A divisor sum problem

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Suppose we have a set Q of x^c primes $\leq x$, where $0 < c < 1$. Let $g(z) = z(3z-1)/2$. We wish to estimate

$$\Sigma \ := \ \frac{1}{|Q|^2} \sum_{p,q \in Q,\ p \neq q} \#\{u, v \in \mathbb{Z}\ :\ p - q = g(u) - g(v)\}.$$

We will show that Σ has size $\gg \log x$.

We begin by noting that for an integer n, counting solutions u, v to

$$g(u) - g(v) \ = \ n$$

is the same as counting solutions to

$$3((u - 1/6)^2 - (v - 1/6)^2)/2 \ = \ n.$$

which is equivalent to computing solutions u, v to

$$(3u - 3v)(3u + 3v - 1) \ = \ ((6u-1)^2 - (6v-1)^2)/2 \ = \ 12n.$$

Or,

$$(u - v)(3u + 3v - 1) \ = \ 4n.$$

Each factorization $4n = ab$, where $b \equiv -1 \pmod 3$, b of opposite parity from a, gives a pair u, v via:

$$u \ = \ (3a + b + 1)/6,\ \ v \ = \ u - a.$$

Certainly, then, a lower bound for the number of solutions to $g(u) - g(v) = n$ is the number of divisors a of n satisfying $a \equiv -1 \pmod 6$. Thus, letting $f(n)$ denote the number of such divisors a, we have

$$\Sigma \ \geq \ \frac{1}{|Q|^2} \sum_{p,q \in Q,\ p \neq q} f(p - q) \ \geq \ \frac{1}{|Q|^2} \sum_{\substack{d < |Q| \\ d \equiv -1 \pmod 6}} \sum_{r=0}^{d-1} (N_Q(r; d)^2 - N_Q(r; d)),$$

where $N_Q(r;q)$ counts the number of elements of Q that are $r \pmod d$. Note that $N_Q(r;q)^2$ counts the number of pairs $p, q \in Q$ that are $r \pmod d$; and so, $d|p-q$. The term $-N_Q(r)$ in the above sum takes care of the "zero solutions" where $p = q$ – such solutions alway give us $d|p-q$.

It turns out that the total contribution of that term $-N_Q(r;d)$ is small enough to where we can ignore it; and so we will get:

$$\Sigma \gg \frac{1}{|Q|^2} \sum_{\substack{d<|Q| \\ d\equiv -1 \pmod 6}} \sum_{r=0}^{d-1} N_Q(r;d)^2.$$

Here, now, we can apply the Cauchy-Schwarz Inequality to produce a lower bound:

$$\sum_{r=0}^{d-1} N_Q(r;d)^2 \geq \frac{1}{d}\left(\sum_{r=0}^{d-1} N_Q(r;d)\right)^2 = \frac{|Q|^2}{d}.$$

So,

$$\Sigma \gg \frac{1}{|Q|^2} \sum_{\substack{d<|Q| \\ d\equiv -1 \pmod 6}} \frac{|Q|^2}{d} = \sum_{\substack{d\leq|Q| \\ d\equiv -1 \pmod 6}} \frac{1}{d} \gg \log|Q| \gg \log x.$$