Slides of Discrete Mathematics based on Susanna Epp's Textbook

Moses A. Boudourides[1]

Visiting Associate Professor of Computer Science
Haverford College

[1] Moses.Boudourides@cs.haverford.edu

Chapter 9

Counting and Probability

November 17, 19, 22, & 24, 2021

9.1 Introduction

Definition

- A **sample space** is the set of all possible outcomes of a random process or experiment.
- An **event** is a subset of a sample space.

Equally Likely Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability** of E, denoted $P(E)$, is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S} = \frac{N(E)}{N(S)},$$

where, for any set X, $N(X)$ denotes *the number of elements of* X, i.e., in another often used notation, $N(X) = |X|$.

Example 9.1.1 Probabilities for a Deck of Cards

An ordinary deck of cards contains 52 cards divided into four *suits*. The *red suits* are diamonds (♦) and hearts (♥) and the *black suits* are clubs (♣) and spades (♠). Each suit contains 13 cards of the following *denominations:* 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king), and A (ace). The cards J, Q, and K are called *face cards*.

Solution

a. The outcomes in the sample space S are the 52 cards in the deck.

b. Let E be the event that a black face card is chosen. The outcomes in E are the jack, queen, and king of clubs and the jack, queen, and king of spades. Symbolically,

$$E = \{\text{J}\clubsuit, \text{Q}\clubsuit, \text{K}\clubsuit, \text{J}\spadesuit, \text{Q}\spadesuit, \text{K}\spadesuit\}.$$

c. By part (b), $N(E) = 6$, and according to the description of the situation, all 52 outcomes in the sample space are equally likely. Therefore, by the equally likely probability formula, the probability that the chosen card is a black face card is

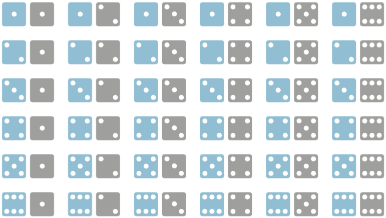
$$P(E) = \frac{N(E)}{N(S)} = \frac{6}{52} \cong 11.5\%.$$

■

9.1 Dice

Example 9.1.2 Rolling a Pair of Dice

A die is one of a pair of dice. It is a cube with six sides, each containing from one to six dots, called *pips*. Suppose a blue die and a gray die are rolled together, and the numbers of dots that occur face up on each are recorded. The possible outcomes can be listed as follows, where in each case the die on the left is blue and the one on the right is gray.



A more compact notation identifies, say,  with the notation 24,  with 53, and so forth.

a. Use the compact notation to write the sample space S of possible outcomes.

b. Use set notation to write the event E that the numbers showing face up have a sum of 6 and find the probability of this event.

Solution

a. $S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$.

b. $E = \{15, 24, 33, 42, 51\}$.

The probability that the sum of the numbers is $6 = P(E) = \dfrac{N(E)}{N(S)} = \dfrac{5}{36}$.

9.1 Exercises (a)

Exercise 9.1.10

In rolling a pair of dice, write the event is that the sum of the numbers showing face–up is at least 9 and compute the probability.

$E = \{36, 45, \ldots\}, N(E) =?, N(S) =?, P(E) =?.$

Exercise 9.1.14 (b) and (c)

Three people have been exposed to a certain illness. Once exposed, a person has a 50–50 chance of actually becoming ill. (b) What is the probability that at least two of the people become ill? (c) What is the probability that none of the three people becomes ill?

The sample space is composed of the following cases: none is ill, one is ill, two, are ill, all are ill. Denoting people by A, B, C, what is $S = \{?\}, N(S) =$? In (b) and (c) what are the events $E_b = \{?\}, E_c = \{?\}$ as subsets of S? $N(E_b) =?, N(E_c) =?, P(E_b) =?, P(E_c) =?$

Exercise 9.1.17

Two faces of a six–sided die are painted red, two are painted blue, and two are painted yellow. The die is rolled three times, and the colors that appear face up on the first, second, and third rolls are recorded. (a) Find the probability of the event that exactly one of the colors that appears face up is red. (b) Find the probability of the event that at least one of the colors that appears face up is red.

Find the sample space $S = \{RRR, \ldots, YYY\}, N(S) = ?$ In (a) $E = \{RBB, \ldots\}, N(E) = ?, P(E) = ?$ In (b), first find the event that none of the faces is red and subsequently the event that at least one is red can be computed by subtraction.

Theorem

If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n, inclusive.

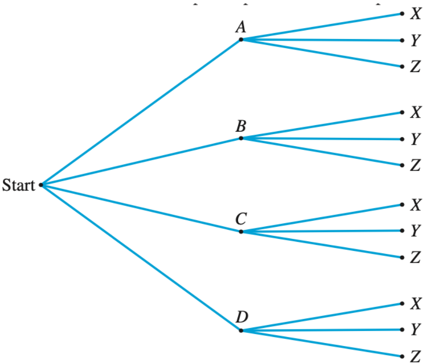
Exercise 9.1.22

(a) How many positive three–digit integers are multipliers of 6? (b) What is the probability that a randomly chosen positive three–digit integer is a multiple of 6? (c) What is the probability that a randomly chosen positive three–digit integer is a multiple of 7?

Apparently three–digit numbers are between 100 and 999, inclusive. In (a), find the smaller integer m such that $6 \cdot m \geq 100$ and the largest integer n such that $6 \cdot n \leq 999$. The rest is obvious, for (b) too. In (c), do the same for multiples of 7.

An Example

Suppose that we have two sets $\mathscr{A} = \{A, B, C, D\}$ and $\mathscr{X} = \{X, Y, Z\}$. We want to count how many ways we can pair an element of $\mathscr{A}$ with an element of $\mathscr{X}$? To do it, just count the branches of the followig **possibility tree** (in order to find 12 ways):

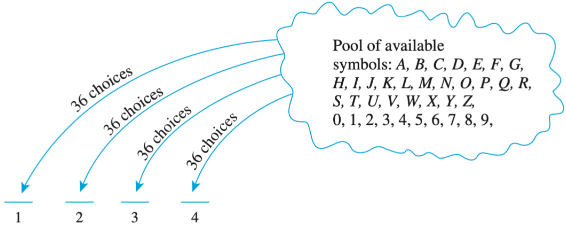


9.2 Possibility Trees and the Multiplication Rule

Theorem (The Multiplication Rule)

If an operation consists of k steps and the first step can be performed in n_1 ways, the second step can be performed in n_2 ways (regardless of how the first step was performed), $\ldots$, the kth step can be performed in n_k ways (regardless of how the preceding steps were performed), then the entire operation can be performed in $n_1 \cdot n_2 \cdots n_k$ ways.

There are 36^4 possible 4–digit PINs (letters and digits):



9.2 Multiplication Rule Examples

Example 1

Three urns contain colored balls: the first 5 red balls, the second 6 green balls, and the third 4 blue balls. Choosing randomly one ball from each urn, how many colored triplets of balls can be chosen?

Example 2

In rolling a pair of dice, each one having a different color, how many outcomes are possible?

9.2 Possibility Trees Exercises (a)

Exercise 9.2.7

One urn contains one blue ball (labeled B_1) and three red balls (labeled R_1, R_2, and R_3). A second urn contains two red balls (R_4 and R_5) and two blue balls (B_2 and B_3). An experiment is performed in which one of the two urns is chosen at random and then two balls are randomly chosen from it, one after the other without replacement. (a) Construct the possibility tree showing all possible outcomes of this experiment. (b) What is the total number of outcomes of this experiment? (c) What is the probability that two red balls are chosen? For the possibility tree, first consider three steps: (1) choose an urn, (2) choose ball 1 and (3) choose ball 2. Apparently, step (1) involves the set of urns, let us denote it as $\{U_1, U_2\}$. Step (2) involves the set of all **single** balls, blue or red, which are $B_1, \ldots, R_1, \ldots$. However, these balls are distributed differently in the two urns: which (single) balls are in U_1 and which in U_2? Finally, step (3) involves different sets of 3 balls which are **complimentary** to the sets in step (2).

9.2 Possibility Trees Exercises (b)

Exercise 9.2.10

Suppose there are three routes from North Point to Boulder Creek, two routes from Boulder Creek to Beaver Dam, two routes from Beaver Dam to Star Lake, and four routes directly from Boulder Creek to Star Lake. (Draw a sketch.) (a) How many routes from North Point to Star Lake pass through Beaver Dam? (b) How many routes from North Point to Star Lake bypass Beaver Dam?

Set locations as 4 points horizontally (North Point, Boulder Creek, Beaver Dam, Start Lake) and draw routes among two locations as arcs joining the corresponding points. When moving between two locations, think of routes as choices and, thus, count arcs joining them. Then use the multiplication rule.

9.2 Possibility Trees Exercises (c)

Exercise 9.2.13

A coin is tossed four time. Each time the outcome is either H(ead) or T(ails). (a) How many distinct outcomes are possible? (b) What is the probability that exactly two heads occur? (c) What is the probability that exactly one head occurs? Do the computations by writing the sample space and each event. Use the multiplication to count the elements of the sample space.

Exercise 9.2.14

Suppose that in a certain state, all automobile license plates have four letters followed by three digits. (a) How many different license plates are possible? (b) How many license plates could begin with A and end in 0? (c) How many license plates could begin with TGIF? (d) How many license plates are possible in which all the letters and digits are distinct? (e) How many license plates could begin with A B and have all letters and digits distinct?

9.2 Strings

Definition

Given a finite set X, a **string over** X is a finite sequence of elements of X (where repetition of elements of X is allowed). The **length of a string** is the number of elements of X that it contains. Usually, a string is written without parentheses or commas separating its elements. A string over $X = \{0, 1\}$ is called **bit string**.

9.2 Permutations

Definition

A **permutation** of n distinct objects is an ordering of these objects (in a row).

Theorem

There are $n!$ permutations of n objects.

Definition

An r–**permutation** of n distinct objects is an ordering of r objects taken from these objects. The number of r–permutations of n objects is denoted $P(n,r)$.

Theorem

$$\begin{aligned} P(n,r) &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n!}{(n-r)!}, \qquad r \leq n. \end{aligned}$$

Exercise 9.2.33

(a) How many ways can three of the letters of the word ALGORITHM be selected and written in a row?

(b) How many ways can six of the letters of the word ALGORITHM be selected and written in a row?

(c) How many ways can six of the letters of the word ALGORITHM be selected and written in a row if the first letter must be A?

(d) How many ways can six of the letters of the word ALGORITHM be selected and written in a row if the first two letters must be OR?

$N(\text{ALGORITHM}) = 9$ and all letters are distinct. (a) $P(9,3) =?$. (b) $P(9,6) =?$. (c) $P(9-1, 6-1) = P(8,5) =?$. (d) $P(9-2, 6-2) = P(7,4) =?$. **Explain!**

9.3 Three Counting Principles

Theorem (The Addition Principle)

If two finite sets A and B are disjoint, then $N(A \cup B) = N(A) + N(B)$. In general, if k finite sets $A_1, A_2, \ldots, A_k$ are mutually disjoint, then $N(A_1 \cup A_2 \cup \cdots \cup A_k) = N(A_1) + N(A_2) + \ldots + N(A_k)$.

Theorem (The Difference Principle)

If A is a finite set and $B \subseteq A$, then $N(A - B) = N(A) - N(B)$.

Theorem (The Inclusion–Exclusion Principle)

If A and B are two finite sets, then $N(A \cup B) = N(A) + N(B) - N(A \cap B)$.
If A, B and C are three finite sets, then $N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$.

9.3 Three Counting Principles Examples

Example 1

A person's age is a two-digits numbers divisible by 3 such that the first digit is odd and the second digit is both less than the first digit and less than 5. What are all possible ages of that person?

Example 2

How many positive integers less than 1000 can have two different digits? (It is 973! Why?)

Example 3

How many positive integers less or equal than 300 can be either even or divisible by 3? (It is 200! Why?)

Exercise 9.3.8

At a certain company, passwords must be from 3–5 symbols long and composed of the 26 letters of the alphabet, the ten digits 0–9, and the 14 symbols !,@,#,$, %, ^, &, $*$, (,), $-$, $+$, {, and }. (a) How many passwords are possible if repetition of symbols is allowed? (b) How many passwords contain no repeated symbols? (c) How many passwords have at least one repeated symbol? (d) What is the probability that a password chosen at random has at least one repeated symbol?

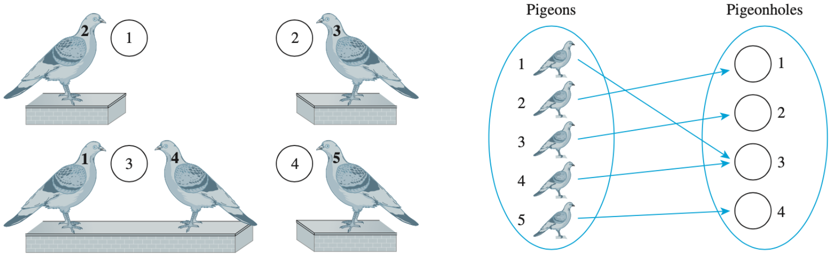
How many symbols are used totally? (a) By the addition principle according to the number of symbols (3-5) and in each case using the multiplication princible. (b) Similarly, but numbers shoud be reduced each time! (c) Use the difference principle. (d) By the formula of probability.

Exercise 9.3.11

(a) How many ways can the letters of the word THEORY be arranged in a row? (b) How many ways can the letters of the word THEORY be arranged in a row if T and H must remain next to each other as either TH or HT?

How many are the letters of this word? Are they distinct? (a) Use permutations! (b) ammounts to the orderings of either TH–E–O–R–Y or HT–E–O–R–Y. How many letters in each case and, thus, how many orderings? As these are two different words, use the addition principle for the total number of orderings.

9.4 The Pigeonhole Principle I



9.4 The Ordinary Pigeonhole Principle II

Theorem (The Ordinary Pigeonhole Principle)

Classical Form: *If n pigeons fly into m pigeonholes, where $m < n$, then there exists a pigeonhole that contains at least two pigeons.*

Function Form: *Given two finite sets X, Y with $|X| = n$ and $|Y| = m$, where $m < n$, and a function $f \colon X \to Y$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$.*

Proof by Contradiction:

Suppose that for any $y \in Y$, there exists at most one $x \in X$ such that $f(x) = y$. This means that either y is not on the range of f or that there exists a unique $x \in X$ such that $f(x) = y$. Notice that Y is trivially partitioned as $Y = Y_1 \cup Y_2$, where $Y_1 = \text{ range}(f)$ and $Y_2 = \text{ range}(f)^c$. Thus, under the previous assumption, $f \colon X \to Y_1$ is one–to–one, which implies that $|Y_1| = |X| = n$ and, consequently, $m = |Y| = |Y_1| + |Y_2| = |X| + |Y_2| \geq |X| = n$. However, deriving $m \geq n$ contradicts the assumption that $m < n$.

Theorem (The Generalized Pigeonhole Principle)

Let two finite sets X, Y with $|X| = n$ and $|Y| = m$, where $m < n$, let a function $f \colon X \to Y$, and let the integer $k = \lceil \frac{n}{m} \rceil$. Then there exist at least k distinct elements of X, $x_1, x_2, \ldots, x_k$ (instead of 2 in the ordinary Pigeonhole Principle), such that $f(x_1) = f(x_2) = \cdots = f(x_k)$.

Proof by Contradiction:

Suppose that for any $y \in Y$, there exist at most $k-1$ distinct $x \in X$ such that $f(x) = y$. Without any loss of generality we may assume that f is onto (**why?**) and let us represent $Y = \{y_1, y_2, \ldots, y_m\}$. This means that there are at most $k-1$ distinct $x \in X$ with $f(x) = y_1$; there are at most $k-1$ distinct $x \in X$ with $f(x) = y_2$; $\ldots$; there are at most $k-1$ distinct $x \in X$ with $f(x) = y_m$. Hence, totally, set X must contain at most $m(k-1)$ elements, which means $n < m(k-1)$. However, since $m < n$, $k = \lceil \frac{n}{m} \rceil < \frac{n}{m} + 1$ or $k - 1 < \frac{n}{m}$. Thus, $n < m(k-1) < m\frac{n}{m} = n$, which is a contradiction.

9.4 The Pigeonhole Principle Examples (a)

Example 1

A drawer contains 10 black and 10 white socks. You reach in and pull two out without looking. What is the least number of socks you must pull out to be guaranteed to get a matched pair?

Consider the set X to be the set of socks pulled out and the set Y to be the set of the two colors of the socks. We know that $|Y| = 2$ but we are not sure what $|X| = n$ should be in order to get a matched pair. The function $f \colon X \to Y$ is the categorization of the color of socks. In other words, a first sock is pulled out and it is placed in the "pigeonhole" of its color. Such placement is repeated with all subsequent draws. Then the question is what is the size of X so that it would be certain that at least two socks might be placed in the same "pigeonhole" and, thus, they might have the same color? According to the Pigeonhole Principle, $n > m = 2$. What is the smaller value of n that implies pair matching?

9.4 The Pigeonhole Principle Examples (b)

Example 2

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many integers should be selected from A so that at least one pair of the integers might have sum of 9?

Let X_n be a set of n distinct numbers selected from A. As far as $n \geq 2$, the elements of X_n may create certain pairs of numbers. Considering the set Y to be the set of all possible pairs of numbers in A with sum 9, i.e., $Y = \{\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}\}$, the question is what the set X_n should be so that at least one pair of distinct elements of X_n would be an element in Y? Notice that each element of A occurs in exactly one element of Y. Observe that for $X_3 = \{1, 2, 3\}$ all the elements of this set correspond to distinct elements of Y and, thus, X_3 does not include pairs with sum 9. Similarly, the elements of $X_4 = \{2, 4, 6, 8\}$ or of $X_4 = \{1, 3, 5, 7\}$ are associated to distinct elements of Y. Notice that for all the previous sets $n = |X_n| \leq 4$. However, when $n = |X_n| \geq 5$, in any such X_n the Pigeonhole Principle implies that there is at least one pairs of elements associated to the same element in Y and, thus, in any X_n, for $n \geq 5$, sums of pairs equal to 9 are always formed.

9.4 The Pigeonhole Principle Examples (c)

Example 3

- In a group of 6 people, must there be at least two who were born in the same month?
- In a group of 13 people, must there be at least two who were born in the same month?
- In a group of 100 people, what would be the minimum number of them who were born in the same month?

Consider X to be the group of people and Y to be the set of 12 months. The function $f\colon X \to Y$ associates the birthday month to a person. If $|X| = 6$, every person can have birthday on a different month, because $12 = |Y| > 6$. Applying the Pigeonhole Principle, when $|X| = 13 > 12 = |Y|$, necessarily two persons were born in the same month. Finally, according to the Generalized Pigeonhole Principle, when $|X| = 100 > 12 = |Y|$, since $\frac{100}{12} = 8.33$, there must be 9 persons who were born in the same month.

Exercise 9.4.19

How many integers from 100 through 999 must you pick in order to be sure that at least two of them have a digit in common? (For example, 256 and 530 have the common digit 5.

All the numbers from 100 to 999 contain at least one of the nine digits 1, 2, 3, 4, 5, 6, 7, 8, or 9. First, let us consider the case of numbers with distinct digits. For instance, the following *nine* numbers have no common digits: 111, 222, 333, 444, 555, 666, 777, 888, and 999. This is the worst case scenario if we were picking nine numbers. However, if we are picking *ten* numbers, the tenth number is going to have at least a digit common with the previously selected numbers.

Exercise 9.4.31

A group of 15 executives are to share 5 assistants. Each executive is assigned exactly 1 assistant, and no assistant is assigned to more than 4 executives. Show that at least 3 assistants are assigned to 3 or more executives.

Let k be the number of assistants assigned to at least three executives. (The target is to show that $k \geq 3$.) These assistants are assigned to at most $4k$ executives (since no assistant is assigned to more than 4 executives). The remaining assistants are $5-k$ and each of them is assigned to 2 executives. Thus, all of the remaining assistants are assigned to at most $2(5-k) = 10-2k$ executives. Therefore, all the assistants are assigned to at most $4k+(10-2k) = 10+2k$ executives. However, since the number of all executives is 15, we should have $15 \leq 10+2k$, i.e., $k \geq 5/2$, and, since k is an integer, we get $k \geq 3$.

9.4 The Pigeonhole Principle Exercises (c)

Exercise 9.4.32

Let A be a set of six (distinct) positive integers each of which is less than 13. Show that there must be two distinct subsets of A whose elements when added up give the same sum. (For example, $A = \{1, 3, 4, 5, 10, 12\}$ and the two sets are $S_1 = \{1, 4, 10\}$ and $S_2 = \{5, 10\}$, both having sum 15.)

Let $\mathscr{X}$ be the set of all nonempty subsets of A. Clearly, $\mathscr{X} \subset \mathscr{P}(A)$. Furthermore, consider the function $F : \mathscr{X} \to \mathbb{Z}^+$ be defined as $F(X) =$ the sum of the elements of X, for any $X \in \mathscr{X}$. We know that the set of all subsets of A has $2^{|A|} = 2^6 = 64$ elements, i.e., $|\mathscr{X}| = 64 - 1 = 63$. Since each element of A is less than 13, the maximum possible sum of elements of any $X \in \mathscr{X}$ is 57 $(= 12 + 11 + 10 + 9 + 8 + 7)$. Since $63 > 57$, the Pigeonhole Principle guarantees that F is not one–to–one. Therefore, there exist distinct sets $A_1, A_2 \in X$ such that $F(A_1) = F(A_2)$.

9.5 Combinations

Definition

A **combination** of k elements from a set of size n is a subset of size k.

Theorem (Counting Combinations = Counting Subsets of Given Size)

The number of combinations of size k from a set of size n, where $k, n \in \mathbb{Z}, 0 \leq k \leq n$, is:

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}.$$

Example 1

1. What is the number of distinct teams of 5 people chosen from a group of 12?
2. If the group includes 7 women and 5 men, how many 5–person teams can be chosen with 3 men and 2 women?
3. How many 5–person teams contain at least 1 man?
4. How many 5–person teams contain at most 1 man?

1. By definition: $\binom{12}{5} =$ do the algebra $= 792$.
2. By the multiplication rule: $\binom{5}{3} \times \binom{7}{2} =$ do the algebra $= 210$.
3. By the addition rule: $\binom{5}{1} \times \binom{7}{4} + \binom{5}{2} \times \binom{7}{3} + \binom{5}{3} \times \binom{7}{2} + \binom{5}{4} \times \binom{7}{1} + \binom{5}{5} \times \binom{7}{0}$ $=$ do the algebra $= 175 + 350 + 210 + 35 + 1 = 771$. Or **better** by the subtraction rule: this is the number of all 5–person teams minus the number of 5–person teams that contain no men $= \binom{12}{5} - \binom{7}{5} =$ do the algebra $= 729 - 21 = 771$.
4. By the addition rule: this is the number of all 5–person teams without any men plus the number of 5–person teams with one man $= \binom{5}{0} \times \binom{7}{5} + \binom{5}{1} \times \binom{7}{4} =$ do the algebra $= 21 + 175 = 196$.

Exercise 9.5.7

A computer programming team has 13 members. (a) How many ways can a group of seven be chosen to work on a project? (b) Suppose seven members are women and six are men. (b1) How many groups of seven can be chosen that contain four women and three men? (b2) How many groups of seven can be chosen that contain at least one man? (b3) How many groups of seven can be chosen that contain at most three women? (c) Suppose two team members refuse to work together on projects. How many groups of seven can be chosen to work on a project? (d) Suppose two team members insist on either working together or not at all on projects. How many groups of seven can be chosen to work on a project?

(c) Let A, B be these persons. Then this is the number of groups with A and 6 others plus the number of groups with B and 6 others plus the number of groups with neither A nor B. (d) Then this is the number of groups with both A and B plus the number of groups with neither A nor B.

Exercise 9.5.10

Two new drugs are to be tested using a group of 60 laboratory mice, each tagged with a number for identification purposes. Drug A is to be given to 22 mice, drug B is to be given to another 22 mice, and the remaining 16 mice are to be used as controls. How many ways can the assignment of treatments to mice be made? (A single assignment involves specifying the treatment for each mouse – whether drug A, drug B, or no drug.)

By the multiplication rule, this is the number of choosing 22 mice out of 60 to receive treatment A times the number of choosing 22 mice out of the remaining 38 to receive treatment B.

Example 2 (**Poker Hands**)

1. How many five–card poker hands cards contain two pairs?
2. What is the probability of being dealt a hand that contains two pairs?

1. Using the multiplication rule, this is the number of choosing 2 pairs from 13 denominations times the number of choosing the two cards of the first pair from the smaller denomination times the number of choosing the two cards of the first pair from the larger denomination (one pair in each suit) times the the number of choosing one card from those remaining
$= \binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times \binom{44}{1} = \text{ do the algebra } = 123,552.$
2. The total number of five–card hands from an ordinary deck of cards is $\binom{52}{5} = 2,598,960$. Thus, the the probability of obtaining a hand with two pairs is $\frac{123,552}{2,598,960} \approx 4.75\%$.

Exercise 9.5.11

Find the probability that a randomly chosen five–card poker hand has the holdings:

(b) straight flush,

(d) full house,

(e) flush,

(g) three of a kind.

You may use the Internet, but you should justify your solutions.

9.5 Combinations: Examples (c)

Example 3 (Strings)

How many eight–bit strings have exactly three 1's?

This is exactly the number of combinations of size 3 from a set of size 8, i.e., it is $= \binom{8}{3} = \ldots = 56$.

Exercise 9.5.13

Tossing a coin ten times, in how many of the possible outcomes the following events are expected to occur?

(b) exactly five heads,

(c) at least eight heads,

(e) at most one head.

These are the answers but you need to justify them (and complete the calculations): (b) $\binom{10}{5}$, (c) $\binom{10}{8} + \binom{10}{9} + \binom{10}{10}$, (e) $\binom{10}{0} + \binom{10}{1}$.

Exercise 9.5.22

How many symbols can be represented in the Braille code?

Solution 1: By the difference rule, this is the total number of subsets of a set of 6 elements minus one (for the empty set): $2^6 - 1 = 63$.

Solution 2: By the addition rule, this is the sum of the numbers of subsets of size n chosen among the elements of a set of size 6, for $n = 1, 2, \ldots, 6$: $\sum_{n=1}^{6} \binom{6}{n} = 6 + 15 + 20 + 15 + 6 + 1 = 63$.

9.5 Generalized Permutations

Theorem

Suppose that a set S contains n elements of which n_1 identical elements are of type 1, n_2 identical elements are of type 2, ..., n_k identical elements are of type k, where $n_1 + n_2 + \ldots + n_k = n$. Then the number of orderings of S is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

9.5 Table of Permutations and Combutations

Permutations	n distinct objects	$n!$
	n repeated objects in k types	$\frac{n!}{n_1!\cdot n_2!\cdots n_k!}$
Strings	k out of n symbols	$\frac{n!}{(n-k)!}$
	n symbols in string of length k	n^k
Combinations	subsets of size k in a set of size n	$\binom{n}{k}$
	subsets of size k with repeated elements in a set of size n	$\binom{n+k-1}{n-1}$

9.6 Pascal's Formula

Theorem

Let n and r be positive integers and suppose $r \leq n$. Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

Exercise 9.6.12

Use Pascal's formula repeatedly to show the following formula:

$$\binom{n+3}{r} = \binom{n}{r-3} + 3\cdot\binom{n}{r-2} + 3\cdot\binom{n}{r-1} + \binom{n}{r}.$$

$\binom{n+3}{r} = \binom{(n+2)+1}{r} = \binom{n+2}{r-1} + \binom{n+2}{r} = \binom{n+1}{r-2} + \binom{n+1}{r-1} + \binom{n+1}{r-1} + \binom{n+1}{r} = \binom{n+1}{r-2} + 2\cdot\binom{n+1}{r-1} + \binom{n+1}{r} = \ldots$ etc.

9.6 The Binomial Theorem

Theorem

For any real numbers a and b any nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \tbinom{n}{k} a^{n-k} b^k =$$

$$= a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n.$$

9.6 Pascal's Triangle

Definition

Pascal's triangle is a triangular array of the binomial coefficients, the borders of which consist of 1's and any interior value is the sum of the two numbers above it.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

9.6 The Binomial Theorem: Exercises

Exercise 9.6.30 and 32

Find the coefficient of the term x^7 in the expansion of $(2x+3)^{10}$ and the coefficient of the term $u^{16}v^4$ in the expansion of $(u^2 - v^2)^{10}$.

For the second, since $u^{16}v^4 = (u^2)^8(-v^2)^2$, the term is $\binom{10}{2}(u^2)^8(-v^2)^2$ and the coefficient is $\binom{10}{2} = \frac{10!}{2!\cdot 8!} \cdot (-1)^2 = 45$. Similarly for the first.

Exercise 9.6.37

For all integers $n \geq 0$,

$$3^n = \binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^n\binom{n}{n}.$$

Apply the Binomial Theorem with $a = 1$ and $b = 2$.