Key Methods of Hypergraph Analysis
Day 4:
Hypergraph Metrics and Directed Hypergraphs

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instats Seminar

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Hypergraph Centralities

Let $G = (V, E)$ a hypergraph with n vertices and $v \in V$.

- The **closeness centrality** of vertex v is:

$$\text{Closeness}(v) = \frac{n-1}{\sum_{u \neq v} d(u, v)}.$$

- The **harmonic closeness centrality** of vertex v is:

$$\text{Harmonic}(v) = \sum_{u \neq v} \frac{1}{d(u, v)} \cdot \frac{2}{(n-1)(n-2)}.$$

- The **betweenness centrality** of vertex v is:

$$\text{Betweenness}(v) = \frac{1}{\binom{n-1}{2}} \sum_{\substack{s,t \in V \\ s \neq t}} \frac{\sigma_{s,t}(v)}{\sigma_{s,t}},$$

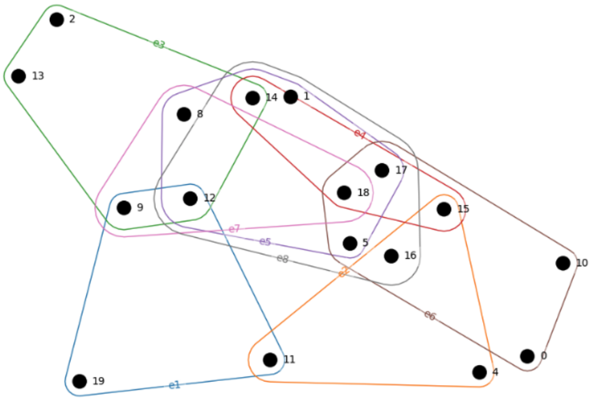
  where $\sigma_{s,t}$ is the total number of shortest paths between vertices s and t, and $\sigma_{s,t}(v)$ is the number of those paths that pass through vertex v

Closeness Centrality

```
{'e1': [9, 11, 12, 19], 'e2': [4, 11, 15, 16], 'e3': [2, 8, 9, 12, 13, 14], 'e4': [14, 15, 17, 18], 'e5': [1, 5, 8,
12, 14, 17], 'e6': [0, 5, 10, 16, 17, 18], 'e7': [8, 9, 18], 'e8': [1, 12, 14, 16, 17, 18]}
```

```
Hyperedge Closeness Centrality:
Closeness centrality of e1: 0.78
Closeness centrality of e2: 0.70
Closeness centrality of e3: 0.78
Closeness centrality of e4: 0.88
Closeness centrality of e5: 0.88
Closeness centrality of e6: 0.78
Closeness centrality of e7: 0.88
Closeness centrality of e8: 1.00

Vertex Closeness Centrality:
Closeness centrality of 0: 0.53
Closeness centrality of 1: 0.64
Closeness centrality of 2: 0.53
Closeness centrality of 4: 0.50
Closeness centrality of 5: 0.70
Closeness centrality of 8: 0.67
Closeness centrality of 9: 0.67
Closeness centrality of 10: 0.53
Closeness centrality of 11: 0.62
Closeness centrality of 12: 0.80
Closeness centrality of 13: 0.53
Closeness centrality of 14: 0.76
Closeness centrality of 15: 0.62
Closeness centrality of 16: 0.76
Closeness centrality of 17: 0.73
Closeness centrality of 18: 0.76
Closeness centrality of 19: 0.52
```

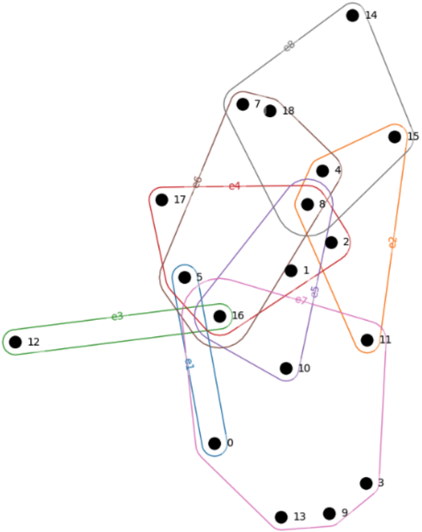


Harmonic Closeness Centrality

```
{'e1': [0, 5], 'e2': [2, 4, 8, 11, 15], 'e3': [12, 16], 'e4': [1, 2, 5, 8, 16, 17], 'e5': [1, 8, 10, 16], 'e6': [4,
5, 7, 16, 18], 'e7': [0, 3, 9, 11, 13, 16], 'e8': [7, 8, 14, 15, 18]}
```

```
Hyperedge Harmonic Closeness Centrality:
Harmonic closeness centrality of e3: 0.26
Harmonic closeness centrality of e8: 0.26
Harmonic closeness centrality of e5: 0.31
Harmonic closeness centrality of e2: 0.29
Harmonic closeness centrality of e7: 0.31
Harmonic closeness centrality of e4: 0.33
Harmonic closeness centrality of e1: 0.24
Harmonic closeness centrality of e6: 0.33

Vertex Harmonic Closeness Centrality:
Harmonic closeness centrality of 0: 0.54
Harmonic closeness centrality of 1: 0.55
Harmonic closeness centrality of 2: 0.60
Harmonic closeness centrality of 3: 0.52
Harmonic closeness centrality of 4: 0.60
Harmonic closeness centrality of 5: 0.62
Harmonic closeness centrality of 7: 0.57
Harmonic closeness centrality of 8: 0.69
Harmonic closeness centrality of 9: 0.52
Harmonic closeness centrality of 10: 0.48
Harmonic closeness centrality of 11: 0.62
Harmonic closeness centrality of 12: 0.41
Harmonic closeness centrality of 13: 0.52
Harmonic closeness centrality of 14: 0.46
Harmonic closeness centrality of 15: 0.56
Harmonic closeness centrality of 16: 0.76
Harmonic closeness centrality of 17: 0.52
Harmonic closeness centrality of 18: 0.57
```

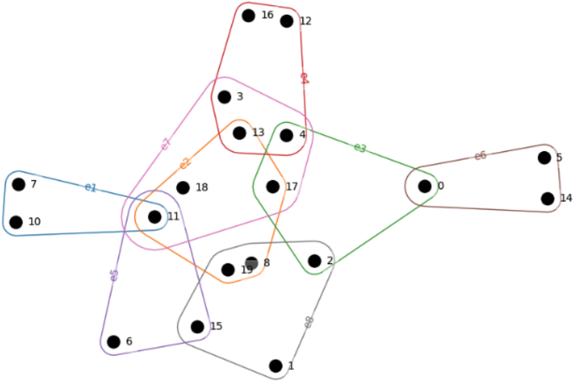


Eccentricity Centrality

```
{'e1': [7, 10, 11], 'e2': [8, 11, 13, 17, 18, 19], 'e3': [0, 2, 4, 17], 'e4': [3, 4, 12, 13, 16], 'e5': [6, 11, 1
5], 'e6': [0, 5, 14], 'e7': [3, 4, 11, 17], 'e8': [1, 2, 8, 15, 19]}
```

```
Hyperedge Eccentricity Centrality:
Eccentricity centrality of e1: 3.00
Eccentricity centrality of e2: 2.00
Eccentricity centrality of e3: 2.00
Eccentricity centrality of e4: 2.00
Eccentricity centrality of e5: 3.00
Eccentricity centrality of e6: 3.00
Eccentricity centrality of e7: 2.00
Eccentricity centrality of e8: 2.00

Vertex Eccentricity Centrality:
Eccentricity centrality of 0: 3.00
Eccentricity centrality of 1: 3.00
Eccentricity centrality of 2: 3.00
Eccentricity centrality of 3: 3.00
Eccentricity centrality of 4: 2.00
Eccentricity centrality of 5: 4.00
Eccentricity centrality of 6: 4.00
Eccentricity centrality of 7: 4.00
Eccentricity centrality of 8: 3.00
Eccentricity centrality of 10: 4.00
Eccentricity centrality of 11: 3.00
Eccentricity centrality of 12: 3.00
Eccentricity centrality of 13: 3.00
Eccentricity centrality of 14: 4.00
Eccentricity centrality of 15: 3.00
Eccentricity centrality of 16: 3.00
Eccentricity centrality of 17: 2.00
Eccentricity centrality of 18: 3.00
Eccentricity centrality of 19: 3.00
```

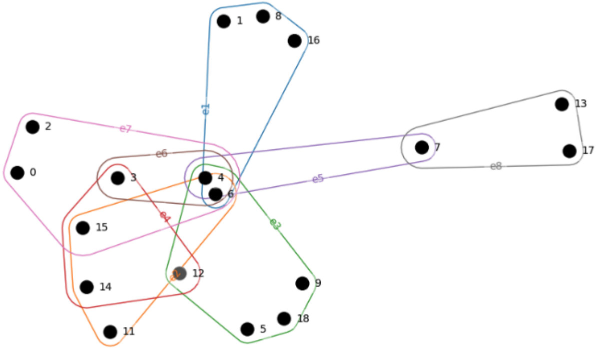


Betweenness Centrality

```
{'e1': [1, 6, 8, 16], 'e2': [6, 11, 14, 15], 'e3': [4, 5, 6, 9, 12, 18], 'e4': [3, 12, 14, 15], 'e5': [4, 7], 'e6':
[3, 4], 'e7': [0, 2, 3, 4, 15], 'e8': [7, 13, 17]}
```

```
Hyperedge Betweenness Centrality:
Betweenness centrality of e1: 0.00
Betweenness centrality of e2: 0.05
Betweenness centrality of e3: 0.29
Betweenness centrality of e4: 0.02
Betweenness centrality of e5: 0.29
Betweenness centrality of e6: 0.03
Betweenness centrality of e7: 0.10
Betweenness centrality of e8: 0.00

Vertex Betweenness Centrality:
Betweenness centrality of 0: 0.00
Betweenness centrality of 1: 0.00
Betweenness centrality of 2: 0.00
Betweenness centrality of 3: 0.02
Betweenness centrality of 4: 0.41
Betweenness centrality of 5: 0.00
Betweenness centrality of 6: 0.37
Betweenness centrality of 7: 0.22
Betweenness centrality of 8: 0.00
Betweenness centrality of 9: 0.00
Betweenness centrality of 11: 0.00
Betweenness centrality of 12: 0.04
Betweenness centrality of 13: 0.00
Betweenness centrality of 14: 0.01
Betweenness centrality of 15: 0.09
Betweenness centrality of 16: 0.00
Betweenness centrality of 17: 0.00
Betweenness centrality of 18: 0.00
```

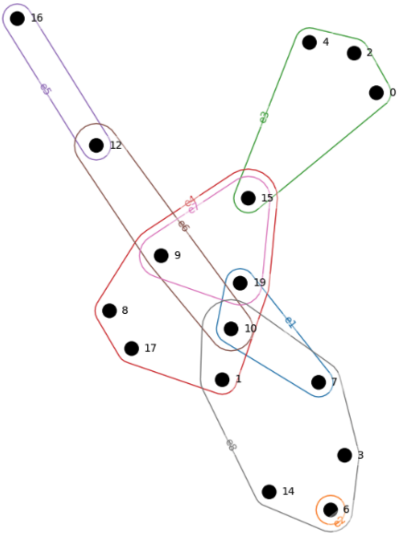


Linear Eigenvector Centrality

```
{'e1': [7, 10, 19], 'e2': [6], 'e3': [0, 2, 4, 15], 'e4': [1, 8, 9, 15, 17, 19], 'e5': [12, 16], 'e6': [9, 10, 12],
'e7': [9, 15, 19], 'e8': [1, 3, 6, 7, 10, 14]}
```

```
T&H Linear Eigenvector Centrality (on hyperedges):
Linear eigenvector centrality of e1: 0.3268
Linear eigenvector centrality of e2: 0.0498
Linear eigenvector centrality of e3: 0.1842
Linear eigenvector centrality of e4: 0.6519
Linear eigenvector centrality of e5: 0.0355
Linear eigenvector centrality of e6: 0.2752
Linear eigenvector centrality of e7: 0.4065
Linear eigenvector centrality of e8: 0.4354

T&H Linear Eigenvector Centrality (on vertices):
Linear eigenvector centrality of 0: 0.0590
Linear eigenvector centrality of 1: 0.3483
Linear eigenvector centrality of 2: 0.0590
Linear eigenvector centrality of 3: 0.1395
Linear eigenvector centrality of 4: 0.0590
Linear eigenvector centrality of 6: 0.1554
Linear eigenvector centrality of 7: 0.2442
Linear eigenvector centrality of 8: 0.2088
Linear eigenvector centrality of 9: 0.4272
Linear eigenvector centrality of 10: 0.3323
Linear eigenvector centrality of 12: 0.0995
Linear eigenvector centrality of 14: 0.1395
Linear eigenvector centrality of 15: 0.3981
Linear eigenvector centrality of 16: 0.0114
Linear eigenvector centrality of 17: 0.2088
Linear eigenvector centrality of 19: 0.4437
```



The Tudisco & Higham Nonlinear Eigenvector Centrality

- Let $G = (V, E)$ be a connected hypergraph with vertex set $V = \{1, \ldots, n\}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a nonlinear real-valued function. Define a centrality score vector $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$, called the **nonlinear singular vector** (**NSV**), as the solution to the nonlinear eigenvalue problem

$$Hf(x) = \lambda x,$$

  where the operator $H : \mathbb{R}^n \to \mathbb{R}^n$ is defined component-wise by

$$(Hf(x))_i = \sum_{\substack{e \in E \\ i \in e}} \sum_{j \in e} f(x_j).$$

- If $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is order-preserving and homogeneous of degree less than 1, then there exists a positive eigenvector $x \in \mathbb{R}^n_{>0}$ corresponding to the eigenvalue $\lambda = 1$, i.e., satisfying the fixed-point equation

$$Hf(x) = x.$$

Convergence of the NSV Centrality

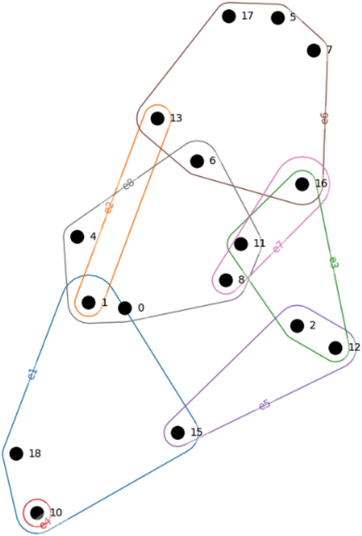
- If, in addition, f is strictly concave, then the eigenvector x is unique up to scaling and it can be calculated through the iteration
$$x^{(k+1)} = \frac{Hf(x^{(k)})}{||Hf(x^{(k)})||_2}$$
starting from a positive initial vector $x^{(0)}$ using the Euclidean L^2–norm $||\cdot||_2$.
- The case $f(x) = x$ reduces to a standard linear eigenvector problem, which generalizes eigenvector centrality from graphs to hypergraphs.
- In the superlinear case, where $f(x) = x^\alpha$ with $\alpha > 1$, centrality scores are biased toward higher-degree vertices, while in the sublinear case, where $f(x) = x^\alpha$ with $0 < \alpha < 1$, centrality is more evenly distributed among vertices.
- Additionally, the logarithmic form $f(x) = \log(x+1)$ is more appropriate for hypergraphs where a small number of vertices dominate the centrality scores.

Nonlinear (sublinear) Eigenvector Centrality

{'e1': [1, 10, 15, 18], 'e2': [1, 13], 'e3': [2, 11, 12, 16], 'e4': [10], 'e5': [2, 12, 15], 'e6': [5, 6, 7, 13, 16, 17], 'e7': [8, 16], 'e8': [0, 1, 4, 6, 8, 11]}

```
T&H Nonlinear (sublinear) Eigenvector Centrality (on hyperedges):
Nonlinear eigenvector centrality of e1: 0.3663
Nonlinear eigenvector centrality of e2: 0.2819
Nonlinear eigenvector centrality of e3: 0.3910
Nonlinear eigenvector centrality of e4: 0.1749
Nonlinear eigenvector centrality of e5: 0.3239
Nonlinear eigenvector centrality of e6: 0.4509
Nonlinear eigenvector centrality of e7: 0.2829
Nonlinear eigenvector centrality of e8: 0.4633

T&H Nonlinear (sublinear) Eigenvector Centrality (on vertices):
Nonlinear eigenvector centrality of 0: 0.2072
Nonlinear eigenvector centrality of 1: 0.3209
Nonlinear eigenvector centrality of 2: 0.2574
Nonlinear eigenvector centrality of 4: 0.2072
Nonlinear eigenvector centrality of 5: 0.2044
Nonlinear eigenvector centrality of 6: 0.2911
Nonlinear eigenvector centrality of 7: 0.2044
Nonlinear eigenvector centrality of 8: 0.2630
Nonlinear eigenvector centrality of 10: 0.2240
Nonlinear eigenvector centrality of 11: 0.2814
Nonlinear eigenvector centrality of 12: 0.2574
Nonlinear eigenvector centrality of 13: 0.2606
Nonlinear eigenvector centrality of 15: 0.2529
Nonlinear eigenvector centrality of 16: 0.3229
Nonlinear eigenvector centrality of 17: 0.2044
Nonlinear eigenvector centrality of 18: 0.1843
```



Nonlinear (superlinear) Eigenvector Centrality

```
{'e1': [1, 2, 9], 'e2': [3, 8, 10, 18], 'e3': [1, 17], 'e4': [5, 15, 17], 'e5': [13, 17, 19], 'e6': [2, 4, 5, 7, 9, 16], 'e7': [0, 9, 10, 12, 15, 19], 'e8': [1, 7]}
```

```
T&H Nonlinear (superlinear) Eigenvector Centrality (on hyperedges):
Nonlinear eigenvector centrality of e1: 0.0926
Nonlinear eigenvector centrality of e2: 0.0000
Nonlinear eigenvector centrality of e3: 0.0000
Nonlinear eigenvector centrality of e4: 0.0095
Nonlinear eigenvector centrality of e5: 0.0000
Nonlinear eigenvector centrality of e6: 0.9955
Nonlinear eigenvector centrality of e7: 0.0171
Nonlinear eigenvector centrality of e8: 0.0095

T&H Nonlinear (superlinear) Eigenvector Centrality (on vertices):
Nonlinear eigenvector centrality of 0: 0.3982
Nonlinear eigenvector centrality of 1: 0.0000
Nonlinear eigenvector centrality of 2: 0.0000
Nonlinear eigenvector centrality of 3: 0.0000
Nonlinear eigenvector centrality of 4: 0.0000
Nonlinear eigenvector centrality of 5: 0.0000
Nonlinear eigenvector centrality of 7: 0.0000
Nonlinear eigenvector centrality of 8: 0.0000
Nonlinear eigenvector centrality of 9: 0.4234
Nonlinear eigenvector centrality of 10: 0.4097
Nonlinear eigenvector centrality of 12: 0.3982
Nonlinear eigenvector centrality of 13: 0.0000
Nonlinear eigenvector centrality of 15: 0.4097
Nonlinear eigenvector centrality of 16: 0.0000
Nonlinear eigenvector centrality of 17: 0.0000
Nonlinear eigenvector centrality of 18: 0.0000
Nonlinear eigenvector centrality of 19: 0.4097
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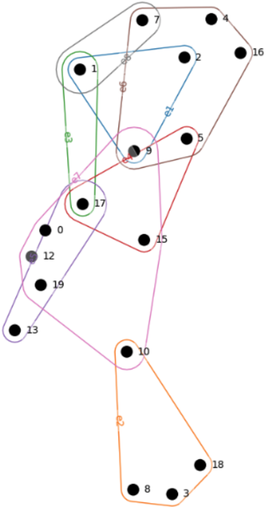

Nonlinear (log plus one) Eigenvector Centrality

```
{'e1': [1, 3, 11], 'e2': [7, 8], 'e3': [3, 7, 13, 14], 'e4': [5, 7, 9], 'e5': [11], 'e6': [0, 3, 9, 12, 18], 'e7':
[0, 3, 4, 6, 8, 15], 'e8': [2, 9, 12, 18]}

 T&H Nonlinear (log plus one) Eigenvector Centrality (on hyperedges):
 Nonlinear eigenvector centrality of e1: 0.2721
 Nonlinear eigenvector centrality of e2: 0.2232
 Nonlinear eigenvector centrality of e3: 0.3649
 Nonlinear eigenvector centrality of e4: 0.2967
 Nonlinear eigenvector centrality of e5: 0.0652
 Nonlinear eigenvector centrality of e6: 0.5168
 Nonlinear eigenvector centrality of e7: 0.4854
 Nonlinear eigenvector centrality of e8: 0.3846

 T&H Nonlinear (log plus one) Eigenvector Centrality (on vertices):
 Nonlinear eigenvector centrality of 0: 0.3259
 Nonlinear eigenvector centrality of 1: 0.1130
 Nonlinear eigenvector centrality of 2: 0.1528
 Nonlinear eigenvector centrality of 3: 0.4556
 Nonlinear eigenvector centrality of 4: 0.1857
 Nonlinear eigenvector centrality of 5: 0.1220
 Nonlinear eigenvector centrality of 6: 0.1857
 Nonlinear eigenvector centrality of 7: 0.2975
 Nonlinear eigenvector centrality of 8: 0.2515
 Nonlinear eigenvector centrality of 9: 0.3697
 Nonlinear eigenvector centrality of 11: 0.1365
 Nonlinear eigenvector centrality of 12: 0.3017
 Nonlinear eigenvector centrality of 13: 0.1460
 Nonlinear eigenvector centrality of 14: 0.1460
 Nonlinear eigenvector centrality of 15: 0.1857
 Nonlinear eigenvector centrality of 18: 0.3017
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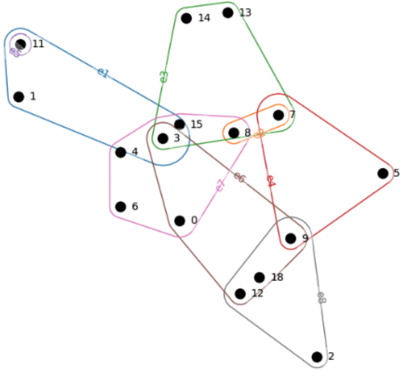


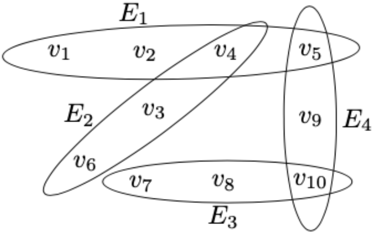
Directed Hypergraphs

Directed hypergraphs are generalizations of hypergraphs, in which hyperedges, called now hyperarcs, consist of pairs of two sets of vertices, the tail and the head.

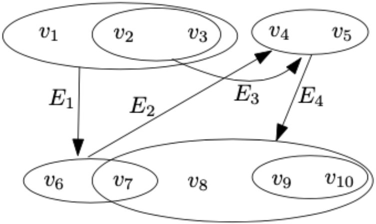
Formal Definition of Directed Hypergraphs

A **directed hypergraph** $G = (V, E)$ is a pair of two finite sets V and E, the set of *vertices* and the set of of **hyperarcs** (or *directed hyperedges*), where E is a family of pairs of subsets of the set of vertices, i.e., $E \subseteq 2^V \times 2^V$, such that every hyperarc $e \in E$ is an ordered pair $e = (\mathrm{tail}(e), \mathrm{head}(e))$, where the two vertex subsets $\mathrm{tail}(e)$ and $\mathrm{head}(e)$ are assumed to be non-empty (or at least one of them should be) and disjoint (i.e., $\varnothing \neq \mathrm{tail}(e), \mathrm{head}(e) \subseteq V, \mathrm{tail}(e) \cap \mathrm{head}(e) = \varnothing$).

Undirected vs. Directed Hypergraphs



(a) Undirected hypergraph.



(b) Directed hypergraph.

The Two Star-Centered Representations of a Directed Hypergraph

Let $G = (V, E)$ be a directed hypergraph with vertex set V and hyperarc set $E \subset 2^V \times 2^V$, where each hyperarc e is a pair $(\text{tail}(e), \text{head}(e))$ of two disjoint non-empty subsets of V.

- Equivalently, e can be expressed either as the union of all pairs $(u, \text{head}(e))$, for all $u \in \text{tail}(e)$,

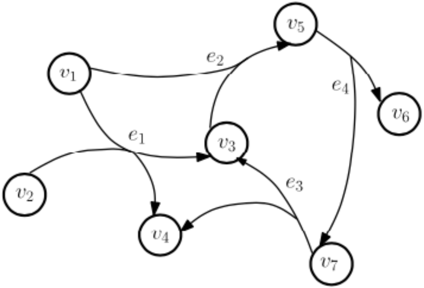
$$e = (\text{tail}(e), \text{head}(e)) = \bigcup_{u \in \text{tail}(e)} (u, \text{head}(e)),$$

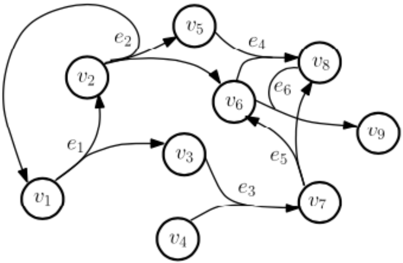
  or as the union of all pairs $(\text{tail}(e), v)$, for all $v \in \text{head}(e)$,

$$e = (\text{tail}(e), \text{head}(e)) = \bigcup_{v \in \text{head}(e)} (\text{tail}(e), v).$$

- The former is the **head–star-centered representation** and the latter is the **tail–star-centered representation** of hypergraph $G = (V, E)$. Both are constructed over the same vertex set V, where hyperarcs consist of pairs between individual vertices and subsets of vertices of V.

Directed Graphs





Incidence Matrices of Directed Hypergraphs

Let $G = (V, E)$ a directed hypergraph of order $|V| = n$ and size $|E| = m$, which means that $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, where $e_j = (\mathrm{tail}(e_j), \mathrm{head}(e_j)),$ for $j = 1, \ldots, m$.

- The **incidence matrix** of the hypergraph G is a $n \times m$ matrix $H = \{h_{ij}\}$, where

$$h_{ij} = \begin{cases} -1, & \text{if } v_i \in \mathrm{tail}(e_j), \\ +1, & \text{if } v_i \in \mathrm{head}(e_j), \\ 0, & \text{otherwise.} \end{cases}$$

- The **head–incidence matrix** of the head–star-centered representation of hypergraph G is a $n \times m$ matrix $H_{\mathrm{head}} = \{h_{\mathrm{head},ij}\}$, where

$$h_{\mathrm{head},ij} = \begin{cases} +1, & \text{if } v_i \in \mathrm{head}(e_j), \\ 0, & \text{otherwise.} \end{cases}$$

- The **tail–incidence matrix** of the tail–star-centered representation of hypergraph G is a $n \times m$ matrix $H_{\mathrm{tail}} = \{h_{\mathrm{tail},ij}\}$, where

$$h_{\mathrm{tail},ij} = \begin{cases} +1, & \text{if } v_i \in \mathrm{tail}(e_j), \\ 0, & \text{otherwise.} \end{cases}$$

Vertex Degrees of Directed Hypergraphs

- The **in-degree**, $\deg^-(v)$, of a vertex v is the number of hyperarcs, in which v appears in the head:

$$\deg^-(v) = \big|\{e \in E : v \in \mathrm{head}(e)\}\big| = \sum_{j=1}^{m} h_{\mathrm{head},ij}.$$

- The **out-degree**, $\deg^+(v)$, of a vertex v is the number of hyperarcs, in which v appears in the tail:

$$\deg^+(v) = \big|\{e \in E : v \in \mathrm{tail}(e)\}\big| = \sum_{j=1}^{m} h_{\mathrm{tail},ij}.$$

- The **total degree**, $\deg(v_i)$, of a vertex v_i is the number of hyperarcs, in which v_i appears either in the head or in the tail:

$$\deg(v_i) = \deg^-(v_i) + \deg^+(v_i) = \sum_{j=1}^{m} |h_{ij}|, \text{ for } i = 1, \ldots, n.$$

Hyperarc Degrees of Directed Hypergraphs

- The **tail degree**, $\delta_{\text{tail}}(e)$, of a hyperarc e is the number of vertices in its tail:

$$\delta_{\text{tail}}(e) = |\,\text{tail}(e)| = \sum_{i=1}^{n} h_{\text{tail},ij}.$$

- The **head degree**, $\delta_{\text{head}}(e)$, of a hyperarc e is the number of vertices in its head:

$$\delta_{\text{head}}(e) = |\,\text{head}(e)| = \sum_{i=1}^{n} h_{\text{head},ij}.$$

- The **total degree**, $\delta(e_j)$, of a hyperarc e_j is the number of vertices in its tail and head:

$$\delta(e_j) = \delta_{\text{tail}}(e_j) + \delta_{\text{head}}(e_j) = \sum_{i=1}^{n} |h_{ij}| = |e_j|, \text{ for } j = 1, \ldots, m.$$

From Directed Graphs to Directed Hypergraphs

Let $G' = (V, E')$ be a directed graph with vertex set V and edge set E'.

- Since each edge $e \in E'$ is a pair of vertices, i.e., $e = (u, v)$ with $u, v \in V$, it can be trivially represented as a hyperarc $e = (\text{tail}(e), \text{head}(e)) \in 2^V \times 2^V$, where $\text{tail}(e) = \{u\}$ and $\text{head}(e) = \{v\}$.
- Beyond this trivial representation, a more structured hypergraph can be constructed by grouping edges with common sources and targets to form hyperarcs. A natural choice is to define the tail of each hyperarc as the set of in-neighbors of a given vertex and the head as the set of its out-neighbors. This approach provides a nontrivial representation of the directed graph as a directed hypergraph, preserving the vertex set while defining a hyperarc set based on nontrivial meaningful tail and head subsets. The following algorithm details this construction.

Representing a Directed Graph as a Directed Hypergraph

Algorithm 1:

Input: Directed graph $G' = (V, E')$
Output: Directed hypergraph $G = (V, E)$

Initialize the directed hypergraph: ;
- Start with the same vertex set V as in G'. ;
- Initialize an empty set of hyperarcs E. ;

for *each vertex* $v \in V$ **do**
    Let T_v be the set of all vertices u such that $(u, v) \in E'$ (i.e., the in-neighbors of v). ;
    Let H_v be the set of all vertices w such that $(v, w) \in E'$ (i.e., the out-neighbors of v). ;
    if T_v *and* H_v *are nonempty* **then**
        Create a hyperarc $e_v = (T_v, H_v)$. ;
        Add e_v to the set of hyperarcs E. ;
    end
end
Construct the directed hypergraph $G = (V, E)$. ;
Output the directed hypergraph G. ;