

Lower bounds on the summatory function of the Möbius function along infinite subsequences

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Abstract

The Mertens function, $M(x) = \sum_{n \leq x} \mu(n)$, is classically defined as the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture stating that $|M(x)| < C \cdot \sqrt{x}$ with some absolute $C > 0$ for all $x \geq 1$ has a well-known disproof due to Odlyzko and té Riele given in the early 1980's by computation of non-trivial zeta function zeros in conjunction with integral formulas expressing $M(x)$. It is conjectured that $M(x)/\sqrt{x}$ changes sign infinitely often and grows unbounded in the direction of both $\pm\infty$ along subsequences of integers $x \geq 1$. We prove a weaker property related to the unboundedness of $|M(x)| \log x/\sqrt{x}$ by showing that

$$\limsup_{x \rightarrow \infty} \frac{|M(x)|(\log x)(\log \log \log \log x)^{\frac{9}{4}}(\log \log \log \log \log x)^{\frac{7}{2}} \times \exp(2(\log \log \log \log x)^2)}{\sqrt{x}(\log \log x)} > 0.$$

There is a distinct stylistic flavor and new element of combinatorial analysis to our proof peppered in with the standard methods from analytic, additive and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

Keywords and Phrases: *Möbius function; Mertens function; summatory function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting functions; Dirichlet generating function; asymptotic lower bounds; Mertens conjecture.*

Math Subject Classifications (MSC 2010): 11N37; 11A25; 11N60; and 11N64.

Glossary of special notation and conventions

| Symbol | Definition |
|------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\mathbb{E}[f(x)], \overset{\mathbb{E}}{\sim}$ | We use the expectation notation $\mathbb{E}[f(x)] = h(x)$, or sometimes write that $f(x) \overset{\mathbb{E}}{\sim} h(x)$, to denote that f has a so-called <i>average order</i> growth rate of $h(x)$. What this means is that $\frac{1}{x} \sum_{n \leq x} f(n) \sim h(x)$, or equivalently that $\lim_{x \rightarrow \infty} \frac{\frac{1}{x} \sum_{n \leq x} f(n)}{h(x)} = 1.$ |
| B | The absolute constant $B \approx 0.2614972128476427837554$ from the statement of Mertens theorem. |
| $o(f), O_\alpha(g)$ | We write that $f = o(g)$ if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0.$ <p>We sometimes adapt the standard big-O notation, writing $f = O_{\alpha_1, \dots, \alpha_k}(g)$ for some parameters $\alpha_1, \dots, \alpha_k$ that do not depend on x, if $f(x) = O(g(x))$ subject only to the stated upper bound on f having an implicit dependence only on x (as usual) and the α_i for $1 \leq i \leq k$.</p> |
| $C_k(n)$ | Auxillary component functions in obtaining asymptotic bounds on $g^{-1}(n)$. These functions are defined recursively for $n \geq 1$ and $1 \leq k \leq \Omega(n)$ according to the formula $C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum_{d n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$ |
| $[q^n]F(q)$ | The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function of some sequence, $\{f_n\}_{n \geq 0}$. |
| DGF | Given a sequence $\{f(n)\}_{n \geq 0}$, its <i>Dirichlet generating function</i> (DGF) is defined by $D_f(s) := \sum_{n \geq 1} f(n)/n^s$ subject to suitable constraints on the real part of the parameter $s \in \mathbb{C}$ that guarantee convergence of $D_f(s)$. |
| $d(n)$ | The divisor function, $d(n) := \sum_{d n} 1$, for $n \geq 1$. |
| $\varepsilon(n)$ | The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) = \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (defined below). |
| $f * g$ | The Dirichlet convolution of f and g , $(f * g)(n) := \sum_{d n} f(d)g(n/d)$, where the sum is taken over the divisors d of n for $n \geq 1$. |
| $f^{-1}(n)$ | The Dirichlet inverse of f with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d n \\ d > 1}} f(d)f^{-1}(n/d)$ for $n \geq 1$ with $f^{-1}(1) = 1/f(1)$. The Dirichlet inverse of f exists if and only if $f(1) \neq 0$. This inverse function, provided it exists, is unique and satisfies the characteristic convolution relations providing that $f^{-1} * f = f * f^{-1} = \varepsilon$. |

| Symbol | Definition |
|---------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\lfloor x \rfloor, [x]$ | The floor function is defined as $\lfloor x \rfloor := x - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$. The floor function is sometimes also written as $\lfloor x \rfloor \equiv [x]$. The corresponding ceiling function $\lceil x \rceil$ denotes the smallest integer $m \geq x$. |
| $g^{-1}(n), G^{-1}(x)$ | The Dirichlet inverse function, $g^{-1}(n) = (\omega + 1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$. |
| H_n | The <i>first-order harmonic numbers</i> , $H_n \equiv H_n^{(1)}$, satisfy the limiting asymptotic relation $\lim_{n \rightarrow \infty} [H_n - \log(n)] = \gamma,$ where $\gamma \approx 0.577216$ denotes Euler's gamma constant. |
| $\mathbb{1}_{\mathbb{S}}, \chi_{\text{cond}(x)}$ | We use the notation $\mathbb{1}, \chi : \mathbb{N} \rightarrow \{0, 1\}$ to denote indicator, or characteristic functions of a set. In particular, $\mathbb{1}_{\mathbb{S}}(n) = 1$ if and only if $n \in \mathbb{S}$, and $\chi_{\text{cond}}(n) = 1$ if and only if n satisfies the boolean-valued condition cond . |
| $[n = k]_{\delta}, [\text{cond}]_{\delta}$ | The notation $[n = k]_{\delta}$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise. For a boolean-valued conditions, cond , $[\text{cond}]_{\delta}$ evaluates to one precisely when cond is true, and to zero otherwise. This notation is called <i>Iverson's convention</i> . |
| $\lambda(n)$ | The Liouville lambda function, $\lambda(n) := (-1)^{\Omega(n)}$, denotes the signed parity of $\Omega(n)$, the number of distinct prime factors of n counting their multiplicity. That is, $\lambda(n) \in \{\pm 1\}$ with $\lambda(n) = +1$ if and only if $\Omega(n) \equiv 0 \pmod{2}$. |
| $\mu(n)$ | The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers, and so that $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree, i.e., n has no prime power divisors with exponent greater than one. |
| $M(x)$ | The Mertens function is the summatory function over $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum_{n \leq x} \mu(n)$. |
| $\nu_p(n)$ | The valuation function that extracts the maximal exponent of p in the prime factorization of n , e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^{\alpha} \parallel n$ (or when p^{α} exactly divides n) for p prime and $n \geq 2$. |
| $\omega(n), \Omega(n)$ | We define the strongly additive function $\omega(n) := \sum_{p n} 1$ and the completely additive function $\Omega(n) := \sum_{p^{\alpha} \parallel n} \alpha$. Equivalently, if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we require that $\omega(1) = \Omega(1) = 0$. |
| $\pi_k(x), \hat{\pi}_k(x)$ | The prime counting function variant $\pi_k(x)$ denotes the number of integers $1 \leq n \leq x$ for $x > 1$ with exactly k distinct prime factors: $\pi_k(x) := \#\{n \leq x : \omega(n) = k\}$. Similarly, the function $\hat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}$ for $x \geq 2$. |
| $\sum_{p \leq x}, \prod_{p \leq x}$ | Unless otherwise specified by context, we use the index variable p to denote that the summation (product) is to be taken only over prime values within the summation bounds. |
| $P(s)$ | For complex s with $\text{Re}(s) > 1$, we define the <i>prime zeta function</i> to be the DGF $P(s) = \sum_{p \text{ prime}} p^{-s}$. |
| $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ | The unsigned Stirling numbers of the first kind, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = (-1)^{n-k} \cdot s(n, k)$. |
| $\zeta(s)$ | The Riemann zeta function, defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\text{Re}(s) > 1$, and by analytic continuation on the entire complex plane with the exception of a simple pole at $s = 1$. |

1 Preface: Notation to express asymptotic relations

We emphasize that the next careful explanation of the subtle distinctions to our usage of what we consider to be traditional notation for asymptotic relations are key to understanding our choices of bounding expressions given throughout the article. Thus, to avoid any confusion that may linger as we begin to state our new results and bounds on the functions we work with in this article, we preface the article starting with this section detailing our precise definitions, meanings and assumptions on the uses of certain symbols, operators, and relations as $x \rightarrow \infty$ [13, cf. §2] [2].

1.1 Average order, similarity and approximation of asymptotic growth rates

1.1.1 Similarity and average order (expectation)

We say that two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if

$$\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} = 1.$$

It is conventional to express the *average order* of an arithmetic function f as $f \sim h$, even when the values of $f(n)$ may actually non-monotonically oscillate in magnitude infinitely often. What the notation $f \sim h$ means when using this notation to express the average order of f is that

$$\frac{1}{x} \cdot \sum_{n \leq x} f(n) \sim h(x).$$

For example, in the acceptably classic language of [4] we would normally write that $\Omega(n) \sim \log \log n$, even though technically, $1 \leq \Omega(n) \leq \frac{\log n}{\log 2}$ where $\Omega(n)$ attains values along this entire bounded range in n infinitely often. To be absolutely clear about notation, we intentionally do not re-use the \sim relation by instead writing $\mathbb{E}[f(x)] = h(x)$ (as in expectation of f), or sometimes $\stackrel{\mathbb{E}}{\sim}$ for convenience, to denote that f has a limiting average order growing at the rate of h .

1.1.2 Abel summation

The formula we prefer for the Abel summation variant of summation by parts to express finite sums of a product of two functions is stated as follows [1, cf. §4.3]*F1.A:

Proposition 1.1 (Abel Summation Integral Formula). *Suppose that $t > 0$ is real-valued, and that $A(t) \sim \sum_{n \leq t} a(n)$ for some weighting arithmetic function $a(n)$ with $A(t)$ continuously differentiable on $(0, \infty)$. Furthermore, suppose that $b(n) \sim f(n)$ with f a differentiable function of $n \geq 0$. That is, $f'(t)$ exists and is smooth for all $t \in (0, \infty)$. Then for $0 \leq y < x$ we have that*

$$\sum_{y < n \leq x} a(n)b(n) \sim A(x)b(x) - A(y)b(y) - \int_y^x A(t)f'(t)dt.$$

1.1.3 Approximation

We adopt the convention that $f(x) \approx g(x)$ if $|f(x) - g(x)| = O(1)$ as $x \rightarrow \infty$. That is, we write $f(x) \approx g(x)$ to denote that f is approximately equal to g at x modulo at most a small constant difference between the functions when x is large.

*F1.A Compare to the exact formula for *summation by parts* of any arithmetic functions, u_n, v_n , stated as in [13, §2.10(ii)] for $U_j := u_1 + u_2 + \cdots + u_j$ when $j \geq 1$:

$$\sum_{j=1}^{n-1} u_j \cdot v_j = U_{n-1}v_n + \sum_{j=1}^{n-1} U_j (v_j - v_{j+1}), n \geq 2.$$

1.1.4 Vinogradov’s notation for asymptotics

We use the relations $f(x) \gg g(x)$ and $h(x) \ll r(x)$ to symbolically express that we should expect f to be “substantially” larger than g , and respectively h to be “significantly” smaller than r , in asymptotic order (e.g., rate of growth when x is large). In practice, we adopt a somewhat looser definition of these symbols which allows $f \gg g$ and $h \ll r$ provided that there are constants $C, D \geq 1$ such that whenever x is sufficiently large we have that $C \cdot f(x) \geq g(x)$ and $h(x) \leq D \cdot r(x)$. This notation is sometimes called *Vinogradov’s asymptotic notation*.

1.2 Asymptotic expansions and uniformity

We introduce the notation for asymptotic expansions of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ from [13, §2.1(iii)] in the next subsections.

1.2.1 Ordinary asymptotic expansions of a function

Let $\sum_n a_n x^{-n}$ denote a formal power series expansion in x where we ignore any necessary conditions to guarantee convergence of the series. For each integer $n \geq 1$, suppose that

$$f(x) = \sum_{s=0}^{n-1} a_s x^{-s} + O(x^{-n}),$$

as $|x| \rightarrow \infty$ where this limiting bound holds for $x \in \mathbb{X}$ in some unbounded set $\mathbb{X} \subseteq \mathbb{C}$. When such a bound holds, we say that $\sum_s a_s x^{-s}$ is a *Poincaré asymptotic expansion*, or just an *asymptotic series expansion*, of $f(x)$ as $|x| \rightarrow \infty$ along the fixed set \mathbb{X} . The condition in the previous equation is equivalent to writing

$$f(x) \sim a_0 + a_1 x^{-1} + a_2 x^{-2} + \cdots; x \in \mathbb{X}, \text{ as } |x| \rightarrow \infty.$$

The prior two characterizations of an asymptotic expansion for f are also equivalent to the statement that

$$x^n \left(f(x) - \sum_{s=0}^{n-1} a_s x^{-s} \right) \xrightarrow{x \rightarrow \infty} a_n.$$

1.2.2 Uniform asymptotic expansions of a function

Let the set \mathbb{X} from the definition in the last subsection correspond to a closed sector of the form

$$\mathbb{X} := \{x \in \mathbb{C} : \alpha \leq \arg(x) \leq \beta\}.$$

Then we say that the asymptotic expansion

$$f(x) = \sum_{s=0}^{n-1} a_s x^{-s} + O(x^{-n}),$$

holds *uniformly* with respect to $\arg(x) \in [\alpha, \beta]$ as $|x| \rightarrow \infty$.

Another useful notion of uniform asymptotic bounds is taken with respect to some parameter u (or set of parameters, respectively) that ranges over the point set (point sets, respectively) $u \in \mathbb{U}$. In this case, if we have that the u -parameterized expressions

$$\left| x^n \left(f(u, x) - \sum_{s=0}^{n-1} a_s(u) x^{-s} \right) \right|,$$

are bounded for all integers $n \geq 1$ with $x \in \mathbb{X}$ as $|x| \rightarrow \infty$, then we say that the asymptotic expansion of f holds *uniformly* for $u \in \mathbb{U}$. Now the function $f \equiv f(u, x)$ and the asymptotic series coefficients $a_s(u)$ may have an implicit dependence on the parameter u . If the previous boundedness condition holds for all positive integers n , we write that

$$f(u, x) \sim \sum_{s=0}^{\infty} a_s(u) x^{-s}; x \in \mathbb{X}, \text{ as } |x| \rightarrow \infty,$$

and say that this asymptotic expansion holds *uniformly with respect to* $u \in \mathbb{U}$. For u taken outside of \mathbb{U} , the stated limiting bound may fail to be valid even for $x \in \mathbb{X}$ as $|x| \rightarrow \infty$.

1.3 Limiting densities of subsets of the integers

In the proofs given in Section 8 of the article, we will require a precise notion of the *asymptotic density* of a set $\mathcal{S} \subseteq \mathbb{Z}^+$. When this limit exists, we denote the asymptotic density of \mathcal{S} by $\alpha_{\mathcal{S}} \in [0, 1]$, defined as follows:

$$\alpha_{\mathcal{S}} := \lim_{x \rightarrow \infty} \frac{1}{x} \cdot \# \{n \leq x : n \in \mathcal{S}\}.$$

In other words, if the set \mathcal{S} has asymptotic density $\alpha_{\mathcal{S}}$, then for all sufficiently large x

$$\alpha_{\mathcal{S}} + o(1) \leq \frac{1}{x} \cdot \# \{n \leq x : n \in \mathcal{S}\} \leq \alpha_{\mathcal{S}} + o(1).$$

When the limit definition of $\alpha_{\mathcal{S}}$ does not exist, or if by some pathology of the way \mathcal{S} is defined we cannot express the exact limit, we are often interested in sets of bounded asymptotic density. We say that \mathcal{S} has *bounded asymptotic density* if for all large x there exist constants $0 \leq B \leq C \leq 1$ such that

$$B + o(1) \leq \frac{1}{x} \cdot \# \{n \leq x : n \in \mathcal{S}\} \leq C + o(1), \text{ as } x \rightarrow \infty.$$

Clearly, finite or bounded subsets of the positive integers have limiting asymptotic density of zero. If the asymptotic density of \mathcal{S} is one, and some property $\mathcal{P}(n)$ holds for all $n \in \mathcal{S}$, then we say that $\mathcal{P}(n)$ is true *almost everywhere* (on the integers), also abbreviated as holding “a.e.” on the positive integers as $n \rightarrow \infty$.

2 An introduction to the Mertens function

2.1 Definitions

Suppose that $n \geq 2$ is a natural number with factorization into distinct primes given by $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $k = \omega(n)$. We define the *Möbius function* to be the signed indicator function of the squarefree integers as follows:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } \alpha_i = 1, \forall 1 \leq i \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

There are many other variants and special properties of the Möbius function and its generalizations [15, cf. §2]. A crucial role of the classical $\mu(n)$ forms an inversion relation for arithmetic functions convolved with one (e.g., Dirichlet convolutions of the form $g = f * 1$) by *Möbius inversion*:

$$g(n) = (f * 1)(n) \iff f(n) = (g * \mu)(n), \forall n \geq 1.$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of the oscillatory values of this summatory function begins as [16, A002321]

$$\{M(x)\}_{x \geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2, \dots\}$$

Clearly, a positive integer $n \geq 1$ is *squarefree*, or contains no (prime power) divisors which are squares, if and only if $\mu^2(n) = 1$. A related summatory function which counts the number of *squarefree* integers $n \leq x$ then satisfies [4, §18.6] [16, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) \sim \frac{6x}{\pi^2} + O(\sqrt{x}).$$

It is known that the asymptotic density of the positively versus negatively weighted sets of squarefree numbers are in fact equal as $x \rightarrow \infty$:

$$\mu_+(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = +1\}}{Q(x)} \underset{\mathbb{E}}{\sim} \mu_-(x) = \frac{\#\{1 \leq n \leq x : \mu(n) = -1\}}{Q(x)} \xrightarrow{x \rightarrow \infty} \frac{3}{\pi^2}.$$

The actual local oscillations between the approximate densities of the sets $\mu_{\pm}(x)$ lend an unpredictable nature to the function and characterize the oscillatory sawtooth shaped plot of $M(x)$ over the positive integers.

2.2 Properties

One conventional approach to evaluating the behavior of $M(x)$ for large $x \rightarrow \infty$ results from a formulation of this summatory function as a predictable exact sum involving x and the non-trivial zeros of the Riemann zeta function for all real $x > 0$. This formula is expressed given the inverse Mellin transformation over the reciprocal zeta function. In particular, we notice that since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \int_1^\infty \frac{s \cdot M(x)}{x^{s+1}} dx,$$

we obtain that

$$M(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{T-i\infty}^{T+i\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

This representation, along with the standard Euler product representation for the reciprocal zeta function cited in the first equation above, leads us to the exact expression for $M(x)$ for any real $x > 0$ given by the next theorem due to Titchmarsh.

Theorem 2.1 (Analytic Formula for $M(x)$). *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any real $x > 0$*

$$M(x) = \lim_{k \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ |\operatorname{Im}(\rho)| < T_k}} \frac{x^\rho}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x} \right)^{2n} + \frac{\mu(x)}{2} [x \in \mathbb{Z}^+]_\delta.$$

A historical unconditional bound on the Mertens function due to Walfisz (1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp \left(-C \cdot \log^{3/5}(x) (\log \log x)^{-3/5} \right).$$

Under the assumption of the RH, Soundararajan proved new updated estimates in 2009 bounding $M(x)$ for large x in the following forms [17]:

$$\begin{aligned} M(x) &\ll \sqrt{x} \cdot \exp \left(\log^{1/2}(x) (\log \log x)^{14} \right), \\ M(x) &= O \left(\sqrt{x} \cdot \exp \left(\log^{1/2}(x) (\log \log x)^{5/2+\epsilon} \right) \right), \quad \forall \epsilon > 0. \end{aligned}$$

2.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O \left(x^{\frac{1}{2}+\epsilon} \right)$ for any $0 < \epsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which posits that

$$|M(x)| < C \cdot \sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens for $C = 1$ and all $x < 10000$. Since its beginnings in 1897, the Mertens conjecture has been disproven by computation of non-trivial simple zeta function zeros with comparatively small imaginary parts in a famous paper by Odlyzko and té Riele from the early 1980's. Since the truth of the conjecture would have implied the RH, more recent attempts at bounding $M(x)$ consider determining the rates at which the function $M(x)/\sqrt{x}$ grows with or without bound towards both $\pm\infty$ along infinite subsequences.

One of the most famous still unanswered questions about the Mertens function concerns whether $|M(x)|/\sqrt{x}$ is in actuality unbounded on the natural numbers. A precise statement of this problem is to produce an affirmative answer whether $\limsup_{x \rightarrow \infty} M(x)/\sqrt{x} = +\infty$ and $\liminf_{x \rightarrow \infty} M(x)/\sqrt{x} = -\infty$, or equivalently whether there are an infinite subsequences of natural numbers $\{x_1, x_2, x_3, \dots\}$ such that the magnitude of $M(x_i)x_i^{-1/2}$ grows without bound towards either $\pm\infty$ along the subsequence. We cite that prior to this point it is only known by computation that [14, cf. §4.1]

$$\limsup_{x \rightarrow \infty} \frac{M(x)}{\sqrt{x}} > 1.060 \quad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x \rightarrow \infty} \frac{M(x)}{\sqrt{x}} < -1.009 \quad (\text{now } \leq -1.837625).$$

Based on work by Odlyzko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [12, 7, 8, 5].

Extensive computational evidence has produced a conjecture due to Gonek (among attempts on limiting bounds by others) that in fact the limiting behavior of $M(x)$ satisfies [11]

$$\limsup_{x \rightarrow \infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log \log x)^{5/4}} = O(1).$$

While it seems to be widely believed that $|M(x)|/\sqrt{x}$ tends to $+\infty$ at some logarithmically scaled rate along subsequences, the infinitely tending factors such as the $(\log \log x)^{\frac{5}{4}}$ in Gonek's conjecture do not appear to readily fall out of work on unconditional bounds for $M(x)$ by existing methods.

3 An overview of the core logical steps and components to the proof

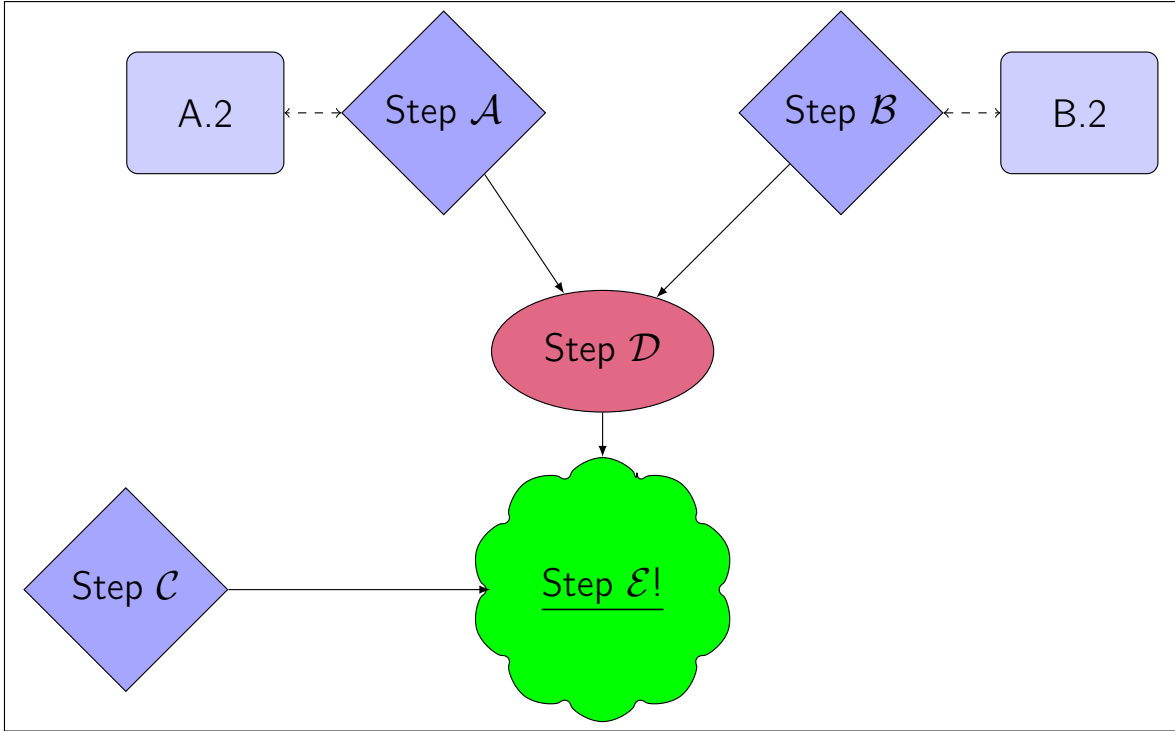
We offer an initial step-by-step summary overview of the critical components to our proof outlined in the next subsections of the introduction. As our proof methodology is new and relies on non-standard elements compared to more traditional methods of bounding $M(x)$, we hope that this sketch of the logical components to this argument makes the article easier to parse.

3.1 Step-by-step overview

- (1) We prove a matrix inversion formula relating the summatory functions of an arithmetic function f and its Dirichlet inverse f^{-1} (for $f(1) \neq 0$). See Theorem 4.1 in Section 5.
- (2) This crucial step provides us with an exact formula for $M(x)$ in terms of $\pi(x)$, the seemingly unconnected prime counting function, and the Dirichlet inverse of the shifted additive function $g(n) := \omega(n) + 1$. This formula is stated in (1).
- (3) We tighten an updated result from [10, §7] providing uniform asymptotic formulas for the summatory functions, $\hat{\pi}_k(x)$, that indicate the parity of $\Omega(n)$ (sign of $\lambda(n)$) for $n \leq x$ and $1 \leq k \leq \log \log x$. These formulas are proved using expansions of more combinatorially motivated Dirichlet series (see Theorem 4.8). We use this result to sum $\sum_{n \leq x} \lambda(n)f(n)$ for particular non-negative arithmetic functions f when x is large.
- (4) We then turn to the average order asymptotics of the quasi-periodic functions, $g^{-1}(n)$, by estimating this inverse function's limiting asymptotics for large $n \leq x$ as $x \rightarrow \infty$ in Section 7. We eventually use these estimates to prove a substantially unique new lower bound formula for the summatory function $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ along certain asymptotically large infinite subsequences (see Theorem 9.5).
- (5) We spend some interim time in Section 8 carefully working out a rigorous justification for why the limiting lower bounds we obtain from average order case analysis of our arithmetic function approximations to $g^{-1}(n)$ are sufficient to prove the corollary on the unboundedness of $M(x)$ below.
- (6) When we return to step (2) with our new lower bounds at hand, we find “magic” in the form of showing the unboundedness of $\frac{|M(x)|}{\sqrt{x}}$ along a very large increasing infinite subsequence of positive natural numbers. What we recover is a quick, and rigorous, proof of Corollary 4.9 given in Section 9.2.

3.2 Schematic flowchart of the proof logic

The next flowchart diagramed below shows how the seemingly disparate components of the proof are organized.



Legend to the diagram stages:

- **Step A:** *Citations and re-statements of existing theorems proved elsewhere.*
 - A.A:** Key results and constructions:
 - Theorem 4.7
 - Corollary 6.5
 - The results, lemmas, and facts cited in Section 5.3
 - A.2:** Lower bounds on the Abel summation based formula for $G^{-1}(x)$:
 - Theorem 4.8 (on page 21)
 - Proposition 6.6
 - Theorem 9.5
- **Step B:** *Constructions of an exact formula for $M(x)$.*
 - B.B:** Key results and constructions:
 - Corollary 4.3 (follows from Theorem 4.1 proved on page 15)
 - Proposition 5.1
 - B.2:** Asymptotics for the component functions $g^{-1}(n)$ and $G^{-1}(x)$:
 - Theorem 4.6 (on page 23)
 - Lemma 7.3
- **Step C:** *A justification for why lower bounds obtained on average suffice.*
 - The results proved in Section 8
- **Step D:** *Interpreting the exact formula for $M(x)$.*
 - Proposition 9.1
 - Theorem 9.5
- **Step E:** *The Holy Grail.* Proving that $\frac{|M(x)|}{\sqrt{x}}$ grows without bound in the limit supremum sense.
 - Corollary 4.9 (on page 40)

4 A concrete new approach for bounding $M(x)$ from below

4.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 4.1 (Summatory functions of Dirichlet convolutions). *Let $f, h : \mathbb{Z}^+ \rightarrow \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n \leq x} f(n)$ and $H(x) := \sum_{n \leq x} h(n)$ denote the summatory functions of f, h , respectively, and that $F^{-1}(x)$ denotes the summatory function of the Dirichlet inverse f^{-1} of f . Then, letting the counting function $\pi_{f*h}(x)$ be defined as in the first equation below, we have the following equivalent expressions for the summatory function of $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned} \pi_{f*h}(x) &= \sum_{n \leq x} \sum_{d|n} f(d)h(n/d) \\ &= \sum_{d \leq x} f(d)H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{k=1}^x H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) \right]. \end{aligned}$$

Moreover, we can invert the linear system determining the coefficients of $H(k)$ for $1 \leq k \leq x$ naturally to express $H(x)$ as a linear combination of the original left-hand-side summatory function as follows:

$$\begin{aligned} H(x) &= \sum_{j=1}^x \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \right] \\ &= \sum_{n=1}^x f^{-1}(n) \pi_{f*h}\left(\left\lfloor \frac{x}{n} \right\rfloor\right). \end{aligned}$$

Corollary 4.2 (Convolutions Arising From Möbius Inversion). *Suppose that g is an arithmetic function on the positive integers such that $g(1) \neq 0$. Define the summatory function of the convolution of g with μ by $\tilde{G}(x) := \sum_{n \leq x} (g * \mu)(n)$. Then the Mertens function equals*

$$M(x) = \sum_{k=1}^x \left(\sum_{j=\lfloor \frac{x}{k+1} \rfloor + 1}^{\lfloor \frac{x}{k} \rfloor} g^{-1}(j) \right) \tilde{G}(k), \forall x \geq 1.$$

Corollary 4.3 (A motivating special case). *We have exactly that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^x (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1 \right]. \quad (1)$$

4.2 An exact expression for $M(x)$ in terms of strongly additive functions

From this point on, we fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and denote its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$. We can compute the first few terms for the Dirichlet inverse of $g(n)$ exactly for the first few sequence values as (see Table T.1 of the appendix section)

$$\{g^{-1}(n)\}_{n \geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \dots\}.$$

The sign of these terms is given by $\text{sgn}(g^{-1}(n)) = \frac{g^{-1}(n)}{|g^{-1}(n)|} = \lambda(n)$ (see Proposition 5.1). This useful property is inherited from the distinctly additive nature of the component function $\omega(n)$ ^{*F4.A}.

^{*F4.A}Indeed, for any non-negative additive arithmetic function $a(n)$, $(a + 1)^{-1}(n)$ has leading sign given by $\lambda(n)$ for any $n \geq 1$. For multiplicative f , we obtain a related condition that $\text{sgn}(f(n)) = (-1)^{\omega(n)}$ for all $n \geq 1$.

There does not appear to be an easy, nor subtle direct recursion between the distinct values of $g^{-1}(n)$, except through auxiliary function sequences. However, the distribution of distinct sets of prime exponents is fairly regular so that $\omega(n)$ and $\Omega(n)$ play a crucial role in the repetition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of $g^{-1}(n)$ over $n \geq 2$:

Heuristic 4.4 (Symmetry in $g^{-1}(n)$ in the exponents in the prime factorization of n). Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for some $r \geq 1$. If $\{\alpha_1, \dots, \alpha_r\} \equiv \{\beta_1, \dots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly two, three, and so on prime factors (compare with Table T.1 starting on page 42).

Conjecture 4.5. *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

- (A) $g^{-1}(1) = 1$;
- (B) For all $n \geq 1$, $\text{sgn}(g^{-1}(n)) = \lambda(n)$;
- (C) For all squarefree integers $n \geq 1$, we have that

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!.$$

We illustrate parts (B)–(C) of the conjecture more clearly using Table T.1 given starting on page 42. The realization that the beautiful and remarkably simple combinatorial form of property (C) in Conjecture 4.5 holds for all squarefree $n \geq 1$ motivates our pursuit of formulas for the inverse functions $g^{-1}(n)$ expressed by sums of auxiliary sequences of arithmetic functions^{*F4.B}.

For natural numbers $n \geq 1, k \geq 0$, let

$$C_k(n) := \begin{cases} \varepsilon(n) = \delta_{n,1}, & \text{if } k = 0; \\ \sum_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$

We have limiting asymptotics on these functions in terms of n and k given by the following theorem:

Theorem 4.6 (Asymptotics for the functions $C_k(n)$). *For $k := 0$, we have by definition that $C_0(n) = \delta_{n,1}$. For all sufficiently large $n > 1$ and any fixed $1 \leq k \leq \Omega(n)$ taken independently of n , we obtain that the dominant asymptotic term for $C_k(n)$ is given uniformly by*

$$\mathbb{E}[C_k(n)] \gg (\log \log n)^{2k-1}, \text{ as } n \rightarrow \infty.$$

For any $n \geq 1$, we can prove that (see Lemma 7.3)

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \quad (2)$$

In light of the fact that (see Proposition 9.1)

$$M(x) \approx G^{-1}(x) - \sum_{k=1}^{x/2} G^{-1}(k) \cdot \frac{x}{k^2 \log(x/k)},$$

the formula in (2) implies that we can establish new *lower bounds* on $M(x)$ along large infinite subsequences by appropriate estimates of the summatory function $G^{-1}(x)$ ^{*F4.C}.

^{*F4.B} A proof of this property is not difficult to give using Lemma 7.3 stated on page 24.

^{*F4.C} We can also prove that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right),$$

4.3 Uniform asymptotics from enumerative counting DGFs in Montgomery and Vaughan

Our inspiration for the new bounds found in the last sections of this article allows us to sum non-negative arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. We utilize a somewhat more general hybrid generating function and enumerative DGF method under which we are able to recover “good enough” asymptotics about the summatory functions that encapsulate the parity of $\Omega(n)$ (or sign of $\lambda(n)$) through the summatory tally functions $\hat{\pi}_k(x)$. The precise statement of the result that we transform to state these new bounds is provided next in Theorem 4.7 (see Section 6.1).

Theorem 4.7 (Montgomery and Vaughan). *Recall that we have defined*

$$\hat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $R < 2$ we have that

$$\hat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log \log x}\right) \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!} \left(1 + O_R\left(\frac{k}{(\log \log x)^2}\right)\right),$$

uniformly for $1 \leq k \leq R \log \log x$ where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, z \geq 0.$$

The next theorem, proved carefully in Section 6, is the primary starting point for our new asymptotic lower bounds. The proof of this result is combinatorially motivated in so much as it interprets lower bounds on a key infinite product factor of $\mathcal{G}(z)$ defined in Theorem 4.7 as corresponding to an ordinary generating function of certain homogeneous symmetric polynomials involving reciprocals of the primes.

Theorem 4.8. *Suppose that $0 \leq z < 1$ is real-valued. We obtain lower bounds of the following form on the function $\mathcal{G}(z)$ from Theorem 4.7 for $A_0 > 0$ an absolute constant and for $C_0(z)$ a strictly linear function only in z :*

$$\mathcal{G}(z) \geq A_0 \cdot (1-z)^3 \cdot C_0(z)^z.$$

It suffices to take the components of the bound given in the previous equation as

$$A_0 = \frac{2^{9/16} \exp\left(-\frac{55}{4} \log^2(2)\right)}{(3e \log 2)^3 \cdot \Gamma\left(\frac{5}{2}\right)} \approx 3.81296 \times 10^{-6}$$

$$C_0(z) = \frac{4(1-z)}{3e \log 2}.$$

In particular, for $1 \leq k \leq \log \log x$ we have that

$$\hat{\pi}_k(x) \gg \frac{A_0 \cdot x}{\log x \cdot (\log \log x)^4 \cdot (k-1)!} \cdot \left(\frac{4}{3e \log 2}\right)^{\frac{k}{\log \log x}} \left(1 + O\left(\frac{k}{(\log \log x)^2}\right)\right).$$

4.4 Cracking the classical unboundedness barrier

In Section 9, we are able to state what forms a culmination of the results we carefully build up to in the proofs established in prior sections of the article. What we eventually obtain at the conclusion of the section is the next important summary corollary that partially resolves the classical question of the unboundedness of the scaled function Mertens function $q(x) := |M(x)|/\sqrt{x}$ scaled by a logarithmic factor in the limit supremum sense.

by inversion since

$$G^{-1}(x) = \sum_{d \leq x} (g^{-1} * 1)(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right),$$

with $(g^{-1} * 1)^{-1} = g * \mu = \chi_{\mathbb{P}} + \varepsilon$ defined such that $\chi_{\mathbb{P}}$ is the characteristic function of the primes.

Corollary 4.9 (Unboundedness of the the Mertens function, $q(x) \log x$). *We have that*

$$\limsup_{x \rightarrow \infty} \frac{|M(x)| \log x}{\sqrt{x}} = +\infty.$$

In establishing the rigorous proof of Corollary 4.9 based on our new methods, we not only show unboundedness of $q(x) \log x$, but also set a minimal rate (along a large infinite subsequence) at which this form of the scaled Mertens function grows without bound.

5 Preliminary proofs of new results

The purpose of this section is to provide proofs and statements of elementary and otherwise well established facts and results. The proof of Theorem 4.1 allows us to easily justify the exact formula in (1). The strong additivity of $\omega(n)$ provides the characteristic signedness of $g^{-1}(n)$ corresponding to the parity of $\Omega(n)$ which we prove precisely in Proposition 5.1. The link relating (1) to canonical additive functions and their distributions then lends a recent distinguishing element to the success of the methods in our proof.

5.1 Establishing the summatory function properties and inversion identities

We will first prove Theorem 4.1 using matrix methods and similarity transforms by shift matrices. Related results on summations of Dirichlet convolutions appear in [1, §2.14; §3.10; §3.12; cf. §4.9, p. 95].

Proof of Theorem 4.1. Let h, g be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of h and g , respectively, by $H(x) = \sum_{n \leq x} h(n)$ and $G(x) = \sum_{n \leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h : $g*h$. Then we can readily see that the following initial formulas hold for all $x \geq 1$:

$$\begin{aligned} \pi_{g*h}(x) &:= \sum_{n=1}^x \sum_{d|n} g(n)h(n/d) = \sum_{d=1}^x g(d)H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &= \sum_{i=1}^x \left[G\left(\left\lfloor \frac{x}{i} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1} \right\rfloor\right) \right] H(i). \end{aligned}$$

We form the matrix of coefficients associated with this linear system defining $H(n)$ for all $n \leq x$. We then invert the system to express an exact solution for $H(x)$ at any $x \geq 1$. Let the matrix entries be denoted by

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j} \right\rfloor\right), \forall 1 \leq j \leq x.$$

The matrix we must invert in this problem is lower triangular, with ones on its diagonals, and hence is invertible. Moreover, if we let $\hat{G} := (G_{x,j})$, then this matrix is expressible by an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

Here, U is a square matrix with finite dimensions whose $(i,j)^{th}$ entries are defined by $(U)_{i,j} = \delta_{i+1,j}$ such that

$$[(I - U^T)^{-1}]_{i,j} = [j \leq i]_{\delta}.$$

It is a useful fact that if we take successive differences in x of the floor of certain fractions, $\left\lfloor \frac{x}{j} \right\rfloor$, we get non-zero behavior at the divisors of x :

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases}$$

We use this property to shift the matrix \hat{G} , and then invert the result to obtain a matrix involving the Dirichlet inverse of g of the following form:

$$[(I - U^T)\hat{G}]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_{\delta}\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_{\delta}\right).$$

Now we can express the inverse of the target matrix $(g_{x,j})$ using a similarity transformation conjugated by shift operators as follows:

$$\begin{aligned} (g_{x,j}) &= (I - U^T)^{-1} \left(g \left(\frac{x}{j} \right) [j|x]_\delta \right) (I - U^T) \\ (g_{x,j})^{-1} &= (I - U^T)^{-1} \left(g^{-1} \left(\frac{x}{j} \right) [j|x]_\delta \right) (I - U^T) \\ &= \left(\sum_{k=1}^{\lfloor \frac{x}{j} \rfloor} g^{-1}(k) \right) (I - U^T) \\ &= \left(\sum_{k=1}^{\lfloor \frac{x}{j} \rfloor} g^{-1}(k) - \sum_{k=1}^{\lfloor \frac{x}{j+1} \rfloor} g^{-1}(k) \right). \end{aligned}$$

Hence, the summatory function $H(x)$ is exactly expressed for any $x \geq 1$ by a vector product with the inverse matrix from the previous equation in the form of

$$\begin{aligned} H(x) &= \sum_{k=1}^x g_{x,k}^{-1} \cdot \pi_{g*h}(k) \\ &= \sum_{k=1}^x \left(\sum_{j=\lfloor \frac{x}{k+1} \rfloor + 1}^{\lfloor \frac{x}{k} \rfloor} g^{-1}(j) \right) \cdot \pi_{g*h}(k). \square \end{aligned}$$

5.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n . Then we can easily prove that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu. \quad (3)$$

When combined with Corollary 4.2 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 4.3.

Proposition 5.1 (The key signedness property of $g^{-1}(n)$). *Let the operator $\text{sgn}(h(n)) = \frac{h(n)}{|h(n)| + [h(n)=0]_\delta} \in \{0, \pm 1\}$ denote the sign of the arithmetic function h at n . For the Dirichlet invertible function, $g(n) := \omega(n) + 1$ defined such that $g(1) = 1$, for all $n \geq 1$, we have that $\text{sgn}(g^{-1}(n)) = \lambda(n)$.*

Proof. Recall that $D_f(s) := \sum_{n \geq 1} f(n)n^{-s}$ denotes the *Dirichlet generating function* (DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\text{Re}(s) > \sigma_f$ for σ_f the abscissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_\mu(s) = 1/\zeta(s)$ and $D_\omega(s) = P(s)\zeta(s)$. Then by (3) and the known property that the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of any invertible arithmetic function f , for all $\text{Re}(s) > 1$ we have

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s) + 1)\zeta(s)}. \quad (4)$$

It follows that $(\omega + 1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + 1$. We first show that $\text{sgn}(h^{-1}) = \lambda$. From this fact, it follows by inspection that $\text{sgn}(h^{-1} * \mu) = \lambda$. The remainder of the proof fills in the precise details needed to make this intuition rigorous.

By the standard recurrence relation we used to define the Dirichlet inverse function of any arithmetic function h , we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ - \sum_{\substack{d|n \\ d>1}} h(d)h^{-1}(n/d), & n \geq 2. \end{cases} \quad (5)$$

For $n \geq 2$, the summands in (5) can be simply indexed over the primes $p|n$ given our definition of h from above. This observation yields that we can inductively expand these sums into nested divisor sums provided the depth of the sums does not exceed the capacity to index summations over the primes dividing n . Namely, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= - \sum_{p|n} h^{-1}(n/p), & \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right), & \text{if } \Omega(n) \geq 2 \\ &= - \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1 p_2}} h^{-1}\left(\frac{n}{p_1 p_2 p_3}\right), & \text{if } \Omega(n) \geq 3. \end{aligned}$$

Then by induction, again with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$h^{-1}(n) = \lambda(n) \times \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1, n \geq 2.$$

If for $n \geq 2$ we write the prime factorization of n as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{\omega(n)}^{\alpha_{\omega(n)}}$ where the exponents $\alpha_i \geq 1$ for all $1 \leq i \leq \omega(n)$, we can see that

$$\begin{aligned} h^{-1}(n) &\geq \lambda(n) \times 1 \cdot 2 \cdot 3 \cdots \omega(n) = \lambda(n) \times (\omega(n))!, & n \geq 2 \\ h^{-1}(n) &\leq \lambda(n) \times (\omega(n))!^{\max(\alpha_1, \alpha_2, \dots, \alpha_{\omega(n)})}, & n \geq 2. \end{aligned}$$

In other words, what these bounds show is that for all $n \geq 1$ (with $\lambda(1) = 1$) the following property holds:

$$\text{sgn}(h^{-1}(n)) = \lambda(n). \quad (6)$$

From (6), we immediately have bounding constants $1 \leq C_{1,n}, C_{2,n} < +\infty$ that exist for each $n \geq 1$ so that

$$C_{1,n} \cdot (\lambda * \mu)(n) \leq (h^{-1} * \mu)(n) \leq C_{2,n} \cdot (\lambda * \mu)(n). \quad (7)$$

Since both λ, μ are multiplicative, the convolution $\lambda * \mu$ is multiplicative. We know that the values of any multiplicative function are uniquely determined by its action at prime powers. So we can compute that for any prime p and non-negative integer exponents $\alpha \geq 1$ that

$$\begin{aligned} (\lambda * \mu)(p^\alpha) &= \sum_{i=0}^{\alpha} \lambda(p^{\alpha-i}) \mu(p^i) \\ &= \lambda(p^\alpha) - \lambda(p^{\alpha-1}) \\ &= (-1)^{\Omega(p^\alpha)} - (-1)^{\Omega(p^{\alpha-1})} = (-1)^\alpha - (-1)^{\alpha-1} = 2\lambda(p^\alpha). \end{aligned}$$

Then by the multiplicativity of $\lambda(n)$, the previous inequalities in (7) are re-stated in the form of

$$2C_{1,n} \cdot \lambda(n) \leq h^{-1}(n) \leq 2C_{2,n} \cdot \lambda(n).$$

Since the absolute constants $C_{1,n}, C_{2,n}$ are always positive, we clearly recover the signedness of $g^{-1}(n)$ as $\lambda(n)$ for all $n \geq 1$. \square

5.3 Statements of other facts and known limiting asymptotics

Theorem 5.2 (Mertens theorem). *For all $x \geq 2$ we have that*

$$P_1(x) := \sum_{p \leq x} \frac{1}{p} = \log \log x + B + o(1),$$

where $B \approx 0.2614972128476427837554$ is an absolute constant^{*F5.A}.

Corollary 5.3 (Product form of Mertens theorem). *We have that for all sufficiently large $x \gg 2$*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-B}}{\log x} (1 + o(1)),$$

where the notation for the absolute constant $0 < B < 1$ coincides with the definition of Mertens constant from Theorem 5.2. Hence, for any real $z \geq 0$ we obtain that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^z = \frac{e^{-Bz}}{(\log x)^z} (1 + o(1))^z \sim \frac{e^{-Bz}}{(\log x)^z}, \text{ as } x \rightarrow \infty.$$

Proofs of Theorem 5.2 and Corollary 5.3 are found in [4, §22.7; §22.8].

Facts 5.4 (Exponential integrals and the incomplete gamma function). The following two variants of the *exponential integral function* are defined by the integral representations [13, §8.19]

$$\begin{aligned} \text{Ei}(x) &:= \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, \\ E_1(z) &:= \int_1^{\infty} \frac{e^{-tz}}{t} dt, \text{Re}(z) \geq 0. \end{aligned}$$

These two functions are related by $\text{Ei}(-kz) = -E_1(kz)$ for real $k, z > 0$. We have the following inequalities providing quasi-polynomial upper and lower bounds on $E_1(z)$ for all large real $z \gg e$ ^{*F5.B}:

$$1 - \frac{3}{4}z \leq E_1(z) - \gamma - \log z \leq 1 - \frac{3}{4}z + \frac{11}{36}z^2. \quad (8a)$$

The (upper) *incomplete gamma function* is defined by [13, §8.4]

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt, \text{Re}(s) > 0.$$

We have the following properties of $\Gamma(s, x)$:

$$\Gamma(s, x) = (s-1)! \cdot e^{-x} \times \sum_{k=0}^{s-1} \frac{x^k}{k!}, s \in \mathbb{Z}^+, \quad (8b)$$

$$\Gamma(s, x) \sim x^{s-1} \cdot e^{-x}, \text{ as } x \rightarrow \infty. \quad (8c)$$

The result in (8b) allows us to form summatory functions over $\hat{\pi}_k(n)$ for $n \leq x$ based on Theorem 4.7 and Theorem 4.8 expressed by the incomplete gamma function (see Corollary 9.4). For large $x \rightarrow \infty$, we can approximate these sums by (8c). The limiting asymptotics in (8c) also allow us to simplify certain definite integral formulas involving fixed powers of $\log t$ (and $\log \log t$) for large, infinitely tending upper limits of integration required by the proofs in Section 8.

^{*F5.A}Exactly, we have that the *Mertens constant* is defined by

$$B = \gamma + \sum_{m \geq 2} \frac{\mu(m)}{m} \log [\zeta(m)],$$

where $\gamma \approx 0.577215664902$ is Euler's gamma constant.

^{*F5.B}Indeed, these inequalities are usually stated to provide bounds on $\text{Ei}(z)$ when $z > 0$ is real-valued. That the bounds are also valid for $E_1(z)$ at all sufficiently large $z \gg e$ is a matter of convenience we state for reference in this way to arrive at the formulas we see in Proposition 6.6 and subsequently obtain in the proof of Theorem 4.8.

6 Components to the asymptotic analysis of lower bounds for sums of arithmetic functions weighted by $\lambda(n)$

In this section, we re-state a couple of key results proved in [10, §7.4] that we rely on to prove Corollary 6.5 stated below. This corollary is important as it shows that (signed) summatory functions over $\hat{\pi}(x)$ capture the dominant asymptotics of the full summatory function formed by taking $1 \leq k \leq \log_2(x)$ when we truncate the range of the index k and instead sum only up to the uniform bound for $1 \leq k \leq \log \log x$ guaranteed by applying Theorem 4.7. We also prove Theorem 4.8 in this section. This key theorem allows us to establish a global minimum we can attain on the function $\mathcal{G}(z)$ from Theorem 4.7. This in turn implies the uniform lower bounds on $\hat{\pi}_k(x)$ guaranteed at the conclusion of the second theorem by a straightforward manipulation of inequalities.

6.1 Results proved by Montgomery and Vaughan

Remark 6.1 (Intuition and constructions in Theorem 4.7). For $|z| < 2$ and $\operatorname{Re}(s) > 1$, let

$$F(s, z) := \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z, \quad (9)$$

and define the DGF coefficients, $a_z(n)$ for $n \geq 1$, by the relation

$$\zeta(s)^z \cdot F(s, z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s}, \operatorname{Re}(s) > 1.$$

Suppose that $A_z(x) := \sum_{n \leq x} a_z(n)$ for $x \geq 1$. Then for the choice of the function $F(s, z)$ defined in (9), we obtain

$$A_z(x) = \sum_{n \leq x} z^{\Omega(n)} = \sum_{k \geq 0} \hat{\pi}_k(x) z^k.$$

Thus for $r < 2$, by Cauchy's integral formula we have

$$\hat{\pi}_k(x) = \frac{1}{2\pi i} \int_{|z|=r} \frac{A_z(x)}{z^{k+1}} dz.$$

Selecting $r := \frac{k-1}{\log \log x}$ leads to the uniform asymptotic formulas for $\hat{\pi}_k(x)$ given in Theorem 4.7.

We also require the next theorems reproduced from [10, §7.4] that handle the relative scarcity of the distribution of the $\Omega(n)$ for $n \leq x$ such that $\Omega(n) > \log \log x$.

Theorem 6.2 (Upper bounds on exceptional values of $\Omega(n)$ for large n). *Let*

$$\begin{aligned} A(x, r) &:= \#\{n \leq x : \Omega(n) \leq r \cdot \log \log x\}, \\ B(x, r) &:= \#\{n \leq x : \Omega(n) \geq r \cdot \log \log x\}. \end{aligned}$$

If $0 < r \leq 1$ and $x \geq 2$, then

$$A(x, r) \ll x(\log x)^{r-1-r \log r}, \quad \text{as } x \rightarrow \infty.$$

If $1 \leq r \leq R < 2$ and $x \geq 2$, then

$$B(x, r) \ll_R x \cdot (\log x)^{r-1-r \log r}, \quad \text{as } x \rightarrow \infty.$$

Theorem 6.3 (Exact bounds on exceptional values of $\Omega(n)$ for large n). *We have that uniformly*

$$\#\{3 \leq n \leq x : \Omega(n) - \log \log n \leq 0\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log \log x}}\right).$$

Remark 6.4. The proofs of Theorem 6.2 and Theorem 6.3 are found in §7.4 of Montgomery and Vaughan. The key interpretation we need is the result stated in the next corollary. The precise way in which the bound stated in the previous theorem depends on the indeterminate parameter R can be reviewed for reference in the proof algebra and relations cited in the reference [10, §7]. The role of the parameter R involved in stating the previous theorem is more notably critical as the scalar factor the upper bound on $k \leq R \log \log x$ in Theorem 4.7 up to which we obtain the valid uniform bounds in x on the asymptotic formulas for $\hat{\pi}_k(x)$.

We have a discrepancy to work out in so much as we can only form summatory functions over the $\hat{\pi}_k(x)$ for $1 \leq k \leq R \log \log x$ using the asymptotic formulas guaranteed by Theorem 4.7, even though we can actually have contributions from values distributed throughout the range $1 \leq \Omega(n) \leq \log_2(n)$ infinitely often. It is then crucial that we can show that the dominant growth of the asymptotic formulas we obtain for these summatory functions is captured by summing only over k in the truncated range where the uniform bounds hold.

Corollary 6.5. *Using the notation for $A(x, r)$ and $B(x, r)$ from Theorem 6.2, we have that for $\delta > 0$,*

$$o(1) \leq \left| \frac{B(x, 1 + \delta)}{A(x, 1)} \right| \ll 2, \text{ as } \delta \rightarrow 0^+, x \rightarrow \infty.$$

Proof. The lower bound stated above should be clear. To show that the asymptotic upper bound is correct, we compute using Theorem 6.2 and Theorem 6.3 that

$$\left| \frac{B(x, 1 + \delta)}{A(x, 1)} \right| \ll \left| \frac{x \cdot (\log x)^{\delta - \log(1 + \delta)}}{O(1) + \frac{x}{2} + O\left(\frac{x}{\sqrt{\log \log x}}\right)} \right| \sim \left| \frac{(\log x)^{\delta - \log(1 + \delta)}}{\frac{1}{2} + o(1)} \right| \xrightarrow{\delta \rightarrow 0^+} 2,$$

as $x \rightarrow \infty$. Notice that since $\mathbb{E}[\Omega(n)] = \log \log n + B$, with $0 < B < 1$ the absolute constant from Mertens theorem, when we denote the range of $k > \log \log x$ as holding in the form of $k > (1 + \delta) \log \log x$ for $\delta > 0$, we can assume that $\delta \rightarrow 0^+$ as $x \rightarrow \infty$. \square

6.2 New results based on refinements of Theorem 4.7

What the enumeratively flavored result in Theorem 4.7 allows us to do is get a sufficient lower bound on sums of positive and asymptotically bounded arithmetic functions weighted by the Liouville lambda function, $\lambda(n) = (-1)^{\Omega(n)}$. We seek to approximate $\mathcal{G}(z)$ defined in this theorem by only taking finite products of the primes in the factor of $\prod_p (1 - z/p)^{-1}$ for $p \leq ux$, e.g., indexing the component products only over those primes $p \in \{2, 3, 5, \dots, ux\}$ for some minimal upper bound depending on the parameter u as $x \rightarrow \infty$.

We first require a handle on partial sums of integer powers of the reciprocal primes as functions of the integral exponent and the upper summation index x . The next statement of Proposition 6.6 effectively provides a coarse rate in x below which the reciprocal prime sums tend to absolute constants given by the prime zeta function, $P(s)$. We make particular use of the property of finite-degree polynomial dependence of these bounds in s to simplify the computations in the theorem below.

Proposition 6.6. *For real $s \geq 1$, let*

$$P_s(x) := \sum_{p \leq x} p^{-s}, x \geq 2.$$

When $s := 1$, we have the asymptotic formula from Mertens theorem (see Theorem 5.2). For all integers $s \geq 2$ there are absolutely defined bounding functions $\gamma_0(s, x), \gamma_1(s, x)$ such that

$$\gamma_0(s, x) + o(1) \leq P_s(x) \leq \gamma_1(s, x) + o(1), \text{ as } x \rightarrow \infty.$$

It suffices to take the bounds in the previous equation as

$$\begin{aligned} \gamma_0(z, x) &= -s \log \left(\frac{\log x}{\log 2} \right) + \frac{3}{4}s(s-1) \log(x/2) - \frac{11}{36}s(s-1)^2 \log^2(x) \\ \gamma_1(s, x) &= -s \log \left(\frac{\log x}{\log 2} \right) + \frac{3}{4}s(s-1) \log(x/2) + \frac{11}{36}s(s-1)^2 \log^2(2). \end{aligned}$$

Proof. Let $s > 1$ be real-valued. By Abel summation with the summatory function $A(x) = \pi(x) \sim \frac{x}{\log x}$ and where our target function smooth function is $f(t) = t^{-s}$ with $f'(t) = -s \cdot t^{-(s+1)}$, we obtain that

$$\begin{aligned} P_s(x) &= \frac{1}{x^s \cdot \log x} + s \cdot \int_2^x \frac{dt}{t^s \log t} \\ &= E_1((s-1) \log 2) - E_1((s-1) \log x) + o(1), \text{ as } x \rightarrow \infty. \end{aligned}$$

Now using the inequalities in Facts 5.4, we obtain that the difference of the exponential integral functions is bounded above and below by

$$\begin{aligned} \frac{P_s(x)}{s} &\geq -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1) \log(x/2) - \frac{11}{36}(s-1)^2 \log^2(x) \\ \frac{P_s(x)}{s} &\leq -\log\left(\frac{\log x}{\log 2}\right) + \frac{3}{4}(s-1) \log(x/2) + \frac{11}{36}(s-1)^2 \log^2(2). \end{aligned}$$

This completes the proof of the bounds cited above in the statement of this lemma. \square

Proof of Theorem 4.8. We have for all integers $0 \leq k \leq m$, and any sequence $\{f(n)\}_{n \geq 1}$ with bounded partial sums, that [9, §2]

$$[z^k] \prod_{1 \leq i \leq m} (1 - f(i)z)^{-1} = [z^k] \exp\left(\sum_{j \geq 1} \left(\sum_{i=1}^m f(i)^j\right) \frac{z^j}{j}\right), |z| < 1. \quad (10)$$

In our case we have that $f(i)$ denotes the i^{th} prime in the generating function expansion of (10). Hence, by summing over all primes $p \leq ux$ in the previous formula and applying Proposition 6.6, we obtain that the logarithm of the corresponding generating function in z is given by

$$\begin{aligned} \log \left[\prod_{p \leq ux} \left(1 - \frac{z}{p}\right)^{-1} \right] &\geq (B + \log \log(ux))z + \sum_{j \geq 2} [a(ux) + b(ux)(j-1) + c(ux)(j-1)^2] z^j \\ &= (B + \log \log(ux))z - a(ux) \left(1 + \frac{1}{z-1} + z\right) \\ &\quad + b(ux) \left(1 + \frac{2}{z-1} + \frac{1}{(z-1)^2}\right) \\ &\quad - c(ux) \left(1 + \frac{4}{z-1} + \frac{5}{(z-1)^2} + \frac{2}{(z-1)^3}\right) \\ &=: \hat{\mathcal{B}}(u, x; z), \end{aligned}$$

for some minimal parameter u that we will determine below.

In the previous equations, the lower bounds formed by the functions (a, b, c) evaluated at ux are given by the corresponding upper bounds from Proposition 6.6 due to the leading sign on the previous expansions as

$$(a_\ell, b_\ell, c_\ell) := \left(-\log\left(\frac{\log(ux)}{\log 2}\right), \frac{3}{4} \log\left(\frac{ux}{2}\right), \frac{11}{36} \log^2 2\right).$$

Now we make a decision to set the uniform bound parameter to a middle ground value of $1 < R < 2$ at $R := \frac{3}{2}$ so that

$$z \equiv z(k, x) = \frac{k-1}{\log \log x} \in [0, R),$$

for $x \gg 1$ sufficiently large. Thus $(z-1)^{-m} \in [(-1)^m, 2^m]$ for integers $m \geq 1$, and so we can obtain the following lower bound:

$$\hat{\mathcal{B}}(u, x; z) \geq (B + \log \log(ux))z - a(ux) \left(1 + \frac{1}{z-1} + z\right)$$

$$+ b(ux) \left(1 + \frac{2}{z-1} + \frac{1}{(z-1)^2} \right) - 45 \cdot c(ux).$$

Since the function $c(ux)$ is constant, we then also obtain the next bounds.

$$\begin{aligned} \frac{e^{-Bz}}{(\log(ux))^z} \times \exp\left(\widehat{\mathcal{B}}(u, x; z)\right) &\geq \exp\left(-\frac{55}{4} \log^2 2\right) \times \left(\frac{\log(ux)}{\log 2}\right)^{1+\frac{1}{z-1}+z} \times \left(\frac{ux}{2}\right)^{\frac{3}{4}\left(1+\frac{2}{z-1}+\frac{1}{(z-1)^2}\right)} \\ &=: \widehat{\mathcal{C}}(u, x; z) \end{aligned} \quad (11)$$

Now we need to determine which values of u minimize the expression for the function defined in (11). For this we will use an elementary method from introductory calculus to determine a global minimum for the product in z . We can symbolically use *Mathematica* to see that

$$\left. \frac{\partial}{\partial u} \left[\widehat{\mathcal{C}}(u, x; z) \right] \right|_{u \rightarrow u_0} = 0 \implies u_0 \in \left\{ \frac{1}{x}, \frac{1}{x} e^{-\frac{4}{3}(z-1)} \right\}.$$

When we substitute this $u_0 =: \hat{u}_0 \mapsto \frac{1}{x} e^{-\frac{4}{3}(z-1)}$ into the next expression for the second derivative of the function $\widehat{\mathcal{C}}(u, x; z)$ we obtain that

$$\begin{aligned} \left. \frac{\partial^2}{\partial u^2} \left[\widehat{\mathcal{C}}(u, x; z) \right] \right|_{u=\hat{u}_0} &= \exp\left(-\frac{55}{4} \log^2 2\right) \times 2^{\frac{8z^3-27z^2+32z-16}{4(z-1)^2}} 3^{-z+\frac{1}{1-z}+1} e^{\frac{5z^2-16z+8}{3(z-1)}} \times \\ &\times (\log 2)^{\frac{z^2}{1-z}} \cdot (xz)^2 (1-z)^{z+\frac{1}{z-1}-2} > 0, \end{aligned}$$

provided that $z < 1$. The restriction to $0 \leq z < 1$ is equivalent to requiring that $1 \leq k \leq \log \log x$ in Theorem 4.7.

After substitution of $u = \frac{1}{x} e^{-\frac{4}{3}(z-1)}$ into the expression for $\widehat{\mathcal{C}}(u, x; z)$ defined above, we have that

$$\widehat{\mathcal{C}}(u, x; z) \geq 2^{\frac{9}{16}} \cdot \exp\left(-\frac{55}{4} \log^2 2\right) \left(\frac{1-z}{3e \log 2}\right)^3 \times \left(\frac{4(1-z)}{3e \log 2}\right)^z.$$

Finally, since $z \equiv z(k, x) = \frac{k-1}{\log \log x}$ and $k \in [0, R \log \log x)$, we obtain for minimal k and all large enough $x \gg 1$ that $\Gamma(z+1) \approx 1$, and for k towards the upper range of its interval that $\Gamma(z+1) \leq \Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$. \square

7 Average case analysis of bounds on the Dirichlet inverse functions, $g^{-1}(n)$

The property in (C) of Conjecture 4.5 along squarefree $n \geq 1$ captures an important characteristic of $g^{-1}(n)$ that holds more globally for all $n \geq 1$. In particular, these functions can be expressed via more simple formulas than inspection of the first few initial values of the repetitive, quasi-periodic sequence otherwise suggests. The pages of tabular data given as Table T.1 in the appendix section starting on page 42 are intended to provide clear insight into why we arrived at the convenient approximations to $g^{-1}(n)$ proved in this section. The table offers illustrative numerical data formed by examining the approximate behavior at hand for the first several cases of $1 \leq n \leq 500$ with *Mathematica*.

7.1 Definitions and basic properties of component function sequences

We define the following sequence for integers $n \geq 1, k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \quad (12)$$

The sequence of important semi-diagonals of these functions begins as [16, A008480]

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, -3, 2, 2, -1, 4, 1, 2, \dots\}.$$

Remark 7.1 (An effective range of k depending on a fixed large n). Notice that by expanding the recursively-based definition in (12) out to its maximal depth by nested divisor sums, for fixed n , $C_k(n)$ is seen to only ever possibly be non-zero for $k \leq \Omega(n)$. Thus, the effective range of k for fixed n is restricted by the conditions of $C_0(n) = \delta_{n,1}$ and that $C_k(n) = 0, \forall k > \Omega(n)$ whenever $n \geq 2$.

Example 7.2 (Special cases of the functions $C_k(n)$ for small k). We cite the following special cases which are verified by explicit computation using (12)*F7.A:

$$\begin{aligned} C_0(n) &= \delta_{n,1} \\ C_1(n) &= \omega(n) \\ C_2(n) &= d(n) \times \sum_{p|n} \frac{\nu_p(n)}{\nu_p(n) + 1} - \gcd(\Omega(n), \omega(n)). \end{aligned}$$

7.2 Uniform asymptotics of $C_k(n)$ for large all n and fixed, bounded k

Theorem 4.6 from the introduction is proved next. The theorem makes precise what these formulas already suggest about the main terms of the growth rates of $C_k(n)$ as functions of k, n for limiting cases of n large and fixed k which is necessarily bounded in n , but still taken as an independent parameter.

Proof of Theorem 4.6. We prove our bounds by induction on k . We can see by Example 7.2 that $C_1(n)$ satisfies the formula we must establish when $k := 1$ since $\mathbb{E}[\omega(n)] = \log \log n$. Suppose that $k \geq 2$ and let the inductive assumption state that for all $1 \leq m < k$

$$\mathbb{E}[C_m(n)] \gg (\log \log n)^{2m-1}.$$

*F7.A For all $n, k \geq 2$, we have the following recurrence relation satisfied by $C_k(n)$ between successive values of k :

$$C_k(n) = \sum_{p|n} \sum_{d|\frac{n}{p\nu_p(n)}} \sum_{i=0}^{\nu_p(n)-1} C_{k-1}(dp^i), n \geq 1.$$

Now using the recursive formula we used to define the sequences of $C_k(n)$ in (12), we have that as $n \rightarrow \infty$ ^{*F7.B}

$$\begin{aligned}
\mathbb{E}[C_k(n)] &= \mathbb{E} \left[\sum_{d|n} \omega(n/d) C_{k-1}(d) \right] \\
&= \frac{1}{n} \times \sum_{d \leq n} C_{k-1}(d) \times \sum_{r=1}^{\lfloor \frac{n}{d} \rfloor} \omega(r) \\
&\sim \sum_{d \leq n} C_{k-1}(d) \left[\frac{\log \log(n/d) \left[d \leq \frac{n}{e} \right]_\delta}{d} + \frac{B}{d} \right] \\
&\sim \sum_{d \leq \frac{n}{e}} \left[\sum_{m < d} \frac{\mathbb{E}[C_{k-1}(m)]}{m} \log \log \left(\frac{n}{m} \right) + B \cdot \mathbb{E}[C_{k-1}(d)] + B \cdot \sum_{m < d} \frac{\mathbb{E}[C_{k-1}(m)]}{m} \right] \\
&\gg B \left[n \log n \cdot (\log \log n)^{2k-3} - \log n \cdot (\log \log n)^{2k-3} \right] \times \left(1 + \frac{\log n}{2} \right) \\
&\gg (\log \log n)^{2k-1}.
\end{aligned}$$

In transitioning to the last equation from the previous step, we have used that $\frac{Bn}{2} \cdot (\log n)^2 \gg (\log \log n)^2$ as $n \rightarrow \infty$. We have also used that for large $n \rightarrow \infty$ and fixed m , we have by an approximation to the incomplete gamma function given by

$$\int_e^n \frac{(\log \log t)^m}{t} \sim (\log n)(\log \log n)^m, \text{ as } n \rightarrow \infty.$$

Thus the claim holds by mathematical induction whenever $n \rightarrow \infty$ is large and $1 \leq k \leq \Omega(n)$. \square

7.3 Relating the auxiliary functions $C_k(n)$ to formulas approximating $g^{-1}(n)$

Lemma 7.3 (An exact formula for $g^{-1}(n)$). *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse of $\omega + 1$ as

$$g^{-1}(n) = - \sum_{\substack{d|n \\ d>1}} (\omega(d) + 1) g^{-1}(n/d) \implies (g^{-1} * 1)(n) = -(\omega * g^{-1})(n).$$

Now by repeatedly expanding the right-hand-side, and removing corner cases in the nested sums with $\omega(1) = 0$, we find inductively that

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n).$$

The statement then follows by Möbius inversion applied to each side of the last equation. \square

Corollary 7.4. *For all squarefree integers $n \geq 1$, we have that*

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \tag{13}$$

^{*F7.B}For all large $x \gg 2$ the summatory function of $\omega(n)$ satisfies [4, §22.10]

$$\sum_{n \leq x} \omega(n) = x \log \log x + Bx + O\left(\frac{x}{\log x}\right).$$

Proof. Since $g^{-1}(1) = 1$, clearly the claim is true for $n = 1$. Suppose that $n \geq 2$ and that n is squarefree. Then $n = p_1 p_2 \cdots p_{\omega(n)}$ where p_i is prime for all $1 \leq i \leq \omega(n)$. So we can transform the exact divisor sum guaranteed for all n in Lemma 7.3 as follows:

$$\begin{aligned} g^{-1}(n) &= \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} (-1)^{\omega(n)-i} (-1)^i \cdot C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{i=0}^{\omega(n)} \sum_{\substack{d|n \\ \omega(d)=i}} C_{\Omega(d)}(d) \\ &= \lambda(n) \times \sum_{d|n} C_{\Omega(d)}(d). \end{aligned}$$

The signed contributions in the first of the previous equations is justified by noting that $\lambda(n) = (-1)^{\omega(n)}$ whenever n is squarefree, and that for d squarefree with $\omega(d) = i$, $\Omega(d) = i$. \square

Corollary 7.5. *We have that*

$$\frac{6}{\pi^2} \log x \ll \mathbb{E}|g^{-1}(n)| \leq \mathbb{E} \left[\sum_{d|n} C_{\Omega(d)}(d) \right].$$

Proof. To prove the lower bound, first notice that by Lemma 7.3, Proposition 5.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain that

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \quad (14)$$

In particular, since $\mu(n)$ is non-zero only at squarefree integers and at any squarefree $n \geq 1$ we have $\mu(n) = (-1)^{\omega(n)} = \lambda(n)$, Lemma 7.3 implies

$$\begin{aligned} |g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \end{aligned}$$

Notice in the above equation that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is necessarily even.

Recall from the introduction that the summatory function of the squarefree integers is given by

$$Q(x) := \sum_{n \leq x} \mu^2(n) = \frac{6}{\pi^2} x + O(\sqrt{x}).$$

Then since $C_{\Omega(d)}(d) \geq 1$ for all $d \geq 1$, we obtain by summing over (14) that as $x \rightarrow \infty$

$$\begin{aligned} \frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| &= \frac{1}{x} \times \sum_{d \leq x} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\ &\sim \sum_{d \leq x} C_{\Omega(d)}(d) \left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dx}}\right) \right] \end{aligned}$$

$$\begin{aligned}
&\geq \sum_{d \leq x} \left[\frac{6}{d \cdot \pi^2} + O\left(\frac{1}{\sqrt{dx}}\right) \right] \\
&\sim \frac{6}{\pi^2} \left(\log x + \gamma + O\left(\frac{1}{x}\right) \right) + O\left(\frac{1}{\sqrt{x}} \times \int_0^x t^{-1/2} dt\right) \\
&= \frac{6}{\pi^2} \log x + O(1).
\end{aligned}$$

To prove the upper bound, notice that by Lemma 7.3 and Corollary 7.4,

$$|g^{-1}(n)| \leq \sum_{d|n} C_{\Omega(d)}(d).$$

Now since both of the above quantities are positive for all $n \geq 1$, we must obtain the upper bound on the average order of $|g^{-1}(n)|$ stated above. \square

8 Establishing initial lower bounds on the summatory functions $G^{-1}(x)$

What we have established so far, and will establish for $G^{-1}(x)$ in Section 9, are lower bound estimates that hold essentially only *on average*. This means that for all sufficiently large $x \rightarrow \infty$, we need to show that the expected value based lower bounds are achieved in asymptotic order more globally within predictably some small window (interval) near x .

Lemma 8.1 (Asymptotic densities of exceptional values of positive arithmetic functions). *Let $F \in C^1(0, \infty)$ be a monotone increasing function such that $F(x) \rightarrow 0$ as $x \rightarrow \infty$. Suppose that f is an arithmetic function such that $f(n) > 0$ for all $n \geq 1$ and so that*

$$\sum_{n \leq x} f(n) \geq x \cdot F(x) + o(xF(x)), \text{ as } x \rightarrow \infty.$$

Let the set defined by

$$\mathcal{F}_- := \{n \geq 1 : f(n) < F(n)\},$$

have corresponding limiting asymptotic density

$$\gamma_- := \lim_{x \rightarrow \infty} \frac{1}{x} \cdot \#\{n \leq x : n \in \mathcal{F}_-\}.$$

Then the limit γ_- exists and $\gamma_- = 0$. In other words, for almost every sufficiently large $n \rightarrow \infty$, $f(n) \geq F(n)$ and $-f(n) \leq -F(n)$ where $\mathbb{E}[f(n)] = F(n)$ as $n \rightarrow \infty$.

Proof. First, suppose that the limit we used to define γ_- exists with $\gamma_- \in [0, 1)$. By the positivity of $f(n)$, we know that $F(x)$ is positive for all sufficiently large x . So we have that as $x \rightarrow \infty$

$$\begin{aligned} \sum_{n \leq x} f(n) &\leq \sum_{\substack{n \leq x \\ n \notin \mathcal{F}_-}} f(n) + \sum_{\substack{n \leq x \\ n \in \mathcal{F}_-}} F(n) \\ &< \sum_{n \leq (1-\gamma_-)x} f(n) + \sum_{(1-\gamma_-)x \leq n \leq x} F(n) \\ &\sim xF(x) - \int_{(1-\gamma_-)x}^x tF'(t)dt \\ &= (1-\gamma_-)x \cdot F((1-\gamma_-)x) + \gamma_-x \cdot F(c), \end{aligned} \tag{15}$$

integrating by parts and for some $c \in [(1-\gamma_-)x, x]$ by the mean value theorem. So by (15), we have that

$$\sum_{n \leq x} f(n) < x \cdot F(x), \text{ as } x \rightarrow \infty.$$

Unless $\gamma_- = 0$ this property contradicts our hypothesis on the limiting behavior of the summatory function of $f(n)$. Notice also that the limiting density cannot be identically one since if $\gamma_- = 1$, then

$$\begin{aligned} \frac{1}{x} \times \sum_{n \leq x} f(n) &< \frac{1}{x} \times \sum_{n \leq x} F(n) \\ &\leq \max_{1 \leq j \leq x} F(j) + o(F(x)) = F(x) + o(F(x)), \end{aligned}$$

by the assumption of the monotonicity of F on $(0, \infty)$. Hence, we conclude that $\gamma_- = 0$ provided that the limit exists.

If a limiting value for γ_- does not exist, then for infinitely many large $x \geq 2$ within a set $\mathcal{M} \subset \mathbb{Z}^+$ where the density of $\mathcal{M} \cap \{n \leq x\}$ is positive, we have that

$$M_x := \frac{1}{x} \cdot \#\{n \leq x : n \in \mathcal{F}_-\} \in (0, 1),$$

where M_x non-monotonically oscillates in value along a subsequence. Using a similar method to what we argued above, in this case there are infinitely many x within a set of eventually always positive limiting densities such that our assumption on the asymptotic lower bound on the summatory function of $f(n)$ also does not hold. This contradiction shows that the limit γ_- must in fact exist, and as we have shown above, is then necessarily zero. \square

Proposition 8.2. *For all sufficiently large n on a set of asymptotic density one, we have that*

$$|g^{-1}(n)| \gg \frac{6}{\pi^2}(\log n)(\log \log n).$$

Proof. Reusing part of the argument given in the proof of Corollary 7.5, recall that we have the following holds:

$$\mathbb{E}|g^{-1}(n)| \sim \sum_{d \leq n} C_{\Omega(d)}(d) \left[\frac{6}{\pi^2 d} + O\left(\frac{1}{\sqrt{nd}}\right) \right].$$

Then we obtain that^{*F8.A}

$$\mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2} \left[\mathbb{E}[C_{\Omega(n)}] + \sum_{d < n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d} \right] + O\left(\mathbb{E}[C_{\Omega(n)}(n)] + \frac{1}{\sqrt{n}} \times \sum_{d < n} \frac{\mathbb{E}[C_{\Omega(d)}(d)]}{\sqrt{d}} \right). \quad (16)$$

Now using Theorem 4.6, we have prove that $\mathbb{E}[C_k(n)]$ is minimized for $k \geq 1$ when $k := 1$. We next argue using the Erdős-Kac type theorem stated as [6, §1.7]

$$\#\left\{n \leq x : \frac{\omega(n) - \log \log x}{\sqrt{\log \log x}} \leq z\right\} = \Phi(z) \cdot x + o(1),$$

for any real $z \in (-\infty, +\infty)$ and where we define $\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-t^2/2} dt$. Then by inclusion-exclusion we have that

$$\begin{aligned} \#\{n \leq x : C_{\Omega(n)}(n) - \log \log n \leq 0\} &= \#\{n \leq x : \Omega(n) = 1 \wedge \omega(n) - \log \log n \leq 0\} \\ &\leq \frac{x}{\log x} (1 + o(1)) + \frac{x}{2} + o(1) - \#\{n \leq x : \omega(n) - \log \log n \leq 0 \vee n \in \mathbb{P}\} \\ &\sim \frac{x}{\log x}, \end{aligned}$$

since $\omega(n) \leq \log \log n$ for almost every prime $n \in \mathbb{P}$. Now by subtracting the results in the above,

$$\#\{n \leq x : C_{\Omega(n)}(n) > \log \log n\} \geq x(1 + o(1)).$$

This means that $C_{\Omega(n)}(n) > \log \log n$ for almost every positive integer n .

Then by (16), we obtain that

$$\begin{aligned} \mathbb{E}|g^{-1}(n)| &\gg \frac{6}{\pi^2} \log \log n + \frac{6}{\pi^2} \times \int_e^n \frac{\log \log t}{t} dt + O\left(\log \log n + \frac{1}{\sqrt{n}} \times \int_e^n \frac{\log \log t}{\sqrt{t}} \right) \\ &\gg \frac{6}{\pi^2} (\log n)(\log \log n) + O(\log \log n). \end{aligned}$$

Finally, to see that our claimed property holds on a set of asymptotic density one, we recall that by Lemma 8.1, we have that since $\log n \cdot \log \log n \rightarrow 0$ as $n \rightarrow \infty$

$$\#\left\{n \leq x : |g^{-1}(n)| \geq \frac{6}{\pi^2} (\log n)(\log \log n)\right\} = x(1 + o(1)), \text{ as } x \rightarrow \infty. \quad (17)$$

In other words, this lower bound for $|g^{-1}(n)|$ holds for almost every sufficiently large $n \in \mathbb{Z}^+$. \square

^{*F8.A}To see why the error term holds observe that

$$\frac{1}{\sqrt{d+1}} - \frac{1}{\sqrt{d}} \sim \frac{1 - \sqrt{1 + \frac{1}{d}}}{\sqrt{d}} = \frac{1}{d} \times \sum_{s \geq 1} \binom{\frac{1}{2}}{s} \frac{1}{d^s} = O\left(\frac{1}{d^{3/2}}\right).$$

Corollary 8.3. *For all sufficiently large n on a set of asymptotic density one, we have that*

$$\sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} - |g^{-1}(n)| \leq 0.$$

Proof. First, we see that we have

$$\begin{aligned} \mathbb{E} \left[\sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} \right] &= \frac{1}{n} \times \sum_{e<d \leq n} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} \left\lfloor \frac{n}{d} \right\rfloor \\ &\sim \int_e^n \frac{(\log t)^{\frac{1}{4}}}{t \cdot \log \log t} dt \\ &= \text{Ei} \left(\frac{5}{4} \log \log n \right) \\ &\ll \frac{11}{36} \left(\frac{5}{4} \right)^2 (\log \log n)^2 = \frac{275}{576} (\log \log n)^2. \end{aligned} \tag{18}$$

Now by applying Proposition 8.2 we have that

$$\begin{aligned} \mathbb{E} \left[\sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} - |g^{-1}(n)| \right] &= \mathbb{E} \left[\sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} \right] - \mathbb{E} |g^{-1}(n)| \\ &\ll \frac{275}{576} (\log \log n)^2 - \frac{6}{\pi^2} (\log n) (\log \log n) \ll 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Since $\log \log n \rightarrow 0$ as $n \rightarrow \infty$, by a reversal of Lemma 8.1, we have that

$$\# \left\{ n \leq x : \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} > \frac{275}{576} (\log \log n)^2 \right\} = x(1 + o(1)), \text{ as } x \rightarrow \infty.$$

We showed a similar property for $\mathbb{E} |g^{-1}(n)|$ in (17) of the proof above. Thus we have that our claimed property holds on a set of asymptotic density one. \square

Let the summatory function $G_E^{-1}(x)$ be defined for $x \geq 1$ by ^{*F8.B}

$$G_E^{-1}(x) := \sum_{n \leq \log \log x} \lambda(n) \times \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d}. \tag{19}$$

Proving another criteria we require, namely that $|G^{-1}(x)| \gg |G_E^{-1}(x)|$ when x is large, requires raises some careful considerations on attainable bounds for these functions (see Theorem 8.5).

Remark 8.4. Given our proof of the result in Corollary 8.3, an analysis by cases on the sign of $G^{-1}(x)$ suggests that nearly everywhere on an infinite set \mathcal{S}_E of asymptotic density one we should have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_E \\ \lambda(n)=+1}} g^{-1}(n) \geq \sum_{\substack{n \leq x \\ n \in \mathcal{S}_E \\ \lambda(n)=+1}} \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} \geq \sum_{\substack{n \leq \log \log x \\ n \in \mathcal{S}_E \\ \lambda(n)=+1}} \sum_{\substack{d|n \\ d>e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d}$$

^{*F8.B}The subscript of E (as in expectation) on the function $G_E^{-1}(x)$ is purely for notation and does not correspond to a formal parameter or any implicit dependence on E in the formula that defines this function.

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_E \\ \lambda(n)=-1}} g^{-1}(n) \leq - \sum_{\substack{n \leq x \\ n \in \mathcal{S}_E \\ \lambda(n)=-1}} \sum_{\substack{d|n \\ d > e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} \leq - \sum_{\substack{n \leq \log \log x \\ n \in \mathcal{S}_E \\ \lambda(n)=-1}} \sum_{\substack{d|n \\ d > e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d}.$$

Notice that the range of n we can sum over to define $G_E^{-1}(x)$ is a somewhat delicate matter to establish without needing much deeper properties of the distribution of the summatory function $G^{-1}(x)$. It turns out that the upper bound of $\log \log x$ in $G_E^{-1}(x)$ compared to the index x in the summatory functions $G^{-1}(x)$ is about the best possible bound we can aim for and still obtain the unboundedness (scaled by a logarithmic factor) in Corollary 4.9. If on the other hand, we were able to prove that infinitely often along predictable subsequences that for

$$\hat{G}_E^{-1}(x) := \sum_{n \leq (\log x)^{1+\delta}} \lambda(n) \times \sum_{\substack{d|n \\ d > e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d},$$

for some absolute $\delta > 0$, we still recover $|G^{-1}(x)| \gg |\hat{G}_E^{-1}(x)|$, then we can adapt the proof we give in Section 9 to show the classical unboundedness conjecture holds, namely that $|M(x)|/\sqrt{x}$ grows without bound to $+\infty$ along subsequences.

Theorem 8.5. *For almost all sufficiently large integers $x \rightarrow \infty$, we have that*

$$|G^{-1}(x)| \gg |G_E^{-1}(x)|.$$

Proof. We bound the magnitude of each respective function respectively below and above in the worst cases of the signage imparted on the otherwise positive terms weighted by $\lambda(n)$. First, consider the following upper bound on $|G_E^{-1}(x)|$ ^{*F8.C}:

$$\begin{aligned} |G_E^{-1}(x)| &= \left| \sum_{e \leq n \leq \log \log x} \lambda(n) \times \sum_{\substack{d|n \\ d > e}} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} \right| \\ &\ll \sum_{d \leq \log \log x} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} \cdot \frac{\log \log x}{d} \\ &\approx \log \log x \times \int_e^{\log \log x} \frac{(\log t)^{\frac{1}{4}}}{t \cdot \log \log t} dt \\ &= \log \log x \times \text{Ei} \left(\frac{5}{4} \log \log n \right) \\ &\ll \frac{11}{36} \cdot \left(\frac{5}{4} \right)^2 \cdot (\log \log x)(\log \log \log x)^2. \end{aligned}$$

Next, we consider bounding the summatory function $|G^{-1}(x)|$ from below. For this, notice that from Lemma 7.3, we can see that $g^{-1}(n) = o(n)$. This observation combined with (17) shows that we can form this lower bound by taking an alternating sum involving the monotone bound of $\mathbb{E}|g^{-1}(n)| \gg \frac{6}{\pi^2}(\log n)(\log \log n)$ from Proposition 8.2 as follows:

$$|G^{-1}(x)| \gg \left| \sum_{n \leq x} (-1)^n \cdot \frac{6}{\pi^2}(\log n)(\log \log n) \right|$$

^{*F8.C}E.g., since we have the following special case of Theorem 4.1 for any arithmetic function f [1, cf. §3.10]:

$$\sum_{n \leq x} \sum_{d|n} f(d) = \sum_{d \leq x} f(d) \left\lfloor \frac{x}{d} \right\rfloor.$$

$$\begin{aligned}
&= \frac{6}{\pi^2} \times \left| \sum_{n \leq \frac{x}{2}} [\log(2n) \log \log(2n) - \log(2n+1) \log \log(2n+1)] \right| \\
&\sim \frac{6}{\pi^2} \times \left| \sum_{n \leq \frac{x}{2}} \left[\log(2n) \log \log(2n) - \left(\log(2n) \log \log(2n) + \frac{1}{2n} \right) \left(\log \log(2n) + \frac{1}{2n \cdot \log(2n)} \right) \right] \right| \\
&= \frac{6}{\pi^2} \times \sum_{n \leq \frac{x}{2}} \left[\frac{\log \log(2n)}{2n} + \frac{1}{2n} + \frac{1}{4n^2 \cdot \log(2n)} \right] \\
&\sim \frac{6}{\pi^2} \times \left[O(1) + \frac{1}{2} \left(\log x - \log 2 + \gamma + O\left(\frac{1}{x}\right) \right) + \int_e^{\frac{x}{2}} \frac{\log \log(2t)}{2t} dt \right] \\
&\gg \frac{3}{\pi^2} (\log x + (\log x)(\log \log x)).
\end{aligned}$$

Now since for all sufficiently large $x \rightarrow \infty$, we have that

$$\frac{3}{\pi^2} (\log x + (\log x)(\log \log x)) \gg \frac{275}{576} \cdot (\log \log x)(\log \log \log x)^2,$$

we have our claimed relation between the two key summatory functions. □

9 Lower bounds for $M(x)$ along infinite subsequences

9.1 The culmination of what we have done so far

Proposition 9.1. *For all sufficiently large x , we have that*

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt. \quad (20)$$

Proof. We know by applying Corollary 4.3 that

$$\begin{aligned} M(x) &= \sum_{k=1}^x g^{-1}(k)(\pi(x/k) + 1) \\ &= G^{-1}(x) + \sum_{k=1}^x g^{-1}(k)\pi(x/k), \end{aligned} \quad (21)$$

where we can drop the asymptotically unnecessary floored integer-valued arguments to $\pi(x)$ in place of its approximation by the monotone non-decreasing asymptotic order, $\pi(x) \sim \frac{x}{\log x}$. Moreover, we can always bound

$$\frac{Ax}{\log x} \leq \pi(x) \leq \frac{Bx}{\log x},$$

for suitably defined absolute constants, $A, B > 0$. Therefore the approximation obtained by replacing $\pi(x)$ by the dominant term in its limiting asymptotic formula is actually valid for all $x > 1$ up to a small constant difference.

What we require to sum and simplify the right-hand-side summation from (21) is a summation by parts argument. In particular, we obtain that for sufficiently large $x \geq 2$ ^{*F9.A}

$$\begin{aligned} \sum_{k=1}^x g^{-1}(k)\pi(x/k) &= G^{-1}(x)\pi(1) - \sum_{k=1}^{x-1} G^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right) \right] \\ &= - \sum_{k=1}^{x/2} G^{-1}(k) \left[\pi\left(\frac{x}{k}\right) - \pi\left(\frac{x}{k+1}\right) \right] \\ &\approx - \sum_{k=1}^{x/2} G^{-1}(k) \left[\frac{x}{k \cdot \log(x/k)} - \frac{x}{(k+1) \cdot \log(x/k)} \right] \\ &\approx - \sum_{k=1}^{x/2} G^{-1}(k) \frac{x}{k^2 \cdot \log(x/k)}. \end{aligned}$$

Since for x large enough the summand factor $\frac{x}{k^2 \cdot \log(x/k)}$ is monotonic as k ranges over $k \in [1, x/2]$ in ascending order, this summand factor is a smooth function of k (and x), and since $G^{-1}(x)$ is a summatory function with jumps only at the positive integers, we can approximate $M(x)$ for any finite $x \geq 2$ by

$$M(x) \approx G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt.$$

We will later only use unsigned lower bound approximations to this function in the next theorems so that the signedness of the summatory function term in the integral formula above as $x \rightarrow \infty$ is a moot point entirely. \square

^{*F9.A} Since $\pi(1) = 0$, the actual range of summation corresponds to $k \in [1, \frac{x}{2}]$.

9.1.1 A few more necessary results

We now use the superscript and subscript notation of (ℓ) not to denote a formal parameter to the functions we define below, but instead to denote that these functions form *lower bound* (rather than exact) approximations to other forms of the functions without the scripted (ℓ) .

Lemma 9.2. *Suppose that $\hat{\pi}_k(x) \geq \hat{\pi}_k^{(\ell)}(x) \geq 0$ with $\hat{\pi}_k^{(\ell)}(x)$ a monotone real-valued function for all sufficiently large x . Let*

$$A_{\Omega}^{(\ell)}(x) := \sum_{k \leq \log \log x} (-1)^k \hat{\pi}_k^{(\ell)}(x)$$

$$A_{\Omega}(x) := \sum_{k \leq \log \log x} (-1)^k \hat{\pi}_k(x).$$

Then for all sufficiently large x , we have that

$$|A_{\Omega}(x)| \gg |A_{\Omega}^{(\ell)}(x)|.$$

Proof. Given an explicit smooth lower bounding function, $\hat{\pi}_k^{(\ell)}(x)$, we define the similarly smooth and monotone residual terms in approximating $\hat{\pi}_k(x)$ using the following notation:

$$\hat{\pi}_k(x) = \hat{\pi}_k^{(\ell)}(x) + \hat{E}_k(x).$$

Then we can form the ordinary exact form of the summatory function A_{Ω} as

$$\begin{aligned} |A_{\Omega}(x)| &= \left| \sum_{k \leq \frac{\log \log x}{2}} [\hat{\pi}_{2k}(x) - \hat{\pi}_{2k-1}(x)] \right| \\ &\geq \left| A_{\Omega}^{(\ell)}(x) - \sum_{k \leq \frac{\log \log x}{2}} \hat{E}_{2k-1}(x) \right| \\ &\geq \left| A_{\Omega}^{(\ell)}(x) \right| - \left| \sum_{k \leq \frac{\log \log x}{2}} \hat{E}_{2k-1}(x) \right|. \end{aligned}$$

If the latter sum, denoted

$$\text{ES}(x) := \sum_{k \leq \frac{\log \log x}{2}} \hat{E}_{2k-1}(x) \rightarrow \infty,$$

as $x \rightarrow \infty$, then we can always find some absolute $C_0 > 0$ (by monotonicity) such that $\text{ES}(x) \leq C_0 \cdot A_{\Omega}(x)$. If on the other hand this sum becomes constant, or is bounded as $x \rightarrow +\infty$, then we also clearly have another absolute $C_1 > 0$ such that $|A_{\Omega}(x)| \geq C_1 \cdot |A_{\Omega}^{(\ell)}(x)|$. In either case, the claimed result holds for all large enough x . \square

Lemma 9.3. *Suppose that $f(n)$ is an arithmetic functions such that $f(n) > 0$ for all $n > u_0$ where $f(n) \gg \hat{\tau}_{\ell}(n)$ as $n \rightarrow \infty$. Assume that the bounding function $\hat{\tau}_{\ell}(t)$ is a continuously differentiable function of t for all large enough $t \gg u_0$. We define the λ -sign-scaled summatory function of f as follows:*

$$F_{\lambda}(x) := \sum_{u_0 < n \leq x} \lambda(n) \cdot f(n).$$

Let

$$A_{\Omega}^{(\ell)}(t) := \sum_{k=1}^{\lfloor \log \log t \rfloor} (-1)^k \hat{\pi}_k^{(\ell)}(t),$$

$$A_\Omega(t) := \sum_{k=1}^{\lfloor \log \log t \rfloor} (-1)^k \widehat{\pi}_k(t),$$

where $\widehat{\pi}_k(x) \geq \widehat{\pi}_k^{(\ell)}(x) \geq 0$ for $\widehat{\pi}_k^{(\ell)}(t)$ some smooth monotone function of t at all sufficiently large $t \rightarrow \infty$. Then we have that

$$|F_\lambda(x)| \gg \left| A_\Omega^{(\ell)}(x) \widehat{\tau}_\ell(x) - \int_{u_0}^x A_\Omega^{(\ell)}(t) \widehat{\tau}'_\ell(t) dt \right|.$$

Proof. We can form an accurate $C^1(\mathbb{R})$ approximation by the smoothness of $\widehat{\pi}_k^{(\ell)}(x)$ that allows us to apply the Abel summation formula using the summatory function $A_\Omega^{(\ell)}(t)$ for t on any bounded connected subinterval of $[1, \infty)$. The second stated formula for $F_\lambda(x)$ above is valid by Abel summation and by applying Lemma 9.2 whenever

$$0 \leq \left| \frac{\sum_{\log \log t < k \leq \frac{\log t}{\log 2}} (-1)^k \widehat{\pi}_k(t)}{A_\Omega(t)} \right| \ll 2, \text{ as } t \rightarrow \infty.$$

What the last equation implies is that the asymptotically dominant terms indicating the parity of $\lambda(n)$ are captured up to a constant factor by the terms in the range over k summed by $A_\Omega(t)$ for sufficiently large $t \rightarrow \infty$. In other words, taking the sum over the summands that defines $A_\Omega(x)$ only over the truncated range of $k \in [1, \log \log x]$ does not non-trivially change the limiting asymptotically dominant terms in the lower bound obtained from using this form of the summatory function in conjunction with the Abel summation formula. This property holds even when we should technically index over all $k \in [1, \log_2(x)]$ to obtain an exact formula for the summatory weight function. By Corollary 6.5, we have that the assertion above holds as $t \rightarrow \infty$. \square

Corollary 9.4. *We have that for almost every sufficiently large x , that “on average” as $x \rightarrow \infty$*

$$|G_E^{-1}(x)| \gg \sqrt{\frac{2}{\pi}} \frac{A_0}{3e \log 2} \cdot \frac{\log \log x}{(\log \log \log \log x)^{\frac{5}{2} + \log \log \log \log x}} \times \left| \sum_{e < d \leq \log \log \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d \cdot (\log \log d)} \right|.$$

Proof. Using the definition in (19), we obtain on average that ^{*F9.B}

$$\begin{aligned} |G_E^{-1}(x)| &= \left| \sum_{n \leq \log \log x} \lambda(n) \times \sum_{\substack{d|n \\ d > e}} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log \log d} \right| \\ &= \left| \sum_{e < d \leq \log \log x} \frac{(\log d)^{\frac{1}{4}}}{\log \log d} \times \sum_{n=1}^{\lfloor \frac{\log x}{d} \rfloor} \lambda(dn) \right|. \end{aligned}$$

We see that by complete additivity of $\Omega(n)$ (multiplicativity of $\lambda(n)$) that

$$\sum_{n=1}^{\lfloor \frac{x}{d} \rfloor} \lambda(dn) = \sum_{n=1}^{\lfloor \frac{x}{d} \rfloor} \lambda(d) \times \lambda(n) = \lambda(d) \times \sum_{n \leq \lfloor \frac{x}{d} \rfloor} \lambda(n).$$

^{*F9.B}For any arithmetic functions f, h , we have that [1, cf. §3.10; §3.12]

$$\sum_{n \leq x} h(n) \times \sum_{d|n} f(d) = \sum_{d \leq x} f(d) \times \sum_{n=1}^{\lfloor \frac{x}{d} \rfloor} h(dn).$$

Now using Theorem 4.8 and Lemma 9.2, we can establish that

$$\begin{aligned} \left| \sum_{n \leq x} \lambda(n) \right| &\gg \left| \sum_{k \leq \log \log x} (-1)^k \cdot \hat{\pi}_k(x) \right| \\ &\gg \sqrt{\frac{2}{\pi}} \cdot \frac{A_0}{3e \log 2} \cdot \frac{x}{(\log \log x)^{\frac{5}{2} + \log \log x}} =: \hat{L}_0(x). \end{aligned} \quad (22)$$

The sign of the sum obtained by taking (22) without an absolute value is given by $(-1)^{\lfloor \log \log x \rfloor}$. The precise formula for the limiting lower bound stated above for $\hat{L}_0(x)$ is computed by symbolic summation in *Mathematica* using the new bounds on $\hat{\pi}_k(x)$ guaranteed by the theorem, and then by applying subsequent standard asymptotic estimates to the resulting formulas for large $x \rightarrow \infty$, e.g., in the form of (8c) and Stirling's formula.

It follows that

$$|G_E^{-1}(x)| \gg \left| \sum_{e < d \leq \log \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log \log d} \times (-1)^{\lfloor \log \log \left(\frac{\log \log x}{d} \right) \rfloor} \cdot \hat{L}_0 \left(\frac{\log \log x}{d} \right) \right|. \quad (23)$$

We will simplify (23) by modifying the limits of summation on d to be bounded by a suitable function of x so that for almost every large x and with the exception of at most finitely many d on this new interval, we obtain a constant sign from the weight on $\hat{L}_0 \left(\frac{\log \log x}{d} \right)$, or equivalently, a constant parity for $\left\lfloor \log \log \left(\frac{\log \log x}{d} \right) \right\rfloor$ at such d . Consider that

$$\log \log \left(\frac{\log \log x}{d} \right) = \log \log \log \log x + \log \left(1 - \frac{\log d}{\log \log \log x \cdot \log \log x} \right), \text{ as } x \rightarrow \infty.$$

If we take $d \in [e, \log \log \log x]$, we have that $\frac{\log d}{\log \log \log x \cdot \log \log x} = o(1)$ as $x \rightarrow \infty$. So for d within this range, we expect that for almost every x there are at most a handful of comparatively small order $d \leq d_0(x)$ such that

$$\left\lfloor \log \log \left(\frac{\log \log x}{d} \right) \right\rfloor \sim \lfloor \log \log \log \log x + o(1) \rfloor,$$

changes parity at $d_0(x)$. To make this argument precise, consider that we have two cases to consider for all sufficiently large x and $\log d \in [1, \log \log \log \log x]$:

(1) If the fractional part $\{\log \log \log \log x\} = 0$, then

$$\left\lfloor \log \log \left(\frac{\log \log x}{d} \right) \right\rfloor = \lfloor \log \log \log \log x \rfloor + \left\lfloor -\frac{\log d}{(\log \log x)(\log \log \log x)} + o \left(\frac{1}{(\log \log x)^2} \right) \right\rfloor.$$

This implies that provided that

$$-1 \leq -\frac{\log d}{(\log \log x)(\log \log \log x)} + o \left(\frac{1}{(\log \log x)^2} \right) < 0,$$

we obtain a constant sign term for $\text{sgn} \left[\hat{L}_0 \left(\frac{\log \log x}{d} \right) \right]$. Since this happens for all sufficiently large x , we obtain the constant sign term.

(2) If the fractional part $\{\log \log \log \log x\} \in (0, 1)$, then

$$\left\lfloor \log \log \left(\frac{\log \log x}{d} \right) \right\rfloor = \lfloor \log \log \log \log x \rfloor + \left\lfloor \{\log \log \log \log x\} - \frac{\log d}{(\log \log x)(\log \log \log x)} + o \left(\frac{1}{(\log \log x)^2} \right) \right\rfloor.$$

Now provided that

$$0 \leq (\log \log x)(\log \log \log x) \{\log \log \log \log x\} - \log d + o(1) < 1,$$

or that

$$-1 \leq (\log \log x)(\log \log \log x) \{\log \log \log \log x\} - \log d + o(1) < 0,$$

we similarly obtain the constant signedness of $\text{sgn} \left[\hat{L}_0 \left(\frac{\log \log x}{d} \right) \right]$ for $d \in [e, \log \log \log x]$ whenever x is sufficiently large.

We do not delve too deeply into the distribution of the fractional parts of $\log \log \log \log x$ in $[0, 1)$ for our purposes here, since a rudimentary analysis by cases argument serves our purposes well. If $\{\log \log \log \log x\} \xrightarrow{x \rightarrow \infty} 0$, then we can proceed with the analysis we used to justify case (1) above as $x \rightarrow \infty$. Otherwise, we can assume that for almost every large enough x , the conditions in case (2) provide us with the necessary constant signedness for $d \in [e, \log \log \log x]$ that we require.

So provided that the sign term involving both d and x from (23) does not change for d within our new interval, we can factor out the dependence of the sign on the monotonically decreasing function $\hat{L}_0(\log \log x/d)$ in the variable d . We can see that the function $\hat{L}_0(\log \log x/d)$ is decreasing on $d \in [e, \log \log \log x]$, as can be viewed by computing its first derivative as

$$\frac{\partial}{\partial d} \left[\left| \hat{L}_0 \left(\frac{\log \log x}{d} \right) \right| \right] = \sqrt{\frac{2}{\pi}} \frac{A_0}{3e \log 2} \cdot \frac{t}{d^2 \log \left(\frac{t}{d} \right)} \times \left[\frac{5 - 2 \log \log \left(\frac{t}{d} \right) (\log \left(\frac{t}{d} \right) - \log \log \log \left(\frac{t}{d} \right) - 1)}{\log \log \left(\frac{t}{d} \right)^{\frac{7}{2} + \log \log \left(\frac{t}{d} \right)}} \right],$$

and observing that the sign on this function is negative for all sufficiently large x for d taken within the larger full interval. So we determine that we should select $d := \log \log \log x$ in (23) to obtain a global lower bound on $|G_E^{-1}(x)|$.

We next show that we can form a lower bound on $G_E^{-1}(x)$ by dropping the contributions from the terms corresponding to the summation indices $d \in (\log \log \log x, (\log \log x)/e)$ from our consideration entirely. Since our analysis of the signed behavior of the function $\hat{L}_0((\log \log x)/d)$ over this upper interval of summation is much less easily determined and so dominated by the analysis of local oscillations depending on both the distribution of $\{\log \log \log \log x\}$ and the choice of d within this range, we choose to simplify our considerations and not consider these upper indexed summation terms at all. We have a couple of arguments to make to show that (i) our bounds make sense as stated when we drop the extraneous terms; and (ii) we cannot arrive at trivial lower bounds that tend to zero as $x \rightarrow \infty$ due to cancellation of the lower bounds based on $d \in [e, \log \log \log x]$ with the signed terms corresponding to the sums over $d > \log \log \log x$ in the original definition of the summatory function $G_E^{-1}(x)$ given in (19).

Let the magnitudes of the oscillatory remainder term sums be defined for all sufficiently large x by

$$R_E(x) := \left| \sum_{\log \log \log x < d < \frac{\log \log x}{e}} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log \log d} \times (-1)^{\lfloor \log \log \left(\frac{\log x}{d} \right) \rfloor} \cdot \hat{L}_0 \left(\frac{\log x}{d} \right) \right|.$$

Next, let the function $T_E(x)$ be defined for all large enough x as

$$T_E(x) := \frac{1}{\log \log \left(\frac{\log \log x}{\log \log \log x} \right)^{\frac{5}{2} + \log \log \left(\frac{\log \log x}{\log \log \log x} \right)}} \gg \frac{1}{(\log \log \log \log x)^{\frac{5}{2} + \log \log \log \log x}}. \quad (24)$$

Then we see that as $x \rightarrow \infty$

$$\begin{aligned} S_{E,1}(x) &:= \left| \sum_{e < d \leq \log \log \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{\log \log d} \times \hat{L}_0 \left(\frac{\log \log x}{d} \right) \right| \\ &\gg \sqrt{\frac{2}{\pi}} \frac{2A_0}{3e \log 2} \times (\log \log x) T_E(x) \times \left| \sum_{e < d \leq \log \log \log x} \frac{\lambda(d)(\log d)^{\frac{1}{4}}}{d \cdot \log \log d} \right| \\ &\gg \sqrt{\frac{2}{\pi}} \frac{2A_0}{3e \log 2} \times (\log \log x) T_E(x) \times \left| A_{\Omega}^{(\ell)}(\log \log \log x) \hat{\tau}_0(\log \log \log x) - \int_e^{\log \log \log x} A_{\Omega}^{(\ell)}(t) \hat{\tau}_0'(t) dt \right|, \end{aligned} \quad (25)$$

where we select the functions $\hat{\tau}_0(t) := \frac{(\log t)^{1/4}}{t \cdot \log \log t}$ and $-\hat{\tau}_0'(t) \gg \frac{(\log t)^{1/4}}{t^2 \cdot \log \log t}$ in the notation of Lemma 9.3.

What we then obtain from (23) and (25) is the following lower bound from the triangle inequality that holds for all sufficiently large x :

$$|G_E^{-1}(x)| \gg \left| S_{E,1}(x) - R_E(x) \right| \gg S_{E,1}(x), \text{ as } x \rightarrow \infty. \quad (26)$$

We have claimed that in fact we can drop the sum terms over upper range of d and still obtain an asymptotic lower bound on $|G_E^{-1}(x)|$ as $x \rightarrow \infty$ on the right-hand-side of (26). To justify this step in the proof, we will provide asymptotic lower bounds on $R_E(x)$ that show that the contribution from these terms in absolute value exceeds the magnitude of the corresponding sums for $d \in [e, \log \log \log x]$ when x is large. In other words, the magnitudes of the terms in the sums defining the difference of the functions in (26) cannot match in approximate order to cause a zero-tending cancellation of the difference of terms when x is large.

In Theorem 9.5 stated below, we prove lower bounds on the sums we used to define $S_{E,1}(x)$ above of the form

$$S_{E,1}(x) \gg \frac{(\log \log x)}{(\log \log \log x)^{\frac{9}{4}} (\log \log \log \log x)^{\frac{7}{2}}} \times \exp(-2(\log \log \log \log x)^2),$$

where the lower bounds on the right-hand-side of the previous equation are clearly $o(\log \log x)$. In contrast, we can bound below to show that $R_E(x)$ is at least on the order of a constant times $\log \log x$. To obtain this lower bound, note that since $\frac{(\log d)^{\frac{1}{4}}}{d \cdot \log \log d}$ is monotone decreasing for all large enough $d > e$, we obtain the smallest possible magnitude on the sum by alternating signs on consecutive terms in the sum. So we have

$$\begin{aligned} R_E(x) &\gg (\log \log x) \times \left| \sum_{\log \log \log x < d < \frac{\log \log x}{2e}} \frac{\log(2d)^{1/4}}{2d \cdot \log \log(2d)} - \frac{\log(2d+1)^{1/4}}{(2d+1) \log \log(2d+1)} \right| \\ &\sim (\log \log x) \times \left| \sum_{\log \log \log x < d < \frac{\log \log x}{2e}} \frac{\log(2d)^{1/4}}{2d \cdot \log \log(2d)} - \frac{1}{(2d+1)} \frac{(\log(2d) + \frac{1}{2d})^{1/4}}{(\log \log(2d) + \frac{1}{2d \cdot \log(2d)})} \right| \\ &\approx (\log \log x) \times \left| \sum_{\log \log \log x < d < \frac{\log \log x}{2e}} \frac{\log(2d)^{1/4}}{\log \log(2d)} \left[\frac{1}{2d} - \frac{\left(1 + \frac{1}{2d \cdot \log(2d)}\right)^{1/4}}{(2d+1) \left(1 + \frac{1}{2d \cdot \log(2d) \log \log(2d)}\right)} \right] \right|. \end{aligned}$$

Then by an appeal to binomial and geometric series expansions, we obtain that the significant terms in the above sum are given by ^{*F9.C}

$$\begin{aligned} \frac{R_E(x)}{\log \log x} &\gg \left| \sum_{\log \log \log x < d < \frac{\log \log x}{2e}} \frac{\log(2d)^{1/4}}{\log \log(2d)} \left[\frac{1}{2d} - \frac{1}{(2d+1)} \left(1 + \frac{1}{8d \cdot \log(2d)} - \frac{1}{2d \cdot \log(2d) \log \log(2d)} \right) \right] \right| \\ &= \left| \sum_{\log \log \log x < d < \frac{\log \log x}{2e}} O\left(\frac{\log(2d)^{1/4}}{2d(2d+1) \cdot \log \log(2d)} \right) \right| \\ &= O(1). \end{aligned}$$

In total, what we obtain is that the magnitude of $R_E(x)$ always exceeds that of the lower bound we establish in Theorem 9.5 for the sums over $d \in [e, \log \log \log x]$. So we obtain the lower bounds on $G_E(x)$ that correspond to the smaller order terms resulting from the first sums above as given by our lower bounds stated in Theorem 9.5 below. \square

^{*F9.C}In particular, with $|2d \log(2d)|^{-1}, |2d \log(2d) \log \log(2d)|^{-1} < 1$ we can compute that

$$\left(1 + \frac{1}{2d \cdot \log(2d)} \right)^{\frac{1}{4}} \times \left(1 + \frac{1}{2d \cdot \log(2d) \cdot \log \log(2d)} \right)^{-1} = 1 + \frac{1}{8d \cdot \log(2d)} - \frac{1}{2d \cdot \log(2d) \cdot \log \log(2d)} + O\left(\frac{1}{d^2} \right).$$

9.1.2 The proof of a central lower bound on the magnitude of $G_E^{-1}(x)$

The next central theorem is the last key barrier required to prove Corollary 4.9 in the next subsection.

Theorem 9.5 (Asymptotics and bounds for the summatory functions $G^{-1}(x)$). *We define a lower summatory function, $G_\ell^{-1}(x)$, to provide bounds on the magnitude of $G_E^{-1}(x)$ such that*

$$|G_\ell^{-1}(x)| \leq |G_E^{-1}(x)|,$$

for all sufficiently large $x \geq e$. Let $C_{\ell,1} > 0$ be the absolute constant defined by

$$C_{\ell,1} = \frac{4A_0^2}{9\pi e^2 \log^2 2} = \frac{128 \cdot 2^{1/8}}{59049 \cdot \pi^2 e^8 \log^8 2} \exp\left(-\frac{55}{2} \log^2 2\right) \approx 2.75594 \times 10^{-12}.$$

We obtain the following limiting estimate for the bounding function $G_\ell^{-1}(x)$ holding as we say “on average” as $x \rightarrow \infty$:

$$|G_\ell^{-1}(x)| \gg \frac{C_{\ell,1} \cdot (\log \log x)}{(\log \log \log x)^{\frac{9}{4}} (\log \log \log \log x)^{\frac{7}{2}}} \times \exp\left(-2(\log \log \log \log x)^2\right).$$

Proof. Recall from our proof of Corollary 4.8 that a lower bound on the variant prime form counting function, $\hat{\pi}_k(x)$, is given by

$$\hat{\pi}_k(x) \gg \frac{A_0 \cdot x}{\log x \cdot (\log \log x)^4 \cdot (k-1)!} \cdot \left(\frac{4}{3e \log 2}\right)^{\frac{k}{\log \log x}} \left(1 + O\left(\frac{k}{(\log \log x)^2}\right)\right), \text{ as } x \rightarrow \infty.$$

So we can then form a lower summatory function indicating the signed contributions over the distinct parity of $\Omega(n)$ for all $n \leq x$ as follows by applying (8b) and Stirling’s approximation:

$$\begin{aligned} |A_\Omega^{(\ell)}(t)| &= \left| \sum_{k \leq \log \log t} (-1)^k \hat{\pi}_k(t) \right| \\ &\gg \sqrt{\frac{2}{\pi}} \cdot \frac{A_0}{3e \log 2} \cdot \frac{t}{(\log \log t)^{\frac{5}{2} + \log \log t}} \left(1 + O\left(\frac{1}{\log \log x}\right)\right) \\ &\gg \sqrt{\frac{2}{\pi}} \cdot \frac{A_0}{3e \log 2} \cdot \frac{t}{(\log \log t)^{\frac{5}{2} + \log \log t}}, \text{ as } t \rightarrow \infty. \end{aligned} \quad (27)$$

The actual sign on this function is given by $\text{sgn}(A_\Omega^{(\ell)}(t)) = (-1)^{|\log \log t|}$ (see Lemma 9.2). By Lemma 9.3 we know that this summatory function forms a lower bound in absolute value for the actual weight of the signed terms indicated by $\lambda(n)$.

As we determined in (25) from the proof of Corollary 9.4, we take the function $\hat{\tau}_0(t) = \frac{(\log t)^{1/4}}{t \cdot \log \log t}$ that satisfies

$$-\hat{\tau}_0'(t) = -\frac{d}{dt} \left[\frac{(\log t)^{1/4}}{t \cdot \log \log t} \right] \gg \frac{(\log t)^{1/4}}{t^2 \cdot \log \log t}.$$

Moreover, we have using the notation from the proof above that we can select the lower bound function $G_\ell^{-1}(x)$ to be defined as follows:

$$G_\ell^{-1}(x) := \sqrt{\frac{2}{\pi}} \frac{2A_0}{3e \log 2} \cdot (\log \log x) \cdot T_E(x) \times \left| A_\Omega^{(\ell)}(\log \log \log x) \hat{\tau}_0(\log \log \log x) - \int_e^{\log \log \log x} A_\Omega^{(\ell)}(t) \hat{\tau}_0'(t) dt \right|. \quad (28)$$

The inner integral term on the rightmost side of (28) is summed approximately by splitting the terms sign weighted by $(-1)^{|\log \log t|}$ in the form of

$$\sqrt{\frac{2}{\pi}} \frac{2A_0}{3e \log 2} \times \left| \int_e^{\log \log \log x} A_\Omega^{(\ell)}(t) \hat{\tau}_0'(t) dt \right| \gg \left| \sum_{k=e+1}^{\frac{1}{2} \log \log \log \log \log x} \left[I_\ell(e^{2k+1}) - I_\ell(e^{2k}) \right] e^{2k} \right| \quad (29)$$

$$\gg \sqrt{\frac{2}{\pi}} \frac{2A_0}{3e \log 2} \times \left| \int_{\frac{\log \log \log \log \log x}{2} - \frac{1}{2}}^{\frac{\log \log \log \log \log x}{2}} I_\ell \left(e^{e^{2k}} \right) e^{e^{2k}} dk \right|.$$

We express the integrand function, $I_\ell(t) := \sqrt{\frac{2}{\pi}} \frac{2A_0}{3e \log 2} \times \hat{\tau}'_0(t) A_\Omega^{(\ell)}(t)$, defined as in the previous equations with some limiting simplifications for the $k \in \left[\frac{\log \log \log \log \log x}{2} - \frac{1}{2}, \frac{\log \log \log \log \log x}{2} \right]$ as

$$I_\ell \left(e^{e^{2k}} \right) e^{e^{2k}} \gg \frac{A_0^2}{9\sqrt{2}\pi e^2 \log^2 2} \cdot \frac{e^{k/2}}{2^{2k} \cdot k^{2k + \frac{7}{2}}} =: \hat{I}_\ell(k). \quad (30)$$

So using the lower bound on the integrand in (30), we find from the mean value theorem that

$$\begin{aligned} & \sqrt{\frac{2}{\pi}} \frac{2A_0}{3e \log 2} \times (\log \log x) \times T_E(x) \times \left| \int_{\frac{\log \log \log \log \log x}{2} - \frac{1}{2}}^{\frac{\log \log \log \log \log x}{2}} I_\ell \left(e^{e^{2k}} \right) e^{e^{2k}} dk \right| \\ & \gg \sqrt{\frac{2}{\pi}} \frac{2A_0}{3e \log 2} \times (\log \log x) \times T_E(x) \times \left| \hat{I}_\ell \left(\frac{\log \log \log \log \log x}{2} \right) \right| \\ & \gg \frac{C_{\ell,1} \cdot (\log \log x)}{(\log \log \log \log x)^{\frac{9}{4} + \log \log \log \log x} (\log \log \log \log \log x)^{\frac{7}{2} + \log \log \log \log \log x}}. \end{aligned} \quad (31)$$

Moreover, by evaluating $\hat{I}_\ell(t)$ at the lower bound on the integral above, we can similarly conclude that

$$\begin{aligned} & \sqrt{\frac{2}{\pi}} \frac{2A_0}{3e \log 2} \times (\log \log x) \times T_E(x) \times \left| \int_{\frac{\log \log \log \log \log x}{2} - \frac{1}{2}}^{\frac{\log \log \log \log \log x}{2}} I_\ell \left(e^{e^{2k}} \right) e^{e^{2k}} dk \right| \\ & \ll \frac{C_{\ell,1} \cdot (\log \log x)}{e^{1/4} \cdot (\log \log \log \log x)^{\frac{9}{4} + \log \log \log \log x} (\log \log \log \log \log x)^{\frac{5}{2} + \log \log \log \log \log x}}. \end{aligned}$$

To make it clear which terms in (28) the limiting lower bounds correspond to, consider the following expansion for the leading term in the Abel summation formula from (28) for comparison with (31):

$$\begin{aligned} & \sqrt{\frac{2}{\pi}} \frac{2A_0}{3e \log 2} \times \log \log x \times T_E(x) \times \left| \hat{\tau}_0(\log \log \log x) A_\Omega^{(\ell)}(\log \log \log x) \right| \\ & \gg \frac{2C_{\ell,1} \cdot (\log \log x)}{(\log \log \log \log x)^{\frac{9}{4} + \log \log \log \log x} (\log \log \log \log \log x)^{\frac{7}{2} + \log \log \log \log \log x}}. \end{aligned} \quad (32)$$

Hence, by Lemma 9.2 and the triangle inequality, we conclude that we can take

$$\begin{aligned} |G_\ell^{-1}(x)| & \gg \frac{C_{\ell,1} \cdot (\log \log x)}{(\log \log \log \log x)^{\frac{9}{4} + \log \log \log \log x} (\log \log \log \log \log x)^{\frac{7}{2} + \log \log \log \log \log x}} \\ & \gg \frac{C_{\ell,1} \cdot (\log \log x)}{(\log \log \log \log x)^{\frac{9}{4}} (\log \log \log \log \log x)^{\frac{7}{2}}} \times \exp \left(-2(\log \log \log \log x)^2 \right). \end{aligned}$$

What is key to observe about these lower bounds is that each of them scaled by $\log x$ monotonically increases without bound as $x \rightarrow \infty$. In particular, the remaining factor after rescaling dominates the asymptotics of the reciprocal powers of iterated logarithms. \square

9.2 Proof of the unboundedness of the scaled Mertens function

We finally address the main conclusion of our arguments given so far with the following proof:

Proof of Corollary 4.9. We break up the integral term in Proposition 9.1 over $t \in [u_0, x/2]$ into two pieces: one that is easily bounded from $u_0 \leq t \leq \sqrt{x}$, and then another that will conveniently give us our slow-growing tendency towards infinity along the subsequence when evaluated using Theorem 9.5. Given a fixed large infinitely tending x , we have some point $x_0 \in [\sqrt{x}, \frac{x}{2}]$ defined such that

$$|G^{-1}(x_0)| := \inf_{\sqrt{x} \leq t \leq \frac{x}{2}} |G^{-1}(t)|,$$

We can then apply Proposition 9.1 to bound

$$\begin{aligned} \frac{|M(x)|}{\sqrt{x}} &= \frac{1}{\sqrt{x}} \left| G^{-1}(x) - x \cdot \int_1^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt \right| \\ &\gg \left| \sqrt{x} \times \int_{\sqrt{x}}^{x/2} \frac{G^{-1}(t)}{t^2 \cdot \log(x/t)} dt \right| \\ &\gg \left| \int_{\sqrt{x_0}}^{\frac{x}{2}} \frac{2\sqrt{x_0}}{t^2 \cdot \log(x_0)} dt \right| \times \left(\min_{\sqrt{x} \leq t \leq \frac{x}{2}} |G^{-1}(t)| \right) \\ &\gg \frac{2|G^{-1}(x_0)|}{\log(x_0)}. \end{aligned} \tag{33}$$

Then by Theorem 8.5 proved in Section 8, (33) implies that

$$\frac{|M(x)|}{\sqrt{x}} \gg \frac{2|G_E^{-1}(x_0)|}{\log(x_0)}. \tag{34}$$

Define the infinite increasing subsequence, $\{x_{0,y}\}_{y \geq Y_0}$, by $x_{0,y} := e^{2e^{e^{e^{2y}}}}$ for some sufficiently large finite integer $Y_0 \gg 1$. When we assume that $x \mapsto x_{0,y}$ is taken along the subsequence defined in the corollary statement given in the introduction, we can transform the bound in the last equation into a statement about a lower bound for $|M(x)| \log x / \sqrt{x}$ along an infinitely tending subsequence in the following form by applying Theorem 9.5 to (34):

$$\frac{|M(x_{0,y})| \log(x_{0,y})}{\sqrt{x_{0,y}}} \gg \frac{2C_{\ell,1} \cdot (\log \log \sqrt{x_{0,y}}) \times \exp(-2(\log \log \log \log \sqrt{x_{0,y}})^2)}{(\log \log \log \log \sqrt{x_{0,y}})^{\frac{9}{4}} (\log \log \log \log \log \sqrt{x_{0,y}})^{\frac{7}{2}}}, \text{ as } y \rightarrow \infty. \tag{35}$$

Notice by a small technicality in stating (35), we are not actually asserting that $|M(x)| \log x / \sqrt{x}$ grows unbounded along the precise subsequence of $x \mapsto x_{0,y}$. Rather, we are asserting that the unboundedness of this function can be witnessed along some subsequence $\hat{x}_{0,y} \in [\sqrt{x_{0,y}}, \frac{x_{0,y}}{2}]$, where we conveniently choose to state the lower bound given on the right-hand-side of (35) using the monotonicity of the lower bound on $|G_E^{-1}(x)|$ we proved in Theorem 9.5. We also can verify that for sufficiently large $y \rightarrow \infty$, this infinitely tending subsequence is well defined as $\hat{x}_{0,y+1} > \hat{x}_{0,y}$ for all sufficiently large $y \geq Y_0$.

Finally, since

$$\exp(-2(\log \log \log \log x)^2) \ll \frac{1}{(\log \log \log \log x)^4}, \text{ as } x \rightarrow \infty,$$

we can compute that

$$\lim_{x \rightarrow \infty} \left[\frac{(\log \log x)}{(\log \log \log \log x)^{\frac{9}{4}} (\log \log \log \log \log x)^{\frac{7}{2}}} \times \exp(-2(\log \log \log \log x)^2) \right] = +\infty.$$

The scaled Mertens function is then unbounded in the limit supremum sense, as we have claimed, since the right-hand-side of (35) tends to positive infinity as $x_{0,y} \rightarrow \infty$, or equivalently as $y \rightarrow \infty$. \square

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T.1 Table: The Dirichlet inverse function $g^{-1}(n)$ and the distribution of its summatory function

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum_{d n} C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|----------------------------------------------|--------|--------|-------------|--------------------------------------|---------------------------------------------------|--------------------|--------------------|-------------|---------------|---------------|
| 1 | 1 ¹ | Y | N | 1 | 0 | 1.000000 | 1.00000 | 0 | 1 | 1 | 0 |
| 2 | 2 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.500000 | 0.500000 | -1 | 1 | -2 |
| 3 | 3 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.333333 | 0.666667 | -3 | 1 | -4 |
| 4 | 2 ² | N | Y | 2 | 0 | 1.500000 | 0.500000 | 0.500000 | -1 | 3 | -4 |
| 5 | 5 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.400000 | 0.600000 | -3 | 3 | -6 |
| 6 | 2 ¹ 3 ¹ | Y | N | 5 | 0 | 1.000000 | 0.500000 | 0.500000 | 2 | 8 | -6 |
| 7 | 7 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.428571 | 0.571429 | 0 | 8 | -8 |
| 8 | 2 ³ | N | Y | -2 | 0 | 2.000000 | 0.375000 | 0.625000 | -2 | 8 | -10 |
| 9 | 3 ² | N | Y | 2 | 0 | 1.500000 | 0.444444 | 0.555556 | 0 | 10 | -10 |
| 10 | 2 ¹ 5 ¹ | Y | N | 5 | 0 | 1.000000 | 0.500000 | 0.500000 | 5 | 15 | -10 |
| 11 | 11 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.454545 | 0.545455 | 3 | 15 | -12 |
| 12 | 2 ² 3 ¹ | N | N | -7 | 2 | 1.2857143 | 0.416667 | 0.583333 | -4 | 15 | -19 |
| 13 | 13 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.384615 | 0.615385 | -6 | 15 | -21 |
| 14 | 2 ¹ 7 ¹ | Y | N | 5 | 0 | 1.000000 | 0.428571 | 0.571429 | -1 | 20 | -21 |
| 15 | 3 ¹ 5 ¹ | Y | N | 5 | 0 | 1.000000 | 0.466667 | 0.533333 | 4 | 25 | -21 |
| 16 | 2 ⁴ | N | Y | 2 | 0 | 2.500000 | 0.500000 | 0.500000 | 6 | 27 | -21 |
| 17 | 17 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.470588 | 0.529412 | 4 | 27 | -23 |
| 18 | 2 ¹ 3 ² | N | N | -7 | 2 | 1.2857143 | 0.444444 | 0.555556 | -3 | 27 | -30 |
| 19 | 19 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.421053 | 0.578947 | -5 | 27 | -32 |
| 20 | 2 ² 5 ¹ | N | N | -7 | 2 | 1.2857143 | 0.400000 | 0.600000 | -12 | 27 | -39 |
| 21 | 3 ¹ 7 ¹ | Y | N | 5 | 0 | 1.000000 | 0.428571 | 0.571429 | -7 | 32 | -39 |
| 22 | 2 ¹ 11 ¹ | Y | N | 5 | 0 | 1.000000 | 0.454545 | 0.545455 | -2 | 37 | -39 |
| 23 | 23 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.434783 | 0.565217 | -4 | 37 | -41 |
| 24 | 2 ³ 3 ¹ | N | N | 9 | 4 | 1.5555556 | 0.458333 | 0.541667 | 5 | 46 | -41 |
| 25 | 5 ² | N | Y | 2 | 0 | 1.500000 | 0.480000 | 0.520000 | 7 | 48 | -41 |
| 26 | 2 ¹ 13 ¹ | Y | N | 5 | 0 | 1.000000 | 0.500000 | 0.500000 | 12 | 53 | -41 |
| 27 | 3 ³ | N | Y | -2 | 0 | 2.000000 | 0.481481 | 0.518519 | 10 | 53 | -43 |
| 28 | 2 ² 7 ¹ | N | N | -7 | 2 | 1.2857143 | 0.464286 | 0.535714 | 3 | 53 | -50 |
| 29 | 29 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.448276 | 0.551724 | 1 | 53 | -52 |
| 30 | 2 ¹ 3 ¹ 5 ¹ | Y | N | -16 | 0 | 1.000000 | 0.433333 | 0.566667 | -15 | 53 | -68 |
| 31 | 31 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.419355 | 0.580645 | -17 | 53 | -70 |
| 32 | 2 ⁵ | N | Y | -2 | 0 | 3.000000 | 0.406250 | 0.593750 | -19 | 53 | -72 |
| 33 | 3 ¹ 11 ¹ | Y | N | 5 | 0 | 1.000000 | 0.424242 | 0.575758 | -14 | 58 | -72 |
| 34 | 2 ¹ 17 ¹ | Y | N | 5 | 0 | 1.000000 | 0.441176 | 0.558824 | -9 | 63 | -72 |
| 35 | 5 ¹ 7 ¹ | Y | N | 5 | 0 | 1.000000 | 0.457143 | 0.542857 | -4 | 68 | -72 |
| 36 | 2 ² 3 ² | N | N | 14 | 9 | 1.3571429 | 0.472222 | 0.527778 | 10 | 82 | -72 |
| 37 | 37 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.459459 | 0.540541 | 8 | 82 | -74 |
| 38 | 2 ¹ 19 ¹ | Y | N | 5 | 0 | 1.000000 | 0.473684 | 0.526316 | 13 | 87 | -74 |
| 39 | 3 ¹ 13 ¹ | Y | N | 5 | 0 | 1.000000 | 0.487179 | 0.512821 | 18 | 92 | -74 |
| 40 | 2 ³ 5 ¹ | N | N | 9 | 4 | 1.5555556 | 0.500000 | 0.500000 | 27 | 101 | -74 |
| 41 | 41 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.487805 | 0.512195 | 25 | 101 | -76 |
| 42 | 2 ¹ 3 ¹ 7 ¹ | Y | N | -16 | 0 | 1.000000 | 0.476190 | 0.523810 | 9 | 101 | -92 |
| 43 | 43 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.465116 | 0.534884 | 7 | 101 | -94 |
| 44 | 2 ² 11 ¹ | N | N | -7 | 2 | 1.2857143 | 0.454545 | 0.545455 | 0 | 101 | -101 |
| 45 | 3 ² 5 ¹ | N | N | -7 | 2 | 1.2857143 | 0.444444 | 0.555556 | -7 | 101 | -108 |
| 46 | 2 ¹ 23 ¹ | Y | N | 5 | 0 | 1.000000 | 0.456522 | 0.543478 | -2 | 106 | -108 |
| 47 | 47 ¹ | Y | Y | -2 | 0 | 1.000000 | 0.446809 | 0.553191 | -4 | 106 | -110 |
| 48 | 2 ⁴ 3 ¹ | N | N | -11 | 6 | 1.8181818 | 0.437500 | 0.562500 | -15 | 106 | -121 |

Table T.1: Computations with $g^{-1}(n) \equiv (\omega + 1)^{-1}(n)$ for $1 \leq n \leq 500$.

- The column labeled **Primes** provides the prime factorization of each n so that the values of $\omega(n)$ and $\Omega(n)$ are easily extracted. The columns labeled **Sqfree** and **PPower**, respectively, list inclusion of n in the sets of squarefree integers and the prime powers.
- The next three columns provide the explicit values of the inverse function $g^{-1}(n)$ and compare its explicit value with other estimates. We define the function $\hat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.
- The last several columns indicate properties of the summatory function of $g^{-1}(n)$. The notation for the densities of the sign weight of $g^{-1}(n)$ is defined as $\mathcal{L}_{\pm}(x) := \frac{1}{x} \cdot \#\{n \leq x : \lambda(n) = \pm 1\}$. The last three columns then show the explicit components to the signed summatory function, $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$, decomposed into its respective positive and negative magnitude sum contributions: $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ where $G_+^{-1}(x) > 0$ and $G_-^{-1}(x) < 0$ for all $x \geq 1$.

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum_{d n} C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|----------------|--------|--------|-------------|--------------------------------------|---------------------------------------------------|--------------------|--------------------|-------------|---------------|---------------|
| 49 | 7^2 | N | Y | 2 | 0 | 1.5000000 | 0.448980 | 0.551020 | -13 | 108 | -121 |
| 50 | $2^1 5^2$ | N | Y | -7 | 2 | 1.2857143 | 0.440000 | 0.560000 | -20 | 108 | -128 |
| 51 | $3^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.450980 | 0.549020 | -15 | 113 | -128 |
| 52 | $2^2 13^1$ | N | N | -7 | 2 | 1.2857143 | 0.442308 | 0.557692 | -22 | 113 | -135 |
| 53 | 53^1 | Y | Y | -2 | 0 | 1.0000000 | 0.433962 | 0.566038 | -24 | 113 | -137 |
| 54 | $2^1 3^3$ | N | N | 9 | 4 | 1.5555556 | 0.444444 | 0.555556 | -15 | 122 | -137 |
| 55 | $5^1 11^1$ | Y | N | 5 | 0 | 1.0000000 | 0.454545 | 0.545455 | -10 | 127 | -137 |
| 56 | $2^3 7^1$ | N | N | 9 | 4 | 1.5555556 | 0.464286 | 0.535714 | -1 | 136 | -137 |
| 57 | $3^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473684 | 0.526316 | 4 | 141 | -137 |
| 58 | $2^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482759 | 0.517241 | 9 | 146 | -137 |
| 59 | 59^1 | Y | Y | -2 | 0 | 1.0000000 | 0.474576 | 0.525424 | 7 | 146 | -139 |
| 60 | $2^2 3^1 5^1$ | N | N | 30 | 14 | 1.1666667 | 0.483333 | 0.516667 | 37 | 176 | -139 |
| 61 | 61^1 | Y | Y | -2 | 0 | 1.0000000 | 0.475410 | 0.524590 | 35 | 176 | -141 |
| 62 | $2^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 40 | 181 | -141 |
| 63 | $3^2 7^1$ | N | N | -7 | 2 | 1.2857143 | 0.476190 | 0.523810 | 33 | 181 | -148 |
| 64 | 2^6 | N | Y | 2 | 0 | 3.5000000 | 0.484375 | 0.515625 | 35 | 183 | -148 |
| 65 | $5^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492308 | 0.507692 | 40 | 188 | -148 |
| 66 | $2^1 3^1 11^1$ | Y | N | -16 | 0 | 1.0000000 | 0.484848 | 0.515152 | 24 | 188 | -164 |
| 67 | 67^1 | Y | Y | -2 | 0 | 1.0000000 | 0.477612 | 0.522388 | 22 | 188 | -166 |
| 68 | $2^2 17^1$ | N | N | -7 | 2 | 1.2857143 | 0.470588 | 0.529412 | 15 | 188 | -173 |
| 69 | $3^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478261 | 0.521739 | 20 | 193 | -173 |
| 70 | $2^1 5^1 7^1$ | Y | N | -16 | 0 | 1.0000000 | 0.471429 | 0.528571 | 4 | 193 | -189 |
| 71 | 71^1 | Y | Y | -2 | 0 | 1.0000000 | 0.464789 | 0.535211 | 2 | 193 | -191 |
| 72 | $2^3 3^2$ | N | N | -23 | 18 | 1.4782609 | 0.458333 | 0.541667 | -21 | 193 | -214 |
| 73 | 73^1 | Y | Y | -2 | 0 | 1.0000000 | 0.452055 | 0.547945 | -23 | 193 | -216 |
| 74 | $2^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.459459 | 0.540541 | -18 | 198 | -216 |
| 75 | $3^1 5^2$ | N | N | -7 | 2 | 1.2857143 | 0.453333 | 0.546667 | -25 | 198 | -223 |
| 76 | $2^2 19^1$ | N | N | -7 | 2 | 1.2857143 | 0.447368 | 0.552632 | -32 | 198 | -230 |
| 77 | $7^1 11^1$ | Y | N | 5 | 0 | 1.0000000 | 0.454545 | 0.545455 | -27 | 203 | -230 |
| 78 | $2^1 3^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.448718 | 0.551282 | -43 | 203 | -246 |
| 79 | 79^1 | Y | Y | -2 | 0 | 1.0000000 | 0.443038 | 0.556962 | -45 | 203 | -248 |
| 80 | $2^4 5^1$ | N | N | -11 | 6 | 1.8181818 | 0.437500 | 0.562500 | -56 | 203 | -259 |
| 81 | 3^4 | N | Y | 2 | 0 | 2.5000000 | 0.444444 | 0.555556 | -54 | 205 | -259 |
| 82 | $2^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.451220 | 0.548780 | -49 | 210 | -259 |
| 83 | 83^1 | Y | Y | -2 | 0 | 1.0000000 | 0.445783 | 0.554217 | -51 | 210 | -261 |
| 84 | $2^2 3^1 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.452381 | 0.547619 | -21 | 240 | -261 |
| 85 | $5^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.458824 | 0.541176 | -16 | 245 | -261 |
| 86 | $2^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.465116 | 0.534884 | -11 | 250 | -261 |
| 87 | $3^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.471264 | 0.528736 | -6 | 255 | -261 |
| 88 | $2^3 11^1$ | N | N | 9 | 4 | 1.5555556 | 0.477273 | 0.522727 | 3 | 264 | -261 |
| 89 | 89^1 | Y | Y | -2 | 0 | 1.0000000 | 0.471910 | 0.528090 | 1 | 264 | -263 |
| 90 | $2^1 3^2 5^1$ | N | N | 30 | 14 | 1.1666667 | 0.477778 | 0.522222 | 31 | 294 | -263 |
| 91 | $7^1 13^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483516 | 0.516484 | 36 | 299 | -263 |
| 92 | $2^2 23^1$ | N | N | -7 | 2 | 1.2857143 | 0.478261 | 0.521739 | 29 | 299 | -270 |
| 93 | $3^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 34 | 304 | -270 |
| 94 | $2^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.489362 | 0.510638 | 39 | 309 | -270 |
| 95 | $5^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.494737 | 0.505263 | 44 | 314 | -270 |
| 96 | $2^5 3^1$ | N | N | 13 | 8 | 2.0769231 | 0.500000 | 0.500000 | 57 | 327 | -270 |
| 97 | 97^1 | Y | Y | -2 | 0 | 1.0000000 | 0.494845 | 0.505155 | 55 | 327 | -272 |
| 98 | $2^1 7^2$ | N | N | -7 | 2 | 1.2857143 | 0.489796 | 0.510204 | 48 | 327 | -279 |
| 99 | $3^2 11^1$ | N | N | -7 | 2 | 1.2857143 | 0.484848 | 0.515152 | 41 | 327 | -286 |
| 100 | $2^2 5^2$ | N | N | 14 | 9 | 1.3571429 | 0.490000 | 0.510000 | 55 | 341 | -286 |
| 101 | 101^1 | Y | Y | -2 | 0 | 1.0000000 | 0.485149 | 0.514851 | 53 | 341 | -288 |
| 102 | $2^1 3^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.480392 | 0.519608 | 37 | 341 | -304 |
| 103 | 103^1 | Y | Y | -2 | 0 | 1.0000000 | 0.475728 | 0.524272 | 35 | 341 | -306 |
| 104 | $2^3 13^1$ | N | N | 9 | 4 | 1.5555556 | 0.480769 | 0.519231 | 44 | 350 | -306 |
| 105 | $3^1 5^1 7^1$ | Y | N | -16 | 0 | 1.0000000 | 0.476190 | 0.523810 | 28 | 350 | -322 |
| 106 | $2^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.481132 | 0.518868 | 33 | 355 | -322 |
| 107 | 107^1 | Y | Y | -2 | 0 | 1.0000000 | 0.476636 | 0.523364 | 31 | 355 | -324 |
| 108 | $2^2 3^3$ | N | N | -23 | 18 | 1.4782609 | 0.472222 | 0.527778 | 8 | 355 | -347 |
| 109 | 109^1 | Y | Y | -2 | 0 | 1.0000000 | 0.467890 | 0.532110 | 6 | 355 | -349 |
| 110 | $2^1 5^1 11^1$ | Y | N | -16 | 0 | 1.0000000 | 0.463636 | 0.536364 | -10 | 355 | -365 |
| 111 | $3^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.468468 | 0.531532 | -5 | 360 | -365 |
| 112 | $2^4 7^1$ | N | N | -11 | 6 | 1.8181818 | 0.464286 | 0.535714 | -16 | 360 | -376 |
| 113 | 113^1 | Y | Y | -2 | 0 | 1.0000000 | 0.460177 | 0.539823 | -18 | 360 | -378 |
| 114 | $2^1 3^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.456140 | 0.543860 | -34 | 360 | -394 |
| 115 | $5^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.460870 | 0.539130 | -29 | 365 | -394 |
| 116 | $2^2 29^1$ | N | N | -7 | 2 | 1.2857143 | 0.456897 | 0.543103 | -36 | 365 | -401 |
| 117 | $3^2 13^1$ | N | N | -7 | 2 | 1.2857143 | 0.452991 | 0.547009 | -43 | 365 | -408 |
| 118 | $2^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.457627 | 0.542373 | -38 | 370 | -408 |
| 119 | $7^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.462185 | 0.537815 | -33 | 375 | -408 |
| 120 | $2^3 3^1 5^1$ | N | N | -48 | 32 | 1.3333333 | 0.458333 | 0.541667 | -81 | 375 | -456 |
| 121 | 11^2 | N | Y | 2 | 0 | 1.5000000 | 0.462810 | 0.537190 | -79 | 377 | -456 |
| 122 | $2^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467213 | 0.532787 | -74 | 382 | -456 |
| 123 | $3^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.471545 | 0.528455 | -69 | 387 | -456 |
| 124 | $2^2 31^1$ | N | N | -7 | 2 | 1.2857143 | 0.467742 | 0.532258 | -76 | 387 | -463 |

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum_{d n} C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|-----------------------------------------------|--------|--------|-------------|--------------------------------------|---------------------------------------------------|--------------------|--------------------|-------------|---------------|---------------|
| 125 | 5 ³ | N | Y | -2 | 0 | 2.0000000 | 0.464000 | 0.536000 | -78 | 387 | -465 |
| 126 | 2 ¹ 3 ² 7 ¹ | N | N | 30 | 14 | 1.1666667 | 0.468254 | 0.531746 | -48 | 417 | -465 |
| 127 | 127 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.464567 | 0.535433 | -50 | 417 | -467 |
| 128 | 2 ⁷ | N | Y | -2 | 0 | 4.0000000 | 0.460938 | 0.539062 | -52 | 417 | -469 |
| 129 | 3 ¹ 43 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.465116 | 0.534884 | -47 | 422 | -469 |
| 130 | 2 ¹ 5 ¹ 13 ¹ | Y | N | -16 | 0 | 1.0000000 | 0.461538 | 0.538462 | -63 | 422 | -485 |
| 131 | 131 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.458015 | 0.541985 | -65 | 422 | -487 |
| 132 | 2 ² 3 ¹ 11 ¹ | N | N | 30 | 14 | 1.1666667 | 0.462121 | 0.537879 | -35 | 452 | -487 |
| 133 | 7 ¹ 19 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.466165 | 0.533835 | -30 | 457 | -487 |
| 134 | 2 ¹ 67 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.470149 | 0.529851 | -25 | 462 | -487 |
| 135 | 3 ³ 5 ¹ | N | N | 9 | 4 | 1.5555556 | 0.474074 | 0.525926 | -16 | 471 | -487 |
| 136 | 2 ³ 17 ¹ | N | N | 9 | 4 | 1.5555556 | 0.477941 | 0.522059 | -7 | 480 | -487 |
| 137 | 137 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.474453 | 0.525547 | -9 | 480 | -489 |
| 138 | 2 ¹ 3 ¹ 23 ¹ | Y | N | -16 | 0 | 1.0000000 | 0.471014 | 0.528986 | -25 | 480 | -505 |
| 139 | 139 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.467626 | 0.532374 | -27 | 480 | -507 |
| 140 | 2 ² 5 ¹ 7 ¹ | N | N | 30 | 14 | 1.1666667 | 0.471429 | 0.528571 | 3 | 510 | -507 |
| 141 | 3 ¹ 47 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.475177 | 0.524823 | 8 | 515 | -507 |
| 142 | 2 ¹ 71 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.478873 | 0.521127 | 13 | 520 | -507 |
| 143 | 11 ¹ 13 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.482517 | 0.517483 | 18 | 525 | -507 |
| 144 | 2 ⁴ 3 ² | N | N | 34 | 29 | 1.6176471 | 0.486111 | 0.513889 | 52 | 559 | -507 |
| 145 | 5 ¹ 29 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.489655 | 0.510345 | 57 | 564 | -507 |
| 146 | 2 ¹ 73 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.493151 | 0.506849 | 62 | 569 | -507 |
| 147 | 3 ¹ 7 ² | N | N | -7 | 2 | 1.2857143 | 0.489796 | 0.510204 | 55 | 569 | -514 |
| 148 | 2 ² 37 ¹ | N | N | -7 | 2 | 1.2857143 | 0.486486 | 0.513514 | 48 | 569 | -521 |
| 149 | 149 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.483221 | 0.516779 | 46 | 569 | -523 |
| 150 | 2 ¹ 3 ¹ 5 ² | N | N | 30 | 14 | 1.1666667 | 0.486667 | 0.513333 | 76 | 599 | -523 |
| 151 | 151 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.483444 | 0.516556 | 74 | 599 | -525 |
| 152 | 2 ³ 19 ¹ | N | N | 9 | 4 | 1.5555556 | 0.486842 | 0.513158 | 83 | 608 | -525 |
| 153 | 3 ² 17 ¹ | N | N | -7 | 2 | 1.2857143 | 0.483660 | 0.516340 | 76 | 608 | -532 |
| 154 | 2 ¹ 7 ¹ 11 ¹ | Y | N | -16 | 0 | 1.0000000 | 0.480519 | 0.519481 | 60 | 608 | -548 |
| 155 | 5 ¹ 31 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 65 | 613 | -548 |
| 156 | 2 ² 3 ¹ 13 ¹ | N | N | 30 | 14 | 1.1666667 | 0.487179 | 0.512821 | 95 | 643 | -548 |
| 157 | 157 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.484076 | 0.515924 | 93 | 643 | -550 |
| 158 | 2 ¹ 79 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.487342 | 0.512658 | 98 | 648 | -550 |
| 159 | 3 ¹ 53 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.490566 | 0.509434 | 103 | 653 | -550 |
| 160 | 2 ⁵ 5 ¹ | N | N | 13 | 8 | 2.0769231 | 0.493750 | 0.506250 | 116 | 666 | -550 |
| 161 | 7 ¹ 23 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.496894 | 0.503106 | 121 | 671 | -550 |
| 162 | 2 ¹ 3 ⁴ | N | N | -11 | 6 | 1.8181818 | 0.493827 | 0.506173 | 110 | 671 | -561 |
| 163 | 163 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.490798 | 0.509202 | 108 | 671 | -563 |
| 164 | 2 ² 41 ¹ | N | N | -7 | 2 | 1.2857143 | 0.487805 | 0.512195 | 101 | 671 | -570 |
| 165 | 3 ¹ 5 ¹ 11 ¹ | Y | N | -16 | 0 | 1.0000000 | 0.484848 | 0.515152 | 85 | 671 | -586 |
| 166 | 2 ¹ 83 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.487952 | 0.512048 | 90 | 676 | -586 |
| 167 | 167 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.485030 | 0.514970 | 88 | 676 | -588 |
| 168 | 2 ³ 3 ¹ 7 ¹ | N | N | -48 | 32 | 1.3333333 | 0.482143 | 0.517857 | 40 | 676 | -636 |
| 169 | 13 ² | N | Y | 2 | 0 | 1.5000000 | 0.485207 | 0.514793 | 42 | 678 | -636 |
| 170 | 2 ¹ 5 ¹ 17 ¹ | Y | N | -16 | 0 | 1.0000000 | 0.482353 | 0.517647 | 26 | 678 | -652 |
| 171 | 3 ² 19 ¹ | N | N | -7 | 2 | 1.2857143 | 0.479532 | 0.520468 | 19 | 678 | -659 |
| 172 | 2 ² 43 ¹ | N | N | -7 | 2 | 1.2857143 | 0.476744 | 0.523256 | 12 | 678 | -666 |
| 173 | 173 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.473988 | 0.526012 | 10 | 678 | -668 |
| 174 | 2 ¹ 3 ¹ 29 ¹ | Y | N | -16 | 0 | 1.0000000 | 0.471264 | 0.528736 | -6 | 678 | -684 |
| 175 | 5 ² 7 ¹ | N | N | -7 | 2 | 1.2857143 | 0.468571 | 0.531429 | -13 | 678 | -691 |
| 176 | 2 ⁴ 11 ¹ | N | N | -11 | 6 | 1.8181818 | 0.465909 | 0.534091 | -24 | 678 | -702 |
| 177 | 3 ¹ 59 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.468927 | 0.531073 | -19 | 683 | -702 |
| 178 | 2 ¹ 89 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.471910 | 0.528090 | -14 | 688 | -702 |
| 179 | 179 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.469274 | 0.530726 | -16 | 688 | -704 |
| 180 | 2 ² 3 ² 5 ¹ | N | N | -74 | 58 | 1.2162162 | 0.466667 | 0.533333 | -90 | 688 | -778 |
| 181 | 181 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.464088 | 0.535912 | -92 | 688 | -780 |
| 182 | 2 ¹ 7 ¹ 13 ¹ | Y | N | -16 | 0 | 1.0000000 | 0.461538 | 0.538462 | -108 | 688 | -796 |
| 183 | 3 ¹ 61 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.464481 | 0.535519 | -103 | 693 | -796 |
| 184 | 2 ³ 23 ¹ | N | N | 9 | 4 | 1.5555556 | 0.467391 | 0.532609 | -94 | 702 | -796 |
| 185 | 5 ¹ 37 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.470270 | 0.529730 | -89 | 707 | -796 |
| 186 | 2 ¹ 3 ¹ 31 ¹ | Y | N | -16 | 0 | 1.0000000 | 0.467742 | 0.532258 | -105 | 707 | -812 |
| 187 | 11 ¹ 17 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.470588 | 0.529412 | -100 | 712 | -812 |
| 188 | 2 ² 47 ¹ | N | N | -7 | 2 | 1.2857143 | 0.468085 | 0.531915 | -107 | 712 | -819 |
| 189 | 3 ³ 7 ¹ | N | N | 9 | 4 | 1.5555556 | 0.470899 | 0.529101 | -98 | 721 | -819 |
| 190 | 2 ¹ 5 ¹ 19 ¹ | Y | N | -16 | 0 | 1.0000000 | 0.468421 | 0.531579 | -114 | 721 | -835 |
| 191 | 191 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.465969 | 0.534031 | -116 | 721 | -837 |
| 192 | 2 ⁶ 3 ¹ | N | N | -15 | 10 | 2.3333333 | 0.463542 | 0.536458 | -131 | 721 | -852 |
| 193 | 193 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.461140 | 0.538860 | -133 | 721 | -854 |
| 194 | 2 ¹ 97 ¹ | Y | N | 5 | 0 | 1.0000000 | 0.463918 | 0.536082 | -128 | 726 | -854 |
| 195 | 3 ¹ 5 ¹ 13 ¹ | Y | N | -16 | 0 | 1.0000000 | 0.461538 | 0.538462 | -144 | 726 | -870 |
| 196 | 2 ² 7 ² | N | N | 14 | 9 | 1.3571429 | 0.464286 | 0.535714 | -130 | 740 | -870 |
| 197 | 197 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.461929 | 0.538071 | -132 | 740 | -872 |
| 198 | 2 ¹ 3 ² 11 ¹ | N | N | 30 | 14 | 1.1666667 | 0.464646 | 0.535354 | -102 | 770 | -872 |
| 199 | 199 ¹ | Y | Y | -2 | 0 | 1.0000000 | 0.462312 | 0.537688 | -104 | 770 | -874 |
| 200 | 2 ³ 5 ² | N | N | -23 | 18 | 1.4782609 | 0.460000 | 0.540000 | -127 | 770 | -897 |

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum_{d n} C_{\Omega(d)}(d)}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|-------------------|--------|--------|-------------|--------------------------------------|---------------------------------------------------|--------------------|--------------------|-------------|---------------|---------------|
| 201 | $3^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.462687 | 0.537313 | -122 | 775 | -897 |
| 202 | $2^1 101^1$ | Y | N | 5 | 0 | 1.0000000 | 0.465347 | 0.534653 | -117 | 780 | -897 |
| 203 | $7^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467980 | 0.532020 | -112 | 785 | -897 |
| 204 | $2^2 3^1 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.470588 | 0.529412 | -82 | 815 | -897 |
| 205 | $5^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473171 | 0.526829 | -77 | 820 | -897 |
| 206 | $2^1 103^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475728 | 0.524272 | -72 | 825 | -897 |
| 207 | $3^2 23^1$ | N | N | -7 | 2 | 1.2857143 | 0.473430 | 0.526570 | -79 | 825 | -904 |
| 208 | $2^4 13^1$ | N | N | -11 | 6 | 1.8181818 | 0.471154 | 0.528846 | -90 | 825 | -915 |
| 209 | $11^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473684 | 0.526316 | -85 | 830 | -915 |
| 210 | $2^1 3^1 5^1 7^1$ | Y | N | 65 | 0 | 1.0000000 | 0.476190 | 0.523810 | -20 | 895 | -915 |
| 211 | 211^1 | Y | Y | -2 | 0 | 1.0000000 | 0.473934 | 0.526066 | -22 | 895 | -917 |
| 212 | $2^2 53^1$ | N | N | -7 | 2 | 1.2857143 | 0.471698 | 0.528302 | -29 | 895 | -924 |
| 213 | $3^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.474178 | 0.525822 | -24 | 900 | -924 |
| 214 | $2^1 107^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476636 | 0.523364 | -19 | 905 | -924 |
| 215 | $5^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.479070 | 0.520930 | -14 | 910 | -924 |
| 216 | $2^3 3^3$ | N | N | 46 | 41 | 1.5000000 | 0.481481 | 0.518519 | 32 | 956 | -924 |
| 217 | $7^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483871 | 0.516129 | 37 | 961 | -924 |
| 218 | $2^1 109^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486239 | 0.513761 | 42 | 966 | -924 |
| 219 | $3^1 73^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488584 | 0.511416 | 47 | 971 | -924 |
| 220 | $2^2 5^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.490909 | 0.509091 | 77 | 1001 | -924 |
| 221 | $13^1 17^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493213 | 0.506787 | 82 | 1006 | -924 |
| 222 | $2^1 3^1 37^1$ | Y | N | -16 | 0 | 1.0000000 | 0.490991 | 0.509009 | 66 | 1006 | -940 |
| 223 | 223^1 | Y | Y | -2 | 0 | 1.0000000 | 0.488789 | 0.511211 | 64 | 1006 | -942 |
| 224 | $2^5 7^1$ | N | N | 13 | 8 | 2.0769231 | 0.491071 | 0.508929 | 77 | 1019 | -942 |
| 225 | $3^2 5^2$ | N | N | 14 | 9 | 1.3571429 | 0.493333 | 0.506667 | 91 | 1033 | -942 |
| 226 | $2^1 113^1$ | Y | N | 5 | 0 | 1.0000000 | 0.495575 | 0.504425 | 96 | 1038 | -942 |
| 227 | 227^1 | Y | Y | -2 | 0 | 1.0000000 | 0.493392 | 0.506608 | 94 | 1038 | -944 |
| 228 | $2^2 3^1 19^1$ | N | N | 30 | 14 | 1.1666667 | 0.495614 | 0.504386 | 124 | 1068 | -944 |
| 229 | 229^1 | Y | Y | -2 | 0 | 1.0000000 | 0.493450 | 0.506550 | 122 | 1068 | -946 |
| 230 | $2^1 5^1 23^1$ | Y | N | -16 | 0 | 1.0000000 | 0.491304 | 0.508696 | 106 | 1068 | -962 |
| 231 | $3^1 7^1 11^1$ | Y | N | -16 | 0 | 1.0000000 | 0.489177 | 0.510823 | 90 | 1068 | -978 |
| 232 | $2^3 29^1$ | N | N | 9 | 4 | 1.5555556 | 0.491379 | 0.508621 | 99 | 1077 | -978 |
| 233 | 233^1 | Y | Y | -2 | 0 | 1.0000000 | 0.489270 | 0.510730 | 97 | 1077 | -980 |
| 234 | $2^1 3^2 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.491453 | 0.508547 | 127 | 1107 | -980 |
| 235 | $5^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493617 | 0.506383 | 132 | 1112 | -980 |
| 236 | $2^2 59^1$ | N | N | -7 | 2 | 1.2857143 | 0.491525 | 0.508475 | 125 | 1112 | -987 |
| 237 | $3^1 79^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493671 | 0.506329 | 130 | 1117 | -987 |
| 238 | $2^1 7^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.491597 | 0.508403 | 114 | 1117 | -1003 |
| 239 | 239^1 | Y | Y | -2 | 0 | 1.0000000 | 0.489540 | 0.510460 | 112 | 1117 | -1005 |
| 240 | $2^4 3^1 5^1$ | N | N | 70 | 54 | 1.5000000 | 0.491667 | 0.508333 | 182 | 1187 | -1005 |
| 241 | 241^1 | Y | Y | -2 | 0 | 1.0000000 | 0.489627 | 0.510373 | 180 | 1187 | -1007 |
| 242 | $2^1 11^2$ | N | N | -7 | 2 | 1.2857143 | 0.487603 | 0.512397 | 173 | 1187 | -1014 |
| 243 | 3^5 | N | Y | -2 | 0 | 3.0000000 | 0.485597 | 0.514403 | 171 | 1187 | -1016 |
| 244 | $2^2 61^1$ | N | N | -7 | 2 | 1.2857143 | 0.483607 | 0.516393 | 164 | 1187 | -1023 |
| 245 | $5^1 7^2$ | N | N | -7 | 2 | 1.2857143 | 0.481633 | 0.518367 | 157 | 1187 | -1030 |
| 246 | $2^1 3^1 41^1$ | Y | N | -16 | 0 | 1.0000000 | 0.479675 | 0.520325 | 141 | 1187 | -1046 |
| 247 | $13^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.481781 | 0.518219 | 146 | 1192 | -1046 |
| 248 | $2^3 31^1$ | N | N | 9 | 4 | 1.5555556 | 0.483871 | 0.516129 | 155 | 1201 | -1046 |
| 249 | $3^1 83^1$ | Y | N | 5 | 0 | 1.0000000 | 0.485944 | 0.514056 | 160 | 1206 | -1046 |
| 250 | $2^1 5^3$ | N | N | 9 | 4 | 1.5555556 | 0.488000 | 0.512000 | 169 | 1215 | -1046 |
| 251 | 251^1 | Y | Y | -2 | 0 | 1.0000000 | 0.486056 | 0.513944 | 167 | 1215 | -1048 |
| 252 | $2^2 3^2 7^1$ | N | N | -74 | 58 | 1.2162162 | 0.484127 | 0.515873 | 93 | 1215 | -1122 |
| 253 | $11^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486166 | 0.513834 | 98 | 1220 | -1122 |
| 254 | $2^1 127^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488189 | 0.511811 | 103 | 1225 | -1122 |
| 255 | $3^1 5^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.486275 | 0.513725 | 87 | 1225 | -1138 |
| 256 | 2^8 | N | Y | 2 | 0 | 4.5000000 | 0.488281 | 0.511719 | 89 | 1227 | -1138 |
| 257 | 257^1 | Y | Y | -2 | 0 | 1.0000000 | 0.486381 | 0.513619 | 87 | 1227 | -1140 |
| 258 | $2^1 3^1 43^1$ | Y | N | -16 | 0 | 1.0000000 | 0.484496 | 0.515504 | 71 | 1227 | -1156 |
| 259 | $7^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486486 | 0.513514 | 76 | 1232 | -1156 |
| 260 | $2^2 5^1 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.488462 | 0.511538 | 106 | 1262 | -1156 |
| 261 | $3^2 29^1$ | N | N | -7 | 2 | 1.2857143 | 0.486590 | 0.513410 | 99 | 1262 | -1163 |
| 262 | $2^1 131^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488550 | 0.511450 | 104 | 1267 | -1163 |
| 263 | 263^1 | Y | Y | -2 | 0 | 1.0000000 | 0.486692 | 0.513308 | 102 | 1267 | -1165 |
| 264 | $2^3 3^1 11^1$ | N | N | -48 | 32 | 1.3333333 | 0.484848 | 0.515152 | 54 | 1267 | -1213 |
| 265 | $5^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486792 | 0.513208 | 59 | 1272 | -1213 |
| 266 | $2^1 7^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.484962 | 0.515038 | 43 | 1272 | -1229 |
| 267 | $3^1 89^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486891 | 0.513109 | 48 | 1277 | -1229 |
| 268 | $2^2 67^1$ | N | N | -7 | 2 | 1.2857143 | 0.485075 | 0.514925 | 41 | 1277 | -1236 |
| 269 | 269^1 | Y | Y | -2 | 0 | 1.0000000 | 0.483271 | 0.516729 | 39 | 1277 | -1238 |
| 270 | $2^1 3^3 5^1$ | N | N | -48 | 32 | 1.3333333 | 0.481481 | 0.518519 | -9 | 1277 | -1286 |
| 271 | 271^1 | Y | Y | -2 | 0 | 1.0000000 | 0.479705 | 0.520295 | -11 | 1277 | -1288 |
| 272 | $2^4 17^1$ | N | N | -11 | 6 | 1.8181818 | 0.477941 | 0.522059 | -22 | 1277 | -1299 |
| 273 | $3^1 7^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.476190 | 0.523810 | -38 | 1277 | -1315 |
| 274 | $2^1 137^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478102 | 0.521898 | -33 | 1282 | -1315 |
| 275 | $5^2 11^1$ | N | N | -7 | 2 | 1.2857143 | 0.476364 | 0.523636 | -40 | 1282 | -1322 |
| 276 | $2^2 3^1 23^1$ | N | N | 30 | 14 | 1.1666667 | 0.478261 | 0.521739 | -10 | 1312 | -1322 |
| 277 | 277^1 | Y | Y | -2 | 0 | 1.0000000 | 0.476534 | 0.523466 | -12 | 1312 | -1324 |

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum_{d n} C_{\Omega(d)}^{(d)}}{ g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|--------------------|--------|--------|-------------|--------------------------------------|------------------------------------------------------|--------------------|--------------------|-------------|---------------|---------------|
| 278 | $2^1 139^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478417 | 0.521583 | -7 | 1317 | -1324 |
| 279 | $3^2 31^1$ | N | N | -7 | 2 | 1.2857143 | 0.476703 | 0.523297 | -14 | 1317 | -1331 |
| 280 | $2^3 5^1 7^1$ | N | N | -48 | 32 | 1.3333333 | 0.475000 | 0.525000 | -62 | 1317 | -1379 |
| 281 | 281^1 | Y | Y | -2 | 0 | 1.0000000 | 0.473310 | 0.526690 | -64 | 1317 | -1381 |
| 282 | $2^1 3^1 47^1$ | Y | N | -16 | 0 | 1.0000000 | 0.471631 | 0.528369 | -80 | 1317 | -1397 |
| 283 | 283^1 | Y | Y | -2 | 0 | 1.0000000 | 0.469965 | 0.530035 | -82 | 1317 | -1399 |
| 284 | $2^2 71^1$ | N | N | -7 | 2 | 1.2857143 | 0.468310 | 0.531690 | -89 | 1317 | -1406 |
| 285 | $3^1 5^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.466667 | 0.533333 | -105 | 1317 | -1422 |
| 286 | $2^1 11^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.465035 | 0.534965 | -121 | 1317 | -1438 |
| 287 | $7^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.466899 | 0.533101 | -116 | 1322 | -1438 |
| 288 | $2^5 3^2$ | N | N | -47 | 42 | 1.7659574 | 0.465278 | 0.534722 | -163 | 1322 | -1485 |
| 289 | 17^2 | N | Y | 2 | 0 | 1.5000000 | 0.467128 | 0.532872 | -161 | 1324 | -1485 |
| 290 | $2^1 5^1 29^1$ | Y | N | -16 | 0 | 1.0000000 | 0.465517 | 0.534483 | -177 | 1324 | -1501 |
| 291 | $3^1 97^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467354 | 0.532646 | -172 | 1329 | -1501 |
| 292 | $2^2 73^1$ | N | N | -7 | 2 | 1.2857143 | 0.465753 | 0.534247 | -179 | 1329 | -1508 |
| 293 | 293^1 | Y | Y | -2 | 0 | 1.0000000 | 0.464164 | 0.535836 | -181 | 1329 | -1510 |
| 294 | $2^1 3^1 7^2$ | N | N | 30 | 14 | 1.1666667 | 0.465986 | 0.534014 | -151 | 1359 | -1510 |
| 295 | $5^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.467797 | 0.532203 | -146 | 1364 | -1510 |
| 296 | $2^3 37^1$ | N | N | 9 | 4 | 1.5555556 | 0.469595 | 0.530405 | -137 | 1373 | -1510 |
| 297 | $3^3 11^1$ | N | N | 9 | 4 | 1.5555556 | 0.471380 | 0.528620 | -128 | 1382 | -1510 |
| 298 | $2^1 149^1$ | Y | N | 5 | 0 | 1.0000000 | 0.473154 | 0.526846 | -123 | 1387 | -1510 |
| 299 | $13^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.474916 | 0.525084 | -118 | 1392 | -1510 |
| 300 | $2^2 3^1 5^2$ | N | N | -74 | 58 | 1.2162162 | 0.473333 | 0.526667 | -192 | 1392 | -1584 |
| 301 | $7^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475083 | 0.524917 | -187 | 1397 | -1584 |
| 302 | $2^1 151^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476821 | 0.523179 | -182 | 1402 | -1584 |
| 303 | $3^1 101^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478548 | 0.521452 | -177 | 1407 | -1584 |
| 304 | $2^4 19^1$ | N | N | -11 | 6 | 1.8181818 | 0.476974 | 0.523026 | -188 | 1407 | -1595 |
| 305 | $5^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478689 | 0.521311 | -183 | 1412 | -1595 |
| 306 | $2^1 3^2 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.480392 | 0.519608 | -153 | 1442 | -1595 |
| 307 | 307^1 | Y | Y | -2 | 0 | 1.0000000 | 0.478827 | 0.521173 | -155 | 1442 | -1597 |
| 308 | $2^2 7^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.480519 | 0.519481 | -125 | 1472 | -1597 |
| 309 | $3^1 103^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482201 | 0.517799 | -120 | 1477 | -1597 |
| 310 | $2^1 5^1 31^1$ | Y | N | -16 | 0 | 1.0000000 | 0.480645 | 0.519355 | -136 | 1477 | -1613 |
| 311 | 311^1 | Y | Y | -2 | 0 | 1.0000000 | 0.479100 | 0.520900 | -138 | 1477 | -1615 |
| 312 | $2^3 3^1 13^1$ | N | N | -48 | 32 | 1.3333333 | 0.477564 | 0.522436 | -186 | 1477 | -1663 |
| 313 | 313^1 | Y | Y | -2 | 0 | 1.0000000 | 0.476038 | 0.523962 | -188 | 1477 | -1665 |
| 314 | $2^1 157^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477707 | 0.522293 | -183 | 1482 | -1665 |
| 315 | $3^2 5^1 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.479365 | 0.520635 | -153 | 1512 | -1665 |
| 316 | $2^2 79^1$ | N | N | -7 | 2 | 1.2857143 | 0.477848 | 0.522152 | -160 | 1512 | -1672 |
| 317 | 317^1 | Y | Y | -2 | 0 | 1.0000000 | 0.476341 | 0.523659 | -162 | 1512 | -1674 |
| 318 | $2^1 3^1 53^1$ | Y | N | -16 | 0 | 1.0000000 | 0.474843 | 0.525157 | -178 | 1512 | -1690 |
| 319 | $11^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476489 | 0.523511 | -173 | 1517 | -1690 |
| 320 | $2^6 5^1$ | N | N | -15 | 10 | 2.3333333 | 0.475000 | 0.525000 | -188 | 1517 | -1705 |
| 321 | $3^1 107^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476636 | 0.523364 | -183 | 1522 | -1705 |
| 322 | $2^1 7^1 23^1$ | Y | N | -16 | 0 | 1.0000000 | 0.475155 | 0.524845 | -199 | 1522 | -1721 |
| 323 | $17^1 19^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476780 | 0.523220 | -194 | 1527 | -1721 |
| 324 | $2^2 3^4$ | N | N | 34 | 29 | 1.6176471 | 0.478395 | 0.521605 | -160 | 1561 | -1721 |
| 325 | $5^2 13^1$ | N | N | -7 | 2 | 1.2857143 | 0.476923 | 0.523077 | -167 | 1561 | -1728 |
| 326 | $2^1 163^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478528 | 0.521472 | -162 | 1566 | -1728 |
| 327 | $3^1 109^1$ | Y | N | 5 | 0 | 1.0000000 | 0.480122 | 0.519878 | -157 | 1571 | -1728 |
| 328 | $2^3 41^1$ | N | N | 9 | 4 | 1.5555556 | 0.481707 | 0.518293 | -148 | 1580 | -1728 |
| 329 | $7^1 47^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483283 | 0.516717 | -143 | 1585 | -1728 |
| 330 | $2^1 3^1 5^1 11^1$ | Y | N | 65 | 0 | 1.0000000 | 0.484848 | 0.515152 | -78 | 1650 | -1728 |
| 331 | 331^1 | Y | Y | -2 | 0 | 1.0000000 | 0.483384 | 0.516616 | -80 | 1650 | -1730 |
| 332 | $2^2 83^1$ | N | N | -7 | 2 | 1.2857143 | 0.481928 | 0.518072 | -87 | 1650 | -1737 |
| 333 | $3^2 37^1$ | N | N | -7 | 2 | 1.2857143 | 0.480480 | 0.519520 | -94 | 1650 | -1744 |
| 334 | $2^1 167^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482036 | 0.517964 | -89 | 1655 | -1744 |
| 335 | $5^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483582 | 0.516418 | -84 | 1660 | -1744 |
| 336 | $2^4 3^1 7^1$ | N | N | 70 | 54 | 1.5000000 | 0.485119 | 0.514881 | -14 | 1730 | -1744 |
| 337 | 337^1 | Y | Y | -2 | 0 | 1.0000000 | 0.483680 | 0.516320 | -16 | 1730 | -1746 |
| 338 | $2^1 13^2$ | N | N | -7 | 2 | 1.2857143 | 0.482249 | 0.517751 | -23 | 1730 | -1753 |
| 339 | $3^1 113^1$ | Y | N | 5 | 0 | 1.0000000 | 0.483776 | 0.516224 | -18 | 1735 | -1753 |
| 340 | $2^2 5^1 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.485294 | 0.514706 | 12 | 1765 | -1753 |
| 341 | $11^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486804 | 0.513196 | 17 | 1770 | -1753 |
| 342 | $2^1 3^2 19^1$ | N | N | 30 | 14 | 1.1666667 | 0.488304 | 0.511696 | 47 | 1800 | -1753 |
| 343 | 7^3 | N | Y | -2 | 0 | 2.0000000 | 0.486880 | 0.513120 | 45 | 1800 | -1755 |
| 344 | $2^3 43^1$ | N | N | 9 | 4 | 1.5555556 | 0.488372 | 0.511628 | 54 | 1809 | -1755 |
| 345 | $3^1 5^1 23^1$ | Y | N | -16 | 0 | 1.0000000 | 0.486957 | 0.513043 | 38 | 1809 | -1771 |
| 346 | $2^1 173^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488439 | 0.511561 | 43 | 1814 | -1771 |
| 347 | 347^1 | Y | Y | -2 | 0 | 1.0000000 | 0.487032 | 0.512968 | 41 | 1814 | -1773 |
| 348 | $2^2 3^1 29^1$ | N | N | 30 | 14 | 1.1666667 | 0.488506 | 0.511494 | 71 | 1844 | -1773 |
| 349 | 349^1 | Y | Y | -2 | 0 | 1.0000000 | 0.487106 | 0.512894 | 69 | 1844 | -1775 |
| 350 | $2^1 5^2 7^1$ | N | N | 30 | 14 | 1.1666667 | 0.488571 | 0.511429 | 99 | 1874 | -1775 |

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum C_{\Omega(d)}(d)}{d n}$ $ g^{-1}(n) $ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|--------------------|--------|--------|-------------|--------------------------------------|------------------------------------------------------|--------------------|--------------------|-------------|---------------|---------------|
| 351 | $3^3 13^1$ | N | N | 9 | 4 | 1.5555556 | 0.490028 | 0.509972 | 108 | 1883 | -1775 |
| 352 | $2^5 11^1$ | N | N | 13 | 8 | 2.0769231 | 0.491477 | 0.508523 | 121 | 1896 | -1775 |
| 353 | 353^1 | Y | Y | -2 | 0 | 1.0000000 | 0.490085 | 0.509915 | 119 | 1896 | -1777 |
| 354 | $2^1 3^1 59^1$ | Y | N | -16 | 0 | 1.0000000 | 0.488701 | 0.511299 | 103 | 1896 | -1793 |
| 355 | $5^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490141 | 0.509859 | 108 | 1901 | -1793 |
| 356 | $2^2 89^1$ | N | N | -7 | 2 | 1.2857143 | 0.488764 | 0.511236 | 101 | 1901 | -1800 |
| 357 | $3^1 7^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.487395 | 0.512605 | 85 | 1901 | -1816 |
| 358 | $2^1 179^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488827 | 0.511173 | 90 | 1906 | -1816 |
| 359 | 359^1 | Y | Y | -2 | 0 | 1.0000000 | 0.487465 | 0.512535 | 88 | 1906 | -1818 |
| 360 | $2^3 3^2 5^1$ | N | N | 145 | 129 | 1.3034483 | 0.488889 | 0.511111 | 233 | 2051 | -1818 |
| 361 | 19^2 | N | Y | 2 | 0 | 1.5000000 | 0.490305 | 0.509695 | 235 | 2053 | -1818 |
| 362 | $2^1 181^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491713 | 0.508287 | 240 | 2058 | -1818 |
| 363 | $3^1 11^2$ | N | N | -7 | 2 | 1.2857143 | 0.490358 | 0.509642 | 233 | 2058 | -1825 |
| 364 | $2^2 7^1 13^1$ | N | N | 30 | 14 | 1.1666667 | 0.491758 | 0.508242 | 263 | 2088 | -1825 |
| 365 | $5^1 73^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493151 | 0.506849 | 268 | 2093 | -1825 |
| 366 | $2^1 3^1 61^1$ | Y | N | -16 | 0 | 1.0000000 | 0.491803 | 0.508197 | 252 | 2093 | -1841 |
| 367 | 367^1 | Y | Y | -2 | 0 | 1.0000000 | 0.490463 | 0.509537 | 250 | 2093 | -1843 |
| 368 | $2^4 23^1$ | N | N | -11 | 6 | 1.8181818 | 0.489130 | 0.510870 | 239 | 2093 | -1854 |
| 369 | $3^2 41^1$ | N | N | -7 | 2 | 1.2857143 | 0.487805 | 0.512195 | 232 | 2093 | -1861 |
| 370 | $2^1 5^1 37^1$ | Y | N | -16 | 0 | 1.0000000 | 0.486486 | 0.513514 | 216 | 2093 | -1877 |
| 371 | $7^1 53^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487871 | 0.512129 | 221 | 2098 | -1877 |
| 372 | $2^2 3^1 31^1$ | N | N | 30 | 14 | 1.1666667 | 0.489247 | 0.510753 | 251 | 2128 | -1877 |
| 373 | 373^1 | Y | Y | -2 | 0 | 1.0000000 | 0.487936 | 0.512064 | 249 | 2128 | -1879 |
| 374 | $2^1 11^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.486631 | 0.513369 | 233 | 2128 | -1895 |
| 375 | $3^1 5^3$ | N | N | 9 | 4 | 1.5555556 | 0.488000 | 0.512000 | 242 | 2137 | -1895 |
| 376 | $2^3 47^1$ | N | N | 9 | 4 | 1.5555556 | 0.489362 | 0.510638 | 251 | 2146 | -1895 |
| 377 | $13^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490716 | 0.509284 | 256 | 2151 | -1895 |
| 378 | $2^1 3^3 7^1$ | N | N | -48 | 32 | 1.3333333 | 0.489418 | 0.510582 | 208 | 2151 | -1943 |
| 379 | 379^1 | Y | Y | -2 | 0 | 1.0000000 | 0.488127 | 0.511873 | 206 | 2151 | -1945 |
| 380 | $2^2 5^1 19^1$ | N | N | 30 | 14 | 1.1666667 | 0.489474 | 0.510526 | 236 | 2181 | -1945 |
| 381 | $3^1 127^1$ | Y | N | 5 | 0 | 1.0000000 | 0.490814 | 0.509186 | 241 | 2186 | -1945 |
| 382 | $2^1 191^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492147 | 0.507853 | 246 | 2191 | -1945 |
| 383 | 383^1 | Y | Y | -2 | 0 | 1.0000000 | 0.490862 | 0.509138 | 244 | 2191 | -1947 |
| 384 | $2^7 3^1$ | N | N | 17 | 12 | 2.5882353 | 0.492188 | 0.507812 | 261 | 2208 | -1947 |
| 385 | $5^1 7^1 11^1$ | Y | N | -16 | 0 | 1.0000000 | 0.490909 | 0.509091 | 245 | 2208 | -1963 |
| 386 | $2^1 193^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492228 | 0.507772 | 250 | 2213 | -1963 |
| 387 | $3^2 43^1$ | N | N | -7 | 2 | 1.2857143 | 0.490956 | 0.509044 | 243 | 2213 | -1970 |
| 388 | $2^2 97^1$ | N | N | -7 | 2 | 1.2857143 | 0.489691 | 0.510309 | 236 | 2213 | -1977 |
| 389 | 389^1 | Y | Y | -2 | 0 | 1.0000000 | 0.488432 | 0.511568 | 234 | 2213 | -1979 |
| 390 | $2^1 3^1 5^1 13^1$ | Y | N | 65 | 0 | 1.0000000 | 0.489744 | 0.510256 | 299 | 2278 | -1979 |
| 391 | $17^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491049 | 0.508951 | 304 | 2283 | -1979 |
| 392 | $2^3 7^2$ | N | N | -23 | 18 | 1.4782609 | 0.489796 | 0.510204 | 281 | 2283 | -2002 |
| 393 | $3^1 131^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491094 | 0.508906 | 286 | 2288 | -2002 |
| 394 | $2^1 197^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492386 | 0.507614 | 291 | 2293 | -2002 |
| 395 | $5^1 79^1$ | Y | N | 5 | 0 | 1.0000000 | 0.493671 | 0.506329 | 296 | 2298 | -2002 |
| 396 | $2^2 3^2 11^1$ | N | N | -74 | 58 | 1.2162162 | 0.492424 | 0.507576 | 222 | 2298 | -2076 |
| 397 | 397^1 | Y | Y | -2 | 0 | 1.0000000 | 0.491184 | 0.508816 | 220 | 2298 | -2078 |
| 398 | $2^1 199^1$ | Y | N | 5 | 0 | 1.0000000 | 0.492462 | 0.507538 | 225 | 2303 | -2078 |
| 399 | $3^1 7^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.491228 | 0.508772 | 209 | 2303 | -2094 |
| 400 | $2^4 5^2$ | N | N | 34 | 29 | 1.6176471 | 0.492500 | 0.507500 | 243 | 2337 | -2094 |
| 401 | 401^1 | Y | Y | -2 | 0 | 1.0000000 | 0.491272 | 0.508728 | 241 | 2337 | -2096 |
| 402 | $2^1 3^1 67^1$ | Y | N | -16 | 0 | 1.0000000 | 0.490050 | 0.509950 | 225 | 2337 | -2112 |
| 403 | $13^1 31^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491315 | 0.508685 | 230 | 2342 | -2112 |
| 404 | $2^2 101^1$ | N | N | -7 | 2 | 1.2857143 | 0.490099 | 0.509901 | 223 | 2342 | -2119 |
| 405 | $3^4 5^1$ | N | N | -11 | 6 | 1.8181818 | 0.488889 | 0.511111 | 212 | 2342 | -2130 |
| 406 | $2^1 7^1 29^1$ | Y | N | -16 | 0 | 1.0000000 | 0.487685 | 0.512315 | 196 | 2342 | -2146 |
| 407 | $11^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488943 | 0.511057 | 201 | 2347 | -2146 |
| 408 | $2^3 3^1 17^1$ | N | N | -48 | 32 | 1.3333333 | 0.487745 | 0.512255 | 153 | 2347 | -2194 |
| 409 | 409^1 | Y | Y | -2 | 0 | 1.0000000 | 0.486553 | 0.513447 | 151 | 2347 | -2196 |
| 410 | $2^1 5^1 41^1$ | Y | N | -16 | 0 | 1.0000000 | 0.485366 | 0.514634 | 135 | 2347 | -2212 |
| 411 | $3^1 137^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486618 | 0.513382 | 140 | 2352 | -2212 |
| 412 | $2^2 103^1$ | N | N | -7 | 2 | 1.2857143 | 0.485437 | 0.514563 | 133 | 2352 | -2219 |
| 413 | $7^1 59^1$ | Y | N | 5 | 0 | 1.0000000 | 0.486683 | 0.513317 | 138 | 2357 | -2219 |
| 414 | $2^1 3^2 23^1$ | N | N | 30 | 14 | 1.1666667 | 0.487923 | 0.512077 | 168 | 2387 | -2219 |
| 415 | $5^1 83^1$ | Y | N | 5 | 0 | 1.0000000 | 0.489157 | 0.510843 | 173 | 2392 | -2219 |
| 416 | $2^5 13^1$ | N | N | 13 | 8 | 2.0769231 | 0.490385 | 0.509615 | 186 | 2405 | -2219 |
| 417 | $3^1 139^1$ | Y | N | 5 | 0 | 1.0000000 | 0.491607 | 0.508393 | 191 | 2410 | -2219 |
| 418 | $2^1 11^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.490431 | 0.509569 | 175 | 2410 | -2235 |
| 419 | 419^1 | Y | Y | -2 | 0 | 1.0000000 | 0.489260 | 0.510740 | 173 | 2410 | -2237 |
| 420 | $2^2 3^1 5^1 7^1$ | N | N | -155 | 90 | 1.1032258 | 0.488095 | 0.511905 | 18 | 2410 | -2392 |
| 421 | 421^1 | Y | Y | -2 | 0 | 1.0000000 | 0.486936 | 0.513064 | 16 | 2410 | -2394 |
| 422 | $2^1 211^1$ | Y | N | 5 | 0 | 1.0000000 | 0.488152 | 0.511848 | 21 | 2415 | -2394 |
| 423 | $3^2 47^1$ | N | N | -7 | 2 | 1.2857143 | 0.486998 | 0.513002 | 14 | 2415 | -2401 |
| 424 | $2^3 53^1$ | N | N | 9 | 4 | 1.5555556 | 0.488208 | 0.511792 | 23 | 2424 | -2401 |
| 425 | $5^2 17^1$ | N | N | -7 | 2 | 1.2857143 | 0.487059 | 0.512941 | 16 | 2424 | -2408 |

| n | Primes | Sqfree | PPower | $g^{-1}(n)$ | $\lambda(n)g^{-1}(n) - \hat{f}_1(n)$ | $\frac{\sum C_{\Omega(d)}(d)}{d n g^{-1}(n) }$ | $\mathcal{L}_+(n)$ | $\mathcal{L}_-(n)$ | $G^{-1}(n)$ | $G_+^{-1}(n)$ | $G_-^{-1}(n)$ |
|-----|--------------------|--------|--------|-------------|--------------------------------------|------------------------------------------------|--------------------|--------------------|-------------|---------------|---------------|
| 426 | $2^1 3^1 71^1$ | Y | N | -16 | 0 | 1.0000000 | 0.485915 | 0.514085 | 0 | 2424 | -2424 |
| 427 | $7^1 61^1$ | Y | N | 5 | 0 | 1.0000000 | 0.487119 | 0.512881 | 5 | 2429 | -2424 |
| 428 | $2^2 107^1$ | N | N | -7 | 2 | 1.2857143 | 0.485981 | 0.514019 | -2 | 2429 | -2431 |
| 429 | $3^1 11^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.484848 | 0.515152 | -18 | 2429 | -2447 |
| 430 | $2^1 5^1 43^1$ | Y | N | -16 | 0 | 1.0000000 | 0.483721 | 0.516279 | -34 | 2429 | -2463 |
| 431 | 431^1 | Y | Y | -2 | 0 | 1.0000000 | 0.482599 | 0.517401 | -36 | 2429 | -2465 |
| 432 | $2^4 3^3$ | N | N | -80 | 75 | 1.5625000 | 0.481481 | 0.518519 | -116 | 2429 | -2545 |
| 433 | 433^1 | Y | Y | -2 | 0 | 1.0000000 | 0.480370 | 0.519630 | -118 | 2429 | -2547 |
| 434 | $2^1 7^1 31^1$ | Y | N | -16 | 0 | 1.0000000 | 0.479263 | 0.520737 | -134 | 2429 | -2563 |
| 435 | $3^1 5^1 29^1$ | Y | N | -16 | 0 | 1.0000000 | 0.478161 | 0.521839 | -150 | 2429 | -2579 |
| 436 | $2^2 109^1$ | N | N | -7 | 2 | 1.2857143 | 0.477064 | 0.522936 | -157 | 2429 | -2586 |
| 437 | $19^1 23^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478261 | 0.521739 | -152 | 2434 | -2586 |
| 438 | $2^1 3^1 73^1$ | Y | N | -16 | 0 | 1.0000000 | 0.477169 | 0.522831 | -168 | 2434 | -2602 |
| 439 | 439^1 | Y | Y | -2 | 0 | 1.0000000 | 0.476082 | 0.523918 | -170 | 2434 | -2604 |
| 440 | $2^3 5^1 11^1$ | N | N | -48 | 32 | 1.3333333 | 0.475000 | 0.525000 | -218 | 2434 | -2652 |
| 441 | $3^2 7^2$ | N | N | 14 | 9 | 1.3571429 | 0.476190 | 0.523810 | -204 | 2448 | -2652 |
| 442 | $2^1 13^1 17^1$ | Y | N | -16 | 0 | 1.0000000 | 0.475113 | 0.524887 | -220 | 2448 | -2668 |
| 443 | 443^1 | Y | Y | -2 | 0 | 1.0000000 | 0.474041 | 0.525959 | -222 | 2448 | -2670 |
| 444 | $2^2 3^1 37^1$ | N | N | 30 | 14 | 1.1666667 | 0.475225 | 0.524775 | -192 | 2478 | -2670 |
| 445 | $5^1 89^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476404 | 0.523596 | -187 | 2483 | -2670 |
| 446 | $2^1 223^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477578 | 0.522422 | -182 | 2488 | -2670 |
| 447 | $3^1 149^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478747 | 0.521253 | -177 | 2493 | -2670 |
| 448 | $2^6 7^1$ | N | N | -15 | 10 | 2.3333333 | 0.477679 | 0.522321 | -192 | 2493 | -2685 |
| 449 | 449^1 | Y | Y | -2 | 0 | 1.0000000 | 0.476615 | 0.523385 | -194 | 2493 | -2687 |
| 450 | $2^1 3^2 5^2$ | N | N | -74 | 58 | 1.2162162 | 0.475556 | 0.524444 | -268 | 2493 | -2761 |
| 451 | $11^1 41^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476718 | 0.523282 | -263 | 2498 | -2761 |
| 452 | $2^2 113^1$ | N | N | -7 | 2 | 1.2857143 | 0.475664 | 0.524336 | -270 | 2498 | -2768 |
| 453 | $3^1 151^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476821 | 0.523179 | -265 | 2503 | -2768 |
| 454 | $2^1 227^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477974 | 0.522026 | -260 | 2508 | -2768 |
| 455 | $5^1 7^1 13^1$ | Y | N | -16 | 0 | 1.0000000 | 0.476923 | 0.523077 | -276 | 2508 | -2784 |
| 456 | $2^3 3^1 19^1$ | N | N | -48 | 32 | 1.3333333 | 0.475877 | 0.524123 | -324 | 2508 | -2832 |
| 457 | 457^1 | Y | Y | -2 | 0 | 1.0000000 | 0.474836 | 0.525164 | -326 | 2508 | -2834 |
| 458 | $2^1 229^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475983 | 0.524017 | -321 | 2513 | -2834 |
| 459 | $3^3 17^1$ | N | N | 9 | 4 | 1.5555556 | 0.477124 | 0.522876 | -312 | 2522 | -2834 |
| 460 | $2^2 5^1 23^1$ | N | N | 30 | 14 | 1.1666667 | 0.478261 | 0.521739 | -282 | 2552 | -2834 |
| 461 | 461^1 | Y | Y | -2 | 0 | 1.0000000 | 0.477223 | 0.522777 | -284 | 2552 | -2836 |
| 462 | $2^1 3^1 7^1 11^1$ | Y | N | 65 | 0 | 1.0000000 | 0.478355 | 0.521645 | -219 | 2617 | -2836 |
| 463 | 463^1 | Y | Y | -2 | 0 | 1.0000000 | 0.477322 | 0.522678 | -221 | 2617 | -2838 |
| 464 | $2^4 29^1$ | Y | N | -11 | 6 | 1.8181818 | 0.476293 | 0.523707 | -232 | 2617 | -2849 |
| 465 | $3^1 5^1 31^1$ | Y | N | -16 | 0 | 1.0000000 | 0.475269 | 0.524731 | -248 | 2617 | -2865 |
| 466 | $2^1 233^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476395 | 0.523605 | -243 | 2622 | -2865 |
| 467 | 467^1 | Y | Y | -2 | 0 | 1.0000000 | 0.475375 | 0.524625 | -245 | 2622 | -2867 |
| 468 | $2^2 3^2 13^1$ | N | N | -74 | 58 | 1.2162162 | 0.474359 | 0.525641 | -319 | 2622 | -2941 |
| 469 | $7^1 67^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475480 | 0.524520 | -314 | 2627 | -2941 |
| 470 | $2^1 5^1 47^1$ | Y | N | -16 | 0 | 1.0000000 | 0.474468 | 0.525532 | -330 | 2627 | -2957 |
| 471 | $3^1 157^1$ | Y | N | 5 | 0 | 1.0000000 | 0.475584 | 0.524416 | -325 | 2632 | -2957 |
| 472 | $2^3 59^1$ | N | N | 9 | 4 | 1.5555556 | 0.476695 | 0.523305 | -316 | 2641 | -2957 |
| 473 | $11^1 43^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477801 | 0.522199 | -311 | 2646 | -2957 |
| 474 | $2^1 3^1 79^1$ | Y | N | -16 | 0 | 1.0000000 | 0.476793 | 0.523207 | -327 | 2646 | -2973 |
| 475 | $5^2 19^1$ | N | N | -7 | 2 | 1.2857143 | 0.475789 | 0.524211 | -334 | 2646 | -2980 |
| 476 | $2^2 7^1 17^1$ | N | N | 30 | 14 | 1.1666667 | 0.476891 | 0.523109 | -304 | 2676 | -2980 |
| 477 | $3^2 53^1$ | N | N | -7 | 2 | 1.2857143 | 0.475891 | 0.524109 | -311 | 2676 | -2987 |
| 478 | $2^1 239^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476987 | 0.523013 | -306 | 2681 | -2987 |
| 479 | 479^1 | Y | Y | -2 | 0 | 1.0000000 | 0.475992 | 0.524008 | -308 | 2681 | -2989 |
| 480 | $2^5 3^1 5^1$ | N | N | -96 | 80 | 1.6666667 | 0.475000 | 0.525000 | -404 | 2681 | -3085 |
| 481 | $13^1 37^1$ | Y | N | 5 | 0 | 1.0000000 | 0.476091 | 0.523909 | -399 | 2686 | -3085 |
| 482 | $2^1 241^1$ | Y | N | 5 | 0 | 1.0000000 | 0.477178 | 0.522822 | -394 | 2691 | -3085 |
| 483 | $3^1 7^1 23^1$ | Y | N | -16 | 0 | 1.0000000 | 0.476190 | 0.523810 | -410 | 2691 | -3101 |
| 484 | $2^2 11^2$ | N | N | 14 | 9 | 1.3571429 | 0.477273 | 0.522727 | -396 | 2705 | -3101 |
| 485 | $5^1 97^1$ | Y | N | 5 | 0 | 1.0000000 | 0.478351 | 0.521649 | -391 | 2710 | -3101 |
| 486 | $2^1 3^5$ | N | N | 13 | 8 | 2.0769231 | 0.479424 | 0.520576 | -378 | 2723 | -3101 |
| 487 | 487^1 | Y | Y | -2 | 0 | 1.0000000 | 0.478439 | 0.521561 | -380 | 2723 | -3103 |
| 488 | $2^3 61^1$ | N | N | 9 | 4 | 1.5555556 | 0.479508 | 0.520492 | -371 | 2732 | -3103 |
| 489 | $3^1 163^1$ | Y | N | 5 | 0 | 1.0000000 | 0.480573 | 0.519427 | -366 | 2737 | -3103 |
| 490 | $2^1 5^1 7^2$ | N | N | 30 | 14 | 1.1666667 | 0.481633 | 0.518367 | -336 | 2767 | -3103 |
| 491 | 491^1 | Y | Y | -2 | 0 | 1.0000000 | 0.480652 | 0.519348 | -338 | 2767 | -3105 |
| 492 | $2^2 3^1 41^1$ | N | N | 30 | 14 | 1.1666667 | 0.481707 | 0.518293 | -308 | 2797 | -3105 |
| 493 | $17^1 29^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482759 | 0.517241 | -303 | 2802 | -3105 |
| 494 | $2^1 13^1 19^1$ | Y | N | -16 | 0 | 1.0000000 | 0.481781 | 0.518219 | -319 | 2802 | -3121 |
| 495 | $3^2 5^1 11^1$ | N | N | 30 | 14 | 1.1666667 | 0.482828 | 0.517172 | -289 | 2832 | -3121 |
| 496 | $2^4 31^1$ | N | N | -11 | 6 | 1.8181818 | 0.481855 | 0.518145 | -300 | 2832 | -3132 |
| 497 | $7^1 71^1$ | Y | N | 5 | 0 | 1.0000000 | 0.482897 | 0.517103 | -295 | 2837 | -3132 |
| 498 | $2^1 3^1 83^1$ | Y | N | -16 | 0 | 1.0000000 | 0.481928 | 0.518072 | -311 | 2837 | -3148 |
| 499 | 499^1 | Y | Y | -2 | 0 | 1.0000000 | 0.480962 | 0.519038 | -313 | 2837 | -3150 |
| 500 | $2^2 5^3$ | N | N | -23 | 18 | 1.4782609 | 0.480000 | 0.520000 | -336 | 2837 | -3173 |