Wilson Coefficient for the effective weak Hamiltonian

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This library is a Mathematica implementation of the Wilson Coefficients for the $\Delta F = 1$ Weak Hamiltonian computed in *Buras et al.* `arXiv:hep-ph/95123801`. The goal for this library is to provide the user with a simple-to-use series of functions which automatically evaluate the Wilson Coefficients at a given mass scale μ and order in α_s. This library contains also all the basic blocks used in the evolution of the Wilson Coefficients, from the strong coupling constant coded up to 4 loops to the evolution matrices U. Starting from this basic functions it is possible to evaluate the Wilson Coefficients for all transitions related to the effective hamiltonian written below.

The documentation is organized as follows. The colored text represents Mathematica code and each function is described according to this structure:

`function[variable1,variable2, ...]`

- `variable1`: explanation
- `variable2`: explanation

Additional comments and explanations go here together with equations to define and describe what `function` does.

1 A brief theoretical introduction

By integrating out the weak bosons from the standard model we obtain an effective theory, called weak effective hamiltonian. As an effective theory it loses the property of being renormalizable, meaning that at each order in perturbation theory a new (finite) set of couplings and operators is needed to properly cancel the remaining divergences. Nevertheless, if we work at a fixed order the theory is finite and in the following we will concentrate only on the leading order effective theory.

Composite operators in the effective theory are expected to mix under renormalization, unless there is a symmetry protecting them. This mixing can be generically expressed through a Z-factor matrix, which can be perturbatively expanded in terms of the so-called anomalous dimension matrices. Once they are known, the operators or alternatively the Wilson Coefficients can be run to low energies and the implementation provided in this library includes effects up to $O(\alpha_\mathrm{s})$ and $O(\alpha_\mathrm{e})$.

The Wilson Coefficients are obtained by comparing (order by order in perturbation theory) amplitudes computed in the standard model against amplitudes computed in the effective theory. Obviously amplitudes are scheme-independent but the two parts of the calculation, the Wilson Coefficients and the low-energy matrix elements, individually depend on the scheme.

1.1 The $\Delta F = 1$ Hamiltonian

We are interested here in weak decays where either the strange or the bottom quantum numbers are violated by one unit. In general these processes can be described by a single effective hamiltonian

$$H_W = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i=1}^{10} \big[z_i(\mu) + \tau y_i(\mu) \big] Q_i(\mu). \tag{1}$$

The 10-operator basis is given below. The scope of this library is to provide the user with a few easy-to-use Mathematica functions to immediately compute the (renormalized) coefficients z_i and y_i at a given renormalization scale μ. The matrix elements Q_i must be computed in the same scheme to obtain consistent result at the end.

Current-Current Operators

$$Q_1 = (\bar{s}_i u_j)_{\mathrm{V-A}} \, (\bar{u}_j d_i)_{\mathrm{V-A}} \qquad Q_2 = (\bar{s} u)_{\mathrm{V-A}} \, (\bar{u} d)_{\mathrm{V-A}} \tag{2}$$

QCD-Penguins Operators

$$Q_3 = (\bar{s} d)_{\mathrm{V-A}} \sum_q (\bar{q} q)_{\mathrm{V-A}} \qquad Q_4 = (\bar{s}_i d_j)_{\mathrm{V-A}} \sum_q (\bar{q}_j q_i)_{\mathrm{V-A}} \tag{3}$$

$$Q_5 = (\bar{s} d)_{\mathrm{V-A}} \sum_q (\bar{q} q)_{\mathrm{V+A}} \qquad Q_6 = (\bar{s}_i d_j)_{\mathrm{V-A}} \sum_q (\bar{q}_j q_i)_{\mathrm{V+A}} \tag{4}$$

Electroweak-Penguins Operators

$$Q_7 = \frac{3}{2}\,(\bar{s}d)_{\mathrm{V-A}} \sum_q e_q\,(\bar{q}q)_{\mathrm{V+A}} \qquad\qquad Q_8 = \frac{3}{2}\,(\bar{s}_i d_j)_{\mathrm{V-A}} \sum_q e_q\,(\bar{q}_j q_i)_{\mathrm{V+A}} \tag{5}$$

$$Q_9 = \frac{3}{2}\,(\bar{s}d)_{\mathrm{V-A}} \sum_q e_q\,(\bar{q}q)_{\mathrm{V-A}} \qquad\qquad Q_{10} = \frac{3}{2}\,(\bar{s}_i d_j)_{\mathrm{V-A}} \sum_q e_q\,(\bar{q}_j q_i)_{\mathrm{V-A}} \tag{6}$$

2 Computation of the Wilson Coefficients

In this section we describe the most important functions of this library: those to compute the Wilson Coefficients at any given energy scale μ between 1GeV and M_{Z}.

2.1 Definition of the Wilson Coefficients

```
C1[a]
C2[a,ae]
C3[a,mt,MW,ae]
C4[a,mt,MW]
C5[a,mt,MW]
C6[a,mt,MW]
C7[mt,MW,ae]
C8
C9[mt,MW,ae]
C10
```

- `a`: value of α_{s}
- `mt`: value of the top quark mass
- `MW`: value of the W boson mass
- `ae`: value of α_{e}

The Wilson Coefficients at the weak scale are

$$C_1(M_\mathrm{W}) = \frac{11}{2}\frac{\alpha_\mathrm{s}(M_\mathrm{W})}{4\pi}\,, \tag{7}$$

$$C_2(M_\mathrm{W}) = 1-\frac{11}{6}\frac{\alpha_\mathrm{s}(M_\mathrm{W})}{4\pi}-\frac{35}{18}\frac{\alpha_\mathrm{e}}{4\pi}\,, \tag{8}$$

$$C_3(M_\mathrm{W}) = -\frac{\alpha_\mathrm{s}(M_\mathrm{W})}{24\pi}\widetilde{E}_0(x_t)+\frac{\alpha_\mathrm{e}}{6\pi}\frac{1}{\sin^2\theta_W}\left[2B_0(x_t)+C_0(x_t)\right]\,, \tag{9}$$

$$C_4(M_\mathrm{W}) = \frac{\alpha_\mathrm{s}(M_\mathrm{W})}{8\pi}\widetilde{E}_0(x_t)\,, \tag{10}$$

$$C_5(M_\mathrm{W}) = -\frac{\alpha_\mathrm{s}(M_\mathrm{W})}{24\pi}\widetilde{E}_0(x_t)\,, \tag{11}$$

$$C_6(M_\mathrm{W}) = \frac{\alpha_\mathrm{s}(M_\mathrm{W})}{8\pi}\widetilde{E}_0(x_t)\,, \tag{12}$$

$$C_7(M_\mathrm{W}) = \frac{\alpha_\mathrm{e}}{6\pi}\left[4C_0(x_t)+\widetilde{D}_0(x_t)\right]\,, \tag{13}$$

$$C_8(M_\mathrm{W}) = 0 \tag{14}$$

$$C_9(M_\mathrm{W}) = \frac{\alpha_\mathrm{e}}{6\pi}\left[4C_0(x_t)+\widetilde{D}_0(x_t)+\frac{1}{\sin^2\theta_W}(10B_0(x_t)-4C_0(x_t))\right]\,, \tag{15}$$

$$C_{10}(M_\mathrm{W}) = 0\,, \tag{16}$$

$$x_t = \frac{m_t^2}{M_\mathrm{W}^2}\,. \tag{17}$$

with the QED-related auxiliary functions

$$B_0(x) = \frac{1}{4}\left[\frac{x}{1-x}+\frac{x\ln x}{(x-1)^2}\right]\,, \tag{18}$$

$$C_0(x) = \frac{x}{8}\left[\frac{x-6}{x-1}+\frac{3x+2}{(x-1)^2}\ln x\right]\,, \tag{19}$$

$$D_0(x) = -\frac{4}{9}\ln x+\frac{-19x^3+25x^2}{36(x-1)^3}+\frac{x^2(5x^2-2x-6)}{18(x-1)^4}\ln x\,, \tag{20}$$

$$\widetilde{D}_0(x_t) = D_0(x_t)-\frac{4}{9}\,. \tag{21}$$

and the QCD-related ones

$$E_0(x) = -\frac{2}{3}\ln x+\frac{x(18-11x-x^2)}{12(1-x)^3}+\frac{x^2(15-16x+4x^2)}{6(1-x)^4}\ln x\,, \tag{22}$$

$$\widetilde{E}_0(x_t) = E_0(x_t)-\frac{2}{3} \tag{23}$$

2.2 Evolution of the Wilson Coefficients

`ComputeZ[mu,initAlphaMZ,loop,MZ,aem,init12]`

- `mu`: mass scale at which the coefficients v_i are computed

- `initAlphaMZ`: initial value of α_{s} at the electroweak scale
- `loop`: number of loops of α_{s}. Accepted values are 1,2,3,4.
- `MZ`: energy scale corresponding to `initAlphaMZ`. Default is `MZ`.
- `ae`: value of α_{e}. Default is 1/129.
- `init12`: bi-dimensional vector containing the values of the Wilson Coefficients C_1 and C_2 at the weak scale. Default corresponds to the results presented in the previous subsection.

The function returns the vector $\vec{z}(\mu)$ by taking into account possible quark thresholds. If $\mu > m_{\mathrm{b}}$

$$z_1(M_{\mathrm{W}}) = C_1(M_{\mathrm{W}})\,, \qquad z_2(M_{\mathrm{W}}) = C_2(M_{\mathrm{W}})\,. \tag{24}$$

$$\begin{pmatrix} z_1(m_{\mathrm{c}}) \\ z_2(m_{\mathrm{c}}) \end{pmatrix} = U_5(\mu, M_{\mathrm{W}}) \begin{pmatrix} z_1(M_{\mathrm{W}}) \\ z_2(M_{\mathrm{W}}) \end{pmatrix}\,, \tag{25}$$

whereas if $\mu = m_{\mathrm{b}}$

$$\begin{pmatrix} z_1(m_{\mathrm{c}}) \\ z_2(m_{\mathrm{c}}) \end{pmatrix} = M(m_{\mathrm{b}})\, U_5(m_{\mathrm{b}}, M_{\mathrm{W}}) \begin{pmatrix} z_1(M_{\mathrm{W}}) \\ z_2(M_{\mathrm{W}}) \end{pmatrix}\,, \tag{26}$$

If $\mu < m_{\mathrm{b}}$

$$\begin{pmatrix} z_1(m_{\mathrm{c}}) \\ z_2(m_{\mathrm{c}}) \end{pmatrix} = U_4(\mu, m_{\mathrm{b}})\, M(m_{\mathrm{b}})\, U_5(m_{\mathrm{b}}, M_{\mathrm{W}}) \begin{pmatrix} z_1(M_{\mathrm{W}}) \\ z_2(M_{\mathrm{W}}) \end{pmatrix}\,, \tag{27}$$

whereas if $\mu = m_{\mathrm{c}}$

$$\begin{pmatrix} z_1(m_{\mathrm{c}}) \\ z_2(m_{\mathrm{c}}) \end{pmatrix} = U_4(m_{\mathrm{c}}, m_{\mathrm{b}})\, M(m_{\mathrm{b}})\, U_5(m_{\mathrm{b}}, M_{\mathrm{W}}) \begin{pmatrix} z_1(M_{\mathrm{W}}) \\ z_2(M_{\mathrm{W}}) \end{pmatrix}\,, \tag{28}$$

$$\vec{z}(m_{\mathrm{c}}) = \begin{pmatrix} z_1(m_{\mathrm{c}}) \\ z_2(m_{\mathrm{c}}) \\ \alpha_{\mathrm{s}}/(36\pi) z_2(m_{\mathrm{c}}) \\ -\alpha_{\mathrm{s}}/(12\pi) z_2(m_{\mathrm{c}}) \\ \alpha_{\mathrm{s}}/(36\pi) z_2(m_{\mathrm{c}}) \\ -\alpha_{\mathrm{s}}/(12\pi) z_2(m_{\mathrm{c}}) \\ \alpha_{\mathrm{e}}/(6\pi) F_{\mathrm{e}}(m_{\mathrm{c}}) \\ 0 \\ \alpha_{\mathrm{e}}/(6\pi) F_{\mathrm{e}}(m_{\mathrm{c}}) \\ 0 \end{pmatrix}\,, \tag{29}$$

with

$$F_{\mathrm{e}}(m_{\mathrm{c}}) = -\frac{4}{9}\,\left(3 z_1(m_{\mathrm{c}}) + z_2(m_{\mathrm{c}})\right)\,. \tag{30}$$

Finally if $\mu < m_{\rm c}$

$$\vec{z}(\mu) = U_3(\mu, m_{\rm c})\vec{z}(m_{\rm c})\,. \tag{31}$$

`ComputeY[z,mu,initAlphaMZ,loop,MZ,aem,init]`

- `z`: 10 dimensional vector containg the values of the Wilson coefficients $z_i(\mu)$
- `mu`: mass scale at which the coefficients v_i are computed
- `initAlphaMZ`: initial value of $\alpha_{\rm s}$ at the electroweak scale
- `loop`: number of loops of $\alpha_{\rm s}$. Accepted values are 1,2,3,4.
- `MZ`: energy scale corresponding to `initAlphaMZ`. Default is `MZ`.
- `ae`: value of $\alpha_{\rm e}$. Default is 1/129.
- `init`: ten dimensional vector containing the values of the Wilson Coefficients C_i at the weak scale. Default corresponds to the results presented in the previous subsection.

$$y_i(\mu) = v_i(\mu) - z_i(\mu)\,. \tag{32}$$

`ComputeV[mu,initAlphaMZ,loop,MZ,aem,init]`

- `mu`: mass scale at which the coefficients v_i are computed
- `initAlphaMZ`: initial value of $\alpha_{\rm s}$ at the electroweak scale
- `loop`: number of loops of $\alpha_{\rm s}$. Accepted values are 1,2,3,4.
- `MZ`: energy scale corresponding to `initAlphaMZ`. Default is `MZ`.
- `ae`: value of $\alpha_{\rm e}$. Default is 1/129.
- `init`: ten dimensional vector containing the values of the Wilson Coefficients C_i at the weak scale. Default corresponds to the results presented in the previous subsection.

The functions return the vector $\vec{v}(\mu)$ by taking into account possible quark thresholds. If $\mu > m_{\rm b}$

$$\vec{v}(\mu) = U_5(\mu, M_{\rm W})\vec{C}(M_{\rm W})\,, \tag{33}$$

whereas if $\mu = m_{\rm b}$

$$\vec{v}(\mu) = M(m_\mathrm{b})U_5(m_\mathrm{b}, M_\mathrm{W})\vec{C}(M_\mathrm{W})\,, \tag{34}$$

If $\mu < m_\mathrm{b}$

$$\vec{v}(\mu) = U_4(\mu, m_\mathrm{b})M(m_\mathrm{b})U_5(m_\mathrm{b}, M_\mathrm{W})\vec{C}(M_\mathrm{W})\,, \tag{35}$$

whereas if $\mu = m_\mathrm{c}$

$$\vec{v}(\mu) = M(m_\mathrm{c})U_4(m_\mathrm{c}, m_\mathrm{b})M(m_\mathrm{b})U_5(m_\mathrm{b}, M_\mathrm{W})\vec{C}(M_\mathrm{W})\,, \tag{36}$$

Finally if $\mu < m_\mathrm{c}$

$$\vec{v}(\mu) = U_3(\mu, m_\mathrm{c})M(m_\mathrm{c})U_4(m_\mathrm{c}, m_\mathrm{b})M(m_\mathrm{b})U_5(m_\mathrm{b}, M_\mathrm{W})\vec{C}(M_\mathrm{W})\,, \tag{37}$$

2.3 LO and NLO prescriptions

`ReduceOrder[expr,order]`

- `expr`: expression of the Wilson Coefficients y_i and z_i
- `order`: accepted values are LO and NLO.

This function takes as input the full expression of the Wilson Coefficients y_i and z_i and reduce it to the desired order.

3 Basic QCD Functions and input parameters

In this section we describe the functions related to the strong coupling constant and the physical constants which are automatically loaded with this library.

3.1 Strong coupling constant

`beta0[Nc,Nf]`
`beta1[Nc,Nf]`

- `Nc`: number of colors
- `Nf`: number of flavors

First coefficients b_0 and b_1 of the QCD β-function.

`alphas[mu,L,Nc,Nf,loop]`

- `mu`: energy scale at which α_s is computed
- `L`: value of the Λ parameter
- `Nc`: number of colors
- `Nf`: number of flavors
- `loop`: number of loops. Accepted values are 1,2,3,4. Default is 2.

`FindLambda[a,mu,Nc,Nf,loop]`

- `a`: input value of α_s
- `mu`: energy scale at which α_s is computed
- `Nc`: number of colors
- `Nf`: number of flavors
- `loop`: number of loops. Accepted values are 1,2,3,4. Default is 2.

The function returns the value of the Λ parameter which solves the matching equation

$$\alpha_s^{\text{input}} = \alpha_s^{(\text{loop})}(\mu, \Lambda, N_c, N_f) \tag{38}$$

3.2 Constants

`MW,MZ,mtop,mbottom,mcharm,alphasMZ`

$$M_{\rm W} = 80.2\text{GeV} \quad , \quad M_{\rm Z} = M_{\rm W}/\sqrt{1-0.23^2}\,, \tag{39}$$

$$m_{\rm t} = 170\text{GeV} \quad , \quad m_{\rm b} = 4.4\text{GeV}\,, m_{\rm c} = 1.3\text{GeV}\,, \tag{40}$$

$$\alpha_{\rm s}(M_{\rm Z}) = 0.117 \tag{41}$$

4 Examples

Example 1 : computation of the LO Wilson Coefficients at $\mu = 4\text{GeV}$ using $\Lambda^4 = 0.325\text{GeV}$ and α_s in LO. Note that to obtain the Wilson Coefficients in the LO approximation we must use the one-loop running of α_s, while for the NLO approximation the two-loop running.

Since we want to start from the Λ parameter in the 4-flavor theory we have to first find the Λ parameter in the 5-flavor theory where we match the standard model with effective weak hamiltonian. Hence we start by computing α_s at the bottom threshold

```
amb = alphas[mbottom,0.325,3,4,1];
```

and then we compute Λ^5 by matching the 4 and 5 flavors theories

```
L = FindLambda[amb,mbottom,3,5,1];
aMW = alphas[mMW,L,3,5,1];
```

Now we can evolve the Wilson Coefficients down to 4GeV

```
z = ComputeZ[4,aMW,1,MW];
y = ComputeY[z,4,aMW,1,MW];
```

To obtain the LO results we simply use

```
ReduceOrder[z,LO]
ReduceOrder[y,LO]
```

Example 2: computation of the NLO Wilson Coefficients at $\mu = 1\text{GeV}$ using $\alpha_s(M_Z)$

```
z = ComputeZ[4,alphasMZ,2];
y = ComputeY[z,4,alphasMZ,2];
```

To obtain the NLO results we simply use

```
ReduceOrder[z,NLO]
ReduceOrder[y,NLO]
```

5 Anomalous Dimension Matrices

In thi section we describe the functions to compute the anomalous dimension matrices used in the evolution of the Wilson Coefficients.

5.1 QCD anomalous dimension matrices

`gammas0[Nc,Nf,size,Nu]`

- `Nc`: number of colors
- `Nf`: number of flavors
- `size`: size of returned matrix. Default is 10
- `Nu`: number of up quarks. Default is 2

$$\gamma_{\mathrm{s}}^{(0)} \tag{42}$$

`gammas1[Nf,size,Nu]`

- `Nf`: number of flavors
- `size`: size of returned matrix. Default is 10
- `Nu`: number of up quarks. Default is 2

$$\gamma_{\mathrm{s}}^{(1)} \tag{43}$$

5.2 QED anomalous dimension matrices

`gammae0[Nc,Nf,size,Nu]`

- `Nc`: number of colors
- `Nf`: number of flavors
- `size`: size of returned matrix. Default is 10
- `Nu`: number of up quarks. Default is 2

$$\gamma_{\mathrm{e}}^{(0)} \tag{44}$$

`gammase1[Nf,size,Nu]`

- `Nf`: number of flavors
- `size`: size of returned matrix. Default is 10
- `Nu`: number of up quarks. Default is 2

$$\gamma_{\rm se}^{(1)} \tag{45}$$

6 Renormalization Group Functions

In this section we describe the core functions used to implement the RG evolution of the Wilson Coefficients.

6.1 QCD RG functions

`U0[a1,a2,b0,g0]`

- `a1,a2`: values of $\alpha_{\rm s}$ at two different scales $\mu_1 > \mu_2$
- `b0`: coefficient b_0 of the QCD β-function
- `g0`: LO anomalous dimension matrix $\gamma_{\rm s}^{(0)}$

$$U^{(0)}(\mu, m) = V \left(\left[\frac{\alpha_{\rm s}(m)}{\alpha_{\rm s}(\mu)} \right]^{\frac{\vec{\gamma}^{(0)}}{2\beta_0}} \right)_D V^{-1} \tag{46}$$

with

$$\gamma_D^{(0)} = V^{-1} \gamma^{(0)T} V \tag{47}$$

`J[b0,b1,g0,g1]`

- `b0`: coefficient b_0 of the QCD β-function
- `b1`: coefficient b_1 of the QCD β-function
- `g0`: LO anomalous dimension matrix $\gamma_{\rm s}^{(0)}$
- `g1`: NLO anomalous dimension matrix $\gamma_{\rm s}^{(1)}$

$$J = V H V^{-1} \tag{48}$$

$$H_{ij} = \delta_{ij} \gamma_i^{(0)} \frac{\beta_1}{2\beta_0^2} - \frac{G_{ij}}{2\beta_0 + \gamma_i^{(0)} - \gamma_j^{(0)}} \tag{49}$$

$$G = V^{-1} \gamma^{(1)T} V \tag{50}$$

$$\gamma_D^{(0)} = V^{-1} \gamma^{(0)T} V \tag{51}$$

`U[a1,a2,b0,g0,J]`

- `a1,a2`: values of α_s at two different scales $\mu_1 > \mu_2$
- `b0`: coefficient b_0 of the QCD β-function
- `g0`: LO anomalous dimension matrix $\gamma_s^{(0)}$
- `J`: J matrix obtained from `J[b0,b1,g0,g1]`

$$U(\mu,m) = U^{(0)}(\mu,m) + \frac{1}{4\pi}\Big[\alpha_s(\mu) J U^{(0)}(\mu,m) - \alpha_s(m) U^{(0)}(\mu,m) J\Big] \tag{52}$$

6.2 QED RG functions

`M1[b0,b1,ge0,gse1,J]`

- `b0`: coefficient b_0 of the QCD β-function
- `b1`: coefficient b_1 of the QCD β-function
- `ge0`: LO anomalous dimension matrix $\geq$
- `gse1`: NLO anomalous dimension matrix $\gamma_{se}^{(1)}$
- `J`: J matrix obtained from `J[b0,b1,g0,g1]`

$$M^{(1)} = V^{-1}\left(\gamma_{se}^{(1)T} - \frac{\beta_1}{\beta_0}\gamma_e^{(0)T} + \left[\gamma_e^{(0)T}, J\right]\right) V\,. \tag{53}$$

`R[a1,a2,b0,g0,ge0,M1,J]`

- `a1,a2`: values of α_s at two different scales $\mu_1 > \mu_2$
- `b0`: coefficient b_0 of the QCD β-function
- `g0`: LO anomalous dimension matrix $\gamma_s^{(0)}$
- `ge0`: LO anomalous dimension matrix $\geq$
- `M1`: M_1 matrix obtained from `M1[b0,b1,ge0,gse1,J]`
- `J`: J matrix obtained from `J[b0,b1,g0,g1]`

$$R(m_1,m_2) \equiv -\frac{2\pi}{\beta_0}\, V\, \left(K^{(0)}(m_1,m_2) + \frac{1}{4\pi}\sum_{i=1}^{3} K_i^{(1)}(m_1,m_2)\right) V^{-1} \tag{54}$$

$$(K^{(0)}(m_1,m_2))_{ij} = \frac{M_{ij}^{(0)}}{a_i - a_j - 1}\left[\left(\frac{\alpha_s(m_2)}{\alpha_s(m_1)}\right)^{a_j}\frac{1}{\alpha_s(m_1)} - \left(\frac{\alpha_s(m_2)}{\alpha_s(m_1)}\right)^{a_i}\frac{1}{\alpha_s(m_2)}\right] \tag{55}$$

$$\left(K_1^{(1)}(m_1,m_2)\right)_{ij} = \begin{cases} \frac{M_{ij}^{(1)}}{a_i-a_j}\left[\left(\frac{\alpha_s(m_2)}{\alpha_s(m_1)}\right)^{a_j} - \left(\frac{\alpha_s(m_2)}{\alpha_s(m_1)}\right)^{a_i}\right] & i \neq j \\ M_{ii}^{(1)}\left(\frac{\alpha_s(m_2)}{\alpha_s(m_1)}\right)^{a_i} \ln\frac{\alpha_s(m_1)}{\alpha_s(m_2)} & i = j \end{cases} \tag{56}$$

$$K_2^{(1)}(m_1,m_2) = -\alpha_s(m_2)\, K^{(0)}(m_1,m_2)\, H\,, \tag{57}$$

$$K_3^{(1)}(m_1,m_2) = \alpha_s(m_1)\, H\, K^{(0)}(m_1,m_2) \tag{58}$$

with

$$M^{(0)} = V^{-1}\,\gamma_e^{(0)T}\,V \tag{59}$$

6.3 Quark threshold matching functions

`Fs[a,z12]`

- `a`: value of α_s at the mass threshold m_c.
- `z12`: vector with the two Wilson Coefficients z_1 and z_2 computed at the quark threshold m_c

The function `Fs` returns the vector below

$$\vec{z}(m_c) = \begin{pmatrix} z_1(m_c) \\ z_2(m_c) \\ \alpha_s/(36\pi)z_2(m_c) \\ -\alpha_s/(12\pi)z_2(m_c) \\ \alpha_s/(36\pi)z_2(m_c) \\ -\alpha_s/(12\pi)z_2(m_c) \end{pmatrix}, \tag{60}$$

`Fse[a,z12,ae]`

- `a`: value of α_s at the mass thresholds m_b and m_c.
- `z12`: vector with the two Wilson Coefficients z_1 and z_2 computed at the quark thresholds m_c and m_b
- `ae`: value of α_e. Default is 0.

The function `Fse` returns the vector below

$$\vec{z}(m_c) = \begin{pmatrix} z_1(m_c) \\ z_2(m_c) \\ \alpha_s/(36\pi) z_2(m_c) \\ -\alpha_s/(12\pi) z_2(m_c) \\ \alpha_s/(36\pi) z_2(m_c) \\ -\alpha_s/(12\pi) z_2(m_c) \\ \alpha_e/(6\pi) F_e(m_c) \\ 0 \\ \alpha_e/(6\pi) F_e(m_c) \\ 0 \end{pmatrix} , \tag{61}$$

with

$$F_e(m_c) = -\frac{4}{9} \, (3z_1(m_c) + z_2(m_c)) \, . \tag{62}$$

`M[mu,a,ae,size]`

- `mu`: energy scale of `a`. Values accepted are m_c and m_b.
- `a`: value of α_s at the mass thresholds m_b and m_c.
- `ae`: value of α_e. Default is 0.
- `size`: size of the returned matrix. Default is 10.

$$M(m) = 1 + \frac{\alpha_s(m)}{4\pi} \, \delta r_s^T + \frac{\alpha_e}{4\pi} \, \delta r_e^T \, . \tag{63}$$

The routine automatically uses

$$\begin{aligned} \delta r_s^T &= \frac{5}{18} P \, (0,0,0,-2,0,-2,0,1,0,1) \\ \delta r_e^T &= \frac{10}{81} \bar{P} \, (0,0,6,2,6,2,-3,-1,-3,-1) \end{aligned} \tag{64}$$

if $\mu = m_b$, whereas if $\mu = m_c$

$$\begin{aligned} \delta r_s^T &= -\frac{5}{9} P \, (0,0,0,1,0,1,0,1,0,1) \\ \delta r_e^T &= -\frac{40}{81} \bar{P} \, (0,0,3,1,3,1,3,1,3,1) \end{aligned} \tag{65}$$

with

$$P^T = (0,0,-\frac{1}{3},1,-\frac{1}{3},1,0,0,0,0) \, , \tag{66}$$

$$\bar{P}^T = (0,0,0,0,0,0,1,0,1,0) \, . \tag{67}$$

6.4 QCD+QED RG functions

`FullU[a1,a2,ae,b0,g0,ge0,M1,J]`

- `a1,a2`: values of α_s at two different scales $\mu_1 > \mu_2$
- `ae`: value of α_e
- `b0`: coefficient b_0 of the QCD β-function
- `g0`: LO anomalous dimension matrix $\gamma_s^{(0)}$
- `ge0`: LO anomalous dimension matrix $\geq$
- `M1`: M_1 matrix obtained from `M1[b0,b1,ge0,gse1,J]`
- `J`: J matrix obtained from `J[b0,b1,g0,g1]`

$$U(m_1, m_2, \alpha_e) = U(m_1, m_2) + \frac{\alpha_e}{4\pi} R(m_1, m_2)\,, \tag{68}$$