New characterizations of the summatory function of the Möbius function

Maxie Dion Schmidt
Georgia Institute of Technology
School of Mathematics

Friday 26th March, 2021

Abstract

The Mertens function, $M(x) \coloneqq \sum_{n\leq x} \mu(n)$, is defined as the summatory function of the classical Möbius function for $x \geq 1$. The inverse function sequence $\{g^{-1}(n)\}_{n\geq 1}$ taken with respect to Dirichlet convolution is defined in terms of the strongly additive function $\omega(n)$ that counts the number of distinct prime factors of any integer $n \geq 2$. For large x and $n \leq x$, we associate a natural combinatorial significance to the magnitude of the distinct values of the function $g^{-1}(n)$ that depends directly on the exponent patterns in the prime factorizations of the integers in $\{2, 3, \ldots, x\}$ viewed as multisets.

We prove an Erdős-Kac theorem analog for the distribution of the unsigned sequence $|g^{-1}(n)|$ over $n \leq x$ with a limiting central limit theorem type tendency towards normal as $x \to \infty$. For all $x \geq 1$, discrete convolutions of $G^{-1}(x) \coloneqq \sum_{n\leq x} \lambda(n)|g^{-1}(n)|$ with the prime counting function $\pi(x)$ determine exact formulas and new characterizations of asymptotic bounds for $M(x)$. In this way, we prove another concrete link of the distribution of $L(x) \coloneqq \sum_{n\leq x} \lambda(n)$ with the Mertens function and connect these classical summatory functions with an explicit normal tending probability distribution at large x. The proofs of these resulting combinatorially motivated new characterizations of $M(x)$ are rigorous and unconditional.

Keywords and Phrases: *Möbius function; Mertens function; Dirichlet inverse; Liouville lambda function; prime omega function; prime counting function; Dirichlet generating function; Erdős-Kac theorem; strongly additive function.*
Math Subject Classifications (MSC 2010): *11N37; 11A25; 11N60; 11N64; and 11-04.*

Index

1 Introduction

1.1 Preliminaries

1.1.1 Definitions

We define the *Möbius function* to be the signed indicator function of the squarefree integers in the form of [25, A008683]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \wedge n \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

The *Mertens function*, or summatory function of $\mu(n)$, is defined on the positive integers as

$$M(x) = \sum_{n \leq x} \mu(n), x \geq 1.$$

The sequence of slow growing oscillatory values of this summatory function begins as follows [25, A002321]:

$$\{M(x)\}_{x \geq 1} = \{1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, \ldots\}.$$

The Mertens function satisfies that $\sum_{n \leq x} M\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = 1$, and is related to the summatory function $L(x) := \sum_{n \leq x} \lambda(n)$ via the relation [6, 12]

$$L(x) = \sum_{d \leq \sqrt{x}} M\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), x \geq 1.$$

A positive integer $n \geq 1$ is *squarefree*, or contains no divisors (other than one when $n \geq 2$) which are squares, if and only if $\mu^2(n) = 1$. The summatory function that counts the number of *squarefree* integers $n \leq x$ satisfies [5, §18.6] [25, A013928]

$$Q(x) = \sum_{n \leq x} \mu^2(n) = \frac{6x}{\pi^2} + O\left(\sqrt{x}\right).$$

1.1.2 Properties

A conventional approach to evaluating the limiting asymptotic behavior of $M(x)$ for large $x \to \infty$ considers an inverse Mellin transformation of the reciprocal of the Riemann zeta function. In particular, since

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = s \cdot \int_1^{\infty} \frac{M(x)}{x^{s+1}} dx, \text{ for } \operatorname{Re}(s) > 1,$$

we obtain that

$$M(x) = \lim_{T \to \infty} \frac{1}{2\pi\imath} \int_{T-\imath\infty}^{T+\imath\infty} \frac{x^s}{s \cdot \zeta(s)} ds.$$

The previous two representations lead us to the exact expression of $M(x)$ for any real $x > 0$ given by the next theorem.

Theorem 1.1 (Analytic Formula for $M(x)$, Titchmarsh)**.** *Assuming the Riemann Hypothesis (RH), there exists an infinite sequence $\{T_k\}_{k \geq 1}$ satisfying $k \leq T_k \leq k+1$ for each k such that for any real $x > 0$*

$$M(x) = \lim_{k \to \infty} \sum_{\substack{\rho : \zeta(\rho) = 0 \\ |\operatorname{Im}(\rho)| < T_k}} \frac{x^{\rho}}{\rho \cdot \zeta'(\rho)} - 2 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n \cdot (2n)! \zeta(2n+1)} \left(\frac{2\pi}{x}\right)^{2n} + \frac{\mu(x)}{2} [x \in \mathbb{Z}^+]_{\delta}.$$

A historical unconditional bound on the Mertens function due to Walfisz (circa 1963) states that there is an absolute constant $C > 0$ such that

$$M(x) \ll x \cdot \exp\left(-C \cdot \log^{\frac{3}{5}}(x)(\log\log x)^{-\frac{3}{5}}\right).$$

Under the assumption of the RH, Soundararajan improved estimates bounding $M(x)$ from above for large x in the following form [26]:

$$M(x) = O\left(\sqrt{x} \cdot \exp\left((\log x)^{\frac{1}{2}}(\log\log x)^{\frac{5}{2}+\epsilon}\right)\right), \ \forall \epsilon > 0.$$

1.1.3 Conjectures on boundedness and limiting behavior

The RH is equivalent to showing that $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$ for any $0 < \epsilon < \frac{1}{2}$. There is a rich history to the original statement of the *Mertens conjecture* which asserts that

$$|M(x)| < C \cdot \sqrt{x}, \quad \text{for some absolute constant } C > 0.$$

The conjecture was first verified by Mertens himself for $C = 1$ and all $x < 10000$ without the benefit of modern computation. Since its beginnings in 1897, the Mertens conjecture has been disproved by computational methods with non-trivial simple zeta function zeros with comparatively small imaginary parts in the famous paper by Odlyzko and té Riele [18]. More recent attempts at bounding $M(x)$ naturally consider determining the rates at which the function $q(x) := M(x)/\sqrt{x}$ grows with or without bound along infinite subsequences, e.g., considering the asymptotics of $q(x)$ in the limit supremum and limit infimum senses.

It is verified by computation that [21, *cf.* §4.1] [25, *cf.* A051400; A051401]

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.060 \qquad (\text{now } \geq 1.826054),$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009 \qquad (\text{now } \leq -1.837625).$$

Based on work by Odlyzyko and té Riele, it seems probable that each of these limits should evaluate to $\pm\infty$, respectively [18, 10, 11, 7]. A famous conjecture due to Gonek asserts that in fact $M(x)$ satisfies [17]

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x} \cdot (\log\log\log x)^{\frac{5}{4}}} = O(1).$$

1.2 A concrete new approach to characterizing $M(x)$

The main interpretation to take away from the article is our rigorous motivation of an equivalent characterization of $M(x)$ formed by constructing combinatorially relevant sequences related to the distribution of the primes through convolutions of strongly additive functions. These sequences and their summatory functions have not yet been studied in the literature surrounding the Mertens function. The prime-related combinatorics at hand are discussed by the remarks given in Section 3.3. This new perspective offers new exact characterizations of $M(x)$ for all $x \geq 1$ through the formulas involving discrete convolutions of $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$ with the prime counting function $\pi(x)$ proved in Section 5.

The sequence $g^{-1}(n)$ defined precisely below and $G^{-1}(x)$ are crucially tied to canonical number theoretic examples of strongly and completely additive functions, e.g., to $\omega(n)$ and $\Omega(n)$, respectively. The definitions of the primary subsequences we define, and the corresponding parameterized bivariate DGF based proof methods that are given in the spirit of Montgomery and Vaughan's work, allow us to reconcile the property of strong additivity with signed sums of multiplicative functions. The proofs of characteristic properties of these new sequences imply a scaled normal tending probability distribution for the unsigned magnitude of $|g^{-1}(n)|$ that is analogous to the Erdős-Kac theorems for $\omega(n)$ and $\Omega(n)$.

Since we prove that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$, it follows that we have a new probabilistic perspective from which to express distributional features of the summatory functions $G^{-1}(x)$ as $x \to \infty$ in terms of the properties of $|g^{-1}(n)|$ and $L(x) := \sum_{n\leq x} \lambda(n)$. Formalizing the properties of the distribution of $L(x)$ is typically viewed as a problem that is equally as difficult as understanding the distribution of $M(x)$ well for large x. The new results in this article then precisely connect the distributions of $L(x)$, a well defined scaled normally tending probability distribution, and $M(x)$ as $x \to \infty$.

1.2.1 Summatory functions of Dirichlet convolutions of arithmetic functions

Theorem 1.2 (Summatory functions of Dirichlet convolutions). *Let $f, h : \mathbb{Z}^+ \to \mathbb{C}$ be any arithmetic functions such that $f(1) \neq 0$. Suppose that $F(x) := \sum_{n\leq x} f(n)$ and $H(x) := \sum_{n\leq x} h(n)$ denote the summatory functions of f and h, respectively, and that $F^{-1}(x) := \sum_{n\leq x} f^{-1}(n)$ denotes the summatory function of the Dirichlet inverse of f for any $x \geq 1$. We have the following exact expressions for the summatory function of the convolution $f * h$ for all integers $x \geq 1$:*

$$\begin{aligned}
\pi_{f*h}(x) &:= \sum_{n\leq x} \sum_{d|n} f(d)h(n/d) \\
&= \sum_{d\leq x} f(d) H\left(\left\lfloor \frac{x}{d} \right\rfloor\right) \\
&= \sum_{k=1}^{x} H(k) \left[F\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - F\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right].
\end{aligned}$$

Moreover, for all $x \geq 1$

$$\begin{aligned}
H(x) &= \sum_{j=1}^{x} \pi_{f*h}(j) \left[F^{-1}\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - F^{-1}\left(\left\lfloor \frac{x}{j+1} \right\rfloor\right)\right] \\
&= \sum_{k=1}^{x} f^{-1}(k) \cdot \pi_{f*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right).
\end{aligned}$$

Corollary 1.3 (Applications of Möbius inversion). *Suppose that h is an arithmetic function such that $h(1) \neq 0$. Define the summatory function of the convolution of h with μ by $\widetilde{H}(x) := \sum_{n\leq x}(h * \mu)(n)$. Then the Mertens function is expressed by the sum*

$$M(x) = \sum_{k=1}^{x} \left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor + 1}^{\left\lfloor \frac{x}{k} \right\rfloor} h^{-1}(j) \right) \widetilde{H}(k), \forall x \geq 1.$$

Corollary 1.4. *We have that for all $x \geq 1$*

$$M(x) = \sum_{k=1}^{x} (\omega + 1)^{-1}(k) \left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right]. \tag{1}$$

1.2.2 An exact expression for $M(x)$ via strongly additive functions

Fix the notation for the Dirichlet invertible function $g(n) := \omega(n) + 1$ and define its inverse with respect to Dirichlet convolution by $g^{-1}(n) = (\omega + 1)^{-1}(n)$ [25, A341444]. We can compute exactly that (see Table B on page 40)

$$\{g^{-1}(n)\}_{n\geq 1} = \{1, -2, -2, 2, -2, 5, -2, -2, 2, 5, -2, -7, -2, 5, 5, 2, -2, -7, -2, -7, 5, 5, -2, 9, \ldots\}.$$

There is not a simple direct recursion between the distinct values of $g^{-1}(n)$ that holds for all $n \geq 1$. The distribution of distinct sets of prime exponents is still clearly quite regular since $\omega(n)$ and $\Omega(n)$ play a crucial role in the repetition of common values of $g^{-1}(n)$. The following observation is suggestive of the quasi-periodicity of the distribution of distinct values of this inverse function over $n \geq 2$:

Heuristic 1.5 (Symmetry in $g^{-1}(n)$ from the prime factorizations of $n \leq x$)**.** Suppose that $n_1, n_2 \geq 2$ are such that their factorizations into distinct primes are given by $n_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $n_2 = q_1^{\beta_1} \cdots q_r^{\beta_r}$ for $\omega(n_i) \geq 1$. If $\{\alpha_1, \ldots, \alpha_r\} \equiv \{\beta_1, \ldots, \beta_r\}$ as multisets of prime exponents, then $g^{-1}(n_1) = g^{-1}(n_2)$. For example, g^{-1} has the same values on the squarefree integers with exactly one, two, three, and so on prime factors.

Conjecture 1.6 (Characteristic properties of the inverse sequence)**.** *We have the following properties characterizing the Dirichlet inverse function $g^{-1}(n)$:*

- ***(A)*** *For all $n \geq 1$, $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$;*
- ***(B)*** *For all squarefree integers $n \geq 2$, we have that*

$$|g^{-1}(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \cdot m!;$$

- ***(C)*** *If $n \geq 2$ and $\Omega(n) = k$, then*

$$2 \leq |g^{-1}(n)| \leq \sum_{j=0}^{k} \binom{k}{j} \cdot j!.$$

The signedness property in (A) is proved precisely in Proposition 2.1. A proof of (B) in fact follows from Lemma 3.1 stated on page 15. The realization that the beautiful and remarkably simple combinatorial form of property (B) in Conjecture 1.6 holds for all squarefree $n \geq 1$ motivates our pursuit of simpler formulas for the inverse functions $g^{-1}(n)$ through the sums of auxiliary subsequences $C_k(n)$ in Section 3. That is, we observe a familiar formula for $g^{-1}(n)$ on an asymptotically dense infinite subset of integers, e.g., that holds for all squarefree $n \geq 2$, and then seek to extrapolate by proving there are regular tendencies of this sequence viewed more generally over any $n \geq 2$. An exact expression for $g^{-1}(n)$ is given by

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d), n \geq 1,$$

where the sequence $\lambda(n) C_{\Omega(n)}(n)$ has DGF $(P(s)+1)^{-1}$ for $\operatorname{Re}(s) > 1$ (see Proposition 2.1). The function $C_{\Omega(n)}(n)$ has been previously considered in [4] with its exact formula given by (*cf.* [8])

$$C_{\Omega(n)}(n) = \begin{cases} 1, & \text{if } n = 1; \\ (\Omega(n))! \times \prod\limits_{p^{\alpha}||n} \frac{1}{\alpha!}, & \text{if } n \geq 2. \end{cases}$$

In Corollary 4.6, we prove that the average order of the unsigned sequence is bounded by by

$$\mathbb{E}|g^{-1}(n)| = \frac{3\sqrt{2}}{\pi^{3/2}} (\log n)^2 \sqrt{\log\log n}(1+o(1)), \text{ as } n \to \infty.$$

In Section 4, we prove a variant of the Erdős-Kac theorem that characterizes the distribution of the sequence $C_{\Omega(n)}(n)$. This leads us to conclude the following statement for any fixed $Y > 0$, with $\mu_x(C) := \log\log x - \log(4\sqrt{2\pi})$ and $\sigma_x(C) := \sqrt{\log\log x}$, that holds uniformly for any $-Y \leq y \leq Y$ as $x \to \infty$ (see Corollary 4.8):

$$\frac{1}{x} \cdot \#\left\{2 \leq n \leq x : |g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq z\right\} = \Phi\left\{\frac{6\sigma_x(C)}{\pi^2}\left(\frac{\pi^2 z}{6} + \sigma_x(C)\right) - \frac{6}{\pi^2}\log(4\sqrt{2\pi})\right\} + O\left(\frac{1}{\sqrt{\log\log x}}\right).$$

The regularity and quasi-periodicity we have alluded to in the remarks above are then quantifiable in so much as the distribution of $|g^{-1}(n)|$ for $n \leq x$ tends to its average order with a non-central normal tendency depending on x as $x \to \infty$. That is, if $x > e$ is sufficiently large and if we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds:

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}(\log\log x)\right) = \frac{1}{2} + o(1) \tag{D}$$

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}(\alpha + \log\log x)\right) = \Phi(\alpha) + o(1), \alpha \in \mathbb{R}. \tag{E}$$

It follows from the last property that as $n \to \infty$,

$$|g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|(1 + o(1)),$$

on an infinite set of the integers with asymptotic density one.

1.2.3 Formulas illustrating the new characterizations of $M(x)$

Let $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$ for integers $x \geq 1$ [25, A341472]. We prove that (see Proposition 5.2)

$$\begin{aligned} M(x) &= G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right] \\ &= G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1. \end{aligned} \tag{2}$$

This formula implies that we can establish new asymptotic bounds on $M(x)$ along large infinite subsequences by sharply bounding the summatory function $G^{-1}(x)$. The take on the regularity of $|g^{-1}(n)|$ is as such imperative to our arguments that formally bound the growth of $M(x)$ by its new identification with $G^{-1}(x)$. A combinatorial approach to summing $G^{-1}(x)$ for large x based on the distribution of the primes is outlined in our remarks in Section 3.3.

Theorem 5.1 proves that for almost every sufficiently large x[A] there exists some $1 \leq t_0 \leq x$ such that

$$G^{-1}(x) = O\left(L(t_0) \cdot \mathbb{E}|g^{-1}(x)|\right).$$

If the RH is true, then for any $\varepsilon > 0$ and sufficiently large $x > e$ we have that

$$G^{-1}(x) = O\left((\log x)^2 \sqrt{\log\log x}\sqrt{x} \times \exp\left(\sqrt{\log x}(\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right).$$

In Corollary 5.4, we also prove that

$$M(x) = O\left(G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) + \frac{x}{\log x} \times \sum_{k \leq \sqrt{x}} \frac{G^{-1}(k)}{k^2} + (\log x)^2(\log\log x)^{3/2}\right).$$

Moving forward, a discussion of the properties of the summatory functions $G^{-1}(x)$ motivates more study in the future to extend the full range of possibilities for viewing the new structure behind $M(x)$ we identify within this article.

1.3 Notation and conventions

The next listing provides a glossary of common notation, conventions and abbreviations used throughout the article.

[A] By *almost every* large integer x, we mean that the result holds for all large x taken within an infinite subset of $\mathbb{Z}^+$ with asymptotic density one.

Symbol	Definition
$\approx, \sim$	We write that $f(x) \approx g(x)$ if $\lvert f(x)-g(x)\rvert = O(1)$ as $x \rightarrow \infty$. Two arithmetic functions $A(x), B(x)$ satisfy the relation $A \sim B$ if $\lim_{x\rightarrow\infty} \frac{A(x)}{B(x)} = 1$.
$\mathbb{E}[f(x)]$	We use the expectation notation of $\mathbb{E}[f(x)] = h(x)$ to denote that f has an *average order* of $h(x)$. This means that $\frac{1}{x}\sum_{n\leq x} f(n) \sim h(x)$.
$\chi_{\mathbb{P}}(n)$	The characteristic (or indicator) function of the primes equals one if and only if $n \in \mathbb{Z}^+$ is prime, and is zero-valued otherwise.
$C_k(n), C_{\Omega(n)}(n)$	The sequence is defined recursively for $n \geq 1$ as follows: $$C_k(n) := \begin{cases} \delta_{n,1}, & \text{if } k = 0; \\ \sum\limits_{d\mid n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases}$$ It represents the multiple, k-fold convolution of the function $\omega(n)$ with itself.
$[q^n]F(q)$, OGF	The coefficient of q^n in the power series expansion of $F(q)$ about zero when $F(q)$ is treated as the ordinary generating function (or OGF) of some sequence, $\{f_n\}_{n\geq 0}$. Namely, for integers $n \geq 0$ we define $[q^n]F(q) = f_n$ whenever $F(q) := \sum_{n\geq 0} f_n q^n$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) := \delta_{n,1}$, defined such that for any arithmetic f we have that $f * \varepsilon = \varepsilon * f = f$ where $*$ denotes Dirichlet convolution (see definition below).
$*, f * g$	The Dirichlet convolution of f and g, $(f * g)(n) := \sum\limits_{d\mid n} f(d) g(n/d)$, where the sum is taken over the divisors of any $n \geq 1$.
$f^{-1}(n)$	The Dirichlet inverse f^{-1} of f exists if and only if $f(1) \neq 0$. The Dirichlet inverse of any f such that $f(1) \neq 0$ with respect to convolution is defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum\limits_{\substack{d\mid n \\ d>1}} f(d) f^{-1}(n/d)$ for $n \geq 2$ with $f^{-1}(1) = 1/f(1)$. When it exists, this inverse function is unique and satisfies the characteristic relations that $f^{-1} * f = f * f^{-1} = \varepsilon$.
$\gg, \ll, \asymp$	For functions A, B, the notation $A \ll B$ implies that $A = O(B)$. Similarly, for $B \geq 0$ the notation $A \gg B$ implies that $B = O(A)$. When we have that $A \ll B$ and $B \ll A$, we write $A \asymp B$.
$g^{-1}(n), G^{-1}(x)$	The Dirichlet inverse function, $g^{-1}(n) = (\omega+1)^{-1}(n)$ with corresponding summatory function $G^{-1}(x) := \sum_{n\leq x} g^{-1}(n)$.
$[n = k]_{\delta}$,$[\mathtt{cond}]_{\delta}$	The symbol $[n = k]_{\delta}$ is a synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and is zero otherwise. For boolean-valued conditions, `cond`, the symbol $[\mathtt{cond}]_{\delta}$ evaluates to one precisely when `cond` is true, and to zero otherwise. This notation is called *Iverson's convention*.
$\lambda(n), L(x), \lambda_*(n)$	The Liouville lambda function is the completely multiplicative function defined by $\lambda(n) := (-1)^{\Omega(n)}$. Its summatory function is defined by $L(x) := \sum_{n\leq x} \lambda(n)$. For positive integers $n \geq 2$, we define $\lambda_*(n) := (-1)^{\omega(n)}$. We have the initial condition that $\lambda_*(1) = 1$.

Symbol	**Definition**
$\mu(n), M(x)$	The Möbius function defined such that $\mu^2(n)$ is the indicator function of the squarefree integers $n \geq 1$ where $\mu(n) = (-1)^{\omega(n)}$ whenever n is squarefree. The Mertens function is the summatory function of $\mu(n)$ defined for all integers $x \geq 1$ by $M(x) := \sum\limits_{n \leq x} \mu(n)$.
$\Phi(z)$	For $x \in \mathbb{R}$, we define the CDF of the standard normal distribution to be $\Phi(z) := \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{z} e^{-t^2/2} dt$.
$\nu_p(n)$	The valuation function that extracts the maximal exponent of p in the prime factorization of n, e.g., $\nu_p(n) = 0$ if $p \nmid n$ and $\nu_p(n) = \alpha$ if $p^{\alpha} \| n$ (e.g., when p^{α} exactly divides n) for p prime, $\alpha \geq 1$ and $n \geq 2$.
$\omega(n)$,$\Omega(n)$	We define the strongly additive function $\omega(n) := \sum_{p \mid n} 1$ and the completely additive function $\Omega(n) := \sum_{p^{\alpha} \| n} \alpha$. This means that if the prime factorization of $n \geq 2$ is given by $n := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \neq p_j$ for all $i \neq j$, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$. By convention, we set $\omega(1) = \Omega(1) = 0$.
$\pi_k(x), \widehat{\pi}_k(x)$	For integers $k \geq 1$, the prime counting function variant $\pi_k(x)$ denotes the number of $2 \leq n \leq x$ with exactly k distinct prime factors: $\pi_k(x) := \#\{2 \leq n \leq x : \omega(n) = k\}$. Similarly, the function $\widehat{\pi}_k(x) := \#\{2 \leq n \leq x : \Omega(n) = k\}$ for $x \geq 2$.
$P(s)$	For complex s with $\mathrm{Re}(s) > 1$, we define the prime zeta function to be the Dirichlet generating function (or DGF) $P(s) = \sum_{n \geq 1} \frac{\chi_{\mathbb{P}}(n)}{n^s} = \sum_{k \geq 2} \frac{\mu(k)}{k} \log \zeta(ks)$.
$Q(x)$	For $x \geq 1$, we define $Q(x)$ to be the summatory function indicating the number of squarefree integers $n \leq x$. That is, $Q(x) := \sum_{n \leq x} \mu^2(n)$.
$W(x)$	For $x, y \in \mathbb{R}_{\geq 0}$, we write that $x = W(y)$ if and only if $xe^x = y$.
$\zeta(s)$	The Riemann zeta function is defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\mathrm{Re}(s) > 1$, and by analytic continuation on the rest of the complex plane with the exception of a simple pole at $s = 1$ of residue one.

2 Initial elementary proofs of new results

2.1 Establishing the summatory function properties and inversion identities

We will offer a proof of Theorem 1.2 suggested by an intuitive construction through matrix based methods. Related results on summations of Dirichlet convolutions and their inversion appear in [1, §2.14; §3.10; §3.12; *cf.* §4.9, p. 95]. It is not difficult to prove the related identity that

$$\sum_{n\leq x} h(n)(f*g)(n) = \sum_{n\leq x} f(n) \times \sum_{k\leq \left\lfloor \frac{x}{n}\right\rfloor} g(k)h(kn).$$

Proof of Theorem 1.2. Let h, g be arithmetic functions such that $g(1) \neq 0$. Denote the summatory functions of h and g, respectively, by $H(x) = \sum_{n\leq x} h(n)$ and $G(x) = \sum_{n\leq x} g(n)$. We define $\pi_{g*h}(x)$ to be the summatory function of the Dirichlet convolution of g with h. We have that the following formulas hold for all $x \geq 1$:

$$\begin{aligned}\pi_{g*h}(x) &:= \sum_{n=1}^{x}\sum_{d\mid n} g(n)h(n/d) = \sum_{d=1}^{x} g(d) H\left(\left\lfloor \frac{x}{d}\right\rfloor\right) \\ &= \sum_{i=1}^{x}\left[G\left(\left\lfloor \frac{x}{i}\right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1}\right\rfloor\right)\right] H(i). \end{aligned} \tag{3}$$

The first formula above is well known in the references. The second formula is justified directly using summation by parts as [19, §2.10(ii)]

$$\begin{aligned}\pi_{g*h}(x) &= \sum_{d=1}^{x} h(d) G\left(\left\lfloor \frac{x}{d}\right\rfloor\right) \\ &= \sum_{i\leq x}\left(\sum_{j\leq i} h(j)\right) \times \left[G\left(\left\lfloor \frac{x}{i}\right\rfloor\right) - G\left(\left\lfloor \frac{x}{i+1}\right\rfloor\right)\right].\end{aligned}$$

We next form the invertible matrix of coefficients associated with this linear system defining $H(j)$ for all $1 \leq j \leq x$ in (3) by setting

$$g_{x,j} := G\left(\left\lfloor \frac{x}{j}\right\rfloor\right) - G\left(\left\lfloor \frac{x}{j+1}\right\rfloor\right) \equiv G_{x,j} - G_{x,j+1},$$

where

$$G_{x,j} := G\left(\left\lfloor \frac{x}{j}\right\rfloor\right), 1 \leq j \leq x.$$

Since $g_{x,x} = G(1) = g(1)$ and $g_{x,j} = 0$ for all $j > x$, the matrix we must work with in this problem is lower triangular with a non-zero constant on its diagonals, and is hence invertible. If we let $\hat{G} := (G_{x,j})$, then this matrix is expressed by applying an invertible shift operation as

$$(g_{x,j}) = \hat{G}(I - U^T).$$

The square matrix U of sufficiently large finite dimensions $N \times N$ has $(i, j)^{th}$ entries for all $1 \leq i, j \leq N$ that are defined by $(U)_{i,j} = \delta_{i+1,j}$ and such that

$$\left[(I - U^T)^{-1}\right]_{i,j} = [j \leq i]_{\delta}.$$

Observe that

$$\left\lfloor \frac{x}{j}\right\rfloor - \left\lfloor \frac{x-1}{j}\right\rfloor = \begin{cases} 1, & \text{if } j\mid x; \\ 0, & \text{otherwise.}\end{cases}$$

The previous property implies that

$$G\left(\left\lfloor \frac{x}{j} \right\rfloor\right) - G\left(\left\lfloor \frac{x-1}{j} \right\rfloor\right) = \begin{cases} g\left(\frac{x}{j}\right), & \text{if } j|x; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

We use the last property in (4) to shift the matrix $\hat{G}$, and then invert the result to obtain a matrix involving the Dirichlet inverse of g in the following form:

$$\left[(I-U^T)\hat{G}\right]^{-1} = \left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)^{-1} = \left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right).$$

In particular, our target matrix in the inversion problem is defined by

$$(g_{x,j}) = (I-U^T)\left(g\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T)^{-1}.$$

We can express its inverse by a similarity transformation conjugated by shift operators as

$$\begin{aligned} (g_{x,j})^{-1} &= (I-U^T)^{-1}\left(g^{-1}\left(\frac{x}{j}\right)[j|x]_\delta\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k)\right)(I-U^T) \\ &= \left(\sum_{k=1}^{\left\lfloor \frac{x}{j} \right\rfloor} g^{-1}(k) - \sum_{k=1}^{\left\lfloor \frac{x}{j+1} \right\rfloor} g^{-1}(k)\right). \end{aligned}$$

Hence, the summatory function $H(x)$ is given exactly for any $x \geq 1$ by a vector product with the inverse matrix from the previous equation by the formula

$$H(x) = \sum_{k=1}^{x}\left(\sum_{j=\left\lfloor \frac{x}{k+1} \right\rfloor+1}^{\left\lfloor \frac{x}{k} \right\rfloor} g^{-1}(j)\right) \cdot \pi_{g*h}(k).$$

We can prove another inversion formula providing the coefficients of the summatory function $G^{-1}(i)$ for $1 \leq i \leq x$ from the last equation by adapting our argument to prove (3) above. This leads to the following equivalent identity expressing $H(x)$:

$$H(x) = \sum_{k=1}^{x} g^{-1}(x) \cdot \pi_{g*h}\left(\left\lfloor \frac{x}{k} \right\rfloor\right). \qquad \square$$

2.2 Proving the characteristic signedness property of $g^{-1}(n)$

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes, let $\varepsilon(n) = \delta_{n,1}$ be the multiplicative identity with respect to Dirichlet convolution, and denote by $\omega(n)$ the strongly additive function that counts the number of distinct prime factors of n. We can easily prove using DGFs (or other elementary methods) that

$$\chi_{\mathbb{P}} + \varepsilon = (\omega + 1) * \mu. \tag{5}$$

When combined with Corollary 1.3 this convolution identity yields the exact formula for $M(x)$ stated in (1) of Corollary 1.4.

Proposition 2.1 (The signedness property of $g^{-1}(n)$)**.** *Let the operator* $\operatorname{sgn}(h(n)) = \frac{h(n)}{|h(n)|+[h(n)=0]_{\delta}} \in \{0, \pm 1\}$ *denote the sign of the arithmetic function h at integers $n \geq 1$. For the Dirichlet invertible function $g(n) := \omega(n) + 1$, we have that $\operatorname{sgn}(g^{-1}(n)) = \lambda(n)$ for all $n \geq 1$.*

Proof. The function $D_f(s) := \sum_{n\geq 1} f(n) n^{-s}$ defines the *Dirichlet generating function* (or DGF) of any arithmetic function $f(n)$ which is convergent for all $s \in \mathbb{C}$ satisfying $\operatorname{Re}(s) > \sigma_f$ with σ_f the abscissa of convergence of the series. Recall that $D_1(s) = \zeta(s)$, $D_{\mu}(s) = \zeta(s)^{-1}$ and $D_{\omega}(s) = P(s)\zeta(s)$ for $\operatorname{Re}(s) > 1$. Then by (5) and the known property that whenever $f(1) \neq 0$, the DGF of $f^{-1}(n)$ is the reciprocal of the DGF of the arithmetic function f, we have for all $\operatorname{Re}(s) > 1$ that

$$D_{(\omega+1)^{-1}}(s) = \frac{1}{(P(s)+1)\zeta(s)}. \tag{6}$$

It follows that $(\omega+1)^{-1}(n) = (h^{-1} * \mu)(n)$ when we take $h := \chi_{\mathbb{P}} + \varepsilon$. We first show that $\operatorname{sgn}(h^{-1}) = \lambda$. This observation then implies that $\operatorname{sgn}(h^{-1} * \mu) = \lambda$.

By the recurrence relation that defines the Dirichlet inverse function of any arithmetic function h such that $h(1) = 1$, we have that [1, §2.7]

$$h^{-1}(n) = \begin{cases} 1, & n = 1; \\ -\sum\limits_{\substack{d|n \\ d>1}} h(d) h^{-1}(n/d), & n \geq 2. \end{cases} \tag{7}$$

For $n \geq 2$, the summands in (7) can be simply indexed over the primes $p|n$ given our definition of h from above. We can inductively unfold these sums into nested divisor sums provided the depth of the expanded divisor sums does not exceed the capacity to index non-trivial summations over the primes dividing n. Namely, notice that for $n \geq 2$

$$\begin{aligned} h^{-1}(n) &= -\sum_{p|n} h^{-1}\left(\frac{n}{p}\right), && \text{if } \Omega(n) \geq 1 \\ &= \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} h^{-1}\left(\frac{n}{p_1 p_2}\right), && \text{if } \Omega(n) \geq 2 \\ &= -\sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \sum_{p_3|\frac{n}{p_1 p_2}} h^{-1}\left(\frac{n}{p_1 p_2 p_3}\right), && \text{if } \Omega(n) \geq 3. \end{aligned}$$

Then by induction with $h^{-1}(1) = h(1) = 1$, we expand these nested divisor sums as above to the maximal possible depth as

$$\lambda(n) \cdot h^{-1}(n) = \sum_{p_1|n} \sum_{p_2|\frac{n}{p_1}} \times \cdots \times \sum_{p_{\Omega(n)}|\frac{n}{p_1 p_2 \cdots p_{\Omega(n)-1}}} 1, n \geq 2. \tag{8}$$

Moreover, by a combinatorial argument related to multinomial coefficient expansions of the sums in (8), we recover exactly that

$$h^{-1}(n) = \lambda(n)(\Omega(n))! \times \prod_{p^{\alpha}||n} \frac{1}{\alpha!}, n \geq 2. \tag{9}$$

The last two expansions imply that the following property holds for all $n \geq 1$:

$$\operatorname{sgn}(h^{-1}(n)) = \lambda(n).$$

Since λ is completely multiplicative we have that $\lambda\left(\frac{n}{d}\right)\lambda(d) = \lambda(n)$ for all divisors $d|n$ when $n \geq 1$. We also know that $\mu(n) = \lambda(n)$ whenever n is squarefree, so that we obtain the following result:

$$g^{-1}(n) = (h^{-1} * \mu)(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) |h^{-1}(n)|, n \geq 1.$$ □

The conclusion of the proof of Proposition 2.1 in fact implies the stronger result that

$$g^{-1}(n) = \lambda(n) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d),$$

where we adopt the notation that for $n \geq 2$, $C_{\Omega(n)}(n) = (\Omega(n))! \times \prod_{p^\alpha \| n} \frac{1}{\alpha!}$, where the sequence is taken to be one at $n := 1$.

2.3 Results on the distribution of exceptional values of $\omega(n)$ and $\Omega(n)$

The next theorems reproduced from [13, §7.4] characterize the relative scarcity of the distributions of $\omega(n)$ and $\Omega(n)$ for $n \leq x$ such that $\omega(n), \Omega(n) > \log\log x$. Since $\mathbb{E}[\omega(n)], \mathbb{E}[\Omega(n)] = \log\log n + B$ for $B \in (0,1)$ an absolute constant in each case, these results imply a regular, normal tendency of these additive arithmetic functions towards their respective average orders.

Theorem 2.2 (Upper bounds on exceptional values of $\Omega(n)$ for large n)**.** *Let*

$$\begin{aligned} A(x,r) &:= \#\left\{n \leq x : \Omega(n) \leq r \cdot \log\log x\right\}, \\ B(x,r) &:= \#\left\{n \leq x : \Omega(n) \geq r \cdot \log\log x\right\}. \end{aligned}$$

If $0 < r \leq 1$ *and* $x \geq 2$*, then*

$$A(x,r) \ll x(\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

If $1 \leq r \leq R < 2$ *and* $x \geq 2$*, then*

$$B(x,r) \ll_R x \cdot (\log x)^{r-1-r\log r}, \quad \text{as } x \to \infty.$$

Theorem 2.3 is a special case analog to the celebrated Erdős-Kac theorem typically stated for the normally distributed values of the scaled-shifted function $\omega(n)$ over $n \leq x$ as $x \to \infty$ [13, *cf.* Thm. 7.21] [9, *cf.* §1.7].

Theorem 2.3 (Exact limiting bounds on exceptional values of $\Omega(n)$ for large n)**.** *We have that as* $x \to \infty$

$$\#\left\{3 \leq n \leq x : \Omega(n) \leq \log\log n\right\} = \frac{x}{2} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Theorem 2.4 (Montgomery and Vaughan)**.** *Recall that we have defined*

$$\widehat{\pi}_k(x) := \#\{n \leq x : \Omega(n) = k\}.$$

For $0 < R < 2$ *we have that uniformly for all* $1 \leq k \leq R \cdot \log\log x$

$$\widehat{\pi}_k(x) = \mathcal{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right],$$

where

$$\mathcal{G}(z) := \frac{1}{\Gamma(z+1)} \times \prod_p \left(1 - \frac{z}{p}\right)^{-1}\left(1 - \frac{1}{p}\right)^z, 0 \leq |z| < R.$$

Remark 2.5. We can extend the work in [13] on the distribution of $\Omega(n)$ to find analogous results bounding the distribution of $\omega(n)$. We have that for $0 < R < 2$

$$\pi_k(x) = \widehat{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O_R\left(\frac{k}{(\log\log x)^2}\right)\right], \text{ unif. for } 1 \leq k \leq R\log\log x. \tag{10}$$

The analogous function to express these bounds for $\omega(n)$ is defined by $\widehat{\mathcal{G}}(z) := \widehat{F}(1,z)/\Gamma(1+z)$ where we take

$$\widehat{F}(s,z) := \prod_p \left(1+\frac{z}{p^s-1}\right)^{-1}\left(1-\frac{1}{p^s}\right)^z, \operatorname{Re}(s) > \frac{1}{2}; |z| \leq R < 2.$$

Let the functions

$$\begin{aligned} C(x,r) &:= \#\{n \leq x : \omega(n) \leq r \log\log x\} \\ D(x,r) &:= \#\{n \leq x : \omega(n) \geq r \log\log x\}. \end{aligned}$$

Then we have the next uniform upper bounds given by

$$\begin{aligned} C(x,r) &\ll x(\log x)^{r-1-r\log r}, \text{ uniformly for } 0 < r \leq 1, \\ D(x,r) &\ll x(\log x)^{r-1-r\log r}, \text{ uniformly for } 1 \leq r \leq R < 2. \end{aligned}$$

3 Auxiliary sequences expressing the Dirichlet inverse function $g^{-1}(n)$

The computational data given as Table B in the appendix section (refer to page 40) is intended to provide clear insight into why we eventually arrived at the approximations to $g^{-1}(n)$ proved in this section. The table provides illustrative numerical data by examining the approximate behavior at hand for the cases of $1 \leq n \leq 500$ with *Mathematica* [24]. In Section 4, we will use these relations between $g^{-1}(n)$ and $C_{\Omega(n)}(n)$ to prove an Erdős-Kac like analog that characterizes the distribution of the unsigned function $|g^{-1}(n)|$.

3.1 Definitions and properties of triangular component function sequences

We define the following sequence for integers $n \geq 1$ and $k \geq 0$:

$$C_k(n) := \begin{cases} \varepsilon(n), & \text{if } k = 0; \\ \sum\limits_{d|n} \omega(d) C_{k-1}(n/d), & \text{if } k \geq 1. \end{cases} \tag{11}$$

By recursively expanding the definition of $C_k(n)$ at any fixed $n \geq 2$, we see that we can form a chain of at most $\Omega(n)$ iterated (or nested) divisor sums by unfolding the definition of (11) inductively. By the same argument, we see that at fixed n, the function $C_k(n)$ is seen to be non-zero only for positive integers $k \leq \Omega(n)$ whenever $n \geq 2$. A sequence of relevant signed semi-diagonals of the functions $C_k(n)$ begins as follows [25, A008480]:

$$\{\lambda(n) \cdot C_{\Omega(n)}(n)\}_{n \geq 1} \mapsto \{1, -1, -1, 1, -1, 2, -1, -1, 1, 2, -1, -3, -1, 2, 2, 1, -1, -3, -1, -3, 2, 2, -1, 4, 1, 2, \ldots\}.$$

We can see that $C_{\Omega(n)}(n) \leq (\Omega(n))!$ for all $n \geq 1$. In fact, $h^{-1}(n) \equiv \lambda(n) C_{\Omega(n)}(n)$ is the same function given by the formula in (9) from Proposition 2.1.

3.2 Relating the function $C_{\Omega(n)}(n)$ to exact formulas for $g^{-1}(n)$

Lemma 3.1 (An initial exact formula for $g^{-1}(n)$)**.** *For all $n \geq 1$, we have that*

$$g^{-1}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d).$$

Proof. We first write out the standard recurrence relation for the Dirichlet inverse as

$$g^{-1}(n) = -\sum_{\substack{d|n \\ d>1}} (\omega(d)+1) g^{-1}(n/d) \quad \Longrightarrow \quad (g^{-1} * 1)(n) = -(\omega * g^{-1})(n). \tag{12}$$

We argue that for $1 \leq m \leq \Omega(n)$, we can inductively expand the implication on the right-hand-side of (12) in the form of $(g^{-1} * 1)(n) = F_m(n)$ where $F_m(n) := (-1)^m \cdot (C_m(-) * g^{-1})(n)$, or so that

$$F_m(n) = -\begin{cases} \sum\limits_{\substack{d|n \\ d>1}} F_{m-1}(d) \times \sum\limits_{\substack{r|\frac{n}{d} \\ r>1}} \omega(r) g^{-1}\left(\frac{n}{dr}\right), & 2 \leq m \leq \Omega(n), \\ (\omega * g^{-1})(n), & m = 1. \end{cases}$$

By repeatedly expanding the right-hand-side of the previous equation, we find that for $m := \Omega(n)$ (i.e., with the expansions taken to a maximal depth in the previous equation)

$$(g^{-1} * 1)(n) = (-1)^{\Omega(n)} C_{\Omega(n)}(n) = \lambda(n) C_{\Omega(n)}(n). \tag{13}$$

The formula then follows from (13) by Möbius inversion applied to each side of the last equation. □

Corollary 3.2. *For all positive integers $n \geq 1$, we have that*

$$|g^{-1}(n)| = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d). \tag{14}$$

Proof. By applying Lemma 3.1, Proposition 2.1 and the complete multiplicativity of $\lambda(n)$, we easily obtain the stated result. In particular, since $\mu(n)$ is non-zero only at squarefree integers and since at any squarefree $d \geq 1$ we have $\mu(d) = (-1)^{\omega(d)} = \lambda(d)$, Lemma 3.1 implies

$$\begin{aligned}|g^{-1}(n)| &= \lambda(n) \times \sum_{d|n} \mu\left(\frac{n}{d}\right) \lambda(d) C_{\Omega(d)}(d) \\ &= \sum_{d|n} \mu^2\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) \lambda(nd) C_{\Omega(d)}(d) \\ &= \lambda(n^2) \times \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_{\Omega(d)}(d).\end{aligned}$$

We see that that $\lambda(n^2) = +1$ for all $n \geq 1$ since the number of distinct prime factors (counting multiplicity) of any square integer is even. ☐

Since $C_{\Omega(n)}(n) = |h^{-1}(n)|$ using the notation defined in the the proof of Proposition 2.1, we can see that $C_{\Omega(n)}(n) = (\omega(n))!$ for squarefree $n \geq 1$. A proof of part (B) of Conjecture 1.6 follows as an immediate consequence.

Remark 3.3. Combined with the signedness property of $g^{-1}(n)$ guaranteed by Proposition 2.1, Corollary 3.2 shows that the summatory function of this sequence satisfies

$$G^{-1}(x) = \sum_{d \leq x} \lambda(d) C_{\Omega(d)}(d) M\left(\left\lfloor \frac{x}{d} \right\rfloor\right).$$

Additionally, equation (5) implies that

$$\lambda(d) C_{\Omega(d)}(d) = (g^{-1} * 1)(d) = (\chi_{\mathbb{P}} + \varepsilon)^{-1}(d),$$

where $\chi_{\mathbb{P}}$ denotes the characteristic function of the primes. We clearly recover by inversion that

$$M(x) = G^{-1}(x) + \sum_{p \leq x} G^{-1}\left(\left\lfloor \frac{x}{p} \right\rfloor\right), x \geq 1.$$

It can in fact be shown that

$$\mu(n) = g^{-1}(n) + \sum_{p|n} g^{-1}\left(\frac{n}{p}\right), n \geq 1.$$

3.3 Another connection to the distribution of the primes

The combinatorial properties of $g^{-1}(n)$ are deeply tied to the distribution of the primes $p \leq n$ as $n \to \infty$. The magnitudes and dispersion of the primes $p \leq n$ certainly restricts the repeating of these distinct sequence values. Nonetheless, we can see that the following is still clear about the relation of the weight functions $|g^{-1}(n)|$ to the distribution of the primes: The value of $|g^{-1}(n)|$ is entirely dependent on the pattern of the *exponents* (viewed as multisets) of the distinct prime factors of $n \geq 2$, rather than on the prime factor weights themselves (*cf.* Heuristic 1.5). This observation implies that $|g^{-1}(n)|$ has an inherently additive, rather than multiplicative, structure behind the distribution of its distinct values over $n \leq x$.

Example 3.4. We have a natural extremal behavior with respect to distinct values of $\Omega(n)$ corresponding to squarefree integers and prime powers. If for integers $k \geq 1$ we define the infinite sets M_k and m_k to correspond to the maximal (minimal) sets of positive integers such that

$$M_k := \left\{ n \geq 2 : |g^{-1}(n)| = \sup_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

$$m_k := \left\{ n \geq 2 : |g^{-1}(n)| = \inf_{\substack{j \geq 2 \\ \Omega(j)=k}} |g^{-1}(j)| \right\} \subseteq \mathbb{Z}^+,$$

then any element of M_k is squarefree and any element of m_k is a prime power. Moreover, for any fixed $k \geq 1$ we have that for any $N_k \in M_k$ and $n_k \in m_k$

$$(-1)^k \cdot g^{-1}(N_k) = \sum_{j=0}^{k} \binom{k}{j} \cdot j!, \qquad \text{and} \qquad (-1)^k \cdot g^{-1}(n_k) = 2.$$

The formula for the function $h^{-1}(n) = (g^{-1} * 1)(n)$ defined in the proof of Proposition 2.1 implies that we can express an exact formula for $g^{-1}(n)$ in terms of symmetric polynomials in the exponents of the prime factorization of n. Namely, for $n \geq 2$ and $0 \leq k \leq \omega(n)$ let

$$\widehat{e}_k(n) := [z^k] \prod_{p|n} (1 + z \cdot \nu_p(n)) = [z^k] \prod_{p^\alpha || n} (1 + \alpha z).$$

Then we can prove using (9) and (14) that we can expand exact formulas for the signed inverse sequence in the following form:

$$g^{-1}(n) = h^{-1}(n) \times \sum_{k=0}^{\omega(n)} \binom{\Omega(n)}{k}^{-1} \frac{\widehat{e}_k(n)}{k!}, n \geq 2.$$

The combinatorial formula[B] for $h^{-1}(n) = \lambda(n) \cdot (\Omega(n))! \times \prod_{p^\alpha || n} (\alpha!)^{-1}$ we discovered in the proof of the key signedness proposition from Section 2 suggests additional patterns and more regularity in the contributions of the distinct weighted terms in the summands of $G^{-1}(x)$. A preliminary analysis suggests that bounds of this type may improve upon those we are able to prove for $G^{-1}(x)$ in Section 5.1.

[B] This sequence is also considered using a different motivation based on the DGFs $(1 \pm P(s))^{-1}$ in [4, §2].

4 The distributions of $C_{\Omega(n)}(n)$ and $|g^{-1}(n)|$

We have already suggested in the introduction that the relation of the component functions, $g^{-1}(n)$ and $C_{\Omega(n)}(n)$, to the canonical additive functions $\omega(n)$ and $\Omega(n)$ leads to the regular properties of these functions cited in Table B. Each of $\omega(n)$ and $\Omega(n)$ satisfies an Erdős-Kac theorem that provides a central limit type theorem for the distributions of these functions over $n \leq x$ as $x \to \infty$ [3, 2, 20] (*cf.* [8]). In the remainder of this section we establish more analytical proofs of related properties of these key sequences used to express $G^{-1}(x)$.

4.1 Analytic proofs and adaptations of DGF methods for summing additive functions

Theorem 4.1. *Let the function $\widehat{F}(s,z)$ be defined in terms of the prime zeta function, $P(s)$, for $\operatorname{Re}(s) \geq 2$ and $|z| < |P(s)|^{-1}$ by*

$$\widehat{F}(s,z) := \frac{1}{1+P(s)z} \times \prod_p \left(1 - \frac{1}{p^s}\right)^z.$$

For $|z| < P(2)^{-1}$, the summatory function of the DGF coefficients of $\widehat{F}(s,z) \cdot \zeta(s)^z$ correspond to

$$\widehat{A}_z(x) := \sum_{n \leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}.$$

We have that for all sufficiently large $x \geq 2$ and any $|z| < P(2)^{-1}$

$$\widehat{A}_z(x) = \frac{x}{\Gamma(z)} \cdot \widehat{F}(2,z) \cdot (\log x)^{z-1} + O_z\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right).$$

Proof. We can see from the proof of Proposition 2.1 that

$$C_{\Omega(n)}(n) = \begin{cases} 1, & n = 1; \\ (\Omega(n))! \times \prod\limits_{p^\alpha \| n} \frac{1}{\alpha!}, & n \geq 2. \end{cases}$$

We can then generate exponentially scaled forms of these terms through a product identity of the following form:

$$\sum_{n\geq 1} \frac{C_{\Omega(n)}(n)}{(\Omega(n))!} \cdot \frac{(-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{z^{\Omega(p^r)}}{r! \cdot p^{rs}}\right)^{-1} = \exp\left(-z \cdot P(s)\right), \operatorname{Re}(s) \geq 2 \wedge \operatorname{Re}(P(s)z) > -1.$$

This product based expansion is similar in construction to the parameterized bivariate DGF used in the reference [13, §7.4]. By computing a Laplace transform on the right-hand-side of the above equation, we obtain

$$\sum_{n \geq 1} \frac{C_{\Omega(n)}(n) \cdot (-1)^{\omega(n)} z^{\Omega(n)}}{n^s} = \int_0^\infty e^{-t} \exp\left(-tz \cdot P(s)\right) dt = \frac{1}{1+P(s)z}, \operatorname{Re}(s) > 1 \wedge \operatorname{Re}(P(s)z) > -1.$$

It follows that

$$\sum_{n \geq 1} \frac{(-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}}{n^s} = \zeta(s)^z \times \widehat{F}(s,z), \operatorname{Re}(s) > 1 \wedge |z| < |P(s)|^{-1}.$$

Since $\widehat{F}(s,z)$ is an analytic function of s for all $\operatorname{Re}(s) \geq 2$ whenever the parameter $|z| < |P(s)|^{-1}$, if the sequence $\{b_z(n)\}_{n \geq 1}$ indexes the coefficients in the DGF expansion of $\widehat{F}(s,z) \cdot \zeta(s)^z$, then

$$\left|\sum_{n \geq 1} \frac{b_z(n)(\log n)^{2R+1}}{n^s}\right| < +\infty, \operatorname{Re}(s) \geq 2$$

is uniformly bounded for $|z| \leq R < +\infty$. This fact follows by repeated termwise differentiation $\lceil 2R+1 \rceil$ times with respect to s.

For fixed $0 < |z| < 2$, let the sequence $d_z(n)$ be generated as the coefficients of the DGF

$$\zeta(s)^z = \sum_{n \geq 1} \frac{d_z(n)}{n^s}, \operatorname{Re}(s) > 1,$$

with corresponding summatory function defined by $D_z(x) := \sum_{n \leq x} d_z(n)$. The theorem proved in the reference [13, Thm. 7.17; §7.4] shows that for any $0 < |z| < 2$ and all integers $x \geq 2$

$$D_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right).$$

We set $b_z(n) := (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)}$, define the convolution function $a_z(n) := \sum_{d|n} b_z(d) d_z(n/d)$, and take its summatory function to be $A_z(x) := \sum_{n \leq x} a_z(n)$. Then we have that

$$\begin{aligned} A_z(x) &= \sum_{m \leq x/2} b_z(m) D_z(x/m) + \sum_{x/2 < m \leq x} b_z(m) \\ &= \frac{x}{\Gamma(z)} \times \sum_{m \leq x/2} \frac{b_z(m)}{m^2} \times m \log\left(\frac{x}{m}\right)^{z-1} + O\left(\sum_{m \leq x} \frac{x|b_z(m)|}{m^2} \times m \cdot \log\left(\frac{2x}{m}\right)^{\operatorname{Re}(z)-2}\right). \qquad (15) \end{aligned}$$

We can sum the coefficients $b_z(m)/m$ for integers $m \leq u$ with $u > e$ taken sufficiently large as follows:

$$\sum_{m \leq u} \frac{b_z(m)}{m} = \left(\widehat{F}(2,z) + O(u^{-2})\right) u - \int_1^u \left(\widehat{F}(2,z) + O(t^{-2})\right) dt = \widehat{F}(2,z) + O(u^{-1}).$$

Suppose that $|z| \leq R < P(2)^{-1} \approx 2.21118$. The error term in (15) satisfies

$$\begin{aligned} \sum_{m \leq x} \frac{x \cdot |b_z(m)|}{m^2} \times m \log\left(\frac{2x}{m}\right)^{\operatorname{Re}(z)-2} &\ll x(\log x)^{\operatorname{Re}(z)-2} \times \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m} \\ &\quad + x(\log x)^{-(R+2)} \times \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m} (\log m)^{2R} \\ &= O_z\left(x \cdot (\log x)^{\operatorname{Re}(z)-2}\right), |z| \leq R. \end{aligned}$$

In the main term estimate for $A_z(x)$ from (15), when $m \leq \sqrt{x}$ we have

$$\log\left(\frac{x}{m}\right)^{z-1} = (\log x)^{z-1} + O\left((\log m)(\log x)^{\operatorname{Re}(z)-2}\right).$$

The total sum over the interval $m \leq x/2$ corresponds to bounding the sum components when we take $|z| \leq R$ as follows:

$$\begin{aligned} \sum_{m \leq x/2} b_z(m) D_z(x/m) &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \times \sum_{m \leq x/2} \frac{b_z(m)}{m} \\ &\quad + O_z\left(x(\log x)^{\operatorname{Re}(z)-2} \times \sum_{m \leq \sqrt{x}} \frac{|b_z(m)|}{m} + x(\log x)^{R-1} \times \sum_{m > \sqrt{x}} \frac{|b_z(m)|}{m}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \widehat{F}(2,z) + O_R\left(x(\log x)^{\operatorname{Re}(z)-2} \times \sum_{m \geq 1} \frac{b_z(m)(\log m)^{2R+1}}{m^2}\right) \\ &= \frac{x}{\Gamma(z)} (\log x)^{z-1} \widehat{F}(2,z) + O_R\left(x(\log x)^{\operatorname{Re}(z)-2}\right). \end{aligned}$$

□

It is necessary to break the next theorem into cases for the $1 \le k < \log\log x$ where we have uniform bounds from Theorem 4.1. This necessity arises from the bounds on the incomplete gamma function we established in Section A to handle the subcases of the asymptotics for $\Gamma(a,z)$ at real $a, z > 0$ as $z \to \infty$. Namely, we are forced to account for an extra factor of $(1 - z/a)^{-1}$ when $z = \lambda a$ for some $\lambda > 1$, whereas we inherit a simpler asymptotic formula to approximate $\Gamma(a,z)$ when a is taken to be a fixed parameter that does not vary, or tend to infinity, when z does.

Theorem 4.2. *For large $x > e$ and integers $k \ge 1$, let*

$$\widehat{C}_{k,*}(x) := \sum_{\substack{n \le x \\ \Omega(n) = k}} (-1)^{\omega(n)} C_k(n)$$

Let the function $\widehat{G}(z) := \widehat{F}(2,z)/\Gamma(z+1)$ for $|z| < P(2)^{-1}$ where the function $\widehat{F}(s,z)$ is defined as in Theorem 4.1 for $\operatorname{Re}(s) > 1$. As $x \to +\infty$, we have uniformly for any $1 \le k \le \log\log x$ that

$$\widehat{C}_{k,*}(x) = -\widehat{G}\left(\frac{k-1}{\log\log x}\right) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \left[1 + O_k\left(\frac{k}{(\log\log x)^2}\right)\right].$$

Proof. When $k = 1$, we have that $\Omega(n) = \omega(n)$ for all $n \le x$ such that $\Omega(n) = k$. The $n \le x$ that satisfy this requirement are precisely the primes $p \le x$. Thus we get that the bound is satisfied as

$$\sum_{p \le x} (-1)^{\omega(p)} C_1(p) = -\sum_{p \le x} 1 = -\frac{x}{\log x}\left[1 + O\left(\frac{1}{\log x}\right)\right].$$

Since $O((\log x)^{-1}) = O((\log\log x)^{-2})$, we obtain the required error term bound when $k := 1$.
For $2 \le k \le \log\log x$, we will apply the error estimate from Theorem 4.1 at $r := \frac{k-1}{\log\log x}$. At large x, the error term from this bound contributes that is bounded from above by

$$\begin{aligned} x(\log x)^{-(r+2)} r^{-(k+1)} &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{k+1}} \cdot \frac{1}{e^{k-1}} \ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k+1}}{(k-1)^{3/2}} \cdot \frac{1}{e^{2k}(k-1)!} \\ &\ll \frac{x}{(\log x)^2} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \ll \frac{x}{\log x} \cdot \frac{k \cdot (\log\log x)^{k-5}}{(k-1)!}. \end{aligned}$$

We find an asymptotically accurate main term approximation to the coefficients of the following contour integral for $r \in [0, z_{\max}]$ where $z_{\max} < P(2)^{-1}$ to satisfy Theorem 4.1:

$$\widetilde{A}_r(x) := -\int_{|v|=r} \frac{x \cdot (\log x)^{-v}}{(\log x)\Gamma(1+v) \cdot v^k (1 - P(2)v)} dv. \tag{16}$$

The main term for the sums $\widehat{C}_{k,*}(x)$ is given by $-\frac{x}{\log x} \cdot I_k(r,x)$, where we take

$$\begin{aligned} I_k(r,x) &= \frac{(-1)^k}{2\pi\imath} \times \int_{|v|=r} \frac{\widehat{G}(v)(\log x)^{-v}}{v^k \cdot (1 - P(2)v)} dv \\ &=: I_{1,k}(r,x) + I_{2,k}(r,x). \end{aligned}$$

The first of the component integrals in the last equation is defined to be

$$I_{1,k}(r,x) := \frac{\widehat{G}(r)}{2\pi\imath} \times \int_{|v|=r} \frac{(\log x)^{-v}}{v^k \cdot (1 - P(2)v)} dv.$$

We can inductively compute the remaining coefficients $[z^k] I_{1,k}(r,x)$ with respect to x for fixed $k \le \log\log x$ to apply the Cauchy integral formula. It is not difficult to see that for any integer $m \ge 0$, we have the m^{th}

partial derivative of the scaled integrand with respect to z has the following limiting expansion by applying (A.3) at large x:

$$\begin{aligned}\frac{(-1)^m}{m!} \times \frac{\partial^{(m)}}{\partial v^{(m)}}\left[\frac{(\log x)^{-v}}{1-P(2)v}\right]\Bigg|_{v=0} &= \sum_{j=0}^{m} \frac{(-1)^{m-j} P(2)^j (\log\log x)^{m-j}}{(m-j)!} \\ &= \frac{P(2)^m (\log x)^{-\frac{1}{P(2)}}}{m!} \times \Gamma\left(m+1, -\frac{\log\log x}{P(2)}\right) \\ &= -\frac{(\log\log x)^m}{m!} + O\left(\frac{(\log\log x)^{m-2}}{m!}\right).\end{aligned}$$

Note that we have restricted the asymptotic analysis of the limiting dominant terms in the above formula to cases of $m+1 < \log\log x$. We see by taking $v = r = \frac{k-1}{\log\log x}$ that (*cf.* (28) in Appendix A for the error term bounds)

$$I_{1,k}(r,x) = \frac{\widehat{G}(r)(\log\log x)^{k-1}}{(k-1)!} + O\left(\frac{(\log\log x)^{k-2}}{(\log x)(k-1)!}\right).$$

The second component integral, $I_{2,k}(r,x)$, corresponds to error terms in our approximation that we must bound. This function is defined by

$$I_{2,k}(r,x) := \frac{1}{2\pi\imath} \times \int_{|v|=r} (\widehat{G}(v) - \widehat{G}(r)) \frac{(\log x)^{-v}}{z^k \cdot (1-P(2)v)} dv.$$

After integrating by parts, we write that

$$I_{2,k}(r,x) := \frac{1}{2\pi\imath} \times \int_{|v|=r} (\widehat{G}(v) - \widehat{G}(r) - \widehat{G}'(r)(v-r))(\log x)^{-v} \left[\sum_{i\geq 0} v^{i-k} P(2)^i\right] dz.$$

Notice that

$$\widehat{G}(v) - \widehat{G}(r) - \widehat{G}'(r)(v-r) = \int_r^v (v-w)\widehat{G}''(w)dw \ll |v-r|^2.$$

We then define component integrands for $I_{2,k}(r,x)$ as follows for any integers $i \geq 0$:

$$T_{k,i}(r,x) := \frac{1}{2\pi\imath} \times \int_{|v|=r} (\widehat{G}(v) - \widehat{G}(r) - \widehat{G}'(r)(v-r))(\log x)^{-v} v^{i-k} dz.$$

With the parameterization $z = re^{2\pi\imath\theta}$ for real $\theta \in [-1/2, 1/2]$, we get that

$$T_{k,i}(r,x) \ll r^{3-k+i} \int_{-1/2}^{1/2} (\sin \pi\theta)^2 e^{(k-i-1)\cos(2\pi\theta)} d\theta.$$

Since $|\sin x| \leq |x|$ for all $|x| < 1$ and $\cos(2\pi\theta) \leq 1 - 8\theta^2$ whenever $-1/2 \leq \theta \leq 1/2$, we obtain bounds of the next forms by setting $r := \frac{k-1}{\log\log x}$.

$$\begin{aligned}T_{k,i}(r,x) &\ll r^{3-k+i} e^{k-i-1} \times \int_0^\infty \theta^2 e^{-8(k-i-1)\theta^2} d\theta \\ &\ll \frac{r^{3+i-k} e^{k-i-1}}{(k-i-1)^{3/2}} \ll \frac{(\log\log x)^{k-3-i} e^{k-i-1}}{(k-1-i)^{3/2} (k-1)^{k-3-i}} \\ &\ll \frac{k \cdot (\log\log x)^{k-3-i}}{(k-1)!}.\end{aligned}$$

Then it follows that with $r := \frac{k-1}{\log\log x}$, the sums

$$\sum_{i\geq 0} |T_{k,i}(r,x)| P(2)^i \ll \frac{k \cdot (\log\log x)^{k-3}}{(k-1)!}(1 + o(1)).$$

Finally, we see that whenever $1 \leq k \leq \log\log x$, we have

$$\widehat{G}\left(\frac{k-1}{\log\log x}\right) = \frac{1}{\Gamma\left(1 + \frac{k-1}{\log\log x}\right)} \cdot \frac{\zeta(2)^{(1-k)/\log\log x}}{\left(1 + \frac{(k-1)}{\log\log x}\right)} \gg 1.$$

In fact, we can show that the the function on the left-hand-side of the last equation is asymptotic to $e^{o(1)}$ as $x \to \infty$. This implies the stated result of our theorem. $\square$

With the next lemma, we can accurately approximate asymptotic order of the sums $\mathcal{A}_\omega(x)$ (defined below) for large x by only considering the truncated sums $\mathcal{D}_\omega(x)$ (defined below) where we have the known uniform bounds on the summands for $1 \leq k \leq \log\log x$.

Lemma 4.3. *Suppose that for $x > e$ we define the following functions:*

$$\mathcal{N}_\omega(x) := \left|\sum_{k > \log\log x} (-1)^k \pi_k(x)\right|$$

$$\mathcal{D}_\omega(x) := \left|\sum_{k \leq \log\log x} (-1)^k \pi_k(x)\right|$$

$$\mathcal{A}_\omega(x) := \left|\sum_{k \geq 1} (-1)^k \pi_k(x)\right|.$$

As $x \to \infty$, we have that $\mathcal{D}_\omega(x)/\mathcal{N}_\omega(x) = o(1)$ and $\mathcal{A}_\omega(x) \sim \mathcal{D}_\omega(x)$.

Proof. First, we sum the main term for the function $\mathcal{D}_\omega(x)$ by applying the limiting asymptotics for the incomplete gamma function derived in Lemma A.3 to obtain that

$$\begin{aligned}\mathcal{D}_\omega(x) &= \left|\sum_{1 \leq k \leq \log\log x} \frac{(-1)^k \cdot x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\right| + O(E_\omega(x)) \\ &= \frac{x}{\sqrt{2\pi \log\log x}} + O(E_\omega(x)),\end{aligned}$$

The error term from the bound in the previous equation is defined according to (10) with $\widehat{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) \gg 1$ for all $1 \leq k \leq \log\log x$ as

$$\begin{aligned}E_\omega(x) &:= \sum_{k \leq \log\log x} \frac{x}{\log x} \cdot \frac{k(\log\log x)^{k-3}}{(k-1)!} \leq \frac{x}{\log x} \times \sum_{1 \leq k \leq \log\log x} \frac{(\log\log x)^{k-2}}{(k-1)!} \\ &\leq \frac{x}{(\log x)(\log\log x)} e^{\log\log x} \leq \frac{x}{\log\log x}.\end{aligned}$$

The right-hand-side expression in the previous equation follows by applying Lemma A.3.

Next, we utilize the notation for and bounds on the function $D(x,r)$ from Remark 2.5 to bound the function $\mathcal{N}_\omega(x)$ as follows:

$$\frac{1}{x} \times |\mathcal{N}_\omega(x)| \leq \sum_{k \geq \log\log x} \frac{\pi_k(x)}{x} = \frac{1}{x} \times \sum_{k \geq \log\log x} \#\{2 \leq n \leq x : \omega(n) = k\} \ll 1.$$

Then we see that

$$\left|\frac{\mathcal{D}_\omega(x)}{\mathcal{N}_\omega(x)}\right| = O\left(\frac{1}{\sqrt{\log\log x}}\right) = o(1), \text{ as } x \to \infty.$$

Equivalently, we have shown that $\mathcal{D}_\omega(x) = o(\mathcal{N}_\omega(x))$. The following results from the triangle inequality when x is large:

$$1 + o(1) = \left(\frac{\mathcal{D}_\omega(x) - \mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)}\right)^{-1} \ll \frac{\mathcal{D}_\omega(x)}{\mathcal{A}_\omega(x)} \ll \left(\frac{\mathcal{D}_\omega(x) + \mathcal{N}_\omega(x)}{\mathcal{D}_\omega(x)}\right)^{-1} = 1 + o(1).$$

The last equation implies that $\mathcal{A}_\omega(x) \sim \mathcal{D}_\omega(x)$ as $x \to \infty$. □

Corollary 4.4. *We have for large $x > e$ and $1 \leq k \leq \log\log x$ that*

$$\widehat{C}_k(x) := \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \sim \frac{4\sqrt{2\pi} \cdot x}{(2k-1)} \cdot \frac{(\log\log x)^{k-1/2}}{(k-1)!}.$$

Proof. We have an integral formula involving the unsigned summand sequence that results by applying Abel summation in the form of the next equations.

$$\sum_{n \leq x} \lambda_*(n) h(n) = \left(\sum_{n \leq x} \lambda_*(n)\right) h(x) - \int_1^x \left(\sum_{n \leq t} \lambda_*(n)\right) h'(t) dt \tag{17a}$$

$$\sim \int_1^x \frac{d}{dt}\left[\sum_{n \leq t} \lambda_*(n)\right] h(t) dt \tag{17b}$$

Let the signed left-hand-side summatory function for our function in (17a) be defined precisely for large $x > e$ and any integers $1 \leq k \leq \log\log x$ by

$$\begin{aligned}\widehat{C}_{k,*}(x) &:= \sum_{\substack{n \leq x \\ \Omega(n)=k}} (-1)^{\omega(n)} C_{\Omega(n)}(n) \\ &\sim \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}\left[1 + O\left(\frac{1}{\log\log x}\right)\right].\end{aligned}$$

The second equation above follows from the proof of Theorem 4.2 where we note that $\widehat{G}((k-1)/\log\log x) \sim e^{o(1)}$ as $x \to \infty$. We adopt the notation that $\lambda_*(n) = (-1)^{\omega(n)}$ for $n \geq 1$ and set $L_*(x) := |\sum_{n \leq x} \lambda_*(n)|$ for $x \geq 1$.

We can then transform our previous results for the partial sums over the signed sequences $\lambda_*(n) \cdot C_{\Omega(n)}(n)$ such that $\Omega(n) = k$ to approximate the same sum over the unsigned summands $C_{\Omega(n)}(n)$. The argument is based on approximating $L_*(t)$ for large t using the following uniform asymptotics for $\pi_k(x)$ that hold when $1 \leq k \leq \log\log x$[C]:

$$\pi_k(x) = \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}(1 + o_k(1)), \text{ as } x \to \infty.$$

We have by Lemma A.3 and Lemma 4.3 that

$$L_*(t) := \left|\sum_{n \leq t} (-1)^{\omega(n)}\right| \sim \left|\sum_{k=1}^{\log\log t} (-1)^k \pi_k(x)\right| = \frac{t}{2\sqrt{2\pi}\log\log t} + O\left(\frac{t}{(\log\log t)^{3/2}}\right), \text{ as } t \to \infty.$$

The main term for the reciprocal of the derivative of the main term approximation of this summatory function is given by computation as

$$\frac{1}{L_*'(t)} \sim 2\sqrt{2\pi \log\log t}.$$

[C] We can in fact show that for any $1 \leq k \leq x$, the function $\widehat{\mathcal{G}}(z)$ defined in Remark 2.5 satisfies

$$\widehat{\mathcal{G}}\left(\frac{k-1}{\log\log x}\right) = e^{o(1)} \xrightarrow{x \to \infty} 1.$$

We apply the formula from (17a), to deduce that the unsigned summatory function variant satisfies

$$\begin{aligned}
\widehat{C}_{k,*}(x) = \int_1^x L_*'(t) C_{\Omega(t)}(t) \left[\Omega(t) = k\right]_\delta dt \qquad &\implies \\
C_{\Omega(x)}(x) \left[\Omega(x) = k\right]_\delta \sim \frac{\widehat{C}_{k,*}'(x)}{L_*'(x)} \qquad &\implies \\
C_{\Omega(x)}(x) \left[\Omega(x) = k\right]_\delta \sim 2\sqrt{2\pi \log\log x} \cdot \widehat{C}_{k,*}'(x)(1+o(1)) =: \widehat{C}_{k,**}(x)&.
\end{aligned}$$

We have that

$$\widehat{C}_{k,**}(x) \sim -2\sqrt{2\pi\log\log x}\left[\frac{(\log\log x)^{k-1}}{(\log x)(k-1)!}\left(1 - \frac{1}{\log x}\right) + \frac{(\log\log x)^{k-2}}{(\log x)^2(k-2)!}\right].$$

Hence, integration by parts yields

$$\begin{aligned}
\sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) &\sim \left| \int \widehat{C}_{k,**}(x) dx \right| \qquad (18) \\
&\sim \frac{4\sqrt{2\pi}(\log\log x)^{k-1/2}}{(2k-1)(k-1)!} + \frac{2\sqrt{2\pi}\Gamma\left(k-\frac{1}{2}, \log\log x\right)}{(k-1)!} - \frac{2\sqrt{2\pi}\Gamma\left(k-\frac{3}{2}, \log\log x\right)}{(k-1)!} \\
&\sim \frac{4\sqrt{2\pi}(\log\log x)^{k-1/2}}{(2k-1)(k-1)!}.
\end{aligned}$$

□

4.2 Average order of the unsigned sequences

Proposition 4.5. *We have that as $n \to \infty$*

$$\mathbb{E}\left[C_{\Omega(n)}(n)\right] = \sqrt{2\pi\log\log n}(\log n)(1+o(1)).$$

Proof. We first compute the following summatory function by applying Corollary 4.4 and Lemma A.4:

$$\sum_{k=1}^{\log\log x} \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \sim x\sqrt{2\pi\log\log x}(\log x). \qquad (19)$$

We claim that

$$\begin{aligned}
\frac{1}{x} \times \sum_{n\leq x} C_{\Omega(n)}(n) &= \frac{1}{x} \times \sum_{k\geq 1} \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \qquad (20) \\
&= \frac{1}{x} \times \sum_{k=1}^{\log\log x} \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n)(1+o(1)), \text{ as } x\to\infty.
\end{aligned}$$

To prove (20) it suffices to show that

$$\frac{1}{x} \times \sum_{k>\log\log x} \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) = o\left((\log x)\sqrt{\log\log x}\right), \text{ as } x\to\infty. \qquad (21)$$

We know from Theorem 4.1 that for all sufficiently large x

$$\sum_{n\leq x} (-1)^{\omega(n)} C_{\Omega(n)}(n) z^{\Omega(n)} = x\frac{\widehat{F}(2,z)}{\Gamma(z)}(\log x)^{z-1} + O_z\left(x(\log x)^{\mathrm{Re}(z)-2}\right).$$

By Lemma A.3, we have that the summatory function

$$\left|\sum_{n\leq x}(-1)^{\omega(n)}\right| = \frac{x}{2\sqrt{2\pi}\log\log x} + O\left(\frac{x}{(\log\log x)^{3/2}}\right).$$

We can argue as in the proof of Corollary 4.4 by integration by parts with the Abel summation formula that whenever $1 < |z| < P(2)^{-1}$ and $x > e$ is sufficiently large

$$\begin{aligned}\sum_{n\leq x} C_{\Omega(n)}(n) z^{\Omega(n)} &\ll \frac{\widehat{F}(2,z)}{\Gamma(z)} \times \int_e^x \frac{\sqrt{\log\log t}}{t} \frac{\partial}{\partial t}\left[t(\log t)^{z-1}\right] dt \\ &\ll \frac{x\widehat{F}(2,z)}{\Gamma(z)}\Bigg[\frac{(\log x)^{z-1}(z+\log x)}{z}\sqrt{\log\log x} - \frac{\sqrt{\pi}}{2\sqrt{z-1}}\operatorname{erfi}\left(\sqrt{(z-1)\log\log x}\right) \\ &\qquad - \frac{\sqrt{\pi}}{2z^{3/2}}\operatorname{erfi}\left(\sqrt{z\log\log x}\right)\Bigg] \\ &\ll \frac{x\widehat{F}(2,z)}{\Gamma(1+z)}(\log x)^z\sqrt{\log\log x}.\end{aligned}$$

For all large enough $x > e$, we define

$$\widehat{B}(x,r) := \sum_{\substack{n\leq x \\ \Omega(n)\geq r\log\log x}} C_{\Omega(n)}(n).$$

We argue as in the proof from the reference [13, Thm. 7.20; §7.4] that for $1 \leq r < P(2)^{-1}$

$$\begin{aligned}\sum_{\substack{n\leq x \\ \Omega(n)\geq r\log\log x}} C_{\Omega(n)}(n) r^{\Omega(n)} &\ll x(\log x)^{-r\log r} \times \sum_{n\leq x} C_{\Omega(n)}(n) r^{\Omega(n)} \\ &\sim \frac{x\widehat{F}(2,z)}{\Gamma(1+z)}\sqrt{\log\log x}(\log x)^{r-r\log r}.\end{aligned}$$

Since $\widehat{F}(2,r) = \frac{\zeta(2)^{-r}}{1+P(2)r} \ll 1$ for $r \in [1, P(2)^{-1})$, and similarly we have that $\frac{1}{\Gamma(1+r)} \gg 1$ for r taken within this same range, we get that

$$\sum_{\substack{n\leq x \\ \Omega(n)\geq r\log\log x}} C_{\Omega(n)}(n) r^{\Omega(n)} \ll x\sqrt{\log\log x} \times (\log x)^{r-r\log r}, \text{ for all } 1 \leq r < P(2)^{-1}.$$

When $1 \leq r < P(2)^{-1}$ we have

$$x\sqrt{\log\log x}(\log x)^{r-r\log r} \gg \sum_{\substack{n\leq x \\ \Omega(n)\geq r\log\log x}} C_{\Omega(n)}(n) r^{\Omega(n)} \gg \sum_{\substack{n\leq x \\ \Omega(n)\geq r\log\log x}} C_{\Omega(n)}(n) r^{r\log\log x}.$$

This implies that for $r \in (1, P(2)^{-1})$ we have

$$\widehat{B}(x,r) \ll x(\log x)^{r-2r\log r}\sqrt{\log\log x} = o\left(x(\log x)\sqrt{\log\log x}\right) \tag{22}$$

We wish to evaluate the limiting asymptotics of the sum

$$S_2(x) := \frac{1}{x\sqrt{\log\log x}} \times \sum_{k\geq\log\log x}\sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \ll \widehat{B}(x,1).$$

We have proved that $S_2(x)\sqrt{\log\log x} = o\left((\log x)\sqrt{\log\log x}\right)$ as $x \to \infty$. $\square$

Corollary 4.6. *We have that as $n \to \infty$, the average order of the unsigned inverse sequence satisfies*

$$\mathbb{E}|g^{-1}(n)| = \frac{3\sqrt{2}}{\pi^{3/2}}(\log n)^2\sqrt{\log\log n}(1+o(1)).$$

Proof. We use the formula from Proposition 4.5 to find $\mathbb{E}[C_{\Omega(n)}(n)]$ as $n \to \infty$. This result implies that for sufficiently large t

$$\int \frac{\mathbb{E}[C_{\Omega(t)}(t)]}{t}dt = \frac{\sqrt{2\pi}}{2}(\log t)^2\sqrt{\log\log t}(1+o(1)).$$

Recall that the summatory function of the squarefree integers is approximated for large x by

$$Q(x) := \sum_{n\leq x}\mu^2(n) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Therefore summing over the formula from (14) we find that

$$\begin{aligned}
\mathbb{E}|g^{-1}(n)| &= \frac{1}{n} \times \sum_{d\leq n} C_{\Omega(d)}(d) Q\left(\left\lfloor \frac{n}{d}\right\rfloor\right) \\
&\sim \sum_{d\leq n} C_{\Omega(d)}(d)\left[\frac{6}{d\cdot\pi^2} + O\left(\frac{1}{\sqrt{dn}}\right)\right] \\
&= \frac{6}{\pi^2}\left(\mathbb{E}[C_{\Omega(n)}(n)] + \sum_{d<n}\frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right) + O(1).
\end{aligned}$$

□

4.3 Erdős-Kac theorem analogs for the distributions of the unsigned sequences

Theorem 4.7 (Central limit theorem for the distribution of $C_{\Omega(n)}(n)$). *Set the mean and variance parameter analogs be defined by*

$$\mu_x(C) := \log\log x - \log\left(4\sqrt{2\pi}\right), \qquad \text{and} \qquad \sigma_x(C) := \sqrt{\log\log x}.$$

Let $Y > 0$ be fixed. We have uniformly for all $-Y \leq z \leq Y$ that

$$\frac{1}{x}\cdot\#\left\{2\leq n\leq x : \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)} \leq z\right\} = \Phi(z) + O\left(\frac{1}{\sqrt{\log\log x}}\right), \text{ as } x \to \infty.$$

Proof. Fix any $Y > 0$ and set $z \in [-Y, Y]$. For large x and $2 \leq n \leq x$, define the following auxiliary variables:

$$\alpha_n := \frac{C_{\Omega(n)}(n) - \mu_n(C)}{\sigma_n(C)}, \quad \text{and} \quad \beta_{n,x} := \frac{C_{\Omega(n)}(n) - \mu_x(C)}{\sigma_x(C)}.$$

Let the corresponding densities be defined by the functions

$$\Phi_1(x,z) := \frac{1}{x}\cdot\#\{n\leq x : \alpha_n \leq z\},$$

and

$$\Phi_2(x,z) := \frac{1}{x}\cdot\#\{n\leq x : \beta_{n,x} \leq z\}.$$

We assert that it suffices to consider the distribution of $\Phi_2(x,z)$ as $x \to \infty$ in place of $\Phi_1(x,z)$ to obtain our desired result. The normalizing terms $\mu_n(C)$ and $\sigma_n(C)$ hardly change over $\sqrt{x} \leq n \leq x$. Namely, we see that for $n \in [\sqrt{x}, x]$

$$|\mu_n(C) - \mu_x(C)| \leq \frac{\log 2}{\log x} + O\left(\frac{1}{(\log x)^2}\right),$$

and

$$|\sigma_n(C) - \sigma_x(C)| \leq \frac{\log 2}{(\log x)\sqrt{\log\log x}} + O\left(\frac{1}{(\log\log x)(\log x)^2}\right).$$

In particular, for $\sqrt{x} \leq n \leq x$ and $C_{\Omega(n)}(n) \leq 2 \cdot \mu_x(C)$ we can show using (22) that the following is true:

$$|\alpha_n - \beta_{n,x}| \xrightarrow{x\to\infty} 0.$$

Thus we can replace α_n by $\beta_{n,x}$ and estimate the limiting densities corresponding to these alternate terms. The rest of our argument follows the method in the proof of the related theorem in [13, Thm. 7.21; §7.4] closely. Readers familiar with the reference will see many parallels to those constructions. The crux of the remainder of the proof emulates the methods from Montgomery and Vaughan. After a change of variable we obtain the limiting CLT statement in analog to their analytic proof of the Erdős-Kac theorem for the distributions of $\omega(n)$ and $\Omega(n)$.

We use the formula proved in Corollary 4.4 to estimate the densities claimed within the ranges bounded by z as $x \to \infty$. Let $k \geq 1$ be a natural number such that $k := t_x + \mu_x(C)$ where $t_x := \frac{t\sqrt{\log\log x}}{(\log x)}$. For fixed large x, we define the small parameter $\delta_{t,x} := \frac{t_x}{\mu_x(C)}$. When $|t| \leq \frac{1}{2}\mu_x(C)$, we have by Stirling's formula that

$$\begin{aligned}\frac{1}{x} \times \sum_{\substack{n\leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) &\sim \frac{4\sqrt{2\pi}(\log\log x)^{k-\frac{1}{2}}}{(2k-1)(k-1)!} \\ &\sim \frac{(\log x)}{\sqrt{2\pi \log\log x} \cdot \sigma_x(C)\left(1-\frac{1}{2k}\right)} \times e^{t_x}(1+o(1))^{k-\frac{1}{2}} \times (1+\delta_{t,x})^{-\mu_x(C)(1+\delta_{t,x})-\frac{1}{2}}.\end{aligned}$$

Notice that

$$\begin{aligned}\frac{1}{1-\frac{1}{2k}} &\sim \sum_{m\geq 0} \frac{1}{(2\mu_x(C))^m(1+\delta)^m} \sim 1 + \frac{1}{2\mu_x(C)}\left(1+\delta+O(\delta^2)\right) \\ &= 1 + o_\delta(1), \text{ for } \delta \approx 0 \text{ as } x \to \infty.\end{aligned}$$

We have the uniform estimate that $\log(1+\delta_{t,x}) = \delta_{t,x} - \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)$ whenever $|\delta_{t,x}| \leq \frac{1}{2}$. Then we can expand the factor involving $\delta_{t,x}$ from the previous equation as follows:

$$\begin{aligned}(1+\delta_{t,x})^{-\mu_x(C)(1+\delta_{t,x})-\frac{1}{2}} &= \exp\left(\left(\frac{1}{2}+\mu_x(C)(1+\delta_{t,x})\right) \times \left(-\delta_{t,x} + \frac{\delta_{t,x}^2}{2} + O(|\delta_{t,x}|^3)\right)\right) \\ &= \exp\left(-t_x - \frac{t_x+t_x^2}{2\mu_x(C)} + \frac{t_x^2}{4\mu_x(C)^2} + O\left(\frac{|t_x|^3}{\mu_x(C)^2}\right)\right).\end{aligned}$$

For both $|t| \leq \mu_x(C)^{1/2}$ and $\mu_x(C)^{1/2} < |t| \leq \mu_x(C)^{2/3}$, we can see that

$$\frac{t}{\mu_x(C)} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Similarly, for both $|t| \leq 1$ and $|t| > 1$, we have that

$$\frac{t^2}{\mu_x(C)^2} \ll \frac{1}{\sqrt{\mu_x(C)}} + \frac{|t|^3}{\mu_x(C)^2}.$$

Let the corresponding error terms in (x,t) be denoted by

$$\widetilde{E}(x,t) := O\left(\frac{1}{\sigma_x(C)} + \frac{|t|^3}{\mu_x(C)^2}\right).$$

Combining these estimates with the previous computations, we can deduce that uniformly for $|t| \leq \mu_x(C)^{2/3}$

$$\frac{4\sqrt{2\pi}(\log\log x)^{k-\frac{1}{2}}}{(2k-1)(k-1)!} \sim \frac{(\log x)\sqrt{\log\log x}}{\sqrt{2\pi}\cdot\sigma_x(C)} \times \exp\left(-\frac{t_x^2}{2\sigma_x(C)^2}\right) \times \left[1 + \widetilde{E}(x, t_x)\right].$$

It follows that for $1 \leq k \leq \log\log x$

$$\begin{aligned} f(k,x) &= \frac{1}{x} \times \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega(n)}(n) \\ &\sim \frac{(\log x)}{\sqrt{2\pi\log\log x}\cdot\sigma_x(C)} \times \exp\left(-\frac{(k-\mu_x(C))^2\sqrt{\log\log x}}{2(\log x)\sigma_x(C)^2}\right) \times \left[1 + \widetilde{E}\left(x, \frac{|k-\mu_x(C)|\sqrt{\log\log x}}{(\log x)}\right)\right]. \end{aligned}$$

Since our target probability density function approximating the PDF (in t) of the normal distribution is given by

$$\frac{f(k,x)\sqrt{\log\log x}}{(\log x)} \to \frac{1}{\sqrt{2\pi}\cdot\sigma_x(C)} \times \exp\left(-\frac{t^2}{2\sigma_x(C)^2}\right),$$

we perform the change of variable $s \mapsto \frac{t\sqrt{\log\log x}}{(\log x)}$ to obtain the normalized form of our theorem stated above. By the same argument utilized in the proof of Proposition 4.5, we see that the contributions of these summatory functions for $k \leq \mu_x(C) - \mu_x(C)^{2/3}$ is negligible. We also require that $k \leq \log\log x$ for all large x as we required by Theorem 4.2. We then sum over a corresponding range of

$$\mu_x(C) - \mu_x(C)^{2/3} \leq k \leq \mu_x(C) + z\cdot\sigma_x(C),$$

to approximate the stated normalized densities. As $x \to \infty$ the three terms that result (one main term and two error terms, respectively) can be considered to each correspond to a Riemann sum for an associated integral whose limiting formula corresponds to a main term given by the standard normal CDF at z. $\square$

Corollary 4.8. *Let $Y > 0$. Suppose that $\mu_x(C)$ and $\sigma_x(C)$ are defined as in Theorem 4.7 for large $x > e$. For $Y > 0$ and all $-Y \leq z \leq Y$ we have uniformly that as $x \to \infty$*

$$\frac{1}{x}\cdot\#\left\{2 \leq n \leq x : |g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq z\right\} = \Phi\left\{\frac{6\sigma_x(C)}{\pi^2}\left(\frac{\pi^2 z}{6} + \sigma_x(C)\right) - \frac{6}{\pi^2}\log(4\sqrt{2\pi})\right\} + o(1).$$

Proof. We claim that

$$|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \sim \frac{6}{\pi^2}C_{\Omega(n)}(n), \text{ as } n \to \infty.$$

As in the proof of Corollary 4.6, we obtain that

$$\frac{1}{x} \times \sum_{n \leq x} |g^{-1}(n)| = \frac{6}{\pi^2}\left[\mathbb{E}[C_{\Omega(x)}(x)] + \sum_{d<x}\frac{\mathbb{E}[C_{\Omega(d)}(d)]}{d}\right] + O(1).$$

Let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) \coloneqq f(x) - f(x-1)$. We see that for large n

$$\begin{aligned} |g^{-1}(n)| &= \Delta_n(n\cdot\mathbb{E}|g^{-1}(n)|) \sim \Delta_n\left(\sum_{d \leq n}\frac{6}{\pi^2}\cdot C_{\Omega(d)}(d)\cdot\frac{n}{d}\right) \\ &= \frac{6}{\pi^2}\left[C_{\Omega(n)}(n) + \sum_{d<n} C_{\Omega(d)}(d)\frac{n}{d} - \sum_{d<n} C_{\Omega(d)}(d)\frac{(n-1)}{d}\right] \\ &\sim \frac{6}{\pi^2}C_{\Omega(n)}(n) + \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n-1)|, \text{ as } n \to \infty. \end{aligned}$$

Since $\mathbb{E}|g^{-1}(n-1)| \sim \mathbb{E}|g^{-1}(n)|$ for all sufficiently large n, the result finally follows by a normalization of Theorem 4.7. $\square$

Lemma 4.9. *For all x sufficiently large, if we pick any integer $n \in [2, x]$ uniformly at random, then each of the following statements holds:*

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}(\log\log x)\right) = \frac{1}{2} + o(1) \tag{A}$$

$$\mathbb{P}\left(|g^{-1}(n)| - \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)| \leq \frac{6}{\pi^2}\left(\alpha + \log\log x\right)\right) = \Phi\left(\alpha\right) + o(1), \alpha \in \mathbb{R}. \tag{B}$$

Proof. Each of these results is a consequence of Corollary 4.8. The result in (A) follows since $\Phi(0) = \frac{1}{2}$ by taking

$$z = \frac{6}{\pi^2} \cdot \frac{(\log(4\sqrt{2\pi}) + \alpha)}{\sigma_x(C)} - \sigma_x(C),$$

in Corollary 4.8 for $\alpha = 0$ and with $\sigma_x(C) \coloneqq \log\log x$. Note that as $\alpha \to +\infty$, we get that the right-hand-side of (B) tends to one for large $x \to +\infty$. □

It follows from Lemma 4.9 and Corollary 4.6 that

$$\lim_{x\to\infty} \frac{1}{x} \cdot \#\left\{n \leq x : |g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|(1 + o(1))\right\} = 1.$$

That is, for almost every sufficiently large integer n we recover that

$$|g^{-1}(n)| \leq \frac{6}{\pi^2}\mathbb{E}|g^{-1}(n)|(1 + o(1)).$$

5 Proofs of new formulas and limiting relations for $M(x)$

5.1 Establishing initial asymptotic bounds on the summatory function $G^{-1}(x)$

Let $L(x) := \sum_{n\leq x} \lambda(n)$ for $x \geq 1$. The most recent known upper bound on $L(x)$ (assuming the RH) is established by Humphries based on Soundararajan's result bounding $M(x)$. It is stated in the following form [6]:

$$L(x) = O\left(\sqrt{x} \cdot \exp\left((\log x)^{\frac{1}{2}} (\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right), \text{ for any } \varepsilon > 0; \text{ as } x \to \infty. \tag{23}$$

Theorem 5.1. *We have that for almost every sufficiently large x, there exists $1 \leq t_0 \leq x$ such that*

$$G^{-1}(x) = O\left(L(t_0) \cdot \mathbb{E}|g^{-1}(x)|\right).$$

If the RH is true, then for any $\varepsilon > 0$ and all large integers $x > e$

$$G^{-1}(x) = O\left((\log x)^2 \sqrt{\log\log x}\sqrt{x} \times \exp\left(\sqrt{\log x}(\log\log x)^{\frac{5}{2}+\varepsilon}\right)\right).$$

Proof. We write the next formulas for $G^{-1}(x)$ at almost every large $x > e$ by Abel summation and applying the mean value theorem:

$$\begin{aligned} G^{-1}(x) &= \sum_{n\leq x} \lambda(n)|g^{-1}(n)| \\ &= L(x)|g^{-1}(x)| - \int L(x) \frac{d}{dx}\left|g^{-1}(x)\right| dx \\ &= O\left(|L(t_0)| \cdot \mathbb{E}|g^{-1}(x)|\right), \text{ for some } t_0 \in [1, x]. \end{aligned} \tag{24}$$

The proof of this result appeals to the last few results we used to establish the probabilistic interpretations of the distribution of $|g^{-1}(n)|$ as $n \to \infty$ in Section 4.

We need to bound the sums of the maximal extreme values of $|g^{-1}(n)|$ over $n \leq x$ as $x \to \infty$ to prove the second bound. We know by a result of Robin that [22]

$$\omega(n) \ll \frac{\log n}{\log\log n}, \text{ as } n \to \infty.$$

Recall that the values of $|g^{-1}(n)|$ are locally maximized when n is squarefree with

$$|g^{-1}(n)| \leq \sum_{j=0}^{\omega(n)} \binom{\omega(n)}{j} j! \ll \Gamma(\omega(n)+1) \ll \left(\frac{\log n}{\log\log n}\right)^{\frac{\log n}{\log\log n}+\frac{1}{2}}.$$

Since we have deduced that the set of $n \leq x$ on which $|g^{-1}(n)|$ is substantially larger than its average order is asymptotically thin, we find the bounds

$$\begin{aligned} \left|\int_{x-o(1)}^{x} L'(t)|g^{-1}(t)|dt\right| &\ll \int_{x-o(1)}^{x} \left(\frac{\log t}{\log\log t}\right)^{\frac{\log t}{\log\log t}+\frac{1}{2}} dt = o\left(\left(\frac{\log x}{\log\log x}\right)^{\frac{\log x}{\log\log x}+\frac{1}{2}}\right) \\ &\ll o\left(\frac{x}{(\log x)^m (\log\log x)^r}\right), \text{ for any } m, r = o\left(\frac{(\log x)(\log\log\log x)}{\log\log x}\right), \text{ as } x \to \infty. \end{aligned}$$

Indeed, we can see that the limit

$$\begin{aligned} \lim_{x\to\infty} \frac{1}{x}\left(\frac{\log x}{\log\log x}\right)^{\frac{\log x}{\log\log x}+\frac{1}{2}} (\log x)^m (\log\log x)^r &\ll \lim_{x\to\infty} x^{-\frac{(\log x)(\log\log\log x)}{\log\log x}} (\log x)^{m+r} \\ = \lim_{x\to\infty} \exp\left((m+r)\log x - (\log x)^2 \frac{\log\log\log x}{\log\log x}\right) &= \lim_{t\to\infty} e^{-t} = 0. \end{aligned}$$

For large x, let $\mathcal{R}_x \coloneqq \{t \leq x : |g^{-1}(t)| \ggg \mathbb{E}|g^{-1}(t)|\}$ such that $|\mathcal{R}_x| = o(1)$. The formula from (17a) implies that for large x and any $m, r = o\left(\frac{(\log x)(\log\log\log x)}{\log\log x}\right)$

$$\begin{aligned} G^{-1}(x) &= O\left(\int L'(x)|g^{-1}(x)|dx\right) = O\left(\mathbb{E}|g^{-1}(x)| \times \int L'(x)dx + \int_{x-|\mathcal{R}_x|}^{x} |L'(t)| \cdot |g^{-1}(t)|dt\right) \\ &= O\left(\mathbb{E}|g^{-1}(x)| \cdot |L(x)| + o\left(\frac{x}{(\log x)^m (\log\log x)^r}\right)\right). \end{aligned}$$

If the RH is true, by applying Humphries' result in (23) in tandem with Corollary 4.6, then for any $\varepsilon > 0$, $m, r = o\left(\frac{(\log x)(\log\log\log x)}{\log\log x}\right)$ and almost every large integer $x \geq 1$ we have that

$$\begin{aligned} G^{-1}(x) &= O\left((\log x)^2 \sqrt{\log\log x} \cdot \sqrt{x} \times \exp\left(\sqrt{\log x} \cdot (\log\log x)^{\frac{5}{2}+\varepsilon}\right) + o\left(\frac{x}{(\log x)^m (\log\log x)^r}\right)\right), \\ &= O\left((\log x)^2 \sqrt{\log\log x} \cdot x^{\frac{1}{2} + \frac{(\log\log x)^{5/2+\varepsilon}}{\sqrt{\log x}}} + o\left(x^{1-\log\log\log x}\right)\right). \end{aligned}$$

To obtain the conclusion of the second result, we take limits as $x \to \infty$ to see that the dominant term is given by the leftmost term in the last equation. ☐

5.2 Bounding $M(x)$ by asymptotics for $G^{-1}(x)$

Proposition 5.2. *For all sufficiently large x, we have that the Mertens function satisfies*

$$M(x) = G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right]. \tag{25}$$

Proof. We know by applying Corollary 1.4 that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x} g^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) + 1\right] \\ &= G^{-1}(x) + \sum_{k=1}^{\frac{x}{2}} g^{-1}(k)\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) \\ &= G^{-1}(x) + G^{-1}\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + \sum_{k=1}^{\frac{x}{2}-1} G^{-1}(k)\left[\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right]. \end{aligned}$$

The upper bound on the sum is truncated to $k \in \left[1, \frac{x}{2}\right]$ in the second equation above due to the fact that $\pi(1) = 0$. The third formula follows from summation by parts. ☐

Lemma 5.3. *For sufficiently large x, integers $k \in \left[1, \sqrt{x}\right]$ and $m \geq 0$, we have that*

$$\frac{x}{k \cdot \log^m\left(\frac{x}{k}\right)} - \frac{x}{(k+1) \cdot \log^m\left(\frac{x}{k+1}\right)} \asymp \frac{x}{(\log x)^m \cdot k(k+1)}, \tag{A}$$

and

$$\sum_{k=\sqrt{x}}^{\frac{x}{2}} \frac{x}{k(k+1)} = \sum_{k=\sqrt{x}}^{\frac{x}{2}} \frac{x}{k^2} + O(1). \tag{B}$$

Proof. The proof of (A) is obvious since for $k_0 \in \left[1, \frac{x}{2}\right]$ we have that

$$\log(2)(1 + o(1)) \leq \log\left(\frac{x}{k_0}\right) \leq \log(x).$$

To prove (B), notice that

$$\frac{x}{k(k+1)} - \frac{x}{k^2} = -\frac{x}{k^2(k+1)}.$$

Then we see that

$$\left|\int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^2(t+1)} dt\right| \leq \left|\int_{\sqrt{x}}^{\frac{x}{2}} \frac{x}{t^3} dt\right| \asymp 1. \qquad \square$$

Corollary 5.4. *We have that as* $x \rightarrow \infty$

$$M(x) = O\left(G^{-1}(x) + G^{-1}\left(\frac{x}{2}\right) + \frac{x}{\log x} \times \sum_{k \leq \sqrt{x}} \frac{G^{-1}(k)}{k^2} + (\log x)^2 (\log\log x)^{3/2}\right).$$

Proof. We need to first bound the prime counting function differences in the formula given by Proposition 5.2. We will require the following known bounds on the prime counting function due to Rosser and Schoenfeld for all large $x > 59$ [23, Thm. 1]:

$$\frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right). \tag{26}$$

The result in (26) together with Lemma 5.3 implies that for $\sqrt{x} \leq k \leq \frac{x}{2}$

$$\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right) = O\left(\frac{x}{k^2 \cdot \log\left(\frac{x}{k}\right)}\right).$$

We will rewrite the intermediate formula from the proof of Proposition 5.2 as a sum of two components with summands taken over disjoint intervals. For large $x > e$, let

$$\begin{aligned} S_1(x) &:= \sum_{1 \leq k \leq \sqrt{x}} g^{-1}(k) \pi\left(\frac{x}{k}\right) \\ S_2(x) &:= \sum_{\sqrt{x} < k \leq \frac{x}{2}} g^{-1}(k) \pi\left(\frac{x}{k}\right). \end{aligned}$$

We assert by the asymptotic formulas for the prime counting function that

$$S_1(x) = O\left(\frac{x}{\log x} \times \sum_{k \leq \sqrt{x}} \frac{G^{-1}(k)}{k^2}\right).$$

To bound the second sum, we perform summation by parts as in the proof of the proposition and apply the bound above for the difference of the summand functions to obtain that

$$\begin{aligned} S_2(x) &= O\left(G^{-1}\left(\frac{x}{2}\right) + \int_{\sqrt{x}}^{\frac{x}{2}} \frac{G^{-1}(t)}{t^2 \log\left(\frac{x}{t}\right)} dt\right) \\ &= O\left(G^{-1}\left(\frac{x}{2}\right) + \max_{\sqrt{x} < k < \frac{x}{2}} \frac{|G^{-1}(k)|}{k} \times \int_{\sqrt{x}}^{\frac{x}{2}} \frac{dt}{t \cdot \log\left(\frac{x}{t}\right)}\right) \\ &= O\left(G^{-1}\left(\frac{x}{2}\right) + (\log\log x) \times \max_{\sqrt{x} < k < \frac{x}{2}} \frac{|G^{-1}(k)|}{k}\right). \end{aligned}$$

The rightmost maximum term in the previous equation is known to satisfy $\frac{|G^{-1}(k)|}{k} \ll \mathbb{E}|g^{-1}(k)|$ as $k \rightarrow \infty$. The conclusion follows since the average order of $|g^{-1}(n)|$ is increasing for sufficiently large n. $\square$

6 Conclusions

We have identified a new sequence, $\{g^{-1}(n)\}_{n\geq 1}$, which is the Dirichlet inverse of the shifted additive function, $g := \omega + 1$. In general, we find that the Dirichlet inverse of any arithmetic function f such that $f(1) \neq 0$ is expressed at each $n \geq 2$ as a signed sum of m-fold convolutions of f with itself for $1 \leq m \leq \Omega(n)$. As we discussed in the remarks in Section 3.3, it happens that there is a natural combinatorial interpretation to the distribution of distinct values of $|g^{-1}(n)|$ for $n \leq x$ involving the distribution of the primes $p \leq x$ at large x. In particular, the magnitude of $|g^{-1}(n)|$ depends only on the pattern of the exponents of the prime factorization of n in so much as $|g^{-1}(n_1)| = |g^{-1}(n_2)|$ whenever $\omega(n_1) = \omega(n_2)$, $\Omega(n_1) = \Omega(n_2)$, and where the is a one-to-one correspondence $\nu_{p_1}(n_1) = \nu_{p_2}(n_2)$ between the distinct primes $p_1|n_1$ and $p_2|n_2$.

The signedness of $g^{-1}(n)$ is given by $\lambda(n)$ for all $n \geq 1$. This leads to a familiar dependence of the summatory functions $G^{-1}(x)$ on the distribution of the summatory function $L(x)$. Section 5 provides equivalent characterizations of the limiting properties of $M(x)$ by exact formulas and asymptotic relations involving $G^{-1}(x)$ and $L(x)$. We emphasize that our new work on the Mertens function proved within this article is significant in providing a new window through which we can view bounding $M(x)$. The computational data generated in Table B suggests numerically that the distribution of $G^{-1}(x)$ may be easier to work with than those of $M(x)$ or $L(x)$. The remarks given in Section 3.3 about the direct combinatorial relation of the distinct (and repetition of) values of $|g^{-1}(n)|$ for $n \leq x$ are suggestive towards bounding main terms for $G^{-1}(x)$ along infinite subsequences.

One topic that we do not touch on in the article is to consider what correlation (if any) exists between $\lambda(n)$ and the unsigned sequence of $|g^{-1}(n)|$ with the limiting distribution proved in Corollary 4.8. Much in the same way that variants of the Erdős-Kac theorem are proved by defining random variables related to $\omega(n)$, we suggest an analysis of the summatory function $G^{-1}(x)$ by scaling the explicitly distributed $|g^{-1}(n)|$ for $n \leq x$ as $x \to \infty$ by its signed weight of $\lambda(n)$ using an initial heuristic along these lines for future work.

Another experiment illustrated in the online computational reference [24] suggests that for many, if not most sufficiently large x, we may consider replacing the summatory function with other summands weighted by $\lambda(n)$. These alternate sums can be seen to average these sequences differently while still preserving the original asymptotic order of $|G^{-1}(x)|$ heuristically. For example, each of the following three summatory functions offer a unique interpretation of an average of sorts that "mixes" the values of $\lambda(n)$ with the unsigned sequence $|g^{-1}(n)|$ over $1 \leq n \leq x$:

$$\begin{aligned}
G_{*}^{-1}(x) &:= \sum_{n\leq x} \frac{1}{2\gamma - 1 + \log n} \times \sum_{d|n} \lambda\left(\frac{n}{d}\right) |g^{-1}(d)| \\
G_{**}^{-1}(x) &:= \sum_{n\leq x} \frac{1}{2\gamma - 1 + \log n} \times \sum_{d|n} \lambda\left(\frac{n}{d}\right) g^{-1}(d) \\
G_{***}^{-1}(x) &:= \sum_{n\leq x} \frac{1}{2\gamma - 1 + \log n} \times \sum_{d|n} g^{-1}(d).
\end{aligned}$$

Then based on preliminary numerical results, a large proportion of the $y \leq x$ for large x satisfy

$$\left|\frac{G_{*}^{-1}(y)}{G^{-1}(y)}\right|^{-1}, \left|\frac{G_{**}^{-1}(y)}{G^{-1}(y)}\right|, \left|\frac{G_{***}^{-1}(y)}{G^{-1}(y)}\right| \in (0,3].$$

Variants of this type of summatory function identity exchange are similarly suggested for future work to extend the topics and new results proved in this article.

Acknowledgments

We thank the following professors that offered discussion, feedback and correspondence while the article was written: Paul Pollack, Steven J. Miller, Gergő Nemes, Ernie Croot, Bruce Reznick and Michael Lacey.

The work on the article was supported in part by funding made available within the School of Mathematics at the Georgia Institute of Technology in 2020. Without this combined support, the article would not have been possible.

References

[1] T. M. Apostol. *Introduction to Analytic Number Theory.* Springer–Verlag, 1976.

[2] P. Billingsley. On the central limit theorem for the prime divisor function. *Amer. Math. Monthly*, 76(2):132–139, 1969.

[3] P. Erdős and M. Kac. The guassian errors in the theory of additive arithmetic functions. *American Journal of Mathematics*, 62(1):738–742, 1940.

[4] C. E. Fröberg. On the prime zeta function. *BIT Numerical Mathematics*, 8:87–202, 1968.

[5] G. H. Hardy and E. M. Wright, editors. *An Introduction to the Theory of Numbers.* Oxford University Press, 2008 (Sixth Edition).

[6] P. Humphries. The distribution of weighted sums of the Liouville function and Pólya's conjecture. *J. Number Theory*, 133:545–582, 2013.

[7] G. Hurst. Computations of the Mertens function and improved bounds on the Mertens conjecture. *Math. Comp.*, 87:1013–1028, 2018.

[8] H. Hwang and S. Janson. A central limit theorem for random ordered factorizations of integers. *Electron. J. Probab.*, 16(12):347–361, 2011.

[9] H. Iwaniec and E. Kowalski. *Analytic Number Theory*, volume 53. AMS Colloquium Publications, 2004.

[10] T. Kotnik and H. té Riele. The Mertens conjecture revisited. *Algorithmic Number Theory*, 7^{th} International Symposium, 2006.

[11] T. Kotnik and J. van de Lune. On the order of the Mertens function. *Exp. Math.*, 2004.

[12] R. S. Lehman. On Liouville's function. *Math. Comput.*, 14:311–320, 1960.

[13] H. L. Montgomery and R. C. Vaughan. *Multiplicative Number Theory: I. Classical Theory.* Cambridge, 2006.

[14] G. Nemes. The resurgence properties of the incomplete gamma function ii. *Stud. Appl. Math.*, 135(1):86–116, 2015.

[15] G. Nemes. The resurgence properties of the incomplete gamma function i. *Anal. Appl. (Singap.)*, 14(5):631–677, 2016.

[16] G. Nemes and A. B. Olde Daalhuis. Asymptotic expansions for the incomplete gamma function in the transition regions. *Math. Comp.*, 88(318):1805–1827, 2019.

[17] N. Ng. The distribution of the summatory function of the Móbius function. *Proc. London Math. Soc.*, 89(3):361–389, 2004.

[18] A. M. Odlyzko and H. J. J. té Riele. Disproof of the Mertens conjecture. *J. Reine Angew. Math*, 1985.

[19] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. *NIST Handbook of Mathematical Functions.* Cambridge University Press, 2010.

[20] A. Renyi and P. Turan. On a theorem of Erdős-Kac. *Acta Arithmetica*, 4(1):71–84, 1958.

[21] P. Ribenboim. *The new book of prime number records.* Springer, 1996.

[22] G. Robin. Estimate of the Chebyshev function θ on the k^{th} prime number and large values of the number of prime divisors function $\omega(n)$ of n. *Acta Arith.*, 42(4):367–389, 1983.

[23] J. B. Rosser and L. Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.*, 6:64–94, 1962.

[24] M. D. Schmidt. SageMath and Mathematica software for computations with the Mertens function, 2021. https://github.com/maxieds/MertensFunctionComputations.

[25] N. J. A. Sloane. The Online Encyclopedia of Integer Sequences, 2021. http://oeis.org.

[26] K. Soundararajan. Partial sums of the Möbius function. *Annals of Mathematics*, 2009.

[27] E. C. Titchmarsh. *The theory of the Riemann zeta function.* Clarendon Press, 1951.

A Appendix: Asymptotic formulas

We thank Gergő Nemes from the Alfréd Rényi Institute of Mathematics for his careful notes on the limiting asymptotics for the sums identified in this section. We have adapted his proofs to establish the next few lemmas.

Facts A.1 (The incomplete gamma function)**.** The (upper) *incomplete gamma function* is defined by [19, §8.4]

$$\Gamma(a,z) = \int_z^{\infty} t^{a-1}e^{-t}dt, a \in \mathbb{R}, |\arg z| < \pi,$$

and where z is defined on the universal covering of $\mathbb{C} \setminus \{0\}$ by analytic continuation. For $a \in \mathbb{Z}^+$, $\Gamma(a,z)$ is an entire function of z. The following properties of $\Gamma(a,z)$ hold:

$$\Gamma(a,z) = (a-1)! \cdot e^{-z} \times \sum_{k=0}^{a-1} \frac{z^k}{k!}, \text{ for } a \in \mathbb{Z}^+, z \in \mathbb{C}, \tag{27a}$$

$$\Gamma(a,z) \sim z^{a-1}e^{-z}, \text{ for fixed } a \in \mathbb{C}, \text{ as } z \to +\infty. \tag{27b}$$

If $z, a \to \infty$ with $z = \lambda a$ for some $\lambda > 1$ such that $(\lambda - 1)^{-1} = o(|a|^{1/2})$, then [14]

$$\Gamma(a,z) = z^a e^{-z} \times \sum_{n \geq 0} \frac{(-a)^n b_n(\lambda)}{(z-a)^{2n+1}}, \tag{27c}$$

where the sequence $b_n(\lambda)$ satisfies the characteristic relation that $b_0(\lambda) = 1$ and[D]

$$b_n(\lambda) = \lambda(1-\lambda)b'_{n-1}(\lambda) + \lambda(2n-1)b_{n-1}(\lambda), n \geq 1.$$

Proposition A.2. *Suppose that $a, z, \lambda > 0$ are such that $z = \lambda a$. If $\lambda > 1$, then as $z \to +\infty$*

$$\Gamma(a,z) = \frac{z^{a-1}e^{-z}}{1-\lambda^{-1}} + O_{\lambda}\left(z^{a-2}e^{-z}\right).$$

If $\lambda > 0.567142 > W(1)$ where $W(x)$ denotes the principal branch of the Lambert W-function, then as $z \to +\infty$

$$\Gamma(a, ze^{\pm \pi \imath}) = -e^{\pm \pi \imath a} \frac{z^{a-1}e^{z}}{1+\lambda^{-1}} + O_{\lambda}\left(z^{a-2}e^{z}\right).$$

Proof. Using the notation from (27c) and [15], we have that

$$\Gamma(a,z) = \frac{z^{a-1}e^{-z}}{1-\lambda^{-1}} + z^a e^{-z} R_1(a,\lambda).$$

From the bounds in [15, §3.1], we get

$$|z^a e^{-z} R_1(a,\lambda)| \leq z^a e^{-z} \times \frac{a \cdot b_1(\lambda)}{(z-a)^3} = \frac{z^{a-2}e^{-z}}{(1-\lambda^{-1})^3}$$

The main and error terms in the previous equation can also be seen by applying the asymptotic series in (27c) directly.

[D] An exact formula for $b_n(\lambda)$ is given in terms of the *second-order Eulerian number triangle* [25, A008517] as follows:

$$b_n(\lambda) = \sum_{k=0}^{n} \left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle \lambda^{k+1}.$$

The proof of the second equation above follows from the following asymptotics [14, Eq. (1.1)]

$$\Gamma(-a,z) \sim z^{-a}e^{-z} \times \sum_{n\geq 0} \frac{a^n b_n(-\lambda)}{(z+a)^{2n+1}},$$

by setting $a \mapsto ae^{\pm\pi\imath}$ and $z \mapsto ze^{\pm\pi\imath}$ with $\lambda = z/a > 0.567142 > W(1)$. Note that we cannot write this expansion as $\Gamma(a,-z)$ directly unless $a \in \mathbb{Z}^+$ as the incomplete gamma function has a branch point at the origin with respect to its second variable. This function becomes a single-valued analytic function of its second input on the universal covering of $\mathbb{C} \setminus \{0\}$. The restriction on the range of λ over which the second formula holds is made to ensure that the last formula from the reference is valid at negative real a. □

Lemma A.3. *For $x \to +\infty$, we have that*

$$S_1(x) := \frac{x}{\log x} \times \left| \sum_{1\leq k\leq \log\log x} \frac{(-1)^k (\log\log x)^{k-1}}{(k-1)!} \right| = \frac{x}{2\sqrt{2\pi\log\log x}} + O\left(\frac{x}{(\log\log x)^{3/2}}\right).$$

Proof. We have that for $n \geq 1$ and any $t > 0$ [19, *cf.* §8.4]

$$\sum_{1\leq k\leq n} \frac{(-1)^k t^{k-1}}{(k-1)!} = -e^{-t} \times \frac{\Gamma(n,-t)}{(n-1)!}.$$

Suppose that $t = n + \xi$ with $\xi = O(1)$. By the second formula in Proposition A.2 with the parameters $(a,z,\lambda) \mapsto \left(n, t, 1 + \frac{\xi}{n}\right)$, we deduce that as $n, t \to +\infty$.

$$\Gamma(n,-t) = (-1)^{n+1} \times \frac{t^n e^t}{t+n} + O\left(\frac{nt^n e^t}{(t+n)^3}\right) = (-1)^{n+1}\frac{t^n e^t}{2n} + O\left(\frac{t^n e^t}{nt}\right). \tag{28}$$

Accordingly, we see that

$$\sum_{1\leq k\leq n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{t^n}{2n!} + O\left(\frac{t^{n-1}}{n!}\right).$$

By a variant of Stirling's formula in [19, *cf.* Eq. (5.11.8)]

$$n! = \Gamma(1+t-\xi) = \sqrt{2\pi}\cdot t^{t-\xi+1/2}e^{-t}\left(1 + O(t^{-1})\right) = \sqrt{2\pi}\cdot t^{n+1/2}e^{-t}\left(1+O(t^{-1})\right).$$

Hence, as $n \to +\infty$ with $t := n + \xi$ and $\xi = O(1)$, we obtain

$$\sum_{k=1}^{n} \frac{(-1)^k t^{k-1}}{(k-1)!} = (-1)^n \frac{e^t}{2\sqrt{2\pi t}} + O\left(\frac{e^t}{t^{3/2}}\right).$$

The conclusion follows by taking $n = \lfloor \log\log x \rfloor$ and $t = \log\log x$ and applying the triangle inequality to obtain the desired result. □

Lemma A.4. *For $x \to +\infty$, we have that*

$$S_3(x) := \sum_{1\leq k\leq \log\log x} \frac{(\log\log x)^{k-1/2}}{(2k-1)(k-1)!} = \frac{(\log x)}{4\sqrt{\log\log x}} + O\left(\frac{\log x}{\log\log x}\right).$$

Proof. Notice that we can write

$$\sum_{k\geq 1} \frac{(\log\log x)^{k-1}}{(2k+1)(k-1)!} - \sum_{k\geq 0} \frac{(\log\log x)^k}{(2k-1)k!} \leq S_3(x) \leq \sum_{k\geq 1} \frac{(\log\log x)^{k-1}}{(2k-1)(k-1)!} - \sum_{k\geq 0} \frac{(\log\log x)^{k-1}}{2\cdot k!}(1+o(1)).$$

As $|z| \to \infty$, the *imaginary error function*, denoted by $\operatorname{erfi}(z)$, has the following asymptotic expansion [19, §7.12]:

$$\operatorname{erfi}(z) := \frac{2}{\sqrt{\pi} \cdot \imath} \times \int_0^{\imath z} e^{t^2} dt = \frac{e^{z^2}}{\sqrt{\pi}} \left(z^{-1} + \frac{z^{-3}}{2} + \frac{3z^{-5}}{4} + \frac{15z^{-7}}{8} + O\left(z^{-9}\right) \right).$$

The symbolic summation procedures in *Mathematica* [24], show that we can arrive at this bound in the form of the following inequalities by applying the asymptotic series formula in the last equation:

$$\frac{\log x}{2\sqrt{\log\log x}} + O\left(\frac{\log x}{(\log\log x)^2}\right) \ll S_3(x) \ll \frac{\log x}{2\sqrt{\log\log x}} + O\left(\frac{\log x}{\log\log x}\right). \tag{29}$$

This implies the bounds given in (29) match the stated formula up to a factor of $\frac{1}{2}$. We will give an exact proof of the result directly using the bounds for the incomplete gamma function established in the recent reference [16]. The reference takes into account the behavior of the incomplete gamma function $\Gamma(a, z)$ near the transition point $\lambda = z/a = 1$ as $a, z \to +\infty$.

For $n \geq 1$ and any $t > 0$, let

$$\widetilde{S}_n(t) := \sum_{1 \leq k \leq n} \frac{t^{k-1}}{(2k-1)(k-1)!}.$$

By the formula in (27b) and a change of variable, we get that

$$\begin{aligned} \widetilde{S}_n(t) &= \int_0^1 \left(\sum_{k=1}^n \frac{(s^2 t)^{k-1}}{(k-1)!} \right) ds \\ &= \frac{1}{(n-1)!} \times \int_0^1 e^{s^2 t} \Gamma(n, s^2 t) ds \\ &= \frac{1}{2t(n-1)!} \times \int_0^1 e^y \Gamma(n, y) dy. \end{aligned}$$

Integration by parts performed one time with

$$\left\{ \begin{array}{ll} u_x = \Gamma(n, y) & v_x' = e^y dy \\ u_x' = -y^{n-1} e^{-y} dy & v_x = e^y \end{array} \right\},$$

implies that

$$\begin{aligned} \widetilde{S}_n(t) &= \frac{1}{2t(n-1)!} \times \left[\Gamma(n, t) e^t - \Gamma(n) + \int_0^t y^{n-1} dy \right] \\ &= \frac{\Gamma(n, t) e^t}{2t(n-1)!} - \frac{1}{2t} + \frac{t^{n-1}}{2n!}. \end{aligned} \tag{30}$$

Suppose that $t = n + \xi$ where $\xi = O(1)$. According to the reference [16, Eq. (2.4)], we have that as $t \to +\infty$

$$\Gamma(n, t) = t^n e^{-t} \sqrt{\frac{\pi}{2t}} \left(1 + O(t^{-1/2})\right).$$

By the variant of Stirling's formula expressed in [19, §5.11(i)], we again have that

$$(n-1)! = \sqrt{2\pi} \cdot t^{n-1/2} e^{-t} \left(1 + O(t^{-1})\right).$$

Whence, with $t = n + O(1)$ and as $n \to +\infty$, we obtain

$$\widetilde{S}_n(t) = \frac{e^t}{4t} + O\left(\frac{e^t}{n t^{3/2}}\right).$$

The conclusion follows taking $n = \lfloor \log\log x \rfloor$, $t = \log\log x$ and mulitplying $\widetilde{S}_n(t)$ by $(\log\log x)^{1/2}$. $\square$

B Table: The Dirichlet inverse function $g^{-1}(n)$

n	Primes	Sqfree	PPower	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
1	1^1	Y	N	1	0	1.0000000	1.000000	0.000000	1	1	0
2	2^1	Y	Y	−2	0	1.0000000	0.500000	0.500000	−1	1	−2
3	3^1	Y	Y	−2	0	1.0000000	0.333333	0.666667	−3	1	−4
4	2^2	N	Y	2	0	1.5000000	0.500000	0.500000	−1	3	−4
5	5^1	Y	Y	−2	0	1.0000000	0.400000	0.600000	−3	3	−6
6	$2^1 3^1$	Y	N	5	0	1.0000000	0.500000	0.500000	2	8	−6
7	7^1	Y	Y	−2	0	1.0000000	0.428571	0.571429	0	8	−8
8	2^3	N	Y	−2	0	2.0000000	0.375000	0.625000	−2	8	−10
9	3^2	N	Y	2	0	1.5000000	0.444444	0.555556	0	10	−10
10	$2^1 5^1$	Y	N	5	0	1.0000000	0.500000	0.500000	5	15	−10
11	11^1	Y	Y	−2	0	1.0000000	0.454545	0.545455	3	15	−12
12	$2^2 3^1$	N	N	−7	2	1.2857143	0.416667	0.583333	−4	15	−19
13	13^1	Y	Y	−2	0	1.0000000	0.384615	0.615385	−6	15	−21
14	$2^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	−1	20	−21
15	$3^1 5^1$	Y	N	5	0	1.0000000	0.466667	0.533333	4	25	−21
16	2^4	N	Y	2	0	2.5000000	0.500000	0.500000	6	27	−21
17	17^1	Y	Y	−2	0	1.0000000	0.470588	0.529412	4	27	−23
18	$2^1 3^2$	N	N	−7	2	1.2857143	0.444444	0.555556	−3	27	−30
19	19^1	Y	Y	−2	0	1.0000000	0.421053	0.578947	−5	27	−32
20	$2^2 5^1$	N	N	−7	2	1.2857143	0.400000	0.600000	−12	27	−39
21	$3^1 7^1$	Y	N	5	0	1.0000000	0.428571	0.571429	−7	32	−39
22	$2^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−2	37	−39
23	23^1	Y	Y	−2	0	1.0000000	0.434783	0.565217	−4	37	−41
24	$2^3 3^1$	N	N	9	4	1.5555556	0.458333	0.541667	5	46	−41
25	5^2	N	Y	2	0	1.5000000	0.480000	0.520000	7	48	−41
26	$2^1 13^1$	Y	N	5	0	1.0000000	0.500000	0.500000	12	53	−41
27	3^3	N	Y	−2	0	2.0000000	0.481481	0.518519	10	53	−43
28	$2^2 7^1$	N	N	−7	2	1.2857143	0.464286	0.535714	3	53	−50
29	29^1	Y	Y	−2	0	1.0000000	0.448276	0.551724	1	53	−52
30	$2^1 3^1 5^1$	Y	N	−16	0	1.0000000	0.433333	0.566667	−15	53	−68
31	31^1	Y	Y	−2	0	1.0000000	0.419355	0.580645	−17	53	−70
32	2^5	N	Y	−2	0	3.0000000	0.406250	0.593750	−19	53	−72
33	$3^1 11^1$	Y	N	5	0	1.0000000	0.424242	0.575758	−14	58	−72
34	$2^1 17^1$	Y	N	5	0	1.0000000	0.441176	0.558824	−9	63	−72
35	$5^1 7^1$	Y	N	5	0	1.0000000	0.457143	0.542857	−4	68	−72
36	$2^2 3^2$	N	N	14	9	1.3571429	0.472222	0.527778	10	82	−72
37	37^1	Y	Y	−2	0	1.0000000	0.459459	0.540541	8	82	−74
38	$2^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	13	87	−74
39	$3^1 13^1$	Y	N	5	0	1.0000000	0.487179	0.512821	18	92	−74
40	$2^3 5^1$	N	N	9	4	1.5555556	0.500000	0.500000	27	101	−74
41	41^1	Y	Y	−2	0	1.0000000	0.487805	0.512195	25	101	−76
42	$2^1 3^1 7^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	9	101	−92
43	43^1	Y	Y	−2	0	1.0000000	0.465116	0.534884	7	101	−94
44	$2^2 11^1$	N	N	−7	2	1.2857143	0.454545	0.545455	0	101	−101
45	$3^2 5^1$	N	N	−7	2	1.2857143	0.444444	0.555556	−7	101	−108
46	$2^1 23^1$	Y	N	5	0	1.0000000	0.456522	0.543478	−2	106	−108
47	47^1	Y	Y	−2	0	1.0000000	0.446809	0.553191	−4	106	−110
48	$2^4 3^1$	N	N	−11	6	1.8181818	0.437500	0.562500	−15	106	−121

Table B: ***Computations with*** $\mathbf{g^{-1}(n) \equiv (\omega+1)^{-1}(n)}$ ***for*** $\mathbf{1 \leq n \leq 500}$.

- *The column labeled* `Primes` *provides the prime factorization of each* n *so that the values of* $\omega(n)$ *and* $\Omega(n)$ *are easily extracted. The columns labeled* `Sqfree` *and* `PPower`*, respectively, list inclusion of* n *in the sets of squarefree integers and the prime powers.*
- *The next three columns provide the explicit values of the inverse function* $g^{-1}(n)$ *and compare its explicit value with other estimates. We define the function* $\widehat{f}_1(n) := \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} \cdot k!$.
- *The last columns indicate properties of the summatory function of* $g^{-1}(n)$*. The notation for the densities of the sign weight of* $g^{-1}(n)$ *is defined as* $\mathcal{L}_{\pm}(x) := \frac{1}{n} \cdot \#\{n \leq x : \lambda(n) = \pm 1\}$*. The last three columns then show the explicit components to the signed summatory function,* $G^{-1}(x) := \sum_{n \leq x} g^{-1}(n)$*, decomposed into its respective positive and negative magnitude sum contributions:* $G^{-1}(x) = G_+^{-1}(x) + G_-^{-1}(x)$ *where* $G_+^{-1}(x) > 0$ *and* $G_-^{-1}(x) < 0$ *for all* $x \geq 1$.

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
49	7^2	N	Y	2	0	1.5000000	0.448980	0.551020	−13	108	−121
50	$2^1 5^2$	N	N	−7	2	1.2857143	0.440000	0.560000	−20	108	−128
51	$3^1 17^1$	Y	N	5	0	1.0000000	0.450980	0.549020	−15	113	−128
52	$2^2 13^1$	N	N	−7	2	1.2857143	0.442308	0.557692	−22	113	−135
53	53^1	Y	Y	−2	0	1.0000000	0.433962	0.566038	−24	113	−137
54	$2^1 3^3$	N	N	9	4	1.5555556	0.444444	0.555556	−15	122	−137
55	$5^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−10	127	−137
56	$2^3 7^1$	N	N	9	4	1.5555556	0.464286	0.535714	−1	136	−137
57	$3^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	4	141	−137
58	$2^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	9	146	−137
59	59^1	Y	Y	−2	0	1.0000000	0.474576	0.525424	7	146	−139
60	$2^2 3^1 5^1$	N	N	30	14	1.1666667	0.483333	0.516667	37	176	−139
61	61^1	Y	Y	−2	0	1.0000000	0.475410	0.524590	35	176	−141
62	$2^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	40	181	−141
63	$3^2 7^1$	N	N	−7	2	1.2857143	0.476190	0.523810	33	181	−148
64	2^6	N	Y	2	0	3.5000000	0.484375	0.515625	35	183	−148
65	$5^1 13^1$	Y	N	5	0	1.0000000	0.492308	0.507692	40	188	−148
66	$2^1 3^1 11^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	24	188	−164
67	67^1	Y	Y	−2	0	1.0000000	0.477612	0.522388	22	188	−166
68	$2^2 17^1$	N	N	−7	2	1.2857143	0.470588	0.529412	15	188	−173
69	$3^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	20	193	−173
70	$2^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.471429	0.528571	4	193	−189
71	71^1	Y	Y	−2	0	1.0000000	0.464789	0.535211	2	193	−191
72	$2^3 3^2$	N	N	−23	18	1.4782609	0.458333	0.541667	−21	193	−214
73	73^1	Y	Y	−2	0	1.0000000	0.452055	0.547945	−23	193	−216
74	$2^1 37^1$	Y	N	5	0	1.0000000	0.459459	0.540541	−18	198	−216
75	$3^1 5^2$	N	N	−7	2	1.2857143	0.453333	0.546667	−25	198	−223
76	$2^2 19^1$	N	N	−7	2	1.2857143	0.447368	0.552632	−32	198	−230
77	$7^1 11^1$	Y	N	5	0	1.0000000	0.454545	0.545455	−27	203	−230
78	$2^1 3^1 13^1$	Y	N	−16	0	1.0000000	0.448718	0.551282	−43	203	−246
79	79^1	Y	Y	−2	0	1.0000000	0.443038	0.556962	−45	203	−248
80	$2^4 5^1$	N	N	−11	6	1.8181818	0.437500	0.562500	−56	203	−259
81	3^4	N	Y	2	0	2.5000000	0.444444	0.555556	−54	205	−259
82	$2^1 41^1$	Y	N	5	0	1.0000000	0.451220	0.548780	−49	210	−259
83	83^1	Y	Y	−2	0	1.0000000	0.445783	0.554217	−51	210	−261
84	$2^2 3^1 7^1$	N	N	30	14	1.1666667	0.452381	0.547619	−21	240	−261
85	$5^1 17^1$	Y	N	5	0	1.0000000	0.458824	0.541176	−16	245	−261
86	$2^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	−11	250	−261
87	$3^1 29^1$	Y	N	5	0	1.0000000	0.471264	0.528736	−6	255	−261
88	$2^3 11^1$	N	N	9	4	1.5555556	0.477273	0.522727	3	264	−261
89	89^1	Y	Y	−2	0	1.0000000	0.471910	0.528090	1	264	−263
90	$2^1 3^2 5^1$	N	N	30	14	1.1666667	0.477778	0.522222	31	294	−263
91	$7^1 13^1$	Y	N	5	0	1.0000000	0.483516	0.516484	36	299	−263
92	$2^2 23^1$	N	N	−7	2	1.2857143	0.478261	0.521739	29	299	−270
93	$3^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	34	304	−270
94	$2^1 47^1$	Y	N	5	0	1.0000000	0.489362	0.510638	39	309	−270
95	$5^1 19^1$	Y	N	5	0	1.0000000	0.494737	0.505263	44	314	−270
96	$2^5 3^1$	N	N	13	8	2.0769231	0.500000	0.500000	57	327	−270
97	97^1	Y	Y	−2	0	1.0000000	0.494845	0.505155	55	327	−272
98	$2^1 7^2$	N	N	−7	2	1.2857143	0.489796	0.510204	48	327	−279
99	$3^2 11^1$	N	N	−7	2	1.2857143	0.484848	0.515152	41	327	−286
100	$2^2 5^2$	N	N	14	9	1.3571429	0.490000	0.510000	55	341	−286
101	101^1	Y	Y	−2	0	1.0000000	0.485149	0.514851	53	341	−288
102	$2^1 3^1 17^1$	Y	N	−16	0	1.0000000	0.480392	0.519608	37	341	−304
103	103^1	Y	Y	−2	0	1.0000000	0.475728	0.524272	35	341	−306
104	$2^3 13^1$	N	N	9	4	1.5555556	0.480769	0.519231	44	350	−306
105	$3^1 5^1 7^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	28	350	−322
106	$2^1 53^1$	Y	N	5	0	1.0000000	0.481132	0.518868	33	355	−322
107	107^1	Y	Y	−2	0	1.0000000	0.476636	0.523364	31	355	−324
108	$2^2 3^3$	N	N	−23	18	1.4782609	0.472222	0.527778	8	355	−347
109	109^1	Y	Y	−2	0	1.0000000	0.467890	0.532110	6	355	−349
110	$2^1 5^1 11^1$	Y	N	−16	0	1.0000000	0.463636	0.536364	−10	355	−365
111	$3^1 37^1$	Y	N	5	0	1.0000000	0.468468	0.531532	−5	360	−365
112	$2^4 7^1$	N	N	−11	6	1.8181818	0.464286	0.535714	−16	360	−376
113	113^1	Y	Y	−2	0	1.0000000	0.460177	0.539823	−18	360	−378
114	$2^1 3^1 19^1$	Y	N	−16	0	1.0000000	0.456140	0.543860	−34	360	−394
115	$5^1 23^1$	Y	N	5	0	1.0000000	0.460870	0.539130	−29	365	−394
116	$2^2 29^1$	N	N	−7	2	1.2857143	0.456897	0.543103	−36	365	−401
117	$3^2 13^1$	N	N	−7	2	1.2857143	0.452991	0.547009	−43	365	−408
118	$2^1 59^1$	Y	N	5	0	1.0000000	0.457627	0.542373	−38	370	−408
119	$7^1 17^1$	Y	N	5	0	1.0000000	0.462185	0.537815	−33	375	−408
120	$2^3 3^1 5^1$	N	N	−48	32	1.3333333	0.458333	0.541667	−81	375	−456
121	11^2	N	Y	2	0	1.5000000	0.462810	0.537190	−79	377	−456
122	$2^1 61^1$	Y	N	5	0	1.0000000	0.467213	0.532787	−74	382	−456
123	$3^1 41^1$	Y	N	5	0	1.0000000	0.471545	0.528455	−69	387	−456
124	$2^2 31^1$	N	N	−7	2	1.2857143	0.467742	0.532258	−76	387	−463

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
125	5^3	N	Y	−2	0	2.0000000	0.464000	0.536000	−78	387	−465
126	$2^1 3^2 7^1$	N	N	30	14	1.1666667	0.468254	0.531746	−48	417	−465
127	127^1	Y	Y	−2	0	1.0000000	0.464567	0.535433	−50	417	−467
128	2^7	N	Y	−2	0	4.0000000	0.460938	0.539062	−52	417	−469
129	$3^1 43^1$	Y	N	5	0	1.0000000	0.465116	0.534884	−47	422	−469
130	$2^1 5^1 13^1$	Y	N	−16	0	1.0000000	0.461538	0.538462	−63	422	−485
131	131^1	Y	Y	−2	0	1.0000000	0.458015	0.541985	−65	422	−487
132	$2^2 3^1 11^1$	N	N	30	14	1.1666667	0.462121	0.537879	−35	452	−487
133	$7^1 19^1$	Y	N	5	0	1.0000000	0.466165	0.533835	−30	457	−487
134	$2^1 67^1$	Y	N	5	0	1.0000000	0.470149	0.529851	−25	462	−487
135	$3^3 5^1$	N	N	9	4	1.5555556	0.474074	0.525926	−16	471	−487
136	$2^3 17^1$	N	N	9	4	1.5555556	0.477941	0.522059	−7	480	−487
137	137^1	Y	Y	−2	0	1.0000000	0.474453	0.525547	−9	480	−489
138	$2^1 3^1 23^1$	Y	N	−16	0	1.0000000	0.471014	0.528986	−25	480	−505
139	139^1	Y	Y	−2	0	1.0000000	0.467626	0.532374	−27	480	−507
140	$2^2 5^1 7^1$	N	N	30	14	1.1666667	0.471429	0.528571	3	510	−507
141	$3^1 47^1$	Y	N	5	0	1.0000000	0.475177	0.524823	8	515	−507
142	$2^1 71^1$	Y	N	5	0	1.0000000	0.478873	0.521127	13	520	−507
143	$11^1 13^1$	Y	N	5	0	1.0000000	0.482517	0.517483	18	525	−507
144	$2^4 3^2$	N	N	34	29	1.6176471	0.486111	0.513889	52	559	−507
145	$5^1 29^1$	Y	N	5	0	1.0000000	0.489655	0.510345	57	564	−507
146	$2^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	62	569	−507
147	$3^1 7^2$	N	N	−7	2	1.2857143	0.489796	0.510204	55	569	−514
148	$2^2 37^1$	N	N	−7	2	1.2857143	0.486486	0.513514	48	569	−521
149	149^1	Y	Y	−2	0	1.0000000	0.483221	0.516779	46	569	−523
150	$2^1 3^1 5^2$	N	N	30	14	1.1666667	0.486667	0.513333	76	599	−523
151	151^1	Y	Y	−2	0	1.0000000	0.483444	0.516556	74	599	−525
152	$2^3 19^1$	N	N	9	4	1.5555556	0.486842	0.513158	83	608	−525
153	$3^2 17^1$	N	N	−7	2	1.2857143	0.483660	0.516340	76	608	−532
154	$2^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.480519	0.519481	60	608	−548
155	$5^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	65	613	−548
156	$2^2 3^1 13^1$	N	N	30	14	1.1666667	0.487179	0.512821	95	643	−548
157	157^1	Y	Y	−2	0	1.0000000	0.484076	0.515924	93	643	−550
158	$2^1 79^1$	Y	N	5	0	1.0000000	0.487342	0.512658	98	648	−550
159	$3^1 53^1$	Y	N	5	0	1.0000000	0.490566	0.509434	103	653	−550
160	$2^5 5^1$	N	N	13	8	2.0769231	0.493750	0.506250	116	666	−550
161	$7^1 23^1$	Y	N	5	0	1.0000000	0.496894	0.503106	121	671	−550
162	$2^1 3^4$	N	N	−11	6	1.8181818	0.493827	0.506173	110	671	−561
163	163^1	Y	Y	−2	0	1.0000000	0.490798	0.509202	108	671	−563
164	$2^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	101	671	−570
165	$3^1 5^1 11^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	85	671	−586
166	$2^1 83^1$	Y	N	5	0	1.0000000	0.487952	0.512048	90	676	−586
167	167^1	Y	Y	−2	0	1.0000000	0.485030	0.514970	88	676	−588
168	$2^3 3^1 7^1$	N	N	−48	32	1.3333333	0.482143	0.517857	40	676	−636
169	13^2	N	Y	2	0	1.5000000	0.485207	0.514793	42	678	−636
170	$2^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.482353	0.517647	26	678	−652
171	$3^2 19^1$	N	N	−7	2	1.2857143	0.479532	0.520468	19	678	−659
172	$2^2 43^1$	N	N	−7	2	1.2857143	0.476744	0.523256	12	678	−666
173	173^1	Y	Y	−2	0	1.0000000	0.473988	0.526012	10	678	−668
174	$2^1 3^1 29^1$	Y	N	−16	0	1.0000000	0.471264	0.528736	−6	678	−684
175	$5^2 7^1$	N	N	−7	2	1.2857143	0.468571	0.531429	−13	678	−691
176	$2^4 11^1$	N	N	−11	6	1.8181818	0.465909	0.534091	−24	678	−702
177	$3^1 59^1$	Y	N	5	0	1.0000000	0.468927	0.531073	−19	683	−702
178	$2^1 89^1$	Y	N	5	0	1.0000000	0.471910	0.528090	−14	688	−702
179	179^1	Y	Y	−2	0	1.0000000	0.469274	0.530726	−16	688	−704
180	$2^2 3^2 5^1$	N	N	−74	58	1.2162162	0.466667	0.533333	−90	688	−778
181	181^1	Y	Y	−2	0	1.0000000	0.464088	0.535912	−92	688	−780
182	$2^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.461538	0.538462	−108	688	−796
183	$3^1 61^1$	Y	N	5	0	1.0000000	0.464481	0.535519	−103	693	−796
184	$2^3 23^1$	N	N	9	4	1.5555556	0.467391	0.532609	−94	702	−796
185	$5^1 37^1$	Y	N	5	0	1.0000000	0.470270	0.529730	−89	707	−796
186	$2^1 3^1 31^1$	Y	N	−16	0	1.0000000	0.467742	0.532258	−105	707	−812
187	$11^1 17^1$	Y	N	5	0	1.0000000	0.470588	0.529412	−100	712	−812
188	$2^2 47^1$	N	N	−7	2	1.2857143	0.468085	0.531915	−107	712	−819
189	$3^3 7^1$	N	N	9	4	1.5555556	0.470899	0.529101	−98	721	−819
190	$2^1 5^1 19^1$	Y	N	−16	0	1.0000000	0.468421	0.531579	−114	721	−835
191	191^1	Y	Y	−2	0	1.0000000	0.465969	0.534031	−116	721	−837
192	$2^6 3^1$	N	N	−15	10	2.3333333	0.463542	0.536458	−131	721	−852
193	193^1	Y	Y	−2	0	1.0000000	0.461140	0.538860	−133	721	−854
194	$2^1 97^1$	Y	N	5	0	1.0000000	0.463918	0.536082	−128	726	−854
195	$3^1 5^1 13^1$	Y	N	−16	0	1.0000000	0.461538	0.538462	−144	726	−870
196	$2^2 7^2$	N	N	14	9	1.3571429	0.464286	0.535714	−130	740	−870
197	197^1	Y	Y	−2	0	1.0000000	0.461929	0.538071	−132	740	−872
198	$2^1 3^2 11^1$	N	N	30	14	1.1666667	0.464646	0.535354	−102	770	−872
199	199^1	Y	Y	−2	0	1.0000000	0.462312	0.537688	−104	770	−874
200	$2^3 5^2$	N	N	−23	18	1.4782609	0.460000	0.540000	−127	770	−897

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
201	$3^1 67^1$	Y	N	5	0	1.0000000	0.462687	0.537313	−122	775	−897
202	$2^1 101^1$	Y	N	5	0	1.0000000	0.465347	0.534653	−117	780	−897
203	$7^1 29^1$	Y	N	5	0	1.0000000	0.467980	0.532020	−112	785	−897
204	$2^2 3^1 17^1$	N	N	30	14	1.1666667	0.470588	0.529412	−82	815	−897
205	$5^1 41^1$	Y	N	5	0	1.0000000	0.473171	0.526829	−77	820	−897
206	$2^1 103^1$	Y	N	5	0	1.0000000	0.475728	0.524272	−72	825	−897
207	$3^2 23^1$	N	N	−7	2	1.2857143	0.473430	0.526570	−79	825	−904
208	$2^4 13^1$	N	N	−11	6	1.8181818	0.471154	0.528846	−90	825	−915
209	$11^1 19^1$	Y	N	5	0	1.0000000	0.473684	0.526316	−85	830	−915
210	$2^1 3^1 5^1 7^1$	Y	N	65	0	1.0000000	0.476190	0.523810	−20	895	−915
211	211^1	Y	Y	−2	0	1.0000000	0.473934	0.526066	−22	895	−917
212	$2^2 53^1$	N	N	−7	2	1.2857143	0.471698	0.528302	−29	895	−924
213	$3^1 71^1$	Y	N	5	0	1.0000000	0.474178	0.525822	−24	900	−924
214	$2^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−19	905	−924
215	$5^1 43^1$	Y	N	5	0	1.0000000	0.479070	0.520930	−14	910	−924
216	$2^3 3^3$	N	N	46	41	1.5000000	0.481481	0.518519	32	956	−924
217	$7^1 31^1$	Y	N	5	0	1.0000000	0.483871	0.516129	37	961	−924
218	$2^1 109^1$	Y	N	5	0	1.0000000	0.486239	0.513761	42	966	−924
219	$3^1 73^1$	Y	N	5	0	1.0000000	0.488584	0.511416	47	971	−924
220	$2^2 5^1 11^1$	N	N	30	14	1.1666667	0.490909	0.509091	77	1001	−924
221	$13^1 17^1$	Y	N	5	0	1.0000000	0.493213	0.506787	82	1006	−924
222	$2^1 3^1 37^1$	Y	N	−16	0	1.0000000	0.490991	0.509009	66	1006	−940
223	223^1	Y	Y	−2	0	1.0000000	0.488789	0.511211	64	1006	−942
224	$2^5 7^1$	N	N	13	8	2.0769231	0.491071	0.508929	77	1019	−942
225	$3^2 5^2$	N	N	14	9	1.3571429	0.493333	0.506667	91	1033	−942
226	$2^1 113^1$	Y	N	5	0	1.0000000	0.495575	0.504425	96	1038	−942
227	227^1	Y	Y	−2	0	1.0000000	0.493392	0.506608	94	1038	−944
228	$2^2 3^1 19^1$	N	N	30	14	1.1666667	0.495614	0.504386	124	1068	−944
229	229^1	Y	Y	−2	0	1.0000000	0.493450	0.506550	122	1068	−946
230	$2^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.491304	0.508696	106	1068	−962
231	$3^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.489177	0.510823	90	1068	−978
232	$2^3 29^1$	N	N	9	4	1.5555556	0.491379	0.508621	99	1077	−978
233	233^1	Y	Y	−2	0	1.0000000	0.489270	0.510730	97	1077	−980
234	$2^1 3^2 13^1$	N	N	30	14	1.1666667	0.491453	0.508547	127	1107	−980
235	$5^1 47^1$	Y	N	5	0	1.0000000	0.493617	0.506383	132	1112	−980
236	$2^2 59^1$	N	N	−7	2	1.2857143	0.491525	0.508475	125	1112	−987
237	$3^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	130	1117	−987
238	$2^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.491597	0.508403	114	1117	−1003
239	239^1	Y	Y	−2	0	1.0000000	0.489540	0.510460	112	1117	−1005
240	$2^4 3^1 5^1$	N	N	70	54	1.5000000	0.491667	0.508333	182	1187	−1005
241	241^1	Y	Y	−2	0	1.0000000	0.489627	0.510373	180	1187	−1007
242	$2^1 11^2$	N	N	−7	2	1.2857143	0.487603	0.512397	173	1187	−1014
243	3^5	N	Y	−2	0	3.0000000	0.485597	0.514403	171	1187	−1016
244	$2^2 61^1$	N	N	−7	2	1.2857143	0.483607	0.516393	164	1187	−1023
245	$5^1 7^2$	N	N	−7	2	1.2857143	0.481633	0.518367	157	1187	−1030
246	$2^1 3^1 41^1$	Y	N	−16	0	1.0000000	0.479675	0.520325	141	1187	−1046
247	$13^1 19^1$	Y	N	5	0	1.0000000	0.481781	0.518219	146	1192	−1046
248	$2^3 31^1$	N	N	9	4	1.5555556	0.483871	0.516129	155	1201	−1046
249	$3^1 83^1$	Y	N	5	0	1.0000000	0.485944	0.514056	160	1206	−1046
250	$2^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	169	1215	−1046
251	251^1	Y	Y	−2	0	1.0000000	0.486056	0.513944	167	1215	−1048
252	$2^2 3^2 7^1$	N	N	−74	58	1.2162162	0.484127	0.515873	93	1215	−1122
253	$11^1 23^1$	Y	N	5	0	1.0000000	0.486166	0.513834	98	1220	−1122
254	$2^1 127^1$	Y	N	5	0	1.0000000	0.488189	0.511811	103	1225	−1122
255	$3^1 5^1 17^1$	Y	N	−16	0	1.0000000	0.486275	0.513725	87	1225	−1138
256	2^8	N	Y	2	0	4.5000000	0.488281	0.511719	89	1227	−1138
257	257^1	Y	Y	−2	0	1.0000000	0.486381	0.513619	87	1227	−1140
258	$2^1 3^1 43^1$	Y	N	−16	0	1.0000000	0.484496	0.515504	71	1227	−1156
259	$7^1 37^1$	Y	N	5	0	1.0000000	0.486486	0.513514	76	1232	−1156
260	$2^2 5^1 13^1$	N	N	30	14	1.1666667	0.488462	0.511538	106	1262	−1156
261	$3^2 29^1$	N	N	−7	2	1.2857143	0.486590	0.513410	99	1262	−1163
262	$2^1 131^1$	Y	N	5	0	1.0000000	0.488550	0.511450	104	1267	−1163
263	263^1	Y	Y	−2	0	1.0000000	0.486692	0.513308	102	1267	−1165
264	$2^3 3^1 11^1$	N	N	−48	32	1.3333333	0.484848	0.515152	54	1267	−1213
265	$5^1 53^1$	Y	N	5	0	1.0000000	0.486792	0.513208	59	1272	−1213
266	$2^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.484962	0.515038	43	1272	−1229
267	$3^1 89^1$	Y	N	5	0	1.0000000	0.486891	0.513109	48	1277	−1229
268	$2^2 67^1$	N	N	−7	2	1.2857143	0.485075	0.514925	41	1277	−1236
269	269^1	Y	Y	−2	0	1.0000000	0.483271	0.516729	39	1277	−1238
270	$2^1 3^3 5^1$	N	N	−48	32	1.3333333	0.481481	0.518519	−9	1277	−1286
271	271^1	Y	Y	−2	0	1.0000000	0.479705	0.520295	−11	1277	−1288
272	$2^4 17^1$	N	N	−11	6	1.8181818	0.477941	0.522059	−22	1277	−1299
273	$3^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−38	1277	−1315
274	$2^1 137^1$	Y	N	5	0	1.0000000	0.478102	0.521898	−33	1282	−1315
275	$5^2 11^1$	N	N	−7	2	1.2857143	0.476364	0.523636	−40	1282	−1322
276	$2^2 3^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−10	1312	−1322
277	277^1	Y	Y	−2	0	1.0000000	0.476534	0.523466	−12	1312	−1324

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G^{-1}_+(n)$	$G^{-1}_-(n)$
278	$2^1 139^1$	Y	N	5	0	1.0000000	0.478417	0.521583	−7	1317	−1324
279	$3^2 31^1$	N	N	−7	2	1.2857143	0.476703	0.523297	−14	1317	−1331
280	$2^3 5^1 7^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−62	1317	−1379
281	281^1	Y	Y	−2	0	1.0000000	0.473310	0.526690	−64	1317	−1381
282	$2^1 3^1 47^1$	Y	N	−16	0	1.0000000	0.471631	0.528369	−80	1317	−1397
283	283^1	Y	Y	−2	0	1.0000000	0.469965	0.530035	−82	1317	−1399
284	$2^2 71^1$	N	N	−7	2	1.2857143	0.468310	0.531690	−89	1317	−1406
285	$3^1 5^1 19^1$	Y	N	−16	0	1.0000000	0.466667	0.533333	−105	1317	−1422
286	$2^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.465035	0.534965	−121	1317	−1438
287	$7^1 41^1$	Y	N	5	0	1.0000000	0.466899	0.533101	−116	1322	−1438
288	$2^5 3^2$	N	N	−47	42	1.7659574	0.465278	0.534722	−163	1322	−1485
289	17^2	N	Y	2	0	1.5000000	0.467128	0.532872	−161	1324	−1485
290	$2^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.465517	0.534483	−177	1324	−1501
291	$3^1 97^1$	Y	N	5	0	1.0000000	0.467354	0.532646	−172	1329	−1501
292	$2^2 73^1$	N	N	−7	2	1.2857143	0.465753	0.534247	−179	1329	−1508
293	293^1	Y	Y	−2	0	1.0000000	0.464164	0.535836	−181	1329	−1510
294	$2^1 3^1 7^2$	N	N	30	14	1.1666667	0.465986	0.534014	−151	1359	−1510
295	$5^1 59^1$	Y	N	5	0	1.0000000	0.467797	0.532203	−146	1364	−1510
296	$2^3 37^1$	N	N	9	4	1.5555556	0.469595	0.530405	−137	1373	−1510
297	$3^3 11^1$	N	N	9	4	1.5555556	0.471380	0.528620	−128	1382	−1510
298	$2^1 149^1$	Y	N	5	0	1.0000000	0.473154	0.526846	−123	1387	−1510
299	$13^1 23^1$	Y	N	5	0	1.0000000	0.474916	0.525084	−118	1392	−1510
300	$2^2 3^1 5^2$	N	N	−74	58	1.2162162	0.473333	0.526667	−192	1392	−1584
301	$7^1 43^1$	Y	N	5	0	1.0000000	0.475083	0.524917	−187	1397	−1584
302	$2^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−182	1402	−1584
303	$3^1 101^1$	Y	N	5	0	1.0000000	0.478548	0.521452	−177	1407	−1584
304	$2^4 19^1$	N	N	−11	6	1.8181818	0.476974	0.523026	−188	1407	−1595
305	$5^1 61^1$	Y	N	5	0	1.0000000	0.478689	0.521311	−183	1412	−1595
306	$2^1 3^2 17^1$	N	N	30	14	1.1666667	0.480392	0.519608	−153	1442	−1595
307	307^1	Y	Y	−2	0	1.0000000	0.478827	0.521173	−155	1442	−1597
308	$2^2 7^1 11^1$	N	N	30	14	1.1666667	0.480519	0.519481	−125	1472	−1597
309	$3^1 103^1$	Y	N	5	0	1.0000000	0.482201	0.517799	−120	1477	−1597
310	$2^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.480645	0.519355	−136	1477	−1613
311	311^1	Y	Y	−2	0	1.0000000	0.479100	0.520900	−138	1477	−1615
312	$2^3 3^1 13^1$	N	N	−48	32	1.3333333	0.477564	0.522436	−186	1477	−1663
313	313^1	Y	Y	−2	0	1.0000000	0.476038	0.523962	−188	1477	−1665
314	$2^1 157^1$	Y	N	5	0	1.0000000	0.477707	0.522293	−183	1482	−1665
315	$3^2 5^1 7^1$	N	N	30	14	1.1666667	0.479365	0.520635	−153	1512	−1665
316	$2^2 79^1$	N	N	−7	2	1.2857143	0.477848	0.522152	−160	1512	−1672
317	317^1	Y	Y	−2	0	1.0000000	0.476341	0.523659	−162	1512	−1674
318	$2^1 3^1 53^1$	Y	N	−16	0	1.0000000	0.474843	0.525157	−178	1512	−1690
319	$11^1 29^1$	Y	N	5	0	1.0000000	0.476489	0.523511	−173	1517	−1690
320	$2^6 5^1$	N	N	−15	10	2.3333333	0.475000	0.525000	−188	1517	−1705
321	$3^1 107^1$	Y	N	5	0	1.0000000	0.476636	0.523364	−183	1522	−1705
322	$2^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.475155	0.524845	−199	1522	−1721
323	$17^1 19^1$	Y	N	5	0	1.0000000	0.476780	0.523220	−194	1527	−1721
324	$2^2 3^4$	N	N	34	29	1.6176471	0.478395	0.521605	−160	1561	−1721
325	$5^2 13^1$	N	N	−7	2	1.2857143	0.476923	0.523077	−167	1561	−1728
326	$2^1 163^1$	Y	N	5	0	1.0000000	0.478528	0.521472	−162	1566	−1728
327	$3^1 109^1$	Y	N	5	0	1.0000000	0.480122	0.519878	−157	1571	−1728
328	$2^3 41^1$	N	N	9	4	1.5555556	0.481707	0.518293	−148	1580	−1728
329	$7^1 47^1$	Y	N	5	0	1.0000000	0.483283	0.516717	−143	1585	−1728
330	$2^1 3^1 5^1 11^1$	Y	N	65	0	1.0000000	0.484848	0.515152	−78	1650	−1728
331	331^1	Y	Y	−2	0	1.0000000	0.483384	0.516616	−80	1650	−1730
332	$2^2 83^1$	N	N	−7	2	1.2857143	0.481928	0.518072	−87	1650	−1737
333	$3^2 37^1$	N	N	−7	2	1.2857143	0.480480	0.519520	−94	1650	−1744
334	$2^1 167^1$	Y	N	5	0	1.0000000	0.482036	0.517964	−89	1655	−1744
335	$5^1 67^1$	Y	N	5	0	1.0000000	0.483582	0.516418	−84	1660	−1744
336	$2^4 3^1 7^1$	N	N	70	54	1.5000000	0.485119	0.514881	−14	1730	−1744
337	337^1	Y	Y	−2	0	1.0000000	0.483680	0.516320	−16	1730	−1746
338	$2^1 13^2$	N	N	−7	2	1.2857143	0.482249	0.517751	−23	1730	−1753
339	$3^1 113^1$	Y	N	5	0	1.0000000	0.483776	0.516224	−18	1735	−1753
340	$2^2 5^1 17^1$	N	N	30	14	1.1666667	0.485294	0.514706	12	1765	−1753
341	$11^1 31^1$	Y	N	5	0	1.0000000	0.486804	0.513196	17	1770	−1753
342	$2^1 3^2 19^1$	N	N	30	14	1.1666667	0.488304	0.511696	47	1800	−1753
343	7^3	N	Y	−2	0	2.0000000	0.486880	0.513120	45	1800	−1755
344	$2^3 43^1$	N	N	9	4	1.5555556	0.488372	0.511628	54	1809	−1755
345	$3^1 5^1 23^1$	Y	N	−16	0	1.0000000	0.486957	0.513043	38	1809	−1771
346	$2^1 173^1$	Y	N	5	0	1.0000000	0.488439	0.511561	43	1814	−1771
347	347^1	Y	Y	−2	0	1.0000000	0.487032	0.512968	41	1814	−1773
348	$2^2 3^1 29^1$	N	N	30	14	1.1666667	0.488506	0.511494	71	1844	−1773
349	349^1	Y	Y	−2	0	1.0000000	0.487106	0.512894	69	1844	−1775
350	$2^1 5^2 7^1$	N	N	30	14	1.1666667	0.488571	0.511429	99	1874	−1775

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n) - \widehat{f}_1(n)$	$\frac{\sum_{d\|n} C_{\Omega(d)}(d)}{\|g^{-1}(n)\|}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
351	$3^3 13^1$	N	N	9	4	1.5555556	0.490028	0.509972	108	1883	−1775
352	$2^5 11^1$	N	N	13	8	2.0769231	0.491477	0.508523	121	1896	−1775
353	353^1	Y	Y	−2	0	1.0000000	0.490085	0.509915	119	1896	−1777
354	$2^1 3^1 59^1$	Y	N	−16	0	1.0000000	0.488701	0.511299	103	1896	−1793
355	$5^1 71^1$	Y	N	5	0	1.0000000	0.490141	0.509859	108	1901	−1793
356	$2^2 89^1$	N	N	−7	2	1.2857143	0.488764	0.511236	101	1901	−1800
357	$3^1 7^1 17^1$	Y	N	−16	0	1.0000000	0.487395	0.512605	85	1901	−1816
358	$2^1 179^1$	Y	N	5	0	1.0000000	0.488827	0.511173	90	1906	−1816
359	359^1	Y	Y	−2	0	1.0000000	0.487465	0.512535	88	1906	−1818
360	$2^3 3^2 5^1$	N	N	145	129	1.3034483	0.488889	0.511111	233	2051	−1818
361	19^2	N	Y	2	0	1.5000000	0.490305	0.509695	235	2053	−1818
362	$2^1 181^1$	Y	N	5	0	1.0000000	0.491713	0.508287	240	2058	−1818
363	$3^1 11^2$	N	N	−7	2	1.2857143	0.490358	0.509642	233	2058	−1825
364	$2^2 7^1 13^1$	N	N	30	14	1.1666667	0.491758	0.508242	263	2088	−1825
365	$5^1 73^1$	Y	N	5	0	1.0000000	0.493151	0.506849	268	2093	−1825
366	$2^1 3^1 61^1$	Y	N	−16	0	1.0000000	0.491803	0.508197	252	2093	−1841
367	367^1	Y	Y	−2	0	1.0000000	0.490463	0.509537	250	2093	−1843
368	$2^4 23^1$	N	N	−11	6	1.8181818	0.489130	0.510870	239	2093	−1854
369	$3^2 41^1$	N	N	−7	2	1.2857143	0.487805	0.512195	232	2093	−1861
370	$2^1 5^1 37^1$	Y	N	−16	0	1.0000000	0.486486	0.513514	216	2093	−1877
371	$7^1 53^1$	Y	N	5	0	1.0000000	0.487871	0.512129	221	2098	−1877
372	$2^2 3^1 31^1$	N	N	30	14	1.1666667	0.489247	0.510753	251	2128	−1877
373	373^1	Y	Y	−2	0	1.0000000	0.487936	0.512064	249	2128	−1879
374	$2^1 11^1 17^1$	Y	N	−16	0	1.0000000	0.486631	0.513369	233	2128	−1895
375	$3^1 5^3$	N	N	9	4	1.5555556	0.488000	0.512000	242	2137	−1895
376	$2^3 47^1$	N	N	9	4	1.5555556	0.489362	0.510638	251	2146	−1895
377	$13^1 29^1$	Y	N	5	0	1.0000000	0.490716	0.509284	256	2151	−1895
378	$2^1 3^3 7^1$	N	N	−48	32	1.3333333	0.489418	0.510582	208	2151	−1943
379	379^1	Y	Y	−2	0	1.0000000	0.488127	0.511873	206	2151	−1945
380	$2^2 5^1 19^1$	N	N	30	14	1.1666667	0.489474	0.510526	236	2181	−1945
381	$3^1 127^1$	Y	N	5	0	1.0000000	0.490814	0.509186	241	2186	−1945
382	$2^1 191^1$	Y	N	5	0	1.0000000	0.492147	0.507853	246	2191	−1945
383	383^1	Y	Y	−2	0	1.0000000	0.490862	0.509138	244	2191	−1947
384	$2^7 3^1$	N	N	17	12	2.5882353	0.492188	0.507812	261	2208	−1947
385	$5^1 7^1 11^1$	Y	N	−16	0	1.0000000	0.490909	0.509091	245	2208	−1963
386	$2^1 193^1$	Y	N	5	0	1.0000000	0.492228	0.507772	250	2213	−1963
387	$3^2 43^1$	N	N	−7	2	1.2857143	0.490956	0.509044	243	2213	−1970
388	$2^2 97^1$	N	N	−7	2	1.2857143	0.489691	0.510309	236	2213	−1977
389	389^1	Y	Y	−2	0	1.0000000	0.488432	0.511568	234	2213	−1979
390	$2^1 3^1 5^1 13^1$	Y	N	65	0	1.0000000	0.489744	0.510256	299	2278	−1979
391	$17^1 23^1$	Y	N	5	0	1.0000000	0.491049	0.508951	304	2283	−1979
392	$2^3 7^2$	N	N	−23	18	1.4782609	0.489796	0.510204	281	2283	−2002
393	$3^1 131^1$	Y	N	5	0	1.0000000	0.491094	0.508906	286	2288	−2002
394	$2^1 197^1$	Y	N	5	0	1.0000000	0.492386	0.507614	291	2293	−2002
395	$5^1 79^1$	Y	N	5	0	1.0000000	0.493671	0.506329	296	2298	−2002
396	$2^2 3^2 11^1$	N	N	−74	58	1.2162162	0.492424	0.507576	222	2298	−2076
397	397^1	Y	Y	−2	0	1.0000000	0.491184	0.508816	220	2298	−2078
398	$2^1 199^1$	Y	N	5	0	1.0000000	0.492462	0.507538	225	2303	−2078
399	$3^1 7^1 19^1$	Y	N	−16	0	1.0000000	0.491228	0.508772	209	2303	−2094
400	$2^4 5^2$	N	N	34	29	1.6176471	0.492500	0.507500	243	2337	−2094
401	401^1	Y	Y	−2	0	1.0000000	0.491272	0.508728	241	2337	−2096
402	$2^1 3^1 67^1$	Y	N	−16	0	1.0000000	0.490050	0.509950	225	2337	−2112
403	$13^1 31^1$	Y	N	5	0	1.0000000	0.491315	0.508685	230	2342	−2112
404	$2^2 101^1$	N	N	−7	2	1.2857143	0.490099	0.509901	223	2342	−2119
405	$3^4 5^1$	N	N	−11	6	1.8181818	0.488889	0.511111	212	2342	−2130
406	$2^1 7^1 29^1$	Y	N	−16	0	1.0000000	0.487685	0.512315	196	2342	−2146
407	$11^1 37^1$	Y	N	5	0	1.0000000	0.488943	0.511057	201	2347	−2146
408	$2^3 3^1 17^1$	N	N	−48	32	1.3333333	0.487745	0.512255	153	2347	−2194
409	409^1	Y	Y	−2	0	1.0000000	0.486553	0.513447	151	2347	−2196
410	$2^1 5^1 41^1$	Y	N	−16	0	1.0000000	0.485366	0.514634	135	2347	−2212
411	$3^1 137^1$	Y	N	5	0	1.0000000	0.486618	0.513382	140	2352	−2212
412	$2^2 103^1$	N	N	−7	2	1.2857143	0.485437	0.514563	133	2352	−2219
413	$7^1 59^1$	Y	N	5	0	1.0000000	0.486683	0.513317	138	2357	−2219
414	$2^1 3^2 23^1$	N	N	30	14	1.1666667	0.487923	0.512077	168	2387	−2219
415	$5^1 83^1$	Y	N	5	0	1.0000000	0.489157	0.510843	173	2392	−2219
416	$2^5 13^1$	N	N	13	8	2.0769231	0.490385	0.509615	186	2405	−2219
417	$3^1 139^1$	Y	N	5	0	1.0000000	0.491607	0.508393	191	2410	−2219
418	$2^1 11^1 19^1$	Y	N	−16	0	1.0000000	0.490431	0.509569	175	2410	−2235
419	419^1	Y	Y	−2	0	1.0000000	0.489260	0.510740	173	2410	−2237
420	$2^2 3^1 5^1 7^1$	N	N	−155	90	1.1032258	0.488095	0.511905	18	2410	−2392
421	421^1	Y	Y	−2	0	1.0000000	0.486936	0.513064	16	2410	−2394
422	$2^1 211^1$	Y	N	5	0	1.0000000	0.488152	0.511848	21	2415	−2394
423	$3^2 47^1$	N	N	−7	2	1.2857143	0.486998	0.513002	14	2415	−2401
424	$2^3 53^1$	N	N	9	4	1.5555556	0.488208	0.511792	23	2424	−2401
425	$5^2 17^1$	N	N	−7	2	1.2857143	0.487059	0.512941	16	2424	−2408

n	**Primes**	**Sqfree**	**PPower**	$g^{-1}(n)$	$\lambda(n)g^{-1}(n)-\widehat{f}_1(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}(d)}{\lvert g^{-1}(n)\rvert}$	$\mathcal{L}_+(n)$	$\mathcal{L}_-(n)$	$G^{-1}(n)$	$G_+^{-1}(n)$	$G_-^{-1}(n)$
426	$2^1 3^1 71^1$	Y	N	−16	0	1.0000000	0.485915	0.514085	0	2424	−2424
427	$7^1 61^1$	Y	N	5	0	1.0000000	0.487119	0.512881	5	2429	−2424
428	$2^2 107^1$	N	N	−7	2	1.2857143	0.485981	0.514019	−2	2429	−2431
429	$3^1 11^1 13^1$	Y	N	−16	0	1.0000000	0.484848	0.515152	−18	2429	−2447
430	$2^1 5^1 43^1$	Y	N	−16	0	1.0000000	0.483721	0.516279	−34	2429	−2463
431	431^1	Y	Y	−2	0	1.0000000	0.482599	0.517401	−36	2429	−2465
432	$2^4 3^3$	N	N	−80	75	1.5625000	0.481481	0.518519	−116	2429	−2545
433	433^1	Y	Y	−2	0	1.0000000	0.480370	0.519630	−118	2429	−2547
434	$2^1 7^1 31^1$	Y	N	−16	0	1.0000000	0.479263	0.520737	−134	2429	−2563
435	$3^1 5^1 29^1$	Y	N	−16	0	1.0000000	0.478161	0.521839	−150	2429	−2579
436	$2^2 109^1$	N	N	−7	2	1.2857143	0.477064	0.522936	−157	2429	−2586
437	$19^1 23^1$	Y	N	5	0	1.0000000	0.478261	0.521739	−152	2434	−2586
438	$2^1 3^1 73^1$	Y	N	−16	0	1.0000000	0.477169	0.522831	−168	2434	−2602
439	439^1	Y	Y	−2	0	1.0000000	0.476082	0.523918	−170	2434	−2604
440	$2^3 5^1 11^1$	N	N	−48	32	1.3333333	0.475000	0.525000	−218	2434	−2652
441	$3^2 7^2$	N	N	14	9	1.3571429	0.476190	0.523810	−204	2448	−2652
442	$2^1 13^1 17^1$	Y	N	−16	0	1.0000000	0.475113	0.524887	−220	2448	−2668
443	443^1	Y	Y	−2	0	1.0000000	0.474041	0.525959	−222	2448	−2670
444	$2^2 3^1 37^1$	N	N	30	14	1.1666667	0.475225	0.524775	−192	2478	−2670
445	$5^1 89^1$	Y	N	5	0	1.0000000	0.476404	0.523596	−187	2483	−2670
446	$2^1 223^1$	Y	N	5	0	1.0000000	0.477578	0.522422	−182	2488	−2670
447	$3^1 149^1$	Y	N	5	0	1.0000000	0.478747	0.521253	−177	2493	−2670
448	$2^6 7^1$	N	N	−15	10	2.3333333	0.477679	0.522321	−192	2493	−2685
449	449^1	Y	Y	−2	0	1.0000000	0.476615	0.523385	−194	2493	−2687
450	$2^1 3^2 5^2$	N	N	−74	58	1.2162162	0.475556	0.524444	−268	2493	−2761
451	$11^1 41^1$	Y	N	5	0	1.0000000	0.476718	0.523282	−263	2498	−2761
452	$2^2 113^1$	N	N	−7	2	1.2857143	0.475664	0.524336	−270	2498	−2768
453	$3^1 151^1$	Y	N	5	0	1.0000000	0.476821	0.523179	−265	2503	−2768
454	$2^1 227^1$	Y	N	5	0	1.0000000	0.477974	0.522026	−260	2508	−2768
455	$5^1 7^1 13^1$	Y	N	−16	0	1.0000000	0.476923	0.523077	−276	2508	−2784
456	$2^3 3^1 19^1$	N	N	−48	32	1.3333333	0.475877	0.524123	−324	2508	−2832
457	457^1	Y	Y	−2	0	1.0000000	0.474836	0.525164	−326	2508	−2834
458	$2^1 229^1$	Y	N	5	0	1.0000000	0.475983	0.524017	−321	2513	−2834
459	$3^3 17^1$	N	N	9	4	1.5555556	0.477124	0.522876	−312	2522	−2834
460	$2^2 5^1 23^1$	N	N	30	14	1.1666667	0.478261	0.521739	−282	2552	−2834
461	461^1	Y	Y	−2	0	1.0000000	0.477223	0.522777	−284	2552	−2836
462	$2^1 3^1 7^1 11^1$	Y	N	65	0	1.0000000	0.478355	0.521645	−219	2617	−2836
463	463^1	Y	Y	−2	0	1.0000000	0.477322	0.522678	−221	2617	−2838
464	$2^4 29^1$	N	N	−11	6	1.8181818	0.476293	0.523707	−232	2617	−2849
465	$3^1 5^1 31^1$	Y	N	−16	0	1.0000000	0.475269	0.524731	−248	2617	−2865
466	$2^1 233^1$	Y	N	5	0	1.0000000	0.476395	0.523605	−243	2622	−2865
467	467^1	Y	Y	−2	0	1.0000000	0.475375	0.524625	−245	2622	−2867
468	$2^2 3^2 13^1$	N	N	−74	58	1.2162162	0.474359	0.525641	−319	2622	−2941
469	$7^1 67^1$	Y	N	5	0	1.0000000	0.475480	0.524520	−314	2627	−2941
470	$2^1 5^1 47^1$	Y	N	−16	0	1.0000000	0.474468	0.525532	−330	2627	−2957
471	$3^1 157^1$	Y	N	5	0	1.0000000	0.475584	0.524416	−325	2632	−2957
472	$2^3 59^1$	N	N	9	4	1.5555556	0.476695	0.523305	−316	2641	−2957
473	$11^1 43^1$	Y	N	5	0	1.0000000	0.477801	0.522199	−311	2646	−2957
474	$2^1 3^1 79^1$	Y	N	−16	0	1.0000000	0.476793	0.523207	−327	2646	−2973
475	$5^2 19^1$	N	N	−7	2	1.2857143	0.475789	0.524211	−334	2646	−2980
476	$2^2 7^1 17^1$	N	N	30	14	1.1666667	0.476891	0.523109	−304	2676	−2980
477	$3^2 53^1$	N	N	−7	2	1.2857143	0.475891	0.524109	−311	2676	−2987
478	$2^1 239^1$	Y	N	5	0	1.0000000	0.476987	0.523013	−306	2681	−2987
479	479^1	Y	Y	−2	0	1.0000000	0.475992	0.524008	−308	2681	−2989
480	$2^5 3^1 5^1$	N	N	−96	80	1.6666667	0.475000	0.525000	−404	2681	−3085
481	$13^1 37^1$	Y	N	5	0	1.0000000	0.476091	0.523909	−399	2686	−3085
482	$2^1 241^1$	Y	N	5	0	1.0000000	0.477178	0.522822	−394	2691	−3085
483	$3^1 7^1 23^1$	Y	N	−16	0	1.0000000	0.476190	0.523810	−410	2691	−3101
484	$2^2 11^2$	N	N	14	9	1.3571429	0.477273	0.522727	−396	2705	−3101
485	$5^1 97^1$	Y	N	5	0	1.0000000	0.478351	0.521649	−391	2710	−3101
486	$2^1 3^5$	N	N	13	8	2.0769231	0.479424	0.520576	−378	2723	−3101
487	487^1	Y	Y	−2	0	1.0000000	0.478439	0.521561	−380	2723	−3103
488	$2^3 61^1$	N	N	9	4	1.5555556	0.479508	0.520492	−371	2732	−3103
489	$3^1 163^1$	Y	N	5	0	1.0000000	0.480573	0.519427	−366	2737	−3103
490	$2^1 5^1 7^2$	N	N	30	14	1.1666667	0.481633	0.518367	−336	2767	−3103
491	491^1	Y	Y	−2	0	1.0000000	0.480652	0.519348	−338	2767	−3105
492	$2^2 3^1 41^1$	N	N	30	14	1.1666667	0.481707	0.518293	−308	2797	−3105
493	$17^1 29^1$	Y	N	5	0	1.0000000	0.482759	0.517241	−303	2802	−3105
494	$2^1 13^1 19^1$	Y	N	−16	0	1.0000000	0.481781	0.518219	−319	2802	−3121
495	$3^2 5^1 11^1$	N	N	30	14	1.1666667	0.482828	0.517172	−289	2832	−3121
496	$2^4 31^1$	N	N	−11	6	1.8181818	0.481855	0.518145	−300	2832	−3132
497	$7^1 71^1$	Y	N	5	0	1.0000000	0.482897	0.517103	−295	2837	−3132
498	$2^1 3^1 83^1$	Y	N	−16	0	1.0000000	0.481928	0.518072	−311	2837	−3148
499	499^1	Y	Y	−2	0	1.0000000	0.480962	0.519038	−313	2837	−3150
500	$2^2 5^3$	N	N	−23	18	1.4782609	0.480000	0.520000	−336	2837	−3173