Slides of Discrete Mathematics based on Susanna Epp's Textbook

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Chapter 2

The Logic of Compound Statements

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2.1 Logical Forms

Definition

A **statement** (or **proposition**) is a sentence that is true or false but not both.

Examples

- Delaware River runs through Pittsburgh. (*False*.)
- $2 + 3 = 7$. (*False*.)
- 4 is a positive number and 3 negative. (*False*.)
- If set S consists of n elements, then it contains 2^n subsets. (*True*.)
- There exists an integer n such that $2^n = n^2$. (*True*.)
- $x + y = y + x$, for every $x, y \in \mathbb{R}$. (*True*.)
- If $A^2 = 0$, then, $A = 0$, for every A. (*Indeterminate*.)
- Every even integer greater than 2 is the sum of two prime numbers. (*Goldbach's Conjecture*.)
- There exist infinitely many integers n such that $2^n + n$ is a prime number. (*Unknown to be true or false*.)

2.1 Logical Connectives

Notation of symbols of common logical connectives

- **Negation (not)** (prefix): $\sim$
- **Conjuction (and)** (prefix): $\wedge$
- **Disjuction (or)** (prefix): $\vee$
- **Conditional (if $\ldots$, then $\ldots$)**: $\longrightarrow$
- **Biconditional (if and only if)**: $\longleftrightarrow$

Connectives	Precedence
$\sim$	1
$\wedge$	2
$\vee$	3
$\longrightarrow$	4
$\longleftrightarrow$	5

Notation of truth values of a statement

- T: True
- F: False

Truth Table for $\sim p$

p	$\sim p$
F	T
T	F

Truth Table for $p \wedge q$, $p \vee q$, $p \longrightarrow q$, $p \longleftrightarrow q$

p	q	$p \wedge q$	$p \vee q$	$p \longrightarrow q$	$p \longleftrightarrow q$
F	F	F	F	T	T
F	T	F	T	T	F
T	F	F	T	F	F
T	T	T	T	T	T

2.1 The Exclusive Or

Definition

The **exclusive or** (or **XOR**) logical connective of two statement variables p, q, denoted $p \oplus q$, is defined by the composite statement:

$$p \oplus q = (p \vee q) \wedge \sim (p \wedge q).$$

Truth Table for $p \oplus q$

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

2.1 An Example of a Statement Form (Composite Statement)

Truth Table for $(p \wedge q) \vee \sim r$

p	q	r	$q \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

2.1 Tautologies and Contradictions

Definition

- A composite statement is called **tautology** if it is always true for all truth values of the individual statements included in it.
- A composite statement is called **contradiction** if it is always false for all truth values of the individual statements included in it.

Example

The statement form $p \vee \sim p$ is a tautology and the statement form $p \wedge \sim p$ is a contradiction.

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

Definition

Two statement forms P and Q are called **logically equivalent**, denoted $P \equiv Q$, if they have identical truth values for each possible substitution of the truth values of all individual statements included in them.

Example: $\sim(\sim p) \equiv p$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

Example: $\sim(p \wedge q) \not\equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

Theorem (De Morgan's Laws)

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$
$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

Example: $\sim(p \wedge q) \equiv \sim p \vee \sim q$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim \boldsymbol{p}$	$\sim \boldsymbol{q}$	$\boldsymbol{p} \wedge \boldsymbol{q}$	$\sim(\boldsymbol{p} \wedge \boldsymbol{q})$	$\sim \boldsymbol{p} \vee \sim \boldsymbol{q}$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

2.1 Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6. *Double negative law:*	${\sim}({\sim}p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* $\mathbf{t}$ *and* $\mathbf{c}$*:*	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

2.2 Conditional Statements (a)

Definition

- If p and q are two statements, the **conditional of q by p** is the statement form "if p then q" or "p implies q" and it is denoted by $p \longrightarrow q$.
- If p is true and q is false, the conditional $p \longrightarrow q$ is false, while in all other truth values of p and q it is true.
- In a conditional $p \longrightarrow q$, p is called **hypothesis** (or **antecedent**) of the conditional and q is called **conclusion** (or **consequent**) of the conditional.

Truth Table for $p \longrightarrow q$

p	q	$p \longrightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

2.2 Conditional Statements (b)

Proposition (Representation of Conditional as Or)

$$p \longrightarrow q \equiv \sim p \vee q.$$

Proposition (Negation of Conditional)

$$\sim (p \longrightarrow q) \equiv p \wedge \sim q.$$

2.2 Conditional Statements (c)

Definition

- The **contrapositive** of $p \longrightarrow q$ is $\sim q \longrightarrow \sim p$.
- The **converse** of $p \longrightarrow q$ is $q \longrightarrow p$.
- The **inverse** of $p \longrightarrow q$ is $\sim p \longrightarrow \sim q$.

Proposition

- *A conditional statement is logically equivalent to its contrapositive.*
- *A conditional statement and its converse are not logically equivalent.*
- *A conditional statement and its inverse are not logically equivalent.*
- *The converse and the inverse of a conditional statement are logically equivalent to each other.*

2.2 The Biconditional Statement

Definition

- If p and q are two statements, the **biconditional of** p **and** q is the statement form "p if and only if q" and it is denoted by $p \longleftrightarrow q$.
- If both p and q have the same truth value, the biconditional $p \longleftrightarrow q$ is true, while otherwise it is false.

Truth Table for $p \longleftrightarrow q$

p	q	$p \longleftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

2.2 Necessary and Sufficient Conditions

Definition

Let r and s be two statements. Then:

- r is a **sufficient condition** for s means "if r then s."
- r is a **necessary condition** for s means "if s then r; it also means "if not r then not s."
- r is a **necessary and sufficient condition** for s means "r if and only if s."

2.3 Arguments (a)

Definition

- An **argument** is a sequence of statements.
- All statements in an argument, except for the final one, are called **premises** (or **assumptions** or **hypotheses**) and the final statement of is called the **conclusion**. The symbol $\therefore$, read "therefore," is normally placed just before the conclusion.
- An argument is **valid** if the conclusion necessarily follows from the premises in the sense that if the premises are all true, then the conclusion is also true.

Example

Is the following argument valid?

$$\begin{aligned} & p \longrightarrow q \vee \sim r \\ & q \longrightarrow p \wedge r \\ \therefore\ & p \longrightarrow r \end{aligned}$$

2.3 Invalid Argument!

						premises		conclusion
p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \to q \vee \sim r$	$q \to p \wedge r$	$p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

Proposition

- **Modus ponens** *(or* **method of affirming***):*

$$\begin{aligned} & p \longrightarrow q \\ & p \\ \therefore \quad & q \end{aligned}$$

- **Modus tollens** *(or* **method of denying***):*

$$\begin{aligned} & p \longrightarrow q \\ & \sim q \\ \therefore \quad & \sim p \end{aligned}$$

2.3 Fallacies

Definition

A **fallacy** is an error in reasoning that results in an invalid argument.

Proposition (Two fallacies)

- **The fallacy of affirming the consequent**:
$$\begin{array}{rl} & p \longrightarrow q \\ & q \\ \therefore & p \end{array}$$

- **The fallacy of denying the antecedent**:
$$\begin{array}{rl} & p \longrightarrow q \\ & \sim p \\ \therefore & \sim q \end{array}$$

2.3 Validity versus Truth

Valid argument with false conclusion

Note that valid and invalid are not synonymous with true and false! The following argument is valid but its conclusion is false:

If John Lennon was a rock star, then John Lennon had red hair.

John Lennon was a rock star.

$\therefore$ John Lennon had red hair.

2.3 Proof by Contradiction

Proposition (Reductio ad impossibile)

If you can show that assuming p is false leads to a contradiction, then you can conclude that p is true. Formally:

$$\begin{array}{rl} & \sim p \longrightarrow c \\ \therefore & p \end{array}$$

Theorem (Euclid's Theorem)

There are infinitely many prime numbers.

Proof: Suppose that $p_1 = 2 < p_2 = 3 < \ldots < p_r$ were all of the primes. Let $P = p_1 p_2 \cdots p_r + 1$ and let p be a prime dividing P. Then p can not be any of $p_1, p_2, \ldots, p_r$, otherwise p would divide the difference $P - p_1 p_2 \cdots p_r = 1$, which is impossible. So this prime p is still another prime, and $p_1, p_2, \ldots, p_r$ would not be all of the primes. ■