Lower bounds on the summatory function of the Möbius function along infinite subsequences

Maxie Dion Schmidt
Georgia Institute of Technology
School of Mathematics

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Abstract

The Mertens function, $M(x) = \sum_{n \leqslant x} \mu(n)$, is classically defined as the summatory function of the Möbius function $\mu(n)$. The Mertens conjecture stating that $|M(x)| < C \cdot \sqrt{x}$ with come absolute $C > 0$ for all $x \geqslant 1$ has a well-known disproof due to Odlyzko and té Riele given in the early 1980's by computation of non-trivial zeta function zeros in conjunction with integral formulas expressing $M(x)$. It is conjectured that $M(x)/\sqrt{x}$ changes sign infinitely often and grows unbounded in the direction of both $\pm\infty$ along subsequences of integers $x \geqslant 1$. We prove a weaker property related to the unboundedness of $|M(x)| \log x/\sqrt{x}$ by showing that

$$\limsup_{x \to \infty} \frac{|M(x)|(\log x)(\log\log\log\log x)^{\frac{9}{4}}(\log\log\log\log\log x)^{\frac{7}{2}} \times \exp\left(2(\log\log\log\log x)^2\right)}{\sqrt{x}(\log\log x)} > 0.$$

There is a distinct stylistic flavor and new element of combinatorial analysis to our proof peppered in with the standard methods from analytic, additive and elementary number theory. This stylistic tendency distinguishes our methods from other proofs of established upper, rather than lower, bounds on $M(x)$.

Keywords and Phrases: *Möbius function; Mertens function; summatory function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting functions; Dirichlet generating function; asymptotic lower bounds; Mertens conjecture.*
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