BREIGER'S DUALITY IN MULTILAYER AND TEMPORAL BIPARTITE GRAPHS

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Abstract. Formally, the framework of Breiger's Duality applies to a bipartite graph of persons and groups, over which two dual projections on the two unipartite graphs of persons and graphs are induced. Here, we are extending the classical framework on multilayer (or temporal) bipartite graphs.

Thus, let H be a multilayer graph of Λ layers (or, as a special case, a temporal graph in Λ time stamps) $H = (H_\alpha, H_\beta, \dots, H_\Lambda)$, where, for $\lambda = \alpha, \beta, \dots, \Lambda$, every H_λ is a bipartite graph with the same bipartition of verticies X, Y, where X consists of n vertices (persons) $x_1, x_2, \dots, x_n$, and Y consists of m vertices (groups) $y_1, y_2, \dots, y_m$. In other words, each one of the layer bipartite graphs can be denoted as $H_\lambda = (X, Y, E_\lambda)$, where the set of edges E_λ is a (different) subset of $X \times Y$, for every $\lambda = \alpha, \beta, \dots, \Lambda$. Apparently, $H_\lambda = \bigcup_{\lambda=\alpha,\beta,\dots,\Lambda} \left(\bigcup_{i=1,\dots,n} (\{x_i\}, Y, E_{\lambda,x_i}) \right)$, where $E_{\lambda,x_i} = \{(x, y) \in E_\lambda \colon x = x_i\}$.

The last decomposition implies that, without any loss of generality, the multilayer (or temporal) bipartite graph H can be represented, after fixing a $x_i \in X$, as $H = \Big((\{x_i\}, Y, E_{\alpha,x_i}), (\{x_i\}, Y, E_{\beta,x_i}), \dots, (\{x_i\}, Y, \ E_{\Lambda,x_i}) \Big)$.

Notice that H (as previously) can be represented as a multilayer (or temporal) biparite hypergraph by $H = \Big((Y, \mathcal{E}_{\alpha,x_i}), (Y, \mathcal{E}_{\beta,x_i}), \dots, (Y, \mathcal{E}_{\Lambda,x_i}) \Big)$, where, for $\lambda = \alpha, \beta, \dots, \Lambda$, every hyperedge $\mathcal{E}$, parametrized over the layer λ and the fixed vertex x_i, is defined by $\mathcal{E}_{\lambda,x_i} = \{(x_i, y) \in E_{\lambda,x_i} \colon y \in Y\}$. Similarly, fixing a $y_j \in Y$, the following dual representations are valid: $H = \Big((X, \{y_j\}, E_{\alpha,y_j}), (X, \{y_j\}, \ E_{\beta,y_j}), \dots, (X, \{y_j\}, E_{\Lambda,y_j}) \Big) = \Big((X, \mathcal{E}_{\alpha,y_j}), (X, \mathcal{E}_{\beta,y_j}), \dots, (X, \mathcal{E}_{\Lambda,y_j}) \Big)$, where now the hyperedges are $\mathcal{E}_{\lambda,y_j} = \{(x, y_j) \in E_{\lambda,y_j} \colon x \in X\}$.

In this way, using elementary properties of hyperedge adjacencies (Aksoy et al., EPJ Data Science (2020) 9:16), we find that there are two kinds of dual projections of a multilayer (or temporal) bipartite graph.

The first kind of projections induces the following two dual multilayer (or temporal) unipartite graphs: $G_X = (G_{X,\alpha}, G_{X,\beta}, \dots, G_{X,\Lambda})$, and $G_Y = (G_{Y,\alpha}, G_{Y,\beta}, \dots, G_{Y,\Lambda})$, where, for $\lambda = \alpha, \beta, \dots, \Lambda$, in the former graph projection $G_{X,\lambda} = (X, I_\lambda)$, while in the latter projection $G_{Y,\lambda} = (Y, J_\lambda)$. In other words, the layer λ of the X–projection (multi)graph G_X has vertices in X and edges in the set I_λ of indirect ties among pairs of X defined as $I_\lambda = \{(x_i, x_j) \in X \times X \colon$ there exist $y \in Y$ such that $(x_i, y), (x_j, y) \in E_\lambda\}$. Similarly, the layer λ of the dual Y–projection (multi)graph G_Y has vertices in Y and edges in the set J_λ of indirect ties among pairs of Y defined as $J_\lambda = \{(y_i, y_j) \in Y \times Y \colon$ there exist $x \in X$ such that $(x, y_i), (x, y_j) \in E_\lambda\}$.

The second kind of projections induces the following two dual trans–layer multi–(Λ–)partite graphs: $M_X = (X_\alpha, X_\beta, \dots, X_\Lambda, T_X)$, and $M_Y = (Y_\alpha, Y_\beta, \dots, Y_\Lambda, T_Y)$, where $X_\alpha, X_\beta, \dots, X_\Lambda$ are the sets of active X–vertices in the layers of H, and $Y_\alpha, Y_\beta, \dots, Y_\Lambda$ are the sets of active Y–vertices in the layers of H (a layer vertex is called active whenever there is at least one edge in that layer which is incident to it). Moreover, the set T_X of edges in the trans–layer projection graph M_X consists of pairs of vertices $(x_{\lambda,i}, x_{\mu,j}) \in X_\lambda \times X_\mu$, called X–transitions among the two layers, such that there exists at least one vertex $y \in Y$, which is active in both layers λ, μ (i.e., $y \in Y_\lambda \cap Y_\mu$), such that $(x_{\lambda,i}, y) \in E_\lambda$, $(x_{\mu,j}, y) \notin E_\lambda$, $(x_{\mu,j}, y) \in E_\mu$ and $(x_{\lambda,i}, y) \notin E_\mu$. Similarly, the set T_Y of edges in the trans–layer projection graph M_Y consists of pairs of vertices $(y_{\lambda,i}, y_{\mu,j}) \in Y_\lambda \times Y_\mu$, called Y–transitions among the two layers, such that there exists at least one vertex $x \in X$, which is active in both layers λ, μ (i.e., $x \in X_\lambda \cap X_\mu$), such that $(x, y_{\lambda,i}) \in E_\lambda$, $(x, y_{\mu,j}) \notin E_\lambda$, $(x, y_{\mu,j}) \in E_\mu$ and $(x, y_{\lambda,i}) \notin E_\mu$.

We are developing the construction of the dual multilayer (or temporal) projections based on the formalism of adjacency matrices and we are also presenting some examples of empirical networks (from biology, diaries and bibliometrics), for which the above projections are constructed and discussed. Our computations (in Python) are available in this github page: `https://github.com/mboudour/var/blob/master/Boudourides_AdjacentNodesTrajectoriesInTemporalBipartiteGraphs%26Hypergraphs.ipynb`.

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