Applications of the Pigeonhole Principle and Ramsey's Theorem in Graph Theory

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A Teaching Demonstration

in 20 minutes

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I. The Pigeonhole Principle

The Pigeonhole Principle

Theorem

If n items are put into m containers, with $n > m$, then at least one container must contain more than

$$\left\lfloor \frac{n-1}{m} \right\rfloor \geq 1$$

items.

Proof. Assuming the opposite, i.e., if the largest number of items in a container is at most $\left\lfloor \frac{n-1}{m} \right\rfloor$, then the total number of items is at most

$$m \left\lfloor \frac{n-1}{m} \right\rfloor \leq n-1 < n,$$

which is a contradiction

II. Graphs: Basic Definitions

Graphs

Definition

- A **graph** G is a tuple (V, E), where
  - V is a set of points, called **nodes** (or **vertices**), and
  - E is a set of lines joining (some) pairs of points, called **links** (or **edges**).
- A **simple undirected graph** $G = (V, E)$ is a graph such that,
  - for any $u, v \in V$ connected by a link $(u, v) \in E$, the pair (u, v) is the unique (possible) link between u and v, and
  - $(u, v) \in E$ if and only if $(v, u) \in E$,

  i.e., a graph in which there exist no multiple links and all links have no particular direction.

  All graphs considered here are simple undirected graphs!
- A graph $G = (V, E)$ with n nodes (i.e., $|V| = n$) is called **complete** and it is denoted as $G = K_n$, when there exists a link $e \in E$ between any two nodes $u, v \in V$ (i.e., $e = (u, v)$).
  - If $|V| = n$ is the number of nodes of a complete (undirected simple) graph $G = (V, E)$, then the number of links is

$$|E| = \frac{1}{2} n(n-1).$$

Degrees and Adjacency Matrices

Definition

Let $G = (V, E)$ a graph of n nodes with $V = \{1, 2, \ldots, n\}$ and let i be a node:

- A node $j \in V$ is **adjacent** to i or a **neighbor** of i if there exists a link (i, j) in the graph.
- The **degree** k_i of node i is the number of neighbors of i in the graph, i.e.,
$$k_i = |\{j \in V \text{ such that } (j, i) \in E\}|.$$
- The **adjacency matrix** of this graph is a *nxn* matrix $A = \{A_{ij}\}$ such that $A_{ij} = 1$, whenever $(i, j) \in E$, while otherwise $A_{ij} = 0$.
  - Notice that, since G is undirected and simple graph, its adjacency matrix A is a *symmetrical* (binary) matrix.
- Computing degrees through the adjacency matrix:
$$k_i = \sum_{j=1}^{n} A_{ij}, \text{ for } i \in V.$$

III. Handshaking

Two Versions of Handshaking

Lemma

In a graph $G = (V, E)$, the sum of degrees of all nodes equals twice the number of all links, i.e.,

$$\sum_{i \in V} k_i = 2|E|.$$

Theorem

In any graph G, there exist two nodes having equal degrees.

Proof. Assume that the graph has n nodes. Since a node can possibly have at most $n-1$ neighbors, the possible degrees of nodes in a graph are $0, 1, 2, \ldots, n-1$. However, no (simple) graph with n vertices can contain both a node of degree 0 and a node of degree $n-1$, which implies that the possible values of degrees are at most $n-1$. Hence, applying the Pigeon Principle for the distribution of n items (nodes) in $n-1$ containers (values of degrees) completes the proof.

IV. Ramsey's Theorem

Ramsey's Theorem – Special Case

Definition

Let $G = (V, E)$ be a graph and k a (fixed) positive integer.

- A k–**coloring** of the edges of G is a mapping $\varphi : E \to \{1, 2, \ldots, k\}$. In other words a k-coloring of edges is an assignment of an attribute (label) on edges from the set $\{1, 2, \ldots, k\}$.
- An 1-coloring of edges is called **monochromatic**.

Theorem

Every 2–coloring of the edges of K_6 generates a monochromatic K_3.

Proof of Ramsey's Theorem – Special Case

Definition

A 2–coloring of edges of a graph $G = (V, E)$ generates two types of pairs of nodes connected by an edge. Indeed, any two nodes $u, v \in V$ such that $(u, v) \in E$ can be called either

- **friends**, when (say) $\varphi((u, v)) = 1$ or
- **strangers**, when (say) $\varphi((u, v)) = 2$.

Proof of Ramsey's Theorem Special Case. Fix a node u and consider two cases:

- If the degree of u is greater than 3, then consider three neighbors, say, x, y, z, of u. By the Pigeonhole Principle for ($n = 3$ and $m = 2$), two of $\{x, y, z\}$ are either friends or strangers to u. In either case, we are done because a monochromatic triangle is formed together with u.
- If the degree of u is less or equal than 2, then there are at least three other nodes, say, x, y, z, which are not neighbors of u. In this case, the argument is complementary to the previous one. Either $\{x, y, z\}$ are mutual friends, in which case we are done. Otherwise, there are two among $\{x, y, z\}$ who are strangers, say, x and y, and then $\{u, x, y\}$ is a triangle of strangers.

Ramsey's Theorem – The General Case

Definition

Let $p, q \geq 2$ be two integers. Then we say that a positive integer n **has the Ramsey property** and call n a **Ramsey number**, writing $n = R(p, q)$, if every p–coloring of edges of K_n has a monochromatic K_q.

Table: **Known Ramsey Numbers**

p	q	$R(p,q)$
2	n	n
3	3	6
3	4	9
3	5	14
3	6	18
3	7	23
3	8	28
3	9	36
4	4	18
4	5	25

Ramsey's Theorem – Alternative Formulation

Definition

Let $G = (V, E)$ be a (general) graph and let $S \subseteq V$ be a set of nodes of size $|S| = s$. Then:

- S is called s–**clique** whenever every two nodes in S form a link, i.e., $(u, v) \in E$, for all $u, v \in S$.
- S is called s–**independent** whenever no two nodes in S form a link, i.e., $(u, v) \notin E$, for all $u, v \in S$.

Theorem

For any integers $p, q \geq 2$, there is a (finite) Ramsey number $n = R(p, q)$ such that any graph with n nodes contains either a p–independent set or a q–clique. In particular,

$$R(p, q) \leq \binom{p+q-2}{p-1}.$$

Frank Ramsey (February 22, 1903 – January 19, 1930)

THANK YOU!