# Experiments of Friedkin–Johnsen Social Influence on Graphs

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# Graph Notations

- Everywhere in here, $G = (V, E)$ is considered to be a graph of $n$ nodes in the set $V = \{1, 2, \ldots, n\}$ with its adjacency matrix denoted by $\mathbf{A}$.
- Obviously, if $G$ is undirected, $\mathbf{A}$ is a symmetric matrix, while, if $G$ is directed, $\mathbf{A}$ is nonsymmetric.
- Given two nodes $i, j \in V$, by $i \sim j$ one denotes that $i, j$ are adjacent, when $G$ is undirected, or that $j$ is an outgoing neighbor of $i$, when $G$ is directed (in which case "$\sim$" is a nonsymmetric relation).
- By $\deg_i$, one denotes the degree of node $i$, when $G$ is undirected, or the out–degree of $i$, when $G$ is directed. Moreover, $\mathbf{D}$ denotes the diagonal matrix of node degrees of $G$.

# Main Graph Assumtions

- Without any loss of generality, we are assuming here:
  - $G$ is connected, when it is undirected, or $G$ is weakly connected, when it is directed.
  - When $G$ is directed, all nodes are assumed to have positive out–degrees *and* postive in–degrees.

- However, when $G$ is a **signed graph** (the case that we are going to consider in the last section), we are only assuming:
  - $G$ is connected, when it is undirected, or $G$ is strongly connected, when it is directed.

# The PageRank

- Let $G = (V, E)$ be a **graph** with $n$ nodes.
- For each node $i \in V$ and each time step $k = 0, 1, 2, \ldots$, the **PageRank** of $i$ at time $k$ is denoted by $x_i^{(k)} \in \mathbb{R} \in [0, 1]$.
- The PageRank of $i$ at time $k$ is updated at the subsequent time step $k + 1$ according to the following iterative scheme:

$$x_i^{(k+1)} = \alpha \, P \, x_i^{(k)} + (1 - \alpha) \, x_i^{(0)},$$

  - where $P \, x_i^{(k)}$ is the sum of the PageRanks of $i$'s neighbors,
  - $\alpha \in (0, 1]$ is the **teleportation constant** and
  - one typically assumes the initial condition $x_i^{(0)} = \frac{1}{n}$.

# Solving the PageRank Problem

- Disregarding the time exponent $(k)$, $P\,x_i$ is defined as:

$$P\,x_i = \sum_{j \in V} \frac{A_{ji}}{\deg_j}\, x_j = \sum_{j \sim i} \frac{x_j}{\deg_j},$$

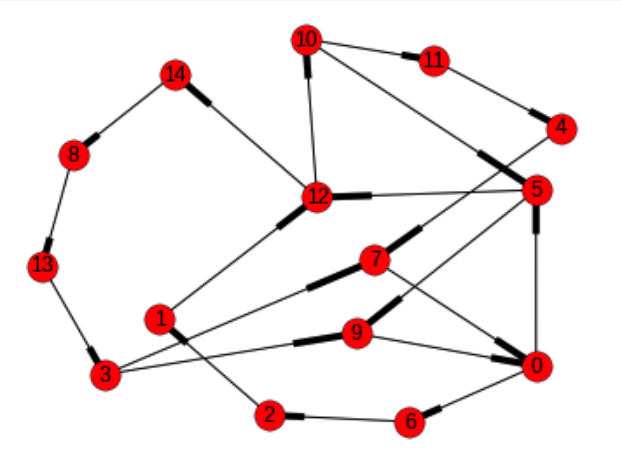
  i.e., $P$ is a **probability transition matrix** of traveling from node $i$ to node $j$ along an edge:

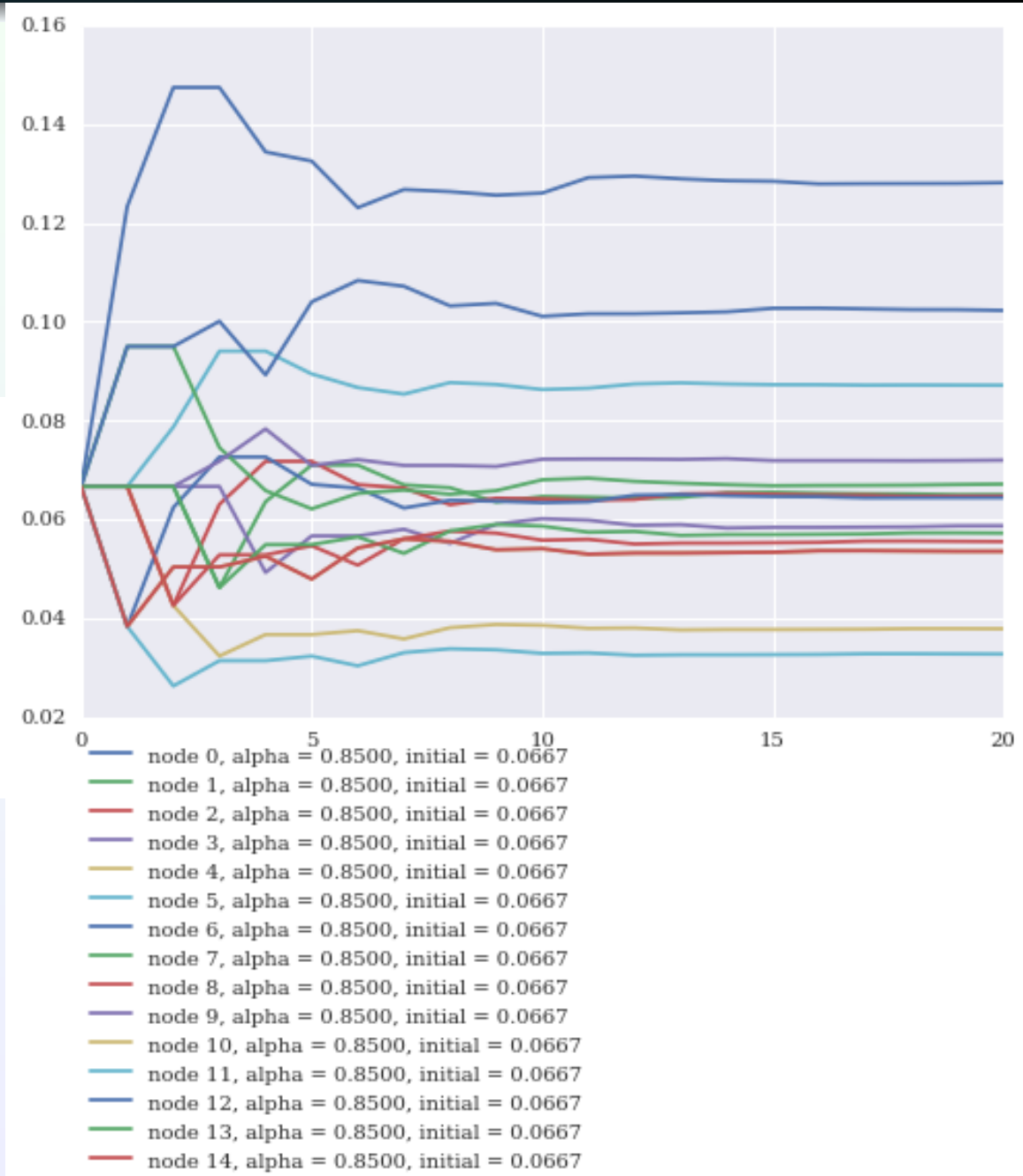
$$\mathbf{P} = \mathbf{A}^T \mathbf{D}^{-1}.$$

- As matrix $\mathbf{P}$ is *column–stochastic*, matrix $\mathbf{I} - \alpha\mathbf{P}$ becomes *diagonally dominant*.
- Therefore, the solution to this problem converges (relatively fast, depending on $\alpha$, typicall taken equal to 0.85) to the vector $\bar{\mathbf{x}} = \bar{\mathbf{x}}(\alpha; \mathbf{x}^{(0)})$ of PageRanks of nodes of $G$:

$$\bar{\mathbf{x}} = (1 - \alpha)\,(\mathbf{I} - \alpha\mathbf{P})^{-1}\mathbf{x}^{(0)}.$$

# PageRank Example



# The Friedkin–Johnsen Model of Social Influence on a Graph

- Let $G = (V, E)$ be a **graph** with $n$ **nodes** $\equiv$ **persons**.
- For each node $i \in V$ and each time step $k = 0, 1, 2, \ldots$, the **opinion** of $i$ at time $k$ is denoted by $x_i^{(k)} \in \mathbb{R}$.
- The opinion of $i$ at time $k$ is updated at the subsequent time step $k+1$ according to the following iterative scheme of the **Friedkin–Johnsen Social Influence Model**:

$$x_i^{(k+1)} = s_i \, N \, x_i^{(k)} + (1 - s_i) \, x_i^{(0)},$$

  - where $N \, x_i^{(k)}$ is the *average* opinion held by $i$'s neighbors at time step $k$ and
  - the scalar parameter $s_i \in [0, 1]$ is called **susceptibility coefficient** of person/node $i$.

# Solving the Friedkin–Johnsen Problem

- Disregarding the time exponent $(k)$, $N\,x_i$ is defined as:

$$N\,x_i = \sum_{j \in V} \frac{A_{ij}}{\deg_i}\, x_j = \sum_{j \sim i} \frac{x_j}{\deg_i},$$

  i.e., $\mathbf{N}$ is a **random walk matrix** on $G$:

$$\mathbf{N} = \mathbf{D}^{-1}\mathbf{A}.$$

- As matrix $\mathbf{N}$ is *row–stochastic*, matrix $\mathbf{I} - \mathbf{S}\,\mathbf{N}$ becomes *diagonally dominant*, where $\mathbf{S}$ denotes the the diagonal matrix of susceptibility coefficients of nodes/persons.
- Therefore, the solution to this problem converges (relatively fast, depending on $\mathbf{S}$) to the vector $\overline{\mathbf{x}} = \overline{\mathbf{x}}(\mathbf{S}; \mathbf{x}^{(0)})$ of equilibrium (steady state) opinions of nodes/persons of $G$:

$$\overline{\mathbf{x}} = (\mathbf{I} - \mathbf{S}\,\mathbf{N})^{-1}(\mathbf{I} - \mathbf{S})\,\mathbf{x}^{(0)}.$$

# A Remark on the Original Formulation of the Friedkin–Johnsen Social Influence Model

- The original equations of the Friedkin–Johnsen model were written as:

$$\mathbf{x}^{(k+1)} = \mathbf{S}\,\mathbf{W}\,\mathbf{x}^{(k)} + (\mathbf{I} - \mathbf{S})\,\mathbf{x}^{(0)},$$

  where $\mathbf{W} = \{W_{ij}\}_{(i,j)\in V\times V}$ was a given row–stochastic matrix, called **influence matrix**. Not that, in this formulation, (diagonal) self–influences are allowed (i.e., $W_{ii}$ can be nonzero).
- Clearly, by limiting social influence interactions only along edges of $G$ (i.e., assuming that $W_{ij} = 0$, if $i \nsim j$),

$$\mathbf{W} = \mathbf{N} = \mathbf{D}^{-1}\mathbf{A},$$

  which is the case that we are considering here.

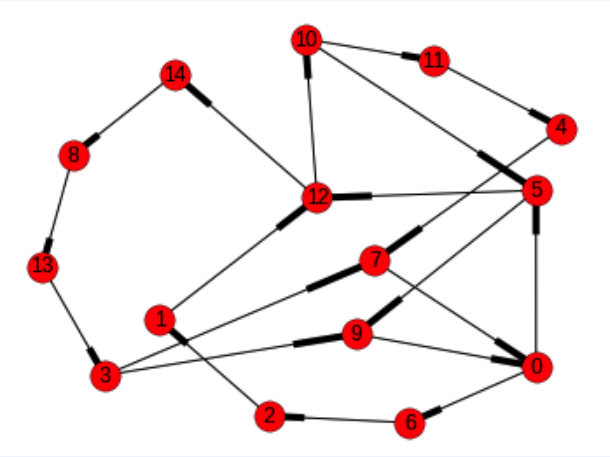
# Remarks on Influence Susceptibility Coefficients

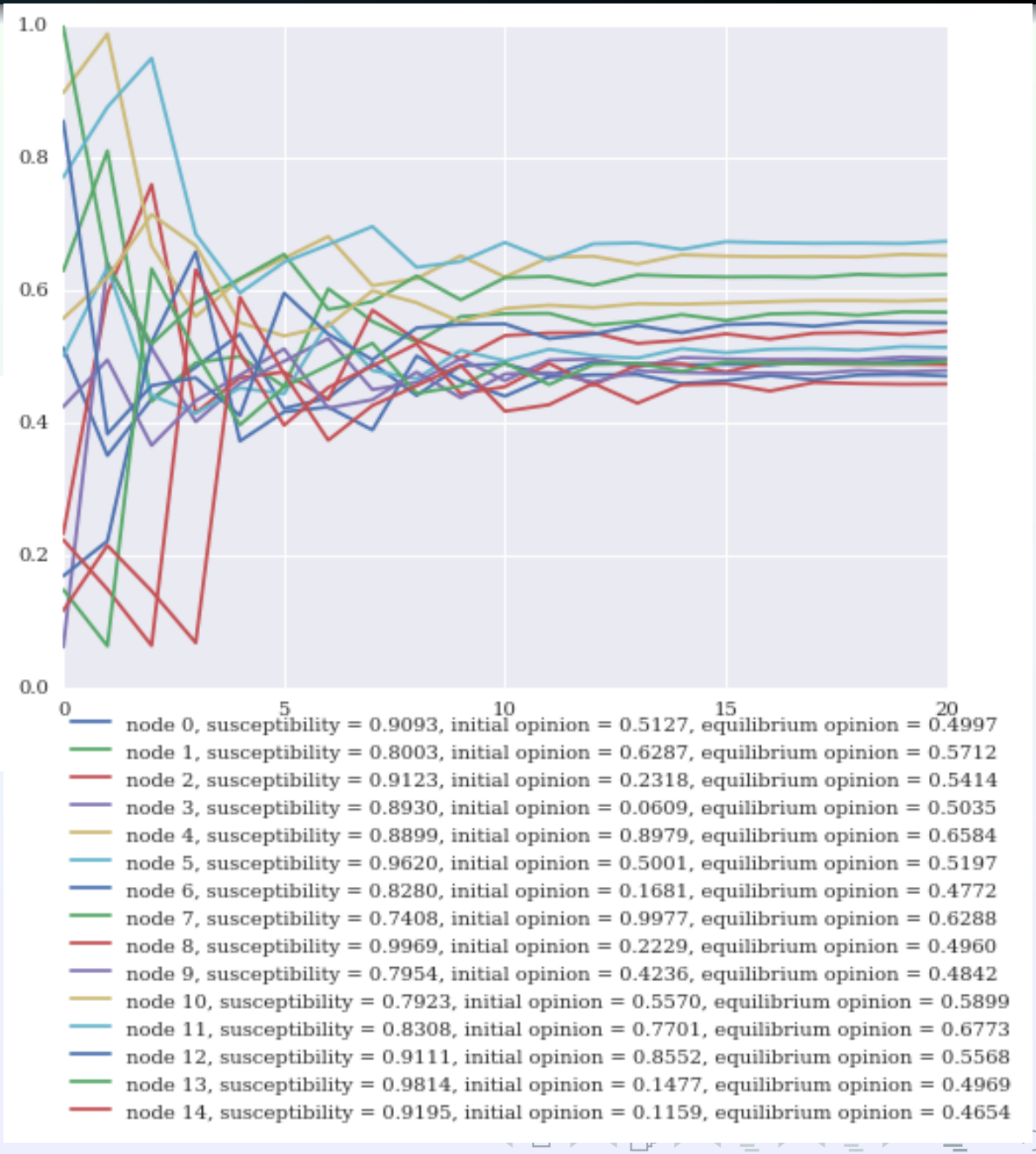
- When $s_i = 0$, $i$'s opinion does not change ($x_i^{(k)} = x_i^{(0)}$, for every time iteration $k = 1, 2, \ldots$). Such a person/node is called **persistent** or **stubborn** in holding the same opinion without being influenced by anybody else.
- When $s_i = 1$, $i$ is always adopting the average neighbors' opinion $N\,x_i^{(k)}$ without taking into account the initial opinion $x_i^{(0)}$. Such a person/node is called **malleable** or fully **conforming** to the neighbors' influences.
- When $0 < s_i < 1$, $i$'s opinion is interpolated in between the average neighbors' opinion $N\,x_i^{(k)}$ and the initial opinion $x_i^{(0)}$, in such a way that the exact position of the resulting $i$'s opinion is weighted as a convex combination through $s_i$.

# Assumptions on Susceptibilities and Initial Conditions

- We are considering here the following assumptions with regards to the susceptibility coefficients $s_i$ of nodes/persons of graph $G = (V, E)$ and their initial opinions $x_i^{(0)}$:
  - $s_i > 0$, for every $i \in V$. (The case that $s_i = 0$, for some nodes/persons, is going to be examined separately in the sequel.)
  - There exists at least one node/person with $s_i < 1$. (The case that $s_i = 1$, for all nodes/persons, would imply the "**consensus**" equilibrium opinion $\overline{x}_i = c$, for all $i \in V$ holding the same constant opinion $c$.)
- Initial opinions $x_i^{(0)}$ are assumed not to exhibit any consensus (i.e., $x_i^{(0)} \neq x_i^{(0)}$, for at least two (distinct) $i, j \in V$), since, if that was the case, all the equilibrium (steady state) opinions would necessarily hold the same consensus opinion with which they were starting, independently of the values of their coefficients of susceptibilities.

# A Friedkin–Johnsen Social Influence Example



# Boundary Conditions of the Friedkin–Johnsen Problem

- Suppose we have the Friedkin–Johnsen social influence system on graph $G = (V, E)$ composed of $n$ persons/nodes, denoted as $i = 1, 2, \ldots, n$ and having susceptibility coefficients $s_i$, respectively.

- Assumptions and Notation:
  - $V = \Omega \cup \partial\Omega$, where $\Omega \cap \partial\Omega = \varnothing$.
  - $\Omega = \{i \in V : s_i > 0\}$ is the domain of the Friedkin–Johnsen social influence system composed *exclusively* of nonpersistent persons/nodes.
  - $\partial\Omega = \{i \in V : s_i = 0\}$ is the  of the domain of the social influence system composed *exclusively* of persistent persons/nodes.
  - $|\partial\Omega| < n$.
  - $\forall j \in \delta\Omega, \exists i \in \Omega, i \sim j$.

# Sources of Boundary Stimulation of the Friedkin–Johnsen Problem

- When $\partial\Omega \neq \varnothing$, any person/node on the social influence boundary $\partial\Omega$ is called **source of a boundary stimulation** or just a **persistent source**. Moreover, we use the notation $\partial\Omega = \{j_a\}_{a=1,\ldots,|\partial\Omega|}$.
- Clearly, the **Initial Boundary Value Problem** (**IBVP**) of the Friedkin–Johnsen social influence system is the following:

$$\begin{aligned} x_i^{(k+1)} &= s_i\, N\, x_i^{(k)} + (1 - s_i) x_i^{(0)}, \text{ for all } i \in \Omega, k = 0, 1, \ldots, \\ x_{j_a}^{(k)} &= 1, \text{ for all } j_a \in \partial\Omega, k = 0, 1, \ldots, \\ x_i^{(0)} &= \phi_i, \text{ for all } i \in \Omega, \end{aligned}$$

where $\phi_i$ is the initial opinion of person/node $i \in \Omega$.

- Notice that, for the null initial condition $\phi_i = 0, i \in \Omega$, the IBVP of the Friedkin–Johnsen social influence system becomes:

$$\begin{aligned} x_i^{(k+1)} &= s_i \, N \, x_i^{(k)} + (1 - s_i)\psi_i, \text{ for all } i \in \Omega, k = 1, 2, \ldots, \\ x_{j_a}^{(k)} &= 1, \text{ for all } j_a \in \partial\Omega, k = 1, 2, \ldots, \\ x_i^{(1)} &= \psi_i, \text{ for all } i \in \Omega, \end{aligned}$$

where

$$\psi_i = \begin{cases} \frac{s_i}{\deg_i}, & \text{when } i \sim j_b, \\ 0, & \text{when } i \nsim j_b. \end{cases}$$

- Therefore, without any loss of generality, from now on, we are assuming that $\phi_i = 0, i \in \Omega$.

# The Steady States of (Equilibrium) Opinions of the Friedkin–Johnsen Problem under Boundary Stimulations

- Let $\overline{\mathbf{x}}(\partial\Omega;\mathbf{S})$[1] be the **steady states** or **equilibrium** (vector of) opinions solving the Friedkin–Johnsen social influence system on the graph $G=(\Omega\cup\partial\Omega,E)$, with regards to the diagonal matrix of susceptibility coefficients of nodes/persons $\mathbf{S}$, under the stimulation of the persistent sources of the boundary $\partial\Omega$. Clearly, the equilibrium opinions are:

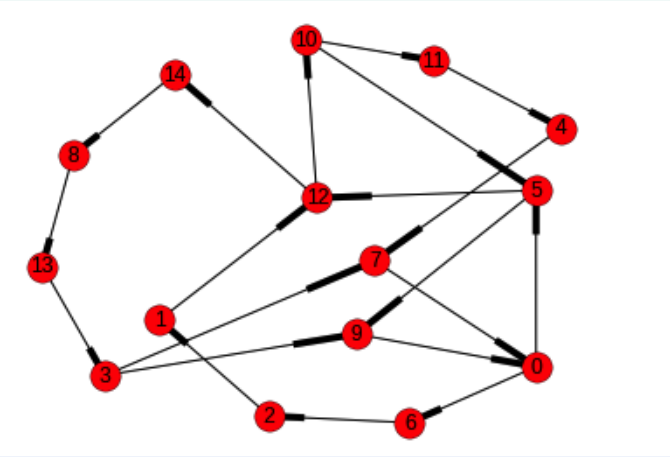
$$\overline{\mathbf{x}}(\partial\Omega;\mathbf{S})=(\mathbf{I}-\mathbf{S}\,\mathbf{N})^{-1}(\mathbf{I}-\mathbf{S})\,\boldsymbol{\psi},\ \text{on }\Omega,$$

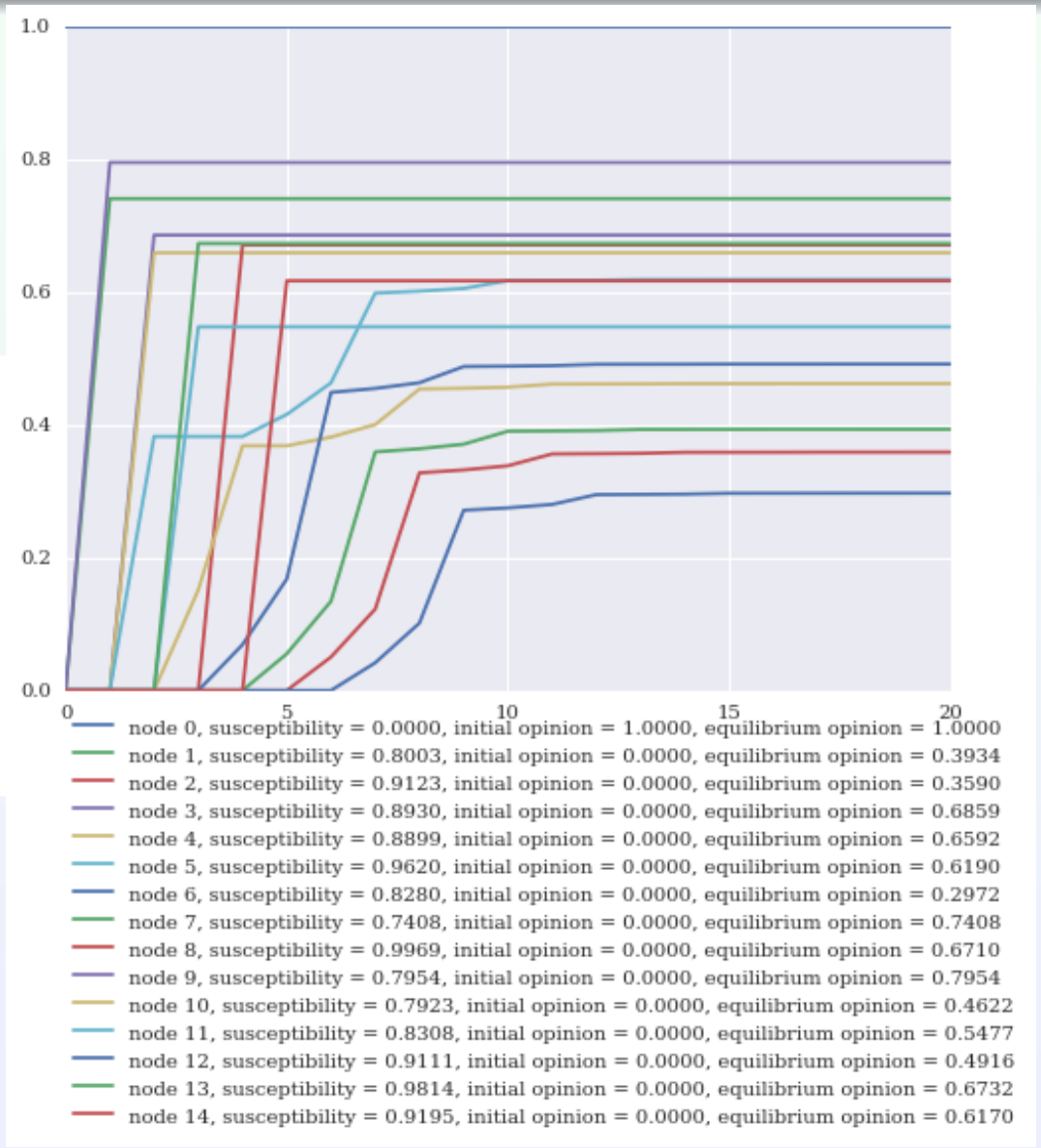
  where the support of $\boldsymbol{\psi}$ is only nodes adjacent to $\partial\Omega$.
- If $s_i=1$, for every $i\in\Omega$, i.e., $\mathbf{S}=\mathbf{1}$, on $\Omega$, then

$$\overline{\mathbf{x}}(\partial\Omega;\mathbf{1})=\mathbf{1},\ \text{on }\Omega.$$

[1] Notice that the diagonal matrix of susceptibilities $\mathbf{S}$ is such that $s_i=S_{ii}=0,\forall i\in\partial\Omega$, while $s_i=S_{ii}\in(0,1],\forall i\in\Omega$. Moreover, as already said, the vector of initial opinions is taken to be $x_i^{(0)}=1,\forall i\in\partial\Omega$, while $x_i^{(0)}=0,\forall i\in\Omega$.

# An Example of One Source of Persistent Boundary Stimulation



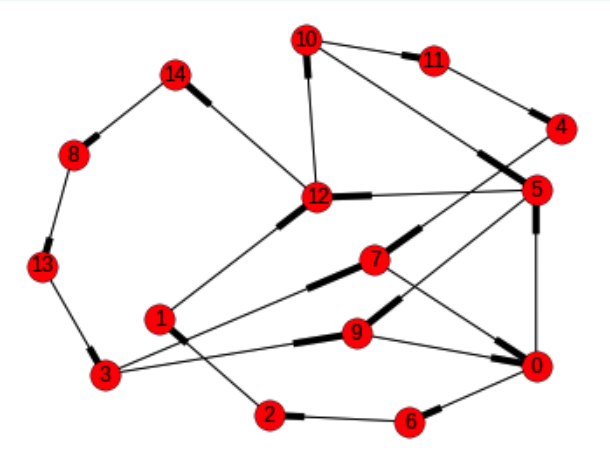
# Synchronization from One Source of Persistent Alternating Boundary Stimulation

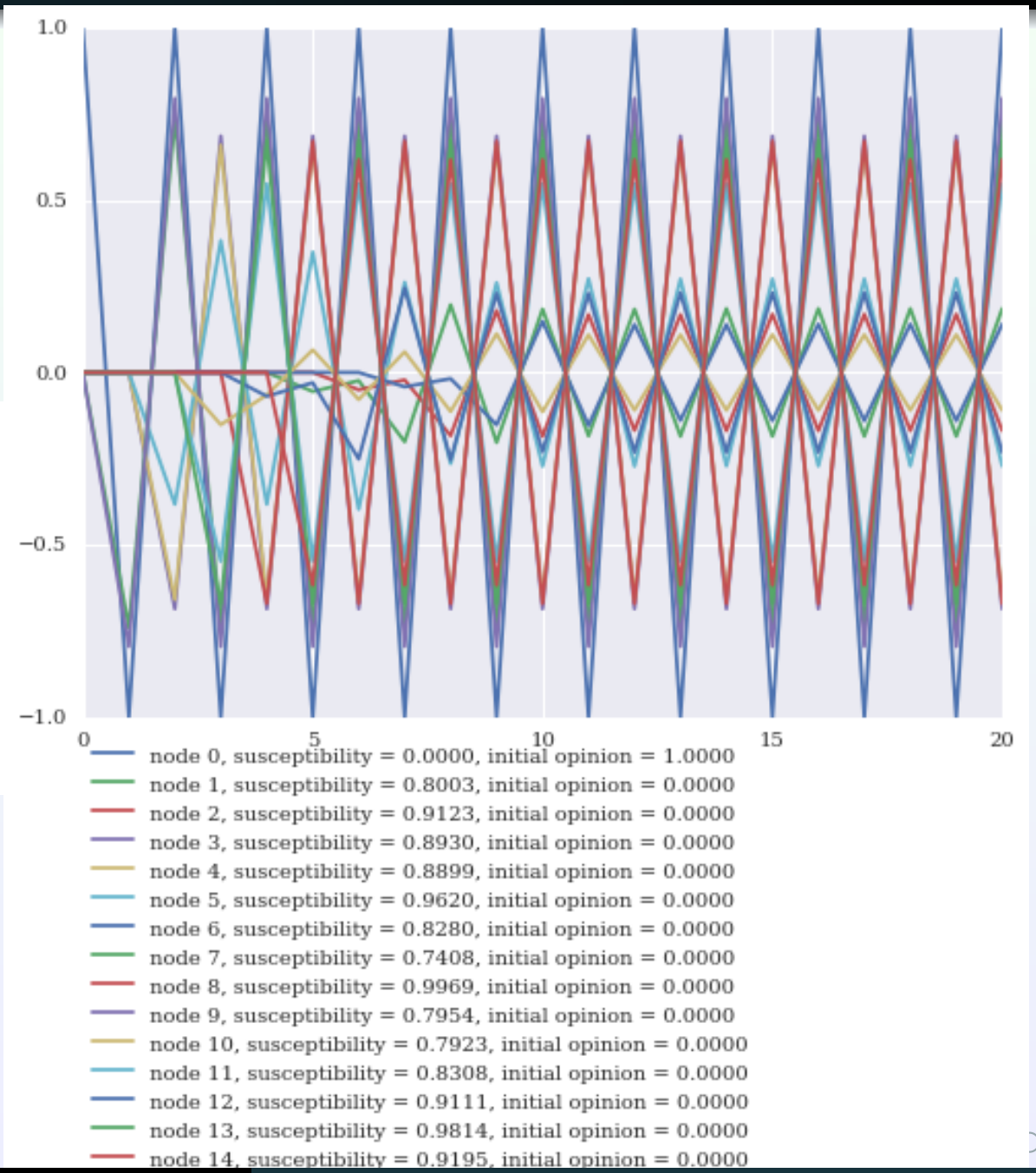
- In the case that $|\partial\Omega| = 1$, i.e., when there exists a single node/person $j_a \in \partial\Omega$, this node/person $j_a$ is said to be a **source of persistent alternating boundary stimulation**. In this case, the IBVP of the Friedkin–Johnsen social influence system is written as:

$$\begin{aligned} x_i^{(k+1)} &= s_i\, N\, v_i^k + (1 - s_i)\, x_i^{(0)}, \text{ on } \Omega, k = 0, 1, \ldots, \\ x_{j_a}^{(k)} &= 1, k = 0, 2, 4, \ldots, \\ x_{j_a}^{(k)} &= -1, k = 1, 3, 5, \ldots, \\ x_i^{(0)} &= 0, \text{ on } \Omega. \end{aligned}$$

- Notice that the opinion of such persistent source of boundary stimulation $j_a$ keeps on oscillating between $+1$ and $-1$ without being influenced at all by its neighbors (although, as a source, it keeps on influencing its neighbors in alternating directions).

# Example


10
11
14
4
8
12
5
13
7
1
9
0
3
2
6

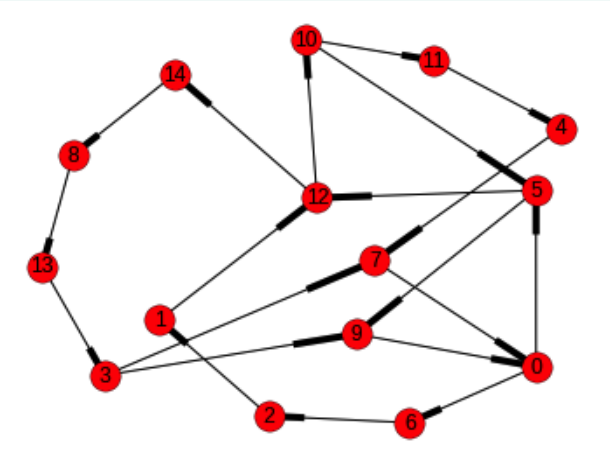

1.0
0.5
0.0
−0.5
−1.0
0
5
10
15
20
node 0, susceptibility = 0.0000, initial opinion = 1.0000
node 1, susceptibility = 0.8003, initial opinion = 0.0000
node 2, susceptibility = 0.9123, initial opinion = 0.0000
node 3, susceptibility = 0.8930, initial opinion = 0.0000
node 4, susceptibility = 0.8899, initial opinion = 0.0000
node 5, susceptibility = 0.9620, initial opinion = 0.0000
node 6, susceptibility = 0.8280, initial opinion = 0.0000
node 7, susceptibility = 0.7408, initial opinion = 0.0000
node 8, susceptibility = 0.9969, initial opinion = 0.0000
node 9, susceptibility = 0.7954, initial opinion = 0.0000
node 10, susceptibility = 0.7923, initial opinion = 0.0000
node 11, susceptibility = 0.8308, initial opinion = 0.0000
node 12, susceptibility = 0.9111, initial opinion = 0.0000
node 13, susceptibility = 0.9814, initial opinion = 0.0000
node 14, susceptibility = 0.9195, initial opinion = 0.0000

# Synchronization from Two Sources of Persistent Alternating Boundary Stimulation

- In the case that $|\partial\Omega| = 2$, i.e., when there exist exactly two nodes/persons $j_a, j_b \in \partial\Omega$, the pair of nodes/persons $j_a, j_b \in \partial\Omega$ is said to be a **pair of sources of persistent alternating boundary stimulations**. In this case, the IBVP of the Friedkin–Johnsen social influence system is written as:

$$
\begin{aligned}
x_i^{(k+1)} &= s_i\, N\, v_i^k + (1 - s_i)\, x_i^{(0)}, \text{ on } \Omega, k = 0, 1, \ldots, \\
x_{j_a}^{(k)} &= 1, k = 0, 2, 4, \ldots, \\
x_{j_a}^{(k)} &= -1, k = 1, 3, 5, \ldots, \\
x_{j_b}^{(k)} &= -1, k = 0, 2, 4, \ldots, \\
x_{j_b}^{(k)} &= 1, k = 1, 3, 5, \ldots, \\
x_i^{(0)} &= 0, \text{ on } \Omega.
\end{aligned}
$$

# Example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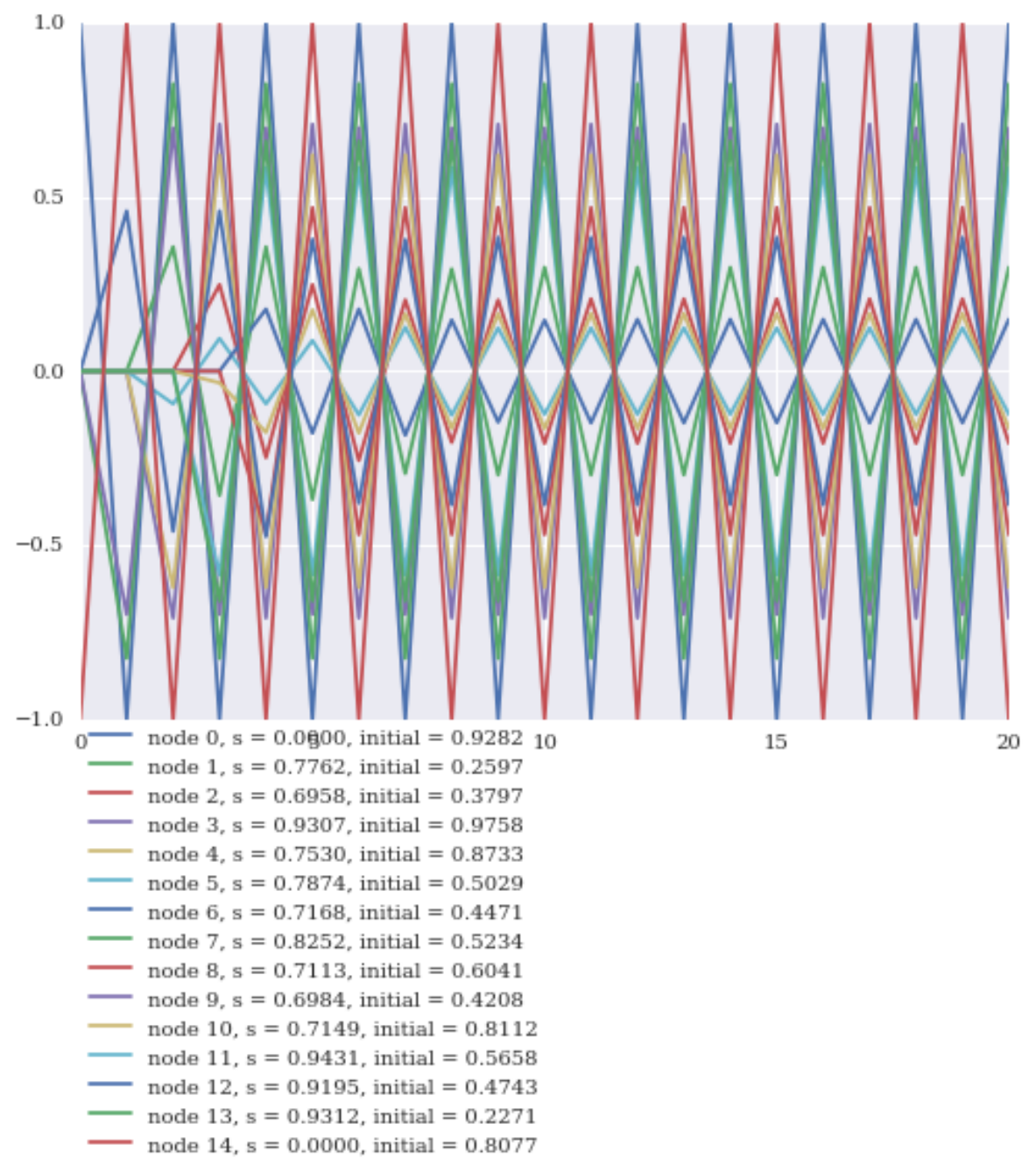

1.0
0.5
0.0
−0.5
−1.0
0
5
10
15
20
node 0, s = 0.0600, initial = 0.9282
node 1, s = 0.7762, initial = 0.2597
node 2, s = 0.6958, initial = 0.3797
node 3, s = 0.9307, initial = 0.9758
node 4, s = 0.7530, initial = 0.8733
node 5, s = 0.7874, initial = 0.5029
node 6, s = 0.7168, initial = 0.4471
node 7, s = 0.8252, initial = 0.5234
node 8, s = 0.7113, initial = 0.6041
node 9, s = 0.6984, initial = 0.4208
node 10, s = 0.7149, initial = 0.8112
node 11, s = 0.9431, initial = 0.5658
node 12, s = 0.9195, initial = 0.4743
node 13, s = 0.9312, initial = 0.2271
node 14, s = 0.0000, initial = 0.8077

# Friedkin–Johnsen Social Influence on a Signed Graph

- Let $G = (V, E)$ be a **signed graph** (undirected or directed), which is assumed to be (**strongly**) **connected**. Of course, now, each edge has a **sign**, which is either *positive* $(+)$ or *negative* $(-)$.
- Given two nodes $i, j \in V$, writing $i \overset{+}{\sim} j$ or $i \overset{-}{\sim} j$, we mean that $i, j$ are adjacent, when $G$ is undirected, or that $j$ is an outgoing neighbor of $i$, when $G$ is directed, and that in both cases the sign of $(i, j)$ is either positive or regative (respectively).
- Thus, the nonzero entries of the adjacency matrix $\mathbf{A}$ are now either $+1$ or $-1$ (i.e., $A_{ij} = +1 \iff i \overset{+}{\sim} j$ and $A_{ij} = -1 \iff i \overset{-}{\sim} j$).
- Then, the system of equations of the **Friedkin–Johnsen Social Influence Model** is:

$$\begin{aligned} x_i^{(k+1)} &= \frac{s_i}{\deg_i} \sum_{j \in V} A_{ij} x_j^{(k)} + (1 - s_i)\, x_i^{(0)}, \\ &= \frac{s_i}{\deg_i} \Big( \sum_{i \overset{+}{\sim} j} x_j^{(k)} - \sum_{i \overset{-}{\sim} j} x_j^{(k)} \Big) + (1 - s_i)\, x_i^{(0)}. \end{aligned}$$

# Remarks on the Signed Friedkin–Johnsen Social Influence Model

- Now, the matrix $\mathbf{N} = \mathbf{D}^{-1}\mathbf{A}$ is *not* row–stochastic. However, the matrix of absolute values of its entries, $|\mathbf{N}| = \mathbf{D}^{-1}|\mathbf{A}|$, *is* row–stochastic.
- Let $\rho = \rho(\mathbf{N})$ be the **spectral radius** of the (square, $n \times n$) matrix $\mathbf{N}$. This means that $\rho = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$, where $\lambda_1, \ldots, \lambda_n$ are the (real or complex) eigenvalues of $\mathbf{N}$.
- Let $s_{\max} = \max\{s_1, \ldots, s_n\}, s_{\min} = \min\{s_1, \ldots, s_n\}$ be the maximum and the minimum susceptibilities (respectively) among the susceptibility coefficients of all nodes of $G$. Without loss of generality, lwe are assuming:

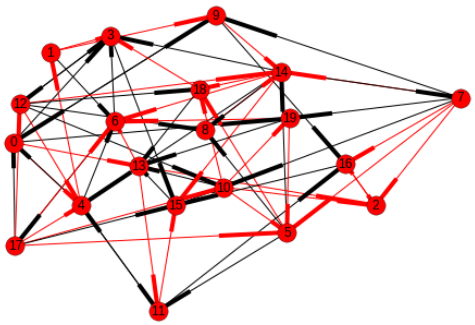
$$0 < s_{\min} \leq s_{\max} \leq 1.$$

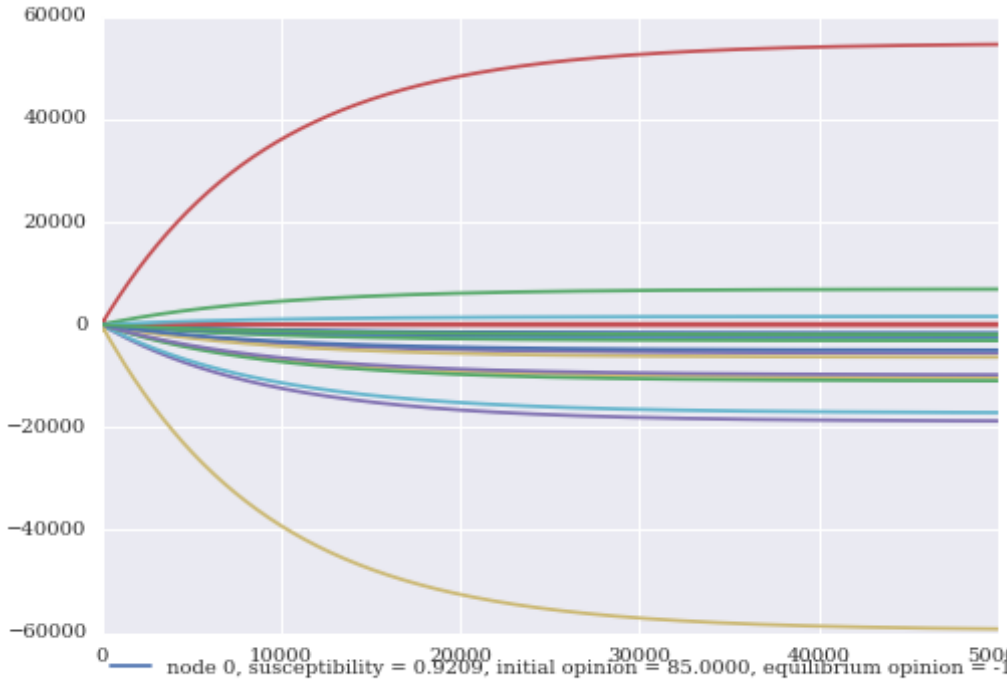
# The Stability of the Signed Friedkin–Johnsen Social Influence Model

- If $s_{\max}\,\rho < 1$, then all the solutions of the signed Friedkin–Johnsen model *converge* to a (steady state) equilibrium vector of opinions $\overline{\mathbf{x}} = (\mathbf{I} - \mathbf{S}\,\mathbf{N})^{-1}(\mathbf{I} - \mathbf{S})\,\mathbf{x}^{(0)}$.
- If $s_{\min}\,\rho \geq 1$, then all the solutions of the signed Friedkin–Johnsen model *diverge*.
- If $s_{\min} = s_{\max} = \rho = 1$, then all the solutions of the signed Friedkin–Johnsen model *oscillate*.

# An Example of Covergence

$$\rho = 1.0858$$

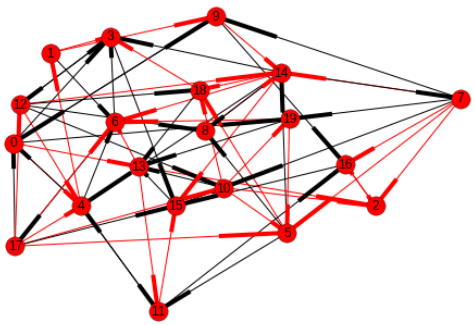


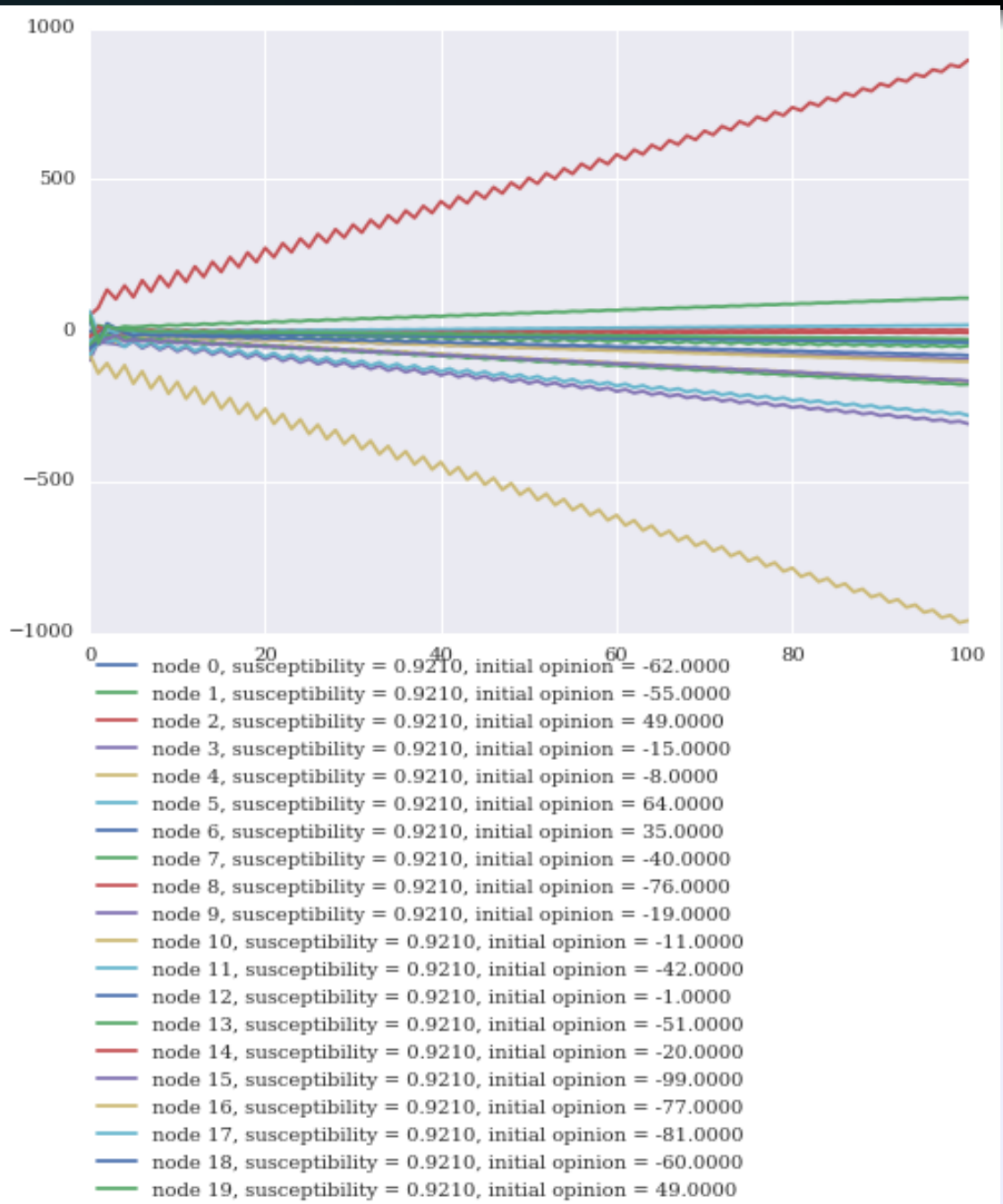


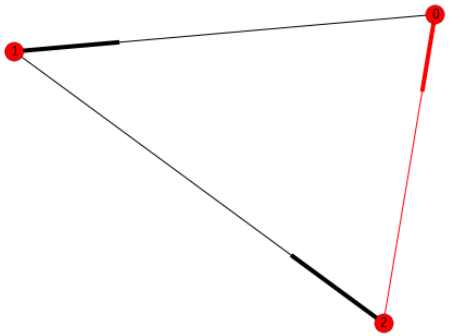
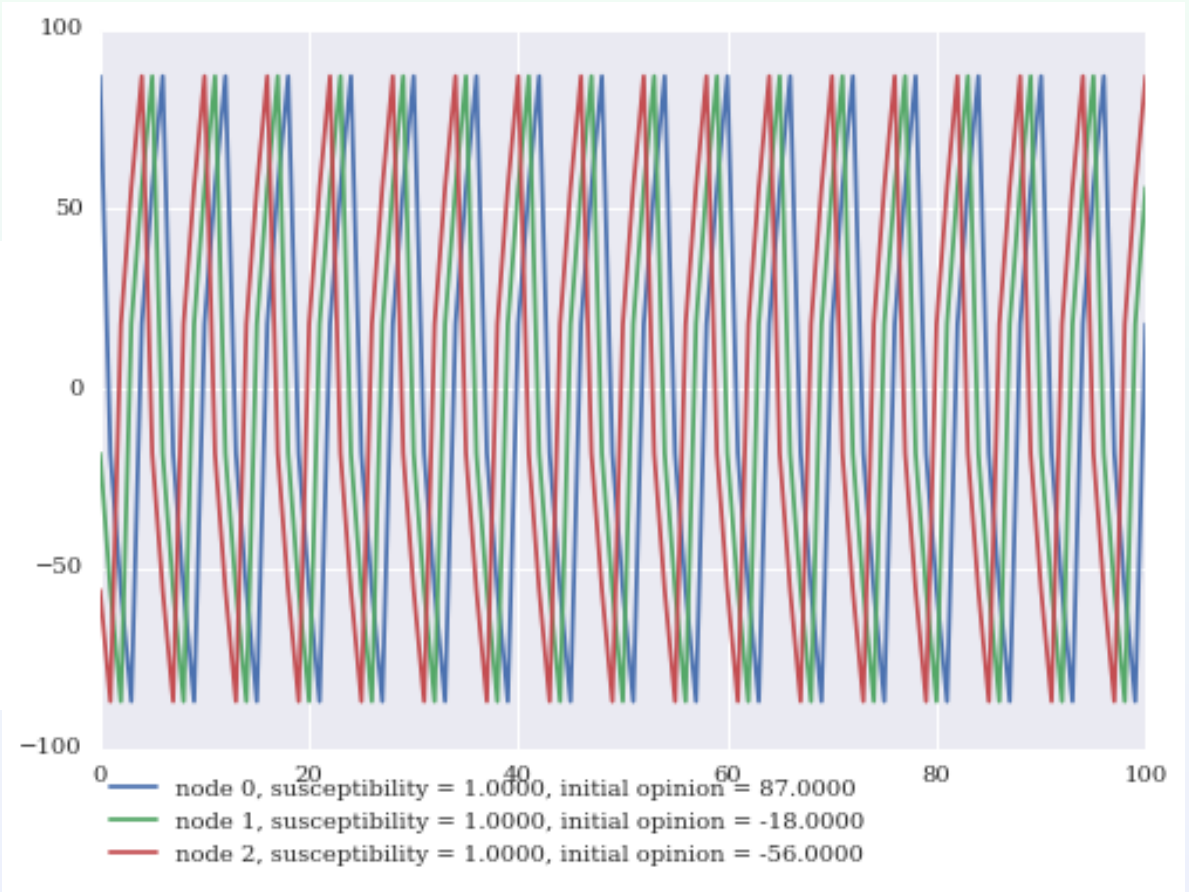

node 0, susceptibility = 0.9209, initial opinion = 85.0000, equilibrium opinion = -1653.0874
node 1, susceptibility = 0.9209, initial opinion = 2.0000, equilibrium opinion = 6934.9467
node 2, susceptibility = 0.9209, initial opinion = 20.0000, equilibrium opinion = 54824.6768
node 3, susceptibility = 0.9209, initial opinion = 51.0000, equilibrium opinion = -5567.7163
node 4, susceptibility = 0.9209, initial opinion = -70.0000, equilibrium opinion = -6363.5252
node 5, susceptibility = 0.9209, initial opinion = 93.0000, equilibrium opinion = -2802.6428
node 6, susceptibility = 0.9209, initial opinion = 86.0000, equilibrium opinion = -1819.6920
node 7, susceptibility = 0.9209, initial opinion = -30.0000, equilibrium opinion = -3133.5426
node 8, susceptibility = 0.9209, initial opinion = -2.0000, equilibrium opinion = -172.6983
node 9, susceptibility = 0.9209, initial opinion = -60.0000, equilibrium opinion = -18844.0521
node 10, susceptibility = 0.9209, initial opinion = -100.0000, equilibrium opinion = -10334.3694
node 11, susceptibility = 0.9209, initial opinion = 60.0000, equilibrium opinion = -17209.5755
node 12, susceptibility = 0.9209, initial opinion = 10.0000, equilibrium opinion = -2284.1444
node 13, susceptibility = 0.9209, initial opinion = 48.0000, equilibrium opinion = -10983.8413
node 14, susceptibility = 0.9209, initial opinion = 60.0000, equilibrium opinion = 187.3664
node 15, susceptibility = 0.9209, initial opinion = -16.0000, equilibrium opinion = -9816.9164
node 16, susceptibility = 0.9209, initial opinion = -95.0000, equilibrium opinion = -59534.0214
node 17, susceptibility = 0.9209, initial opinion = 57.0000, equilibrium opinion = 1600.8215
node 18, susceptibility = 0.9209, initial opinion = -19.0000, equilibrium opinion = -4989.0505
node 19, susceptibility = 0.9209, initial opinion = 62.0000, equilibrium opinion = -1947.8008

# An Example of Divergence

$$\rho = 1.0858$$





# An Example of Oscillation

$\rho = 1$