A SIMULATION OF A VOTING SYSTEM AMONG THREE PARTIES EMBEDDED ON A SOCIAL INFLUENCE NETWORK

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Let $G = (V, E)$ be an undirected graph with set of nodes (or vertices) $V = \{1, 2, \ldots, |V|\}$ and set of links (or lines) among nodes $E \subset V \times V$. We assume that G is simple.[1] Let $V(i)$ denote the set of neighbors of node $i \in V$, i.e.,

$$V(i) = \{j \in V \colon (i, j) \in E\}$$

and let $\upsilon(i)$ denote the number of neighbors of i (which is the degree of i), i.e.,

$$\upsilon(i) = |V(i)|.$$

Furthermore, let us assume that graph G is connected, i.e., all actors are non-isolated, which means that $\upsilon(i) > 0$, for all $i \in V$.

We suppose that nodes represent actors, each one preferring a party from a set of parties $\mathcal{P}$, which is assumed to contain three parties (although the two parties system might be considered as a special case, when one of the three parties is missing). So, typically, let $\mathcal{P} = \{R, C, L\}$, where R denotes "right," C "center" and L "left."

Let the assignment of parties on actors be given by a mapping $\pi \colon V \to \mathcal{P}$. Furthermore, for each party $p \in \mathcal{P}$, let $V_p(i)$ be the set of neighbors of actor $i \in V$ preferring party p, i.e.,

$$V_p(i) = \{j \in V(i) \colon \pi(j) = p\}$$

and let $\upsilon_p(i)$ be the number of such neighbors of i, i.e.,

$$\upsilon_p(i) = |V_p(i)|.$$

Remark 0.1. For any two parties $p, q \in \mathcal{P}$,

$$\begin{aligned} \sum_{\pi(i)=p} \upsilon_q(i) &= \sum_{\pi(j)=q} \upsilon_p(j) = |E_{pq}|, \text{ when } p \neq q, \\ &\sum_{\pi(i)=p} \upsilon_p(i) = 2|E_{pp}|, \end{aligned}$$

where

$$E_{pq} = \{(i, j) \in E \colon \pi(i) = p, \pi(j) = q\}.$$

Definition 0.2. For each actor $i \in V$, let $\theta(i) \geq 0$ be a *threshold* value for the number of neighbors of actor i needed so that this actor might change her party preference and adopt a "majority" party in her neighborhood. To define a notion of local majority, let us first consider those parties preferred by a number of neighbors of i exceeding the threshold $\theta(i)$, which form the set (of parties):

$$P(i) = \{p \in \mathcal{P} \colon \upsilon_p(i) \geq \theta(i)\},$$

[1] This means that, for any $i, j \in V$, there exists at most one $(i, j) \in E$, which is symmetric (i.e., $(i, j) \in E$ if and only if $(j, i) \in E$ and $(i, j) = (j, i)$) and not reflexive (i.e., $(i, i) \notin E$).

called set of *locally eligible* parties. (Of course, $P(i)$ might be empty.) When $P(i) \neq \varnothing$, a party $q \in P(i)$ is called *locally maximal* if

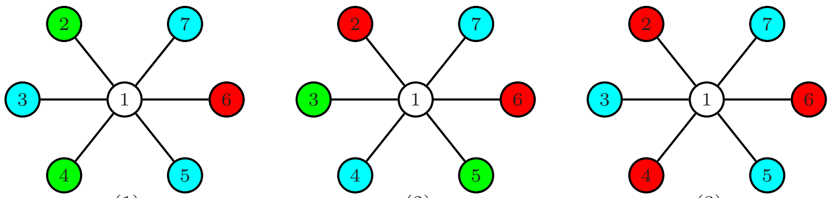
$$\upsilon_q(i) > \upsilon_p(i), \text{ for any } p \in P(i), p \neq q.$$

Let $Q(i) \subseteq P(i)$ denote the set of all locally maximal eligible parties.

- If $|Q(i)| = 1$, the unique party $p_{\max} = p_{\max}(i) \in Q(i)$ is said to form a *local single majority*.
- If $|Q(i)| \geq 2$, the parties in $Q(i)$ are said to form a *local tied majority*.
- If $|Q(i)| = 0$ (equivalently, $P(i) = \varnothing$), all the parties in $\mathcal{P}$ are said to be *locally noneligible*.

In the graphical representations of the examples, a node in $\boldsymbol{R}$ will be painted cyan, $\boldsymbol{C}$ green and $\boldsymbol{L}$ red.

Example 0.3. Consider the following three neighborhoods of actor 1 ($\upsilon(1) = 6$):



In case (1), $\upsilon_R(1) = 3, \upsilon_C(1) = 2, \upsilon_L(1) = 1$ and, so, if $\theta(1) = 2$, then $P(1) = \{R, C\}$ and $Q(i) = \{R\}$, i.e., we have a local single majority around 1 with $p_{\max}(1) = R$. In case (2), $\upsilon_C(1) = 2, \upsilon_R(1) = 2, \upsilon_L(1) = 2$ and, so, if $\theta(1) = 1$, then $P(1) = \{R, C, L\}$, i.e., we have a local tied majority around 1 with $Q(i) = \{R, C, L\}$. In case (3), $\upsilon_R(1) = 3, \upsilon_L(1) = 3, \upsilon_C(1) = 0$ and, so, if $\theta(1) = 4$, $P(1) = \varnothing$, i.e., we have local noneligibility around 1.

As for the set of parties $\mathcal{P} = \{R, C, L\}$, we assume that two of the three parties are antithetical to each other, while the third party is equally indifferent to these two. Typically, we consider that the antithetical pair of parties is (R, L) and we denote by $(\cdot)^{-1}$ an operator of *party-transposition*, which is a mapping $(\cdot)^{-1} : \mathcal{P} \to \mathcal{P}$ such that $(R)^{-1} = L$, $(L)^{-1} = R$ and $(C)^{-1} = C$.

Now, the specific ways in which actors are influenced by their neighbors and, thus, possibly change their party preferences are given by the following rules. The basic idea is that an actor might change her party preference only whenever there is a local majority at her neighborhood and only as far as no interchange among antithetical parties is involved. By the latter we mean that if an actor prefers R (or L), then she is never going to change directly to L (or R, respectively), although if she prefers C, then she might change directly to R or L. In other words, R-actors can convert to L-actors and vice-versa only indirectly, first becoming C-actors. In this way, actors are influened to adopt their party preferences by their neighbors as follows:

- If there is a local single majority at an actor's neighborhood, an actor preferring C might change to R or L or keep on C, depending on whether the local majoritarian preference is R, L or C, respectively.
- Also, in this case of local single majority, an actor preferring R (or L) is going to keep on R (or L), whenever the local majority is R (or L), or

change to C, whenever C is the local majority. However, when the local majority is L (or R), there are two possibilities depending on the number of C-neighbors around this actor: if her C-neighbors exceed the threshold and are more than the neighbors in R (or L), then the actor changes to C, while otherwise keeps on preferring R (or L).

- If there is a local tied majority at an actor's neighborhood, then the actor sticks to her own first preference, whenever this is included in the tie, or otherwise she follows another one of the tied parties, which might be chosen randomly (but always under the constraint that R and L do not interchange).
- If there is a local noneligibility at an actor's neighborhood, then the actor does not change her party preference.

To write more formally the rules of party change, let us first introduce some notation:

Notation 0.4. When $\pi(i) \in \mathcal{P}$ is the party preference of actor $i \in V$, we denote by $\hat{\pi}(i) \in \mathcal{P}$ her party preference after she is influenced by her neighbors.

0.1. **Rules of Party Change.**

(1) When there is local noneligibility around i (i.e., $P(i) = \varnothing$),

$$\hat{\pi}(i) = \pi(i).$$

(2) When there is a local single majority around i and $Q(i) = \{p_{\max}(i)\}$,
- if $\pi(i) = C$, then $\hat{\pi}(i) = p_{\max}(i)$,
- if $\pi(i) = R$ (or L) and $p_{\max}(i) = \pi(i)$, then $\hat{\pi}(i) = \pi(i)$,
- if $\pi(i) = R$ (or L) and $p_{\max}(i) = C$, then $\hat{\pi}(i) = C$,
- if $\pi(i) = R$ (or L) and $p_{\max}(i) = \pi^{-1}(i)$, then

$$\hat{\pi}(i) = \begin{cases} \pi(i), & \text{whenever } \upsilon_{\pi(i)}(i) > \upsilon_C(i), \\ C, & \text{whenever } \upsilon_C(i) > \upsilon_{\pi(i)}(i) \text{ and } \upsilon_C(i) \geq \theta(i). \end{cases}$$

(3) When there is a local tied majority around i and $|Q(i)| = 2$,
- if $\pi(i) \in Q(i)$, then $\hat{\pi}(i) = \pi(i)$,
- if $\pi(i) = C \notin Q(i)$, then $\hat{\pi}(i) = \rho(i)$, where $\rho(i) \in Q(i)$ is randomly chosen,
- if $\pi(i) = R$ (or L) $\notin Q(i)$, then $\hat{\pi}(i) = C$.

(4) When there is a local tied majority around i and $|Q(i)| = 3$,

$$\hat{\pi}(i) = \pi(i).$$

Definition 0.5. An actor $i \in V$ is called:

(1) *party-invariant* if $\hat{\pi}(i) = \pi(i)$ and
(2) *party-variable* if $\hat{\pi}(i) \neq \pi(i)$.

Remark 0.6. An actor $i \in V$ is party-invariant, if one of the following conditions holds:

Inv1: $\pi(i) \in Q(i)$,
Inv2: $\pi(i) \notin Q(i) = \{\pi^{-1}(i)\}$ and $\upsilon_{\pi(i)}(i) > \upsilon_C(i)$,
Inv3: $Q(i) = \varnothing$.

Actor i is party-variable, if one of the following conditions holds:

Var1: $\pi(i) \notin Q(i) \neq \varnothing$ (and $\hat{\pi}(i) \in Q(i)$),

Var2: $\pi(i) \notin Q(i) = \{\pi^{-1}(i)\}$ and $\upsilon_C(i) > \upsilon_{\pi(i)}(i)$, $\upsilon_C(i) \geq \theta(i)$ (and $\hat{\pi}(i) = C$).

0.2. **Simulation.** We start with an undirected graph $G = (V, E)$ and let $\boldsymbol{\pi}^{(0)} = \{\pi^{(0)}(i)\}_{i \in V} \in \mathcal{P}^{|V|}$ be an (arbitrary or random) initial configuration of parties over V. At each iteration $n > 0$, we choose randomly an actor $i \in V$ – called "interacting actor" – and we apply the above rules of party change. We denote by $\boldsymbol{\pi}^{(n)} = \{\pi^{(n)}(i)\}_{i \in V}$ the configuration of parties over V at iteration $n > 0$. We note that, because of the above rules, at each iteration, only the party of the interacting actor might possibly change. Let $\boldsymbol{\pi}^{(\infty)} = \{\pi^{(\infty)}(i)\}_{i \in V}$ be the stationary configuration of parties over V, which traps the iterated configurations as the iteration "time" n increases. Since the system constitutes a finite Markov chain without any recurrent states, every initial configuration $\boldsymbol{\pi}^{(0)}$ should converge to a stationary configuration $\boldsymbol{\pi}^{(\infty)}$, which is not unique, since the chain is reducible. (In fact, the stationary configuration is "path-dependent," since it depends on the random selections of interacting actors at each iteration).

Definition 0.7. We say that the simulation:

(1) *fixates* if the stationary configuration contains more than two parties and
(2) *clusters* if the stationary configuration contains a single party.

Definition 0.8. An actor $i \in V$ is called:

(1) *compliant* if $0 < \theta(i) \leq \upsilon(i)$ and
(2) *incompliant* if $\theta(i) > \upsilon(i)$.

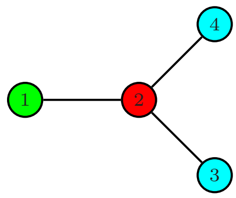
If actor i is compliant, her *degree of compliance* is defined as the number $\sigma \in (0, 1]$ such that $\theta(i) = \sigma(i)\upsilon(i)$. A compliant actor i is said to be:

(1) *easily compliant* if $\sigma(i) \leq \frac{1}{2}$ and
(2) *hardly compliant* if $\sigma(i) > \frac{1}{2}$.

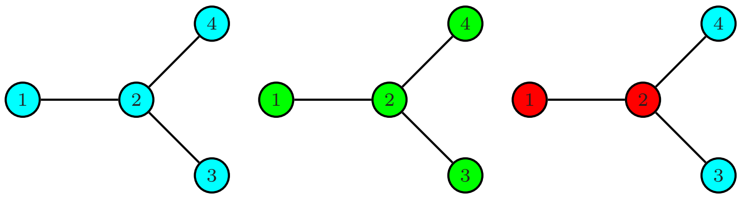
Clearly, any incompliant actor i has $P(i) = \varnothing$ and, so, she is party-invariant. Hence, only compliant actors might be party-variable depending on whether they are included or not in their $Q(i)$ sets.

When the initial configuration of parties over V contains all three parties, in general, any party could survive or vanish in the stationary configuration, depending on the structure of the graph G, the initial configuration, the actors' compliances and the random selections of interacting actors in the simulation. In other words, the possible outcomes of the simulation include clusterings in any party and fixations in any two or all the three parties.

Example 0.9. Consider the following graph of actors and the initial configuration of parties $\pi(1) = C, \pi(2) = L, \pi(3) = \pi(4) = R$:



If $\theta(i) = 1$, for all actors i, the stationary configuration is one of the following three:



Definition 0.10. The set of actors V is called *isotropic* if all thresholds $\theta(i)$ are independent of i, i.e., $\theta(i) = \theta$, for some $\theta \geq 0$.

Proposition 0.11. *Let $G = (V, E)$ be an incomplete graph, which is regular of degree $\upsilon \geq 2$. If V is isotropic and contains only two parties p, q such that $p \neq q^{-1}$, then the simulation:*

(1) *fixates if all actors are hardly compliant (i.e., $\frac{\upsilon}{2} < \theta \leq \upsilon$) and it*
(2) *clusters if all actors are easily compliant (i.e., $0 < \theta \leq \frac{\upsilon}{2}$).*

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