

# Notes on Linear Panel Model

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## 1 Fixed Effect Model

$$y_{it} = \mathbf{x}'_{it}\boldsymbol{\beta} + \alpha_i + \varepsilon_{it} \quad \text{Level 1}$$

$$\begin{aligned} \begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} &= \begin{pmatrix} \mathbf{x}'_{i1} \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \alpha_i + \begin{pmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{pmatrix} \\ \mathbf{y}_i &= \mathbf{X}_i \boldsymbol{\beta} + \underbrace{(\mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i)}_{\mathbf{u}_i} \end{aligned} \quad \text{Level 2}$$

$$\begin{aligned} \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{pmatrix} &= \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_N \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_1 \\ \vdots \\ \boldsymbol{\varepsilon}_N \end{pmatrix} \\ \mathbf{y} &= \mathbf{X} \boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{e}) \boldsymbol{\alpha} + \boldsymbol{\varepsilon} \end{aligned} \quad \text{Level 3}$$

where  $\alpha_i$  is unobserved heterogeneity,  $\boldsymbol{\varepsilon}_i$  is idiosyncratic error,  $\mathbf{u}_i$  is composite error.

### 1.1 Assumption

#### 1.1.1 Strong/strict exogeneity of regressors

For all  $t$ ,

$$\mathbb{E}(\varepsilon_{it} | \mathbf{x}_{i1}, \dots, \mathbf{x}_{iT}) = 0$$

Equivalently,

$$\mathbb{E}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \mathbf{0}$$

### 1.2 OLS estimator is inconsistent and biased

The necessary condition for OLS estimator to be consistent is  $\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) = \mathbf{0}$ .

$$\begin{aligned} \mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) &= \mathbb{E}(\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i | \mathbf{X}_i)) \\ &= \mathbb{E}(\mathbf{X}'_i \mathbb{E}(\mathbf{u}_i | \mathbf{X}_i)) \\ &= \mathbb{E}(\mathbf{X}'_i \mathbb{E}(\mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i | \mathbf{X}_i)) \\ &= \mathbb{E}(\mathbf{X}'_i \mathbb{E}(\mathbf{e} \alpha_i | \mathbf{X}_i) + \underbrace{\mathbf{X}'_i \mathbb{E}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i)}_{\mathbf{0}}) && \text{because of strict exogeneity} \\ &= \mathbb{E}(\mathbf{X}'_i \mathbf{e} \mathbb{E}(\alpha_i | \mathbf{X}_i)) \\ &= \mathbb{E}(\mathbf{X}'_i \mathbf{e} \alpha_i) \end{aligned}$$

$\mathbb{E}(\mathbf{X}'_i \mathbf{e} \alpha_i) \neq 0 \iff \mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) \neq \mathbf{0}$ . Thus, OLS estimator is inconsistent if  $\mathbb{E}(\mathbf{X}'_i \mathbf{e} \alpha_i) \neq \mathbf{0}$ .

The necessary condition for OLS estimator to be unbiased is  $\mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) = \mathbf{0}$ . However,  $\mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) = \mathbf{0} \implies \mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) = \mathbf{0}$  as  $\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) = \mathbb{E}(\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i | \mathbf{X}_i)) = \mathbb{E}(\mathbf{X}'_i \mathbb{E}(\mathbf{u}_i | \mathbf{X}_i)) = \mathbb{E}(\mathbf{X}'_i \mathbf{0}) = \mathbf{0}$ . Thus,  $\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) \neq \mathbf{0} \implies \mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) \neq \mathbf{0}$ . As a result, OLS estimator is biased if  $\mathbb{E}(\mathbf{X}'_i \mathbf{e} \alpha_i) \neq \mathbf{0}$ .

$\mathbb{E}(\alpha_i|\mathbf{X}_i) = 0 \implies \mathbb{E}(\mathbf{X}_i' e \alpha_i) = \mathbf{0}$  as  $\mathbb{E}(\mathbf{X}_i' e \alpha_i) = \mathbb{E}(\mathbb{E}(\mathbf{X}_i' e \alpha_i | \mathbf{X}_i)) = \mathbb{E}(\mathbf{X}_i' e \mathbb{E}(\alpha_i | \mathbf{X}_i)) = \mathbb{E}(\mathbf{X}_i' e \mathbf{0}) = \mathbf{0}$ . Thus,  $\mathbb{E}(\mathbf{X}_i' e \alpha_i) \neq \mathbf{0} \implies \mathbb{E}(\alpha_i | \mathbf{X}_i) \neq 0$

OLS estimator of  $\beta$  is inconsistent and biased if  $\alpha_i$  is correlated with  $\mathbf{X}_i$  ( $u_i$  is also correlated with  $\mathbf{X}_i$ ). This is called omitted variable bias. To tackle this, we simply eliminate  $\alpha_i$  by using different methods.

### 1.3 Fixed Effect Estimator / Within Estimator

#### 1.3.1 Demean operator

$$\mathbf{Q} = \mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}'$$

This  $\mathbf{Q}$  is symmetric and idempotent,

$$\begin{aligned} \mathbf{Q}' &= (\mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}')' \\ &= \mathbf{I}_T' - T^{-1} \mathbf{e}'' \mathbf{e}' \\ &= \mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}' = \mathbf{Q} \end{aligned}$$

$$\begin{aligned} \mathbf{Q} \mathbf{Q}' &= \mathbf{Q} \mathbf{Q} \\ &= (\mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}')(\mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}') \\ &= \mathbf{I}_T \mathbf{I}_T - \mathbf{I}_T T^{-1} \mathbf{e} \mathbf{e}' - T^{-1} \mathbf{e} \mathbf{e}' \mathbf{I}_T + T^{-1} \mathbf{e} \mathbf{e}' T^{-1} \mathbf{e} \mathbf{e}' \\ &= \mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}' - T^{-1} \mathbf{e} \mathbf{e}' + T^{-2} \mathbf{e} \mathbf{e}' \mathbf{e} \mathbf{e}' \\ &= \mathbf{I}_T - 2T^{-1} \mathbf{e} \mathbf{e}' + T^{-2} \mathbf{e} T \mathbf{e}' \\ &= \mathbf{I}_T - 2T^{-1} \mathbf{e} \mathbf{e}' + T^{-1} \mathbf{e} \mathbf{e}' \\ &= \mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}' = \mathbf{Q} \end{aligned}$$

#### 1.3.2 Demean transformed model

$$\begin{aligned} \mathbf{Q} \mathbf{y}_i &= \mathbf{Q}(\mathbf{X}_i \beta + \mathbf{e} \alpha_i + \varepsilon_i) \\ &= \mathbf{Q} \mathbf{X}_i \beta + \mathbf{Q} \mathbf{e} \alpha_i + \mathbf{Q} \varepsilon_i \\ &= \mathbf{Q} \mathbf{X}_i \beta + \mathbf{0} \alpha_i + \mathbf{Q} \varepsilon_i \\ &= \mathbf{Q} \mathbf{X}_i \beta + \mathbf{Q} \varepsilon_i \end{aligned}$$

Level 2

It is because

$$\begin{aligned} \mathbf{Q} \mathbf{e} &= (\mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}') \mathbf{e} \\ &= \mathbf{I}_T \mathbf{e} - T^{-1} \mathbf{e} \mathbf{e}' \mathbf{e} \\ &= \mathbf{e} - T^{-1} \mathbf{e} T \\ &= \mathbf{e} - \mathbf{e} = \mathbf{0} \end{aligned}$$

It can be written as  $\mathbf{y}_i - \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}_i') \beta + (\varepsilon_i - \mathbf{e} \bar{\varepsilon}_i)$  because

$$\begin{aligned} \mathbf{Q} \mathbf{X}_i &= (\mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}') \mathbf{X}_i \\ &= \mathbf{I}_T \mathbf{X}_i - T^{-1} \mathbf{e} \mathbf{e}' \mathbf{X}_i \\ &= \mathbf{X}_i - T^{-1} \mathbf{e} \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x}'_{i1} \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} \\ &= \mathbf{X}_i - \mathbf{e} T^{-1} \sum_{t=1}^T \mathbf{x}'_{it} \\ &= \mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}_i' \end{aligned}$$

$$\begin{aligned}
Qy_i &= (I_T - T^{-1}ee')y_i \\
&= I_T y_i - T^{-1}ee'y_i \\
&= y_i - T^{-1}e(1 \quad \cdots \quad 1) \begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} \\
&= y_i - eT^{-1} \sum_{t=1}^T y_{it} \\
&= y_i - e\bar{y}_i
\end{aligned}$$

$y_i - e\bar{y}_i = (X_i - e\bar{x}'_i)\beta + (\varepsilon_i - e\bar{\varepsilon}_i)$  can be written as

$$\begin{aligned}
\begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{y}_i &= \left( \begin{pmatrix} x'_{i1} \\ \vdots \\ x'_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{x}'_i \right) \beta + \left( \begin{pmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{\varepsilon}_i \right) \\
\begin{pmatrix} y_{i1} - \bar{y}_i \\ \vdots \\ y_{iT} - \bar{y}_i \end{pmatrix} &= \begin{pmatrix} x'_{i1} - \bar{x}'_i \\ \vdots \\ x'_{iT} - \bar{x}'_i \end{pmatrix} \beta + \begin{pmatrix} \varepsilon_{i1} - \bar{\varepsilon}_i \\ \vdots \\ \varepsilon_{iT} - \bar{\varepsilon}_i \end{pmatrix} \\
\begin{pmatrix} y_{i1} - \bar{y}_i \\ \vdots \\ y_{iT} - \bar{y}_i \end{pmatrix} &= \begin{pmatrix} (x_{i1} - \bar{x}_i)' \\ \vdots \\ (x_{iT} - \bar{x}_i)' \end{pmatrix} \beta + \begin{pmatrix} \varepsilon_{i1} - \bar{\varepsilon}_i \\ \vdots \\ \varepsilon_{iT} - \bar{\varepsilon}_i \end{pmatrix} \\
y_{it} - \bar{y}_i &= (x_{it} - \bar{x}_i)' \beta + (\varepsilon_{it} - \bar{\varepsilon}_i)
\end{aligned}$$

Level 1

### 1.3.3 OLS estimator of the demean transformed model / Fixed Effect (FE) estimator

$$\begin{aligned}
\hat{\beta}_{within}^{ols} &= \left[ \sum_{i=1}^N (QX_i)' QX_i \right]^{-1} \sum_{i=1}^N (QX_i)' Qy_i \\
&= \left[ \sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_i)' \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i)
\end{aligned}$$

Level 2

Level 1

It is because

$$\begin{aligned}
(QX_i)' QX_i &= (X_i - e\bar{x}'_i)' (X_i - e\bar{x}'_i) \\
&= \left( \begin{pmatrix} x'_{i1} \\ \vdots \\ x'_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{x}'_i \right) \left( \begin{pmatrix} x'_{i1} \\ \vdots \\ x'_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{x}'_i \right) \\
&= \begin{pmatrix} x'_{i1} - \bar{x}'_i \\ \vdots \\ x'_{iT} - \bar{x}'_i \end{pmatrix}' \begin{pmatrix} x'_{i1} - \bar{x}'_i \\ \vdots \\ x'_{iT} - \bar{x}'_i \end{pmatrix} \\
&= \begin{pmatrix} (x_{i1} - \bar{x}_i)' \\ \vdots \\ (x_{iT} - \bar{x}_i)' \end{pmatrix}' \begin{pmatrix} (x_{i1} - \bar{x}_i)' \\ \vdots \\ (x_{iT} - \bar{x}_i)' \end{pmatrix} \\
&= ((x_{i1} - \bar{x}_i) \quad \cdots \quad (x_{iT} - \bar{x}_i)) \begin{pmatrix} (x_{i1} - \bar{x}_i)' \\ \vdots \\ (x_{iT} - \bar{x}_i)' \end{pmatrix} \\
&= \sum_{t=1}^T (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_i)'
\end{aligned}$$

$$\begin{aligned}
(QX_i)'Qy_i &= (X_i - e\bar{x}_i)'(y_i - e\bar{y}_i) \\
&= \left( \begin{pmatrix} x'_{i1} \\ \vdots \\ x'_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{x}_i' \right)' \left( \begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{y}_i \right) \\
&= \begin{pmatrix} x'_{i1} - \bar{x}_i' \\ \vdots \\ x'_{iT} - \bar{x}_i' \end{pmatrix}' \begin{pmatrix} y_{i1} - \bar{y}_i \\ \vdots \\ y_{iT} - \bar{y}_i \end{pmatrix} \\
&= ((x_{i1} - \bar{x}_i) \quad \cdots \quad (x_{iT} - \bar{x}_i)) \begin{pmatrix} y_{i1} - \bar{y}_i \\ \vdots \\ y_{iT} - \bar{y}_i \end{pmatrix} \\
&= \sum_{t=1}^T (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i)
\end{aligned}$$

$$\begin{aligned}
\hat{\beta}_{within}^{ols} &= \left[ \sum_{i=1}^N (QX_i)'QX_i \right]^{-1} \sum_{i=1}^N (QX_i)'Qy_i && \text{Level 2} \\
&= \left[ \sum_{i=1}^N (X_i'X_i - \bar{x}_i T \bar{x}_i') \right]^{-1} \sum_{i=1}^N (X_i'y_i - \bar{x}_i T \bar{y}_i) \\
&= \left[ \sum_{i=1}^N X_i'X_i - T \sum_{i=1}^N \bar{x}_i \bar{x}_i' \right]^{-1} \left( \sum_{i=1}^N X_i'y_i - T \sum_{i=1}^N \bar{x}_i \bar{y}_i \right) \\
&= \left[ (X_1' \quad \cdots \quad X_N') \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} - T (\bar{x}_1 \quad \cdots \quad \bar{x}_N) \begin{pmatrix} \bar{x}_1' \\ \vdots \\ \bar{x}_N' \end{pmatrix} \right]^{-1} \left( (X_1' \quad \cdots \quad X_N') \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} - T (\bar{x}_1 \quad \cdots \quad \bar{x}_N) \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_N \end{pmatrix} \right) \\
&= [X'X - T\bar{X}'\bar{X}]^{-1} (X'y - T\bar{X}'\bar{y}) && \text{Level 3}
\end{aligned}$$

It is because

$$\begin{aligned}
(QX_i)'QX_i &= (X_i - e\bar{x}_i)'(X_i - e\bar{x}_i) \\
&= (X_i' - \bar{x}_i' e')(X_i - e\bar{x}_i) \\
&= X_i'X_i - X_i'e\bar{x}_i' - \bar{x}_i e'X_i + \bar{x}_i e' e \bar{x}_i' \\
&= X_i'X_i - X_i'e\bar{x}_i' - \bar{x}_i e'X_i + \bar{x}_i T \bar{x}_i' \\
&= X_i'X_i - (e'X_i)' \bar{x}_i' - \bar{x}_i e'X_i + \bar{x}_i T \bar{x}_i' \\
&= X_i'X_i - \left( \sum_{t=1}^T x'_{it} \right)' \bar{x}_i' - \bar{x}_i \sum_{t=1}^T x'_{it} + \bar{x}_i T \bar{x}_i' \\
&= X_i'X_i - \left( \sum_{t=1}^T x_{it}/T \right) T \bar{x}_i' - \bar{x}_i T \sum_{t=1}^T x'_{it}/T + \bar{x}_i T \bar{x}_i' \\
&= X_i'X_i - \bar{x}_i T \bar{x}_i' - \bar{x}_i T \bar{x}_i' + \bar{x}_i T \bar{x}_i' \\
&= X_i'X_i - \bar{x}_i T \bar{x}_i' \\
\\
(QX_i)'Qy_i &= (X_i - e\bar{x}_i)'(y_i - e\bar{y}_i) \\
&= X_i'y_i - X_i'e\bar{y}_i - \bar{x}_i e'y_i + \bar{x}_i e' e \bar{y}_i \\
&= X_i'y_i - (e'X_i)' \bar{y}_i - \bar{x}_i e'y_i + \bar{x}_i T \bar{y}_i \\
&= X_i'y_i - \left( \sum_{t=1}^T x'_{it} \right)' \bar{y}_i - \bar{x}_i \sum_{t=1}^T y_{it} + \bar{x}_i T \bar{y}_i \\
&= X_i'y_i - \left( \sum_{t=1}^T x_{it}/T \right) T \bar{y}_i - \bar{x}_i T \sum_{t=1}^T y_{it}/T + \bar{x}_i T \bar{y}_i \\
&= X_i'y_i - \bar{x}_i T \bar{y}_i - \bar{x}_i T \bar{y}_i + \bar{x}_i T \bar{y}_i \\
&= X_i'y_i - \bar{x}_i T \bar{y}_i
\end{aligned}$$

### 1.3.4 The necessary condition for consistency and unbiasedness

The necessary condition for FE estimator (OLS estimator of the demean transformed model) to be consistent is  $\mathbb{E}((\mathbf{Q}\mathbf{X}_i)' \mathbf{Q}\varepsilon_i) = \mathbf{0}$ .

$$\begin{aligned}
\mathbb{E}((\mathbf{Q}\mathbf{X}_i)' \mathbf{Q}\varepsilon_i) &= \mathbb{E}(\mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \varepsilon_i) \\
&= \mathbb{E}(\mathbf{X}_i' \mathbf{Q} \varepsilon_i) && \text{as } \mathbf{Q} \text{ is idempotent and symmetric} \\
&= \mathbb{E}(\mathbb{E}(\mathbf{X}_i' \mathbf{Q} \varepsilon_i | \mathbf{X}_i)) \\
&= \mathbb{E}(\mathbf{X}_i' \mathbf{Q} \underbrace{\mathbb{E}(\varepsilon_i | \mathbf{X}_i)}_{\mathbf{0}}) && \text{because of strict exogeneity} \\
&= \mathbf{0}
\end{aligned}$$

Thus, FE estimator satisfies the necessary condition for consistency given strict exogeneity assumption. Indeed, strict exogeneity is stronger than what is required. To see this, first note that for any  $t$ ,

$$\mathbb{E}(\varepsilon_i | \mathbf{X}_i) = \mathbb{E}(\varepsilon_{it} | \mathbf{x}_{i1}, \dots, \mathbf{x}_{iT}) = \mathbf{0} \implies \mathbb{E}(\mathbf{x}_{is} \varepsilon_{it}) = \mathbf{0} \quad \text{for all } s$$

It is because for any  $t$  and  $s$ ,

$$\begin{aligned}
\mathbb{E}(\mathbf{x}_{is} \varepsilon_{it}) &= \mathbb{E}(\mathbb{E}(\mathbf{x}_{is} \varepsilon_{it} | \mathbf{x}_{i1}, \dots, \mathbf{x}_{iT})) \\
&= \mathbb{E}(\mathbf{x}_{is} \underbrace{\mathbb{E}(\varepsilon_{it} | \mathbf{x}_{i1}, \dots, \mathbf{x}_{iT})}_{\mathbf{0}}) \\
&= \mathbf{0}
\end{aligned}$$

The necessary condition for FE estimator to be consistent can also be written as  $\mathbb{E}((\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\varepsilon_{it} - \bar{\varepsilon}_i)) = \mathbf{0}$ .

$$\mathbb{E}((\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\varepsilon_{it} - \bar{\varepsilon}_i)) = \mathbb{E}(\mathbf{x}_{it} \varepsilon_{it}) - \mathbb{E}(\mathbf{x}_{it} \bar{\varepsilon}_i) - \mathbb{E}(\bar{\mathbf{x}}_i \varepsilon_{it}) + \mathbb{E}(\bar{\mathbf{x}}_i \bar{\varepsilon}_i) = \mathbf{0}$$

It is because  $\mathbb{E}(\mathbf{x}_{is} \varepsilon_{it}) = \mathbf{0}$  for any  $t$  and  $s$  implies

$$\begin{aligned}
\mathbb{E}(\mathbf{x}_{it} \varepsilon_{it}) &= \mathbf{0} \\
\mathbb{E}(\mathbf{x}_{it} \bar{\varepsilon}_i) &= \mathbb{E}(\mathbf{x}_{it} T^{-1} \sum_{s=1}^T \varepsilon_{is}) = T^{-1} \sum_{s=1}^T \underbrace{\mathbb{E}(\mathbf{x}_{it} \varepsilon_{is})}_{\mathbf{0}} = \mathbf{0} \\
\mathbb{E}(\bar{\mathbf{x}}_i \varepsilon_{it}) &= \mathbb{E}(T^{-1} \sum_{s=1}^T \mathbf{x}_{is} \varepsilon_{it}) = T^{-1} \sum_{s=1}^T \underbrace{\mathbb{E}(\mathbf{x}_{is} \varepsilon_{it})}_{\mathbf{0}} = \mathbf{0} \\
\mathbb{E}(\bar{\mathbf{x}}_i \bar{\varepsilon}_i) &= \mathbb{E}(T^{-1} \sum_{s=1}^T \mathbf{x}_{is} T^{-1} \sum_{t=1}^T \varepsilon_{it}) = T^{-2} \sum_{s=1}^T \sum_{t=1}^T \underbrace{\mathbb{E}(\mathbf{x}_{is} \varepsilon_{it})}_{\mathbf{0}} = \mathbf{0}
\end{aligned}$$

Thus, the weaker assumption  $\mathbb{E}(\mathbf{x}_{is} \varepsilon_{it}) = \mathbf{0}$  for any  $t$  and  $s$  is sufficient for  $\mathbb{E}((\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\varepsilon_{it} - \bar{\varepsilon}_i)) = \mathbf{0}$

The necessary condition for FE estimator to be unbiased is  $\mathbb{E}(\mathbf{Q}\varepsilon_i | \mathbf{Q}\mathbf{X}_i) = \mathbf{0}$ .

$$\begin{aligned}
\mathbb{E}(\mathbf{Q}\varepsilon_i | \mathbf{Q}\mathbf{X}_i) &= \mathbf{Q} \underbrace{\mathbb{E}(\varepsilon_i | \mathbf{X}_i)}_{\mathbf{0}} && \text{as } \mathbf{Q} \text{ is constant and strict exogeneity} \\
&= \mathbf{0}
\end{aligned}$$

### 1.3.5 Conditional variance of $\hat{\beta}_{within}^{ols}$

Given independence of  $i$ ,

$$\begin{aligned}
Var(\hat{\beta}_{within}^{ols} | \mathbf{X}_i) &= Var([\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{y}_i | \mathbf{X}_i) \\
&= [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} Var(\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{y}_i | \mathbf{X}_i) [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1'} \\
&= [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' Var(\mathbf{Q}\varepsilon_i | \mathbf{X}_i) \mathbf{Q} \mathbf{X}_i [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1}
\end{aligned}$$

It is because

$$\begin{aligned}
Var\left(\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{y}_i | \mathbf{X}_i\right) &= \sum_{i=1}^N Var(\mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{y}_i | \mathbf{X}_i) \\
&= \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' Var(\mathbf{Q} \mathbf{y}_i | \mathbf{X}_i) (\mathbf{X}_i' \mathbf{Q}')' \\
&= \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' Var(\mathbf{Q} \mathbf{X}_i \beta + \mathbf{Q} \varepsilon_i | \mathbf{X}_i) \mathbf{Q}'' \mathbf{X}_i'' \\
&= \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' Var(\mathbf{Q} \varepsilon_i | \mathbf{X}_i) \mathbf{Q} \mathbf{X}_i
\end{aligned}$$

**1.3.6**  $Var(\varepsilon_i | \mathbf{X}_i) = \sigma_\varepsilon^2 \mathbf{I}_T$

If  $\varepsilon_{it}$  is homoskedasticity and serially uncorrelated across  $t$  i.e.,  $Var(\varepsilon_i | \mathbf{X}_i) = \sigma_\varepsilon^2 \mathbf{I}_T$  (further assume independence of  $i$  and strict exogeneity), we have  $\varepsilon_i | \mathbf{X}_i \sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T]$ .

$$Var(\mathbf{Q} \varepsilon_i | \mathbf{X}_i) = \mathbf{Q} Var(\varepsilon_i | \mathbf{X}_i) \mathbf{Q}' = \mathbf{Q} \sigma_\varepsilon^2 \mathbf{I}_T \mathbf{Q}' = \sigma_\varepsilon^2 \mathbf{Q} \mathbf{Q}' = \sigma_\varepsilon^2 \mathbf{Q} = \sigma_\varepsilon^2 (\mathbf{I}_T - T^{-1} \mathbf{e} \mathbf{e}') = \sigma_\varepsilon^2 \begin{pmatrix} 1 - \frac{1}{T} & -\frac{1}{T} & \cdots & -\frac{1}{T} \\ -\frac{1}{T} & 1 - \frac{1}{T} & \cdots & -\frac{1}{T} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{T} & -\frac{1}{T} & \cdots & 1 - \frac{1}{T} \end{pmatrix}.$$

Thus,  $\mathbf{Q} \varepsilon_i$  is homoskedasticity but negatively serially correlated. For any  $t$ ,

$$\begin{aligned}
Var(\varepsilon_{it} - \bar{\varepsilon}_i) &= \sigma_\varepsilon^2 (1 - \frac{1}{T}) \iff \sigma_\varepsilon^2 = \frac{T}{T-1} Var(\varepsilon_{it} - \bar{\varepsilon}_i) \\
\hat{\sigma}_\varepsilon^2 &= \frac{T}{T-1} \widehat{Var}(\varepsilon_{it} - \bar{\varepsilon}_i) \\
&= \frac{T}{T-1} \frac{\sum_{i=1}^N \sum_{t=1}^T (y_{it} - \bar{y}_i - (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' \hat{\beta}_{within}^{ols})^2}{NT - (K + N)} \\
&= \frac{T}{T-1} \frac{\sum_{i=1}^N \sum_{t=1}^T (y_{it} - \mathbf{x}_{it}' \hat{\beta}_{within}^{ols} - \overbrace{(\bar{y}_i - \bar{\mathbf{x}}_i' \hat{\beta}_{within}^{ols})}^{\hat{\alpha}_i})^2}{\underbrace{NT - (K + N)}_{N(T-1) - K}} \quad \frac{T}{T-1} \approx 1 \text{ if } T \text{ is large.}
\end{aligned}$$

$$\begin{aligned}
Var(\hat{\beta}_{within}^{ols} | \mathbf{X}_i) &= [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \sigma_\varepsilon^2 \underbrace{\mathbf{Q} \mathbf{Q}'}_{\mathbf{Q}} \mathbf{X}_i [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} \\
&= \sigma_\varepsilon^2 [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} \\
&= \sigma_\varepsilon^2 \mathbf{I}_T [\sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i]^{-1} \\
&= \sigma_\varepsilon^2 [\sum_{i=1}^N (\mathbf{Q} \mathbf{X}_i)' \mathbf{Q} \mathbf{X}_i]^{-1} \quad \text{Level 2} \\
&= \sigma_\varepsilon^2 [\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)']^{-1} \quad \text{Level 1}
\end{aligned}$$

### 1.3.7 $Var(\varepsilon_i|\mathbf{X}_i) = \Omega_i$

We have  $\varepsilon_i|\mathbf{X}_i \sim iid [\mathbf{0}, \Omega_i]$ .

$$\begin{aligned}
Var(\hat{\beta}_{within}^{ols}|\mathbf{X}_i) &= \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbb{E}[(\mathbf{Q}\varepsilon_i - \mathbb{E}(\mathbf{Q}\varepsilon_i|\mathbf{X}_i))(\mathbf{Q}\varepsilon_i - \mathbb{E}(\mathbf{Q}\varepsilon_i|\mathbf{X}_i))'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \\
&= \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbb{E}[(\mathbf{Q}\varepsilon_i - \mathbf{Q}\mathbb{E}(\varepsilon_i|\mathbf{X}_i))(\mathbf{Q}\varepsilon_i - \mathbf{Q}\mathbb{E}(\varepsilon_i|\mathbf{X}_i))'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \\
&= \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbb{E}[(\mathbf{Q}\varepsilon_i - \mathbf{Q}\mathbf{0})(\mathbf{Q}\varepsilon_i - \mathbf{Q}\mathbf{0})'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \\
&= \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbb{E}[\mathbf{Q}\varepsilon_i(\mathbf{Q}\varepsilon_i)'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \\
&= \left[ \sum_{i=1}^N (\mathbf{Q} \mathbf{X}_i)' \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N (\mathbf{Q} \mathbf{X}_i)' \mathbb{E}[\mathbf{Q}\varepsilon_i(\mathbf{Q}\varepsilon_i)'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i \left[ \sum_{i=1}^N (\mathbf{Q} \mathbf{X}_i)' \mathbf{Q} \mathbf{X}_i \right]^{-1} \\
&= \left[ \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)'\right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) \mathbb{E}[\dot{\varepsilon}_{it} \dot{\varepsilon}_{is}|\mathbf{X}_i] (\mathbf{x}_{is} - \bar{\mathbf{x}}_i)' \left[ \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)'\right]^{-1}
\end{aligned}$$

It is because

$$\begin{aligned}
\sum_{i=1}^N (\mathbf{Q} \mathbf{X}_i)' \mathbb{E}[\mathbf{Q}\varepsilon_i(\mathbf{Q}\varepsilon_i)'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i &= \sum_{i=1}^N (\mathbf{Q} \mathbf{X}_i)' \mathbb{E}[\dot{\varepsilon}_i \dot{\varepsilon}_i'|\mathbf{X}_i] \mathbf{Q} \mathbf{X}_i \\
&= \sum_{i=1}^N \begin{pmatrix} (\mathbf{x}_{i1} - \bar{\mathbf{x}}_i)' \\ \vdots \\ (\mathbf{x}_{iT} - \bar{\mathbf{x}}_i)' \end{pmatrix}' \begin{pmatrix} \mathbb{E}[\dot{\varepsilon}_{i1}^2|\mathbf{X}_i] & \cdots & \mathbb{E}[\dot{\varepsilon}_{i1} \dot{\varepsilon}_{iT}|\mathbf{X}_i] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[\dot{\varepsilon}_{iT} \dot{\varepsilon}_{i1}|\mathbf{X}_i] & \cdots & \mathbb{E}[\dot{\varepsilon}_{iT}^2|\mathbf{X}_i] \end{pmatrix} \begin{pmatrix} (\mathbf{x}_{i1} - \bar{\mathbf{x}}_i)' \\ \vdots \\ (\mathbf{x}_{iT} - \bar{\mathbf{x}}_i)' \end{pmatrix} \\
&= \sum_{i=1}^N ((\mathbf{x}_{i1} - \bar{\mathbf{x}}_i) \quad (\mathbf{x}_{iT} - \bar{\mathbf{x}}_i)) \begin{pmatrix} \mathbb{E}[\dot{\varepsilon}_{i1}^2|\mathbf{X}_i] & \cdots & \mathbb{E}[\dot{\varepsilon}_{i1} \dot{\varepsilon}_{iT}|\mathbf{X}_i] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[\dot{\varepsilon}_{iT} \dot{\varepsilon}_{i1}|\mathbf{X}_i] & \cdots & \mathbb{E}[\dot{\varepsilon}_{iT}^2|\mathbf{X}_i] \end{pmatrix} \begin{pmatrix} (\mathbf{x}_{i1} - \bar{\mathbf{x}}_i)' \\ \vdots \\ (\mathbf{x}_{iT} - \bar{\mathbf{x}}_i)' \end{pmatrix} \\
&= \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) \mathbb{E}[\dot{\varepsilon}_{it} \dot{\varepsilon}_{is}|\mathbf{X}_i] (\mathbf{x}_{is} - \bar{\mathbf{x}}_i)'
\end{aligned}$$

Finite sample adjustment can also be added. In Stata,  $\frac{N}{N-1} \frac{T-1}{T-(K-1)}$  is multiplied.

### 1.3.8 GLS estimator of the demean transformed model if $Var(\varepsilon_i|\mathbf{X}_i) = \sigma_\varepsilon^2 \mathbf{I}_T$

$\varepsilon_i|\mathbf{X}_i \sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T]$  implies  $\mathbf{Q}\varepsilon_i|\mathbf{X}_i \sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{Q}]$ , we want to find a GLS transformer  $\mathbf{T}_{GLS}$  such that

$$\begin{aligned}
Var(\mathbf{T}_{GLS} \mathbf{Q}\varepsilon_i|\mathbf{X}_i) &= \sigma_\varepsilon^2 \mathbf{I}_T \\
\mathbf{T}_{GLS} Var(\mathbf{Q}\varepsilon_i|\mathbf{X}_i) \mathbf{T}_{GLS}' &= \sigma_\varepsilon^2 \mathbf{I}_T \\
\mathbf{T}_{GLS} \sigma_\varepsilon^2 \mathbf{Q} \mathbf{T}_{GLS}' &= \sigma_\varepsilon^2 \mathbf{I}_T \\
\mathbf{T}_{GLS} \mathbf{Q}^{1/2} \mathbf{Q}^{1/2} \mathbf{T}_{GLS}' &= \mathbf{I}_T \\
\mathbf{T}_{GLS} \mathbf{Q}^{1/2} \mathbf{Q}'^{1/2} \mathbf{T}_{GLS}' &= \mathbf{I}_T \\
\mathbf{T}_{GLS} \mathbf{Q}^{1/2} (\mathbf{T}_{GLS} \mathbf{Q}^{1/2})' &= \mathbf{I}_T
\end{aligned}$$

So,  $\mathbf{T}_{GLS} = \mathbf{Q}^{-1/2}$

$$\mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{y}_i = \mathbf{Q}^{-1/2} (\mathbf{Q} \mathbf{X}_i \beta + \mathbf{Q} \varepsilon_i) = \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{X}_i \beta + \mathbf{Q}^{-1/2} \mathbf{Q} \varepsilon_i$$

Thus, we have  $Var(\mathbf{Q}^{-1/2} \mathbf{Q} \varepsilon_i|\mathbf{X}_i) = \mathbf{Q}^{-1/2} Var(\mathbf{Q} \varepsilon_i|\mathbf{X}_i) \mathbf{Q}'^{-1/2} = \mathbf{Q}^{-1/2} \sigma_\varepsilon^2 \mathbf{Q} \mathbf{Q}^{-1/2} = \sigma_\varepsilon^2 \mathbf{Q}^{-1/2} \mathbf{Q}^{1/2} \mathbf{Q}^{1/2} \mathbf{Q}^{-1/2} = \sigma_\varepsilon^2 \mathbf{I}_T$ .

By Gauss-Markov Theorem, GLS estimator is efficient.

$$\begin{aligned}
\hat{\beta}_{within}^{gl} &= \left[ \sum_{i=1}^N (\mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{X}_i)' \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N (\mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{X}_i)' \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{y}_i \\
&= \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q}'^{-1/2} \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q}'^{-1/2} \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{y}_i \\
&= \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q}^{-1/2} \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q}^{-1/2} \mathbf{Q}^{-1/2} \mathbf{Q} \mathbf{y}_i \\
&= \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q}^- \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q}^- \mathbf{Q} \mathbf{y}_i \\
&= \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \mathbf{Q}' \mathbf{Q} \mathbf{y}_i = \hat{\beta}_{within}^{ols}
\end{aligned}$$

So, FE estimator is also efficient

For generalized inverse,  $\mathbf{Q}' \mathbf{Q}^- \mathbf{Q} = \mathbf{Q}$ . As  $\mathbf{Q}$  is idempotent and symmetry,  $\mathbf{Q} = \mathbf{Q} \mathbf{Q}' = \mathbf{Q}' \mathbf{Q}$ . Therefore,  $\mathbf{Q}' \mathbf{Q}^- \mathbf{Q} = \mathbf{Q}' \mathbf{Q}$ .

## 1.4 First-Difference Estimator

### 1.4.1 First-difference operator

$$\begin{aligned}
\Delta &= \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \\
\Delta \Delta' &= \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}' \\
&= \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}
\end{aligned}$$

### 1.4.2 First difference transformed model

$$\begin{aligned}
\Delta \mathbf{y}_i &= \Delta (\mathbf{X}_i \boldsymbol{\beta} + \mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i) \\
&= \Delta \mathbf{X}_i \boldsymbol{\beta} + \Delta \mathbf{e} \alpha_i + \Delta \boldsymbol{\varepsilon}_i \\
&= \Delta \mathbf{X}_i \boldsymbol{\beta} + \mathbf{0} \alpha_i + \Delta \boldsymbol{\varepsilon}_i \\
&= \Delta \mathbf{X}_i \boldsymbol{\beta} + \Delta \boldsymbol{\varepsilon}_i
\end{aligned}$$

Level 2



It is because

$$\Delta \mathbf{e} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}$$

It can be written as

$$\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \\ \vdots \\ \vdots \\ y_{iT} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x}'_{i1} \\ \mathbf{x}'_{i2} \\ \mathbf{x}'_{i3} \\ \vdots \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} \beta + \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \varepsilon_{i3} \\ \vdots \\ \vdots \\ \varepsilon_{iT} \end{pmatrix}$$

$$\begin{pmatrix} y_{i2} - y_{i1} \\ y_{i3} - y_{i2} \\ y_{i4} - y_{i3} \\ \vdots \\ \vdots \\ y_{iT} - y_{i,T-1} \end{pmatrix} = \begin{pmatrix} \mathbf{x}'_{i2} - \mathbf{x}'_{i1} \\ \mathbf{x}'_{i3} - \mathbf{x}'_{i2} \\ \mathbf{x}'_{i4} - \mathbf{x}'_{i3} \\ \vdots \\ \vdots \\ \mathbf{x}'_{iT} - \mathbf{x}'_{i,T-1} \end{pmatrix} \beta + \begin{pmatrix} \varepsilon_{i2} - \varepsilon_{i1} \\ \varepsilon_{i3} - \varepsilon_{i2} \\ \varepsilon_{i4} - \varepsilon_{i3} \\ \vdots \\ \vdots \\ \varepsilon_{iT} - \varepsilon_{i,T-1} \end{pmatrix}$$

$$\begin{pmatrix} y_{i2} - y_{i1} \\ y_{i3} - y_{i2} \\ y_{i4} - y_{i3} \\ \vdots \\ \vdots \\ y_{iT} - y_{i,T-1} \end{pmatrix} = \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1})' \\ (\mathbf{x}_{i3} - \mathbf{x}_{i2})' \\ (\mathbf{x}_{i4} - \mathbf{x}_{i3})' \\ \vdots \\ \vdots \\ (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1})' \end{pmatrix} \beta + \begin{pmatrix} \varepsilon_{i2} - \varepsilon_{i1} \\ \varepsilon_{i3} - \varepsilon_{i2} \\ \varepsilon_{i4} - \varepsilon_{i3} \\ \vdots \\ \vdots \\ \varepsilon_{iT} - \varepsilon_{i,T-1} \end{pmatrix}$$

$$y_{it} - y_{i,t-1} = (\mathbf{x}_{it} - \mathbf{x}_{i,t-1})' \beta + (\varepsilon_{it} - \varepsilon_{i,t-1})$$

Level 1

### 1.4.3 OLS estimator of the first difference transformed model

$$\hat{\beta}_{fd}^{ols} = \left[ \sum_{i=1}^N (\Delta \mathbf{X}_i)' \Delta \mathbf{X}_i \right]^{-1} \sum_{i=1}^N (\Delta \mathbf{X}_i)' \Delta \mathbf{y}_i$$

Level 2

$$= \left[ \sum_{i=1}^N \sum_{t=2}^T (\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(\mathbf{x}_{it} - \mathbf{x}_{i,t-1})' \right]^{-1} \sum_{i=1}^N \sum_{t=2}^T (\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(y_{it} - y_{i,t-1})$$

Level 1

It is because

$$\begin{aligned}
(\Delta \mathbf{X}_i)' \Delta \mathbf{X}_i &= \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1})' \\ (\mathbf{x}_{i3} - \mathbf{x}_{i2})' \\ (\mathbf{x}_{i4} - \mathbf{x}_{i3})' \\ \vdots \\ \vdots \\ (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1})' \end{pmatrix}' \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1})' \\ (\mathbf{x}_{i3} - \mathbf{x}_{i2})' \\ (\mathbf{x}_{i4} - \mathbf{x}_{i3})' \\ \vdots \\ \vdots \\ (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1})' \end{pmatrix} \\
&= \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1}) & (\mathbf{x}_{i3} - \mathbf{x}_{i2}) & (\mathbf{x}_{i4} - \mathbf{x}_{i3}) & \cdots & \cdots & (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1}) \end{pmatrix} \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1})' \\ (\mathbf{x}_{i3} - \mathbf{x}_{i2})' \\ (\mathbf{x}_{i4} - \mathbf{x}_{i3})' \\ \vdots \\ \vdots \\ (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1})' \end{pmatrix} \\
&= \sum_{t=2}^T (\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(\mathbf{x}_{it} - \mathbf{x}_{i,t-1})' \\
\\
(\Delta \mathbf{X}_i)' \Delta \mathbf{y}_i &= \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1})' \\ (\mathbf{x}_{i3} - \mathbf{x}_{i2})' \\ (\mathbf{x}_{i4} - \mathbf{x}_{i3})' \\ \vdots \\ \vdots \\ (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1})' \end{pmatrix}' \begin{pmatrix} y_{i2} - y_{i1} \\ y_{i3} - y_{i2} \\ y_{i4} - y_{i3} \\ \vdots \\ \vdots \\ y_{iT} - y_{i,T-1} \end{pmatrix} \\
&= \begin{pmatrix} (\mathbf{x}_{i2} - \mathbf{x}_{i1}) & (\mathbf{x}_{i3} - \mathbf{x}_{i2}) & (\mathbf{x}_{i4} - \mathbf{x}_{i3}) & \cdots & \cdots & (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1}) \end{pmatrix} \begin{pmatrix} y_{i2} - y_{i1} \\ y_{i3} - y_{i2} \\ y_{i4} - y_{i3} \\ \vdots \\ \vdots \\ y_{iT} - y_{i,T-1} \end{pmatrix} \\
&= \sum_{t=2}^T (\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(y_{it} - y_{i,t-1})
\end{aligned}$$

#### 1.4.4 The necessary condition for consistency and unbiasedness

The necessary condition for FD estimator (OLS estimator of the FD transformed model) to be consistent is  $\mathbb{E}(\Delta \mathbf{X}_i)' \Delta \boldsymbol{\varepsilon}_i = \mathbf{0}$

$$\begin{aligned}
\mathbb{E}((\Delta \mathbf{X}_i)' \Delta \boldsymbol{\varepsilon}_i) &= \mathbb{E}(\mathbf{X}_i' \Delta \Delta' \Delta \boldsymbol{\varepsilon}_i) \\
&= \mathbb{E}(\mathbb{E}(\mathbf{X}_i' \Delta \Delta' \boldsymbol{\varepsilon}_i | \mathbf{X}_i)) \\
&= \mathbb{E}(\mathbf{X}_i' \Delta \Delta' \underbrace{\mathbb{E}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i)}_{\mathbf{0}}) && \text{because of strict exogeneity} \\
&= \mathbf{0}
\end{aligned}$$

Thus, FD estimator satisfies the necessary condition for consistency given strict exogeneity assumption. Indeed, strict exogeneity is stronger than what is required. To see this, first note that for any  $t$

$$\mathbb{E}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \mathbb{E}(\boldsymbol{\varepsilon}_{it} | \mathbf{x}_{i1}, \dots, \mathbf{x}_{iT}) = \mathbf{0} \implies \mathbb{E}(\mathbf{x}_{is} \boldsymbol{\varepsilon}_{it}) = \mathbf{0} \quad \text{for all } s$$

The necessary condition for FD estimator to be consistent can also be written as  $\mathbb{E}((\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(\boldsymbol{\varepsilon}_{it} - \boldsymbol{\varepsilon}_{i,t-1})) = \mathbf{0}$

$$\mathbb{E}((\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(\boldsymbol{\varepsilon}_{it} - \boldsymbol{\varepsilon}_{i,t-1})) = \mathbb{E}(\mathbf{x}_{it} \boldsymbol{\varepsilon}_{it}) - \mathbb{E}(\mathbf{x}_{it} \boldsymbol{\varepsilon}_{i,t-1}) - \mathbb{E}(\mathbf{x}_{i,t-1} \boldsymbol{\varepsilon}_{it}) + \mathbb{E}(\mathbf{x}_{i,t-1} \boldsymbol{\varepsilon}_{i,t-1}) = \mathbf{0}$$

It is because  $\mathbb{E}(\mathbf{x}_{is} \boldsymbol{\varepsilon}_{it}) = \mathbf{0}$  for any  $t$  and  $s$  implies

$$\mathbb{E}(\mathbf{x}_{it} \boldsymbol{\varepsilon}_{it}) = \mathbb{E}(\mathbf{x}_{it} \boldsymbol{\varepsilon}_{i,t-1}) = \mathbb{E}(\mathbf{x}_{i,t-1} \boldsymbol{\varepsilon}_{it}) = \mathbb{E}(\mathbf{x}_{i,t-1} \boldsymbol{\varepsilon}_{i,t-1}) = \mathbf{0}$$

Thus, the weaker assumption  $\mathbb{E}(\mathbf{x}_{is}\varepsilon_{it}) = \mathbf{0}$  for any  $t$  and  $s$  is sufficient for  $\mathbb{E}((\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(\varepsilon_{it} - \varepsilon_{i,t-1})) = \mathbf{0}$

The necessary condition for FD estimator to be unbiased is  $\mathbb{E}(\Delta\boldsymbol{\varepsilon}_i|\Delta\mathbf{X}_i) = \mathbf{0}$

$$\begin{aligned}\mathbb{E}(\Delta\boldsymbol{\varepsilon}_i|\Delta\mathbf{X}_i) &= \Delta \underbrace{\mathbb{E}(\boldsymbol{\varepsilon}_i|\mathbf{X}_i)}_{\mathbf{0}} \\ &= \mathbf{0}\end{aligned}\quad \text{as } \Delta \text{ is constant and strict exogeneity}$$

#### 1.4.5 Conditional variance of $\hat{\boldsymbol{\beta}}_{fd}^{ols}$

$$\begin{aligned}Var(\hat{\boldsymbol{\beta}}_{fd}^{ols}|\mathbf{X}_i) &= Var([\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} \sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{y}_i|\mathbf{X}_i) \\ &= [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} Var(\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{y}_i|\mathbf{X}_i) [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1'} \\ &= [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} \sum_{i=1}^N (\Delta\mathbf{X}_i)' Var(\Delta\boldsymbol{\varepsilon}_i|\mathbf{X}_i) \Delta\mathbf{X}_i [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1}\end{aligned}$$

#### 1.4.6 $Var(\boldsymbol{\varepsilon}_i|\mathbf{X}_i) = \sigma_\varepsilon^2 \mathbf{I}_T$

If  $\varepsilon_{it}$  is homoskedasticity and serially uncorrelated across  $t$  i.e.,  $Var(\boldsymbol{\varepsilon}_i|\mathbf{X}_i) = \sigma_\varepsilon^2 \mathbf{I}_T$  (further assume independence of  $i$  and strict exogeneity), we have  $\boldsymbol{\varepsilon}_i|\mathbf{X}_i \sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T]$ .

$$Var(\Delta\boldsymbol{\varepsilon}_i|\mathbf{X}_i) = \Delta Var(\boldsymbol{\varepsilon}_i|\mathbf{X}_i) \Delta' = \Delta \sigma_\varepsilon^2 \mathbf{I}_T \Delta' = \sigma_\varepsilon^2 \Delta \Delta' = \sigma_\varepsilon^2 \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} \quad \text{Thus, } \Delta\boldsymbol{\varepsilon}_i \text{ is homoskedastic-}$$

ity but not serially uncorrelated e.g.  $Cov(\varepsilon_{it} - \varepsilon_{i,t-1}, \varepsilon_{i,t-1} - \varepsilon_{i,t-2}|\mathbf{X}_i) = -\sigma_\varepsilon^2 < 0$ . Therefore, we cannot apply Gauss-Markov Theorem.

$$\begin{aligned}Var(\hat{\boldsymbol{\beta}}_{fd}^{ols}|\mathbf{X}_i) &= [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} \sum_{i=1}^N (\Delta\mathbf{X}_i)' \sigma_\varepsilon^2 \Delta \Delta' \Delta\mathbf{X}_i [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} \\ &= \sigma_\varepsilon^2 [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} \sum_{i=1}^N \mathbf{X}_i' \Delta' \Delta \Delta' \Delta\mathbf{X}_i [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1}\end{aligned}$$

#### 1.4.7 $Var(\boldsymbol{\varepsilon}_i|\mathbf{X}_i) = \boldsymbol{\Omega}_i$

We have  $\boldsymbol{\varepsilon}_i|\mathbf{X}_i \sim inid [\mathbf{0}, \boldsymbol{\Omega}_i]$ ,

$$\begin{aligned}Var(\Delta\boldsymbol{\varepsilon}_i|\mathbf{X}_i) &= \Delta Var(\boldsymbol{\varepsilon}_i|\mathbf{X}_i) \Delta' = \Delta \mathbb{E}[(\boldsymbol{\varepsilon}_i - \mathbb{E}[\boldsymbol{\varepsilon}_i|\mathbf{X}_i])(\boldsymbol{\varepsilon}_i - \mathbb{E}[\boldsymbol{\varepsilon}_i|\mathbf{X}_i])'|\mathbf{X}_i] \Delta' = \Delta \mathbb{E}[(\boldsymbol{\varepsilon}_i - \mathbf{0})(\boldsymbol{\varepsilon}_i - \mathbf{0})'|\mathbf{X}_i] \Delta' = \Delta \mathbb{E}[\boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}_i'|\mathbf{X}_i] \Delta' = \\ &= \mathbb{E}[\Delta \boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}_i' \Delta'|\mathbf{X}_i] = \mathbb{E}[\Delta \boldsymbol{\varepsilon}_i (\Delta \boldsymbol{\varepsilon}_i)'|\mathbf{X}_i]\end{aligned}$$

$$Var(\hat{\boldsymbol{\beta}}_{fd}^{ols}|\mathbf{X}_i) = [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1} \sum_{i=1}^N (\Delta\mathbf{X}_i)' E[\Delta \boldsymbol{\varepsilon}_i (\Delta \boldsymbol{\varepsilon}_i)'|\mathbf{X}_i] \Delta\mathbf{X}_i [\sum_{i=1}^N (\Delta\mathbf{X}_i)' \Delta\mathbf{X}_i]^{-1}$$

If  $\varepsilon_{it}$  follows random walk process i.e.,  $\varepsilon_{it} = \varepsilon_{i,t-1} + v_{it}$  where  $v_{it}$  follows white noise process,  $\varepsilon_{it} - \varepsilon_{i,t-1} = v_{it}$  follows white noise process. Thus,  $\varepsilon_{it} - \varepsilon_{i,t-1}$  is homoskedasticity and serially uncorrelated as they are the properties of white noise process. As a result, FD estimator is efficient in this case by applying Gauss-Markov Theorem.

## 1.5 Least-Squares Dummy Variable Estimator

Level 3

$$\mathbf{y} = (\mathbf{I}_N \otimes \mathbf{e})\boldsymbol{\alpha} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} = \begin{pmatrix} (\mathbf{I}_N \otimes \mathbf{e}) & \mathbf{X} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} + \boldsymbol{\varepsilon}$$

$$\begin{aligned} \begin{pmatrix} \hat{\boldsymbol{\alpha}}_{dv}^{ols} \\ \hat{\boldsymbol{\beta}}_{dv}^{ols} \end{pmatrix} &= [((\mathbf{I}_N \otimes \mathbf{e}) \quad \mathbf{X})' ((\mathbf{I}_N \otimes \mathbf{e}) \quad \mathbf{X})]^{-1} ((\mathbf{I}_N \otimes \mathbf{e}) \quad \mathbf{X})' \mathbf{y} \\ &= \begin{pmatrix} (\mathbf{I}_N \otimes \mathbf{e})'(\mathbf{I}_N \otimes \mathbf{e}) & (\mathbf{I}_N \otimes \mathbf{e})'\mathbf{X} \\ \mathbf{X}'(\mathbf{I}_N \otimes \mathbf{e}) & \mathbf{X}'\mathbf{X} \end{pmatrix}^{-1} \begin{pmatrix} (\mathbf{I}_N \otimes \mathbf{e})'\mathbf{y} \\ \mathbf{X}'\mathbf{y} \end{pmatrix} \\ &= \begin{pmatrix} T\mathbf{I}_N & T\bar{\mathbf{X}} \\ T\bar{\mathbf{X}}' & \mathbf{X}'\mathbf{X} \end{pmatrix}^{-1} \begin{pmatrix} T\bar{\mathbf{y}} \\ \mathbf{X}'\mathbf{y} \end{pmatrix} \\ \hat{\boldsymbol{\beta}}_{dv}^{ols} &= [\mathbf{X}'\mathbf{X} - T\bar{\mathbf{X}}'\bar{\mathbf{X}}]^{-1}(\mathbf{X}'\mathbf{y} - T\bar{\mathbf{X}}'\bar{\mathbf{y}}) = \hat{\boldsymbol{\beta}}_{within}^{ols} \end{aligned}$$

It is because

$$\begin{aligned} ((\mathbf{I}_N \otimes \mathbf{e}) \quad \mathbf{X})' ((\mathbf{I}_N \otimes \mathbf{e}) \quad \mathbf{X}) &= \begin{pmatrix} (\mathbf{I}_N \otimes \mathbf{e})' \\ \mathbf{X}' \end{pmatrix} \begin{pmatrix} (\mathbf{I}_N \otimes \mathbf{e}) & \mathbf{X} \end{pmatrix} \\ &= \begin{pmatrix} (\mathbf{I}_N \otimes \mathbf{e})'(\mathbf{I}_N \otimes \mathbf{e}) & (\mathbf{I}_N \otimes \mathbf{e})'\mathbf{X} \\ \mathbf{X}'(\mathbf{I}_N \otimes \mathbf{e}) & \mathbf{X}'\mathbf{X} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} (\mathbf{I}_N \otimes \mathbf{e})'(\mathbf{I}_N \otimes \mathbf{e}) &= \begin{pmatrix} \mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e} \end{pmatrix}' \begin{pmatrix} \mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{e}' & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e}' \end{pmatrix} \begin{pmatrix} \mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{e}'\mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e}'\mathbf{e} \end{pmatrix} \\ &= \begin{pmatrix} T & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & T \end{pmatrix} \\ &= T\mathbf{I}_N \end{aligned}$$

$$\begin{aligned} (\mathbf{I}_N \otimes \mathbf{e})'\mathbf{X} &= \begin{pmatrix} \mathbf{e}' & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e}' \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_N \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{e}'\mathbf{X}_1 \\ \vdots \\ \mathbf{e}'\mathbf{X}_N \end{pmatrix} \\ &= \begin{pmatrix} \sum_{t=1}^T \mathbf{x}'_{1t} \\ \vdots \\ \sum_{t=1}^T \mathbf{x}'_{Nt} \end{pmatrix} \\ &= \begin{pmatrix} T \sum_{t=1}^T \mathbf{x}'_{1t}/T \\ \vdots \\ T \sum_{t=1}^T \mathbf{x}'_{Nt}/T \end{pmatrix} \\ &= \begin{pmatrix} T\bar{\mathbf{x}}'_1 \\ \vdots \\ T\bar{\mathbf{x}}'_N \end{pmatrix} \\ &= T\bar{\mathbf{X}} \end{aligned}$$

$$\begin{aligned}
((I_N \otimes e) \quad X)' y &= \begin{pmatrix} (I_N \otimes e)' \\ X' \end{pmatrix} y \\
&= \begin{pmatrix} (I_N \otimes e)' y \\ X' y \end{pmatrix}
\end{aligned}$$

Another way to show the equivalence of within estimator and dummy variable estimator by using Frisch-Waugh-Lovell Theorem,

$$y = X\beta + (I_N \otimes e)\alpha + \varepsilon$$

$$y = X\beta + E\alpha + \varepsilon$$

$$X = E\alpha_{XE} + \varepsilon_{XE}$$

$$y = E\alpha_{yE} + \varepsilon_{yE}$$

$$\begin{aligned}
\hat{\alpha}_{yE} &= (E'E)^{-1} E'y \\
&= \begin{pmatrix} e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e \end{pmatrix}' \begin{pmatrix} e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e \end{pmatrix}^{-1} \begin{pmatrix} e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e \end{pmatrix}' \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \\
&= \begin{pmatrix} e' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e' \end{pmatrix} \begin{pmatrix} e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e \end{pmatrix}^{-1} \begin{pmatrix} e' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e' \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \\
&= \begin{pmatrix} e'e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e'e \end{pmatrix}^{-1} \begin{pmatrix} e'y_1 \\ \vdots \\ e'y_N \end{pmatrix} \\
&= \begin{pmatrix} (e'e)^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (e'e)^{-1} \end{pmatrix} \begin{pmatrix} e'y_1 \\ \vdots \\ e'y_N \end{pmatrix} \\
&= \begin{pmatrix} (e'e)^{-1} e'y_1 \\ \vdots \\ (e'e)^{-1} e'y_N \end{pmatrix} = \begin{pmatrix} T^{-1} \sum_{t=1}^T y_{1t} \\ \vdots \\ T^{-1} \sum_{t=1}^T y_{Nt} \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_N \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\hat{\varepsilon}_{yE} &= y - E\hat{\alpha}_{yE} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} - \begin{pmatrix} e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_N \end{pmatrix} \\
&= \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} - \begin{pmatrix} e\bar{y}_1 \\ \vdots \\ e\bar{y}_N \end{pmatrix} \\
&= \begin{pmatrix} y_1 - e\bar{y}_1 \\ \vdots \\ y_N - e\bar{y}_N \end{pmatrix} = \begin{pmatrix} Qy_1 \\ \vdots \\ Qy_N \end{pmatrix} = Qy
\end{aligned}$$

Similarly,

$$\hat{\varepsilon}_{XE} = QX$$

By Frisch-Waugh-Lovell Theorem,

$$\begin{aligned}
\hat{\beta}_{dv}^{ols} &= (\hat{\varepsilon}_{XE}' \hat{\varepsilon}_{XE})^{-1} \hat{\varepsilon}_{XE}' \hat{\varepsilon}_{yE} \\
&= [(QX)' QX]^{-1} (QX)' Qy = \hat{\beta}_{within}^{ols}
\end{aligned}$$

If  $N \rightarrow \infty$ , the number of  $\alpha_i$  estimated goes to infinity. If  $T$  is fixed, the LSDV estimates are consistent for  $\beta$  (as FE estimates for  $\beta$  is consistent for fixed  $T$  and  $N \rightarrow \infty$ ) but inconsistent for  $\alpha$ . (There is no incidental parameters problem as the estimates for  $\beta$  are not contaminated). If  $T$  also  $\rightarrow \infty$ , then the LSDV estimates of  $\alpha$  are also consistent.

## 2 Random Effect Model

$$y_{it} = \mathbf{x}'_{it}\boldsymbol{\beta} + \alpha_i + \varepsilon_{it}$$

$$\begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} = \begin{pmatrix} \mathbf{x}'_{i1} \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \alpha_i + \begin{pmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{pmatrix}$$

$$\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta} + \underbrace{(\mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i)}_{\mathbf{u}_i}$$

$$\begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_N \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_1 \\ \vdots \\ \boldsymbol{\varepsilon}_N \end{pmatrix}$$

$$\mathbf{y} = \mathbf{X} \boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{e}) \boldsymbol{\alpha} + \boldsymbol{\varepsilon}$$

### 2.1 Assumptions

#### 2.1.1 Strong/strict exogeneity of regressors

For all  $t$ ,

$$E(\varepsilon_{it} | \mathbf{x}_{i1}, \dots, \mathbf{x}_{iT}) = 0$$

Equivalently,

$$E(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \mathbf{0}$$

#### 2.1.2 Covariance structure

$$\begin{aligned} \boldsymbol{\varepsilon}_i | \mathbf{X}_i &\sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T] \\ \alpha_i | \mathbf{X}_i &\sim iid [0, \sigma_\alpha^2] \\ \boldsymbol{\varepsilon}_i &\perp \alpha_i | \mathbf{X}_i \end{aligned}$$

### 2.2 Moments of $\mathbf{u}_i | \mathbf{X}_i$

$$\begin{aligned} \boldsymbol{\Omega} &:= Var(\mathbf{u}_i | \mathbf{X}_i) = Var(\mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i | \mathbf{X}_i) && \text{because of } \boldsymbol{\varepsilon}_i \perp \alpha_i | \mathbf{X}_i \\ &= Var(\mathbf{e} \alpha_i | \mathbf{X}_i) + Var(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) \\ &= \mathbf{e} Var(\alpha_i | \mathbf{X}_i) \mathbf{e}' + Var(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) \\ &= \sigma_\alpha^2 \mathbf{e} \mathbf{e}' + \sigma_\varepsilon^2 \mathbf{I}_T \\ &= \begin{pmatrix} \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 \\ \vdots & \ddots & \vdots \\ \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 \end{pmatrix} + \begin{pmatrix} \sigma_\varepsilon^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_\varepsilon^2 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_\alpha^2 + \sigma_\varepsilon^2 & \cdots & \sigma_\alpha^2 \\ \vdots & \ddots & \vdots \\ \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 + \sigma_\varepsilon^2 \end{pmatrix} \\ \mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) &= \mathbb{E}(\mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i | \mathbf{X}_i) \\ &= \mathbf{e} \underbrace{\mathbb{E}(\alpha_i | \mathbf{X}_i)}_0 + \underbrace{\mathbb{E}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i)}_0 \\ &= \mathbf{0} \end{aligned}$$

Note that  $\mathbb{E}(\alpha_i | \mathbf{X}_i) = \mathbb{E}(\alpha_i | \mathbf{x}_{i1}, \dots, \mathbf{x}_{iT}) = 0$  called orthogonality assumption.  $\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0 \implies Cov(\alpha_i, \mathbf{X}_i) = \mathbf{0}$ . It is because  $Cov(\alpha_i, \mathbf{X}_i) = \mathbb{E}(\alpha_i \mathbf{X}_i) - \mathbb{E}(\alpha_i) \mathbb{E}(\mathbf{X}_i) = \mathbb{E}(\mathbb{E}(\alpha_i \mathbf{X}_i | \mathbf{X}_i)) - \mathbb{E}(\mathbb{E}(\alpha_i | \mathbf{X}_i)) \mathbb{E}(\mathbf{X}_i) = \mathbb{E}(\mathbb{E}(\alpha_i | \mathbf{X}_i) \mathbf{X}_i) = \mathbf{0}$ . There is no OVB i.e.,  $\mathbf{u}_i$  is not correlated with  $\mathbf{X}_i$ .

$\mathbb{E}(\mathbf{u}_i|\mathbf{X}_i) = \mathbf{0}$  means that the necessary condition for OLS estimator to be unbiased is satisfied. Moreover,  $\mathbb{E}(\mathbf{u}_i|\mathbf{X}_i) = \mathbf{0} \implies \mathbb{E}(\mathbf{X}_i'\mathbf{u}_i) = \mathbf{0}$  which means that the necessary condition for OLS estimator to be consistent is also satisfied. However,  $\mathbf{u}_i|\mathbf{X}_i$  is homoskedasticity but serially correlated. Thus, the necessary condition for OLS estimator to be efficient is not satisfied. As a result, it is not efficient.

As we know the covariance structure of  $\mathbf{u}_i|\mathbf{X}_i$  due to the strong assumptions in random effect model, we can use GLS estimation, which yields efficient estimates.

## 2.3 Random Effect Estimator (GLS Estimator)

### 2.3.1 GLS transformed model

We want to find a  $\mathbf{T}_{GLS}$  such that

$$\begin{aligned} Var(\mathbf{T}_{GLS}\mathbf{u}_i|\mathbf{X}_i) &= \sigma_\varepsilon^2 \mathbf{I}_T \\ \mathbf{T}_{GLS} Var(\mathbf{u}_i|\mathbf{X}_i) \mathbf{T}_{GLS}' &= \sigma_\varepsilon^2 \mathbf{I}_T \\ \mathbf{T}_{GLS} \boldsymbol{\Omega} \mathbf{T}_{GLS}' &= \sigma_\varepsilon^2 \mathbf{I}_T \\ \mathbf{T}_{GLS} \boldsymbol{\Omega}^{1/2} \boldsymbol{\Omega}^{1/2} \mathbf{T}_{GLS}' &= \sigma_\varepsilon^2 \mathbf{I}_T \\ \mathbf{T}_{GLS} \boldsymbol{\Omega}^{1/2} \boldsymbol{\Omega}^{1/2} \mathbf{T}_{GLS}' &= \sigma_\varepsilon^2 \mathbf{I}_T \\ \mathbf{T}_{GLS} \boldsymbol{\Omega}^{1/2} (\mathbf{T}_{GLS} \boldsymbol{\Omega}^{1/2})' &= \sigma_\varepsilon^2 \mathbf{I}_T \end{aligned}$$

So,  $\mathbf{T}_{GLS} = \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2}$ . Define  $\psi^2 := \frac{\sigma_\varepsilon^2}{T\sigma_\alpha^2 + \sigma_\varepsilon^2}$ .

$$\begin{aligned} \boldsymbol{\Omega} &= \sigma_\varepsilon^2 \mathbf{I}_T + \sigma_\alpha^2 \mathbf{e}\mathbf{e}' \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T + \frac{\sigma_\alpha^2}{\sigma_\varepsilon^2} \mathbf{e}\mathbf{e}') \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T + \frac{T\sigma_\alpha^2}{\sigma_\varepsilon^2} T^{-1} \mathbf{e}\mathbf{e}') \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T + \frac{T\sigma_\alpha^2 + \sigma_\varepsilon^2 - \sigma_\varepsilon^2}{\sigma_\varepsilon^2} T^{-1} \mathbf{e}\mathbf{e}') \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T + (\frac{T\sigma_\alpha^2 + \sigma_\varepsilon^2}{\sigma_\varepsilon^2} - 1) T^{-1} \mathbf{e}\mathbf{e}') \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T + (\frac{1}{\psi^2} - 1) T^{-1} \mathbf{e}\mathbf{e}') \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T + \frac{1}{\psi^2} T^{-1} \mathbf{e}\mathbf{e}' - T^{-1} \mathbf{e}\mathbf{e}') \\ &= \sigma_\varepsilon^2 (\mathbf{I}_T - T^{-1} \mathbf{e}\mathbf{e}' + \frac{1}{\psi^2} (T^{-1} \mathbf{e}\mathbf{e}' - \mathbf{I}_T + \mathbf{I}_T)) \\ &= \sigma_\varepsilon^2 (\mathbf{Q} + \frac{1}{\psi^2} (\mathbf{I}_T - \mathbf{Q})) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Omega}^{-1} &= [\sigma_\varepsilon^2 (\mathbf{Q} + \frac{1}{\psi^2} (\mathbf{I}_T - \mathbf{Q}))]^{-1} \\ &= \sigma_\varepsilon^{-2} (\mathbf{Q}^{-1} + \psi^2 (\mathbf{I}_T^{-1} - \mathbf{Q}^{-1})) \\ &= \sigma_\varepsilon^{-2} (\mathbf{Q} + \psi^2 (\mathbf{I}_T - \mathbf{Q})) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Omega}^{-1/2} &= \sigma_\varepsilon^{-1} (\mathbf{Q} + \psi (\mathbf{I}_T - \mathbf{Q})) \\ \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} &= (\mathbf{Q} + \psi (\mathbf{I}_T - \mathbf{Q})) \end{aligned}$$

$$\sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{y}_i = \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} (\mathbf{X}_i \boldsymbol{\beta} + (\mathbf{e}\alpha_i + \varepsilon_i)) = \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} (\mathbf{X}_i \boldsymbol{\beta} + \mathbf{u}_i) = \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{X}_i \boldsymbol{\beta} + \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{u}_i$$

$$\text{So, } Var(\sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{u}_i | \mathbf{X}_i) = \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} Var(\mathbf{u}_i | \mathbf{X}_i) \sigma_\varepsilon \boldsymbol{\Omega}'^{-1/2} = \sigma_\varepsilon^2 \boldsymbol{\Omega}^{-1/2} \boldsymbol{\Omega} \boldsymbol{\Omega}^{-1/2} = \sigma_\varepsilon^2 \boldsymbol{\Omega}^{-1/2} \boldsymbol{\Omega}^{1/2} \boldsymbol{\Omega}^{1/2} \boldsymbol{\Omega}^{-1/2} = \sigma_\varepsilon^2 \mathbf{I}_T$$

$$(\mathbf{Q} + \psi (\mathbf{I}_T - \mathbf{Q})) \mathbf{y}_i = (\mathbf{Q} + \psi (\mathbf{I}_T - \mathbf{Q})) \mathbf{X}_i \boldsymbol{\beta} + (\mathbf{Q} + \psi (\mathbf{I}_T - \mathbf{Q})) \mathbf{e}\alpha_i + (\mathbf{Q} + \psi (\mathbf{I}_T - \mathbf{Q})) \varepsilon_i$$

Level 2

It can also be written as

$$\mathbf{y}_i - \lambda \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \lambda \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + (1 - \lambda) \mathbf{e} \alpha_i + (\boldsymbol{\varepsilon}_i - \lambda \mathbf{e} \bar{\varepsilon}_i)$$

Level 2

where  $\lambda = 1 - \psi = 1 - \frac{\sigma_\varepsilon}{\sqrt{T\sigma_\alpha^2 + \sigma_\varepsilon^2}}$ . It is because

$$\begin{aligned} \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{y}_i &= (\mathbf{Q} + \psi(\mathbf{I}_T - \mathbf{Q})) \mathbf{y}_i = \mathbf{Q} \mathbf{y}_i + \psi(\mathbf{I}_T \mathbf{y}_i - \mathbf{Q} \mathbf{y}_i) \\ &= \mathbf{y}_i - \mathbf{e} \bar{y}_i + \psi(\mathbf{y}_i - \mathbf{y}_i + \mathbf{e} \bar{y}_i) \\ &= \mathbf{y}_i - \mathbf{e} \bar{y}_i + \psi \mathbf{e} \bar{y}_i \\ &= \mathbf{y}_i - \mathbf{e} \bar{y}_i (1 - \psi) \\ &= \mathbf{y}_i - \lambda \mathbf{e} \bar{y}_i \end{aligned}$$

$$\begin{aligned} \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{X}_i \boldsymbol{\beta} &= (\mathbf{Q} + \psi(\mathbf{I}_T - \mathbf{Q})) \mathbf{X}_i \boldsymbol{\beta} = \mathbf{Q} \mathbf{X}_i \boldsymbol{\beta} + \psi(\mathbf{I}_T \mathbf{X}_i \boldsymbol{\beta} - \mathbf{Q} \mathbf{X}_i \boldsymbol{\beta}) \\ &= (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + \psi(\mathbf{X}_i \boldsymbol{\beta} - (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta}) \\ &= (\mathbf{X}_i \boldsymbol{\beta} - \mathbf{e} \bar{\mathbf{x}}'_i \boldsymbol{\beta}) + \psi(\mathbf{X}_i \boldsymbol{\beta} - \mathbf{X}_i \boldsymbol{\beta} + \mathbf{e} \bar{\mathbf{x}}'_i \boldsymbol{\beta}) \\ &= \mathbf{X}_i \boldsymbol{\beta} - \mathbf{e} \bar{\mathbf{x}}'_i \boldsymbol{\beta} + \psi \mathbf{e} \bar{\mathbf{x}}'_i \boldsymbol{\beta} \\ &= (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}'_i + \psi \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} \\ &= (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}'_i (1 - \psi)) \boldsymbol{\beta} \\ &= (\mathbf{X}_i - \lambda \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} \end{aligned}$$

$$\begin{aligned} \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{e} \alpha_i &= (\mathbf{Q} + \psi(\mathbf{I}_T - \mathbf{Q})) \mathbf{e} \alpha_i = \mathbf{Q} \mathbf{e} \alpha_i + \psi(\mathbf{I}_T \mathbf{e} \alpha_i - \mathbf{Q} \mathbf{e} \alpha_i) \\ &= \mathbf{0} \alpha_i + \psi(\mathbf{e} \alpha_i - \mathbf{0} \alpha_i) \\ &= \psi \mathbf{e} \alpha_i \\ &= (1 - \lambda) \mathbf{e} \alpha_i \end{aligned}$$

Random effect estimator is the OLS estimator of the beta in the transformed model  $\mathbf{y}_i - \lambda \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \lambda \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + (1 - \lambda) \mathbf{e} \alpha_i + (\boldsymbol{\varepsilon}_i - \lambda \mathbf{e} \bar{\varepsilon}_i)$ .

Fixed effect / within estimator is the OLS estimator of the beta in the transformed model  $\mathbf{y}_i - \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + (\boldsymbol{\varepsilon}_i - \mathbf{e} \bar{\varepsilon}_i)$ .

Pooled OLS estimator is the OLS estimator of the beta in the original model  $\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta} + \mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i$ .

As  $T \rightarrow \infty$ ,  $\lambda \rightarrow 1$ ,  $\mathbf{y}_i - \lambda \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \lambda \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + (1 - \lambda) \mathbf{e} \alpha_i + (\boldsymbol{\varepsilon}_i - \lambda \mathbf{e} \bar{\varepsilon}_i)$  converges to  $\mathbf{y}_i - \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + (\boldsymbol{\varepsilon}_i - \mathbf{e} \bar{\varepsilon}_i)$ . Thus, random effect estimator converges to fixed effect / within estimator as  $T \rightarrow \infty$ .

As  $\sigma_\alpha^2 \rightarrow 0$ ,  $\lambda \rightarrow 0$ ,  $\mathbf{y}_i - \lambda \mathbf{e} \bar{y}_i = (\mathbf{X}_i - \lambda \mathbf{e} \bar{\mathbf{x}}'_i) \boldsymbol{\beta} + (1 - \lambda) \mathbf{e} \alpha_i + (\boldsymbol{\varepsilon}_i - \lambda \mathbf{e} \bar{\varepsilon}_i)$  converges to  $\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta} + \mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i$ . Thus, random effect estimator converges to pooled OLS estimator as  $\sigma_\alpha^2 \rightarrow 0$ .

$$\begin{aligned} \begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{y}_i &= \left( \begin{pmatrix} \mathbf{x}'_{i1} \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{\mathbf{x}}'_i \right) \boldsymbol{\beta} + (1 - \lambda) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \alpha_i + \left( \begin{pmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \bar{\varepsilon}_i \right) \\ \begin{pmatrix} y_{i1} - \lambda \bar{y}_i \\ \vdots \\ y_{iT} - \lambda \bar{y}_i \end{pmatrix} &= \begin{pmatrix} \mathbf{x}'_{i1} - \lambda \bar{\mathbf{x}}'_i \\ \vdots \\ \mathbf{x}'_{iT} - \lambda \bar{\mathbf{x}}'_i \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} (1 - \lambda) \alpha_i \\ \vdots \\ (1 - \lambda) \alpha_i \end{pmatrix} + \begin{pmatrix} \varepsilon_{i1} - \lambda \bar{\varepsilon}_i \\ \vdots \\ \varepsilon_{iT} - \lambda \bar{\varepsilon}_i \end{pmatrix} \\ \begin{pmatrix} y_{i1} - \lambda \bar{y}_i \\ \vdots \\ y_{iT} - \lambda \bar{y}_i \end{pmatrix} &= \begin{pmatrix} (\mathbf{x}_{i1} - \lambda \bar{\mathbf{x}}_i)' \\ \vdots \\ (\mathbf{x}_{iT} - \lambda \bar{\mathbf{x}}_i)' \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} (1 - \lambda) \alpha_i \\ \vdots \\ (1 - \lambda) \alpha_i \end{pmatrix} + \begin{pmatrix} \varepsilon_{i1} - \lambda \bar{\varepsilon}_i \\ \vdots \\ \varepsilon_{iT} - \lambda \bar{\varepsilon}_i \end{pmatrix} \\ y_{it} - \lambda \bar{y}_i &= (\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i)' \boldsymbol{\beta} + \underbrace{(1 - \lambda) \alpha_i + (\varepsilon_{it} - \lambda \bar{\varepsilon}_i)}_{v_{it}} \end{aligned}$$

Level 1



### 2.3.2 OLS estimator of the GLS transformed model i.e., Random Effect / GLS estimator

$$\begin{aligned}\hat{\beta}_{re}^{ols} &= \left[ \sum_{i=1}^N (\mathbf{X}_i - \lambda e \bar{\mathbf{x}}_i)' (\mathbf{X}_i - \lambda e \bar{\mathbf{x}}_i) \right]^{-1} \sum_{i=1}^N (\mathbf{X}_i - \lambda e \bar{\mathbf{x}}_i)' (\mathbf{y}_i - \lambda e \bar{y}_i) && \text{Level 2} \\ &= \left[ \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i) (\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i)' \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i) (y_{it} - \lambda \bar{y}_i) && \text{Level 1}\end{aligned}$$

If  $\mathbf{x}_{it}$  is replaced by  $\mathbf{x}_{it} - \bar{\mathbf{x}}$  and  $\bar{\mathbf{x}}_i$  is replaced by  $\bar{\mathbf{x}}_i - \bar{\mathbf{x}}$ ,

$$\begin{aligned}(\mathbf{x}_{it} - \bar{\mathbf{x}}) - \lambda(\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) &= \mathbf{x}_{it} - \bar{\mathbf{x}} - \lambda \bar{\mathbf{x}}_i + \lambda \bar{\mathbf{x}} \\ &= \mathbf{x}_{it} - \bar{\mathbf{x}} - (1 - \psi) \bar{\mathbf{x}}_i + (1 - \psi) \bar{\mathbf{x}} \\ &= \mathbf{x}_{it} - \bar{\mathbf{x}} - \bar{\mathbf{x}}_i + \psi \bar{\mathbf{x}}_i + \bar{\mathbf{x}} - \psi \bar{\mathbf{x}} \\ &= (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) + \psi(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^N \sum_{t=1}^T ((\mathbf{x}_{it} - \bar{\mathbf{x}}) - \lambda(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})) ((\mathbf{x}_{it} - \bar{\mathbf{x}}) - \lambda(\bar{\mathbf{x}}_i - \bar{\mathbf{x}}))' &= \sum_{i=1}^N \sum_{t=1}^T ((\mathbf{x}_{it} - \bar{\mathbf{x}}_i) + \psi(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})) ((\mathbf{x}_{it} - \bar{\mathbf{x}}_i) + \psi(\bar{\mathbf{x}}_i - \bar{\mathbf{x}}))' \\ &= \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \psi \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \\ &\quad + \psi \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \psi^2 \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \\ &= \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \psi^2 \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})'\end{aligned}$$

It is because

$$\begin{aligned}\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' &= \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' - \sum_{i=1}^N \sum_{t=1}^T \bar{\mathbf{x}}_i (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \\ &= \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' - \sum_{i=1}^N T \bar{\mathbf{x}}_i (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \\ &= \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' - \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \\ &= \mathbf{0}\end{aligned}$$

Similarly, if  $y_{it}$  is replaced by  $y_{it} - \bar{y}$  and  $\bar{y}_i$  is replaced by  $\bar{y}_i - \bar{y}$

$$\sum_{i=1}^N \sum_{t=1}^T ((y_{it} - \bar{y}) - \lambda(\bar{y}_i - \bar{y})) ((y_{it} - \bar{y}) - \lambda(\bar{y}_i - \bar{y})) = \sum_{i=1}^N \sum_{t=1}^T (y_{it} - \bar{y}_i) (y_{it} - \bar{y}_i) + \psi^2 \sum_{i=1}^N \sum_{t=1}^T (\bar{y}_i - \bar{y}) (\bar{y}_i - \bar{y})$$

$$\begin{aligned}\hat{\beta}_{re}^{ols} &= \overbrace{\left( \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' \right)}^{\text{Within}} + \psi^2 \overbrace{\left( \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \right)}^{\text{Between}}^{-1} \\ &\quad \left( \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i) (y_{it} - \bar{y}_i) + \psi^2 \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) (\bar{y}_i - \bar{y}) \right)\end{aligned}$$

If  $T \rightarrow \infty$ ,  $\psi^2 \rightarrow 0$ ,  $\hat{\beta}_{re}^{ols} \rightarrow \hat{\beta}_{within}^{ols}$

If  $\sigma_\alpha^2 \rightarrow 0$ ,  $\psi^2 \rightarrow 1$ ,  $\hat{\beta}_{re}^{ols} \rightarrow \hat{\beta}_{pool}^{ols}$  It is because

$$\begin{aligned}
\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}})(\mathbf{x}_{it} - \bar{\mathbf{x}})' &= \sum_{i=1}^N \sum_{t=1}^T ((\mathbf{x}_{it} - \bar{\mathbf{x}}_i) + (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}))((\mathbf{x}_{it} - \bar{\mathbf{x}}_i) + (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}))' \\
&= \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' + \\
&\quad \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \\
&= \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})'
\end{aligned}$$

Similarly,

$$\sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}})(y_{it} - \bar{y}) = \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(y_{it} - \bar{y}_i) + \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{y}_i - \bar{y})$$

Thus,

$$\begin{aligned}
\hat{\beta}_{pool}^{ols} &= \left( \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}})(\mathbf{x}_{it} - \bar{\mathbf{x}})' \right)^{-1} \left( \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}})(y_{it} - \bar{y}) \right) \\
&= \left( \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' + \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{\mathbf{x}}_i - \bar{\mathbf{x}})' \right)^{-1} \\
&\quad \left( \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(y_{it} - \bar{y}_i) + \sum_{i=1}^N \sum_{t=1}^T (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})(\bar{y}_i - \bar{y}) \right)
\end{aligned}$$

So, pooled OLS estimator is an inefficient weighted average of within and between effects. RE estimator is an efficient weighted average of within and between effects. As RE model assumes  $\varepsilon_i | \mathbf{X}_i \sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T]$ ,

$$Var(\hat{\beta}_{re}^{ols}) = \sigma_\varepsilon^2 \left[ \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i)' \right]^{-1}$$

### 2.3.3 Between effect model and estimation of $\sigma_\alpha^2$

$$\begin{aligned}
\bar{y}_i &= \bar{\mathbf{x}}_i' \boldsymbol{\beta} + \overbrace{\alpha_i + \bar{\varepsilon}_i}^{v_i} \\
\sigma_B^2 &= Var(v_i) = Var(\alpha_i + \bar{\varepsilon}_i) \\
&= Var(\alpha_i) + Var(\bar{\varepsilon}_i) \\
&= Var(\alpha_i) + T^{-1} Var(\varepsilon_{it})
\end{aligned}$$

as  $\varepsilon_{it}$  is serially uncorrelated

$$\overbrace{Var(\alpha_i)}^{\sigma_\alpha^2} = \overbrace{Var(v_i)}^{\sigma_B^2} - \overbrace{Var(\varepsilon_{it})}^{\sigma_\varepsilon^2}$$

### 3 GMM Estimation of Linear Panel Model

#### 3.1 Linear panel model

$$\begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} = \begin{pmatrix} \mathbf{x}'_{i1} \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix}$$

$$\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta} + \mathbf{u}_i$$

#### 3.2 Exogeneity assumption

$$\mathbb{E}(\mathbf{Z}'_i \mathbf{u}_i) = \mathbf{0}$$

$\mathbf{Z}_i$  is a  $T \times r$  matrix.  $r$  is the number of exogeneous variables in  $\mathbf{X}_i$  plus the number of instrumental variables for endogeneous variables in  $\mathbf{X}_i$ . In GMM context,  $r$  is also the number of moment conditions.

$K$  is the number of parameters.

$r \geq K$ . If  $r = K$ , the model is just-identified, GMM is the same as MM; if  $r > K$ , the model is over-identified.

##### 3.2.1 Summation assumption

The weakest exogeneity assumption

$$\mathbf{Z}_i = \begin{pmatrix} \mathbf{z}'_{i1} \\ \vdots \\ \mathbf{z}'_{iT} \end{pmatrix}$$

$$\mathbb{E}(\mathbf{Z}'_i \mathbf{u}_i) = \mathbb{E}\left( \begin{pmatrix} \mathbf{z}'_{i1} \\ \vdots \\ \mathbf{z}'_{iT} \end{pmatrix}' \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix} \right) = \mathbb{E}\left( \begin{pmatrix} \mathbf{z}_{i1} & \cdots & \mathbf{z}_{iT} \end{pmatrix} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix} \right) = \mathbb{E}\left( \sum_{t=1}^T \mathbf{z}_{it} u_{it} \right) = \mathbf{0}$$

##### 3.2.2 Contemporaneous exogeneity assumption

Stronger

$$\mathbf{Z}_i = \begin{pmatrix} \mathbf{z}'_{i1} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{z}'_{iT} \end{pmatrix}$$

$$\begin{aligned} \mathbb{E}(\mathbf{Z}'_i \mathbf{u}_i) &= \mathbb{E}\left( \begin{pmatrix} \mathbf{z}'_{i1} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{z}'_{iT} \end{pmatrix}' \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix} \right) \\ &= \mathbb{E}\left( \begin{pmatrix} \mathbf{z}_{i1} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{z}_{iT} \end{pmatrix} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix} \right) \\ &= \mathbb{E}\left( \begin{pmatrix} \mathbf{z}_{i1} u_{i1} \\ \vdots \\ \mathbf{z}_{iT} u_{iT} \end{pmatrix} \right) \\ &= \begin{pmatrix} \mathbb{E}(\mathbf{z}_{i1} u_{i1}) \\ \vdots \\ \mathbb{E}(\mathbf{z}_{iT} u_{iT}) \end{pmatrix} = \mathbf{0} \end{aligned}$$

### 3.2.3 Weak/sequential exogeneity assumption

Stronger

$$\begin{aligned}
\mathbf{Z}_i &= \begin{pmatrix} \mathbf{z}'_{i1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & (\mathbf{z}'_{i1} & \mathbf{z}'_{i2}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & (\mathbf{z}'_{i1} \cdots \mathbf{z}'_{iT}) \end{pmatrix} \\
\mathbb{E}(\mathbf{Z}'_i \mathbf{u}_i) &= \mathbb{E} \left( \begin{pmatrix} \mathbf{z}'_{i1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & (\mathbf{z}'_{i1} & \mathbf{z}'_{i2}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & (\mathbf{z}'_{i1} \cdots \mathbf{z}'_{iT}) \end{pmatrix}' \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix} \right) \\
&= \mathbb{E} \left( \begin{pmatrix} \mathbf{z}_{i1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & (\mathbf{z}_{i1} \\ \mathbf{z}_{i2}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \begin{pmatrix} \mathbf{z}_{i1} \\ \vdots \\ \mathbf{z}_{iT} \end{pmatrix} \end{pmatrix} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix} \right) \\
&= \mathbb{E} \left( \begin{pmatrix} \mathbf{z}_{i1} u_{i1} \\ (\mathbf{z}_{i1} u_{i2} \\ \mathbf{z}_{i2} u_{i2}) \\ \vdots \\ (\mathbf{z}_{i1} u_{iT} \\ \vdots \\ \mathbf{z}_{iT} u_{iT}) \end{pmatrix} \right) = \begin{pmatrix} \mathbb{E}(\mathbf{z}_{i1} u_{i1}) \\ \mathbb{E}(\mathbf{z}_{i1} u_{i2}) \\ \mathbb{E}(\mathbf{z}_{i2} u_{i2}) \\ \vdots \\ \mathbb{E}(\mathbf{z}_{i1} u_{iT}) \\ \vdots \\ \mathbb{E}(\mathbf{z}_{iT} u_{iT}) \end{pmatrix} = \mathbf{0}
\end{aligned}$$

which is equivalent as  $\mathbb{E}(\mathbf{z}_{is} u_{it}) = \mathbf{0}$  for  $s \leq t$ .

Strong form of sequential exogeneity  $\mathbb{E}(u_{it} | \mathbf{z}_{it}, \dots, \mathbf{z}_{i1}) = 0$  implies weak form of sequential exogeneity  $\mathbb{E}(\mathbf{z}_{is} u_{it}) = \mathbf{0}$  for  $s \leq t$  as  $\mathbb{E}(\mathbf{z}_{is} u_{it}) = \mathbb{E}(\mathbb{E}(\mathbf{z}_{is} u_{it} | \mathbf{z}_{it}, \dots, \mathbf{z}_{i1})) = \mathbb{E}(\mathbf{z}_{is} \underbrace{\mathbb{E}(u_{it} | \mathbf{z}_{it}, \dots, \mathbf{z}_{i1})}_{0}) = \mathbf{0}$  for  $s \leq t$ .

It also implies  $Cov(\mathbf{z}_{is}, u_{it}) = \mathbf{0}$  for  $s \leq t$  as  $Cov(\mathbf{z}_{is}, u_{it}) = \underbrace{\mathbb{E}(\mathbf{z}_{is} u_{it})}_{0} - \mathbb{E}(\mathbf{z}_{is}) \mathbb{E}(u_{it}) = -\mathbb{E}(\mathbf{z}_{is}) \underbrace{\mathbb{E}(\mathbb{E}(u_{it} | \mathbf{z}_{it}, \dots, \mathbf{z}_{i1}))}_{0} = \mathbf{0}$  for  $s \leq t$ .

### 3.2.4 Strong/strict exogeneity assumption

The strongest exogeneity assumption

$$\mathbf{Z}_i = \begin{pmatrix} (\mathbf{z}'_{i1} \cdots \mathbf{z}'_{iT}) & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & (\mathbf{z}'_{i1} \cdots \mathbf{z}'_{iT}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & (\mathbf{z}'_{i1} \cdots \mathbf{z}'_{iT}) \end{pmatrix}$$

$$\begin{aligned}
\mathbb{E}(\mathbf{Z}'_i \mathbf{u}_i) &= \mathbb{E} \left( \begin{pmatrix} (\mathbf{z}'_{i1} & \cdots & \mathbf{z}'_{iT}) & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & (\mathbf{z}'_{i1} & \cdots & \mathbf{z}'_{iT}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & (\mathbf{z}'_{i1} & \cdots & \mathbf{z}'_{iT}) \end{pmatrix}' \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix} \right) \\
&= \mathbb{E} \left( \begin{pmatrix} \begin{pmatrix} \mathbf{z}_{i1} \\ \vdots \\ \mathbf{z}_{iT} \end{pmatrix} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \begin{pmatrix} \mathbf{z}_{i1} \\ \vdots \\ \mathbf{z}_{iT} \end{pmatrix} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \begin{pmatrix} \mathbf{z}_{i1} \\ \vdots \\ \mathbf{z}_{iT} \end{pmatrix} \end{pmatrix} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iT} \end{pmatrix} \right) \\
&= \mathbb{E} \left( \begin{pmatrix} \begin{pmatrix} \mathbf{z}_{i1} u_{i1} \\ \vdots \\ \mathbf{z}_{iT} u_{i1} \end{pmatrix} \\ \begin{pmatrix} \mathbf{z}_{i1} u_{i2} \\ \vdots \\ \mathbf{z}_{iT} u_{i2} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \mathbf{z}_{i1} u_{iT} \\ \vdots \\ \mathbf{z}_{iT} u_{iT} \end{pmatrix} \end{pmatrix} \right) \\
&= \begin{pmatrix} \begin{pmatrix} \mathbb{E}(\mathbf{z}_{i1} u_{i1}) \\ \vdots \\ \mathbb{E}(\mathbf{z}_{iT} u_{i1}) \end{pmatrix} \\ \begin{pmatrix} \mathbb{E}(\mathbf{z}_{i1} u_{i2}) \\ \vdots \\ \mathbb{E}(\mathbf{z}_{iT} u_{i2}) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \mathbb{E}(\mathbf{z}_{i1} u_{iT}) \\ \vdots \\ \mathbb{E}(\mathbf{z}_{iT} u_{iT}) \end{pmatrix} \end{pmatrix} = \mathbf{0}
\end{aligned}$$

which is equivalent as  $\mathbb{E}(\mathbf{z}_{is} u_{it}) = \mathbf{0}$  for  $s = 1, \dots, T$

Strong form of strict exogeneity  $\mathbb{E}(u_{it} | \mathbf{z}_{i1}, \dots, \mathbf{z}_{iT}) = 0$  implies weak form of strict exogeneity  $\mathbb{E}(\mathbf{z}_{is} u_{it}) = \mathbf{0}$  for  $s = 1, \dots, T$ . Since for  $s = 1, \dots, T$ ,

$$\begin{aligned}
\mathbb{E}(\mathbf{z}_{is} u_{it}) &= \mathbb{E}(\mathbb{E}(\mathbf{z}_{is} u_{it} | \mathbf{z}_{i1}, \dots, \mathbf{z}_{iT})) \\
&= \mathbb{E}(\mathbf{z}_{is} \underbrace{\mathbb{E}(u_{it} | \mathbf{z}_{i1}, \dots, \mathbf{z}_{iT})}_0) \\
&= \mathbf{0}
\end{aligned}$$

### 3.3 GMM Estimator of Linear Panel Model

#### 3.3.1 Unconditional moment condition

$$\mathbb{E}(\mathbf{Z}'_i \mathbf{u}_i) = \mathbb{E}(\mathbf{Z}'_i (\mathbf{y}_i - \mathbf{X}_i \beta_0)) = \mathbf{0}$$

where  $\beta_0$  is the true population parameter. So,  $g(\mathbf{d}_i; \theta_0) = \mathbf{Z}'_i \mathbf{u}_i = \mathbf{Z}'_i (\mathbf{y}_i - \mathbf{X}_i \beta_0)$

### 3.3.2 Objective/loss function

We want to find  $\beta$  from the parameter space such that the squared distance between  $\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N$  and  $\mathbb{E}(\mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta_0))$  i.e.,

$$\begin{aligned}
& [\rho(\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N, \mathbb{E}(\mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta_0)))]^2 && \text{where } \rho(\cdot) \text{ is a metric function} \\
& = \|\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N - \mathbb{E}(\mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta_0))\|^2 \\
& = (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N - \underbrace{\mathbb{E}(\mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta_0))}_{\mathbf{0}})' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N - \underbrace{\mathbb{E}(\mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta_0))}_{\mathbf{0}}) \\
& = (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N)' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N) \geq 0 && \text{as distance cannot be negative}
\end{aligned}$$

is as close to the zero as possible. The distance is a function of  $\beta$  i.e.,

$$Q_N(\beta) := (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N)' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N) \geq 0$$

If  $\mathbf{W}_N$  is symmetric and positive definite, then  $Q_N(\beta)$  is strictly convex. So, first order condition becomes sufficient and there is an unique minimizer.

### 3.3.3 Gradient vector

$$\begin{aligned}
\nabla Q_N(\beta) &= \frac{\partial Q_N(\beta)}{\partial \beta} = \frac{\partial (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N)' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N)}{\partial \beta} \\
&= 2(\frac{\partial (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N)}{\partial \beta'})' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N) \\
&= 2[\sum_{i=1}^N (\frac{\partial \mathbf{Z}'_i \mathbf{y}_i}{\partial \beta'} - \frac{\partial \mathbf{Z}'_i \mathbf{X}_i \beta}{\partial \beta'})/N]' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N) \\
&= 2[\sum_{i=1}^N -\frac{\partial \mathbf{Z}'_i \mathbf{X}_i \beta}{\partial \beta'}/N]' \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N) \\
&= -2(1/N^2) \sum_{i=1}^N (\mathbf{Z}'_i \mathbf{X}_i)' \mathbf{W}_N \sum_{i=1}^N (\mathbf{Z}'_i \mathbf{y}_i - \mathbf{Z}'_i \mathbf{X}_i \beta) \\
&= -2(1/N^2) \sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}''_i \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i - \sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i \beta) \\
&= -2(1/N^2) [(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) - (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i \beta)]
\end{aligned}$$

If  $r = K$ , both  $(\frac{\partial (\sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i\beta)/N)}{\partial \beta'})'$  and  $\mathbf{W}_N$  are square matrixes and invertible. In this case, FOC is  $\nabla Q_N(\hat{\beta}_{pmm}) = \sum_{i=1}^N \mathbf{Z}'_i(\mathbf{y}_i - \mathbf{X}_i \hat{\beta}_{pmm})/N = \mathbf{0}$  which is MM estimation.

### 3.3.4 First order condition

$$\begin{aligned}
-2(1/N^2)[(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) - (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i) \hat{\beta}_{pgmm}] &= \mathbf{0} \\
(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) - (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i) \hat{\beta}_{pgmm} &= \mathbf{0} \\
(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) &= (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i) \hat{\beta}_{pgmm} \\
[(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i)]^{-1} (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) &= \hat{\beta}_{pgmm}
\end{aligned}$$

Special case: if  $\mathbf{W}_N = (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{Z}_i)^{-1}$ ,

$$\begin{aligned}
\hat{\beta}_{pgmm} &= [(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \overbrace{(\sum_{i=1}^N \mathbf{Z}'_i \mathbf{Z}_i)^{-1} (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i)}^{\hat{\Gamma}_{2SLs}}]^{-1} \overbrace{(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{Z}_i)^{-1} (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i)}^{\hat{\Gamma}'_{2SLs}} \\
&= [\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i \hat{\Gamma}_{2SLs}]^{-1} \hat{\Gamma}'_{2SLs} \sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i \\
&= [\sum_{i=1}^N \mathbf{X}'_i \underbrace{\mathbf{Z}_i \hat{\Gamma}_{2SLs}}_{\hat{\mathbf{X}}_i}]^{-1} \sum_{i=1}^N \underbrace{(\mathbf{Z}_i \hat{\Gamma}_{2SLs})'}_{\hat{\mathbf{X}}'_i} \mathbf{y}_i = \hat{\beta}_{p2SLs}
\end{aligned}$$

Special case: if  $r = K$ , the model is just-identified, GMM is the same as MM,

$$\begin{aligned}
\hat{\beta}_{pmm} &= \hat{\beta}_{pgmm} = [(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i)]^{-1} (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) \\
&= (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i)^{-1} \mathbf{W}_N^{-1} (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i)^{-1} (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) \\
&= (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i)^{-1} (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) = \hat{\beta}_{piv}
\end{aligned}$$

Special case: if all regressors are exogeneous:  $\mathbf{Z}_i = \mathbf{X}_i$  (which implies  $r = K$ ),

$$\begin{aligned}
\hat{\beta}_{pgmm} &= \hat{\beta}_{piv} \\
&= (\sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i)^{-1} (\sum_{i=1}^N \mathbf{X}'_i \mathbf{y}_i) = \hat{\beta}_{pols}
\end{aligned}$$

$$\begin{aligned}
\hat{\beta}_{pgmm} &= [(\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{X}_i)]^{-1} (\sum_{i=1}^N \mathbf{X}'_i \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{y}_i) \\
&= [(\mathbf{X}'_1 \quad \cdots \quad \mathbf{X}'_N) \begin{pmatrix} \mathbf{Z}_1 \\ \vdots \\ \mathbf{Z}_N \end{pmatrix} \mathbf{W}_N (\mathbf{Z}'_1 \quad \cdots \quad \mathbf{Z}'_N) \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_N \end{pmatrix}]^{-1} (\mathbf{X}'_1 \quad \cdots \quad \mathbf{X}'_N) \begin{pmatrix} \mathbf{Z}_1 \\ \vdots \\ \mathbf{Z}_N \end{pmatrix} \mathbf{W}_N (\mathbf{Z}'_1 \quad \cdots \quad \mathbf{Z}'_N) \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{pmatrix} \\
&= [\mathbf{X}' \mathbf{Z} \mathbf{W}_N \mathbf{Z}' \mathbf{X}]^{-1} \mathbf{X}' \mathbf{Z} \mathbf{W}_N \mathbf{Z}' \mathbf{y}
\end{aligned}$$

### 3.4 Conditional variance of $\hat{\beta}_{pgmm}$

$$\begin{aligned}
Var(\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{y}|\mathbf{X}, \mathbf{Z}) &= \mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'Var(\mathbf{y}|\mathbf{X}, \mathbf{Z})(\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}')' \\
&= \mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'Var(\mathbf{X}\beta + \mathbf{u}|\mathbf{X}, \mathbf{Z})(\mathbf{Z}''\mathbf{W}_N'\mathbf{Z}'\mathbf{X}'') \\
&= \mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'Var(\mathbf{u}|\mathbf{X}, \mathbf{Z})(\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}) \\
&= \mathbf{X}'\mathbf{Z}\mathbf{W}_NVar(\mathbf{Z}'\mathbf{u}|\mathbf{X}, \mathbf{Z})\mathbf{W}_N\mathbf{Z}'\mathbf{X} \\
&= \mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbb{E}((\mathbf{Z}'\mathbf{u} - \mathbb{E}(\mathbf{Z}'\mathbf{u}|\mathbf{X}, \mathbf{Z}))(\mathbf{Z}'\mathbf{u} - \mathbb{E}(\mathbf{Z}'\mathbf{u}|\mathbf{X}, \mathbf{Z}))'|\mathbf{X}, \mathbf{Z})\mathbf{W}_N\mathbf{Z}'\mathbf{X} \\
&= \mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbb{E}((\mathbf{Z}'\mathbf{u})(\mathbf{Z}'\mathbf{u})'|\mathbf{X}, \mathbf{Z})\mathbf{W}_N\mathbf{Z}'\mathbf{X} \\
&= \mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbb{E}(\mathbf{Z}'\mathbf{u}\mathbf{u}'\mathbf{Z}'|\mathbf{X}, \mathbf{Z})\mathbf{W}_N\mathbf{Z}'\mathbf{X}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1'} &= [\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]'^{-1} \\
&= [\mathbf{X}'\mathbf{Z}''\mathbf{W}_N'\mathbf{Z}'\mathbf{X}'']^{-1} \\
&= [\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1}
\end{aligned}$$

$$\begin{aligned}
Var(\hat{\beta}_{pgmm}|\mathbf{X}, \mathbf{Z}) &= Var([\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{y}|\mathbf{X}, \mathbf{Z}) \\
&= [\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1}Var(\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{y}|\mathbf{X}, \mathbf{Z})[\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1'} \\
&= [\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbb{E}(\mathbf{Z}'\mathbf{u}\mathbf{u}'\mathbf{Z}'|\mathbf{X}, \mathbf{Z})\mathbf{W}_N\mathbf{Z}'\mathbf{X}[\mathbf{X}'\mathbf{Z}\mathbf{W}_N\mathbf{Z}'\mathbf{X}]^{-1}
\end{aligned}$$



## 4 GMM Estimation of Fixed Effect Model

$$\begin{aligned}
y_{it} &= \mathbf{x}_{it}'\boldsymbol{\beta} + \alpha_i + \varepsilon_{it} \\
\mathbf{y}_i &= \mathbf{X}_i\boldsymbol{\beta} + \underbrace{(\mathbf{e}\alpha_i + \boldsymbol{\varepsilon}_i)}_{\mathbf{u}_i} \\
\mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{e})\boldsymbol{\alpha} + \boldsymbol{\varepsilon}
\end{aligned}$$

### 4.1 Assumption

$\alpha_i$  is potentially correlated with  $\mathbf{X}_i$ , so  $\mathbf{u}_i$  is potentially correlated with  $\mathbf{X}_i$

$\boldsymbol{\varepsilon}_i$  is also potentially correlated with  $\mathbf{X}_i$ , so  $\mathbf{u}_i$  is potentially correlated with  $\mathbf{X}_i$

Even after eliminating  $\alpha_i$  by using any arbitrary operators  $\mathbf{T}$ ,  $\tilde{\mathbf{u}}_i := \mathbf{T}\mathbf{u}_i$  is still potentially correlated with  $\tilde{\mathbf{X}}_i := \mathbf{T}\mathbf{X}_i$  because of the potential correlation between  $\boldsymbol{\varepsilon}_i$  and  $\mathbf{X}_i$ . Thus,  $\tilde{\mathbf{X}}_i$  is potentially endogenous.

If  $\tilde{\mathbf{X}}_i$  is endogeneous, OLS estimation is inconsistent and biased. We should use IV estimation (for just-identified case) and 2SLS estimation (for over-identified case). IV and 2SLS estimation are special cases of GMM estimation.

### 4.2 GMM estimator of fixed effect model

There exists a  $\mathbf{T}$  such that  $\mathbf{T}\mathbf{e} = \mathbf{0}$ .

#### 4.2.1 Transformed model

$$\begin{aligned}
\tilde{\mathbf{y}}_i &:= \mathbf{T}\mathbf{y}_i = \mathbf{T}(\mathbf{X}_i\boldsymbol{\beta} + \mathbf{u}_i) = \mathbf{T}\mathbf{X}_i\boldsymbol{\beta} + \mathbf{T}\mathbf{u}_i := \tilde{\mathbf{X}}_i\boldsymbol{\beta} + \tilde{\mathbf{u}}_i \\
\tilde{\mathbf{u}}_i &:= \mathbf{T}\mathbf{u}_i = \mathbf{T}(\mathbf{e}\alpha_i + \boldsymbol{\varepsilon}_i) = \mathbf{T}\mathbf{e}\alpha_i + \mathbf{T}\boldsymbol{\varepsilon}_i = \mathbf{0} + \mathbf{T}\boldsymbol{\varepsilon}_i = \mathbf{T}\boldsymbol{\varepsilon}_i =: \tilde{\boldsymbol{\varepsilon}}_i
\end{aligned}$$

It is obvious that  $\tilde{\mathbf{u}}_i = \mathbf{T}\boldsymbol{\varepsilon}_i$  is correlated with  $\tilde{\mathbf{X}}_i := \mathbf{T}\mathbf{X}_i$  if  $\boldsymbol{\varepsilon}_i$  is correlated with  $\mathbf{X}_i$ .

If  $\mathbf{T} = \mathbf{Q} = \mathbf{I}_T - \mathbf{T}^{-1}\mathbf{e}\mathbf{e}'$ ,

$$\begin{aligned}
\tilde{\mathbf{y}}_i &= \tilde{\mathbf{X}}_i\boldsymbol{\beta} + \tilde{\boldsymbol{\varepsilon}}_i \\
(\mathbf{y}_i - \mathbf{e}\bar{y}_i) &= (\mathbf{X}_i - \mathbf{e}\bar{\mathbf{x}}_i')\boldsymbol{\beta} + (\boldsymbol{\varepsilon}_i - \mathbf{e}\bar{\boldsymbol{\varepsilon}}_i) \\
(y_{it} - \bar{y}_i) &= (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)'\boldsymbol{\beta} + (\varepsilon_{it} - \bar{\varepsilon}_i)
\end{aligned}$$

Under weak form of weak/sequential exogeneity assumption  $\mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) = \mathbf{0}$  for  $s \leq t$ .

For  $s \leq t$ , we have

$$\begin{aligned}
\mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \bar{\varepsilon}_i)) &= \mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) - \mathbb{E}(\mathbf{z}_{is}\bar{\varepsilon}_i) \\
&= \mathbf{0} - \mathbb{E}(\mathbf{z}_{is} \sum_{t=1}^T \varepsilon_{it} / T) \\
&= -\frac{1}{T} \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1} + \cdots + \mathbf{z}_{is}\varepsilon_{i,s-1} + \mathbf{z}_{is}\varepsilon_{is} + \cdots + \mathbf{z}_{is}\varepsilon_{iT}) \\
&= -\frac{1}{T} (\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,s-1}) + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{is}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{iT})) \\
&= -\frac{1}{T} (\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,s-1}) + \mathbf{0} + \cdots + \mathbf{0}) \\
&= -\frac{1}{T} (\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,s-1}))
\end{aligned}$$

So  $\mathbb{E}(\mathbf{z}_{it}(\varepsilon_{it} - \bar{\varepsilon}_i))$  is not necessarily equal to zero under weak form of weak/sequential exogeneity assumption. If weak form of strong/strict exogeneity is assumed  $\mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) = \mathbf{0} \forall s$ , then  $\mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \bar{\varepsilon}_i)) = \mathbf{0} \forall s$ . So,  $\mathbf{z}_{is}$ ,  $s = 1, \dots, T$  satisfy the exclusion restriction (exogeneity) requirement of valid instrument since  $\text{Cov}(\mathbf{z}_{is}, \varepsilon_{it} - \bar{\varepsilon}_i) = \underbrace{\mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \bar{\varepsilon}_i))}_{\mathbf{0}} - \mathbb{E}(\mathbf{z}_{is})\mathbb{E}(\varepsilon_{it} - \bar{\varepsilon}_i) =$

$$-\mathbb{E}(\mathbf{z}_{is}) \underbrace{(\mathbb{E}(\varepsilon_{it}) - T^{-1} \sum_{t=1}^T \mathbb{E}(\varepsilon_{it}))}_0 = \mathbf{0} \quad \forall s \text{ (additionally assume } \mathbb{E}(\varepsilon_{it}) = 0). \text{ So, we have}$$

$$\begin{aligned} \mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \bar{\varepsilon}_i)) &= \mathbf{0} \\ \iff \mathbb{E}(\mathbf{Z}'_i(\boldsymbol{\varepsilon}_i - \mathbf{e}\bar{\varepsilon}_i)) &= \mathbf{0} \\ \iff \mathbb{E}(\mathbf{Z}'_i\tilde{\boldsymbol{\varepsilon}}_i) &= \mathbf{0} \end{aligned} \quad \text{for } \forall s$$

We can then apply IV estimation in GMM framework.

If  $\mathbf{T} = \boldsymbol{\Delta}$

$$\begin{aligned} \tilde{\mathbf{y}}_i &= \tilde{\mathbf{X}}_i\boldsymbol{\beta} + \tilde{\boldsymbol{\varepsilon}}_i \\ \boldsymbol{\Delta}\mathbf{y}_i &= \boldsymbol{\Delta}\mathbf{X}_i\boldsymbol{\beta} + \boldsymbol{\Delta}\boldsymbol{\varepsilon}_i \\ (y_{it} - y_{i,t-1}) &= (\mathbf{x}_{it} - \mathbf{x}_{i,t-1})'\boldsymbol{\beta} + (\varepsilon_{it} - \varepsilon_{i,t-1}) \end{aligned}$$

Under weak form of weak/sequential exogeneity assumption  $\mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) = \mathbf{0}$  for  $s \leq t$ .

For  $s < t$ , we have

$$\begin{aligned} \mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \varepsilon_{i,t-1})) &= \mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) - \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,t-1}) \\ &= \mathbf{0} - \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,t-1}) \\ &= \mathbf{0} \end{aligned} \quad \begin{aligned} &\text{as } s < t \implies s \leq t \\ &\text{as } s < t \iff s \leq t-1 \end{aligned}$$

So,  $\mathbf{z}_{is}$  for  $s < t$  satisfy the exclusion restriction (exogeneity) requirement of valid instrument since  $Cov(\mathbf{z}_{is}, \varepsilon_{it} - \varepsilon_{i,t-1}) = 0$  for  $s < t$  (additionally assume  $\mathbb{E}(\varepsilon_{it}) = 0$ ). Equivalently,

$$\mathbf{Z}_i = \begin{pmatrix} t=2; \mathbf{z}'_{i1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & t=3; (\mathbf{z}'_{i1} \quad \mathbf{z}'_{i2}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & t=T; (\mathbf{z}'_{i1} \quad \cdots \quad \mathbf{z}'_{i,T-1}) \end{pmatrix}$$

So, we have

$$\begin{aligned} \mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \varepsilon_{i,t-1})) &= \mathbf{0} \\ \iff \mathbb{E}(\mathbf{Z}'_i\boldsymbol{\Delta}\boldsymbol{\varepsilon}_i) &= \mathbf{0} \\ \iff \mathbb{E}(\mathbf{Z}'_i\tilde{\boldsymbol{\varepsilon}}_i) &= \mathbf{0} \end{aligned} \quad \text{for } s < t$$

We can then apply IV estimation in GMM framework.

## 5 GMM Estimation of Random Effect Model

$$\begin{aligned}
y_{it} &= \mathbf{x}_{it}'\boldsymbol{\beta} + \alpha_i + \varepsilon_{it} \\
\mathbf{y}_i &= \mathbf{X}_i\boldsymbol{\beta} + \underbrace{(\mathbf{e}\alpha_i + \boldsymbol{\varepsilon}_i)}_{\mathbf{u}_i} \\
\mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{e})\boldsymbol{\alpha} + \boldsymbol{\varepsilon}
\end{aligned}$$

### 5.1 Assumption

$\alpha_i$  is not correlated with  $\mathbf{X}_i$ .

$\varepsilon_i$  is potentially correlated with  $\mathbf{X}_i$ , so  $\mathbf{u}_i$  is potentially correlated with  $\mathbf{X}_i$ . Thus,  $\mathbf{X}_i$  is potentially endogenous.

If  $\mathbf{X}_i$  is endogenous, OLS estimation is inconsistent and biased. We should use IV estimation (for just-identified case) and 2SLS estimation (for over-identified case). IV and 2SLS estimations are special cases of GMM estimation.

Assume

$$\mathbb{E}(\mathbf{u}_i | \mathbf{Z}_i) = \mathbf{0}$$

Which is stronger than  $\mathbb{E}(\mathbf{Z}_i'\mathbf{u}_i) = \mathbf{0}$  as  $\mathbb{E}(\mathbf{u}_i | \mathbf{Z}_i) = \mathbf{0}$  implies  $\mathbb{E}(\mathbf{Z}_i'\mathbf{u}_i) = \mathbf{0}$

And assume

$$Var(\mathbf{u}_i | \mathbf{Z}_i) = \boldsymbol{\Omega}_i = \begin{pmatrix} \sigma_\alpha^2 + \sigma_\varepsilon^2 & \cdots & \sigma_\alpha^2 \\ \vdots & \ddots & \vdots \\ \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 + \sigma_\varepsilon^2 \end{pmatrix}$$

#### 5.1.1 Optimal moment condition

$$\begin{aligned}
\mathbf{D}_i &= \mathbb{E}\left(\frac{\partial \mathbf{u}_i'}{\partial \boldsymbol{\beta}} | \mathbf{Z}_i\right) Var(\mathbf{u}_i | \mathbf{Z}_i)^{-1} \\
&= \mathbb{E}\left(\frac{\partial (\mathbf{Z}_i\boldsymbol{\beta})'}{\partial \boldsymbol{\beta}} | \mathbf{Z}_i\right) \boldsymbol{\Omega}_i^{-1} \\
&= \mathbb{E}(\mathbf{Z}_i' | \mathbf{Z}_i) \boldsymbol{\Omega}_i^{-1} \\
&= \mathbf{Z}_i' \boldsymbol{\Omega}_i^{-1}
\end{aligned}$$

Optimal unconditional moment is

$$\begin{aligned}
\mathbb{E}(\mathbf{D}_i \mathbf{u}_i) &= \mathbf{0} \\
\mathbb{E}(\mathbf{Z}_i' \boldsymbol{\Omega}_i^{-1} \mathbf{u}_i) &= \mathbf{0} \\
\mathbb{E}(\mathbf{Z}_i' \boldsymbol{\Omega}_i^{-1/2} \boldsymbol{\Omega}_i^{-1/2} \mathbf{u}_i) &= \mathbf{0} \\
\mathbb{E}(\mathbf{Z}_i' \boldsymbol{\Omega}_i'^{-1/2} \boldsymbol{\Omega}_i^{-1/2} \mathbf{u}_i) &= \mathbf{0} \\
\mathbb{E}(\mathbf{Z}_i' \boldsymbol{\Omega}_i^{-1/2'} \boldsymbol{\Omega}_i^{-1/2} \mathbf{u}_i) &= \mathbf{0} \\
\mathbb{E}((\boldsymbol{\Omega}_i^{-1/2} \mathbf{Z}_i)' \boldsymbol{\Omega}_i^{-1/2} \mathbf{u}_i) &= \mathbf{0} \\
\sigma_\varepsilon^2 \mathbb{E}((\boldsymbol{\Omega}_i^{-1/2} \mathbf{Z}_i)' \boldsymbol{\Omega}_i^{-1/2} \mathbf{u}_i) &= \sigma_\varepsilon^2 \mathbf{0} \\
\mathbb{E}((\sigma_\varepsilon \boldsymbol{\Omega}_i^{-1/2} \mathbf{Z}_i)' \sigma_\varepsilon \boldsymbol{\Omega}_i^{-1/2} \mathbf{u}_i) &= \mathbf{0}
\end{aligned}$$

This implies that the model should be transformed by  $\sigma_\varepsilon \boldsymbol{\Omega}_i^{-1/2}$

## 5.2 GMM Estimator of Random Effect Model

### 5.2.1 Transformed model

$$\begin{aligned}
\sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{y}_i &= \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} (\mathbf{X}_i \boldsymbol{\beta} + (\mathbf{e}\alpha_i + \boldsymbol{\varepsilon}_i)) = \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} (\mathbf{X}_i \boldsymbol{\beta} + \mathbf{u}_i) = \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{X}_i \boldsymbol{\beta} + \sigma_\varepsilon \boldsymbol{\Omega}^{-1/2} \mathbf{u}_i \\
(\mathbf{y}_i - \lambda \mathbf{e} \bar{y}_i) &= (\mathbf{X}_i - \lambda \mathbf{e} \bar{\mathbf{x}}_i') \boldsymbol{\beta} + [(1 - \lambda) \mathbf{e} \alpha_i + (\boldsymbol{\varepsilon}_i - \lambda \mathbf{e} \bar{\boldsymbol{\varepsilon}}_i)] & \lambda = 1 - \psi = 1 - \frac{\sigma_\varepsilon}{\sqrt{T\sigma_\alpha^2 + \sigma_\varepsilon^2}} \\
(y_{it} - \lambda \bar{y}_i) &= (\mathbf{x}_{it} - \lambda \bar{\mathbf{x}}_i)' \boldsymbol{\beta} + [(1 - \lambda) \alpha_i + (\varepsilon_{it} - \lambda \bar{\varepsilon}_i)]
\end{aligned}$$

Under weak form of weak/sequential exogeneity assumption  $\mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) = \mathbf{0}$  for  $s \leq t$ .

For  $s \leq t$ , we have

$$\begin{aligned}
\mathbb{E}(\mathbf{z}_{is}[(1-\lambda)\alpha_i + (\varepsilon_{it} - \lambda\bar{\varepsilon}_i)]) &= \mathbb{E}(\mathbf{z}_{is}(1-\lambda)\alpha_i + \mathbf{z}_{is}(\varepsilon_{it} - \lambda\bar{\varepsilon}_i)) \\
&= (1-\lambda)\mathbb{E}(\mathbf{z}_{is}\alpha_i) + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) - \lambda\mathbb{E}(\mathbf{z}_{is}\bar{\varepsilon}_i) \\
&= (1-\lambda)\mathbf{0} + \mathbf{0} - \lambda E(\mathbf{z}_{is} \sum_{t=1}^T \varepsilon_{it}/T) \\
&= -\frac{\lambda}{T}\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1} + \cdots + \mathbf{z}_{is}\varepsilon_{i,s-1} + \mathbf{z}_{is}\varepsilon_{is} + \cdots + \mathbf{z}_{is}\varepsilon_{iT}) \\
&= -\frac{\lambda}{T}(\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,s-1}) + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{is}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{iT})) \\
&= -\frac{\lambda}{T}(\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,s-1}) + \mathbf{0} + \cdots + \mathbf{0}) \\
&= -\frac{\lambda}{T}(\mathbb{E}(\mathbf{z}_{is}\varepsilon_{i1}) + \cdots + \mathbb{E}(\mathbf{z}_{is}\varepsilon_{i,s-1}))
\end{aligned}$$

So  $\mathbb{E}(\mathbf{z}_{it}(\varepsilon_{it} - \bar{\varepsilon}_i))$  is not necessarily equal to zero under weak form of weak/sequential exogeneity assumption.

If weak form of strong/strict exogeneity assumption is assumed  $\mathbb{E}(\mathbf{z}_{is}\varepsilon_{it}) = \mathbf{0} \forall s$ , then  $\mathbb{E}(\mathbf{z}_{is}(\varepsilon_{it} - \bar{\varepsilon}_i)) = \mathbf{0} \forall s$ . So,  $\mathbf{z}_{is}$ ,  $s = 1, \dots, T$  satisfy the exclusion restriction (exogeneity) requirement of valid instrument.

So, we have

$$\begin{aligned}
&\mathbb{E}(\mathbf{z}_{is}[(1-\lambda)\alpha_i + (\varepsilon_{it} - \lambda\bar{\varepsilon}_i)]) = \mathbf{0} && \text{for } \forall s \\
\iff &\mathbb{E}(\mathbf{Z}'_i[(1-\lambda)\mathbf{e}\alpha_i + (\boldsymbol{\varepsilon}_i - \lambda\mathbf{e}\bar{\varepsilon}_i)]) = \mathbf{0}
\end{aligned}$$

We can then apply IV estimation in GMM framework.

## 6 Dynamic Linear Panel Model

### 6.1 Assumption

#### 6.1.1 Weak/sequential Exogeneity

For  $t = 2, \dots, T$

$$\mathbb{E}(\varepsilon_{it} | y_{i,t-1}, \dots, y_{i1}, \alpha_i) = 0$$

This implies

$$\mathbb{E}(y_{is}\varepsilon_{it}) = 0, \quad \mathbb{E}(\varepsilon_{it}) = 0 \quad \text{and} \quad \mathbb{E}(\alpha_i\varepsilon_{it}) = 0 \quad \text{for } s < t$$

And

$$\text{Cov}(y_{is}, \varepsilon_{it}) = 0 \quad \text{and} \quad \text{Cov}(\alpha_i, \varepsilon_{it}) = 0 \quad \text{for } s < t$$

It is because

$$\begin{aligned} \text{Cov}(y_{is}, \varepsilon_{it}) &= \mathbb{E}(y_{is}\varepsilon_{it}) - \mathbb{E}(y_{is})\mathbb{E}(\varepsilon_{it}) \\ &= \mathbb{E}(\mathbb{E}(y_{is}\varepsilon_{it} | y_{i,t-1}, \dots, y_{i1}, \alpha_i)) - \mathbb{E}(y_{is})\mathbb{E}(\mathbb{E}(\varepsilon_{it} | y_{i,t-1}, \dots, y_{i1}, \alpha_i)) \\ &= \mathbb{E}(y_{is} \underbrace{\mathbb{E}(\varepsilon_{it} | y_{i,t-1}, \dots, y_{i1}, \alpha_i)}_0) - \mathbb{E}(y_{is})\mathbb{E}(\underbrace{\mathbb{E}(\varepsilon_{it} | y_{i,t-1}, \dots, y_{i1}, \alpha_i)}_0) \quad \text{as } s < t \\ &= 0 \end{aligned}$$

### 6.2 Model

#### 6.2.1 No transformation

$$y_{it} = \gamma y_{i,t-1} + \mathbf{x}'_{it}\boldsymbol{\beta} + \underbrace{(\alpha_i + \varepsilon_{it})}_{u_{it}}$$

$$\begin{aligned} \text{Cov}(y_{i,t-1}, \alpha_i) &= \text{Cov}(\gamma y_{i,t-2} + \mathbf{x}'_{i,t-1}\boldsymbol{\beta} + \alpha_i + \varepsilon_{i,t-1}, \alpha_i) \\ &= \gamma \text{Cov}(y_{i,t-2}, \alpha_i) + \text{Cov}(\mathbf{x}'_{i,t-1}\boldsymbol{\beta}, \alpha_i) + \text{Var}(\alpha_i) + \underbrace{\text{Cov}(\varepsilon_{i,t-1}, \alpha_i)}_0 \\ &= \gamma \text{Cov}(y_{i,t-2}, \alpha_i) + \boldsymbol{\beta}' \text{Cov}(\mathbf{x}_{i,t-1}, \alpha_i) + \text{Var}(\alpha_i) \\ &\neq 0 \quad \text{assume } \text{Cov}(\mathbf{x}_{i,t-1}, \alpha_i) \neq 0 \text{ and } \text{Var}(\alpha_i) > 0 \end{aligned}$$

so that

$$\begin{aligned} \text{Cov}(y_{i,t-1}, u_{it}) &= \text{Cov}(y_{i,t-1}, \alpha_i + \varepsilon_{it}) \\ &= \underbrace{\text{Cov}(y_{i,t-1}, \alpha_i)}_{\neq 0} + \underbrace{\text{Cov}(y_{i,t-1}, \varepsilon_{it})}_0 \\ &\neq 0 \end{aligned}$$

The necessary condition for OLS estimator to be unbiased is  $\mathbb{E}(u_{it} | y_{i,t-1}, \mathbf{x}_{it}) = 0$ . As  $\mathbb{E}(u_{it} | y_{i,t-1}, \mathbf{x}_{it}) = 0 \implies \text{Cov}(y_{i,t-1}, u_{it}) = 0$ . As a result,  $\text{Cov}(y_{i,t-1}, u_{it}) \neq 0 \implies \mathbb{E}(u_{it} | y_{i,t-1}, \mathbf{x}_{it}) \neq 0$ . Thus, OLS estimator is biased.

#### 6.2.2 Special case: no $\mathbf{x}_{it}$

$$y_{it} = \gamma y_{i,t-1} + \underbrace{(\alpha_i + \varepsilon_{it})}_{u_{it}}$$

The necessary condition for OLS estimator to be consistent is  $\mathbb{E}(y_{i,t-1}u_{it}) = 0$ . However,

$$\begin{aligned}\mathbb{E}(y_{i,t-1}u_{it}) &= \mathbb{E}(y_{i,t-1}(\alpha_i + \varepsilon_{it})) \\ &= \mathbb{E}(y_{i,t-1}\alpha_i) + \underbrace{\mathbb{E}(y_{i,t-1}\varepsilon_{it})}_0 > 0\end{aligned}$$

$$\begin{aligned}\mathbb{E}(y_{i,t-1}\alpha_i) &= \mathbb{E}((\gamma y_{i,t-2} + \alpha_i + \varepsilon_{i,t-1})\alpha_i) \\ &= \gamma \mathbb{E}(y_{i,t-2}\alpha_i) + \mathbb{E}(\alpha_i^2) + \mathbb{E}(\varepsilon_{i,t-1}\alpha_i) \\ &= \gamma \mathbb{E}((\gamma y_{i,t-3} + \alpha_i + \varepsilon_{i,t-2})\alpha_i) + \mathbb{E}(\alpha_i^2) + \mathbb{E}(\mathbb{E}(\varepsilon_{i,t-1}\alpha_i | y_{i,t-2}, \dots, y_{i1}, \alpha_i)) \\ &= \gamma^2 \mathbb{E}(y_{i,t-3}\alpha_i) + \gamma \mathbb{E}(\alpha_i^2) + \gamma \mathbb{E}(\varepsilon_{i,t-2}\alpha_i) + \mathbb{E}(\alpha_i^2) + \underbrace{\mathbb{E}(\alpha_i \mathbb{E}(\varepsilon_{i,t-1} | y_{i,t-2}, \dots, y_{i1}, \alpha_i))}_0 \\ &= \gamma^2 \mathbb{E}(y_{i,t-3}\alpha_i) + \gamma \mathbb{E}(\alpha_i^2) + \mathbb{E}(\alpha_i^2) \\ &\dots \\ &= \gamma^{t-2} \mathbb{E}(y_{i,t-(t-2+1)}) + \gamma^{t-2-1} \mathbb{E}(\alpha_i^2) + \dots + \mathbb{E}(\alpha_i^2) \\ &= \gamma^{t-2} \mathbb{E}(y_{i1}) + \gamma^{t-3} \mathbb{E}(\alpha_i^2) + \dots + \mathbb{E}(\alpha_i^2) \\ &= \gamma^{t-2} y_{i1} + \gamma^{t-3} \text{Var}(\alpha_i) + \dots + \text{Var}(\alpha_i) \quad y_{i1} \text{ is initial value and assume } \mathbb{E}(\alpha_i) = 0 \\ &> 0 \quad \text{assume } \text{Var}(\alpha_i) > 0, y_{i1} > 0 \text{ and } 0 < \gamma < 1\end{aligned}$$

Thus, OLS estimator is inconsistent. The necessary condition for OLS estimator to be unbiased is  $\mathbb{E}(u_{it} | y_{i,t-1}) = 0$ . As  $\mathbb{E}(u_{it} | y_{i,t-1}) = 0 \implies \mathbb{E}(y_{i,t-1}u_{it}) = 0$ ,  $\mathbb{E}(y_{i,t-1}u_{it}) \neq 0 \implies \mathbb{E}(u_{it} | y_{i,t-1}) \neq 0$ . Thus, OLS estimator is biased. It can also be seen by OVB formula.

$$\begin{aligned}\gamma_{short} &= \frac{\text{Cov}(y_{it}, y_{i,t-1})}{\text{Var}(y_{i,t-1})} \\ &= \frac{\text{Cov}(\gamma_{long} y_{i,t-1} + \alpha_i + \varepsilon_{it}, y_{i,t-1})}{\text{Var}(y_{i,t-1})} \\ &= \gamma_{long} + \frac{\text{Cov}(\alpha_i, y_{i,t-1})}{\text{Var}(y_{i,t-1})} + \frac{\overbrace{\text{Cov}(\varepsilon_{it}, y_{i,t-1})}^0}{\text{Var}(y_{i,t-1})} \\ &= \gamma_{long} + \frac{\text{Cov}(\alpha_i, y_{i,t-1})}{\text{Var}(y_{i,t-1})}\end{aligned}$$

$$\gamma_{short} - \gamma_{long} = \frac{\text{Cov}(\alpha_i, y_{i,t-1})}{\text{Var}(y_{i,t-1})} > 0 \quad \text{if } \text{Var}(y_{i,t-1}) > 0$$

$$\text{Cov}(\alpha_i, y_{i,t-1}) = \mathbb{E}(\alpha_i y_{i,t-1}) - \mathbb{E}(\alpha_i) \mathbb{E}(y_{i,t-1}) > 0 \quad \text{see above for } \mathbb{E}(\alpha_i y_{i,t-1}) > 0 \text{ and assume } \mathbb{E}(\alpha_i) = 0$$

Thus, OLS estimator is biased upward / over-estimate.

### 6.2.3 Within transformation

$$y_{it} - \bar{y}_i = \gamma(y_{i,t-1} - \bar{y}_{i,-1}) + (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)' \boldsymbol{\beta} + (\varepsilon_{it} - \bar{\varepsilon}_i)$$

$$\text{Cov}(y_{i,t-1}, \bar{\varepsilon}_i) = \text{Cov}(\gamma y_{i,t-2} + \mathbf{x}_{i,t-1}' \boldsymbol{\beta} + \alpha_i + \varepsilon_{i,t-1}, T^{-1} \sum_{t=1}^T \varepsilon_{it})$$

$$\neq 0$$

since  $\varepsilon_{i,t-1}$  is correlated with  $T^{-1} \sum_{t=1}^T \varepsilon_{it}$

so that

$$\text{Cov}(y_{i,t-1} - \bar{y}_{i,-1}, \varepsilon_{it} - \bar{\varepsilon}_i) \neq 0$$

The necessary condition for FE estimator to be unbiased is  $\mathbb{E}(\varepsilon_{it} - \bar{\varepsilon}_i | y_{i,t-1} - \bar{y}_{i,-1}, \mathbf{x}_{it} - \bar{\mathbf{x}}_i) = 0$ . As  $\mathbb{E}(\varepsilon_{it} - \bar{\varepsilon}_i | y_{i,t-1} - \bar{y}_{i,-1}, \mathbf{x}_{it} - \bar{\mathbf{x}}_i) = 0 \implies \text{Cov}(y_{i,t-1} - \bar{y}_{i,-1}, \varepsilon_{it} - \bar{\varepsilon}_i) = 0$ . As a result,  $\text{Cov}(y_{i,t-1} - \bar{y}_{i,-1}, \varepsilon_{it} - \bar{\varepsilon}_i) \neq 0 \implies \mathbb{E}(\varepsilon_{it} - \bar{\varepsilon}_i | y_{i,t-1} - \bar{y}_{i,-1}, \mathbf{x}_{it} - \bar{\mathbf{x}}_i) \neq 0$ . Thus, FE estimator is biased.

### 6.2.4 Special case: no $x_{it}$

$$y_{it} - \bar{y}_i = \gamma(y_{i,t-1} - \bar{y}_{i,-1}) + (\varepsilon_{it} - \bar{\varepsilon}_i)$$

The bias is called Nickell (1981) bias / dynamic panel bias. If  $\gamma > 0$ , the bias must be negative. The bias converges to zero when  $T \rightarrow \infty$ .

### 6.2.5 First difference transformation

$$\begin{aligned} \tilde{y}_i &= \tilde{X}_i \delta + \tilde{\varepsilon}_i \\ \begin{pmatrix} y_{i3} - y_{i2} \\ \vdots \\ y_{iT} - y_{i,T-1} \end{pmatrix} &= \begin{pmatrix} y_{i2} - y_{i1} & (x_{i3} - x_{i2})' \\ \vdots & \\ y_{i,T-1} - y_{i,T-2} & (x_{iT} - x_{i,T-1})' \end{pmatrix} \begin{pmatrix} \gamma \\ \beta \end{pmatrix} + \begin{pmatrix} \varepsilon_{i3} - \varepsilon_{i2} \\ \vdots \\ \varepsilon_{iT} - \varepsilon_{i,T-1} \end{pmatrix} \\ y_{it} - y_{i,t-1} &= \gamma(y_{i,t-1} - y_{i,t-2}) + (x_{it} - x_{i,t-1})' \beta + (\varepsilon_{it} - \varepsilon_{i,t-1}) \end{aligned} \quad t \geq 3$$

$$\begin{aligned} Cov(y_{i,t-1} - y_{i,t-2}, \varepsilon_{it} - \varepsilon_{i,t-1}) &= Cov(y_{i,t-1}, \varepsilon_{it}) - Cov(y_{i,t-1}, \varepsilon_{i,t-1}) - Cov(y_{i,t-2}, \varepsilon_{it}) + Cov(y_{i,t-2}, \varepsilon_{i,t-1}) \\ &= 0 - Cov(y_{i,t-1}, \varepsilon_{i,t-1}) - 0 + 0 \quad \text{as } Cov(y_{is}, \varepsilon_{it}) = 0 \text{ for } s < t \\ &= -Cov(\gamma y_{i,t-2} + x'_{i,t-1} \beta + \alpha_i + \varepsilon_{i,t-1}, \varepsilon_{i,t-1}) \\ &= -\gamma \underbrace{Cov(y_{i,t-2}, \varepsilon_{i,t-1})}_0 - \beta' \underbrace{Cov(x_{i,t-1}, \varepsilon_{i,t-1})}_0 - \underbrace{Cov(\alpha_i, \varepsilon_{i,t-1})}_0 - Var(\varepsilon_{i,t-1}) \\ &< 0 \quad \text{assume } Var(\varepsilon_{i,t-1}) > 0 \end{aligned}$$

The necessary condition for FD estimator to be unbiased is  $\mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1} | y_{i,t-1} - y_{i,t-2}, x_{it} - x_{i,t-1}) = 0$ . As  $\mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1} | y_{i,t-1} - y_{i,t-2}, x_{it} - x_{i,t-1}) = 0 \implies Cov(y_{i,t-1} - y_{i,t-2}, \varepsilon_{it} - \varepsilon_{i,t-1}) = 0$ . As a result,  $Cov(y_{i,t-1} - y_{i,t-2}, \varepsilon_{it} - \varepsilon_{i,t-1}) \neq 0 \implies \mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1} | y_{i,t-1} - y_{i,t-2}, x_{it} - x_{i,t-1}) \neq 0$ . Thus, FD estimator is biased.

### 6.2.6 Special case: no $x_{it}$

$$y_{it} - y_{i,t-1} = \gamma(y_{i,t-1} - y_{i,t-2}) + (\varepsilon_{it} - \varepsilon_{i,t-1})$$

The necessary condition for FD estimator to be consistent is  $\mathbb{E}((y_{i,t-1} - y_{i,t-2})(\varepsilon_{it} - \varepsilon_{i,t-1})) = 0$ . However,

$$\begin{aligned} \mathbb{E}((y_{i,t-1} - y_{i,t-2})(\varepsilon_{it} - \varepsilon_{i,t-1})) &= \mathbb{E}(y_{i,t-1} \varepsilon_{it}) - \mathbb{E}(y_{i,t-1} \varepsilon_{i,t-1}) - \mathbb{E}(y_{i,t-2} \varepsilon_{it}) + \mathbb{E}(y_{i,t-2} \varepsilon_{i,t-1}) \\ &= 0 - \mathbb{E}(y_{i,t-1} \varepsilon_{i,t-1}) - 0 + 0 \quad \text{as } \mathbb{E}(y_{is} \varepsilon_{it}) = 0 \text{ for } s < t \\ &= -\mathbb{E}((\gamma y_{i,t-2} + \alpha_i + \varepsilon_{i,t-1}) \varepsilon_{i,t-1}) \\ &= -\gamma \underbrace{\mathbb{E}(y_{i,t-2} \varepsilon_{i,t-1})}_0 - \underbrace{\mathbb{E}(\alpha_i \varepsilon_{i,t-1})}_0 - \mathbb{E}(\varepsilon_{i,t-1}^2) \\ &= -Var(\varepsilon_{i,t-1}) \quad \text{as } \mathbb{E}(\varepsilon_{i,t-1}) = 0 \\ &< 0 \quad \text{assume } Var(\varepsilon_{i,t-1}) > 0 \end{aligned}$$

Thus, FD estimator is inconsistent. The necessary condition for FD estimator to be unbiased is  $\mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1} | y_{i,t-1} - y_{i,t-2}) = 0$ . As  $\mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1} | y_{i,t-1} - y_{i,t-2}) = 0 \implies \mathbb{E}((y_{i,t-1} - y_{i,t-2})(\varepsilon_{it} - \varepsilon_{i,t-1})) = 0$ ,  $\mathbb{E}((y_{i,t-1} - y_{i,t-2})(\varepsilon_{it} - \varepsilon_{i,t-1})) \neq 0 \implies \mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1} | y_{i,t-1} - y_{i,t-2}) \neq 0$ . Thus, FD estimator is biased.

Thus, IV estimation (for just-identified case) or 2SLS estimation (for over-identified case) is applied. IV and 2SLS estimations are special cases of GMM estimation.

Under weak/sequential exogeneity,  $Cov(y_{is}, \varepsilon_{it}) = 0$  for  $s < t$ . This implies for  $s < t - 1 \iff s \leq t - 2$

$$\begin{aligned} Cov(y_{is}, \varepsilon_{it} - \varepsilon_{i,t-1}) &= Cov(y_{is}, \varepsilon_{it}) - Cov(y_{is}, \varepsilon_{i,t-1}) \\ &= 0 - Cov(y_{is}, \varepsilon_{i,t-1}) \quad \text{as } s < t - 1 \implies s < t \\ &= 0 \quad \text{as } s < t - 1 \end{aligned}$$

Note that  $Cov(y_{is}, \varepsilon_{it} - \varepsilon_{i,t-1}) = 0 \implies \mathbb{E}(y_{is}(\varepsilon_{it} - \varepsilon_{i,t-1})) = 0$  as  $\mathbb{E}(\varepsilon_{it} - \varepsilon_{i,t-1}) = 0$  under weak/sequential exogeneity. So,  $y_{is}$  for  $s \leq t-2$  satisfy the exclusion restriction (exogeneity) requirement of valid instrument. i.e.,

$$\tilde{z}'_{i3} = (y_{i1}, \Delta \mathbf{x}'_{i3}) \quad \text{at } t = 3$$

$$\tilde{z}'_{i4} = (y_{i1}, y_{i2}, \Delta \mathbf{x}'_{i4}) \quad \text{at } t = 4$$

...

$$\tilde{z}'_{iT} = (y_{i1}, \dots, y_{i,T-2}, \Delta \mathbf{x}'_{iT}) \quad \text{at } t = T$$

That is,  $\tilde{z}'_{it} = [y_{i1}, \dots, y_{i,t-2}, \Delta \mathbf{x}'_{it}]$ .  $\mathbf{Z}_i = \begin{pmatrix} \tilde{z}'_{i3} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \tilde{z}'_{i4} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \tilde{z}'_{iT} \end{pmatrix}$

So, we have

$$\begin{aligned} \mathbb{E}(\tilde{\mathbf{z}}_{it}(\varepsilon_{it} - \varepsilon_{i,t-1})) &= \mathbf{0} \\ \iff \mathbb{E}(\mathbf{Z}'_i \Delta \varepsilon_i) &= \mathbf{0} \end{aligned}$$

We can then apply 2SLS estimation in GMM framework. This is the same as Arellano-Bond estimator with 2SLS weight.

### 6.2.7 Anderson-Hsiao estimator

Anderson & Hsiao (1981) considers a special case  $y_{is}$  for  $s = t-2$  i.e.,  $y_{i,t-2}$  as the instrument since they not only satisfy the exclusion restriction (exogeneity) requirement but also satisfy the relevancy requirement of valid instrument i.e., correlates with  $y_{i,t-1} - y_{i,t-2}$ . Thus,  $\tilde{z}'_{it} = [y_{i,t-2}, \Delta \mathbf{x}'_{it}]$

$$\mathbf{Z}_i = \begin{pmatrix} (y_{i1} \quad \Delta \mathbf{x}'_{i3}) & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & (y_{i2} \quad \Delta \mathbf{x}'_{i4}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & (y_{i,T-2} \quad \Delta \mathbf{x}'_{iT}) \end{pmatrix}$$

and

$$\tilde{z}'_{it} = [ \underbrace{\Delta y_{i,t-2}}_{y_{i,t-2} - y_{i,t-3}}, \Delta \mathbf{x}'_{it}]$$

$$\mathbf{Z}_i = \begin{pmatrix} (\Delta y_{i2} \quad \Delta \mathbf{x}'_{i4}) & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & (\Delta y_{i3} \quad \Delta \mathbf{x}'_{i5}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & (\Delta y_{i,T-2} \quad \Delta \mathbf{x}'_{iT}) \end{pmatrix}$$

As only one instrument is used at each  $t$ , the number of moments is equal to the number of parameters i.e.,  $r = K$ . In such case, GMM estimation = MM estimation = IV estimation.

$$\hat{\delta}_{AH}^{pgmm} = [\sum_{i=1}^N \mathbf{Z}'_i \tilde{\mathbf{X}}_i]^{-1} \sum_{i=1}^N \mathbf{Z}'_i \tilde{\mathbf{y}}_i = \hat{\delta}_{AH}^{piv}$$

### 6.2.8 Arellano-Bond estimator

Arellano & Bond (1991) considers all the possible cases i.e.,  $y_{is}$  for  $s \leq t-2$ . Except  $t = 3$ , more than one instruments are used, number of moments is larger than the number of parameters i.e.,  $r > K$ . GMM estimation is 2SLS estimation if  $\mathbf{W}_N = (\sum_{i=1}^N \mathbf{Z}'_i \mathbf{Z}_i)^{-1}$ .

$$\tilde{z}'_{it} = [y_{i1}, \dots, y_{i,t-2}, \Delta \mathbf{x}'_{it}]$$

$$\mathbf{Z}_i = \begin{pmatrix} (y_{i1} \quad \Delta \mathbf{x}'_{i3}) & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & (y_{i1} \quad y_{i2} \quad \Delta \mathbf{x}'_{i4}) & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & (y_{i1} \quad \dots \quad y_{i,T-2} \quad \Delta \mathbf{x}'_{iT}) \end{pmatrix}$$



$$\hat{\boldsymbol{\delta}}_{AB}^{pgmm} = [(\sum_{i=1}^N \tilde{\mathbf{X}}_i' \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}_i' \tilde{\mathbf{X}}_i)]^{-1} (\sum_{i=1}^N \tilde{\mathbf{X}}_i' \mathbf{Z}_i) \mathbf{W}_N (\sum_{i=1}^N \mathbf{Z}_i' \tilde{\mathbf{y}}_i)$$

If  $\mathbf{W}_N = (\sum_{i=1}^N \mathbf{Z}_i' \mathbf{Z}_i)^{-1}$ ,  $\hat{\boldsymbol{\delta}}_{AB}^{pgmm} = \hat{\boldsymbol{\delta}}_{AB}^{2SLS}$

If  $\mathbf{W}_N = \hat{\mathbf{S}}^{-1}$ ,  $\hat{\boldsymbol{\delta}}_{AB}^{pgmm} = \hat{\boldsymbol{\delta}}_{AB}^{opgmm}$

## 7 Pooled Model and Clustered Standard Error

$$y_{it} = \mathbf{x}'_{it}\boldsymbol{\beta} + \alpha_i + \varepsilon_{it} \quad \text{Level 1}$$

$$\begin{aligned} \begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix} &= \begin{pmatrix} \mathbf{x}'_{i1} \\ \vdots \\ \mathbf{x}'_{iT} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \alpha_i + \begin{pmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{pmatrix} \\ \mathbf{y}_i &= \mathbf{X}_i \boldsymbol{\beta} + \underbrace{(\mathbf{e}_i \alpha_i + \boldsymbol{\varepsilon}_i)}_{\mathbf{u}_i} \end{aligned} \quad \text{Level 2}$$

$$\begin{aligned} \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{pmatrix} &= \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_N \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \mathbf{e} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{e} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_1 \\ \vdots \\ \boldsymbol{\varepsilon}_N \end{pmatrix} \\ \mathbf{y} &= \mathbf{X} \boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{e}) \boldsymbol{\alpha} + \boldsymbol{\varepsilon} \end{aligned} \quad \text{Level 3}$$

where  $\alpha_i$  is unobserved heterogeneity,  $\varepsilon_i$  is idiosyncratic error,  $\mathbf{u}_i$  is composite error.

If  $\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0$ , OLS estimator is likely to be unbiased and consistent.  $\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0 \implies \mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) = \mathbf{0}$  as  $\mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) = \mathbb{E}(\mathbf{X}'_i \mathbf{e} \alpha_i + \boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \mathbf{X}'_i \mathbf{e} \mathbb{E}(\alpha_i | \mathbf{X}_i) + \mathbb{E}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \mathbf{0}$ . Thus, the necessary condition for OLS estimator to be unbiased is satisfied if  $\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0$ .

$\mathbb{E}(\mathbf{u}_i | \mathbf{X}_i) = \mathbf{0} \implies \mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) = \mathbf{0}$  as  $\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i) = \mathbb{E}(\mathbb{E}(\mathbf{X}'_i \mathbf{u}_i | \mathbf{X}_i)) = \mathbb{E}(\mathbf{X}'_i \mathbb{E}(\mathbf{u}_i | \mathbf{X}_i)) = \mathbb{E}(\mathbf{X}'_i \mathbf{0}) = \mathbf{0}$ . Thus, the necessary condition for OLS estimator to be consistent is satisfied if  $\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0$ .

$\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0 \implies \mathbb{E}(\alpha_i \mathbf{X}_i) = \mathbf{0}$  and  $\mathbb{E}(\alpha_i) = 0$ . Thus,  $\mathbb{E}(\alpha_i | \mathbf{X}_i) = 0 \implies \text{Cov}(\alpha_i, \mathbf{X}_i) = \mathbf{0}$  as  $\text{Cov}(\alpha_i, \mathbf{X}_i) = \mathbb{E}(\alpha_i \mathbf{X}_i) - \mathbb{E}(\alpha_i) \mathbb{E}(\mathbf{X}_i) = \mathbf{0}$ .

$$\begin{aligned} \hat{\boldsymbol{\beta}}_{pooled}^{ols} &= \left[ \sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}'_i \mathbf{y}_i \quad \text{Level 2} \\ &= \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}'_{it} \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} y_{it} \quad \text{Level 1} \end{aligned}$$

From now on in this section, we assume  $\alpha_i = 0$  for  $\forall i$ . Therefore,  $\mathbf{u}_i = \boldsymbol{\varepsilon}_i$ .

$$\text{Var}(\hat{\boldsymbol{\beta}}_{pooled}^{ols} | \mathbf{X}_i) = \left[ \sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}'_i \text{Var}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) \mathbf{X}_i \left[ \sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1}$$

If  $\varepsilon_{it}$  is homoskedasticity and serially uncorrelated across  $t$  i.e.,  $\text{Var}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \sigma_\varepsilon^2 \mathbf{I}_T$  (further assume independence of  $i$  and strict exogeneity), we have  $\boldsymbol{\varepsilon}_i | \mathbf{X}_i \sim iid [\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_T]$

$$\begin{aligned} &= \sigma_\varepsilon^2 \left[ \sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1} \\ &= \sigma_\varepsilon^2 \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}'_{it} \right]^{-1} \end{aligned}$$

If  $\text{Var}(\boldsymbol{\varepsilon}_i | \mathbf{X}_i) = \boldsymbol{\Omega}_i$ , we have  $\boldsymbol{\varepsilon}_i | \mathbf{X}_i \sim inid [\mathbf{0}, \boldsymbol{\Omega}_i]$

$$\begin{aligned} &= \left[ \sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}'_i \overbrace{\mathbb{E}[\boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}'_i | \mathbf{X}_i]}^{\boldsymbol{\Omega}_i} \mathbf{X}_i \left[ \sum_{i=1}^N \mathbf{X}'_i \mathbf{X}_i \right]^{-1} \\ &= \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}'_{it} \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbf{x}_{it} \mathbb{E}[\varepsilon_{it} \varepsilon_{is} | \mathbf{X}_i] \mathbf{x}'_{is} \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}'_{it} \right]^{-1} \end{aligned}$$

## 7.1 Block bootstrapping

Suggested by MacKinnon, Nielsen, & Webb (2022).

## 7.2 Clustered standard error with independence of $i$

To be more precise, clustered covariance matrix is discussed here.

### 7.2.1 Liang & Zeger (1986) and Arellano (1987)

Clustered standard error can handle both heteroscedasticity and serial correlation within a cluster/group.

$$\widehat{Var}(\hat{\beta}_{pooled}^{ols} | \mathbf{X}_i) = \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \hat{\Omega}_i \mathbf{X}_i \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{X}_i \right]^{-1}$$

$$\hat{\Omega}_i = \hat{\varepsilon}_i \hat{\varepsilon}_i' = \begin{pmatrix} \hat{\varepsilon}_{i1}^2 & \hat{\varepsilon}_{i1}\hat{\varepsilon}_{i2} & \cdots & \hat{\varepsilon}_{i1}\hat{\varepsilon}_{iT} \\ \vdots & \hat{\varepsilon}_{i2}^2 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\varepsilon}_{iT}\hat{\varepsilon}_{i1} & \cdots & \hat{\varepsilon}_{iT}\hat{\varepsilon}_{i,T-1} & \hat{\varepsilon}_{iT}^2 \end{pmatrix}$$

$$\hat{\Omega} = \begin{pmatrix} \hat{\Omega}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \hat{\Omega}_2 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \hat{\Omega}_N \end{pmatrix}$$

$$\begin{aligned} \widehat{Var}(\hat{\beta}_{pooled}^{ols} | \mathbf{X}_i) &= \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbf{x}_{it} \hat{\varepsilon}_{it} \hat{\varepsilon}_{is} \mathbf{x}_{is}' \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \\ &= \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \left( \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \hat{\varepsilon}_{it}^2 \mathbf{x}_{it}' + \right. \\ &\quad \left. \sum_{l=1}^T \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \hat{\varepsilon}_{it} \hat{\varepsilon}_{i,t-l} \mathbf{x}_{i,t-l}' \right] + \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it} \hat{\varepsilon}_{it} \hat{\varepsilon}_{i,t-l} \mathbf{x}_{i,t-l}')' \right) \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \end{aligned}$$

It is the panel generalization of Eicker-Huber-White estimator (White, 1980). If there is no serial correlation within the cluster/group, clustered standard error reduces to the exact form of Eicker-Huber-White estimator i.e.,

$$\begin{aligned} \widehat{Var}(\hat{\beta}_{pooled}^{ols} | \mathbf{X}_i) &= \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{X}_i \right]^{-1} \sum_{i=1}^N \mathbf{X}_i' \text{diag}(\hat{\varepsilon}_{it}^2) \mathbf{X}_i \left[ \sum_{i=1}^N \mathbf{X}_i' \mathbf{X}_i \right]^{-1} \\ &= \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \hat{\varepsilon}_{it}^2 \mathbf{x}_{it}' \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \end{aligned}$$

### 7.2.2 Panel Newey-West (Petersen, 2009)

A weight can also be added to clustered standard error, this generalizes the Newey-West estimator (Newey & West, 1987).

$$\widehat{Var}(\hat{\beta}_{pooled}^{ols} | \mathbf{X}_i) = \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbf{x}_{it} w_{t,s} \hat{\varepsilon}_{it} \hat{\varepsilon}_{is} \mathbf{x}_{is}' \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1}$$

where

$$w_{t,s} = \begin{cases} 1 - \frac{|s-t|}{L+1} & \text{if } |s-t| \leq L \\ 0 & \text{otherwise} \end{cases}$$

This can also be written as

$$\widehat{Var}(\hat{\beta}_{pooled}^{ols}|\mathbf{X}_i) = [\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it}\mathbf{x}'_{it}]^{-1} (\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it}\hat{\varepsilon}_{it}^2\mathbf{x}'_{it} + \sum_{l=1}^L w_l [\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it}\hat{\varepsilon}_{it}\hat{\varepsilon}_{i,t-l}\mathbf{x}'_{i,t-l} + \sum_{i=1}^N \sum_{t=1}^T (\mathbf{x}_{it}\hat{\varepsilon}_{it}\hat{\varepsilon}_{i,t-l}\mathbf{x}'_{i,t-l})']) [\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it}\mathbf{x}'_{it}]^{-1}$$

where  $w_l = 1 - \frac{l}{L+1}$ . Petersen (2009) finds that this adjustment is worse than the one without weight.

### 7.2.3 Bias-Reduced Linearization (BRL) adjustment (Bell & McCaffrey, 2002)

Finite sample adjustment e.g.,  $\frac{N}{N-1} \frac{T-1}{T-K}$  is multiplied in Stata (generalization of HC1 in MacKinnon & White (1985)). If  $N$  (the number of cluster) is small, clustered standard error is inconsistent because law of large number cannot be applied (even  $T \rightarrow \infty$ ). However, we can adjust it by Bell & McCaffrey (2002)'s Bias-Reduced Linearization (BRL) adjustment (generalization of HC2) and use  $t$ -distribution with  $N - K$  degree of freedom, instead of standard normal distribution.

In BRL adjustment, we replace  $\hat{\varepsilon}_i$  by

$$\tilde{\varepsilon}_i = \mathbf{A}_i \hat{\varepsilon}_i$$

where  $\mathbf{A}'_i \mathbf{A}_i = (\mathbf{I}_T - \mathbf{H}_i)^{-1}$  where  $\mathbf{H}_i = \mathbf{X}_i(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'_i$  the projection/hat matrix.

There are many possible  $\mathbf{A}_i$ , Bell & McCaffrey (2002) uses eigen-decomposition of the inverse of the residual marker  $\mathbf{I}_T - \mathbf{H}_i$  i.e.,

$$\begin{aligned} (\mathbf{I}_T - \mathbf{H}_i)^{-1} &= \mathbf{P}\mathbf{\Lambda}\mathbf{P}' \\ &= \mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}\mathbf{P}' \\ &= \mathbf{P}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2'}\mathbf{P}' \\ &= \mathbf{P}\mathbf{\Lambda}^{1/2}(\mathbf{P}\mathbf{\Lambda}^{1/2})' \\ &= \mathbf{A}'\mathbf{A}'' \end{aligned}$$

where  $\mathbf{P}$  is a matrix in which vectors are eigenvectors;  $\mathbf{\Lambda}$  is a diagonal matrix with eigenvalues items.

## 7.3 Clustered standard error with dependence of $i$

### 7.3.1 Spatial Correlation Consistent (SCC) estimator (Driscoll & Kraay, 1998)

$$\widehat{Var}(\hat{\beta}_{pooled}^{ols}|\mathbf{X}_i) = [\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it}\mathbf{x}'_{it}]^{-1} \sum_{i=1}^N \sum_{j=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbf{x}_{it}w_{t,s}\hat{\varepsilon}_{it}\hat{\varepsilon}_{js}\mathbf{x}'_{js} [\sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it}\mathbf{x}'_{it}]^{-1}$$

where

$$w_{t,s} = \begin{cases} 1 - \frac{|s-t|}{L+1} & \text{if } |s-t| \leq L \\ 0 & \text{otherwise} \end{cases}$$

This can also be written as

$$\widehat{Var}(\hat{\beta}_{pooled}^{ols}|\mathbf{X}_t) = [\sum_{t=1}^T \mathbf{X}'_t \mathbf{X}_t]^{-1} (\sum_{t=1}^T \mathbf{X}'_t \hat{\varepsilon}_t \tilde{\varepsilon}'_t \mathbf{X}_t + \sum_{l=1}^L w_l [\sum_{t=1}^T \mathbf{X}'_t \hat{\varepsilon}_t \tilde{\varepsilon}'_{t-l} \mathbf{X}_{t-l} + \sum_{t=1}^T (\mathbf{X}'_t \hat{\varepsilon}_t \tilde{\varepsilon}'_{t-l} \mathbf{X}_{t-l})']) [\sum_{t=1}^T \mathbf{X}'_t \mathbf{X}_t]^{-1}$$

where  $w_l = 1 - \frac{l}{L+1}$ . It requires large  $T$  while  $L$  is up to you.

## 7.4 Fama-Macbeth estimation

Fama-Macbeth estimation (Fama & Macbeth, 1973) was invented before the development of linear panel model in Econometrics. It is still widely applied in the areas of empirical asset pricing. Petersen (2009) finds that FM standard error performs

well when there is only time fixed effect and no OVB in his simulation setting.

$$y_{ti} = \mathbf{x}'_{ti}\boldsymbol{\beta} + \varepsilon_{ti} \quad \text{Level 1}$$

$$\begin{aligned} \begin{pmatrix} y_{t1} \\ \vdots \\ y_{tN} \end{pmatrix} &= \begin{pmatrix} \mathbf{x}'_{t1} \\ \vdots \\ \mathbf{x}'_{tN} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \varepsilon_{t1} \\ \vdots \\ \varepsilon_{tN} \end{pmatrix} \\ \mathbf{y}_t &= \mathbf{X}_t \boldsymbol{\beta} + \boldsymbol{\varepsilon}_t \quad \text{Level 2} \end{aligned}$$

Fama-Macbeth estimator is

$$\hat{\boldsymbol{\beta}}_{FM} = \frac{1}{T} \sum_{t=1}^T [(\mathbf{X}'_t \mathbf{X}_t)^{-1} \mathbf{X}'_t \mathbf{y}_t]$$

Its covariance matrix is

$$\begin{aligned} \text{Var}(\hat{\boldsymbol{\beta}}_{FM} | \mathbf{X}_t) &= \text{Var}\left(\frac{1}{T} \sum_{t=1}^T [(\mathbf{X}'_t \mathbf{X}_t)^{-1} \mathbf{X}'_t \mathbf{y}_t] | \mathbf{X}_t\right) \\ &= \frac{1}{T^2} \text{Var}\left(\sum_{t=1}^T [(\mathbf{X}'_t \mathbf{X}_t)^{-1} \mathbf{X}'_t \mathbf{y}_t] | \mathbf{X}_t\right) \\ &= \frac{1}{T^2} \sum_{t=1}^T \text{Var}([(\mathbf{X}'_t \mathbf{X}_t)^{-1} \mathbf{X}'_t \mathbf{y}_t] | \mathbf{X}_t) \quad \text{due to independence} \\ &= \frac{1}{T^2} \sum_{t=1}^T (\mathbf{X}'_t \mathbf{X}_t)^{-1} \mathbf{X}'_t \text{Var}(\mathbf{y}_t | \mathbf{X}_t) \mathbf{X}_t (\mathbf{X}'_t \mathbf{X}_t)^{-1} \\ &= \frac{1}{T^2} \sum_{t=1}^T (\mathbf{X}'_t \mathbf{X}_t)^{-1} \mathbf{X}'_t \underbrace{\text{Var}(\boldsymbol{\varepsilon}_t | \mathbf{X}_t)}_{\boldsymbol{\Omega}_t} \mathbf{X}_t (\mathbf{X}'_t \mathbf{X}_t)^{-1} \end{aligned}$$

Cochrane (2005) demonstrates that Fama-Macbeth estimator is equivalent to pooled OLS estimator if  $\mathbf{X}_t = \mathbf{X}$  i.e., not time changing. FM variance is same as clustered standard error if  $\boldsymbol{\Omega}_t = \boldsymbol{\Sigma}$  in addition to the assumption just mentioned.

$$\begin{aligned} \hat{\boldsymbol{\beta}}_{pooled}^{ols} &= \left(\sum_{t=1}^T \mathbf{X}' \mathbf{X}\right)^{-1} \sum_{t=1}^T \mathbf{X}' \mathbf{y}_t \\ &= (T \mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \sum_{t=1}^T \mathbf{y}_t \\ &= (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \frac{1}{T} \sum_{t=1}^T \mathbf{y}_t = \hat{\boldsymbol{\beta}}_{FM} \end{aligned}$$

$$\begin{aligned} \text{Var}(\hat{\boldsymbol{\beta}}_{pooled}^{ols} | \mathbf{X}) &= \left[\sum_{t=1}^T \mathbf{X}' \mathbf{X}\right]^{-1} \sum_{t=1}^T \mathbf{X}' \boldsymbol{\Sigma} \mathbf{X} \left[\sum_{t=1}^T \mathbf{X}' \mathbf{X}\right]^{-1} \\ &= [T \mathbf{X}' \mathbf{X}]^{-1} T \mathbf{X}' \boldsymbol{\Sigma} \mathbf{X} [T \mathbf{X}' \mathbf{X}]^{-1} \\ &= \frac{1}{T} [\mathbf{X}' \mathbf{X}]^{-1} \mathbf{X}' \boldsymbol{\Sigma} \mathbf{X} [\mathbf{X}' \mathbf{X}]^{-1} \end{aligned}$$

$$\text{Var}(\hat{\boldsymbol{\beta}}_{FM} | \mathbf{X}) = \frac{1}{T^2} T (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \boldsymbol{\Sigma} \mathbf{X} (\mathbf{X}' \mathbf{X})^{-1} = \text{Var}(\hat{\boldsymbol{\beta}}_{pooled}^{ols} | \mathbf{X})$$

## 7.5 Petersen (2009) Simulation Result

### 7.5.1 Only individual fixed effect

$$y_{it} = \mathbf{x}'_{it} \boldsymbol{\beta} + \alpha_i + \varepsilon_{it}$$

If there is only  $\alpha_i$  (individual fixed effect) and  $\alpha_i$  is not correlated with  $\mathbf{x}_{it}$  (so no OVB), OLS estimator is unbiased and clustered standard error clustered by individual i.e.,

$$\left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbf{x}_{it} \hat{\varepsilon}_{it} \hat{\varepsilon}_{is} \mathbf{x}_{is}' \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1}$$

is unbiased. In contrast, conventional standard error i.e.,

$$\hat{\sigma}_\varepsilon^2 \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1}$$

Eicker-Huber-White standard error i.e.,

$$\left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \hat{\varepsilon}_{it}^2 \mathbf{x}_{it}' \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1}$$

Newey-West standard error i.e.,

$$\left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbf{x}_{it} w_{t,s} \hat{\varepsilon}_{it} \hat{\varepsilon}_{is} \mathbf{x}_{is}' \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1}$$

Fama-Macbeth standard error are biased downward (over-rejection).

### 7.5.2 Only time fixed effect

$$y_{it} = \mathbf{x}_{it}' \boldsymbol{\beta} + \gamma_t + \varepsilon_{it}$$

If there is only  $\gamma_t$  (time fixed effect) and  $\gamma_t$  is not correlated with  $\mathbf{x}_{it}$  (so no OVB), OLS estimator is unbiased and Fama-Macbeth standard error i.e.,

$$\frac{1}{T^2} \sum_{t=1}^T (\mathbf{X}_t' \mathbf{X}_t)^{-1} \mathbf{X}_t' \boldsymbol{\Omega}_t \mathbf{X}_t (\mathbf{X}_t' \mathbf{X}_t)^{-1}$$

and clustered standard error clustered by time (only if  $T$  is large) is i.e.,

$$\left[ \sum_{t=1}^T \sum_{i=1}^N \mathbf{x}_{ti} \mathbf{x}_{ti}' \right]^{-1} \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^N \mathbf{x}_{ti} \hat{\varepsilon}_{ti} \hat{\varepsilon}_{tj} \mathbf{x}_{tj}' \left[ \sum_{t=1}^T \sum_{i=1}^N \mathbf{x}_{ti} \mathbf{x}_{ti}' \right]^{-1}$$

unbiased. In contrast, conventional standard error are biased downward (over-rejection).

### 7.5.3 Both individual and time fixed effect

$$y_{it} = \mathbf{x}_{it}' \boldsymbol{\beta} + \alpha_i + \gamma_t + \varepsilon_{it}$$

Cameron, Gelbach & Miller (2011), and Thompson (2011) suggests Clustered standard error clustered by individual + Clustered standard error clustered by time - Eicker-Huber-White standard error. i.e.,

$$\begin{aligned} & \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbf{x}_{it} \hat{\varepsilon}_{it} \hat{\varepsilon}_{is} \mathbf{x}_{is}' \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} + \left[ \sum_{t=1}^T \sum_{i=1}^N \mathbf{x}_{ti} \mathbf{x}_{ti}' \right]^{-1} \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^N \mathbf{x}_{ti} \hat{\varepsilon}_{ti} \hat{\varepsilon}_{tj} \mathbf{x}_{tj}' \left[ \sum_{t=1}^T \sum_{i=1}^N \mathbf{x}_{ti} \mathbf{x}_{ti}' \right]^{-1} - \\ & \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \hat{\varepsilon}_{it}^2 \mathbf{x}_{it}' \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \\ & = \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \left( \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbf{x}_{it} \hat{\varepsilon}_{it} \hat{\varepsilon}_{is} \mathbf{x}_{is}' + \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^N \mathbf{x}_{ti} \hat{\varepsilon}_{ti} \hat{\varepsilon}_{tj} \mathbf{x}_{tj}' - \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \hat{\varepsilon}_{it}^2 \mathbf{x}_{it}' \right) \left[ \sum_{i=1}^N \sum_{t=1}^T \mathbf{x}_{it} \mathbf{x}_{it}' \right]^{-1} \end{aligned}$$

The last term is subtracted in order to prevent double-counting of diagonal items. Simulation shows it works.

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## 9 Appendix - R Code

```
library(lmtest)
library(plm)
library(fastDummies)
library(tidyverse)

set.seed(15)

gen_long_data <- function (N, T, var_gamma_i, static = TRUE) {
  df <- tibble(.rows = N) %>% mutate(i = seq_len(N))
  df[["gamma_i"]] <- rnorm(N, mean = 0, sd = sqrt(var_gamma_i))

  for (t in seq_len(T)) {
    df[str_c("x_i", t)] <-
      df[["gamma_i"]] + rnorm(N, mean = 0, sd = sqrt(1 - var_gamma_i))

    if (t == 1 | static) {
      df[str_c("y_i", t)] <-
        2 + df[[str_c("x_i", t)]] * 2 + df[["gamma_i"]] + rnorm(N, mean = 0, sd = sqrt(4 - var_gamma_i))
    } else {
      df[str_c("ly_i", t)] <- df[[str_c("y_i", t - 1)]]
      df[str_c("y_i", t)] <-
        2 + df[[str_c("y_i", t - 1)]] * 0.5 + df[[str_c("x_i", t)]] * 2 + df[["gamma_i"]] +
          rnorm(N, mean = 0, sd = sqrt(4 - var_gamma_i))
    }
  }

  if (static) {
    df %>%
      select(-gamma_i) %>%
      pivot_longer(
        cols = starts_with(c("y_i", "x_i")),
        names_to = c(".value", "t"),
        names_sep = "_i"
      )
  } else {
    df %>%
      select(-gamma_i) %>%
      pivot_longer(
        cols = starts_with(c("y_i", "ly_i", "x_i")),
        names_to = c(".value", "t"),
        names_sep = "_i"
      )
  }
}

long_data_static <-
  gen_long_data(100, 5, 0.1, static = TRUE) %>%
  mutate(t = as.numeric(t)) %>%
  dummy_cols(select_columns = "i")

long_data_static_plm <- long_data_static %>% pdata.frame(index = c("i", "t"))

# within / FE estimator
plm(y ~ x, model = "within", effect = "individual", data = long_data_static_plm) %>%
  coeftest(., vcov = plm::vcovHC(., type = "HC1", cluster = "group"))

# first difference estimator
plm(y ~ x, model = "fd", effect = "individual", data = long_data_static_plm) %>%
  coeftest(., vcov = plm::vcovHC(., type = "HC1", cluster = "group"))

# LSDV estimator
plm(y ~ . + 0, model = "pooling", data = long_data_static_plm %>% select(-i, -t)) %>%
  coeftest(., vcov = plm::vcovHC(., type = "HC1", cluster = "group"))

# GLS / RE estimator
plm(y ~ x, model = "random", effect = "individual", data = long_data_static_plm) %>% coeftest()

# Pooled OLS with clustered standard error
plm(y ~ x, model = "pooling", data = long_data_static_plm) %>%
  coeftest(., vcov = plm::vcovHC(., type = "HC1", cluster = "group"))

# Pooled OLS with BRL adjusted clustered standard error
plm(y ~ x, model = "pooling", data = long_data_static_plm) %>%
```



```

coeftest(., vcov = plm::vcovHC(., type = "HC2", cluster = "group"))

long_data_dynamic<-
  gen_long_data(100, 5, 0.1, static = FALSE) %>%
  mutate(t = as.numeric(t))

long_data_dynamic_plm <- long_data_dynamic %>% pdata.frame(index = c("i", "t"))

# Anderson-Hsiao estimator
pgmm(
  y ~ lag(y, 1) + x | lag(y, 2) | x,
  effect = "individual",
  model = "onestep",
  transformation = "d",
  data = long_data_dynamic_plm
) %>% summary(robust = TRUE)

# Arellano-Bond estimator
pgmm(
  y ~ lag(y, 1) + x | lag(y, 2:4) | x,
  effect = "individual",
  model = "twosteps",
  transformation = "d",
  data = long_data_dynamic_plm
) %>% summary(robust = TRUE)

```