Slides of Discrete Mathematics based on Susanna Epp's Textbook

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Chapter 3

The Logic of Quantified Statements

September 3, 8, & 10, 2021

3.1 Predicates

Definition

A **predicate** is an expression that contains a finite number of variables, each one of them defined on some specific **domain**. By assigning a value to each variable (or quantifying it) from the domain of the corresponding value, a predicate becomes a statement (proposition) and as such it may be true or false.

Examples

- Let $P(x)$ be the predicate "$x^2 > x$" (written: $P(x)\colon x^2 > x$), where the domain of variable x is the set $\mathbb{R}$ of real numbers. Apparently, when assigning the value $x = 2$, this predicate is reduced to the statement $4 > 2$, which is true, while for the value $x = \frac{1}{2}$ one gets the statement $\frac{1}{4} > \frac{1}{2}$, which is false.
- Let $P(x, y)$ be the predicate "$x = y$", where the domain of both variables x and y is $\mathbb{R}$. Trivially, $x = 2$ and $y = 2$ give a true statement, while $x = 2$ and $y = 3$ a false one.

3.1 Predicates

Definition

Let $P(x)$ be a predicate with a variable x defined on a domain D. Then the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true, when assigned to the predicate. In other words, the truth set of $P(x)$ is denoted $\{x \in D \mid P(x)\}$.

Examples

- Let the predicate $Q(n)$: n is a factor of 8 with domain the set $\mathbb{Z}$ of all integers. Then the truth set of $Q(n)$ is the set $\{-8, -4, -2, -1, 1, 2, 4, 8\}$.
- The truth set of the predicate $P(x)$: $x^2 > x$ with domain $\mathbb{R}$ is the set $\{x \in \mathbb{R} : x > 1\}$.
- The truth set of the predicate $P(x, y)$: $x = y$ with domain $\mathbb{R}^2$ is the diagonal line in $\mathbb{R}^2$ $\{(x, y) \in \mathbb{R}^2 \mid y = x\}$.

3.1 The Universal Quantifier $\forall$

Definition

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$" (read "for all x in $D, Q(x)$ holds"). This statement is defined to be true if and only if $Q(x)$ is true, for every x in D, and false if and only if $Q(x)$ is false, for at least one x in D. A value for x, for which $Q(x)$ is false, is called **counterexample** to the universal statement. Using the existential quantifier, the universal statement "$\forall x \in D, Q(x)$" is false if and only if "$\exists x$ such that $\sim Q(x)$."

Examples

- Let the predicate $Q(x)$: $x^2 \geq 0$ with domain $\mathbb{R}$. Then the statement "$\forall x \in \mathbb{R}, Q(x)$" is true.
- Let the predicate $P(x)$: $x^2 > x$ with domain $\mathbb{R}$. Then the statement "$\forall x \in \mathbb{R}, P(x)$" is false and a counterexample to this statement is $x = \frac{1}{2}$.

3.1 The Existential Quantifier $\exists$

Definition

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$" (read "there exists x in D such that $Q(x)$ holds"). This statement is defined to be true if and only if $Q(x)$ is true, for at least one x in D, and false if and only if $Q(x)$ is false, for all x in D.

Examples

- Let the predicate $Q(x)$: $x > 0$ with domain $\mathbb{R}$. Then the statement "$\exists x \in \mathbb{R}$ such that $Q(x)$" is true, for $x = 1$, and false, for $x = -1$.
- Let the predicate $P(x)$: $x^2 = x$ with domain $\mathbb{Z}^+$. Then the statement "$\exists x \in \mathbb{Z}^+$ such that $P(x)$" is true, for $x = 1$, and false, for $x = 2$.

3.2 Negations of Quantified Statements

Theorem (Negation of a Universal Statement)

$$\sim (\forall x \in D, Q(x)) \equiv \exists x \in D \textit{ such that } \sim Q(x).$$

Theorem (Negation of an Existential Statement)

$$\sim (\exists x \in D \textit{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x).$$

Theorem (Negation of a Universal Conditional Statement)

$$\sim (\forall x, \textit{ if } P(x) \textit{ and } Q(x)) \equiv \exists x \textit{ such that } P(x) \textit{ and } \sim Q(x).$$

3.2 The Relation among $\forall, \exists, \wedge$ and $\vee$

Theorem

If $Q(x)$ is a predicate and the domain D of x is the set $\{x_1, x_2, \ldots, x_n\}$, then

$$\forall x \in D, Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \cdots \wedge Q(x_n),$$

$$\exists x \in D \textit{ such that } Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \cdots \vee Q(x_n).$$

Definition

A statement of the form

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x)$$

is called **vacuously true** or **true by default** if and only if

$$\forall x \in D, \sim (P(x)).$$

3.3 Statements with Multiple Quantifiers

Possible Statements with Two Quantifiers

$$\forall x \in D, \exists y \in E \text{ such that } P(x, y),$$

$$\exists x \in D \text{ such that } \forall y \in E, P(x, y).$$

Examples

- Let $D = E$ be the set of married persons.
  - $\forall x \in D, \exists y \in D$ such that x is married to y. (True)
  - $\exists x \in D$ such that $\forall y \in D, x$ is married to y. (False)
- Let $D = E = \{1, 2, 3, 4, 5\}$.
  - $\forall x \in D, \exists y \in D$ such that $x + y$ is even. (True)
  - $\exists x \in D$ such that $\forall y \in D, x + y$ is odd. (False)
- Let $D = E = \mathbb{R}$.
  - $\forall x \in D, \exists y \in D$ such that $x > y$. (True)
  - $\exists x \in D$ such that $\forall y \in D, x > y$. (False)
- Let $D = E = \mathbb{R}$.
  - $\forall x \in D, \exists y \in D$ such that $y^2 = x$. (False)
  - $\exists x \in D$ such that $\forall y \in D, y^2 = x$. (False)

3.3 Negation and Order of Multiply–Quantified Statements

Negations of Multiply–Quantified Statements

$\sim(\forall x \in D, \exists y \in E \text{ such that } P(x,y)) \equiv \exists x \in D \text{ such that } \forall y \in E, \sim P(x,y)$,

$\sim(\exists x \in D \text{ such that } \forall y \in E, P(x,y)) \equiv \forall x \in D, \exists y \in E \text{ such that } \sim P(x,y)$.

Order of Multiply–Quantified Statements

- In a statement containing both $\forall$ and $\exists$, changing the order of the quantifiers usually changes the meaning of the statement.
- However, if one quantifier immediately follows another quantifier *of the same type*, then the order of the quantifiers does not affect the meaning.

3.3 Limit of a Sequence

Definition

- A sequence of real numbers $a_1, a_2, a_3, \ldots$ has a limit $L \in \mathbb{R}$ whenever:

  $$\forall \varepsilon > 0, \exists N \in \mathbb{Z}^+ \text{ such that } \forall i > N, L - \varepsilon < a_i < L + \varepsilon.$$

- The negation that a sequence $a_1, a_2, a_3, \ldots$ has no limit $L \in \mathbb{R}$ is written symbolically as:

  $$\exists \varepsilon > 0 \text{ such that } \forall N \in \mathbb{Z}^+, \exists i > N \text{ such that } a_i \leq L - \varepsilon \vee L + \varepsilon \leq a_i.$$

3.4 Arguments with Quantified Statements

Rule of Universal Instantiation

If some property is true of *everything* in a set, then it is true of *any particular* thing in the set.

$$(\forall x \in D, P(x)) \longrightarrow (a \in D \longrightarrow P(a))$$

Proposition (Universal Modus Ponens)

$$\begin{aligned} & \forall x,\ P(x) \longrightarrow Q(x). \\ & P(a) \textit{ for a particular } a. \\ \therefore\ & Q(a). \end{aligned}$$

Proposition (Universal Modus Tollens)

$$\begin{aligned} & \forall x,\ P(x) \longrightarrow Q(x). \\ & \sim Q(a) \textit{ for a particular } a. \\ \therefore\ & \sim P(a). \end{aligned}$$

Example

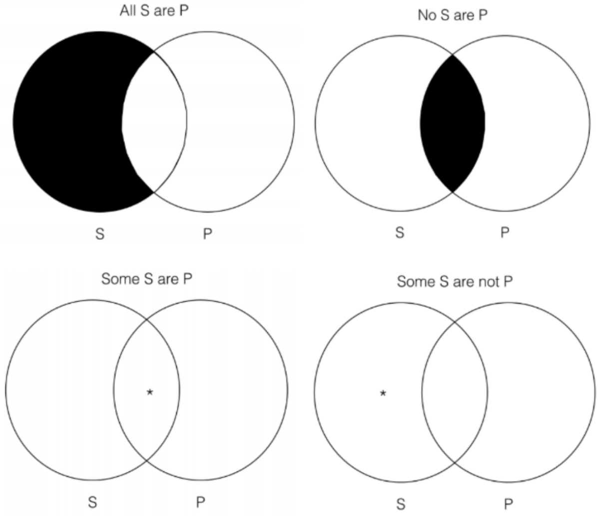
Question: Is this argument valid?

$$\begin{array}{rl} & \text{All humans are mortals.} \\ & \text{Felix is mortal.} \\ \therefore & \text{Felix is human.} \end{array}$$

Answer: It is false because of a converse error. You may also see it using a Venn diagram.

3.4 Using Venn Diagrams to Test for Validity

Any shaded portions of the Venn diagram (by “shaded” one means “blacked out”) represent that there is nothing in that area of the category.



3.4 Universal Transitivity

Proposition (Universal Transitivity)

$$
\begin{aligned}
& \forall x,\ P(x) \longrightarrow Q(x). \\
& \forall x,\ Q(x) \longrightarrow R(x). \\
\therefore\quad & \forall x,\ P(x) \longrightarrow R(x).
\end{aligned}
$$