# Key Methods of Hypergraph Analysis Day 2: From Graphs to Hypergraphs

Moses Boudourides

Moses.Boudourides@gmail.com

instats Seminar

Monday, March 10, 2025, from 5:00 PM to 6:30 PM UTC

# Table of Contents

# Hypergraph Connectedness: Traversals

**Definition**

- A **walk** of length $\ell$ from vertex $u$ to vertex $v$ is an alternating vertex–hyperedge sequence
$$u = v_1, e_1, v_2, e_2, \ldots, v_\ell, e_\ell, v_{\ell+1} = v$$
such that consecutive vertices are distinct ($v_i \neq v_{i+1}$), and each hyperedge $e_i$ is incident to its adjacent vertices $v_i$ and $v_{i+1}$.
- If the sequence of pairs of consecutive vertices–hyperedges, $(v_1, e_1), (v_2, e_1), (v_2, e_2), \ldots, (v_\ell, e_\ell), (v_{\ell+}1,\ e_\ell)$, has no repeated hyperedges, then the walk is called a **trail**; note that, in graphs, this corresponds to a walk in which no edge appears more than once, though vertices may be revisited.
- A walk in which all the vertices are distinct is called a **path**. Unlike in simple graphs, where the requirement for distinct vertices ensures that all traversed edges are also distinct, this constraint does not apply to hypergraphs, where paths may include repeated hyperedges.
- Finally, if $u = v_1 = v_{\ell+1} = v$ the path is called a **cycle**.

# Walks, Trails, Paths, and Cycles

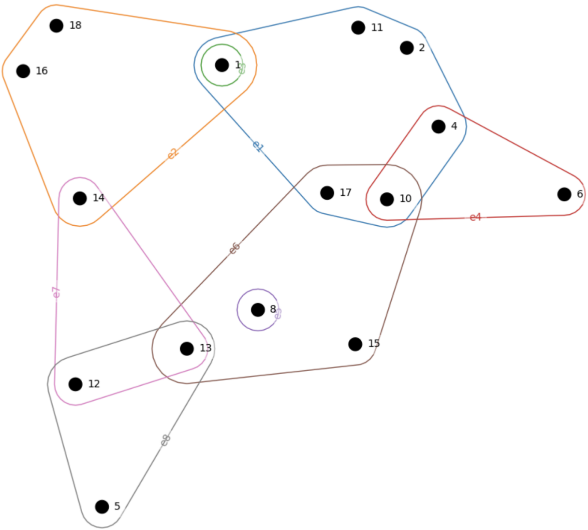
```
{'e1': [1, 2, 4, 10, 11, 17], 'e2': [1, 14, 16, 18], 'e3': [1], 'e4': [4, 6, 10], 'e5': [8], 'e6': [8, 10, 13, 15,
17], 'e7': [12, 13, 14], 'e8': [5, 12, 13]}
diameter= 3
```

```
A Random Walk:
(4, 'e1')
(2, 'e1')
(4, 'e4')
(6, 'e4')
(10, 'e1')
(17, 'e1')
(1, 'e2')
(14, 'e7')
(12, 'e7')
(14, 'e2')

A Random Trail:
(8, 'e5')
(8, 'e6')
(17, 'e1')
(1, 'e3')
(1, 'e2')
(14, 'e7')

A Random Path:
(10, 'e6')
(13, 'e7')
(12, 'e8')

A Random Cycle:
(1, 'e3')
(4, 'e1')
(10, 'e4')
(13, 'e6')
(14, 'e7')
(1, 'e2')
```



# Connected Hypergraphs, Distances, Eccentricity, Diameter, Connected Components

## Definitions

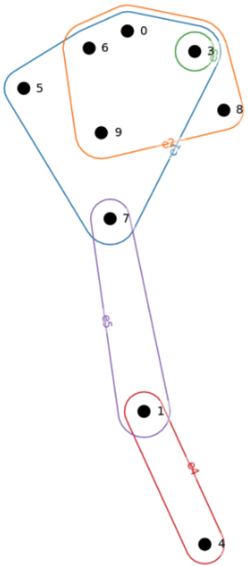
- A hypergraph is **connected** if, for every pair of vertices, there exists at least one path connecting them.
- The **distance** between two distinct vertices $u$ and $v$ is the length of the shortest path connecting them, and it is denoted by $d(u, v)$. If no such path exists, their distance is defined as infinite.
- The **eccentricity** of a vertex is the maximum distance from that vertex to any other vertex.
- The **diameter** of the hypergraph is the maximum eccentricity of any vertex, i.e., it is the maximum distance between any pair of vertices.
- If a hypergraph is not connected, its **connected components** are the equivalence classes induced by the equivalence relation $\mathscr{R}$ on the set of vertices, defined as:

$$(u, v) \in \mathscr{R} \iff \exists \text{ a path between } u \text{ and } v.$$

# Distances

```
{'e1': [0, 3, 5, 6, 7, 9], 'e2': [0, 3, 6, 8, 9], 'e3': [3], 'e4': [1, 4], 'e5': [1, 7]}
```

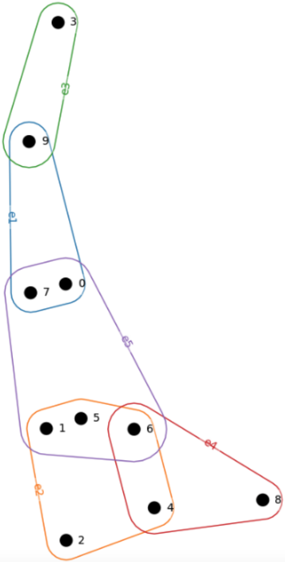
```
All Distances:
Distance of 0, 1: 2
Distance of 0, 3: 1
Distance of 0, 4: 3
Distance of 0, 5: 1
Distance of 0, 6: 1
Distance of 0, 7: 1
Distance of 0, 8: 1
Distance of 0, 9: 1
Distance of 1, 3: 2
Distance of 1, 4: 1
Distance of 1, 5: 2
Distance of 1, 6: 2
Distance of 1, 7: 1
Distance of 1, 8: 3
Distance of 1, 9: 2
Distance of 3, 4: 3
Distance of 3, 5: 1
Distance of 3, 6: 1
Distance of 3, 7: 1
Distance of 3, 8: 1
Distance of 3, 9: 1
Distance of 4, 5: 3
Distance of 4, 6: 3
Distance of 4, 7: 2
Distance of 4, 8: 4
Distance of 4, 9: 3
Distance of 5, 6: 1
Distance of 5, 7: 1
Distance of 5, 8: 2
Distance of 5, 9: 1
Distance of 6, 7: 1
Distance of 6, 8: 1
Distance of 6, 9: 1
Distance of 7, 8: 2
Distance of 7, 9: 1
Distance of 8, 9: 1
```



# Eccentricities and Diameter

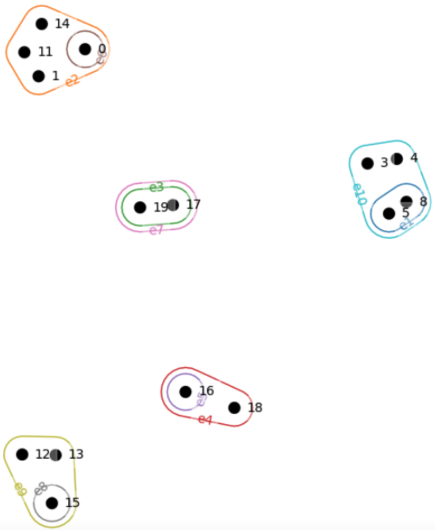
```
{'e1': [0, 7, 9], 'e2': [1, 2, 4, 5, 6], 'e3': [3, 9], 'e4': [4, 6, 8], 'e5': [0, 1, 5, 6, 7]}
diameter= 4
```

```
All Eccentricities:
Eccentricity of 0: 2
Eccentricity of 1: 3
Eccentricity of 2: 4
Eccentricity of 3: 4
Eccentricity of 4: 4
Eccentricity of 5: 3
Eccentricity of 6: 3
Eccentricity of 7: 2
Eccentricity of 8: 4
Eccentricity of 9: 3
```



# A Disconnected Hypergraph with 5 Connected Components

```
{'e1': [5, 8], 'e2': [0, 1, 11, 14], 'e3': [17, 19], 'e4': [16, 18], 'e5': [16], 'e6': [0], 'e7': [17, 19], 'e8':
[15], 'e9': [12, 13, 15], 'e10': [3, 4, 5, 8]}
```



# s-Based Terminology

## Definitions

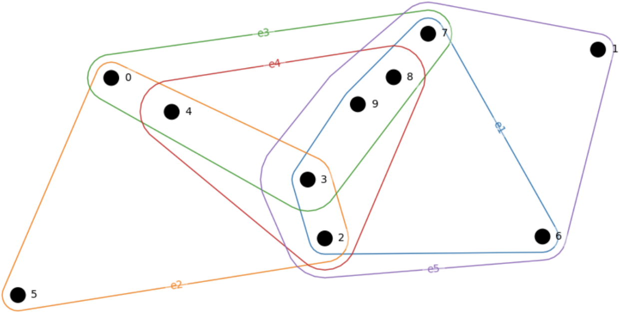
Let $G = (V, E)$ hypergraph and $s$ integer (typically, $1 \leq s \leq |E|$).

- Two vertices are said to be $s$-**adjacent** if they are contained in at least $s$ common hyperedges.
- Two hyperedges are said to be $s$-**incident** if their intersection contains at least $s$ vertices.
- An $s$-**walk** ($s$-**trail**, $s$-**path**) is a walk (trail, path, resp.), in which each pair of consequtive vertices is $s$-adjacent and each pair of consecutive hyperedges is $s$-incident.
- The $s$-**distance** between two vertices is the length of the shortest $s$-path connecting these two vertices.
- The $s$-**eccentricity** of a vertex is the maximum $s$-distance from that vertex to any other vertex.
- The $s$-**diameter** of the hypergraph is the maximum $s$-distance between any two vertices.
- The $s$-**connected components** are the equivalence classes induced by the relation that there exists an $s$ path between any two vertices.
- A hypergraph is $s$-**connected** if it has exactly one $s$-connected component.

# 2-Distances

```
{'e1': [2, 3, 6, 7, 8, 9], 'e2': [0, 2, 3, 4, 5], 'e3': [0, 3, 4, 7, 8, 9], 'e4': [2, 3, 4, 8, 9], 'e5': [1, 2, 3,
6, 7, 9]}
```

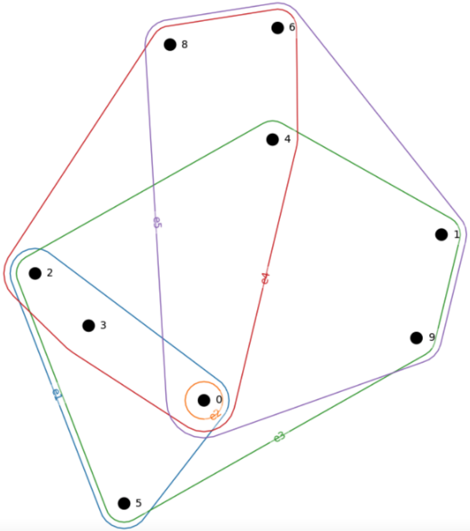
```
All Finite 2-Distances:
2-Distance of 0, 2: 2
2-Distance of 0, 3: 1
2-Distance of 0, 4: 1
2-Distance of 0, 6: 2
2-Distance of 0, 7: 2
2-Distance of 0, 8: 2
2-Distance of 0, 9: 2
2-Distance of 2, 3: 1
2-Distance of 2, 4: 1
2-Distance of 2, 6: 1
2-Distance of 2, 7: 1
2-Distance of 2, 8: 1
2-Distance of 2, 9: 1
2-Distance of 3, 4: 1
2-Distance of 3, 6: 1
2-Distance of 3, 7: 1
2-Distance of 3, 8: 1
2-Distance of 3, 9: 1
2-Distance of 4, 6: 2
2-Distance of 4, 7: 2
2-Distance of 4, 8: 1
2-Distance of 4, 9: 1
2-Distance of 6, 7: 1
2-Distance of 6, 8: 2
2-Distance of 6, 9: 1
2-Distance of 7, 8: 1
2-Distance of 7, 9: 1
2-Distance of 8, 9: 1
```



# 2-Eccentricities and 2-Diameter

```
{'e1': [0, 2, 3, 5], 'e2': [0], 'e3': [1, 2, 3, 4, 5, 9], 'e4': [0, 2, 3, 4, 6, 8], 'e5': [0, 1, 4, 6, 8, 9]}
2–diameter = 3
```

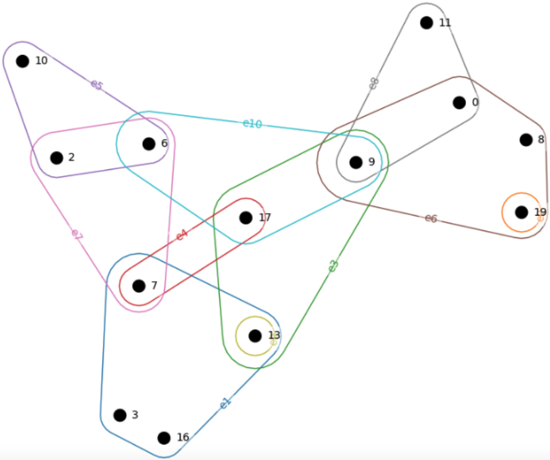
```
All 2-Eccentricities:
2-Eccentricity of 0: 1
2-Eccentricity of 1: 1
2-Eccentricity of 2: 1
2-Eccentricity of 3: 1
2-Eccentricity of 4: 1
2-Eccentricity of 5: 2
2-Eccentricity of 6: 2
2-Eccentricity of 8: 2
2-Eccentricity of 9: 1
```



# A Connected Hypergraph with 3 2-Connected Components

```
{'e1': [3, 7, 13, 16], 'e2': [19], 'e3': [9, 13, 17], 'e4': [7, 17], 'e5': [2, 6, 10], 'e6': [0, 8, 9, 19], 'e7':
[2, 6, 7], 'e8': [0, 9, 11], 'e9': [13], 'e10': [6, 9, 17]}
```

```
3 2-connected components:
{'e10': [6, 9, 17], 'e3': [9, 13, 17]}
{'e5': [2, 6, 10], 'e7': [2, 6, 7]}
{'e6': [0, 8, 9, 19], 'e8': [0, 9, 11]}
```



# Simplicial Complexes: $k$-Simplices

## Definition

- An **abstract $k$-simplex** $\sigma$ is a set of $k+1$ distinct vertices, $\sigma = \{v_0, v_1, \ldots, v_k\} \subseteq V$, for a nonnegative integer $k$.
- A **geometric $k$-simplex** $\sigma$ embedded in $\mathbb{R}^2$ is the convex hull of a set of $k+1$ affinely independent points in $\mathbb{R}^2$. For example,
    - 0-dimensional simplex is a point,
    - 1-dimensional simplex is a line segment,
    - 2-dimensional simplex is a triangle,
    - 3-dimensional simplex is a tetrahedron,
    - 4-dimensional simplex is a 5-cell, etc.
- A **face** of a simplex $\sigma$ is any subset of $\sigma$.

# Simplicial Complexes

**Definition**

A **simplicial complex** $K$ is a collection of simplices over the vertex set $V$ that satisfies the following two conditions:

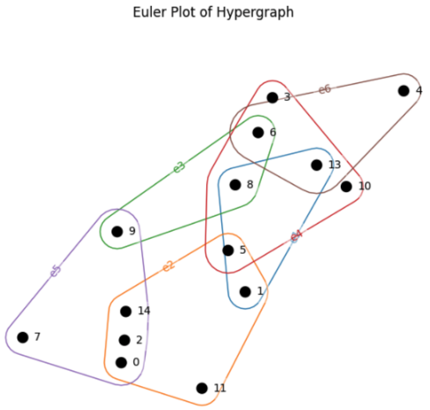
- *Inclusion of Faces*: For every simplex $\sigma \in K$, every face of $\sigma$ is also in $K$.
- *Intersection Condition*: For any two simplices $\sigma_1, \sigma_2 \in K$, the intersection $\sigma_1 \cap \sigma_2$ is either empty or a face of both $\sigma_1$ and $\sigma_2$. [a]

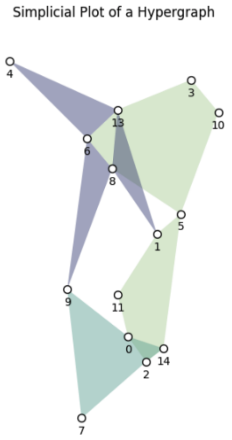
[a] The intersection condition is redunadant for abstract simplicial complexes and is only necessary for geometric simplicial complexes.

- Every abstract simplicial complex over the vertex set $V$ can be viewed as a hypergraph, where the set of hyperedges $E$ consists of the simplices in the simplicial complex.
- A hypergraph is an abstract simplicial complex if and only if it is *downward-closed*, meaning that every subset of a hyperedge is also considered a hyperedge.

# Euler and Simplicial Plot of a Hypergraph

```
{'e1': [1, 8, 13], 'e2': [0, 1, 2, 5, 11, 14], 'e3': [6, 8, 9], 'e4': [3, 5, 6, 8, 10, 13], 'e5': [0, 2, 7, 9, 14],
'e6': [4, 6, 13]}
```



# Hypergraphs and Bipartite Graphs: Star Expansion (or Bipartite Representation) of a Hypergraph

- In the star expansion of a hypergraph, the resulting star-expanded graph is a bipartite graph, where one set of verices represents the vertices and the other represents the hyperedges. Furthermore, in this bipartite graph, an edge connects a vertex to a hyperedge if the vertex is incident to that hyperedge in the original hypergraph. More formally:

**Definition**

Let $G = (V, E)$ be a hypergraph. The **star-expanded graph** (or **incidence graph**) of hypergraph $G$, denoted $\text{Star}(G)$, is the bipartite graph $\text{Star}(G) = (V \cup E, E')$, where the set of edges $E'$ of this bipartite graph is defined as:

$$E' = \{(v, e) : e \in E, v \in e\}.$$

- $H = B$, where $H$ is the incidence matrix of the hypergraph and $B$ is the biadjacency matrix of the biparite graph of its star-expansion.
- $\text{Star}(G) = \text{Star}(G^*)$.

# The Hypergraph Derived from a Bipartite Graph

- Conversely to the star expansion of a hypergraph, any bipartite graph can generate a hypergraph by taking one part of the bipartition as the set of vertices and defining the hyperedges based on the bipartite graph connections induced by the vertices in the other part. More formally:

**Definition**

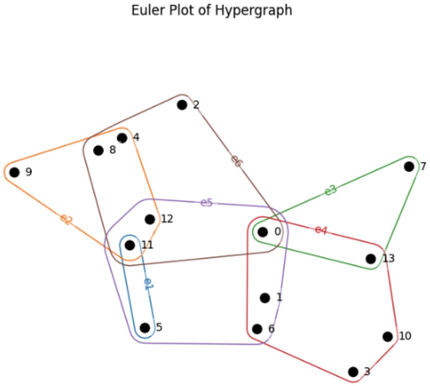
Let $G' = (U, V, E')$ be a bipartite graph, where $U$ and $V$ are two disjoint finite sets and $E' \subseteq U \times V$. The **hypergraph derived from the bipartite graph** $G'$ (or **incidence hypergraph**) is defined as the hypergraph $G = (V, E)$ where:

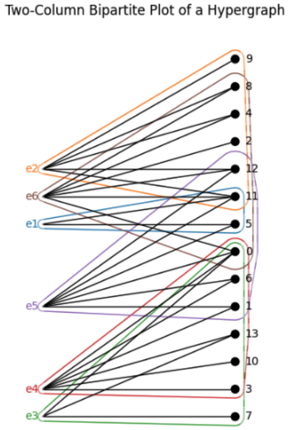
$$E = \{e_u : u \in U\}, \text{ with } e_u = \{v \in V : (u, v) \in E'\} \subseteq V.$$

- Consequently, there is a bijection between hypergraphs and bipartite graphs, as each hypergraph corresponds uniquely to a bipartite graph and vice versa.

# Euler and Two-Column Bipartite Plot of a Hypergraph

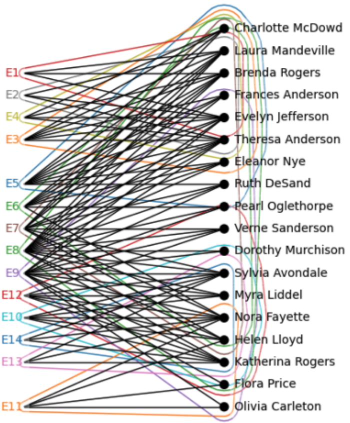
{'e1': [5, 11], 'e2': [4, 8, 9, 11, 12], 'e3': [0, 7, 13], 'e4': [0, 1, 3, 6, 10, 13], 'e5': [0, 1, 5, 6, 11, 12],
'e6': [0, 2, 4, 8, 11, 12]}



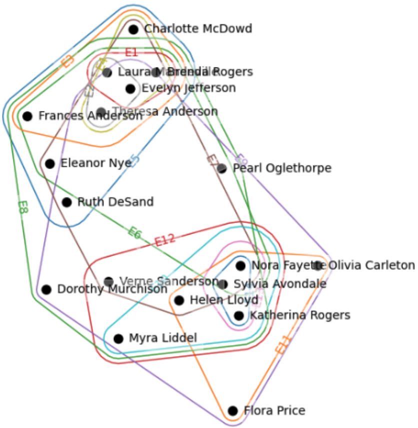



# Davis Southern Women Bipartite Graph and Hypergraph

Davis Southern Women Two-Column Bipartite Graph




Davis Southern Women Hypergraph



# Hypergraphs from Graphs

- As already seen, *there is a bijection between hypergraphs and bipartite graphs*†. Beyond that, **what is the relationship between graphs and hypergraphs?**
- As already seen, **a hypergraph can be associated to multiple graphs**: the clique-expanded graph (or 2-section), the line graph (or intersection graph), and the star-expanded graph (or bipartite reperesentation). Conversely, **a graph can be associated with multiple hypergraphs** too: a trivial one, defined below, and several more sophisticated ones, explored in the next slide.

**Definition**

Let $G' = (V, E')$ be a graph. The **edge-based hypergraph** (or **canonical pairwise hypergraph representation**) of graph $G'$ is the 2-uniform hypergraph $G = (V, E)$, where the set of hyperedges $E$ consists of all vertex pairs corresponding to the edges of $G'$, formally defined as:

$$E = \{\{v_i, v_j\} : (v_i, v_j) \in E'\}.$$

† In more formal graph-theoretic terms, there is a bijection between hypergraphs and *bicolored graphs*. However, without loss of generality, bipartite graphs can be identified with bicolored graphs.