Combinatorial methods for approximating the Mertens function and sums of the Möbius function

Maxie D. Schmidt

School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332

maxieds@gmail.com
mschmidt34@gatech.edu

Last Revised: September 25, 2019

Abstract

The Mertens function, $M(x) = \sum_{n \leq x} \mu(n)$, is classically defined to be the summatory function of the Möbius function $\mu(n)$. In some sense, the Móbius function can be viewed as a signed indicator function of the squarefree integers which have asymptotic density of $6/\pi^2 \approx 0.607927$ and a corresponding well-known asymptotic average order formula. The signed terms in the sums in the definition of the Mertens function introduce complications in the form of semi-randomness and cancellation inherent to the distribution of the Möbius function over the natural numbers. The Mertens conjecture which states that $|M(x)| < C \cdot \sqrt{x}$ for all $x \geq 1$ has a well-known disproof due to Odlyzko et. al. It is widely believed that $M(x)/\sqrt{x}$ is an unbounded function which changes sign infinitely often and exhibits a negative bias over all natural numbers $x \geq 1$. We focus on obtaining new bounds for $M(x)$ by methods that generalize to handle other related cases of special number theoretic summatory functions.

Keywords. *Mertens function; Möbius function; summatory function; Ramanujan's sum.*
MSC (2010). *11A25; 11N37; 11N56.* .

1 Introduction

1.1 Summatory functions of the Móbius function

Suppose that $n \geq 1$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2 \alpha_2 \cdots p_k^{\alpha_k}$. We define the *Möebius function* to be the signed indicator function of the squarefree integers given by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leq i \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many known variants and special properties of the Möebius function and its generalizations [21, *cf.* §2], however, for our purposes we seek to explore the properties and asymptotics of weighted summatory functions over $\mu(n)$. The Mertens summatory function, or *Mertens function*, is defined as

$$\begin{aligned} M(x) &= \sum_{n \leq x} \mu(n), \ x \geq 1, \\ &\longmapsto \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2\} \end{aligned}$$

A related function which counts the number of *squarefree* integers than x sums the average order of the Möbius function as

$$Q(n) = \sum_{n \leq x} |\mu(n)| \sim \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \rightarrow \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{Q(x)} = \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{Q(x)} \xrightarrow[n \rightarrow \infty]{} \frac{3}{\pi^2}.$$

While this limiting law suggests an even bias for the Mertens function, in practice $M(x)$ has a noted negative bias in its values, and the actual local oscillations between the approximate densities of the sets $\mu_\pm(x)$ lend an unpredictable nature to the function and its characteistic oscillatory sawtooth shaped plot − even over asymptotically large and variable intervals. As we consider next, these local oscillations and irregularity in growth lead to many natural unsolved questions about the eventual boundedness (or lack thereof) along subsequences of the natural numbers.

We define the notion of a *generalized, or weighted, Mertens summatory function* for fixed $\alpha \in \mathbb{C}$ as

$$M_\alpha^*(x) = \sum_{n \leq x} n^\alpha \mu(n), \ x \geq 1, \tag{1}$$

where the special case of $M_0^*(x)$ corresponds to the definition of the classical Mertens function $M(x)$ given above. The plots shown in Figure 1.1 illustrate the chaotic behavior of the growth of these functions for x in small intervals when $\alpha \in \{-1, 0, 1, 2\}$. Related questions are often posed in relation to the strikingly similar properties of the summatory functions over the Liouville lambda function, $L_\alpha(x) := \sum_{n \leq x} \lambda(n) n^{-\alpha}$ [?, ?].

1.2 Properties and bounds on $M(x)$

1.2.1 Exact formulae

The well-known approach to evaluating the behavior of $M(x)$ for large $x \rightarrow \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is easily expressed via an inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since by Perron's formula we have

$$\frac{1}{\zeta(s)} = \int_1^\infty \frac{s \cdot M(x)}{x^{s+1}} dx,$$

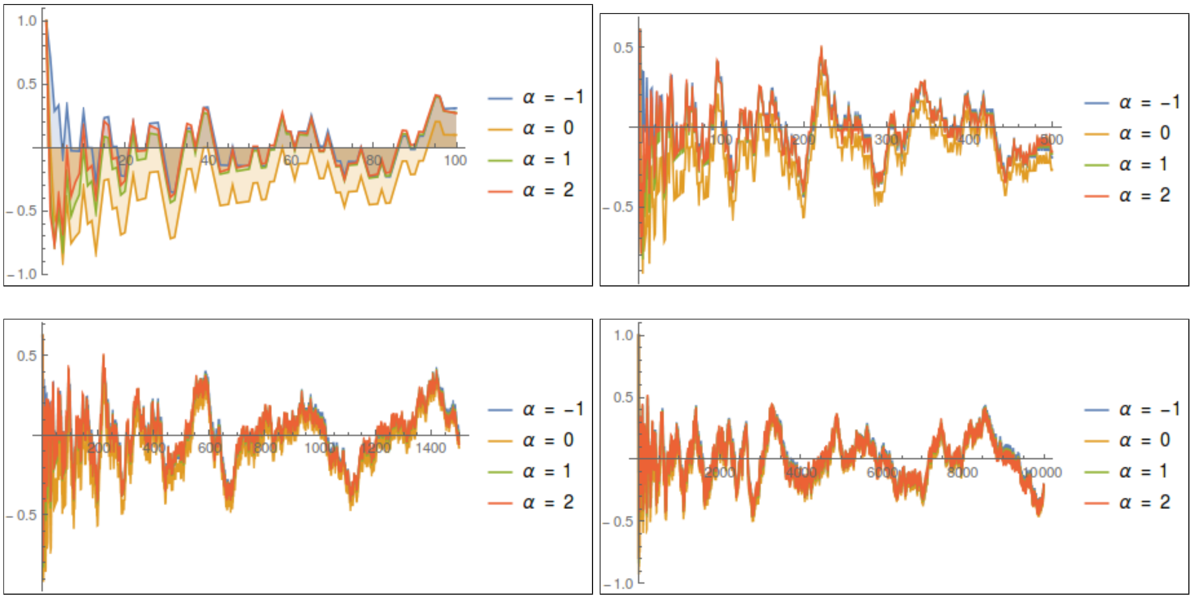


Figure 1.1: Comparison of the Mertens Summatory Functions $M_\alpha(x)/x^{\frac{1}{2}+\alpha}$ for Small x and α

we then obtain that

$$M(x) = \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation along with the standard Euler product representation for the reciprocal zeta function leads us to the exact expression for $M(x)$ when $x > 0$ given by the next theorem [?].

Theorem 1 (Analytic Formula for $M(x)$). If the RH is true, the there exists an infinite sequence $\{T_k\}_{k\geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any $x \in \mathbb{R}_{>0}$

$$M(x) = \lim_{k\to\infty} \sum_{\substack{\rho:\zeta(\rho)=0 \\ |\operatorname{Im}(\rho)|<T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n\geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)!\zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} \left[x \in \mathbb{Z}^+\right]_\delta.$$

1.2.2 Explicit bounds for large x

An unconditional bound on the Mertens function due to Walfisz [?] states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \exp\left(-C \cdot \log^{3/5}(x) (\log\log x)^{-3/5}\right).$$

Under the assumption of the RH, Soundarajan proved new updated estimates bounding $M(x)$ for large x in 2007 of the following forms:

$$\begin{aligned} M(x) &\ll \sqrt{x} \exp\left(\log^{1/2}(x) (\log\log x)^{14}\right), \\ M(x) &= O\left(\sqrt{x} \exp\left(\log^{1/2}(x) (\log\log x)^{5/2+\epsilon}\right)\right), \ \forall \epsilon > 0. \end{aligned}$$

Other explicit bounds due to the article by Kotnik include the following simpler estimates for the Mertens function when x is sufficiently large:

$$\begin{aligned} |M(x)| &< \frac{x}{4345}, \ \forall x > 2160535, \\ |M(x)| &< \frac{0.58782 \cdot x}{\log^{11/9}(x)}, \ \forall x > 685. \end{aligned}$$

1.3 Open problems

The Riemann Hypothesis is equivalent to showing that $M(x) = O\left(x^{1/2+\varepsilon}\right)$ for any $0 < \varepsilon < \frac{1}{2}$. For $\operatorname{Re}(\alpha) < 1$, we know the limiting absolute behavior of these functions as $x \to \infty$ as the Dirichlet generating function

$$\frac{1}{\zeta(\alpha)} = \lim_{x \longrightarrow \infty} M_{-\alpha}^{*}(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}},$$

which is definitively bounded for all large x. It is still unresolved whether

$$\limsup_{x \to \infty} |M(x)|/\sqrt{x} = \infty,$$

although computational evidence suggests that this is a likely conjecture [9, 7]. There is a rich history to the original statement of the *Mertens conjecture* which states that

$$|M(x)| < c \cdot x^{1/2}, \quad \text{some constant } c > 0,$$

which was first verified by Mertens for $c = 1$ and $x < 10000$, although since its beginnings in 1897 has since been disproved by computation.

There are a number of other interesting unsolved and at least somewhat accessible open problems related to the asymptotic behavior of $M(x)$ at large x. It is believed that the sign of $M(x)$ changes infinitely often. That is to say that it is widely believed that $M(x)$ is ocsillatory and exhibits a negative bias in so much as $M(x) < 0$ more frequently than $M(x) > 0$ over all $x \in \mathbb{N}$[1] One of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ is unbounded on the natural numbers. In particular, the precise statement of this problem is to produce an affirmative answer whether $\limsup_{x \to \infty} |M(x)|/\sqrt{x} = +\infty$, or equivalently whether there is an infinite sequence of natural numbers $\{x_1, x_2, x_3, \ldots\}$ such that $M(x_i) x_i^{-1/2}$ grows without bound along this subsequence.

Extensive computational evidence has produced a conjecture due to Gonek that in fact the limiting behavior of $M(x)$ satisfies that

$$\limsup_{x \to \infty} \frac{|M(x)|}{\sqrt{x}(\log \log x)^{5/4}},$$

corresponds to some bounded constant. A probabilistic proof along these lines has been given by Ng in 2008, though to date an exact rigorous proof (rather than somewhat heuristic argument) that $M(x)/\sqrt{x}$ is unbounded still remains elusive. We cite that prior to this point it is known that [16, *cf.* §4.1]

$$\limsup_{x \to \infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } 1.826054),$$

and

$$\liminf_{x \to \infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } -1.837625),$$

although based on work by Odlyzyko and te Riele it seems probable that each of these limits should be $\pm\infty$, respectively [14, 10, 9, 7]. It is also known that $M(x) = \Omega_{\pm}(\sqrt{x})$ and $M(x)/\sqrt{x} = \Omega_{\pm}(1)$.

1.4 The weighted summatory functions of the Liouville lambda function

2 A common language for problems we are interested in asymptotically estimating

The Mertens function case is the motivation for exploring the setup of a more general construction for relating summatory functions of special sequences via natural recurrence relations which arise for these functions. As it turns out, all of the machinery we will need to formulate new bounds for the classical Mertens function case can really just be re-worked and re-stated up front to provide theorems for analogous new bounds on other summatory

[1] See for example the discussion in the following thread: https://mathoverflow.net/questions/98174/is-mertens-function-negatively-biased.

functions. We might as well sketch out this more general scenario before diving head first into the initially very, very interesting classical special case. Then, later, once we have rigorously divined proofs of new bounds for this exceptionally interesting case for $M(x)$, we will have a loose path backwards which will allow us to cover the existence of related bounds for other summatory functions which we can show show up in a variety of contexts.

2.1 Introducing several key problems of interest to our study

Before we dive into definitions that formulate the necessary complication that this more general framework brings with it, let's be precise in stating the nature of the results about a general summatory function, denoted $M^*(x)$, which we are interested in investigating. Here, the assumption is that $M^*(x)$ inherits some signed oscillatory nature from its constituent summands – just as $M(x)$ classically varies according to localized behaviors of the Möbius function. Thus our problem is substantially more interesting (and difficult) than only estimating and best-error-bounding a monotone non-decreasing sum of arithmetic functions.

Problem of Interest 2.1 (Proving Scaled Unboundedness). We let the smooth positive function $\delta(x)$ take the place of $\sqrt{x}$ in the classical Mertens function problem statements. The question at hand is whether for a fixed, appropriately (and application dependent) chosen $\delta(x)$, we have unboundedness towards both $\pm\infty$. That is to say, is it true that each of

$$\limsup_{x\to\infty} \frac{M^*(x)}{\delta(x)} = +\infty,$$
$$\liminf_{x\to\infty} \frac{M^*(x)}{\delta(x)} = -\infty,$$

hold in the limiting case? If these two questions assert a "YES" given some optimally chosen function $\delta(x)$, we next should ask along which subsequences of the positive integers is this unboundedness achieved. That is to say, given any real $M > 0$, can we identify (natural) infinite subsequences $(x_n(M))_{n\geq 1}, (y_m(M))_{m\geq 1}$ such that

$$\frac{M^*(x_n(M))}{\delta(x_n(M))} > M, \forall n \geq 1,$$
$$\frac{M^*(y_m(M))}{\delta(y_m(M))} < -M, \forall m \geq 1?$$

Is there a limiting measure for the form of these subsequences, i.e., is there a limiting distribution ν on $\mathbb{R}$ for either of $M^*(x_n(M))/\delta(x_n(M))$ or $M^*(y_m(M))/\delta(y_m(M))$ (*cf.* Ng's probabilistic approach to $M(x)$)?

Problem of Interest 2.2 (Proving Best Possible Limit Supremum and Infimum Growth Rates). This is essentially the generalized form of verifying Gonek's conjecture for $M(x)$. That is to say, we seek positive smooth functions $\delta(x)$ and $\xi(x)$, and some absolute constants $C_1, C_2; D_\ell, D_u > 0$, such that

$$\limsup_{x\to\infty} \frac{M^*(x)}{\delta(x) \times \xi(x)^{C_1}} = D_u,$$
$$\liminf_{x\to\infty} \frac{M^*(x)}{\delta(x) \times \xi(x)^{C_2}} = D_\ell.$$

Is it true that we should expect that $C_1 = C_2$, or otherwise any symmetry between the limiting constants D_ℓ and D_u?

Problem of Interest 2.3 (Expected Time to Exceed a Preset Bound).

Problem of Interest 2.4 (Frequency of Visitation to a Preset Bound).

Problem of Interest 2.5 (Distribution of Zeros of the Summatory Function).

2.2 Motivating, non-classical applications to summatory functions of Dirichlet convolutions

Note that Section B goes into significantly more background and preparation to generalize the procedure for the classical Mertens function $M(x)$ to Mertens (and Möbius) functions which are constructed in a slightly different manner. In addition to this solid application, there are a couple of more instances which are worth enumerating where our general approach and the construction offered by Theorem 25 arise in useful applications. Let's enumerate some attention one example case by case below.

2.2.1 Example I: Sums over Dirichlet convolution divisor sums

Let the summatory function $G(x) := \sum_{n\leq x} g(n)$. We then consider the summatory functions of the Dirichlet convolution between two arithmetic functions, f, g: $(f * g)(n) = \sum_{d|n} f(d)g(n/d)$. Suppose that this summatory function is given by $\hat{\pi}(x)$. Then we see that

$$\begin{aligned}\hat{\pi}(x) &= \sum_{n\leq x}\sum_{d|n} f(d)g(n/d) \\ &= \sum_{d=1}^{x} f(d) G\left(\left\lfloor \frac{x}{d}\right\rfloor\right).\end{aligned}$$

Note that this is actually a special case of the Mertens function recurrences which lead to the form of Theorem 25. When the left-hand-side of the previous equations is not easily expressed in terms of the summatory function $G(x)$, we have the following related theorem which allows us to exactly express the formula for $G(x)$ in terms of the forms of f and $\hat{\pi}$.

Theorem 2. Let the arithmetic functions f, g be given with respective summatory functions $F(x)$ and $G(x)$ where $f(1) \neq 0$. Suppose that $\hat{\pi}(x)$ is a non-identically zero function defined on the positive integers. Furthermore, suppose that we have a recurrence relation for G of the form

$$\hat{\pi}(x) = \sum_{d=1}^{x} f(d) G\left(\left\lfloor \frac{x}{d}\right\rfloor\right), \forall x \geq 1.$$

Then we have the exact expressions

$$G(x) = \sum_{k=1}^{x} t_{x,k}(f)\hat{\pi}(k), \forall x \geq 1,$$

where the lower triangular sequence $t_{x,j}(f)$ depends only on f (and F) as the inverse matrix entries of

$$(t_{i,j}(f))_{1\leq i,j\leq x} = \left(F\left(\left\lfloor \frac{i}{j}\right\rfloor\right) - F\left(\left\lfloor \frac{i}{j+1}\right\rfloor\right)\right)^{-1}_{i\leq i,j\leq x}.$$

Since $f(1) \neq 0$, it's Dirichlet inverse, $f^{-1}(n)$, exists and provides us with an expansion of the inverses of $(t_{i,j}(f))$ of the form

$$t_{x,j}^{-1}(f) = \sum_{k=\left\lfloor \frac{x}{j+1}\right\rfloor+1}^{\left\lfloor \frac{x}{j}\right\rfloor} f^{-1}(k) = F^{-1}\left(\left\lfloor \frac{x}{j}\right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1}\right\rfloor\right),$$

if $F^{-1}(x)$ denotes the summatory function of f^{-1}.

Given the interpretation of the summatory functions over an arbitrary Dirichlet convolution (and the vast number of such identities for special number theoretic functions – *cf.* [?]), it is not surprising that this formulation of the first theorem may well provide many fruitful applications, indeed! In addition to those cited in the compendia of the catalog reference, we have notable identities of the form: $(f*1)(n) = [q^n]\sum_{m\geq 1} f(m)q^m/(1-q^m)$, $\sigma_k = \mathrm{Id}_k * 1$, $\mathrm{Id}_1 = \phi * \sigma_0$, $\chi_{\mathrm{sq}} = \lambda * 1$ (see sections below), $\mathrm{Id}_k = J_k * 1$, $\log = \Lambda * 1$, and of course $2^{\omega} = \mu^2 * 1$.

The result in Theorem 2 is natural and displays a quite beautiful form of symmetry between the initial matrix terms,

$$t_{x,j}(f) = \sum_{k=\left\lfloor \frac{x}{j+1}\right\rfloor+1}^{\left\lfloor \frac{x}{j}\right\rfloor} f(k),$$

and the corresponding inverse matrix,

$$t_{x,j}^{-1}(f) = \sum_{k=\left\lfloor \frac{x}{j+1}\right\rfloor+1}^{\left\lfloor \frac{x}{j}\right\rfloor} f^{-1}(k),$$

as expressed by the duality of f and its Dirichlet inverse function f^{-1}. Since the recurrence relations for the summatory functions $G(x)$ arise naturally in applications where we have established bounds on sums of Dirichlet convolutions of arithmetic functions, we will go ahead and prove it here before moving along to some motivating examples of the use of this theorem.

Proof of Theorem 2. Let h, g be arithmetic functions where $g(1) \neq 1$ has a Dirichlet inverse. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $S_{g,h}(x)$ to be the summatory function of the Dirichlet convolution of g with h: $g * h$. Then we can easily see that the following expansions hold:

$$\begin{aligned} S_{g,h}(x) &:= \sum_{n=1}^{x} \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^{x} \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] H(i). \end{aligned}$$

Thus we have an implicit statement of a recurrence relation for the summatory function H, weighted by g and G, whose non-homogeneous term is the summatory function, $S_{g,h}(x)$, of the Dirichlet convolutions $g * h$. We form the matrix of coefficients associated with this system for $H(x)$, and proceed to invert it to express an exact solution for this function over all $x \geq 1$. Let the ordinary (initial, non-inverse) matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1}.$$

Then the matrix we must invert in this problem is lower triangular, with ones on its diagonals – and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressable by an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T); U = (\delta_{i,j+1}), (I - U^T)^{-1} = ([j \leq i]_\delta).$$

It is a nice round fact that if we take successive differences of floor functions, we get non-zero behavior at divisors:

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

We use this property to invertibly shift the matrix $\hat{G}$, and then invert the result, to obtain a matrix involving the Dirichlet inverse of g:

$$\left[(I - U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

Now we can express the inverse of the target matrix $(g_{x,j})$ in terms of these Dirichlet inverse functions as follows:

$$\begin{aligned} (g_{x,j}) &= (I - U^T)^{-1}\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I - U^T) \\ (g_{x,j})^{-1} &= (I - U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I - U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k)\right)(I - U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} g^{-1}(k)\right). \end{aligned}$$

Thus the summatory function H is exactly expressed by the inverse vector product of the form

$$H(x) = \sum_{k=1}^{x} g_{x,k}^{-1} \cdot S_{g,h}(k)$$

$$= \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \cdot S_{g,h}(k).$$

It stands to reason that explicit (accurate) bounds on the initial summatory function $S_{g,h}(x)$ ought dictate quite a bit about limiting behaviors and tendencies of the function $H(x)$, such as unboundedness in the limit supremum or infimum sense, when $x \gg 1$. We will tackle these topics in later sections. □

There are many natural ways given the above theorem for expressing new formulas for the Mertens function, $M(x)$. We will simply motivate our theorem's methodology and approach by giving one example, of which many others, and some perhaps more optimal, exist.

Corollary 3 (Convolutions Arising From Möbius Inversion). *Suppose that g is an arithmetic functions with $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\widetilde{G}(x) := \sum_{n \leq x} (g * \mu)(n)$. Then*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j) \right) \widetilde{G}(k), \forall x \geq 1.$$

Remark 2.6. Ideally, we can use the corollary above to express a definite, usable formula by which we may attempt to derive new bounds for the Mertens function. If this is the goal in applying Corollary 3, the the Dirichlet inverse of g should not depend on μ in any obvious ways, so as to avoid recursion in computing $M(x)$ exactly. This excludes some better known divisor sum convolution identities whose left-hand-side summatory functions have well established bounds with error terms, for example, $1 = d * \mu$, $\phi = \mathrm{Id}_1 * \mu$, and $\sigma = \phi * d$. However, the following two special cases illustrate how our new results allow us to express $M(x)$ in terms of classically well known, and well studied interesting function cases:

- Using $\lambda = \chi_{\mathrm{sq}} * \mu$, where χ_{sq} denotes the characteristic function of the squares, we obtain that $\widetilde{G}(x) \equiv L_0(x)$, the non-weighted summatory function of $\lambda(n)$. The corresponding inverse sequence, g^{-1} for $g(n) := \chi_{\mathrm{sq}}$, can be calculated numerically, leading to the following sequence values:

  $$\left\{\chi_{\mathrm{sq}}^{-1}\right\}_{n \geq 1} = \{1, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, \ldots\}.$$

  It appears that the inverse function depends on the values of the Möebius function, so we move along.

- Using $\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu$ where $\chi_{\mathbb{P}}$ is the characteristic function of the primes, we have that $\widetilde{G}(x) = \pi(x) + 1$, and can compute the inverse sequence of $g(n) := \omega(n) + 1$ numerically for the first few sequence values as

  $$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

  The sign of these terms is apparently dictated by $\lambda(n) = (-1)^{\Omega(n)}$, though no formula for the unsigned magnitudes is immediately obvious.

Example 2.7 (A Recurrence Relating Squares of the Mertens Function). Let

$$\hat{B}_k(x) := \sum_{n \leq x} n^k, k \in \mathbb{R}.$$

When $k \geq 0$ is integer-valued, Faulhaber's formula provides an explicit summation in powers of x to express the function $\hat{B}_k(x)$ involving the Bernoulli numbers. For $k < 1$, the sequence of $\hat{B}_k(x)$ corresponds to the partial sums of a convergent Riemann zeta function value.

We have a known divisor sum convolution for the k^{th} powers of the identity function given by $\mathrm{Id}_k = \sigma_k * \mu$, where $\sigma_k(n) = \sum_{d|n} d^k$ denotes the generalized sum-of-divisors function. It has a known Dirichlet inverse function expressed by the convolution

$$\sigma_k^{-1}(n) = \sum_{d|n} d^k \mu(d) \mu\left(\frac{n}{d}\right).$$

By Theorem 2, we obtain that

$$\begin{aligned}
\hat{B}_k(x) &= \sum_{d=1}^{x} \sigma_k(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \qquad \Longrightarrow \\
M(x) &= \sum_{m=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{m+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{m} \right\rfloor} \sigma_k^{-1}(j) \right) \hat{B}_k(m).
\end{aligned}$$

The inner summation terms in the last equation correspond to a difference of summatory function inputs for $S_k^{-1}(x) := \sum_{n \leq x} \sigma_k^{-1}(n)$. Since the Dirichlet inverse of $\sigma_k(n)$ is given as a divisor sum, we can further transform it with the theorems at hand as

$$\begin{aligned}
S_k^{-1}(x) &= \sum_{d=1}^{x} d^k \mu(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{i=1}^{x} \left[M_k\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - M_k\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] M(i).
\end{aligned}$$

Now by shifting the index of summation, and re-interpreting with our sum of divisor sums construction, we re-expand the last formula for $M(x)$ by writing

$$\begin{aligned}
M(x) &= \sum_{i=1}^{x} \left[\hat{B}_k\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - \hat{B}_k\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] S_k^{-1}(i) \\
&= \sum_{j=1}^{x} \sum_{i=j}^{x} \left[\hat{B}_k\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - \hat{B}_k\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] \left[M_k\left(\left\lfloor \frac{i}{j} \right\rfloor\right) - M_k\left(\left\lfloor \frac{i}{j+1} \right\rfloor\right) \right] M(j).
\end{aligned}$$

We can of course take $k := 0$ in this setup to see a two-index, square-like dependence of the classical Mertens function terms. In this special case, we can simplify according to the observation that $\hat{B}_0(x) = x$ for all $x \geq 1$.

Proposition 4 (Expressing the Mertens Function by Any Arithmetic Function). *Suppose that f is any Dirichlet invertible multiplicative arithmetic function. Let $\widetilde{F}(x)$ be the summatory function of $f * 1$, where by convention we denote $F(x) = \sum_{n \leq x} f(n)$ and $F^{-1}(x) = \sum_{n \leq x} f^{-1}(n)$. Then we have that*

(A) For all $x \geq 1$,

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} (\mu * f^{-1})(j) \right) F(k);$$

(B) For all $x \geq 1$, we have a recurrence relation for $M(x)$ given by

$$M(x) = \sum_{d=1}^{x} \sum_{k=d}^{x} \left[F^{-1}\left(\left\lfloor \frac{k}{d} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{k}{d+1} \right\rfloor\right) \right] \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right] M(d);$$

(C) Moreover, we can write

$$M(x) = \sum_{k=1}^{x} h_{x,k}^{-1} \cdot M(k), \forall x \geq 1,$$

where the inverse matrices $(h_{x,j})^{-1}$ have the formula

$$\begin{aligned}
h_{x,j}^{-1} = &\sum_{r=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} f^{-1}(r) \left[f(j) - F\left(\left\lfloor \frac{x}{r(j+1)} \right\rfloor\right) \right] \\
&+ \sum_{r=\left\lfloor \frac{x}{j+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{j} \right\rfloor} f^{-1}(r) \left[F\left(\left\lfloor \frac{x}{rj} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{r(j+1)} \right\rfloor\right) + F(j-1) \right], \forall 1 \leq j \leq x.
\end{aligned}$$

Proof. □

2.2.2 Example II: Generalized α-power scaled Mertens functions

This is really just a way of re-writing the results of either Theorem 25 or Theorem 2 to express a formula for the weighted Mertens functions, $M_\alpha^*(x)$, defined by (1). In particular, observe that for any arithmetic function f_0, we can expand

$$\begin{aligned} S_{0,\alpha}(x) &:= \sum_{n \leq x} \frac{1}{n^\alpha} \sum_{d|n} f_0(d)\mu(n/d) \\ &= \sum_{d=1}^{x} \frac{f_0(d)}{d^\alpha} M_\alpha^* \left(\left\lfloor \frac{x}{d} \right\rfloor \right). \end{aligned}$$

Now to put these sums in the class of recurrence relations we handle by Theorem 25, we must identify functions such that

$$S_{0,\alpha}(x) := M_\alpha^*(x) - \pi_\alpha^*(x), \forall x \geq 1.$$

Notice that given a fixed choice of f_0, the function π_α^* can be recovered from the formula

$$f_0(x) = \sum_{d|x} x^\alpha \cdot \nabla[S_{0,\alpha}](d), x \geq 1.$$

Otherwise, if $S_{0,\alpha}(x)$ does not depend on the weighted summatory function for general x, we are firmly in the territory covered by the case of Theorem 2. Notice that the free parameter in the form of the function $f_0(n)$ not only determines the non-homogeneous function in the recurrence relation above, but it allows us some flexibility to play with the bounds on the summatory functions in question.

2.2.3 Example III: Weighted summatory functions for the Liouville lambda function

We define the weighted summatory function of the *Liouville lambda function*, $\lambda(n) = (-1)^{\Omega(n)}$ where $\Omega(n)$ counts the total number of prime factors of n (counting multiplicity), for a parameter $\alpha \geq 0$ as follows:

$$L_\alpha(x) := \sum_{n \leq x} \frac{\lambda(n)}{n^\alpha}, x \geq 1.$$

The special cases of the functions $L(n) \equiv L_0(n)$ and $T(n) \equiv L_1(n)$ are well studied in the literature [**?**, **?**, **?**]. For example, it is known that

$$L_0(x) = O\left(\sqrt{x} \exp\left(C \cdot \log^{1/2}(x) \left(\log\log x\right)^{5/2+\varepsilon}\right)\right),$$

for any $\varepsilon > 0$; and that

$$L_\alpha(x) = O\left(x^{1-\alpha} \exp\left(-C_\alpha \frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right),$$

for absolute constants $C_\alpha > 0$ which depend only on the parameter $\alpha \in \mathbb{R}_{\geq 0}$. There are analogous expressions for these summatory functions as we gave for $M(x)$ in Section 1 which involve sums over the non-trivial zeros of the Riemann zeta function. Such formulas result from an inverse Mellin transform and the known Dirichlet series representations over $\lambda(n)$:

$$\int_1^\infty \frac{L_\alpha(x)}{x^{s+1}} dx = \frac{\zeta(2\alpha + 2s)}{s \cdot \zeta(\alpha + s)}, \operatorname{Re}(s) > 1 - \alpha.$$

For our purposes, we have the fundamental identity that $\chi_{\rm sq}(n) = (\lambda * 1)(n)$ where $\chi_{\rm sq}(n)$ is the characteristic function of the squares, i.e., $\chi_{\rm sq}(n) = [\sqrt{n} \in \mathbb{N}]_\delta$. Now by summing over $n^{-\alpha} \times \chi_{\rm sq}(n)$ from $1 \leq n \leq x$, we can similarly obtain the following recursion, which corresponds to a case of Theorem 2:

$$H_x^{(2\alpha)} = \sum_{d \leq x^2} \frac{1}{d^\alpha} \cdot L_\alpha\left(\left\lfloor \frac{x^2}{d} \right\rfloor\right), x \geq 1.$$

The left-hand-side sequence corresponds to the generalized 2α-order harmonic number sequence for $\alpha \geq 0$. Some special case formulas for these sequences are given by $H_x^{(0)} = x$, $H_x^{(1)} = \log x + \gamma + O(x^{-1})$. For $\alpha \neq 1/2$, we can approximate these sequences by the integral

$$H_x^{(2\alpha)} \approx \int_1^x \frac{dt}{t^{2\alpha}} = \frac{1}{2\alpha - 1}\left(1 - x^{1-2\alpha}\right),$$

or perhaps more precisely when $\operatorname{Re}(\alpha) > 1/2$ by the Riemann zeta function minus its tail as

$$H_x^{(2\alpha)} = \zeta(2\alpha) + O(x^{-2\alpha}).$$

We can use our new result in Theorem 2 to enumerate some almost immediate properties and corollaries that tie $L_\alpha(x)$ to the corresponding weighted Mertens functions, $M_{-\alpha}(x)$. In particular, since the Dirichlet inverse of $\mathrm{Id}_\beta^{-1}(n) = \mu(n)\,\mathrm{Id}_\beta(n)$ for any fixed $\beta \in \mathbb{C}$, we quickly work our theorem magic to obtain that

$$\begin{aligned} L_\alpha(x) &= \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} \frac{\mu(j)}{j^\alpha}\right) H_{\sqrt{k}}^{(2\alpha)} \\ &= \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} \frac{1}{j^\alpha}\right) M_{-\alpha}(k). \end{aligned} \tag{2}$$

We will hit on a few notable special cases of α to make a representative point about what the theorem provides us information on off the bat. Namely, we will consider $\alpha := 1$ (later we should try $\alpha = 0, 1/2$).

Corollary 5 (Relating $L_1(x)$ to the Mertens Function $M_{-1}(x)$). *For all sufficiently large $x \geq 1$, we have that*

$$L_1(x) \approx \frac{\pi^2}{6} M_{-1}(x) + o(1).$$

Thus if one of these two functions grows without bound, or approaches zero, then so must the other.

Proof. We use the first form of (2) together with the bound

$$H_{\sqrt{k}}^{(2)} = \frac{\pi^2}{6} + O\left(\frac{1}{k}\right),$$

to write that

$$L_1(x) = \frac{\pi^2}{6} M_{-1}(x) + O\left(\sum_{k=1}^{x}\left[M_{-1}\left(\left\lfloor \frac{x}{k}\right\rfloor\right) - M_{-1}\left(\left\lfloor \frac{x}{k+1}\right\rfloor\right)\right]\frac{1}{k}\right).$$

In the previous equation we make use of the fact that the intervals

$$\bigcup_{1 \leq k \leq x}\left[\left\lfloor \frac{x}{k+1}\right\rfloor + 1, \left\lfloor \frac{x}{k}\right\rfloor\right] \cap \mathbb{Z} = [1, x] \cap \mathbb{Z}.$$

Now to change the order of summatory functions we are considering, notices that if $S(k) = \frac{1}{k}$, $S(k) - S(k-1) \approx -\frac{1}{k^2}$, so that is our new approximate recurrence weight function when we exchange the sums. We now bound the error term by summing

$$\begin{aligned} E_1(x) &= \left|\sum_{k=1}^{x}\left[\sum_{j=\left\lfloor \frac{x}{k+1}\right\rfloor+1}^{\left\lfloor \frac{x}{k}\right\rfloor} \frac{1}{j^2}\right] M_{-1}(k)\right| \\ &\approx \left|\sum_{k=1}^{x} \frac{2k+1}{x^2} M_{-1}(k)\right|. \end{aligned}$$

If we perform summation by parts on the numerator of the error sum, we will see that it has lesser order than x^2. So we obtain that the error is $o(1)$. ∎

3 New bounds for the classical Mertens function

3.1 Approach and problem setup

Lemma 6 (An Exact Representation of the Dirichlet Series Over $\omega(n)$). *Whenever* $\operatorname{Re}(s) > 1$, *we have that for* $\omega^*(n) := \omega(n) + \delta_{n,1}$,

$$D_{\omega^*}(s) := \sum_{n \geq 1} \frac{\omega^*(n)}{n^s} = 1 + \zeta(s) P(s),$$

where $P(s) := \sum_{p \text{ prime}} p^{-s}$ *denotes the prime zeta function* [2]. *As a consequence, the DGF of* $f(n) := \omega(n) + 1$ *is given by* $D_{\omega+1}(s) = \zeta(s)(P(s) + 1)$.

Proof. The first result is justified by noticing that

$$\prod_{p \text{ prime}} \left(1 - \frac{u}{1 - p^s}\right) = \sum_{n \geq 1} \frac{u^{\omega(n)}}{n^s},$$

so that

$$\begin{aligned} D_\omega(s) &= \frac{\partial}{\partial u}\left[\prod_{p \text{ prime}} \left(1 - \frac{u}{1-p^s}\right)\right]\Bigg|_{u=1} \\ &= \prod_{p \text{ prime}} \left(1 - \frac{1}{1-p^s}\right) \sum_p \frac{\left(1 - \frac{1}{1-p^s}\right)^{-1}}{1 - p^s} \\ &= \zeta(s) \sum_p p^{-s}. \end{aligned}$$

Thus the result is proved. □

Proposition 7 (An Antique Divisor Sum Identity). *Let* $\chi_{\mathbb{P}}$ *denote the characteristic function of the primes,* $\varepsilon(n) = \delta_{n,1}$ *be the identity with respect to Dirichlet convolution, and denote by* $\omega(n)$ *the additive function that counts the number of distinct prime factors of* n. *Then*

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu.$$

The summatory function of the LHS is $\widetilde{G}(x) = \pi(x) + 1$.

Proof. The the crux of the stated identity is to prove that for all $n \geq 1$, $\chi_{\mathbb{P}}(n) = (\mu * \omega)(n)$ – our claim. We notice that the Mellin transform of $\pi(x)$ – the summatory function of $\chi_{\mathbb{P}}(n)$ – at $-s$ is given by

$$\begin{aligned} s \cdot \int_1^\infty \frac{\pi(x)}{x^{s+1}} dx &= \sum_{n \geq 1} \frac{\nabla[\pi](n-1)}{n^s} \\ &= \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s} = P(s). \end{aligned}$$

[2] By a Mellin transform, we have that

$$P(s) = s \cdot \int_1^\infty \pi(x) x^{-(s+1)} dx.$$

By expanding out the logarithm of the Euler product for $\zeta(s)$, and then performing a Möbius series inversion, we also have that

$$P(s) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log \zeta(ks),$$

which implies that $P(s)$ has (logarithmic) singularities at $s := 1/k$ for all $k \geq 1$. We can then show that $s := 0$ is a natural boundary for $P(s)$. Near $s := 1$, we have that $P(1 + \varepsilon) \sim -\log(\varepsilon) + C_1 + O(\varepsilon)$ as $\varepsilon \to 0^+$ where $C_1 \approx -0.315718452$ is an absolute constant.

This is typical fodder which more generally relates the Mellin transform $\mathcal{M}[S_f](-s)$ to the DGF of the sequence $f(n)$ as cited, for example, in [1, §11]. Now we consider the DGF of the right-hand-side function, $f(n) := (\mu * \omega)(n)$, as

$$D_f(s) = \frac{1}{\zeta(s)} \times \sum_{n\geq 1} \frac{\omega(n)}{n^s} = P(s),$$

by Lemma 6. Thus for any $\mathrm{Re}(s) > 1$, the DGFs of each side of the claimed equation coincide. So by uniqueness of Dirichlet series, we see that in fact the claim holds. To obtain the full result, we add to each side of this equation a term of $\varepsilon(n) \equiv (\mu * 1)(n)$, and then factor the resulting convolution identity. ☐

Remark 3.1 (Exact Statement of Our New Formulas for $M(x)$). Let $S^{-1}(x) := \sum_{n\leq x} (\omega+1)^{-1}(n)$ denote the summatory function of the invertible function $\omega(n)+1$. We use the construction following from Corollary 3 applied to the divisor sum identity $\chi_{\mathbb{P}} + \varepsilon = (\omega+1) * \mu$. Then we obtain the following exact formulas for the Mertens function $M(x)$:

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} (\omega+1)^{-1}(j) \right) (\pi(x)+1) \qquad (3) \\ &= S_{(\omega+1)^{-1}}(x) + \sum_{\substack{p \leq x \\ \chi_{\mathbb{P}}(p)=1}} S_{(\omega+1)^{-1}}\left(\left\lfloor \frac{x}{p} \right\rfloor \right). \end{aligned}$$

The second expansion of $M(x)$ in the previous equation follows from an application of summation by parts to the first formula. Exact formulas for $(\omega+1)^{-1}(n)$ for all $n \geq 1$ and the DGF of this inverse sequence are proved next.

Heuristic 3.2 (Why These New Formulas Are Promising, All Things Considering). In Lemma 8 below, we prove that

$$\mathrm{sgn}[(\omega+1)^{-1}(n)] = \mathrm{sgn}[\lambda(n)] = (-1)^{\Omega(n)}, \forall n \geq 1.$$

In other words, we can express this Dirichlet inverse function as $(\omega+1)^{-1}(n) \equiv \lambda(n) \cdot f_0^{-1}(n)$ where $f_0^{-1}(n) \geq 0$ for all positive natural numbers n. Since our new formulas for $M(x)$ expanded in (3), depend naturally on the summatory functions of $(\omega+1)^{-1}$, partial summation implies a representation for the main terms of $M(x)$ which is written in terms of $L_0(x)$, the summatory function of the Liouville lambda function, $\lambda(n)$.

We take heed by the references [?, ?, ?] and see that compared to the properties of $M(x)$, much, much more is known about the properties and bounds satisfied by $L_0(x)$. Thus it stands to reason that we ought be more optimistic than not about discovering new bounds for $M(x)$, as our formulas implicitly rely on the summatory functions $L_0(x)$, about which classically considerably more has been rigorously proved. This of course does not mean that proving new bounds on the Mertens function via these formulas will be immediate, nor easy, but does offer a glimmer of new insight into these functions based on past results.

Lemma 8 (Decompositions of the Dirichlet Inverse Function, $(\omega+1)^{-1}(n)$). *We have the following properties of $(\omega+1)^{-1}(n)$ for any $n \geq 1$:*

(i) The sign of $(\omega+1)^{-1}(n)$ is given by $\mathrm{sgn}[\lambda(n)] = (-1)^{\Omega(n)}$;

(ii) If we decompose the inverse function into the product $(\omega+1)^{-1}(n) := \lambda(n) \cdot f_0^{-1}(n)$, then $f_0^{-1}(n) \geq 0, \forall n \geq 1$, and the DGF of this non-negative weight sequence is given by

$$D_{f_0^{-1}}(s) = TODO, \mathrm{Re}(s) > 1.$$

Proof of (i). We will utilize the iterated convolutions expansion for the Dirichlet inverse of a function f proved in [?]. First, for $j \geq 1$ we can expand the convolution of the function $\omega + 1 - \varepsilon$ with itself j times as we would in a trinomial coefficient expansion:

$$\underbrace{(\omega+1-\varepsilon) * \cdots * (\omega+1-\varepsilon)}_{j \text{ times}} = \sum_{\substack{i_1+i_2+i_3=j \\ i_1,i_2,i_3 \geq 0}} \binom{j}{i_1, i_2, i_3} \underbrace{(\omega * \cdots * \omega)}_{i_1 \text{ times}} * \underbrace{(1 * \cdots * 1)}_{i_2 \text{ times}} \times (-1)^{i_3}.$$

Since the multiple convolutions of the non-negative functions $f(n) = \omega(n), 1$ are always non-negative, the signage in the inverse function comes from the interplay of the index i_3 when we sum over $j \geq 0$ to express the full formula for the Dirichlet inverse function. Thus we can actually express just the form of the signs of the inverse function by performing the sum

$$\begin{aligned}\operatorname{sgn}[(\omega+1)^{-1}(n)] &= \sum_{j\geq 0}\sum_{i_3\geq 0}\left[\binom{2j}{i_3} - 2\binom{2j+1}{i_3}\right](-1)^{i_3} \\ &= \left[(x-1)^{\Omega(n)} + (-1)^{\Omega(n)}(x+1)^{\Omega(n)} - 2(x-1)^{\Omega(n)} + 2\cdot(-1)^{\Omega(n)}(x+1)^{\Omega(n)}\right]\Big|_{x\mapsto 1} \\ &= 3\cdot 2^{\Omega(n)}\cdot\lambda(n).\end{aligned}$$

The leading terms in the previous equation that do not depend on $\lambda(n)$ are positive for any $n \geq 1$. Hence, we obtain the claim. □

Proof of (ii). □

Lemma 9 (Dirichlet Series of a Dirichlet Inverse With Respect to Convolution). *Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. We adopt the following notation for the Dirichlet series over f with respect to the (complex-valued) parameter $s \in \mathbb{C}$:*

$$\mathcal{D}[f](s) := \sum_{n\geq 1}\frac{f(n)}{n^s},\ |s| < \sigma_f.$$

If $f(1) \neq 0$, then there is a unique multiplicative Dirichlet inverse with respect to Dirichlet convolution, i.e., another unique arithmetic function $f^{-1} : \mathbb{N} \to \mathbb{C}$ such that $f^{-1} * f = \varepsilon$, or equivalently such that $(f^{-1} * f)(n) = \delta_{n,1}$. Suppose that $f(1) \neq 0$ and that f^{-1} is its corresponding Dirichlet inverse function. Then for all $s \in \mathbb{C}$ such that $|s| < \sigma_f$ and such that $|\mathcal{D}[f](s)/f(1)| < 1$, we have that

$$\mathcal{D}[f^{-1}](s) = \frac{1}{\mathcal{D}[f](s)}.$$

Proof. This result follows from the inversion relations derived in [**?**, §3.2]. In particular, we notice that since we can write

$$f_{\pm}(n) = f(n) - 2f(1)\varepsilon(n),$$

we have that

$$\mathrm{ds}_j(n) = (f - f(1)\varepsilon) *_j - f(1)\varepsilon * (f - f(1)\varepsilon) *_{j-1},$$

where the notation $g*_j$ denotes the j-fold multiple Dirichlet convolution of the function g with itself. Then by the formulas proved in the reference, we can write

$$\begin{aligned}f(1)\mathcal{D}[f](s) &= 1 + \sum_{n\geq 1}\sum_{j\geq 1}\left(\frac{(f-f(1)\varepsilon)*_{2j}}{f(1)^{2j}} - \frac{(f-f(1)\varepsilon)*_{2j-1}}{f(1)^{2j-1}}\right) \\ &= \sum_{j\geq 0}\frac{(-1)^j\left(\mathcal{D}[f](s) - f(1)\right)^j}{f(1)^j} = \frac{1}{\left(1 + \frac{\mathcal{D}[f](s)-f(1)}{f(1)}\right)}.\end{aligned}$$

Dividing through by $f(1) \neq 0$ completes the proof of the result. □

Theorem 10 (Perron's Formula and a DGF Inversion Formula). Suppose that $\mathcal{D}[f](s)$ is absolutely convergent for $\mathrm{Re}(s) > \sigma_f$. Let $c, x > 0$ be such that $c > \sigma_f$. Then

$$F(x) := {\sum_{n\leq x}}' f(n) = \frac{1}{2\pi\imath}\int_{c-\imath\infty}^{c+\imath\infty}\mathcal{D}[f](s)\frac{x^s}{s}ds,$$

where the notation $\sum'$ denotes that the last term in the sum is multiplied by $\frac{1}{2}$ whenever $x \in \mathbb{Z}$. Moreover, for any $b > c > \sigma_f$, we can write the non-limiting form of the integral in the previous equation as

$$\int_{c-\imath T}^{c+\imath T}\mathcal{D}[f](s)\frac{x^s}{s}ds = \sum_{\rho\in\mathcal{R}_f(c,b)}\frac{1}{2\pi\imath}\operatorname{Res}_{s=\rho}\left[\frac{x^s D[f](s)}{s}\right] + \frac{1}{2\pi\imath}\left(\int_{b-\imath T}^{b+\imath T} - \int_{c+\imath T}^{b+\imath T} - \int_{c-\imath T}^{b-\imath T}\right)\left\{\frac{x^s D[f](s)}{2\pi\imath\cdot s}ds\right\}.$$

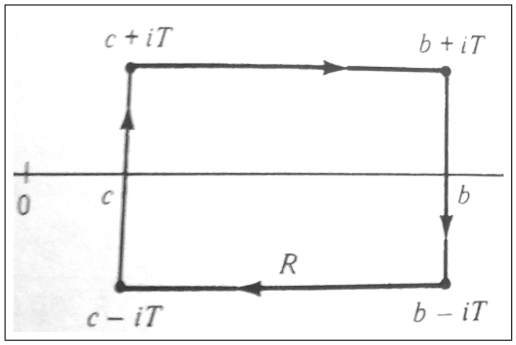
where $\mathcal{R}_f(c,b) = \{s \in \mathbb{C} : c < \operatorname{Re}(s) < b \wedge \mathcal{D}[f](s)/s = \pm\infty\}$ denotes the set of poles of the integrand within the closed box bounded by the points $s = b \pm \imath T, c \pm \imath T$.

We also have the complex-valued real line integral which provides that for integers $x \geq 1$,

$$f(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathcal{D}[f](\sigma + \imath t) x^{\sigma+\imath t} dt, \tag{4}$$

whenever $\sigma > \sigma_f$. This is a sort of analog to the coefficient extraction operation obtained by applying an inverse Z-transform to a generating function given by an analytic function in some z.

Proof. For a proof of *Perron's formula* in a slightly different form, see the reference [1, §11.12]. The stated formulation of Perron's formula is standard in applications. The contour we typically use to evaluate and bound Perron's first integral formula is shown below (reproduced from Apostol's book):



The re-statement of Perron's formula using this closed contour follows by applying the standard residue theorem to the integrand. ☐

Corollary 11 (An Initial Integral Formula for the Summatory Function of $(\omega+1)^{-1}(n)$). *For any positive integers $x \geq 1$, we have that*

$$S_{(\omega+1)^{-1}}(x) = \frac{(\omega+1)^{-1}(x)}{2} + \frac{1}{2\pi\imath} \int_{c-\imath\infty}^{c+\imath\infty} \frac{x^s}{s\zeta(s)(P(s)+1)} ds,$$

where we can take the real-valued $c > 1$. Set $b > c > 1$. Let $\mathcal{R}_P(c,b)$ denote the set of zeros of $P(s)+1$[3] with real part bounded by $c \leq \operatorname{Re}(\rho) \geq b$. Then we have that

$$\begin{aligned} S_{(\omega+1)^{-1}}(x) = \frac{(\omega+1)^{-1}(x)}{2} &+ \frac{1}{2\pi\imath} \sum_{\rho \in \mathcal{R}_P(c,b)} \operatorname{Res}_{s=\rho}\left[\frac{x^s}{s\zeta(s)(P(s)+1)}\right] \\ &+ \frac{1}{2\pi\imath}\left(\int_{b-\imath T}^{b+\imath T} - \int_{c+\imath T}^{b+\imath T} - \int_{c-\imath T}^{b-\imath T}\right)\left[\frac{x^s}{s\zeta(s)(P(s)+1)} ds\right], \end{aligned}$$

where the three component integrals follow the path shown in the proof of Perron's formulas.

Proof. We can apply Lemma 6 and Lemma 9 to see that the DGF of the Dirichlet inverse function, $(\omega+1)^{-1}(n)$ is given by $D_{(\omega+1)^{-1}}(s) = [\zeta(s)(P(s)+1)]^{-1}$. Next, we consider the closed contour shown in the diagram from the proof of Perron's theorem. By the residue theorem, the integral of our function along this contour corresponds to a sum over the residues of its poles. In our case, since we are sufficiently to the right of one, the only contributions to the residue sum occur at the zeros of $P(s)+1$. All that remains is to re-write and re-arrange the integral terms from the statement Perron's formula given in Theorem 10 to arrive at our result. ☐

[3] This is actually a tricky, more subtle subject matter than it might seem from the outset. According to [?], the only known root of $P(s) = -1$ when $\operatorname{Re}(s) > 1$ is given by $s := \sigma + \imath\gamma$ for $(\sigma,\gamma) \approx (1.15339687, 14.01661169)$. It has really not been well studied whether, and of what approximate distribution, we might find other inconvenient others that pop up in our calculations. We attempt to find a sufficiently large real part, beyond which $P(s)+1$ has a zero-free region in Lemma 13 below.

Lemma 12 (Basic Bounds on $\zeta(s)$ and $P(s)$ to the Right of One). *If we write $s := \sigma + \imath t$, we obtain the following more basic bounds for the functions $\zeta(s), P(s)$ when $\operatorname{Re}(s) > 1$ and $t \in (-\infty, \infty)$:*

$$\begin{aligned}
\operatorname{Re}[\zeta(s)] &\sim \frac{\sigma-1}{(\sigma-1)^2+t^2} \\
\operatorname{Im}[\zeta(s)] &\sim \frac{t}{(\sigma-1)^2+t^2} \\
\operatorname{Re}[P(s)] &= -\frac{1}{2^{\sigma-1}\left((\sigma-1)^2+t^2\right)^2}\Bigg[\left((\sigma^2+t^2-\sigma)\cos(t\log 2) - t\sin(t\log 2)\right)\left((\sigma-1)^2+t^2-(\sigma-1)+O(1)\right) \\
&\qquad - \left((\sigma^2+t^2-\sigma)\sin(t\log 2)+t\cos(t\log 2)\right)(t+O(1))\Bigg] \\
\operatorname{Im}[P(s)] &= -\left[\frac{\left((\sigma^2+t^2-\sigma)\sin(t\log 2)+t\cos(t\log 2)\right)}{2^{\sigma-1}\left((\sigma-1)^2+t^2\right)}\left(1-\frac{\sigma-1}{(\sigma-1)^2+t^2}+O\left(\frac{1}{\min(\sigma-1,t)^2}\right)\right)\right].
\end{aligned}$$

In particular, we have that

$$\frac{1}{|s|\cdot|\zeta(s)|} \sim \frac{\sqrt{(\sigma-1)^2+t^2}}{\sqrt{\sigma^2+t^2}} = \sqrt{1+\frac{1-2t(\sigma-1)}{(\sigma-1)^2+t^2}}.$$

Proof. The procedure is similar to that used in the proof of Lemma 24. Namely, we will use direct integration to approximate $\zeta(s)$, and the limiting form of Abel summation to approximate $P(s)$ with precise upper bounds. The case for $\zeta(s)$ is straightforward:

$$\begin{aligned}
\operatorname{Re}[\zeta(s)] &= \sum_{n\geq 1}\frac{\cos(t\log n)}{n^\sigma} \approx \int_1^\infty \frac{\cos(t\log u)}{u^\sigma}du \\
&= \frac{u^{1-\sigma}}{(\sigma-1)^2+t^2}\left((1-\sigma)\cos(t\log u)+t\sin(t\log u)\right)\Bigg|_{u=1}^{u=+\infty} \\
&= \frac{\sigma-1}{(\sigma-1)^2+t^2} \\
\operatorname{Im}[\zeta(s)] &= \sum_{n\geq 1}\frac{\sin(t\log n)}{n^\sigma} \approx \int_1^\infty \frac{\sin(t\log u)}{u^\sigma}du \\
&= -\frac{u^{1-\sigma}}{(\sigma-1)^2+t^2}\left(t\cos(t\log u)+(\sigma-1)\sin(t\log u)\right)\Bigg|_{u=1}^{u=+\infty} \\
&= \frac{t}{(\sigma-1)^2+t^2}.
\end{aligned}$$

To evaluate the bounds for the prime zeta function, we proceed as before by expanding

$$\begin{aligned}
\sum_{p\leq x}\frac{\cos(t\log p)}{p^\sigma} &\approx \frac{\pi(x)\cos(t\log x)}{x^\sigma} + \int_2^x \frac{\pi(r)}{r^{\sigma+1}}\left(\sigma\cos(t\log r)+t\sin(t\log r)\right)dr \\
&= \frac{\pi(x)\cos(t\log x)}{x^\sigma} - \frac{1}{2}\sum_{b=\pm 1}E_1((\sigma-1+\imath bt)\log r)(\sigma+\imath bt)\Bigg|_{r=2}^{r=x} \\
&\xrightarrow{x\rightarrow\infty} -\frac{1}{2}\sum_{b=\pm 1}E_1((\sigma-1+\imath bt)\log 2)(\sigma+\imath bt).
\end{aligned}$$

Since we expect the real part of our DGF to be in fact real, we take the real part of the previous expression, together with the approximation for $E_1(z)$ truncated at upper index $N := 1$. This leads to the expansions:

$$\operatorname{Re}[P(s)] \approx -\frac{1}{2^\sigma\left((\sigma-1)^2+t^2\right)}\sum_{b=\pm 1}\operatorname{Re}\Bigg[(\sigma+\imath bt)(\sigma-1+\imath bt)\left(\cos(t\log 2)-b\imath\sin(t\log 2)\right)\times$$

$$\times \left(1 - \frac{(\sigma-1) - \imath bt}{(\sigma-1)^2+t^2} + O\left(\frac{(\sigma-1-\imath bt)^2}{((\sigma-1)^2+t^2)^2}\right)\right)\Bigg].$$

We approximate the error terms that result by simlifying the last expansion by

$$O\left(\frac{(\sigma-1)^2-t^2}{((\sigma-1)^2+t^2)^2}\right) \mapsto O\left(\frac{1}{(\sigma-1)^2+t^2}\right), O\left(\frac{2t(\sigma-1)}{((\sigma-1)^2+t^2)^2}\right) \mapsto O\left(\frac{1}{\min(\sigma-1,t)^2}\right).$$

Similarly, we approximate the related imaginary parts of $P(s)$ by Abel summation as

$$\begin{aligned}\sum_{p \leq x} \frac{\sin(t\log p)}{p^\sigma} &\approx \frac{\pi(x)\sin(t\log x)}{x^\sigma} + \int_2^x \frac{\pi(r)}{r^{\sigma+1}}\left(\sigma\sin(t\log r) - t\cos(t\log r)\right)dr \\ &= \frac{\pi(x)\sin(t\log x)}{x^\sigma} - -\frac{\imath}{2}\sum_{b=\pm1} E_1((\sigma-1+\imath bt)\log r)(\sigma+\imath bt)\Bigg|_{r=2}^{r=x} \\ &\xrightarrow{x\rightarrow\infty} \frac{\imath}{2}\sum_{b=\pm1} E_1((\sigma-1+\imath bt)\log 2)(\sigma+\imath bt).\end{aligned}$$

Thus we proceed as before, this time by taking the imaginary parts that we omitted in the last calculations. So we see that

$$\begin{aligned}\operatorname{Im}[P(s)] &= -\frac{1}{2}\operatorname{Im}\left[\sum_{b=\pm1} E_1((\sigma-1+\imath bt)\log 2)(\sigma+\imath bt)\right] \\ &= \frac{1}{2^\sigma\left((\sigma-1)^2+t^2\right)}\sum_{b=\pm1}\operatorname{Im}\Bigg[(\sigma+\imath bt)(\sigma-1+\imath bt)\left(\cos(t\log 2) - b\imath\sin(t\log 2)\right)\times \\ &\qquad \times\left(1 - \frac{(\sigma-1)-\imath bt}{(\sigma-1)^2+t^2} + O\left(\frac{(\sigma-1-\imath bt)^2}{((\sigma-1)^2+t^2)^2}\right)\right)\Bigg].\end{aligned}$$

In fact, we note a subtle point is that we are actually taking the *real part* of the expansion we obtain for the imaginary parts – which is supposed to be real-valued modulo approximation induced variants. $\square$

Lemma 13 (Constructing a Zero-Free Region for $P(s)+1$ to the Right of One). *We denote the known zero of $P(s)+1$ to the right of one by $s_0 := \beta_0 + \imath\gamma_0$ with $(\beta_0,\gamma_0) \approx (1.15339687, 14.01661169)$ where $|s_0| \approx 14.064$. We claim that there exists finite, real $\sigma_0 > 1$ and an absolute function $z_0(t)$ defined on the real line such that*

$$P(\sigma+\imath t) + 1 \neq 0, \forall(\sigma,t) \in [\sigma_0 - z_0(t), \sigma_0 + z_0(t)] \times (-\infty,\infty) \setminus \{(\beta_0,\gamma_0)\}.$$

In other words, we prove the existence of a strip of real values, or real parts of s, such that there is exactly one zero of $P(s)+1$ in this strip – and it corresponds to a simple zero at $s_0 := \beta_0 + \imath\gamma_0$. It suffices to take

$$\sigma_0 > TODO,$$

and the absolute function

$$z_0(t) := TODO.$$

Proof. $\square$

Remark 3.3 (Bounds for $\omega(n)$). Since we are inverting a translation of $\omega(n)$, it is henceforth important to understand its maximal, minimal, and average orders of growth. We have that $\exists C > 0$ such that

$$1 \leq \omega(x) \leq C\frac{\log(x)}{\log\log x},$$

where the upper bound is attained when x is primorial. Otherwise, the average order of $\omega(n) \sim \log\log n + B_1$, where $B_1 > 0$ is an absolute constant. In particular, $\omega(n)$ has sub-polynomial, logarithmic growth on the positive integers.

We can calculate due to Erdös' work that the prime counting function of natural numbers with exactly k factors, denoted by $\pi_k(x)$, has asymptotic formula

$$\pi_k(x) \sim (1+o(1)) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

This function is in some sense an indicator of the growth of the natural numbers n such that $\omega(n) = k$ for $k \geq 1$.

Proposition 14 (Bounds on the Dirichlet Inverse Function, $(\omega+1)^{-1}(n)$).

Proof. ☐

Theorem 15 (Strict Bounds on the Summatory Function of $(\omega+1)^{-1}(n)$). Suppose that we can pick positive reals $\sigma_0 + \delta > b > c > \sigma_0 > 1$ such that for any $b \geq \operatorname{Re}(s) \geq c$, the only zero of $P(s)+1$ occurs at $s_0 = \beta_0 + \imath\gamma_0$ for $(\beta_0, \gamma_0) \approx (1.15339687, 14.01661169)$. Furthermore, assume that the pole of $\frac{1}{P(s)+1}$ at $s = s_0$ is simple. For all $s \in \mathbb{C}$ satisfying $b \geq \operatorname{Re}(s) \geq c$ (for these fixed choices of the b, c), we require that $|P(s)| \in (-1, 1)$. Then we have that the magnitude of the summatory function of $(\omega+1)^{-1}(n)$ is bounded by

$$\begin{aligned}
\operatorname{Re}[S_{(\omega+1)^{-1}}(x)] - F_0^{-1}(x) - R_0(x) &\geq I_{1,\ell}(x) - I_{2,\ell}(x) - I_{3,\ell}(x), \forall x \geq 1 \\
\operatorname{Re}[S_{(\omega+1)^{-1}}(x)] - F_0^{-1}(x) - R_0(x) &\leq I_{1,u}(x) - I_{2,u}(x) - I_{3,u}(x), \forall x \geq 1.
\end{aligned}$$

where we define

$$\begin{aligned}
F_0^{-1}(x) &= \frac{(\omega+1)^{-1}(x)}{2} \\
R_0(x) &= [C_2(s_0)\sin(\gamma_0 \log x) - C_1(s_0)\cos(\gamma_0 \log x)] \cdot x^{\beta_0} \\
I_{1,\ell}(x) &= \\
I_{1,u}(x) &= \\
I_{2,\ell}(x) &= \\
I_{2,u}(x) &= \\
I_{3,\ell}(x) &= \\
I_{3,u}(x) &=,
\end{aligned}$$

for absolute real constants $C_1(s_0) \approx 0.00526526$ and $C_2(s_0) \approx 0.136651$[4]. Upper and lower bounds on the term $F_0^{-1}(x)$ are stated in Lemma 14 for all $x \geq 1$.

Proof Preliminaries: Verifying Assumptions. TODO ... ☐

Proof Part I: Assembling and Separating Signed Components. We have to take a delicate touch with the signedness of the terms in our integrand functions from Corollary 11 in order to preserve upper, but most importantly a harder-to-obtain class of lower bounds, for the summatory function. Since $|P(s)| < 1$, we can expand the integrand functions in s as

$$\operatorname{Re}\left[\frac{x^s}{s\zeta(s)(P(s)+1)}\right] = \sum_{m\geq 0} (-1)^m \operatorname{Re}\left[\frac{P(s)^m x^s}{s\zeta(s)}\right]. \tag{5}$$

Now for the delicate tour du force of from which an intricate case-by-case analysis of the relevant signed terms must be processed. We use the notation of Lemma 24 to write $\alpha \equiv \alpha(\sigma,t) := \operatorname{Re}[P(s)]$ and $\beta \equiv \beta(\sigma,t) := \operatorname{Im}[P(s)]$. Then we have

$$P(s)^m = \alpha^m \times \sum_{r\geq 0} \binom{m}{2r} (-1)^r \left(\frac{\beta}{\alpha}\right)^{2r} - \imath \cdot \alpha^{m-1}\beta \times \sum_{r\geq 0} \binom{m}{2r+1} (-1)^r \left(\frac{\beta}{\alpha}\right)^{2r}.$$

[4] The terms resulting from the residue calculations, $R_0(x)$, have been considered before in the note on the prime zeta function by Fröberg (1968) [**?**, §]. The considerations in that article, which were backed by numerical evidence when they were first presented, consider the sequence generated by the DGF $D_f(s) = 1/(P(s)+1)$, though in a much more combinatorial sense than what we have interpreted here. Nonetheless, it is encouraging that some of these new structures that take hold in our approximations for $M(x)$ have come up and in fact been studied before – even if years ago in some other realm.

One idea to keep in mind is that we need to keep track of the signed nature of the terms we are working with as our integrand functions. This is essential because when the sign of the real-valued terms are positive, we need a lower (upper) bound on the integral over these terms, and when the terms are negative we need an upper (lower) bound to express the inequalities stated above. With this in mind, let

$$\begin{aligned}
I_e^+(m) &:= \sum_{r\geq 0}\binom{m}{4r}\left(\frac{\beta}{\alpha}\right)^{4r} \geq 0 \\
I_e^-(m) &:= \sum_{r\geq 0}\binom{m}{4r+2}\left(\frac{\beta}{\alpha}\right)^{4r+2} \geq 0 \\
I_o^+(m) &:= \sum_{r\geq 0}\binom{m}{4r+1}\left(\frac{\beta}{\alpha}\right)^{4r} \geq 0 \\
I_o^-(m) &:= \sum_{r\geq 0}\binom{m}{4r+3}\left(\frac{\beta}{\alpha}\right)^{4r+2} \geq 0.
\end{aligned}$$

Then each of the functions above is unsigned, and we have that

$$\begin{aligned}
\operatorname{Re}[P(s)^m] &= \alpha^m \times \left(I_e^+(m) - I_e^-(m)\right) \\
\operatorname{Im}[P(s)^m] &= \alpha^{m-1}\beta \times \left(I_o^-(m) - I_o^+(m)\right).
\end{aligned}$$

Since we can decompose $\zeta(s)$ as a sum of its real and imaginary parts via what we worked so hard to prove in Lemma 12, and since $x^s = x^\sigma\left(\cos(t\log x) + \imath\sin(t\log x)\right)$, it follows that

$$\begin{aligned}
I(m,x,s) &:= \operatorname{Re}\left[\frac{P(s)^m x^s}{s\zeta(s)}\right] \\
&= \frac{x^\sigma\alpha^m}{(\sigma^2+t^2)}\left[\widehat{\alpha}_2\cdot I_e^+(m) + \widehat{\beta}\cdot I_o^+(m) + \widehat{\gamma}\cdot I_o^- - \widehat{\alpha}_2\cdot I_e^-(m) - \widehat{\beta}\cdot I_o^-(m) - \widehat{\gamma}\cdot I_o^+\right], \qquad (6)
\end{aligned}$$

where $\widehat{\alpha},\widehat{\beta},\widehat{\gamma}$ are the functions of s (σ and t) and x given by

$$\begin{aligned}
\widehat{\alpha}(\sigma,t,x) &:= (\sigma(\sigma-1)+t^2)\cos(t\log x) + (\sigma t + (\sigma-1)t)\sin(t\log x) \\
\widehat{\beta}(\sigma,t,x) &:= \frac{\beta}{\alpha}(\sigma(\sigma-1)+t^2)\sin(t\log x) \\
\widehat{\gamma}(\sigma,t,x) &:= \frac{\beta}{\alpha}(\sigma t + (\sigma-1)t)\cos(t\log x).
\end{aligned}$$

Let $\mathcal{Q}_i$ for $i \in \{1,2,3,4\}$ denote the i^{th} quartile of the complex plane. That is, $\mathcal{Q}_1 := \{s : \operatorname{Arg}(s) \in \left(0, \frac{\pi}{2}\right]\}$, $\mathcal{Q}_2 := \{s : \operatorname{Arg}(s) \in \left(\frac{\pi}{2}, \pi\right]\}$, $\mathcal{Q}_3 := \{s : \operatorname{Arg}(s) \in \left(\pi, \frac{3\pi}{2}\right]\}$, and $\mathcal{Q}_4 := \{s : \operatorname{Arg}(s) \in \left(\frac{3\pi}{2}, 2\pi\right]\}$. Since the bounds we obtained in Lemma 12 depend on local oscillations of the sine and cosine functions over any fixed compact interval $[-T,T]$, we will need to break down the regions of $\vartheta_t := t\log(2) \pmod{2\pi}$ according to the signs of these trigonometric functions on each quartile.

We can write bounds for the integral powers of α using the lemma:

$$\begin{aligned}
\alpha &\leq \frac{1}{2^{\sigma-1}\left((\sigma-1)^2+t^2\right)^2}\Big|\left((\sigma-1)^2+t^2\right)\left((\sigma-1)^2+t^2-(\sigma-1)+O(1)\right) - (t+\sigma+O(1)) \\
&\qquad - \sigma(t+1)\left((\sigma-1)^2+t^2-(\sigma-1)+O(1)\right)\Big| \\
&\sim 0, \text{as } |t| \to \infty \\
&\sim TODO, \text{as } |t| \to 0^+
\end{aligned}$$

$$\alpha \geq \begin{cases} , & \vartheta_t \in \mathcal{Q}_1; \\ , & \vartheta_t \in \mathcal{Q}_2; \\ , & \vartheta_t \in \mathcal{Q}_3; \\ , & \vartheta_t \in \mathcal{Q}_4; \end{cases}$$

$$\sim \begin{cases} , & \vartheta_t \in \mathcal{Q}_1; \\ , & \vartheta_t \in \mathcal{Q}_2; \\ , & \vartheta_t \in \mathcal{Q}_3; \\ , & \vartheta_t \in \mathcal{Q}_4, \end{cases} \text{as } |t| \to \infty.$$

We might as well also simplify some of the estimates we will need for the ratios of the component constants β/α. We use the shorthand notation of α_ℓ, α_u to denote the respective forms of the lower (upper) bounds for α we obtained in the previous formulas.

$$\frac{\beta}{\alpha} \leq \frac{\beta}{\alpha_u} \leq .$$

Formulating a lower bound from the expansions given in Lemma 12 is somewhat trickier in so much as we need to again break down the formulas for our bound according to the quartile location of ϑ_t:

$$\frac{\beta}{\alpha} \geq \frac{\beta}{\alpha_\ell} \geq \begin{cases} , & \vartheta_t \in \mathcal{Q}_1; \\ , & \vartheta_t \in \mathcal{Q}_2; \\ , & \vartheta_t \in \mathcal{Q}_3; \\ , & \vartheta_t \in \mathcal{Q}_4, \end{cases}.$$

The next step is to consider case-by-case the upper and lower bounds that result from each of the three components of the contour given by the integral formula from Corollary 11. □

Proof Part II: Bounding the Three Components of the Contour. We seek upper and lower bounds on the integrand for (σ, t) along the following lines:

1. $(\sigma, t) \in \{(b, v) : v \in [-T, T]\}$;

2. $(\sigma, t) \in \{(v, T) : v \in [c, b]\}$;

3. $(\sigma, t) \in \{(v, -T) : v \in [c, b]\}$.

Case I. We first look at obtaining lower bounds for the integrals

$$\begin{aligned} \int_{b-\imath T}^{b+\imath T} I(m, x, s) ds &= \int_{-T}^{T} I(m, x, b + \imath t) dt \\ &\geq TODO. \end{aligned}$$

The last equation defines the bound function, $I_{1,\ell}^{(m)}(x)$, defined above. Similarly, an upper bound is easier to obtain by expanding

$$\int_{b-\imath T}^{b+\imath T} I(m, x, s) ds \leq TODO.$$

The last equation leads to the bound function of $I_{1,u}^{(m)}(x)$. Now we have to combine these results and sum over the signed weights of $(-1)^m$ for all $m \geq 0$. Then

$$\begin{aligned} I_{1,\ell}(x) &= \sum_{m \geq 0} I_{1,\ell}^{(2m)}(x) - \sum_{m \geq 0} I_{1,u}^{(2m+1)}(x) \\ &= TODO \\ I_{1,u}(x) &= \sum_{m \geq 0} I_{1,u}^{(2m)}(x) - \sum_{m \geq 0} I_{1,\ell}^{(2m+1)}(x) \\ &= TODO \end{aligned}$$

This procedure is indicative of the method we will employ to bound the remaining two integrals in the contour.
Case II.

Case III.

□

Corollary 16 (New Bounds for $M(x)$).

Proof. We apply Theorem 15 to (3) in two places. The obvious leading term in the formula for $M(x)$ does not require any additional handling other than the statement of the theorem. The sum over the summatory function at primes p is transformed via an appeal to Abel summation with the summatory function $A(x) := \pi(x) \sim \frac{x}{\log x}$. In particular, we expand the prime sum as

$$\sum_{p \leq x} S_{(\omega+1)^{-1}}\left(\left\lfloor \frac{x}{p} \right\rfloor\right) = \pi(x) + x \int_2^x \frac{\pi(r)}{r^2} S'_{(\omega+1)^{-1}}\left(\frac{x}{r}\right) dr.$$

The theorem implies that we have lower bounds on this prime sum given by

$$TODO.$$

Similarly, the theorem provides us with statements of upper bounds for the prime sum of the form

$$TODO.$$

$\square$

3.2 Applications and discussion

Application 3.4 (Recovering Soundarajan's 2009 Annals Paper Bounds for $M(x)$). We need to see just how far we can push these new results. In 2009, Soundarajan published a mid-length paper in the *Annals of Mathematics* which used creative new zeta function bounds to obtain the current best known unconditional upper bound on $M(x)$:

$$\frac{M(x)}{\sqrt{x}} = O\left(\exp\left\{(\log x)^{1/2}(\log\log x)^{5/2+\varepsilon}\right\}\right), \forall \varepsilon > 0.$$

To verify that out formulas work in the correct direction, we use Corollary 16 to verify and recover a bound equivalent to Soundarajan's result:

$$TODO...$$

Heuristic 3.5 (Application to Logarithmic Averages). In light of the unboundedness problem of $M(x)/\sqrt{x}$ towards $+\infty$ having to rely on an explicit increasing certificate sequence, $\{x_0, x_1, x_2, \ldots\}$, whose terms increase to infinity along which the limit supremum bound for the Mertens function is attained, an alternate method is suggested to us. The explicit certificate sequence requirement is not an easy one, even given some of our better formulas for expressing $M(x)$.

So we believe that if there are infinitely many large N for which

$$\frac{1}{\log N} \sum_{x=1}^{N} \left(\frac{M(x)}{\sqrt{x}\alpha(x)}\right)^2 \frac{1}{x} > C,$$

then the proof of such a result is equivalent to showing that

$$\limsup_{x \to \infty} \frac{|M(x)|}{\sqrt{x} \cdot \alpha(x)} = +\infty.$$

The scaling function $\alpha(x)$ in our case is imprecise. In fact, one of our ideal applications in the direction of progress forwards would be to formulate a "best possible" such factor so that the limit supremum of $M(x)x^{-1/2}/\alpha(x)$ is precisely constant. Gonek's conjecture observed in the introduction posits that $\alpha(x) \sim (\log\log x)^{5/4}$ ought be our target optimal function.