Slides of Discrete Mathematics based on Susanna Epp's Textbook

Moses A. Boudourides[1]

Visiting Associate Professor of Computer Science
Haverford College

[1] Moses.Boudourides@cs.haverford.edu

Chapter 5c

Sequences, Mathematical Induction, and Recursion, V, VI

October 18, 20, & 22, 2021

5.5 Defining Sequences Recursively (a)

Definition

A **recurrence relation** for a sequence $a_0, a_1, a_2, \ldots$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$ through a function F, i.e.,

$$a_k = F(a_{k-1}, a_{k-2}, \ldots, a_{k-i})$$

where i is an integer with $k - i \geq 0$. The **initial conditions** for such a recurrence relation specify the values of $a_0, a_1, a_2, \ldots, a_{i-1}$.

The Fibonacci Sequence

$$F_k = F_{k-1} + F_{k-2}, \text{ for all integers } k \geq 2,$$
$$F_0 = 1, F_1 = 1.$$

The next four terms are easily found to be: $F_2 = F_1 + F_0 = 1 + 1 = 2$, $F_3 = F_2 + F_1 = 2 + 1 = 3, F_4 = F_3 + F_2 = 3 + 2 = 5, F_5 = F_4 + F_3 = 5 + 3 = 8, F_6 = F_5 + F_4 = 8 + +5 = 13.$

5.5 Defining Sequences Recursively (b)

The Catalan Sequence

The **Catalan numbers** are defined as

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \text{ for all integers } n \geq 1.$$

Show that the sequence satisfies the recurrence relation $C_k = \frac{4k-2}{k+1} C_{k-1}$, for all integers $k \geq 2$.

First, notice that setting $n = k-1$ in the definition of this sequence, we get $C_{k-1} = \frac{1}{k-1+1}\binom{2(k-1)}{k-1} = \frac{1}{k}\binom{2k-2}{k-1}$. Therefore, to verify the recurence relation, start with the right hand side and replace the previous expression:

$$\begin{aligned} \frac{4k-2}{k+1} C_{k-1} &= \frac{4k-2}{k+1}\frac{1}{k}\binom{2k-2}{k-1} \\ &= \text{ do the algebra as in the book p. 225} \\ &= \frac{1}{k+1}\binom{2k}{k} = C_k. \end{aligned}$$

Exercise 5.5.14

Let a sequence be defined as

$$d_n = 3^n - 2^n, \text{ for all integers } n \geq 0.$$

Show that the sequence satisfies the recurrence relation $d_k = 5d_{k-1} - 6d_{k-2}$, for all integers $k \geq 2$.

First, notice that by the definition of this sequence $d_{k-1} = 3^{k-1} - 2^{k-1}$ and $d_{k-2} = 3^{k-2} - 2^{k-2}$. Therefore, starting from the right hand side of the recurrence relation that we want to show:

$$\begin{aligned} 5d_{k-1} - 6d_{k-2} &= 5(3^{k-1} - 2^{k-1}) - 6(3^{k-2} - 2^{k-2}) \\ &= \textbf{do the algebra to get} \\ &= 3^k - 2^k = d_k. \end{aligned}$$

Exercise 5.5.19

Show that in the **four–pole tower of Hanoi**, $s_k \leq 2s_{k-2} + 3$, for all integers $k \geq 3$, where s_k denotes the minimum number of moves needed to transfer the top k disks from the left–most to the right–most pole.

Name the poles A, B, C, and D from left to right. To transfer a tower of k disks from A to D, proceed according to the following successive steps: (1) transfer the top $k-2$ disks from A to B; (2) transfer the second largest disc from A to C; (3) transfer the largest disc from A to D; (4) transfer the second largest disc from C to D; (5) transfer the top $k-2$ disks from B to D. Thus, we obtain (**justify why the following inequalities are true**):

$$\begin{aligned} s_k \leq\ & s_{k-2} && \text{[Step (1)]} \\ & + 1 && \text{[Step (2)]} \\ & + 1 && \text{[Step (3)]} \\ & + 1 && \text{[Step (4)]} \\ & + s_{k-2} && \text{[Step (5)]} \\ \leq\ & 2s_{k-2} + 3. \end{aligned}$$

5.5 Defining Sequences Recursively (e)

Exercise 5.5.28

For the Fibonacci sequence, prove that

$$F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_kF_{k-1}, \text{ for all integers } k \geq 1.$$

By definition,

$$\begin{aligned} F_{k+1}^2 - F_k^2 - F_{k-1}^2 &= (F_k + F_{k-1})^2 - F_k^2 - F_{k-1}^2 \\ &= \textbf{do the algebra to get} \\ &= 2F_kF_{k-1} \end{aligned}$$

Exercise 5.5.31

For the Fibonacci sequence, prove that

$$F_n < 2^n, \text{ for all integers } n \geq 1.$$

Proof by Strong Induction (Sketch):

1. **Basic Steps (for $a = 1, b = 2$)**: By definition, $F_1 = 1 < 2 = 2^1, F_2 = 2 = 2^1$.
2. **Inductive Step**: Suppose $k \geq 1$ and that for all integers i with $1 \leq i \leq k$, $F_i < 2^i$. (Goal: $F_{k+1} < 2^{k+1}$.) By definition, $F_{k+1} = F_k + F_{k-1}$, where, according to the inductive step, $F_k < 2^k$ and $F_{k-1} = 2^{k-1}$. Then, writing $2^k = 2 \cdot 2^{k-1}$, do the algebra to show the goal.

Compound Interest

A person invests a_0 dollars at p percernt interest compounded annually. If A_n represents the amount at the end of n years, find a recurrence relation and initial conditions that define the sequence $A_1, A_2, \ldots$.

At the end of $n-1$ years, the amount is A_{n-1}. At the next year, we will have the amount A_{n-1} plus the interest. Thus,

$$A_n = A_{n-1} + pA_{n-1} = (1+p)A_{n-1}, \text{ for all integers } n \geq 1.$$

Clearly, the initai condition is given as $A_0 = a_0$. Therefore, we obtain:

$$\begin{aligned} A_1 &= (1+p)A_0 = (1+p)a_0, \\ A_2 &= (1+p)A_1 = (1+p)(1+p)a_0 = (1+p)^2 a_0, \\ A_3 &= (1+p)A_2 = (1+p)(1+p)(1+p)a_0 = (1+p)^3 a_0, \\ &\text{and so on.} \end{aligned}$$

In other words, the recurrence rlation is

$$A_n = (1+p)^n a_0, \text{ for all integers } n \geq 1.$$

Exercise 5.5.37

Suppose a certain amount of money is deposited in an account paying 3% annual interest compounded monthly. For each positive integer n, let S_n = the amount on deposit at the end of the nth month, and let S_0 be the initial amount deposited. Find a recurrence relation for $S_0, S_1, S_2, \ldots$, assuming no additional deposits or withdrawals during the year.

When 3% interest is compounded monthly, th interest rate per month is $0.03/12 = 0.0025$. If S_k is the amount on deposit at the end of month k, then $S_k = S_{k-1} + 0.0025S_{k-1} = (1 + 0.0025)S_{k-1} = (1.0025)S_{k-1}$, for each integer $k \geq 1$.

Exercise 5.5.39

A set of blocks contains blocks of heights 1, 2, and 4 centimeters. Imagine constructing towers by piling blocks of different heights directly on top of one another. (A tower of height 6 cm could be obtained using six 1-cm blocks, three 2-cm blocks, one 2-cm block with one 4-cm block on top, one 4-cm block with one 2-cm block on top, and so forth.) Let t_n be the number of ways to construct a tower of height n cm using blocks from the set. (Assume an unlimited supply of blocks of each size.) Find a recurrence relation for $t_1, t_2, t_3, \ldots$.

Let a tower have (total) height k cm and let t_k be the number of ways to construct it. There are three cases for the height h of the bottom block of any tower: (i) $h = 1$ cm, (ii) $h = 2$ cm and (iii) $h = 4$ cm. In any case, the remaining blocks (save the bottom one) make up a tower of height $(k - h)$ cm and, hence, there are t_{k-h} ways to construct it. Apparently, the total number of ways to construct the tower of k cm is equal to the sum of ways in each one of the three cases. In other words, the recurrence relation is $t_k = t_{k-1} + t_{k-2} + t_{k-4}$, for all integers $k \geq 5$.

5.6 Solving Recurrence Relations by Iteration (a)

Definition

A sequence $a_0, a_1, a_2, \ldots$ is called an **arithmetic sequence** if and only if there is a constant d such that

$$a_k = a_{k-1} + d, \text{ for all integers } k \geq 1.$$

It follows that,

$$a_n = a_0 + dn, \text{ for all integers } n \geq 0.$$

Definition

A sequence $a_0, a_1, a_2, \ldots$ is called a **geometric sequence** if and only if there is a constant r such that

$$a_k = ra_{k-1}, \text{ for all integers } k \geq 1.$$

It follows that,

$$a_n = a_0 r^n, \text{ for all integers } n \geq 1.$$

5.6 Solving Recurrence Relations by Iteration (b)

Exercise 5.5.8 & 33

Guess the formula of the sequence and use induction to verify it:

$$f_k = f_{k-1} + 2^k, \text{ for all integers } k \geq 2,$$
$$f_1 = 1.$$

$$\begin{aligned}
f_1 &= 1, \\
f_2 &= f_1 + 2^2 = 1 + 2^2, \\
f_3 &= f_2 + 2^3 = 1 + 2^2 + 2^3, \\
f_4 &= f_3 + 2^4 = 1 + 2^2 + 2^3 + 2^4, \\
f_5 &= f_4 + 2^5 = 1 + 2^2 + 2^3 + 2^4 + 2^5, \\
&\vdots
\end{aligned}$$

Guess: $f_n = 1 + 2^2 + 2^3 + \ldots + 2^n = \left(\frac{2^{n+1}-1}{2-1}\right) - 2 = 2^{n+1} - 3, \forall n \geq 1.$

Proof of the inductive step: $f_{k+1} = f_k + 2^{k+1} = 2^{k+1} - 3 + 2^{k+1} = 2 \cdot 2^{k+1} - 3 = 2^{k+2} - 3.$ **Fill out all the remaining details**.

Exercise 5.5.10 & 35

Guess the formula of the sequence and use induction to verify it:

$$h_k = 2^k - h_{k-1}, \text{ for all integers } k \geq 1,$$
$$h_0 = 1.$$

$$h_0 = 1,$$
$$h_1 = 2^1 - h_0 = 2^1 - 1,$$
$$h_2 = 2^2 - h_1 = 2^2 - (2^1 - 1) = 2^2 - 2^1 + 1,$$
$$h_3 = 2^3 - h_2 = 2^3 - (2^2 - 2^1 + 1) = 2^3 - 2^2 + 2^1 - 1,$$
$$h_4 = 2^4 - h_3 = 2^4 - (2^3 - 2^2 + 2^1 - 1) = 2^4 - 2^3 + 2^2 - 2^1 + 1,$$
$$\vdots$$

Guess:

$$h_n = 2^n - 2^{n-1} + \ldots + (-1)^n \cdot 1 = (-1)^n[1 - 2 + 2^2 - \ldots + (-1)^n \cdot 2^n]$$
$$= (-1)^n[1 + (-2) + (-2)^2 - \ldots + (-2)^n]$$
$$= (-1)^n \left[\frac{(-2)^{n+1} - 1}{(-2) - 1}\right] = (-1)^n \frac{((-2)^{n+1} - 1)}{(-3)}$$
$$= \frac{(-1)^{n+1}}{(-1)} \cdot \frac{((-2)^{n+1} - 1)}{(-3)} = \frac{1}{3}\left[2^{n+1} - (-1)^{n+1}\right], \forall n \geq 1.$$

Exercise 5.5.10 & 35 (cont.)

Proof of the inductive step:

$$
\begin{aligned}
h_{k+1} &= 2^{k+1} - h_k \\
&= 2^{k+1} - \frac{1}{3}\left[2^{k+1} - (-1)^{k+1}\right] \\
&= \frac{1}{3}\left[3 \cdot 2^{k+1} - 2^{k+1} + (-1)^{k+1}\right] \\
&= \frac{1}{3}\left[2 \cdot 2^{k+1} - (-1)^{k+2}\right] \\
&= \frac{1}{3}\left[2^{k+2} - (-1)^{k+2}\right] \\
&= \frac{1}{3}\left[2^{(k+1)+1} - (-1)^{(k+1)+1}\right].
\end{aligned}
$$

5.6 Solving Recurrence Relations by Iteration (d)

Exercise 5.5.48

Guess the formula of the sequence and use induction to verify it:

$$u_k = u_{k-2} \cdot u_{k-1}, \text{ for all integers } k \geq 2,$$
$$u_0 = u_1 = 2.$$

$$\begin{aligned}
u_0 &= 2,\\
u_1 &= 2,\\
u_2 &= u_0 \cdot u_1 = 2 \cdot 2 = 2^{1+1} = 2^2,\\
u_3 &= u_1 \cdot u_2 = 2 \cdot 2^2 = 2^{1+2} = 2^3,\\
u_4 &= u_2 \cdot u_3 = 2^2 \cdot 2^3 = 2^{2+3} = 2^5,\\
u_5 &= u_3 \cdot u_4 = 2^3 \cdot 2^5 = 2^{3+5} = 2^8,\\
u_6 &= u_4 \cdot u_5 = 2^5 \cdot 2^8 = 2^{5+8} = 2^{13},\\
&\vdots
\end{aligned}$$

Guess:

$u_n = 2^{F_n}$, where F_n is the nth Fibonacci number, for all integers $n \geq 0$.

Exercise 5.5.49 (cont.)

Proof of the inductive step:

$$\begin{aligned} u_{k+1} &= u_{k-1} \cdot u_k \\ &= 2^{Fk-1} \cdot 2^{Fk} \\ &= 2^{Fk-1+Fk} \\ &= 2^{Fk+1}. \end{aligned}$$

Exercise 5.5.23

Suppose the population of a country increases at a steady rate of 3% per year. If the population is 50 million at a certain time, what will it be 25 years later?

Let, for each integer $n \geq 1$, P_n denote the population at the end of year n. Then, for all integers $k \geq 1$, $k \geq 1$, $P_k = P_{k-1} + (0.03)P_{k-1} = (1.033)P_{k-1}$. Hence, $P_0, P_1, P_2, \ldots$ is a geometric sequence with constant multiplier 1.03 and, so, $P_n = (1.033)^n P_0$, for all integers $n \geq 0$. Since $P_0 = 50$ million, it follows that at the end of 25 years it would be $P_{25} = (1.033)^{25} 50 \cong 104.7$ million.

Exercise 5.5.53

A single line divides a plane into two regions. Two lines (by crossing) can divide a plane into four regions; three lines can divide it into seven regions (see the figure in the book). Let P_n be the maximum number of regions into which n lines divide a plane, where n is a positive integer. (i) Derive a recurrence relation for P_k in terms of P_{k-1}, for all integers $k \geq 2$. (ii) Use iteration to guess an explicit formula for P_n.

Let us suppose that there are $k-1$ lines already drawn on the plane in such a way that they divide the plane into a maximum number P_{k-1} of regions. If addition of a new line is to create a maximum number of regions, it must cross all the $k-1$ lines that are already drawn. Furthermore, in this case, one can imagine traveling along the new line from a point before it reaches the first line it crosses to a point after it reaches the last line it crosses. This means that the new line is going to create k new regions. In other words, $P_{k-1} = P_k + k$, for all integers $k \geq 1$ and, thus, we get:

Exercise 5.5.53 (cont.)

$$
\begin{aligned}
P_1 &= 2,\\
P_2 &= P_1 + 2 = 2 + 2,\\
P_3 &= P_2 + 2 = 2 + 2 + 3,\\
P_4 &= P_3 + 2 = 2 + 2 + 3 + 4,\\
P_5 &= P_4 + 2 = 2 + 2 + 3 + 4 + 5,\\
P_6 &= P_5 + 2 = 2 + 2 + 3 + 4 + 5 + 6,\\
&\vdots
\end{aligned}
$$

Guess:

$$
\begin{aligned}
P_n &= 2 + 2 + 3 + 4 + \cdots + n = 1 + 1 + 2 + 3 + 4 + + \cdots + n\\
&= 1 + \frac{1}{2}n(n+1) = \frac{1}{2}\left(n^2 + n + 2\right).
\end{aligned}
$$