Slides of Discrete Mathematics based on Susanna Epp's Textbook

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Chapter 5a

Sequences, Mathematical Induction, and Recursion, I, II

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5.1 Sequences

Definition

A **sequence** of numbers is a finite or infinite set of numbers S. Typically, we understand that the elements of the set S (or the values of the sequence) to be numbers in $\mathbb{Z}$ or in $\mathbb{Q}$ or in $\mathbb{R}$. All the elements of a sequence are called **terms** and the representative form of terms is called **general term** and it is written as a_k (read "a sub k"), where the subscript k in a_k is an integer which is called **index** (of the sequence). A **finite** sequence of n elements is written as $\{a_1, a_2, \ldots, a_n\}$ and an **infinite** sequence as $\{a_1, a_2, \ldots\}$. The set of indices of a sequence is called **domain** (of the sequence) and it is either a finite or an infinite set of integers, depending on whether the sequence is finite or infinite (respectively). The domain of a finite sequence is taken to be the set of all integers between two given $m, n \in \mathbb{Z}$ such that $m \leq n$, while the domain of an infinite sequence is usually taken to be the set of positive integers $\mathbb{Z}^+$. In other words, a sequence is a function with domain either an interval of integers $[m, n]$ or all positive $\mathbb{Z}^+$ and with range, typically, in $\mathbb{R}$. If we know such a function for the general term a_k of a sequence, the formula of this function is said to be the **explicit formula** or **general formula** (for the sequence).

5.1 Finding Sequences

Examples

- *Finding terms of a sequence given its general formula*:
  Example: If $a_k = \frac{k}{10+k}$, for all $k \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, then the sequence is infinite with values $a_1 = \frac{1}{11}, a_2 = \frac{2}{12} = \frac{1}{6}, a_3 = \frac{3}{13}, \ldots$, i.e., the sequence is $\frac{1}{11}, \frac{1}{6}, \frac{3}{13}, \ldots$.
- **An alternating sequence** has general formula $c_j = (-1)^j$, for all integers $j \geq 0$, i.e., it is the sequence $1, -1, 1, -1, \ldots$.
- *Finding the general formula of a sequence given its terms*:
  Example: For the finite sequence $0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}$, the general formula is $a_k = (-1)^{k-1}(\frac{k-1}{k})$, for all integers k from 1 to 7.
  Why?

5.1 Summation of Terms of a Finite Sequence

Definition

Let $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$ be a finite sequence with domain all integers between integer m and integer n, where $m \leq n$. Then $\sum_{k=m}^{n} a_k$, read **sum(mation) from** k **equals** m **to** n **of sequence** a**–sub–**k is defineed as the sum of terms of the sequence:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

We call k **index** of the summation, m the **lower limit** of the summation and n the **upper limit** of the summation.

Notice that the summation of a sequence from m to n is a function of m and n.

Examples

- *Finding sum of a finite sequence from its general formula*:
  Example: $\sum_{i=1}^{k+1} i(i!) =$
  $1(1!)+2(2!)+3(3!)+\cdots+(k+1)\big((k+1)!\big) = 1+4+18+\cdots+(k+1)^2k!$.
- **A telescoping sum**: For any integer $n \geq 1$,
  $$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{1+n} \ (\textbf{Why?})$$
- *Expressing expanded summation to its general formula*:
  Example:
  $(1^3-1)-(2^3-1)+(3^3-1)-(4^3-1)+(5^3-1) = \sum_{k=1}^{5}(-1)^{k+1}(k^3-1)$
  (**Why?**)

5.1 Product Notation

Definition

Let $a_m, a_{m+1}, a_{m+2}, \dots, a_n$ be a finite sequence between m and n, where m, n are integers and $m \leq n$. Then the symbol $\prod_{k=m}^{n} a_k$, read the **product from** k **equals** m **to** n **of** a–**sub**–k, is the product of all terms of this finite sequence, i.e.:

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

Example

$$(1-t)\cdot(1-t^2)\cdot(1-t^3)\cdot(1-t^4) = \prod_{j=1}^{4}(1-t^j).$$

5.1 Properties of Summations and Products

Theorem

1. $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$,
2. $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$,
3. $\displaystyle\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$.

5.1 Transforming Sums by Change of Variables, 1

- The index of sequence in summation is a **dummy variable**:

$$\sum_{k=1}^{n} a_k = \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j \text{ and so on.}$$

- **Index change of variable transformation**: Let m, n two integers, $m \leq n$, and suppose that the index k of the sum $\sum_{k=m}^{n} a_k$ changes to a new index j by a transformation $j = \varphi(k)$, which is assumed to be a nondecreasing function with inverse $k = \varphi^{-1}(j)$. Then:

$$\sum_{k=m}^{n} a_k = \sum_{j=\varphi(m)}^{\varphi(n)} a_{\varphi^{-1}(j)}.$$

Example of Index Transformations

Show that

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}.$$

Proof:

First, to transform the limits of summation, do the following change of variables in the left–hand sum:

$$j = k - 1 \text{ or } k = j + 1$$

to get

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{j=0}^{n} \frac{j+1}{n+(j+1)}.$$

Next, denoting the dummy variable j as k (in the right–hand side of the last equation), we get:

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{k=0}^{n} \frac{j+1}{n+(k+1)}.$$

5.1 Factorial

Definition

For each positive integer n, the quantity n **factorial**, denoted $n!$, is defined as the following product from k equals 1 to n of the sequence $a_k = k$:

$$n! = \prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1.$$

Zero factorial, denoted $0!$, is defined to be 1:

$$0! = 1.$$

A (alternative) recursive definition for factorial

$$n! = \begin{cases} 1, & \text{if } n = 0, \\ n \cdot (n-1)!, & \text{if } n \geq 1. \end{cases}$$

5.1 The "n Choose r" Notation

Definition

Let n and r be integers with $0 \leq r \leq n$. The symbol

$$\binom{n}{r},$$

read "n **choose** r", represents the number of subsets of size r that can be chosen from a set with n elements.

Formula for computing $\binom{n}{r}$

For all integers n and r with $0 \leq r \leq n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Exercise 5.1.73

For all nonnegative integers n and r with $r + 1 \leq n$,

$$\binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}.$$

Solution:

$$\begin{aligned}
\frac{n-r}{r+1}\binom{n}{r} &= \frac{n-r}{r+1}\frac{n!}{r!(n-r)!} \\
&= \frac{n-r}{r+1}\frac{n!}{r!(n-r)(n-r-1)!} \\
&= \frac{n!}{(r+1)!(n-r-1)!} \\
&= \frac{n!}{(r+1)!(n-(r+1))!} \\
&= \binom{n}{r+1}.
\end{aligned}$$

Exercise 5.1.74

If p is a prime number and r an integer such that $0 < r < p$, then $\binom{p}{r}$ is divisible by p.

Solution:

Since

$$\binom{p}{r} = \frac{p!}{r!(p-r)!} = \frac{p(p-1)!}{r!(p-r)!},$$

we get

$$p(p-1)! = \binom{p}{r}(r!(p-r)!).$$

Now, $\binom{p}{r}$ is an integer because it equals the number of subsets of size r that can be formed from a set with p elements. Thus, according to the theorem of unique factorization of integers, the right–hand side of the above equation can be expressed as a product of prime numbers. Moreover, since p is a factor of the left–hand side, p should be a factor of the right–hand side too. However, since $0 < r < p$, p cannot be a factor of either $r!$ or $(p-r)!$. Therefore, p must be a factor 0f $\binom{p}{r}$, which means that $\binom{p}{r}$ should be divisible by p.

5.2 Mathematical Induction I, 1

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true.

Then the statement

$$\text{for all integers } n \geq a, P(n)$$

is true.

Exercise 5.2.2

Use mathematical induction to show that any postage of at least 12¢ can be obtained using 3¢ and 7¢ stamps.

Solution:

Let $P(n) = \{$posting of n¢ can be obtained using 3¢ and 7¢ stamps$\}$.

Show that $P(12)$ is true: It is, because $12 = 4 \cdot 3 = 3 + 3 + 3 + 3$.

Show that for all integers $k \geq 12$, if $P(k)$ is true, then $P(k+1)$ is true: It is, because we have two cases: Either there are at least two 3¢ stamps among the k¢ stamps (where $k \geq 12$) (case 1); or there are at least two 7¢ stamps among the k¢ stamps (case 2) (**Why?**). In case 1, replace the two 3¢ stamps with one 7¢ stamp, and in case 2, remove the two 7¢ stamps and replace them with five 3¢ stamps.

5.2 Mathematical Induction I, 2

Theorem (Sum of the First n Integers)

For all integers $n \geq 1$,

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof: As in book pp. 190-1.

Theorem (Sum of the Squares of the First n Integers)

For all integers $n \geq 1$,

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof:

It is true for $k = 1$. Assume that, for $k \geq 1$, $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$. Then $\sum_{j=1}^{k+1} j^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{1}{6}(k+1)\big(k(2k+1) + 6(k+1)\big) = \frac{1}{6}(k+1)(2k^2+k+6k+6) = \frac{1}{6}(k+1)(2k^2+7k+6) = \frac{1}{6}(k+1)(2k^2+4k+3k+6) = \frac{1}{6}(k+1)\big(2k(k+2)+3(k+2)\big) = \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$.

Exercise 5.2.11

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^2 = \left[\frac{n(n+1)}{2}\right]^2.$$

Solution: It is true for $k = 1$. Assume that, for $k \geq 1$, $\sum_{j=1}^{k} j^3 = \frac{1}{4}\Big[n(n+1)\Big]^2$. Then $\sum_{j=1}^{k+1} j^3 = \frac{1}{4}\Big[k(k+1)\Big]^2 + (k+1)^3 = \frac{1}{4}(k+1)^2\Big[k^2 + 4(k+1)\Big] = \frac{1}{4}(k+1)^2\Big[k^2 + 4k + 4\Big] = \frac{1}{4}(k+1)^2(k+2)^2 = \frac{1}{4}\Big[(k+1)(k+2)\Big]^2 = \frac{1}{4}\Big[(k+1)\big((k+1)+1\big)\Big]^2$.

5.2 Mathematical Induction I, 4

Theorem (Sum of a Geometric Sequence)

For any real number r except 1 and for any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

Proof: As in book pp. 194-5.

Exercise 5.2.29

Find $1 - 2 + 2^2 - 2^3 + \cdots + (-1)^n 2^n$, where n is a positive integer.

Solution: $1 - 2 + 2^2 - 2^3 + \cdots + (-1)^n 2^n = 1 + (-2) + (-2)^2 + (-2)^3 + \cdots + (-2)^n$. Therefore, for $r = -2 \neq 1$, the formula of the sum of a geometric sequence yields $1 - 2 + 2^2 - 2^3 + \cdots + (-1)^n 2^n = \frac{(-2)^{n+1} - 1}{(-2) - 1} = \frac{(-2)^{n+1} - 1}{-3} = \frac{1}{3}\left(1 + (-1)^n 2^{n+1}\right)$.