Slides of Discrete Mathematics based on Susanna Epp's Textbook

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Chapter 6

Set Theory

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6.1 Subsets I

Definition

- $A \subseteq B \iff \forall x,$ if $x \in A,$ then $x \in B.$
- $A \not\subseteq B \iff \exists x$ such that $x \in A$ and $x \notin B.$
- A is a **proper subset** of B, written as $A \subset B$, $\iff$
  1. $A \subseteq B$ and
  2. there is at least one element in B that is not in A.

Proving That One Set Is a Subset of Another

Let X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrary chosen element of X,
2. **show** that x is an element of Y.

Definition

Given sets A and B, A **equals** B, written $A = B$, if and only if every element of A is in B and every element of B is in A. Symbolically:

$$A = B \iff A \subseteq B \text{ and } B \subseteq A.$$

Exercise 6.1.1(b)

If $A = \{3, \sqrt{5^2 - 4^2}, 24 \bmod 7\}, B = \{8 \bmod 5\}$, how are sets A, B related?

Compute $\sqrt{5^2 - 4^2}, 24 \bmod 7, 8 \bmod 5$ and find the elements of A, B. Notice that repeated elements do not count.

Exercise 6.1.7 (a) and (b)

Let $A = \{x \in \mathbb{Z} \mid x = 6a + 4,\ \text{for } a \in \mathbb{Z}\}, B = \{y \in \mathbb{Z} \mid x = 18b - 2,\ \text{for } b \in \mathbb{Z}\}$. Prove or disprove (a) $A \subseteq B$, (b) $B \subseteq A$.

First, find the integers a, b for which $x = y$. Is this happenning for all $a, b \in \mathbb{Z}$? If it is not, there exist $a, b \in \mathbb{Z}$ such that $x = 6a + b \neq 18b + 2 = y$. Which are these pairs of $a, b \in \mathbb{Z}$? Let us denote them as the set $D = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid 6a + b \neq 18b + 2\}$. After we find the elements of D, we have the following cases:

1. if, for every $a \in \mathbb{Z}$, there exist $b \in \mathbb{Z}$ such that $(a, b) \in D$, then $A \subseteq B$;
2. if there exist $a \in \mathbb{Z}$, for which there exist no $b \in \mathbb{Z}$ such that $(a, b) \in D$, then $A \nsubseteq B$;
3. if, for every $b \in \mathbb{Z}$, there exist $a \in \mathbb{Z}$ such that $(a, b) \in D$, then $B \subseteq A$;
4. if there exist $b \in \mathbb{Z}$, for which there exist no $a \in \mathbb{Z}$ such that $(a, b) \in D$, then $B \nsubseteq A$.

To which case do (a) and (b) correspond?

For instance, the relationship $A \subseteq B$ can be pictured in one of two ways, as shown in Figure 6.1.1.

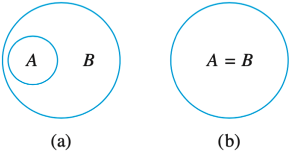


Figure 6.1.1 $A \subseteq B$

The relationship $A \nsubseteq B$ can be represented in three different ways with Venn diagrams, as shown in Figure 6.1.2.

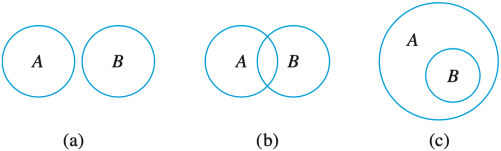


Figure 6.1.2 $A \nsubseteq B$

Exercise 6.1.14(b)

Draw the Venn diagrams for three sets A, B, C such that $C \subseteq A, B \cap C = \varnothing$.

Definition

Let A, B subsets of a universal set U.

- The **union** of A and B, denoted $A \cup B$, is the set of all elements of U that are in A or/and in B.
- The **intersection** of A and B, denoted $A \cap B$, is the set of all elements of U that are both in A and in B.
- The **difference** of A minus B, denoted $A - B$, is the set of all elements of U that are in B but not in A.
- The **complement** of A, denoted A^c, is the set of all elements of U that are not in A.

Proposition

$$A - B = A \cap B^c$$

Exercise 6.1.16

Let $A = \{a, b, c\}, B = \{b, c, d\}$ and $C = \{b, c, e\}$. Find $(A - B) - C$ and $A - (B - C)$. Are they equal?

Definition

The **empty set** is the unique set $\{\} = \varnothing$ with no members.

Definition

Two sets A, B are called **disjoint** if they have no elements in common, i.e., if and only if $A \cap B = \varnothing$.

6.1 Operations on Sets III

Definition

A finite or infinite number of sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if and only if any pair of two different sets is disjoint, i.e., if and only if, for all $i, j = 1, 2, 3, \ldots, A_i \cap A_j = \varnothing$, whenever $i \neq j$.

Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is **partition** of a set A if and only if:

1. A is the union of all the A_i, written $A = \bigcup_{i=1,2,\ldots} A_i$, and
2. the sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

Definition

Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times A_3 \times \cdots \times A_n$, is the set of all ordered n–tuples $\{(a_1, a_2, \ldots, a_n)\}$, where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$. Symbolically:

$$A_1 \times A_2 \times A_3 \times \cdots \times A_n = \{(a_1, \ldots, a_n) | a_i \in A_i, i = 1, \ldots, n\}.$$

In particular, the Cartesian product of A, B is:

$$A \times B = \{(a, b) | a \in A, b \in B\}.$$

Exercise 6.1.35 (c) and (d)

Let $A = \{a, b\}, B = \{1, 2\}$ and $C = \{2, 3\}$. Find $A \times (B \cap C)$ and $(A \times B) \cap (A \times C)$.

Theorem (Some Subset Relations)

1. *Inclusion of Intersection:* $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$.
2. *Inclusion in Union:* $A \subseteq A \cup B$ *and* $B \subseteq A \cup B$.
3. *Transitive Property of Sets: If* $A \subseteq B$ *and* $B \subseteq C$*, then* $A \subseteq C$.

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

6.2 Properies of Sets II

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

Exercise 6.2.10

$$(A - B) \cap (C - B) = (A \cap C) - B.$$

First, we'll show that $(A - B) \cap (C - B) \subseteq (A \cap C) - B$.

Let $x \in (A - B) \cap (C - B)$. Thus, by definition of intersection, $x \in A - B$ and $x \in C - B$ and, then, by definition of difference, $x \in A$ and $x \notin B$ and $x \in C$ and $x \notin B$. Hence, $x \in A \cap C$ (**why?**), which implies that $x \in (A \cap B) - B$ (**why?**).

Next, we'll show that $(A \cap C) - B \subseteq (A - B) \cap (C - B)$. (Why do these two inclusions suffice?)

Let $x \in (A \cap C) - B$. Then, by definition of difference, $x \in A \cap C$ and $x \notin B$, which also implies that $x \in A$ and $x \in C$ (**why?**). But then $x \in A - B$ and $x \in C - B$ (**why?**), which is what we wanted to show (**why?**).

6.2 Properies of Sets IV

Exercise 6.2.19

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

First, we'll show that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

Let $(x, y) \in A \times (B \cap C)$. Thus, by definition of Cartesian product, $x \in A$ and $y \in B$ and $y \in C$ (**why?**). But this means that both statements "$x \in A$ and $y \in B$" and "$x \in A$ and $y \in C$" are true. Therefore (**why?**), $(x, y) \in A \times B$ and $(x, y) \in A \times C$ and thus (**why?**), $(x, y) \in (A \times B) \cap (A \times C)$.

Next, we'll show that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. (Why do these two inclusions suffice?)

Let $(x, y) \in (A \times B) \cap (A \times C)$. Then, by definition of intersection, $(x, y) \in A \times B$ and $(x, y) \in A \times C$, which imply (**why?**) that $x \in A$ and $y \in B$ and $y \in C$. Consequently, the statement "$x \in A$ and both $y \in B$ and $y \in C$" is true, which is translated in saying "$x \in A$ and $y \in B \cap C$" (**why?**). Therefore (**why?**), $(x, y) \in A \times (B \cap C)$.

Exercise 6.2.34

$$\text{If } B \cap C \subseteq A, \text{ then } (C - A) \cap (B - A) = \varnothing.$$

Suppose the opposite is true: $(C-A)\cap(B-A) \neq \varnothing$. This means that there exists a $x \in (C - A) \cap (B - A)$. Then $x \in C$ and $x \notin A$ and $x \in B$ (**why?**). Consequently, $x \in B \cap C$ (**why?**), but, since by hypothesis $B \cap C \subseteq A$, then we get that $x \in A$, which contradicts the previous finding that $x \notin A$.

Lemma

$$(A \times B) \cap (A \times C) = A \times (B \cap C).$$

$(A \times B) \cap (A \times C) = \{(x, y) \mid x \in A, y \in B\} \cap \{(x, y) \mid x \in A, y \in C\} =$
$\{(x, y) \mid x \in A, y \in B \textit{ and } y \in C\} = \{(x, y) \mid x \in A, y \in B \cap C\} = A \times (B \cap C).$

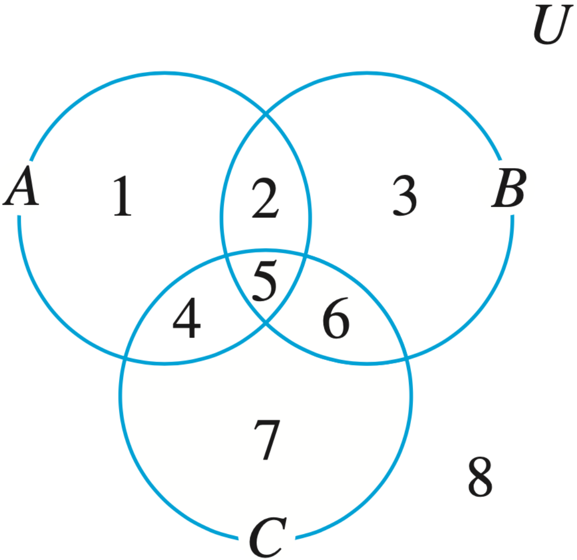
Exercise 6.2.41

For all integers $n \geq 1$, if A and $B_1, B_2, B_3, \ldots$ are any sets, then

$$\bigcap_{i=1}^{n}(A \times B_i) = A \times \left(\bigcap_{i=1}^{n} B_i\right).$$

We will prove it by induction. Certainly it is true for $n = 1$ ($A \times B_1 = A \times B_1$). Assume that, for some $k \geq 1$, $\bigcap_{i=1}^{k}(A \times B_i) = A \times \left(\bigcap_{i=1}^{k} B_i\right)$. Then $\bigcap_{i=1}^{k+1}(A \times B_i) = \left(\bigcap_{i=1}^{k}(A \times B_i)\right) \cap (A \times B_{k+1})$. Hence, by the inductive hypothesis and applying the previous Lemma, $\bigcap_{i=1}^{k+1}(A \times B_i) = \left(A \times \left(\bigcap_{i=1}^{k} B_i\right)\right) \cap (A \times B_{k+1}) = A \times \left(\left(\bigcap_{i=1}^{k} B_i\right) \cap B_{k+1}\right) = A \times \left(\bigcap_{i=1}^{k+1} B_i\right)$.

6.3 Venn Diagram for Three Sets



6.3 Counterexamples

Exercise 6.3.4

Find a counterexample to show that this is false:

$$\text{If } B \cap C \subseteq A, \text{ then } (A - B) \cap (A - C) = \varnothing.$$

Using the Venn diagramm for three sets, let us consider an example such that $B \cap C = \varnothing$. Notice that then the condition $B \cap C \subseteq A$ is satisfied, because $B \cap C = \varnothing \subseteq A$. Such an example would be when $A = \{1, 2, 3\}$, $B = \{2\}$ and $C = \{1\}$. But then $A - B = \{1, 3\}$ (**why?**) and $A - C = \{1, 2\}$ (**why?**) and, thus, $(A - B) \cap (A - C) \neq \varnothing$ (**why?**).

Theorem

For all integers $n \geq 0$, if a set X has n elements, then its power set $\mathscr{P}(X)$ has 2^n elements.

Exercise 6.3.19

$$\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B).$$

Let $X \in \mathscr{P}(A) \cup \mathscr{P}(B)$. Then $X \in \mathscr{P}(A)$ or $X \in \mathscr{P}(B)$. In the former case, $X \subseteq A$ and, thus, $X \subseteq A \cup B$. What about in the latter case? Then, what will we get in either case and what does it mean?

Exercise 6.3.23

Let $S = \{a, b, c\}$ and, for each integer $i = 0, 1, 2, 3$, let S_i be the set of all subsets of S that have i elements. List the elements in S_0, S_1, S_2, and S_3. Is $\{S_0, S_1, S_2, S_3\}$ a partition of $\mathscr{P}(S)$?

$S_0 = \{\varnothing\}, S_1 = \{\{a\}, \ldots\}, S_2 = \{\{a, b\}, \ldots\}, S_3 = \{\ldots\}$. Fill up these sets and explain whether they form a partition of $\mathscr{P}(S)$.

6.3 “Algebraic” Proofs of Set Identities

Set Identities

(a) Commutative Laws
(b) Assosiative Laws
(c) Distributive Laws
(d) Identity Laws
(e) Complement Laws
(f) Double Complement Law
(g) Idempotend Laws
(h) Universal Bound Laws
(i) De Morgan’s Laws
(j) Absorption Laws
(k) Complements of U and $\varnothing$
(l) Set Difference Law

Exercise 6.3.28

Solution hint: (a) By Set Difference Law, (b) By Set Difference Law, (c) By Commutative Laws, (d) By De Morgan’s Laws, etc. **Fill it up!**

Exercise 6.3.40

Solution hint: (a) By Set Difference Law (used three times), (b) By De Morgan’s Laws, (c) By Commutative Laws, etc. **Fill it up!**

6.3 Symmetric Difference

Definition of **Symmetric Difference**

$$A \triangle B = (A - B) \cup (B - A)$$

Exercise 6.3.46 (d)

Apply the above formula. The final solution should be a set of 4 elements.

Exercise 6.3.52

$$(A \triangle B) \triangle C = A \triangle (B \triangle C)$$

Method 1. Consider $x \in (A \triangle B) \triangle C$ and then show that x is either exactly in one of the sets A, B, C or x is iall three of them. Start with $x \in A \triangle (B \triangle C)$ and show the same thing as previously.

Method 2. By "algebraic" proof (harder).

6.4 Boolean Algebras I

Logical Equivalences	Set Properties
For all statement variables p, q, and r:	For all sets A, B, and C:
a. $p \vee q \equiv q \vee p$ b. $p \wedge q \equiv q \wedge p$	a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$ b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$	a. $A \cup (B \cup C) = A \cup (B \cup C)$ b. $A \cap (B \cap C) = A \cap (B \cap C)$
a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	a. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
a. $p \vee \mathbf{c} \equiv p$ b. $p \wedge \mathbf{t} \equiv p$	a. $A \cup \emptyset = A$ b. $A \cap U = A$
a. $p \vee \sim p \equiv \mathbf{t}$ b. $p \wedge \sim p \equiv \mathbf{c}$	a. $A \cup A^c = U$ b. $A \cap A^c = \emptyset$
$\sim(\sim p) \equiv p$	$(A^c)^c = A$
a. $p \vee p \equiv p$ b. $p \wedge p \equiv p$	a. $A \cup A = A$ b. $A \cap A = A$
a. $p \vee \mathbf{t} \equiv \mathbf{t}$ b. $p \wedge \mathbf{c} \equiv \mathbf{c}$	a. $A \cup U = U$ b. $A \cap \emptyset = \emptyset$
a. $\sim(p \vee q) \equiv \sim p \wedge \sim q$ b. $\sim(p \wedge q) \equiv \sim p \vee \sim q$	a. $(A \cup B)^c = A^c \cap B^c$ b. $(A \cap B)^c = A^c \cup B^c$
a. $p \vee (p \wedge q) \equiv p$ b. $p \wedge (p \vee q) \equiv p$	a. $A \cup (A \cap B) = A$ b. $A \cap (A \cup B) = A$
a. $\sim\mathbf{t} \equiv \mathbf{c}$ b. $\sim\mathbf{c} \equiv \mathbf{t}$	a. $U^c = \emptyset$ b. $\emptyset^c = U$

6.4 Boolean Algebras II

• Definition: Boolean Algebra

A **Boolean algebra** is a set B together with two operations, generally denoted $+$ and $\cdot$, such that for all a and b in B both $a+b$ and $a \cdot b$ are in B and the following properties hold:

1. *Commutative Laws:* For all a and b in B,

$$\text{(a) } a+b=b+a \quad \text{and} \quad \text{(b) } a \cdot b = b \cdot a.$$

2. *Associative Laws:* For all a, b, and c in B,

$$\text{(a) } (a+b)+c=a+(b+c) \quad \text{and} \quad \text{(b) } (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

3. *Distributive Laws:* For all a, b, and c in B,

$$\text{(a) } a+(b \cdot c)=(a+b) \cdot (a+c) \quad \text{and} \quad \text{(b) } a \cdot (b+c) = (a \cdot b)+(a \cdot c).$$

4. *Identity Laws:* There exist distinct elements 0 and 1 in B such that for all a in B,

$$\text{(a) } a+0=a \quad \text{and} \quad \text{(b) } a \cdot 1 = a.$$

5. *Complement Laws:* For each a in B, there exists an element in B, denoted $\overline{a}$ and called the **complement** or **negation** of a, such that

$$\text{(a) } a+\overline{a}=1 \quad \text{and} \quad \text{(b) } a \cdot \overline{a} = 0.$$

6.4 Boolean Algebras III

Theorem 6.4.1 Properties of a Boolean Algebra

Let B be any Boolean algebra.

1. *Uniqueness of the Complement Law:* For all a and x in B, if $a + x = 1$ and $a \cdot x = 0$ then $x = \overline{a}$.
2. *Uniqueness of* 0 *and* 1: If there exists x in B such that $a + x = a$ for all a in B, then $x = 0$, and if there exists y in B such that $a \cdot y = a$ for all a in B, then $y = 1$.
3. *Double Complement Law:* For all $a \in B$, $\overline{(\overline{a})} = a$.
4. *Idempotent Law:* For all $a \in B$,

$$\text{(a) } a + a = a \quad \text{and} \quad \text{(b) } a \cdot a = a.$$

5. *Universal Bound Law:* For all $a \in B$,

$$\text{(a) } a + 1 = 1 \quad \text{and} \quad \text{(b) } a \cdot 0 = 0.$$

6. *De Morgan's Laws:* For all a and $b \in B$,

$$\text{(a) } \overline{a + b} = \overline{a} \cdot \overline{b} \quad \text{and} \quad \text{(b) } \overline{a \cdot b} = \overline{a} + \overline{b}.$$

7. *Absorption Laws:* For all a and $b \in B$,

$$\text{(a) } (a + b) \cdot a = a \quad \text{and} \quad \text{(b) } (a \cdot b) + a = a.$$

8. *Complements of* 0 *and* 1:

$$\text{(a) } \overline{0} = 1 \quad \text{and} \quad \text{(b) } \overline{1} = 0.$$

Exercise 6.4.2

Solution hint: Use the complement and the associative laws for $+$. Which ones when?

Exercise 6.4.10

Solution hint: Use the result of Exercise 6.4.3, the commutative and distributive laws for $+$ and $\cdot$, and the hypothesis. Which ones when?

6.4 Russell's Paradox I

Exercise 6.4.22

Can there exist a book that refers to all those books and only those books that do not refer to themselves? Explain your answer.

Solution hint: The answer is no. To find it, you need to consider two cases. In the case that such a book did not refer to itself, then it would refer to the set of all books that do not refer to themselves. But this is impossible (**why?**). In the case that the book referred to itself, then it would belong to the set of books to which it refers and this set contains only books which do not refer to themselves. Obviously, this is again impossible (**why?**).

Exercise 6.4.25

For any set A, $\mathscr{P}(A) \not\subseteq A$.

Solution hint: Suppose that there exists a set A such that $\mathscr{P}(A) \subseteq A$. Let $B = \{x \in A \mid x \notin x\}$. Then $B \subseteq A$ and, thus, $B \in \mathscr{P}(A)$. Consequently, by what we have assumed in the beginning, i.e., that $\mathscr{P}(A) \subseteq A$, it follows that $B \in A$. Now, either $B \in B$ or $B \notin B$. In the former case, by definition of B, $B \notin B$, but if $B \notin B$, then B satisfies the defining property of B and, so, $B \in B$. Therfeore, both $B \notin B$ and $B \in B$ are true, which is a contradiction.