Black Scholes Notes
Expectations

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1 Normal options

Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$. Then

$$\begin{aligned} \mathrm{E}(X-K)^+ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} (x-K)^+ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\sigma u + \mu - K)^+ e^{-\frac{1}{2}u^2} \, du \\ &= \frac{1}{\sqrt{2\pi}} \int_{(K-\mu)/\sigma}^{\infty} (\sigma u + \mu - K) e^{-\frac{1}{2}u^2} \, du \\ &= (\mu - K)\Phi\left(\frac{\mu-K}{\sigma}\right) + \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\mu-K}{\sigma}\right)^2} \end{aligned} \tag{1}$$

If $\sigma = s\sqrt{t}$ and $\mathrm{E}\, X = \mu = F$ is the forward price, then this can be written

$$\mathrm{E}(X-K)^+ = (F-K)\,\Phi(d) + \frac{s\sqrt{t}}{\sqrt{2\pi}} e^{-\frac{1}{2}d^2} \qquad \text{where} \qquad d = \frac{F-K}{s\sqrt{t}} \tag{2}$$

2 Lognormal options

Now suppose $Y = e^X$ where $Y \sim \mathcal{N}(\mu, \sigma)$. Then

$$\begin{aligned} \mathrm{E}(Y-K)^+ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} (e^x - K)^+ \, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (e^{\sigma u + \mu} - K)^+ \, e^{-\frac{1}{2}u^2} \, du \\ &= \frac{1}{\sqrt{2\pi}} \int_{(\log K - \mu)/\sigma}^{\infty} (e^{\sigma u + \mu} - K) \, e^{-\frac{1}{2}u^2} \, du \\ &= -K\,\Phi\left(\frac{\mu - \log K}{\sigma}\right) + e^{\mu + \frac{1}{2}\sigma^2} \frac{1}{\sqrt{2\pi}} \int_{(\log K - \mu)/\sigma}^{\infty} e^{-\frac{1}{2}(u-\sigma)^2} \, dx \\ &= -K\,\Phi\left(\frac{\mu - \log K}{\sigma}\right) + e^{\mu + \frac{1}{2}\sigma^2} \Phi\left(\frac{\mu - \log K + \sigma^2}{\sigma}\right) \end{aligned} \tag{3}$$

When $K \to -\infty$, then $\mathrm{E}(Y-K)^+ \to \mathrm{E}\,Y = e^{\mu+\frac{1}{2}\sigma^2}$. If we write $F = \mathrm{E}\,Y$ and $\sigma = s\sqrt{t}$, this can be written more symetrically as

$$\mathrm{E}(Y-K)^+ = F\,\Phi(d_+) - K\,\Phi(d_-) \tag{4}$$

where

$$d_- = \frac{\log(F/K) - \frac{1}{2}s^2 t}{s\sqrt{t}} \qquad d_+ = d_+ + s\sqrt{t} = \frac{\log(F/K) + \frac{1}{2}s^2 t}{s\sqrt{t}} \tag{5}$$

Let the forward call price be given by $c_F = \mathrm{E}(Y-K)^+$ and the forward put price be given by $p_F = \mathrm{E}(K-Y)^+$. Then put call pairity is

$$c_F - p_F = \mathrm{E}(Y-K) = F - K \tag{6}$$

Thus

$$p_F = \mathrm{E}(K-Y)^+ = -F\,\Phi(-d_+) + K\,\Phi(-d_-) \tag{7}$$

3 *Greeks*

Let's compute sensitivties. Also, assume the forward price is given by $e^{(r-q)t}S$, where r is the riskfree interest rate and q is the convenience yield or dividend yield. As before, let the forward call price be given by $c_F = \mathrm{E}(X-K)^+$ and the spot option price be given by $c = e^{-rt}c_F$, and let's compute the generic partial derivative with respect ot u– we'll specialize later.

$$\frac{\partial c_F}{\partial u} = \frac{\partial F}{\partial u}\,\Phi(d_+) + F\Phi'(d_+)\frac{\partial d_+}{\partial u} - \frac{\partial K}{\partial u}\,\Phi(d_-) - K\Phi'(d_-)\frac{\partial d_-}{\partial u} \tag{8}$$

Note that

$$d_+^2 - d_-^2 = (d_+ + d_-)(d_+ - d_-) = \frac{2\log(F/K)}{s\sqrt{t}} \cdot s\sqrt{t} = 2\log(F/K) \tag{9}$$

Therefore, we can define

$$\Psi := F\Phi'(d_+) = \frac{F}{\sqrt{2\pi}}e^{-\frac{1}{2}d_+^2} = \frac{K}{\sqrt{2\pi}}e^{-\frac{1}{2}d_-^2} = K\Phi'(d_-) \tag{10}$$

This allows us to simplify (8)

$$\frac{\partial c}{\partial u} = \frac{\partial F}{\partial u}\,\Phi(d_+) - \frac{\partial K}{\partial u}\,\Phi(d_-) + \Psi\,\frac{\partial(d_+ - d_-)}{\partial u} \tag{11}$$

First let's consider sensitivities to variables on which $d_+ - d_- = s\sqrt{t}$ does not depend, so the last term is 0.

$$\begin{aligned} m &= \frac{\partial c_F}{\partial K} = -\Phi(d_-) = -\Pr(Y \geq K) \\ \Delta_F &= \frac{\partial c_F}{\partial S} = \frac{\partial F}{\partial S}\Phi(d_+) = e^{(r-q)t}\Phi(d_+) \\ \Delta &= \frac{\partial c}{\partial S} = e^{-rt}\frac{\partial c_F}{\partial S} = e^{-qt}\Phi(d_+) \\ \rho_F &= \frac{\partial c_F}{\partial r} = \frac{\partial F}{\partial r}\Phi(d_+) = tF\Phi(d_+) \\ \rho &= \frac{\partial c}{\partial r} = -tc + e^{-rt}tF\Phi(d_+) = tK\Phi(d_-) \end{aligned} \tag{12}$$

The quantity m is sometimes called moneyness. Now let's compute greeks which do include the last term. Define a new quantity

$$\Upsilon = e^{-rt}\Psi = e^{-rt}K\Phi'(d_-) = e^{-qt}S\Phi'(d_+) \tag{13}$$

Then we can compute

$$\begin{aligned} \mathrm{vega}_F &= \frac{\partial c_F}{\partial s} = \sqrt{t}\Psi \\ \mathrm{vega} &= \frac{\partial c}{\partial s} = e^{-rt}\frac{\partial c_F}{\partial s} = \sqrt{t}\Upsilon \\ \tau_F &= -\frac{\partial c_F}{\partial t} = (q-r)F\Phi(d_+) - \frac{s\Psi}{2\sqrt{t}} \\ \tau &= -\frac{\partial c}{\partial t} = rc - e^{-rt}\frac{\partial c_F}{\partial t} = qF\Phi(d_+) - rK\Phi(d_-) - \frac{s\Upsilon}{2\sqrt{t}} \end{aligned} \tag{14}$$

Finally we compute

$$\begin{aligned} \Gamma_F &= \frac{\partial^2 c_F}{\partial S^2} = \frac{F}{S}\Phi'(d_+)\frac{\partial d_+}{\partial S} = \frac{\Psi}{S^2 s\sqrt{t}} \\ \Gamma &= \frac{\partial^2 c}{\partial S^2} = e^{-rt}\Gamma_F = \frac{\Upsilon}{S^2 s\sqrt{t}} \end{aligned} \tag{15}$$