CAP5415
Computer Vision

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HEC-241

Linear Algebra Basics

Lecture 2

Outline

- Vectors
  - Operations
- Matrix
  - Operations
- Transformations
  - Scaling
  - Rotation
  - Translation
- Singular value decomposition

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Vector

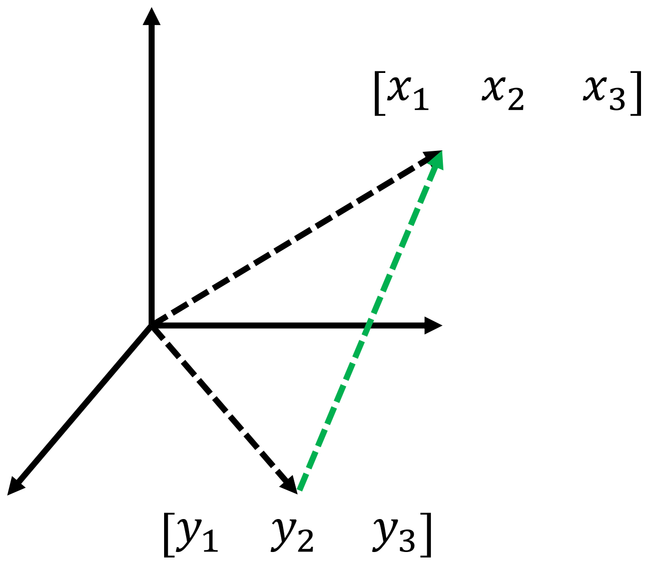
- Scalar: $x \in \mathbb{R}$
- Vector: $\boldsymbol{x} \in \mathbb{R}^N$
  - Row Vector v $\in \mathbb{R}^{1\times n}$

$$\boldsymbol{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]$$

  - Column vector v $\in \mathbb{R}^{n\times 1}$: $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \quad x_2 \quad \cdots \quad x_n]^{\boldsymbol{T}}$
  - Transpose

Vectors - use

- Store data in memory
  - Feature vectors
  - Pixel values
  - Any other data for processing
- Any point in coordinate system
  - Can be n dimensional
- Difference between two points



$$[x_1 - y_1 \quad x_2 - y_2 \quad x_3 - y_3]$$

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Vector operations

- Norm – size of the vector
- p-norm $$\|x\|_p = \left(\sum_i |a_i|^p \right)^{\frac{1}{p}} \qquad p \geq 1$$
- Euclidean norm $$\|x\|_2 = \left(\sum_i |a_i|^2 \right)^{1/2}$$
- L1-norm $$\|x\|_1 = \left(\sum_i |a_i| \right)$$

Vector operations

- Inner product (dot product)
  - Scalar number
  - Multiply corresponding entries and add

$$\boldsymbol{x}^T \boldsymbol{y} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_k^n x_k \; y_k$$

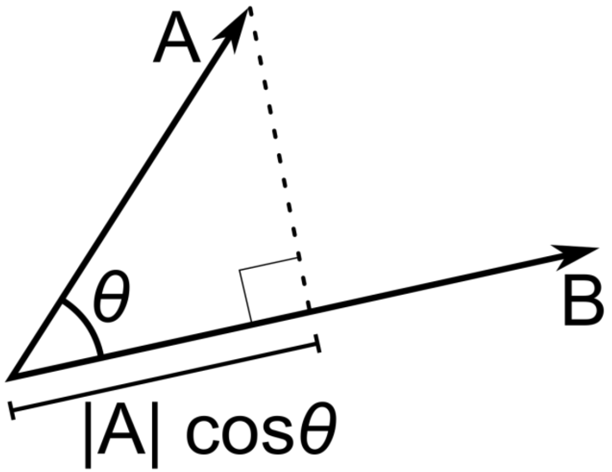
Vector operations

- Inner product (dot product)

$$\boldsymbol{x}_i^T \boldsymbol{x}_i = \sum_k^n \left(x_k^i\right)^2 = \text{squared norm of } \boldsymbol{x}_i$$

- x.y is also |x||y|cos(angle between x and y)

- If B is a unit vector, A.B gives projection of A on B



Vector operations

- Outer product

$$\boldsymbol{x}_i \boldsymbol{x}_j^T = \begin{bmatrix} x_1^i x_1^j & x_1^i x_2^j & \cdots & x_1^i x_n^j \\ x_2^i x_1^j & x_2^i x_2^j & \cdots & x_2^i x_2^j \\ \vdots & \vdots & \ddots & \vdots \\ x_n^i x_1^j & x_n^i x_2^j & \cdots & x_n^i x_m^j \end{bmatrix} \text{ (a matrix)}$$

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Matrix

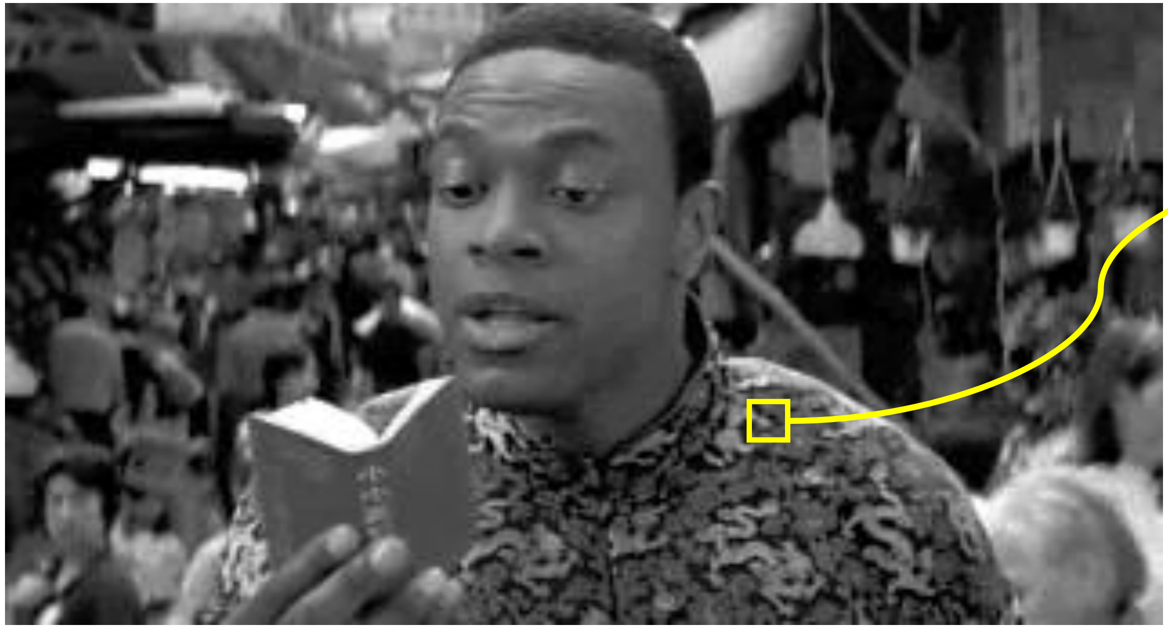
- Array $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ of numbers with shape m by n,
  - m rows and n columns

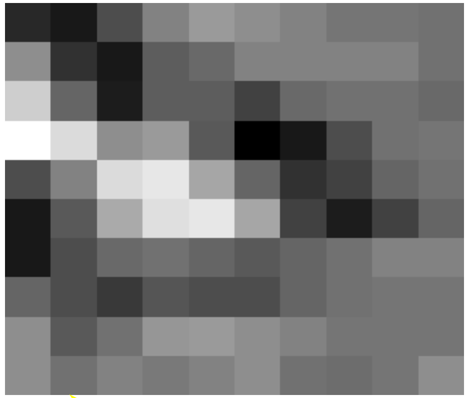
$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A row vector is a matrix with single row
- A column vector is a matric with single column

Matrix - use

- Image representation – grayscale
  - One number per pixel
  - Stored as nxm matrix





0	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0

Matrix - use

- Image representation
  - RGB
  - 3 numbers per pixel
  - Stored as nxmx3 matrix

0	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0

0	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0

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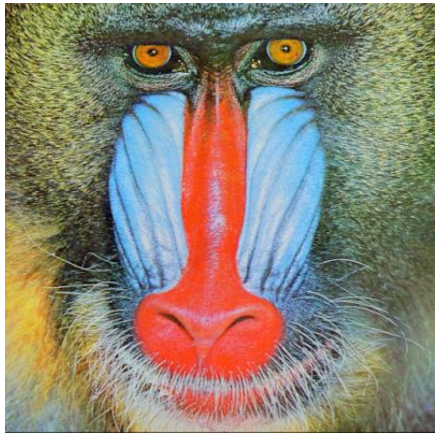
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0



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Matrix operations

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

- Both matrices should have same shape
  - except with a scalar

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 2 = \begin{bmatrix} a+2 & b+2 \\ c+2 & d+2 \end{bmatrix}$$

- Same with subtraction

Matrix operations

- Scaling

$$s \text{ x } \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} s\text{x}a & s\text{x}b \\ s\text{x}c & s\text{x}d \end{bmatrix}$$

- Hadamard product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \odot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a\text{x}e & b\text{x}f \\ c\text{x}g & d\text{x}h \end{bmatrix}$$

Matrix operation

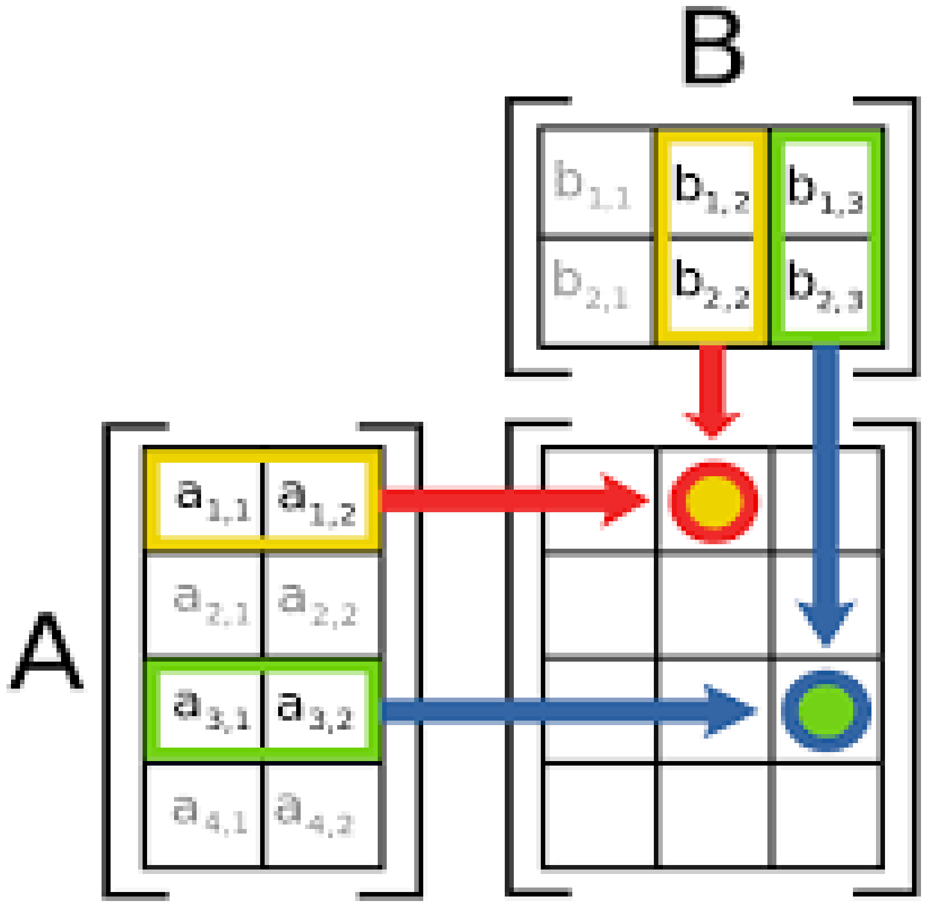
- Transpose

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\boldsymbol{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Matrix operation

- Matrix Multiplication
  - Compatibility?
  - mxn and nxp
  - Results in mxp matrix



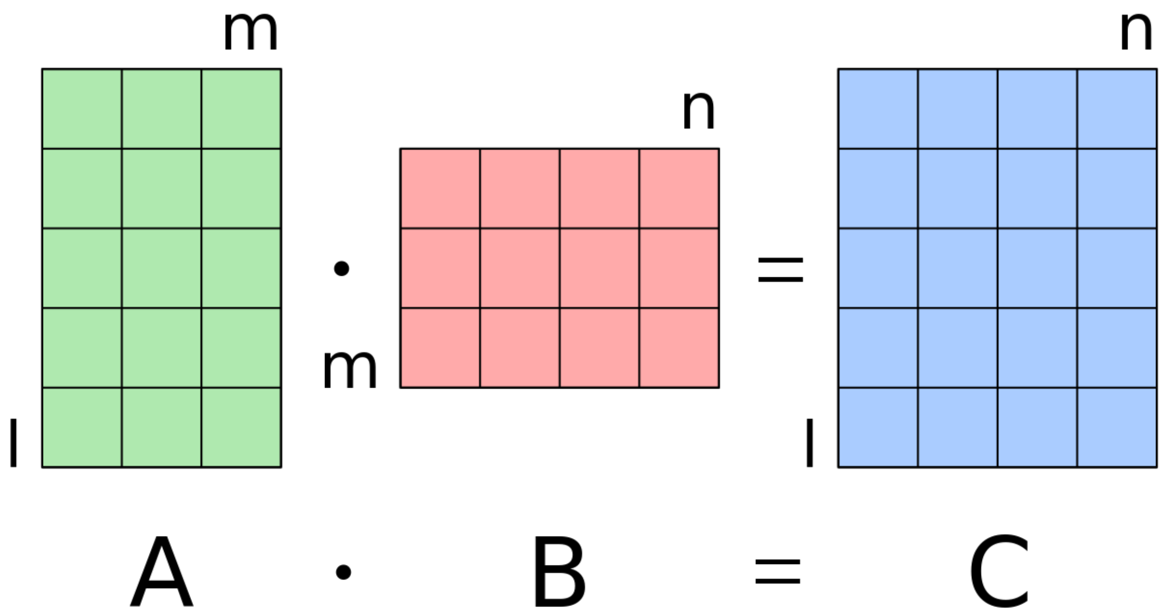
Matrix operation

- Matrix Multiplication – another interpretation (my favorite)
  - Let $\boldsymbol{a}_i$ denote the *i-th* column of the matrix $\boldsymbol{A}$, and
  - $\boldsymbol{b}_j$ denote the *j-th* column of the matrix ***B***.

    $\boldsymbol{A} = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \cdots \quad \boldsymbol{a}_n]$, and $\boldsymbol{B} = [\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \cdots \quad \boldsymbol{b}_{\boldsymbol{m}}]$
  - The first column of ***AB***
    - Linear combination of all the columns in ***A***

$$[\boldsymbol{a}_{\mathbf{1}} b_{11} + \boldsymbol{a}_{\mathbf{2}} b_{12} + \ \ldots + \boldsymbol{a}_{\boldsymbol{n}} b_{1n}]$$

Matrix operation

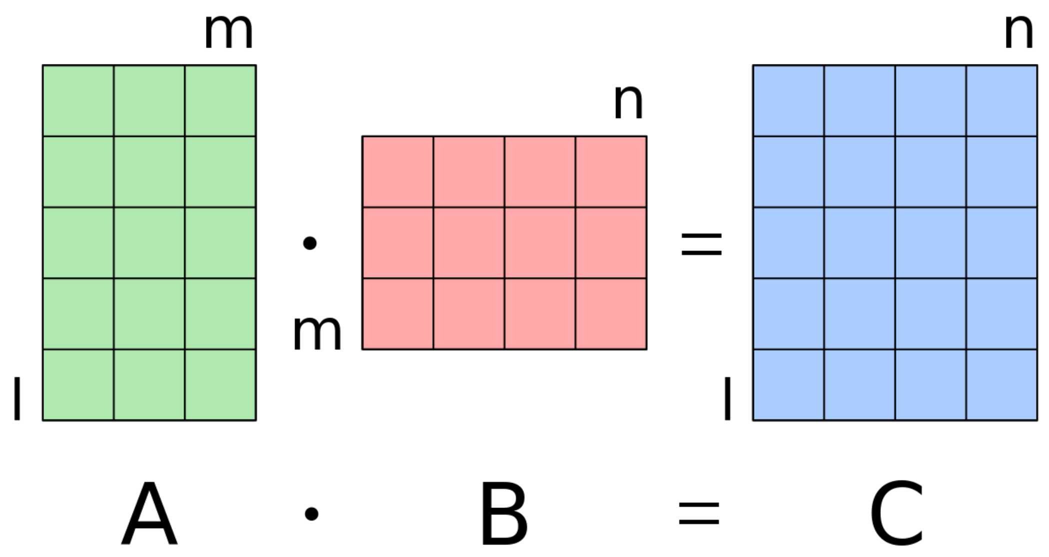


$$[\boldsymbol{a_1} b_{11} + \boldsymbol{a_2} b_{12} + \ \dots + \boldsymbol{a_n} b_{1n}]$$

- Similarly, we can get other columns…

Matrix operation

- How about linear combination of all rows of ***B***?
  - Each row of **C** = **AB** is a linear combination of rows of **B**

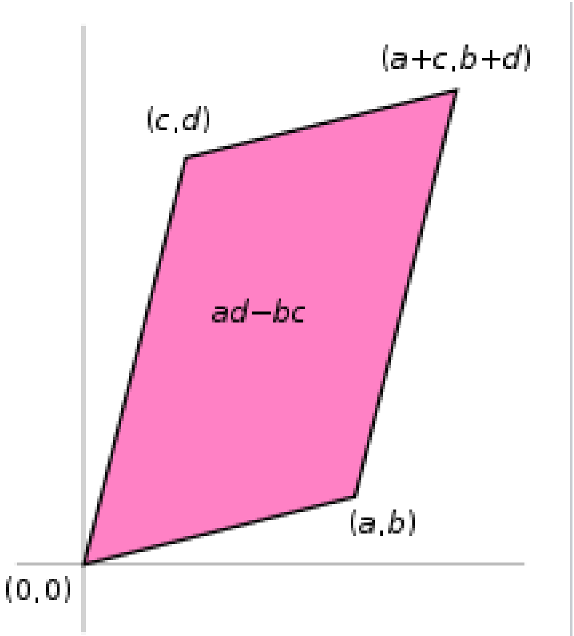


$$[\boldsymbol{b_1} a_{11} + \boldsymbol{b_2} a_{21} + \ \ldots + \boldsymbol{b_n} a_{n1}]$$

Matrix operation

- Determinant
  - A scalar

- For A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, det(A) = ad - bc



- Represents area of the parallelogram described by the vectors in the rows of the matrix

Matrix operation

- Determinant
  - The determinant of a matrix $\boldsymbol{A}$ is denoted by

    $|\boldsymbol{A}| = \sum_{i=1}^{K} a_{ij} C_{ij}$

  where C_{ij} is the cofactor of a_{ij} defined by

    $C_{ij} = (-1)^{i+1} |\boldsymbol{M}_{ij}|$, and

    $\boldsymbol{M}_{ij}$ is the minor of matrix $\boldsymbol{A}$ formed by eliminating row i and column j of $\boldsymbol{A}$

- Some properties
  - $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$
  - $|\boldsymbol{AB}| = |\boldsymbol{BA}|$
  - $|\boldsymbol{A^T}| = |\boldsymbol{A}|$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Matrix operation

- Trace
  - Tr(A) = sum of diagonal elements

$$Tr\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = a_{00} + a_{11} + a_{22}$$

- Properties
  - tr(AB) = tr(BA)
  - tr(A + B) = tr(A) + tr (B)

Matrix operation

- Inverse
  - Given a matrix A, its inverse A^{-1} is a matrix such that

$$\mathbf{AA^{-1} = A^{-1}A = I}$$

- Inverse does not always exist
  - Singular vs non-singular
- Properties
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$

Special matrices

- Symmetric matrix

$$A^T = A$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

- Skew-symmetric matrix

$$A^T = -A$$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

Special matrices

- Diagonal matrix
  - Used for row scaling

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$$

- Identity matrix
  - Special diagonal matrix
  - 1 along diagonals

$$\boldsymbol{I}.\boldsymbol{A} = \boldsymbol{A}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Questions?

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Transformations

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Transformation - scaling

- Matrices are useful for vector transformations
- Matrix multiplication

x’ = Ax

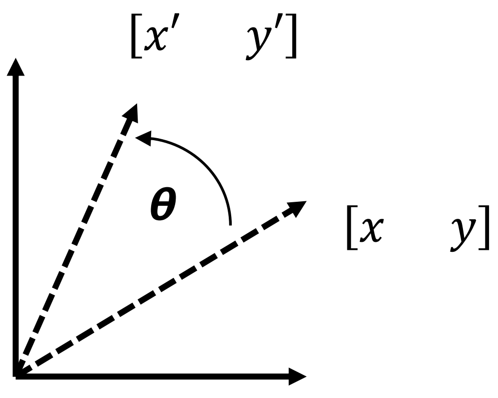
$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \text{x} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

- Linear combination of columns

Transformation

- Rotation
  - Matrix multiplication to rotate a vector
- Rotation matrix

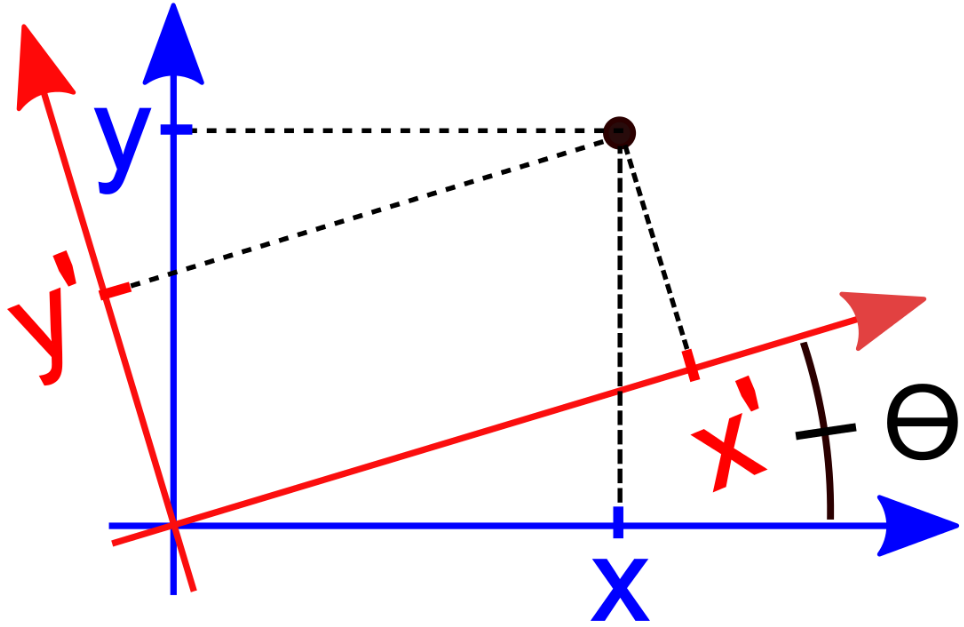
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\boldsymbol{\theta} & -sin\boldsymbol{\theta} \\ sin\boldsymbol{\theta} & cos\boldsymbol{\theta} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



x' = $\cos\boldsymbol{\theta}$x $-$ $\sin\boldsymbol{\theta}$y

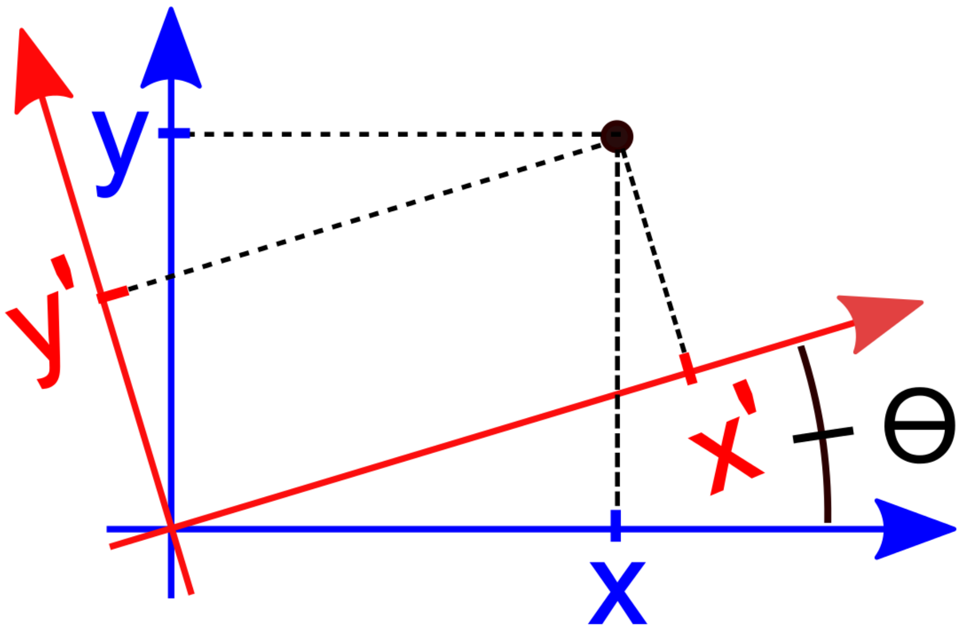
y' = sin$\boldsymbol{\theta}$x + cos$\boldsymbol{\theta}$y

Transformation - rotation

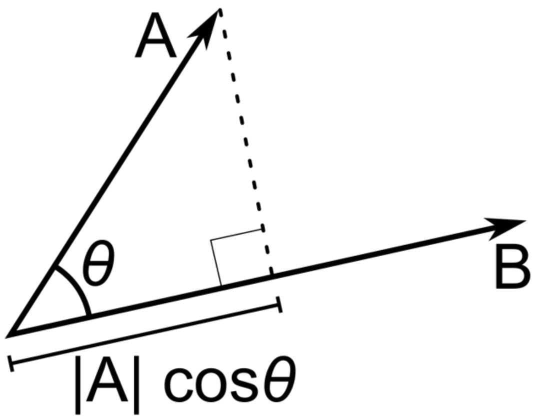


- Rotating axis first
- Vector v = [x y]
  - x – projection of v on x axis
  - y – projection of v on y axis

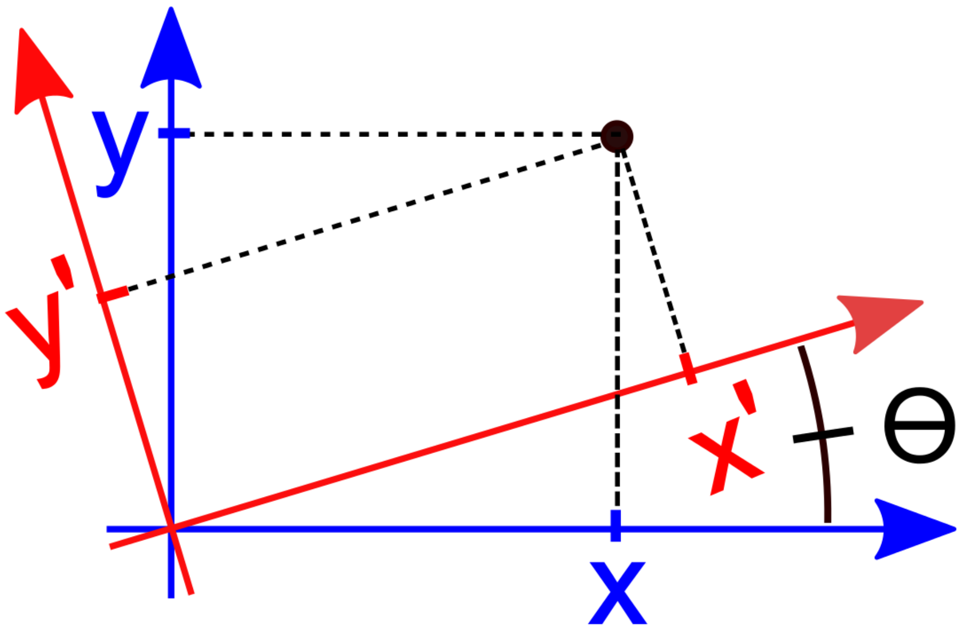
Transformation - rotation



- To rotate the axis, we need to find
  - x component in new x-axis
  - y component in new y-axis
- Remember vector dot product?



Transformation - rotation



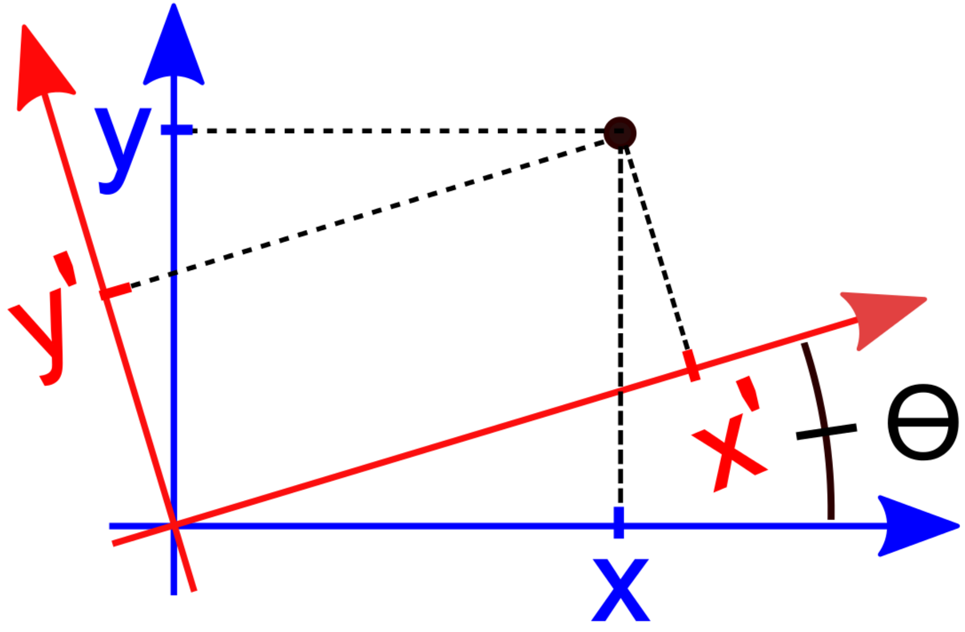
- Now we need new x and y axis

  x’ = v **dot** new x-axis

  y’ = v **dot** new y-axis

- For rotation of $\boldsymbol{\theta}$
  - New x-axis = [cos$\boldsymbol{\theta}$, sin$\boldsymbol{\theta}$]
  - New y-axis = [-sin$\boldsymbol{\theta}$, cos$\boldsymbol{\theta}$]

Transformation - rotation



x’ = v **dot** new x-axis

y’ = v **dot** new y-axis

New x-axis = [cos$\boldsymbol{\theta}$, sin$\boldsymbol{\theta}$]

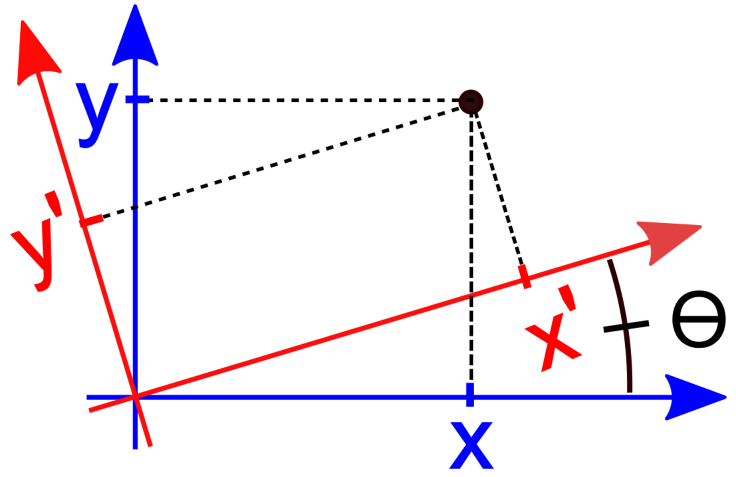
New y-axis = [-sin$\boldsymbol{\theta}$, cos$\boldsymbol{\theta}$]

We can form a matrix using new axis

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\boldsymbol{\theta} & sin\boldsymbol{\theta} \\ -sin\boldsymbol{\theta} & cos\boldsymbol{\theta} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Transformation - rotation

- When we rotate the axis to left
- We are rotating the vector to right



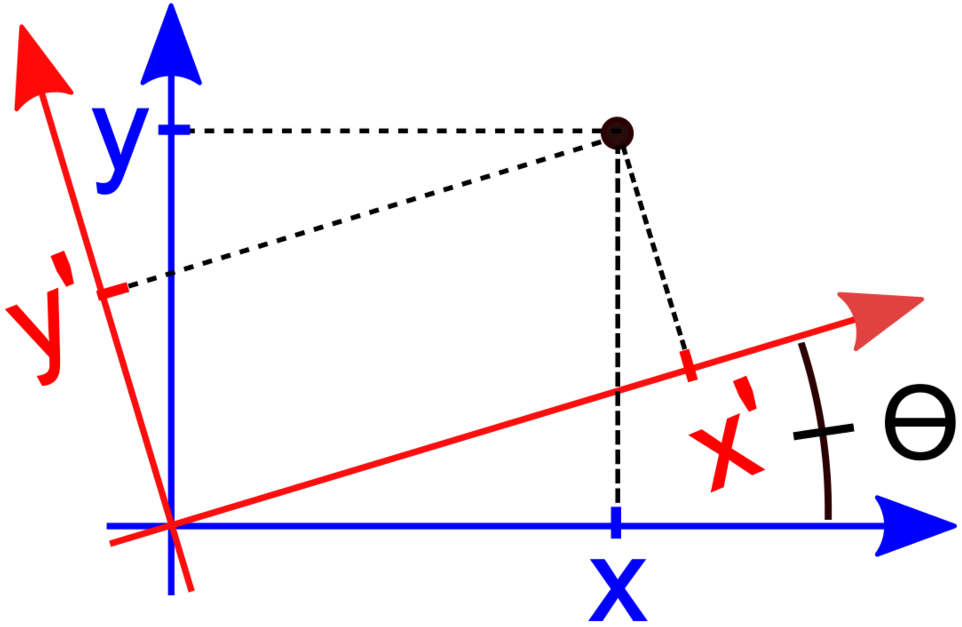
- We can use rotation matrix to rotate the vector

Transformation - rotation

- We need new x y axis coordinates
  - When we rotate the axis right

- Updated rotation matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\boldsymbol{\theta} & -sin\boldsymbol{\theta} \\ sin\boldsymbol{\theta} & cos\boldsymbol{\theta} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Transformation

- Linear combination

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{x} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

- Sufficient for
  - scaling
  - rotating and
  - skew transformations
- But no shifting

Transformation

- Linear combination

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix}$$

- Still be able to
  - scaling
  - rotation and
  - skew transformations

Transformation

- Homogenous system

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \text{x} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- We can also perform shifting now
- This is called homogenous coordinates

Scaling + rotation + translation

- Careful about the order

$$V' = (TRS)V$$

$$\begin{bmatrix} 1 & 0 & t_y \\ 0 & 1 & t_x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\boldsymbol{\theta} & -sin\boldsymbol{\theta} & 0 \\ sin\boldsymbol{\theta} & cos\boldsymbol{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{x} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Questions?

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Linear independence

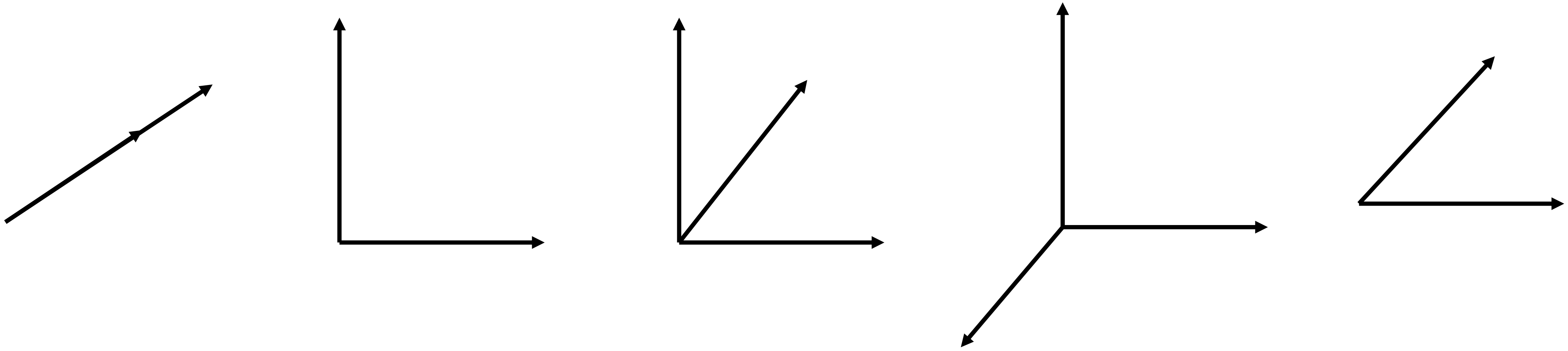
- Linear independence

  $\{\boldsymbol{x}_1, \boldsymbol{x}_2 \cdots, \boldsymbol{x}_M\}$ is a set of linearly independent vectors provided

  $a_1\boldsymbol{x}_1 + a_2\boldsymbol{x}_2 + \cdots + a_M\boldsymbol{x}_M = \mathbf{0} \Rightarrow a_1 = 0 = a_2 = a_M$

- In other words, none of the vectors can be expressed as a linear combination of the other vectors
  - Each vector is perpendicular to every other vector
  - For example, axis in cartesian coordinate system

Intuition



- In terms of features
  - Person recognition – [height, hair color, weight, specs, eye color, etc.]

Matrix factorization

- Singular value decomposition (SVD)

$$A = U\Sigma V^T$$

- If A is mxn matrix, then
  - U will be mxm,
  - Σ will be mxn, and
  - V^T will be nxn
- U and V are unitary matrices
  - Each column is a unit vector
- Σ is a diagonal matrix

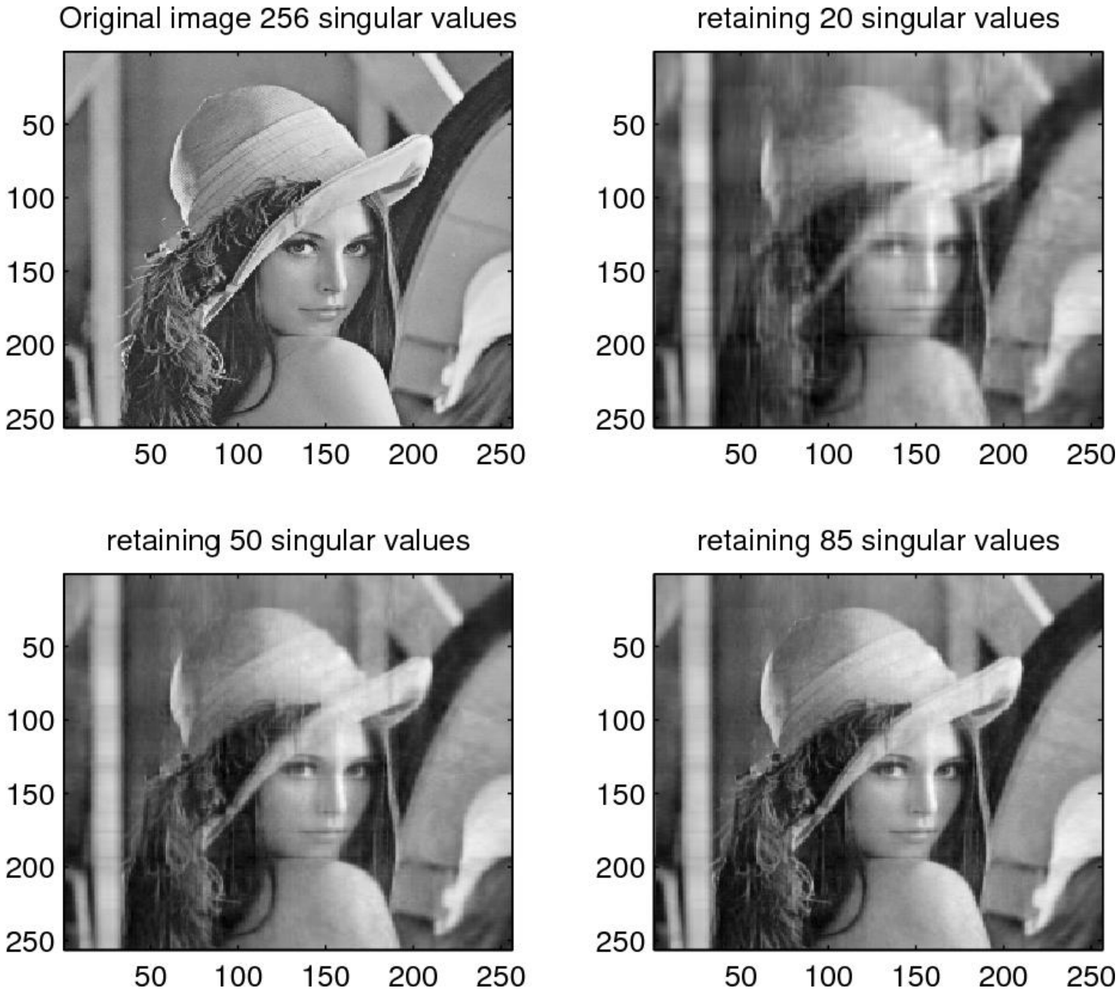
Singular value decomposition

- Interpretation

$$A = U\Sigma V^T$$

- Columns of U are scaled by values in Σ
- The resultant columns are linearly combined by V
- **A** is formed as a linear combination of columns of U
- If we use all column, we will get original **A**
- We can just use few columns of U and we get an approximation
  - We call these columns principal components

SVD application



Questions?

Sources for this lecture include materials from works by Abhijit Mahalanobis and Fei Fei Li