By the calculus of residues we may write

$$I = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(G(z)(\log x)^z \right)\Big|_{z=0}$$
$$= \sum_{\nu=0}^{k-1} \frac{G^{(\nu)}(0)}{\nu!} \frac{(\log\log x)^{k-1-\nu}}{(k-1-\nu)!}.$$

This gives a more accurate, but more complicated, main term.

In Section 2.3 we saw that $\Omega(n)$ rarely differs very much from $\log\log n$. In particular, from Theorem 2.12 we see that if $r < 1$, then the number of $n \le x$ for which $\Omega(n) < r \log\log n$ is $\ll_r x/\log\log x$. We now give a much sharper upper bound for the number of occurrences of such large deviations.

Theorem 7.20 *Let $A(x, r)$ denote the number of $n \le x$ such that $\Omega(n) \le r \log\log x$, and let $B(x, r)$ denote the number of $n \le x$ for which $\Omega(n) \ge r \log\log x$. If $0 < r \le 1$ and $x \ge 2$, then*

$$A(x, r) \ll x(\log x)^{r-1-r\log r}.$$

If $1 \le r \le R < 2$ and $x \ge 2$, then

$$B(x, r) \ll_R x(\log x)^{r-1-r\log r}.$$

Proof We argue directly from Theorem 7.18, using a modified form of Rankin's method. If $0 \le r \le 1$ and $\Omega(n) \le r \log\log x$, then $r^{r\log\log x} \le r^{\Omega(n)}$. Hence

$$A(x, r) \le (\log x)^{-r\log r} \sum_{n \le x} r^{\Omega(n)}.$$

By Theorem 7.18 this is

$$\sim \frac{F(1, r)}{\Gamma(r)} x(\log x)^{r-1-r\log r}$$

where $F(s, z)$ is taken as in (7.60). This gives the result since $F(1, r) \ll 1$ and $\Gamma(r) \gg 1$ uniformly for $0 < r \le 1$.

Now suppose that $1 \le r \le R < 2$ and that $\Omega(n) \ge r \log\log x$. Then $r^{\Omega(n)} \ge r^{r\log\log x}$, and hence

$$B(x, r) \le (\log x)^{-r\log r} \sum_{n \le x} r^{\Omega(n)}.$$

Thus we have only to proceed as before to obtain the result. □

In discussing Theorem 2.12 we proposed a probabilistic model, which in conjunction with the Central Limit Theorem would predict that the quantity

$$\alpha_n = \frac{\Omega(n) - \log\log n}{\sqrt{\log\log n}} \tag{7.64}$$

is asymptotically normally distributed. We now confirm this.

Theorem 7.21 *Let α_n be given by* (7.64) *and suppose that $Y > 0$. Then the number of n, $3 \le n \le x$, such that $\alpha_n \le y$ is*

$$\Phi(y)x + O_Y\left(\frac{x}{\sqrt{\log\log x}}\right)$$

uniformly for $-Y \le y \le Y$ where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\, dt.$$

Proof Let

$$\beta_n = \frac{\Omega(n) - \log\log x}{\sqrt{\log\log x}}.$$

Since $\Phi'(y) \ll 1$ and $\alpha_n - \beta_n \ll 1/\sqrt{\log\log x}$ when $x^{1/2} \le n \le x$ and $\Omega(n) \le 2\log\log x$, it suffices to consider β_n in place of α_n. We may of course also suppose that x is large.

Let k be a natural number and let u be defined by writing $k = u + \log\log x$. If $|u| \le \frac{1}{2}\log\log x$, then by Stirling's formula (see (B.26) or the more general Theorem C.1) we see that

$$\frac{(\log\log x)^{k-1}}{(k-1)!}$$
$$= \frac{e^u \log x}{\sqrt{2\pi \log\log x}} \left(1 + \frac{u}{\log\log x}\right)^{\frac{1}{2} - \log\log x - u} \left(1 + O\left(\frac{1}{\log\log x}\right)\right).$$

The estimate $\log(1+\delta) = \delta - \delta^2/2 + O(|\delta|^3)$ holds uniformly for $|\delta| \le 1/2$. By taking $\delta = u/\log\log x$ we find that

$$\left(1 + \frac{u}{\log\log x}\right)^{\frac{1}{2} - \log\log x - u}$$
$$= \exp\left(-u + \frac{u - u^2}{2\log\log x} - \frac{u^2}{4(\log\log x)^2} + O\left(\frac{|u|^3}{(\log\log x)^2}\right)\right).$$

Suppose now that $|u| \le (\log\log x)^{2/3}$. By considering separately $|u| \le (\log\log x)^{1/2}$ and $(\log\log x)^{1/2} < |u| \le (\log\log x)^{2/3}$ we see that

$$\frac{u}{\log\log x} \ll \frac{1}{\sqrt{\log\log x}} + \frac{|u|^3}{(\log\log x)^2}.$$

Similarly, by considering $|u| \le 1$ and $|u| > 1$ we see that

$$\frac{u^2}{(\log\log x)^2} \ll \frac{1}{\sqrt{\log\log x}} + \frac{|u|^3}{(\log\log x)^2}.$$

On combining these estimates we deduce that

$$\begin{aligned}\frac{(\log\log x)^{k-1}}{(k-1)!} &= \frac{\log x}{\sqrt{2\pi\log\log x}} \exp\left(\frac{-u^2}{2\log\log x}\right) \\ &\quad \times \left(1 + O\left(\frac{1}{\sqrt{\log\log x}}\right) + O\left(\frac{|u|^3}{(\log\log x)^2}\right)\right)\end{aligned}$$

uniformly for $|u| \le (\log\log x)^{2/3}$. In Theorem 7.19 we have $G(1) = 1$ and

$$G\left(\frac{k-1}{\log\log x}\right) = G(1) + O\left(\frac{1+|u|}{\log\log x}\right).$$

Hence by Theorem 7.19,

$$\begin{aligned}\sigma_k(x) &= \frac{x\exp\left(\frac{-(k-\log\log x)^2}{2\log\log x}\right)}{\sqrt{2\pi\log\log x}} \\ &\quad \times \left(1 + O\left(\frac{1}{\sqrt{\log\log x}}\right) + O\left(\frac{|k-\log\log x|^3}{(\log\log x)^2}\right)\right).\end{aligned}$$

By Theorem 7.20 we know that the contribution of $k \le \log\log x - (\log\log x)^{2/3}$ is negligible. We sum over the range

$$\log\log x - (\log\log x)^{2/3} \le k \le \log\log x + y(\log\log x)^{1/2}.$$

This gives rise to three sums, one for the main term and two for error terms. Each of these sums can be considered to be a Riemann sum for an associated integral, and the stated result follows. □

7.4.1 Exercises

1. Let $p_1, p_2, \ldots, p_K$ be distinct primes. Show that the number of $n \le x$ composed entirely of the p_k is

$$\frac{(\log x)^K}{K!\prod_{k=1}^K \log p_k} + O\big((\log x)^{K-1}\big).$$

2. (a) Let $d_z(n)$ be defined as in (7.56), and suppose that $|z| \le R$. Show that $|d_z(n)| \le d_{|z|}(n) \le d_R(n)$.
   (b) Let $F(s, z)$ be defined as in (7.60). Show that if $0 < r < 1$ and $\sigma > 1/2$, then $0 < F(\sigma, r) < 1$.
   (c) Let $F(s, z)$ be defined as in (7.60). Show that if $1 < r < 2$, then the Dirichlet series coefficients of $F(s, r)$ are all non-negative.
3. (a) Show that if
$$F(s, z) = \prod_p \left(1 + \frac{z}{p^s - 1}\right)\left(1 - \frac{1}{p^s}\right)^z,$$
then $F(s, z)$ converges for $\sigma > 1/2$, uniformly for $|z| \le R$.
   (b) Show that if $F(s, z)$ is taken as above, and if $a_z(n)$ is defined as in Theorem 7.18, then $a_z(n) = z^{\omega(n)}$.
   (c) Let $\rho_k(x)$ denote the number of $n \le x$ for which $\omega(n) = k$. Show that if $x \ge 2$, then
$$\rho_k(x) = G\left(\frac{k-1}{\log\log x}\right)\frac{x(\log\log x)^{k-1}}{(k-1)!\,\log x}\left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right)$$
uniformly for $1 \le k \le R\log\log x$ where $G(z) = F(1, z)/\Gamma(z+1)$.
   (d) Show that $G(0) = G(1) = 1$.
   (e) Let $A(x, r)$ denote the number of $n \le x$ for which $\omega(n) \le r\log\log x$. Show that
$$A(x, r) \ll x(\log x)^{r-1-r\log r}$$
uniformly for $0 < r \le 1$.
   (f) Let $B(x, r)$ denote the number of $n \le x$ for which $\omega(n) \ge r\log\log x$. Show that
$$B(x, r) \ll x(\log x)^{r-1-r\log r}$$
uniformly for $1 \le r \le R$.
4. (a) Show that if
$$F(s, z) = \prod_p \left(1 + \frac{z}{p^s}\right)\left(1 - \frac{1}{p^s}\right)^z,$$
then $F(s, z)$ converges for $\sigma > 1/2$, uniformly for $|z| \le R$.
   (b) Show that if $F(s, z)$ is taken as above, and if $a_z(n)$ is defined as in Theorem 7.18, then $a_z(n) = \mu(n)^2 z^{\omega(n)}$.
   (c) Let $\pi_k(x)$ denote the number of square-free $n \le x$ for which $\omega(n) = k$. Show that if $x \ge 2$, then
$$\pi_k(x) = G\left(\frac{k-1}{\log\log x}\right)\frac{x(\log\log x)^{k-1}}{(k-1)!\,\log x}\left(1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right)$$
uniformly for $1 \le k \le R\log\log x$ where $G(z) = F(1, z)/\Gamma(z+1)$.