Slides of Discrete Mathematics based on Susanna Epp's Textbook

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Chapter 1

Speaking Mathematically

August 30, 2021

1.1 Variables and Statements

Intuitive definition

A **variable** is a carrier for something, i.e., it is identified to or represented by a symbol which works as a placeholder for expressions or quantities that may vary.

Kinds of statements in mathematics

- **Universal statement** is an expression that something is true for all possible cases to which it refers.
- **Conditional statement** is an expression saying that if one thing is true then some other thing should be necessarily true.
- **Existential statement** is an expression about a given property saying that there is at least something for which the property is true, though there is no universal statement guarantying a priori the truth of the property.

1.2 Sets

Notation

- In *Set Theory*, according to the **axiom of extension**, a **set** S is completely defined by describing what its elements are, i.e., describing a property that the elements of the set should satisfy.
- $x \in S$ denotes that x is an element of S.
- $x \notin S$ denotes that x is not an element of S.
- **Set–roster notation of sets**:
  - for a **finite** set, $S = \{x_1, x_2, \ldots, x_n\}$;
  - for an **infinite** set, $S = \{x_1, x_2, \ldots\}$.

Notation of special sets

- $\mathbb{R}$ denotes the set or all real numbers.
- $\mathbb{Z}$ denotes the set or all integers.
- $\mathbb{Q}$ denotes the set or all rational numbers, i.e., quotients of integers.

1.2 Sets: The set–builder notation

Set–builder notation

Let S be a set and, for $x \in S$, let $P(x)$ be a universal statement that prescribes the membership property of x in S, i.e., the property P that elements x of S need to satisfy in order to be elements of S. Then S can be denoted as follows:

$$S = \{x \in S \mid P(x)\},$$

where by writing "$P(x)$," for $x \in S$, it is meant that "x satisfies property P."

1.2 Sets: Subsets

Definition

- If A and B are two sets, then A is called a **subset** of B or A is said to **be contained** in B, written $A \subseteq B$, if and only if every element of A is also an element of B, i.e.,

$$A \subseteq B \iff \forall x \in A:\ \text{if } x \in A,\ \text{then } x \in B.$$

- $A \nsubseteq B \iff \exists$ at least one $x \in A$ such that $x \notin B$.
- A is called a **proper subset** of B, if and only if every element of A is also in B but there is at least one element of B that is not in A.

1.2 Sets: Cartesian products

Ordered pairs of elements of two or one set

Let A and B two sets; it could be one single set, i.e., $B = A$. Then, given the two elements $a \in A$ and $b \in B$, the symbol (a, b) denotes an **ordered pair** of elements of A and B (or just A, when $B = A$) consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Given two other elements $c \in A$ and $d \in B$, the two ordered pairs (a, b) and (c, d) are **equal**, if and only if $a = c$ and $b = d$ (i.e., $(a, b) = (c, d) \iff a = c$ and $b = d$).

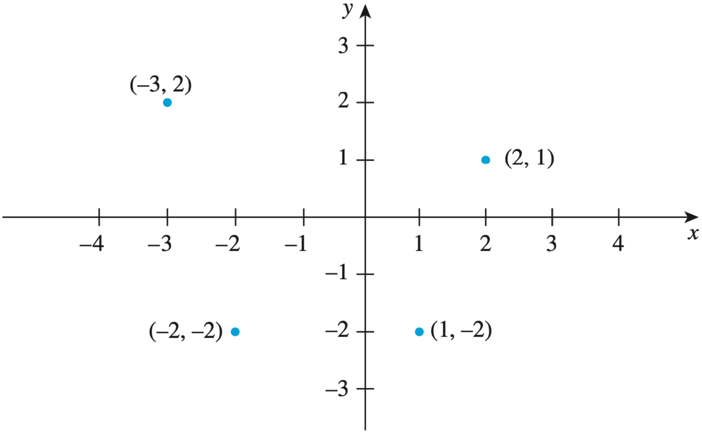
Definition

Let A and B two sets; it could be $B = A$. Then the **Cartesian product of A and B**, denoted $A \times B$, is defined as:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

In case $B = A$, we get the **Cartesian product of A and itself** $A^2 = A \times A = \{(a, b) \mid a, b \in A\}$.

1.2 Sets: $\mathbb{R}^2$ as the **Cartesian plane** of $\mathbb{R}$ and itself



1.3 Relations

Definition

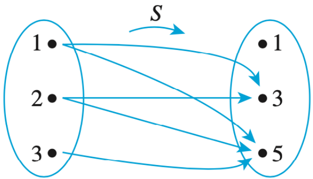
Let A and B two sets; it could be $B = A$.

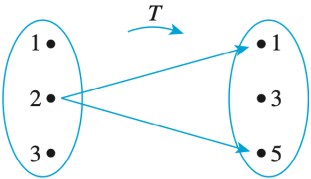
- A **relation** R **from** A **and** B is a subset of $A \times B$.
- Given two elements $x \in A$ and $y \in B$, x **is said to be related to** y **by the relation** R, written $x\,R\,y$, if and only if $(x, y) \in R$ $(\subseteq A \times B)$.
- Moreover, the set A is called the **domain** of relation R and the set B is called the **co–domain** of R.

The arrow diagram of a relation

Let R be a relation from a set A to a set B (it could be $B = A$). The **arrow diagram for** R is obtained as follows:

1. Represent the elements of A as points in one region and the elements of B as points in another region.
2. For each $x \in A$ and $y \in B$, draw an arrow from x to y, if and only if x is related to y (i.e., symbolically, $(x, y) \in R$).





Definition

Let A and B two sets; it could be $B = A$. A **function from A to B** is a relation with domain A and co–domain B that satisfies the following two properties:

1. $\forall x \in A, \exists y \in B$ such that $(x, y) \in F$ (in other words, every element of A is the first element of an ordered pair of F).
2. $\forall x \in A$ and $y, z \in B,$ if $(x, y) \in F$ and $(x, z) \in F$, then $y = z$ (in other words, no two distinct ordered pairs in F have the same first element).

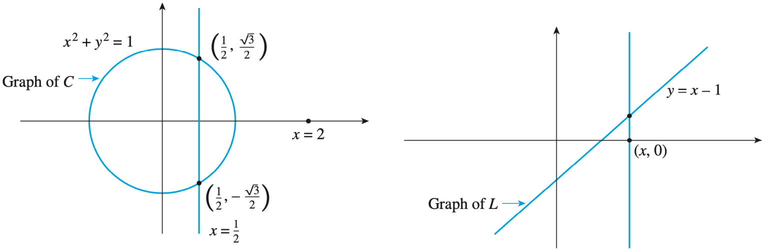
Notation of a function as a mapping

Let A and B two sets; it could be $B = A$. If F is a function from A to be B, then given any element $x \in A$, the unique element in B that is related to x by F is denoted as $F(x)$ (read "F of x") and the function F is denoted as a **mapping** $F : A \longrightarrow B$.

Definition

Let A and B two sets (it could be $B = A$) and let F be a function from A to B. The **graph of function** F, denoted $G(F)$, is defined as the corresponding relation from A to B in the definition of F:

$$G(F) = \{(x, F(x)) \mid x \in A\}.$$



1.3 Functions as mappings: Function machines