Is White's QMLE a special case of GMM estimation?

Max Leung

April 22, 2024

White's Quasi Maximum Likelihood Estimation (QMLE) can be a special case of GMM estimation if we make a strong assumption on the unconditional moment.

The original formulation of QMLE in White (1982) only converges to θ^*, which is the minimizer of KLIC given the specified density f and the true density g. In the rare case that $f(z,\theta_0)=g(z)$, we have $\theta^*=\theta_0$ where the latter is the true population parameter. If $\theta^* \neq \theta_0$, we know that $f(z,\theta_0) \neq g(z)$.

$\theta^* \neq \theta_0$ means that QMLE does not converge to the true parameter vector θ_0. As both θ_0 and g is unknown to the econometricians ex-ante, we cannot specify f such that $f(z,\theta_0)=g(z)$ in advance.

In contrast, QMLE under GMM framework converges to θ_0 if we assume $E_g[\frac{\partial lnf(y_i|x_i;\theta)}{\partial\theta}|_{\theta_0}]=0$ where E_g is the expectation operator with respect to the true density g. In general, $E_g[\frac{\partial lnf(y_i|x_i;\theta)}{\partial\theta}|_{\theta_0}]$ may not be equal to zero. For the case f is correctly specified i.e., $f=g$, we have $E_g[\frac{\partial lng(y_i|x_i;\theta)}{\partial\theta}|_{\theta_0}]=0$ which can be derived easily by the property of density function.

If we make a strong assumption that $E_g[\frac{\partial lnf(y_i|x_i;\theta)}{\partial\theta}|_{\theta_0}]=0$, we can derive that QMLE is consistent to the true parameter θ_0 and asymptotic normal by using GMM framework. This is the case found in Verbeek (2018, p.208-209) and Abadie, Imbens, and Zheng (2014, assumption 2 and example 3). This assumption implies that $\theta^*=\theta_0$. Thus, the difficulty mentioned above is assumed away.

In contrast, White (1982)'s treatment of misspecification of density in MLE is more general in the sense that $E_g[\frac{\partial lnf(y_i|x_i;\theta)}{\partial\theta}|_{\theta_0}]$ need not to be zero. He only assumes that there exists a unique θ^* in the parameter space such that $E_g[\frac{\partial lnf(y_i|x_i;\theta)}{\partial\theta}|_{\theta^*}]=0$ which is equivalent to $\theta^*=argmin_\theta KLIC(f,g;\theta)$ and θ^* may or may be equal to θ_0. He then derives the consistency and asymptotic normality properties of QMLE.

One well-known example is that if we specify or guess the error term in the linear regression model follows normal distribution (i.e., $f=$ gaussian density with parameter μ_0 and σ_0^2 where $\mu_0=E[y_i|x_i]=x_i'\beta_0$. Assuming σ_0^2 is known, we have $\theta_0=\beta_0$ and $E_g[\frac{\partial lnf(y_i|x_i;\beta)}{\partial\beta}|_{\beta_0}] \propto E_g[x_i(y_i-x_i'\beta_0)]$. The latter is zero if we assume regressors and error are independent.

Under GMM framework, such model is just-identified because $E_g[x_i(y_i-x_i'\beta_0)]$ means there are $dim(x)$ moments and $dim(\beta_0)$ parameters. $dim(x)=dim(\beta_0)$ means number of moments and parameters are the same. In this case, GMM reduces to MM. Thus, $N^{-1}\sum_{i=1}^N x_i(y_i-x_i'\widehat{\beta}_{gmm})=0$. It is easy to verify that $\widehat{\beta}_{gmm}=\widehat{\beta}_{ols}$ in such case. Given the fact that GMM estimator is consistent and asymptotic normal (Hansen, 1982), the large sample properties of OLS estimator of linear regression model can also be proved by using GMM estimation.

Even the true density of error follows non-normal distribution e.g., student t distribution, the GMM estimator under normal density of error (which is our guess) is still consistent to the true parameter and asymptotic normal given $E_g[x_i(y_i-x_i'\beta_0)]=0$ is correct.

1 Reference

Abadie, A., Imbens, G. W., & Zheng, F. (2014). Inference for Misspecified Models With Fixed Regressors. Journal of the American Statistical Association, 109(508), 1601–1614. https://doi.org/10.1080/01621459.2014.928218

Hansen, L. P. (1982). Large Sample Properties of Generalized Method of Moments Estimators. Econometrica, 50(4), 1029–1054. https://doi.org/10.2307/1912775

Verbeek, M. (2018). A Guide to Modern Econometrics. 5th Edition, Erasmus University Rotterdam, John Wiley & Sons Ltd., Hoboken.

White, H. (1982). Maximum Likelihood Estimation of Misspecified Models. Econometrica, 50(1), 1–25.