Math 830 Assignment Three Group Work

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1. We wish to show $(E \mid X \mid^s)^{\frac{1}{s}} \leq (E \mid X \mid^p)^{\frac{1}{p}}$. By the Holder inequality, we have $E(\mid XY \mid) \leq (E \mid X \mid^q)^{\frac{1}{q}}(E \mid Y \mid^p)^{\frac{1}{p}}$. Let $Y = 1$ with probability $P(Y = 1) = 1$. Then we have will have

$$E \mid X \mid \leq (E \mid X \mid^p)^{\frac{1}{p}}.$$

For $1 < s < p$,

$$E \mid X \mid^s \leq (E \mid X \mid^{ps})^{\frac{1}{p}}.$$

Letting $r = ps > s$ yields $E \mid X \mid^s \leq (E \mid X \mid^r)^{\frac{s}{r}}$, which leads to

$$(E \mid X \mid^s)^{\frac{1}{s}} \leq (E \mid X \mid^r)^{\frac{1}{r}}.$$

Therefore

$$\| X \|_s \leq \| X \|_r \ .$$

Hence, if X_n converges to X in L_r, then X_n converges to X in L_S as $E \mid X_n - X \mid^2 \leq E \mid X_n - X \mid^r \rightarrow 0$.

Alternate Proof: Assume $X_n \xrightarrow{L_r} X$, then $\lim_{n\to\infty} E(\mid X_n - X \mid^r) = 0$. For s, we have

$$\begin{aligned} E(\mid X_n - X \mid^s) &= \int \mid X_n - X \mid^s dP \\ &= (\int \mid X_n - X \mid^r)^{s/r} dP \\ &= E(\mid X_n - X \mid^r)^{s/r}. \end{aligned}$$

Taking the limit, since $\lim_{n\to\infty} E(\mid X_n - X \mid^r) = 0$, then

$$\lim_{n\to\infty} E(\mid X_n - X \mid^r)^{s/r} = 0.$$

Thus $X_n \xrightarrow{L_s} X$.

2. Assume $X_n \xrightarrow{L_1} X$. Let $\epsilon > 0$. Then

$$P(\mid X_n - X \mid > \epsilon) \leq \frac{E(\mid X_n - X \mid)}{\epsilon},$$

by Markov's Theorem. Therefore

$$lim_{n\to\infty} \frac{E(\mid X_n - X \mid^p)}{\epsilon^p} = 0,$$

by convergence in L_1.

3. a) Define X_n as follows:

$$\left\{ \begin{array}{cc} n & \text{with probability } = n^{-\frac{s+r}{2}} \\ 0 & \text{with probability } = 1 - n^{-\frac{s+r}{2}}. \end{array} \right\}$$

Then

$$E \mid X_n \mid^s = \mid n \mid^s n^{-\frac{s-r}{2}} = n^{\frac{s-r}{2}} \tag{1}$$

$$E \mid X_n \mid^r = \mid n \mid^r n^{-\frac{s+r}{2}} = n^{\frac{r-s}{2}} \tag{2}$$

Clearly (1) converges to 0 when $n \to \infty$, however (2) converges to ∞ when $n \to \infty$. Therefore $X_n \xrightarrow{L_s} X$, but X_n does not converge to X in L_r space.

b) Define

$$X_n = \left\{ \begin{array}{cc} n^2 & \text{with probability} = n^{-1} \\ 0 & \text{with probability} = 1 - n^{-1}. \end{array} \right\}$$

Clearly, $P(\mid X_n - 0 \mid \geq \epsilon) = 0 \to 0$ as $n \to \infty$, but $E \mid X_n \mid = n^2 \frac{1}{n} = n \to \infty$. Therefore $X_n \xrightarrow{P} X$, but X_n does not converge in L_1 space to X.