Slides of Discrete Mathematics based on Susanna Epp's Textbook

Moses A. Boudourides[1]

Visiting Associate Professor of Computer Science
Haverford College

[1] Moses.Boudourides@cs.haverford.edu

Chapter 7

Functions

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7.1 Functions I

Definition

A **function** f **from a set** X **to a set** Y, denoted $f: X \to Y$, is a relation from X, the **domain** of f, to Y, the **co–domain**, that satisfies two properties:

1. every element in X is related to some element in Y and
2. no element in X is related to more than one element in Y.

The unique element to which f sends an element x in its domain is denoted as $f(x)$ and is called the **value of** f **at** x, or the **image of** x **under** f.

Definition (continue)

The set of all values of f taken together is called the **range of** f or the **image of** X **under** f. Symbolically:

$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \\ = \{y \in Y \mid y = f(x), \text{ for some } x \in X\}.$$

Given an element $y \in Y$, there may exist elements $x \in X$ with y as their images. For all these x's, $f(x) = y$, and any such x is called a **preimage of** y or an **inverse image of** y. The set of all inverse images of y is called the **inverse image of** y. Symbolically:

$$\textbf{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$

Theorem

If $F\colon X \to Y$ and $G\colon X \to Y$ are functions, then $F = G$ if and only if $F(x) = G(x)$, for all $x \in X$.

Exercise 7.1.14

Let $J_5 = \{0, 1, 2, 3, 4\}$ and define functions $h\colon J_5 \to J_5$ and $k\colon J_5 \to J_5$ as follows: for each $x \in J_5$,

$$h(x) = (x+3)^3 \mod 5,$$
$$k(x) = (x^3 + 4x^2 + 2x + 2) \mod 5.$$

Is $h = k$? Explain.

Complete the following table and then use the definition of set equality.

x	$(x+3)^3$	$h(x)$	x^3+4x^2+2x+2	$k(x)$
0	27	$27\ mod\ 5 = 2$	2	$2\ mod\ 5 = 2$
1	$4^3 =$	$64\ mod\ 5 =$	$1^3 + 4\cdot 1^2 + 2\cdot 1 + 2 =$	$9\ mod\ 5 =$
2	$5^3 =$	$125\ mod\ 5 =$	$2^3 + 4\cdot 2^2 + 2\cdot 2 + 2 =$	$0\ mod\ 5 =$
3	$6^3 =$	$216\ mod\ 5 =$	$3^3 + 4\cdot 3^2 + 2\cdot 3 + 2 =$	$71\ mod\ 5 =$
4	$7^3 =$	$343\ mod\ 5 =$	$4^3 + 4\cdot 4^2 + 2\cdot 4 + 2 =$	$138\ mod\ 5 =$

7.1 Functions III: Logarithms and Logarithmic Functions (a)

Definition (Logaithms and Logarithmic Functions)

Let b be a positive real number with $b \neq 1$. For each positive real x, the **logarithm with base b of x**, written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically:

$$\log_b x = y \iff b^y = x.$$

The **logarithmic function with base b** is the function $\log_b \colon \mathbb{R}^+ \to \mathbb{R}$ that takes each positive real number x to its logarithm with base b, i.e., $\log_b(x) = \log_b x$.

7.1 Functions III: Logarithms and Logarithmic Functions (b)

Theorem (Properties of Logarithms)

For any $a, b, c, x, y \in \mathbb{R}, b \neq 1, c \neq 1,$ *the following hold:*

$$\begin{aligned}
&\text{(a)} \log_b(xy) = \log_b x + \log_b y, \\
&\text{(b)} \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y, \\
&\text{(c)} \log_b(x^a) = a\log_b x, \\
&\text{(d)} \log_c x = \frac{\log_b x}{\log_b c}.
\end{aligned}$$

7.1 Functions III: Logarithms and Logarithmic Functions (c)

Exercise 7.1.22

Use the unique factorization for the integers theorem and the definition of logarithm to prove that $\log_3(7)$ is irrational.

Suppose that $\log_3(7)$ is rational, i.e., suppose that $\log_3(7) = \frac{a}{b}$, for some integers a, b with $b \neq 0$. By the definition of logarithm, $\frac{a}{b} > 0$ (**explain!**) and, thus, we can take both $a, b > 0$. Thus, $3^{\frac{a}{b}} = 7$ or $3^a = 7^b$ (**why?**). Let $N = 3^a = 7^b$. Clearly, N is an integer and it is expressed either as $N = 3^a$ or as $N = 7^b$. But then the uniqueness of the integer factorization theorem leads to a contradiction. **Why?**

7.1 Functions IV: Functions Acting on Sets (a)

Definition

If $f\colon X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$$

and

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of** A, and $f^{-1}(C)$ is called the **inverse image of** C.

Exercise 7.1.32

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$. Define $g \colon X \to Y$ as follows: $g(1) = a, g(2) = a, g(3) = a$ and $g(4) = d$.

(a) Draw an arrow diagram for g.

(b) Let $A = \{2, 3\}, C = \{a\}$ and $D = \{b, c\}$. Find $g(A), g(X), g^{-1}(C), g^{-1}(D)$ and $g^{-1}(Y)$.

Apply definitions!

Exercise 7.1.42

Let $F\colon X \to Y$ be a function and $C \subseteq Y$. Show that

$$F(F^{-1}(C)) \subseteq C.$$

Let $y \in F(F^{-1}(C))$. Then, by definition of image of a set, there exists $x \in F^{-1}(C)$ such that $F(x) = y$. Moreover, because $x \in F^{-1}(C)$, by definition of inverse image, $F(x) \in C$. Thus, since $F(x) = y$ and $F(x) \in C$, we conclude that $y \in C$.

7.1 Functions IV: (d)

Exercise 7.1.43

Given a set S and a subset A, the **characteristic function** of A, denoted χ_A, is the function defined from S to $\mathbb{Z}$ with the property that, for all $u \in S$,

$$\chi_A(u) = \begin{cases} 1, & \text{if } u \in A, \\ 0, & \text{if } u \notin A. \end{cases}$$

Show that each of the following holds for all subsets A and B of S and all $u \in S$.

(a) $\chi_{A \cap B}(u) = \chi_A(u) \cdot \chi_B(u)$.

(b) $\chi_{A \cup B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$.

a.

$$
\begin{aligned}
\chi_A(u)\cdot\chi_B(u) &= \begin{cases} 1\cdot 1 & \text{if } u\in A \text{ and } u\in B \\ 1\cdot 0 & \text{if } u\in A \text{ and } u\notin B \\ 0\cdot 1 & \text{if } u\notin A \text{ and } u\in B \\ 0\cdot 0 & \text{if } u\notin A \text{ and } u\notin B \end{cases} \\
&= \begin{cases} 1 & \text{if } u\in A\cap B \\ 0 & \text{if } u\notin A\cap B \end{cases} \\
&= \chi_{A\cap B}(u)
\end{aligned}
$$

b.

$$
\begin{aligned}
\chi_A(u)+\chi_B(u)-\chi_A(u)\cdot\chi_B(u) &= \begin{cases} 1+1-1\cdot 1 & \text{if } u\in A \text{ and } u\in B \\ 1+0-1\cdot 0 & \text{if } u\in A \text{ and } u\notin B \\ 0+1-0\cdot 1 & \text{if } u\notin A \text{ and } u\in B \\ 0+0-0\cdot 0 & \text{if } u\notin A \text{ and } u\notin B \end{cases} \\
&= \begin{cases} 1 & \text{if } u\in A \text{ and } u\in B \\ 1 & \text{if } u\in A \text{ and } u\notin B \\ 1 & \text{if } u\notin A \text{ and } u\in B \\ 0 & \text{if } u\notin A \text{ and } u\notin B \end{cases} \\
&= \begin{cases} 1 & \text{if } u\in A\cup B \\ 0 & \text{if } u\notin A\cup B \end{cases} \\
&= \chi_{A\cup B}(u)
\end{aligned}
$$

7.2 One–to–One Functions (a)

Definition

A function $F : X \to Y$ is called **one–to–one** (or **injective**) if and only if, for all $x_1, x_2 \in X$,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently,

$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

Notice that F is **not one–to–one** if and only if there exist $x_1, x_2 \in X$ such that

$$x_1 \neq x_2 \text{ and } F(x_1) = F(x_2).$$

Exercise 7.2.17

Show that the following function is one–to–one:

$$f(x) = \frac{3x-1}{x}.$$

Let x_1, x_2 be any non–zero real numbers such that $f(x_1) = f(x_2)$. This means that $\frac{3x_1-1}{x} = \frac{3x_2-1}{x}$. Then, do the algebra to get $x_1 = x_2$.

7.2 One–to–One Functions (c)

Exercise 7.2.25

Define $F : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ and $G : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ as follows: for all $(n, m) \in \mathbb{Z}^+ \times \mathbb{Z}^+$,

$$F(n, m) = 3^n 5^m \text{ and } G(n, m) = 3^n 6^m.$$

Prove that F and G are one–to–one.

Suppose $F(a, b) = F(c, d)$, for some $(a, b), (c, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Then, by definition, $3^a 5^b = 3^c 5^d$. Using the unique integers factorization theorem (**explain how**), we should have $a = c$ and $b = d$, which means $(a, b) = (c, d)$. Similarly, for G, if $G(a, b) = G(c, d)$, we would have $3^a 6^b = 3^c 6^b$ or $3^{a+b} 2^b = 3^{c+d} 2^d$ (**why?**), which again using the unique integers factorization theorem (**explain how**) would imply that $a + b = c + d$ and $b = d$ and, thus, eventually $(a, b) = (c, d)$ (**why?**).

Exercise 7.2.26 (b)

Show that $\log_{16} 9 = \log_4 3$.

Let $x = \log_{16} 9$ and $y = \log_4 3$. Then, by definition of the logarithm, $16^x = 9$ and $4^y = 3$. Now, use the facts that $16 = 4^2$ and $9 = 3^2$ to get $(4^2)^x = (4^y)^2$. Then, **explain how** the last equality would reduce to $x = y$.

7.2 Onto Functions (a)

Definition

A function $F : X \to Y$ is called **onto** (or **surjective**) if and only if, given any $y \in Y$, it is possible to find a $x \in X$ such that $y = F(x)$.

Notice that F is **not onto** if and only if there exists a $y \in Y$ such that, for every $x \in X, F(x) \neq y$.

Exercise 7.2.35

If $F\colon X \to Y$ is onto, then for all $B \subseteq Y, F(F^{-1}(B)) = B$.

First, let $y \in F(F^{-1}(B))$. Then, by definition of the image set, there exists $x \in F^{-1}(B)$ such that $F(x) = y$. Moreover, by definition of the inverse image, since $x \in F^{-1}(B)$, $F(x) \in B$. But $F(x) = y$ and, thus, $y \in B$.

On the other hand, consider a $y \in B$. Because F is onto, there exists $x \in X$ such that $F(x) = y$ and, thus, by definition of inverse image, $x \in F^{-1}(B)$. Therefore, by definition of the image of a set, $F(x) \in F(F^{-1}(B))$ and, since $y = F(x), y \in F(F^{-1}(B))$.

Definition

A function $F\colon X \to Y$ which is both one–to–one and onto is called **one–to–one correspondence** (or **bijection**).

Exercise 7.2.49

Show that the following function is one–to–one and onto, for all $x \in \mathbb{R}, x \neq 1$,

$$y = \frac{x+1}{x-1}.$$

First, let $x_1, x_2 \in \mathbb{R}, x_1 \neq 1, x_2 \neq 1$, be such that $\frac{x_1+1}{x_1-1} = \frac{x_2+1}{x_2-1}$. **Do the algebra** to deduce that $x_1 = x_2$ and, thus, the function is one–to–one.

Next, for any $y \in \mathbb{R}, y \neq 1$, consider the number $x = \frac{y+1}{y-1}$. Obviously, $x \in \mathbb{R}$ and then, **do the algebra** to find that $\frac{x+1}{x-1} = \frac{\frac{y+1}{y-1}+1}{\frac{y+1}{y-1}-1} = \cdots = y$. Therefore, the function is also onto.

7.2 Inverse Functions

Definition (Theorem)

If $F\colon X \to Y$ is one–to–one and onto, then, for any $y \in Y$ there exists an $x \in X$ such that $F(x) = y$ (because F is onto), and this x is unique (because F is one-to-one). This means that there exists a function $F^{-1}\colon Y \to X$, called **inverse function** for F, which is defined as follows: for any $y \in Y$,

$$F^{-1}(y) = \text{ the unique } x \in X \text{ such that } F(x) = y.$$

In other words,

$$F^{-1}(y) = x \iff y = F(x).$$

Theorem

If $F\colon X \to Y$ is one–to–one and onto, then $F^{-1}\colon Y \to X$ is also one–to–one and onto.

7.3 Composition of Functions (a)

Definition

Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that range$(f) \subseteq$ domain(g), then the **composition function** of f and g, denoted as $f \circ g$, is defined as a function $g \circ f : X \to Z$ such that

$$(g \circ f)(x) = g(f(x)), \text{ for all } x \in X.$$

Exercise 7.3.2

Use arrow diagrams, to determine equality of functions.

Exercise 7.3.4

$F(x) = x^5, G(x) = x^{1/5}, x \in \mathbb{R}.$

Do the algebra to show that $(G \circ F)(x) = G(F(x)) = \cdots = x = \cdots = F(G(x)) = (F \circ G)(x)$.

7.3 Composition of Functions (b)

Exercise 7.3.11

$H, H^{-1} : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ are defined as $H(x) = H^{-1}(x) = \frac{x+1}{x-1}$, for all $x \in \mathbb{R} - \{1\}$. **Do the algebra** to find that $(H^{-1} \circ H)(x) = (H \circ H^{-1})(x) = H(\frac{x+1}{x-1}) = \cdots = x$.

Exercise 7.3.20

If $f : W \to X, g : X \to Y, h : Y \to Z$ are three functions, then $h \circ (g \circ f) = (h \circ g) \circ f$.

For every $w \in W$, $[h \circ (g \circ f)](w) = h((g \circ f)(w)) = h(g(f(w))) = h \circ g(f(w)) = [(h \circ g) \circ f](w)$. Thus, by the definition of equality of functions $h \circ (g \circ f) = (h \circ g) \circ f$.

7.3 Composition of Functions (c)

Definition of the Identity Function

The **identity function** on a set X, denoted as I_X, is defined as the function $I_X : X \to X$ such that $I_X(x) = x$, for all $x \in X$.

Theorem (Composition of a Function and its Inverse)

If $f : X \to Y$ is a one–to–one and onto function with inverse function $f^{-1} : Y \to X$, then

$$f^{-1} \circ f = I_X \text{ and}$$
$$f \circ f^{-1} = I_Y.$$

7.3 Composition of Functions (d)

Theorem

If $f\colon X \to Y$ and $g\colon Y \to X$ are two functions which are both one–to–one or onto, then their composition $g \circ f\colon X \to Y$ is one–to–one or onto (respectively).

Exercise 7.3.25

If $f\colon X \to Y$ and $g\colon Y \to X$ are two functions such that $g \circ f = I_X$ and $f \circ g = I_Y$, then show that both f and g are one–to–one and onto and $g = f^{-1}$.

Since I_X and I_Y are one–to–one and onto and by hypothesis $g \circ f = I_X$ and $f \circ g = I_Y$, the previous Theorem implies that both f and g are one–to–one and onto. Therefore, both f and g have inverse functions f^{-1} and g^{-1} (respectively). Thus, $f \circ f^{-1} = I_Y = f \circ g$. In other words, for all $y \in Y, f(f^{-1}(y)) = (f \circ f^{-1})(y) = (f \circ g)(y) = f(g(y))$. Now, since f is one–to–one, it follows that $f^{-1}(y) = g(y)$, for all $y \in Y$, and, therefore, by the definition of equality of functions $f^{-1} = g$.

7.4 Cardinlity and Sizes of Sets of Numbers (a)

Definition

Let A and B be sets. We say that A **has the same cardinality as** B if and only if there is a function $f\colon A \to B$, which is one–to–one and onto.

Definition

Let X be a set.

- X is called **finite** if and only if there is a positive integer n such that X has the same cardinality with the set $[n] = \{1, 2, \ldots, n\}$.
- X is called **countably infinite** if and only if X has the same cardinality with the set $\mathbb{Z}^+$, i.e., the set of positive integers.
- X is called **countable** if and only if it is either finite or countably infinite.
- X is called **uncountable** if and only if it is not countable.

7.4 Cardinlity and Sizes of Sets of Numbers (b)

Theorem (Cantor)

- *The set of all real numbers between 0 and 1 is uncountable.*
- $\mathbb{R}$ *has the same cardinality as the set of all real numbers between 0 and 1.*