Contents

System Consider a network of oscillators

$$\dot{\theta}_i = \omega + \frac{K_1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \frac{K_2}{N^2} \sum_{j,k=1}^{N} \sin(\theta_j + \theta_k - 2\theta_i) \tag{1}$$

1 Generalised order parameters and clustered states

Order parameter of order n:

$$Z_n = R_n e^{i\Phi_n} = \frac{1}{N} \sum_{j=1}^{N} e^{in\theta_j} \tag{2}$$

Z_1 is the usual Kuramoto order parameter. Its length R_1 is 0 when the oscillators are incoherent, and 1 when they reach maximum coherence. Higher-order order parameters Z_n with $n \geq 2$ can be used to identify clustered states. Indeed for a 2-cluster state, R_2 is close to 1 but R_1 is close to 0 (if the clusters are balanced, i.e. they contain the same number of oscillators). Note that for a 1-cluster state, all R_n are close 1. Similarly, for a splay state, all R_n are close to 0.

We can see this analytically:

- Full sync: $\theta_i = \theta_j = \theta$

$$Z_1 = R_1 e^{i\Phi_1} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta} = e^{i\theta} \tag{3}$$

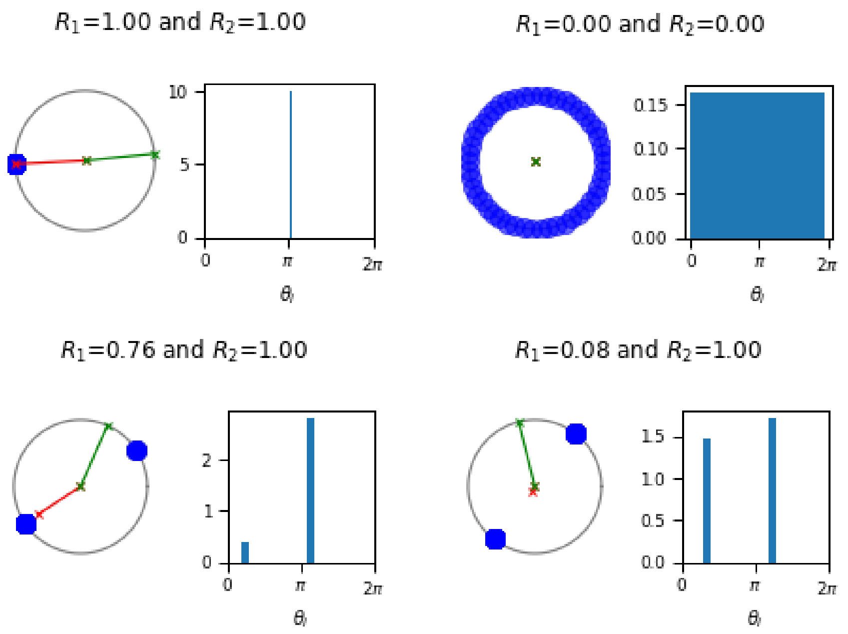

Figure 1: R_2 detects a 2-cluster state, regardless of the size of each cluster.

$$Z_2 = R_2 e^{i\Phi_2} = \frac{1}{N}\sum_{j=1}^{N} e^{i2\theta} = e^{i2\theta} \tag{4}$$

so $R_1 = 1$, $\Phi_1 = \theta$ and $R_2 = 1$, $\Phi_2 = 2\theta$.

- Splay state: $\theta_i = 2\pi i/N$

$$Z_1 = R_1 e^{i\Phi_1} = \frac{1}{N}\sum_{j=1}^{N} e^{i2\pi j/N} = 0 \tag{5}$$

$$Z_2 = R_2 e^{i\Phi_2} = \frac{1}{N}\sum_{j=1}^{N} e^{i4\pi j/N} = 0 \tag{6}$$

(check how to show it) so $R_1 = 0$,and $R_2 = 0$.

- 2 cluster state: $\theta_{1,..N_1} = \theta$ and $\theta_{N_1,..N} = \theta + \pi$

$$Z_1 = R_1 e^{i\Phi_1} = \frac{1}{N}\sum_{j=1}^{N_1} e^{i\theta} + \frac{1}{N}\sum_{j=N_1}^{N} e^{i(\theta+\pi)} = \frac{1}{N}\sum_{j=1}^{N_1} e^{i\theta} - \frac{1}{N}\sum_{j=N_1}^{N} e^{i\theta} = (\frac{2N_1}{N} - 1)e^{i\theta} \tag{7}$$

$$Z_2 = R_2 e^{i\Phi_2} = \frac{1}{N}\sum_{j=1}^{N_1} e^{i2\theta} + \frac{1}{N}\sum_{j=N_1}^{N} e^{i(2\theta+2\pi)} = \frac{1}{N}\sum_{j=1}^{N_1} e^{i2\theta} + \frac{1}{N}\sum_{j=N_1}^{N} e^{i2\theta} = e^{i2\theta} \quad (8)$$

so $R_1 = \frac{2N_1}{N} - 1$, $\Phi_1 = \theta$ and $R_2 = 1$, $\Phi_2 = 2\theta$. For balanced clusters, $N_1 = N/2$ and $R_1 = 0$. For extremely unbalanced clusters, $N_1 = N$ and $R_1 = 1$.

state	R_1	R_2	Φ_1	Φ_2
full sync	1	1	θ	2θ
splay	0	0	/	/
2-cluster	$\frac{2N_1}{N} - 1$	1	θ	2θ

Is there a bijection between a set of those macroscopic numbers and a give solution?

R_1 is 1 if and only if there is full sync, and 0 otherwise. So $R_1 = 1$ alone implies full sync. No other state is implied by any other value of R_1, however. Also, $R_2 = 1$ alone does not imply a unique solution: both full sync and the 2-cluster state have $R_2 = 1$. However, the combination of both R_1 and R_2 uniquely distingues between those three states. (Add smething for even higher orders).

2 Beyond pairwise interactions and nonlinear meanfield coupling

Eq. (1) can be rewritten in terms of the order parameter Z_1. First, for ease of calculation we introduce the complex notation

$$\dot{\theta}_i = \omega + \frac{K_1}{N}\sum_{j=1}^{N} Im[e^{i(\theta_j-\theta_i)}] + \frac{K_2}{N^2}\sum_{j,k=1}^{N} Im[e^{i(\theta_j+\theta_k-2\theta_i)}] \quad (9)$$

$$= \omega + K_1 Im[\frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)}] + K_2 Im[\frac{1}{N^2}\sum_{j,k=1}^{N} e^{i(\theta_j+\theta_k-2\theta_i)}] \quad (10)$$

Now, starting from the definition of Z_1

$$Z_1 e^{-i\theta_i} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)} \quad (11)$$

and when squared,

$$Z_1^2 = \frac{1}{N^2}\sum_{j=1}^{N} e^{i\theta_j}\sum_{k=1}^{N} e^{i\theta_k} \quad (12)$$

so that

$$Z_1^2 e^{-i2\theta_i} = \frac{1}{N^2}\sum_{j,k=1}^{N} e^{i(\theta_j+\theta_k-2\theta_i)} \quad (13)$$

Hence, our original network of oscillators can be rewritten

$$\dot{\theta}_i = \omega + K_1 Im[Z_1 e^{-i\theta_i}] + K_2 Im[Z_1^2 e^{-i2\theta_i}] \tag{14}$$

$$= \omega + K_1 Im[R_1 e^{i(\Phi_1 - \theta_i)}] + K_2 Im[R_1^2 e^{i2(\Phi_1 - \theta_i)}] \tag{15}$$

or without the complex notation

$$\dot{\theta}_i = \omega + K_1 R_1 \sin(\Phi_1 - \theta_i) + K_2 R_1^2 \sin[2(\Phi_1 - \theta_i)] \tag{16}$$

For different order and schemes of coupling functions, we write down the three equivalent notations:

$\frac{1}{N}\sum_j \sin(\theta_j - \theta_i)$	$Z_1 e^{-i\theta_i}$	$R_1 \sin[(\Phi_1 - \theta_i)]$
$\frac{1}{N^2}\sum_{j,k} \sin(\theta_j + \theta_k - 2\theta_i)$	$Z_1^2 e^{-i2\theta_i}$	$R_1^2 \sin[2(\Phi_1 - \theta_i)]$
$\frac{1}{N^3}\sum_{j,k,l} \sin(\theta_j + \theta_k + \theta_l - 3\theta_i)$	$Z_1^3 e^{-i3\theta_i}$	$R_1^3 \sin[3(\Phi_1 - \theta_i)]$
$\frac{1}{N^3}\sum_{j,k,l} \sin(\theta_j + \theta_k - \theta_l - \theta_i)$	$\lvert Z_1\rvert^2 Z_1 e^{-i\theta_i}$	$R_1^3 \sin[(\Phi_1 - \theta_i)]$
$\frac{1}{N^2}\sum_{j,k} \sin(2\theta_j - \theta_k - \theta_i)$	$Z_2 Z_1^* e^{-i\theta_i}$	$R_1 R_2 \sin[(\Phi_2 - \Phi_1 - \theta_i)]$

Is there a case where the second-order order parameter Z_2 appears? When the microscopic coupling between each phase contains a harmonic of order 2, even in pairwise coupling:

$\frac{1}{N}\sum_j \sin[2(\theta_j - \theta_i)]$	$Z_2 e^{-i2\theta_i}$	$R_2 \sin[2(\Phi_2 - \theta_i)]$

3 Beyond pairwise interactions and cluster states

Depending on the coupling function, we saw above that higher-order interactions can yield higher-harmonics in the mean-field coupling. For examples, terms like $R_1^2 \sin[2(\Phi_1 - \theta_i)]$ and $R_1^3 \sin[3(\Phi_1 - \theta_i)]$ for triplet and quadruplet interactions. Note however, that a quadruplet coupling scheme like $\sin(\theta_j + \theta_k - \theta_l - \theta_i)$ yields mean-field coupling of the form $R_1^3 \sin[(\Phi_1 - \theta_i)]$ which does not contain any harmonics.

We know that harmonics in the coupling function are directly linked stable cluster states. This can be understood intuitively from the mean-field coupling notation. Take $\dot{\theta}_i = \omega + K_1 R_1 \sin(\Phi_1 - \theta_i)$. The mean-field phase Φ_1 can be seen as an external driver of each phase θ_i. This sine coupling function has two fixed points, one stable and one unstable, as shown in Fig. 2(a). The stable fixed point corresponds to a stable synchronised (one cluster) state.

When triplet interactions yield second harmonics as in $\theta_i = \omega + R_1^2 \sin[2(\Phi_1 - \theta_i)]$, the coupling function as four fixed points. Two are stable and two are unstable. The two stable fixed ponts are separated by a distance of π. This is shown in Fig. 2(b).

A similar reasoning suggests that coupling schemes yielding harmonics of order n will favour stable n-cluster states. This is only an intuitive presentation and the argument has not been made so that it is self-consistent between the order parameter and the microscopic phases. A rigourous analysis of existence and stability of such cluster states needs to be carried out.

(No te: repulsive coupling can also lead to clustering)

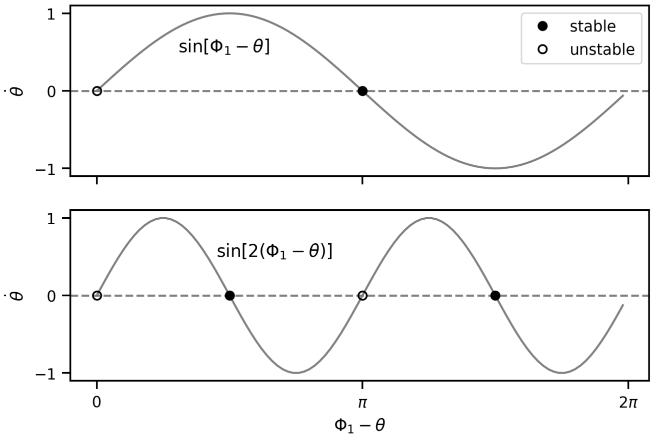

Figure 2: Harmonics in the mean-field coupling and cluster states.

$\frac{1}{N}\sum_j \sin(\theta_j - \theta_i)$	$R_1 \sin[(\Phi_1 - \theta_i)]$
$\frac{1}{N^2}\sum_{j,k} \sin(\theta_j + \theta_k - 2\theta_i)$	$R_1^2 \sin[2(\Phi_1 - \theta_i)]$
$\frac{1}{N}\sum_j \sin[2(\theta_j - \theta_i)]$	$R_2 \sin[2(\Phi_2 - \theta_i)]$

4 Zoology of solutions: existence and stability

Identities that we use in the calculations of this section:

$$\sum_{j=1}^{N} \sin[\frac{2\pi}{N}(j-i)] = 0 = \sum_{j=1}^{N} \sin[\frac{2\pi}{N}(j)] \tag{17}$$

4.1 Pairwise

$$\dot{\theta}_i = \omega + \frac{K_1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) \tag{18}$$

$$\dot{\theta}_i = \omega + K_1 R_1 \sin(\Phi_1 - \theta_i) \tag{19}$$

$$Z_1 = R_1 e^{i\Phi_1} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \tag{20}$$

Existence of solutions can most of the time be checked quite easily from (17) or (18). For stability, I see 3 mains methods: "heavy duty" but general Laplacian, more intuitive and simpler but correct? mean-field, and phase difference for cluster states. Each method is probably more suited for some system and/or some type of state. I am trying them "all" below to see what I can do and what is best and if they are consistent between themselves.

Note: the method most useful for a given state in the pairwise case will probably be the most useful in the triplet case too. So it is worth reading the following by jumping from say 'full sync' in pairwise to 'full sync' in triplet because the analysis is similar.

4.1.1 Full sync

Full sync is the in-phase solution $\theta_i(t) = \theta(t) = \omega t$. This yields $R_1 = 1$ and $\Phi_1 = \theta$.

Existence We inject our ansatz into (17). This yields $\dot{\theta}_i = \omega$, which is consistent, and hence the solution exists.

Equivalently, existence can be checked from (18). Injecting the solution into it, yields the same result.

Stability We want to asses linear stability of the full sync solution.

- A first way to do so is by considering a tiny heterogeneous perturbation around the solution $\theta_i(t) = \theta(t) + \delta\theta_i(t)$. Injecting this into (17) yields

$$\delta\dot{\theta}_i = \frac{K_1}{N}\sum_{j=1}^{N}\sin(\delta\theta_j - \delta\theta_i) \tag{21}$$

$$\simeq \frac{K_1}{N}\sum_{j=1}^{N}(\delta\theta_j - \delta\theta_i) \tag{22}$$

$$= \frac{K_1}{N}\sum_{j=1}^{N}(1 - N\delta_{ij})\delta\theta_j \tag{23}$$

$$= \frac{K_1}{N}\sum_{j=1}^{N}L^0_{ij}\delta\theta_j \tag{24}$$

  where we end up with a Laplacian matrix, that is degenerate in the all-to-all scheme: $L^0_{ij} = 1 - N\delta_{ij}$. It has eigenvalues $\Lambda_1 = 0$ (corresponding to eigenvector $(1, ..., 1)$) and $\Lambda_{\geq 2} = -N$. The following line was used in the above derivation.

$$\sum_{j=1}^{N}(\delta\theta_j - \delta\theta_i) = \sum_{j=1}^{N}(\delta\theta_j) - N\delta\theta_i = \sum_{j=1}^{N}(\delta\theta_j) - N\sum_{j=1}^{N}\delta\theta_j\delta_{ij} = \sum_{j=1}^{N}(1 - N\delta_{ij})\delta\theta_j \tag{25}$$

  Finally, this yields Lyapunov exponents

$$\lambda_1 = 0 \qquad \lambda_{\geq 2} = -K_1 \tag{26}$$

  The zero exponent means that the global phase of the full sync state is neutrally stable (equivalently, any initial could be chosen). The other negative exponents (for positive K_1) mean that the full sync state is stable (up to a phase shift).

This is already known of course, but was done to apply which methods are best here and for the generalised systems and other solutions. The Laplacian method used here seems a bit heavy-duty in the all-to-all scheme. However, it has the advantage of being very precise and clear.

- Can we compute the stability from the meanfield version of our system (18)? Eq. (18) is identical to that of the simple and classical case of a single driven oscillator. The only difference is that the "external" driving here is the mean-field of the systems, like in a feedback loop.

  First try: we treat the system as the simple externally driven oscillator case with external driving $\dot{\Phi}_1 = \omega$. Also, $R_1 = 1$. Note that in (18), R_1 does not affect whether a state is stable or not (as long as it is non-zero), it only modulates the strength of the forcing). Eq (18) is the same for any i. We treat the equation as in textbooks: we go to the rotating frame of the forcing, $\psi_i = \theta_i - \Phi_1$, so that

  $$\dot{\psi}_i = -K_1 R_1 \sin(\psi_i) \tag{27}$$

  This equation has 2 fixed points: 0 is the stable one, and π is unstable. $\psi_i = 0$ is the only self-consistent solution, since it means $\theta_i = \Phi_1$. Stability can computed by look at the derivative at the fixed point:

  $$\lambda_i = \frac{d\dot{\psi}_i}{d\psi_i} = -K_1 R_1 = -K_1 < 0 \tag{28}$$

  (this is the same as the Lyapunov exponent). This is consistent with the Laplacian method. Note that these are the exponents of the phase difference between each phase and the meanfield phase. The extra degree of freedom (zero Lyapunov exponent) found with the Laplacian method is not lost: it can be obtained by looking at the stability of the meanfield phase $\dot{\Phi}_1 = \omega$ which has Lyapunov exponent zero.

  Some questions regarding the method: *Note: have I done it fully correctly?* Computing the stability means that we look at what happens with a perturbation near the solution. To be self consistent, should we also consider perturbed versions of R_1 and Φ_1 when we compute the stability? So far, we have considered Φ_1 as an external independent system, but perturbing the system perturbs the meanfield (implicitely) too. Maybe we have taken it into account implicitely?

  This last "meanfield" has advantages over the Laplacian method, if we are sure it is 100% correct. First, it is intuitively clearer from the equation as we can port our knowledge from the case of a single driven oscillator. Second, this equation will look very similar for all the other cases. Third, it makes direct use of the all-to-all situation: there is only a single-dimensional equation for the whole system, the situation is the same for all oscillators. This is in contrast with the Laplacian method.

4.1.2 Splay state

State of the form $\theta_j(t) = \omega t + \frac{2\pi j}{N}$. This yields $R_1 = 0$ and Φ_1 is undefined.

Existence We inject our ansatz into (17). This yields

$$\dot{\theta}_i = \omega + \frac{K_1}{N}\sum_{j=1}^{N}\sin(\frac{2\pi(j-i)}{N}) = \omega \tag{29}$$

Note: check how, so it is consistent, so that the solution exists.

Equivalently, existence can be checked from (18). Injecting $R_1 = 0$ into it, yields the same result.

Stability We want to asses linear stability of the full sync solution.

- A first way to do so is by considering a tiny heterogeneous perturbation around the solution $\theta_i(t) = \theta^*(t) + \delta\theta_i(t)$. Injecting this into (17) yields

$$\delta\dot{\theta}_i = \frac{K_1}{N}\sum_{j=1}^{N}\sin\left[\frac{2\pi}{N}(j-i) + (\delta\theta_j - \delta\theta_i)\right] \tag{30}$$

$$\simeq \frac{K_1}{N}\sum_{j=1}^{N}\left[\sin[\frac{2\pi}{N}(j-i)] + \cos[\frac{2\pi}{N}(j-i)](\delta\theta_j - \delta\theta_i)\right] \tag{31}$$

$$= \frac{K_1}{N}\left[0 + \sum_{j=1}^{N}\cos[\frac{2\pi}{N}(j-i)](\delta\theta_j - \delta\theta_i)\right] \tag{32}$$

$$= \frac{K_1}{N}\left[\sum_{j=1}^{N}\cos[\frac{2\pi}{N}(j-i)]\delta\theta_j - \delta\theta_i\sum_{j=1}^{N}\cos[\frac{2\pi}{N}(j-i)]\right] \tag{33}$$

$$= \frac{K_1}{N}\sum_{j=1}^{N}\cos[\frac{2\pi}{N}(j-i)]\delta\theta_j \tag{34}$$

$$= \frac{K_1}{N}\sum_{j=1}^{N}L'_{ij}\delta\theta_j \tag{35}$$

  where we used the identities $\sum_{j=1}^{N}\sin[\frac{2\pi}{N}(j-i)] = 0 = \sum_{j=1}^{N}\sin[\frac{2\pi}{N}(j)]$ (also true with cosine) and we denoted $L'_{ij} = \cos[\frac{2\pi}{N}(j-i)]$. Note that L'_{ij} is the *the* classical Laplacian from above, but it has properties of *a* Laplacian, its columns add up to zero: $\sum_{j=1}^{N}L'_{ij} = 0$.

  I have looked at L'_{ij} numerically. Its eigenvalues normed by N, seem to be

$$\Lambda_{1,2}/N = 1/2 \qquad \Lambda_{\geq 3}/N = 0 \tag{36}$$

  so that the Lyapunov exponents $\lambda_i = K_1\Lambda_i/N$ are

$$\lambda_{1,2} = K_1/2 > 0 \qquad \lambda_{\geq 3} = 0 \tag{37}$$

So that the splay state is unstable (consistent with the literature, for example Watanabe Strogatz). *Note: Check the signs at each step!!! Very easy to get the signs wrong, also in def of Laplacian and eigenvalues and then conclusion changes!*. Note the eigenvectors of L'_{ij} are those of the usual Laplacian and are not directly related to the structure of the network *Note: right?*.

4.1.3 2-cluster

We consider a two cluster state that is symmetric in it location, but asymmetric/unbalanced in its distribution of oscillator among the cluster. Formally, we define the state as $\theta_{1,..,N_1}(t) = \theta(t) = \omega t$ and $\theta_{N_1+1,..,N}(t) = \theta(t) + \pi$, for $0 \leq N_1 \leq N$. We define $p = N_1/N$ the proportion of oscillators in the first cluster. If $p = 0.5$ the cluster are balanced, if $p = 0$ or 1, there is only one cluster. Note that the position of the cluster is up to the initial phase of the first cluster.

This state implies $R_1 = 2p - 1$, and $\Phi_1 = \theta$. *Note: check asymmetry could be the phase of the other cluster*

Existence In (17), the distance between two oscillators is either 0 or π so that $\dot{\theta}_i = \omega$. It is consistent, hence the solution exists.

Stability

- Laplacian.

  A first way to do so is by considering a tiny heterogeneous perturbation around the solution $\theta_i(t) = \theta^*(t) + \delta\theta_i(t)$. Injecting this into (17) yields for oscillators in the first cluster

$$\delta\dot{\theta}_{i=1,..,N_1} = \frac{K_1}{N}\sum_{j=1}^{N_1}\sin(\delta\theta_j - \delta\theta_i) + \frac{K_1}{N}\sum_{j=N_1+1}^{N}\sin[\pi + (\delta\theta_j - \delta\theta_i)] \tag{38}$$

$$= \frac{K_1}{N}\sum_{j=1}^{N_1}\sin(\delta\theta_j - \delta\theta_i) - \frac{K_1}{N}\sum_{j=N_1+1}^{N}\sin(\delta\theta_j - \delta\theta_i) \tag{39}$$

$$= \frac{K_1}{N}\sum_{j=1}^{N}B_{ij}(1 - N\delta_{ij})\delta\theta_j \tag{40}$$

$$= \frac{K_1}{N}\sum_{j=1}^{N}L'_{ij}\delta\theta_j \tag{41}$$

  where $B_{ij} = +1$ if i and j in same cluster, and -1 if in different clusters. And for oscillators in the second cluster

$$\delta\dot{\theta}_{i=N_1+1,..,N} = \frac{K_1}{N}\sum_{j=1}^{N_1}\sin(-\pi+\delta\theta_j-\delta\theta_i)+\frac{K_1}{N}\sum_{j=N_1+1}^{N}\sin(\delta\theta_j-\delta\theta_i) \tag{42}$$

$$= -\frac{K_1}{N}\sum_{j=1}^{N_1}\sin(\delta\theta_j-\delta\theta_i)+\frac{K_1}{N}\sum_{j=N_1+1}^{N}\sin(\delta\theta_j-\delta\theta_i) \tag{43}$$

$$= -\frac{K_1}{N}\sum_{j=1}^{N}B_{ij}(1-N\delta_{ij})\delta\theta_j \tag{44}$$

$$= -\frac{K_1}{N}\sum_{j=1}^{N}L'_{ij}\delta\theta_j \tag{45}$$

which has opposite sign compare to the first cluster.

Here are matrices B, L^0, and L' for $N=5$ and $N_1=2$:

$$B = \left(\begin{array}{cc|ccc} 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 \\ \hline -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \end{array}\right) \quad L^0 = \begin{pmatrix} 1-N & 1 & \cdots & & 1 \\ 1 & 1-N & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & & 1 & 1-N \end{pmatrix} \tag{46}$$

$$L' = \left(\begin{array}{cc|ccc} 1-N & 1 & -1 & -1 & -1 \\ 1 & 1-N & -1 & -1 & -1 \\ \hline -1 & -1 & 1-N & 1 & 1 \\ -1 & -1 & 1 & 1-N & 1 \\ -1 & -1 & 1 & 1 & 1-N \end{array}\right) \tag{47}$$

The sum of each row or column of L' is $N_1-(N-N_1)-N=2(N_1-N)\leq 0$.

From numerics, it seems that L' has eigenvalues

$$\Lambda_1/N = 0 \qquad \Lambda_{\geq 2}/N = -1 \tag{48}$$

so that the Lyapunov exponents $\lambda_i = K_1\Lambda_i/N$ are

$$\lambda_1 = 0 \qquad \lambda_{\geq 3} = -K_1 < 0 \tag{49}$$

for oscillators in the first cluster, but with $+K_1$ for the second cluster. Hence the first cluster is stable and the second unstable. This means that the 2-cluster state is unstable.

- Meanfield.

  For balanced clusters, $p = 0.5$, $R_1 = 0$ so that the meanfield phase Φ_1 does not drive the oscillators anymore. *Note: stability then?*

  For unbalanced cluster, $p \neq 0.5$, $R_1 > 0$ and the meanfield phase $\Phi = \theta$ drives the oscillators. we go to the rotating frame of the forcing, $\psi_i = \theta_i - \Phi_1$, so that

  $$\dot{\psi}_i = -K_1 R_1 \sin(\psi_i) \tag{50}$$

  Eq. (18) has 2 fixed points: 0 is the stable one, whereas π is unstable. This indicates that the first cluster for which $\psi_i = 0$ is stable with exponent $-K_1 R_1$ but the second cluster is unstable with exponent $K_1 R_1$. Hence, the 2-cluster state cannot be stable.

  This is consistent with the Laplacian method qualitatively (stable/unstable). But quantitatively, the exponent in the Laplacian method $-K_1$ does not depend on the relative size of the clusters p whereas here the exponent $-K_1 R_1$ does. *Note: which one is right??*

4.2 Triplet

$$\dot{\theta}_i = \omega + \frac{K_2}{N^2} \sum_{j,k=1}^{N} \sin(\theta_j + \theta_k - 2\theta_i) \tag{51}$$

$$\dot{\theta}_i = \omega + K_2 R_1^2 \sin[2(\Phi_1 - \theta_i)] \tag{52}$$

$$Z_1 = R_1 e^{i\Phi_1} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \tag{53}$$

4.2.1 Full sync

Full sync is the in-phase solution $\theta_i(t) = \theta(t) = \omega t$. This yields $R_1 = 1$ and $\Phi_1 = \theta$.

Existence We inject our ansatz into (50). This yields $\dot{\theta}_i = \omega$, which is consistent, and hence the solution exists.

Equivalently, existence can be checked from (51). Injecting the solution into it, yields the same result.

Stability We want to assess linear stability of the full sync solution.

- A first way to do so is by considering a tiny heterogeneous perturbation around the solution $\theta_i(t) = \theta(t) + \delta\theta_i(t)$. Injecting this into (17) yields

$$\delta\dot{\theta}_i = \frac{K_2}{N^2} \sum_{j,k=1}^{N} \sin(\delta\theta_j + \delta\theta_k - 2\delta\theta_i) \tag{54}$$

$$\simeq \frac{K_2}{N^2} \sum_{j,k=1}^{N} (\delta\theta_j + \delta\theta_k - 2\delta\theta_i) \tag{55}$$

$$= \frac{K_2}{N^2} [\sum_{j=1}^{N} (\delta\theta_j - \delta\theta_i) \sum_{k=1}^{N} + \sum_{k=1}^{N} (\delta\theta_k - \delta\theta_i) \sum_{j=1}^{N}] \tag{56}$$

$$= \frac{2K_2 N}{N^2} \sum_{j=1}^{N} (1 - N\delta_{ij}) \delta\theta_j \tag{57}$$

$$= \frac{2K_2}{N} \sum_{j=1}^{N} L_{ij}^0 \delta\theta_j \tag{58}$$

  This is the equation of the pairwise case, multiplied by two. So either we consider that the stability is twice larger, or we divide by two the original equation because each oscillator receives the input from two rather than one other oscillator.

  Lyapunov exponents

$$\lambda_1 = 0 \qquad \lambda_{\geq 2} = -2K_2(/N) \tag{59}$$

Note: Check if consistent with our Laplacian from March?

- A second way is to do it from the meanfield equation (51). External driving $\dot{\Phi}_1 = \omega$. Also, $R_1 = 1$. Note that in (51), R_1 does not affect whether a state is stable or not (as long as it is non-zero), it only modulates the strength of the forcing. Eq (51) is the same for any i. We treat the equation as in textbooks: we go to the rotating frame of the forcing, $\psi_i = \theta_i - \Phi_1$, so that

$$\dot{\psi}_i = -K_1 R_1^2 \sin(2\psi_i) \tag{60}$$

This equation has 4 fixed points: 0 and π are stable, $\pi/2$ and $3\pi/2$ are unstable. $\psi_i = 0$ is the only self-consistent solution, since it means $\theta_i = \Phi_1$. *Note: Right? Check!* Stability can be computed by looking at the derivative at the fixed point:

$$\lambda_i = \frac{d\dot{\psi}_i}{d\psi_i} = -2K_2 R_1^2 = -2K_2 < 0 \tag{61}$$

which is consistent with the Laplacian method. *Note: Should we take into account the perturbation on R_1 and Φ_1?*

4.2.2 Splay state

State of the form $\theta_j(t) = \omega t + \frac{2\pi j}{N}$. This yields $R_1 = 0$ and Φ_1 is undefined.

Existence We inject our ansatz into (17). This yields

$$\dot{\theta}_i = \omega + \frac{K_2}{N^2} \sum_{j,k=1}^{N} \sin\left(\frac{2\pi(j+k-2i)}{N}\right) = \omega \tag{62}$$

Note: check how, so it is consistent, so that the solution exists.

Equivalently, existence can be checked from (18). Injecting $R_1 = 0$ into it, yields the same result.

Stability We want to asses linear stability of the full sync solution.

- A first way to do so is by considering a tiny heterogeneous perturbation around the solution $\theta_i(t) = \theta^*(t) + \delta\theta_i(t)$. Injecting this into (17) yields

$$\delta\dot{\theta}_i = \frac{K_2}{N^2}\sum_{j,k=1}^{N}\sin\left[\frac{2\pi}{N}(j+k-2i) + (\delta\theta_j + \delta\theta_k - 2\delta\theta_i)\right] \tag{63}$$

$$\simeq \frac{K_2}{N^2}\sum_{j,k=1}^{N}\left[\sin[\frac{2\pi}{N}(j+k-2i)] + \cos[\frac{2\pi}{N}(j+k-2i)](\delta\theta_j + \delta\theta_k - 2\delta\theta_i)\right] \tag{64}$$

$$= \frac{K_2}{N^2}\left[0 + \sum_{j,k=1}^{N}\cos[\frac{2\pi}{N}(j+k-2i)](\delta\theta_j + \delta\theta_k - 2\delta\theta_i)\right] \tag{65}$$

$$= \frac{K_2}{N^2}\left[\sum_{j,k=1}^{N}\cos[\frac{2\pi}{N}(j+k-2i)](\delta\theta_j + \delta_k) - 2\delta\theta_i\sum_{j,k=1}^{N}\cos[\frac{2\pi}{N}(j+k-2i)]\right] \tag{66}$$

$$= \frac{K_2}{N^2}\sum_{j,k=1}^{N}\cos[\frac{2\pi}{N}(j+k-2i)](\delta\theta_j + \delta_k) \tag{67}$$

$$= \frac{K_2}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\left[\cos[\frac{2\pi}{N}(j-i)]\cos[\frac{2\pi}{N}(k-i)] + ...\right](\delta\theta_j + \delta_k) \tag{68}$$

Where the last line is obtained via the trigonometric identity for the sine of a sum: $\cos[\frac{2\pi}{N}(j+k-2i)] = \cos[\frac{2\pi}{N}(j-i) + \frac{2\pi}{N}(k-i)] = \cos[\frac{2\pi}{N}(j-i)]\cos[\frac{2\pi}{N}(k-i)] - \sin[\frac{2\pi}{N}(j-i)]\sin[\frac{2\pi}{N}(k-i)]$. Also, similarly as in the pairwise case, the identity (verified with Wofram) $\sum_{j,k=1}^{N}\sin[\frac{2\pi}{N}(j+k-2i)] = 0$ was used. We inspect the four terms separately

$$\frac{K_2}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\left[\cos[\frac{2\pi}{N}(j-i)]\cos[\frac{2\pi}{N}(k-i)]\right]\delta\theta_j \tag{69}$$

$$= \frac{K_2}{N^2}\sum_{j=1}^{N}\cos[\frac{2\pi}{N}(j-i)]\delta\theta_j\sum_{k=1}^{N}\left[\cos[\frac{2\pi}{N}(k-i)]\right] \tag{70}$$

$$= 0 \tag{71}$$

$$\frac{K_2}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\left[\cos[\frac{2\pi}{N}(j-i)]\cos[\frac{2\pi}{N}(k-i)]\right]\delta\theta_k \tag{72}$$

$$= \frac{K_2}{N^2}\sum_{k=1}^{N}\cos[\frac{2\pi}{N}(k-i)]\delta\theta_k\sum_{j=1}^{N}\left[\cos[\frac{2\pi}{N}(j-i)]\right] \tag{73}$$

$$= 0 \tag{74}$$

and the same with sine for the last two terms. Hence,

$$\delta\dot{\theta}_i = 0 \tag{75}$$

which means that the splay state is neutrally stable. This is difference than in the pairwise case, where the splay state was unstable.

Note: Check the calculations!!! Is it correct? Also, check against the literature: shouldn't there be some negative exponents for stability instead of all zero?

- I don't think there is a second way based on the meanfield for the splay state since $R_1 = 0$. Is there any other method? Maybe based on phase differences.

4.2.3 2-cluster

Formally, we define the state as $\theta_{1,..,N_1}(t) = \theta(t) = \omega t$ and $\theta_{N_1+1,..,N}(t) = \theta(t) + \pi$, for $0 \leq N_1 \leq N$. We define $p = N_1/N$ the proportion of oscillators in the first cluster.

This state implies $R_1 = 2p - 1$, and $\Phi_1 = \theta$. *Note: check asymmetry could be the phase of the other cluster*

Existence In (50), the distance between two oscillators is either 0 or π. There are 3 different configurations for the repartition of a given triplet among the 2 clusters: (i) all three in same cluster, (ii) one in a different cluster than i, (iii) both in a different cluster than i.

- (i): say i, j, k in cluster 1. Then, $g_i(j,k) = 0$
- (ii): say i, j in cluster 1, k in cluster 2. Then, $g_i(j,k) = \sin(\pi) = 0$
- (iii): say i in cluster 1, j and k in cluster 2. Then, $g_i(j,k) = \sin(2\pi) = 0$

So the coupling function for all configurations so that $\dot{\theta}_i = \omega$ for all i and the 2-cluster solution exists.

Stability

- Laplacian.

  A first way to do so is by considering a tiny heterogeneous perturbation around the solution $\theta_i(t) = \theta^*(t) + \delta\theta_i(t)$. Injecting this into (50) yields for oscillators in the first cluster

$$\begin{aligned}
\delta\dot{\theta}_{i=1,..,N_1} = \;& \frac{K_2}{N^2}\sum_{j,k=1}^{N_1}\sin(\delta\theta_j+\delta\theta_k-2\delta\theta_i) \\
& + 2\frac{K_2}{N^2}\sum_{j=1}^{N_1}\sum_{k=N_1+1}^{N}\sin(\pi+\delta\theta_j+\delta\theta_k-2\delta\theta_i) \\
& + \frac{K_2}{N^2}\sum_{j,k=N_1+1}^{N}\sin(2\pi+\delta\theta_j+\delta\theta_k-2\delta\theta_i) && (76) \\
\simeq \;& \frac{K_2}{N^2}\sum_{j,k=1}^{N_1}(\delta\theta_j+\delta\theta_k-2\delta\theta_i) \\
& - 2\frac{K_2}{N^2}\sum_{j=1}^{N_1}\sum_{k=N_1+1}^{N}(\delta\theta_j+\delta\theta_k-2\delta\theta_i) \\
& + \frac{K_2}{N^2}\sum_{j,k=N_1+1}^{N}(\delta\theta_j+\delta\theta_k-2\delta\theta_i) && (77) \\
= \;& \frac{2K_2}{N}\sum_{j=1}^{N_1}(\delta\theta_j-\delta\theta_i) \\
& - 2\frac{K_2}{N^2}\left[(N-N_1)\sum_{j=1}^{N_1}(\delta\theta_j-\delta\theta_i)+N_1\sum_{k=N_1+1}^{N}(\delta\theta_k-\delta\theta_i)\right] \\
& + \frac{2K_2}{N}\sum_{j=N_1+1}^{N}(\delta\theta_j-\delta\theta_i) && (78) \\
= \;& \frac{2K_2}{N}\sum_{j=1}^{N}(\delta\theta_j-\delta\theta_i) \\
& - 2\frac{K_2}{N^2}\left[N\sum_{j=1}^{N_1}(\delta\theta_j-\delta\theta_i)-N_1\sum_{j=1}^{N_1}(\delta\theta_j-\delta\theta_i)+N_1\sum_{j=N_1+1}^{N}(\delta\theta_j-\delta\theta_i)\right] && (79) \\
= \;& \frac{2K_2}{N}\sum_{j=N_1+1}^{N}(\delta\theta_j-\delta\theta_i)-2p\frac{K_2}{N}\left[\sum_{j=1}^{N}\sigma_j(\delta\theta_j-\delta\theta_i)\right] && (80) \\
= \;& \frac{2K_2}{N}\left[\sum_{j=N_1+1}^{N}(\delta\theta_j-\delta\theta_i)-p\sum_{j=1}^{N}\sigma_j(\delta\theta_j-\delta\theta_i)\right] && (81) \\
= \;& \frac{2K_2}{N}\sum_{j=1}^{N}\rho_j(\delta\theta_j-\delta\theta_i) && (82)
\end{aligned}$$

where ρ_j is $1-p$ if j in cluster 1, p if j in cluster 2. Indeedm it is defined as $\rho_j = -\sigma_j p + \beta_j$ where β_j is 0 in cluster 1 and 1 in cluster 2.

$$2\frac{K_2}{N^2}\left[(N-N_1)\sum_{j=1}^{N_1}(\delta\theta_j-\delta\theta_i)+N_1\sum_{k=N_1+1}^{N}(\delta\theta_k-\delta\theta_i)\right] \tag{83}$$

$$=2\frac{K_2}{N^2}\left[N\sum_{j=1}^{N_1}(\delta\theta_j-\delta\theta_i)-N_1\sum_{j=1}^{N_1}(\delta\theta_j-\delta\theta_i)+N_1\sum_{j=N_1+1}^{N}(\delta\theta_j-\delta\theta_i)\right] \tag{84}$$

$$=2\frac{K_2}{N}\sum_{j=1}^{N_1}(\delta\theta_j-\delta\theta_i)+2p\frac{K_2}{N}\left[-\sum_{j=1}^{N_1}(\delta\theta_j-\delta\theta_i)+\sum_{j=N_1+1}^{N}(\delta\theta_j-\delta\theta_i)\right] \tag{85}$$

$$=2\frac{K_2}{N}\sum_{j=1}^{N_1}(\delta\theta_j-\delta\theta_i)+2p\frac{K_2}{N}\left[\sum_{j=1}^{N_1}\sigma_j(\delta\theta_j-\delta\theta_i)\right] \tag{86}$$

$$\tag{87}$$

where σ_j is -1 if j is in cluster 1, and +1 if in cluster 2. We can go further and write the evolution of the perturbation in terms of a modified Laplacian. For this we need the following identity:

$$\sum_{j=1}^{N}\rho_j=\sum_{j=1}^{N_1}(1-p)+\sum_{j=N_1+1}^{N}p=Np(1-p)+N(1-p)p=2Np(1-p) \tag{88}$$

Hence

$$\dot{\delta\theta_i}=2\frac{K_2}{N}\left[\sum_{j=1}^{N}\rho_j\delta\theta_j-\delta\theta_i\sum_{j=1}^{N}\rho_j\right] \tag{89}$$

$$=2\frac{K_2}{N}\sum_{j=1}^{N}\left[\rho_j-2Np(1-p)\delta_{ij}\right]\delta\theta_j \tag{90}$$

$$=2\frac{K_2}{N}\sum_{j=1}^{N}L'_{ij}\delta\theta_j \tag{91}$$

Note that again, this new modified Laplacian $L'_{ij}=\rho_j-2Np(1-p)$ has its rows summing to zero. Also, it depends only on j.

Note that this equation is only for the perurbation of an oscillator in cluster 1. However, it is the same as that for cluster 2 (which is different from the pairwise case where the sign is different). This can be seen from the first line where considering cluster 2 adds 2π in each coupling function, coming from the $2\theta_i$, and that has no effect.

From numerics, it seems that L' has eigenvalues *Note: check!!*

$$\Lambda_1/N = 0 \qquad \Lambda_{\geq 2}/N = -2p(1-p) \leq 0 \tag{92}$$

so that the Lyapunov exponents $\lambda_i = 2K_2\Lambda_i/N$ are

$$\lambda_1 = 0 \qquad \lambda_{\geq 2} = -4p(1-p)K_1 \leq 0 \tag{93}$$

which means that both clusters are stable. Hence the 2-cluster state is stable. This is in opposition with the pairwise case.

- Meanfield

  For balanced clusters, $p = 0.5$, $R_1 = 0$ so that the meanfield phase Φ_1 does not drive the oscillators anymore. *Note: stability then?*

  For unbalanced cluster, $p \neq 0.5$, $R_1 = 2p - 1 > 0$ and the meanfield phase $\Phi = \theta$ drives the oscillators. We go to the rotating frame of the forcing, $\psi_i = \theta_i - \Phi_1$, so that

  $$\dot{\psi}_i = -K_1 R_1^2 \sin(2\psi_i) \tag{94}$$

  Eq. (18) has 4 fixed points: 0 and π are stable, whereas $\pi/2$ and $3\pi/2$ are unstable. This indicates that the first cluster for which $\psi_i = 0$ is stable with exponent $-K_1 R_1^2 = -K_1(2p-1)^2$ and the same for the second cluster $\psi_i = \pi$. Hence, the 2-cluster state is be stable.

  This is consistent with the Laplacian method qualitatively (stable/unstable). But quantitatively, the exponent in the Laplacian method $-4p(1-p)K_1 = -K_1(-4p^2 + 4p)$ is different from $-(2p-1)^2 K_1 = -K_1(4p^2 - 4p + 2)$ does. *Note: which one is right??*

4.3 Results summary

Summary table

	pos	zero	neg	stable?
full sync 2		λ_1	$\lambda_{\geq 2} = -K_1$	yes
full sync 3		λ_1	$\lambda_{\geq 2} = -2K_2$	yes
splay 2	$\lambda_1 = K_1 \ / \ 2$	$\lambda_{\geq 3}$		no
splay 3		λ_i		semi
2-cluster 2	$\lambda_{\geq 2}^{(2)} = K_1(R_1?)$	λ_1	$\lambda_{\geq 2}^{(1)} = -K_1(R_1?)$	no
2-cluster 3		λ_1	$\lambda_{\geq 2} = -4p(1-p)K_1$	yes