	1.0000000	0.476780	0.523220	−194	1527	−1721
324	$2^2 3^4$	N	N	34	29	1.6176471	0.478395	0.521605	−160	1561	−1721
325	$5^2 13^1$	N	N	−7	2	1.2857143	0.476923	0.523077	−167	1561	−1728
326	$2^1 163^1$	Y	N	5	0	1.0000000	0.478528	0.521472	−162	1566	−1728
327	$3^1 109^1$	Y	N	5	0	1.0000000	0.480122	0.519878	−157	1571	−1728
328	$2^3 41^1$	N	N	9	4	1.5555556	0.481707	0.518293	−148	1580	−1728
329	$7^1 47^1$	Y	N	5	0	1.0000000	0.483283	0.516717	−143	1585	−1728
330	$2^1 3^1 5^1 11^1$	Y	N	65	0	1.0000000	0.484848	0.515152	−78	1650	−1728
331	331^1	Y	Y	−2	0	1.0000000	0.483384	0.516616	−80	1650	−1730
332	$2^2 83^1$	N	N	−7	2	1.2857143	0.481928	0.518072	−87	1650	−1737
333	$3^2 37^1$	N	N	−7	2	1.2857143	0.480480	0.519520	−94	1650	−1744
334	$2^1 167^1$	Y	N	5	0	1.0000000	0.482036	0.517964	−89	1655	−1744
335	$5^1 67^1$	Y	N	5	0	1.0000000	0.483582	0.516418	−84	1660	−1744
336	$2^4 3^1 7^1$	N	N	70	54	1.5000000	0.485119	0.514881	−14	1730	−1744
337	337^1	Y	Y	−2	0	1.0000000	0.483680	0.516320	−16	1730	−1746
338	$2^1 13^2$	N	N	−7	2	1.2857143	0.482249	0.517751	−23	1730	−1753
339	$3^1 113^1$	Y	N	5	0	1.0000000	0.483776	0.516224	−18	1735	−1753
340	$2^2 5^1 17^1$	N	N	30	14	1.1666667	0.485294	0.514706	12	1765	−1753
341	$11^1 31^1$	Y	N	5	0	1.0000000	0.486804	0.513196	17	1770	−1753
342	$2^1 3^2 19^1$	N	N	30	14	1.1666667	0.488304	0.511696	47	1800	−1753
343	7^3	N	Y	−2	0	2.0000000	0.486880	0.513120	45	1800	−1755
344	$2^3 43^1$	N	N	9	4	1.5555556	0.488372	0.511628	54	1809	−1755
345	$3^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.486957	0.513043	38	1809	−1771
346	$2^1 173^1$	Y	N	5	0	1.0000000	0.488439	0.511561	43	1814	−1771
347	347^1	Y	Y	−2	0	1.0000000	0.487032	0.512968	41	1814	−1773
348	$2^2 3^1 29^1$	N	N	30	14	1.1666667	0.488506	0.511494	71	1844	−1773
349	349^1	Y	Y	−2	0	1.0000000	0.487106	0.512894	69	1844	−1775
350	$2^1 5^2 7^1$	N	N	30	14	1.1666667	0.488571	0.511429	99	1874	−1775

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
351	$3^3 13^1$	N	N	9	4	1.5555556	0.490028	0.509972	108	1883	−1775
352	$2^5 11^1$	N	N	13	8	2.0769231	0.491477	0.508523	121	1896	−1775
353	353^1	Y	Y	−2	0	1.0000000	0.490085	0.509915	119	1896	−1777
354	$2^1 3^1 59^1$	Y	N	−16	0	1.0000000	0.488701	0.511299	103	1896	−1793
355	$5^1 71^1$	Y	N	5	0	1.0000000	0.490141	0.509859	108	1901	−1793
356	$2^2 89^1$	N	N	−7	2	1.2857143	0.488764	0.511236	101	1901	−1800
357	$3^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.487395	0.512605	85	1901	−1816
358	$2^1 179^1$	Y	N	5	0	1.0000000	0.488827	0.511173	90	1906	−1816
359	359^1	Y	Y	−2	0	1.0000000	0.487465	0.512535	88	1906	−1818
360	$2^3 3^2 5^1$	N	N	145	129	1.3034483	0.488889	0.511111	233	2051	−1818
361	19^2	N	Y	2	0	1.5000000	0.490305	0.509695	235	2053	−1818
362	$2^1 181^1$	Y	N	5	0	1.0000000	0.491713	0.508287	240	2058	−1818
363	$3^1 11^2$	N	N	−7	2	1.2857143	0.490358	0.509642	233	2058	−1825
364	$2^2 7^1 13^1$	N	N	30	14	1.1666667	0.491758	0.508242	263	2088	−1825
365	$5^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	268	2093	−1825
366	$2^1 3^1 61^1$	Y	N	−16	0	1.0000000	0.491803	0.508197	252	2093	−1841
367	367^1	Y	Y	−2	0	1.0000000	0.490463	0.509537	250	2093	−1843
368	$2^4 23^1$	N	N	−11	6	1.8181818	0.489130	0.510870	239	2093	−1854
369	$3^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	232	2093	−1861
370	$2^1 5^1 37^1$	Y	N	−16	0	1.0000000	0.486486	0.513514	216	2093	−1877
371	$7^1 53^1$	Y	N	5	0	1.0000000	0.487871	0.512129	221	2098	−1877
372	$2^2 3^1 31^1$	N	N	30	14	1.1666667	0.489247	0.510753	251	2128	−1877
373	373^1	Y	Y	−2	0	1.0000000	0.487936	0.512064	249	2128	−1879
374	$2^1 11^1 17^1$	Y	N	−16	0	1.0000000	0.486631	0.513369	233	2128	−1895
375	$3^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	242	2137	−1895
376	$2^3 47^1$	N	N	9	4	1.5555556	0.489362	0.510638	251	2146	−1895
377	$13^1 29^1$	Y	N	5	0	1.0000000	0.490716	0.509284	256	2151	−1895
378	$2^1 3^3 7^1$	N	N	−48	32	1.3333333	0.489418	0.510582	208	2151	−1943
379	379^1	Y	Y	−2	0	1.0000000	0.488127	0.511873	206	2151	−1945
380	$2^2 5^1 19^1$	N	N	30	14	1.1666667	0.489474	0.510526	236	2181	−1945
381	$3^1 127^1$	Y	N	5	0	1.0000000	0.490814	0.509186	241	2186	−1945
382	$2^1 191^1$	Y	N	5	0	1.0000000	0.492147	0.507853	246	2191	−1945
383	383^1	Y	Y	−2	0	1.0000000	0.490862	0.509138	244	2191	−1947
384	$2^7 3^1$	N	N	17	12	2.5882353	0.492188	0.507812	261	2208	−1947
385	$5^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.490909	0.509091	245	2208	−1963
386	$2^1 193^1$	Y	N	5	0	1.0000000	0.492228	0.507772	250	2213	−1963
387	$3^2 43^1$	N	N	−7	2	1.2857143	0.490956	0.509044	243	2213	−1970
388	$2^2 97^1$	N	N	−7	2	1.2857143	0.489691	0.510309	236	2213	−1977
389	389^1	Y	Y	−2	0	1.0000000	0.488432	0.511568	234	2213	−1979
390	$2^1 3^1 5^1 13^1$	Y	N	65	0	1.0000000	0.489744	0.510256	299	2278	−1979
391	$17^1 23^1$	Y	N	5	0	1.0000000	0.491049	0.508951	304	2283	−1979
392	$2^3 7^2$	N	N	−23	18	1.4782609	0.489796	0.510204	281	2283	−2002
393	$3^1 131^1$	Y	N	5	0	1.0000000	0.491094	0.508906	286	2288	−2002
394	$2^1 197^1$	Y	N	5	0	1.0000000	0.492386	0.507614	291	2293	−2002
395	$5^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	296	2298	−2002
396	$2^2 3^2 11^1$	N	N	−74	58	1.2162162	0.492424	0.507576	222	2298	−2076
397	397^1	Y	Y	−2	0	1.0000000	0.491184	0.508816	220	2298	−2078
398	$2^1 199^1$	Y	N	5	0	1.0000000	0.492462	0.507538	225	2303	−2078
399	$3^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.491228	0.508772	209	2303	−2094
400	$2^4 5^2$	N	N	34	29	1.6176471	0.492500	0.507500	243	2337	−2094
401	401^1	Y	Y	−2	0	1.0000000	0.491272	0.508728	241	2337	−2096
402	$2^1 3^1 67^1$	Y	N	−16	0	1.0000000	0.490050	0.509950	225	2337	−2112
403	$13^1 31^1$	Y	N	5	0	1.0000000	0.491315	0.508685	230	2342	−2112
404	$2^2 101^1$	N	N	−7	2	1.2857143	0.490099	0.509901	223	2342	−2119
405	$3^4 5^1$	N	N	−11	6	1.8181818	0.488889	0.511111	212	2342	−2130
406	$2^1 7^1 29^1$	Y	N	−16	0	1.0000000	0.487685	0.512315	196	2342	−2146
407	$11^1 37^1$	Y	N	5	0	1.0000000	0.488943	0.511057	201	2347	−2146
408	$2^3 3^1 17^1$	N	N	−48	32	1.3333333	0.487745	0.512255	153	2347	−2194
409	409^1	Y	Y	−2	0	1.0000000	0.486553	0.513447	151	2347	−2196
410	$2^1 5^1 41^1$	Y	N	−16	0	1.0000000	0.485366	0.514634	135	2347	−2212
411	$3^1 137^1$	Y	N	5	0	1.0000000	0.486618	0.513382	140	2352	−2212
412	$2^2 103^1$	N	N	−7	2	1.2857143	0.485437	0.514563	133	2352	−2219
413	$7^1 59^1$	Y	N	5	0	1.0000000	0.486683	0.513317	138	2357	−2219
414	$2^1 3^2 23^1$	N	N	30	14	1.1666667	0.487923	0.512077	168	2387	−2219
415	$5^1 83^1$	Y	N	5	0	1.0000000	0.489157	0.510843	173	2392	−2219
416	$2^5 13^1$	N	N	13	8	2.0769231	0.490385	0.509615	186	2405	−2219
417	$3^1 139^1$	Y	N	5	0	1.0000000	0.491607	0.508393	191	2410	−2219
418	$2^1 11^1 19^1$	Y	N	−16	0	1.0000000	0.490431	0.509569	175	2410	−2235
419	419^1	Y	Y	−2	0	1.0000000	0.489260	0.510740	173	2410	−2237
420	$2^2 3^1 5^1 7^1$	N	N	−155	90	1.1032258	0.488095	0.511905	18	2410	−2392
421	421^1	Y	Y	−2	0	1.0000000	0.486936	0.513064	16	2410	−2394
422	$2^1 211^1$	Y	N	5	0	1.0000000	0.488152	0.511848	21	2415	−2394
423	$3^2 47^1$	N	N	−7	2	1.2857143	0.486998	0.513002	14	2415	−2401
424	$2^3 53^1$	N	N	9	4	1.5555556	0.488208	0.511792	23	2424	−2401
425	$5^2 17^1$	N	N	−7	2	1.2857143	0.487059	0.512941	16	2424	−2408

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
426	$2^1 3^1 71^1$	Y	N	-16	0	1.0000000	0.485915	0.514085	0	2424	-2424
427	$7^1 61^1$	Y	N	5	0	1.0000000	0.487119	0.512881	5	2429	-2424
428	$2^2 107^1$	N	N	-7	2	1.2857143	0.485981	0.514019	-2	2429	-2431
429	$3^1 11^1 13^1$	Y	N	-16	0	1.0000000	0.484848	0.515152	-18	2429	-2447
430	$2^1 5^1 43^1$	Y	N	-16	0	1.0000000	0.483721	0.516279	-34	2429	-2463
431	431^1	Y	Y	-2	0	1.0000000	0.482599	0.517401	-36	2429	-2465
432	$2^4 3^3$	N	N	-80	75	1.5625000	0.481481	0.518519	-116	2429	-2545
433	433^1	Y	Y	-2	0	1.0000000	0.480370	0.519630	-118	2429	-2547
434	$2^1 7^1 31^1$	Y	N	-16	0	1.0000000	0.479263	0.520737	-134	2429	-2563
435	$3^1 5^1 29^1$	Y	N	-16	0	1.0000000	0.478161	0.521839	-150	2429	-2579
436	$2^2 109^1$	N	N	-7	2	1.2857143	0.477064	0.522936	-157	2429	-2586
437	$19^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	-152	2434	-2586
438	$2^1 3^1 73^1$	Y	N	-16	0	1.0000000	0.477169	0.522831	-168	2434	-2602
439	439^1	Y	Y	-2	0	1.0000000	0.476082	0.523918	-170	2434	-2604
440	$2^3 5^1 11^1$	N	N	-48	32	1.3333333	0.475000	0.525000	-218	2434	-2652
441	$3^2 7^2$	N	N	14	9	1.3571429	0.476190	0.523810	-204	2448	-2652
442	$2^1 13^1 17^1$	Y	N	-16	0	1.0000000	0.475113	0.524887	-220	2448	-2668
443	443^1	Y	Y	-2	0	1.0000000	0.474041	0.525959	-222	2448	-2670
444	$2^2 3^1 37^1$	N	N	30	14	1.1666667	0.475225	0.524775	-192	2478	-2670
445	$5^1 89^1$	Y	N	5	0	1.0000000	0.476404	0.523596	-187	2483	-2670
446	$2^1 223^1$	Y	N	5	0	1.0000000	0.477578	0.522422	-182	2488	-2670
447	$3^1 149^1$	Y	N	5	0	1.0000000	0.478747	0.521253	-177	2493	-2670
448	$2^6 7^1$	N	N	-15	10	2.3333333	0.477679	0.522321	-192	2493	-2685
449	449^1	Y	Y	-2	0	1.0000000	0.476615	0.523385	-194	2493	-2687
450	$2^1 3^2 5^2$	N	N	-74	58	1.2162162	0.475556	0.524444	-268	2493	-2761
451	$11^1 41^1$	Y	N	5	0	1.0000000	0.476718	0.523282	-263	2498	-2761
452	$2^2 113^1$	N	N	-7	2	1.2857143	0.475664	0.524336	-270	2498	-2768
453	$3^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	-265	2503	-2768
454	$2^1 227^1$	Y	N	5	0	1.0000000	0.477974	0.522026	-260	2508	-2768
455	$5^1 7^1 13^1$	Y	N	-16	0	1.0000000	0.476923	0.523077	-276	2508	-2784
456	$2^3 3^1 19^1$	N	N	-48	32	1.3333333	0.475877	0.524123	-324	2508	-2832
457	457^1	Y	Y	-2	0	1.0000000	0.474836	0.525164	-326	2508	-2834
458	$2^1 229^1$	Y	N	5	0	1.0000000	0.475983	0.524017	-321	2513	-2834
459	$3^3 17^1$	N	N	9	4	1.5555556	0.477124	0.522876	-312	2522	-2834
460	$2^2 5^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	-282	2552	-2834
461	461^1	Y	Y	-2	0	1.0000000	0.477223	0.522777	-284	2552	-2836
462	$2^1 3^1 7^1 11^1$	Y	N	65	0	1.0000000	0.478355	0.521645	-219	2617	-2836
463	463^1	Y	Y	-2	0	1.0000000	0.477322	0.522678	-221	2617	-2838
464	$2^4 29^1$	N	N	-11	6	1.8181818	0.476293	0.523707	-232	2617	-2849
465	$3^1 5^1 31^1$	Y	N	-16	0	1.0000000	0.475269	0.524731	-248	2617	-2865
466	$2^1 233^1$	Y	N	5	0	1.0000000	0.476395	0.523605	-243	2622	-2865
467	467^1	Y	Y	-2	0	1.0000000	0.475375	0.524625	-245	2622	-2867
468	$2^2 3^2 13^1$	N	N	-74	58	1.2162162	0.474359	0.525641	-319	2622	-2941
469	$7^1 67^1$	Y	N	5	0	1.0000000	0.475480	0.524520	-314	2627	-2941
470	$2^1 5^1 47^1$	Y	N	-16	0	1.0000000	0.474468	0.525532	-330	2627	-2957
471	$3^1 157^1$	Y	N	5	0	1.0000000	0.475584	0.524416	-325	2632	-2957
472	$2^3 59^1$	N	N	9	4	1.5555556	0.476695	0.523305	-316	2641	-2957
473	$11^1 43^1$	Y	N	5	0	1.0000000	0.477801	0.522199	-311	2646	-2957
474	$2^1 3^1 79^1$	Y	N	-16	0	1.0000000	0.476793	0.523207	-327	2646	-2973
475	$5^2 19^1$	N	N	-7	2	1.2857143	0.475789	0.524211	-334	2646	-2980
476	$2^2 7^1 17^1$	N	N	30	14	1.1666667	0.476891	0.523109	-304	2676	-2980
477	$3^2 53^1$	N	N	-7	2	1.2857143	0.475891	0.524109	-311	2676	-2987
478	$2^1 239^1$	Y	N	5	0	1.0000000	0.476987	0.523013	-306	2681	-2987
479	479^1	Y	Y	-2	0	1.0000000	0.475992	0.524008	-308	2681	-2989
480	$2^5 3^1 5^1$	N	N	-96	80	1.6666667	0.475000	0.525000	-404	2681	-3085
481	$13^1 37^1$	Y	N	5	0	1.0000000	0.476091	0.523909	-399	2686	-3085
482	$2^1 241^1$	Y	N	5	0	1.0000000	0.477178	0.522822	-394	2691	-3085
483	$3^1 7^1 23^1$	Y	N	-16	0	1.0000000	0.476190	0.523810	-410	2691	-3101
484	$2^2 11^2$	N	N	14	9	1.3571429	0.477273	0.522727	-396	2705	-3101
485	$5^1 97^1$	Y	N	5	0	1.0000000	0.478351	0.521649	-391	2710	-3101
486	$2^1 3^5$	N	N	13	8	2.0769231	0.479424	0.520576	-378	2723	-3101
487	487^1	Y	Y	-2	0	1.0000000	0.478439	0.521561	-380	2723	-3103
488	$2^3 61^1$	N	N	9	4	1.5555556	0.479508	0.520492	-371	2732	-3103
489	$3^1 163^1$	Y	N	5	0	1.0000000	0.480573	0.519427	-366	2737	-3103
490	$2^1 5^1 7^2$	N	N	30	14	1.1666667	0.481633	0.518367	-336	2767	-3103
491	491^1	Y	Y	-2	0	1.0000000	0.480652	0.519348	-338	2767	-3105
492	$2^2 3^1 41^1$	N	N	30	14	1.1666667	0.481707	0.518293	-308	2797	-3105
493	$17^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	-303	2802	-3105
494	$2^1 13^1 19^1$	Y	N	-16	0	1.0000000	0.481781	0.518219	-319	2802	-3121
495	$3^2 5^1 11^1$	N	N	30	14	1.1666667	0.482828	0.517172	-289	2832	-3121
496	$2^4 31^1$	N	N	-11	6	1.8181818	0.481855	0.518145	-300	2832	-3132
497	$7^1 71^1$	Y	N	5	0	1.0000000	0.482897	0.517103	-295	2837	-3132
498	$2^1 3^1 83^1$	Y	N	-16	0	1.0000000	0.481928	0.518072	-311	2837	-3148
499	499^1	Y	Y	-2	0	1.0000000	0.480962	0.519038	-313	2837	-3150
500	$2^2 5^3$	N	N	-23	18	1.4782609	0.480000	0.520000	-336	2837	-3173

T.2 Table: Approximations of the summatory functions of $\lambda(n)$ and $\lambda_*(n)$

x	$L(x)$	$R_\pm(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$	x	$L(x)$	$R_\pm(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$
100000	−401	1	0.0320	−1.28	100000	−720	−0.0282	100045	−389	1	0.0310	−1.24	100045	−711	−0.0278
100001	−400	1	0.0319	−1.28	100001	−719	−0.0282	100046	−388	1	0.0309	−1.24	100046	−710	−0.0278
100002	−398	1	0.0318	−1.27	100002	−718	−0.0281	100047	−387	1	0.0308	−1.24	100047	−709	−0.0278
100003	−399	1	0.0318	−1.28	100003	−719	−0.0282	100048	−395	1	0.0315	−1.26	100048	−710	−0.0278
100004	−398	1	0.0318	−1.27	100004	−720	−0.0282	100049	−396	1	0.0316	−1.27	100049	−711	−0.0278
100005	−397	1	0.0317	−1.27	100005	−719	−0.0282	100050	−392	1	0.0312	−1.25	100050	−712	−0.0279
100006	−398	1	0.0318	−1.27	100006	−720	−0.0282	100051	−391	1	0.0312	−1.25	100051	−711	−0.0278
100007	−397	1	0.0317	−1.27	100007	−719	−0.0282	100052	−392	1	0.0312	−1.25	100052	−710	−0.0278
100008	−403	1	0.0322	−1.29	100008	−720	−0.0282	100053	−394	1	0.0314	−1.26	100053	−709	−0.0278
100009	−400	1	0.0319	−1.28	100009	−721	−0.0283	100054	−395	1	0.0315	−1.26	100054	−710	−0.0278
100010	−399	1	0.0318	−1.28	100010	−720	−0.0282	100055	−394	1	0.0314	−1.26	100055	−709	−0.0278
100011	−398	1	0.0317	−1.27	100011	−719	−0.0282	100056	−393	1	0.0313	−1.26	100056	−708	−0.0277
100012	−397	1	0.0317	−1.27	100012	−720	−0.0282	100057	−394	1	0.0314	−1.26	100057	−709	−0.0278
100013	−396	1	0.0316	−1.27	100013	−719	−0.0282	100058	−391	1	0.0312	−1.25	100058	−710	−0.0278
100014	−395	1	0.0315	−1.26	100014	−718	−0.0281	100059	−390	1	0.0311	−1.25	100059	−709	−0.0278
100015	−396	1	0.0316	−1.27	100015	−719	−0.0282	100060	−388	1	0.0309	−1.24	100060	−710	−0.0278
100016	−396	1	0.0316	−1.27	100016	−718	−0.0281	100061	−389	1	0.0310	−1.24	100061	−711	−0.0278
100017	−398	1	0.0317	−1.27	100017	−717	−0.0281	100062	−388	1	0.0309	−1.24	100062	−710	−0.0278
100018	−399	1	0.0318	−1.28	100018	−718	−0.0281	100063	−387	1	0.0308	−1.24	100063	−709	−0.0278
100019	−400	1	0.0319	−1.28	100019	−719	−0.0282	100064	−389	1	0.0310	−1.24	100064	−710	−0.0278
100020	−405	1	0.0323	−1.30	100020	−718	−0.0281	100065	−388	1	0.0309	−1.24	100065	−709	−0.0278
100021	−404	1	0.0322	−1.29	100021	−717	−0.0281	100066	−387	1	0.0308	−1.24	100066	−708	−0.0277
100022	−405	1	0.0323	−1.30	100022	−718	−0.0281	100067	−389	1	0.0310	−1.24	100067	−707	−0.0277
100023	−404	1	0.0322	−1.29	100023	−717	−0.0281	100068	−391	1	0.0312	−1.25	100068	−706	−0.0276
100024	−403	1	0.0321	−1.29	100024	−716	−0.0280	100069	−392	1	0.0312	−1.25	100069	−707	−0.0277
100025	−406	1	0.0324	−1.30	100025	−715	−0.0280	100070	−393	1	0.0313	−1.26	100070	−708	−0.0277
100026	−404	1	0.0322	−1.29	100026	−716	−0.0280	100071	−395	1	0.0315	−1.26	100071	−707	−0.0277
100027	−403	1	0.0321	−1.29	100027	−715	−0.0280	100072	−394	1	0.0314	−1.26	100072	−708	−0.0277
100028	−402	1	0.0321	−1.29	100028	−716	−0.0280	100073	−395	1	0.0315	−1.26	100073	−709	−0.0277
100029	−401	1	0.0320	−1.28	100029	−715	−0.0280	100074	−394	1	0.0314	−1.26	100074	−708	−0.0277
100030	−400	1	0.0319	−1.28	100030	−714	−0.0280	100075	−397	1	0.0316	−1.27	100075	−707	−0.0277
100031	−399	1	0.0318	−1.28	100031	−713	−0.0279	100076	−396	1	0.0316	−1.27	100076	−708	−0.0277
100032	−394	1	0.0314	−1.26	100032	−714	−0.0280	100077	−395	1	0.0315	−1.26	100077	−707	−0.0277
100033	−393	1	0.0313	−1.26	100033	−713	−0.0279	100078	−396	1	0.0316	−1.27	100078	−708	−0.0277
100034	−394	1	0.0314	−1.26	100034	−714	−0.0280	100079	−394	1	0.0314	−1.26	100079	−709	−0.0277
100035	−397	1	0.0317	−1.27	100035	−713	−0.0279	100080	−384	1	0.0306	−1.23	100080	−708	−0.0277
100036	−396	1	0.0316	−1.27	100036	−714	−0.0280	100081	−383	1	0.0305	−1.22	100081	−707	−0.0277
100037	−397	1	0.0317	−1.27	100037	−715	−0.0280	100082	−384	1	0.0306	−1.23	100082	−708	−0.0277
100038	−398	1	0.0317	−1.27	100038	−716	−0.0280	100083	−385	1	0.0307	−1.23	100083	−709	−0.0277
100039	−397	1	0.0317	−1.27	100039	−715	−0.0280	100084	−384	1	0.0306	−1.23	100084	−710	−0.0278
100040	−396	1	0.0316	−1.27	100040	−714	−0.0280	100085	−385	1	0.0307	−1.23	100085	−711	−0.0278
100041	−395	1	0.0315	−1.26	100041	−713	−0.0279	100086	−383	1	0.0305	−1.22	100086	−710	−0.0278
100042	−394	1	0.0314	−1.26	100042	−712	−0.0279	100087	−382	1	0.0305	−1.22	100087	−709	−0.0277
100043	−395	1	0.0315	−1.26	100043	−713	−0.0279	100088	−381	1	0.0304	−1.22	100088	−708	−0.0277
100044	−390	1	0.0311	−1.25	100044	−712	−0.0279	100089	−383	1	0.0305	−1.22	100089	−709	−0.0277

Table T.2: ***Approximations to the summatory functions of*** $\lambda(n)$ ***and*** $\lambda_*(n)$.

- *We define the exact summatory functions over these sequences by* $L(x) := \sum_{n \leq x} \lambda(n)$ *and* $L_*(n) := \sum_{n \leq x} \lambda_*(n)$.
- *Let the expected sign ratio function be defined by* $R_\pm(x) := \frac{\operatorname{sgn}(L(x))}{(-1)^{\lceil \log\log x \rceil}}$.
- *We compare the ratios of the following two functions with* $L(x)$: $L_{\approx,1}(x) := \sum_{k=1}^{\log\log x} \frac{x}{\log x} \cdot \frac{(-\log\log x)^{k-1}}{(k-1)!}$ *and* $L_{\approx,2}(x) := \frac{x^{3/4}}{(\log x)\sqrt{\log\log x}}$.
- *Finally, we compare the approximations (very accurate) to* $L_*(x)$ *by the summatory function* $\sum_{k \leq x} (-1)^{\omega(n)}$ *using the approximation* $L_{\approx}^*(x) := \frac{x}{\sqrt{2\pi}\sqrt{\log\log x}}$.

We are expecting to see and verify numerically that for sufficiently large x *the following properties:*

- *Almost always we have that* $R_\pm(x) = -1$.
- *The ratio* $\frac{L(x)}{L_{\approx,1}(x)}$ *should be bounded by a constant approximately equal to one, and the ratio* $\frac{L(x)}{L_{\approx,2}(x)}$ *should be at least one.*
- *The ratio* $\frac{L_*(x)}{L_{\approx}^*(x)}$ *tends towards an absolute constant.*

The summatory functions $L(x)$ *and* $L_*(x)$ *are numerically intensive to compute directly using standard packages for large* x. *We have written a software package in [19] in* `Python3` *for use with the* `SageMath` *platform that employs known algorithms for more efficiently computing these functions.*

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L^*_{\approx}(x)}$	x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L^*_{\approx}(x)}$
100090	−384	1	0.0306	−1.23	100090	−710	−0.0278	100165	−370	1	0.0295	−1.18	100165	−707	−0.0277
100091	−383	1	0.0305	−1.22	100091	−709	−0.0277	100166	−369	1	0.0294	−1.18	100166	−706	−0.0276
100092	−385	1	0.0307	−1.23	100092	−708	−0.0277	100167	−370	1	0.0295	−1.18	100167	−707	−0.0277
100093	−386	1	0.0308	−1.23	100093	−709	−0.0277	100168	−371	1	0.0296	−1.19	100168	−708	−0.0277
100094	−385	1	0.0307	−1.23	100094	−708	−0.0277	100169	−372	1	0.0296	−1.19	100169	−709	−0.0277
100095	−386	1	0.0308	−1.23	100095	−709	−0.0277	100170	−383	1	0.0305	−1.22	100170	−710	−0.0278
100096	−382	1	0.0305	−1.22	100096	−710	−0.0278	100171	−382	1	0.0304	−1.22	100171	−709	−0.0277
100097	−381	1	0.0304	−1.22	100097	−709	−0.0277	100172	−386	1	0.0307	−1.23	100172	−710	−0.0278
100098	−383	1	0.0305	−1.22	100098	−708	−0.0277	100173	−385	1	0.0307	−1.23	100173	−709	−0.0277
100099	−382	1	0.0305	−1.22	100099	−707	−0.0277	100174	−384	1	0.0306	−1.23	100174	−708	−0.0277
100100	−389	1	0.0310	−1.24	100100	−708	−0.0277	100175	−387	1	0.0308	−1.24	100175	−707	−0.0276
100101	−390	1	0.0311	−1.25	100101	−709	−0.0277	100176	−384	1	0.0306	−1.23	100176	−708	−0.0277
100102	−389	1	0.0310	−1.24	100102	−708	−0.0277	100177	−385	1	0.0307	−1.23	100177	−709	−0.0277
100103	−390	1	0.0311	−1.25	100103	−709	−0.0277	100178	−386	1	0.0307	−1.23	100178	−710	−0.0278
100104	−388	1	0.0309	−1.24	100104	−708	−0.0277	100179	−388	1	0.0309	−1.24	100179	−709	−0.0277
100105	−387	1	0.0308	−1.24	100105	−707	−0.0277	100180	−386	1	0.0307	−1.23	100180	−710	−0.0278
100106	−386	1	0.0308	−1.23	100106	−706	−0.0276	100181	−387	1	0.0308	−1.24	100181	−711	−0.0278
100107	−393	1	0.0313	−1.26	100107	−707	−0.0277	100182	−386	1	0.0307	−1.23	100182	−710	−0.0278
100108	−392	1	0.0312	−1.25	100108	−708	−0.0277	100183	−387	1	0.0308	−1.24	100183	−711	−0.0278
100109	−393	1	0.0313	−1.26	100109	−709	−0.0277	100184	−386	1	0.0307	−1.23	100184	−712	−0.0278
100110	−390	1	0.0311	−1.25	100110	−710	−0.0278	100185	−387	1	0.0308	−1.24	100185	−713	−0.0279
100111	−391	1	0.0312	−1.25	100111	−711	−0.0278	100186	−386	1	0.0307	−1.23	100186	−712	−0.0278
100112	−393	1	0.0313	−1.26	100112	−710	−0.0278	100187	−385	1	0.0307	−1.23	100187	−711	−0.0278
100113	−392	1	0.0312	−1.25	100113	−709	−0.0277	100188	−397	1	0.0316	−1.27	100188	−710	−0.0278
100114	−393	1	0.0313	−1.26	100114	−710	−0.0278	100189	−398	1	0.0317	−1.27	100189	−711	−0.0278
100115	−392	1	0.0312	−1.25	100115	−709	−0.0277	100190	−397	1	0.0316	−1.27	100190	−710	−0.0278
100116	−385	1	0.0307	−1.23	100116	−710	−0.0278	100191	−396	1	0.0315	−1.27	100191	−709	−0.0277
100117	−384	1	0.0306	−1.23	100117	−709	−0.0277	100192	−393	1	0.0313	−1.26	100192	−710	−0.0278
100118	−385	1	0.0307	−1.23	100118	−710	−0.0278	100193	−394	1	0.0314	−1.26	100193	−711	−0.0278
100119	−386	1	0.0308	−1.23	100119	−711	−0.0278	100194	−395	1	0.0314	−1.26	100194	−712	−0.0278
100120	−388	1	0.0309	−1.24	100120	−712	−0.0279	100195	−396	1	0.0315	−1.27	100195	−713	−0.0279
100121	−387	1	0.0308	−1.24	100121	−711	−0.0278	100196	−395	1	0.0314	−1.26	100196	−714	−0.0279
100122	−388	1	0.0309	−1.24	100122	−712	−0.0279	100197	−398	1	0.0317	−1.27	100197	−713	−0.0279
100123	−387	1	0.0308	−1.24	100123	−711	−0.0278	100198	−397	1	0.0316	−1.27	100198	−712	−0.0278
100124	−388	1	0.0309	−1.24	100124	−710	−0.0278	100199	−396	1	0.0315	−1.27	100199	−711	−0.0278
100125	−383	1	0.0305	−1.22	100125	−711	−0.0278	100200	−401	1	0.0319	−1.28	100200	−710	−0.0278
100126	−384	1	0.0306	−1.23	100126	−712	−0.0279	100201	−400	1	0.0318	−1.28	100201	−709	−0.0277
100127	−383	1	0.0305	−1.22	100127	−711	−0.0278	100202	−399	1	0.0318	−1.27	100202	−708	−0.0277
100128	−381	1	0.0304	−1.22	100128	−710	−0.0278	100203	−400	1	0.0318	−1.28	100203	−709	−0.0277
100129	−382	1	0.0304	−1.22	100129	−711	−0.0278	100204	−401	1	0.0319	−1.28	100204	−708	−0.0277
100130	−383	1	0.0305	−1.22	100130	−712	−0.0279	100205	−398	1	0.0317	−1.27	100205	−709	−0.0277
100131	−382	1	0.0304	−1.22	100131	−711	−0.0278	100206	−398	1	0.0317	−1.27	100206	−708	−0.0277
100132	−383	1	0.0305	−1.22	100132	−710	−0.0278	100207	−399	1	0.0318	−1.27	100207	−709	−0.0277
100133	−382	1	0.0304	−1.22	100133	−709	−0.0277	100208	−401	1	0.0319	−1.28	100208	−708	−0.0277
100134	−380	1	0.0303	−1.21	100134	−710	−0.0278	100209	−400	1	0.0318	−1.28	100209	−707	−0.0276
100135	−381	1	0.0304	−1.22	100135	−711	−0.0278	100210	−399	1	0.0318	−1.27	100210	−706	−0.0276
100136	−380	1	0.0303	−1.21	100136	−710	−0.0278	100211	−398	1	0.0317	−1.27	100211	−705	−0.0276
100137	−381	1	0.0304	−1.22	100137	−711	−0.0278	100212	−401	1	0.0319	−1.28	100212	−704	−0.0275
100138	−380	1	0.0303	−1.21	100138	−710	−0.0278	100213	−402	1	0.0320	−1.28	100213	−705	−0.0276
100139	−379	1	0.0302	−1.21	100139	−709	−0.0277	100214	−403	1	0.0321	−1.29	100214	−706	−0.0276
100140	−384	1	0.0306	−1.23	100140	−708	−0.0277	100215	−405	1	0.0322	−1.29	100215	−705	−0.0276
100141	−383	1	0.0305	−1.22	100141	−707	−0.0277	100216	−404	1	0.0322	−1.29	100216	−704	−0.0275
100142	−382	1	0.0304	−1.22	100142	−706	−0.0276	100217	−408	1	0.0325	−1.30	100217	−703	−0.0275
100143	−380	1	0.0303	−1.21	100143	−705	−0.0276	100218	−409	1	0.0326	−1.31	100218	−704	−0.0275
100144	−378	1	0.0301	−1.21	100144	−706	−0.0276	100219	−410	1	0.0326	−1.31	100219	−705	−0.0276
100145	−377	1	0.0300	−1.21	100145	−705	−0.0276	100220	−408	1	0.0325	−1.30	100220	−706	−0.0276
100146	−378	1	0.0301	−1.21	100146	−706	−0.0276	100221	−409	1	0.0326	−1.31	100221	−707	−0.0276
100147	−379	1	0.0302	−1.21	100147	−707	−0.0277	100222	−408	1	0.0325	−1.30	100222	−706	−0.0276
100148	−380	1	0.0303	−1.21	100148	−706	−0.0276	100223	−409	1	0.0326	−1.31	100223	−707	−0.0276
100149	−379	1	0.0302	−1.21	100149	−705	−0.0276	100224	−422	1	0.0336	−1.35	100224	−708	−0.0277
100150	−376	1	0.0300	−1.20	100150	−706	−0.0276	100225	−419	1	0.0334	−1.34	100225	−709	−0.0277
100151	−377	1	0.0300	−1.21	100151	−707	−0.0277	100226	−420	1	0.0334	−1.34	100226	−710	−0.0277
100152	−381	1	0.0304	−1.22	100152	−706	−0.0276	100227	−419	1	0.0334	−1.34	100227	−709	−0.0277
100153	−382	1	0.0304	−1.22	100153	−707	−0.0277	100228	−420	1	0.0334	−1.34	100228	−708	−0.0277
100154	−381	1	0.0304	−1.22	100154	−706	−0.0276	100229	−419	1	0.0334	−1.34	100229	−707	−0.0276
100155	−380	1	0.0303	−1.21	100155	−705	−0.0276	100230	−422	1	0.0336	−1.35	100230	−708	−0.0277
100156	−375	1	0.0299	−1.20	100156	−706	−0.0276	100231	−421	1	0.0335	−1.35	100231	−707	−0.0276
100157	−374	1	0.0298	−1.20	100157	−705	−0.0276	100232	−420	1	0.0334	−1.34	100232	−706	−0.0276
100158	−375	1	0.0299	−1.20	100158	−706	−0.0276	100233	−422	1	0.0336	−1.35	100233	−705	−0.0276
100159	−374	1	0.0298	−1.20	100159	−705	−0.0276	100234	−423	1	0.0337	−1.35	100234	−706	−0.0276
100160	−369	1	0.0294	−1.18	100160	−706	−0.0276	100235	−422	1	0.0336	−1.35	100235	−705	−0.0276
100161	−367	1	0.0292	−1.17	100161	−707	−0.0277	100236	−420	1	0.0334	−1.34	100236	−706	−0.0276
100162	−368	1	0.0293	−1.18	100162	−708	−0.0277	100237	−421	1	0.0335	−1.35	100237	−707	−0.0276
100163	−369	1	0.0294	−1.18	100163	−709	−0.0277	100238	−420	1	0.0334	−1.34	100238	−706	−0.0276
100164	−371	1	0.0296	−1.19	100164	−708	−0.0277	100239	−419	1	0.0334	−1.34	100239	−705	−0.0275

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$	x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$
100240	−420	1	0.0334	−1.34	100240	−704	−0.0275	100315	−410	1	0.0326	−1.31	100315	−699	−0.0273
100241	−419	1	0.0333	−1.34	100241	−703	−0.0275	100316	−409	1	0.0325	−1.31	100316	−700	−0.0273
100242	−417	1	0.0332	−1.33	100242	−704	−0.0275	100317	−408	1	0.0324	−1.30	100317	−699	−0.0273
100243	−418	1	0.0333	−1.34	100243	−705	−0.0275	100318	−407	1	0.0324	−1.30	100318	−698	−0.0273
100244	−417	1	0.0332	−1.33	100244	−706	−0.0276	100319	−406	1	0.0323	−1.30	100319	−697	−0.0272
100245	−416	1	0.0331	−1.33	100245	−705	−0.0275	100320	−415	1	0.0330	−1.32	100320	−698	−0.0273
100246	−415	1	0.0330	−1.33	100246	−704	−0.0275	100321	−414	1	0.0329	−1.32	100321	−697	−0.0272
100247	−414	1	0.0329	−1.32	100247	−703	−0.0275	100322	−415	1	0.0330	−1.32	100322	−698	−0.0273
100248	−416	1	0.0331	−1.33	100248	−704	−0.0275	100323	−413	1	0.0328	−1.32	100323	−699	−0.0273
100249	−415	1	0.0330	−1.33	100249	−703	−0.0275	100324	−412	1	0.0328	−1.32	100324	−700	−0.0273
100250	−418	1	0.0333	−1.34	100250	−704	−0.0275	100325	−415	1	0.0330	−1.32	100325	−699	−0.0273
100251	−420	1	0.0334	−1.34	100251	−705	−0.0275	100326	−414	1	0.0329	−1.32	100326	−698	−0.0273
100252	−419	1	0.0333	−1.34	100252	−706	−0.0276	100327	−413	1	0.0328	−1.32	100327	−697	−0.0272
100253	−418	1	0.0333	−1.34	100253	−705	−0.0275	100328	−412	1	0.0328	−1.32	100328	−696	−0.0272
100254	−417	1	0.0332	−1.33	100254	−706	−0.0276	100329	−413	1	0.0328	−1.32	100329	−697	−0.0272
100255	−416	1	0.0331	−1.33	100255	−705	−0.0275	100330	−412	1	0.0328	−1.32	100330	−696	−0.0272
100256	−418	1	0.0333	−1.34	100256	−706	−0.0276	100331	−413	1	0.0328	−1.32	100331	−697	−0.0272
100257	−419	1	0.0333	−1.34	100257	−707	−0.0276	100332	−409	1	0.0325	−1.31	100332	−698	−0.0273
100258	−418	1	0.0333	−1.34	100258	−706	−0.0276	100333	−410	1	0.0326	−1.31	100333	−699	−0.0273
100259	−417	1	0.0332	−1.33	100259	−705	−0.0275	100334	−409	1	0.0325	−1.31	100334	−698	−0.0273
100260	−411	1	0.0327	−1.31	100260	−704	−0.0275	100335	−410	1	0.0326	−1.31	100335	−699	−0.0273
100261	−410	1	0.0326	−1.31	100261	−703	−0.0275	100336	−412	1	0.0328	−1.32	100336	−698	−0.0273
100262	−409	1	0.0325	−1.31	100262	−702	−0.0274	100337	−411	1	0.0327	−1.31	100337	−697	−0.0272
100263	−410	1	0.0326	−1.31	100263	−703	−0.0275	100338	−409	1	0.0325	−1.31	100338	−696	−0.0272
100264	−411	1	0.0327	−1.31	100264	−704	−0.0275	100339	−408	1	0.0324	−1.30	100339	−695	−0.0271
100265	−412	1	0.0328	−1.32	100265	−705	−0.0275	100340	−410	1	0.0326	−1.31	100340	−694	−0.0271
100266	−411	1	0.0327	−1.31	100266	−704	−0.0275	100341	−412	1	0.0328	−1.32	100341	−693	−0.0271
100267	−412	1	0.0328	−1.32	100267	−705	−0.0275	100342	−413	1	0.0328	−1.32	100342	−694	−0.0271
100268	−411	1	0.0327	−1.31	100268	−706	−0.0276	100343	−414	1	0.0329	−1.32	100343	−695	−0.0271
100269	−409	1	0.0325	−1.31	100269	−707	−0.0276	100344	−412	1	0.0328	−1.32	100344	−694	−0.0271
100270	−408	1	0.0325	−1.30	100270	−706	−0.0276	100345	−411	1	0.0327	−1.31	100345	−693	−0.0271
100271	−409	1	0.0325	−1.31	100271	−707	−0.0276	100346	−412	1	0.0328	−1.32	100346	−694	−0.0271
100272	−406	1	0.0323	−1.30	100272	−708	−0.0277	100347	−411	1	0.0327	−1.31	100347	−693	−0.0271
100273	−405	1	0.0322	−1.29	100273	−707	−0.0276	100348	−412	1	0.0328	−1.32	100348	−692	−0.0270
100274	−406	1	0.0323	−1.30	100274	−708	−0.0277	100349	−411	1	0.0327	−1.31	100349	−691	−0.0270
100275	−409	1	0.0325	−1.31	100275	−707	−0.0276	100350	−405	1	0.0322	−1.29	100350	−690	−0.0269
100276	−407	1	0.0324	−1.30	100276	−706	−0.0276	100351	−404	1	0.0321	−1.29	100351	−689	−0.0269
100277	−406	1	0.0323	−1.30	100277	−705	−0.0275	100352	−422	1	0.0336	−1.35	100352	−688	−0.0269
100278	−403	1	0.0321	−1.29	100278	−706	−0.0276	100353	−423	1	0.0336	−1.35	100353	−689	−0.0269
100279	−404	1	0.0322	−1.29	100279	−707	−0.0276	100354	−422	1	0.0336	−1.35	100354	−688	−0.0269
100280	−402	1	0.0320	−1.28	100280	−706	−0.0276	100355	−421	1	0.0335	−1.34	100355	−687	−0.0268
100281	−401	1	0.0319	−1.28	100281	−705	−0.0275	100356	−419	1	0.0333	−1.34	100356	−688	−0.0269
100282	−405	1	0.0322	−1.29	100282	−706	−0.0276	100357	−420	1	0.0334	−1.34	100357	−689	−0.0269
100283	−407	1	0.0324	−1.30	100283	−705	−0.0275	100358	−416	1	0.0331	−1.33	100358	−690	−0.0269
100284	−409	1	0.0325	−1.31	100284	−704	−0.0275	100359	−419	1	0.0333	−1.34	100359	−691	−0.0270
100285	−410	1	0.0326	−1.31	100285	−705	−0.0275	100360	−417	1	0.0331	−1.33	100360	−690	−0.0269
100286	−411	1	0.0327	−1.31	100286	−706	−0.0276	100361	−418	1	0.0332	−1.33	100361	−691	−0.0270
100287	−409	1	0.0325	−1.31	100287	−707	−0.0276	100362	−417	1	0.0331	−1.33	100362	−690	−0.0269
100288	−413	1	0.0329	−1.32	100288	−706	−0.0276	100363	−418	1	0.0332	−1.33	100363	−691	−0.0270
100289	−412	1	0.0328	−1.32	100289	−705	−0.0275	100364	−417	1	0.0331	−1.33	100364	−692	−0.0270
100290	−409	1	0.0325	−1.31	100290	−704	−0.0275	100365	−418	1	0.0332	−1.33	100365	−693	−0.0270
100291	−410	1	0.0326	−1.31	100291	−705	−0.0275	100366	−417	1	0.0331	−1.33	100366	−692	−0.0270
100292	−411	1	0.0327	−1.31	100292	−704	−0.0275	100367	−416	1	0.0331	−1.33	100367	−691	−0.0270
100293	−412	1	0.0328	−1.32	100293	−705	−0.0275	100368	−408	1	0.0324	−1.30	100368	−690	−0.0269
100294	−411	1	0.0327	−1.31	100294	−704	−0.0275	100369	−407	1	0.0323	−1.30	100369	−689	−0.0269
100295	−412	1	0.0328	−1.32	100295	−705	−0.0275	100370	−408	1	0.0324	−1.30	100370	−690	−0.0269
100296	−417	1	0.0332	−1.33	100296	−704	−0.0275	100371	−407	1	0.0323	−1.30	100371	−689	−0.0269
100297	−418	1	0.0332	−1.33	100297	−705	−0.0275	100372	−406	1	0.0323	−1.30	100372	−690	−0.0269
100298	−417	1	0.0332	−1.33	100298	−704	−0.0275	100373	−407	1	0.0323	−1.30	100373	−691	−0.0270
100299	−418	1	0.0332	−1.33	100299	−705	−0.0275	100374	−408	1	0.0324	−1.30	100374	−692	−0.0270
100300	−414	1	0.0329	−1.32	100300	−704	−0.0275	100375	−411	1	0.0327	−1.31	100375	−693	−0.0270
100301	−413	1	0.0329	−1.32	100301	−703	−0.0275	100376	−410	1	0.0326	−1.31	100376	−692	−0.0270
100302	−412	1	0.0328	−1.32	100302	−702	−0.0274	100377	−408	1	0.0324	−1.30	100377	−693	−0.0270
100303	−409	1	0.0325	−1.31	100303	−703	−0.0275	100378	−409	1	0.0325	−1.31	100378	−694	−0.0271
100304	−411	1	0.0327	−1.31	100304	−702	−0.0274	100379	−410	1	0.0326	−1.31	100379	−695	−0.0271
100305	−413	1	0.0329	−1.32	100305	−703	−0.0275	100380	−402	1	0.0320	−1.28	100380	−696	−0.0272
100306	−412	1	0.0328	−1.32	100306	−702	−0.0274	100381	−401	1	0.0319	−1.28	100381	−695	−0.0271
100307	−411	1	0.0327	−1.31	100307	−701	−0.0274	100382	−402	1	0.0320	−1.28	100382	−696	−0.0272
100308	−411	1	0.0327	−1.31	100308	−700	−0.0273	100383	−401	1	0.0319	−1.28	100383	−695	−0.0271
100309	−413	1	0.0329	−1.32	100309	−699	−0.0273	100384	−399	1	0.0317	−1.27	100384	−694	−0.0271
100310	−412	1	0.0328	−1.32	100310	−698	−0.0273	100385	−400	1	0.0318	−1.28	100385	−695	−0.0271
100311	−413	1	0.0329	−1.32	100311	−699	−0.0273	100386	−404	1	0.0321	−1.29	100386	−694	−0.0271
100312	−412	1	0.0328	−1.32	100312	−698	−0.0273	100387	−403	1	0.0320	−1.29	100387	−693	−0.0270
100313	−413	1	0.0329	−1.32	100313	−699	−0.0273	100388	−404	1	0.0321	−1.29	100388	−692	−0.0270
100314	−411	1	0.0327	−1.31	100314	−700	−0.0273	100389	−405	1	0.0322	−1.29	100389	−693	−0.0270

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$	x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$
100390	−406	1	0.0323	−1.30	100390	−694	−0.0271	100465	−400	1	0.0318	−1.28	100465	−685	−0.0267
100391	−407	1	0.0323	−1.30	100391	−695	−0.0271	100466	−401	1	0.0318	−1.28	100466	−686	−0.0268
100392	−405	1	0.0322	−1.29	100392	−694	−0.0271	100467	−413	1	0.0328	−1.32	100467	−685	−0.0267
100393	−406	1	0.0323	−1.30	100393	−695	−0.0271	100468	−414	1	0.0329	−1.32	100468	−684	−0.0267
100394	−405	1	0.0322	−1.29	100394	−694	−0.0271	100469	−415	1	0.0329	−1.32	100469	−685	−0.0267
100395	−407	1	0.0323	−1.30	100395	−693	−0.0270	100470	−418	1	0.0332	−1.33	100470	−686	−0.0268
100396	−406	1	0.0323	−1.30	100396	−694	−0.0271	100471	−419	1	0.0332	−1.34	100471	−687	−0.0268
100397	−405	1	0.0322	−1.29	100397	−693	−0.0270	100472	−420	1	0.0333	−1.34	100472	−688	−0.0268
100398	−404	1	0.0321	−1.29	100398	−692	−0.0270	100473	−421	1	0.0334	−1.34	100473	−689	−0.0269
100399	−403	1	0.0320	−1.29	100399	−691	−0.0270	100474	−422	1	0.0335	−1.35	100474	−690	−0.0269
100400	−412	1	0.0327	−1.31	100400	−692	−0.0270	100475	−425	1	0.0337	−1.36	100475	−689	−0.0269
100401	−409	1	0.0325	−1.31	100401	−693	−0.0270	100476	−429	1	0.0341	−1.37	100476	−690	−0.0269
100402	−410	1	0.0326	−1.31	100402	−694	−0.0271	100477	−430	1	0.0341	−1.37	100477	−691	−0.0270
100403	−411	1	0.0327	−1.31	100403	−695	−0.0271	100478	−431	1	0.0342	−1.38	100478	−692	−0.0270
100404	−415	1	0.0330	−1.32	100404	−696	−0.0271	100479	−430	1	0.0341	−1.37	100479	−691	−0.0270
100405	−416	1	0.0331	−1.33	100405	−697	−0.0272	100480	−435	1	0.0345	−1.39	100480	−692	−0.0270
100406	−417	1	0.0331	−1.33	100406	−698	−0.0272	100481	−434	1	0.0345	−1.39	100481	−691	−0.0269
100407	−416	1	0.0331	−1.33	100407	−697	−0.0272	100482	−435	1	0.0345	−1.39	100482	−692	−0.0270
100408	−417	1	0.0331	−1.33	100408	−696	−0.0271	100483	−436	1	0.0346	−1.39	100483	−693	−0.0270
100409	−418	1	0.0332	−1.33	100409	−697	−0.0272	100484	−437	1	0.0347	−1.39	100484	−692	−0.0270
100410	−415	1	0.0330	−1.32	100410	−696	−0.0271	100485	−435	1	0.0345	−1.39	100485	−693	−0.0270
100411	−416	1	0.0331	−1.33	100411	−697	−0.0272	100486	−436	1	0.0346	−1.39	100486	−694	−0.0271
100412	−415	1	0.0330	−1.32	100412	−698	−0.0272	100487	−437	1	0.0347	−1.39	100487	−695	−0.0271
100413	−413	1	0.0328	−1.32	100413	−697	−0.0272	100488	−435	1	0.0345	−1.39	100488	−694	−0.0271
100414	−412	1	0.0327	−1.31	100414	−696	−0.0271	100489	−428	1	0.0340	−1.37	100489	−695	−0.0271
100415	−411	1	0.0327	−1.31	100415	−695	−0.0271	100490	−427	1	0.0339	−1.36	100490	−694	−0.0271
100416	−406	1	0.0323	−1.30	100416	−696	−0.0271	100491	−426	1	0.0338	−1.36	100491	−693	−0.0270
100417	−407	1	0.0323	−1.30	100417	−697	−0.0272	100492	−427	1	0.0339	−1.36	100492	−692	−0.0270
100418	−406	1	0.0323	−1.30	100418	−696	−0.0271	100493	−428	1	0.0340	−1.37	100493	−693	−0.0270
100419	−405	1	0.0322	−1.29	100419	−695	−0.0271	100494	−430	1	0.0341	−1.37	100494	−694	−0.0271
100420	−403	1	0.0320	−1.29	100420	−696	−0.0271	100495	−431	1	0.0342	−1.38	100495	−695	−0.0271
100421	−402	1	0.0320	−1.28	100421	−695	−0.0271	100496	−429	1	0.0340	−1.37	100496	−696	−0.0271
100422	−405	1	0.0322	−1.29	100422	−694	−0.0271	100497	−430	1	0.0341	−1.37	100497	−697	−0.0272
100423	−404	1	0.0321	−1.29	100423	−693	−0.0270	100498	−431	1	0.0342	−1.38	100498	−698	−0.0272
100424	−403	1	0.0320	−1.29	100424	−692	−0.0270	100499	−428	1	0.0340	−1.37	100499	−697	−0.0272
100425	−406	1	0.0323	−1.30	100425	−691	−0.0270	100500	−435	1	0.0345	−1.39	100500	−696	−0.0271
100426	−407	1	0.0323	−1.30	100426	−692	−0.0270	100501	−436	1	0.0346	−1.39	100501	−697	−0.0272
100427	−406	1	0.0323	−1.30	100427	−691	−0.0270	100502	−437	1	0.0347	−1.39	100502	−698	−0.0272
100428	−404	1	0.0321	−1.29	100428	−692	−0.0270	100503	−435	1	0.0345	−1.39	100503	−699	−0.0273
100429	−403	1	0.0320	−1.29	100429	−691	−0.0270	100504	−436	1	0.0346	−1.39	100504	−700	−0.0273
100430	−405	1	0.0322	−1.29	100430	−690	−0.0269	100505	−435	1	0.0345	−1.39	100505	−699	−0.0273
100431	−407	1	0.0323	−1.30	100431	−689	−0.0269	100506	−433	1	0.0344	−1.38	100506	−698	−0.0272
100432	−409	1	0.0325	−1.31	100432	−688	−0.0268	100507	−432	1	0.0343	−1.38	100507	−697	−0.0272
100433	−408	1	0.0324	−1.30	100433	−687	−0.0268	100508	−433	1	0.0344	−1.38	100508	−696	−0.0271
100434	−407	1	0.0323	−1.30	100434	−686	−0.0268	100509	−432	1	0.0343	−1.38	100509	−695	−0.0271
100435	−408	1	0.0324	−1.30	100435	−687	−0.0268	100510	−436	1	0.0346	−1.39	100510	−694	−0.0271
100436	−409	1	0.0325	−1.31	100436	−686	−0.0268	100511	−437	1	0.0347	−1.39	100511	−695	−0.0271
100437	−408	1	0.0324	−1.30	100437	−685	−0.0267	100512	−428	1	0.0340	−1.37	100512	−696	−0.0271
100438	−409	1	0.0325	−1.30	100438	−686	−0.0268	100513	−429	1	0.0340	−1.37	100513	−697	−0.0272
100439	−408	1	0.0324	−1.30	100439	−685	−0.0267	100514	−430	1	0.0341	−1.37	100514	−698	−0.0272
100440	−415	1	0.0330	−1.32	100440	−684	−0.0267	100515	−431	1	0.0342	−1.38	100515	−699	−0.0273
100441	−416	1	0.0330	−1.33	100441	−685	−0.0267	100516	−430	1	0.0341	−1.37	100516	−700	−0.0273
100442	−415	1	0.0330	−1.32	100442	−684	−0.0267	100517	−431	1	0.0342	−1.38	100517	−701	−0.0273
100443	−416	1	0.0330	−1.33	100443	−685	−0.0267	100518	−430	1	0.0341	−1.37	100518	−700	−0.0273
100444	−417	1	0.0331	−1.33	100444	−684	−0.0267	100519	−431	1	0.0342	−1.38	100519	−701	−0.0273
100445	−416	1	0.0330	−1.33	100445	−683	−0.0266	100520	−431	1	0.0342	−1.38	100520	−700	−0.0273
100446	−417	1	0.0331	−1.33	100446	−684	−0.0267	100521	−428	1	0.0340	−1.37	100521	−701	−0.0273
100447	−418	1	0.0332	−1.33	100447	−685	−0.0267	100522	−427	1	0.0339	−1.36	100522	−700	−0.0273
100448	−420	1	0.0334	−1.34	100448	−686	−0.0268	100523	−428	1	0.0340	−1.37	100523	−701	−0.0273
100449	−422	1	0.0335	−1.35	100449	−685	−0.0267	100524	−426	1	0.0338	−1.36	100524	−702	−0.0274
100450	−411	1	0.0326	−1.31	100450	−684	−0.0267	100525	−429	1	0.0340	−1.37	100525	−701	−0.0273
100451	−410	1	0.0326	−1.31	100451	−683	−0.0266	100526	−428	1	0.0340	−1.37	100526	−700	−0.0273
100452	−411	1	0.0326	−1.31	100452	−682	−0.0266	100527	−429	1	0.0340	−1.37	100527	−701	−0.0273
100453	−412	1	0.0327	−1.31	100453	−683	−0.0266	100528	−427	1	0.0339	−1.36	100528	−702	−0.0274
100454	−411	1	0.0326	−1.31	100454	−682	−0.0266	100529	−426	1	0.0338	−1.36	100529	−701	−0.0273
100455	−410	1	0.0326	−1.31	100455	−681	−0.0266	100530	−429	1	0.0340	−1.37	100530	−700	−0.0273
100456	−411	1	0.0326	−1.31	100456	−682	−0.0266	100531	−428	1	0.0340	−1.37	100531	−699	−0.0272
100457	−412	1	0.0327	−1.31	100457	−683	−0.0266	100532	−427	1	0.0339	−1.36	100532	−700	−0.0273
100458	−410	1	0.0326	−1.31	100458	−684	−0.0267	100533	−426	1	0.0338	−1.36	100533	−699	−0.0272
100459	−411	1	0.0326	−1.31	100459	−685	−0.0267	100534	−425	1	0.0337	−1.36	100534	−698	−0.0272
100460	−409	1	0.0325	−1.30	100460	−686	−0.0267	100535	−424	1	0.0336	−1.35	100535	−697	−0.0272
100461	−408	1	0.0324	−1.30	100461	−685	−0.0267	100536	−422	1	0.0335	−1.35	100536	−696	−0.0271
100462	−407	1	0.0323	−1.30	100462	−684	−0.0267	100537	−423	1	0.0336	−1.35	100537	−697	−0.0272
100463	−406	1	0.0322	−1.30	100463	−683	−0.0266	100538	−424	1	0.0336	−1.35	100538	−698	−0.0272
100464	−399	1	0.0317	−1.27	100464	−684	−0.0267	100539	−426	1	0.0338	−1.36	100539	−697	−0.0272

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L^*_{\approx}(x)}$	x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L^*_{\approx}(x)}$
100540	−428	1	0.0340	−1.37	100540	−696	−0.0271	100615	−419	1	0.0332	−1.34	100615	−687	−0.0268
100541	−429	1	0.0340	−1.37	100541	−697	−0.0272	100616	−418	1	0.0331	−1.33	100616	−686	−0.0267
100542	−428	1	0.0340	−1.37	100542	−696	−0.0271	100617	−419	1	0.0332	−1.34	100617	−687	−0.0268
100543	−427	1	0.0339	−1.36	100543	−695	−0.0271	100618	−420	1	0.0333	−1.34	100618	−688	−0.0268
100544	−431	1	0.0342	−1.37	100544	−694	−0.0271	100619	−419	1	0.0332	−1.34	100619	−687	−0.0268
100545	−432	1	0.0343	−1.38	100545	−695	−0.0271	100620	−429	1	0.0340	−1.37	100620	−688	−0.0268
100546	−431	1	0.0342	−1.37	100546	−694	−0.0271	100621	−430	1	0.0341	−1.37	100621	−689	−0.0268
100547	−432	1	0.0343	−1.38	100547	−695	−0.0271	100622	−429	1	0.0340	−1.37	100622	−688	−0.0268
100548	−420	1	0.0333	−1.34	100548	−694	−0.0271	100623	−430	1	0.0341	−1.37	100623	−689	−0.0268
100549	−421	1	0.0334	−1.34	100549	−695	−0.0271	100624	−428	1	0.0339	−1.36	100624	−690	−0.0269
100550	−418	1	0.0332	−1.33	100550	−696	−0.0271	100625	−425	1	0.0337	−1.35	100625	−691	−0.0269
100551	−416	1	0.0330	−1.33	100551	−697	−0.0272	100626	−424	1	0.0336	−1.35	100626	−690	−0.0269
100552	−415	1	0.0329	−1.32	100552	−696	−0.0271	100627	−423	1	0.0335	−1.35	100627	−689	−0.0268
100553	−414	1	0.0329	−1.32	100553	−695	−0.0271	100628	−422	1	0.0334	−1.35	100628	−690	−0.0269
100554	−415	1	0.0329	−1.32	100554	−696	−0.0271	100629	−420	1	0.0333	−1.34	100629	−689	−0.0268
100555	−419	1	0.0332	−1.34	100555	−695	−0.0271	100630	−419	1	0.0332	−1.34	100630	−688	−0.0268
100556	−418	1	0.0331	−1.33	100556	−696	−0.0271	100631	−418	1	0.0331	−1.33	100631	−687	−0.0267
100557	−420	1	0.0333	−1.34	100557	−695	−0.0271	100632	−417	1	0.0331	−1.33	100632	−686	−0.0267
100558	−421	1	0.0334	−1.34	100558	−696	−0.0271	100633	−416	1	0.0330	−1.33	100633	−685	−0.0267
100559	−422	1	0.0335	−1.35	100559	−697	−0.0272	100634	−417	1	0.0331	−1.33	100634	−686	−0.0267
100560	−427	1	0.0339	−1.36	100560	−696	−0.0271	100635	−418	1	0.0331	−1.33	100635	−687	−0.0267
100561	−426	1	0.0338	−1.36	100561	−695	−0.0271	100636	−417	1	0.0331	−1.33	100636	−688	−0.0268
100562	−425	1	0.0337	−1.36	100562	−694	−0.0270	100637	−416	1	0.0330	−1.33	100637	−687	−0.0267
100563	−424	1	0.0336	−1.35	100563	−693	−0.0270	100638	−414	1	0.0328	−1.32	100638	−688	−0.0268
100564	−423	1	0.0335	−1.35	100564	−694	−0.0270	100639	−415	1	0.0329	−1.32	100639	−689	−0.0268
100565	−422	1	0.0335	−1.35	100565	−693	−0.0270	100640	−412	1	0.0327	−1.31	100640	−688	−0.0268
100566	−424	1	0.0336	−1.35	100566	−692	−0.0270	100641	−411	1	0.0326	−1.31	100641	−687	−0.0267
100567	−425	1	0.0337	−1.36	100567	−693	−0.0270	100642	−410	1	0.0325	−1.31	100642	−686	−0.0267
100568	−426	1	0.0338	−1.36	100568	−694	−0.0270	100643	−409	1	0.0324	−1.30	100643	−685	−0.0267
100569	−427	1	0.0339	−1.36	100569	−695	−0.0271	100644	−407	1	0.0323	−1.30	100644	−686	−0.0267
100570	−426	1	0.0338	−1.36	100570	−694	−0.0270	100645	−406	1	0.0322	−1.29	100645	−685	−0.0267
100571	−425	1	0.0337	−1.36	100571	−693	−0.0270	100646	−412	1	0.0327	−1.31	100646	−684	−0.0266
100572	−418	1	0.0331	−1.33	100572	−692	−0.0270	100647	−410	1	0.0325	−1.31	100647	−685	−0.0267
100573	−419	1	0.0332	−1.34	100573	−693	−0.0270	100648	−411	1	0.0326	−1.31	100648	−686	−0.0267
100574	−418	1	0.0331	−1.33	100574	−692	−0.0270	100649	−412	1	0.0327	−1.31	100649	−687	−0.0267
100575	−413	1	0.0327	−1.32	100575	−693	−0.0270	100650	−407	1	0.0323	−1.30	100650	−688	−0.0268
100576	−413	1	0.0327	−1.32	100576	−694	−0.0270	100651	−406	1	0.0322	−1.29	100651	−687	−0.0267
100577	−412	1	0.0327	−1.31	100577	−693	−0.0270	100652	−407	1	0.0323	−1.30	100652	−686	−0.0267
100578	−413	1	0.0327	−1.32	100578	−694	−0.0270	100653	−408	1	0.0323	−1.30	100653	−687	−0.0267
100579	−412	1	0.0327	−1.31	100579	−693	−0.0270	100654	−409	1	0.0324	−1.30	100654	−688	−0.0268
100580	−414	1	0.0328	−1.32	100580	−692	−0.0270	100655	−410	1	0.0325	−1.31	100655	−689	−0.0268
100581	−415	1	0.0329	−1.32	100581	−693	−0.0270	100656	−401	1	0.0318	−1.28	100656	−690	−0.0269
100582	−414	1	0.0328	−1.32	100582	−692	−0.0270	100657	−402	1	0.0319	−1.28	100657	−691	−0.0269
100583	−413	1	0.0327	−1.32	100583	−691	−0.0269	100658	−401	1	0.0318	−1.28	100658	−690	−0.0269
100584	−418	1	0.0331	−1.33	100584	−690	−0.0269	100659	−400	1	0.0317	−1.27	100659	−689	−0.0268
100585	−417	1	0.0331	−1.33	100585	−689	−0.0268	100660	−402	1	0.0319	−1.28	100660	−688	−0.0268
100586	−418	1	0.0331	−1.33	100586	−690	−0.0269	100661	−401	1	0.0318	−1.28	100661	−687	−0.0267
100587	−417	1	0.0331	−1.33	100587	−689	−0.0268	100662	−400	1	0.0317	−1.27	100662	−686	−0.0267
100588	−418	1	0.0331	−1.33	100588	−688	−0.0268	100663	−399	1	0.0316	−1.27	100663	−685	−0.0267
100589	−419	1	0.0332	−1.34	100589	−689	−0.0268	100664	−398	1	0.0315	−1.27	100664	−684	−0.0266
100590	−421	1	0.0334	−1.34	100590	−690	−0.0269	100665	−396	1	0.0314	−1.26	100665	−685	−0.0267
100591	−422	1	0.0334	−1.35	100591	−691	−0.0269	100666	−395	1	0.0313	−1.26	100666	−684	−0.0266
100592	−424	1	0.0336	−1.35	100592	−690	−0.0269	100667	−396	1	0.0314	−1.26	100667	−685	−0.0267
100593	−426	1	0.0338	−1.36	100593	−689	−0.0268	100668	−394	1	0.0312	−1.26	100668	−686	−0.0267
100594	−425	1	0.0337	−1.36	100594	−688	−0.0268	100669	−395	1	0.0313	−1.26	100669	−687	−0.0267
100595	−424	1	0.0336	−1.35	100595	−687	−0.0268	100670	−396	1	0.0314	−1.26	100670	−688	−0.0268
100596	−426	1	0.0338	−1.36	100596	−686	−0.0267	100671	−397	1	0.0315	−1.27	100671	−689	−0.0268
100597	−429	1	0.0340	−1.37	100597	−685	−0.0267	100672	−410	1	0.0325	−1.31	100672	−690	−0.0268
100598	−430	1	0.0341	−1.37	100598	−686	−0.0267	100673	−411	1	0.0325	−1.31	100673	−691	−0.0269
100599	−429	1	0.0340	−1.37	100599	−685	−0.0267	100674	−409	1	0.0324	−1.30	100674	−692	−0.0269
100600	−425	1	0.0337	−1.35	100600	−686	−0.0267	100675	−412	1	0.0327	−1.31	100675	−691	−0.0269
100601	−424	1	0.0336	−1.35	100601	−685	−0.0267	100676	−413	1	0.0327	−1.32	100676	−690	−0.0268
100602	−427	1	0.0338	−1.36	100602	−686	−0.0267	100677	−414	1	0.0328	−1.32	100677	−691	−0.0269
100603	−426	1	0.0338	−1.36	100603	−685	−0.0267	100678	−415	1	0.0329	−1.32	100678	−692	−0.0269
100604	−425	1	0.0337	−1.35	100604	−686	−0.0267	100679	−414	1	0.0328	−1.32	100679	−691	−0.0269
100605	−424	1	0.0336	−1.35	100605	−685	−0.0267	100680	−409	1	0.0324	−1.30	100680	−690	−0.0268
100606	−423	1	0.0335	−1.35	100606	−684	−0.0267	100681	−410	1	0.0325	−1.31	100681	−691	−0.0269
100607	−424	1	0.0336	−1.35	100607	−685	−0.0267	100682	−409	1	0.0324	−1.30	100682	−690	−0.0268
100608	−419	1	0.0332	−1.34	100608	−686	−0.0267	100683	−406	1	0.0322	−1.29	100683	−691	−0.0269
100609	−420	1	0.0333	−1.34	100609	−687	−0.0268	100684	−407	1	0.0322	−1.30	100684	−690	−0.0268
100610	−421	1	0.0334	−1.34	100610	−688	−0.0268	100685	−408	1	0.0323	−1.30	100685	−691	−0.0269
100611	−419	1	0.0332	−1.34	100611	−689	−0.0268	100686	−407	1	0.0322	−1.30	100686	−690	−0.0268
100612	−420	1	0.0333	−1.34	100612	−688	−0.0268	100687	−406	1	0.0322	−1.29	100687	−689	−0.0268
100613	−421	1	0.0334	−1.34	100613	−689	−0.0268	100688	−406	1	0.0322	−1.29	100688	−688	−0.0268
100614	−420	1	0.0333	−1.34	100614	−688	−0.0268	100689	−405	1	0.0321	−1.29	100689	−687	−0.0267

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$
100690	−406	1	0.0322	−1.29	100690	−688	−0.0268
100691	−405	1	0.0321	−1.29	100691	−687	−0.0267
100692	−409	1	0.0324	−1.30	100692	−688	−0.0268
100693	−410	1	0.0325	−1.31	100693	−689	−0.0268
100694	−407	1	0.0322	−1.30	100694	−688	−0.0268
100695	−410	1	0.0325	−1.31	100695	−687	−0.0267
100696	−411	1	0.0325	−1.31	100696	−688	−0.0268
100697	−410	1	0.0325	−1.31	100697	−687	−0.0267
100698	−409	1	0.0324	−1.30	100698	−686	−0.0267
100699	−410	1	0.0325	−1.31	100699	−687	−0.0267
100700	−409	1	0.0324	−1.30	100700	−686	−0.0267
100701	−407	1	0.0322	−1.30	100701	−687	−0.0267
100702	−408	1	0.0323	−1.30	100702	−688	−0.0268
100703	−409	1	0.0324	−1.30	100703	−689	−0.0268
100704	−412	1	0.0326	−1.31	100704	−690	−0.0268
100705	−413	1	0.0327	−1.32	100705	−691	−0.0269
100706	−414	1	0.0328	−1.32	100706	−692	−0.0269
100707	−413	1	0.0327	−1.32	100707	−691	−0.0269
100708	−412	1	0.0326	−1.31	100708	−692	−0.0269
100709	−411	1	0.0325	−1.31	100709	−691	−0.0269
100710	−408	1	0.0323	−1.30	100710	−690	−0.0268
100711	−409	1	0.0324	−1.30	100711	−691	−0.0269
100712	−408	1	0.0323	−1.30	100712	−690	−0.0268
100713	−409	1	0.0324	−1.30	100713	−691	−0.0269
100714	−410	1	0.0325	−1.31	100714	−692	−0.0269
100715	−409	1	0.0324	−1.30	100715	−691	−0.0269
100716	−410	1	0.0325	−1.31	100716	−692	−0.0269
100717	−411	1	0.0325	−1.31	100717	−693	−0.0270
100718	−410	1	0.0325	−1.31	100718	−692	−0.0269
100719	−419	1	0.0332	−1.33	100719	−693	−0.0270
100720	−416	1	0.0329	−1.33	100720	−694	−0.0270
100721	−415	1	0.0329	−1.32	100721	−693	−0.0270
100722	−416	1	0.0329	−1.33	100722	−694	−0.0270
100723	−415	1	0.0329	−1.32	100723	−693	−0.0270
100724	−418	1	0.0331	−1.33	100724	−694	−0.0270
100725	−421	1	0.0333	−1.34	100725	−693	−0.0270
100726	−420	1	0.0332	−1.34	100726	−692	−0.0269
100727	−419	1	0.0332	−1.33	100727	−691	−0.0269
100728	−413	1	0.0327	−1.32	100728	−692	−0.0269
100729	−412	1	0.0326	−1.31	100729	−691	−0.0269
100730	−411	1	0.0325	−1.31	100730	−690	−0.0268
100731	−410	1	0.0325	−1.31	100731	−689	−0.0268
100732	−411	1	0.0325	−1.31	100732	−688	−0.0268
100733	−412	1	0.0326	−1.31	100733	−689	−0.0268
100734	−411	1	0.0325	−1.31	100734	−688	−0.0268
100735	−410	1	0.0325	−1.31	100735	−687	−0.0267
100736	−406	1	0.0322	−1.29	100736	−686	−0.0267
100737	−404	1	0.0320	−1.29	100737	−685	−0.0267
100738	−403	1	0.0319	−1.28	100738	−684	−0.0266
100739	−402	1	0.0318	−1.28	100739	−683	−0.0266
100740	−397	1	0.0314	−1.26	100740	−684	−0.0266
100741	−398	1	0.0315	−1.27	100741	−685	−0.0267
100742	−399	1	0.0316	−1.27	100742	−686	−0.0267
100743	−398	1	0.0315	−1.27	100743	−685	−0.0267
100744	−395	1	0.0313	−1.26	100744	−686	−0.0267
100745	−394	1	0.0312	−1.25	100745	−685	−0.0267
100746	−396	1	0.0314	−1.26	100746	−684	−0.0266
100747	−397	1	0.0314	−1.26	100747	−685	−0.0267
100748	−396	1	0.0314	−1.26	100748	−686	−0.0267
100749	−395	1	0.0313	−1.26	100749	−685	−0.0267
100750	−392	1	0.0310	−1.25	100750	−684	−0.0266
100751	−393	1	0.0311	−1.25	100751	−685	−0.0267
100752	−390	1	0.0309	−1.24	100752	−686	−0.0267
100753	−389	1	0.0308	−1.24	100753	−685	−0.0267
100754	−388	1	0.0307	−1.24	100754	−684	−0.0266
100755	−386	1	0.0305	−1.23	100755	−685	−0.0266
100756	−387	1	0.0306	−1.23	100756	−684	−0.0266
100757	−386	1	0.0305	−1.23	100757	−683	−0.0266
100758	−384	1	0.0304	−1.22	100758	−682	−0.0265
100759	−383	1	0.0303	−1.22	100759	−681	−0.0265
100760	−381	1	0.0302	−1.21	100760	−680	−0.0264
100761	−380	1	0.0301	−1.21	100761	−679	−0.0264
100762	−381	1	0.0302	−1.21	100762	−680	−0.0264
100763	−382	1	0.0302	−1.22	100763	−681	−0.0265
100764	−389	1	0.0308	−1.24	100764	−682	−0.0265

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$
100765	−390	1	0.0309	−1.24	100765	−683	−0.0266
100766	−389	1	0.0308	−1.24	100766	−682	−0.0265
100767	−388	1	0.0307	−1.24	100767	−681	−0.0265
100768	−390	1	0.0309	−1.24	100768	−682	−0.0265
100769	−391	1	0.0309	−1.24	100769	−683	−0.0266
100770	−388	1	0.0307	−1.24	100770	−682	−0.0265
100771	−387	1	0.0306	−1.23	100771	−681	−0.0265
100772	−388	1	0.0307	−1.24	100772	−680	−0.0264
100773	−390	1	0.0309	−1.24	100773	−679	−0.0264
100774	−389	1	0.0308	−1.24	100774	−678	−0.0264
100775	−386	1	0.0305	−1.23	100775	−679	−0.0264
100776	−389	1	0.0308	−1.24	100776	−680	−0.0264
100777	−388	1	0.0307	−1.24	100777	−679	−0.0264
100778	−389	1	0.0308	−1.24	100778	−680	−0.0264
100779	−390	1	0.0309	−1.24	100779	−681	−0.0265
100780	−388	1	0.0307	−1.24	100780	−682	−0.0265
100781	−387	1	0.0306	−1.23	100781	−681	−0.0265
100782	−389	1	0.0308	−1.24	100782	−680	−0.0264
100783	−388	1	0.0307	−1.24	100783	−679	−0.0264
100784	−390	1	0.0309	−1.24	100784	−678	−0.0264
100785	−391	1	0.0309	−1.24	100785	−679	−0.0264
100786	−390	1	0.0309	−1.24	100786	−678	−0.0264
100787	−391	1	0.0309	−1.24	100787	−679	−0.0264
100788	−393	1	0.0311	−1.25	100788	−678	−0.0264
100789	−392	1	0.0310	−1.25	100789	−677	−0.0263
100790	−393	1	0.0311	−1.25	100790	−678	−0.0264
100791	−391	1	0.0309	−1.24	100791	−677	−0.0263
100792	−392	1	0.0310	−1.25	100792	−678	−0.0264
100793	−402	1	0.0318	−1.28	100793	−679	−0.0264
100794	−401	1	0.0317	−1.28	100794	−678	−0.0264
100795	−402	1	0.0318	−1.28	100795	−679	−0.0264
100796	−401	1	0.0317	−1.28	100796	−680	−0.0264
100797	−400	1	0.0317	−1.27	100797	−679	−0.0264
100798	−401	1	0.0317	−1.28	100798	−680	−0.0264
100799	−402	1	0.0318	−1.28	100799	−681	−0.0265
100800	−434	1	0.0343	−1.38	100800	−680	−0.0264
100801	−435	1	0.0344	−1.39	100801	−681	−0.0265
100802	−436	1	0.0345	−1.39	100802	−682	−0.0265
100803	−435	1	0.0344	−1.39	100803	−681	−0.0265
100804	−436	1	0.0345	−1.39	100804	−680	−0.0264
100805	−435	1	0.0344	−1.39	100805	−679	−0.0264
100806	−440	1	0.0348	−1.40	100806	−678	−0.0264
100807	−439	1	0.0347	−1.40	100807	−677	−0.0263
100808	−438	1	0.0347	−1.39	100808	−676	−0.0263
100809	−436	1	0.0345	−1.39	100809	−677	−0.0263
100810	−435	1	0.0344	−1.38	100810	−676	−0.0263
100811	−436	1	0.0345	−1.39	100811	−677	−0.0263
100812	−438	1	0.0347	−1.39	100812	−676	−0.0263
100813	−437	1	0.0346	−1.39	100813	−675	−0.0262
100814	−436	1	0.0345	−1.39	100814	−674	−0.0262
100815	−437	1	0.0346	−1.39	100815	−675	−0.0262
100816	−439	1	0.0347	−1.40	100816	−674	−0.0262
100817	−438	1	0.0347	−1.39	100817	−673	−0.0262
100818	−440	1	0.0348	−1.40	100818	−674	−0.0262
100819	−439	1	0.0347	−1.40	100819	−673	−0.0262
100820	−447	1	0.0354	−1.42	100820	−674	−0.0262
100821	−448	1	0.0354	−1.43	100821	−675	−0.0262
100822	−447	1	0.0354	−1.42	100822	−674	−0.0262
100823	−448	1	0.0354	−1.43	100823	−675	−0.0262
100824	−450	1	0.0356	−1.43	100824	−676	−0.0263
100825	−447	1	0.0354	−1.42	100825	−677	−0.0263
100826	−448	1	0.0354	−1.43	100826	−678	−0.0264
100827	−446	1	0.0353	−1.42	100827	−679	−0.0264
100828	−450	1	0.0356	−1.43	100828	−678	−0.0263
100829	−451	1	0.0357	−1.44	100829	−679	−0.0264
100830	−448	1	0.0354	−1.43	100830	−678	−0.0263
100831	−447	1	0.0354	−1.42	100831	−677	−0.0263
100832	−449	1	0.0355	−1.43	100832	−678	−0.0263
100833	−448	1	0.0354	−1.43	100833	−677	−0.0263
100834	−447	1	0.0354	−1.42	100834	−676	−0.0263
100835	−446	1	0.0353	−1.42	100835	−675	−0.0262
100836	−450	1	0.0356	−1.43	100836	−676	−0.0263
100837	−451	1	0.0357	−1.44	100837	−677	−0.0263
100838	−452	1	0.0358	−1.44	100838	−678	−0.0263
100839	−451	1	0.0357	−1.44	100839	−677	−0.0263

x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$	x	$L(x)$	$R_{\pm}(x)$	$\frac{L(x)}{L_{\approx,1}(x)}$	$\frac{L(x)}{L_{\approx,2}(x)}$	x	$L_*(x)$	$\frac{L_*(x)}{L_{\approx}^*(x)}$
100840	−453	1	0.0358	−1.44	100840	−678	−0.0263	100915	−463	1	0.0366	−1.47	100915	−679	−0.0264
100841	−452	1	0.0358	−1.44	100841	−677	−0.0263	100916	−464	1	0.0367	−1.48	100916	−678	−0.0263
100842	−459	1	0.0363	−1.46	100842	−678	−0.0263	100917	−466	1	0.0368	−1.48	100917	−677	−0.0263
100843	−458	1	0.0362	−1.46	100843	−677	−0.0263	100918	−465	1	0.0368	−1.48	100918	−676	−0.0262
100844	−457	1	0.0361	−1.45	100844	−678	−0.0263	100919	−466	1	0.0368	−1.48	100919	−677	−0.0263
100845	−460	1	0.0364	−1.46	100845	−679	−0.0264	100920	−474	1	0.0375	−1.51	100920	−676	−0.0262
100846	−459	1	0.0363	−1.46	100846	−678	−0.0263	100921	−473	1	0.0374	−1.51	100921	−675	−0.0262
100847	−460	1	0.0364	−1.46	100847	−679	−0.0264	100922	−472	1	0.0373	−1.50	100922	−674	−0.0262
100848	−462	1	0.0365	−1.47	100848	−678	−0.0263	100923	−471	1	0.0372	−1.50	100923	−673	−0.0261
100849	−461	1	0.0365	−1.47	100849	−677	−0.0263	100924	−470	1	0.0372	−1.50	100924	−674	−0.0262
100850	−458	1	0.0362	−1.46	100850	−678	−0.0263	100925	−467	1	0.0369	−1.49	100925	−675	−0.0262
100851	−457	1	0.0362	−1.45	100851	−677	−0.0263	100926	−471	1	0.0372	−1.50	100926	−674	−0.0262
100852	−456	1	0.0361	−1.45	100852	−678	−0.0263	100927	−472	1	0.0373	−1.50	100927	−675	−0.0262
100853	−457	1	0.0362	−1.45	100853	−679	−0.0264	100928	−468	1	0.0370	−1.49	100928	−676	−0.0262
100854	−459	1	0.0363	−1.46	100854	−678	−0.0263	100929	−469	1	0.0371	−1.49	100929	−677	−0.0263
100855	−460	1	0.0364	−1.46	100855	−679	−0.0264	100930	−470	1	0.0371	−1.50	100930	−678	−0.0263
100856	−459	1	0.0363	−1.46	100856	−680	−0.0264	100931	−471	1	0.0372	−1.50	100931	−679	−0.0264
100857	−458	1	0.0362	−1.46	100857	−679	−0.0264	100932	−471	1	0.0372	−1.50	100932	−678	−0.0263
100858	−459	1	0.0363	−1.46	100858	−680	−0.0264	100933	−470	1	0.0371	−1.50	100933	−677	−0.0263
100859	−460	1	0.0364	−1.46	100859	−681	−0.0265	100934	−471	1	0.0372	−1.50	100934	−678	−0.0263
100860	−448	1	0.0354	−1.43	100860	−680	−0.0264	100935	−469	1	0.0371	−1.49	100935	−679	−0.0264
100861	−450	1	0.0356	−1.43	100861	−679	−0.0264	100936	−468	1	0.0370	−1.49	100936	−678	−0.0263
100862	−449	1	0.0355	−1.43	100862	−678	−0.0263	100937	−469	1	0.0371	−1.49	100937	−679	−0.0264
100863	−447	1	0.0354	−1.42	100863	−679	−0.0264	100938	−470	1	0.0371	−1.50	100938	−680	−0.0264
100864	−443	1	0.0350	−1.41	100864	−678	−0.0263	100939	−469	1	0.0371	−1.49	100939	−679	−0.0264
100865	−442	1	0.0350	−1.41	100865	−677	−0.0263	100940	−463	1	0.0366	−1.47	100940	−678	−0.0263
100866	−443	1	0.0350	−1.41	100866	−678	−0.0263	100941	−462	1	0.0365	−1.47	100941	−677	−0.0263
100867	−442	1	0.0350	−1.41	100867	−677	−0.0263	100942	−463	1	0.0366	−1.47	100942	−678	−0.0263
100868	−441	1	0.0349	−1.40	100868	−678	−0.0263	100943	−464	1	0.0367	−1.48	100943	−679	−0.0264
100869	−440	1	0.0348	−1.40	100869	−677	−0.0263	100944	−473	1	0.0374	−1.51	100944	−680	−0.0264
100870	−441	1	0.0349	−1.40	100870	−678	−0.0263	100945	−474	1	0.0375	−1.51	100945	−681	−0.0264
100871	−440	1	0.0348	−1.40	100871	−677	−0.0263	100946	−475	1	0.0375	−1.51	100946	−682	−0.0265
100872	−446	1	0.0353	−1.42	100872	−678	−0.0263	100947	−476	1	0.0376	−1.51	100947	−683	−0.0265
100873	−445	1	0.0352	−1.42	100873	−677	−0.0263	100948	−477	1	0.0377	−1.52	100948	−682	−0.0265
100874	−446	1	0.0353	−1.42	100874	−678	−0.0263	100949	−482	1	0.0381	−1.53	100949	−681	−0.0264
100875	−449	1	0.0355	−1.43	100875	−679	−0.0264	100950	−487	1	0.0385	−1.55	100950	−680	−0.0264
100876	−450	1	0.0356	−1.43	100876	−678	−0.0263	100951	−486	1	0.0384	−1.55	100951	−679	−0.0264
100877	−449	1	0.0355	−1.43	100877	−677	−0.0263	100952	−485	1	0.0383	−1.54	100952	−678	−0.0263
100878	−450	1	0.0356	−1.43	100878	−678	−0.0263	100953	−483	1	0.0382	−1.54	100953	−677	−0.0263
100879	−449	1	0.0355	−1.43	100879	−677	−0.0263	100954	−484	1	0.0383	−1.54	100954	−678	−0.0263
100880	−452	1	0.0358	−1.44	100880	−676	−0.0263	100955	−485	1	0.0383	−1.54	100955	−679	−0.0264
100881	−450	1	0.0356	−1.43	100881	−677	−0.0263	100956	−487	1	0.0385	−1.55	100956	−678	−0.0263
100882	−449	1	0.0355	−1.43	100882	−676	−0.0263	100957	−488	1	0.0386	−1.55	100957	−679	−0.0264
100883	−448	1	0.0354	−1.43	100883	−675	−0.0262	100958	−487	1	0.0385	−1.55	100958	−678	−0.0263
100884	−451	1	0.0357	−1.44	100884	−674	−0.0262	100959	−488	1	0.0386	−1.55	100959	−679	−0.0264
100885	−450	1	0.0356	−1.43	100885	−673	−0.0261	100960	−491	1	0.0388	−1.56	100960	−680	−0.0264
100886	−451	1	0.0357	−1.44	100886	−674	−0.0262	100961	−490	1	0.0387	−1.56	100961	−679	−0.0264
100887	−450	1	0.0356	−1.43	100887	−673	−0.0261	100962	−492	1	0.0389	−1.56	100962	−678	−0.0263
100888	−449	1	0.0355	−1.43	100888	−672	−0.0261	100963	−491	1	0.0388	−1.56	100963	−677	−0.0263
100889	−448	1	0.0354	−1.43	100889	−671	−0.0261	100964	−490	1	0.0387	−1.56	100964	−678	−0.0263
100890	−447	1	0.0354	−1.42	100890	−672	−0.0261	100965	−489	1	0.0386	−1.56	100965	−677	−0.0263
100891	−444	1	0.0351	−1.41	100891	−673	−0.0261	100966	−490	1	0.0387	−1.56	100966	−678	−0.0263
100892	−443	1	0.0350	−1.41	100892	−674	−0.0262	100967	−489	1	0.0386	−1.56	100967	−677	−0.0263
100893	−439	1	0.0347	−1.40	100893	−675	−0.0262	100968	−488	1	0.0386	−1.55	100968	−676	−0.0262
100894	−440	1	0.0348	−1.40	100894	−676	−0.0263	100969	−489	1	0.0386	−1.56	100969	−677	−0.0263
100895	−441	1	0.0349	−1.40	100895	−677	−0.0263	100970	−488	1	0.0386	−1.55	100970	−676	−0.0262
100896	−444	1	0.0351	−1.41	100896	−678	−0.0263	100971	−486	1	0.0384	−1.55	100971	−677	−0.0263
100897	−443	1	0.0350	−1.41	100897	−677	−0.0263	100972	−487	1	0.0385	−1.55	100972	−676	−0.0262
100898	−444	1	0.0351	−1.41	100898	−678	−0.0263	100973	−486	1	0.0384	−1.55	100973	−675	−0.0262
100899	−446	1	0.0353	−1.42	100899	−679	−0.0264	100974	−487	1	0.0385	−1.55	100974	−676	−0.0262
100900	−450	1	0.0356	−1.43	100900	−680	−0.0264	100975	−484	1	0.0382	−1.54	100975	−677	−0.0263
100901	−451	1	0.0357	−1.44	100901	−681	−0.0265	100976	−486	1	0.0384	−1.55	100976	−676	−0.0262
100902	−450	1	0.0356	−1.43	100902	−680	−0.0264	100977	−487	1	0.0385	−1.55	100977	−677	−0.0263
100903	−449	1	0.0355	−1.43	100903	−679	−0.0264	100978	−488	1	0.0386	−1.55	100978	−678	−0.0263
100904	−448	1	0.0354	−1.43	100904	−678	−0.0263	100979	−487	1	0.0385	−1.55	100979	−677	−0.0263
100905	−454	1	0.0359	−1.44	100905	−677	−0.0263	100980	−495	1	0.0391	−1.57	100980	−678	−0.0263
100906	−455	1	0.0360	−1.45	100906	−678	−0.0263	100981	−496	1	0.0392	−1.58	100981	−679	−0.0263
100907	−456	1	0.0361	−1.45	100907	−679	−0.0264	100982	−497	1	0.0393	−1.58	100982	−680	−0.0264
100908	−460	1	0.0364	−1.46	100908	−680	−0.0264	100983	−498	1	0.0394	−1.58	100983	−681	−0.0264
100909	−461	1	0.0365	−1.47	100909	−681	−0.0264	100984	−499	1	0.0394	−1.59	100984	−682	−0.0265
100910	−462	1	0.0365	−1.47	100910	−682	−0.0265	100985	−500	1	0.0395	−1.59	100985	−683	−0.0265
100911	−461	1	0.0365	−1.47	100911	−681	−0.0264	100986	−501	1	0.0396	−1.59	100986	−684	−0.0265
100912	−461	1	0.0365	−1.47	100912	−680	−0.0264	100987	−502	1	0.0397	−1.60	100987	−685	−0.0266
100913	−462	1	0.0365	−1.47	100913	−681	−0.0264	100988	−503	1	0.0397	−1.60	100988	−684	−0.0265
100914	−464	1	0.0367	−1.48	100914	−680	−0.0264	100989	−510	1	0.0403	−1.62	100989	−685	−0.0266