Oct 27, 2022 (updated on Oct 30, 2024)

Engineering Statistics

Week 3: Probability

Basic definitions of probability

Basic Definitions

Sample space	The sample space of an experiment is the collection of all its sample points.
Sample point	A sample point is the most basic outcome of an experiment.
Experiment	An experiment is an act or process of observation that leads to a single outcome that cannot be predicted with certainty.

Example #1

Experiment: Observe the up face on a coin

Sample points: Head, tail

Sample space: $S = \{Head, Tail\}$

Example #2

Experiment: Observe the up face on a die

Sample points: 1, 2, 3, 4, 5, 6

Sample space: $S = \{1,2,3,4,5,6\}$

Example #3

Experiment: Observe the up faces on two coins

Sample points: HH, HT, TH, TT

Sample space: $S = \{HH, HT, TH, TT\}$

Counting rules

Counting rule for multiple-step experiments

If an experiment can be described as a sequence of k steps with n_1 possible outcomes on the first step, n_2 possible outcomes on the second step, and so on, then the total number of experimental outcomes is given by

$$(n_1)(n_2)\ldots(n_k)$$

Example #1

In the city of Milford, applications for zoning changes go through a two-step process: a review by the planning commission and a final decision by the city council. At step 1 the planning commission reviews the zoning change request and makes a positive or negative recommendation concerning the change. At step 2 the city council reviews the planning commission's recommendation and then votes to approve or to disapprove the zoning change. Suppose the developer of an apartment complex submits an application for a zoning change. Consider the application process as an experiment.

(1) How many sample points are there for this experiment? List the sample points.

(2) Construct a tree diagram for the experiment.

Example #2

The outcomes of two variables are (low, medium high) and (on, off), respectively. An experiment is conducted in which the outcomes of each of the two variables are observed. The probabilities associated with each of the six possible outcome pairs are given the following table:

	low	**medium**	**high**
on	0.50	0.10	0.05
off	0.25	0.07	0.03

(1) How many sample points are there for this experiment? List the sample points.

(2) Construct a tree diagram for the experiment.

Counting rule for combinations

The number of combinations of N objects taken n at a time is:

$$C_n^N = \binom{N}{n} = \frac{N!}{(N-n)!n!}$$

Example #1

Consider a quality control procedure in which an inspector randomly selects two of five parts to test for defects. In a group of five parts, how many combinations of two parts can be selected?

Example #2

Consider that the Florida lottery system uses the random selection of six integers from a group of 53 to determine the weekly winner. How many experimental outcomes are possible in the lottery drawing?

Counting rule for permutations

The number of permutations of N objects taken n at a time is:

$$P_n^N = n!\binom{N}{n} = \frac{N!}{(N-n)!}$$

Example

Consider a quality control procedure in which an inspector randomly selects two of five parts to test for defects. In a group of five parts, how many permutations of two parts can be selected?

Assigning probabilities

Basic probability rules

1. The probability assigned to each experimental outcome must be between 0 and 1, inclusively.

Rule 1. If we let E_i denote the i^{th} sample point (experimental outcome) and $P(E_i)$ its probability, then

$$0 \leq P(E_i) \leq 1 \text{ for all } i$$

Basic probability rules

2. The sum of the probabilities for all the experimental outcomes must equal to 1.

Rule 2. For n experimental outcomes ($S = \{E_1, E_2, \ldots, E_n\}$), this requirement can be written as

$$P(E_1) + P(E_2) + \ldots + P(E_n) = 1$$

Classical definition of probability

- The classical method of assigning probabilities is appropriate when all the experimental outcomes are equally likely.

- If n experimental outcomes are possible, a probability of 1/n is assigned to each experimental outcome:

$$P(E) = \frac{1}{n},\ i = 1,2,...,n$$

- When using this approach, the two basic requirements for assigning probabilities are automatically satisfied.

The relative frequency definition of probability

The relative frequency method of assigning probabilities is appropriate when data are available to estimate the proportion of the time the experimental outcome will occur if the experiment is repeated a large number of times:

$$P = \frac{f}{N}$$

here, f and N are number of the observed outcomes and repetition of the experiment.

Example

The National Highway Traffic Safety Administration (NHTSA) conducted a survey to learn about how drivers throughout the US are using seat belts (Associated Press, August 25, 2003). Sample data consistent with the NHTSA survey are given in the table.

	Yes	No
Northest	148	52
Midwest	162	54
South	296	74
West	252	48

For US, what is the probability that a driver is using a seat belt?

Probability of an event

Probability of an event

Event: An event is a specific collection of sample point.

Example #1

Experiment: Observe the up faces on two coins

Sample points: HH, HT, TH, TT

Sample space: $S = \{HH, HT, TH, TT\}$

Event A: At least one heads on top of coins. ($A = \{HH, TH, HT\}$)

Probability of an event

If A be an event in S, say $A = \{E_1, E_2, \ldots, E_k\}$. Then,

$$P(A) = P(E_1) + P(E_2) + \ldots + P(E_k)$$

If the outcomes $E_1, E_2, \ldots, E_k$ are equally likely, then

$$P(A) = \frac{\#(A)}{\#(S)} = \frac{k}{n}$$

Example #2

Consider the experiment of rolling a pair of dice. Suppose that we are interested in the sum of the face values showing on the dice. How many sample points are possible?

(a) List the sample points.

(b) What is the probability of obtaining a value of 7?

(c) What is the probability of obtaining a value of 9 or greater?

Example #3

Consider the experiment of tossing two balanced coins and the following events:

$A =$ {Observe exactly one head.}

$B =$ {Observe at least one head.}

Calculate the probability of A and the probability of B.

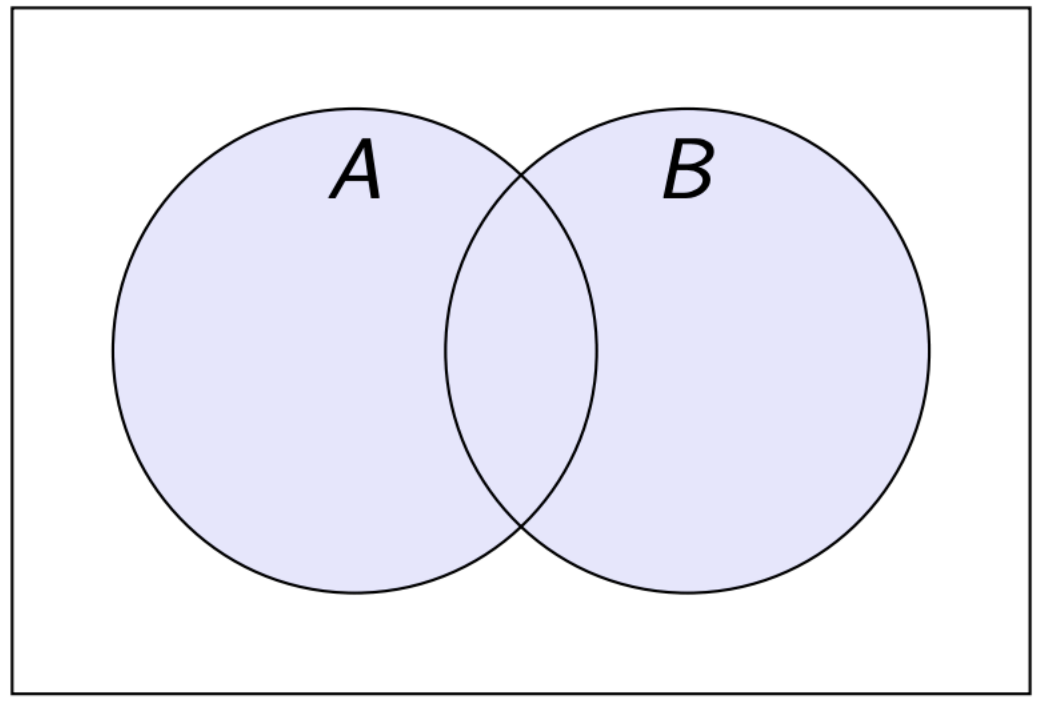
Unions and intersections

Union

The union of two events A and B is the event that occurs if either A and B (or both) occurs on a single performance of the experiment.

We denote the union of events A and B by the symbol $A \cup B$.

$A \cup B$ consists of all the sample points that belong to A or B or both.

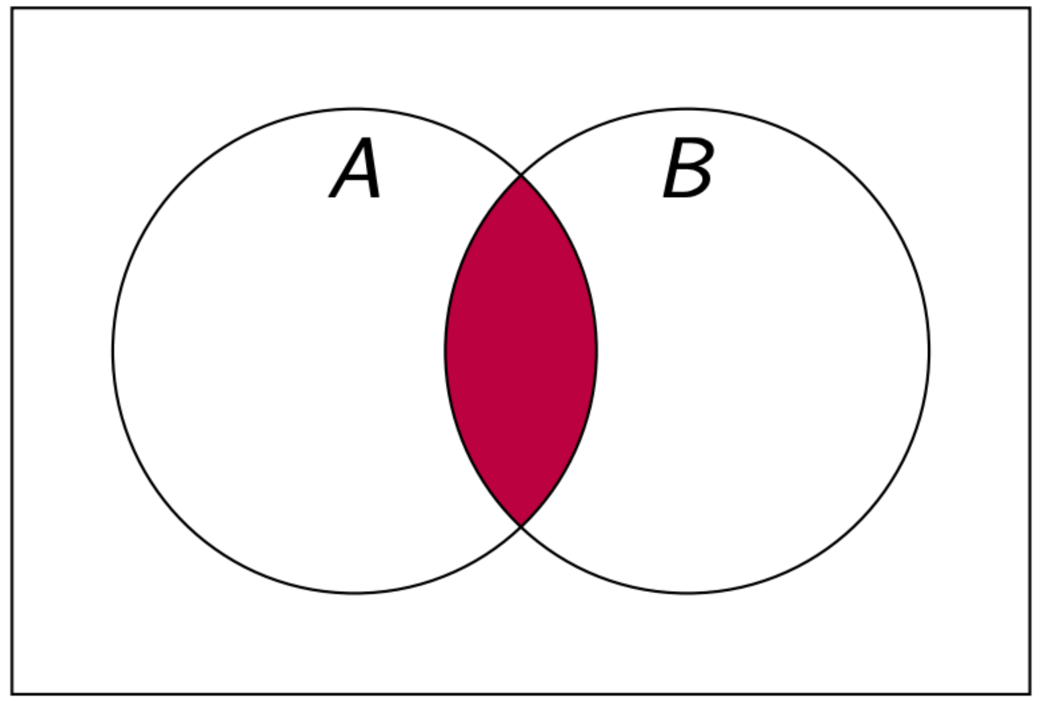


Intersection

The intersection of two events A and B is the event that occurs if both A or B occurs on a single performance of the experiment.

We denote the union of event A and B by the symbol $A \cap B$.

$A \cap B$ consists of all the sample points that belong to both A and B.



Additive rule of probability

The probability of the union of events A and B is the sum of the probability of event A and the probability of event B, minus the probability of the intersection of events A and B.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example #1

Hospital records show that 12% of all patients are admitted for surgical treatment, 16% are admitted for obstetrics, and 2% receive both obstetrics and surgical treatment. If a new patient is admitted to the hospital, what is the probability that the patient will be admitted for surgery, for obstetrics, or for both?

Example #2

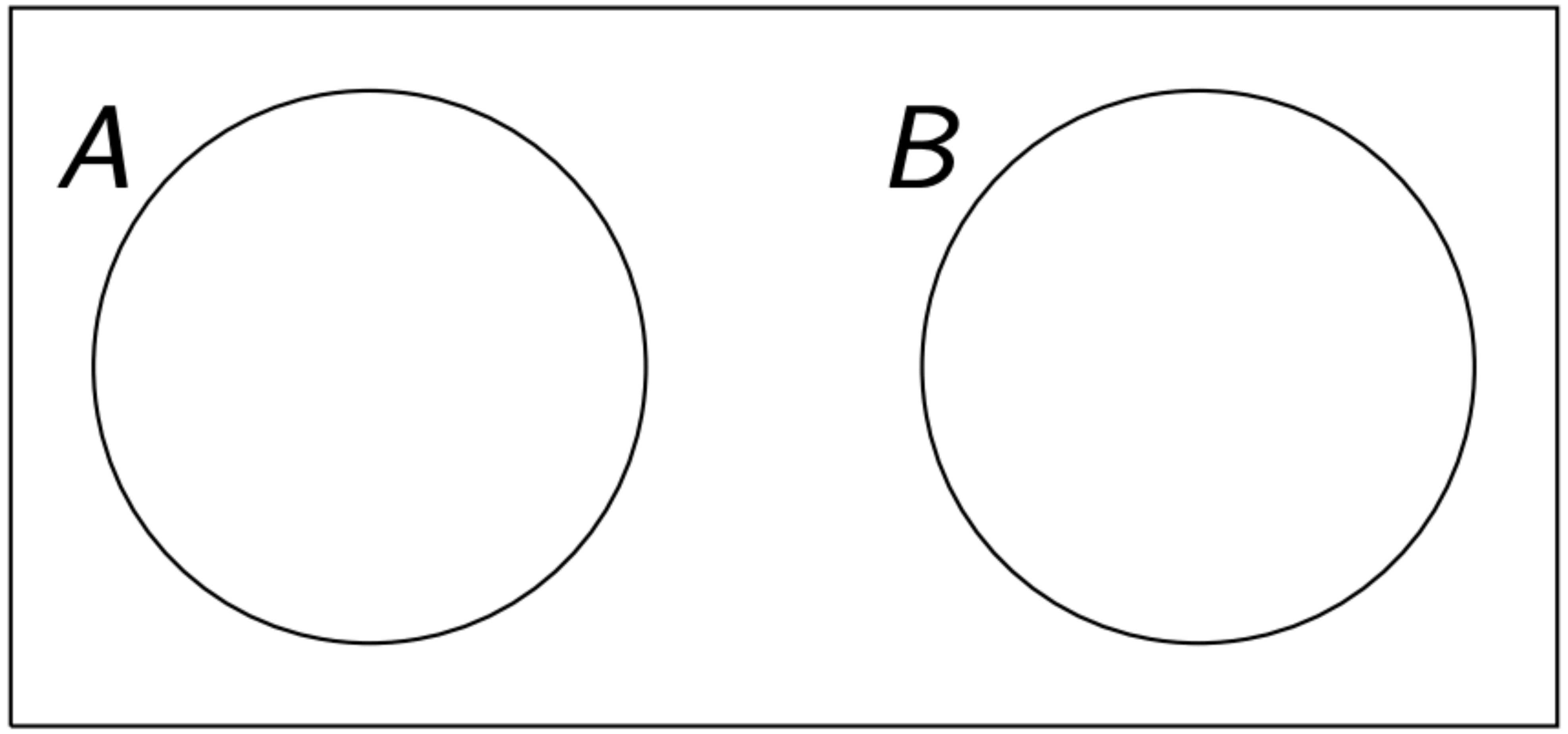
The outcomes of two variables are (low, medium high) and (on, off), respectively. An experiment is conducted in which the outcomes of each of the two variables are observed. The probabilities associated with each of the six possible outcome pairs are given the following table:

	low	**medium**	**high**
on	0.50	0.10	0.05
off	0.25	0.07	0.03

Consider the following events: $A = \{on\}$ and $B = \{high\}$. Calculate the probability that A or B?

Mutually exclusive events

Events A and B are mutually exclusive (disjoint) if $A \cap B$ contains no sample points - that is, if A and B have no sample points in common.



For disjoint events, $A \cap B = \varnothing$ therefore $P(A \cap B) = 0$.

Probability of union of two mutually exclusive events

If two events A and B are disjoint, the probability of the union of A and B equals the sum of the probability of A and the probability of B.

$$P(A \cup B) = P(A) + P(B) \text{ when } A \cap B = \varnothing$$

Complementary events

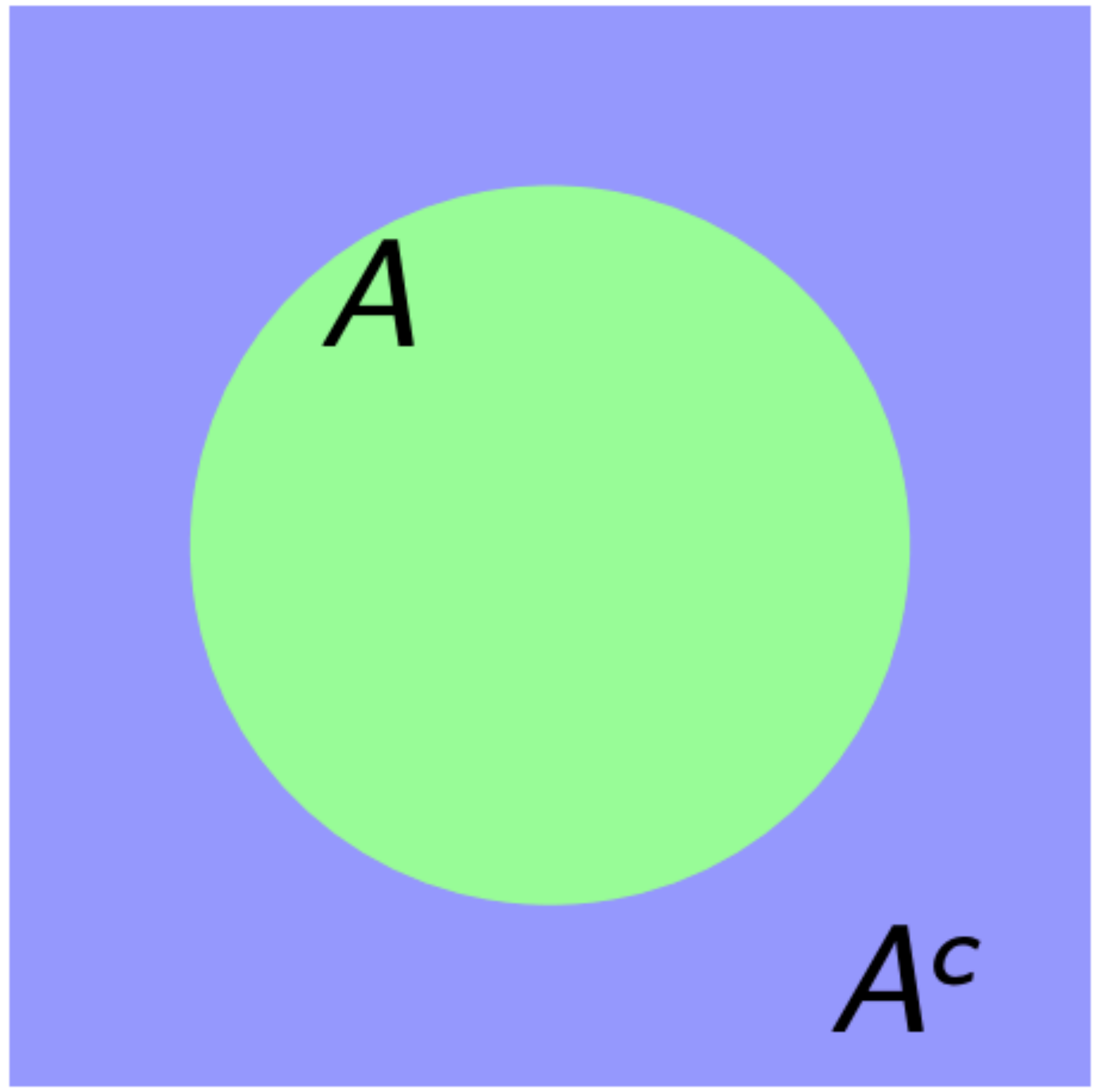
Rule of complements

- The complement of an event A is the event that A does not occur - that is, the event consisting of all sample points that are not in event A.

- We denote the complement of A by A^c.

- The sum of the probabilities of complementary events equals to 1.

$$P(A) + P(A^c) = 1$$



Example #1

A survey of magazine subscribers showed that 45.8% rented a car during the past 12 months for business reasons, 54% rented a car during the past 12 months for personal reasons, 30% rented a car during the past 12 months for both business and personal reasons.

(a) What is the probability that a subscriber rented a car during the past 12 months for business or personal reasons?

(b) What is the probability that a subscriber did not rent a car during the past 12 months for either business or personal reasons?

Example #2

The outcomes of two variables are (low, medium high) and (on, off), respectively. An experiment is conducted in which the outcomes of each of the two variables are observed. The probabilities associated with each of the six possible outcome pairs are given the following table:

	low	**medium**	**high**
on	0.50	0.10	0.05
off	0.25	0.07	0.03

Consider the following events: $A = \{on\}$, $B = \{medium\}$, $C = \{off\}$, and $D = \{high\}$. Find $P(A \cap B)$, $P(A \cup D)$, $P(B^c \cap C)$

Example #3

If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he/she will service at least 5 cars on his/her next day at work?

The multiplicative rule and independent events

Conditional probability

The event probabilities we have been discussing give the relative frequencies of the occurrences of the events when the experiment is repeated a very large number of times. Such probabilities are often called **unconditional probabilities**, because no special conditions are assumed other than those which define the experiment.

Often however, we have additional knowledge that might affect the outcome of an experiment, so we may need to alter the probability of an event of interest. A probability that reflects such additional knowledge is called the **conditional probability** of the event.

Conditional probability

To find the conditional probability that event A occurs given that event B occurs, divide the probability that both A and B occur by the probability that B occurs, **<u>or vice-versa.</u>**

$$P(A\,|\,B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

$$P(B\,|\,A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$$

Example

The Federal Trade Commission's investigation of consumer product complaints has generated much interest on the part of manufacturers in the quality of their products. A manufacturer of an electromechanical kitchen utensil conducted an analysis of a large number of consumer complaints and found that they fell into the six categories shown in the following table. If a consumer complaint is received, what is the probability that the cause of the complaint was the appearance of the product given that the complaint originated during the guarantee period?

	Electrical	**Mechanical**	**Appearance**	**Totals**
During guarantee	%18	%13	%32	%63
After guarantee period	%12	%22	%3	%37
Totals	%30	%35	%35	%100

Multiplicative rule of probability

The probability of an intersection of two events can be calculated with the multiplicative rule, which employs the conditional probabilities.

$$P(A \cap B) = P(A \mid B)P(B)$$

or

$$P(A \cap B) = P(B \mid A)P(A)$$

Example

Consider a newspaper circulation department where it is known that 84% of the households in a particular neighborhood subscribe to the daily edition of the paper. If we let D denote the event that a household subscribers to the daily edition, $P(D) = 0.84$. In addition, it is known that the probability that a household that already holds a daily subscription also subscribers to the sunday edition (event S) is 0.75.

What is the probability that a household subscribes to both the sunday and daily editions of the newspaper?

Independent events

Events A and B are independent events if the occurrence of B does not alter the probability that A has occurred; that is, events A and B are independent if

$$P(A \mid B) = P(A)$$

When events A and B are independent, it is also true that

$$P(A \mid B) = P(B)$$

Events that are not independent are said to be dependent.

Probability of intersection of two independent events

If events A and B are independent, then the probability of the intersection of A and B equals the product of the probabilities of A and B; that is,

$$P(A \cap B) = P(A)P(B)$$

The converse is also true, if

$$P(A \cap B) = P(A)P(B)$$

then events A and B are independent.

Example

A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

Note (added on Oct 30, 2024)

Conditional probability is based on the occurrence of one event and gives the probability of another event happening within a specific subset.

Intersection probability shows the occurrence of both events together.

To distinguish between these two concepts: **conditional probability operates within a subset**, while **intersection probability reflects the likelihood of both events occurring** in the entire population.

Probability of intersection of two independent events

Suppose that A and B are event on the same sample space S and they are independent. Then

- A^c and B^c are independent.
- A^c and B are independent.
- A and B^c are independent.

Example

The outcomes of two variables are (low, medium high) and (on, off), respectively. An experiment is conducted in which the outcomes of each of the two variables are observed. The probabilities associated with each of the six possible outcome pairs are given the following table:

	low	**medium**	**high**
on	0.50	0.10	0.05
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Consider the following events: $A = \{on\}$, $B = \{medium\}$, $C = \{off\}$, and $D = \{high\}$.

(a) Are the events A and D^c independent?

(b) Are the events B^C and C^c independent?

Course materials

You can download the notes and codes from:

https://github.com/mcavs/ESTUMatse_2022Fall_EngineeringStatistics

SCAN ME

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