On the information disclosed by the difference from hamiltonian evolution

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Abstract

A dissipative version of hamiltonian mechanics is proposed via a principle of minimal information content disclosed by the deviation from hamiltonian evolution.

1 Introduction

In hamiltonian mechanics a physical system is described by a state vector $q \in X$ and a momentum vector $p \in Y$. The evolution in time of the system is governed by a hamiltonian function $H = H(q,p,t)$, via the equations:

$$\left\{ \begin{array}{rcl} \dot{q} & = & \frac{\partial H}{\partial p}(q,p,t) \\ -\dot{p} & = & \frac{\partial H}{\partial q}(q,p,t) \end{array} \right. \quad (1)$$

where $\dot{q}$, $\dot{p}$ denote derivatives with respect to time. The evolution is reversible.

We propose a dissipative modification of these equations.

Definition 1.1 *The physical system is described by the hamiltonian $H : X \times Y \times \mathbb{R} \rightarrow \mathbb{R}$ and a likelihood function*

$$\pi : (X \times Y)^3 \rightarrow [0,1]$$

The vector $\eta = (\eta_q, \eta_p) \in X \times Y$ is called deviation (or gap) from hamiltonian evolution (driven by the hamiltonian H):

$$\left\{ \begin{array}{rcl} \dot{q} & = & \frac{\partial H}{\partial p}(q,p,t) \, + \, \eta_q \\ -\dot{p} & = & \frac{\partial H}{\partial q}(q,p,t) \, + \, \eta_p \end{array} \right. \quad (2)$$

The evolution of the system maximizes the likelihood:

$$\pi(z, \dot{z}, \eta) \;=\; 1 \tag{3}$$

with the notations $z = (q, p) \in X \times Y$, $\dot{z} = (\dot{p}, \dot{q}) \in X \times Y$.

The likelihood function π has an associated information content function

$$I : (X \times Y)^3 \to [0, +\infty] \;,\; I(z, z', z") \;=\; -\ln \pi(z, z', z") \tag{4}$$

with the convention that $-\ln 0 \;=\; +\infty$.

The likelihood function and the associated information content function are not, otherwise the generalization would be too vague. We collect in the next defintion the properties of likelihood and information content functions.

Definition 1.2 *The likelihood and information content satisfy:*

(a) The information content function (4) is convex in each of the 2nd and 3rd variables,

(b) for any $z, z' \in X \times Y$, *if any of the functions* $\pi(z, z', \cdot)$, $\pi(z, \cdot, z')$ *have a maximum then the maximum equals* 0 *or* 1.

The equation (3) can be rephrased as: given the hamiltonian H and the information content function I, the physical system evolves such that at any moment it minimizes the information content of the gap from a hamiltonian evolution. Indeed, an evolution of the system is a curve $c_0 : [0, T] \to X \times Y$ with the property that it minimizes the information content gap functional:

$$G(c) = \int_0^T I\left(q(t), p(t), \dot{q}(t), \dot{p}(t), \dot{q}(t) - \frac{\partial H}{\partial p}(q(t), p(t), t), -\dot{p}(t) - \frac{\partial H}{\partial q}(q(t), p(t), t)\right) \mathrm{d}t \tag{5}$$

among all admissible evolution curves $c(t) = (q(t), p(t))$.

2 Examples

The notations and useful general definitions are in section 3 . The mentioned section contains as well a description of the algebraic structures (like duality products), convex analysis notions and topological structure which is needed sometimes.

Pure Hamiltonian evolution. Let's pick the information content (4)to be:

$$I(z, z', z") \ = \ \chi_0(z") = \left\{ \begin{array}{ll} 0 & \mbox{if } z" = 0 \\ +\infty & \mbox{otherwise} \end{array} \right.$$

This corresponds to a likelihood function:

$$\pi(z, z', z") \ = \ \left\{ \begin{array}{ll} 1 & \mbox{if } z" = 0 \\ 0 & \mbox{otherwise} \end{array} \right.$$

The maximization of the likelihood (3) implies that the gap vector $\eta = 0$, therefore the evolution equations (2) reduce to the pure Hamiltonian evolution equations (1).

This example is trivial, we need a method to construct more interesting ones. One such method is based on the following observation, adapted from [10], section 2. We use the notations explained in section 3, in particular we use the duality $\langle\langle \cdot, \cdot \rangle\rangle$ and we supose that we have on X, Y a topology compatible with it, so that the information content function I is lower semicontinuous (lsc).

Proposition 2.1 *The information content function (4) satisfies the conditions from Definition 1.2 (and it is lsc) if and only if the function*

$$b : (X \times Y)^3 \rightarrow \mathbb{R} \cup \{+\infty\} \ , \ b(z, z', z") \ = \ I(z, z', z") \ + \ \langle\langle z', z" \rangle\rangle \tag{6}$$

satisfies: for any $z \in X \times Y$,

(a) for any $z', z" \in X \times Y$ the functions $b(z, z', \cdot)$ and $b(z, \cdot, z")$ are convex (and lsc),

(b) for any $z', z" \in X \times Y$ we have the equivalences

$$z' \in \partial b(z, z', \cdot)(z") \iff z" \in \partial b(z, \cdot, z")(z') \iff I(z, z', z") = 0 \tag{7}$$

where "∂" denotes a subgradient, see section 3 for notations.

By concentrating our attention to the function b, instead of the information content I, we can build a host of examples. Indeed, for any lsc and convex function

$$\Phi : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$$

the associated function

$$b(z, z', z") \ = \ \Phi(z') + \Phi^*(z") \tag{8}$$

satisfies the conditions (a), (b) from Proposition 2.1 , where Φ^* is the polar, or Fenchel conjugate of Φ. Indeed, the mentioned condition (b) is just a reformulation of the Fenchel inequality.

Corollary 2.2 *For an information content function of the form*

$$I(z, z', z") = \Phi(z') + \Phi^*(z") - \langle\langle z', z" \rangle\rangle \tag{9}$$

the equation(3) is equivalent with the symplectic Brezis-Ekeland-Nayroles principle [7] definition 4.1, [8] definition 1.1:

$$\eta \in \partial\Phi(\dot{z}) \tag{10}$$

Viscosity, Rayleigh dissipation. In particular, let's pick

$$\Phi(z') = \Phi(q', p') = \phi(q')$$

where $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is a convex, lsc function. A straightforward computation of Φ^* gives:

$$\Phi^*(z") = \Phi^*(q", p") = \sup \left\{ \langle\langle (q', p'), (q", p") \rangle\rangle - \phi(q') \mid q' \in X, p' \in Y \right\} =$$

$$= \sup \left\{ \langle q', p" \rangle + \langle q", p' \rangle - \phi(q') \mid q' \in X, p' \in Y \right\} =$$

$$= \chi_0(q") + \phi^*(p")$$

therefore the information content has the expression:

$$I(z, z', z") = \phi(q') + \phi^*(p") + \chi_0(q") - \langle q', p" \rangle$$

By the corollary 2.2 we obtain the equations:

$$\left\{ \begin{array}{rcl} \dot{q} & = & \frac{\partial H}{\partial p}(q, p, t) \\ -\dot{p} & = & \frac{\partial H}{\partial q}(q, p, t) + \eta_p \\ \eta_p & \in & \partial\phi(\dot{q}) \end{array} \right. \tag{11}$$

This shows that ϕ is a Rayleigh dissipation potential.

Plasticity. Take a hamiltonian system with space state X and momentum space Y and supplement the state and the momentum with a new pair of state and momentum spaces:

$$(q, q_I) \in X \times X_I \,,\ (p, p_I) \in Y \times Y_I$$

Suppose further that the pair of spaces (X_I, Y_I) are in duality, so that we can define

$$\langle (q, q_I), (p, p_I) \rangle \ = \ \langle q, p \rangle \ + \ \langle q_I, p_I \rangle$$

which leads us to a duality product of $(X \times X_I) \times (Y \times Y_I)$ with itself:

$$\langle\langle (q', q'_I, p', p'_I), (q", q_I", p", p_I") \rangle\rangle \ = \ \langle q', p" \rangle + \langle q", p' \rangle + \langle q'_I, p_I" \rangle + \langle q_I", p'_I \rangle$$

In this setting, we take $N = (X \times X_I) \times (Y \times Y_I)$ and

$$\Phi \left(q', q'_I, p', p'_I \right) \ = \ \phi \left(p'_I \right)$$

where $\phi : Y_I \to \mathbb{R} \cup \{+\infty\}$ is a convex lsc function.

There are two differences with respect to the viscosity example: there is a cartesian decomposition of the state space and momentum space, and the dissipation potential depends on (a component of) the momentum variable, while previously the dependence was on the state variable.

By a computation analoguous with the one from the previous example we obtain:

$$\Phi^* \left(q", q_I", p", p_I" \right) \ = \ \phi^* \left(q_I" \right) \ + \ \chi_0 \left(p" \right) \ + \ \chi_0 \left(p_I" \right) \ + \ \chi_0 \left(q" \right)$$

Via the corollary 2.2 we obtain the equations:

$$\left\{ \begin{array}{rcl} \dot{q} & = & \frac{\partial H}{\partial p}(q, p, q_I, p_I, t) \\ \dot{q}_I & = & \frac{\partial H}{\partial p_I}(q, p, q_I, p_I, t) \ + \ \eta_{q,I} \\ -\dot{p} & = & \frac{\partial H}{\partial q}(q, p, q_I, p_I, t) \\ -\dot{p}_I & = & \frac{\partial H}{\partial q_I}(q, p, q_I, p_I, t) \\ \eta_{q,I} & \in & \partial \phi \left(\right) \end{array} \right. \tag{12}$$

In particular, let's take $X = X_I$, $Y = Y_I$, Hilbert spaces, and a hamiltonian of the form:

$$H(q, p, q_I, p_I, t) \ = \ K(p) \ + \ E \left(q - q_I \right) \ - \ \langle q, f(t) \rangle$$

where $K(p)$ is the kinetic energy and $E \left(q - q_I \right)$ is the elastic energy. We denote the elastic force by

$$\sigma \ = \ \frac{\partial E}{\partial q}(q - q_I)$$

The system of equations (12) becomes:

$$\left\{ \begin{array}{rcl} \dot{q} & = & \frac{\partial K}{\partial p}(p) \\ \dot{p} & = & f(t) \ - \ \sigma \\ \dot{q}_I & \in & \partial \phi \left(\sigma \right) \end{array} \right. \tag{13}$$

Damage.

3 Notations and useful definitions

General notations. The space X of states $q \in X$ and the space Y of momenta $p \in Y$ are real topological vector spaces in duality:

$$(q, p) \in X \times Y \mapsto \langle q, p \rangle \in \mathbb{R}$$

We suppose the usual: the duality is bilinear, continuous and for any linear and continuous functions $L : X \to \mathbb{R}$, $G : Y \to \mathbb{R}$ there exist $q \in X$, $p \in Y$ such that $L(\cdot) = \langle \cdot, p \rangle$ and $G(\cdot) = \langle q, \cdot \rangle$.

The space $N = X \times Y$ is in duality $\langle\langle \cdot, \cdot \rangle\rangle : N \times N \to \mathbb{R}$ with itself by:

$$\langle\langle (q_1, p_1), (q_2, p_2) \rangle\rangle \ = \ \langle q_1, p_2 \rangle \ + \ \langle q_2, p_1 \rangle$$

The space N is also symplectic, with the symplectic form defined by: for any $z_1 = (q_1, p_1)$, $z_2 = (q_2, p_2)$

$$\omega(z_1, z_2) \ = \ \langle\langle Jz_1, z_2 \rangle\rangle \ = \ \langle q_1, p_2 \rangle \ - \ \langle q_2, p_1 \rangle$$

For any differentiable function $H : N \to \mathbb{R}$ the gradient of H at a point $z \in N$ is the element $DH(z) \in N$ with the property that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(H\left(z + \varepsilon z' \right) - H(z) \right) \ = \ \langle\langle DH(z), z' \rangle\rangle$$

and the symplectic gradient of H is $XH(z) \in N$ is defined in a similar way by the equality

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(H\left(z + \varepsilon z' \right) - H(z) \right) \ = \ \omega(XH(z), z')$$

If we use the partial derivatives notation

$$DH(q, p) \ = \ \left(\frac{\partial}{\partial p} H(q, p), \frac{\partial}{\partial q} H(q, p) \right)$$

then

$$XH(q, p) \ = \ \left(\frac{\partial}{\partial p} H(q, p), - \frac{\partial}{\partial q} H(q, p) \right)$$

With the introduction of the linear conjugation

$$\bar{\cdot} : N \to N \ , \ \overline{(q, p)} \ =, (q, -p)$$

we have $XH(z) \ = \ \overline{DH(z)}$.

Convex analysis notations. These are the classical ones from Moreau [15]. We add $+\infty$ the field of reals $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. The addition operation is extended with $a + (+\infty) = +\infty$ for any $a \in \bar{\mathbb{R}}$. The multiplication with positive numbers is extended with: if $a > 0$ then $a\,(+\infty) = +\infty$.

For any function $\phi : X \rightarrow \bar{\mathbb{R}}$, it's domain is $dom\,\phi = \{x \in X \,:\, \phi(x) \in \mathbb{R}\}$.

The set of lower semicontinuous (lsc), convex functions defined on X, with non-empty domain is $\Gamma_0(X)$. The indicator function $\chi_A \in \Gamma_0(X)$ of a convex and closed set $A \subset X$ is

$$\chi_A(x) = \left\{ \begin{array}{ll} 0 & \text{if } x \in A \\ +\infty & \text{otherwise} \end{array} \right.$$

For any natural number $n \geq 1$, a non empty set $A \subset X \times Y$ is n-monotone monotone if for any collection $\{(x_k, y_k) \in A \,:\, k = 0, 1, ..., n\}$ we have the inequality:

$$\langle x_n - x_0, y_n \rangle + \sum_{1}^{n} \langle x_{k-1} - x_k, y_{k-1} \rangle \, \geq 0 \quad .$$

The set $A \subset X \times Y$ is maximally n-monotone if it is n-monotone and maximal with respect to the inclusion of sets.

The set A is cyclically monotone if it is n monotone for any natural number $n \geq 1$. It is cyclically maximal monotone if it cyclically monotone and maximal with respect to the inclusion of sets.

The subdifferential of a function $\phi : X \rightarrow \bar{\mathbb{R}}$ at a point $x \in X$ is the set:

$$\partial \phi(x) = \{ u \in Y \mid \forall z \in X \ \langle z - x, u \rangle \leq \phi(z) - \phi(x) \} \ .$$

The polar of a function $\phi : X \rightarrow \bar{\mathbb{R}}$ is $\phi^* : Y \rightarrow \bar{\mathbb{R}}$

$$\phi^*(y) \ = \ \sup \{ \langle x, y \rangle - \phi(x) \ \mid \ x \in X \}$$

The polar is always convex and lsc.

Polars and subgradients are related by the Fenchel inequality. For any function $\phi : X \rightarrow \bar{\mathbb{R}}$ which is convex, lsc, we define

$$c(x,y) \ = \ \phi(x) \ + \ \phi^*(y) \ - \ \langle x, y \rangle$$

for any $x \in X$, $y \in Y$. The Fenchel inequality has two parts:

(a) $c(x, y) \geq 0$

(b) $c(x, y) = 0 \iff y \in \partial\phi(x) \iff x \in \partial\phi^*(y)$

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