Experiments of Friedkin–Johnsen Social Influence on Graphs

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Contents

Graph Notations

- Everywhere in here, $G = (V, E)$ is considered to be a graph of n nodes in the set $V = \{1, 2, \ldots, n\}$ with its adjacency matrix denoted by $\mathbf{A}$.
- Obviously, if G is undirected, $\mathbf{A}$ is a symmetric matrix, while, if G is directed, $\mathbf{A}$ is nonsymmetric.
- Given two nodes $i, j \in V$, by $i \sim j$ one denotes that i, j are adjacent, when G is undirected, or that j is an outgoing neighbor of i, when G is directed (in which case "$\sim$" is a nonsymmetric relation).
- By $\deg_i$, one denotes the degree of node i, when G is undirected, or the out–degree of i, when G is directed. Moreover, $\mathbf{D}$ denotes the diagonal matrix of node degrees of G.

Main Graph Assumtions

- Without any loss of generality, we are assuming here:
  - G is connected, when it is undirected, or G is weakly connected, when it is directed.
  - When G is directed, all nodes are assumed to have positive out–degrees *and* postive in–degrees.

- However, when G is a **signed graph** (the case that we are going to consider in the last section), we are only assuming:
  - G is connected, when it is undirected, or G is strongly connected, when it is directed.

- Let $G = (V, E)$ be a **graph** with n nodes.
- For each node $i \in V$ and each time step $k = 0, 1, 2, \ldots$, the **PageRank** of i at time k is denoted by $x_i^{(k)} \in \mathbb{R} \in [0, 1]$.
- The PageRank of i at time k is updated at the subsequent time step $k + 1$ according to the following iterative scheme:

$$x_i^{(k+1)} = \alpha \, P \, x_i^{(k)} + (1 - \alpha) \, x_i^{(0)},$$

  - where $P \, x_i^{(k)}$ is the sum of the PageRanks of i's neighbors,
  - $\alpha \in (0, 1]$ is the **teleportation constant** and
  - one typically assumes the initial condition $x_i^{(0)} = \frac{1}{n}$.

Solving the PageRank Problem

- Disregarding the time exponent (k), $P\,x_i$ is defined as:

$$P\,x_i = \sum_{j \in V} \frac{A_{ji}}{\deg_j} x_j = \sum_{j \sim i} \frac{x_j}{\deg_j},$$

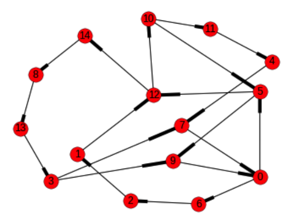
  i.e., P is a **probability transition matrix** of traveling from node i to node j along an edge:

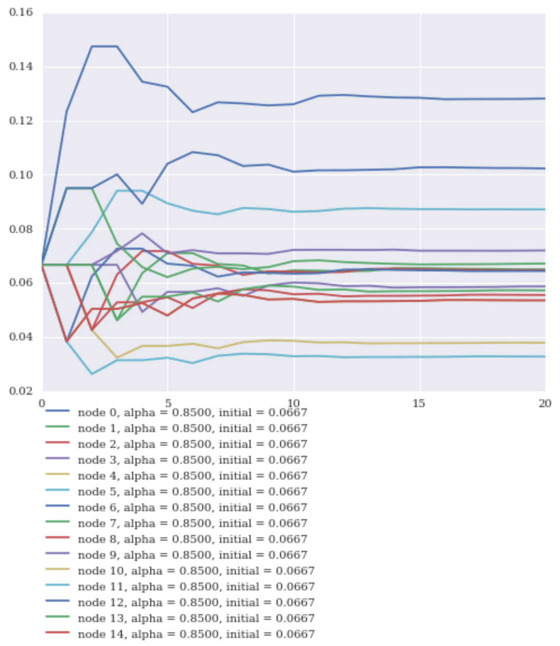
$$\mathbf{P} = \mathbf{A}^T \mathbf{D}^{-1}.$$

- As matrix $\mathbf{P}$ is *column–stochastic*, matrix $\mathbf{I} - \alpha\mathbf{P}$ becomes *diagonally dominant*.
- Therefore, the solution to this problem converges (relatively fast, depending on α, typicall taken equal to 0.85) to the vector $\bar{\mathbf{x}} = \bar{\mathbf{x}}(\alpha; \mathbf{x}^{(0)})$ of PageRanks of nodes of G:

$$\bar{\mathbf{x}} = (1-\alpha)\,(\mathbf{I} - \alpha\mathbf{P})^{-1}\mathbf{x}^{(0)}.$$

PageRank Example





The Friedkin–Johnsen Model of Social Influence on a Graph

- Let $G = (V, E)$ be a **graph** with n **nodes** $\equiv$ **persons**.
- For each node $i \in V$ and each time step $k = 0, 1, 2, \ldots$, the **opinion** of i at time k is denoted by $x_i^{(k)} \in \mathbb{R}$.
- The opinion of i at time k is updated at the subsequent time step $k + 1$ according to the following iterative scheme of the **Friedkin–Johnsen Social Influence Model**:

$$x_i^{(k+1)} = s_i \, N \, x_i^{(k)} + (1 - s_i) \, x_i^{(0)},$$

  - where $N \, x_i^{(k)}$ is the *average* opinion held by i's neighbors at time step k and
  - the scalar parameter $s_i \in [0, 1]$ is called **susceptibility coefficient** of person/node i.

Solving the Friedkin–Johnsen Problem

- Disregarding the time exponent (k), $N\,x_i$ is defined as:

$$N\,x_i = \sum_{j\in V} \frac{A_{ij}}{\deg_i}\,x_j = \sum_{j\sim i} \frac{x_j}{\deg_i},$$

i.e., $\mathbf{N}$ is a **random walk matrix** on G:

$$\mathbf{N} = \mathbf{D}^{-1}\mathbf{A}.$$

- As matrix $\mathbf{N}$ is *row–stochastic*, matrix $\mathbf{I} - \mathbf{S}\,\mathbf{N}$ becomes *diagonally dominant*, where $\mathbf{S}$ denotes the the diagonal matrix of susceptibility coefficients of nodes/persons.
- Therefore, the solution to this problem converges (relatively fast, depending on $\mathbf{S}$) to the vector $\bar{\mathbf{x}} = \bar{\mathbf{x}}(\mathbf{S};\mathbf{x}^{(0)})$ of equilibrium (steady state) opinions of nodes/persons of G:

$$\bar{\mathbf{x}} = (\mathbf{I} - \mathbf{S}\,\mathbf{N})^{-1}(\mathbf{I} - \mathbf{S})\,\mathbf{x}^{(0)}.$$

A Remark on the Original Formulation of the Friedkin–Johnsen Social Influence Model

- The original equations of the Friedkin–Johnsen model were written as:

$$\mathbf{x}^{(k+1)} = \mathbf{S}\,\mathbf{W}\,\mathbf{x}^{(k)} + (\mathbf{I} - \mathbf{S})\,\mathbf{x}^{(0)},$$

  where $\mathbf{W} = \{W_{ij}\}_{(i,j) \in V \times V}$ was a given row–stochastic matrix, called **influence matrix**. Not that, in this formulation, (diagonal) self–influences are allowed (i.e., W_{ii} can be nonzero).

- Clearly, by limiting social influence interactions only along edges of G (i.e., assuming that $W_{ij} = 0$, if $i \nsim j$),

$$\mathbf{W} = \mathbf{N} = \mathbf{D}^{-1}\mathbf{A},$$

  which is the case that we are considering here.

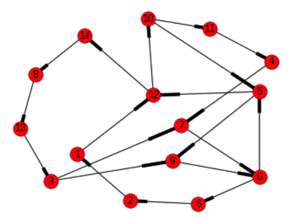
Remarks on Influence Susceptibility Coefficients

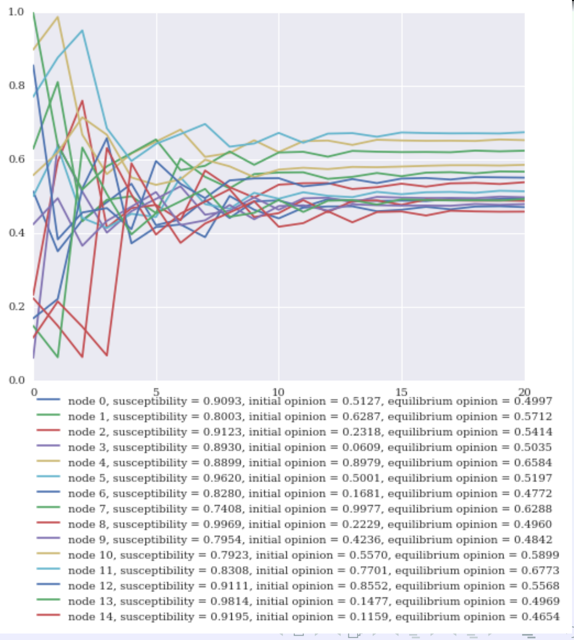
- When $s_i = 0, i$'s opinion does not change ($x_i^{(k)} = x_i^{(0)}$, for every time iteration $k = 1, 2, \ldots$). Such a person/node is called **persistent** or **stubborn** in holding the same opinion without being influenced by anybody else.
- When $s_i = 1, i$ is always adopting the average neighbors' opinion $N x_i^{(k)}$ without taking into account the initial opinion $x_i^{(0)}$. Such a person/node is called **malleable** or fully **conforming** to the neighbors' influences.
- When $0 < s_i < 1, i$'s opinion is interpolated in between the average neighbors' opinion $N x_i^{(k)}$ and the initial opinion $x_i^{(0)}$, in such a way that the exact position of the resulting i's opinion is weighted as a convex combination through s_i.

Assumptions on Susceptibilities and Initial Conditions

- We are considering here the following assumptions with regards to the susceptibility coefficients s_i of nodes/persons of graph $G = (V, E)$ and their initial opinions $x_i^{(0)}$:
    - $s_i > 0$, for every $i \in V$. (The case that $s_i = 0$, for some nodes/persons, is going to be examined separately in the sequel.)
    - There exists at least one node/person with $s_i < 1$. (The case that $s_i = 1$, for all nodes/persons, would imply the "**consensus**" equilibrium opinion $\overline{x}_i = c$, for all $i \in V$ holding the same constant opinion c.)
- Initial opinions $x_i^{(0)}$ are assumed not to exhibit any consensus (i.e., $x_i^{(0)} \neq x_i^{(0)}$, for at least two (distinct) $i, j \in V$), since, if that was the case, all the equilibrium (steady state) opinions would necessarily hold the same consensus opinion with which they were starting, independently of the values of their coefficients of susceptibilities.

A Friedkin–Johnsen Social Influence Example





Boundary Conditions of the Friedkin–Johnsen Problem

- Suppose we have the Friedkin–Johnsen social influence system on graph $G = (V, E)$ composed of n persons/nodes, denoted as $i = 1, 2, \ldots, n$ and having susceptibility coefficients s_i, respectively.

- Assumptions and Notation:
  - $V = \Omega \cup \partial\Omega$, where $\Omega \cap \partial\Omega = \varnothing$.
  - $\Omega = \{i \in V : s_i > 0\}$ is the domain of the Friedkin–Johnsen social influence system composed *exclusively* of nonpersistent persons/nodes.
  - $\partial\Omega = \{i \in V : s_i = 0\}$ is the of the domain of the social influence system composed *exclusively* of persistent persons/nodes.
  - $|\partial\Omega| < n$.
  - $\forall j \in \delta\Omega, \exists i \in \Omega, i \sim j$.

Sources of Boundary Stimulation of the Friedkin–Johnsen Problem

- When $\partial\Omega \neq \varnothing$, any person/node on the social influence boundary $\partial\Omega$ is called **source of a boundary stimulation** or just a **persistent source**. Moreover, we use the notation $\partial\Omega = \{j_a\}_{a=1,\ldots,|\partial\Omega|}$.
- Clearly, the **Initial Boundary Value Problem** (**IBVP**) of the Friedkin–Johnsen social influence system is the following:

$$
\begin{aligned}
x_i^{(k+1)} &= s_i\, N\, x_i^{(k)} + (1 - s_i) x_i^{(0)}, \text{ for all } i \in \Omega, k = 0, 1, \ldots, \\
x_{j_a}^{(k)} &= 1, \text{ for all } j_a \in \partial\Omega, k = 0, 1, \ldots, \\
x_i^{(0)} &= \phi_i, \text{ for all } i \in \Omega,
\end{aligned}
$$

where ϕ_i is the initial opinion of person/node $i \in \Omega$.

- Notice that, for the null initial condition $\phi_i = 0, i \in \Omega$, the IBVP of the Friedkin–Johnsen social influence system becomes:

$$
\begin{aligned}
x_i^{(k+1)} &= s_i \, N \, x_i^{(k)} + (1 - s_i)\psi_i, \text{ for all } i \in \Omega, k = 1, 2, \ldots, \\
x_{j_a}^{(k)} &= 1, \text{ for all } j_a \in \partial\Omega, k = 1, 2, \ldots, \\
x_i^{(1)} &= \psi_i, \text{ for all } i \in \Omega,
\end{aligned}
$$

where

$$
\psi_i = \begin{cases} \frac{s_i}{\deg_i}, & \text{when } i \sim j_b, \\ 0, & \text{when } i \nsim j_b. \end{cases}
$$

- Therefore, without any loss of generality, from now on, we are assuming that $\phi_i = 0, i \in \Omega$.

The Steady States of (Equilibrium) Opinions of the Friedkin–Johnsen Problem under Boundary Stimulations

- Let $\overline{\mathbf{x}}(\partial\Omega; \mathbf{S})$[1] be the **steady states** or **equilibrium** (vector of) opinions solving the Friedkin–Johnsen social influence system on the graph $G = (\Omega \cup \partial\Omega, E)$, with regards to the diagonal matrix of susceptibility coefficients of nodes/persons $\mathbf{S}$, under the stimulation of the persistent sources of the boundary $\partial\Omega$. Clearly, the equilibrium opinions are:

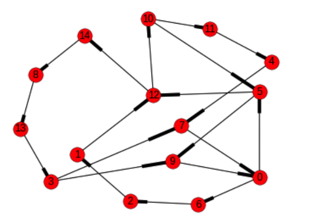
$$\overline{\mathbf{x}}(\partial\Omega; \mathbf{S}) = (\mathbf{I} - \mathbf{S}\,\mathbf{N})^{-1}\,(\mathbf{I} - \mathbf{S})\,\psi, \text{ on } \Omega,$$

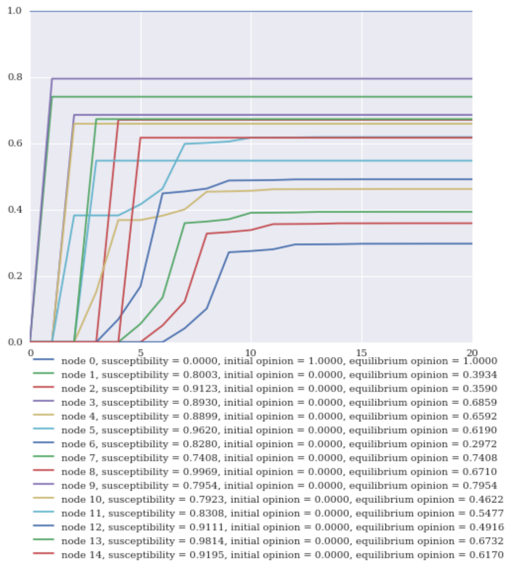
  where the support of ψ is only nodes adjacent to $\partial\Omega$.
- If $s_i = 1$, for every $i \in \Omega$, i.e., $\mathbf{S} = \mathbf{1}$, on Ω, then

$$\overline{\mathbf{x}}(\partial\Omega; \mathbf{1}) = \mathbf{1}, \text{ on } \Omega.$$

[1] Notice that the diagonal matrix of susceptibilities $\mathbf{S}$ is such that $s_i = S_{ii} = 0, \forall i \in \partial\Omega$, while $s_i = S_{ii} \in (0, 1], \forall i \in \Omega$. Moreover, as already said, the vector of initial opinions is taken to be $x_i^{(0)} = 1, \forall i \in \partial\Omega$, while $x_i^{(0)} = 0, \forall i \in \Omega$.

An Example of One Source of Persistent Boundary Stimulation





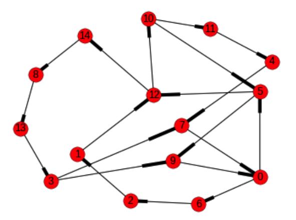
Synchronization from One Source of Persistent Alternating Boundary Stimulation

- In the case that $|\partial\Omega| = 1$, i.e., when there exists a single node/person $j_a \in \partial\Omega$, this node/person j_a is said to be a **source of persistent alternating boundary stimulation**. In this case, the IBVP of the Friedkin–Johnsen social influence system is written as:

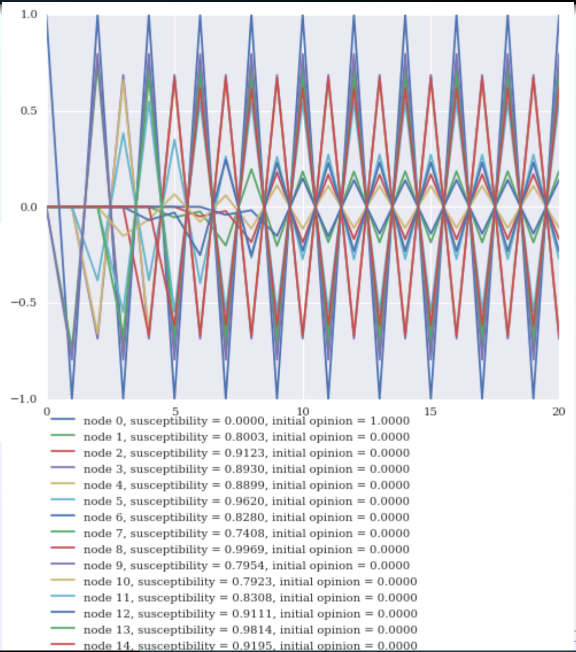
$$\begin{aligned} x_i^{(k+1)} &= s_i\, N\, v_i^k + (1 - s_i)\, x_i^{(0)}, \text{ on } \Omega, k = 0, 1, \ldots, \\ x_{j_a}^{(k)} &= 1, k = 0, 2, 4, \ldots, \\ x_{j_a}^{(k)} &= -1, k = 1, 3, 5, \ldots, \\ x_i^{(0)} &= 0, \text{ on } \Omega. \end{aligned}$$

- Notice that the opinion of such persistent source of boundary stimulation j_a keeps on oscillating between $+1$ and -1 without being influenced at all by its neighbors (although, as a source, it keeps on influencing its neighbors in alternating directions).

Example



10
11
14
4
8
12
5
13
7
1
9
3
0
2
6



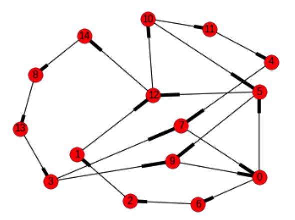
1.0
0.5
0.0
−0.5
−1.0
0
5
10
15
20
node 0, susceptibility = 0.0000, initial opinion = 1.0000
node 1, susceptibility = 0.8003, initial opinion = 0.0000
node 2, susceptibility = 0.9123, initial opinion = 0.0000
node 3, susceptibility = 0.8930, initial opinion = 0.0000
node 4, susceptibility = 0.8899, initial opinion = 0.0000
node 5, susceptibility = 0.9620, initial opinion = 0.0000
node 6, susceptibility = 0.8280, initial opinion = 0.0000
node 7, susceptibility = 0.7408, initial opinion = 0.0000
node 8, susceptibility = 0.9969, initial opinion = 0.0000
node 9, susceptibility = 0.7954, initial opinion = 0.0000
node 10, susceptibility = 0.7923, initial opinion = 0.0000
node 11, susceptibility = 0.8308, initial opinion = 0.0000
node 12, susceptibility = 0.9111, initial opinion = 0.0000
node 13, susceptibility = 0.9814, initial opinion = 0.0000
node 14, susceptibility = 0.9195, initial opinion = 0.0000

Synchronization from Two Sources of Persistent Alternating Boundary Stimulation

- In the case that $|\partial\Omega| = 2$, i.e., when there exist exactly two nodes/persons $j_a, j_b \in \partial\Omega$, the pair of nodes/persons $j_a, j_b \in \partial\Omega$ is said to be a **pair of sources of persistent alternating boundary stimulations**. In this case, the IBVP of the Friedkin–Johnsen social influence system is written as:

$$
\begin{aligned}
x_i^{(k+1)} &= s_i\, N\, v_i^k + (1 - s_i)\, x_i^{(0)}, \text{ on } \Omega, k = 0, 1, \ldots, \\
x_{j_a}^{(k)} &= 1, k = 0, 2, 4, \ldots, \\
x_{j_a}^{(k)} &= -1, k = 1, 3, 5, \ldots, \\
x_{j_b}^{(k)} &= -1, k = 0, 2, 4, \ldots, \\
x_{j_b}^{(k)} &= 1, k = 1, 3, 5, \ldots, \\
x_i^{(0)} &= 0, \text{ on } \Omega.
\end{aligned}
$$

Example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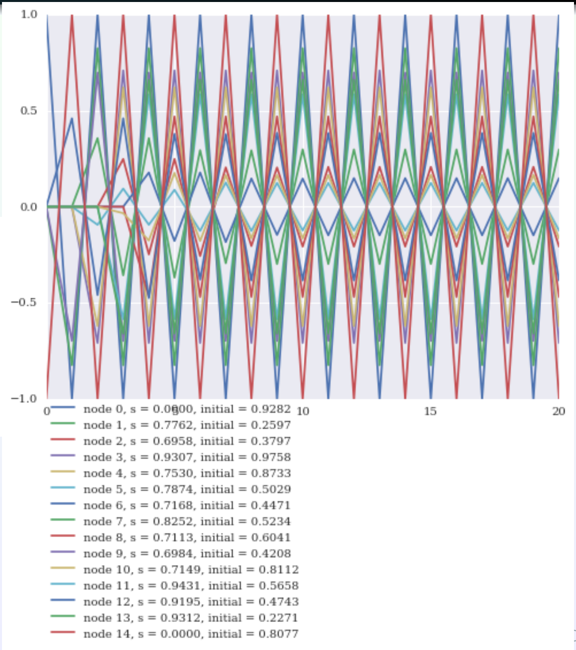

1.0
0.5
0.0
−0.5
−1.0
0
5
10
15
20
node 0, s = 0.0000, initial = 0.9282
node 1, s = 0.7762, initial = 0.2597
node 2, s = 0.6958, initial = 0.3797
node 3, s = 0.9307, initial = 0.9758
node 4, s = 0.7530, initial = 0.8733
node 5, s = 0.7874, initial = 0.5029
node 6, s = 0.7168, initial = 0.4471
node 7, s = 0.8252, initial = 0.5234
node 8, s = 0.7113, initial = 0.6041
node 9, s = 0.6984, initial = 0.4208
node 10, s = 0.7149, initial = 0.8112
node 11, s = 0.9431, initial = 0.5658
node 12, s = 0.9195, initial = 0.4743
node 13, s = 0.9312, initial = 0.2271
node 14, s = 0.0000, initial = 0.8077

Influenciability

- Given a Friedkin–Johnsen social influence system on graph $G = (V, E)$, with regards to the diagonal matrix of susceptibility coefficients $\mathbf{S}$ (where $s_i = S_{ii} \in (0, 1], \forall i \in V$), the **influenciability matrix** is the $(n \times n)$ matrix $\boldsymbol{\mathcal{U}}^\infty = \boldsymbol{\mathcal{U}}^\infty(\mathbf{S}) = \{\mathcal{U}_{ij}^\infty(\mathbf{S})\}_{i,j \in V}$, with entries:

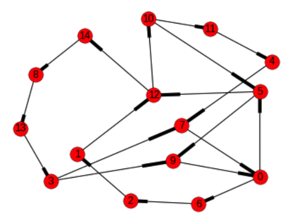
$$\mathcal{U}_{ij}^\infty(\mathbf{S}) = \begin{cases} 1, & i = j, \\ \overline{v}_j(i; \mathbf{S}^{[i]}), & \text{for } i \neq j, \end{cases}$$

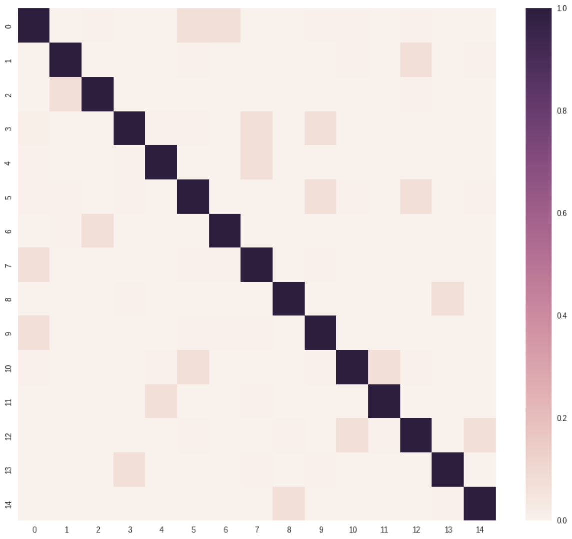
  where

$$\overline{v}_j(i; \mathbf{S}^{[i]}) = \overline{v}_j(\partial\Omega = \{i\}; \mathbf{S}^{[i]})$$

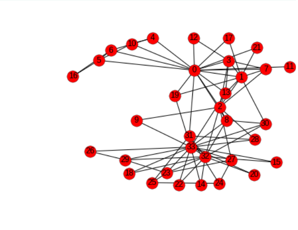
  and $\mathbf{S}^{[i]}$ is a diagonal matrix such that $S_{kk}^{[i]} = s_k, \forall k \neq i$, $S_{ii}^{[i]} = 0$.
- In other words, the entry $\mathcal{U}_{ij}^\infty(\mathbf{S})$ is the value of the (steady) equilibrium opinion on j, when there is a single source of persistent boundary stimulation at i, provided, of course, that the coefficient of susceptibility of this source is $s_i = 0$, while the coefficients of susceptibilities of all other nodes/persons are given my the corresponding diagonal elements of $\mathbf{S}$.

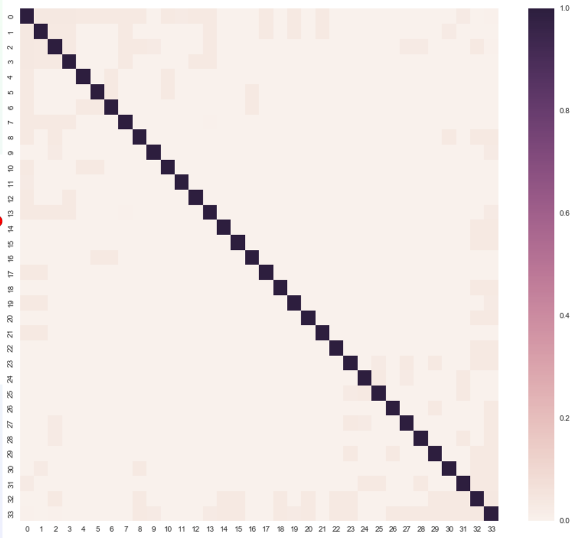
The Influenciability Matrix of the Example with $s_i = c_{\text{degree}}(i)$





The Influenciability Matrix of the Karate Network with $s_i = c_{\text{degree}}(i)$





Influence–Degree Centrality

- Given a Friedkin–Johnsen social influence system on graph $G = (V, E)$, with regards to the diagonal matrix of susceptibility coefficients $\mathbf{S}$, the ndex of **influence–degree centrality** of node $i \in V$ is defined as follows:

$$\begin{aligned} c_{\text{influence-degree}}(i; \mathbf{S}) &= \frac{1}{n-1} \sum_{j \sim i} \mathcal{U}^{\infty}_{i,j}(\mathbf{S}) \\ &= \frac{1}{n-1} \sum_{j \sim i} \overline{v}_j(i; \mathbf{S}). \end{aligned}$$

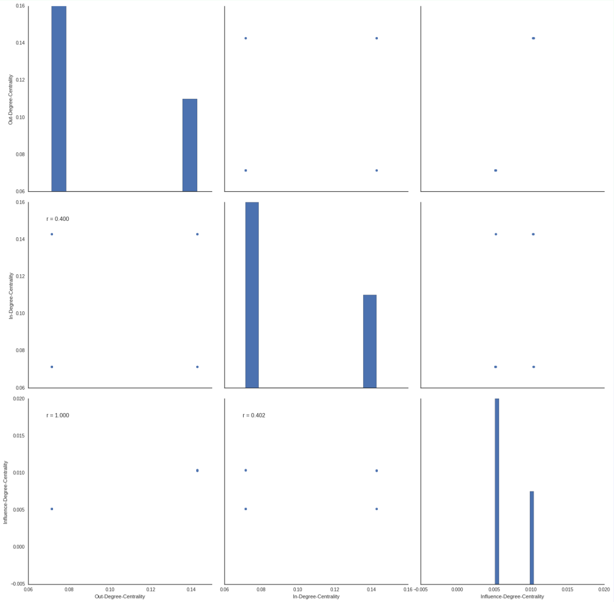
- Clearly, in an undirected graph,[2] one gets:

$$c_{\text{influence-degree}}(i; \mathbf{1}) = \frac{1}{n-1} \deg_i = c_{\text{degree}}(i),$$

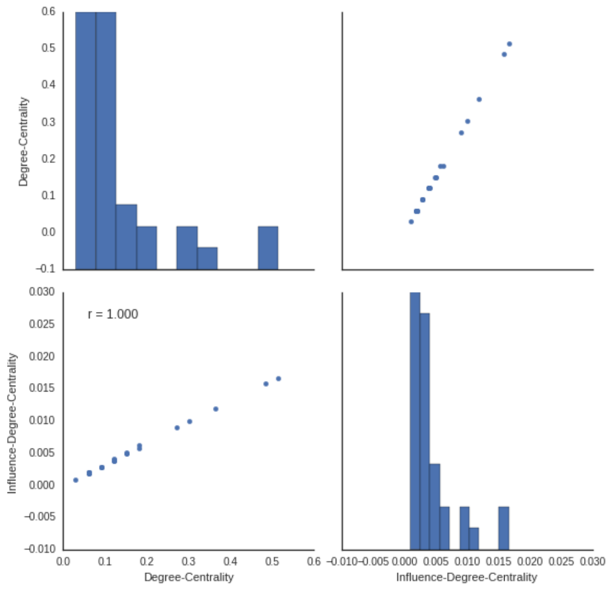
where $c_{\text{degree}}(i)$ is the index of **degree centrality** of node $i \in V$.

[2] The usual modifications are needed for a directed graph.

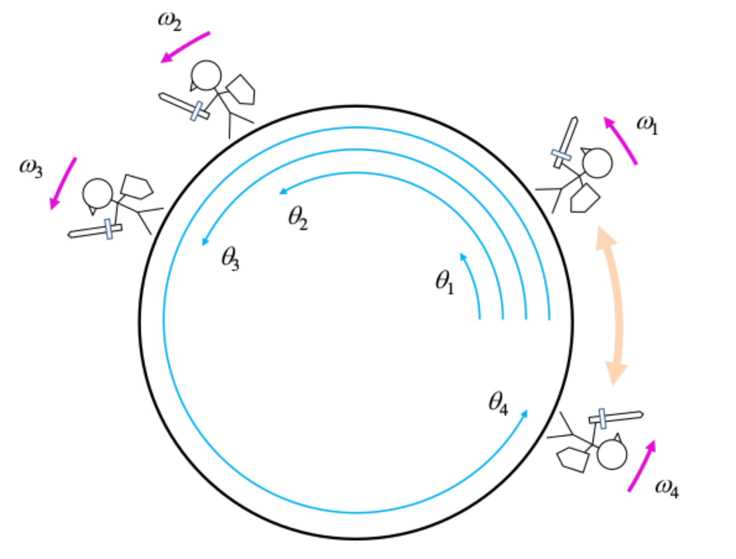
Pearson Correlation between Degree Centrality and Influence–Degree Centrality of the Example when $s_i = c_{degree}(i)$



Pearson Correlation between Degree Centrality and Influence–Degree Centrality of the Karate graph when $s_i = c_{degree}(i)$



Kuramoto Synchronization



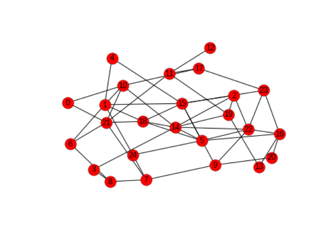
The Kuramoto Model of Synchronization of Coupled Oscillators on a Graph

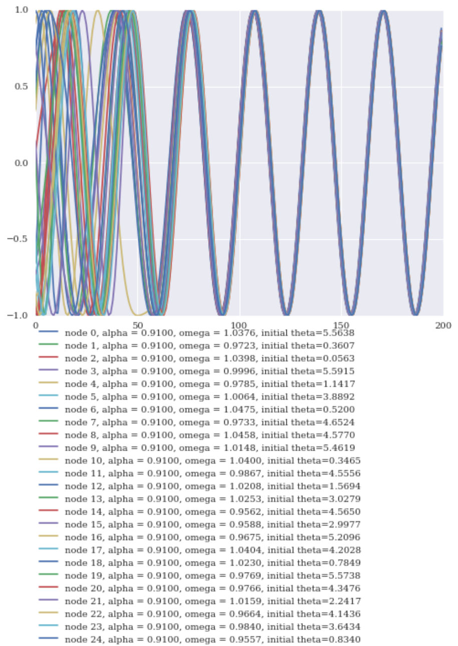
- n **oscillators** are placed on the nodes of graph $G = (V, E)$. Each oscillator $i \in V$ is represented as a marching soldier on a circular track placed at **angle** θ_i and having his/her own preferred **angular speed** ω_i.
- As the oscillators are considered **coupled** along the graph edges, the **Kuramoto model** assumes that the angular position θ_i of each oscillator (or marching soldier) depends on the angular position of his/her neighbors according to the following iterative scheme (at each time step):

$$\begin{aligned} \theta_i^{(k+1)} - \theta_i^{(k)} &= \omega_i + \alpha \frac{\sum_{j \sim i} \sin(\theta_j^{(k)} - \theta_i^{(k)})}{\deg_i}, \quad V, k = 0, 1, \ldots, \\ \theta_i^{(0)} &= \phi_i, \text{ on } V, \end{aligned}$$

where α is the coupling parameter (typically in $(0, 1]$) and ϕ_i's are the initial (angular) positions of the oscillators.

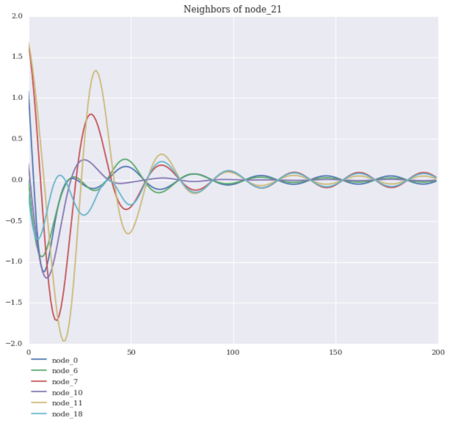
Example

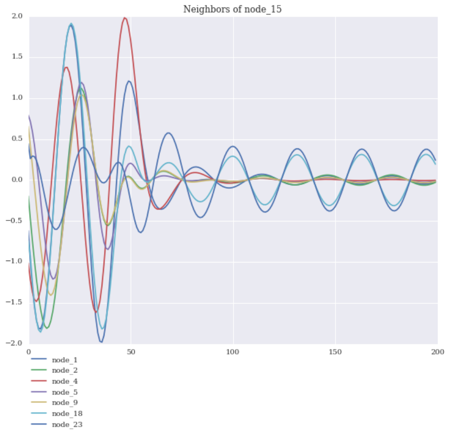



1.0
0.5
0.0
−0.5
−1.0
0
50
100
150
200
node 0, alpha = 0.9100, omega = 1.0376, initial theta=5.5638
node 1, alpha = 0.9100, omega = 0.9723, initial theta=0.3607
node 2, alpha = 0.9100, omega = 1.0398, initial theta=0.0563
node 3, alpha = 0.9100, omega = 0.9996, initial theta=5.5915
node 4, alpha = 0.9100, omega = 0.9785, initial theta=1.1417
node 5, alpha = 0.9100, omega = 1.0064, initial theta=3.8892
node 6, alpha = 0.9100, omega = 1.0475, initial theta=0.5200
node 7, alpha = 0.9100, omega = 0.9733, initial theta=4.6524
node 8, alpha = 0.9100, omega = 1.0458, initial theta=4.5770
node 9, alpha = 0.9100, omega = 1.0148, initial theta=5.4619
node 10, alpha = 0.9100, omega = 1.0400, initial theta=0.3465
node 11, alpha = 0.9100, omega = 0.9867, initial theta=4.5556
node 12, alpha = 0.9100, omega = 1.0208, initial theta=1.5694
node 13, alpha = 0.9100, omega = 1.0253, initial theta=3.0279
node 14, alpha = 0.9100, omega = 0.9562, initial theta=4.5650
node 15, alpha = 0.9100, omega = 0.9588, initial theta=2.9977
node 16, alpha = 0.9100, omega = 0.9675, initial theta=5.2096
node 17, alpha = 0.9100, omega = 1.0404, initial theta=4.2028
node 18, alpha = 0.9100, omega = 1.0230, initial theta=0.7849
node 19, alpha = 0.9100, omega = 0.9769, initial theta=5.5738
node 20, alpha = 0.9100, omega = 0.9766, initial theta=4.3476
node 21, alpha = 0.9100, omega = 1.0159, initial theta=2.2417
node 22, alpha = 0.9100, omega = 0.9664, initial theta=4.1436
node 23, alpha = 0.9100, omega = 0.9840, initial theta=3.6434
node 24, alpha = 0.9100, omega = 0.9557, initial theta=0.8340

Example (contin.)

- Synchronization between two nodes and their neighbors (by plotting $\theta_i^{(k)} - \theta_j^{(k)}$, for $j \sim i$):





Kuramoto–Friedkin–Johnsen “Synchronization”

- Writing the Friedkin–Johnsen model in angular space, we get the following system of equations:

$$\begin{aligned}\theta_i^{(k+1)} - \theta_i^{(k)} &= \omega_i + \frac{1}{\deg_i} \sum_{j \sim i} \left[s_i(\theta_j^{(k)} - \theta_i^{(k)}) + \right. \\ &\qquad \left. (1 - s_i)(\theta_i^{(0)} - \theta_i^{(k)}) \right], \text{ on } V, k = 0, 1, \ldots, \\ \theta_i^{(0)} &= \phi_i, \text{ on } V.\end{aligned}$$

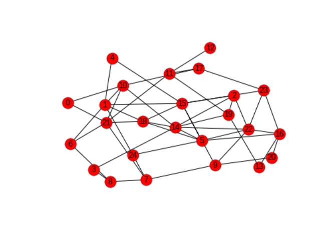
- Thus, denoting

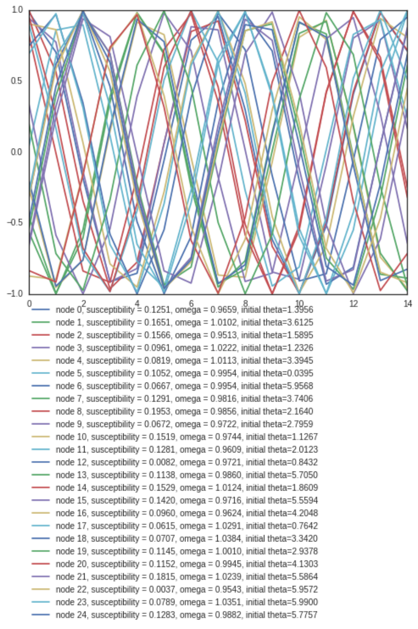
$$S(\theta_i^{(k)}) = \sin \left(\sum_{j \sim i} \left[s_i(\theta_j^{(k)} - \theta_i^{(k)}) + (1 - s_i)(\theta_i^{(0)} - \theta_i^{(k)}) \right] \right),$$

the equations of the **Kuramoto–Friedkin–Johnsen model** become:

$$\begin{aligned}\theta_i^{(k+1)} - \theta_i^{(k)} &= \omega_i + \frac{S(\theta_i^{(k)})}{\deg_i}, \text{ on } V, k = 0, 1, \ldots, \\ \theta_i^{(0)} &= \phi_i, \text{ on } V.\end{aligned}$$

Example

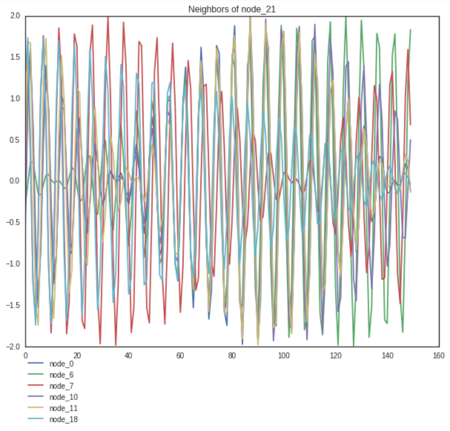


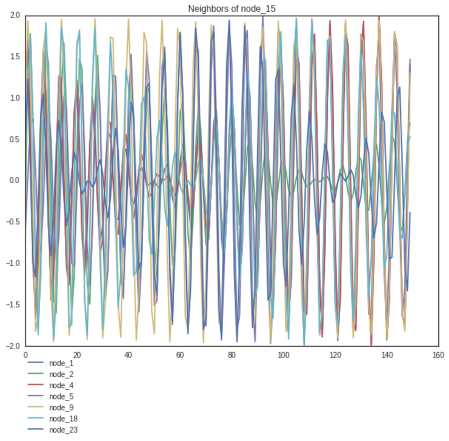


1.0
0.5
0.0
−0.5
−1.0
0
2
4
6
8
10
12
14
node 0, susceptibility = 0.1251, omega = 0.9659, initial theta=1.3956
node 1, susceptibility = 0.1651, omega = 1.0102, initial theta=3.6125
node 2, susceptibility = 0.1566, omega = 0.9513, initial theta=1.5895
node 3, susceptibility = 0.0961, omega = 1.0222, initial theta=1.2326
node 4, susceptibility = 0.0819, omega = 1.0113, initial theta=3.3945
node 5, susceptibility = 0.1052, omega = 0.9954, initial theta=0.0395
node 6, susceptibility = 0.0667, omega = 0.9954, initial theta=5.9568
node 7, susceptibility = 0.1291, omega = 0.9816, initial theta=3.7406
node 8, susceptibility = 0.1953, omega = 0.9856, initial theta=2.1640
node 9, susceptibility = 0.0672, omega = 0.9722, initial theta=2.7959
node 10, susceptibility = 0.1519, omega = 0.9744, initial theta=1.1267
node 11, susceptibility = 0.1281, omega = 0.9609, initial theta=2.0123
node 12, susceptibility = 0.0082, omega = 0.9721, initial theta=0.8432
node 13, susceptibility = 0.1138, omega = 0.9860, initial theta=5.7050
node 14, susceptibility = 0.1529, omega = 1.0124, initial theta=1.8609
node 15, susceptibility = 0.1420, omega = 0.9716, initial theta=5.5594
node 16, susceptibility = 0.0960, omega = 0.9624, initial theta=4.2048
node 17, susceptibility = 0.0615, omega = 1.0291, initial theta=0.7642
node 18, susceptibility = 0.0707, omega = 1.0384, initial theta=3.3420
node 19, susceptibility = 0.1145, omega = 1.0010, initial theta=2.9378
node 20, susceptibility = 0.1152, omega = 0.9945, initial theta=4.1303
node 21, susceptibility = 0.1815, omega = 1.0239, initial theta=5.5864
node 22, susceptibility = 0.0037, omega = 0.9543, initial theta=5.9572
node 23, susceptibility = 0.0789, omega = 1.0351, initial theta=5.9900
node 24, susceptibility = 0.1283, omega = 0.9882, initial theta=5.7757

Example (contin.)

- "Synchronization," exhibiting *phase offsets*, between two nodes and their neighbors (by plotting $\theta_i^{(k)} - \theta_j^{(k)}$, for $j \sim i$):





Friedkin–Johnsen Social Influence on a Signed Graph

- Let $G = (V, E)$ be a **signed graph** (undirected or directed), which is assumed to be (**strongly**) **connected**. Of course, now, each edge has a **sign**, which is either *positive* $(+)$ or *negative* $(-)$.
- Given two nodes $i, j \in V$, writing $i \overset{+}{\sim} j$ or $i \overset{-}{\sim} j$, we mean that i, j are adjacent, when G is undirected, or that j is an outgoing neighbor of i, when G is directed, and that in both cases the sign of (i, j) is either positive or regative (respectively).
- Thus, the nonzero entries of the adjacency matrix $\mathbf{A}$ are now either $+1$ or -1 (i.e., $A_{ij} = +1 \iff i \overset{+}{\sim} j$ and $A_{ij} = -1 \iff i \overset{-}{\sim} j$).
- Then, the system of equations of the **Friedkin–Johnsen Social Influence Model** is:

$$
\begin{aligned}
x_i^{(k+1)} &= \frac{s_i}{\deg_i} \sum_{j \in V} A_{ij} x_j^{(k)} + (1 - s_i)\, x_i^{(0)}, \\
&= \frac{s_i}{\deg_i} \Big(\sum_{i \overset{+}{\sim} j} x_j^{(k)} - \sum_{i \overset{-}{\sim} j} x_j^{(k)} \Big) + (1 - s_i)\, x_i^{(0)}.
\end{aligned}
$$

Remarks on the Signed Friedkin–Johnsen Social Influence Model

- Now, the matrix $\mathbf{N} = \mathbf{D}^{-1}\mathbf{A}$ is *not* row–stochastic. However, the matrix of absolute values of its entries, $|\mathbf{N}| = \mathbf{D}^{-1}|\mathbf{A}|$, *is* row–stochastic.
- Let $\rho = \rho(\mathbf{N})$ be the **spectral radius** of the (square, $n \times n$) matrix $\mathbf{N}$. This means that $\rho = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$, where $\lambda_1, \ldots, \lambda_n$ are the (real or complex) eigenvalues of $\mathbf{N}$.
- Let $s_{\max} = \max\{s_1, \ldots, s_n\}, s_{\min} = \min\{s_1, \ldots, s_n\}$ be the maximum and the minimum susceptibilities (respectively) among the susceptibility coefficients of all nodes of G. Without loss of generality, Iwe are assuming:

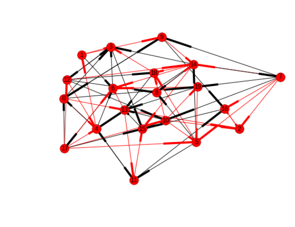
$$0 < s_{\min} \leq s_{\max} \leq 1.$$

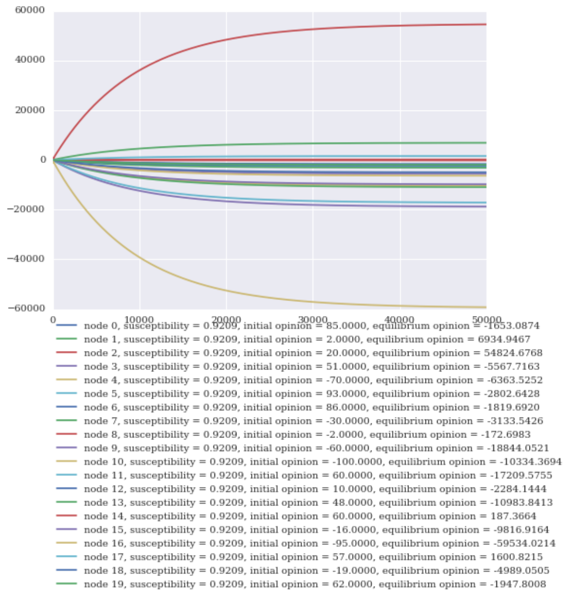
The Stability of the Signed Friedkin–Johnsen Social Influence Model

- If $s_{\max}\,\rho < 1$, then all the solutions of the signed Friedkin–Johnsen model *converge* to a (steady state) equilibrium vector of opinions $\bar{\mathbf{x}} = (\mathbf{I} - \mathbf{S}\,\mathbf{N})^{-1}(\mathbf{I} - \mathbf{S})\,\mathbf{x}^{(0)}$.
- If $s_{\min}\,\rho \geq 1$, then all the solutions of the signed Friedkin–Johnsen model *diverge*.
- If $s_{\min} = s_{\max} = \rho = 1$, then all the solutions of the signed Friedkin–Johnsen model *oscillate*.

An Example of Covergence

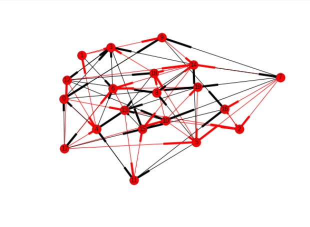
$\rho = 1.0858$

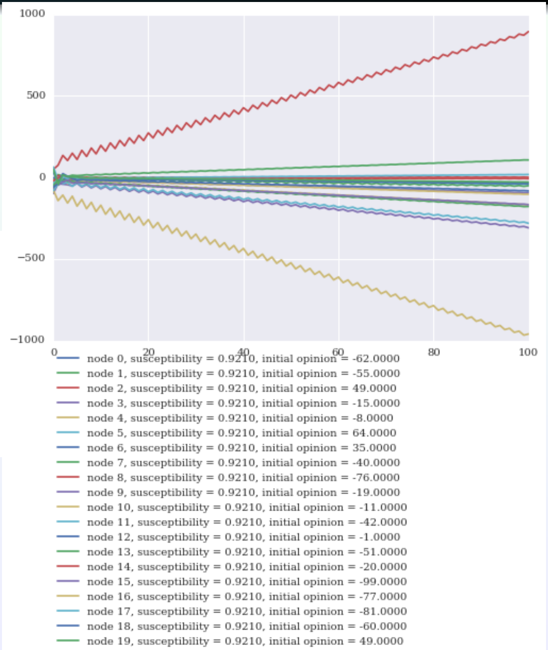




An Example of Divergence

$\rho = 1.0858$





An Example of Oscillation

$\rho = 1$