Slides of Discrete Mathematics based on Susanna Epp's Textbook

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Chapter 5b

Sequences, Mathematical Induction, and Recursion, III, IV

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5.3 Mathematical Induction II: Proving a Divisibility Property

Proposition

For all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

Proof: As in the book pp. 201-2.

Exercise 5.3.14

For all integers $n \geq 0$, $n^3 - n$ is divisible by 6.

Sketch of Proof: Let the property $P(n) = \{n^3 - n \text{ is divisible by } 6\}$. First, we observe that $P(0)$ is true (**Why?**). Next, we will prove that, if, for all integers $k \geq 0$, $P(k)$ is true (i.e., if $k^3 - k$ is divisible by 6, which means that $k^3 - k = 6p$, for some integer p), then $P(k+1)$ should be true too (i.e., we need to show that $(k+1)^3 - (k+1)$ is divisible by 6 too). Doing the algebra (**fill up all details**), we get $(k+1)^3 - (k+1) = \ldots = 6p + 3(k(k+1))$. Since $k(k+1)$ is the product of two consequtive integers, necessarily one of the integers has to be even, and, thus, their product has to be even too, i.e., $k(k+1) = 2q$, for some integers q. Therefore, we get $(k+1)^3 - (k+1) = 6(p+q)$, which implies the wanted divisibility property.

5.3 Mathematical Induction II: Proving an Inequality

Proposition

For all integers $n \geq 3$, $2n + 1 < 2^n$.

Proof: As in the book pp. 202-4.

Exercise 5.3.17

For all integers $n \geq 0$, $1 + 3n \leq 4^n$.

Sketch of Proof: Let the property $P(n) = \{1 + 3n \leq 4^n\}$. First, we observe that $P(0)$ is true (**Why?**). Next, we will prove that, if, for all integers $k \geq 0$, $P(k)$ is true (i.e., if $1 + 3k \leq 4^k$), then $P(k+1)$ should be true too (i.e., we need to show that $1 + 3(k+1) \leq 4^{k+1}$ too). Apparently, $1 + 3(k+1) = 1 + 3k + 3 =$ $4 + 3k \leq 4 + 12k$, since $k \geq 0$. The reason that we have taken $12k$ is that because multiplying the inductive hypothesis $(1 + 3k \leq 4^k)$ with 4 gives $4 + 12 \leq 4 \cdot 4^k = 4^{k+1}$. Therefore, the transitivity property of order implies that $1 + 3(k+1) \leq 4 + 12k \leq 4^{k+1}$, which is what we wanted to prove and it completes the induction.

5.3 Mathematical Induction II: Proving a Property of a Sequence

Example

Let the sequence $a_1, a_2, a_3, \ldots$ be defined as follows:

$$a_1 = 2,$$
$$a_k = 5a_{k-1}, \text{ for all integers } k \geq 2.$$

Show that $a_n = 2 \cdot 5^{n-1}$, for any integer $n \geq 1$.

Proof: As in the book pp. 204-5.

Exercise 5.3.26

A sequence $c_0, c_1, c_2, \ldots$ is defined by letting $c_0 = 3$ and $c_k = (c_{k-1})^2$, for all integers $k \geq 1$. Show that $c_n = 3^{2^n}$, for all integers $n \geq 0$.

Proof Sketch: Let $P(n) = \{c_n = 3^{2^n}\}$. First, we note that $P(0)$ is true (**Why?**). Next, we will prove that, if, for all integers $k \geq 0$, $P(k)$ is true (i.e., if $c_k = 3^{2^k}$), then $P(k+1)$ should be true (i.e., we need to show that $c_{k+1} = 3^{2^{k+1}}$). Apparently, by the definition of this sequence, $c_{k+1} = (c_k)^2 = (3^{2^k})^2 = 3^{2^k \cdot 2} = 3^{2^{k+1}}$, which is what we wanted to prove and it completes the induction.

Exercise 5.3.37

On the outside rim of a circular disk the integers from 1 through 30 are painted in random order. Show that no matter what this order is, there must be three successive integers whose sum is at least 45.

Sketch of Proof: Suppose it is impossible to find three successive integers on the rim of the disk whose sum is at least 45. Then there is some ordering of the integers from 1 to 30, say $x_1, x_2, \dots, x_{30}$, such that

$$x_k + x_{k+1} + x_{k+2} < 45, \text{ for any } k = 1, 2, \dots 30,$$

where, by periodicity $x_{31} = x_1, x_{32} = x_2$. Adding these inequalities, $\sum_{k=1}^{30}(x_k + x_{k+1} + x_{k+2}) < 30 \cdot 45 = 1350$, while we note that every term of this sequence appears three times, i.e., $\sum_{k=1}^{30}(x_k + x_{k+1} + x_{k+2}) = 3\sum_{i=1}^{30} x_i$, and, thus, $3\sum_{i=1}^{30} x_i < 1350$. However, $\sum_{i=1}^{30} x_i = \sum_{i=1}^{30} i = \frac{30 \cdot 31}{2} = 465$, which means that $3 \cdot 465 = 1395 < 1350$, which is a contradiction.

5.4 Strong Mathematical Induction: Steps

Proof by Induction

To show that $\forall$ integers $n \geq a, P(n)$:

1. **Basic Step**: Show that $P(a)$ is true.
2. **Inductive Step**: Show that, $\forall$ integers $k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true. That is,
   - Suppose $k \geq a$ and that $P(k)$ is true.
   - Show: $P(k+1)$ is true.

Proof by Strong Induction

To show that $\forall$ integers $n \geq a, P(n)$, where $a \leq b$:

1. **Basic Steps**: Show that $P(a), \ldots, P(b)$ are true.
2. **Inductive Step**: Show that, $\forall$ integers $k \geq b$, if $P(a), \ldots, P(k)$ are true, then $P(k+1)$ is true. That is,
   - Suppose $k \geq b$ and that $P(i)$ is true, for all integers $a \leq i \leq k$.
   - Show: $P(k+1)$ is true.

5.4 Strong Mathematical Induction: Application I

Theorem

Every integer greater than 1 has a prime divisor. In symbols:

$$\forall \textit{ integers } n \geq 2, \exists\, p \textit{ a prime such that } p|n.$$

Proof by Strong Induction:

1. **Basic Step (for $b = a = 2$)**: Certainly, 2 is prime and divides itself.
2. **Inductive Step**: Suppose $k \geq 2$ and that each integer i with $2 \leq i \leq k$ has a prime divisor. (Goal: $k + 1$ has a prime divisor.) Two cases:
   - Either $k + 1$ is prime: Since, it obviously it divides itself, we have shown the goal.
   - Or $k + 1$ is composite: Then $k + 1 = rs$, where $2 \leq r \leq k$ and $2 \leq s \leq k$. In particular, the inductive hypothesis applies to the integer r, i.e., r has a prime divisor and, since r is a factor of $k + 1$, so does $k + 1$ and the goal is shown.

5.4 Strong Mathematical Induction: Application II

Theorem (Fundamental Theorem of Arithmetic)

Every integer n greater than 1 has a (unique) factorization of the form

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot \cdots \cdot p_m^{e_m},$$

where m is a positive integer, $p_1 < p_2 < \cdots < p_m$ are primes and $e_1, e_2, \ldots, e_m$ are positive integers (called **standard factorization***).*

Proof by Strong Induction:

1. **Basic Step (for $b = a = 2$)**: Certainly, 2 is prime and $2 = 2^1$ is a standard factorization.
2. **Inductive Step**: Suppose $k \geq 2$ and that each integer i with $2 \leq i \leq k$ has a standard factorization. (Goal: $k+1$ does too.) Two cases:
   - Either $k+1$ is prime: Since, $k+1$ is prime and $k+1 = (k+1)^1$, we have shown the goal.
   - Or $k+1$ is composite: Then $k+1 = rs$, where $2 \leq r \leq k$ and $2 \leq s \leq k$. By the inductive hypothesis, r and s have standard factorizations and, by appropriately grouping the primes in the product rs, so does $k+1$ and the goal is shown.

5.4 Strong Mathematical Induction: Application III

Exercise 5.4.7

Let $g_1, g_2, g_3, \ldots$ be a sequence defined as follows:

$$g_1 = 3, g_2 = 5,$$
$$g_k = 3g_{k-1} - 2g_{k-2}, \text{ for all integers } k \geq 3.$$

Prove that $g_n = 2^n + 1$, for all integers $n \geq 1$.

Proof by Strong Induction (Sketch):

1. **Basic Steps (for $a = 1, b = 2$)**: By definition, $g_1 = 3 = 2^1 + 1, g_2 = 5 = 2^2 + 1$.
2. **Inductive Step**: Suppose $k \geq 1$ and that for all integers i with $1 \leq i \leq k$, $g_i = 2^i + 1$. (Goal: $g_{k+1} = 2^{k+1} + 1$.)

   By definition, $g_{k+1} = 3g_k - 2g_{k-1}$, where, according to the inductive step, $g_k = 2^k + 1$ and $g_{k-1} = 2^{k-1} + 1$. Then, writing $2^k = 2 \cdot 2^{k-1}$, do the algebra to show the goal.

5.4 Strong Mathematical Induction: Application IV

Exercise 5.4.8

Let $h_0, h_1, h_2, \ldots$ be a sequence defined as follows:

$$h_0 = 1, h_1 = 2, h_3 = 3,$$
$$h_k = h_{k-1} + k_{k-2} + h_{k-3}, \text{ for all integers } k \geq 3.$$

Prove that $h_n \leq 3^n$, for all integers $n \geq 0$.

Proof by Strong Induction (Sketch):

1. **Basic Steps (for** $a = 0, b = 2$**)**: By definition, $h_0 = 1 = 3^0, h_1 = 2 < 3 = 3^1, h_2 = 3 < 9 = 3^2$.
2. **Inductive Step**: Suppose $k \geq 0$ and that for all integers i with $0 \leq i \leq k$, $h_i \leq 3^i$. (Goal: $h_{k+1} \leq 3^{k+1}$.)

   By definition, $h_{k+1} = h_k + k_{k-1} + h_{k-2}$, where, according to the inductive step, $h_k \leq 3^k$, $h_{k-1} \leq 3^{k-1}$ and $h_{k-2} \leq 3^{k-2}$. Then do the algebra to show the goal.

The Fibonacci Sequence

The sequence of numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots,$$

is characterized by the fact that, after the first two terms, each term is obtained as the sum of the previous two. It is called **Fibonacci sequence** and, formally, it is defined as follows:

$$\begin{aligned} &F_0 = 1, F_1 = 1, \\ &F_n = F_{n-2} + F_{n-1}, \text{ for all integers } n \geq 2. \end{aligned}$$

Show that the general formula of the Fibonacci sequence is

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$$

5.4 Strong Mathematical Induction: Application Vb

Proof the general form of the Fibonacci sequence by Strong Induction:

1. **Basic Steps (for $a = 0, b = 1$)**: Clearly, substituting $n = 0$ and $n = 1$ in the expression to be shown, we get the true values of the first two terms.
2. **Inductive Step**: Suppose that $k \geq 1$ and that

$$F_i = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{i+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{i+1}\right], \text{ for each } 0 \leq i \leq k.$$

(Goal: $F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right]$.) Notice that $k+1 \geq 2$ and that both $k-1$ and k lie in the interval $[0, k]$. Thus, we obtain:

$$\begin{aligned}
F_{k+1} &= F_{k-1} + F_k \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} + \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]
\end{aligned}$$

5.4 Strong Mathematical Induction: Application Vc

Proof the general form of the Fibonacci sequence by Strong Induction (continuation from previous slide):

$$
\begin{aligned}
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(1+\frac{1+\sqrt{5}}{2}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^k\left(1+\frac{1-\sqrt{5}}{2}\right)\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(\frac{3+\sqrt{5}}{2}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^k\left(\frac{3-\sqrt{5}}{2}\right)\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(\frac{1+\sqrt{5}}{2}\right)^2-\left(\frac{1-\sqrt{5}}{2}\right)^k\left(\frac{1-\sqrt{5}}{2}\right)^2\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right],
\end{aligned}
$$

where we have used the identities:

$$
\begin{aligned}
\left(\frac{1+\sqrt{5}}{2}\right)^2 &= \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{2\cdot(3+\sqrt{5})}{2\cdot 2} = \frac{3+\sqrt{5}}{2},\\
\left(\frac{1-\sqrt{5}}{2}\right)^2 &= \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{2\cdot(3-\sqrt{5})}{2\cdot 2} = \frac{3-\sqrt{5}}{2}.
\end{aligned}
$$

Theorem (**The Number of Multiplications Needed to Multiply n Numbers**)

Prove that for any integer $n \geq 1$, if $x_1, x_2, \ldots, x_n$ are n numbers, then no matter how the parentheses are inserted into their product, the number of multiplications used to compute the product is $n-1$.

Proof by Strong Induction (Sketch): For instance, the product of two numbers involves one multiplication, the product of three numbers involves two multiplications, the product of four numbers involves three multiplications and so one. The truth of the basis step follows immediately from the convention about a product with one factor. The inductive step is based on the fact that when several numbers are multiplied together, each step of the process involves multiplying two individual quantities. For instance, the final step for computing $((x_1x_2)x_3)(x_4x_5)$ is to multiply $(x_1x_2)x_3$ and x_4x_5. In general, if $k+1$ numbers are multiplied, the two quantities in the final step each consist of fewer than $k+1$ factors. This is what makes it possible to use the inductive hypothesis. For the rest of the proof, see pp. 213–4 in the book.

Theorem (Existence and Uniqueness of Binary Integer representations)

Given any positive integer n, n has a unique representation in the form

$$n = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0,$$

where r is a nonnegative integer, $c_r = 1$, and $c_j = 1$ or 0 for all $j = 0, 1, 2, \ldots, r-1$.

Proof by Strong Induction: As in the book pp. 216–7.

Exercise 5.4.29

Convert in decimal notation: (a) 1110_2, (b) 10111_2, (c) 110110_2, (d) 1100101_2, (e) 1000111_2, (f) 1011011_2.

Solutions In each case, $r =$ number of binary digits $- 1$.

(a) $r = 4 - 1 = 3 \Rightarrow 1110_2 = 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 8 + 4 + 2 = 14_{10}$.

(b) $r = 5 - 1 = 4 \Rightarrow 10111_2 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 16 + 4 + 2 + 1$
$= 23_{10}$.

(c) $r = 6 - 1 = 5 \Rightarrow 110110_2 = 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$
$= 32 + 16 + 4 + 2 = 54_{10}$.

(d) $r = 7 - 1 = 6 \Rightarrow 1100101_2 = 1 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$
$= 64 + 32 + 4 + 1 = 101_{10}$.

(e) $r = 7 - 1 = 6 \Rightarrow 1000111_2 = 1 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$
$= 64 + 4 + 2 + 1 = 71_{10}$.

(f) $r = \ ? - 1 = \ ? \ \Rightarrow 1011011_2 = \ ?$.

Theorem (Well–Ordering Principle for the Integers)

Let S be a set of integers containing one or more integers all of which are greater than some fixed integer. Then S has a least element.

Proof by Strong Induction: As in the book pp. 217–8.