Lower bounds on the Mertens function $M(x)$ along infinite subsequences for large $x \gg 2.3315 \times 10^{1656520}$

Maxie Dion Schmidt

Georgia Institute of Technology

School of Mathematics

Last Revised: Friday 22nd May, 2020 – Compiled with LaTeX2e

Abstract

The Mertens function, $M(x) = \sum_{n \leqslant x} \mu(n)$, is classically defined to be the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture stating that $|M(x)| < C \cdot \sqrt{x}$ with $C > 0$ for all $x \geqslant 1$ has a well-known disproof due to Odlyzko and té Riele given in the early 1980's by computation of non-trivial zeta function zeros in conjunction with integral formulas for expressions of $M(x)$. It is conjectured and widely believed that $M(x)/\sqrt{x}$ changes sign infinitely often and grows unbounded in the direction of both $\pm\infty$ along subsequences of integers $x \geqslant 1$. Our proof of a result close to this property of $M(x)/\sqrt{x}$, e.g., showing that

$$\limsup_{x \to \infty} \frac{|M(x)| \log x}{\sqrt{x}} = +\infty,$$

is not based on standard estimates of $M(x)$ by Mellin inversion, which are intimately tied to the intricate distribution of the non-trivial zeros of the Riemann zeta function. There is a distinct stylistic flavor and new element of combinatorial analysis peppered in with the standard methods from analytic and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

Keywords and Phrases: *Möbius function sums; Mertens function; summatory function; arithmetic functions; Dirichlet inverse; Liouville lambda function; prime omega functions; prime counting functions; Dirichlet series and DGFs; asymptotic lower bounds; Mertens conjecture.*

Primary Math Subject Classifications (2010): *11N37; 11A25; 11N60; 11N64; and 11-04 (TBD).*

Reference on special notation and other conventions

Symbol	Definition
$\mathbb{E}[f(x)]$	We use the expectation notation $\mathbb{E}[f(x)] = h(x)$ to denote that f has a so-called average order growth rate of $h(x)$. What this means is that $\frac{1}{x}\sum_{n \leqslant x} f(n) \sim h(x)$, or equivalently that $$\lim_{x\to\infty} \frac{\frac{1}{x}\sum_{n\leqslant x} f(n)}{h(x)} = 1.$$
$o(f), O_\alpha(g)$	Using standard notation, we write that $f = o(g)$ if $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$ We adapt the stock big-O notation, writing $f = O_{\alpha_1,\ldots,\alpha_k}(g)$ for some parameters $\alpha_1, \ldots, \alpha_k$ that do not depend on x, if $f(x) = O(g(x))$ subject only to the upper bounds having an implicit dependence only on x (as usual) and the α_i.
$C_k(n)$	Auxillary component functions in obtaining asymptotic bounds on $g^{-1}(n)$. These functions are defined for $k \geqslant 0$ and $n \geqslant 1$ by the formula $$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d\mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geqslant 1. \end{cases}$$
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n\geqslant 0}$.
DGF	*Dirichlet generating function.* Given a sequence $\{f(n)\}_{n\geqslant 0}$, its DGF is given by $D_f(s) := \sum_{n\geqslant 1} f(n)/n^s$ subject to suitable constraints on the real part of the parameter $s \in \mathbb{C}$.
$\sigma_0(n), d(n)$	The ordinary divisor function, $d(n) := \sum_{d\mid n} 1$. The arithmetic functions $d(n) \equiv \sigma_0(n)$ for all $n \geqslant 1$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) = \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (defined below).
$f * g$	The Dirichlet convolution of f and g, $(f * g)(n) := \sum\limits_{d\mid n} f(d)g(n/d)$, where the sum is taken over the divisors d of n for $n \geqslant 1$.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ for $n \geqslant 1$ with $f^{-1}(1) = 1/f(1)$ and exists if and only if $f(1) \neq 0$. The inverse function, when it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$.
$\lfloor x \rfloor, \lceil x \rceil$	The floor function is defined as $\lfloor x \rfloor := x - \{x\}$ where $0 \leqslant \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$. The corresponding ceiling function $\lceil x \rceil$ denotes the smallest integer $m \geqslant x$. The floor function is sometimes also written as $[x] \equiv \lfloor x \rfloor$.

Symbol	Definition
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega+1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$.
$\mathbb{1}_{\mathbb{S}}$, $\chi_{\mathtt{cond}(x)}$	We use the notation $\mathbb{1}, \chi : \mathbb{N} \to \{0,1\}$ to denote indicator, or characteristic functions. In paticular, $\mathbb{1}_{\mathbb{S}}(n) = 1$ if and only if $n \in \mathbb{S}$, and $\chi_{\mathtt{cond}}(n) = 1$ if and only if n satisfies the boolean-valued condition `cond`.
$\log_*^m(x)$	The iterated logarithm function defined recursively for integers $m \geqslant 0$ and any $x > 0$ taken so that the function is non-negative (e.g., with $x \geqslant e^e$ if $m = 2$, $x \geqslant e^{e^e}$ if $m = 3$, and so on) by $$\log_*^m(x) := \begin{cases} x, & \text{if } m = 0; \\ \log x, & \text{if } m = 1; \\ \log\left[\log_*^{m-1}(x)\right], & \text{if } m \geqslant 2. \end{cases}$$
$[n = k]_\delta$	Synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise.
$[\mathtt{cond}]_\delta$	For a boolean-valued conditions, `cond`, $[\mathtt{cond}]_\delta$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is sometimes called *Iverson's convention*.
$\lambda(n)$	The Liouville lambda function, $\lambda(n) := (-1)^{\Omega(n)}$, denotes the parity of $\Omega(n)$, the number of distinct prime factors of n counting their multiplicity. That is, $\lambda(n) \in \{\pm 1\}$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \bmod 2$.
$\mu(n)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree, i.e., has no prime power divisors with exponent greater than one.
$M(x)$	The Mertens function is the summatory function over $\mu(n)$ defined for all integers $x \geqslant 1$ by $M(x) := \sum\limits_{n \leqslant x} \mu(n)$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^\alpha \vert\vert n$ (p^α exactly divides n) for p prime and $n \geqslant 2$.
$\omega(n)$,$\Omega(n)$	We define these distinct prime factor counting functions as the strongly additive function $\omega(n) := \sum_{p \vert n} 1$ and the completely additive function $\Omega(n)$ by $\Omega(n) := \sum_{p^\alpha \vert\vert n} \alpha$. Equivalently, if the factorization of $n \geqslant 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leqslant n \leqslant x$ for $x > 1$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leqslant x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{n \leqslant x : \Omega(n) = k\}$ for $x \geqslant 2$. Montgomery and Vaughan use the alternate notation of $\sigma_k(x)$, which we intentionally avoid due to conflicting notation with other special arithmetic functions used in this article, in place of $\widehat{\pi}_k(x)$.
$\sum_{p \leqslant x}, \prod_{p \leqslant x}$	Unless otherwise specified by context, we use the index variable p to denote that the summation (product) is to be taken only over prime values within the summation bounds.
$P(s)$	For complex s with $\Re(s) > 1$, we define the prime zeta function to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$.

Symbol	**Definition**
$\sigma_\alpha(n)$	The generalized sum-of-divisors function, $\sigma_\alpha(n) := \sum\limits_{d\mid n} d^\alpha$, is defined for any $n \geqslant 1$ and $\alpha \in \mathbb{R}$.
$\left[{n \atop k}\right]$	The unsigned Stirling numbers of the first kind, $\left[{n \atop k}\right] = (-1)^{n-k} \cdot s(n,k)$.
$\sim, \approx, \gtrsim, \lesssim, \gg, \ll$	See the first section of the introduction to the article for clarification of the asymptotic notation we employ in the article including precise definitions of our usage of these limiting asymptotic relation symbols.
$\zeta(s)$	The Riemann zeta function, defined by $\zeta(s) := \sum_{n\geqslant 1} n^{-s}$ when $\Re(s) > 1$, and by analytic continuation to the entire complex plane with the exception of a simple pole at $s = 1$.

1 Preface: Explanations of unconventional notions and preconceptions of asymptotics and notation for asymptotic relations

We exphasize that the next itemized careful explanation of the subtle distinctions to our usage of what we consider to be traditional notation for asymptotic relations are key to understanding our choices of upper and lower bound expressions given throughout the article. Thus, to avoid any confusion that may linger as we begin to state our new results and bounds on the functions we work with in this article, we preface the article starting with this section detailing our precise definitions, meanings and assumptions on the uses of certain symbols, operators, and relations. The interpretation of this notation forms the core of how we choose to convey the growth rates of arithmetic functions on their domain of x within this article when x is taken to be very large as $x \to \infty$ [13, *cf.* §2] [3].

1.1 Average order, similarity and approximation of asymptotic growth rates of quantities

1.1.1 Similarity and average order (expectation)

We say that two functions $A(x), B(x)$ satisfy the relation $A \sim B$ if

$$\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1.$$

It is sometimes standard to express the *average order* of an arithmetic function f as $f \sim h$, even when the values of $f(n)$ may actually non-monotonically oscillate in magnitude infinitely often. What the notation $f \sim h$ means when expressing the average order of f is that $\frac{1}{x} \cdot \sum_{n\leqslant x} f(n) \sim h(x)$.

For example, in the acceptably classic language of [6] we would normally write that $\Omega(n) \sim \log\log n$, even though technically, $1 \leqslant \Omega(n) \leqslant \frac{\log n}{\log 2}$. To be absolutely clear about notation, we intentionally do not re-use the $\sim$ relation by instead writing $\mathbb{E}[f(x)] = h(x)$ (as in expectation of f) to denote that f has a limiting average order growing at the rate of h.

A related conception of f having a so-called *normal order* of g holds whenever

$$f(n) = (1 + o(1))g(n), \text{a.e.}$$

1.1.2 Approximation

We choose to adpot the convention to write that $f(x) \approx g(x)$ if $|f(x) - g(x)| = O(1)$. That is, we write $f(x) \approx g(x)$ to denote that f is approximately equal to g at x modulo at most a small constant difference between the functions.

The formula we prefer for the Abel summation variant of summation by parts to express finite sums of a product of two functions is stated as follows [1, *cf.* §4.3] *:

Proposition 1.1 (Abel Summation Integral Formula)**.** *Suppose that $t > 0$ is real-valued, and that $A(t) \sim \sum_{n\leqslant t} a(n)$ for some weighting arithmetic function $a(n)$ with $A(t)$ continuously differentiable on $(0, \infty)$. Furthermore, suppose that $b(n) \sim f(n)$ with f a differentiable function of $n \geqslant 0$ – that is, $f'(t)$ exists and is smooth for all $t \in (0, \infty)$. Then for $0 \leqslant y < x$, where we typcially assume that the bounds of summation satisfy $x, y \in \mathbb{Z}^+$, we have that*

$$\sum_{y<n\leqslant x} a(n)b(n) \sim A(x)b(x) - A(y)b(y) - \int_y^x A(t)f'(t)dt.$$

*Compare to the exact formula for *summation by parts* of any arithmetic functions, u_n, v_n, stated as in [13, §2.10(ii)] for $U_j := u_1 + u_2 + \cdots + u_j$ when $j \geqslant 1$:

$$\sum_{j=1}^{n-1} u_j \cdot v_j = U_{n-1}v_n + \sum_{j=1}^{n-1} U_j\left(v_j - v_{j+1}\right), n \geqslant 2.$$

1.1.3 Vinogradov's notation for asymptotics

We use the conventional relations $f(x) \gg g(x)$ and $h(x) \ll r(x)$ to symbolically express that we should expect f to be "substantially" larger than g, and h to be "significantly" smaller than r, in asymptotic order (e.g., rate of growth when x is large). In practice, we adopt a somewhat looser definition of these symbols which allows $f \gg g$ and $h \ll r$ provided that there are constants $C, D > 0$ such that whenever x is sufficiently large we have that $f(x) \geqslant C \cdot g(x)$ and $h(x) \leqslant D \cdot r(x)$. This notation is sometimes called *Vinogradov's asymptotic notation*.

Another way of expressing our precise meaning of these relations is by writing

$$f \gg g \iff g = O(f),$$

and

$$h \ll r \iff r = \Omega(h),$$

using Knuth's well-trodden style of big-O (and Landau notation) and big-Ω (Hardy-Littlewood notation) symbols from the language of theoretical computer science and in the analysis of algorithms.

1.2 An unconventional pair of asymptotic relations employed to drop lower-order terms in upper and lower bounds on arithmetic functions

We define two new defintions of relations for expressing limiting asymptotic bounds on functions by adapting notation for existing operators for clarity of the way we use them here. Namely, we say that $h(x) \overset{\blacktriangle}{\succsim} r(x)$ if $h \gg r$ as $x \rightarrow \infty$, and define the relation $\overset{\blacktriangle}{\precsim}$ similarly as $h(x) \overset{\blacktriangle}{\precsim} r(x)$ if $h \ll r$ as $x \rightarrow \infty$. This usage of the notation of $\overset{\blacktriangle}{\succsim}, \overset{\blacktriangle}{\precsim}$ intentionally breaks with the usual conventions for the use of the relations $\succsim, \precsim$. Our intentional distinct usage of these new relations is intended to simplify notation for limiting upper and lower bounds that are valid as $x \rightarrow \infty$.

The use of the new (modified) notation for $\overset{\blacktriangle}{\succsim}$ is intended to capture both that we are conveying a lower (upper) bound for the function, and crucially that this lower bound is valid only when x is very large, i.e., in some sense that the lower bound holds in the same sense as the relation $\sim$ for equality. This is a subtle distinction that comes into play when we use it later to state lower bounds within our new results.

An intentionally mock example motivating this usage of these relations thats clarifies the point of this new use of notation appears below:

Example 1.2. Suppose that exactly for all $x \geqslant 1$ we have

$$f(x) \geqslant -(\log\log\log x)^2 + 3 \times 10^{1000000} \cdot (\log\log\log x)^{1.999999999} + E(x),$$

where $E(x) = o\left((\log\log\log x)^2\right)$ and there is a complicated expression for $E(x)$ that requires more than 100000 ascii characters to typeset accurately, e.g., is too exceedingly complicated to write down and include as a component of our expression for the terms in the primary bound. Then naturally, we prefer to work with only the expression for the asymptotically dominant main term in the lower bounds on $f(x)$ stated above.

Note that in this case the main term contribution does not dominate the bound until x is very large, so that replacing the right-hand-side expression with just this term yields an invalid inequality except for in limiting cases. In this instance, we prefer to write

$$f(x) \overset{\blacktriangle}{\succsim} -(\log\log\log x)^2, \text{ as } x \rightarrow \infty,$$

or more conventionally applying this notation only to unsigned functions by writing

$$|f(x)| \overset{\blacktriangle}{\succsim} (\log\log\log x)^2, \text{ as } x \rightarrow \infty.$$

It is problematic to only write a lower bound expression for $f(x)$ that states

$$f(x) \geqslant -(\log\log\log x)^2.$$

The problem with the bound as stated in the previous equation is that there is a substantial (however, asymptotically negligible) initial range of $x \geqslant 1$ where the given lower bound is invalid.

Remark 1.3 (Emphasizing the rationale of the use of the new notation)**.** We emphasize that our new uses of the traditional symbols as asymptotic relations are defined to simplify our results by dropping expressions involving more precise, exact terms that are nonetheless asymptotically insignificant. Instead, we choose to express upper and lower bounds in x that form accurate statements in limiting cases as $x \to \infty$. This convention allows us to write out simplified bounds that still capture the most simple essence of the upper or lower bound as we choose to view it in this article without regard to technical corner cases of initial intervals of x where the bound is invalid.

1.3 Asymptotic expansions and uniformity

Because a subset of the results we cite that are proved in the references provide statements of asymptotic bounds that hold *uniformly* for x large depending on parameters, we need to briefly make precise what our preconceptions are of this terminology. We introduce the notation for asymptotic expansions of a function $f : \mathbb{R} \to \mathbb{R}$ from [13, §2.1(iii)].

1.3.1 Ordinary asymptotic expansions of a function

Let $\sum_n a_n x^{-n}$ denote a formal power series expansion in x where we ignore any necessary conditions to guarantee convergence of the series. For each integer $n \geqslant 1$, suppose that

$$f(x) = \sum_{s=0}^{n-1} a_s x^{-s} + O(x^{-n}),$$

as $|x| \to \infty$ where this limiting bound holds for $x \in \mathbb{X}$ in some unbounded set $\mathbb{X} \subseteq \mathbb{R}, \mathbb{C}$. When such a bound holds, we say that $\sum_s a_s x^{-s}$ is a *Poincaré asymptotic expansion*, or just *asymptotic series expansion*, of $f(x)$ as $x \to \infty$ along the fixed set $\mathbb{X}$. The condition in the previous equation is equivalent to writing

$$f(x) \sim a_0 + a_1 x^{-1} + a_2 x^{-2} + \cdots ; x \in \mathbb{X}, \text{ for } |x| \to \infty.$$

The prior two characterizations of an asymptotic expansion for f are also equivalent to the statement that

$$x^n \left(f(x) - \sum_{s=0}^{n-1} a_s x^{-s}\right) \xrightarrow{x\to\infty} a_n.$$

1.3.2 Uniform asymptotic expansions of a function

Let the set $\mathbb{X}$ from the definition in the last subsection correspond to a closed sector of the form

$$\mathbb{X} := \{x \in \mathbb{C} : \alpha \leqslant \arg(x) \leqslant \beta\}.$$

Then we say that the asymptotic property

$$f(x) = \sum_{s=0}^{n-1} a_s x^{-s} + O(x^{-n}),$$

from before holds *uniformly* with respect to $\arg(x) \in [\alpha, \beta]$ as $|x| \to \infty$.

Another useful, important notion of uniform asymptotic bounds is taken with respect to some parameter u (or set of parameters, respectively) that ranges over the point set (point sets, respectively) $u \in \mathbb{U}$. In this case, if we have that the u-parameterized expressions

$$\left| x^n \left(f(u, x) - \sum_{s=0}^{n-1} a_s(u) x^{-s} \right) \right|,$$

are bounded for all integers $n \geqslant 1$ for $x \in \mathbb{X}$ as $|x| \to \infty$, then we say that the asymptotic expansion of f holds *uniformly* for $u \in \mathbb{U}$. Now the function $f \equiv f(u, x)$ and the asymptotic series coefficients $a_s(u)$ may have an implicit dependence on the parameter u. If the previous boundedness condition holds for all positive integers n, we write that

$$f(u, x) \sim \sum_{s=0}^{\infty} a_s(u) x^{-s}; x \in \mathbb{X}, \text{ as } |x| \to \infty,$$

and say that this asymptotic expansion, or bound, holds *uniformly with respect to* $u \in \mathbb{U}$. For u taken outside of $\mathbb{U}$, the stated bound may fail to be valid even for $x \in \mathbb{X}$ as $|x| \to \infty$.

2 An introduction to the Mertens function

Suppose that $n \geqslant 1$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. We define the *Möebius function* to be the signed indicator function of the squarefree integers:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leqslant i \leqslant k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many known variants and special properties of the Möebius function and its generalizations [15, *cf.* §2]. For our purposes we seek to explore the properties and asymptotics of weighted summatory functions over $\mu(n)$. The Mertens summatory function, or *Mertens function*, is defined as [16, A002321]

$$\begin{aligned} M(x) &= \sum_{n \leqslant x} \mu(n), \ x \geqslant 1, \\ &\longmapsto \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2, \ldots\} \end{aligned}$$

A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as [16, A013928]

$$Q(n) = \sum_{n \leqslant x} \mu^2(n) \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \rightarrow \infty$:

$$\mu_+(x) = \frac{\#\{1 \leqslant n \leqslant x : \mu(n) = +1\}}{Q(x)} \sim \mu_-(x) = \frac{\#\{1 \leqslant n \leqslant x : \mu(n) = -1\}}{Q(x)} \xrightarrow[x \rightarrow \infty]{} \frac{3}{\pi^2}.$$

While this limiting law suggests an even bias for the Mertens function, in practice $M(x)$ has an apparent unproven negative bias in its values. Moroever, the actual local oscillations between the approximate densities of the sets $\mu_\pm(x)$ lend an unpredictable nature to the function and characterize the function's oscillatory sawtooth shaped plot when viewed over the positive integers.

2.1 Properties

The conventional approach to evaluating the behavior of $M(x)$ for large $x \rightarrow \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is expressed given the inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since

$$\frac{1}{\zeta(s)} = \int_1^\infty \frac{s \cdot M(x)}{x^{s+1}} dx,$$

we obtain that

$$M(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation along with the standard Euler product representation for the reciprocal zeta function leads us to the exact expression for $M(x)$ when $x > 0$ given by the next theorem.

Theorem 2.1 (Analytic Formula for $M(x)$)**.** *Assuming the Riemann Hypothesis (RH), we can show that there exists an infinite sequence $\{T_k\}_{k \geqslant 1}$ satisfying $k \leqslant T_k \leqslant k+1$ for each k such that for any real $x > 0$*

$$M(x) = \lim_{k \rightarrow \infty} \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |\Im(\rho)| < T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geqslant 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (1963) states that there is an absolute constant $C > 0$ such that
$$M(x) \ll x \cdot \exp\left(-C \cdot \log^{3/5}(x)(\log\log x)^{-3/5}\right).$$
Under the assumption of the RH, Soundararajan proved new updated estimates in 2009 bounding $M(x)$ for large x in the following forms [17]:
$$\begin{aligned} M(x) &\ll \sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x} \cdot \exp\left(\log^{1/2}(x)(\log\log x)^{5/2+\epsilon}\right)\right), \ \forall \epsilon > 0. \end{aligned}$$
To date, considerably less has been conjectured about explicit lower bounds on $|M(x)|$ along subsequences.

2.2 Conjectures

The RH is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. It is still unresolved whether
$$\limsup_{x\to\infty} |M(x)|/\sqrt{x} = \infty,$$
although computational evidence suggests that this is a likely conjecture [10, 7]. There is a rich history to the original statement of the *Mertens conjecture* which states that
$$|M(x)| < c \cdot x^{1/2}, \quad \text{some absolute constant } c > 0.$$
Mertens conjecture was first verified by Mertens for $c = 1$ and $x < 10000$. Since its beginnings in 1897, the conjecture has been disproven by computation of low-lying zeta function zeros in a famous paper by Odlyzko and té Riele from the early 1980's. Since the truth of the Mertens conjecture would have implied the RH, more recent attempts at bounding $M(x)$ favor determining the rate at which the function $M(x)/\sqrt{x}$ grows without bound towards both $\pm\infty$ along infinite subsequences.

One of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ is in actuality unbounded on the natural numbers. A precise statement of this problem is to produce an affirmative answer whether $\limsup_{x\to\infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x\to\infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there is an infinite sequence of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound along the subsequence. Currently, an exact rigorous proof that $M(x)/\sqrt{x}$ is unbounded still remains elusive, though there is suggestive probabilistic evidence of this property established by Ng in 2008. We cite that prior to this point it is known that [14, *cf.* §4.1]
$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } 1.826054),$$
and
$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } -1.837625),$$
although based on work by Odlyzyko and té Riele it seems probable that each of these limits should be $\pm\infty$, respectively [12, 9, 10, 7].

Extensive computational evidence has produced a conjecture due to Gonek (among attempts on limiting bounds by others) that in fact the limiting behavior of $M(x)$ satisfies that
$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}(\log\log x)^{5/4}} = O(1).$$
While it seems to be widely believed that $|M(x)|/\sqrt{x}$ tends to $+\infty$ at a logarithmic rate along subsequences, infinitely tending factors such as the $(\log\log x)^{\frac{5}{4}}$ in Gonek's conjecture do not appear to readily fall out of work on bounds for $M(x)$ by existing methods.

3 A summary outline: Listing the core logical steps and critical components to the proof

3.1 Step-by-step overview

We offer another brief step-by-step summary overview of the critical components to our proof outlined in the next section of the introduction below that are proved piece-by-piece in the following sections of the article. This outline is provided to help the reader see our logic and proof methodology as easily and quickly as possible. As the proof methodology is new and relies on non-standard elements compared to more traditional methods of bounding $M(x)$, we hope that this sketch of the logical components to our new argument makes the article easier to parse.

(1) We prove a matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). See Theorem 4.1 in Section 5.

(2) This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the seemingly unconnected prime counting function, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is stated in (1).

(3) We tighten an updated result from [11, §7] providing summatory functions that indicate the parity of $\lambda(n)$ using elementary arguments and offer more combinatorially flavored expansions of Dirichlet series (see Theorem 4.8). We use this result to sum $\sum_{n \leqslant x} \lambda(n) f(n)$ for particular non-negative arithmetic functions f when x is large.

(4) We then turn to the average order asymptotics of the quasi-periodic $g^{-1}(n)$, estimating this inverse function's limiting asymptotics for large $n \leqslant x$ as $x \to \infty$ in Section 7. We eventually use these estimates to prove a substantially unique new lower bound formula for the summatory function $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$ along prescribed asymptotically large infinite subsequences (see Theorem 9.5).

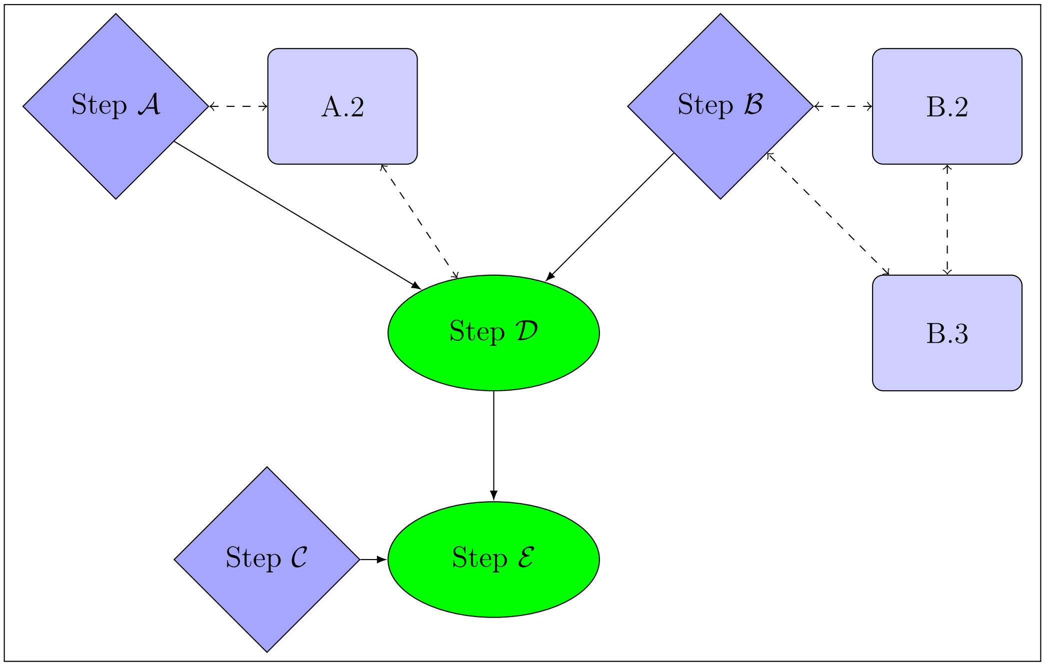
(5) We spend some interim time in Section 8 carefully working out a rigorous justification for why the limiting lower bounds we obtain from average order case analysis of certain arithmetic function approximations we define are sufficient to prove the limit supremum corollary below (our primary new significant result established the article).

(6) When we return to step **(2)** with our new lower bounds at hand, and bootstrap, we find "magic" in the form of showing the unboundedness of $\frac{|M(x)| \log x}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. What we recover is a quick, and rigorous, proof of Corollary 4.12 given in Section 9.2.

3.2 Diagramatic flowchart of the proof logic with references to results

Flowchart schematic diagram:

The next flowchart diagramed below shows how the seemingly disparate components of the proof are organized. It also indicates how the separate initial "lands" of material and corresponding sets of requisite results forming the connected components to steps $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ (as viewed below) combine to form the next core stages of the proof.



Key to the diagram stages:

Step A: *Citations and re-statements of existing theorems proved elsewhere*: E.g., statements of non-trivial theorems and key results we need that are proved in the references.

A.A Key results and constructions:
- Theorem 4.7
- Theorem 6.1
- Corollary 6.4
- The results, lemmas, and facts cited in Section 5.3

A.2 Lower bounds on the Abel summation based formula for $G^{-1}(x)$:
- Theorem 4.8 (on page 25)
- Proposition 6.5
- Theorem 9.5
- Lemma 9.3
- Lemma 9.4

Step B: *Constructions of an exact formula for $M(x)$*: The exact formula we prove uses special arithmetic functions with particularly "nice" properties and bounds. This choice of the expression from Theorem 4.1 is key to how far we have traveled along the new approaches in this article. In particular, the additivity of $\omega(n)$ and the easily integrable logarithmically weighted bound on $\pi(x)$ for large x are indispensible components to why this proof works well.

B.B Key results and constructions:
- Corollary 4.3 (follows from Theorem 4.1 proved on page 19)
- Conjecture 4.4 (to a lesser expository only extent)
- Proposition 5.1

B.2 Asymptotics for the component functions $g^{-1}(n)$ and $G^{-1}(x)$:
- Theorem 4.6 (on page 29)
- Lemma 7.1

B.3 Simplifying formulas for $g^{-1}(n)$ and $G^{-1}(x)$:
- Corollary 9.2

Step C: *A justification for why lower bounds holding on average suffice*:
- Theorem 4.9 (proved on page 31)

Step D: *Re-writing the exact formula for $M(x)$*: Key interpretations used in formulating the lower bounds based on the re-phrased integral formula.
- Proposition 9.1

Step E: *The Holy Grail*: A big leap twoards proving that $\frac{|M(x)| \log x}{\sqrt{x}}$ is unbounded in the limit supremum sense.
– Corollary 4.12 (on page 43)

4 An introduction to our new methodology: A concrete approach to bounding $M(x)$ from below

4.1 Summing series over Dirichlet convolutions

Theorem 4.1 (Summatory functions of Dirichlet convolutions)**.** *Let $f, g : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leqslant x} f(n)$ and $H(x) := \sum_{n \leqslant x} h(n)$ denote the summatory functions of f, g, respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse $f^{-1}(n)$ of f. Then, letting the counting function $\pi_{f*h}(x)$ be defined as in the first equation below, we have the following equivalent expressions for the summatory function of $f * h$ for integers $x \geqslant 1$:*

$$\begin{aligned}\pi_{f*h}(x) &= \sum_{n \leqslant x} \sum_{d|n} f(d) h(n/d) \\ &= \sum_{d \leqslant x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].\end{aligned}$$

Moreover, we can invert the linear system determining the coefficients of $H(k)$ for $1 \leqslant k \leqslant x$ naturally to express $H(x)$ as a linear combination of the original left-hand-side summatory function as follows:

$$\begin{aligned}H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\ &= \sum_{n=1}^{x} f^{-1}(n) \pi_{f*h}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).\end{aligned}$$

Corollary 4.2 (Convolutions Arising From Möbius Inversion)**.** *Suppose that g is an arithmetic function with $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n \leqslant x} (g * \mu)(n)$. Then the Mertens function equals*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geqslant 1.$$

Corollary 4.3 (A motivating special case)**.** *We have exactly that for all $x \geqslant 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

4.2 Elaborating on the construction behind the motivating special case formula for $M(x)$

We can compute the first few terms for the Dirichlet inverse sequence of the arithmetic function $g(n) := \omega(n) + 1$ from Corollary 4.3 numerically for the first few sequence values as

$$\{g^{-1}(n)\}_{n \geqslant 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

The sign of these terms is given by $\operatorname{sgn}(g^{-1}(n)) = \frac{g^{-1}(n)}{|g^{-1}(n)|} = \lambda(n)$ (see Proposition 5.1). This useful property is inherited from the distinctly additive nature of the component function $\omega(n)$. We will still require substantially simpler asymptotic formulae for $g^{-1}(n)$ than what complications are suggested by inspection of the initial numerical calculations of this sequence.

Consider first the following motivating conjecture:

Conjecture 4.4. *Suppose that $n \geqslant 1$ is a squarefree integer. We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n) = (\omega+1)^{-1}(n)$ over these integers:*

(A) $g^{-1}(1) = 1$;

(B) $\operatorname{sgn}(g^{-1}(n)) = \mu(n) \equiv \lambda(n)$;

(C) We can write the inverse function at squarefree n as

$$g^{-1}(n) = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!.$$

We illustrate parts (B)–(C) of this conjecture clearly using Table T.1 given on page 45 of the appendix section.

The realization that the beautiful and remarkably simple form of property (C) in Conjecture 4.4 holds for all squarefree $n \geqslant 1$ motivates our pursuit of formulas for the inverse functions $g^{-1}(n)$ based on the configuration of the exponents in the prime factorization of any $n \geqslant 2$. The summation methods we employ in Section 7 to weight sums of our arithmetic functions according to the sign of $\lambda(n)$ (or parity of $\Omega(n)$) is also reminiscent of the combinatorially motivated sieve methods in [4, §17].

Remark 4.5 (Comparison to formative methods for bounding $M(x)$)**.** Note that since the DGF of $\omega(n)$ is given by $D_\omega(s) = P(s)\zeta(s)$ where $P(s)$ is the *prime zeta function*, we do have a Dirichlet series for the inverse functions to invert coefficient-wise using more classical contour integral methods*. However, the uniqueness to our new methods is that our new approach does not rely on typical constructions for bounding $M(x)$ based on estimates of the non-trivial zeros of the Riemann zeta function that have so far been employed to bound the Mertens function from above. That is, we will instead take a more combinatorial tack to investigating bounds on this inverse function sequence in the coming sections. By Corollary 4.3, once we have established bounds on this $g^{-1}(n)$ and its summatory function, we will be able to formulate new lower bounds (in the limit supremum sense) on $M(x)$.

4.3 Fixing an exact expression for $M(x)$ using additive functions

From this point on, we fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and denote its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega+1)^{-1}(n)$. For natural numbers $n \geqslant 1, k \geqslant 0$, let

$$C_k(n) := \begin{cases} \varepsilon(n) = \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geqslant 1. \end{cases}$$

We have limiting asymptotics on these functions in terms of n and k within a fixed range depending on n given by the following theorem:

Theorem 4.6 (Asymptotics for the functions $C_k(n)$)**.** *For $k := 0$, we have by definition that $C_0(n) = \delta_{n,1}$. For all sufficiently large $n > 1$ and any fixed $1 \leqslant k \leqslant \Omega(n)$ taken independently of n, we obtain that the dominant asymptotic term for $C_k(n)$ is given uniformly by*

$$\mathbb{E}[C_k(n)] \geqslant (\log\log n)^{2k-1}, \text{ as } n \to \infty.$$

*E.g., using contour integration or the following integral formula for Dirichlet series inversion [1, §11]:

$$f(n) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{n^{\sigma+\imath t}}{\zeta(\sigma+\imath t)(P(\sigma+\imath t)+1)}, \sigma > 1.$$

Fröberg has also previously done some preliminary investigation as to the properties of the inversion to find the coefficients of $(1+P(s))^{-1}$ in [5].

Since we have that

$$(g^{-1} * 1)(n) = \lambda(n) \cdot C_{\Omega(n)}(n), \forall n \geqslant 1, \tag{2}$$

Möbius inversion provides us with an exact divisor sum based expression for $g^{-1}(n)$ (see Lemma 7.1). Then we can prove (see Corollary 9.2) that we can obtain lower bounds on the magnitude of $g^{-1}(n)$ by approximating it by the simpler divisor sums

$$\lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

Specifically, the last result in turn implies that

$$|G^{-1}(x)| \overset{\blacktriangle}{\gtrsim} \left| \sum_{n \leqslant x} \lambda(n) \cdot C_{\Omega(n)}(n) \times \sum_{d=1}^{\left\lfloor \frac{x}{n} \right\rfloor} \lambda(d) \right|. \tag{3}$$

In light of the fact that (see Proposition 9.1)

$$M(x) \sim G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)},$$

the formula in (3) implies that we can establish new finite *lower bounds* on $M(x)$ along large infinite subsequences by appropriate estimates of the summatory function $G^{-1}(x)$. As explicit lower bounds for $M(x)$ along particular subsequences are not obvious, and are historically ellusive non-trivial features of the function to obtain as we expect sign changes of this function infinitely often, we find this approach to be an effective one.

4.4 Uniform asymptotics from enumerative counting based DGFs from Mongomery and Vaughan

The precise formulations of the inverse function asymptotics proved in Section 7 depend on being able to form significant lower bounds on the summatory functions of an always positive arithmetic function weighted by $\lambda(n)$. Our inspiration for the new bounds found in the last sections of this article allows us to sum non-negative arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. In particular, it uses a hybrid generating function and enumerative DGF method under which we are able to recover "good enough" asymptotics about the summatory functions that encapsulate the parity of $\lambda(n)$ through the summatory count functions $\widehat{\pi}_k(x)$. The precise statement of the theorem that we transform to state these new bounds is re-stated as Theorem 4.7 below.

Theorem 4.7 (Montgomery and Vaughan, §7.4)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leqslant x : \Omega(n) = k\}.$$

For $R < 2$ we have that

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right),$$

uniformly for $1 \leqslant k \leqslant R \log\log x$ where

$$\mathcal{G}(z) := \frac{F(1,z)}{\Gamma(z+1)} = \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, z \geqslant 0.$$

The next theorem, proved carefully in Section 6, is the primary starting point for our new asymptotic lower bounds.

Theorem 4.8 (Generating functions of symmetric functions)**.** *We obtain lower bounds of the following form on the function $\mathcal{G}(z)$ from Theorem 4.7 for $A_0 > 0$ an absolute constant, for $C_0(z)$ a strictly linear function only in z, and where we must take $0 \leqslant z \leqslant 1$, or equivalently $1 \leqslant k \leqslant \log\log x$ for x large:*

$$\mathcal{G}(z) \geqslant A_0 \cdot (1-z)^3 \cdot C_0(z)^z.$$

It suffices to take the components to the bound in the previous equation as

$$\begin{aligned} A_0 &= \frac{2^{9/16} \exp\left(-\frac{55}{4} \log^2(2)\right)}{(3e \log 2)^3 \cdot \Gamma\left(\frac{5}{2}\right)} \approx 3.81296 \times 10^{-6} \\ C_0(z) &= \frac{4(1-z)}{3e \log 2}. \end{aligned}$$

In particular, with $0 \leqslant z \leqslant 1$ and $z \equiv z(k,x) = \frac{k-1}{\log\log x}$, by Theorem 4.7, we have that

$$\widehat{\pi}_k(x) \overset{\blacktriangle}{\gtrsim} \frac{A_0 \cdot x}{\log x \cdot (\log\log x)^4 \cdot (k-1)!} \cdot \left(\frac{4}{3e \log 2}\right)^k.$$

4.5 Rigorous proofs jusifying that so-called average order lower bounds are meaningful with respect to our problem

Theorem 4.9. *Let the summatory function $G_E^{-1}(x)$ be defined for $x \geqslant 1$ by*

$$G_E^{-1}(x) := \sum_{n \leqslant x} \lambda(n) \times \sum_{\substack{d \mid n \\ d > e^e}} \mathbb{E}[C_{\Omega(d)}(d)]. \tag{4}$$

If for some respectively minimally and maximally defined absolute constants $B, C \in [0,1)$, we have that as $x \rightarrow \infty$

$$B + o(1) \leqslant \frac{1}{x} \cdot \#\left\{n \leqslant x : |G^{-1}(n)| - |G_E^{-1}(n)| \leqslant 0\right\} \leqslant C + o(1),$$

then there is some $\varepsilon \in (0,1)$ (depending on B, C) with $0 < B - \varepsilon, C + \varepsilon < 1$ such that for all sufficiently large x we have some $x_0 \in [(B-\varepsilon)x, (C+\varepsilon)x]$ such that

$$|G^{-1}(x_0)| \geqslant \left|G_E^{-1}(x_0)\right|.$$

We prove Theorem 4.9, and rigorously justify that its hypothesis holds, in Section 8. This result combines to allow us to take lower bounds based on average order estimates of certain arithmetic functions we have defined to approximate $g^{-1}(n)$ and still recover an infinite subsequence along which we can witness the unboundedness in Corollary 4.12 stated below.

The following observation that is suggestive of the quasi-periodicity at play with the distinct values of $g^{-1}(n)$ distributed over $n \geqslant 2$:

Heuristic 4.10 (Symmetry in $g^{-1}(n)$ in the exponents in the prime factorization of n)**.** Suppose that $n_1, n_2 \geqslant 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly two, three, and so on prime factors (see Table T.1 starting on page 45).

There does not appear to be an easy, nor subtle direct recursion between the distinct g^{-1} values, except through auxiliary function sequences. However, the distribution of distinct sets of prime exponents is fairly regular with $\omega(n)$ and $\Omega(n)$ playing a crucial role in the repitition of common values of $g^{-1}(n)$. The next remark makes clear what our intuiton ought suggest about the relation of the actual function values to the average case expectation of $g^{-1}(n)$ for $n \leqslant x$ when x is large.

Remark 4.11 (Essential components of the proof)**.** Given that we have chosen to work with a representation for $M(x)$ that depends critically on the distribution of the values of the additive functions, $\omega(n)$ and $\Omega(n)$, there is substantial intuition involved á priori that suggests our sums over these functions ought behave regularly on average. Notably, we have an Erdös-Kac like theorem for each of $\omega(n)$ and $\Omega(n)$, which when the bounding parameter is set to $z := 0$, we provably can expect these sums involving the classically "nice" functions to tend towards their average case asymptotic nature infinitely often, and predictably near any large x [8, §1.7] (*cf.* Theorem 6.1). Thus the choice in stating (1) as it depends on the cannonical additive function examples we have cited is *absolutely essential* to the success of our proof making "magic" happen out of the average case scenario we easily bound from below.

4.6 Nearly cracking the classical unboundedness barrier

In Section 9, we provide the culmination of what we build up to in the proofs established in prior sections of the article. What we obtain at the conclusion of the section is the following important summary corollary that comes close (by a factor of $\log x$) to resolving the classical question of the unboundedness of the scaled function Mertens function $|M(x)|/\sqrt{x}$ in the limit supremum sense:

Corollary 4.12 (Lower Bounds for the Mertens function)**.** *Let* $u_0 := e^{e^{e^{e^{e^{e^{e}}}}}}$ *and define the infinite increasing subsequence,* $\{x_{0,n}\}_{n\geqslant 1}$*, by* $x_{0,n} := e^{e^{e^{e^{4n\cdot\left\lceil e^{e^{4n}}\right\rceil}}}}$. *We have that along the increasing subsequence* x_y*, for some* $x_y \in (x_{0,y-1}, x_{0,y+1})$*, for large all sufficiently large* $y \gg \max\left(\lceil x_{0,1}\rceil + 1, u_0 + 2\right)$ *the following bound holds:*

$$\frac{|M(x_y)|\log\sqrt{x_y}}{\sqrt{x_y}} \overset{\blacktriangle}{\gtrsim} 2C_{\ell,1}\cdot(\log\log\sqrt{x_y})\frac{(\log\log\log\sqrt{x_y})^{2\log 2+\frac{1}{3\log 2}-1}}{(\log\log\log\log\sqrt{x_y})^{\frac{5}{2}}}\cdot\frac{\log_*^5(\sqrt{x_y})^{2\log 2+\frac{1}{3\log 2}}}{\log_*^6(\sqrt{x_y})^{\frac{5}{2}}}, \text{ as } y\rightarrow\infty.$$

In the previous equation, we adopt the notation for the absolute constant $C_{\ell,1} > 0$ *defined more precisely by*

$$C_{\ell,1} := \frac{128\cdot 2^{1/8}}{6561\cdot e^6\pi\log^6(2)}\exp\left(-\frac{55}{2}\log^2(2)\right) \approx 2.76631\times 10^{-10}.$$

This is all to say that in establishing the rigorous proof of Corollary 4.12 based on our new methods, we not only show that

$$\limsup_{x\rightarrow\infty}\frac{|M(x)|\log x}{\sqrt{x}} = +\infty,$$

but also set a minimal rate (along a large infinite subsequence) at which this form of the scaled Mertens function grows without bound.

5 Preliminary proofs of lemmas and new results

The purpose of this section is to provide proofs and statements of elementary and otherwise well established facts and results. In particular, the proof of Theorem 4.1 allows us to easily justify the formula in (1). This formula is the crucial formulation that constiutes an exact expression for $M(x)$. The indispensible property inherent to the arithmetic functions, $\omega(n)$ and $g^{-1}(n)$, that are used to state the formula are strong additivity, which leads to the sign of the inverse function $g^{-1}(n)$ being given by $\lambda(n)$. Hence the summatory function of $g^{-1}(n)$ is intimately tied to the exact limiting distribution of the values of $\Omega(n)$.

5.1 Establishing the summatory function inversion identities

We will prove Theorem 4.1, a crucial component to our new more combinatorial formulations used to bound $M(x)$ in later sections, using matrix methods before moving on. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95].

Proof of Theorem 4.1. Let h, g be arithmetic functions where $g(1) \neq 0$ necessarily has a Dirichlet inverse. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leqslant x} h(n)$ and $G(x) = \sum_{n \leqslant x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h: $g * h$. Then we can easily see that the following expansions hold:

$$\begin{aligned}\pi_{g*h}(x) := \sum_{n=1}^{x} \sum_{d|n} g(n)h(n/d) &= \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right)\right] H(i).\end{aligned}$$

We form the matrix of coefficients associated with this system for $H(x)$, and proceed to invert it to express an exact solution for this function over all $x \geqslant 1$. Let the ordinary (initial, non-inverse) matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1}.$$

The matrix we must invert in this problem is lower triangular, with ones on its diagonals – and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T); \qquad U = (\delta_{i,j+1}), (I - U^T)^{-1} = ([j \leqslant i]_\delta).$$

Here, U is the $N \times N$ matrix whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$.

It is a useful fact that if we take successive differences of floor functions, we get non-zero behavior at divisors:

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

We use this property to shift the matrix $\hat{G}$, and then invert the result, to obtain a matrix involving the Dirichlet inverse of g:

$$\left[(I - U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Now we can express the inverse of the target matrix $(g_{x,j})$ in terms of these Dirichlet inverse functions as follows:

$$(g_{x,j}) = (I - U^T)^{-1}\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I - U^T)$$

$$
\begin{aligned}
(g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_{\delta}\right)(I-U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k)\right)(I-U^T) \\
&= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j}\right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1}\right\rfloor} g^{-1}(k)\right).
\end{aligned}
$$

Thus the summatory function H is exactly expressed by the inverse vector product of the form

$$
\begin{aligned}
H(x) &= \sum_{k=1}^{x} g_{x,k}^{-1}\cdot \pi_{g*h}(k) \\
&= \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} g^{-1}(j)\right)\cdot \pi_{g*h}(k). \square
\end{aligned}
$$

5.2 Proving the crucial signedness property from the conjecture

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n. Then we can easily prove that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu. \tag{5}$$

When combined with Corollary 4.2, an immediate consequence of Theorem 4.1, this convolution identity yields the necessary convolution identity that yields the exact formula for $M(x)$ stated in (1) of Corollary 4.3.

The proof of the next proposition is essential to our argument given in later sections. We try to keep the argument brief while sketching all relevant details to rigorously justifying the key parts to the proof of our claim.

Proposition 5.1 (The key signedness property of $g^{-1}(n)$)**.** *For the Dirichlet invertible function, $g(n) := \omega(n)+1$ defined such that $g(1) = 1$, at any $n \geqslant 1$, we have that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$. The notation for the operation given by $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_{\delta}} \in \{0, \pm 1\}$ denotes the sign of the arithmetic function h at n.*

Proof. Recall that $D_f(s) := \sum_{n\geqslant 1} f(n)n^{-s}$ denotes the Dirichlet generating function (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\Re(s) > \sigma_f$. In particular, recall that $D_1(s) = \zeta(s)$, $D_{\mu}(s) = 1/\zeta(s)$ and $D_{\omega}(s) = P(s)\zeta(s)$. Then by (5) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of the original arithmetic function f, for all $\Re(s) > 1$ we have

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{6}$$

It follows that $(\omega + 1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + 1$. We show that $\operatorname{sgn}(h^{-1}) = \lambda$. From this fact, it follows by inspection that $\operatorname{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make this intuition precise.

By the standard recurrence relation we used to define the Dirichlet inverse function of any arithmetic function h such that $h(1) = 1 \neq 0$, we have that

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d|n \\ d>1}} h(d)h^{-1}(n/d), & n \geqslant 2. \end{cases} \tag{7}$$

For $n \geqslant 2$, the summands in (7) can be simply indexed over the primes $p|n$. This observation yields that we can inductively expand these sums into nested divisor sums provided the depth of the sums does not exceed the capacity to index summations over the primes dividing n. Namely, notice that for $n \geqslant 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p|n} h^{-1}(n/p), && \text{if } \Omega(n) \geqslant 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1p_2}\right), && \text{if } \Omega(n) \geqslant 2 \\ &= -\sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1p_2}} h^{-1}\left(\frac{n}{p_1p_2p_3}\right), && \text{if } \Omega(n) \geqslant 3. \end{aligned}$$

Then by induction, again with $h^{-1}(1) = 1$, we obtain by expanding the nested divisor sums as above to their maximal depth as

$$h^{-1}(n) = \lambda(n) \times \sum_{p_1|n} \sum_{p_2|\frac{n}{p_2}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1p_2\cdots p_{\Omega(n)-1}}} 1, n \geqslant 2.$$

If for $n \geqslant 2$ we write the prime factorization of n as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{\omega(n)}^{\alpha_{\omega(n)}}$ where the exponents $\alpha_i \geqslant 1$ are all non-zero for $1 \leqslant i \leqslant \omega(n)$, we can see that

$$\begin{aligned} h^{-1}(n) &\geqslant \lambda(n) \times 1 \cdot 2 \cdot 3 \cdots \omega(n) = \lambda(n) \times (\omega(n))!, && n \geqslant 2 \\ h^{-1}(n) &\leqslant \lambda(n) \times (\omega(n))!^{\max(\alpha_1, \alpha_2, \ldots, \alpha_{\omega(n)})}, && n \geqslant 2. \end{aligned}$$

In other words, what these bounds show is that for all $n \geqslant 1$ (with $\lambda(1) = 1$) the following property holds:

$$\operatorname{sgn}(h^{-1}(n)) = \lambda(n). \tag{8}$$

By (8), we immediately have bounding constants $1 \leqslant C_{1,n}, C_{2,n} < +\infty$ that exist for each $n \geqslant 1$ so that

$$C_{1,n} \cdot (\lambda * \mu)(n) \leqslant (h^{-1} * \mu)(n) \leqslant C_{2,n} \cdot (\lambda * \mu)(n). \tag{9}$$

Since both λ, μ are multiplicative, the convolution $\lambda * \mu$ is multiplicative. We know that the values of any multiplicative function are uniquely determined by its action at prime powers. So we can compute that for any prime p and non-negative integer exponents $\alpha \geqslant 1$ that

$$\begin{aligned} (\lambda * \mu)(p^\alpha) &= \sum_{i=0}^{\alpha} \lambda(p^{\alpha-i})\mu(p^i) \\ &= \lambda(p^\alpha) - \lambda(p^{\alpha-1}) \\ &= (-1)^{\Omega(p^\alpha)} - (-1)^{\Omega(p^{\alpha-1})} = (-1)^\alpha - (-1)^{\alpha-1} = 2\lambda(p^\alpha). \end{aligned}$$

Then by the multiplicativity of $\lambda(n)$, the previous inequalities derived in (9) are re-stated in the form of

$$2C_{1,n} \cdot \lambda(n) \leqslant h^{-1}(n) \leqslant 2C_{2,n} \cdot \lambda(n).$$

Since the absolute constants $C_{1,n}, C_{2,n}$ are always positive for all $n \geqslant 1$, we clearly recover the signedness of $g^{-1}(n)$ as $\lambda(n)$. □

5.3 Other facts and listings of results we will need in our proofs

Theorem 5.2 (Mertens theorem)**.** *For all $x \geqslant 2$ we have that*

$$P_1(x) := \sum_{p \leqslant x} \frac{1}{p} = \log\log x + B + o(1),$$

where $B \approx 0.2614972128476427837554$ is an explicitly defined absolute constant.

Corollary 5.3. *We have that for sufficiently large $x \gg 1$*

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) = \frac{e^{-B}}{\log x} \left(1 + o(1)\right).$$

Hence, for $1 < |z| < 2$ we obtain that

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right)^z = \frac{e^{-Bz}}{(\log x)^z} \left(1 + o(1)\right)^z.$$

Facts 5.4 (Exponential Integrals and Incomplete Gamma Functions)**.** The following two variants of the *exponential integral function* are defined by [13, §8.19]

$$\begin{aligned} \mathrm{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, \\ E_1(z) &:= \int_1^{\infty} \frac{e^{-tz}}{t} dt, \Re(z) \geqslant 0, \end{aligned}$$

where $\mathrm{Ei}(-kz) = -E_1(kz)$ for real $k > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $E_1(z)$:

$$1 - \frac{3}{4} z \leqslant E_1(z) - \gamma - \log z \leqslant 1 - \frac{3}{4} z + \frac{11}{36} z^2. \tag{10a}$$

A related function is the (upper) *incomplete gamma function* defined by [13, §8.4]

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt, \Re(s) > 0.$$

We have the following properties of $\Gamma(s, x)$:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, \tag{10b}$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, |x| \to +\infty. \tag{10c}$$

6 Summing arithmetic functions weighted by $\lambda(n)$

In this section, we re-state a couple of key results proved in [11, §7.4] that we rely on to state and prove Corollary 6.4 stated below. This corollary is important as it shows that (signed) summatory functions over $\widehat{\pi}(x)$ capture the dominant asymptotics of the full summatory function formed by taking $1 \leqslant k \leqslant \log_2(x)$ when we truncate and instead sum only up to the uniform bound of $1 \leqslant k \leqslant \log\log x$ guaranteed by applying Theorem 4.7.

We also prove Theorem 4.8 in this section. This key theorem allows us to establish a global minimum we can attain on the function $\mathcal{G}(z)$ from Theorem 4.7 by truncating the formerly stated infinite range of the primes p over which we take a component product in the definition of this function. This in turn implies the uniform lower bounds on $\widehat{\pi}_k(x)$ guaranteed by that theorem by a straightforward manipulation of inequalities.

6.1 Discussion: The enumerative DGF result from Montgomery and Vaughan

What the enumeratively-flavored result of Montgomery and Vaughan in Theorem 4.7 allows us to do is get a "good enough" lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. For comparison, we already have known, more classical bounds due to Erdös (and earlier) that we can tightly bound [2, 11]

$$\pi_k(x) = (1 + o(1)) \cdot \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

We seek to approximate the right-hand-side of $\mathcal{G}(z)$ by only taking the products of the primes $p \leqslant u$, e.g., indexing the component products only over those primes $p \in \{2, 3, 5, \ldots, u\}$ for some minimal upper bound u (with respect to x) as $x \to \infty$.

We also state the following theorems reproduced from [11, §7.4] that handle the relative scarcity of the distribution of the $\Omega(n)$ for $n \leqslant x$ such that $\Omega(n) > \log\log x$.

Theorem 6.1 (Bounds on exceptional values of $\Omega(n)$ for large n)**.** *Let*

$$\begin{aligned} A(x, r) &:= \#\left\{n \leqslant x : \Omega(n) \leqslant r \cdot \log\log x\right\}, \\ B(x, r) &:= \#\left\{n \leqslant x : \Omega(n) \geqslant r \cdot \log\log x\right\}. \end{aligned}$$

If $0 < r \leqslant 1$ and $x \geqslant 2$, then

$$A(x, r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

If $1 \leqslant r \leqslant R < 2$ and $x \geqslant 2$, then

$$B(x, r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

Theorem 6.2 (Bounds on exceptional values of $\Omega(n)$ for large n, MV 7.21)**.** *We have that uniformly*

$$\#\left\{3 \leqslant n \leqslant x : \frac{\Omega(n) - \log\log n}{\sqrt{\log\log n}} \leqslant 0\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Remark 6.3. The proofs of Theorem 6.1 and Theorem 6.2 are found in Chapter 7 of Montgomery and Vaughan. The key interpretation we need is the result stated in the next corollary. In the previous theorem, the dependence on R, and the necessity of using the conditional relation $\ll_R$, serves to denote this R as a bounding (maximally limiting) parameter on the input $r \in (1, R)$ to the functions $B(x, r)$. The precise way in which the bound stated in this cited theorem depends on this bounded, indeterminate paramater R can be reviewed for reference in the proof algebra and relations cited in the reference [11, §7]. The role of the parameter R involved in stating the previous theorem is notably important as a scalar factor the upper bound on $k \leqslant R\log\log x$ in Theorem 4.7 up to which we obtain the valid uniform bounds in x on the asymptotics for $\widehat{\pi}_k(x)$.

We have a discrepancy to work out in so much as we can only form summatory functions over the $\widehat{\pi}_k(x)$ for $1 \leqslant k \leqslant R \log\log x$ using the desirable, or "nice", asymptotic formulas guaranteed by Theorem 4.7, even though we can actually have contributions from values distributed throughout the range $1 \leqslant \Omega(n) \leqslant \log_2(n)$. It is then crucial that we can show that the dominant growth of the asymptotic formulas we obtain for these summatory functions is captured by summing only over k in the truncated range where the uniform formulas hold. In particular, we will require a proof that we can discard the terms in the full summatory function asymptotic formulas as negligible (up to at most a constant) for large x when they happen to fall in the limiting exceptional range of $\Omega(n) > R \log\log x$ for $n \leqslant x$.

Corollary 6.4. *Using the notation for $A(x, r)$ and $B(x, r)$ from Theorem 6.1, we have that for $\delta > 0$,*

$$0 \leqslant \left| \frac{B(x, 1+\delta)}{A(x, 1)} \right| \ll 2, \text{ as } \delta \to 0^+, x \to \infty.$$

Proof. The lower bound stated above should be clear. To show that the asymptotic upper bound is correct, we compute using Theorem 6.1 and Theorem 6.2 that

$$\begin{aligned}
\left| \frac{B(x, 1+\delta)}{A(x, 1)} \right| &\ll \left| \frac{x \cdot (\log x)^{\delta - \delta \log(1+\delta)}}{\widehat{\pi}_1(x) + \widehat{\pi}_2(x) + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right)} \right| \\
&\sim \left| \frac{x \cdot (\log x)^{\delta - \delta \log(1+\delta)}}{\frac{x}{\log x} + \frac{x \cdot (\log\log x)}{\log x} + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right)} \right| \\
&= \left| \frac{(\log x)^{1+\delta - \delta \log(1+\delta)}}{1 + \log\log x + \frac{\log x}{2} + o(1)} \right| \\
&\xrightarrow{\delta \to 0^+} \left| \frac{(\log x)}{1 + \log\log x + \frac{\log x}{2} + o(1)} \right| \\
&\sim 2,
\end{aligned}$$

as $x \to \infty$. Notice that since $\mathbb{E}[\Omega(n)] = \log\log n + B$ for $0 < B < 1$ an absolute constant, when we apply this theorem, the range $k > \log\log x$ denoted as above by $k > (1+\delta)\log\log x$ lets us assume that $\delta \to 0^+$ as $x \to \infty$. $\square$

We again emphasize that Corollary 6.4 implies that for sums involving $\widehat{\pi}_k(x)$ indexed by k, we can capture the dominant asymptotic behavior of these sums by taking k in the truncated range $1 \leqslant k \leqslant \log\log x$, e.g., with $0 \leqslant z \leqslant 1$ in Theorem 4.7. This fact will be important when we prove Theorem 9.5 in Section 9 using a sign-weighted summatory function in Abel summation that depends on these functions (see Lemma 9.3).

6.2 The key new results utilizing Theorem 4.7

We will require a handle on partial sums of integer powers of the reciprocal primes as functions of the integral exponent and the upper summation index x. The next corollary is not a triviality as it comes in handy when we take to the next task of proving the bound in Theorem 4.8. The statement of Proposition 6.5 effectively provides a coarse rate in x below which the reciprocal prime sums tend to absolute constants given by the prime zeta function, $P(s)$. We also require the finite-degree polynomial dependence of these bounds on s to simplify the computations in the theorem below.

Proposition 6.5. *For real $s \geqslant 1$, let*

$$P_s(x) := \sum_{p \leqslant x} p^{-s}, x \gg 2.$$

When $s := 1$, we have the known bound in Mertens theorem (see Theorem 5.2). For $s > 1$, we obtain that

$$P_s(x) \approx E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1).$$

For integers $s \geqslant 2$ we have that

$$P_s(x) \leqslant \gamma_1(s, x) + o(1).$$

It suffices to take the bounding function in the previous equation as

$$\gamma_1(s, x) = -s \log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4} s(s-1)\log(x/2) + \frac{11}{36} s(s-1)^2 \log^2(2).$$

Proof. Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$ and where our target function $f(t) = t^{-s}$ with $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= E_1((s-1)\log 2) - E_1((s-1)\log x) + o(1), |x| \to \infty. \end{aligned}$$

Now using the inequalities in Facts 5.4, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned} \frac{P_s(x)}{s} &\geqslant -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1)\log(x/2) - \frac{11}{36}(s-1)^2 \log^2(x) \\ \frac{P_s(x)}{s} &\leqslant -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1)\log(x/2) + \frac{11}{36}(s-1)^2 \log^2(2). \end{aligned}$$

This completes the proof of the bounds cited above in the statement of this lemma. □

Proof of Theorem 4.8. We have that for all integers $0 \leqslant k \leqslant m$

$$[z^k] \prod_{1 \leqslant i \leqslant m} (1 - f(i)z)^{-1} = [z^k] \exp\left(\sum_{j \geqslant 1} \left(\sum_{i=1}^{m} f(i)^j\right) \frac{z^j}{j}\right). \tag{11}$$

In our case we have that $f(i)$ denotes the i^{th} prime. Hence, summing over all $p \leqslant ux$ in place of $0 \leqslant k \leqslant m$ in the previous formula in tandem with Proposition 6.5, we obtain that the logarithm of the generating function in z obtained when we sum over all $p \leqslant ux$ for some minimal parameter u is given by

$$\begin{aligned} \log\left[\prod_{p \leqslant ux} \left(1 - \frac{z}{p}\right)^{-1}\right] &\geqslant (B + \log\log(ux))z + \sum_{j \geqslant 2} \left[a(ux) + b(ux)(j-1) + c(ux)(j-1)^2\right] z^j \\ &= (B + \log\log(ux))z - a(ux)\left(1 + \frac{1}{z-1} + z\right) \\ &\qquad + b(ux)\left(1 + \frac{2}{z-1} + \frac{1}{(z-1)^2}\right) \\ &\qquad - c(ux)\left(1 + \frac{4}{z-1} + \frac{5}{(z-1)^2} + \frac{2}{(z-1)^3}\right) \\ &=: \widehat{\mathcal{B}}(u, x; z). \end{aligned}$$

In the previous equations, the lower bounds formed by the functions (a, b, c) evaluated at ux are given by the corresponding upper bounds from Proposition 6.5 due to the leading sign on the previous expansions as

$$(a_\ell, b_\ell, c_\ell) := \left(-\log\left(\frac{\log(ux)}{\log 2}\right), \frac{3}{4}\log\left(\frac{ux}{2}\right), \frac{11}{36}\log^2 2\right).$$

Now we make a decision to set the uniform bound parameter to a middle ground value of $1 < R < 2$ at $R := \frac{3}{2}$ (practically, to be truncated and taken as though $R \equiv 1$ in sums by the restriction that $z \leqslant 1$) so that

$$z \equiv z(k,x) = \frac{k}{\log\log x} \in (0, R),$$

for $x \gg 1$ very large. Thus $(z-1)^{-m} \in [(-1)^m, 2^m]$ for integers $m \geqslant 1$, and so we can obtain the lower bound stated below. Namely, these bounds on the signed reciprocals of $z-1$ lead to an effective bound of the following form:

$$\begin{aligned}\widehat{\mathcal{B}}(u,x;z) \geqslant (B + \log\log(ux))z - a(ux)\left(1 + \frac{1}{z-1} + z\right) \\ + b(ux)\left(1 + \frac{2}{z-1} + \frac{1}{(z-1)^2}\right) - 45 \cdot c(ux).\end{aligned}$$

Since the function $c(ux)$ is constant, we then also obtain the next bounds.

$$\begin{aligned}\frac{e^{-Bz}}{(\log(ux))^z} \times \exp\left(\widehat{\mathcal{B}}(u,x;z)\right) &\geqslant \exp\left(-\frac{55}{4}\log^2(2)\right) \times \left(\frac{\log(ux)}{\log 2}\right)^{1+\frac{1}{z-1}+z} \\ &\quad \times \left(\frac{ux}{2}\right)^{\frac{3}{4}\left(1+\frac{2}{z-1}+\frac{1}{(z-1)^2}\right)} \\ &=: \widehat{\mathcal{C}}(u,x;z) \end{aligned} \tag{12}$$

Now we need to determine which values of u minimize the expression for the function defined in (12). For this we will use a somewhat limited elementary method from introdutory calculus to determine a global minimum for the products. We can symbolically use *Mathematica* to see that

$$\frac{\partial}{\partial u}\left[\widehat{\mathcal{C}}(u,x;z)\right]\Bigg|_{u \mapsto u_0} = 0 \implies u_0 \in \left\{\frac{1}{x}, \frac{1}{x}e^{-\frac{4}{3}(z-1)}\right\}.$$

When we substitute this outstanding parameter value of $u_0 =: \hat{u}_0 \mapsto \frac{1}{x}e^{-\frac{4}{3}(z-1)}$ into the next expression for the second derivative of the same function $\widehat{\mathcal{C}}(u,x;z)$ we obtain

$$\begin{aligned}\frac{\partial^2}{\partial u^2}\left[\widehat{\mathcal{C}}(u,x;z)\right]\Bigg|_{u=\hat{u}_0} &= \exp\left(-\frac{55}{4}\log^2(2)\right) x^2 2^{\frac{8z^3-27z^2+32z-16}{4(z-1)^2}} 3^{-z+\frac{1}{1-z}+1} e^{\frac{5z^2-16z+8}{3(z-1)}} \times \\ &\quad \times (1-z)^{z+\frac{1}{z-1}-2} z^2 \log^{\frac{z^2}{1-z}}(2) > 0,\end{aligned}$$

provided that $z < 1$. The restriction to $0 \leqslant z < 1$ is equivalent to requiring that $1 \leqslant k \leqslant \log\log x$ in Theorem 4.7. This restriction on k to note leads to a minimum value on the partial product, or lower bound, at this $u = \hat{u}_0$ since the second derivative is positive at this critical value.
After substitution of $u = \frac{1}{x}e^{-\frac{4}{3}(z-1)}$ into the expression for $\widehat{\mathcal{C}}(u,x;z)$ defined above, we have that

$$\widehat{\mathcal{C}}(u,x;z) \geqslant \exp\left(-\frac{55}{4}\log^2(2)\right) \cdot 2^{\frac{9}{16}} \left(\frac{1-z}{3e\log 2}\right)^3 \times \left(\frac{4(1-z)}{3e\log 2}\right)^z.$$

Finally, since $z \equiv z(k,x) = \frac{k}{\log\log x}$ and $k \in [0, R\log\log x)$, we obtain that for small k and $x \gg 1$ large $\Gamma(z+1) \approx 1$, and for k towards the upper range of its interval that $\Gamma(z+1) \approx \Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$. In total, what we get out of these formulas is stated up to accurate constant factor as in the theorem bounds. □

7 Precisely bounding the Dirichlet inverse functions, $g^{-1}(n)$

This section is essential because we prove key results that allow us to bound the oscillatory Dirichlet inverse functions $g^{-1}(n)$ from the exact formula for $M(x)$ given in (1). Using summation by parts, we eventually show that this formula can be approximated with a clear dependency on the summatory functions $G^{-1}(x)$ of $g^{-1}(n)$ by the integral formula we later state and prove as Proposition 9.1.

The pages of tabular data given as Table T.1 given in the appendix section starting on page 45 provide insight into why we arrived at the convenient approximations to $g^{-1}(n)$ proved in this section. The table data offers clear numerical data formed by examining the approximate behavior at work here for the asymptotically small order cases of $1 \leqslant n \leqslant 350$ with *Mathematica*.

It happens that Conjecture 4.4 is not the most simple accurate way to express the limiting behavior of the Dirichlet inverse functions $g^{-1}(n)$ we can formulate, though it does capture an important characteristic. Namely, that these functions can be expressed via more simple formulas than inspection of the initial repetitive, quasi-periodic sequence properties in the table might otherwise suggest.

With all of this in mind, we define the following sequence for integers $n \geqslant 1, k \geqslant 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geqslant 1. \end{cases} \tag{13}$$

The sequence of important semi-diagonals of these functions begins as [16, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n\geqslant 1} \mapsto \{1,-1,-1,1,-1,2,-1,-1,1,2,-1,-3,-1,2,2,1,-1,-3,-1,\ -3,2,2,-1,4,1,2,\ldots\}.$$

Notice that by expanding the recursively-based definition in (13) out to its maximal depth by nested divisor sums, for fixed n, $C_k(n)$ is seen to only ever possibly be non-zero for $k \leqslant \Omega(n)$. This observation follows from the fact that a minimal condition on the forms of divisors $d > 1$ of n requires that d have at least a single prime factor. Thus, the effective range of k for fixed n is restricted by the conditions that $C_0(n) = \delta_{n,1}$ and $C_k(n) = 0\ \forall k > \Omega(n)$. That is, for $n \geqslant 2$, the contributions from summations over $C_k(n)$ are only significant when $1 \leqslant k \leqslant \Omega(n)$.

Lemma 7.1 (An exact formula for $g^{-1}(n)$)**.** *For all $n \geqslant 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu(n/d)\lambda(d)C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$\begin{aligned} g^{-1}(n) &= -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) g^{-1}(n/d) \qquad \Longrightarrow \\ (g^{-1} * 1)(n) &= -(\omega * g^{-1})(n). \end{aligned}$$

Now by repeatedly expanding the right-hand-side, and removing corner cases in the nested sums since $\omega(1) = 0$ by convention, we find that

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n).$$

The statement follows by Möbius inversion applied to each side of the last equation. □

Corollary 7.2. *For all squarefree integers $n \geqslant 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d).$$

Proof. Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geqslant 2$ and that n is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where p_i is prime for all $1 \leqslant i \leqslant \omega(n)$. So we can transform the exact divisor sum guaranteed for all n in Lemma 7.1 into the following:

$$\begin{aligned} g^{-1}(n) &= \sum_{i=1}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i}(-1)^i \cdot C_{\Omega(d)}(d) + \mu(1)\lambda(n)C_1(1) \\ &= \lambda(n) \left[\sum_{i=1}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d) + 1 \right] \\ &= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \end{aligned}$$

The signed computations in the first of the previous equations is justified by noting that $\lambda(n) = (-1)^{\omega(n)}$ whenever n is squarefree, and that for d squarefree with $\omega(d) = i$, $\Omega(d) = i$. ☐

Example 7.3 (Special cases of the functions $C_k(n)$ for small k)**.** We cite the following special cases which should be easy enough to see on paper by explicit computation using (13):

$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= d(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n)+1} - \gcd\left(\Omega(n), \omega(n)\right). \end{aligned}$$

We have a recurrence relation between successive $C_k(n)$ values over k of the form given in Lemma 7.4.

Lemma 7.4 (Recurrence relation between the $C_k(n)$)**.** *For $k \geqslant 2$, we have that*

$$C_k(n) = \sum_{p|n} \sum_{d \left| \frac{n}{p^{\nu_p(n)}} \right.} \sum_{i=0}^{\nu_p(n)-1} C_{k-1}\left(d \cdot p^i\right). \tag{14}$$

Proof. First, we re-write the formula for $C_k(n)$ given directly by the recursion implicit to its definition stated in (13) for $k \geqslant 2$:

$$\begin{aligned} C_k(n) &= \sum_{d|n} \omega(d) C_{k-1}(n/d) \\ &= \sum_{d|n} \sum_{p|d} C_{k-1}(n/d) \\ &= \sum_{d|n} \sum_{p|\frac{n}{d}} C_{k-1}(d). \end{aligned} \tag{15}$$

We wish to interchange the inner and outer divisor sums in the last equation. Since $p|\frac{n}{d}$ where d ranges over the divisors of n, the exchanged outer sum should clearly be indexed over $p|n$.

We claim that in fact $A_n = B_n$ where these two sets are defined for all $n \geqslant 2$ as follows:

$$\begin{aligned} A_n &:= \left\{ d : d|n, p\Big|\frac{n}{d} \right\} \\ B_n &:= \left\{ d \cdot p^i : 0 \leqslant i < \nu_p(n), p|n, d \left| \frac{n}{p^{\nu_p(n)}} \right. \right\}. \end{aligned}$$

The truth of this claim then establishes (14) based on (15). We prove the claim by subset inclusion below.
Proof that $A_n \subseteq B_n$. Suppose that $d|n$ and $p|\frac{n}{d}$ with p prime. This implies that $\exists m \in \mathbb{Z}^+$ such that $dm = n$. The assumptions also imply that $\exists j \in \mathbb{Z}^+$ such that $pj = \frac{n}{d} = \frac{n}{\frac{n}{m}} = m$. So $p|m$ which implies that $p|n$. Since $d|n$ with $p|n$, clearly there is some $0 \leqslant i < \nu_p(n)$ such that $d = d_0 p^i$ where $d_0 \Big| \frac{n}{p^{\nu_p(n)}}$.
Proof that $B_n \subseteq A_n$. First suppose that $i = 0$, let $p|n$, and suppose that $dp^i = d$ satisfies $d \Big| \frac{n}{p^{\nu_p(n)}}$. We have to show that $d \in A_n$. Clearly, as $p|n$, $\nu_p(n) \geqslant 1$. So $d \Big| \frac{n}{p^{\nu_p(n)}}$ implies also that $d|n$. This assumption also shows that $\exists j \in \mathbb{Z}^+$ with $dj \cdot p^{\nu_p(n)} = n \implies p \cdot \left(jp^{\nu_p(n)-1}\right) = \frac{n}{d}$, or eqivalently that $p|\frac{n}{d}$ since $jp^{\nu_p(n)-1} \in \mathbb{Z}^+$. Now suppose that $1 \leqslant i < \nu_p(n)$, $p|n$, and that $d \Big| \frac{n}{p^{\nu_p(n)}}$. We have to show that $dp^i \in A_n$. Clearly, we have that $\nu_p(n) \geqslant 2$ in this case. The third assumption implies that $\exists j \in \mathbb{Z}^+$ such that $dj \cdot p^{\nu_p(n)} = n$, which in turn implies that $dp^i|n$. Moreover, since $jp^{\nu_p(n)-1-i} \in \mathbb{Z}^+$ here, we get that $p \Big| \frac{n}{dp^i}$. Thus, $B_n \subseteq A_n$. □

Theorem 4.6 from the introduction is proved next. The theorem makes precise what these formulas already suggest about the main terms of the growth rates of $C_k(n)$ as functions of k, n for limiting cases of n large for fixed k which is bounded in n, but taken as an independent parameter.

Proof of Theorem 4.6. We can see by Example 7.3 that $C_1(n)$ satsfies the formula we must establish when $k := 1$. We prove our bounds by induction on k. In particular, suppose that $k \geqslant 2$ and let the inductive assumption for all $1 \leqslant m < k$ be that

$$\mathbb{E}[C_m(n)] \geqslant (\log\log n)^{2m-1}.$$

Now we have by the recursive formula in (14) that

$$\begin{aligned} C_k(n) &= \sum_{p|n} \sum_{d \Big| \frac{n}{p^{\nu_p(n)}}} \sum_{i=1}^{\nu_p(n)} \log\log(dp^i)^{2k-3} \\ &\sim \sum_{p|n} \sum_{d \Big| \frac{n}{p^{\nu_p(n)}}} \left[\int \log\log(dp^{\alpha})^{2k-3} d\alpha\right]\Bigg|_{\alpha=\nu_p(n)}. \end{aligned} \tag{16}$$

The inner integral in the previous equation can be evaluated using the limiting asymptotic expansions for the incomplete gamma function stated in Section 5.3. In particular, for $p|n$ and $n \geqslant 2$ large, we let the parameters assume average order values of

$$\mathbb{E}[\nu_p(n)] = \log\log n, \mathbb{E}[p] = \frac{n}{\log n}.$$

Then we evaluate the integral from above as

$$\begin{aligned} \int (\log\log(dp^{\alpha}))^{2k-3} d\alpha &\sim \alpha \times \log\log(dp^{\alpha})^{2k-3} + \frac{\log d}{\log p} \times \log\log(dp^{\alpha})^{2k-3} \\ &\overset{\blacktriangle}{\gtrsim} \alpha \times \log\log(dp^{\alpha})^{2k-3}. \end{aligned}$$

We know that the average order of the number of primes $p|n$ is given by $\mathbb{E}[\omega(n)] = \log\log n$, so approximating p as the cited function of n initially allows us to take a factor of $\log\log n$ and remove the outer divisor sum in (16). So we obtain that

$$\begin{aligned} \mathbb{E}[C_k(n)] &\sim (\log\log n)^2 \mathbb{E}\left[\frac{d(n)}{(\nu_p(n)+1)}\right] \times \left[\log\log n + \log\log\log n + \frac{\log d}{\log n \cdot \log\log n} - \frac{\log\log n}{\log n}\right]^{2k+3} \\ &\sim (\log n) \cdot (\log\log n) \times [\log\log n + \log\log\log n + o(1)]^{2k-3} \\ &\sim (\log n) \cdot (\log\log n)^{2k-2} \\ &\geqslant (\log\log n)^{2k-1}, \text{ as } n \to \infty. \end{aligned}$$

In the previous equation, we have used that the average order of the divisor function, $d(n)$, is given by $\mathbb{E}[d(n)] = \log n$ [13, §27.11]. Thus the claim holds by mathematical induction. $\square$

Note that in Section 8 (proof of fact (A)) we show that when $k := \Omega(n)$ depends on n, then

$$\mathbb{E}[C_{\Omega(n)}(n)] \gg (\log n)(\log\log n)^{2\log\log n - 1} \geqslant \log n \cdot \log\log n.$$

8 A rigorous justification for using average order lower bounds to prove Corollary 4.12

The point of proving the results in this section before moving onto the core results needed in the next section is to provide a rigorous justification for the intuition we sketched in Section 4.5 of the introduction. That is, we expect our arithmetic functions that are closely tied to the additive functions, $\omega(n)$ and $\Omega(n)$, to similarly behave regularly (and infinitely often) in accordance with their values being close to the average case for large x.

What we have established so far, and will establish for $G^{-1}(x)$ in Section 9, are lower bound estimates that hold essentially *on average*. This means that for limiting cases of x, we need to show that the expected value lower bounds are achieved in asymptotic order predictably often within some small window depending linearly on x that we will determine precisely in this section.

Proof of Theorem 4.9. The result is obtained simply by contradiction. Suppose that x is so large that the inequalities in the hypothesis hold. Also, suppose that for all $x_0 \in [(B-\varepsilon)x, (C+\varepsilon)x]$, we have that

$$|G^{-1}(x_0)| < |G_E^{-1}(x_0)|. \tag{17}$$

We have supposed that the constants $B, C \in (0,1)$ are the tightest possible. That is, for all sufficiently large x if

$$\mathcal{G}_0(x) := \frac{1}{x} \cdot \#\left\{n \leqslant x : |G^{-1}(x_0)| - |G_E^{-1}(x_0)| \leqslant 0\right\},$$

denotes the density at x, then there is no larger $B > 0$ such that $B + o(1) \leqslant \mathcal{G}_0(x)$ and no smaller $C < 1$ such that $\mathcal{G}_0(x) \leqslant C + o(1)$. Let $\varepsilon \in (0,1)$ satisfy $0 < B - \varepsilon, C + \varepsilon < 1$. We need to show that such a concrete fixed ε satisfying the conditions in the theorem exists (depending only on B, C).

For $n \geqslant 1$, we have the disjoint interval decomposition of $\{n \leqslant x\}$ given by

$$\{n \leqslant x\} = \{1 \leqslant n < (B-\varepsilon)x\} \oplus \{(B-\varepsilon)x \leqslant n \leqslant (C+\varepsilon)x\} \oplus \{(C+\varepsilon)x < n \leqslant x\},$$

where the three disjoint sets denoted by $\mathcal{D}_i(x)$ for $i = 1,2,3$ yield that as $x \to \infty$, if (17) is true:

$$\begin{aligned}
\mathcal{G}_1(x) &:= \frac{1}{x} \cdot \#\left\{n \in \mathcal{D}_1(x) : |G^{-1}(x_0)| - |G_E^{-1}(x_0)| \leqslant 0\right\} \in [(B-\varepsilon)^2, (B-\varepsilon)(C+\varepsilon)] \\
\mathcal{G}_2(x) &:= \frac{1}{x} \cdot \#\left\{n \in \mathcal{D}_2(x) : |G^{-1}(x_0)| - |G_E^{-1}(x_0)| \leqslant 0\right\} = C - B \\
\mathcal{G}_3(x) &:= \frac{1}{x} \cdot \#\left\{n \in \mathcal{D}_3(x) : |G^{-1}(x_0)| - |G_E^{-1}(x_0)| \leqslant 0\right\} \in [(B-\varepsilon) - (B-\varepsilon)(C+\varepsilon), (C+\varepsilon) - (C+\varepsilon)^2].
\end{aligned}$$

So we obtain that

$$(B-\varepsilon)^2 + (1+\varepsilon-B)(C+\varepsilon) + o(1) \leqslant \mathcal{G}_0(x) \leqslant (C+\varepsilon)(2-C-\varepsilon) + (B-\varepsilon)(C+\varepsilon-1) + o(1).$$

We show that we can pick any $\varepsilon > 0$ that satisfies $B - \varepsilon < C < 1 - \varepsilon$. In fact, given such a choice of this parameter, we have that

$$C + \varepsilon - [(C+\varepsilon)(2-C-\varepsilon) + (B-\varepsilon)(C+\varepsilon-1)] = (C+\varepsilon-B)(C+\varepsilon-1) < 0.$$

This implies a contradiction to the maximality of our bound $C \in (0,1)$. Then we must have that our contrary assumption on x_0 is invalid as $x \to \infty$. Indeed, there is such a fixed $\varepsilon > 0$ and such a $x_0 \in [(B-\varepsilon)x, (C+\varepsilon)x]$ such that $|G^{-1}(x_0)| \geqslant |G_E^{-1}(x_0)|$. □

8.1 Proving the necessary hypothesis in Theorem 4.9

Facts 8.1. To prove the hypothesis assumed by the conclusion of Theorem 4.9, we require the following two facts of our notation for average order:

(A) $\mathbb{E}[C_{\Omega(n)}(n)] = (\log n) \cdot (\log\log n)^{2\mathbb{E}[\Omega(n)]-1} \gg \log n \cdot \log\log n$; and

(B) $\mathbb{E}\left[\sum_{d|n} C_{\Omega(d)}(d)\right] \geqslant \sum_{d|n} \mathbb{E}[C_{\Omega(d)}(d)]$ for all n in some set $\mathcal{C}_E$ of such that $\mathcal{C}_E$ has asymptotic density one.

Proof of fact (A). We utilize Theorem 6.1 to show each of the following:

$$\begin{aligned} \frac{1}{x} \cdot \#\{n \leqslant x : \Omega(n) \geqslant (1+\delta)\log\log x\} &\ll (\log x)^{\delta-(1+\delta)\log(1+\delta)} & &= o(1), \forall \delta > 0, \delta \approx 0 \\ \frac{1}{x} \cdot \#\{n \leqslant x : \Omega(n) \leqslant (1-\rho)\log\log x\} &\ll (\log x)^{-\rho-(1-\rho)\log(1-\rho)} & &= o(1), \forall \rho > 0, \rho \approx 0. \end{aligned} \tag{18}$$

Thus with our result for fixed $1 \leqslant k \leqslant \Omega(n)$ from Theorem 4.6, we can conclude that

$$\begin{aligned} \mathbb{E}[C_{\Omega(n)}(n)] &\sim \frac{1}{n} \sum_{d \leqslant n} (\log\log d)^{2\Omega(d)-1} \\ &\sim (\log n) \cdot (\log\log n)^{2\log\log n - 1} \\ &\gg (\log\log n)^{2\log\log n-1}, \text{ as } n \to \infty \\ \mathbb{E}[C_{\Omega(n)}(n)] &\gg \log n \cdot \log\log n \end{aligned}$$

The results expanded in (18) shows that we can expect the asymptotic density of the $n \leqslant x$ where $\Omega(n) \not\approx \mathbb{E}[\Omega(n)]$ to be small, tending to zero as $n \to \infty$. The last implication follows by an expansion by the binomial series where

$$\begin{aligned} \frac{1}{t} \times \int_{e^e}^{n} (\log\log t)^{2\log\log t - 1} dt &\sim \frac{1}{t} \times \int_{e^e}^{n} \frac{(1+\log\log t)^{2\log\log t}}{\log\log t} dt \\ &= \frac{1}{t} \times \int_{e^e}^{n} \sum_{s \geqslant 0} \sum_{k=0}^{s} \begin{bmatrix} s \\ k \end{bmatrix} (2\log\log t)^k (-1)^{s-k} \times \frac{(\log\log t)^{s-1}}{s!} dt. \end{aligned}$$

Then since for any fixed m we have integrating by parts that

$$\begin{aligned} \frac{1}{t} \times \int (\log\log t)^m dt &= \frac{1}{t}\left(t \cdot (\log t)(\log\log t)^m - (\log t)(\log\log t)^m\right) \\ &\sim (\log t)(\log\log t)^m \\ &\gg (\log\log t)^m, \end{aligned}$$

we obtain our lower bound on the average order of $C_{\Omega(n)}(n)$. □

Proof of fact (B). We begin by proving a subclaim. Let the set defined by

$$\mathcal{D}_+ := \{n \geqslant 1 : d(n) > \log n\},$$

have corresponding asymptotic density

$$\alpha_+ := \lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{D}_+\}.$$

We claim that $\alpha_+ = 0$. To prove this we first note the following classical result on the asymptotic tendencies of the summatory function of $d(n)$ [13, §27.11]:

$$\sum_{n \leqslant x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}). \tag{19}$$

Then we see that as $x \rightarrow \infty$, by Abel summation we have that

$$\begin{aligned} \sum_{\substack{n \leqslant x \\ n \in \mathcal{D}_+}} \log n \geqslant \sum_{n \leqslant \alpha_+ x} \log n & \\ &= (\alpha_+ x) \cdot \log(\alpha_+ x) - \int_0^{\alpha_+ x} dt \\ &\sim (\alpha_+ x) \log x. \end{aligned}$$

Thus we can bound the summatory function of $d(n)$ for $n \leqslant x$ as $x \rightarrow \infty$ by

$$\begin{aligned} \sum_{n \leqslant x} d(n) \geqslant \sum_{\substack{n \leqslant x \\ n \notin \mathcal{D}_+}} d(n) + \sum_{\substack{n \leqslant x \\ n \notin \mathcal{D}_+}} \log n & \\ &\geqslant (1 - \alpha_+)x \left(\log x + \log(1 - \alpha_+) + 2\gamma - 1\right) + (\alpha_+ x) \log x + O(\sqrt{x}), \end{aligned}$$

which can only be true if $\alpha_+ = 0$ through the formula given by (19). By a similar argument reversing inequalities, we can see that indeed $\alpha_- = 0$ as well.

We next bound the average order expectations we see in the upper bound of the inequality using the known identity

$$\sum_{n \leqslant x} \sum_{d|n} f(d) = \sum_{d \leqslant x} f(d) \left\lfloor \frac{x}{d} \right\rfloor,$$

and summation by parts:

$$\begin{aligned} \mathbb{E}\left[\sum_{d|n} C_{\Omega(d)}(d)\right] &\sim \sum_{d \leqslant x} \frac{C_{\Omega(d)}(d)}{d} \\ &\sim \mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d < n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d} \\ &\sim (\log n) \cdot (\log \log n)^{2 \log \log n - 1} + \frac{(\log n)^2}{2} \cdot (\log \log n)^{2 \log \log n - 1}. \end{aligned} \tag{20}$$

The rightmost term in the previous equation follows by integration on the monotone non-decreasing summand (for all large enough $d > e^e$) as we justified using a close binomial series approximation in the proof of fact (A). The result is calculated according to the following indefinite integral formula when m is fixed and where we approximate the result using the asymptotic formula for the incomplete gamma function from Section 5.3:

$$\int \frac{(\log t)(\log \log t)^m}{t} dt = \frac{(\log t)^2}{2} \times (\log \log t)^m.$$

Next, we bound the lower portion of the expected inequality from above to agree at a middleground where the $n \in \mathcal{C}_E$ converge:

$$\sum_{d|n} \mathbb{E}[C_{\Omega(d)}(d)] \leqslant \sum_{d|n} (\log d) \cdot (\log \log d)^{2 \log \log n - 1}. \tag{21}$$

Consider that for n large, $\forall \sqrt{n} \leqslant j < n$, we have that $\pi(j) = \pi(\sqrt{n})$. This inequality implies that for some $p|n$, the smallest $d|n$ smaller than n is of the form $d = \frac{n}{p} \leqslant \sqrt{n}$ for some prime divisor p of n. Then we have from (21) that

$$\sum_{d|n} \mathbb{E}[C_{\Omega(d)}(d)] \leqslant (\log n) \cdot (\log\log n)^{2\log\log n - 1} + d(\sqrt{n})(\log n)(\log\log n)^{2\log\log n-1}. \tag{22}$$

As we have shown above, the set of positive integers on which $d(n) > \log n$ has thin asymptotic density $\alpha_+ = 0$. Now what this implies from (21) is that on an infinite set $\mathcal{A}_0 \subseteq \mathcal{C}_E$ of asymptotic density one, we have that

$$\sum_{d|n} \mathbb{E}[C_{\Omega(d)}(d)] \leqslant (\log n) \cdot (\log\log n)^{2\log\log n-1} + \frac{(\log n)^2}{2} (\log\log n)^{2\log\log n-1}.$$

Thus by (20), we have that

$$\mathbb{E}\left[\sum_{d|n} C_{\Omega(d)}(d)\right] \geqslant \sum_{d|n} \mathbb{E}[C_{\Omega(d)}(d)], \forall n \in \mathcal{A}_0.$$

Since $\mathcal{A}_0$ has limiting density of one, and $\mathcal{A}_0 \subseteq \mathcal{C}_E$, i.e., the $\mathcal{C}_E$ has an asymptotic density at least as great as that of $\mathcal{A}_0$, we conclude that the limiting density of $\mathcal{C}_E$ is also one. That is,

$$\lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{C}_E\} = 1. \qquad \square$$

Remark 8.2. What we have actually argued in less generality for the divisor function above holds for the expectation of positive arithmetic functions in general. That is, if $f(n) > 0$ for all $n \geqslant 1$, and if there exists a function F such that

$$\sum_{n\leqslant x} f(n) \sim x \cdot F(x),$$

where $F(x) \nrightarrow 0$ as $x \to \infty$, then the asymptotic density of the set of exeptional values of f is zero:

$$\lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leqslant x : f(n) > F(n)\} = 0.$$

Facts 8.3 (Bounds on the asymptotic densities of the sets over the parity of $\lambda(n) = \pm 1$)**.** Let the asymptotic densities for the distinct parity of $\Omega(n)$ (sign of $\lambda(n)$) be denoted by

$$\begin{aligned} \lambda_+ &:= \lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leqslant x : \lambda(n) = +1\} \\ \lambda_- &:= \lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leqslant x : \lambda(n) = -1\}. \end{aligned}$$

Then we can prove that $\lambda_+ = \lambda_- = \frac{1}{2}$.

Lemma 8.4 (Bounded expectation of $g^{-1}(n)$)**.** *For all large enough n, we have that*

$$2\sqrt{2} \leqslant \mathbb{E}[|g^{-1}(n)|] \leqslant \log(n) + o(1).$$

Proof. Consider that using Lemma 7.1 we have

$$\begin{aligned} \frac{1}{x}\sum_{n\leqslant x} |g^{-1}(n)| &\geqslant \left|\frac{1}{x}\sum_{n\leqslant x} g^{-1}(n)\right| \\ &= \left|\frac{1}{x}\sum_{n\leqslant x}\sum_{d|n} \mu(n/d)\lambda(d) C_{\Omega(d)}(d)\right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{1}{x} \sum_{d \leqslant x} \lambda(d) C_{\Omega(d)}(d) M\left(\frac{x}{d}\right) \right| \\
&\geqslant \left| \frac{1}{\sqrt{x}} \sum_{d \leqslant x} \frac{\lambda(d)}{\sqrt{d}} C_{\Omega(d)}(d) \right| \\
&\gtrsim \left| \frac{1}{\sqrt{x}} \sum_{d \leqslant \lambda_+ \cdot x} \frac{2}{\sqrt{d}} \right| \\
&\approx \left| \frac{1}{\sqrt{x}} \int_0^{\lambda_+ \cdot x} \frac{1}{t^{1/2}} dt \right| \\
&= 2\sqrt{2}.
\end{aligned}
$$

In the previous equations we used that $\lambda_+ = \lambda_- = \frac{1}{2}$, that the function $1/\sqrt{d}$ is monotone decreasing, and that $C_{\Omega(n)}(n) \geqslant 2$ for all $n \geqslant 1$. We also used that $|M(x)| \geqslant \sqrt{x}$.
Notice that since $|M(x)| \leqslant x$ for all $x \geqslant 1$, we can follow similar steps to show instead that

$$\mathbb{E}[|g^{-1}(n)|] \leqslant \log(\lambda_+ \cdot x) + \gamma + O\left(\frac{1}{x}\right),$$

which implies our upper bound. □

Proposition 8.5. *Let the set where $G^{-1}(x)$ is non-negative be defined as*

$$\mathcal{G}_+ := \left\{n \leqslant x : G^{-1}(x) \geqslant 0\right\}.$$

We claim that for all large $x \to \infty$, the density of this set is positive:

$$0 < \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{G}_+\} < 1.$$

Moreover, if a limiting asymptotic density for $\mathcal{G}_+$ exists, it does not tend to zero as $x \to \infty$:

$$\lim_{x \to \infty} \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{G}_+\} \neq 0.$$

Note that the proposition above also implies that the corresponding set $\mathcal{G}_-$ over which $G^{-1}(x) < 0$ has positive density for all x sufficiently large, and that this density does not tend to zero as $x \to \infty$. We will prove Proposition 8.5 after we prove Proposition 9.1 in the next section.

Proof of the hypothesis of Theorem 4.9. Let $G_E^{-1}(x)$ be defined as in (4) of the theorem. We need to find some absolute tight (tightest possible) constants $B, C \in (0, 1]$ such that as $x \to \infty$

$$B + o(1) \leqslant \frac{1}{x} \cdot \#\left\{n \leqslant x : |G^{-1}(n)| - |G_E^{-1}(n)| \leqslant 0\right\} \leqslant C + o(1). \tag{23}$$

Our proof of fact (B) above implies that on the set $n \in \mathcal{C}_E$, which has asymptotic density of one, we have

$$\mathbb{E}\left[\sum_{d|n} C_{\Omega(d)}(d)\right] - |g^{-1}(n)| \geqslant \sum_{d|n} \mathbb{E}[C_{\Omega(d)}(d)] - |g^{-1}(n)|. \tag{24a}$$

Moreover, by Remark 8.2 and Lemma 8.4, for n within a set $\mathcal{S}_E$ of asymptotic density also one,

$$\mathbb{E}[C_{\Omega(d)}(d)] - |g^{-1}(n)| \geqslant \mathbb{E}\left[\sum_{d|n} (\log d)(\log\log d)\right] - \mathbb{E}[|g^{-1}(n)|]$$

$$\begin{aligned} &\sim \frac{(\log n)^2}{2}(\log\log n) - (\log n) \\ &\geqslant 0, \text{ as } n \to \infty, \end{aligned}$$

where we have used that

$$\int \frac{(\log t)(\log\log t)}{t} dt = \frac{(\log t)^2}{2}(\log\log t) - \frac{(\log t)^2}{4}.$$

Now we aim to sum the functions $G^{-1}(x)$ and $G_E^{-1}(x)$ weighted by the same signs on the terms at each n that respectively satisfy the following condition:

$$\mathbb{E}\left[\sum_{d|n} C_{\Omega(d)}(d)\right] - |g^{-1}(n)| \geqslant \sum_{d|n} \mathbb{E}[C_{\Omega(d)}(d)] - |g^{-1}(n)| \geqslant 0. \tag{24b}$$

Notice that the intersection of the sets $\mathcal{C}_E \cap \mathcal{S}_E$ necessarily also has asymptotic density of one. So (24b) holds for all n on a set of full asymptotic density one (e.g., almost everywhere on the integers).

Since the sign of $g^{-1}(n)$ is $\lambda(n)$ as given by Proposition 5.1, on the set satisfying (24b), we have that both

$$\begin{aligned} \sum_{\substack{n \leqslant x \\ \lambda(n)=+1}} g^{-1}(n) &\leqslant \sum_{\substack{n \leqslant x \\ \lambda(n)=+1}} \sum_{d|n} \mathbb{E}[C_{\Omega(d)}(d)] \\ \sum_{\substack{n \leqslant x \\ \lambda(n)=-1}} g^{-1}(n) &\leqslant -\sum_{\substack{n \leqslant x \\ \lambda(n)=-1}} \sum_{d|n} \mathbb{E}[C_{\Omega(d)}(d)]. \end{aligned}$$

Hence, on this set we have that

$$G^{-1}(x) \leqslant \sum_{n \leqslant x} \lambda(n) \sum_{\substack{d|n \\ d > e^e}} \mathbb{E}[C_{\Omega(d)}(d)], \forall x \in \mathcal{S}_E \cap \mathcal{C}_E. \tag{25}$$

Now noticing that the right-hand-side of (25) corresponds to the definition of the function $G_E^{-1}(x)$, we observe that if $G^{-1}(x) \geqslant 0$ on this set, then also $G_E^{-1}(x) \geqslant 0$, and so for $x \in \mathcal{S}_E \cap \mathcal{C}_E$ such that this holds (almost everywhere) we have that $|G^{-1}(x)| - |G_E^{-1}(x)| \leqslant 0$.

Finally, using Proposition 8.5, we can see that there are constants $B, C \in (0,1)$ such that there is a set $\mathcal{H}_T$ with limiting asymptotic density bounded between these constants such that the condition $|G^{-1}(x)| - |G_E^{-1}(x)| \leqslant 0$ holds:

$$B + o(1) \leqslant \frac{1}{x} \cdot \#\left\{n \leqslant x : |G^{-1}(n)| - |G_E^{-1}(n)| \leqslant 0\right\} \leqslant C + o(1), \text{ as } x \to \infty.$$

Hence, we have shown that the necessary condition in Theorem 4.9 can in fact be attained. □

9 Establishing the lower bounds for $M(x)$ by cases along infinite subsequences

9.1 The culmination of what we have done so far

Proposition 9.1. *For all sufficiently large x, we have that*

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt, \tag{26}$$

where $G^{-1}(x) := \sum_{n \leqslant x} g^{-1}(n)$ is the summatory function of $g^{-1}(n)$.

Proof. We know by applying Corollary 4.3 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)(\pi(x/k) + 1) \\ &= G^{-1}(x) + \sum_{k=1}^{x} g^{-1}(k)\pi(x/k), \end{aligned} \tag{27}$$

where we can drop the asymptotically unnecessary floored integer-valued arguments to $\pi(x)$ in place of its approximation by the monotone non-decreasing $\pi(x) \sim \frac{x}{\log x}$. Moreover, we can always bound

$$\frac{Ax}{\log x} \leqslant \pi(x) \leqslant \frac{Bx}{\log x},$$

for suitably defined absolute constants, $A, B > 0$. Therefore the approximation obtained is valid for all $x > 1$ up to a small constant difference.

What we now require to sum and simplify the right-hand-side summation from (27) is an ordinary summation by parts argument. Namely, we obtain that for sufficiently large $x \geqslant 2$ *

$$\begin{aligned} \sum_{k=1}^{x} g^{-1}(k)\pi(x/k) &= G^{-1}(x)\pi(1) - \sum_{k=1}^{x-1} G^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)\right] \\ &= -\sum_{k=1}^{x/2} G^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right)\right] \\ &\approx -\sum_{k=1}^{x/2} G^{-1}(k) \left[\frac{x}{k \cdot \log(x/k)} - \frac{x}{(k+1) \cdot \log(x/k)}\right] \\ &\approx -\sum_{k=1}^{x/2} G^{-1}(k) \frac{x}{k^2 \cdot \log(x/k)}. \end{aligned}$$

Since for x large enough the summand is monotonic as k ranges in order over $k \in [1, x/2]$, and since the summands in the last equation are smooth functions of k (and x), and also since $G^{-1}(x)$ is a summatory function with jumps at the positive integers, we can approximate $M(x)$ for any finite $x \geqslant 2$ by

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt.$$

We will later only use unsigned lower bound approximations to this function in the next theorems so that the signedness of the summatory function term in the integral formula above as $x \to \infty$ is a moot point entirely. □

*Since $\pi(1) = 0$, the actual range of summation corresponds to $k \in \left[1, \frac{x}{2}\right]$.

Proof of Proposition 8.5. Suppose to the contrary that

$$\lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leqslant x : n \in \mathcal{G}_+\} = 0,$$

i.e., that $G^{-1}(x) \leqslant 0$ almost everywhere for all integers x sufficiently large. We will utilize (27) from Proposition 9.1 to derive a contradiction under this assumption. In particular, assuming the above limiting density is zero, we have that

$$|M(x)| \approx \left| x \cdot \int_1^{x/2} \frac{|G^{-1}(t)|}{t^2 \cdot \log(x/t)} - |G^{-1}(x)| \right|, \text{ a.e., as } x \to \infty. \tag{28}$$

So since we expect by Lemma 8.4 (see Remark 8.2) that almost everywhere on the suffficiently large integers x, we have

$$2x \leqslant |G^{-1}(x)| \leqslant x \log x.$$

From (28), we then obtain that

$$\begin{aligned}|M(x)| &\geqslant \left| 2x \cdot \int_{e^e}^{x/2} \frac{dt}{t \cdot \log(x/t)} - x \log x \right| + C_M \\ &= |2 \log\log(2) \cdot x + x \log x| + C_M,\end{aligned}$$

for some bounded constant $C_M \in (-\infty, +\infty)$. Here, we have computed that

$$\int \frac{dt}{t \cdot \log(x/t)} = -\log\log(x/t) + C.$$

This inequality clearly violates the known (obvious) bound that $|M(x)| \leqslant x$ for all $x \geqslant 1$. A similarly phrased argument shows the corresponding result is true for the set $\mathcal{G}_-$. Thus, combined, these two consequences show that the limiting density of $\mathcal{G}_+$ is positive, and in particular, that it cannot tend to zero as $x \to \infty$. □

9.1.1 From the routine: Proofs of a few cut-and-dry lemmas

Corollary 9.2. *We have that for sufficiently large x, as $x \to \infty$ that*

$$\left|G_E^{-1}(x)\right| \overset{\blacktriangle}{\gtrsim} \left| \widehat{L}_0\left(\log\log x\right) \times \sum_{n \leqslant \log\log x} \lambda(n) \cdot \mathbb{E}[C_{\Omega(n)}(n)] \right|,$$

where the function

$$\left|\widehat{L}_0(x)\right| \overset{\blacktriangle}{\gtrsim} \sqrt{\frac{2}{\pi}} A_0 \cdot x \frac{(\log x)^{2\log 2 + \frac{1}{3\log 2} - 1}}{(\log\log x)^{\frac{5}{2} + \log\log x}},$$

and such that $\operatorname{sgn}(\widehat{L}_0(x)) = (-1)^{\lfloor \log\log x \rfloor}$ *(as the function is defined inline below), and where the exponent* $2\log 2 + \frac{1}{3\log 2} - 1 \approx 0.867193$.

Proof. Using the definition in (4), we have that †

$$\left|G_E^{-1}(x)\right| \gg \left| \sum_{n \leqslant x} \lambda(n) \sum_{d|n} \mathbb{E}[C_{\Omega(d)}(d)] \right|$$

†For any arithmetic functions f, h, we have that [1, *cf.* §3.10; §3.12]

$$\sum_{n \leqslant x} h(n) \times \sum_{d|n} f(d) = \sum_{d \leqslant x} f(d) \times \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} h(dn).$$

$$= \left| \sum_{d \leqslant \log\log x} \mathbb{E}[C_{\Omega(d)}(d)] \times \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn) \right|.$$

Now we see that by complete additivity of $\Omega(n)$ (multiplicativity of $\lambda(n)$) that

$$\sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(dn) = \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \lambda(d)\lambda(n) = \lambda(d) \sum_{n \leqslant \left\lfloor \frac{x}{d} \right\rfloor} \lambda(n).$$

Using the result proved in Section 6 as (see Theorem 4.8 and Corollary 6.4) we can establish that

$$\left| \sum_{n \leqslant x} \lambda(n) \right| \gg \left| \sum_{k \leqslant \log\log x} (-1)^k \cdot \widehat{\pi}_k(x) \right| =: \left| \widehat{L}_0(x) \right|.$$

Then since for large enough x and $d \ll x$,

$$\log(x/d) \sim \log x, \log\log(x/d) \sim \log\log x,$$

we have that

$$\left| \widehat{L}_0(\log\log x) \right| \sim \left| \widehat{L}_0(\log\log(x/d)) \right|,$$

with $d \leqslant \log\log x$ and for all large $x \to \infty$. The limiting lower bound stated above for $\widehat{L}_0(x)$ is computed by symbolic summation in *Mathematica* using the new bounds on $\widehat{\pi}_k(x)$ guaranteed by Theorem 4.8. □

Lemma 9.3. *Suppose that $f_k(n)$ is a sequence of arithmetic functions such that $f_k(n) > 0$ for all $n \geqslant 1$, $f_0(n) = \delta_{n,1}$, and $f_{\Omega(n)}(n) \overset{\blacktriangle}{\gtrsim} \widehat{\tau}_\ell(n)$ as $n \to \infty$. We suppose that $\widehat{\tau}_\ell(t)$ is a continuously differentiable function of t for all large enough $t \gg 1$ [‡]. We define the λ-sign-scaled summatory function of f as follows:*

$$F_\lambda(x) := \sum_{\substack{n \leqslant x \\ \Omega(n) \leqslant x}} \lambda(n) \cdot f_{\Omega(n)}(n).$$

Let

$$A_\Omega^{(\ell)}(t) := \sum_{k=1}^{\lfloor \log\log t \rfloor} (-1)^k \widehat{\pi}_k^{(\ell)}(t),$$

where $\widehat{\pi}_k(x) \geqslant \widehat{\pi}_k^{(\ell)}(x) \geqslant 0$ for $\widehat{\pi}_k^{(\ell)}(t)$ some a smooth monotone non-decreasing function of t whenever t sufficiently large. Then we have that

$$F_\lambda(\log\log x) \overset{\blacktriangle}{\gtrsim} A_\Omega^{(\ell)}(\log\log x)\widehat{\tau}_\ell(\log\log x) - \int_1^{\log\log x} A_\Omega^{(\ell)}(t)\widehat{\tau}_\ell'(t)dt.$$

Proof. We can form an accurate $C^1(\mathbb{R})$ approximation by the smoothness of $\widehat{\pi}_k^{(\ell)}(x)$ that allows us to apply the Abel summation formula using the summatory function $A_\Omega^{(\ell)}(t)$ for t on any connected subinterval of $[1, \infty)$. The second stated formula for $F_\lambda(\log\log x)$ is valid by Abel summation whenever

$$0 \leqslant \left| \frac{\sum\limits_{\log\log t < k \leqslant \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_\Omega^{(\ell)}(t)} \right| \leqslant 2, \text{ as } t \to \infty,$$

[‡] We will require that $\widehat{\tau}_\ell(t) \in C^1(\mathbb{R})$ when we apply the Abel summation formula in the proof of Theorem 9.5. At this point, it is technically an unnecessary condition that is vacously satisfied by assumption (by requirement) and will importantly need to hold only when we specialize to the actual functions employed to form our new bounds in the theorem below.

What the last equation implies is that the asymptotically dominant terms indicating the parity of $\lambda(n)$ are captured up to a constant factor by the terms in the range over k summed by $A_{\Omega}^{(\ell)}(t)$ for sufficiently large $t \to \infty$.

In other words, taking the sum over the summands that defines $A_{\Omega}(x)$ only over the truncated range of $k \in [1, \log\log x]$ does not affect the limiting asymptotically dominant terms in the lower bound obtained from using this form of the summatory function in conjunction with the Abel summation formula. This property holds even when we should technically index over all $k \in [1, \log_2(x)]$ to obtain an exact formula for this function. Using the arguments in Montgomery and Vaughan [11, §7; Thm. 7.20] (see Corollary 6.4), we can see that the assertion above holds in the limit as $t \to \infty$. □

The results in Corollary 9.2 and in Lemma 9.3 combine to provide a key formula used in the proof of Theorem 9.5 to bound $G^{-1}(x)$ from below in the average order sense. We require one more sanity check to our approximations used in that proof explored in the next subsection in the form of the next lemma. Observe that we now use the superscript and subscript notation of (ℓ) not to denote a formal parameter to the functions we define below, but instead to denote that these functions form *lower bound* approximations to other forms of the functions without the scripted (ℓ).

Lemma 9.4. *Suppose that $\widehat{\pi}_k(x) \geqslant \widehat{\pi}_k^{(\ell)}(x) \geqslant 0$ with $\widehat{\pi}_k^{(\ell)}(x)$ a monotone non-decreasing real-valued function for all sufficiently large x. Let*

$$\begin{aligned} A_{\Omega}^{(\ell)}(x) &:= \sum_{k \leqslant \log\log x} (-1)^k \widehat{\pi}_k^{(\ell)}(x) \\ A_{\Omega}(x) &:= \sum_{k \leqslant \log\log x} (-1)^k \widehat{\pi}_k(x). \end{aligned}$$

Then for all sufficiently large x, we have that

$$|A_{\Omega}(x)| \gg |A_{\Omega}^{(\ell)}(x)|.$$

Proof. Given an explicit smooth lower bounding function, $\widehat{\pi}_k^{(\ell)}(x)$, we define the similarly smooth and monotone residual terms in approximating $\widehat{\pi}_k(x)$ using the following notation:

$$\widehat{\pi}_k(x) = \widehat{\pi}_k^{(\ell)}(x) + \widehat{E}_k(x).$$

Then we can form the ordinary form of the summatory function A_{Ω} as

$$\begin{aligned} |A_{\Omega}(x)| &= \left| \sum_{k \leqslant \frac{\log\log x}{2}} \left[\widehat{\pi}_{2k}(x) - \widehat{\pi}_{2k-1}(x)\right] \right| \\ &\geqslant \left| A_{\Omega}^{(\ell)}(x) - \sum_{k \leqslant \frac{\log\log x}{2}} \widehat{E}_{2k-1}(x) \right| \\ &\geqslant \left| A_{\Omega}^{(\ell)}(x) \right| - \left| \sum_{k \leqslant \frac{\log\log x}{2}} \widehat{E}_{2k-1}(x) \right|. \end{aligned}$$

If the latter sum,

$$\mathrm{ES}(x) := \sum_{k \leqslant \frac{\log\log x}{2}} \widehat{E}_{2k-1}(x) \to \infty,$$

as $x \to \infty$, then we can always find some absolute (by monotonicity) $C_0 > 0$ such that $\mathrm{ES}(x) \leqslant C_0 \cdot A_{\Omega}(x)$. If on the other hand this sum becomes constant as $x \to +\infty$, then we also clearly have another absolute $C_1 > 0$ such that $|A_{\Omega}(x)| \geqslant C_1 \cdot |A_{\Omega}^{(\ell)}(x)|$. In either case, the claimed result holds for all large enough x. □

9.1.2 A proof of the key bound from below on $G^{-1}(x)$

The following central theorem is the last key barrier required to prove Corollary 9.2 in the subsection:

Theorem 9.5 (Asymptotics and bounds for the summatory functions $G^{-1}(x)$)**.** *We define a lower summatory function, $G_\ell^{-1}(x)$, to provide bounds on the magnitude of $G_E^{-1}(x)$ such that*

$$|G_\ell^{-1}(x)| \ll |G_E^{-1}(x)|,$$

for all sufficiently large $x \geqslant e^e$ as follows:

$$G_\ell^{-1}(x) := \sum_{n \leqslant x} \lambda(n) \times \sum_{\substack{d|n \\ d>e^e}} (\log d) \cdot (\log\log d)^{2\log\log d - 1}.$$

Then we have the next asymptotic approximations for the lower summatory function where $C_{\ell,1}$ is the absolute constant defined by

$$C_{\ell,1} = 4A_0^2 = \frac{128 \cdot 2^{1/8}}{6561 \cdot e^6 \pi \log^6(2)} \exp\left(-\frac{55}{2}\log^2(2)\right) \approx 2.76631 \times 10^{-10}.$$

That is, we have on average §

$$\left|G_\ell^{-1}(x)\right| \overset{\blacktriangle}{\gtrsim} C_{\ell,1} \cdot (\log\log x) \frac{(\log\log\log x)^{2\log 2 + \frac{1}{3\log 2} - 2}}{(\log\log\log\log x)^{\frac{5}{2}}} \cdot \frac{\log_*^5(x)^{2\log 2 + \frac{1}{3\log 2}}}{\log_*^6(x)^{\frac{5}{2}}}.$$

The exponent in the previous equation is numerically approximated as $2\log 2 + \frac{1}{3\log 2} - 2 \approx -0.402633 < 0$.

Initial Sketch: Logarithmic scaling of parameters to the accurate order. For the sums given by

$$S_{g^{-1}}(x) := \sum_{n \leqslant x} \lambda(n) \cdot C_{\Omega(n)}(n),$$

we notice that using the average case asymptotic bounds (rather than the complicated exact formulas) for the functions $C_{\Omega(n)}(n)$ from Theorem 4.6, we recover an observation that have over-summed on the range over n by quite a bit. In particular, following from the intent behind the constructions in the last sections, we are really summing only over all $n \leqslant x$ with $\Omega(n) \leqslant x$. Since $\Omega(n) \leqslant \lfloor \log_2 n \rfloor$ maximally, many of the terms in the previous equation are actually zero (recall that $C_0(n) = \delta_{n,1}$).

So we are actually only summing on n up to the average order of $\mathbb{E}[\Omega(n)] = \log\log n$ in practice. Hence, the sum that we are really interested in bounding is bounded below in magnitude by $S_{g^{-1}}(\log\log x)$ as bounded in Corollary 4.6. After noting this adjustment, we can then safely apply the asymptotic formulas for the functions $C_k(n)$ from Theorem 4.6 that hold once we have verified these important constraints on k, n, x. □

Proof. Recall from our proof of Corollary 4.8 that a lower bound on the variant prime form counting function is given by

$$\widehat{\pi}_k(x) \overset{\blacktriangle}{\gtrsim} \frac{A_0 \cdot x}{\log x \cdot (\log\log x)^4 \cdot (k-1)!} \cdot \left(\frac{4}{3e\log 2}\right)^k, \text{ as } x \to \infty.$$

So we can then form a lower summatory function indicating the parity of all $\Omega(n)$ for $n \leqslant x$ as

$$\left|A_\Omega^{(\ell)}(t)\right| = \left|\sum_{k \leqslant \log\log t} (-1)^k \widehat{\pi}_k(t)\right| \tag{29}$$

§E.g., within a predictably bounded interval around each x sufficiently large. This distinction in the statement is necessary since our limiting lower bounds have so far depended on average order estimates of certain sums and arithmetic functions as $n \to \infty$. We will rely on the results proved in Section 8 to justify that these lower bounds that hold on average can still be reconciled to prove the key corollary in the next subsection using an infinitely tending subsequence defined pointwise within intervals.

$$\overset{\blacktriangle}{\gtrsim} \sqrt{\frac{2}{\pi}} A_0 \cdot (\log\log t) \frac{(\log\log\log t)^{2\log 2+\frac{1}{3\log 2}-1}}{(\log\log\log\log t)^{\frac{5}{2}+\log\log\log\log t}},$$

where the actual sign on this function is given by $\operatorname{sgn}(A_\Omega^{(\ell)}(t)) = (-1)^{\lfloor \log\log\log\log t \rfloor}$ (see Lemma 9.4).

Next, by Corollary 4.6 we recover from the main term approximation to $C_{\Omega(n)}(n)$ proved in Section 8, denoted here by $\widehat{\tau}_0(t) = (\log t) \cdot (\log\log t)^{2\log\log t - 1}$, that

$$\widehat{\tau}_0'(t) = \frac{d}{dt}\left[(\log t) \cdot (\log\log t)^{2\log\log t-1}\right] \overset{\blacktriangle}{\gtrsim} \frac{2(\log\log t)^{2\log\log t-1}(\log\log\log t)}{t}.$$

As in Lemma 9.3 and Corollary 9.2, we apply Abel summation to obtain that we have

$$G_\ell^{-1}(x) = \widehat{L}_0(\log\log x)\left[\widehat{\tau}_0(\log\log x) A_\Omega^{(\ell)}(\log\log x) - \widehat{\tau}_0(u_0) A_\Omega^{(\ell)}(u_0) - \int_{u_0}^{\log\log x} \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t) dt\right]. \tag{30}$$

The inner integral term on the rightmost side of (30) is summed approximately in the form of

$$\begin{aligned}\int_{u_0}^{\log\log x} \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t) dt &\sim \sum_{k=u_0+1}^{\frac{1}{2}\log\log\log\log x} \left(I_\ell\left(e^{e^{2k+1}}\right) - I_\ell\left(e^{e^{2k}}\right)\right) e^{e^{2k}} \\ &\approx C_0(u_0) + (-1)^{\left\lfloor \frac{\log\log\log\log x}{2} \right\rfloor} \times \int_{\frac{\log\log\log\log x}{2}-1}^{\frac{\log\log\log\log x}{2}} I_\ell\left(e^{e^{2k}}\right) e^{e^{2k}} dk.\end{aligned} \tag{31}$$

We define the integrand function, $I_\ell(t) := \widehat{\tau}_0'(t) A_\Omega^{(\ell)}(t)$, from the previous equations with some limiting simplifications for the $k \in \left[\frac{\log\log\log\log x}{2} - 1, \frac{\log\log\log\log x}{2}\right]$ as

$$I_\ell\left(e^{e^{2k}}\right) e^{e^{2k}} \overset{\blacktriangle}{\gtrsim} \frac{2^{3/2} A_0 \cdot (2k)^{4k}}{\sqrt{\pi}} \frac{\log(2k)^{2\log 2+\frac{1}{3\log 2}}}{\log\log(2k)^{\frac{5}{2}+\log\log(2k)}}. \tag{32}$$

So using the lower bound on the integrand in (32), we find that [¶]

$$\begin{aligned}&\widehat{L}_0(\log\log x) \times \int_{\frac{\log\log\log\log x}{2}-1}^{\frac{\log\log\log\log x}{2}} I_\ell\left(e^{e^{2k}}\right) e^{e^{2k}} dk \\ &\quad \overset{\blacktriangle}{\gtrsim} \frac{4B_\ell A_0^2}{\pi}(\log\log x) \frac{(\log\log\log x)^{2\log 2+\frac{1}{3\log 2}-1}}{(\log\log\log\log x)^{\frac{5}{2}}} \cdot \frac{\log_*^5(x)^{2\log 2+\frac{1}{3\log 2}}}{\log_*^6(x)^{\frac{5}{2}}}.\end{aligned} \tag{33}$$

It is clear from our prior computations of the growth of $A_\Omega^{(\ell)}(x)$ and $\widehat{\tau}_0(x)$ that the asymptotically dominant behavior of the lower bound for $|G_\ell^{-1}(x)|$ comes from the integral term calculated in the last equation of (33). To make this observation precise, consider the following expansion for comparison with (33):

$$\begin{aligned}\widehat{L}_0(\log\log x)\widehat{\tau}_0(\log\log x) A_\Omega^{\ell}(\log\log x) &\overset{\blacktriangle}{\gtrsim} \frac{4A^2}{\pi\log\log x} \cdot \frac{\log_*^4(x)^{2\log_*^4(x)+1}}{\log_*^6(x)^{2\log_*^6(x)+5}} \cdot \log_*^5(x)^{4\log 2+\frac{2}{3\log 2}-1} \\ &\overset{\blacktriangle}{\gtrsim} \frac{4A^2}{\pi\log\log x} \cdot \frac{\log_*^4(x)^3}{\log_*^{4\log 2+\frac{2}{3\log 2}+4}(x)^4 \cdot \log_*^6(x)^5}.\end{aligned}$$

[¶] We have invoked the simplification that for sufficiently large x,

$$(\log\log\log\log\log x)^2 \overset{\blacktriangle}{\gtrsim} \exp\left(-(\log\log\log\log\log\log x)^2\right).$$

The simplifications in arriving at the last equation follow from the bounds

$$\begin{aligned}\exp\left(2\log_*^4(x)\cdot\log_*^5(x)\right) &\overset{\blacktriangle}{\gtrsim} \exp\left(\log_*^5(x)^2\right) \overset{\blacktriangle}{\gtrsim} \log_*^4(x)^2 \\ \exp\left(-2\log_*^6(x)\cdot\log_*^7(x)\right) &\overset{\blacktriangle}{\gtrsim} \exp\left(-2\log_*^6(x)^2\right) \overset{\blacktriangle}{\gtrsim} \log_*^5(x)^4.\end{aligned}$$

So clearly, in absolute value the integral term dominates the asymptotics. □

9.2 Lower bounds on the scaled Mertens function along an infinite subsequence

What we will have shown in total concluding the proof of Corollary 4.12 below is a logarithmically scaled lesser form of the classically conjectured unboundedness property of $M(x)$ in the form of

$$\limsup_{x\to\infty}\frac{|M(x)|\log x}{\sqrt{x}} = +\infty.$$

This statement still comprises a better than previously known rate of the minimal asymptotic tendencies of $|M(x)|/\sqrt{x}$ towards unboundedness along an infinite subsequence, e.g., progress on the classical problem. This is still a much weaker condition than the RH as stated, and moreover, we must emphasize that its construction is much differently motivated by the encouraging combinatorial structures we have observed.

Now we finally address the conclusion of our argument:

Proof of Corollary 4.12. It suffices to take $u_0 = e^{e^{e^e}}$. Now, we break up the integral over $t \in [u_0, x/2]$ into two pieces: one that is easily bounded from $u_0 \leqslant t \leqslant \sqrt{x}$, and then another that will conveniently give us our slow-growing tendency towards infinity along the subsequence. In the next calculations, we assume that $x \mapsto x_y$ is taken along the subsequence defined inexplicitly over intervals as stated above.

First, since $\pi(j) = \pi(\sqrt{x})$ for all $\sqrt{x} \leqslant j < x$, we can take the first chunk of the interval of integration and bound it using (26) as

$$-\int_{u_0}^{\sqrt{x}} \frac{2\sqrt{x}}{t^2\log(x)} G_\ell^{-1}(t)dt \overset{\blacktriangle}{\gtrsim} B_{\ell,2}\times\frac{2}{\log(x)}\cdot\left(\min_{u_0\leqslant t\leqslant\sqrt{x}} G_\ell^{-1}(t)\right) = o(1),$$

where $B_{\ell,2}$ can be taken as an indefinite, but still some absolute constant with respect to u_0. The maximum in the previous equation is clearly attained by taking $t := \sqrt{x}$.

We next have to prove a related bound over the second portion of the interval from $\sqrt{x} \leqslant t \leqslant x/2$:

$$\begin{aligned}-\int_{\sqrt{x}}^{x/2} \frac{2x}{t^2\log(x)}\cdot G_\ell^{-1}(t)dt &\overset{\blacktriangle}{\gtrsim} \frac{2\sqrt{x}}{\log x}\cdot\left(\min_{\sqrt{x}\leqslant t\leqslant x/2} G_\ell^{-1}(t)\right) \\ &= 2C_{\ell,1}\cdot\sqrt{x}\cdot\frac{(\log\log\sqrt{x})}{\log x}\frac{(\log\log\log\sqrt{x})^{2\log 2+\frac{1}{3\log 2}-1}}{(\log\log\log\log\sqrt{x})^{\frac{5}{2}}}\cdot\frac{\log_*^5(\sqrt{x})^{2\log 2+\frac{1}{3\log 2}}}{\log_*^6(\sqrt{x})^{\frac{5}{2}}} + o(1).\end{aligned}$$

Finally, since $G_\ell^{-1}(x) = o(\sqrt{x})$, we obtain in total that as $x\to\infty$ along this infinite subsequence:

$$|M(x)| \overset{\blacktriangle}{\gtrsim} 2C_{\ell,1}\cdot\sqrt{x}\cdot\frac{(\log\log\sqrt{x})}{\log x}\frac{(\log\log\log\sqrt{x})^{2\log 2+\frac{1}{3\log 2}-1}}{(\log\log\log\log\sqrt{x})^{\frac{5}{2}}}\cdot\frac{\log_*^5(\sqrt{x})^{2\log 2+\frac{1}{3\log 2}}}{\log_*^6(\sqrt{x})^{\frac{5}{2}}}. \square$$

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T.1 Table: Computations with a signed Dirichlet inverse function and its summatory function

n	Primes		Sqfree	PPower	$\bar{\mathbb{S}}$		$g^{-1}(n)$	$\lambda(n)\operatorname{sgn}(g^{-1}(n))$	$\lambda(n)g^{-1}(n) - \hat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$		$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1^1	–	Y	N	N	–	1	1	0	1.0000000	–	1	1	0
2	2^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	−1	1	−2
3	3^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	−3	1	−4
4	2^2	–	N	Y	N	–	2	1	0	1.5000000	–	−1	3	−4
5	5^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	−3	3	−6
6	$2^1 3^1$	–	Y	N	N	–	5	1	0	1.0000000	–	2	8	−6
7	7^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	0	8	−8
8	2^3	–	N	Y	N	–	−2	1	0	2.0000000	–	−2	8	−10
9	3^2	–	N	Y	N	–	2	1	0	1.5000000	–	0	10	−10
10	$2^1 5^1$	–	Y	N	N	–	5	1	0	1.0000000	–	5	15	−10
11	11^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	3	15	−12
12	$2^2 3^1$	–	N	N	Y	–	−7	1	2	1.2857143	–	−4	15	−19
13	13^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	−6	15	−21
14	$2^1 7^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−1	20	−21
15	$3^1 5^1$	–	Y	N	N	–	5	1	0	1.0000000	–	4	25	−21
16	2^4	–	N	Y	N	–	2	1	0	2.5000000	–	6	27	−21
17	17^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	4	27	−23
18	$2^1 3^2$	–	N	N	Y	–	−7	1	2	1.2857143	–	−3	27	−30
19	19^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	−5	27	−32
20	$2^2 5^1$	–	N	N	Y	–	−7	1	2	1.2857143	–	−12	27	−39
21	$3^1 7^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−7	32	−39
22	$2^1 11^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−2	37	−39
23	23^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	−4	37	−41
24	$2^3 3^1$	–	N	N	Y	–	9	1	4	1.5555556	–	5	46	−41
25	5^2	–	N	Y	N	–	2	1	0	1.5000000	–	7	48	−41
26	$2^1 13^1$	–	Y	N	N	–	5	1	0	1.0000000	–	12	53	−41
27	3^3	–	N	Y	N	–	−2	1	0	2.0000000	–	10	53	−43
28	$2^2 7^1$	–	N	N	Y	–	−7	1	2	1.2857143	–	3	53	−50
29	29^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	1	53	−52
30	$2^1 3^1 5^1$	–	Y	N	N	–	−16	1	0	1.0000000	–	−15	53	−68
31	31^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	−17	53	−70
32	2^5	–	N	Y	N	–	−2	1	0	3.0000000	–	−19	53	−72
33	$3^1 11^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−14	58	−72
34	$2^1 17^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−9	63	−72
35	$5^1 7^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−4	68	−72
36	$2^2 3^2$	–	N	N	Y	–	14	1	9	1.3571429	–	10	82	−72
37	37^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	8	82	−74
38	$2^1 19^1$	–	Y	N	N	–	5	1	0	1.0000000	–	13	87	−74
39	$3^1 13^1$	–	Y	N	N	–	5	1	0	1.0000000	–	18	92	−74
40	$2^3 5^1$	–	N	N	Y	–	9	1	4	1.5555556	–	27	101	−74
41	41^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	25	101	−76
42	$2^1 3^1 7^1$	–	Y	N	N	–	−16	1	0	1.0000000	–	9	101	−92
43	43^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	7	101	−94
44	$2^2 11^1$	–	N	N	Y	–	−7	1	2	1.2857143	–	0	101	−101
45	$3^2 5^1$	–	N	N	Y	–	−7	1	2	1.2857143	–	−7	101	−108
46	$2^1 23^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−2	106	−108
47	47^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	−4	106	−110
48	$2^4 3^1$	–	N	N	Y	–	−11	1	6	1.8181818	–	−15	106	−121

Table T.1: ***Computations of*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for small*** $\mathbf{1 \leqslant n \leqslant 350}$.
The column labeled `Primes` *provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled, respectively,* `Sqfree`, `PPower` *and $\bar{\mathbb{S}}$ list inclusion of n in the sets of squarefree integers, prime powers, and the set $\bar{\mathbb{S}}$ that denotes the positive integers n which are neither squarefree nor prime powers. The next two columns provide the explicit values of the inverse function $g^{-1}(n)$ and indicate that the sign of this function at n is given by $\lambda(n)$.*
The next column shows the small-ish magnitude differences between the unsigned magnitude of $g^{-1}(n)$ and the summations $\widehat{f}_1(n) := \sum_{k\geqslant 0} \binom{\omega(n)}{k} \cdot k!$. The following column in order shows the ratio of $\sum_{d\mid n} C_{\Omega(d)}(d)/|g^{-1}(n)|$.
The last three columns show the summatory function of $g^{-1}(n)$, $G^{-1}(x) := \sum_{n\leqslant x} g^{-1}(n)$, decomposed into its respective positive and negative summatory function components: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$.

n	**Primes**		**Sqfree**	**PPower**	$\bar{\mathbb{S}}$		$g^{-1}(n)$	$\lambda(n)\operatorname{sgn}(g^{-1}(n))$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$		$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
49	7^2	–	N	Y	N	–	2	1	0	1.5000000	–	-13	108	-121
50	$2^1 5^2$	–	N	N	Y	–	-7	1	2	1.2857143	–	-20	108	-128
51	$3^1 17^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-15	113	-128
52	$2^2 13^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-22	113	-135
53	53^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-24	113	-137
54	$2^1 3^3$	–	N	N	Y	–	9	1	4	1.5555556	–	-15	122	-137
55	$5^1 11^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-10	127	-137
56	$2^3 7^1$	–	N	N	Y	–	9	1	4	1.5555556	–	-1	136	-137
57	$3^1 19^1$	–	Y	N	N	–	5	1	0	1.0000000	–	4	141	-137
58	$2^1 29^1$	–	Y	N	N	–	5	1	0	1.0000000	–	9	146	-137
59	59^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	7	146	-139
60	$2^2 3^1 5^1$	–	N	N	Y	–	30	1	14	1.1666667	–	37	176	-139
61	61^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	35	176	-141
62	$2^1 31^1$	–	Y	N	N	–	5	1	0	1.0000000	–	40	181	-141
63	$3^2 7^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	33	181	-148
64	2^6	–	N	Y	N	–	2	1	0	3.5000000	–	35	183	-148
65	$5^1 13^1$	–	Y	N	N	–	5	1	0	1.0000000	–	40	188	-148
66	$2^1 3^1 11^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	24	188	-164
67	67^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	22	188	-166
68	$2^2 17^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	15	188	-173
69	$3^1 23^1$	–	Y	N	N	–	5	1	0	1.0000000	–	20	193	-173
70	$2^1 5^1 7^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	4	193	-189
71	71^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	2	193	-191
72	$2^3 3^2$	–	N	N	Y	–	-23	1	18	1.4782609	–	-21	193	-214
73	73^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-23	193	-216
74	$2^1 37^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-18	198	-216
75	$3^1 5^2$	–	N	N	Y	–	-7	1	2	1.2857143	–	-25	198	-223
76	$2^2 19^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-32	198	-230
77	$7^1 11^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-27	203	-230
78	$2^1 3^1 13^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-43	203	-246
79	79^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-45	203	-248
80	$2^4 5^1$	–	N	N	Y	–	-11	1	6	1.8181818	–	-56	203	-259
81	3^4	–	N	Y	N	–	2	1	0	2.5000000	–	-54	205	-259
82	$2^1 41^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-49	210	-259
83	83^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-51	210	-261
84	$2^2 3^1 7^1$	–	N	N	Y	–	30	1	14	1.1666667	–	-21	240	-261
85	$5^1 17^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-16	245	-261
86	$2^1 43^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-11	250	-261
87	$3^1 29^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-6	255	-261
88	$2^3 11^1$	–	N	N	Y	–	9	1	4	1.5555556	–	3	264	-261
89	89^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	1	264	-263
90	$2^1 3^2 5^1$	–	N	N	Y	–	30	1	14	1.1666667	–	31	294	-263
91	$7^1 13^1$	–	Y	N	N	–	5	1	0	1.0000000	–	36	299	-263
92	$2^2 23^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	29	299	-270
93	$3^1 31^1$	–	Y	N	N	–	5	1	0	1.0000000	–	34	304	-270
94	$2^1 47^1$	–	Y	N	N	–	5	1	0	1.0000000	–	39	309	-270
95	$5^1 19^1$	–	Y	N	N	–	5	1	0	1.0000000	–	44	314	-270
96	$2^5 3^1$	–	N	N	Y	–	13	1	8	2.0769231	–	57	327	-270
97	97^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	55	327	-272
98	$2^1 7^2$	–	N	N	Y	–	-7	1	2	1.2857143	–	48	327	-279
99	$3^2 11^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	41	327	-286
100	$2^2 5^2$	–	N	N	Y	–	14	1	9	1.3571429	–	55	341	-286
101	101^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	53	341	-288
102	$2^1 3^1 17^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	37	341	-304
103	103^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	35	341	-306
104	$2^3 13^1$	–	N	N	Y	–	9	1	4	1.5555556	–	44	350	-306
105	$3^1 5^1 7^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	28	350	-322
106	$2^1 53^1$	–	Y	N	N	–	5	1	0	1.0000000	–	33	355	-322
107	107^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	31	355	-324
108	$2^2 3^3$	–	N	N	Y	–	-23	1	18	1.4782609	–	8	355	-347
109	109^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	6	355	-349
110	$2^1 5^1 11^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-10	355	-365
111	$3^1 37^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-5	360	-365
112	$2^4 7^1$	–	N	N	Y	–	-11	1	6	1.8181818	–	-16	360	-376
113	113^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-18	360	-378
114	$2^1 3^1 19^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-34	360	-394
115	$5^1 23^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-29	365	-394
116	$2^2 29^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-36	365	-401
117	$3^2 13^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-43	365	-408
118	$2^1 59^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-38	370	-408
119	$7^1 17^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-33	375	-408
120	$2^3 3^1 5^1$	–	N	N	Y	–	-48	1	32	1.3333333	–	-81	375	-456
121	11^2	–	N	Y	N	–	2	1	0	1.5000000	–	-79	377	-456
122	$2^1 61^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-74	382	-456
123	$3^1 41^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-69	387	-456
124	$2^2 31^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-76	387	-463

n	**Primes**		**Sqfree**	**PPower**	$\bar{\mathbb{S}}$		$g^{-1}(n)$	$\lambda(n)\,\mathrm{sgn}(g^{-1}(n))$	$\lambda(n)g^{-1}(n) - \hat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$		$G^{-1}(n)$	$G_{+}^{-1}(n)$	$G_{-}^{-1}(n)$
125	5^3	–	N	Y	N	–	-2	1	0	2.0000000	–	-78	387	-465
126	$2^1 3^2 7^1$	–	N	N	Y	–	30	1	14	1.1666667	–	-48	417	-465
127	127^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-50	417	-467
128	2^7	–	N	Y	N	–	-2	1	0	4.0000000	–	-52	417	-469
129	$3^1 43^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-47	422	-469
130	$2^1 5^1 13^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-63	422	-485
131	131^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-65	422	-487
132	$2^2 3^1 11^1$	–	N	N	Y	–	30	1	14	1.1666667	–	-35	452	-487
133	$7^1 19^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-30	457	-487
134	$2^1 67^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-25	462	-487
135	$3^3 5^1$	–	N	N	Y	–	9	1	4	1.5555556	–	-16	471	-487
136	$2^3 17^1$	–	N	N	Y	–	9	1	4	1.5555556	–	-7	480	-487
137	137^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-9	480	-489
138	$2^1 3^1 23^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-25	480	-505
139	139^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-27	480	-507
140	$2^2 5^1 7^1$	–	N	N	Y	–	30	1	14	1.1666667	–	3	510	-507
141	$3^1 47^1$	–	Y	N	N	–	5	1	0	1.0000000	–	8	515	-507
142	$2^1 71^1$	–	Y	N	N	–	5	1	0	1.0000000	–	13	520	-507
143	$11^1 13^1$	–	Y	N	N	–	5	1	0	1.0000000	–	18	525	-507
144	$2^4 3^2$	–	N	N	Y	–	34	1	29	1.6176471	–	52	559	-507
145	$5^1 29^1$	–	Y	N	N	–	5	1	0	1.0000000	–	57	564	-507
146	$2^1 73^1$	–	Y	N	N	–	5	1	0	1.0000000	–	62	569	-507
147	$3^1 7^2$	–	N	N	Y	–	-7	1	2	1.2857143	–	55	569	-514
148	$2^2 37^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	48	569	-521
149	149^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	46	569	-523
150	$2^1 3^1 5^2$	–	N	N	Y	–	30	1	14	1.1666667	–	76	599	-523
151	151^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	74	599	-525
152	$2^3 19^1$	–	N	N	Y	–	9	1	4	1.5555556	–	83	608	-525
153	$3^2 17^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	76	608	-532
154	$2^1 7^1 11^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	60	608	-548
155	$5^1 31^1$	–	Y	N	N	–	5	1	0	1.0000000	–	65	613	-548
156	$2^2 3^1 13^1$	–	N	N	Y	–	30	1	14	1.1666667	–	95	643	-548
157	157^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	93	643	-550
158	$2^1 79^1$	–	Y	N	N	–	5	1	0	1.0000000	–	98	648	-550
159	$3^1 53^1$	–	Y	N	N	–	5	1	0	1.0000000	–	103	653	-550
160	$2^5 5^1$	–	N	N	Y	–	13	1	8	2.0769231	–	116	666	-550
161	$7^1 23^1$	–	Y	N	N	–	5	1	0	1.0000000	–	121	671	-550
162	$2^1 3^4$	–	N	N	Y	–	-11	1	6	1.8181818	–	110	671	-561
163	163^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	108	671	-563
164	$2^2 41^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	101	671	-570
165	$3^1 5^1 11^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	85	671	-586
166	$2^1 83^1$	–	Y	N	N	–	5	1	0	1.0000000	–	90	676	-586
167	167^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	88	676	-588
168	$2^3 3^1 7^1$	–	N	N	Y	–	-48	1	32	1.3333333	–	40	676	-636
169	13^2	–	N	Y	N	–	2	1	0	1.5000000	–	42	678	-636
170	$2^1 5^1 17^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	26	678	-652
171	$3^2 19^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	19	678	-659
172	$2^2 43^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	12	678	-666
173	173^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	10	678	-668
174	$2^1 3^1 29^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-6	678	-684
175	$5^2 7^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-13	678	-691
176	$2^4 11^1$	–	N	N	Y	–	-11	1	6	1.8181818	–	-24	678	-702
177	$3^1 59^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-19	683	-702
178	$2^1 89^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-14	688	-702
179	179^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-16	688	-704
180	$2^2 3^2 5^1$	–	N	N	Y	–	-74	1	58	1.2162162	–	-90	688	-778
181	181^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-92	688	-780
182	$2^1 7^1 13^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-108	688	-796
183	$3^1 61^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-103	693	-796
184	$2^3 23^1$	–	N	N	Y	–	9	1	4	1.5555556	–	-94	702	-796
185	$5^1 37^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-89	707	-796
186	$2^1 3^1 31^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-105	707	-812
187	$11^1 17^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-100	712	-812
188	$2^2 47^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-107	712	-819
189	$3^3 7^1$	–	N	N	Y	–	9	1	4	1.5555556	–	-98	721	-819
190	$2^1 5^1 19^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-114	721	-835
191	191^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-116	721	-837
192	$2^6 3^1$	–	N	N	Y	–	-15	1	10	2.3333333	–	-131	721	-852
193	193^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-133	721	-854
194	$2^1 97^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-128	726	-854
195	$3^1 5^1 13^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-144	726	-870
196	$2^2 7^2$	–	N	N	Y	–	14	1	9	1.3571429	–	-130	740	-870
197	197^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-132	740	-872
198	$2^1 3^2 11^1$	–	N	N	Y	–	30	1	14	1.1666667	–	-102	770	-872
199	199^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-104	770	-874
200	$2^3 5^2$	–	N	N	Y	–	-23	1	18	1.4782609	–	-127	770	-897

n	**Primes**		**Sqfree**	**PPower**	$\bar{\mathbb{S}}$		$g^{-1}(n)$	$\lambda(n)\operatorname{sgn}(g^{-1}(n))$	$\lambda(n)g^{-1}(n)-\hat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$		$G^{-1}(n)$	$G_{+}^{-1}(n)$	$G_{-}^{-1}(n)$
201	$3^1 67^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−122	775	−897
202	$2^1 101^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−117	780	−897
203	$7^1 29^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−112	785	−897
204	$2^2 3^1 17^1$	–	N	N	Y	–	30	1	14	1.1666667	–	−82	815	−897
205	$5^1 41^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−77	820	−897
206	$2^1 103^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−72	825	−897
207	$3^2 23^1$	–	N	N	Y	–	−7	1	2	1.2857143	–	−79	825	−904
208	$2^4 13^1$	–	N	N	Y	–	−11	1	6	1.8181818	–	−90	825	−915
209	$11^1 19^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−85	830	−915
210	$2^1 3^1 5^1 7^1$	–	Y	N	N	–	65	1	0	1.0000000	–	−20	895	−915
211	211^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	−22	895	−917
212	$2^2 53^1$	–	N	N	Y	–	−7	1	2	1.2857143	–	−29	895	−924
213	$3^1 71^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−24	900	−924
214	$2^1 107^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−19	905	−924
215	$5^1 43^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−14	910	−924
216	$2^3 3^3$	–	N	N	Y	–	46	1	41	1.5000000	–	32	956	−924
217	$7^1 31^1$	–	Y	N	N	–	5	1	0	1.0000000	–	37	961	−924
218	$2^1 109^1$	–	Y	N	N	–	5	1	0	1.0000000	–	42	966	−924
219	$3^1 73^1$	–	Y	N	N	–	5	1	0	1.0000000	–	47	971	−924
220	$2^2 5^1 11^1$	–	N	N	Y	–	30	1	14	1.1666667	–	77	1001	−924
221	$13^1 17^1$	–	Y	N	N	–	5	1	0	1.0000000	–	82	1006	−924
222	$2^1 3^1 37^1$	–	Y	N	N	–	−16	1	0	1.0000000	–	66	1006	−940
223	223^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	64	1006	−942
224	$2^5 7^1$	–	N	N	Y	–	13	1	8	2.0769231	–	77	1019	−942
225	$3^2 5^2$	–	N	N	Y	–	14	1	9	1.3571429	–	91	1033	−942
226	$2^1 113^1$	–	Y	N	N	–	5	1	0	1.0000000	–	96	1038	−942
227	227^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	94	1038	−944
228	$2^2 3^1 19^1$	–	N	N	Y	–	30	1	14	1.1666667	–	124	1068	−944
229	229^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	122	1068	−946
230	$2^1 5^1 23^1$	–	Y	N	N	–	−16	1	0	1.0000000	–	106	1068	−962
231	$3^1 7^1 11^1$	–	Y	N	N	–	−16	1	0	1.0000000	–	90	1068	−978
232	$2^3 29^1$	–	N	N	Y	–	9	1	4	1.5555556	–	99	1077	−978
233	233^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	97	1077	−980
234	$2^1 3^2 13^1$	–	N	N	Y	–	30	1	14	1.1666667	–	127	1107	−980
235	$5^1 47^1$	–	Y	N	N	–	5	1	0	1.0000000	–	132	1112	−980
236	$2^2 59^1$	–	N	N	Y	–	−7	1	2	1.2857143	–	125	1112	−987
237	$3^1 79^1$	–	Y	N	N	–	5	1	0	1.0000000	–	130	1117	−987
238	$2^1 7^1 17^1$	–	Y	N	N	–	−16	1	0	1.0000000	–	114	1117	−1003
239	239^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	112	1117	−1005
240	$2^4 3^1 5^1$	–	N	N	Y	–	70	1	54	1.5000000	–	182	1187	−1005
241	241^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	180	1187	−1007
242	$2^1 11^2$	–	N	N	Y	–	−7	1	2	1.2857143	–	173	1187	−1014
243	3^5	–	N	Y	N	–	−2	1	0	3.0000000	–	171	1187	−1016
244	$2^2 61^1$	–	N	N	Y	–	−7	1	2	1.2857143	–	164	1187	−1023
245	$5^1 7^2$	–	N	N	Y	–	−7	1	2	1.2857143	–	157	1187	−1030
246	$2^1 3^1 41^1$	–	Y	N	N	–	−16	1	0	1.0000000	–	141	1187	−1046
247	$13^1 19^1$	–	Y	N	N	–	5	1	0	1.0000000	–	146	1192	−1046
248	$2^3 31^1$	–	N	N	Y	–	9	1	4	1.5555556	–	155	1201	−1046
249	$3^1 83^1$	–	Y	N	N	–	5	1	0	1.0000000	–	160	1206	−1046
250	$2^1 5^3$	–	N	N	Y	–	9	1	4	1.5555556	–	169	1215	−1046
251	251^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	167	1215	−1048
252	$2^2 3^2 7^1$	–	N	N	Y	–	−74	1	58	1.2162162	–	93	1215	−1122
253	$11^1 23^1$	–	Y	N	N	–	5	1	0	1.0000000	–	98	1220	−1122
254	$2^1 127^1$	–	Y	N	N	–	5	1	0	1.0000000	–	103	1225	−1122
255	$3^1 5^1 17^1$	–	Y	N	N	–	−16	1	0	1.0000000	–	87	1225	−1138
256	2^8	–	N	Y	N	–	2	1	0	4.5000000	–	89	1227	−1138
257	257^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	87	1227	−1140
258	$2^1 3^1 43^1$	–	Y	N	N	–	−16	1	0	1.0000000	–	71	1227	−1156
259	$7^1 37^1$	–	Y	N	N	–	5	1	0	1.0000000	–	76	1232	−1156
260	$2^2 5^1 13^1$	–	N	N	Y	–	30	1	14	1.1666667	–	106	1262	−1156
261	$3^2 29^1$	–	N	N	Y	–	−7	1	2	1.2857143	–	99	1262	−1163
262	$2^1 131^1$	–	Y	N	N	–	5	1	0	1.0000000	–	104	1267	−1163
263	263^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	102	1267	−1165
264	$2^3 3^1 11^1$	–	N	N	Y	–	−48	1	32	1.3333333	–	54	1267	−1213
265	$5^1 53^1$	–	Y	N	N	–	5	1	0	1.0000000	–	59	1272	−1213
266	$2^1 7^1 19^1$	–	Y	N	N	–	−16	1	0	1.0000000	–	43	1272	−1229
267	$3^1 89^1$	–	Y	N	N	–	5	1	0	1.0000000	–	48	1277	−1229
268	$2^2 67^1$	–	N	N	Y	–	−7	1	2	1.2857143	–	41	1277	−1236
269	269^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	39	1277	−1238
270	$2^1 3^3 5^1$	–	N	N	Y	–	−48	1	32	1.3333333	–	−9	1277	−1286
271	271^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	−11	1277	−1288
272	$2^4 17^1$	–	N	N	Y	–	−11	1	6	1.8181818	–	−22	1277	−1299
273	$3^1 7^1 13^1$	–	Y	N	N	–	−16	1	0	1.0000000	–	−38	1277	−1315
274	$2^1 137^1$	–	Y	N	N	–	5	1	0	1.0000000	–	−33	1282	−1315
275	$5^2 11^1$	–	N	N	Y	–	−7	1	2	1.2857143	–	−40	1282	−1322
276	$2^2 3^1 23^1$	–	N	N	Y	–	30	1	14	1.1666667	–	−10	1312	−1322
277	277^1	–	Y	Y	N	–	−2	1	0	1.0000000	–	−12	1312	−1324

n	**Primes**		**Sqfree**	**PPower**	$\bar{\mathbb{S}}$		$g^{-1}(n)$	$\lambda(n)\,\mathrm{sgn}(g^{-1}(n))$	$\lambda(n)g^{-1}(n) - \hat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$		$G^{-1}(n)$	$G_{+}^{-1}(n)$	$G_{-}^{-1}(n)$
278	$2^1 139^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-7	1317	-1324
279	$3^2 31^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-14	1317	-1331
280	$2^3 5^1 7^1$	–	N	N	Y	–	-48	1	32	1.3333333	–	-62	1317	-1379
281	281^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-64	1317	-1381
282	$2^1 3^1 47^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-80	1317	-1397
283	283^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-82	1317	-1399
284	$2^2 71^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-89	1317	-1406
285	$3^1 5^1 19^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-105	1317	-1422
286	$2^1 11^1 13^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-121	1317	-1438
287	$7^1 41^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-116	1322	-1438
288	$2^5 3^2$	–	N	N	Y	–	-47	1	42	1.7659574	–	-163	1322	-1485
289	17^2	–	N	Y	N	–	2	1	0	1.5000000	–	-161	1324	-1485
290	$2^1 5^1 29^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-177	1324	-1501
291	$3^1 97^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-172	1329	-1501
292	$2^2 73^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-179	1329	-1508
293	293^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-181	1329	-1510
294	$2^1 3^1 7^2$	–	N	N	Y	–	30	1	14	1.1666667	–	-151	1359	-1510
295	$5^1 59^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-146	1364	-1510
296	$2^3 37^1$	–	N	N	Y	–	9	1	4	1.5555556	–	-137	1373	-1510
297	$3^3 11^1$	–	N	N	Y	–	9	1	4	1.5555556	–	-128	1382	-1510
298	$2^1 149^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-123	1387	-1510
299	$13^1 23^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-118	1392	-1510
300	$2^2 3^1 5^2$	–	N	N	Y	–	-74	1	58	1.2162162	–	-192	1392	-1584
301	$7^1 43^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-187	1397	-1584
302	$2^1 151^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-182	1402	-1584
303	$3^1 101^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-177	1407	-1584
304	$2^4 19^1$	–	N	N	Y	–	-11	1	6	1.8181818	–	-188	1407	-1595
305	$5^1 61^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-183	1412	-1595
306	$2^1 3^2 17^1$	–	N	N	Y	–	30	1	14	1.1666667	–	-153	1442	-1595
307	307^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-155	1442	-1597
308	$2^2 7^1 11^1$	–	N	N	Y	–	30	1	14	1.1666667	–	-125	1472	-1597
309	$3^1 103^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-120	1477	-1597
310	$2^1 5^1 31^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-136	1477	-1613
311	311^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-138	1477	-1615
312	$2^3 3^1 13^1$	–	N	N	Y	–	-48	1	32	1.3333333	–	-186	1477	-1663
313	313^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-188	1477	-1665
314	$2^1 157^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-183	1482	-1665
315	$3^2 5^1 7^1$	–	N	N	Y	–	30	1	14	1.1666667	–	-153	1512	-1665
316	$2^2 79^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-160	1512	-1672
317	317^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-162	1512	-1674
318	$2^1 3^1 53^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-178	1512	-1690
319	$11^1 29^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-173	1517	-1690
320	$2^6 5^1$	–	N	N	Y	–	-15	1	10	2.3333333	–	-188	1517	-1705
321	$3^1 107^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-183	1522	-1705
322	$2^1 7^1 23^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	-199	1522	-1721
323	$17^1 19^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-194	1527	-1721
324	$2^2 3^4$	–	N	N	Y	–	34	1	29	1.6176471	–	-160	1561	-1721
325	$5^2 13^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-167	1561	-1728
326	$2^1 163^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-162	1566	-1728
327	$3^1 109^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-157	1571	-1728
328	$2^3 41^1$	–	N	N	Y	–	9	1	4	1.5555556	–	-148	1580	-1728
329	$7^1 47^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-143	1585	-1728
330	$2^1 3^1 5^1 11^1$	–	Y	N	N	–	65	1	0	1.0000000	–	-78	1650	-1728
331	331^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-80	1650	-1730
332	$2^2 83^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-87	1650	-1737
333	$3^2 37^1$	–	N	N	Y	–	-7	1	2	1.2857143	–	-94	1650	-1744
334	$2^1 167^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-89	1655	-1744
335	$5^1 67^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-84	1660	-1744
336	$2^4 3^1 7^1$	–	N	N	Y	–	70	1	54	1.5000000	–	-14	1730	-1744
337	337^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	-16	1730	-1746
338	$2^1 13^2$	–	N	N	Y	–	-7	1	2	1.2857143	–	-23	1730	-1753
339	$3^1 113^1$	–	Y	N	N	–	5	1	0	1.0000000	–	-18	1735	-1753
340	$2^2 5^1 17^1$	–	N	N	Y	–	30	1	14	1.1666667	–	12	1765	-1753
341	$11^1 31^1$	–	Y	N	N	–	5	1	0	1.0000000	–	17	1770	-1753
342	$2^1 3^2 19^1$	–	N	N	Y	–	30	1	14	1.1666667	–	47	1800	-1753
343	7^3	–	N	Y	N	–	-2	1	0	2.0000000	–	45	1800	-1755
344	$2^3 43^1$	–	N	N	Y	–	9	1	4	1.5555556	–	54	1809	-1755
345	$3^1 5^1 23^1$	–	Y	N	N	–	-16	1	0	1.0000000	–	38	1809	-1771
346	$2^1 173^1$	–	Y	N	N	–	5	1	0	1.0000000	–	43	1814	-1771
347	347^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	41	1814	-1773
348	$2^2 3^1 29^1$	–	N	N	Y	–	30	1	14	1.1666667	–	71	1844	-1773
349	349^1	–	Y	Y	N	–	-2	1	0	1.0000000	–	69	1844	-1775
350	$2^1 5^2 7^1$	–	N	N	Y	–	30	1	14	1.1666667	–	99	1874	-1775