Notes: A tale of higher-order interactions and harmonics in coupling functions

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I. INTRODUCTION

When studying higher-order interactions among oscillators, people make different choices for the coupling functions without really justifying them, and no systematic study exists on the effect of this choice.

II. PROLOGUE: GENERALISED ORDER PARAMETERS

Generalised parameter of order n:

$$Z_n = R_n e^{i\Phi n} = \frac{1}{N}\sum_{j=1}^{N} e^{in\theta_j} \tag{1}$$

which at order 1 reduces to the usual parameter. By definition, for a 1-cluster state with phase θ^*, we have $R_n = 1$ and $\Phi_n = n\theta^*$ for all n.

For $n = 2$, Z_2 is typically used to detect evenly spaced 2-cluster states, i.e. with a distance of π: for these, no matter if the 2 clusters are balanced or not, $R_2 = 1$. In general, for $n \geq 2$, these Z_n typically detect evenly spaced n-cluster states, see the following table:

	R_1	R_2	Φ_1	Φ_2
splay	0	0	/	/
1-cluster	1	1	θ^*	$2\theta^*$
2-cluster, balanced	0	1	/	$2\theta^*$
2-cluster, unbalanced	$2\eta - 1$	1	θ^*	$2\theta^*$

So that $R_2 = 1$ for all evenly spaced 2-cluster states, balanced or unbalanced. This is contrary to R_1 which takes values between 0 and 1 depending on the unbalance: 0 for totally balande $\eta = 0.5$, and 1 when totally unbalanced $\eta = 0$ when it is actually a 1-cluster.

III. MODELS

For simplicity, we start with all-to-all pure triplet interactions

$$\dot{\theta}_i = \omega + \frac{K_2}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N} g_2(\theta_i, \theta_j, \theta_k) \tag{2}$$

and we compare the two simplest choices of function g: symmetric and asymmetric with respect to i. The symmetric choice is

$$g_2(\theta_i, \theta_j, \theta_k) = \sin(\theta_j + \theta_k - 2\theta_i), \tag{3}$$

since any permutation of it indices (other than i) leaves the function invariant. The asymmetric choice is

$$g_2(\theta_i, \theta_j, \theta_k) = \sin(2\theta_j - \theta_k - \theta_i) \tag{4}$$

since that invariance does not hold anymore.

We will see that higher-order interactions can be linked to higher harmonics in the coupling function, and so we also introduce this known systems of pairwise interactions with a l-th harmonic

$$\dot{\theta}_i = \omega + \frac{K_1}{N} \sum_{j=1}^{N} \sin[l(\theta_j - \theta_i)] \tag{5}$$

To compare these models, we rewrite them in three different forms, each making different similutes apparent:

	Coupling function	**Complex mean-field**	**Real mean-field**
A. 3-body sym.	$\sin(\theta_j + \theta_k - 2\theta_i)$	$\mathrm{Im}[Z_1^2 e^{-i2\theta_i}]$	$R_1^2 \sin[2(\Phi_1 - \theta_i)]$
B. 3-body asym.	$\sin(2\theta_j - \theta_k - \theta_i)$	$\mathrm{Im}[Z_2 Z_1^* e^{-i\theta_i}]$	$R_1 R_2 \sin[(\Phi_2 - \Phi_1 - \theta_i)]$
C. 2-body 2nd harm.	$\sin[2(\theta_j - \theta_i)]$	$\mathrm{Im}[Z_2 e^{-i2\theta_i}]$	$R_2 \sin(\Phi_2 - 2\theta_i)$
D. 2-body 1st harm.	$\sin(\theta_j - \theta_i)$	$\mathrm{Im}[Z_1 e^{-i\theta_i}]$	$R_1 \sin(\Phi_1 - \theta_i)$

From the coupling function point of view, the 2- and 3-body cases are clearly apart. But it is interesting to see that the real meanfield notation of A has a structure similar to the 'microscopic' one of C: it is driven by Φ_1 with a 2nd harmonic, and strength R_1^2. in addition, Gong and Pikovsky consider them both 2nd harmonic cases, because their complex meanfield notation if of the form $\mathrm{Im}[He^{-i2\theta_i}]$.

IV. MODELS COMPARISON

We begin with just a self-consitency reasoning to gain intuition of the phenomenology. We will look into three possible states: splay (incoherence), 1-cluster, and 2-cluster states, where the 2 clusters have a distance of π, and their relative size is given by $0 \leq \eta \leq 1$.

A. 3-body symmetric

Real meanfield equation is same as each oscillator unidirectionally driven by external oscillator Φ_1 with second harmonic, and strength R_1^2. It has two stable fixed points (because 2nd harmonic) with distance π.

- **splay state:** $R_1 = 0$ so that there is no meanfield driving, each oscillator keeps going at its own frequency
- **1-cluster:** Start with slightly perturbed 1-cluster, $R_1 \approx 1$ and drives the oscillators to the closest stable fixed point, R_1 increases to 1, state is stable.
- **2-cluster:**
  - $\eta = 0.5$: Start with slightly perturbed balanced 2-cluster, $R_1 \approx 0$ and there is no driving. Unstable (or neutrally).
  - $\eta \neq 0.5$: Start with slightly perturbed unbalanced 2-cluster, $0 \leq R_1 \leq 1$ and the meanfield drives oscillators to both stable fixed points (depedning on their basin of attraction), R_1 increases and drives even stonger. Stable.

B. 3-body asymmetric

Real meanfield equation is a bit more complicated, 2 different (meanfield) phases are driving each oscillators, with strenght $R_1 R_2$. However, it can be simplified, because those two phases can be related at the 1- and 2-cluster states: (see table above).

It has two stable fixed points (because 2nd harmonic) with distance π.

- **splay state:** $R_1 = 0$ so that there is no meanfield driving, each oscillator keeps going at its own frequency
- **1-cluster:**
- **2-cluster:**
  - $\eta = 0.5$:
  - $\eta \neq 0.5$: