Dec 22, 2022

Engineering Statistics

Week 9: Hypothesis testing-II

Introduction

Introduction

- In virtually every area of human activity, new procedures are invented and existing techniques revised. Advances occur whenever a new technique proves to be better than the old.

- To compare them, we conduct experiments, collect data about their performance, and then draw conclusions from statistical analyses.

- The manner in which sample data are collected, called an experimental design or sampling design, is crucial to an investigation.

- Here, we introduce two experimental designs that are most fundamental to a comparative study:

  (1) Independent samples

  (2) A matched pairs sample.

Introduction

Treatment: Treatment refers to the things that are being compared.

Experimental unit: The basic unit that is exposed to one treatment or another are called an experimental unit or experimental subject.

Response: The characteristic that is recorded after the application of a treatment to a subject is called the response.

Introduction

Experimental design refers to the manner in which subjects are chosen and assigned to treatments.

For comparing two treatments, the two basic types of design are:

1. Independent samples

2. Matched pairs sample

Independent samples

- The case of independent samples arises when the subjects are randomly divided into two groups, one group is assigned to treatment 1 and the other to treatment 2.

- The response measurements for the two treatments are then unrelated because they arise from separate and unrelated groups of subjects.

- Consequently, each set of response measurements can be considered a sample from a population, and we can speak in terms of a comparison between two population distributions.

Matched pairs sample

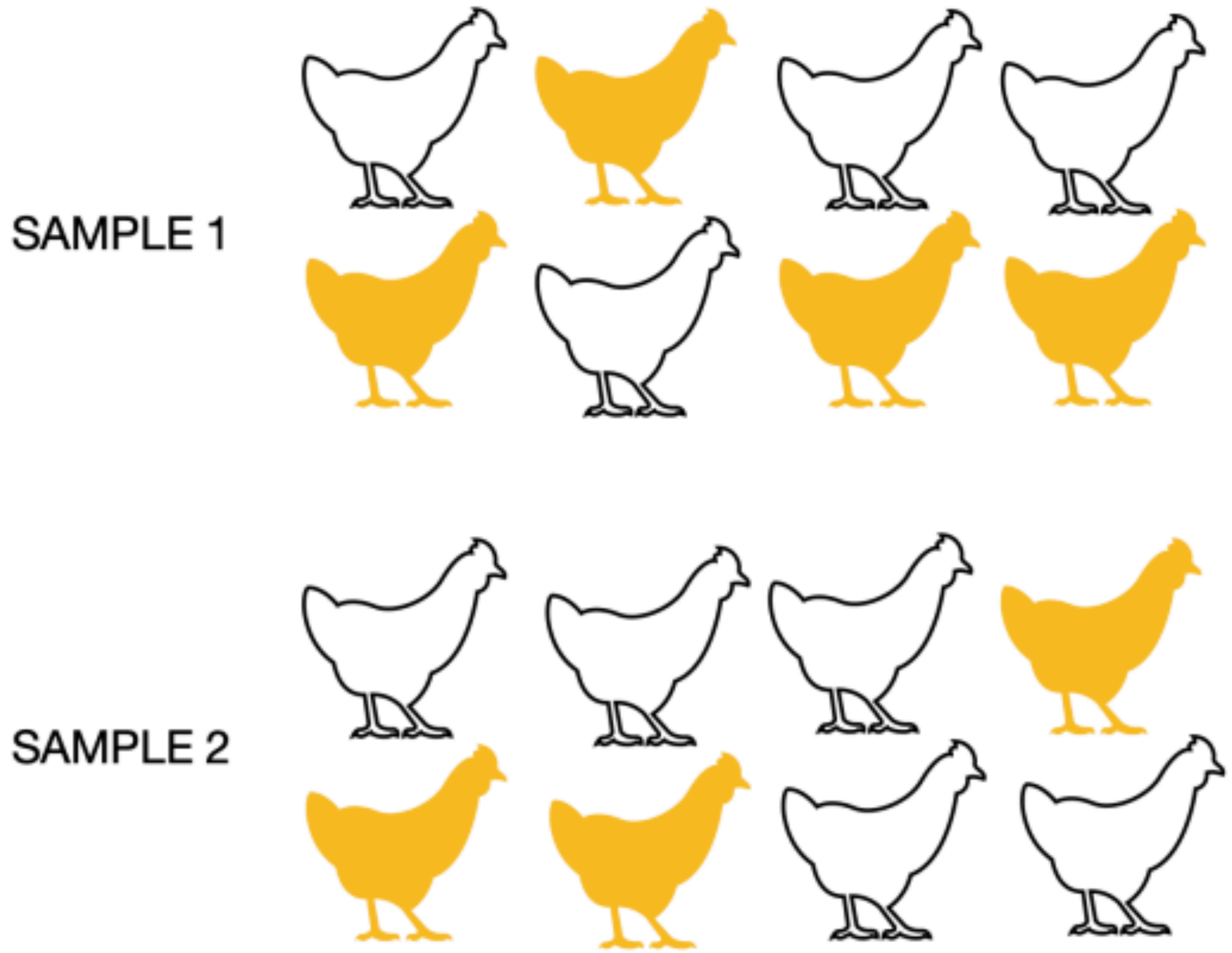
- In the matched pairs design, the experimental subjects are chosen in pairs so that the members in each pair are alike, whereas those in different pairs may be substantially dissimilar.
- One member of each pair receives treatment 1 and the other treatment 2.

Example

To compare the effectiveness of two different feeds called A and B are used for chickens. The response here is kg gain of the chickens. Suppose that the chickens are selected from two different species. For illustrative purposes, we denote the chickens in different species with yellow and white colors. There are two ways to proceed for testing which feed is more effective on kg gain of the chickens.

Independent samples vs. matched pairs design

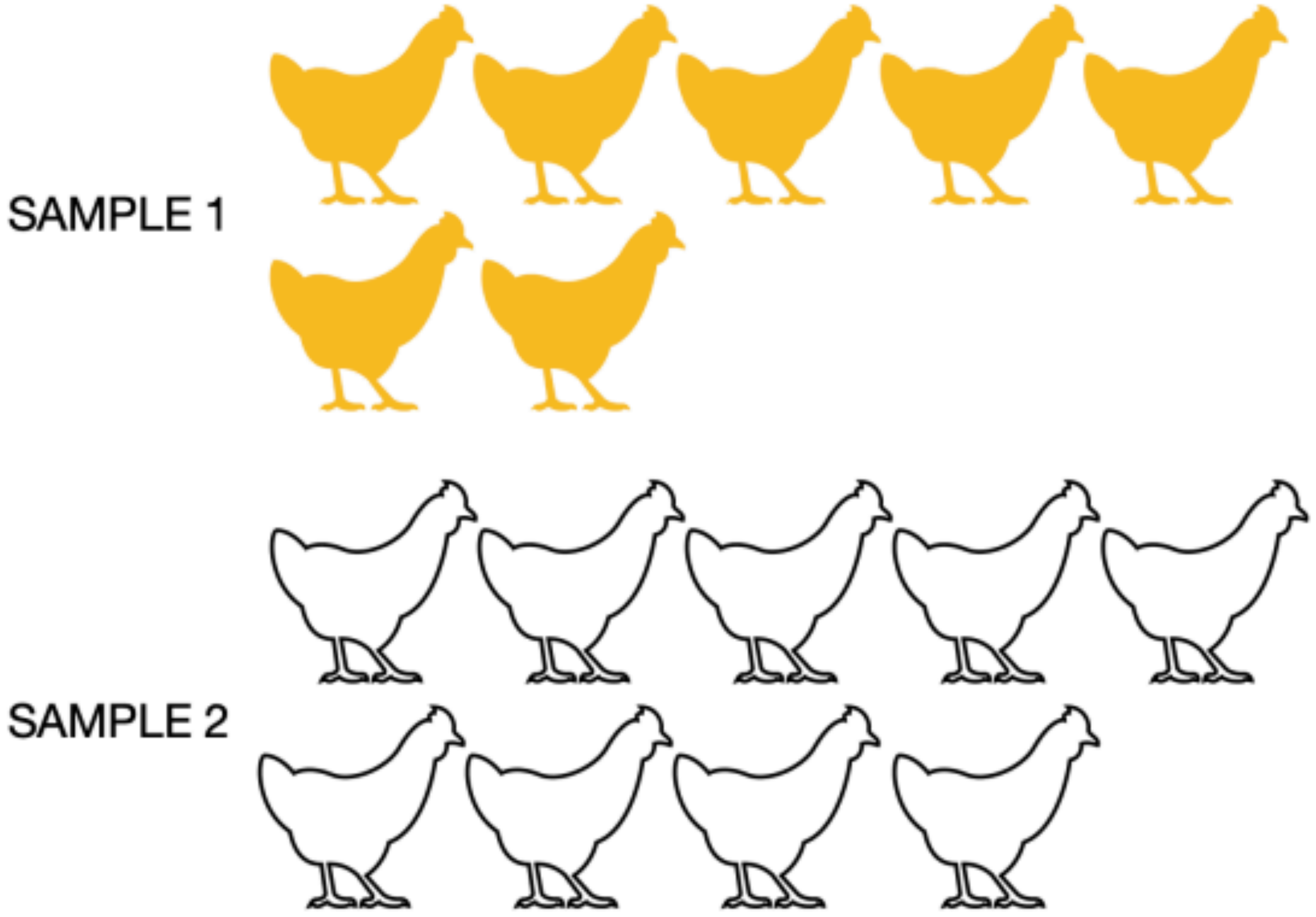
(i) 16 chickens are randomly divided into two different samples. Then, sample 1 and sample 2 are assigned to feed A and feed B, respectively.



Independent samples

Independent samples vs. matched pairs design

(ii) Chickens from the same species should be in the same sample as shown in the figure. Then, sample 1 and sample 2 are assigned to feed A and feed B, respectively.



Matched pairs

Independent samples vs. matched pairs design

For this example, matched pairs, given in (ii), should be used since the chickens in each sample are alike and those from the other sample are different.

Comparing two population means based on independent samples

Assumptions

(i) Let $X_1, X_2, \ldots, X_{n_1}$ be a sample from a population having normal distribution with mean μ_1 and variance σ_1^2.

(ii) Let $Y_1, Y_2, \ldots, Y_{n_2}$ be a sample from a population having normal distribution with mean μ_2 and variance σ_2^2.

(iii) σ_1^2 and σ_2^2 are known.

(iv) The samples are independent. In other words, the response measurements under one treatment are unrelated to the response measurements under the other treatment.

Aim

Our goal is drawing a comparison between the mean responses of the two treatments or populations. In statistical language, we are interested in making inferences about the parameter $\mu_1 - \mu_2$.

Notations

Sample	Sample mean	Sample variance
$X_1, X_2, \ldots, X_{n_1}$	$\bar{X} = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i$	$s_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (X_i - \bar{X})^2$
$Y_1, Y_2, \ldots, Y_{n_2}$	$\bar{Y} = \frac{1}{n_2} \sum_{i=1}^{n_2} Y_i$	$s_2^2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2$

Sampling distribution of $\bar{X} - \bar{Y}$

The most convenience point estimator of $\mu_1 - \mu_2$ is $\bar{X} - \bar{Y}$.

It can be shown that

$$E(\bar{X} - \bar{Y}) = \mu_1 - \mu_2, \quad Var(\bar{X} - \bar{Y}) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

and

$$SE(\bar{X} - \bar{Y}) = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Sampling distribution of $\bar{X} - \bar{Y}$

The sampling distribution of $\bar{X} - \bar{Y}$ is normal with mean $\mu_1 - \mu_2$ and variance $\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}$. In other words,

$$\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

or

$$\frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

Sampling distribution of $\bar{X} - \bar{Y}$

The assumption that is σ_1^2 and σ_2^2 are known is not realistic in real life problems.

If sample sizes n_1 and n_2 are large enough, we can replace σ_1^2 and σ_2^2 by their estimators s_1^2 and s_2^2, respectively. Then,

$$Z = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \text{ is approximately } N(0,1)$$

Confidence interval for $\mu_1 - \mu_2$

Suppose that we have two independent samples and n_1 and n_2 are large enough.

$100(1-\alpha)\,\%$ confidence interval for $\mu_1 - \mu_2$:

$$(\bar{X} - \bar{Y}) - z_{\alpha/2} SE(\bar{X} - \bar{Y}) < \mu_1 - \mu_2 < (\bar{X} - \bar{Y}) + z_{\alpha/2} SE(\bar{X} - \bar{Y})$$

where

$$SE(\bar{X} - \bar{Y}) = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Example

A considerable proportion of a person's time is spent working, and satisfaction with the job and satisfaction with life tend to be related. Job satisfaction is typically measured on a four point scale:

Very dissatisfied, a little dissatisfied, moderately satisfied, very satisfied

A numerical scale is created by assigning 1 to very dissatisfied, 2 to a little dissatisfied, 3 to moderately satisfied, and 4 to very satisfied. The responses ıf 226 firefighters and 247 office supervisors yielded the summary statistics as follows:

	Fire fighter	Office supervisor
Mean	3.673	3.547
Standard deviation	0.7235	0.6089

Construct a 95% confidence interval for difference in mean job satisfaction scores.

Example

We are given following sample statistics:

Sample	Sample size	Sample mean	Sample standard
Sample 1 (fire fighters)	226	3.673	0.7235
Sample 2 (office supervisors)	247	0.7235	0.6089

It is clear that $n_1 = 226$ and $n_2 = 247$ are large enough.

Example

$100(1-\alpha)\,\% = 95\,\% \rightarrow z_{\alpha/2} = z_{0.05/2} = z_{0.025} = 1.96$

$$SE(\bar{X} - \bar{Y}) = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{0.7235^2}{226} + \frac{0.6089^2}{247}} = 0.0618$$

$95\,\%$ confidence interval for $\mu_1 - \mu_2$ is:

$$(\bar{X} - \bar{Y}) - z_{\alpha/2} SE(\bar{X} - \bar{Y}) < \mu_1 - \mu_2 < (\bar{X} - \bar{Y}) + z_{\alpha/2} SE(\bar{X} - \bar{Y})$$

$$(3.673 - 0.7235) - 1.96(0.0618) < \mu_1 - \mu_2 < (3.673 - 0.7235) + 1.96(0.0618)$$

$$2.8284 < \mu_1 - \mu_2 < 3.0706$$

Hypothesis testing for $\mu_1 - \mu_2$ (Z-test)

We want to test $H_0 : \mu_1 - \mu_2 = 0$ versus one of the following alternatives at given level of significance α:

$$H_1 : \mu_1 - \mu_2 < 0, \quad H_1 : \mu_1 - \mu_2 > 0, \quad H_1 : \mu_1 - \mu_2 \neq 0$$

Hypothesis testing for $\mu_1 - \mu_2$ (Z-test)

If sample sizes n_1 and n_2 are large enough, the following Z-test is used to test H_0:

$$Z_0 = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

We know that the distribution of test statistic Z_0 is approximately $N(0,1)$.

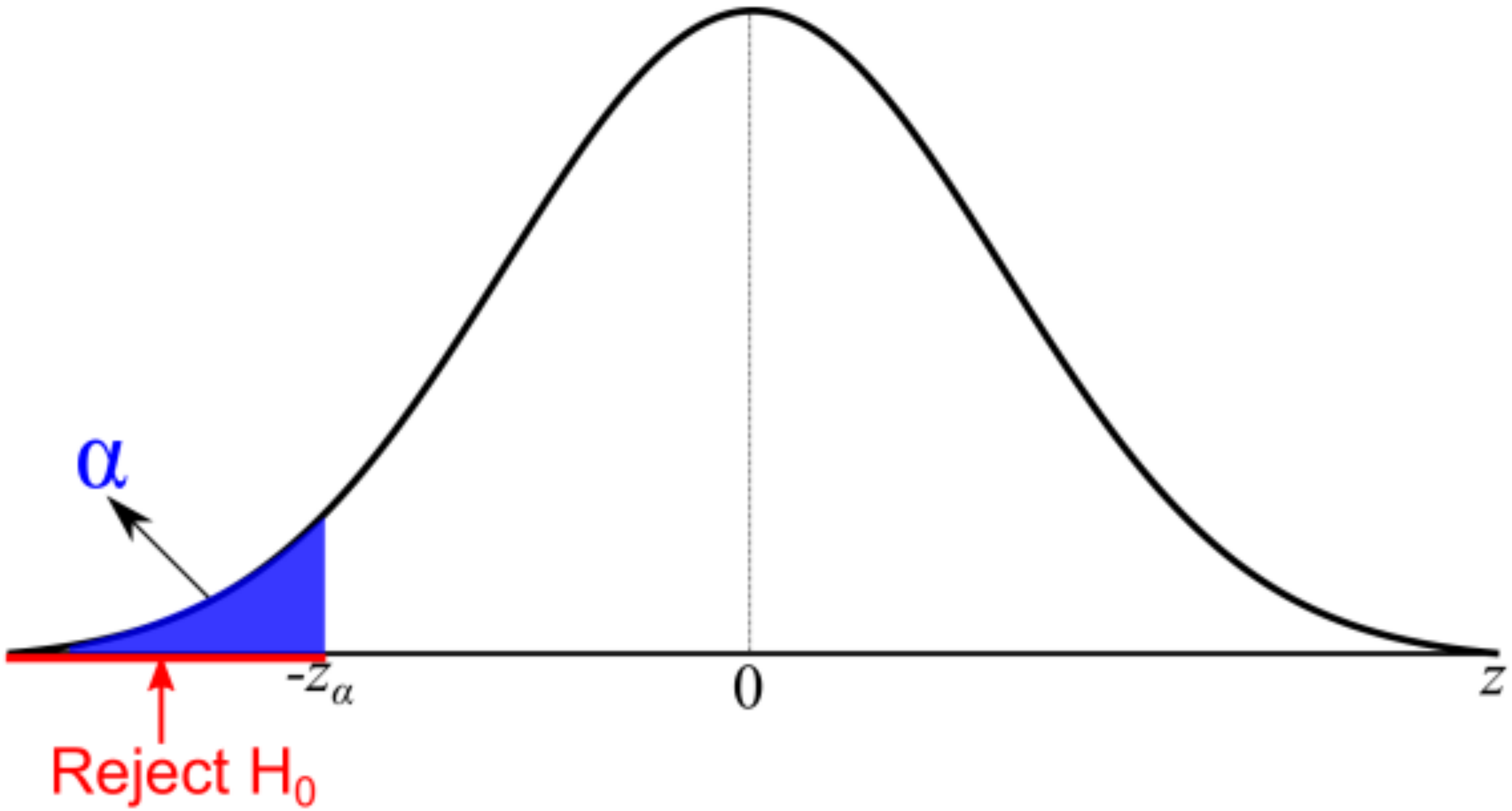
Hypothesis testing for $\mu_1 - \mu_2$ (Z-test)

The rejection region is one or two-sided depending on the alternative hypothesis. Specifically,

Alternative hypothesis	Rejection region	p-value
$H_1 : \mu_1 - \mu_2 < 0$	$Z_0 < -Z_\alpha$	$p < \alpha$
$H_1 : \mu_1 - \mu_2 > 0$	$Z_0 > Z_\alpha$	$p < \alpha$
$H_1 : \mu_1 - \mu_2 \neq 0$	$\lvert Z_0 \rvert > Z_{\alpha/2}$	$p < \alpha/2$

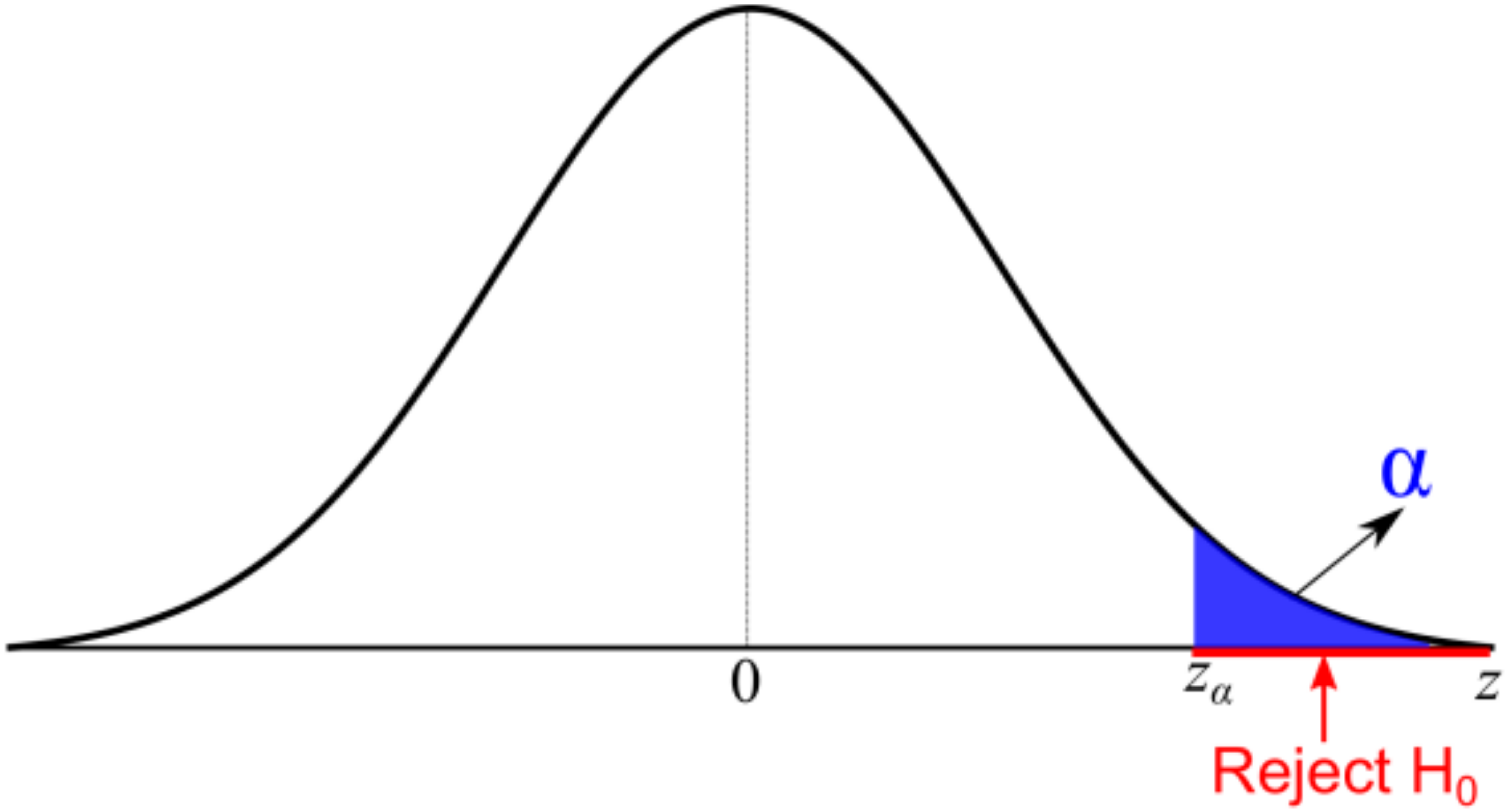
Hypothesis testing for $\mu_1 - \mu_2$ (Z-test)

$H_1 : \mu_1 - \mu_2 < 0 \rightarrow$ Reject H_0 if $Z_0 < -z_\alpha$



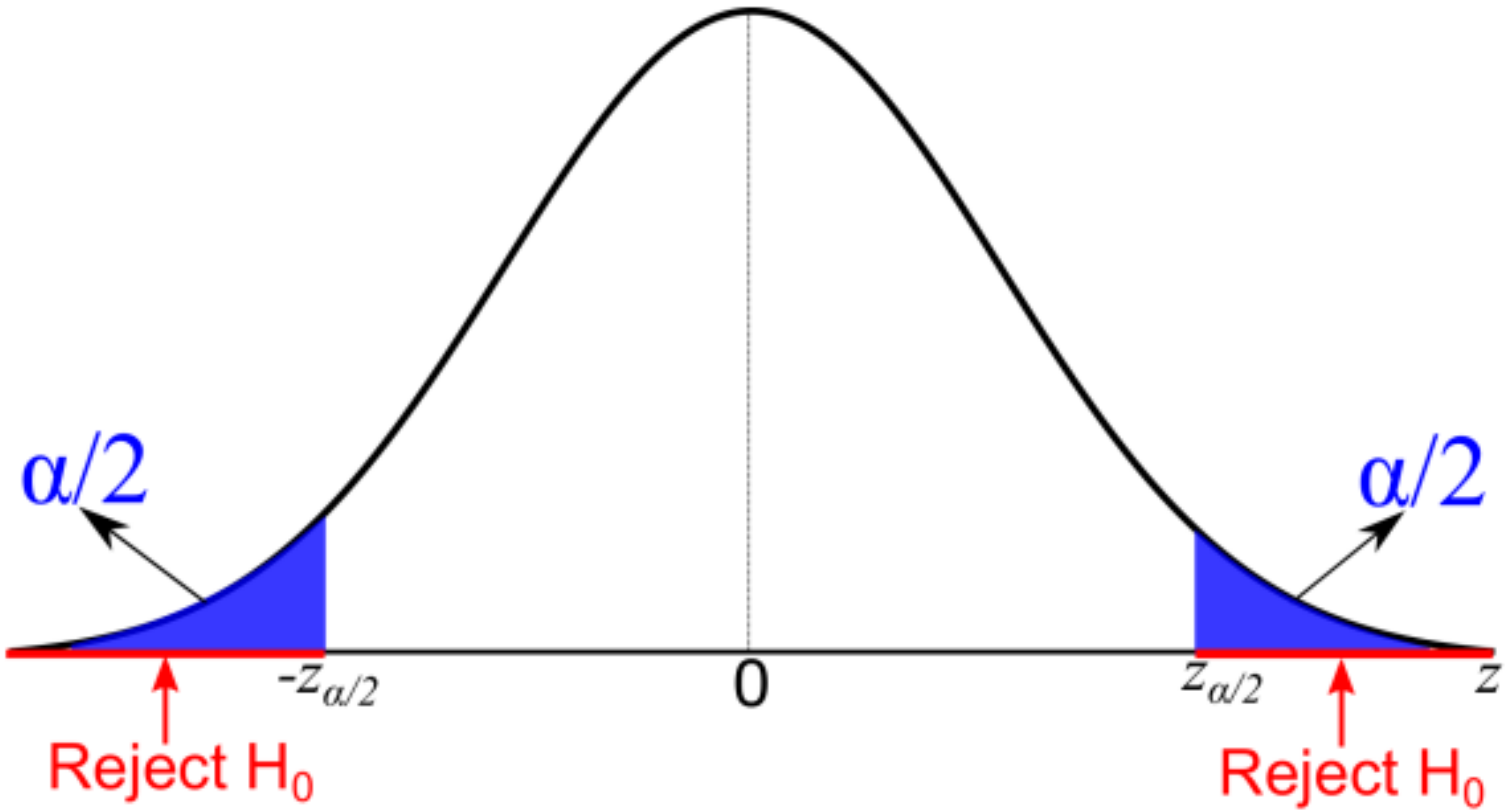
Hypothesis testing for $\mu_1 - \mu_2$ (Z-test)

$$H_1 : \mu_1 - \mu_2 > 0 \rightarrow \text{ Reject } H_0 \text{ if } Z_0 > z_\alpha$$



Hypothesis testing for $\mu_1 - \mu_2$ (Z-test)

$H_1 : \mu_1 - \mu_2 \neq 0 \rightarrow$ Reject H_0 if $|Z_0| > z_\alpha$



Example

Do the data in the previous example provide strong evidence that the mean job satisfaction of firefighters is different from the mean job satisfaction of office supervisors? Test at $\alpha = 0.01$.

Example

We are given following sample statistics:

Sample	Sample size	Sample mean	Sample standard
Sample 1 (fire fighters)	226	3.673	0.7235
Sample 2 (office supervisors)	247	0.7235	0.6089

It is clear that $n_1 = 226$ and $n_2 = 247$ are large enough.

Example

- We should investigate whether the mean job satisfaction of firefighters is different from the mean job satisfaction of office supervisors. Therefore, the null and alternative hypotheses should be in the following:

$$H_0 : \mu_1 - \mu_2 = 0 \text{ vs. } H_1 : \mu_1 - \mu_2 \neq 0$$

where μ_1 and μ_2 stand for the mean satisfaction of firefighters and office supervisors, respectively. (Here, “different” refers to “not equal” in expression of H_1.)

- Alternative hypothesis determines the rejection region. Since it is two-sided, the null hypothesis H_0 is rejected when $|Z_0| > z_{\alpha/2}$.

- $\alpha = 0.01 \rightarrow z_{\alpha/2} = z_{0.01/2} = z_{0.005} = 2.58$

Example

$$Z_0 = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{3.673 - 0.7235}{\sqrt{\frac{0.7235^2}{226} + \frac{0.6089^2}{247}}} = \frac{2.9494}{0.0618} = 47.7392$$

$|Z_0| = 47.7392 < z_{0.01/2} = 2.58$ implies that the null hypothesis should be rejected. In other words, the mean job satisfaction of firefighters is different from the mean job satisfaction of office supervisors.

Example

One semester, an instructor taught the same computer course at two universities located in different cities. He was able to give the same final at both locations. The student's scores provided the summary statistics.

Sample	Sample size	Sample mean	Sample variance
Sample 1 (University 1)	44	66	142
Sample 2 (University 2)	52	73	151

Test whether the mean score of university 1 is less than that of university 2 at 10% level of significance.

Example

- It is clear that $n_1 = 44$ and $n_2 = 52$ are large enough, i.e. both are greater than 30.
- We should investigate whether the mean score of University 1 is less than the mean score of University 2. Therefore, the null and alternative hypotheses should be in the following form:

$$H_0 : \mu_1 - \mu_2 \geq 0 \text{ vs. } H_1 : \mu_1 - \mu_2 < 0$$

where μ_1 and μ_2 stand for the mean scores of University 1 and University 2, respectively. (Here, “less” refers to “<” in expression of H_1.)

- Alternative hypothesis determines the rejection region. Since it is one-sided and in the form $H_1 : \mu_1 - \mu_2 < 0$, the null hypothesis H_0 is rejected when $Z_0 < -z_\alpha$.
- $\alpha = 0.10 \rightarrow z_\alpha = z_{0.10} = 1.64$

Example

$$Z_0 = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{44 - 52}{\sqrt{\frac{142}{44} + \frac{151}{52}}} = \frac{-8}{2.4761} = -3.2309$$

$Z_0 = -3.2309 < -z_{0.10} = -1.64$ implies that the null hypothesis should be rejected. In other words, the mean score of University 1 is less than the mean score of University 2.

Assumptions

(i) Let $X_1, X_2, \ldots, X_{n_1}$ be a sample from a population having normal distribution with mean μ_1 and variance σ_1^2.

(ii) Let $Y_1, Y_2, \ldots, Y_{n_2}$ be a sample from a population having normal distribution with mean μ_2 and variance σ_2^2.

(iii) σ_1^2 and σ_2^2 are unknown but they are equal: $\sigma_1^2 = \sigma_2^2 = \sigma^2$.

(iv) The samples are independent.

Our goal is drawing a comparison between the mean responses of the two treatments or populations. In statistical language, we are interested in making inferences about the parameter $\mu_1 - \mu_2$.

Sampling distribution of $\bar{X} - \bar{Y}$

- The most convenience point estimator of $\mu_1 - \mu_2$ is $\bar{X} - \bar{Y}$.
- It can be shown that

$$E(\bar{X} - \bar{Y}) = \mu_1 - \mu_2, \quad Var(\bar{X} - \bar{Y}) = \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

- Since variances are unknown, the common variance σ^2 should also be estimated The pooled estimator of common variance is

$$s_p^2 = \frac{(n_1 + 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

Sampling distribution of $\bar{X} - \bar{Y}$

- The sampling distribution of

$$T = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{2}{n_2}}}$$

is Student's t distributed with degrees of freedom $n_1 + n_2 - 2$. In other words,

$$T = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{2}{n_2}}} \sim t_{n_1+n_2-2}$$

Confidence interval for $\mu_1 - \mu_2$

Suppose that we have two independent samples and **population variances are unknown but assumed equal.**

$100(1-\alpha)\,\%$ confidence interval for $\mu_1 - \mu_2$:

$$(\bar{X} - \bar{Y}) - t_{n_1+n_2-2,\alpha/2} SE(\bar{X} - \bar{Y}) < \mu_1 - \mu_2 < (\bar{X} - \bar{Y}) + t_{n_1+n_2-2,\alpha/2} SE(\bar{X} - \bar{Y})$$

where

$$SE(\bar{X} - \bar{Y}) = s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Example

Suppose you wish to compare a new method of teaching reading to “slow learners” with the current standard method. You decide to base your comparison on the results of a reading test given at the end of a learning period of six months. Of a random sample of 22 “slow learners”, 10 are taught by the new methods and 12 are thought by the standard method. All 22 children are taught by qualified instructors under similar conditions for the designated six-month period. The results of the reading test at the end of this period are given in the following table:

New method				Standard method			
80	80	79	81	79	62	70	68
76	66	71	76	73	76	86	73
70	85			72	68	75	66

Use the data in the table to estimate the true mean difference between the test scores for the new method and the standard method. Use a 95% confidence interval.

Example

We are given following sample statistics:

Sample	Sample size	Sample mean	Sample standard deviation
Sample 1 (New method)	10	76.4	5.8348
Sample 2 (Standard method)	12	72.33	6.3437

It is clear that $n_1 = 10 < 30$ and $n_2 < 30$ so we use the following formula:

$$(\bar{X} - \bar{Y}) - t_{n_1+n_2-2,\alpha/2} SE(\bar{X} - \bar{Y}) < \mu_1 - \mu_2 < (\bar{X} - \bar{Y}) + t_{n_1+n_2-2,\alpha/2} SE(\bar{X} - \bar{Y})$$

Example

$100(1-\alpha)\,\% = 95\,\% \rightarrow t_{n_1+n_2-2,\alpha/2} = t_{20-1,0.025} = 2.086$

$$s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} = \frac{(10-1)5.8348^2 + (12-1)6.3437^2}{10+12-2} = 37.4536$$

$s_p = 6.1199$

$$SE(\bar{X}-\bar{Y}) = s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 6.1199\sqrt{\frac{1}{10}+\frac{1}{12}} = 2.6204$$

$95\,\%$ confidence interval for $\mu_1 - \mu_2$ is

$$(\bar{X}-\bar{Y}) - t_{n_1+n_2-2,\alpha/2}SE(\bar{X}-\bar{Y}) < \mu_1-\mu_2 < (\bar{X}-\bar{Y}) + t_{n_1+n_2-2,\alpha/2}SE(\bar{X}-\bar{Y})$$

$$(76.4-72.3333) - 2.086(2.6204) < \mu_1-\mu_2 < (76.4-72.3333) + 2.086(2.6204)$$

$$-1.3995 < \mu_1-\mu_2 < 9.5329$$

Hypothesis testing for $\mu_1 - \mu_2$ (t-test)

We want to test $H_0 : \mu_1 - \mu_2 = 0$ versus one of the following alternatives at given level of significance α:

$$H_1 : \mu_1 - \mu_2 < 0, \quad H_1 : \mu_1 - \mu_2 > 0, \quad H_1 : \mu_1 - \mu_2 \neq 0$$

Hypothesis testing for $\mu_1 - \mu_2$ (t-test)

The following t-test is used to test H_0:

$$T_0 = \frac{\bar{X} - \bar{Y}}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}$$

We know that the distribution of test statistic T_0 is Student's t with degrees of freedom $n_1 + n_2 - 2$.

Hypothesis testing for $\mu_1 - \mu_2$ (t-test)

The rejection region is one or two-sided depending on the alternative hypothesis. Specifically,

Alternative hypothesis	Rejection region	p-value
$H_1 : \mu_1 - \mu_2 < 0$	$T_0 < -t_{n_1+n_2-2,\alpha}$	$p < \alpha$
$H_1 : \mu_1 - \mu_2 > 0$	$T_0 > t_{n_1+n_2-2,\alpha}$	$p < \alpha$
$H_1 : \mu_1 - \mu_2 \neq 0$	$\lvert T_0 \rvert > t_{n_1+n_2-2,\alpha/2}$	$p < \alpha/2$

Example #1

Do the data in the previous example provide strong evidence that the mean of new method learners is larger from the mean of standard method learners? Test at $\alpha = 0.10$.

Solution #1

We are given following sample statistics:

Sample	Sample size	Sample mean	Sample standard deviation
Sample 1 (New method)	10	76.4	5.8348
Sample 2 (Standard method)	12	72.33	6.3437

It is clear that $n_1 = 10 < 30$ and $n_2 < 30$ so we use the following test statistic here:

$$T_0 = \frac{\bar{X} - \bar{Y}}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Solution #1

- We should investigate whether the mean of new learners is larger than the mean of standard learners. Therefore, the null and alternative hypotheses should be in the following form:

$$H_0 : \mu_1 - \mu_2 \leq 0 \text{ vs. } H_1 : \mu_1 - \mu_2 > 0$$

where μ_1 and μ_2 stand for the mean of new learners and standard learners, respectively. (Here, "larger" refers to "$>$" in expression of H_1.)

- Alternative hypothesis determines the rejection region. Since it is one-sided in the form $H_1 : \mu_1 - \mu_2 > 0$, the null hypothesis is rejected when $T_0 > t_{n_1+n_2-2,\alpha}$

- $\alpha = 0.10 \rightarrow t_{n_1+n_2-2,\alpha} = t_{20,0.10} = 1.325$

Solution #1

$$s_p^2 = \frac{(n_1 - 1)s_2^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{(10 - 1)5.8348^2 + (12 - 1)6.3437^2}{10 + 12 - 2} = 37.4536$$

$$s_p = 6.1199$$

$$SE(\bar{X} - \bar{Y}) = s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 6.1199\sqrt{\frac{1}{10} + \frac{1}{12}} = 2.6204$$

$$T_0 = \frac{\bar{X} - \bar{Y}}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{76.4 - 72.3333}{6.1199\sqrt{\frac{1}{10} + \frac{1}{12}}} = 1.5519$$

$T_0 = 1.5519 > t_{20,0.10} = 1.325$ implies that the null hypothesis should be rejected. In other words, the mean of new learners is larger than the mean of standard learners.

Example #2

Students can bike to a park on the other side of a lake by going around one side of the lake or the other. After much discussion about which was faster, they decided to perform an experiment. Among the 12 students available, 6 were randomly selected to follow Path A on one side of the lake and rest followed Path B on the other side. They all went on different days so the conclusion would apply to a variety of conditions.

Path A	10	12	15	11	16	11
Path B	12	15	17	13	18	16

Is there a significant difference between the mean travel times between the two paths? Test at level of significance $\alpha = 0.05$

Solution #2

Sample	Sample size	Sample mean	Sample standard deviation
Sample 1 (Path A)	6	12.5	2.4290
Sample 2 (Path B)	6	15.1667	2.3166

$n_1 = 6$ and $n_2 = 6 < 30$ so we use the following test statistic here:

$$T_0 = \frac{\bar{X} - \bar{Y}}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Solution #2

We should investigate whether the mean travel times for path A and path B is different. Therefore, the null and alternative hypotheses should be in the following form:

$H_0 : \mu_1 - \mu_2 = 0$ vs. $H_1 : \mu_1 - \mu_2 \neq 0$

where μ_1 and μ_2 stand for the mean travel times for path A and B, respectively. (Here, “different” refers to “not equal” in expression of H_1.)

Alternative hypothesis determines the rejection region. Since it is two-sided, the null hypothesis H_0 is rejected when $|T_0| > t_{n_1+n_2-2,\alpha/2}$

$$\alpha = 0.05 \rightarrow t_{n_1+n_2-2,\alpha/2} = t_{10,0.05/2} = t_{10,0.025} = 2.228$$

Solution #2

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{(6 - 1)2.4290^2 + (6 - 1)2.3166^2}{6 + 6 - 2} = 5.6333$$

$$s_p = 2.3735$$

$$SE(\bar{X} - \bar{Y}) = s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 2.3735\sqrt{\frac{1}{6} + \frac{1}{6}} = 1.3703$$

$$T_0 = \frac{\bar{X} - \bar{Y}}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{12.5 - 15.1667}{2.3735\sqrt{\frac{1}{6} + \frac{1}{6}}} = -1.9461$$

Since $|T_0| = 1.9461 < t_{10,0.025} = 2.228$, the null hypothesis cannot be rejected. In other words, there is no significant difference between the mean travel times between the two paths.

Z-test vs. t-test

- Both tests are essentially based on a normal distributed population and testing the difference between the population means.
- If population variances are unknown, we can use either Z-test or t-test, but we should be careful.
    - If n is large enough ($n \geq 30$), use Z-test.
    - If n is small ($n < 30$), use t-test.

Comparing two population means based on matched pairs sampling

Matched pair design

- The experiment, in which observations are paired and the differences are analyzed, is called a matched pair design or sampling.
- The idea of matched pairs is that the members of these pairs should resemble one another as closely as possible so that the comparison of interest can be made directly.
- Making comparisons within groups of similar experimental units is called blocking, and the paired difference experiment is a simple example of a randomized block experiment.
- Pairing similar experimental units according to some identifiable characteristic(s) serves to remove this source of variation from the experiment.

Matched pair design

In a matched pairs design, the response of an experimental unit is influenced by:

1. The conditions prevailing in the block (pair).
2. A treatment effect.

By taking the difference between the two observations in a block, we can filter out the common block effect. These differences then permit us to focus on the effects of treatments that are freed from undesirable sources of variation.

Matched pair design

Pair	Treatment 1	Treatment 2	Difference (D)
1	X_1	Y_1	$D_1 = X_1 - Y_1$
2	X_2	Y_2	$D_2 = X_2 - Y_2$
...	...	...	...
n	X_n	Y_n	$D_n = X_n - Y_n$

$$\bar{D} = \frac{1}{n}\sum_{i=1}^{n} D_i \quad \text{and} \quad s_D = \frac{1}{n-1}\sum_{i=1}^{n}(D_i - \bar{D})^2$$

Note

The pairs (X_i, Y_i) are independent of one another but X_i and Y_i within the i.pair will usually be dependent.

Matched pair design

Let the following n matched pairs of observations,

$$(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$$

are selected from populations having means μ_1 and μ_2.

Our goal is drawing a comparison between the mean responses of the two treatments or populations based on matched pairs. In other words, we would like to make inferences about the parameter $\mu_1 - \mu_2$.

Inferences based on matched pairs

Assume that the population distribution of the differences is assumed to be normal.

$100(1-\alpha)$ confidence interval for $\mu_1 - \mu_2$:

$$\bar{D} - t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}} < \mu_1 - \mu_2 < \bar{D} + t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}$$

where $t_{n-1,\alpha/2}$ is the upper quantile of Student's t distribution with degrees of freedom $n-1$.

Inferences based on matched pairs

We want to test $H_0 : \mu_1 - \mu_2 = 0$ versus one of the following alternatives at given level of significance α:

$$H_1 : \mu_1 - \mu_2 < 0, H_1 : \mu_1 - \mu_2 > 0, H_1 : \mu_1 - \mu_2 \neq 0$$

Inferences based on matched pairs

The following t-test is used to test H_0:

$$T_0 = \frac{\bar{D}}{s_D/\sqrt{n}} \sim t_{n-1}$$

The distribution of test statistic T_0 is Student's t with degrees of freedom $n - 1$.

Inferences based on matched pairs

The rejection region is one or two-sided depending on the alternative hypothesis. Specifically,

Alternative hypothesis	**Rejection region**	**p-value**
$H_1 : \mu_1 - \mu_2 < 0$	$T_0 < -t_{n_1+n_2-2,\alpha}$	$p < \alpha$
$H_1 : \mu_1 - \mu_2 > 0$	$T_0 > t_{n_1+n_2-2,\alpha}$	$p < \alpha$
$H_1 : \mu_1 - \mu_2 \neq 0$	$\lvert T_0 \rvert > t_{n_1+n_2-2,\alpha/2}$	$p < \alpha/2$

Example

Two methods of memorizing difficult material are being tested to determine if one procedures better retention. Nine pairs of students are included in the study. The students in each pair are matched according to IQ and academic background and then assigned to the two methods at random. A memorization test is given to all the students, and the following scores are obtained:

	Pairs								
	1	2	3	4	5	6	7	8	9
Method A	90	86	72	65	44	52	66	38	83
Method B	85	87	70	62	44	53	62	35	86

(a) Calculate a 95% confidence interval for the mean score.

(b) At $\alpha = 0.05$, test to determine if there is a significant difference in the effectiveness of the two methods

Solution

As given in the question, here we have matched pairs. Summary statistics can be obtained as follows:

Method A	Method B	D
90	85	90 - 85 = 5
86	87	86 - 87 = -1
72	70	72 - 70 = 2
65	62	65 - 62 = 3
44	44	44 - 44 = 0
52	53	52 - 53 = -1
66	62	66 - 64 = 2
38	35	38 - 35 = 3
83	86	83 - 86 = -3

$\bar{D} = 1.3333$ and $s_D = 2.6926$

Solution

(a) $100(1-\alpha)\,\% = 95\,\% \rightarrow t_{n-1,\alpha/2} = t_{9-1,0.025} = 2.306$

$95\,\%$ confidence interval for $\mu_1 - \mu_2$ is

$$\bar{D} - t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}} < \mu_1 - \mu_2 < \bar{D} + t_{n-1,\alpha/2}\frac{s_D}{\sqrt{n}}$$

$$1.3333 - (2.306)\frac{2.6926}{\sqrt{9}} < \mu_1 - \mu_2 < 1.3333 + (2.306)\frac{2.6926}{\sqrt{9}}$$

$$-0.7363 < \mu_1 - \mu_2 < 3.4029$$

Solution

(b)

We should investigate whether there is a significant difference in the effectiveness of the two methods. Therefore, the null and alternative hypothesis should be in following form:

$H_0 : \mu_1 - \mu_2 = 0$ vs. $H_1 : \mu_1 - \mu_2 \neq 0$

where μ_1 and μ_2 stand for the means of method A and B, respectively. (Here, “different” refers to “not equal” in expression of H_1.)

Alternative hypothesis determines the rejection region. Since it is two-sided, the null hypothesis H_0 is rejected when $|T_0| > t_{n-1,\alpha/2}$

Solution

$$\alpha = 0.05 \rightarrow t_{n-1,\alpha/2} = t_{8,0.05/2} = t_{8,0.025} = 2.306$$

$$T_0 = \frac{\bar{D}}{s_D/\sqrt{n}} = \frac{1.3333}{2.6926/\sqrt{9}} = \frac{1.3333}{0.8975} = 1.4856$$

Since $|T_0| = 1.4856 < t_{8,0.025} = 2.306$, the null hypothesis cannot be rejected. In other words, there is no significant difference in the effectiveness of the two methods.

Comparing two population proportions

Assumptions

(i) A random sample of n_1 observations from a population with proportion p_1 of "success" yields sample proportion $\hat{p}_1$.

(ii) A random sample of n_2 observations from a population with proportion p_2 of "success" yields sample proportion $\hat{p}_2$.

(iii) Samples are independent.

(iv) n_1 and n_2 are large enough.

Data structure

The form of the data can be displayed as follows:

	Number of success	Number of failures	Sample size
Sample 1	X	$n_1 - X$	n_1
Sample 2	Y	$n_2 - Y$	n_2

Our goal is drawing a comparison about the difference between the populations proportions. In statistical language, we are interested in making inferences about the parameter

$$p_1 - p_2$$

Sampling distribution of $\hat{p}_1 - \hat{p}_2$

The most convenience point estimator of $p_1 - p_2$ is $\hat{p}_1 - \hat{p}_2$ where

$\hat{p}_1 = \dfrac{X}{n_1}$ and $\hat{p}_2 = \dfrac{Y}{n_2}$

It can be shown that

$E(\hat{p}_1 = \dfrac{X}{n_1}) = p_1 - p_2$ and $Var(\hat{p}_1 = \dfrac{X}{n_1}) = \dfrac{p_1(1-p_1)}{n_1} + \dfrac{p_2(1-p_2)}{n_2}$

Estimated standard error is therefore

$$Estimated\ SE(\hat{p}_1 - \hat{p}_2) = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

Sampling distribution of $\hat{p}_1 - \hat{p}_2$

If sample sizes n_1 and n_2 are large enough,

$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}$$ is approximately $N(0,1)$

Confidence interval of $p_1 - p_2$

If sample sizes n_1 and n_2 are large enough,

$100(1-\alpha)$ confidence interval for $p_1 - p_2$:

$$(\hat{p}_1 - \hat{p}_2) - z_{\alpha/2}\hat{SE}(\hat{p}_1 - \hat{p}_2) < p_1 - p_2 < (\hat{p}_1 - \hat{p}_2) + z_{\alpha/2}\hat{SE}(\hat{p}_1 - \hat{p}_2)$$

where

$$\hat{SE}(\hat{p}_1 - \hat{p}_2) = \text{Estimated SE}(\hat{p}_1 - \hat{p}_2) = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

Example

Supermarket shoppers were observed and questioned immediately after putting an item in their cart. Of a random sample of 510 choosing a product at the regular price, 320 cşaimed to check the price before putting the item in their cart. Of an independent random sample of 332 choosing a product at a special price, 200 made this claim. Find a 90% confidence interval for the difference between the two population proportions.

Solution

We are given following sample statistics:

Sample	Sample size	Number of successes	Sample proportion
Sample 1	510	320	$\hat{p}_1 = 320/510 = 0.6275$
Sample 2	332	200	$\hat{p}_2 = 200/332 = 0.6024$

It is clear that n_1 and n_2 are large enough so we use the following formula:

$$(\hat{p}_1 - \hat{p}_2) - z_{\alpha/2}\hat{SE}(\hat{p}_1 - \hat{p}_2) < p_1 - p_2 < (\hat{p}_1 - \hat{p}_2) + z_{\alpha/2}\hat{SE}(\hat{p}_1 - \hat{p}_2)$$

where

$$\hat{SE}(\hat{p}_1 - \hat{p}_2) = \sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}$$

Solution

$$100(1-\alpha)\,\% = 90\,\% \rightarrow z_{\alpha/2} = z_{0.05} = 1.64$$

$$\hat{SE}(\hat{p}_1 - \hat{p}_2) = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} = \sqrt{\frac{0.6275(1-0.6275)}{510} + \frac{0.6024(1-0.6024)}{332}} = 0.0343$$

90% confidence interval for $p_1 - p_2$ is

$$(\hat{p}_1 - \hat{p}_2) - z_{\alpha/2}\hat{SE}(\hat{p}_1 - \hat{p}_2) < p_1 - p_2 < (\hat{p}_1 - \hat{p}_2) + z_{\alpha/2}\hat{SE}(\hat{p}_1 - \hat{p}_2)$$

$$(0.6275 - 0.6024) - (1.64)0.0343 < p_1 - p_2 < (0.6275 - 0.6024) + (1.64)0.0343$$

$$-0.0312 < p_1 - p_2 < 0.0814$$

Hypothesis testing for $p_1 - p_2$

We want to test $H_0 : p_1 - p_2 = 0$ versus one of the following alternatives at given level of significance α:

Hypothesis testing for $p_1 - p_2$

If sample sizes n_1 and n_2 are large enough, the following Z-test is used to test H_0:

$$Z_0 = \frac{\hat{p}_1 - \hat{p}_2}{\hat{SE}(\hat{p}_1 - \hat{p}_2)}$$

where

$$\hat{SE}(\hat{p}_1 - \hat{p}_2) = \sqrt{\hat{p}(1 - \hat{p})}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \quad \text{and} \quad \hat{p} = \frac{X + Y}{n_1 + n_2}$$

The distribution of test statistic Z_0 is approximately $N(0,1)$.

Hypothesis testing for $\mu_1 - \mu_2$ (Z-test)

The rejection region is one or two-sided depending on the alternative hypothesis. Specifically,

Alternative hypothesis	Rejection region	p-value
$H_1 : p_1 - p_2 < 0$	$Z_0 < -Z_\alpha$	$p < \alpha$
$H_1 : p_1 - p_2 > 0$	$Z_0 > Z_\alpha$	$p < \alpha$
$H_1 : p_1 - p_2 \neq 0$	$\lvert Z_0 \rvert > Z_{\alpha/2}$	$p < \alpha/2$

Example #1

Small-business telephone users were surveyed 6 months after access to carriers other than AT&T became available for wide-area telephone service. Of a random sample of 268 users, 92 said they were attempting to learn more about their options, as did 37 of an independent random sample of 116 users of alternative carriers. **Test, at the 5% significance level against a two-sided alternative**, the null hypothesis that the two population proportions are the same.

Solution #1

We are given following sample statistics:

Sample	Sample size	Number of successes	Sample proportion
Sample 1	368	92	92/368 = 0.25
Sample 2	116	37	37/116=0.3190

It is clear that n_1 and n_2 are large enough so we use the following test statistic here:

$$Z_0 = \frac{\hat{p}_1 - \hat{p}_2}{\hat{SE}(\hat{p}_1 - \hat{p}_2)}$$

where

$$\hat{SE}(\hat{p}_1 - \hat{p}_2) = \sqrt{\hat{p}(1-\hat{p})}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \quad \text{and} \quad \hat{p} = \frac{X+Y}{n_1+n_2}$$

Solution #1

We are asked to test, at the 5% significance level against a two-sided alternative, the null hypothesis that the two population proportions are same. Therefore, the null and alternative hypotheses should be in the following form:

$H_0 : p_1 - p_2 = 0$ vs. $H_1 : p_1 - p_2 \neq 0$

where p_1 and p_2 stand for the population proportions.

Alternative hypothesis determines the rejection region. Since it is two-sided in the form $H_! : p_1 - p_2 \neq 0$, the null hypothesis is rejected when $|Z_0| > z_{\alpha/2}$.

$\alpha = 0.05 \rightarrow z_{\alpha/2} = z_{0.05/2} = z_{0.025} = 1.96$

Solution #1

$$\hat{p} = \frac{X + Y}{n_1 + n_2} = \frac{92 + 37}{368 + 116} = 0.2665$$

$$\hat{SE}(\hat{p}_1 - \hat{p}_2) = \sqrt{\hat{p}(1 - \hat{p})}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = \sqrt{0.2665(1 - 0.2665)}\sqrt{\frac{1}{368} + \frac{1}{116}} = 0.0471$$

$$Z_0 = \frac{\hat{p}_1 - \hat{p}_2}{\hat{SE}(\hat{p}_1 - \hat{p}_2)} = \frac{0.25 - 0.3190}{0.0471} = -1.4650$$

$|Z_0| = 1.4650 < z_{0.025} = 1.96$ implies that the null hypothesis cannot be rejected. In other words, the population proportions are different.

Example #2

Random samples of 900 people in the US and in UK indicated that 60% of the people in US were positive about the future economy, whereas 66% of the people in UK were positive about the future economy. Does this provide strong evidence that the people in UK are more optimistic about the economy? Use $\alpha = 0.01$

Solution #2

We are given following sample statistics:

Sample	Sample size	Sample proportion
Sample 1 (US)	900	0.60
Sample 2 (UK)	900	0.66

It is clear that n_1 and n_2 are large enough so we use the following test statistic here:

$$Z_0 = \frac{\hat{p}_1 - \hat{p}_2}{\hat{SE}(\hat{p}_1 - \hat{p}_2)}$$

where

$$\hat{SE}(\hat{p}_1 - \hat{p}_2) = \sqrt{\hat{p}(1-\hat{p})}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \quad \text{and} \quad \hat{p} = \frac{X+Y}{n_1+n_2}$$

Solution #2

We should investigate whether the people in UK are more optimistic about the economy. In other words, the population proportion of UK is larger than that of US. Therefore, the null and alternative hypotheses should be in the following form:

$H_0 : p_1 - p_2 \geq 0$ vs. $H_1 : p_1 - p_2 < 0$

where p_1 and p_2 stand for the population proportions of US and UK respectively.

Alternative hypothesis determines the rejection region. Since it is one-sided in the form $H_1 : p_1 - p_2 < 0$, the null hypothesis is rejected when $Z_0 < -z_{\alpha/2}$

$\alpha = 0.01 \rightarrow z_\alpha = z_{0.01} = 2.33$

Solution #2

$$\hat{p} = \frac{X+Y}{n_1+n_2} = \frac{\hat{p}_1 n_1 + \hat{p}_2 n_2}{n_1+n_2} = \frac{(0.60)900 + (0.66)900}{900+900} = 0.63$$

$$\hat{SE}(\hat{p}_1 - \hat{p}_2) = \sqrt{\hat{p}(1-\hat{p})}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = \sqrt{0.63(1-0.63)}\sqrt{\frac{1}{900}+\frac{1}{900}} = 0.0228$$

$$Z_0 = \frac{\hat{p}_1 - \hat{p}_2}{\hat{SE}(\hat{p}_1 - \hat{p}_2)} = \frac{0.60-0.66}{0.0228} = -2.6316$$

$Z_0 = -2.6316 < -z_{0.01} = -2.33$ implies that the null hypothesis should be rejected. In other words, the people in UK are more optimistic about the economy.

Course materials

You can download the notes and codes from:

https://github.com/mcavs/ESTUMatse_2022Fall_EngineeringStatistics

SCAN ME

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