1D Two Stream Plasma Instability Simulation

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Abstract. Two-Stream Instability was simulated and it's phase-space evolution plotted using the Particle in Cell method. This yielded results in accordance to alternative methods explored in previous papers, particularly using the Vlasov equation.

Keywords: Plasma, Particle in Cell (PIC), Two-Stream Instability, Simulation

1 Introduction

Plasma is one of the four fundamental states of matter. It's created by adding energy to a gas, either by heating it or subjecting it to a strong electromagnetic field. This will make some electrons leave their atoms, creating a soup of positively charged particles (ions) and negatively charged particles (electrons) that its called plasma.

The presence of charged particles makes plasma electrically conductive. The dynamics of individual particles and macroscopic plasma motion are governed by a collective of electromagnetic fields and are very sensitive to externally applied fields.

Plasma applications include fluorescent lightbulbs, neon signs, plasma displays used for television or computer screens, plasma lamps, and globes. Currently, even a new kind of energy production called fusion is being developed to generate electricity by using the heat from nuclear fusion reactions, making plasma research a scorching topic.

Plasma behaves more like a collection of discrete particles at low density rather than a single continuous fluid. And in high-density plasmas, the opposite happens, and they are usually simulated using the extension of computational fluid dynamics. This splits the plasma flows simulations categories into two: Kinetic and Fluid.

For the scope of this report, only the Kinetic approaches will be aborded. One of the most popular approaches is to use the Particle in Cell (PIC) method [1], which follows the motions of individual particles in a self-consistent electromagnetic field. It is a technique applicable to studies of solar wind propagation or even analysis of electric thruster plumes since low plasma densities characterize these discharges.

For this report, the PIC method will simulate two counter-streaming plasma flows in a very common instability problem in plasma physics called two-stream instability. This phenomena plays an essential role from topics related to electrostatic collisionless shock formation [10], to the coronal heating problem [11], or even inertial confinement fusion [12].

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2 Particle-In-Cell (PIC)

As referred previously, plasma is a collection of charged particles (ions and electrons), and non-ionized atoms with a global charge neutral. And so, plasma is just charged particles interacting with each other by attracting particles of opposite charge and repelling those with the same charge. For simulating this, and how this collection of particles evolves over time, one could have made the case of using the Coulomb force given by Equation 1.

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\vec{r}_{12} \tag{1}$$

Conceptually, it's possible to simulate plasma by directly computing this force. But since Coulomb force leads to an $O(N^2)$ problem, computation of a single time step would require at least the number of particles in the simulation squared operations.

Introducing the Particle-In-Cell (PIC) method, which now uses computational particles to represent the real ions, electrons, and neutrons, leads to a more efficient simulation. Instead of computing the Coulomb force directly, it will use the electric field as the force acting on the particles (Lorentz Force). By the end of this section, it will be deduced to be a $O(N)$ problem.

2.1 Lorentz force

The Lorentz force combines electric and magnetic forces on a point charge due to electromagnetic fields. And so, a particle of charge q moving with a velocity v in an electric field E and a magnetic field B experiences a force given by Equation 2.

$$\vec{F} = q(\vec{E} + (\vec{v} \times \vec{B})) \tag{2}$$

2.2 Maxwell's Equations

The evolution of the electric and magnetic fields is given by four fundamental equations of electromagnetics know as *Maxwell's equations*:

$$\text{Gauss' Law} \qquad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{3a}$$

$$\text{Gauss' Law } (\vec{B} \text{ Fields}) \qquad \nabla \cdot \vec{B} = 0 \tag{3b}$$

$$\text{Faraday's Law} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3c}$$

$$\text{Ampere's Law} \qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{3d}$$

Faraday's Law tells us that if $\frac{\partial \vec{B}}{\partial t} = 0$ (Constant $\vec{B}$), $\nabla \times \vec{E} = 0$, and a zero curl is said to be irrotational.

Helmholtz decomposition, also known as the fundamental law of vector calculus, further tells us that any sufficiently smooth vector field $\vec{F}$ can be decomposed into an irrotational (curl-free) and a solenoidal (divergence-free) part. These two parts can be defined in terms of a scalar (for irrotational) and a vector (for solenoidal) potential, such as:

$$\vec{E} = -\nabla\phi + \nabla \times \vec{A} \tag{4}$$

And since $\nabla \times \vec{E} = 0$, $\nabla \times \vec{A}$ must also be equal to zero, leaving:

$$\vec{E} = -\nabla\phi \tag{5}$$

Note that $\frac{\partial \vec{B}}{\partial t} = 0$ is known as the *electrostatic assumption*. It assumes the self-induced magnetic fields are negligible. In other words, the current generated by the plasma medium is assumed to be low enough so that the self-induced magnetic field can be ignored.

2.3 Poisson's Equation

After the considerations made above, the derivation of Poisson's equation is straightforward. Substituting the potential gradient for the electric field into Gauss' law directly produces Poisson's equation for electrostatics:

$$\nabla \cdot (-\nabla\phi) = \frac{\rho}{\varepsilon_0} \Leftrightarrow \nabla^2\phi = -\frac{\rho}{\varepsilon_0} \tag{6}$$

Poisson's equation provides a clear relationship between plasma potential and charge density. In terms of ion and electron number densities, charge density is written as $\rho = e(Z_i n_i - n_e)$. The subscripts i and e denote ions and electrons, respectively, and $Z_i$is the average ion charge number.

2.4 Grid

Using Poisson's equation to simulate plasma is quite attractive because the charge density is treated as macroscopic propriety. This means it won't be calculated everywhere but just at the intersections of a discretized simulation mesh.

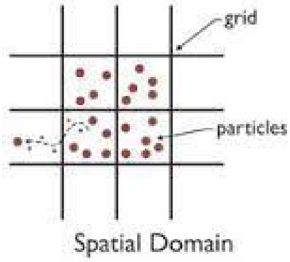


Figure 1: The grid overlay over the cell (D=2) [2]

The number of mesh nodes (intersections on the grid), G, is many times smaller than the number of particles, N. Instead of the $O(N^2)$ complexity required with the direct method using the Coulomb force, the operation count is reduced roughly to:

- N operations to compute charge density (loop over particles)
- G log(G) operations to solve Poisson's equation (loop over mesh nodes)
- G operations to compute electric field (loop over mesh nodes)
- N[1] operations to integrate particle positions (loop over particles)

For example, for $N = 10^5$ and $G = 10^3$, using the direct method requires $N^2 + N = (10^5)^2 + 10^5 \approx 10^{10}$ operations. Where using the techniques explained above with Lorentz force would just require $N + G\log(G) + G + N = 10^5 + 10^3\log(10^3) + 10^3 + 10^5 \approx 10^5$ operations. This is why PIC method is advantageous.

2.5 Simulation Loop

As the previous section hints, the computational domain is first discretized into a simulation mesh. Using the particle positions, the number density is computed at each time step. Then, the corresponding charge density is used to compute the plasma electric potential using Equation 6. Once the potential is known, the electric field is obtained from Equation 5. This electric field is used to update particle velocities according to the Equation 2. The new velocity pushes particles to new positions, and the whole process repeats.

3 Simulation Methodology

Two opposing plasma streams will be simulated. The goal is to visualize phase-space contours for the two-stream instability, it should be possible to see something somewhat resembling "water bags", just like in Figure 2.

[1]Note that integrating particles positions can be greatly optimized using vectorization techniques, making this operation far less computationally expensive.

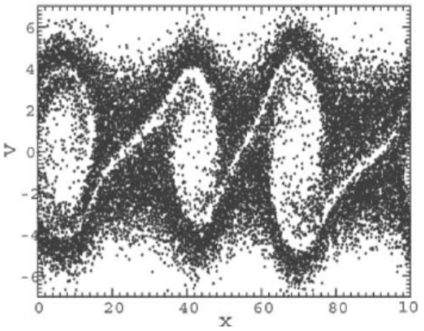


Figure 2: Two-stream instability in [3]

3.1 Initialization

As specified in subsection 2.5, the first step shall be initializing the particles and grid.

The simulation space will have one dimension with a length of 100 units. 100k particles will be initialized and distributed along two beams. One, moving from left to right; and another moving from right to left. Both streams are moving at 2 units/sec.

Note that both streams have an added small perturbance, which was accomplished by adding a *sin* wave at 0.1 amplitude and a period equal to the length of the simulation itself. The velocities of each particle also have a small perturbance added similar to the previous one, but now with an amplitude of 0.2.

There would be a total of 1k grid cells.

3.2 Charge Density Computation

Charge density defines the number of charge units per unit of volume (D=3). But since a simulation mesh is being used, the charge of all particles will be distributed into the nodes of the corresponding computational cell and then divided by the corresponding cell length (D=1). Hence the name for this method, particle-in-cell.

And so, in each node, there will be a "macro-particle", whose charge is equal to the sum of the surrounding particle's charge, divided by its length (D=1).

Note that in this report, is assumed that ions form a fixed neutralizing background with $n_i = 1$. In other words, an uniform plasma in which the ions are stationary and the electrons have a velocity relative to the ions will be used. Charge density will thus be given by $\rho = e(1 - n_e)$.

3.3 Potential Computation

The next step will be solving Poisson's Equation (Equation 6). In each computational node, i:

$$\frac{\partial^2 \phi}{\partial x^2} = -\rho_i \tag{7}$$

Note that for simplicity, $\epsilon_o = 1$ in the equation above. To solve this inhomogeneous differential equation, the FDM Method (subsubsection 6.1.1) will be used, obtaining:

$$\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{(\Delta x)^2} = -\rho_i \tag{8}$$

And so, the sparce system to obtain all the potentials for all the computational nodes is:

$$\begin{aligned}
\phi_0 &= \rho_0 \\
\frac{1}{(\Delta x)^2}(\phi_0 - 2\phi_1 + \phi_2) &= -\rho_1 \\
\frac{1}{(\Delta x)^2}(\phi_1 - 2\phi_2 + \phi_3) &= -\rho_2 \\
&\dots \\
\frac{1}{(\Delta x)^2}(\phi_{n-3} - 2\phi_{n-2} + \phi_{n-1}) &= -\rho_{n-2} \\
\frac{1}{(\Delta x)^2}(\phi_{n-2} - 2\phi_{n-1} + \phi_n) &= -\rho_{n-1} \\
\phi_n &= \rho_n
\end{aligned} \tag{9}$$

Where the potential on the boundaries of the domain is equal to the charge density on that node.

The computation of the solution for the system above will be done using the *banded* function, provided by the professor.

3.4 Electric Field Computation

After solving the potential using the FDM method, the next step is to compute the electric field using Equation 5:

$$E = -\frac{\partial \phi}{\partial x} \tag{10}$$

And so, a numerically obtained estimation of the first derivative is needed. Using the equation obtained in subsubsection 6.1.2, for each computational node, i:

$$E_i = -\frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} \tag{11}$$

Analogous to the previous section, this problem is described by an sparse system of equations:

$$\begin{aligned}
-E_0 &= -\frac{1}{2\Delta x}\phi_1 \\
-E_1 &= \frac{1}{2\Delta x}(\phi_2 - \phi_0) \\
-E_2 &= \frac{1}{2\Delta x}(\phi_3 - \phi_1) \\
&\dots \\
-E_{n-2} &= \frac{1}{2\Delta x}(\phi_{n-1} - \phi_{n-3}) \\
-E_{n-1} &= \frac{1}{2\Delta x}(\phi_n - \phi_{n-2}) \\
-E_n &= \frac{1}{2\Delta x}\phi_{n-1}
\end{aligned} \tag{12}$$

To solve this, a sparse matrix will be simply multiplied by the potential vector, returning the desired electric field vector.

3.5 Interpolation of the Electric Field

The goal is to use the electric field as the force acting on each particle to then evolve the positions and velocities of each particle according to that. But note that there is a problem: The computed electric field is located on the computational nodes, not the particles.

Interpolation is then needed to evaluate the field on the particle's position. This will be done with a linear interpolation using *interp* from *NumPy's* library.

3.6 Leapfrog Method

Now that the electric field was computed, the next step is to define a strategy used to describe the evolution of the motion of particles.

As deduced previously, on each particle, acts the Lorentz force:

$$F_x = E_x \tag{13}$$

In order to know each particles evolution, an integration of the electric field is required. Using leapfrog integration, the equations for updating position and velocity will be:

$$\begin{aligned} a_x &= E_x \\ v_{i+\frac{1}{2}} &= v_{i-\frac{1}{2}} + a_i \Delta t \\ x_{i+1} &= xi + vi + \frac{1}{2}\Delta t \end{aligned} \tag{14}$$

Computationally, what will be done is at the beginning of the program, the velocity will be shifted backwards by half a time step. Then, the velocities and positions on the main program loop will be updated with a regular time step.

Note: For simplicity, q and m are assumed to be 1 for each particle.

4 Results & Discussion

In Figure 3 the phase-space of the particle stream at different time steps obtained using the PIC method is plotted. The x-axis is the particle's position and the y-axis is its velocity. Blue is the population of particles moving from left to right, and in red the ones moving from right to left.

Analyzing the results, initially, the simulation consists of two distinct beams with no (apparent) variation in spatial density. After 10 time steps, a wave develops, slight wiggles arise from the two populations starting to interact via the electric field. After 18 iterations, this perturbance consequently makes the two populations merge together to form a vortex in the velocity phase space.

Analogous to this, this problem could be described using *Vlasov Equation* (subsection 6.2), which serves as an introduction to MHD (Magnetohydrodynamics, [9]).

Comparing the results obtained to Figure 4, it's possible to infer that the electrons follow the contours of $f(v)$ for brief moments. But unfortunately, the instability progressively distorts f(v) in a way that would be hard to describe analytically [6], just as seen in Figure 5 and in the results obtained in this report, Figure 6.

This instability can be seen as the inverse of Landau damping, just as explained in subsection 6.3. The instability grows until the electrons are trapped in the electric field of the wave. E. g. the simulation starts with multiple vortex's (Figure 3). As the simulation unrolls, energy is transferred from the particles to the waves, giving rise to exponential wave growth. In the end, when the instability is said to saturate, only one vortex survives, trapping the electrons on the electrostatic potential of the wave - In Figure 7 the beginning of the saturation can be seen.

This vortex will then continue to move, however, it will dissipate due to the numerical diffusion introduced by the first order interpolation scheme[14] (Figure 8).

Additionally, the authors at [7], for the same dimensionality, following the spatial and temporal development of distribution functions in the position-velocity phase space described by the *Vlasov Equation* for the two-stream instability problem, obtained a similar phase-space evolution. This further validated the PIC method explored during this report since similar results were obtained.

5 Conclusion

This report explored the two-stream instability problem, which involved the simulation of two opposing plasma streams using the PIC method. This provided results in accordance to an alternative approach that uses the *Vlasov Equation* to describe the phase-space evolution.

The resulting phenomena also agreed with the velocity distribution function analysis, using the Landau damping effect.

6 Appendix

6.1 Finite Difference Method (FDM)

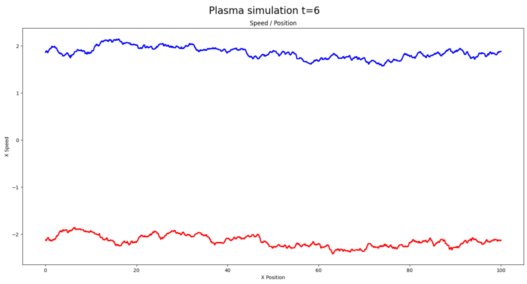
6.1.1 Second-Order Derivative

The first step is to use Taylor Series to provide an expression for the value of some function f at a point offset by x from another point at which the values of the function and its derivatives are known:

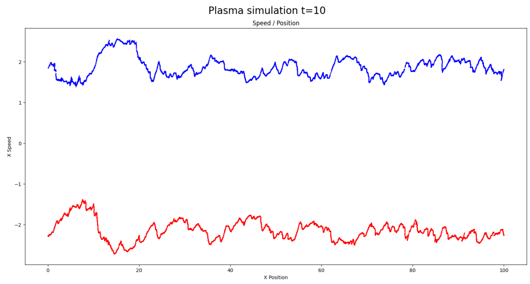
$$f(x+\Delta x) = f(x) + \frac{\Delta x}{1!}\frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3} + ... \tag{15}$$

And a similar expression can be written for a point offset by $-\Delta x$,

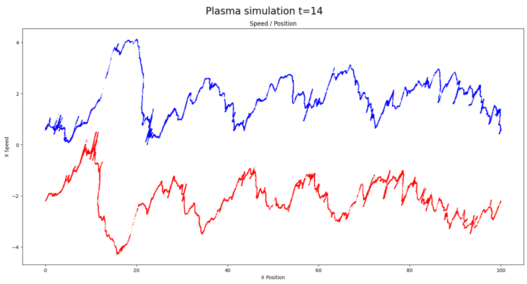
$$f(x-\Delta x) = f(x) - \frac{\Delta x}{1!}\frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3} + ... \tag{16}$$



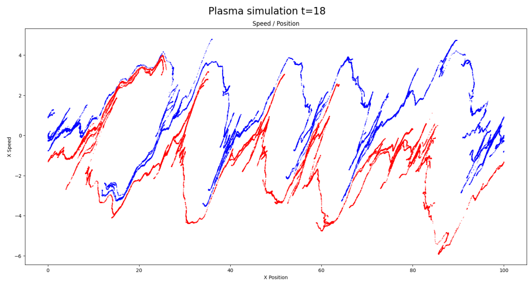
(a) t=6



(b) t=10



(c) t=14



(d) t=18

Figure 3: Results obtained using the PIC method for the two-stream instability.

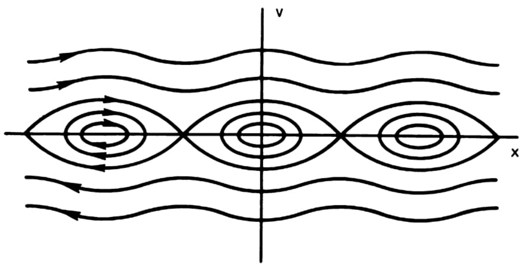


Figure 4: Electron trajectories, or contours of constant $f(v)$, as seen in the wave frame, in which the pattern is stationary. [6]

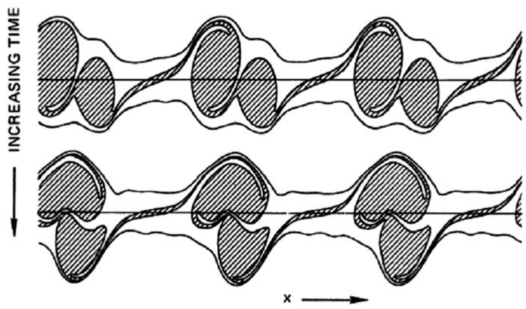


Figure 5: Phase-space contours for electrons in a two-stream instability. The shaded region is devoid of electrons. [6]

Considering $f_{i+1} \equiv f(x + \Delta x)$, these two equations can be added together, obtaining:

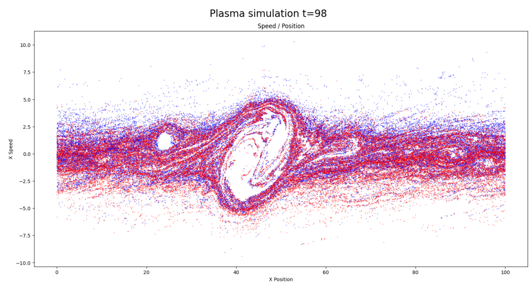


Figure 6: Results obtained using the PIC method for the two-stream instability at t=98.

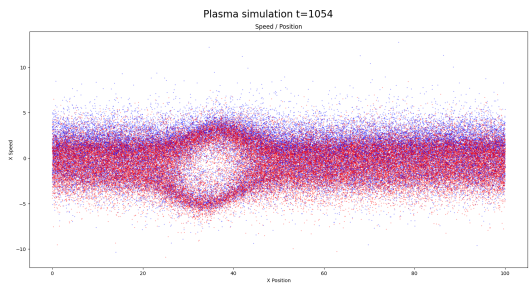


Figure 7: Results obtained using the PIC method for the two-stream instability at t=1054.

$$f_{i+1} + f_{i-1} = 2f_i + (\Delta x)^2 \frac{\partial^2 f}{\partial x^2} + ... \quad (17)$$

Ignoring the higher order terms, since generally $\Delta x <<$ 1 and the factorial in the denominator grows rapidly, these

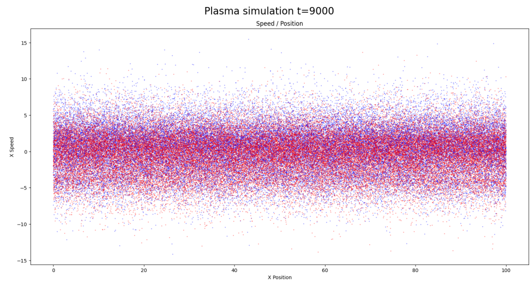


Figure 8: Results obtained using the PIC method for the two-stream instability at t=9000.

terms quickly become very small, its possible to re-write the equation above to isolate the derivative term:

$$\frac{\partial^2 f}{\partial x^2} = \frac{f_{i-1} - 2f_i + f_{i+1}}{(\Delta x)^2} + O(3) \tag{18}$$

6.1.2 First Order Derivative

Instead of adding both equations on subsubsection 6.1.1, if an subtraction of the second by the first is done, this is obtained:

$$f_{i+1} - f_{i-1} = 2\Delta x \frac{\partial f}{\partial x} + ... \tag{19}$$

Ignoring once again the higher order terms and rearranging the equation above:

$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_{i-1}}{2\Delta x} wit \tag{20}$$

6.2 Vlasov Equation

One alternative to describing the evolution of the plasma is with the *Vlasov equation*, which uses an distribution function $f(v)$.

The deduction starts with the *Boltzmann equation* [4]:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{\vec{F}}{m} \nabla_v \cdot f = \left(\frac{\partial f}{\partial t}\right)_{col} \tag{21}$$

Here F is the force acting on the particles, and $\left(\frac{\partial f}{\partial t}\right)_{col}$ is the time rate of change of f due to collisions. The symbol ∇_v stands for the gradient in velocity space.

If collisions are ignored, and acceleration arises solely from the Lorentz force, the *Vlasov equation* emerges.

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{q(\vec{E} + \vec{v} \times \vec{B})}{m} \nabla_v \cdot f = 0 \tag{22}$$

For the case with no magnetic field, the equation reduces further to:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{q\vec{E}}{m} \nabla_v \cdot f = 0 \tag{23}$$

Here $f = f(\vec{x}, \vec{v}, t)$ is value of the distribution function evaluated at some time t for some spatial and velocity coordinate $(\vec{x}, \vec{v})$. In general, there is no analytical equation for f and thus its to represented as an discretized form in the velocity phase space.

6.3 Inverse Landau Damping

Landau Damping, is the effect of damping plasma oscillations. In other words, this phenomenon prevents an instability from developing, and creates a region of stability in the parameter space [13].

The effect is connected with those particles in the distribution that have a velocity nearly equal to the phase velocity. These particles travel along with the wave and do not see a rapidly fluctuating electric field: They can, therefore, exchange energy with the wave effectively [6].

The easiest way to understand this exchange of energy is to picture a surfer trying to catch an ocean wave. If the surfer is moving on the water surface at a velocity slightly less than the waves he will eventually be caught and pushed along the wave (gaining energy), while a surfer moving slightly faster than a wave will be pushing on the wave as he moves uphill (losing energy to the wave).

In a plasma, there are electrons both faster and slower than the wave. Landau Damping is an effect that describes the evolution of particles when particle velocities are often taken to be approximately a Maxwellian distribution function. But in the case of the two-stream instability, the particles' velocity distribution function is the same has Figure 9, having a "bump" on its tail.

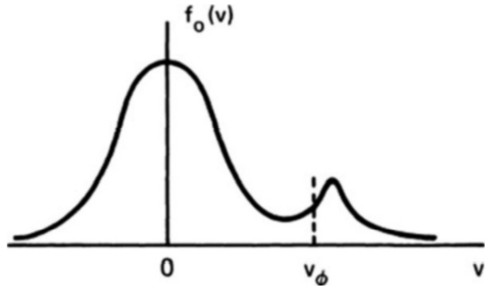


Figure 9: A double-humped distribution and the region where instabilities will develop [6]

Since the wave has phase velocity in the region where the slope is positive, there is a greater number of faster particles ($v > v_\phi$) than slower particles, and so there is a greater amount of energy being transferred from the fast particles to the wave. This will lead to a unstable regime, giving rise to exponential wave growth.

The instability grows until the beam particles are trapped in the electric field of the wave. This is when the instability is said to saturate.

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