Slides of Discrete Mathematics based on Susanna Epp's Textbook

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Chapter 8

Relations

November 8, 10, 12, 15 & 17, 2021

8.1 Relations on Sets

Definition

- If A and B are two sets, a **relation** R from A to B is defined as a subset of the Cartesian product $A \times B$. Moreover, given an ordered pair $(x, y) \in A \times B$, we say that x **is related to** y **by** R, written $x \, R \, y$, if and only if $(x, y) \in R$.
- Given a relation R from A to B, the **inverse relation** R^{-1} is defined as the following relation from B to A:

$$R^{-1} = \{(y, x) \in B \times A \,|\, (x, y) \in R\}.$$

- In other words,

$$x \, R^{-1} \, y \iff y \, R \, x.$$

8.1 Relations on Sets: Exercises

Exercise 8.1.11

Let $A = \{3,4,5\}$ and $B = \{4,5,6\}$ and let S be the "divides" relation. This is, for all $(x, y) \in A \times B, x\, S\, y \iff x \mid y$. Find explicitly which ordered pairs belong to S and S^{-1}.

Exercise 8.1.17

Let $A = \{2,3,4,5,6,7,8\}$ and define a relation T on A as: for all $x, y \in A, x\, T\, y \iff 3 \mid (x - y)$. Find the direct graph of T.

Exercise 8.1.20

Let $A = \{-1,1,2,4\}$ and $B = \{1,2\}$ and define relations R and S as: for all $(x, y) \in A \times B, x\, R\, y \iff |x| = |y|$ and $x\, S\, y \iff x - y$ is even. Find explicitly which ordered pairs belong to $A \times B, R, S, R \cup S$ and $R \cap S$.

8.2 Reflexivity, Symmetry and Transitivity

Definition

Let R be a relation on a set A.

1. R is **reflexive** if and only if, for all $x \in A, x\,R\,x$.
2. R is **symmetric** if and only if, for all $x, y \in A,$ if $x\,R\,y,$ then $y\,R\,x$.
3. R is **transitive** if and only if, for all $x, y, z \in A,$ if $x\,R\,y$ and $y\,R\,z,$ then $x\,R\,z$.

8.2 Reflexivity, Symmetry and Transitivity: Exercises (a)

Exercise 8.2.17

A relation P is defined on $\mathbb{Z}$ as follows: For all $m, n \in \mathbb{Z}$, $m\,P\,n \iff \exists$ a prime number p such that $p \mid m$ and $p \mid n$. Is P reflexive, symmetric, transitive?

P is not reflexive: Otherwise, there would exist a prime divisor of any integer. Counterexample: there is no prime dividing 1.

P is symmetric: Trivial. **Why?**

P is not transitive: Counterexample: find three integers m, n, k such that both pairs m, n and n, k have a common prime divisor, but the pair m, k does not.

8.2 Reflexivity, Symmetry and Transitivity: Exercises (b)

Exercise 8.2.19

Define a relation I on $\mathbb{R}$ as follows: For all real numbers x and y, $x\, I, y \iff x - y$ is irrational. Is I reflexive, symmetric, transitive?

I is not reflexive: For all $x \in \mathbb{R}, x - x = 0$, which is not irrational.

I is symmetric: Trivial. **Why?**

I is not transitive: Counterexample: find three $x, y, z \in \mathbb{R}$ such that

$$x - y \notin \mathbb{Q}, y - z \notin \mathbb{Q}, \text{ but } x - z \in \mathbb{Q}.$$

8.2 Reflexivity, Symmetry and Transitivity: Exercises (c)

Exercise 8.2.22

Let $X = \{a, b, c\}$ and $\mathscr{P}(X)$ be the power set of X. A relation N is defined on $\mathscr{P}(X)$ as follows: For all $A, B \in \mathscr{P}(X)$, $A\,N\,B \iff$ the number of elements in A is not equal to the number of elements in B. Is N reflexive, symmetric, transitive?

N **is not reflexive**: Denoting by $|S|$ the number of elements of set S, for all $A \in \mathscr{P}(X)$, it is false to say that $|A| \neq |A|$.

N **is symmetric**: Trivial. **Why?**

N **is not transitive**: Counterexample: find three sets such that A, B, C such that $|A| \neq |B|, |B| \neq |C|,$ but $|A| = |C|$.

8.3 Equivalence Relations I

Definition

- A **partition** of a set A is a collection of nonempty, mutually disjoint subsets of A, whose union is A.
- Given a partition of A, the **relation induced by the partition**, R, is defined on A as follows: For all $x, y \in A, x\,R\,y \iff$ there is a subset A_i of the partition suth that both x and y are in A_i.
- A relation on a set that satisfies the three properties of reflexivity, symmetry and transitivity is called an **equivalence relation**.

Theorem

Any relation on a set induced by a partition is an equivalence relation.

8.3 Equivalence Relations II, (a)

Definition

Let R be an equivalence relation on a set A. Then, for each $a \in A$, the **equivalence class of** a, denoted $[a]$ and called the **class of** a for short, is defined as the set of $x \in A$ such that $x \, R \, a$.

Theorem

Let R be an equivalence relation on a set A. Then the following are true:

- *For any $a, b \in A$, if $a \, R \, b$, then $[a] = [b]$.*
- *For any $a, b \in A$, either $[a] \cap [b] = \varnothing$ or $[a] = [b]$.*
- *The distinct equivalence classes of R form a partition of A.*
- *A* **representative** *of a class S of R is any $a \in A$ such that $[a] = S$.*

8.3 Equivalence Relations II, (b)

Definition

Let m and n be integers and let d be a positive integer. We say that m **is congruent to** n **modulo** d and write $m \equiv n \pmod{d}$ if and only if $d \mid (m - n)$.

Exercise 8.3.2 (b) and (c)

In $A = \{0, 1, 2, 3, 4\}$, find the relation R for the partitions (b) $\{0\}, \{1, 3, 4\}, \{2\}$ and (c) $\{0\}, \{1, 2, 3, 4\}$.

Exercise 8.3.4

Let $A = \{a, b, c, d\}$ be a set and $R = \{(a, a), (b, b), (b, d), (c, c), (d, b), (d, d)\}$ be an equivalence relation on A. Find the distinct equivalence classes of R.

Use the definition $[a] = \{x \in A \mid x \, R \, a\}$ for all $a \in A$.

Exercise 8.3.10

Let $A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ and the equivalence relation R is defined on A as follows: For all $m, n \in \mathbb{Z}$, $m\,R\,n \iff 3\,|\,(m^2 - n^2)$. Find the distinct equivalence classes of R.

Use the definition $[a] = \{x \in A \,|\, x\,R\,a\}$ for all $a \in A$.

8.3 Equivalence Relations: Exercises (b)

Exercise 8.3.15 (b)

Prove that, for all integers m and n and any positive integer d, $m \equiv n \pmod{d}$ if and only if $m \bmod d = n \bmod d$.

First, suppose that $m \equiv n \pmod{d}$. By definition of congruence, $d \mid (m-n)$ and, thus, $m - n = dk$, for some integer k. Furthermore, assume that $m \bmod d = r$ or $m = dl + r$, for some integer l. Therefore, after a simple substitution $n = d(lik) + r$ (**why exactly?**), i.e., $n \bmod d = r = m \bmod d$.

Next, suppose that $m \bmod d = n \bmod d$ and set $r = m \bmod d = n \bmod d$. Then, by definition of mod, $m = dp + r$ and $n = dq + r$, for some integers p and q. Then compute $m - n$ and why would this imply that $d \mid (m - n)$, which is the definition of congruence?

8.3 Equivalence Relations: Exercises (c)

Exercise 8.3.22

Let the relation D be defined on $\mathbb{Z}$ as follows: For all $m, n \in \mathbb{Z}$, $m\,D\,n \iff 3 \mid (m^2 - n^2)$. Prove that D is an equivalence relation and find its distinct equivalence classes.

Reflexivity: Trivial. **Why?**
Symmetry: Notice that $3 \mid (m^2 - n^2)$ means that $m^2 - n^2 = 3k$, for some integer k. Then, what about $n^2 - m^2$?
Transitivity: Let $m\,D\,n$ and $n\,D\,p$. Then use the definition of divisibility and some simple manipulation in order to find that $3 \mid (m^2 - p^2)$. **Fill in the details!**

To find the equivalence classes of D, first, notice that $m^2 - n^2 = (m - n)(m + n)$, which would imply that $m\,D\,n \iff$ which two divisibility conditions should occur? Subsequently, using the definition of divisibility, express m in terms of n in two ways, which are going to generate two equivalence classes. Which ones?