

Notes on Linear Panel Model

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1 Fixed Effect Model

$$y_{it} = \mathbf{x}'_{it}\boldsymbol{\beta} + \alpha_i + \varepsilon_{it}$$

Level 1

$$\begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} = \begin{pmatrix} \mathbf{x}'_{i1} \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \alpha_i + \begin{pmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{pmatrix}$$
$$\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta} + \underbrace{(\mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i)}_{\mathbf{u}_i}$$

Level 2

$$\begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_N \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_1 \\ \vdots \\ \boldsymbol{\varepsilon}_N \end{pmatrix}$$
$$\mathbf{y} = \mathbf{X} \boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{e}) \boldsymbol{\alpha} + \boldsymbol{\varepsilon}$$

Level 3

where α_i is unobserved heterogeneity, $\boldsymbol{\varepsilon}_i$ is idiosyncratic error, \mathbf{u}_i is composite error.

1.1 Assumption

1.1.1 Strong/strict exogeneity of regressors

For all t ,

$$\mathbb{E}(\varepsilon_{it} | \mathbf{x}_{i1}, \dots, \mathbf{x}_{iT}) = 0$$

Equivalently,

$$\mathbb{E}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \mathbf{0}$$

1.2 OLS estimator is inconsistent and biased

The necessary condition for OLS estimator to be consistent is $\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) = \mathbf{0}$.

$$\begin{aligned} \mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) &= \mathbb{E}(\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i | \mathbf{X}_i)) \\ &= \mathbb{E}(\mathbf{X}'_i \mathbb{E}(\mathbf{u}_i | \mathbf{X}_i)) \\ &= \mathbb{E}(\mathbf{X}'_i \mathbb{E}(\mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i | \mathbf{X}_i)) \\ &= \mathbb{E}(\mathbf{X}'_i \mathbb{E}(\mathbf{e} \alpha_i | \mathbf{X}_i) + \underbrace{\mathbf{X}'_i \mathbb{E}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i)}_{\mathbf{0}}) \\ &= \mathbb{E}(\mathbf{X}'_i \mathbf{e} \mathbb{E}(\alpha_i | \mathbf{X}_i)) \\ &= \mathbb{E}(\mathbf{X}'_i \mathbf{e} \alpha_i) \end{aligned}$$

because of strict exogeneity

$\mathbb{E}(\mathbf{X}'_i \mathbf{e} \alpha_i) \neq 0 \iff \mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) \neq \mathbf{0}$. Thus, OLS estimator is inconsistent if $\mathbb{E}(\mathbf{X}'_i \mathbf{e} \alpha_i) \neq \mathbf{0}$.

The necessary condition for OLS estimator to be unbiased is $\mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) = \mathbf{0}$. However, $\mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) = \mathbf{0} \implies \mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) = \mathbf{0}$ as $\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) = \mathbb{E}(\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i | \mathbf{X}_i)) = \mathbb{E}(\mathbf{X}'_i \mathbb{E}(\mathbf{u}_i | \mathbf{X}_i)) = \mathbb{E}(\mathbf{X}'_i \mathbf{0}) = \mathbf{0}$. Thus, $\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) \neq \mathbf{0} \implies \mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) \neq \mathbf{0}$. As a result, OLS estimator is biased if $\mathbb{E}(\mathbf{X}'_i \mathbf{e} \alpha_i) \neq \mathbf{0}$.

$\mathbb{E}(\alpha_i|\mathbf{X}_i) = 0 \implies \mathbb{E}(\mathbf{X}_i' e \alpha_i) = \mathbf{0}$ as $\mathbb{E}(\mathbf{X}_i' e \alpha_i) = \mathbb{E}(\mathbb{E}(\mathbf{X}_i' e \alpha_i | \mathbf{X}_i)) = \mathbb{E}(\mathbf{X}_i' e \mathbb{E}(\alpha_i | \mathbf{X}_i)) = \mathbb{E}(\mathbf{X}_i' e \mathbf{0}) = \mathbf{0}$. Thus, $\mathbb{E}(\mathbf{X}_i' e \alpha_i) \neq \mathbf{0} \implies \mathbb{E}(\alpha_i | \mathbf{X}_i) \neq 0$

OLS estimator of β is inconsistent and biased if α_i is correlated with \mathbf{X}_i (u_i is also correlated with \mathbf{X}_i). This is called omitted variable bias. To tackle this, we simply eliminate α_i by using different methods.

1.3 Fixed Effect Estimator / Within Estimator

1.3.1 Demean operator

$$\mathbf{Q} = \mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}'$$

This \mathbf{Q} is symmetric and idempotent,

$$\begin{aligned} \mathbf{Q}' &= (\mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}')' \\ &= \mathbf{I}_T' - T^{-1} \mathbf{e}'' \mathbf{e}' \\ &= \mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}' = \mathbf{Q} \end{aligned}$$

$$\begin{aligned} \mathbf{Q} \mathbf{Q}' &= \mathbf{Q} \mathbf{Q} \\ &= (\mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}')(\mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}') \\ &= \mathbf{I}_T \mathbf{I}_T - \mathbf{I}_T T^{-1} \mathbf{e} \mathbf{e}' - T^{-1} \mathbf{e} \mathbf{e}' \mathbf{I}_T + T^{-1} \mathbf{e} \mathbf{e}' T^{-1} \mathbf{e} \mathbf{e}' \\ &= \mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}' - T^{-1} \mathbf{e} \mathbf{e}' + T^{-2} \mathbf{e} \mathbf{e}' \mathbf{e} \mathbf{e}' \\ &= \mathbf{I}_T - 2T^{-1} \mathbf{e} \mathbf{e}' + T^{-2} \mathbf{e} T \mathbf{e}' \\ &= \mathbf{I}_T - 2T^{-1} \mathbf{e} \mathbf{e}' + T^{-1} \mathbf{e} \mathbf{e}' \\ &= \mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}' = \mathbf{Q} \end{aligned}$$

1.3.2 Demean transformed model

$$\begin{aligned} \mathbf{Q} \mathbf{y}_i &= \mathbf{Q}(\mathbf{X}_i \beta + \mathbf{e} \alpha_i + \varepsilon_i) \\ &= \mathbf{Q} \mathbf{X}_i \beta + \mathbf{Q} \mathbf{e} \alpha_i + \mathbf{Q} \varepsilon_i \\ &= \mathbf{Q} \mathbf{X}_i \beta + \mathbf{0} \alpha_i + \mathbf{Q} \varepsilon_i \\ &= \mathbf{Q} \mathbf{X}_i \beta + \mathbf{Q} \varepsilon_i \end{aligned}$$

Level 2

It is because

$$\begin{aligned} \mathbf{Q} \mathbf{e} &= (\mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}') \mathbf{e} \\ &= \mathbf{I}_T \mathbf{e} - T^{-1} \mathbf{e} \mathbf{e}' \mathbf{e} \\ &= \mathbf{e} - T^{-1} \mathbf{e} T \\ &= \mathbf{e} - \mathbf{e} = \mathbf{0} \end{aligned}$$

It can be written as $\mathbf{y}_i - \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}_i') \beta + (\varepsilon_i - \mathbf{e} \bar{\varepsilon}_i)$ because

$$\begin{aligned} \mathbf{Q} \mathbf{X}_i &= (\mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}') \mathbf{X}_i \\ &= \mathbf{I}_T \mathbf{X}_i - T^{-1} \mathbf{e} \mathbf{e}' \mathbf{X}_i \\ &= \mathbf{X}_i - T^{-1} \mathbf{e} \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x}'_{i1} \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} \\ &= \mathbf{X}_i - \mathbf{e} T^{-1} \sum_{t=1}^T \mathbf{x}'_{it} \\ &= \mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}_i' \end{aligned}$$

$$\begin{aligned}
Qy_i &= (I_T - T^{-1}ee')y_i \\
&= I_T y_i - T^{-1}ee'y_i \\
&= y_i - T^{-1}e(1 \quad \cdots \quad 1) \begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} \\
&= y_i - eT^{-1} \sum_{t=1}^T y_{it} \\
&= y_i - e\bar{y}_i
\end{aligned}$$

$y_i - e\bar{y}_i = (X_i - e\bar{x}'_i)\beta + (\varepsilon_i - e\bar{\varepsilon}_i)$ can be written as

$$\begin{aligned}
\begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{y}_i &= \left(\begin{pmatrix} x'_{i1} \\ \vdots \\ x'_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{x}'_i \right) \beta + \left(\begin{pmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{\varepsilon}_i \right) \\
\begin{pmatrix} y_{i1} - \bar{y}_i \\ \vdots \\ y_{iT} - \bar{y}_i \end{pmatrix} &= \begin{pmatrix} x'_{i1} - \bar{x}'_i \\ \vdots \\ x'_{iT} - \bar{x}'_i \end{pmatrix} \beta + \begin{pmatrix} \varepsilon_{i1} - \bar{\varepsilon}_i \\ \vdots \\ \varepsilon_{iT} - \bar{\varepsilon}_i \end{pmatrix} \\
\begin{pmatrix} y_{i1} - \bar{y}_i \\ \vdots \\ y_{iT} - \bar{y}_i \end{pmatrix} &= \begin{pmatrix} (x_{i1} - \bar{x}_i)' \\ \vdots \\ (x_{iT} - \bar{x}_i)' \end{pmatrix} \beta + \begin{pmatrix} \varepsilon_{i1} - \bar{\varepsilon}_i \\ \vdots \\ \varepsilon_{iT} - \bar{\varepsilon}_i \end{pmatrix} \\
y_{it} - \bar{y}_i &= (x_{it} - \bar{x}_i)' \beta + (\varepsilon_{it} - \bar{\varepsilon}_i)
\end{aligned}$$

Level 1

1.3.3 OLS estimator of the demean transformed model / Fixed Effect (FE) estimator

$$\begin{aligned}
\hat{\beta}_{within}^{ols} &= \left[\sum_{i=1}^N (QX_i)' QX_i \right]^{-1} \sum_{i=1}^N (QX_i)' Qy_i \\
&= \left[\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_i)' \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i)
\end{aligned}$$

Level 2

Level 1

It is because

$$\begin{aligned}
(QX_i)' QX_i &= (X_i - e\bar{x}'_i)' (X_i - e\bar{x}'_i) \\
&= \left(\begin{pmatrix} x'_{i1} \\ \vdots \\ x'_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{x}'_i \right) \left(\begin{pmatrix} x'_{i1} \\ \vdots \\ x'_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{x}'_i \right) \\
&= \begin{pmatrix} x'_{i1} - \bar{x}'_i \\ \vdots \\ x'_{iT} - \bar{x}'_i \end{pmatrix}' \begin{pmatrix} x'_{i1} - \bar{x}'_i \\ \vdots \\ x'_{iT} - \bar{x}'_i \end{pmatrix} \\
&= \begin{pmatrix} (x_{i1} - \bar{x}_i)' \\ \vdots \\ (x_{iT} - \bar{x}_i)' \end{pmatrix}' \begin{pmatrix} (x_{i1} - \bar{x}_i)' \\ \vdots \\ (x_{iT} - \bar{x}_i)' \end{pmatrix} \\
&= ((x_{i1} - \bar{x}_i) \quad \cdots \quad (x_{iT} - \bar{x}_i)) \begin{pmatrix} (x_{i1} - \bar{x}_i)' \\ \vdots \\ (x_{iT} - \bar{x}_i)' \end{pmatrix} \\
&= \sum_{t=1}^T (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_i)'
\end{aligned}$$

$$\begin{aligned}
(QX_i)'Qy_i &= (X_i - e\bar{x}_i)'(y_i - e\bar{y}_i) \\
&= \left(\begin{pmatrix} x'_{i1} \\ \vdots \\ x'_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{x}_i' \right)' \left(\begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{y}_i \right) \\
&= \begin{pmatrix} x'_{i1} - \bar{x}_i' \\ \vdots \\ x'_{iT} - \bar{x}_i' \end{pmatrix}' \begin{pmatrix} y_{i1} - \bar{y}_i \\ \vdots \\ y_{iT} - \bar{y}_i \end{pmatrix} \\
&= ((x_{i1} - \bar{x}_i) \quad \cdots \quad (x_{iT} - \bar{x}_i)) \begin{pmatrix} y_{i1} - \bar{y}_i \\ \vdots \\ y_{iT} - \bar{y}_i \end{pmatrix} \\
&= \sum_{t=1}^T (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i)
\end{aligned}$$

$$\begin{aligned}
\hat{\beta}_{within}^{ols} &= \left[\sum_{i=1}^N (QX_i)'QX_i \right]^{-1} \sum_{i=1}^N (QX_i)'Qy_i && \text{Level 2} \\
&= \left[\sum_{i=1}^N (X_i'X_i - \bar{x}_i T \bar{x}_i') \right]^{-1} \sum_{i=1}^N (X_i'y_i - \bar{x}_i T \bar{y}_i) \\
&= \left[\sum_{i=1}^N X_i'X_i - T \sum_{i=1}^N \bar{x}_i \bar{x}_i' \right]^{-1} \left(\sum_{i=1}^N X_i'y_i - T \sum_{i=1}^N \bar{x}_i \bar{y}_i \right) \\
&= \left[(X_1' \quad \cdots \quad X_N') \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} - T (\bar{x}_1 \quad \cdots \quad \bar{x}_N) \begin{pmatrix} \bar{x}_1' \\ \vdots \\ \bar{x}_N' \end{pmatrix} \right]^{-1} \left((X_1' \quad \cdots \quad X_N') \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} - T (\bar{x}_1 \quad \cdots \quad \bar{x}_N) \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_N \end{pmatrix} \right) \\
&= [X'X - T\bar{X}'\bar{X}]^{-1} (X'y - T\bar{X}'\bar{y}) && \text{Level 3}
\end{aligned}$$

It is because

$$\begin{aligned}
(QX_i)'QX_i &= (X_i - e\bar{x}_i)'(X_i - e\bar{x}_i) \\
&= (X_i' - \bar{x}_i' e')(X_i - e\bar{x}_i) \\
&= X_i'X_i - X_i'e\bar{x}_i' - \bar{x}_i e'X_i + \bar{x}_i e' e \bar{x}_i' \\
&= X_i'X_i - X_i'e\bar{x}_i' - \bar{x}_i e'X_i + \bar{x}_i T \bar{x}_i' \\
&= X_i'X_i - (e'X_i)' \bar{x}_i' - \bar{x}_i e'X_i + \bar{x}_i T \bar{x}_i' \\
&= X_i'X_i - \left(\sum_{t=1}^T x'_{it} \right)' \bar{x}_i' - \bar{x}_i \sum_{t=1}^T x'_{it} + \bar{x}_i T \bar{x}_i' \\
&= X_i'X_i - \left(\sum_{t=1}^T x_{it}/T \right) T \bar{x}_i' - \bar{x}_i T \sum_{t=1}^T x'_{it}/T + \bar{x}_i T \bar{x}_i' \\
&= X_i'X_i - \bar{x}_i T \bar{x}_i' - \bar{x}_i T \bar{x}_i' + \bar{x}_i T \bar{x}_i' \\
&= X_i'X_i - \bar{x}_i T \bar{x}_i' \\
\\
(QX_i)'Qy_i &= (X_i - e\bar{x}_i)'(y_i - e\bar{y}_i) \\
&= X_i'y_i - X_i'e\bar{y}_i - \bar{x}_i e'y_i + \bar{x}_i e' e \bar{y}_i \\
&= X_i'y_i - (e'X_i)' \bar{y}_i - \bar{x}_i e'y_i + \bar{x}_i T \bar{y}_i \\
&= X_i'y_i - \left(\sum_{t=1}^T x'_{it} \right)' \bar{y}_i - \bar{x}_i \sum_{t=1}^T y_{it} + \bar{x}_i T \bar{y}_i \\
&= X_i'y_i - \left(\sum_{t=1}^T x_{it}/T \right) T \bar{y}_i - \bar{x}_i T \sum_{t=1}^T y_{it}/T + \bar{x}_i T \bar{y}_i \\
&= X_i'y_i - \bar{x}_i T \bar{y}_i - \bar{x}_i T \bar{y}_i + \bar{x}_i T \bar{y}_i \\
&= X_i'y_i - \bar{x}_i T \bar{y}_i
\end{aligned}$$

1.3.4 The necessary condition for consistency and unbiasedness

The necessary condition for FE estimator (OLS estimator of the demean transformed model) to be consistent is $\mathbb{E}((\mathbf{Q}\mathbf{X}_i)'\mathbf{Q}\varepsilon_i) = \mathbf{0}$.

$$\begin{aligned}
\mathbb{E}((\mathbf{Q}\mathbf{X}_i)'\mathbf{Q}\varepsilon_i) &= \mathbb{E}(\mathbf{X}_i'\mathbf{Q}'\mathbf{Q}\varepsilon_i) \\
&= \mathbb{E}(\mathbf{X}_i'\mathbf{Q}\varepsilon_i) && \text{as } \mathbf{Q} \text{ is idempotent and symmetric} \\
&= \mathbb{E}(\mathbb{E}(\mathbf{X}_i'\mathbf{Q}\varepsilon_i|\mathbf{X}_i)) \\
&= \mathbb{E}(\mathbf{X}_i'\mathbf{Q}\underbrace{\mathbb{E}(\varepsilon_i|\mathbf{X}_i)}_{\mathbf{0}}) && \text{because of strict exogeneity} \\
&= \mathbf{0}
\end{aligned}$$

Thus, FE estimator satisfies the necessary condition for consistency given strict exogeneity assumption. Indeed, strict exogeneity is stronger than what is required. To see this, first note that for any t ,

$$\mathbb{E}(\varepsilon_i|\mathbf{X}_i) = \mathbb{E}(\varepsilon_{it}|\mathbf{x}_{i1}, \dots, \mathbf{x}_{iT}) = \mathbf{0} \implies \mathbb{E}(\mathbf{x}_{is}\varepsilon_{it}) = \mathbf{0} \quad \text{for all } s$$

It is because for any t and s ,

$$\begin{aligned}
\mathbb{E}(\mathbf{x}_{is}\varepsilon_{it}) &= \mathbb{E}(\mathbb{E}(\mathbf{x}_{is}\varepsilon_{it}|\mathbf{x}_{i1}, \dots, \mathbf{x}_{iT})) \\
&= \mathbb{E}(\mathbf{x}_{is}\underbrace{\mathbb{E}(\varepsilon_{it}|\mathbf{x}_{i1}, \dots, \mathbf{x}_{iT})}_{\mathbf{0}}) \\
&= \mathbf{0}
\end{aligned}$$

The necessary condition for FE estimator to be consistent can also be written as $\mathbb{E}((\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\varepsilon_{it} - \bar{\varepsilon}_i)) = \mathbf{0}$.

$$\mathbb{E}((\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\varepsilon_{it} - \bar{\varepsilon}_i)) = \mathbb{E}(\mathbf{x}_{it}\varepsilon_{it}) - \mathbb{E}(\mathbf{x}_{it}\bar{\varepsilon}_i) - \mathbb{E}(\bar{\mathbf{x}}_i\varepsilon_{it}) + \mathbb{E}(\bar{\mathbf{x}}_i\bar{\varepsilon}_i) = \mathbf{0}$$

It is because $\mathbb{E}(\mathbf{x}_{is}\varepsilon_{it}) = \mathbf{0}$ for any t and s implies

$$\begin{aligned}
\mathbb{E}(\mathbf{x}_{it}\varepsilon_{it}) &= \mathbf{0} \\
\mathbb{E}(\mathbf{x}_{it}\bar{\varepsilon}_i) &= \mathbb{E}(\mathbf{x}_{it}T^{-1}\sum_{s=1}^T\varepsilon_{is}) = T^{-1}\sum_{s=1}^T\underbrace{\mathbb{E}(\mathbf{x}_{it}\varepsilon_{is})}_{\mathbf{0}} = \mathbf{0} \\
\mathbb{E}(\bar{\mathbf{x}}_i\varepsilon_{it}) &= \mathbb{E}(T^{-1}\sum_{s=1}^T\mathbf{x}_{is}\varepsilon_{it}) = T^{-1}\sum_{s=1}^T\underbrace{\mathbb{E}(\mathbf{x}_{is}\varepsilon_{it})}_{\mathbf{0}} = \mathbf{0} \\
\mathbb{E}(\bar{\mathbf{x}}_i\bar{\varepsilon}_i) &= \mathbb{E}(T^{-1}\sum_{s=1}^T\mathbf{x}_{is}T^{-1}\sum_{t=1}^T\varepsilon_{it}) = T^{-2}\sum_{s=1}^T\sum_{t=1}^T\underbrace{\mathbb{E}(\mathbf{x}_{is}\varepsilon_{it})}_{\mathbf{0}} = \mathbf{0}
\end{aligned}$$

Thus, the weaker assumption $\mathbb{E}(\mathbf{x}_{is}\varepsilon_{it}) = \mathbf{0}$ for any t and s is sufficient for $\mathbb{E}((\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\varepsilon_{it} - \bar{\varepsilon}_i)) = \mathbf{0}$

The necessary condition for FE estimator to be unbiased is $\mathbb{E}(\mathbf{Q}\varepsilon_i|\mathbf{Q}\mathbf{X}_i) = \mathbf{0}$.

$$\begin{aligned}
\mathbb{E}(\mathbf{Q}\varepsilon_i|\mathbf{Q}\mathbf{X}_i) &= \mathbf{Q}\underbrace{\mathbb{E}(\varepsilon_i|\mathbf{X}_i)}_{\mathbf{0}} && \text{as } \mathbf{Q} \text{ is constant and strict exogeneity} \\
&= \mathbf{0}
\end{aligned}$$

1.3.5 Conditional Variance of $\hat{\beta}_{within}^{ols}$

Given independence of i ,

$$\begin{aligned}
Var(\hat{\beta}_{within}^{ols}|\mathbf{X}_i) &= Var([\sum_{i=1}^N\mathbf{X}_i'\mathbf{Q}'\mathbf{Q}\mathbf{X}_i]^{-1}\sum_{i=1}^N\mathbf{X}_i'\mathbf{Q}'\mathbf{Q}\mathbf{y}_i|\mathbf{X}_i) \\
&= [\sum_{i=1}^N\mathbf{X}_i'\mathbf{Q}'\mathbf{Q}\mathbf{X}_i]^{-1}Var(\sum_{i=1}^N\mathbf{X}_i'\mathbf{Q}'\mathbf{Q}\mathbf{y}_i|\mathbf{X}_i)[\sum_{i=1}^N\mathbf{X}_i'\mathbf{Q}'\mathbf{Q}\mathbf{X}_i]^{-1'} \\
&= [\sum_{i=1}^N\mathbf{X}_i'\mathbf{Q}'\mathbf{Q}\mathbf{X}_i]^{-1}\sum_{i=1}^N\mathbf{X}_i'\mathbf{Q}'Var(\mathbf{Q}\varepsilon_i|\mathbf{X}_i)\mathbf{Q}\mathbf{X}_i[\sum_{i=1}^N\mathbf{X}_i'\mathbf{Q}'\mathbf{Q}\mathbf{X}_i]^{-1}
\end{aligned}$$

It is because

$$\begin{aligned}
Var\left(\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{y}_i | \mathbf{X}_i\right) &= \sum_{i=1}^N Var(\mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{y}_i | \mathbf{X}_i) \\
&= \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' Var(\mathbf{Q} \mathbf{y}_i | \mathbf{X}_i) (\mathbf{X}_i' \mathbf{Q}')' \\
&= \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' Var(\mathbf{Q} \mathbf{X}_i \beta + \mathbf{Q} \varepsilon_i | \mathbf{X}_i) \mathbf{Q}'' \mathbf{X}_i'' \\
&= \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' Var(\mathbf{Q} \varepsilon_i | \mathbf{X}_i) \mathbf{Q} \mathbf{X}_i
\end{aligned}$$

1.3.6 $Var(\varepsilon_i | \mathbf{X}_i) = \sigma_\varepsilon^2 \mathbf{I}_T$

If ε_{it} is homoskedasticity and serially uncorrelated across t i.e., $Var(\varepsilon_i | \mathbf{X}_i) = \sigma_\varepsilon^2 \mathbf{I}_T$ (further assume independence of i and strict exogeneity), we have $\varepsilon_i | \mathbf{X}_i \sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T]$.

$$Var(\mathbf{Q} \varepsilon_i | \mathbf{X}_i) = \mathbf{Q} Var(\varepsilon_i | \mathbf{X}_i) \mathbf{Q}' = \mathbf{Q} \sigma_\varepsilon^2 \mathbf{I}_T \mathbf{Q}' = \sigma_\varepsilon^2 \mathbf{Q} \mathbf{Q}' = \sigma_\varepsilon^2 \mathbf{Q} = \sigma_\varepsilon^2 (\mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}') = \sigma_\varepsilon^2 \begin{pmatrix} 1 - \frac{1}{T} & -\frac{1}{T} & \cdots & -\frac{1}{T} \\ -\frac{1}{T} & 1 - \frac{1}{T} & \cdots & -\frac{1}{T} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{T} & -\frac{1}{T} & \cdots & 1 - \frac{1}{T} \end{pmatrix}.$$

Thus, $\mathbf{Q} \varepsilon_i$ is homoskedasticity but negatively serially correlated. For any t ,

$$\begin{aligned}
Var(\varepsilon_{it} - \bar{\varepsilon}_i) &= \sigma_\varepsilon^2 (1 - \frac{1}{T}) \iff \sigma_\varepsilon^2 = \frac{T}{T-1} Var(\varepsilon_{it} - \bar{\varepsilon}_i) \\
\hat{\sigma}_\varepsilon^2 &= \frac{T}{T-1} \widehat{Var}(\varepsilon_{it} - \bar{\varepsilon}_i) \\
&= \frac{T}{T-1} \frac{\sum_{i=1}^N \sum_{t=1}^T (y_{it} - \bar{y}_i - (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' \hat{\beta}_{within}^{ols})^2}{NT - (K + N)} \\
&= \frac{T}{T-1} \frac{\sum_{i=1}^N \sum_{t=1}^T (y_{it} - \mathbf{x}_{it}' \hat{\beta}_{within}^{ols} - \overbrace{(\bar{y}_i - \bar{\mathbf{x}}_i' \hat{\beta}_{within}^{ols})}^{\hat{\alpha}_i})^2}{\underbrace{NT - (K + N)}_{N(T-1) - K}} \quad \frac{T}{T-1} \approx 1 \text{ if } T \text{ is large.}
\end{aligned}$$

$$\begin{aligned}
Var(\hat{\beta}_{within}^{ols} | \mathbf{X}_i) &= [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \sigma_\varepsilon^2 \underbrace{\mathbf{Q} \mathbf{Q}'}_{\mathbf{Q}} \mathbf{X}_i [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} \\
&= \sigma_\varepsilon^2 [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} \\
&= \sigma_\varepsilon^2 \mathbf{I}_T [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} \\
&= \sigma_\varepsilon^2 [\sum_{i=1}^N (\mathbf{Q} \mathbf{X}_i)' \mathbf{Q} \mathbf{X}_i]^{-1} \quad \text{Level 2} \\
&= \sigma_\varepsilon^2 [\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)']^{-1} \quad \text{Level 1}
\end{aligned}$$

1.3.7 $Var(\varepsilon_i|\mathbf{X}_i) = \Omega_i$

We have $\varepsilon_i|\mathbf{X}_i \sim iid [\mathbf{0}, \Omega_i]$.

$$\begin{aligned}
Var(\hat{\beta}_{within}^{ols}|\mathbf{X}_i) &= \left[\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbb{E}[(\mathbf{Q}\varepsilon_i - \mathbb{E}(\mathbf{Q}\varepsilon_i|\mathbf{X}_i))(\mathbf{Q}\varepsilon_i - \mathbb{E}(\mathbf{Q}\varepsilon_i|\mathbf{X}_i))'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i \left[\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \\
&= \left[\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbb{E}[(\mathbf{Q}\varepsilon_i - \mathbf{Q}\mathbb{E}(\varepsilon_i|\mathbf{X}_i))(\mathbf{Q}\varepsilon_i - \mathbf{Q}\mathbb{E}(\varepsilon_i|\mathbf{X}_i))'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i \left[\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \\
&= \left[\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbb{E}[(\mathbf{Q}\varepsilon_i - \mathbf{Q}\mathbf{0})(\mathbf{Q}\varepsilon_i - \mathbf{Q}\mathbf{0})'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i \left[\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \\
&= \left[\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbb{E}[\mathbf{Q}\varepsilon_i(\mathbf{Q}\varepsilon_i)'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i \left[\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \\
&= \left[\sum_{i=1}^N (\mathbf{Q} \mathbf{X}_i)' \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N (\mathbf{Q} \mathbf{X}_i)' \mathbb{E}[\mathbf{Q}\varepsilon_i(\mathbf{Q}\varepsilon_i)'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i \left[\sum_{i=1}^N (\mathbf{Q} \mathbf{X}_i)' \mathbf{Q} \mathbf{X}_i \right]^{-1} \\
&= \left[\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)'\right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) \mathbb{E}[\dot{\varepsilon}_{it} \dot{\varepsilon}_{is}|\mathbf{X}_i] (\mathbf{x}_{is} - \bar{\mathbf{x}}_i)' \left[\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)'\right]^{-1}
\end{aligned}$$

It is because

$$\begin{aligned}
\sum_{i=1}^N (\mathbf{Q} \mathbf{X}_i)' \mathbb{E}[\mathbf{Q}\varepsilon_i(\mathbf{Q}\varepsilon_i)'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i &= \sum_{i=1}^N (\mathbf{Q} \mathbf{X}_i)' \mathbb{E}[\dot{\varepsilon}_i \dot{\varepsilon}_i'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i \\
&= \sum_{i=1}^N \begin{pmatrix} (\mathbf{x}_{i1} - \bar{\mathbf{x}}_i)' \\ \vdots \\ (\mathbf{x}_{iT} - \bar{\mathbf{x}}_i)' \end{pmatrix}' \begin{pmatrix} \mathbb{E}[\dot{\varepsilon}_{i1}^2|\mathbf{X}_i] & \cdots & \mathbb{E}[\dot{\varepsilon}_{i1} \dot{\varepsilon}_{iT}|\mathbf{X}_i] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[\dot{\varepsilon}_{iT} \dot{\varepsilon}_{i1}|\mathbf{X}_i] & \cdots & \mathbb{E}[\dot{\varepsilon}_{iT}^2|\mathbf{X}_i] \end{pmatrix} \begin{pmatrix} (\mathbf{x}_{i1} - \bar{\mathbf{x}}_i)' \\ \vdots \\ (\mathbf{x}_{iT} - \bar{\mathbf{x}}_i)' \end{pmatrix} \\
&= \sum_{i=1}^N ((\mathbf{x}_{i1} - \bar{\mathbf{x}}_i) \quad (\mathbf{x}_{iT} - \bar{\mathbf{x}}_i)) \begin{pmatrix} \mathbb{E}[\dot{\varepsilon}_{i1}^2|\mathbf{X}_i] & \cdots & \mathbb{E}[\dot{\varepsilon}_{i1} \dot{\varepsilon}_{iT}|\mathbf{X}_i] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[\dot{\varepsilon}_{iT} \dot{\varepsilon}_{i1}|\mathbf{X}_i] & \cdots & \mathbb{E}[\dot{\varepsilon}_{iT}^2|\mathbf{X}_i] \end{pmatrix} \begin{pmatrix} (\mathbf{x}_{i1} - \bar{\mathbf{x}}_i)' \\ \vdots \\ (\mathbf{x}_{iT} - \bar{\mathbf{x}}_i)' \end{pmatrix} \\
&= \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) \mathbb{E}[\dot{\varepsilon}_{it} \dot{\varepsilon}_{is}|\mathbf{X}_i] (\mathbf{x}_{is} - \bar{\mathbf{x}}_i)'
\end{aligned}$$

Finite sample adjustment can also be added. In Stata, $\frac{N}{N-1} \frac{T-1}{T-(K-1)}$ is multiplied.

1.3.8 GLS estimator of the demean transformed model if $Var(\varepsilon_i|\mathbf{X}_i) = \sigma_\varepsilon^2 \mathbf{I}_T$

$\varepsilon_i|\mathbf{X}_i \sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T]$ implies $\mathbf{Q}\varepsilon_i|\mathbf{X}_i \sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{Q}]$, we want to find a GLS transformer \mathbf{T}_{GLS} such that

$$\begin{aligned}
Var(\mathbf{T}_{GLS} \mathbf{Q}\varepsilon_i|\mathbf{X}_i) &= \sigma_\varepsilon^2 \mathbf{I}_T \\
\mathbf{T}_{GLS} Var(\mathbf{Q}\varepsilon_i|\mathbf{X}_i) \mathbf{T}_{GLS}' &= \sigma_\varepsilon^2 \mathbf{I}_T \\
\mathbf{T}_{GLS} \sigma_\varepsilon^2 \mathbf{Q} \mathbf{T}_{GLS}' &= \sigma_\varepsilon^2 \mathbf{I}_T \\
\mathbf{T}_{GLS} \mathbf{Q}^{1/2} \mathbf{Q}^{1/2} \mathbf{T}_{GLS}' &= \mathbf{I}_T \\
\mathbf{T}_{GLS} \mathbf{Q}^{1/2} \mathbf{Q}'^{1/2} \mathbf{T}_{GLS}' &= \mathbf{I}_T \\
\mathbf{T}_{GLS} \mathbf{Q}^{1/2} (\mathbf{T}_{GLS} \mathbf{Q}^{1/2})' &= \mathbf{I}_T
\end{aligned}$$

So, $\mathbf{T}_{GLS} = \mathbf{Q}^{-1/2}$

$$\mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{y}_i = \mathbf{Q}^{-1/2} (\mathbf{Q} \mathbf{X}_i \beta + \mathbf{Q} \varepsilon_i) = \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{X}_i \beta + \mathbf{Q}^{-1/2} \mathbf{Q} \varepsilon_i$$

Thus, we have $Var(\mathbf{Q}^{-1/2} \mathbf{Q} \varepsilon_i|\mathbf{X}_i) = \mathbf{Q}^{-1/2} Var(\mathbf{Q} \varepsilon_i|\mathbf{X}_i) \mathbf{Q}'^{-1/2} = \mathbf{Q}^{-1/2} \sigma_\varepsilon^2 \mathbf{Q} \mathbf{Q}^{-1/2} = \sigma_\varepsilon^2 \mathbf{Q}^{-1/2} \mathbf{Q}^{1/2} \mathbf{Q}^{1/2} \mathbf{Q}^{-1/2} = \sigma_\varepsilon^2 \mathbf{I}_T$.

By Gauss-Markov Theorem, GLS estimator is efficient.

$$\begin{aligned}
\hat{\beta}_{within}^{gl} &= \left[\sum_{i=1}^N (\mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{X}_i)' \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N (\mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{X}_i)' \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{y}_i \\
&= \left[\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q}'^{-1/2} \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q}'^{-1/2} \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{y}_i \\
&= \left[\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q}^{-1/2} \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q}^{-1/2} \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{y}_i \\
&= \left[\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q}^- \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q}^- \mathbf{Q} \mathbf{y}_i \\
&= \left[\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{y}_i = \hat{\beta}_{within}^{ols}
\end{aligned}$$

So, FE estimator is also efficient

For generalized inverse, $\mathbf{Q}' \mathbf{Q}^- \mathbf{Q} = \mathbf{Q}$. As \mathbf{Q} is idempotent and symmetry, $\mathbf{Q} = \mathbf{Q} \mathbf{Q}' = \mathbf{Q}' \mathbf{Q}$. Therefore, $\mathbf{Q}' \mathbf{Q}^- \mathbf{Q} = \mathbf{Q}' \mathbf{Q}$.

1.4 First-Difference Estimator

1.4.1 First-difference operator

$$\begin{aligned}
\Delta &= \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \\
\Delta \Delta' &= \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}' \\
&= \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}
\end{aligned}$$

1.4.2 First difference transformed model

$$\begin{aligned}
\Delta \mathbf{y}_i &= \Delta (\mathbf{X}_i \boldsymbol{\beta} + \mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i) \\
&= \Delta \mathbf{X}_i \boldsymbol{\beta} + \Delta \mathbf{e} \alpha_i + \Delta \boldsymbol{\varepsilon}_i \\
&= \Delta \mathbf{X}_i \boldsymbol{\beta} + \mathbf{0} \alpha_i + \Delta \boldsymbol{\varepsilon}_i \\
&= \Delta \mathbf{X}_i \boldsymbol{\beta} + \Delta \boldsymbol{\varepsilon}_i
\end{aligned}$$

Level 2

It is because

$$\Delta \mathbf{e} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}$$

It can be written as

$$\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \\ \vdots \\ \vdots \\ y_{iT} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x}'_{i1} \\ \mathbf{x}'_{i2} \\ \mathbf{x}'_{i3} \\ \vdots \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} \beta + \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \varepsilon_{i3} \\ \vdots \\ \vdots \\ \varepsilon_{iT} \end{pmatrix}$$

$$\begin{pmatrix} y_{i2} - y_{i1} \\ y_{i3} - y_{i2} \\ y_{i4} - y_{i3} \\ \vdots \\ \vdots \\ y_{iT} - y_{i,T-1} \end{pmatrix} = \begin{pmatrix} \mathbf{x}'_{i2} - \mathbf{x}'_{i1} \\ \mathbf{x}'_{i3} - \mathbf{x}'_{i2} \\ \mathbf{x}'_{i4} - \mathbf{x}'_{i3} \\ \vdots \\ \vdots \\ \mathbf{x}'_{iT} - \mathbf{x}'_{i,T-1} \end{pmatrix} \beta + \begin{pmatrix} \varepsilon_{i2} - \varepsilon_{i1} \\ \varepsilon_{i3} - \varepsilon_{i2} \\ \varepsilon_{i4} - \varepsilon_{i3} \\ \vdots \\ \vdots \\ \varepsilon_{iT} - \varepsilon_{i,T-1} \end{pmatrix}$$

$$\begin{pmatrix} y_{i2} - y_{i1} \\ y_{i3} - y_{i2} \\ y_{i4} - y_{i3} \\ \vdots \\ \vdots \\ y_{iT} - y_{i,T-1} \end{pmatrix} = \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1})' \\ (\mathbf{x}_{i3} - \mathbf{x}_{i2})' \\ (\mathbf{x}_{i4} - \mathbf{x}_{i3})' \\ \vdots \\ \vdots \\ (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1})' \end{pmatrix} \beta + \begin{pmatrix} \varepsilon_{i2} - \varepsilon_{i1} \\ \varepsilon_{i3} - \varepsilon_{i2} \\ \varepsilon_{i4} - \varepsilon_{i3} \\ \vdots \\ \vdots \\ \varepsilon_{iT} - \varepsilon_{i,T-1} \end{pmatrix}$$

$$y_{it} - y_{i,t-1} = (\mathbf{x}_{it} - \mathbf{x}_{i,t-1})' \beta + (\varepsilon_{it} - \varepsilon_{i,t-1})$$

Level 1

1.4.3 OLS estimator of the first difference transformed model

$$\hat{\beta}_{fd}^{ols} = \left[\sum_{i=1}^N (\Delta \mathbf{X}_i)' \Delta \mathbf{X}_i \right]^{-1} \sum_{i=1}^N (\Delta \mathbf{X}_i)' \Delta \mathbf{y}_i \quad \text{Level 2}$$

$$= \left[\sum_{i=1}^N \sum_{t=2}^T (\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(\mathbf{x}_{it} - \mathbf{x}_{i,t-1})' \right]^{-1} \sum_{i=1}^N \sum_{t=2}^T (\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(y_{it} - y_{i,t-1}) \quad \text{Level 1}$$

It is because

$$\begin{aligned}
(\Delta \mathbf{X}_i)' \Delta \mathbf{X}_i &= \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1})' \\ (\mathbf{x}_{i3} - \mathbf{x}_{i2})' \\ (\mathbf{x}_{i4} - \mathbf{x}_{i3})' \\ \vdots \\ \vdots \\ (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1})' \end{pmatrix}' \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1})' \\ (\mathbf{x}_{i3} - \mathbf{x}_{i2})' \\ (\mathbf{x}_{i4} - \mathbf{x}_{i3})' \\ \vdots \\ \vdots \\ (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1})' \end{pmatrix} \\
&= \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1}) & (\mathbf{x}_{i3} - \mathbf{x}_{i2}) & (\mathbf{x}_{i4} - \mathbf{x}_{i3}) & \cdots & \cdots & (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1}) \end{pmatrix} \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1})' \\ (\mathbf{x}_{i3} - \mathbf{x}_{i2})' \\ (\mathbf{x}_{i4} - \mathbf{x}_{i3})' \\ \vdots \\ \vdots \\ (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1})' \end{pmatrix} \\
&= \sum_{t=2}^T (\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(\mathbf{x}_{it} - \mathbf{x}_{i,t-1})' \\
\\
(\Delta \mathbf{X}_i)' \Delta \mathbf{y}_i &= \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1})' \\ (\mathbf{x}_{i3} - \mathbf{x}_{i2})' \\ (\mathbf{x}_{i4} - \mathbf{x}_{i3})' \\ \vdots \\ \vdots \\ (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1})' \end{pmatrix}' \begin{pmatrix} y_{i2} - y_{i1} \\ y_{i3} - y_{i2} \\ y_{i4} - y_{i3} \\ \vdots \\ \vdots \\ y_{iT} - y_{i,T-1} \end{pmatrix} \\
&= \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1}) & (\mathbf{x}_{i3} - \mathbf{x}_{i2}) & (\mathbf{x}_{i4} - \mathbf{x}_{i3}) & \cdots & \cdots & (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1}) \end{pmatrix} \begin{pmatrix} y_{i2} - y_{i1} \\ y_{i3} - y_{i2} \\ y_{i4} - y_{i3} \\ \vdots \\ \vdots \\ y_{iT} - y_{i,T-1} \end{pmatrix} \\
&= \sum_{t=2}^T (\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(y_{it} - y_{i,t-1})
\end{aligned}$$

1.4.4 The necessary condition for consistency and unbiasedness

The necessary condition for FD estimator (OLS estimator of the FD transformed model) to be consistent is $\mathbb{E}(\Delta \mathbf{X}_i)' \Delta \boldsymbol{\varepsilon}_i = \mathbf{0}$

$$\begin{aligned}
\mathbb{E}((\Delta \mathbf{X}_i)' \Delta \boldsymbol{\varepsilon}_i) &= \mathbb{E}(\mathbf{X}_i' \Delta \boldsymbol{\Delta}' \Delta \boldsymbol{\varepsilon}_i) \\
&= \mathbb{E}(\mathbb{E}(\mathbf{X}_i' \Delta \boldsymbol{\Delta}' \boldsymbol{\varepsilon}_i | \mathbf{X}_i)) \\
&= \mathbb{E}(\mathbf{X}_i' \Delta \boldsymbol{\Delta}' \underbrace{\mathbb{E}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i)}_{\mathbf{0}}) && \text{because of strict exogeneity} \\
&= \mathbf{0}
\end{aligned}$$

Thus, FD estimator satisfies the necessary condition for consistency given strict exogeneity assumption. Indeed, strict exogeneity is stronger than what is required. To see this, first note that for any t

$$\mathbb{E}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \mathbb{E}(\boldsymbol{\varepsilon}_{it} | \mathbf{x}_{i1}, \dots, \mathbf{x}_{iT}) = \mathbf{0} \implies \mathbb{E}(\mathbf{x}_{is} \boldsymbol{\varepsilon}_{it}) = \mathbf{0} \quad \text{for all } s$$

The necessary condition for FD estimator to be consistent can also be written as $\mathbb{E}((\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(\boldsymbol{\varepsilon}_{it} - \boldsymbol{\varepsilon}_{i,t-1})) = \mathbf{0}$

$$\mathbb{E}((\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(\boldsymbol{\varepsilon}_{it} - \boldsymbol{\varepsilon}_{i,t-1})) = \mathbb{E}(\mathbf{x}_{it} \boldsymbol{\varepsilon}_{it}) - \mathbb{E}(\mathbf{x}_{it} \boldsymbol{\varepsilon}_{i,t-1}) - \mathbb{E}(\mathbf{x}_{i,t-1} \boldsymbol{\varepsilon}_{it}) + \mathbb{E}(\mathbf{x}_{i,t-1} \boldsymbol{\varepsilon}_{i,t-1}) = \mathbf{0}$$

It is because $\mathbb{E}(\mathbf{x}_{is} \boldsymbol{\varepsilon}_{it}) = \mathbf{0}$ for any t and s implies

$$\mathbb{E}(\mathbf{x}_{it} \boldsymbol{\varepsilon}_{it}) = \mathbb{E}(\mathbf{x}_{it} \boldsymbol{\varepsilon}_{i,t-1}) = \mathbb{E}(\mathbf{x}_{i,t-1} \boldsymbol{\varepsilon}_{it}) = \mathbb{E}(\mathbf{x}_{i,t-1} \boldsymbol{\varepsilon}_{i,t-1}) = \mathbf{0}$$

Thus, the weaker assumption $\mathbb{E}(\mathbf{x}_{it}\varepsilon_{it}) = \mathbf{0}$ for any t and s is sufficient for $\mathbb{E}((\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(\varepsilon_{it} - \varepsilon_{i,t-1})) = \mathbf{0}$

The necessary condition for FD estimator to be unbiased is $\mathbb{E}(\Delta\boldsymbol{\varepsilon}_i|\Delta\mathbf{X}_i) = \mathbf{0}$

$$\begin{aligned}\mathbb{E}(\Delta\boldsymbol{\varepsilon}_i|\Delta\mathbf{X}_i) &= \Delta \underbrace{\mathbb{E}(\boldsymbol{\varepsilon}_i|\mathbf{X}_i)}_{\mathbf{0}} \\ &= \mathbf{0}\end{aligned}\quad \text{as } \Delta \text{ is constant and strict exogeneity}$$

1.4.5 Conditional Variance of $\hat{\beta}_{fd}^{ols}$

$$\begin{aligned}Var(\hat{\beta}_{fd}^{ols}|\mathbf{X}_i) &= Var([\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} \sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{y}_i | \mathbf{X}_i) \\ &= [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} Var(\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{y}_i | \mathbf{X}_i) [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1'} \\ &= [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} \sum_{i=1}^N (\Delta\mathbf{X}_i)' Var(\Delta\boldsymbol{\varepsilon}_i | \mathbf{X}_i) \Delta\mathbf{X}_i [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1}\end{aligned}$$

1.4.6 $Var(\boldsymbol{\varepsilon}_i|\mathbf{X}_i) = \sigma_\varepsilon^2 \mathbf{I}_T$

If ε_{it} is homoskedasticity and serially uncorrelated across t i.e., $Var(\boldsymbol{\varepsilon}_i|\mathbf{X}_i) = \sigma_\varepsilon^2 \mathbf{I}_T$ (further assume independence of i and strict exogeneity), we have $\boldsymbol{\varepsilon}_i|\mathbf{X}_i \sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T]$.

$$Var(\Delta\boldsymbol{\varepsilon}_i|\mathbf{X}_i) = \Delta Var(\boldsymbol{\varepsilon}_i|\mathbf{X}_i) \Delta' = \Delta \sigma_\varepsilon^2 \mathbf{I}_T \Delta' = \sigma_\varepsilon^2 \Delta \Delta' = \sigma_\varepsilon^2 \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} \quad \text{Thus, } \Delta\boldsymbol{\varepsilon}_i \text{ is homoskedastic-}$$

ity but not serially uncorrelated e.g. $Cov(\varepsilon_{it} - \varepsilon_{i,t-1}, \varepsilon_{i,t-1} - \varepsilon_{i,t-2} | \mathbf{X}_i) = -\sigma_\varepsilon^2 < 0$. Therefore, we cannot apply Gauss-Markov Theorem.

$$\begin{aligned}Var(\hat{\beta}_{fd}^{ols}|\mathbf{X}_i) &= [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} \sum_{i=1}^N (\Delta\mathbf{X}_i)' \sigma_\varepsilon^2 \Delta \Delta' \Delta\mathbf{X}_i [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} \\ &= \sigma_\varepsilon^2 [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} \sum_{i=1}^N \mathbf{X}_i' \Delta' \Delta \Delta' \Delta\mathbf{X}_i [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1}\end{aligned}$$

1.4.7 $Var(\boldsymbol{\varepsilon}_i|\mathbf{X}_i) = \boldsymbol{\Omega}_i$

We have $\boldsymbol{\varepsilon}_i|\mathbf{X}_i \sim inid [\mathbf{0}, \boldsymbol{\Omega}_i]$,

$$\begin{aligned}Var(\Delta\boldsymbol{\varepsilon}_i|\mathbf{X}_i) &= \Delta Var(\boldsymbol{\varepsilon}_i|\mathbf{X}_i) \Delta' = \Delta \mathbb{E}[(\boldsymbol{\varepsilon}_i - \mathbb{E}[\boldsymbol{\varepsilon}_i|\mathbf{X}_i])(\boldsymbol{\varepsilon}_i - \mathbb{E}[\boldsymbol{\varepsilon}_i|\mathbf{X}_i])' | \mathbf{X}_i] \Delta' = \Delta \mathbb{E}[(\boldsymbol{\varepsilon}_i - \mathbf{0})(\boldsymbol{\varepsilon}_i - \mathbf{0})' | \mathbf{X}_i] \Delta' = \Delta \mathbb{E}[\boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}_i' | \mathbf{X}_i] \Delta' = \\ &= \mathbb{E}[\Delta \boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}_i' \Delta' | \mathbf{X}_i] = \mathbb{E}[\Delta \boldsymbol{\varepsilon}_i (\Delta \boldsymbol{\varepsilon}_i)' | \mathbf{X}_i]\end{aligned}$$

$$Var(\hat{\beta}_{fd}^{ols}|\mathbf{X}_i) = [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} \sum_{i=1}^N (\Delta\mathbf{X}_i)' E[\Delta \boldsymbol{\varepsilon}_i (\Delta \boldsymbol{\varepsilon}_i)' | \mathbf{X}_i] \Delta\mathbf{X}_i [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1}$$

If ε_{it} follows random walk process i.e., $\varepsilon_{it} = \varepsilon_{i,t-1} + v_{it}$ where v_{it} follows white noise process, $\varepsilon_{it} - \varepsilon_{i,t-1} = v_{it}$ follows white noise process. Thus, $\varepsilon_{it} - \varepsilon_{i,t-1}$ is homoskedasticity and serially uncorrelated as they are the properties of white noise process. As a result, FD estimator is efficient in this case by applying Gauss-Markov Theorem.

1.5 Least-Squares Dummy Variable Estimator

Level 3

$$\mathbf{y} = (\mathbf{I}_N \otimes \mathbf{e})\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} = \begin{pmatrix} (\mathbf{I}_N \otimes \mathbf{e}) & \mathbf{X} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} + \boldsymbol{\varepsilon}$$

$$\begin{aligned} \begin{pmatrix} \hat{\boldsymbol{\alpha}}_{dv}^{ols} \\ \hat{\boldsymbol{\beta}}_{dv}^{ols} \end{pmatrix} &= [((\mathbf{I}_N \otimes \mathbf{e}) \quad \mathbf{X})' ((\mathbf{I}_N \otimes \mathbf{e}) \quad \mathbf{X})]^{-1} ((\mathbf{I}_N \otimes \mathbf{e}) \quad \mathbf{X})' \mathbf{y} \\ &= \begin{pmatrix} (\mathbf{I}_N \otimes \mathbf{e})'(\mathbf{I}_N \otimes \mathbf{e}) & (\mathbf{I}_N \otimes \mathbf{e})'\mathbf{X} \\ \mathbf{X}'(\mathbf{I}_N \otimes \mathbf{e}) & \mathbf{X}'\mathbf{X} \end{pmatrix}^{-1} \begin{pmatrix} (\mathbf{I}_N \otimes \mathbf{e})'\mathbf{y} \\ \mathbf{X}'\mathbf{y} \end{pmatrix} \\ &= \begin{pmatrix} T\mathbf{I}_N & T\bar{\mathbf{X}} \\ T\bar{\mathbf{X}}' & \mathbf{X}'\mathbf{X} \end{pmatrix}^{-1} \begin{pmatrix} T\bar{\mathbf{y}} \\ \mathbf{X}'\mathbf{y} \end{pmatrix} \\ \hat{\boldsymbol{\beta}}_{dv}^{ols} &= [\mathbf{X}'\mathbf{X} - T\bar{\mathbf{X}}'\bar{\mathbf{X}}]^{-1}(\mathbf{X}'\mathbf{y} - T\bar{\mathbf{X}}'\bar{\mathbf{y}}) = \hat{\boldsymbol{\beta}}_{within}^{ols} \end{aligned}$$

It is because

$$\begin{aligned} ((\mathbf{I}_N \otimes \mathbf{e}) \quad \mathbf{X})' ((\mathbf{I}_N \otimes \mathbf{e}) \quad \mathbf{X}) &= \begin{pmatrix} (\mathbf{I}_N \otimes \mathbf{e})' \\ \mathbf{X}' \end{pmatrix} \begin{pmatrix} (\mathbf{I}_N \otimes \mathbf{e}) & \mathbf{X} \end{pmatrix} \\ &= \begin{pmatrix} (\mathbf{I}_N \otimes \mathbf{e})'(\mathbf{I}_N \otimes \mathbf{e}) & (\mathbf{I}_N \otimes \mathbf{e})'\mathbf{X} \\ \mathbf{X}'(\mathbf{I}_N \otimes \mathbf{e}) & \mathbf{X}'\mathbf{X} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} (\mathbf{I}_N \otimes \mathbf{e})'(\mathbf{I}_N \otimes \mathbf{e}) &= \begin{pmatrix} \mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e} \end{pmatrix}' \begin{pmatrix} \mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{e}' & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e}' \end{pmatrix} \begin{pmatrix} \mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{e}'\mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e}'\mathbf{e} \end{pmatrix} \\ &= \begin{pmatrix} T & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & T \end{pmatrix} \\ &= T\mathbf{I}_N \end{aligned}$$

$$\begin{aligned} (\mathbf{I}_N \otimes \mathbf{e})'\mathbf{X} &= \begin{pmatrix} \mathbf{e}' & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e}' \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_N \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{e}'\mathbf{X}_1 \\ \vdots \\ \mathbf{e}'\mathbf{X}_N \end{pmatrix} \\ &= \begin{pmatrix} \sum_{t=1}^T \mathbf{x}'_{1t} \\ \vdots \\ \sum_{t=1}^T \mathbf{x}'_{Nt} \end{pmatrix} \\ &= \begin{pmatrix} T \sum_{t=1}^T \mathbf{x}'_{1t}/T \\ \vdots \\ T \sum_{t=1}^T \mathbf{x}'_{Nt}/T \end{pmatrix} \\ &= \begin{pmatrix} T\bar{\mathbf{x}}'_1 \\ \vdots \\ T\bar{\mathbf{x}}'_N \end{pmatrix} \\ &= T\bar{\mathbf{X}} \end{aligned}$$

$$\begin{aligned}
((I_N \otimes e) \quad X)' y &= \begin{pmatrix} (I_N \otimes e)' \\ X' \end{pmatrix} y \\
&= \begin{pmatrix} (I_N \otimes e)' y \\ X' y \end{pmatrix}
\end{aligned}$$

Another way to show the equivalence of within estimator and dummy variable estimator by using Frisch-Waugh-Lovell Theorem,

$$y = X\beta + (I_N \otimes e)\alpha + \varepsilon$$

$$y = X\beta + E\alpha + \varepsilon$$

$$X = E\alpha_{XE} + \varepsilon_{XE}$$

$$y = E\alpha_{yE} + \varepsilon_{yE}$$

$$\begin{aligned}
\hat{\alpha}_{yE} &= (E'E)^{-1} E'y \\
&= \begin{pmatrix} e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e \end{pmatrix}' \begin{pmatrix} e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e \end{pmatrix}^{-1} \begin{pmatrix} e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e \end{pmatrix}' \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \\
&= \begin{pmatrix} e' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e' \end{pmatrix} \begin{pmatrix} e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e \end{pmatrix}^{-1} \begin{pmatrix} e' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e' \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \\
&= \begin{pmatrix} e'e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e'e \end{pmatrix}^{-1} \begin{pmatrix} e'y_1 \\ \vdots \\ e'y_N \end{pmatrix} \\
&= \begin{pmatrix} (e'e)^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (e'e)^{-1} \end{pmatrix} \begin{pmatrix} e'y_1 \\ \vdots \\ e'y_N \end{pmatrix} \\
&= \begin{pmatrix} (e'e)^{-1} e'y_1 \\ \vdots \\ (e'e)^{-1} e'y_N \end{pmatrix} = \begin{pmatrix} T^{-1} \sum_{t=1}^T y_{1t} \\ \vdots \\ T^{-1} \sum_{t=1}^T y_{Nt} \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_N \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\hat{\varepsilon}_{yE} &= y - E\hat{\alpha}_{yE} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} - \begin{pmatrix} e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_N \end{pmatrix} \\
&= \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} - \begin{pmatrix} e\bar{y}_1 \\ \vdots \\ e\bar{y}_N \end{pmatrix} \\
&= \begin{pmatrix} y_1 - e\bar{y}_1 \\ \vdots \\ y_N - e\bar{y}_N \end{pmatrix} = \begin{pmatrix} Qy_1 \\ \vdots \\ Qy_N \end{pmatrix} = Qy
\end{aligned}$$

Similarly,

$$\hat{\varepsilon}_{XE} = QX$$

By Frisch-Waugh-Lovell Theorem,

$$\begin{aligned}
\hat{\beta}_{dv}^{ols} &= (\hat{\varepsilon}_{XE}' \hat{\varepsilon}_{XE})^{-1} \hat{\varepsilon}_{XE}' \hat{\varepsilon}_{yE} \\
&= [(QX)' QX]^{-1} (QX)' Qy = \hat{\beta}_{within}^{ols}
\end{aligned}$$

If $N \rightarrow \infty$, the number of α_i estimated goes to infinity. If T is fixed, the LSDV estimates are consistent for β (as FE estimates for β is consistent for fixed T and $N \rightarrow \infty$) but inconsistent for α . (There is no incidental parameters problem as the estimates for β are not contaminated). If T also $\rightarrow \infty$, then the LSDV estimates of α are also consistent.

2 Random Effect Model

$$y_{it} = \mathbf{x}'_{it}\boldsymbol{\beta} + \alpha_i + \varepsilon_{it}$$

$$\begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} = \begin{pmatrix} \mathbf{x}'_{i1} \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \alpha_i + \begin{pmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{pmatrix}$$

$$\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta} + \underbrace{(\mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i)}_{\mathbf{u}_i}$$

$$\begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_N \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_1 \\ \vdots \\ \boldsymbol{\varepsilon}_N \end{pmatrix}$$

$$\mathbf{y} = \mathbf{X} \boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{e}) \boldsymbol{\alpha} + \boldsymbol{\varepsilon}$$

2.1 Assumptions

2.1.1 Strong/Strict Exogeneity of Regressors

For all t ,

$$E(\varepsilon_{it} | \mathbf{x}_{i1}, \dots, \mathbf{x}_{iT}) = 0$$

Equivalently,

$$E(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \mathbf{0}$$

2.1.2 Covariance Structure

$$\begin{aligned} \boldsymbol{\varepsilon}_i | \mathbf{X}_i &\sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T] \\ \alpha_i | \mathbf{X}_i &\sim iid [0, \sigma_\alpha^2] \\ \boldsymbol{\varepsilon}_i &\perp \alpha_i | \mathbf{X}_i \end{aligned}$$

2.2 Moments of $\mathbf{u}_i | \mathbf{X}_i$

$$\begin{aligned} \boldsymbol{\Omega} &:= Var(\mathbf{u}_i | \mathbf{X}_i) = Var(\mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i | \mathbf{X}_i) && \text{because of } \boldsymbol{\varepsilon}_i \perp \alpha_i | \mathbf{X}_i \\ &= Var(\mathbf{e} \alpha_i | \mathbf{X}_i) + Var(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) \\ &= \mathbf{e} Var(\alpha_i | \mathbf{X}_i) \mathbf{e}' + Var(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) \\ &= \sigma_\alpha^2 \mathbf{e} \mathbf{e}' + \sigma_\varepsilon^2 \mathbf{I}_T \\ &= \begin{pmatrix} \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 \\ \vdots & \ddots & \vdots \\ \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 \end{pmatrix} + \begin{pmatrix} \sigma_\varepsilon^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_\varepsilon^2 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_\alpha^2 + \sigma_\varepsilon^2 & \cdots & \sigma_\alpha^2 \\ \vdots & \ddots & \vdots \\ \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 + \sigma_\varepsilon^2 \end{pmatrix} \\ \\ \mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) &= \mathbb{E}(\mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i | \mathbf{X}_i) \\ &= \mathbf{e} \underbrace{\mathbb{E}(\alpha_i | \mathbf{X}_i)}_0 + \underbrace{\mathbb{E}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i)}_0 \\ &= \mathbf{0} \end{aligned}$$

Note that $\mathbb{E}(\alpha_i | \mathbf{X}_i) = \mathbb{E}(\alpha_i | \mathbf{x}_{i1}, \dots, \mathbf{x}_{iT}) = 0$ called orthogonality assumption. $\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0 \implies Cov(\alpha_i, \mathbf{X}_i) = \mathbf{0}$. It is because $Cov(\alpha_i, \mathbf{X}_i) = \mathbb{E}(\alpha_i \mathbf{X}_i) - \mathbb{E}(\alpha_i) \mathbb{E}(\mathbf{X}_i) = \mathbb{E}(\mathbb{E}(\alpha_i \mathbf{X}_i | \mathbf{X}_i)) - \mathbb{E}(\mathbb{E}(\alpha_i | \mathbf{X}_i)) \mathbb{E}(\mathbf{X}_i) = \mathbb{E}(\mathbb{E}(\alpha_i | \mathbf{X}_i) \mathbf{X}_i) = \mathbf{0}$. There is no OVB i.e., \mathbf{u}_i is not correlated with \mathbf{X}_i .

$\mathbb{E}(\mathbf{u}_i|\mathbf{X}_i) = \mathbf{0}$ means that the necessary condition for OLS estimator to be unbiased is satisfied. Moreover, $\mathbb{E}(\mathbf{u}_i|\mathbf{X}_i) = \mathbf{0} \implies \mathbb{E}(\mathbf{X}_i'\mathbf{u}_i) = \mathbf{0}$ which means that the necessary condition for OLS estimator to be consistent is also satisfied. However, $\mathbf{u}_i|\mathbf{X}_i$ is homoskedasticity but serially correlated. Thus, the necessary condition for OLS estimator to be efficient is not satisfied. As a result, it is not efficient.

As we know the covariance structure of $\mathbf{u}_i|\mathbf{X}_i$ due to the strong assumptions in random effect model, we can use GLS estimation, which yields efficient estimates.

2.3 Random Effect Estimator (GLS Estimator)

2.3.1 GLS transformed model

We want to find a \mathbf{T}_{GLS} such that

$$\begin{aligned} Var(\mathbf{T}_{GLS}\mathbf{u}_i|\mathbf{X}_i) &= \sigma_\varepsilon^2 \mathbf{I}_T \\ \mathbf{T}_{GLS} Var(\mathbf{u}_i|\mathbf{X}_i) \mathbf{T}_{GLS}' &= \sigma_\varepsilon^2 \mathbf{I}_T \\ \mathbf{T}_{GLS} \boldsymbol{\Omega} \mathbf{T}_{GLS}' &= \sigma_\varepsilon^2 \mathbf{I}_T \\ \mathbf{T}_{GLS} \boldsymbol{\Omega}^{1/2} \boldsymbol{\Omega}^{1/2} \mathbf{T}_{GLS}' &= \sigma_\varepsilon^2 \mathbf{I}_T \\ \mathbf{T}_{GLS} \boldsymbol{\Omega}^{1/2} \boldsymbol{\Omega}^{1/2} \mathbf{T}_{GLS}' &= \sigma_\varepsilon^2 \mathbf{I}_T \\ \mathbf{T}_{GLS} \boldsymbol{\Omega}^{1/2} (\mathbf{T}_{GLS} \boldsymbol{\Omega}^{1/2})' &= \sigma_\varepsilon^2 \mathbf{I}_T \end{aligned}$$

So, $\mathbf{T}_{GLS} = \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2}$. Define $\psi^2 := \frac{\sigma_\varepsilon^2}{T\sigma_\alpha^2 + \sigma_\varepsilon^2}$.

$$\begin{aligned} \boldsymbol{\Omega} &= \sigma_\varepsilon^2 \mathbf{I}_T + \sigma_\alpha^2 \mathbf{e}\mathbf{e}' \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T + \frac{\sigma_\alpha^2}{\sigma_\varepsilon^2} \mathbf{e}\mathbf{e}') \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T + \frac{T\sigma_\alpha^2}{\sigma_\varepsilon^2} T^{-1} \mathbf{e}\mathbf{e}') \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T + \frac{T\sigma_\alpha^2 + \sigma_\varepsilon^2 - \sigma_\varepsilon^2}{\sigma_\varepsilon^2} T^{-1} \mathbf{e}\mathbf{e}') \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T + (\frac{T\sigma_\alpha^2 + \sigma_\varepsilon^2}{\sigma_\varepsilon^2} - 1) T^{-1} \mathbf{e}\mathbf{e}') \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T + (\frac{1}{\psi^2} - 1) T^{-1} \mathbf{e}\mathbf{e}') \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T + \frac{1}{\psi^2} T^{-1} \mathbf{e}\mathbf{e}' - T^{-1} \mathbf{e}\mathbf{e}') \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T - T^{-1} \mathbf{e}\mathbf{e}' + \frac{1}{\psi^2} (T^{-1} \mathbf{e}\mathbf{e}' - \mathbf{I}_T + \mathbf{I}_T)) \\ &= \sigma_\varepsilon^2 (\mathbf{Q} + \frac{1}{\psi^2} (\mathbf{I}_T - \mathbf{Q})) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Omega}^{-1} &= [\sigma_\varepsilon^2 (\mathbf{Q} + \frac{1}{\psi^2} (\mathbf{I}_T - \mathbf{Q}))]^{-1} \\ &= \sigma_\varepsilon^{-2} (\mathbf{Q}^{-1} + \psi^2 (\mathbf{I}_T^{-1} - \mathbf{Q}^{-1})) \\ &= \sigma_\varepsilon^{-2} (\mathbf{Q} + \psi^2 (\mathbf{I}_T - \mathbf{Q})) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Omega}^{-1/2} &= \sigma_\varepsilon^{-1} (\mathbf{Q} + \psi (\mathbf{I}_T - \mathbf{Q})) \\ \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} &= (\mathbf{Q} + \psi (\mathbf{I}_T - \mathbf{Q})) \end{aligned}$$

$$\sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{y}_i = \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} (\mathbf{X}_i \boldsymbol{\beta} + (\mathbf{e}\alpha_i + \varepsilon_i)) = \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} (\mathbf{X}_i \boldsymbol{\beta} + \mathbf{u}_i) = \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{X}_i \boldsymbol{\beta} + \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{u}_i$$

$$\text{So, } Var(\sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{u}_i | \mathbf{X}_i) = \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} Var(\mathbf{u}_i | \mathbf{X}_i) \sigma_\varepsilon \boldsymbol{\Omega}'^{-1/2} = \sigma_\varepsilon^2 \boldsymbol{\Omega}^{-1/2} \boldsymbol{\Omega} \boldsymbol{\Omega}^{-1/2} = \sigma_\varepsilon^2 \boldsymbol{\Omega}^{-1/2} \boldsymbol{\Omega}^{1/2} \boldsymbol{\Omega}^{1/2} \boldsymbol{\Omega}^{-1/2} = \sigma_\varepsilon^2 \mathbf{I}_T$$

$$(\mathbf{Q} + \psi (\mathbf{I}_T - \mathbf{Q})) \mathbf{y}_i = (\mathbf{Q} + \psi (\mathbf{I}_T - \mathbf{Q})) \mathbf{X}_i \boldsymbol{\beta} + (\mathbf{Q} + \psi (\mathbf{I}_T - \mathbf{Q})) \mathbf{e}\alpha_i + (\mathbf{Q} + \psi (\mathbf{I}_T - \mathbf{Q})) \varepsilon_i$$

Level 2

It can also be written as

$$\mathbf{y}_i - \lambda \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \lambda \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + (1 - \lambda) \mathbf{e} \alpha_i + (\boldsymbol{\varepsilon}_i - \lambda \mathbf{e} \bar{\varepsilon}_i) \quad \text{Level 2}$$

where $\lambda = 1 - \psi = 1 - \frac{\sigma_\varepsilon}{\sqrt{T\sigma_\alpha^2 + \sigma_\varepsilon^2}}$. It is because

$$\begin{aligned} \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{y}_i &= (\mathbf{Q} + \psi(\mathbf{I}_T - \mathbf{Q})) \mathbf{y}_i = \mathbf{Q} \mathbf{y}_i + \psi(\mathbf{I}_T \mathbf{y}_i - \mathbf{Q} \mathbf{y}_i) \\ &= \mathbf{y}_i - \mathbf{e} \bar{y}_i + \psi(\mathbf{y}_i - \mathbf{y}_i + \mathbf{e} \bar{y}_i) \\ &= \mathbf{y}_i - \mathbf{e} \bar{y}_i + \psi \mathbf{e} \bar{y}_i \\ &= \mathbf{y}_i - \mathbf{e} \bar{y}_i (1 - \psi) \\ &= \mathbf{y}_i - \lambda \mathbf{e} \bar{y}_i \end{aligned}$$

$$\begin{aligned} \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{X}_i \boldsymbol{\beta} &= (\mathbf{Q} + \psi(\mathbf{I}_T - \mathbf{Q})) \mathbf{X}_i \boldsymbol{\beta} = \mathbf{Q} \mathbf{X}_i \boldsymbol{\beta} + \psi(\mathbf{I}_T \mathbf{X}_i \boldsymbol{\beta} - \mathbf{Q} \mathbf{X}_i \boldsymbol{\beta}) \\ &= (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + \psi(\mathbf{X}_i \boldsymbol{\beta} - (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta}) \\ &= (\mathbf{X}_i \boldsymbol{\beta} - \mathbf{e} \bar{\mathbf{x}}'_i \boldsymbol{\beta}) + \psi(\mathbf{X}_i \boldsymbol{\beta} - \mathbf{X}_i \boldsymbol{\beta} + \mathbf{e} \bar{\mathbf{x}}'_i \boldsymbol{\beta}) \\ &= \mathbf{X}_i \boldsymbol{\beta} - \mathbf{e} \bar{\mathbf{x}}'_i \boldsymbol{\beta} + \psi \mathbf{e} \bar{\mathbf{x}}'_i \boldsymbol{\beta} \\ &= (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}'_i + \psi \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} \\ &= (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}'_i (1 - \psi)) \boldsymbol{\beta} \\ &= (\mathbf{X}_i - \lambda \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} \end{aligned}$$

$$\begin{aligned} \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{e} \alpha_i &= (\mathbf{Q} + \psi(\mathbf{I}_T - \mathbf{Q})) \mathbf{e} \alpha_i = \mathbf{Q} \mathbf{e} \alpha_i + \psi(\mathbf{I}_T \mathbf{e} \alpha_i - \mathbf{Q} \mathbf{e} \alpha_i) \\ &= \mathbf{0} \alpha_i + \psi(\mathbf{e} \alpha_i - \mathbf{0} \alpha_i) \\ &= \psi \mathbf{e} \alpha_i \\ &= (1 - \lambda) \mathbf{e} \alpha_i \end{aligned}$$

Random effect estimator is the OLS estimator of the beta in the transformed model $\mathbf{y}_i - \lambda \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \lambda \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + (1 - \lambda) \mathbf{e} \alpha_i + (\boldsymbol{\varepsilon}_i - \lambda \mathbf{e} \bar{\varepsilon}_i)$.

Fixed effect / within estimator is the OLS estimator of the beta in the transformed model $\mathbf{y}_i - \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + (\boldsymbol{\varepsilon}_i - \mathbf{e} \bar{\varepsilon}_i)$.

Pooled OLS estimator is the OLS estimator of the beta in the original model $\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta} + \mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i$.

As $T \rightarrow \infty$, $\lambda \rightarrow 1$, $\mathbf{y}_i - \lambda \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \lambda \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + (1 - \lambda) \mathbf{e} \alpha_i + (\boldsymbol{\varepsilon}_i - \lambda \mathbf{e} \bar{\varepsilon}_i)$ converges to $\mathbf{y}_i - \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + (\boldsymbol{\varepsilon}_i - \mathbf{e} \bar{\varepsilon}_i)$. Thus, random effect estimator converges to fixed effect / within estimator as $T \rightarrow \infty$.

As $\sigma_\alpha^2 \rightarrow 0$, $\lambda \rightarrow 0$, $\mathbf{y}_i - \lambda \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \lambda \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + (1 - \lambda) \mathbf{e} \alpha_i + (\boldsymbol{\varepsilon}_i - \lambda \mathbf{e} \bar{\varepsilon}_i)$ converges to $\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta} + \mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i$. Thus, random effect estimator converges to pooled OLS estimator as $\sigma_\alpha^2 \rightarrow 0$.

$$\begin{aligned} \begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{y}_i &= \left(\begin{pmatrix} \mathbf{x}'_{i1} \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{\mathbf{x}}'_i \right) \boldsymbol{\beta} + (1 - \lambda) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \alpha_i + \left(\begin{pmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{\varepsilon}_i \right) \\ \begin{pmatrix} y_{i1} - \lambda \bar{y}_i \\ \vdots \\ y_{iT} - \lambda \bar{y}_i \end{pmatrix} &= \begin{pmatrix} \mathbf{x}'_{i1} - \lambda \bar{\mathbf{x}}'_i \\ \vdots \\ \mathbf{x}'_{iT} - \lambda \bar{\mathbf{x}}'_i \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} (1 - \lambda) \alpha_i \\ \vdots \\ (1 - \lambda) \alpha_i \end{pmatrix} + \begin{pmatrix} \varepsilon_{i1} - \lambda \bar{\varepsilon}_i \\ \vdots \\ \varepsilon_{iT} - \lambda \bar{\varepsilon}_i \end{pmatrix} \\ \begin{pmatrix} y_{i1} - \lambda \bar{y}_i \\ \vdots \\ y_{iT} - \lambda \bar{y}_i \end{pmatrix} &= \begin{pmatrix} (\mathbf{x}_{i1} - \lambda \bar{\mathbf{x}}_i)' \\ \vdots \\ (\mathbf{x}_{iT} - \lambda \bar{\mathbf{x}}_i)' \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} (1 - \lambda) \alpha_i \\ \vdots \\ (1 - \lambda) \alpha_i \end{pmatrix} + \begin{pmatrix} \varepsilon_{i1} - \lambda \bar{\varepsilon}_i \\ \vdots \\ \varepsilon_{iT} - \lambda \bar{\varepsilon}_i \end{pmatrix} \\ y_{it} - \lambda \bar{y}_i &= (\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i)' \boldsymbol{\beta} + \underbrace{(1 - \lambda) \alpha_i + (\varepsilon_{it} - \lambda \bar{\varepsilon}_i)}_{v_{it}} \end{aligned}$$

Level 1

2.3.2 OLS estimator of the GLS transformed model i.e., Random Effect / GLS estimator

$$\begin{aligned}\hat{\beta}_{re}^{ols} &= \left[\sum_{i=1}^N (\mathbf{X}_i - \lambda e \bar{\mathbf{x}}_i)' (\mathbf{X}_i - \lambda e \bar{\mathbf{x}}_i) \right]^{-1} \sum_{i=1}^N (\mathbf{X}_i - \lambda e \bar{\mathbf{x}}_i)' (\mathbf{y}_i - \lambda e \bar{y}_i) && \text{Level 2} \\ &= \left[\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i) (\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i)' \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i) (y_{it} - \lambda \bar{y}_i) && \text{Level 1}\end{aligned}$$

If \mathbf{x}_{it} is replaced by $\mathbf{x}_{it} - \bar{\mathbf{x}}$ and $\bar{\mathbf{x}}_i$ is replaced by $\bar{\mathbf{x}}_i - \bar{\mathbf{x}}$,

$$\begin{aligned}(\mathbf{x}_{it} - \bar{\mathbf{x}}) - \lambda(\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) &= \mathbf{x}_{it} - \bar{\mathbf{x}} - \lambda \bar{\mathbf{x}}_i + \lambda \bar{\mathbf{x}} \\ &= \mathbf{x}_{it} - \bar{\mathbf{x}} - (1 - \psi) \bar{\mathbf{x}}_i + (1 - \psi) \bar{\mathbf{x}} \\ &= \mathbf{x}_{it} - \bar{\mathbf{x}} - \bar{\mathbf{x}}_i + \psi \bar{\mathbf{x}}_i + \bar{\mathbf{x}} - \psi \bar{\mathbf{x}} \\ &= (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) + \psi(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^N \sum_{t=1}^T ((\mathbf{x}_{it} - \bar{\mathbf{x}}) - \lambda(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})) ((\mathbf{x}_{it} - \bar{\mathbf{x}}) - \lambda(\bar{\mathbf{x}}_i - \bar{\mathbf{x}}))' &= \sum_{i=1}^N \sum_{t=1}^T ((\mathbf{x}_{it} - \bar{\mathbf{x}}_i) + \psi(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})) ((\mathbf{x}_{it} - \bar{\mathbf{x}}_i) + \psi(\bar{\mathbf{x}}_i - \bar{\mathbf{x}}))' \\ &= \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \psi \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \\ &\quad + \psi \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \psi^2 \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \\ &= \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \psi^2 \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})'\end{aligned}$$

It is because

$$\begin{aligned}\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' &= \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' - \sum_{i=1}^N \sum_{t=1}^T \bar{\mathbf{x}}_i (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \\ &= \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' - \sum_{i=1}^N T \bar{\mathbf{x}}_i (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \\ &= \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' - \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \\ &= \mathbf{0}\end{aligned}$$

Similarly, if y_{it} is replaced by $y_{it} - \bar{y}$ and \bar{y}_i is replaced by $\bar{y}_i - \bar{y}$

$$\sum_{i=1}^N \sum_{t=1}^T ((y_{it} - \bar{y}) - \lambda(\bar{y}_i - \bar{y})) ((y_{it} - \bar{y}) - \lambda(\bar{y}_i - \bar{y})) = \sum_{i=1}^N \sum_{t=1}^T (y_{it} - \bar{y}_i) (y_{it} - \bar{y}_i) + \psi^2 \sum_{i=1}^N \sum_{t=1}^T (\bar{y}_i - \bar{y}) (\bar{y}_i - \bar{y})$$

$$\begin{aligned}\hat{\beta}_{re}^{ols} &= \overbrace{\left(\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' \right)}^{\text{Within}} + \psi^2 \overbrace{\left(\sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \right)}^{\text{Between}}^{-1} \\ &\quad \left(\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (y_{it} - \bar{y}_i) + \psi^2 \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) (\bar{y}_i - \bar{y}) \right)\end{aligned}$$

If $T \rightarrow \infty$, $\psi^2 \rightarrow 0$, $\hat{\beta}_{re}^{ols} \rightarrow \hat{\beta}_{within}^{ols}$

If $\sigma_\alpha^2 \rightarrow 0$, $\psi^2 \rightarrow 1$, $\hat{\beta}_{re}^{ols} \rightarrow \hat{\beta}_{pool}^{ols}$ It is because

$$\begin{aligned}
\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}})(\mathbf{x}_{it} - \bar{\mathbf{x}})' &= \sum_{i=1}^N \sum_{t=1}^T ((\mathbf{x}_{it} - \bar{\mathbf{x}}_i) + (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}))((\mathbf{x}_{it} - \bar{\mathbf{x}}_i) + (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}))' \\
&= \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' + \\
&\quad \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \\
&= \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})'
\end{aligned}$$

Similarly,

$$\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}})(y_{it} - \bar{y}) = \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(y_{it} - \bar{y}_i) + \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{y}_i - \bar{y})$$

Thus,

$$\begin{aligned}
\hat{\beta}_{pool}^{ols} &= \left(\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}})(\mathbf{x}_{it} - \bar{\mathbf{x}})' \right)^{-1} \left(\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}})(y_{it} - \bar{y}) \right) \\
&= \left(\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \right)^{-1} \\
&\quad \left(\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(y_{it} - \bar{y}_i) + \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{y}_i - \bar{y}) \right)
\end{aligned}$$

So, pooled OLS estimator is an inefficient weighted average of within and between effects. RE estimator is an efficient weighted average of within and between effects. As RE model assumes $\varepsilon_i | \mathbf{X}_i \sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T]$,

$$Var(\hat{\beta}_{re}^{ols}) = \sigma_\varepsilon^2 \left[\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i)' \right]^{-1}$$

2.3.3 Between Effect Model and estimation of σ_α^2

$$\begin{aligned}
\bar{y}_i &= \bar{\mathbf{x}}_i' \boldsymbol{\beta} + \overbrace{\alpha_i + \bar{\varepsilon}_i}^{v_i} \\
\sigma_B^2 &= Var(v_i) = Var(\alpha_i + \bar{\varepsilon}_i) \\
&= Var(\alpha_i) + Var(\bar{\varepsilon}_i) \\
&= Var(\alpha_i) + T^{-1} Var(\varepsilon_{it})
\end{aligned}$$

as ε_{it} is serially uncorrelated

$$\underbrace{Var(\alpha_i)}_{\sigma_\alpha^2} = \underbrace{Var(v_i)}_{\sigma_B^2} - T^{-1} \underbrace{Var(\varepsilon_{it})}_{\sigma_\varepsilon^2}$$

3 GMM Estimation of Linear Panel Model

3.1 Linear Panel Model

$$\begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} = \begin{pmatrix} \mathbf{x}'_{i1} \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix}$$

$$\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta} + \mathbf{u}_i$$

3.2 Exogeneity Assumption

$$\mathbb{E}(\mathbf{Z}'_i \mathbf{u}_i) = \mathbf{0}$$

\mathbf{Z}_i is a $T \times r$ matrix. r is the number of exogeneous variables in \mathbf{X}_i plus the number of instrumental variables for endogeneous variables in \mathbf{X}_i . In GMM context, r is also the number of moment conditions.

K is the number of parameters.

$r \geq K$. If $r = K$, the model is just-identified, GMM is the same as MM; if $r > K$, the model is over-identified.

3.2.1 Summation Assumption

The weakest exogeneity assumption

$$\mathbf{Z}_i = \begin{pmatrix} \mathbf{z}'_{i1} \\ \vdots \\ \mathbf{z}'_{iT} \end{pmatrix}$$

$$\mathbb{E}(\mathbf{Z}'_i \mathbf{u}_i) = \mathbb{E}\left(\begin{pmatrix} \mathbf{z}'_{i1} \\ \vdots \\ \mathbf{z}'_{iT} \end{pmatrix}' \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix}\right) = \mathbb{E}\left(\begin{pmatrix} \mathbf{z}_{i1} & \cdots & \mathbf{z}_{iT} \end{pmatrix} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix}\right) = \mathbb{E}\left(\sum_{t=1}^T \mathbf{z}_{it} u_{it}\right) = \mathbf{0}$$

3.2.2 Contemporaneous Exogeneity Assumption

Stronger

$$\mathbf{Z}_i = \begin{pmatrix} \mathbf{z}'_{i1} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{z}'_{iT} \end{pmatrix}$$

$$\begin{aligned} \mathbb{E}(\mathbf{Z}'_i \mathbf{u}_i) &= \mathbb{E}\left(\begin{pmatrix} \mathbf{z}'_{i1} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{z}'_{iT} \end{pmatrix}' \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix}\right) \\ &= \mathbb{E}\left(\begin{pmatrix} \mathbf{z}_{i1} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{z}_{iT} \end{pmatrix} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix}\right) \\ &= \mathbb{E}\left(\begin{pmatrix} \mathbf{z}_{i1} u_{i1} \\ \vdots \\ \mathbf{z}_{iT} u_{iT} \end{pmatrix}\right) \\ &= \begin{pmatrix} \mathbb{E}(\mathbf{z}_{i1} u_{i1}) \\ \vdots \\ \mathbb{E}(\mathbf{z}_{iT} u_{iT}) \end{pmatrix} = \mathbf{0} \end{aligned}$$

3.2.3 Weak/Sequential Exogeneity Assumption

Stronger

$$\begin{aligned}
\mathbf{Z}_i &= \begin{pmatrix} \mathbf{z}'_{i1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & (\mathbf{z}'_{i1} & \mathbf{z}'_{i2}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & (\mathbf{z}'_{i1} \cdots \mathbf{z}'_{iT}) \end{pmatrix} \\
\mathbb{E}(\mathbf{Z}'_i \mathbf{u}_i) &= \mathbb{E} \left(\begin{pmatrix} \mathbf{z}'_{i1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & (\mathbf{z}'_{i1} & \mathbf{z}'_{i2}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & (\mathbf{z}'_{i1} \cdots \mathbf{z}'_{iT}) \end{pmatrix}' \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix} \right) \\
&= \mathbb{E} \left(\begin{pmatrix} \mathbf{z}_{i1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & (\mathbf{z}_{i1} \\ \mathbf{z}_{i2}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \begin{pmatrix} \mathbf{z}_{i1} \\ \vdots \\ \mathbf{z}_{iT} \end{pmatrix} \end{pmatrix} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix} \right) \\
&= \mathbb{E} \left(\begin{pmatrix} \mathbf{z}_{i1} u_{i1} \\ (\mathbf{z}_{i1} u_{i2} \\ \mathbf{z}_{i2} u_{i2}) \\ \vdots \\ (\mathbf{z}_{i1} u_{iT} \\ \vdots \\ \mathbf{z}_{iT} u_{iT}) \end{pmatrix} \right) = \begin{pmatrix} \mathbb{E}(\mathbf{z}_{i1} u_{i1}) \\ \mathbb{E}(\mathbf{z}_{i1} u_{i2}) \\ \mathbb{E}(\mathbf{z}_{i2} u_{i2}) \\ \vdots \\ \mathbb{E}(\mathbf{z}_{i1} u_{iT}) \\ \vdots \\ \mathbb{E}(\mathbf{z}_{iT} u_{iT}) \end{pmatrix} = \mathbf{0}
\end{aligned}$$

which is equivalent as $\mathbb{E}(\mathbf{z}_{is} u_{it}) = \mathbf{0}$ for $s \leq t$.

Strong form of sequential exogeneity $\mathbb{E}(u_{it} | \mathbf{z}_{it}, \dots, \mathbf{z}_{i1}) = 0$ implies weak form of sequential exogeneity $\mathbb{E}(\mathbf{z}_{is} u_{it}) = \mathbf{0}$ for $s \leq t$ as $\mathbb{E}(\mathbf{z}_{is} u_{it}) = \mathbb{E}(\mathbb{E}(\mathbf{z}_{is} u_{it} | \mathbf{z}_{it}, \dots, \mathbf{z}_{i1})) = \mathbb{E}(\mathbf{z}_{is} \underbrace{\mathbb{E}(u_{it} | \mathbf{z}_{it}, \dots, \mathbf{z}_{i1})}_0) = \mathbf{0}$ for $s \leq t$.

It also implies $Cov(\mathbf{z}_{is}, u_{it}) = \mathbf{0}$ for $s \leq t$ as $Cov(\mathbf{z}_{is}, u_{it}) = \underbrace{\mathbb{E}(\mathbf{z}_{is} u_{it})}_0 - \mathbb{E}(\mathbf{z}_{is}) \mathbb{E}(u_{it}) = -\mathbb{E}(\mathbf{z}_{is}) \underbrace{\mathbb{E}(\mathbb{E}(u_{it} | \mathbf{z}_{it}, \dots, \mathbf{z}_{i1}))}_0 = \mathbf{0}$ for $s \leq t$.

3.2.4 Strong/Strict Exogeneity Assumption

The strongest exogeneity assumption

$$\mathbf{Z}_i = \begin{pmatrix} (\mathbf{z}'_{i1} \cdots \mathbf{z}'_{iT}) & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & (\mathbf{z}'_{i1} \cdots \mathbf{z}'_{iT}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & (\mathbf{z}'_{i1} \cdots \mathbf{z}'_{iT}) \end{pmatrix}$$

$$\begin{aligned}
\mathbb{E}(\mathbf{Z}'_i \mathbf{u}_i) &= \mathbb{E} \left(\begin{pmatrix} (\mathbf{z}'_{i1} & \cdots & \mathbf{z}'_{iT}) & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & (\mathbf{z}'_{i1} & \cdots & \mathbf{z}'_{iT}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & (\mathbf{z}'_{i1} & \cdots & \mathbf{z}'_{iT}) \end{pmatrix}' \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix} \right) \\
&= \mathbb{E} \left(\begin{pmatrix} \begin{pmatrix} \mathbf{z}_{i1} \\ \vdots \\ \mathbf{z}_{iT} \end{pmatrix} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \begin{pmatrix} \mathbf{z}_{i1} \\ \vdots \\ \mathbf{z}_{iT} \end{pmatrix} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \begin{pmatrix} \mathbf{z}_{i1} \\ \vdots \\ \mathbf{z}_{iT} \end{pmatrix} \end{pmatrix} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix} \right) \\
&= \mathbb{E} \left(\begin{pmatrix} \begin{pmatrix} \mathbf{z}_{i1} u_{i1} \\ \vdots \\ \mathbf{z}_{iT} u_{i1} \end{pmatrix} \\ \begin{pmatrix} \mathbf{z}_{i1} u_{i2} \\ \vdots \\ \mathbf{z}_{iT} u_{i2} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \mathbf{z}_{i1} u_{iT} \\ \vdots \\ \mathbf{z}_{iT} u_{iT} \end{pmatrix} \end{pmatrix} \right) \\
&= \begin{pmatrix} \begin{pmatrix} \mathbb{E}(\mathbf{z}_{i1} u_{i1}) \\ \vdots \\ \mathbb{E}(\mathbf{z}_{iT} u_{i1}) \end{pmatrix} \\ \begin{pmatrix} \mathbb{E}(\mathbf{z}_{i1} u_{i2}) \\ \vdots \\ \mathbb{E}(\mathbf{z}_{iT} u_{i2}) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \mathbb{E}(\mathbf{z}_{i1} u_{iT}) \\ \vdots \\ \mathbb{E}(\mathbf{z}_{iT} u_{iT}) \end{pmatrix} \end{pmatrix} = \mathbf{0}
\end{aligned}$$

which is equivalent as $\mathbb{E}(\mathbf{z}_{is} u_{it}) = \mathbf{0}$ for $s = 1, \dots, T$

Strong form of strict exogeneity $\mathbb{E}(u_{it} | \mathbf{z}_{i1}, \dots, \mathbf{z}_{iT}) = 0$ implies weak form of strict exogeneity $\mathbb{E}(\mathbf{z}_{is} u_{it}) = \mathbf{0}$ for $s = 1, \dots, T$. Since for $s = 1, \dots, T$,

$$\begin{aligned}
\mathbb{E}(\mathbf{z}_{is} u_{it}) &= \mathbb{E}(\mathbb{E}(\mathbf{z}_{is} u_{it} | \mathbf{z}_{i1}, \dots, \mathbf{z}_{iT})) \\
&= \mathbb{E}(\mathbf{z}_{is} \underbrace{\mathbb{E}(u_{it} | \mathbf{z}_{i1}, \dots, \mathbf{z}_{iT})}_0) \\
&= \mathbf{0}
\end{aligned}$$

3.3 GMM Estimator of Linear Panel Model

3.3.1 Unconditional Moment Condition

$$\mathbb{E}(\mathbf{Z}'_i \mathbf{u}_i) = \mathbb{E}(\mathbf{Z}'_i (\mathbf{y}_i - \mathbf{X}_i \beta_0)) = \mathbf{0}$$

where β_0 is the true population parameter. So, $g(\mathbf{d}_i; \theta_0) = \mathbf{Z}'_i \mathbf{u}_i = \mathbf{Z}'_i (\mathbf{y}_i - \mathbf{X}_i \beta_0)$

3.3.2 Objective / Loss Function

We want to find β from the parameter space such that the squared distance between $\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N$ and $\mathbb{E}(\mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta_0))$ i.e.,

$$\begin{aligned}
& [\rho(\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N, \mathbb{E}(\mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta_0)))]^2 && \text{where } \rho(\cdot) \text{ is a metric function} \\
& = \|\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N - \mathbb{E}(\mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta_0))\|^2 \\
& = (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N - \underbrace{\mathbb{E}(\mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta_0))}_{\mathbf{0}})' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N - \underbrace{\mathbb{E}(\mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta_0))}_{\mathbf{0}}) \\
& = (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N)' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N) \geq 0 && \text{as distance cannot be negative}
\end{aligned}$$

is as close to the zero as possible. The distance is a function of β i.e.,

$$Q_N(\beta) := (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N)' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N) \geq 0$$

If \mathbf{W}_N is symmetric and positive definite, then $Q_N(\beta)$ is strictly convex. So, first order condition becomes sufficient and there is an unique minimizer.

3.3.3 Gradient Vector

$$\begin{aligned}
\nabla Q_N(\beta) &= \frac{\partial Q_N(\beta)}{\partial \beta} = \frac{\partial (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N)' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N)}{\partial \beta} \\
&= 2(\frac{\partial (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N)}{\partial \beta'})' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N) \\
&= 2[\sum_{i=1}^N (\frac{\partial \mathbf{Z}'_i \mathbf{y}_i}{\partial \beta'} - \frac{\partial \mathbf{Z}'_i \mathbf{X}_i \beta}{\partial \beta'})/N]' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N) \\
&= 2[\sum_{i=1}^N -\frac{\partial \mathbf{Z}'_i \mathbf{X}_i \beta}{\partial \beta'}/N]' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N) \\
&= -2(1/N^2) \sum_{i=1}^N (\mathbf{Z}'_i \mathbf{X}_i)' \mathbf{W}_N \sum_{i=1}^N (\mathbf{Z}'_i \mathbf{y}_i - \mathbf{Z}'_i \mathbf{X}_i \beta) \\
&= -2(1/N^2) \sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}''_i \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i - \sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i \beta) \\
&= -2(1/N^2) [(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) - (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i \beta)]
\end{aligned}$$

If $r = K$, both $(\frac{\partial (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N)}{\partial \beta'})'$ and \mathbf{W}_N are square matrixes and invertible. In this case, FOC is $\nabla Q_N(\hat{\beta}_{pmm}) = \sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\hat{\beta}_{pmm})/N = \mathbf{0}$ which is MM estimation.

3.3.4 First Order Condition

$$\begin{aligned}
-2(1/N^2)[(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) - (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i) \hat{\beta}_{pgmm}] &= \mathbf{0} \\
(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) - (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i) \hat{\beta}_{pgmm} &= \mathbf{0} \\
(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) &= (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i) \hat{\beta}_{pgmm} \\
[(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i)]^{-1} (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) &= \hat{\beta}_{pgmm}
\end{aligned}$$

Special case: if $\mathbf{W}_N = (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{Z}_i)^{-1}$,

$$\begin{aligned}
\hat{\beta}_{pgmm} &= [(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \overbrace{(\sum_{i=1}^N \mathbf{Z}'_i \mathbf{Z}_i)^{-1} (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i)}^{\hat{\Gamma}_{2SLs}}]^{-1} \overbrace{(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{Z}_i)^{-1} (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i)}^{\hat{\Gamma}'_{2SLs}} \\
&= [\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i \hat{\Gamma}_{2SLs}]^{-1} \hat{\Gamma}'_{2SLs} \sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i \\
&= [\sum_{i=1}^N \mathbf{X}'_i \underbrace{\mathbf{Z}_i \hat{\Gamma}_{2SLs}}_{\hat{\mathbf{X}}_i}]^{-1} \sum_{i=1}^N \underbrace{(\mathbf{Z}_i \hat{\Gamma}_{2SLs})'}_{\hat{\mathbf{X}}'_i} \mathbf{y}_i = \hat{\beta}_{p2SLs}
\end{aligned}$$

Special case: if $r = K$, the model is just-identified, GMM is the same as MM,

$$\begin{aligned}
\hat{\beta}_{pmm} = \hat{\beta}_{pgmm} &= [(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i)]^{-1} (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) \\
&= (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i)^{-1} \mathbf{W}_N^{-1} (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i)^{-1} (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) \\
&= (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i)^{-1} (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) = \hat{\beta}_{piv}
\end{aligned}$$

Special case: if all regressors are exogeneous: $\mathbf{Z}_i = \mathbf{X}_i$ (which implies $r = K$),

$$\begin{aligned}
\hat{\beta}_{pgmm} &= \hat{\beta}_{piv} \\
&= (\sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i)^{-1} (\sum_{i=1}^N \mathbf{X}'_i \mathbf{y}_i) = \hat{\beta}_{pols}
\end{aligned}$$

$$\begin{aligned}
\hat{\beta}_{pgmm} &= [(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i)]^{-1} (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) \\
&= [(\mathbf{X}'_1 \quad \cdots \quad \mathbf{X}'_N) \begin{pmatrix} \mathbf{Z}_1 \\ \vdots \\ \mathbf{Z}_N \end{pmatrix} \mathbf{W}_N (\mathbf{Z}'_1 \quad \cdots \quad \mathbf{Z}'_N) \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_N \end{pmatrix}]^{-1} (\mathbf{X}'_1 \quad \cdots \quad \mathbf{X}'_N) \begin{pmatrix} \mathbf{Z}_1 \\ \vdots \\ \mathbf{Z}_N \end{pmatrix} \mathbf{W}_N (\mathbf{Z}'_1 \quad \cdots \quad \mathbf{Z}'_N) \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{pmatrix} \\
&= [\mathbf{X}' \mathbf{Z} \mathbf{W}_N \mathbf{Z}' \mathbf{X}]^{-1} \mathbf{X}' \mathbf{Z} \mathbf{W}_N \mathbf{Z}' \mathbf{y}
\end{aligned}$$

3.4 Conditional Variance of $\hat{\beta}_{pgmm}$

$$\begin{aligned}
Var(\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{y}|\mathbf{X}, \mathbf{Z}) &= \mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'Var(\mathbf{y}|\mathbf{X}, \mathbf{Z})(\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}')' \\
&= \mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'Var(\mathbf{X}\beta + \mathbf{u}|\mathbf{X}, \mathbf{Z})(\mathbf{Z}''\mathbf{W}_N'\mathbf{Z}'\mathbf{X}'') \\
&= \mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'Var(\mathbf{u}|\mathbf{X}, \mathbf{Z})(\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}) \\
&= \mathbf{X}'\mathbf{Z}\mathbf{W}_NVar(\mathbf{Z}'\mathbf{u}|\mathbf{X}, \mathbf{Z})\mathbf{W}_N\mathbf{Z}'\mathbf{X} \\
&= \mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbb{E}((\mathbf{Z}'\mathbf{u} - \mathbb{E}(\mathbf{Z}'\mathbf{u}|\mathbf{X}, \mathbf{Z}))(\mathbf{Z}'\mathbf{u} - \mathbb{E}(\mathbf{Z}'\mathbf{u}|\mathbf{X}, \mathbf{Z}))'|\mathbf{X}, \mathbf{Z})\mathbf{W}_N\mathbf{Z}'\mathbf{X} \\
&= \mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbb{E}((\mathbf{Z}'\mathbf{u})(\mathbf{Z}'\mathbf{u})'|\mathbf{X}, \mathbf{Z})\mathbf{W}_N\mathbf{Z}'\mathbf{X} \\
&= \mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbb{E}(\mathbf{Z}'\mathbf{u}\mathbf{u}'\mathbf{Z}'|\mathbf{X}, \mathbf{Z})\mathbf{W}_N\mathbf{Z}'\mathbf{X}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1'} &= [\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]'^{-1} \\
&= [\mathbf{X}'\mathbf{Z}''\mathbf{W}_N'\mathbf{Z}'\mathbf{X}'']^{-1} \\
&= [\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1}
\end{aligned}$$

$$\begin{aligned}
Var(\hat{\beta}_{pgmm}|\mathbf{X}, \mathbf{Z}) &= Var([\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{y}|\mathbf{X}, \mathbf{Z}) \\
&= [\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1}Var(\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{y}|\mathbf{X}, \mathbf{Z})[\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1'} \\
&= [\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbb{E}(\mathbf{Z}'\mathbf{u}\mathbf{u}'\mathbf{Z}'|\mathbf{X}, \mathbf{Z})\mathbf{W}_N\mathbf{Z}'\mathbf{X}[\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1}
\end{aligned}$$

4 GMM Estimation of Fixed Effect Model

$$\begin{aligned}
y_{it} &= \mathbf{x}_{it}'\boldsymbol{\beta} + \alpha_i + \varepsilon_{it} \\
\mathbf{y}_i &= \mathbf{X}_i\boldsymbol{\beta} + \underbrace{(\mathbf{e}\alpha_i + \boldsymbol{\varepsilon}_i)}_{\mathbf{u}_i} \\
\mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{e})\boldsymbol{\alpha} + \boldsymbol{\varepsilon}
\end{aligned}$$

4.1 Assumption

α_i is potentially correlated with \mathbf{X}_i , so \mathbf{u}_i is potentially correlated with \mathbf{X}_i

$\boldsymbol{\varepsilon}_i$ is also potentially correlated with \mathbf{X}_i , so \mathbf{u}_i is potentially correlated with \mathbf{X}_i

Even after eliminating α_i by using any arbitrary operators \mathbf{T} , $\tilde{\mathbf{u}}_i := \mathbf{T}\mathbf{u}_i$ is still potentially correlated with $\tilde{\mathbf{X}}_i := \mathbf{T}\mathbf{X}_i$ because of the potential correlation between $\boldsymbol{\varepsilon}_i$ and \mathbf{X}_i . Thus, $\tilde{\mathbf{X}}_i$ is potentially endogenous.

If $\tilde{\mathbf{X}}_i$ is endogeneous, OLS estimation is inconsistent and biased. We should use IV estimation (for just-identified case) and 2SLS estimation (for over-identified case). IV and 2SLS estimation are special cases of GMM estimation.

4.2 GMM Estimator of Fixed Effect Model

There exists a \mathbf{T} such that $\mathbf{T}\mathbf{e} = \mathbf{0}$.

4.2.1 Transformed Model

$$\begin{aligned}
\tilde{\mathbf{y}}_i &:= \mathbf{T}\mathbf{y}_i = \mathbf{T}(\mathbf{X}_i\boldsymbol{\beta} + \mathbf{u}_i) = \mathbf{T}\mathbf{X}_i\boldsymbol{\beta} + \mathbf{T}\mathbf{u}_i := \tilde{\mathbf{X}}_i\boldsymbol{\beta} + \tilde{\mathbf{u}}_i \\
\tilde{\mathbf{u}}_i &:= \mathbf{T}\mathbf{u}_i = \mathbf{T}(\mathbf{e}\alpha_i + \boldsymbol{\varepsilon}_i) = \mathbf{T}\mathbf{e}\alpha_i + \mathbf{T}\boldsymbol{\varepsilon}_i = \mathbf{0} + \mathbf{T}\boldsymbol{\varepsilon}_i = \mathbf{T}\boldsymbol{\varepsilon}_i =: \tilde{\boldsymbol{\varepsilon}}_i
\end{aligned}$$

It is obvious that $\tilde{\mathbf{u}}_i = \mathbf{T}\boldsymbol{\varepsilon}_i$ is correlated with $\tilde{\mathbf{X}}_i := \mathbf{T}\mathbf{X}_i$ if $\boldsymbol{\varepsilon}_i$ is correlated with \mathbf{X}_i .

If $\mathbf{T} = \mathbf{Q} = \mathbf{I}_T - \mathbf{T}^{-1}\mathbf{e}\mathbf{e}'$,

$$\begin{aligned}
\tilde{\mathbf{y}}_i &= \tilde{\mathbf{X}}_i\boldsymbol{\beta} + \tilde{\boldsymbol{\varepsilon}}_i \\
(\mathbf{y}_i - \mathbf{e}\bar{y}_i) &= (\mathbf{X}_i - \mathbf{e}\bar{\mathbf{x}}_i')\boldsymbol{\beta} + (\boldsymbol{\varepsilon}_i - \mathbf{e}\bar{\boldsymbol{\varepsilon}}_i) \\
(y_{it} - \bar{y}_i) &= (\mathbf{x}_{it} - \bar{\mathbf{x}}_i')\boldsymbol{\beta} + (\varepsilon_{it} - \bar{\varepsilon}_i)
\end{aligned}$$

Under weak form of weak/sequential exogeneity assumption $\mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) = \mathbf{0}$ for $s \leq t$.

For $s \leq t$, we have

$$\begin{aligned}
\mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \bar{\varepsilon}_i)) &= \mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) - \mathbb{E}(\mathbf{z}_{is}\bar{\varepsilon}_i) \\
&= \mathbf{0} - \mathbb{E}(\mathbf{z}_{is} \sum_{t=1}^T \varepsilon_{it} / T) \\
&= -\frac{1}{T} \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1} + \cdots + \mathbf{z}_{is}\varepsilon_{i,s-1} + \mathbf{z}_{is}\varepsilon_{is} + \cdots + \mathbf{z}_{is}\varepsilon_{iT}) \\
&= -\frac{1}{T} (\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,s-1}) + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{is}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{iT})) \\
&= -\frac{1}{T} (\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,s-1}) + \mathbf{0} + \cdots + \mathbf{0}) \\
&= -\frac{1}{T} (\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,s-1}))
\end{aligned}$$

So $\mathbb{E}(\mathbf{z}_{it}(\varepsilon_{it} - \bar{\varepsilon}_i))$ is not necessarily equal to zero under weak form of weak/sequential exogeneity assumption. If weak form of strong/strict exogeneity is assumed $\mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) = \mathbf{0} \forall s$, then $\mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \bar{\varepsilon}_i)) = \mathbf{0} \forall s$. So, \mathbf{z}_{is} , $s = 1, \dots, T$ satisfy the exclusion restriction (exogeneity) requirement of valid instrument since $\text{Cov}(\mathbf{z}_{is}, \varepsilon_{it} - \bar{\varepsilon}_i) = \underbrace{\mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \bar{\varepsilon}_i))}_{\mathbf{0}} - \mathbb{E}(\mathbf{z}_{is})\mathbb{E}(\varepsilon_{it} - \bar{\varepsilon}_i) =$

$$-\mathbb{E}(\mathbf{z}_{is}) \underbrace{(\mathbb{E}(\varepsilon_{it}) - T^{-1} \sum_{t=1}^T \mathbb{E}(\varepsilon_{it}))}_0 = \mathbf{0} \quad \forall s \text{ (additionally assume } \mathbb{E}(\varepsilon_{it}) = 0). \text{ So, we have}$$

$$\begin{aligned} \mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \bar{\varepsilon}_i)) &= \mathbf{0} && \text{for } \forall s \\ \iff \mathbb{E}(\mathbf{Z}'_i(\boldsymbol{\varepsilon}_i - \mathbf{e}\bar{\varepsilon}_i)) &= \mathbf{0} \\ \iff \mathbb{E}(\mathbf{Z}'_i\tilde{\boldsymbol{\varepsilon}}_i) &= \mathbf{0} \end{aligned}$$

We can then apply IV estimation in GMM framework.

If $\mathbf{T} = \boldsymbol{\Delta}$

$$\begin{aligned} \tilde{\mathbf{y}}_i &= \tilde{\mathbf{X}}_i\boldsymbol{\beta} + \tilde{\boldsymbol{\varepsilon}}_i \\ \boldsymbol{\Delta}\mathbf{y}_i &= \boldsymbol{\Delta}\mathbf{X}_i\boldsymbol{\beta} + \boldsymbol{\Delta}\boldsymbol{\varepsilon}_i \\ (y_{it} - y_{i,t-1}) &= (\mathbf{x}_{it} - \mathbf{x}_{i,t-1})'\boldsymbol{\beta} + (\varepsilon_{it} - \varepsilon_{i,t-1}) \end{aligned}$$

Under weak form of weak/sequential exogeneity assumption $\mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) = \mathbf{0}$ for $s \leq t$.

For $s < t$, we have

$$\begin{aligned} \mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \varepsilon_{i,t-1})) &= \mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) - \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,t-1}) \\ &= \mathbf{0} - \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,t-1}) && \text{as } s < t \implies s \leq t \\ &= \mathbf{0} && \text{as } s < t \iff s \leq t-1 \end{aligned}$$

So, \mathbf{z}_{is} for $s < t$ satisfy the exclusion restriction (exogeneity) requirement of valid instrument since $Cov(\mathbf{z}_{is}, \varepsilon_{it} - \varepsilon_{i,t-1}) = 0$ for $s < t$ (additionally assume $\mathbb{E}(\varepsilon_{it}) = 0$). Equivalently,

$$\mathbf{Z}_i = \begin{pmatrix} t=2; \mathbf{z}'_{i1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & t=3; (\mathbf{z}'_{i1} \quad \mathbf{z}'_{i2}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & t=T; (\mathbf{z}'_{i1} \quad \cdots \quad \mathbf{z}'_{i,T-1}) \end{pmatrix}$$

So, we have

$$\begin{aligned} \mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \varepsilon_{i,t-1})) &= \mathbf{0} && \text{for } s < t \\ \iff \mathbb{E}(\mathbf{Z}'_i\boldsymbol{\Delta}\boldsymbol{\varepsilon}_i) &= \mathbf{0} \\ \iff \mathbb{E}(\mathbf{Z}'_i\tilde{\boldsymbol{\varepsilon}}_i) &= \mathbf{0} \end{aligned}$$

We can then apply IV estimation in GMM framework.

5 GMM Estimation of Random Effect Model

$$\begin{aligned}
y_{it} &= \mathbf{x}_{it}'\boldsymbol{\beta} + \alpha_i + \varepsilon_{it} \\
\mathbf{y}_i &= \mathbf{X}_i\boldsymbol{\beta} + \underbrace{(\mathbf{e}\alpha_i + \boldsymbol{\varepsilon}_i)}_{\mathbf{u}_i} \\
\mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{e})\boldsymbol{\alpha} + \boldsymbol{\varepsilon}
\end{aligned}$$

5.1 Assumption

α_i is not correlated with \mathbf{X}_i .

ε_i is potentially correlated with \mathbf{X}_i , so \mathbf{u}_i is potentially correlated with \mathbf{X}_i . Thus, \mathbf{X}_i is potentially endogeneous.

If \mathbf{X}_i is endogeneous, OLS estimation is inconsistent and biased. We should use IV estimation (for just-identified case) and 2SLS estimation (for over-identified case). IV and 2SLS estimations are special cases of GMM estimation.

Assume

$$\mathbb{E}(\mathbf{u}_i|\mathbf{Z}_i) = \mathbf{0}$$

Which is stronger than $\mathbb{E}(\mathbf{Z}_i'\mathbf{u}_i) = \mathbf{0}$ as $\mathbb{E}(\mathbf{u}_i|\mathbf{Z}_i) = \mathbf{0}$ implies $\mathbb{E}(\mathbf{Z}_i'\mathbf{u}_i) = \mathbf{0}$

And assume

$$Var(\mathbf{u}_i|\mathbf{Z}_i) = \boldsymbol{\Omega}_i = \begin{pmatrix} \sigma_\alpha^2 + \sigma_\varepsilon^2 & \cdots & \sigma_\alpha^2 \\ \vdots & \ddots & \vdots \\ \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 + \sigma_\varepsilon^2 \end{pmatrix}$$

5.1.1 Optimal Moment Condition

$$\begin{aligned}
\mathbf{D}_i &= \mathbb{E}\left(\frac{\partial \mathbf{u}_i'}{\partial \boldsymbol{\beta}}|\mathbf{Z}_i\right)Var(\mathbf{u}_i|\mathbf{Z}_i)^{-1} \\
&= \mathbb{E}\left(\frac{\partial (\mathbf{Z}_i\boldsymbol{\beta})'}{\partial \boldsymbol{\beta}}|\mathbf{Z}_i\right)\boldsymbol{\Omega}_i^{-1} \\
&= \mathbb{E}(\mathbf{Z}_i'|\mathbf{Z}_i)\boldsymbol{\Omega}_i^{-1} \\
&= \mathbf{Z}_i'\boldsymbol{\Omega}_i^{-1}
\end{aligned}$$

Optimal unconditional moment is

$$\begin{aligned}
\mathbb{E}(\mathbf{D}_i\mathbf{u}_i) &= \mathbf{0} \\
\mathbb{E}(\mathbf{Z}_i'\boldsymbol{\Omega}_i^{-1}\mathbf{u}_i) &= \mathbf{0} \\
\mathbb{E}(\mathbf{Z}_i'\boldsymbol{\Omega}_i^{-1/2}\boldsymbol{\Omega}_i^{-1/2}\mathbf{u}_i) &= \mathbf{0} \\
\mathbb{E}(\mathbf{Z}_i'\boldsymbol{\Omega}_i'^{-1/2}\boldsymbol{\Omega}_i^{-1/2}\mathbf{u}_i) &= \mathbf{0} \\
\mathbb{E}(\mathbf{Z}_i'\boldsymbol{\Omega}_i^{-1/2'}\boldsymbol{\Omega}_i^{-1/2}\mathbf{u}_i) &= \mathbf{0} \\
\mathbb{E}((\boldsymbol{\Omega}_i^{-1/2}\mathbf{Z}_i)'\boldsymbol{\Omega}_i^{-1/2}\mathbf{u}_i) &= \mathbf{0} \\
\sigma_\varepsilon^2\mathbb{E}((\boldsymbol{\Omega}_i^{-1/2}\mathbf{Z}_i)'\boldsymbol{\Omega}_i^{-1/2}\mathbf{u}_i) &= \sigma_\varepsilon^2\mathbf{0} \\
\mathbb{E}((\sigma_\varepsilon\boldsymbol{\Omega}_i^{-1/2}\mathbf{Z}_i)'\sigma_\varepsilon\boldsymbol{\Omega}_i^{-1/2}\mathbf{u}_i) &= \mathbf{0}
\end{aligned}$$

This implies that the model should be transformed by $\sigma_\varepsilon\boldsymbol{\Omega}_i^{-1/2}$

5.2 GMM Estimator of Random Effect Model

5.2.1 Transformed Model

$$\begin{aligned}
\sigma_\varepsilon\boldsymbol{\Omega}_i^{-1/2}\mathbf{y}_i &= \sigma_\varepsilon\boldsymbol{\Omega}_i^{-1/2}(\mathbf{X}_i\boldsymbol{\beta} + (\mathbf{e}\alpha_i + \boldsymbol{\varepsilon}_i)) = \sigma_\varepsilon\boldsymbol{\Omega}_i^{-1/2}(\mathbf{X}_i\boldsymbol{\beta} + \mathbf{u}_i) = \sigma_\varepsilon\boldsymbol{\Omega}_i^{-1/2}\mathbf{X}_i\boldsymbol{\beta} + \sigma_\varepsilon\boldsymbol{\Omega}_i^{-1/2}\mathbf{u}_i \\
(\mathbf{y}_i - \lambda\mathbf{e}\bar{y}_i) &= (\mathbf{X}_i - \lambda\mathbf{e}\bar{\mathbf{x}}_i')\boldsymbol{\beta} + [(1-\lambda)\mathbf{e}\alpha_i + (\boldsymbol{\varepsilon}_i - \lambda\mathbf{e}\bar{\boldsymbol{\varepsilon}}_i)] & \lambda = 1 - \psi = 1 - \frac{\sigma_\varepsilon}{\sqrt{T\sigma_\alpha^2 + \sigma_\varepsilon^2}} \\
(y_{it} - \lambda\bar{y}_i) &= (\mathbf{x}_{it} - \lambda\bar{\mathbf{x}}_i')\boldsymbol{\beta} + [(1-\lambda)\alpha_i + (\varepsilon_{it} - \lambda\bar{\varepsilon}_i)]
\end{aligned}$$

Under weak form of weak/sequential exogeneity assumption $\mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) = \mathbf{0}$ for $s \leq t$.

For $s \leq t$, we have

$$\begin{aligned}
\mathbb{E}(\mathbf{z}_{is}[(1-\lambda)\alpha_i + (\varepsilon_{it} - \lambda\bar{\varepsilon}_i)]) &= \mathbb{E}(\mathbf{z}_{is}(1-\lambda)\alpha_i + \mathbf{z}_{is}(\varepsilon_{it} - \lambda\bar{\varepsilon}_i)) \\
&= (1-\lambda)\mathbb{E}(\mathbf{z}_{is}\alpha_i) + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) - \lambda\mathbb{E}(\mathbf{z}_{is}\bar{\varepsilon}_i) \\
&= (1-\lambda)\mathbf{0} + \mathbf{0} - \lambda E(\mathbf{z}_{is} \sum_{t=1}^T \varepsilon_{it}/T) \\
&= -\frac{\lambda}{T} \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1} + \cdots + \mathbf{z}_{is}\varepsilon_{i,s-1} + \mathbf{z}_{is}\varepsilon_{is} + \cdots + \mathbf{z}_{is}\varepsilon_{iT}) \\
&= -\frac{\lambda}{T} (\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,s-1}) + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{is}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{iT})) \\
&= -\frac{\lambda}{T} (\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,s-1}) + \mathbf{0} + \cdots + \mathbf{0}) \\
&= -\frac{\lambda}{T} (\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,s-1}))
\end{aligned}$$

So $\mathbb{E}(\mathbf{z}_{it}(\varepsilon_{it} - \bar{\varepsilon}_i))$ is not necessarily equal to zero under weak form of weak/sequential exogeneity assumption.

If weak form of strong/strict exogeneity assumption is assumed $\mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) = \mathbf{0} \forall s$, then $\mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \bar{\varepsilon}_i)) = \mathbf{0} \forall s$. So, \mathbf{z}_{is} , $s = 1, \dots, T$ satisfy the exclusion restriction (exogeneity) requirement of valid instrument.

So, we have

$$\begin{aligned}
&\mathbb{E}(\mathbf{z}_{is}[(1-\lambda)\alpha_i + (\varepsilon_{it} - \lambda\bar{\varepsilon}_i)]) = \mathbf{0} && \text{for } \forall s \\
\iff &\mathbb{E}(\mathbf{Z}'_i[(1-\lambda)\mathbf{e}\alpha_i + (\boldsymbol{\varepsilon}_i - \lambda\mathbf{e}\bar{\varepsilon}_i)]) = \mathbf{0}
\end{aligned}$$

We can then apply IV estimation in GMM framework.

6 Dynamic Linear Panel Model

6.1 Assumption

6.1.1 Weak/Sequential Exogeneity

For $t = 2, \dots, T$

$$\mathbb{E}(\varepsilon_{it}|y_{i,t-1}, \dots, y_{i1}, \alpha_i) = 0$$

This implies

$$\mathbb{E}(y_{is}\varepsilon_{it}) = 0, \quad \mathbb{E}(\varepsilon_{it}) = 0 \quad \text{and} \quad \mathbb{E}(\alpha_i\varepsilon_{it}) = 0 \quad \text{for } s < t$$

And

$$\text{Cov}(y_{is}, \varepsilon_{it}) = 0 \quad \text{and} \quad \text{Cov}(\alpha_i, \varepsilon_{it}) = 0 \quad \text{for } s < t$$

It is because

$$\begin{aligned} \text{Cov}(y_{is}, \varepsilon_{it}) &= \mathbb{E}(y_{is}\varepsilon_{it}) - \mathbb{E}(y_{is})\mathbb{E}(\varepsilon_{it}) \\ &= \mathbb{E}(\mathbb{E}(y_{is}\varepsilon_{it}|y_{i,t-1}, \dots, y_{i1}, \alpha_i)) - \mathbb{E}(y_{is})\mathbb{E}(\mathbb{E}(\varepsilon_{it}|y_{i,t-1}, \dots, y_{i1}, \alpha_i)) \\ &= \mathbb{E}(y_{is} \underbrace{\mathbb{E}(\varepsilon_{it}|y_{i,t-1}, \dots, y_{i1}, \alpha_i)}_0) - \mathbb{E}(y_{is})\mathbb{E}(\underbrace{\mathbb{E}(\varepsilon_{it}|y_{i,t-1}, \dots, y_{i1}, \alpha_i)}_0) \quad \text{as } s < t \\ &= 0 \end{aligned}$$

6.2 Model

6.2.1 No Transformation

$$y_{it} = \gamma y_{i,t-1} + \mathbf{x}'_{it}\boldsymbol{\beta} + \underbrace{(\alpha_i + \varepsilon_{it})}_{u_{it}}$$

$$\begin{aligned} \text{Cov}(y_{i,t-1}, \alpha_i) &= \text{Cov}(\gamma y_{i,t-2} + \mathbf{x}'_{i,t-1}\boldsymbol{\beta} + \alpha_i + \varepsilon_{i,t-1}, \alpha_i) \\ &= \gamma \text{Cov}(y_{i,t-2}, \alpha_i) + \text{Cov}(\mathbf{x}'_{i,t-1}\boldsymbol{\beta}, \alpha_i) + \text{Var}(\alpha_i) + \underbrace{\text{Cov}(\varepsilon_{i,t-1}, \alpha_i)}_0 \\ &= \gamma \text{Cov}(y_{i,t-2}, \alpha_i) + \boldsymbol{\beta}' \text{Cov}(\mathbf{x}_{i,t-1}, \alpha_i) + \text{Var}(\alpha_i) \\ &\neq 0 \quad \text{assume } \text{Cov}(\mathbf{x}_{i,t-1}, \alpha_i) \neq 0 \text{ and } \text{Var}(\alpha_i) > 0 \end{aligned}$$

so that

$$\begin{aligned} \text{Cov}(y_{i,t-1}, u_{it}) &= \text{Cov}(y_{i,t-1}, \alpha_i + \varepsilon_{it}) \\ &= \underbrace{\text{Cov}(y_{i,t-1}, \alpha_i)}_{\neq 0} + \underbrace{\text{Cov}(y_{i,t-1}, \varepsilon_{it})}_0 \\ &\neq 0 \end{aligned}$$

The necessary condition for OLS estimator to be unbiased is $\mathbb{E}(u_{it}|y_{i,t-1}, \mathbf{x}_{it}) = 0$. As $\mathbb{E}(u_{it}|y_{i,t-1}, \mathbf{x}_{it}) = 0 \implies \text{Cov}(y_{i,t-1}, u_{it}) = 0$. As a result, $\text{Cov}(y_{i,t-1}, u_{it}) \neq 0 \implies \mathbb{E}(u_{it}|y_{i,t-1}, \mathbf{x}_{it}) \neq 0$. Thus, OLS estimator is biased.

6.2.2 Special case: no \mathbf{x}_{it}

$$y_{it} = \gamma y_{i,t-1} + \underbrace{(\alpha_i + \varepsilon_{it})}_{u_{it}}$$

The necessary condition for OLS estimator to be consistent is $\mathbb{E}(y_{i,t-1}u_{it}) = 0$. However,

$$\begin{aligned}\mathbb{E}(y_{i,t-1}u_{it}) &= \mathbb{E}(y_{i,t-1}(\alpha_i + \varepsilon_{it})) \\ &= \mathbb{E}(y_{i,t-1}\alpha_i) + \underbrace{\mathbb{E}(y_{i,t-1}\varepsilon_{it})}_0 > 0\end{aligned}$$

$$\begin{aligned}\mathbb{E}(y_{i,t-1}\alpha_i) &= \mathbb{E}((\gamma y_{i,t-2} + \alpha_i + \varepsilon_{i,t-1})\alpha_i) \\ &= \gamma \mathbb{E}(y_{i,t-2}\alpha_i) + \mathbb{E}(\alpha_i^2) + \mathbb{E}(\varepsilon_{i,t-1}\alpha_i) \\ &= \gamma \mathbb{E}((\gamma y_{i,t-3} + \alpha_i + \varepsilon_{i,t-2})\alpha_i) + \mathbb{E}(\alpha_i^2) + \mathbb{E}(\mathbb{E}(\varepsilon_{i,t-1}\alpha_i | y_{i,t-2}, \dots, y_{i1}, \alpha_i)) \\ &= \gamma^2 \mathbb{E}(y_{i,t-3}\alpha_i) + \gamma \mathbb{E}(\alpha_i^2) + \gamma \mathbb{E}(\varepsilon_{i,t-2}\alpha_i) + \mathbb{E}(\alpha_i^2) + \underbrace{\mathbb{E}(\alpha_i \mathbb{E}(\varepsilon_{i,t-1} | y_{i,t-2}, \dots, y_{i1}, \alpha_i))}_0 \\ &= \gamma^2 \mathbb{E}(y_{i,t-3}\alpha_i) + \gamma \mathbb{E}(\alpha_i^2) + \mathbb{E}(\alpha_i^2) \\ &\dots \\ &= \gamma^{t-2} \mathbb{E}(y_{i,t-(t-2+1)}) + \gamma^{t-2-1} \mathbb{E}(\alpha_i^2) + \dots + \mathbb{E}(\alpha_i^2) \\ &= \gamma^{t-2} \mathbb{E}(y_{i1}) + \gamma^{t-3} \mathbb{E}(\alpha_i^2) + \dots + \mathbb{E}(\alpha_i^2) \\ &= \gamma^{t-2} y_{i1} + \gamma^{t-3} \text{Var}(\alpha_i) + \dots + \text{Var}(\alpha_i) \quad y_{i1} \text{ is initial value and assume } \mathbb{E}(\alpha_i) = 0 \\ &> 0 \quad \text{assume } \text{Var}(\alpha_i) > 0, y_{i1} > 0 \text{ and } 0 < \gamma < 1\end{aligned}$$

Thus, OLS estimator is inconsistent. The necessary condition for OLS estimator to be unbiased is $\mathbb{E}(u_{it} | y_{i,t-1}) = 0$. As $\mathbb{E}(u_{it} | y_{i,t-1}) = 0 \implies \mathbb{E}(y_{i,t-1}u_{it}) = 0$, $\mathbb{E}(y_{i,t-1}u_{it}) \neq 0 \implies \mathbb{E}(u_{it} | y_{i,t-1}) \neq 0$. Thus, OLS estimator is biased. It can also be seen by OVB formula.

$$\begin{aligned}\gamma_{short} &= \frac{\text{Cov}(y_{it}, y_{i,t-1})}{\text{Var}(y_{i,t-1})} \\ &= \frac{\text{Cov}(\gamma_{long} y_{i,t-1} + \alpha_i + \varepsilon_{it}, y_{i,t-1})}{\text{Var}(y_{i,t-1})} \\ &= \gamma_{long} + \frac{\text{Cov}(\alpha_i, y_{i,t-1})}{\text{Var}(y_{i,t-1})} + \underbrace{\frac{\text{Cov}(\varepsilon_{it}, y_{i,t-1})}{\text{Var}(y_{i,t-1})}}_0 \\ &= \gamma_{long} + \frac{\text{Cov}(\alpha_i, y_{i,t-1})}{\text{Var}(y_{i,t-1})}\end{aligned}$$

$$\gamma_{short} - \gamma_{long} = \frac{\text{Cov}(\alpha_i, y_{i,t-1})}{\text{Var}(y_{i,t-1})} > 0 \quad \text{if } \text{Var}(y_{i,t-1}) > 0$$

$$\text{Cov}(\alpha_i, y_{i,t-1}) = \mathbb{E}(\alpha_i y_{i,t-1}) - \mathbb{E}(\alpha_i) \mathbb{E}(y_{i,t-1}) > 0 \quad \text{see above for } \mathbb{E}(\alpha_i y_{i,t-1}) > 0 \text{ and assume } \mathbb{E}(\alpha_i) = 0$$

Thus, OLS estimator is biased upward / over-estimate.

6.2.3 Within Transformation

$$y_{it} - \bar{y}_i = \gamma(y_{i,t-1} - \bar{y}_{i,-1}) + (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' \boldsymbol{\beta} + (\varepsilon_{it} - \bar{\varepsilon}_i)$$

$$\text{Cov}(y_{i,t-1}, \bar{\varepsilon}_i) = \text{Cov}(\gamma y_{i,t-2} + \mathbf{x}_{i,t-1}' \boldsymbol{\beta} + \alpha_i + \varepsilon_{i,t-1}, T^{-1} \sum_{t=1}^T \varepsilon_{it})$$

$$\neq 0$$

since $\varepsilon_{i,t-1}$ is correlated with $T^{-1} \sum_{t=1}^T \varepsilon_{it}$

so that

$$\text{Cov}(y_{i,t-1} - \bar{y}_{i,-1}, \varepsilon_{it} - \bar{\varepsilon}_i) \neq 0$$

The necessary condition for FE estimator to be unbiased is $\mathbb{E}(\varepsilon_{it} - \bar{\varepsilon}_i | y_{i,t-1} - \bar{y}_{i,-1}, \mathbf{x}_{it} - \bar{\mathbf{x}}_i) = 0$. As $\mathbb{E}(\varepsilon_{it} - \bar{\varepsilon}_i | y_{i,t-1} - \bar{y}_{i,-1}, \mathbf{x}_{it} - \bar{\mathbf{x}}_i) = 0 \implies \text{Cov}(y_{i,t-1} - \bar{y}_{i,-1}, \varepsilon_{it} - \bar{\varepsilon}_i) = 0$. As a result, $\text{Cov}(y_{i,t-1} - \bar{y}_{i,-1}, \varepsilon_{it} - \bar{\varepsilon}_i) \neq 0 \implies \mathbb{E}(\varepsilon_{it} - \bar{\varepsilon}_i | y_{i,t-1} - \bar{y}_{i,-1}, \mathbf{x}_{it} - \bar{\mathbf{x}}_i) \neq 0$. Thus, FE estimator is biased.

6.2.4 Special case: no x_{it}

$$y_{it} - \bar{y}_i = \gamma(y_{i,t-1} - \bar{y}_{i,-1}) + (\varepsilon_{it} - \bar{\varepsilon}_i)$$

The bias is called Nickell (1981) bias / dynamic panel bias. If $\gamma > 0$, the bias must be negative. The bias converges to zero when $T \rightarrow \infty$.

6.2.5 First Difference Transformation

$$\begin{aligned} \tilde{y}_i &= \tilde{X}_i \delta + \tilde{\varepsilon}_i \\ \begin{pmatrix} y_{i3} - y_{i2} \\ \vdots \\ y_{iT} - y_{i,T-1} \end{pmatrix} &= \begin{pmatrix} y_{i2} - y_{i1} & (x_{i3} - x_{i2})' \\ \vdots & \\ y_{i,T-1} - y_{i,T-2} & (x_{iT} - x_{i,T-1})' \end{pmatrix} \begin{pmatrix} \gamma \\ \beta \end{pmatrix} + \begin{pmatrix} \varepsilon_{i3} - \varepsilon_{i2} \\ \vdots \\ \varepsilon_{iT} - \varepsilon_{i,T-1} \end{pmatrix} \\ y_{it} - y_{i,t-1} &= \gamma(y_{i,t-1} - y_{i,t-2}) + (x_{it} - x_{i,t-1})' \beta + (\varepsilon_{it} - \varepsilon_{i,t-1}) \end{aligned} \quad t \geq 3$$

$$\begin{aligned} Cov(y_{i,t-1} - y_{i,t-2}, \varepsilon_{it} - \varepsilon_{i,t-1}) &= Cov(y_{i,t-1}, \varepsilon_{it}) - Cov(y_{i,t-1}, \varepsilon_{i,t-1}) - Cov(y_{i,t-2}, \varepsilon_{it}) + Cov(y_{i,t-2}, \varepsilon_{i,t-1}) \\ &= 0 - Cov(y_{i,t-1}, \varepsilon_{i,t-1}) - 0 + 0 \quad \text{as } Cov(y_{is}, \varepsilon_{it}) = 0 \text{ for } s < t \\ &= -Cov(\gamma y_{i,t-2} + x'_{i,t-1} \beta + \alpha_i + \varepsilon_{i,t-1}, \varepsilon_{i,t-1}) \\ &= -\gamma \underbrace{Cov(y_{i,t-2}, \varepsilon_{i,t-1})}_0 - \beta' \underbrace{Cov(x_{i,t-1}, \varepsilon_{i,t-1})}_0 - \underbrace{Cov(\alpha_i, \varepsilon_{i,t-1})}_0 - Var(\varepsilon_{i,t-1}) \\ &< 0 \quad \text{assume } Var(\varepsilon_{i,t-1}) > 0 \end{aligned}$$

The necessary condition for FD estimator to be unbiased is $\mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1} | y_{i,t-1} - y_{i,t-2}, x_{it} - x_{i,t-1}) = 0$. As $\mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1} | y_{i,t-1} - y_{i,t-2}, x_{it} - x_{i,t-1}) = 0 \implies Cov(y_{i,t-1} - y_{i,t-2}, \varepsilon_{it} - \varepsilon_{i,t-1}) = 0$. As a result, $Cov(y_{i,t-1} - y_{i,t-2}, \varepsilon_{it} - \varepsilon_{i,t-1}) \neq 0 \implies \mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1} | y_{i,t-1} - y_{i,t-2}, x_{it} - x_{i,t-1}) \neq 0$. Thus, FD estimator is biased.

6.2.6 Special case: no x_{it}

$$y_{it} - y_{i,t-1} = \gamma(y_{i,t-1} - y_{i,t-2}) + (\varepsilon_{it} - \varepsilon_{i,t-1})$$

The necessary condition for FD estimator to be consistent is $\mathbb{E}((y_{i,t-1} - y_{i,t-2})(\varepsilon_{it} - \varepsilon_{i,t-1})) = 0$. However,

$$\begin{aligned} \mathbb{E}((y_{i,t-1} - y_{i,t-2})(\varepsilon_{it} - \varepsilon_{i,t-1})) &= \mathbb{E}(y_{i,t-1} \varepsilon_{it}) - \mathbb{E}(y_{i,t-1} \varepsilon_{i,t-1}) - \mathbb{E}(y_{i,t-2} \varepsilon_{it}) + \mathbb{E}(y_{i,t-2} \varepsilon_{i,t-1}) \\ &= 0 - \mathbb{E}(y_{i,t-1} \varepsilon_{i,t-1}) - 0 + 0 \quad \text{as } \mathbb{E}(y_{is} \varepsilon_{it}) = 0 \text{ for } s < t \\ &= -\mathbb{E}((\gamma y_{i,t-2} + \alpha_i + \varepsilon_{i,t-1}) \varepsilon_{i,t-1}) \\ &= -\gamma \underbrace{\mathbb{E}(y_{i,t-2} \varepsilon_{i,t-1})}_0 - \underbrace{\mathbb{E}(\alpha_i \varepsilon_{i,t-1})}_0 - \mathbb{E}(\varepsilon_{i,t-1}^2) \\ &= -Var(\varepsilon_{i,t-1}) \quad \text{as } \mathbb{E}(\varepsilon_{i,t-1}) = 0 \\ &< 0 \quad \text{assume } Var(\varepsilon_{i,t-1}) > 0 \end{aligned}$$

Thus, FD estimator is inconsistent. The necessary condition for FD estimator to be unbiased is $\mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1} | y_{i,t-1} - y_{i,t-2}) = 0$. As $\mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1} | y_{i,t-1} - y_{i,t-2}) = 0 \implies \mathbb{E}((y_{i,t-1} - y_{i,t-2})(\varepsilon_{it} - \varepsilon_{i,t-1})) = 0$, $\mathbb{E}((y_{i,t-1} - y_{i,t-2})(\varepsilon_{it} - \varepsilon_{i,t-1})) \neq 0 \implies \mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1} | y_{i,t-1} - y_{i,t-2}) \neq 0$. Thus, FD estimator is biased.

Thus, IV estimation (for just-identified case) or 2SLS estimation (for over-identified case) is applied. IV and 2SLS estimations are special cases of GMM estimation.

Under weak/sequential exogeneity, $Cov(y_{is}, \varepsilon_{it}) = 0$ for $s < t$. This implies for $s < t - 1 \iff s \leq t - 2$

$$\begin{aligned} Cov(y_{is}, \varepsilon_{it} - \varepsilon_{i,t-1}) &= Cov(y_{is}, \varepsilon_{it}) - Cov(y_{is}, \varepsilon_{i,t-1}) \\ &= 0 - Cov(y_{is}, \varepsilon_{i,t-1}) \quad \text{as } s < t - 1 \implies s < t \\ &= 0 \quad \text{as } s < t - 1 \end{aligned}$$

Note that $Cov(y_{is}, \varepsilon_{it} - \varepsilon_{i,t-1}) = 0 \implies \mathbb{E}(y_{is}(\varepsilon_{it} - \varepsilon_{i,t-1})) = 0$ as $\mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1}) = 0$ under weak/sequential exogeneity. So, y_{is} for $s \leq t-2$ satisfy the exclusion restriction (exogeneity) requirement of valid instrument. i.e.,

$$\tilde{z}'_{i3} = (y_{i1}, \Delta \mathbf{x}'_{i3}) \quad \text{at } t = 3$$

$$\tilde{z}'_{i4} = (y_{i1}, y_{i2}, \Delta \mathbf{x}'_{i4}) \quad \text{at } t = 4$$

...

$$\tilde{z}'_{iT} = (y_{i1}, \dots, y_{i,T-2}, \Delta \mathbf{x}'_{iT}) \quad \text{at } t = T$$

$$\text{That is, } \tilde{z}'_{it} = [y_{i1}, \dots, y_{i,t-2}, \Delta \mathbf{x}'_{it}]. \quad \mathbf{Z}_i = \begin{pmatrix} \tilde{z}'_{i3} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \tilde{z}'_{i4} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \tilde{z}'_{iT} \end{pmatrix}$$

So, we have

$$\begin{aligned} \mathbb{E}(\tilde{z}_{it}(\varepsilon_{it} - \varepsilon_{i,t-1})) &= \mathbf{0} \\ \iff \mathbb{E}(\mathbf{Z}'_i \Delta \varepsilon_i) &= \mathbf{0} \end{aligned}$$

We can then apply 2SLS estimation in GMM framework. This is the same as Arellano-Bond estimator with 2SLS weight.

6.2.7 Anderson-Hsiao Estimator

Anderson & Hsiao (1981) considers a special case y_{is} for $s = t-2$ i.e., $y_{i,t-2}$ as the instrument since they not only satisfy the exclusion restriction (exogeneity) requirement but also satisfy the relevancy requirement of valid instrument i.e., correlates with $y_{i,t-1} - y_{i,t-2}$. Thus, $\tilde{z}'_{it} = [y_{i,t-2}, \Delta \mathbf{x}'_{it}]$

$$\mathbf{Z}_i = \begin{pmatrix} (y_{i1} \quad \Delta \mathbf{x}'_{i3}) & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & (y_{i2} \quad \Delta \mathbf{x}'_{i4}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & (y_{i,T-2} \quad \Delta \mathbf{x}'_{iT}) \end{pmatrix}$$

and

$$\tilde{z}'_{it} = [\underbrace{\Delta y_{i,t-2}}_{y_{i,t-2} - y_{i,t-3}}, \Delta \mathbf{x}'_{it}]$$

$$\mathbf{Z}_i = \begin{pmatrix} (\Delta y_{i2} \quad \Delta \mathbf{x}'_{i4}) & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & (\Delta y_{i3} \quad \Delta \mathbf{x}'_{i5}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & (\Delta y_{i,T-2} \quad \Delta \mathbf{x}'_{iT}) \end{pmatrix}$$

As only one instrument is used at each t , the number of moments is equal to the number of parameters i.e., $r = K$. In such case, GMM estimation = MM estimation = IV estimation.

$$\hat{\delta}_{AH}^{pgmm} = [\sum_{i=1}^N \mathbf{Z}'_i \tilde{\mathbf{X}}_i]^{-1} \sum_{i=1}^N \mathbf{Z}'_i \tilde{\mathbf{y}}_i = \hat{\delta}_{AH}^{piv}$$

6.2.8 Arellano-Bond Estimator

Arellano & Bond (1991) considers all the possible cases i.e., y_{is} for $s \leq t-2$. Except $t = 3$, more than one instruments are used, number of moments is larger than the number of parameters i.e., $r > K$. GMM estimation is 2SLS estimation if $\mathbf{W}_N = (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{Z}_i)^{-1}$.

$$\tilde{z}'_{it} = [y_{i1}, \dots, y_{i,t-2}, \Delta \mathbf{x}'_{it}]$$

$$\mathbf{Z}_i = \begin{pmatrix} (y_{i1} \quad \Delta \mathbf{x}'_{i3}) & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & (y_{i1} \quad y_{i2} \quad \Delta \mathbf{x}'_{i4}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & (y_{i1} \quad \dots \quad y_{i,T-2} \quad \Delta \mathbf{x}'_{iT}) \end{pmatrix}$$

$$\hat{\boldsymbol{\delta}}_{AB}^{pgmm} = [(\sum_{i=1}^N \tilde{\mathbf{X}}_i' \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}_i' \tilde{\mathbf{X}}_i)]^{-1} (\sum_{i=1}^N \tilde{\mathbf{X}}_i' \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}_i' \tilde{\mathbf{y}}_i)$$

If $\mathbf{W}_N = (\sum_{i=1}^N \mathbf{Z}_i' \mathbf{Z}_i)^{-1}$, $\hat{\boldsymbol{\delta}}_{AB}^{pgmm} = \hat{\boldsymbol{\delta}}_{AB}^{2SLS}$

If $\mathbf{W}_N = \hat{\mathbf{S}}^{-1}$, $\hat{\boldsymbol{\delta}}_{AB}^{pgmm} = \hat{\boldsymbol{\delta}}_{AB}^{opgmm}$

7 Pooled Model and Clustered Standard Error

$$y_{it} = \mathbf{x}'_{it}\boldsymbol{\beta} + \alpha_i + \varepsilon_{it} \quad \text{Level 1}$$

$$\begin{aligned} \begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} &= \begin{pmatrix} \mathbf{x}'_{i1} \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \alpha_i + \begin{pmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{pmatrix} \\ \mathbf{y}_i &= \mathbf{X}_i \boldsymbol{\beta} + \underbrace{(\mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i)}_{\mathbf{u}_i} \end{aligned} \quad \text{Level 2}$$

$$\begin{aligned} \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{pmatrix} &= \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_N \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_1 \\ \vdots \\ \boldsymbol{\varepsilon}_N \end{pmatrix} \\ \mathbf{y} &= \mathbf{X} \boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{e}) \boldsymbol{\alpha} + \boldsymbol{\varepsilon} \end{aligned} \quad \text{Level 3}$$

where α_i is unobserved heterogeneity, $\boldsymbol{\varepsilon}_i$ is idiosyncratic error, \mathbf{u}_i is composite error.

If $\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0$, OLS estimator is likely to be unbiased and consistent.

$\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0 \implies \mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) = \mathbf{0}$ as $\mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) = \mathbb{E}(\mathbf{X}'_i \mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \mathbf{X}'_i \mathbf{e} \mathbb{E}(\alpha_i | \mathbf{X}_i) + \mathbb{E}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \mathbf{0}$. Thus, the necessary condition for OLS estimator to be unbiased is satisfied if $\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0$

$\mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) = \mathbf{0} \implies \mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) = \mathbf{0}$ as $\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) = \mathbb{E}(\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i | \mathbf{X}_i)) = \mathbb{E}(\mathbf{X}'_i \mathbb{E}(\mathbf{u}_i | \mathbf{X}_i)) = \mathbb{E}(\mathbf{X}'_i \mathbf{0}) = \mathbf{0}$. Thus, the necessary condition for OLS estimator to be consistent is satisfied if $\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0$.

$\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0 \implies \mathbb{E}(\alpha_i \mathbf{X}_i) = \mathbf{0}$ and $\mathbb{E}(\alpha_i) = 0$. Thus, $\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0 \implies \text{Cov}(\alpha_i, \mathbf{X}_i) = \mathbf{0}$ as $\text{Cov}(\alpha_i, \mathbf{X}_i) = \mathbb{E}(\alpha_i \mathbf{X}_i) - \mathbb{E}(\alpha_i) \mathbb{E}(\mathbf{X}_i) = \mathbf{0}$.

$$\hat{\boldsymbol{\beta}}_{pooled}^{ols} = \left[\sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}'_i \mathbf{y}_i \quad \text{Level 2}$$

$$= \left[\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}'_{it} \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} y_{it} \quad \text{Level 1}$$

As pooled model does not perform any transformation before OLS estimation, so $\mathbf{Q} = \mathbf{I}$.

$$\begin{aligned} \text{Var}(\hat{\boldsymbol{\beta}}_{pooled}^{ols} | \mathbf{X}_i) &= \left[\sum_{i=1}^N \mathbf{X}'_i \mathbf{I}' \mathbf{I} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}'_i \mathbf{I}' \text{Var}(\mathbf{I} \boldsymbol{\varepsilon}_i | \mathbf{X}_i) \mathbf{I} \mathbf{X}_i \left[\sum_{i=1}^N \mathbf{X}'_i \mathbf{I}' \mathbf{I} \mathbf{X}_i \right]^{-1} \\ &= \left[\sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}'_i \text{Var}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) \mathbf{X}_i \left[\sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1} \end{aligned}$$

If ε_{it} is homoskedasticity and serially uncorrelated across t i.e., $\text{Var}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \sigma_\varepsilon^2 \mathbf{I}_T$ (further assume independence of i and strict exogeneity), we have $\boldsymbol{\varepsilon}_i | \mathbf{X}_i \sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T]$

$$\begin{aligned} &= \sigma_\varepsilon^2 \left[\sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1} \\ &= \sigma_\varepsilon^2 \left[\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}'_{it} \right]^{-1} \end{aligned}$$

If $\text{Var}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \boldsymbol{\Omega}_i$, we have $\boldsymbol{\varepsilon}_i | \mathbf{X}_i \sim inid [\mathbf{0}, \boldsymbol{\Omega}_i]$

$$\begin{aligned} &= \left[\sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}'_i \overbrace{\mathbb{E}[\boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}'_i | \mathbf{X}_i]}^{\boldsymbol{\Omega}_i} \mathbf{X}_i \left[\sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1} \\ &= \left[\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}'_{it} \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbf{x}_{it} \mathbb{E}[\varepsilon_{it} \varepsilon_{is} | \mathbf{X}_i] \mathbf{x}'_{is} \left[\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}'_{it} \right]^{-1} \end{aligned}$$

7.1 Petersen (2009) Simulation Result

7.1.1 Only individual fixed effect

$$y_{it} = \mathbf{x}'_{it}\boldsymbol{\beta} + \alpha_i + \varepsilon_{it}$$

If there is only α_i (individual fixed effect) and α_i is not correlated with \mathbf{x}_{it} (so no OVB), OLS estimator is unbiased and clustered standard error clustered by individual is unbiased. In contrast, conventional standard error, White standard error, Newey-West standard error, Fama-Macbeth standard error are biased downward (over-rejection).

7.1.2 Only time fixed effect

$$y_{it} = \mathbf{x}'_{it}\boldsymbol{\beta} + \gamma_t + \varepsilon_{it}$$

If there is only γ_t (time fixed effect) and γ_t is not correlated with \mathbf{x}_{it} (so no OVB), OLS estimator is unbiased and Fama-Macbeth standard error and clustered standard error clustered by time (only if T is large) is unbiased. In contrast, conventional standard error are biased downward (over-rejection).

7.1.3 Both individual and time fixed effect

$$y_{it} = \mathbf{x}'_{it}\boldsymbol{\beta} + \alpha_i + \gamma_t + \varepsilon_{it}$$

Clustered standard error clustered by both individual and time is unbiased. Cameron, Gelbach, & Miller (2011), and Thompson (2011) suggests clustered standard error clustered by both individual and time = Clustered standard error clustered by individual + Clustered standard error clustered by time - White standard error. Simulation result seems work.

7.2 Clustering Problem

7.2.1 Block bootstrapping

Suggested by MacKinnon, Nielsen, & Webb (2022).

7.2.2 Clustered standard error with independence of i (Liang & Zeger, 1986)

Clustered standard error can handle both heteroscedasticity and serial correlation within a cluster / group.

$$\widehat{Var}(\hat{\boldsymbol{\beta}}_{pooled}^{ols} | \mathbf{X}_i) = \left[\sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}'_i \hat{\boldsymbol{\Omega}}_i \mathbf{X}_i \left[\sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1}$$

$$\hat{\boldsymbol{\Omega}}_i = \hat{\boldsymbol{\varepsilon}}_i \hat{\boldsymbol{\varepsilon}}'_i = \begin{pmatrix} \hat{\varepsilon}_{i1}^2 & \hat{\varepsilon}_{i1}\hat{\varepsilon}_{i2} & \cdots & \hat{\varepsilon}_{i1}\hat{\varepsilon}_{iT} \\ \vdots & \hat{\varepsilon}_{i2}^2 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\varepsilon}_{iT}\hat{\varepsilon}_{i1} & \cdots & \hat{\varepsilon}_{iT}\hat{\varepsilon}_{i,T-1} & \hat{\varepsilon}_{iT}^2 \end{pmatrix}$$

$$\hat{\boldsymbol{\Omega}} = \begin{pmatrix} \hat{\boldsymbol{\Omega}}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \hat{\boldsymbol{\Omega}}_2 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \hat{\boldsymbol{\Omega}}_N \end{pmatrix}$$

$$\widehat{Var}(\hat{\boldsymbol{\beta}}_{pooled}^{ols} | \mathbf{X}_i) = \left[\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}'_{it} \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbf{x}_{it} \hat{\varepsilon}_{it} \hat{\varepsilon}_{is} \mathbf{x}'_{is} \left[\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}'_{it} \right]^{-1}$$

Finite sample adjustment can also be added, $\frac{N}{N-1} \frac{T-1}{T-K}$ is multiplied in Stata. It is the HC1 generalization of MacKinnon & White (1985). If N (the number of cluster) is small, it is biased because law of large number cannot be applied (even $T \rightarrow \infty$). However, we can adjust it by Bell & McCaffrey (2002)'s Bias-Reduced Linearization (BRL) adjustment (analogous to HC2 in cross-sectional case), and using t-distribution with $N-K$ degree of freedom, instead of standard normal distribution.

In BRL adjustment, we replace $\hat{\boldsymbol{\varepsilon}}_i$ by

$$\tilde{\boldsymbol{\varepsilon}}_i = \mathbf{A}_i \hat{\boldsymbol{\varepsilon}}_i$$

where $\mathbf{A}'_i \mathbf{A}_i = (\mathbf{I}_T - \mathbf{H}_i)^{-1}$ where $\mathbf{H}_i = \mathbf{X}_i(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'_i$ the projection/hat matrix.

There are many possible \mathbf{A}_i , Bell & McCaffrey (2002) uses eigen-decomposition of the inverse of the residual marker $\mathbf{I}_T - \mathbf{H}_i$ i.e.,

$$\begin{aligned} (\mathbf{I}_T - \mathbf{H}_i)^{-1} &= \mathbf{P}\mathbf{\Lambda}\mathbf{P}' \\ &= \mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}\mathbf{P}' \\ &= \mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2'}\mathbf{P}' \\ &= \mathbf{P}\mathbf{\Lambda}^{1/2}(\mathbf{P}\mathbf{\Lambda}^{1/2})' \\ &= \mathbf{A}'\mathbf{A}'' \end{aligned}$$

where \mathbf{P} is a matrix in which vectors are eigenvectors; $\mathbf{\Lambda}$ is a diagonal matrix with eigenvalues items.

A weight can also be added, this generalizes the Newey-West estimator.

$$\widehat{Var}(\hat{\beta}_{pooled}^{ols}|\mathbf{X}_i) = [\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it}\mathbf{x}'_{it}]^{-1} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbf{x}_{it}w_{t,s}\hat{\varepsilon}_{it}\hat{\varepsilon}_{is}\mathbf{x}'_{is}[\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it}\mathbf{x}'_{it}]^{-1}$$

where

$$w_{t,s} = \begin{cases} 1 - \frac{|s-t|}{L+1} & \text{if } |s-t| \leq L \\ 0 & \text{otherwise} \end{cases}$$

Petersen (2009) finds that this adjustment is worse than the one without weight.

7.2.3 Clustered standard error with dependence of i (Driscoll & Kraay, 1998)

$$\widehat{Var}(\hat{\beta}_{pooled}^{ols}|\mathbf{X}_i) = [\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it}\mathbf{x}'_{it}]^{-1} \sum_{i=1}^N \sum_{j=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbf{x}_{it}w_{t,s}\hat{\varepsilon}_{it}\hat{\varepsilon}_{js}\mathbf{x}'_{js}[\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it}\mathbf{x}'_{it}]^{-1}$$

where

$$w_{t,s} = \begin{cases} 1 - \frac{|s-t|}{L+1} & \text{if } |s-t| \leq L \\ 0 & \text{otherwise} \end{cases}$$

It requires large T . L is up to you.

8 References

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9 Appendix - R Code

```
library(lmtest)
library(plm)
library(fastDummies)
library(tidyverse)

set.seed(15)

gen_long_data <- function (N, T, var_gamma_i) {
  df <- tibble(.rows = N) %>% mutate(i = seq_len(N))
  df["gamma_i"] <- rnorm(N, mean = 0, sd = sqrt(var_gamma_i))

  for (t in seq_len(T)) {
    df[str_c("x_i", t)] <-
      df[["gamma_i"]] + rnorm(N, mean = 0, sd = sqrt(1 - var_gamma_i))

    df[str_c("y_i", t)] <-
      2 + df[[str_c("x_i", t)]] * 2 + df[["gamma_i"]] + rnorm(N, mean = 0, sd = sqrt(4 - var_gamma_i))
  }

  df %>%
    select(-gamma_i) %>%
    pivot_longer(
      cols = starts_with(c("y_i", "x_i")),
      names_to = c(".value", "t"),
      names_sep = "_i"
    )
}

long_data <-
  gen_long_data(100, 5, 0.1) %>%
  mutate(t = as.numeric(t)) %>%
  dummy_cols(select_columns = "i")

long_data_plm <- long_data %>% pdata.frame(index = c("i", "t"))

# within / FE estimator
plm(y ~ x, model = "within", effect = "individual", data = long_data_plm) %>%
  coeftest(., vcov = plm::vcovHC(., type = "HC1", cluster = "group"))

# first difference estimator
plm(y ~ x, model = "fd", effect = "individual", data = long_data_plm) %>%
  coeftest(., vcov = plm::vcovHC(., type = "HC1", cluster = "group"))

# LSDV estimator
plm(y ~ . + 0, model = "pooling", data = long_data_plm %>% select(-i, -t)) %>%
  coeftest(., vcov = plm::vcovHC(., type = "HC1", cluster = "group"))

# GLS / RE estimator
plm(y ~ x, model = "random", effect = "individual", data = long_data_plm) %>% coeftest()

# Pooled OLS with clustered standard error
plm(y ~ x, model = "pooling", data = long_data_plm) %>%
  coeftest(., vcov = plm::vcovHC(., type = "HC1", cluster = "group"))

# Pooled OLS with BRL adjusted clustered standard error
plm(y ~ x, model = "pooling", data = long_data_plm) %>%
  coeftest(., vcov = plm::vcovHC(., type = "HC2", cluster = "group"))
```