SUPER SQUARE ROOT LOWER BOUNDS ON THE PARITY OF THE PARTITION FUNCTION

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ABSTRACT. We define the function $N_e(x)$ to be the number of times the parity partition function $p(n)$ is even for $n \leq x$. We expect that on average the value of $N_e(x)$ is approximately $x/2$. However, currently the best known lower bound for the function is given by $N_e(x) \geq C \cdot \sqrt{x} \log(x)$ for sufficiently large x. Based on our intuition we expect that we can find a constant $C_2 > 0$ such that for all sufficiently large x we have that $N_e(x) \geq C_2 \cdot x^{0.51} + o(\sqrt{x})$, a small-order, but significant improvement on the known lower bound for the function. Based on a heuristic argument and computational evidence, we reduce the proof of our conjecture to showing a certain criterion about the distribution of the integers $q - 1 - \omega(\pm j)$ where $\omega(n) = n(3n-1)/2$ denotes a pentagonal number and q is a prime in the range $x^2 < q < (x+1)^2$ for large x. Rather than phrasing our expectation to hold for all large $x \geq x_0$, we conjecture that the improved bound for $N_e(x)$ holds infinitely often.

1. Introduction

Talk about the problems and known bounds;
We (hope to) prove super square root bounds on the number of times the partition function $p(n)$ is even for $n \leq x$ by improving the bound to $x^{0.51} + o(\sqrt{n})$. This is a significant improvement that departs from the current best known bound involving a power of x which is previously known to satisfy

$$N_e(x) := \#\{n \leq x : p(n) \equiv 0 \bmod 2\} \geq C_1 \cdot \sqrt{x},$$

for some positive constant $C_1 > 0$.

2. A new identity for the Möbius function

For natural numbers $n \geq 0$, we use the following notation to extract the index j of the sequence of interleaved pentagonal numbers listed in order, $\{0, 1, 2, 5, 7, 12, 26, \ldots\}$, [3, A001318] as follows:

$$\widehat{G}_n = \begin{cases} j, & \text{if } n = \frac{1}{2} \left\lfloor \frac{j}{2} \right\rfloor \left\lfloor \frac{3j+1}{2} \right\rfloor; \\ 0, & \text{otherwise.} \end{cases}$$

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We similarly define the next notation for the function G_j generating these interleaved pentagonal numbers.

$$G_j = \frac{1}{2}\left\lfloor \frac{j}{2} \right\rfloor \left\lfloor \frac{3j+1}{2} \right\rfloor$$

We note then that Euler's pentagonal theorem expressed as a unilateral power series is equivalent to the following expansions for $|q|<1$:

$$\begin{aligned}
(q;q)_\infty &= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} + \cdots \\
&= 1 + \sum_{j\geq 1} (-1)^j \left(q^{j(3j-1)/2} + q^{j(3j+1)/2}\right) \\
&= \sum_{j\geq 0} (-1)^{\left\lceil \frac{j}{2} \right\rceil} q^{G_j}.
\end{aligned}$$

With this notation in mind we state the next result.

Lemma 2.1 (Lambert Series Factorizations)**.** *For natural numbers $n \geq 1$, we have the identity that*[1]

$$\mu(n) = \sum_{d|n} \sum_{k=1}^{n} p(k-1)\mu(n/d) \left([G_{d-k} \neq 0]_\delta \, (-1)^{\left\lceil \frac{G_{d-k}}{2} \right\rceil} + [d-k=0]_\delta \right),$$

where $\mu(n)$ is the Möbius function.

Proof. The proof follows from the Lambert series factorization theorems recently proved by Merca and Schmidt in 2017. In particular, if we let $s_{n,k} := p_o(n,k) - p_e(n,k)$ where $p_{o/e}(n,k)$ respectively count the number of k's in all partitions of n into an *odd* (*even*) number of distinct parts[2], then we have the following matrix-product-like factorization of the Lambert series generating function of any arithmetic function a_n:

$$L_a(q) := \sum_{n\geq 1} \frac{a_n q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n \geq 1} \sum_{k=1}^{n} s_{n,k} a_k \cdot q^n, \ \ |q|<1.$$

We can invert the matrix product corresponding to the coefficients of q^n on the right-hand-side of the previous equation by the matrix whose $(i,j)^{th}$ entry is the triangular sequence value defined by the function $s_{n,k}^{(-1)}$. We have a specific and unusual explicit expansion formula for this inverse sequence involving products of the partition function $p(n)$, the series coefficients of $L_(q)$ with respect to q with $[q^n]L_a(q) = \sum_{d|n} a_d$ for all $n \geq 1$, and a discrete convolution with the generating function $(q;q)_\infty$. More precisely, in [1, Thm. 2.9] we prove that

$$s_{n,k}^{(-1)} = \sum_{k=1}^{n} \sum_{d|n} p(d-k)\mu(n/d) \times \sum_{\substack{j \geq 0 \\ 0 \leq G_j \leq k}} (-1)^{\lceil j/2 \rceil} [q^{k-G_j}] L_a(q). \tag{1}$$

[1] $\underline{\text{Notation}}$: *Iverson's convention* compactly specifies boolean-valued conditions and is equivalent to the *Kronecker delta function*, $\delta_{i,j}$, as $[n=k]_\delta \equiv \delta_{n,k}$. Similarly, $[\texttt{cond} = \texttt{True}]_\delta \equiv \delta_{\texttt{cond},\texttt{True}}$ in the remainder of the article.

[2] A generating function for $s_{n,k}$ is given for all $n,k \geq 1$ by

$$s_{n,k} = [q^n](q;q)_\infty \frac{q^k}{1-q^k}.$$

In particular, since we express the known convolution of the Möbius function with the constant 1 to be

$$(\mu * 1)(n) = \delta_{n,1},$$

we can see that when $a_n := \mu(n)$ for all n, the expansion in (1) implies that

$$\begin{aligned}
\mu(n) &= \sum_{k=1}^{n} s_{n,k}^{(-1)} \times \sum_{\substack{j\geq 0 \\ 0\leq G_j\leq k}} (-1)^{\lceil j/2\rceil} \sum_{d|k-G_j} \mu(d) \\
&= \sum_{d|n} \sum_{1\leq k\leq d} p(d-k)\mu(n/d) \times \left(\sum_{\substack{j\geq 0 \\ 0\leq G_j\leq k}} (-1)^{\lceil j/2\rceil} \left[k-G_j=1\right]_\delta + \left[k-1=0\right]_\delta \right) \\
&= \sum_{d|n} \sum_{1\leq k\leq d} p(k)\mu(n/d) \times \left(\sum_{\substack{j\geq 0 \\ 0\leq G_j\leq k}} (-1)^{\lceil \widehat{G}_{d-k}/2\rceil} \left[\widehat{G}_{d-k}\neq 0\right]_\delta + \left[d-k=0\right]_\delta \right).
\end{aligned}$$

We can also reinterpret the inner sum in the last formula in form of

$$\mu(n) = \sum_{d|n} \left(\sum_{m=1}^{\mu_q} p(d-1-G_m)\mu(n/d)\cdot(-1)^{\lceil m/2\rceil} + p(d-1)\mu(n/d) \right), \tag{2}$$

where by solving the inequality $d-1-G_m \geq 0$ for all possible integer indices m we obtain the upper bound $\mu_d := \left\lfloor \frac{\sqrt{24(d-1)+1}+1}{6} \right\rfloor$. The last expression in (2) will be of particular interest in formulating minimally quadratic and then super-quadratic bounds in the next section. □

Theorem 2.2 (An Exact Identity for the Möbius Function). *For $q \geq 2$ prime, we have that*

$$0 \equiv p(q-1) + \sum_{k=1}^{\mu(q)} p(q-1-G_k) \pmod{2}.$$

In particular, when q is prime and μ_q is even, we see that at least one of the elements in the set

$$\{p(q-1-G_k) : 1 \leq k \leq \mu_q\} \cup \{p(q-1)\},$$

is even.

Proof. We can prove this theorem using the known identities related to sums of multiplicative functions and the partition function $p(n)$ stated in Lemma 2.1. In particular, if the integer n is prime then it has only the divisors $d \in \{1, n\}$ and $\mu(n) = -1$ in the expansion guaranteed by (2) from the lemma. Then plugging in these explicit and simple forms of the divisors $d|n$ to the formula implies the stated result above in the form of

$$-1 = p(q-1) - p(0) + \sum_{k=1}^{\mu(n)} p(n-1-G_k) \pmod{2}. \qquad \square$$

We note that this result provides us with a sum over approximately $\sqrt{n}$ partition function terms which whose parity is even. We will see the utility to this construction in the next section.

3. A new approach the the super square root power bound problem

3.1. **Constructions of the new approach.** A particular consequence of Theorem 2.2 is that when q is prime and μ_q is even, we have a sum of an odd number of distinct values of the partition function which are congruent to zero modulo 2. Thus whenever q is prime and μ_q is even we must have that at least one of these partition function values is even [2, *cf.* §1]. We note that similar prime factor identities expanding the result from Lemma 2.1 can be stated whenever n is a squarefree integer. However, for now we will focus on the case where n is prime where for sufficiently large x the number of prime $n \leq x$ we can count to bound $N_e(x)$ from below is given by

$$\pi(x) \sim \frac{x}{\log(x)}.$$

If we further restrict ourselves to the cases of prime $n \leq x$ such that μ_q is even, then we define the modified prime counting functions counting the number of such n to be

$$\begin{aligned} \pi_Q(x) &= \#\{n \leq x : n \text{ is prime and } \mu_n \equiv 0 \bmod 2\} \qquad (3) \\ \pi_S(x) &= \#\{n \leq x : n \in S \text{ and } \mu_n \equiv 0 \bmod 2\}. \end{aligned}$$

We conjecture experimentally that $\pi_Q(x) \approx \frac{2}{3}\pi(n)$. A somewhat weaker result providing that for large x $\pi_Q(x) \sim C \cdot \frac{x}{\log(x)}$ for some limiting constant $C \in (0,1)$ is in fact proved in the next proposition.

Proposition 3.1 (An Asymptotic Formula for $\pi_Q(x)$)**.** *As $x \longrightarrow \infty$ the prime counting function $\pi_Q(x)$ asymptotically tends to the formula*

$$\pi_Q(x) \sim \varepsilon \cdot \frac{x}{\log(x)},$$

for some exact limiting non-zero constant $\varepsilon \in (0,1]$.

Proof. We notice that by construction we always have that $\pi_Q(x) = \varepsilon_x \pi(x)$ where $\varepsilon_x \in (1,0]$ for all $x \geq 1$. Since the parity of μ_q alternates parity on the intervals $[G_j, G_{j+1})$ for each $j \geq 0$, we can produce the following lower bound for the prime tally given by $\pi_Q(x)$:

$$\begin{aligned} \pi_Q(x) &\geq \frac{1}{2} \cdot \#\{j : G_j \leq x\} \\ &= \frac{1}{2}\left(\#\left\{j : \frac{j(3j+1)}{2} \leq x\right\} + \#\left\{j : \frac{j(3j-1)}{2} \leq x\right\}\right) \\ &= \frac{1}{2}\left(\left\lfloor \frac{\sqrt{24x+1}-1}{6} \right\rfloor + \left\lfloor \frac{\sqrt{24x+1}+1}{6} \right\rfloor\right) \\ &\geq C_1 \cdot \sqrt{24x+1}, \quad \text{for some } C_1 > 0 \\ &> C_2 \cdot \sqrt{x}, \quad \text{for some } C_2 > 0. \end{aligned}$$

Then we compute that for all x we have some constant $\varepsilon_x \in [0,1)$ satisfying

$$\varepsilon_x \cdot \frac{x}{\log(x)} > C_2\sqrt{x} \qquad \Longleftrightarrow \qquad \varepsilon_x > C_1 \cdot \frac{\log(x)}{\sqrt{x}} \xrightarrow{x\to\infty} 0. \qquad (4)$$

Hence there is a limiting constant $\varepsilon_x \to \varepsilon \in (0,1)$, which is what we needed to show. $\square$

3.2. **Bounds on the size of the $I_{e,S}(x)$ count of overlap.**

Definition 3.2. Let the set operation $\ddot{\cup}$ denote set union counting multiplicity of the elements in the union. Then for a fixed prime subset $S \subset \{q : q \text{ is prime and } \mu_q \equiv 0 \bmod 2\}$, we define the index overlap counting function to be

$$I_{e,S}(x) := \# \left[\ddot{\bigcup} - \bigcup\right]_{q \neq r \in S} \left(\{q-1-G_j : j \leq \mu_q\} \cap \{r-1-G_j : j \leq \mu_r\}\right).$$

For the set $S_x^* := \{q \leq x : q \text{ is prime and } \mu_q \equiv 0 \bmod 2\}$, we define the special notation of $I_{e,Q}(x) \equiv I_{e,S_x^*}(x)$.

By a counting arument, i.e., by removing overcounted terms via the convenient definition of $I_{e,Q}(x)$, we can see that

$$N_e(x) \geq \pi_Q(x) - I_{e,Q}(x).$$

It turns out numerically that $I_{e,Q}(x)$ in the form defined above appears to be super-$\frac{x}{\log(x)}$, so we need to further restrict our attention to subsets of the primes $q \leq x$ such that μ_q is even to attempt to remove more of the overlap between repeated partition function values. To reproduce the order of known bounds on $N_e(x)$, it is sufficient to find a subset $S \subseteq \{q : q \text{ prime }, \mu_q \text{ even }\}$ of order $\lfloor \sqrt{x} \rfloor$ such that $I_{e,S}(x) = o(\sqrt{x})$, that is the overcount of overlapping partition values is of sub-square-root order on the set S. We conjecture that this is always possible by choosing our S such that it contains exactly one prime, say the first such prime, in the interval $[G_j, G_{j+1}]$ for each $1 \leq j \leq \mu_x$.

Proposition 3.3 (Bounds on the Overlap of a Standard Set)**.** *We define the (technically variable) set S_x^{**} to be*

$$S_x^{**} := \{G_s < q_s < G_{s+1} : q_s \text{ prime}, 1 \leq s \leq \mu_x, \mu_{q_s} \equiv 0 \bmod 2\},$$

and the corresponding overlap function to be $I_e^{**}(x) := I_{e,S_x^{**}}(x)$. Then for all $x \geq 2$, we have the upper bound

$$I_{e,S_x^{**}}(x) \leq \sum_{t=1}^{\mu_x} \left(\left\lceil \frac{\sqrt{\frac{\sqrt{24(t+1)^2+1}-1}{12} + 2(t+1)^2 - \frac{23}{12}}}{\sqrt{3}} - \frac{1}{18} \right\rceil - \left\lfloor \frac{\sqrt{\frac{\sqrt{24t^2+1}-1}{12} + 2t^2 + \frac{1}{12}}}{\sqrt{3}} - \frac{1}{18} \right\rfloor \right) - \mu_x.$$

Proof. □

3.3. **A sufficient condition and reduction of the proof of new bounds.** Now we turn to an approach to proving the conjectured super-square-root power bound that

$$N_e(x) \geq C \cdot x^{0.51} + o(\sqrt{x}), \tag{5}$$

happens infinitely often for some limiting constant $C > 0$ independent of x. In particular, we have the next theorem.

Theorem 3.4 (A Sufficient Condition for the Improved Power Bound)**.** *Suppose that for some sufficiently large $x \geq x_0$ we have that*

$$N_e(G_x) \geq \pi_S(x) - I_{e,S}(x) \geq C \cdot G_x^{0.51} + o(\sqrt{G_x})$$

for some subset S of the primes $q \leq G_x$ with the parity of μ_q even, and if we can find a prime $q_x \in (G_x, G_{x+1}]$ satisfying $\mu_{q_x} \equiv 0 \bmod 2$ such that the overlap

$$\{q_x - 1 - G_j : j \leq \mu_{q_x}\} \cap \{q - 1 - G_j : q \in S, j \leq \mu_q\} = \emptyset,$$

Then $N_e(G_{x+1}) \geq C \cdot G_{x+k}^{0.51} + o(\sqrt{G_x})$. In other words, if these conditions hold for all large enough $x \geq x_0$, then the bound $N_e(x) \geq C \cdot x^{0.51} + o(\sqrt{x})$ occurs infinitely often.

Proof. □

Based on extensive *upcoming* computations, we conjecture that the criteria in the statement of Theorem 3.4 is true for all large $x \geq TODO$ (see Table **??**). Thus we have reduced the problem of proving that the lower bound in (5) is true infinitely often to showing that the assumptions in the theorem hold for some large enough x_0. It should be noted that while it may well be another difficult problem to show the conditions in the theorem, it is a definite improvement on the existing methods for obtaining the known square root power bounds on the parity of $p(n)$. Thus we believe our work so far is of significance to a general mathematical audience.

4. Conclusions

4.1. Summary.

4.2. Topics for future research and investigation.

Acknowledgments.

References

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