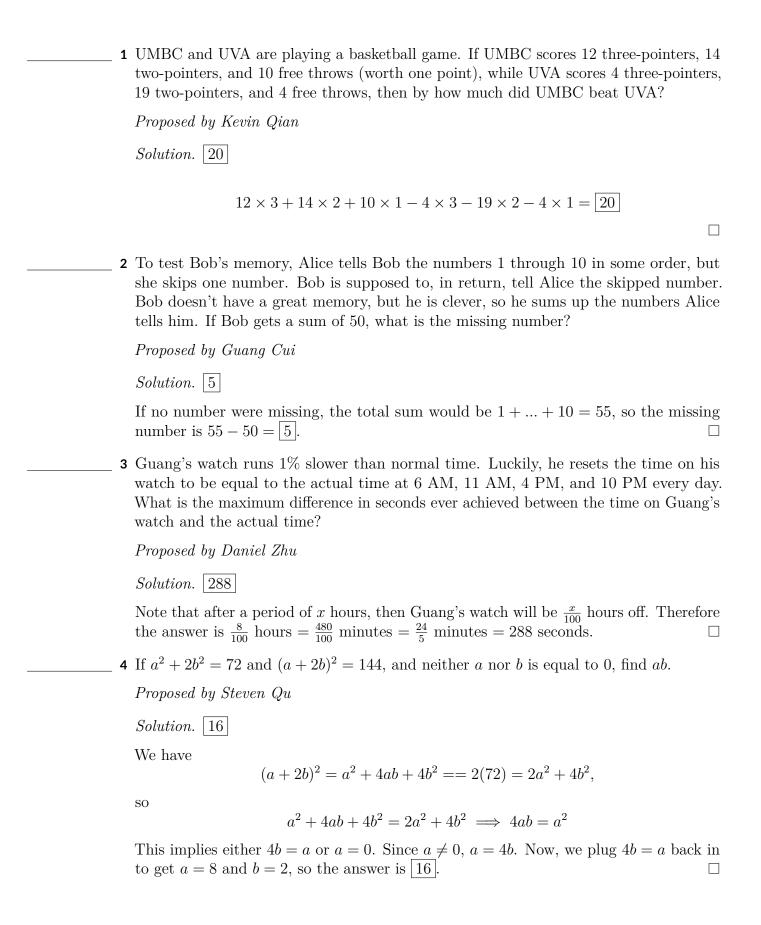
MBMT Algebra Round — Gauss

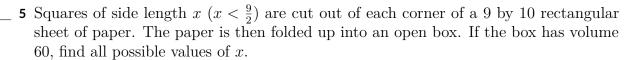
April 7, 2018

Full Name		
	Team Number	

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered correctly by fewer competitors will be weighted more heavily. Please write your answers in a reasonably simplified form.





Proposed by Guang Cui

Solution.
$$2, \frac{15 - \sqrt{105}}{4}$$

The side lengths would be x, 10 - 2x, and 9 - 2x, so x(10 - 2x)(9 - 2x) = 60, or x(5-x)(9-2x) = 30, so $2x^3 - 19x^2 + 45x - 30 = 0$. Notice that x = 2 is a solution, so $(x-2)(2x^2 - 15x + 15) = 0$. The quadratic has solutions $\frac{15 \pm \sqrt{105}}{4}$. $\frac{15 - \sqrt{105}}{4}$ is between 0

and 4.5, but
$$\frac{15+\sqrt{105}}{4}$$
 is not. Therefore, x can be $2, \frac{15-\sqrt{105}}{4}$.

6 Find the minimum value of $(x-y+1)^2 + (xy+y+1)^2$ over all pairs of real numbers (x,y).

Proposed by Kevin Qian

Solution.
$$\boxed{1}$$

Expanding gives
$$(x^2 + 2x + 2)(y^2 + 1) = ((x + 1)^2 + 1)(y^2 + 1)$$
, so the answer is 1, given when $x = -1$ and $y = 0$.

7 Find the unique positive solution to $x\lceil x\lfloor x\rfloor\rceil\lceil x\rceil=130$. Here, $\lfloor x\rfloor$ is the largest integer less than or equal to x, and $\lceil x\rceil$ is the smallest integer greater than or equal to x.

Proposed by Kevin Qian

Solution.
$$\boxed{\frac{13}{4}}$$

First, observe that the function is monotonically increasing over positive x, so there is only one value of x which satisfies this. Notice that x = 3 is too small and x = 4 is too big, so our value is between 3 and 4. In particular, $\lceil x \rceil = 4$ and $\lceil x \rceil = 3$.

Then we can reduce the equation to

$$x\lceil 3x\rceil 4 = 130$$

Let x = 3 + y where 0 < y < 1. Then

$$(3+y)[9+3y]4 = 130$$

Now we do casework on the value of [9+3y].

Case 1: $\lceil 9+3y \rceil = 10$ Then we get $3+y = \frac{130}{4\cdot 10} = \frac{13}{4}$. This gives $y = \frac{13}{4}$, which is in fact our answer.

We'll do the rest of the cases for completeness.

Case 2: $\lceil 9+3y \rceil = 11$ Then we get $3+y = \frac{130}{4\cdot 11} < 3$, so y < 0, a contradiction.

Case 3:
$$[9+3y] = 12$$
 Then we get $3 + y = \frac{130}{4 \cdot 12} < 3$, so $y < 0$, a contradiction.

8 Find the minimum value of

$$a + \frac{8}{a} + \frac{8}{a + \frac{8}{a}}$$

where a is a positive real number.

Proposed by Jyotsna Rao

Solution. $5\sqrt{2}$

Let $f(a) = a + \frac{8}{a}$. We see that f(a) is smallest when a and $\frac{1}{a}$ are as close together as possible. When $a = 2\sqrt{2}$, $\frac{8}{a}$ is also $2\sqrt{2}$. This is the closest that a and $\frac{1}{a}$ can be to each other, so we know that $\min(f(a)) = 4\sqrt{2}$. By the same logic, f(f(a)) is smallest when f(a) and $\frac{1}{f(a)}$ are as close together as possible. We would like to plug in $f(a) = 2\sqrt{2}$ to get the minimum value of f(f(a)), but the smallest value of f(a) is $4\sqrt{2}$. Plugging in $f(a) = 4\sqrt{2}$ makes f(a) and $\frac{1}{f(a)}$ as close as possible, so the minimum value of f(f(a)) is $f(a) = 5\sqrt{2}$.