defs:

$$\hat{F}(s,z) := \prod_p \left[\left(1+\frac{z}{p^s-1}\right)\left(1-\frac{1}{p^s}\right)^z\right], \quad |z|<2,\ \mathrm{Re}(z)\geq 1$$

so that $|\frac{z}{p}|<1$

$$\hat{G}(z) := \hat{F}(1,z)/\Gamma(1+z)$$

claim that: $\hat{G}\left(\frac{k-1}{\log\log x}\right) \asymp 1$ (e.g., $\gg$ and $\ll 1$) as $x\to\infty$ whenever $1\leq k\leq \log\log x$.

recall Merten's (Second) Theorem:

$$\sum_{p\leq x}\frac{1}{p} = \log\log x + B + O\left(\frac{1}{\log x}\right), \quad B\simeq 0.26\cdots,$$

an abs. constant

steps: ① $\prod_p \left(1+\frac{z}{p-1}\right)^{-1} \in e^{o(1)}[1-o(1),1]$;

② $\prod_p \left(1-\frac{1}{p}\right)^z \in e^{o(1)}[1-o(1),1]$;

③ For $z := \frac{k-1}{\log\log x}$, $1\leq k\leq \log\log x$,

$\Gamma(1+z)\asymp 1$. (easiest step first)

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③ / : $\Gamma(1+z) = (1+z)^{-1}\Gamma(2+z)$

$\geq (1+z)^{-1}\Gamma\left(2-\frac{1}{\log\log x}\right) \gg 1$

(at the upper end, where $k\to \ell\ell x$)

$\Gamma(1+z) \leq \Gamma(1) = 1 \ll 1$ (at the lower end, as $k\to 1^+$).