Newton v. Laguerre Numerical Solutions of Kepler's Equation

Michael Bernard

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1. Introduction

We would like to be able to determine the position, $\vec{r}$, and velocity, $\vec{v}$, of an object on orbit as a function of time in a two-body problem. Given the position vector, $\vec{r_0}$, and velocity vector, $\vec{v_0}$, at some initial time t_0, we can use the Lagrange coefficient equations to determine the position and velocity vectors at some time $t = t_0 + \Delta t$:

$$\vec{r} = f\vec{r_0} + g\vec{v_0} \tag{1}$$

$$\vec{v} = \dot{f}\vec{r_0} + \dot{g}\vec{v_0} \tag{2}$$

where

$$f = 1 - \frac{\chi^2}{r_0}C(\alpha\chi^2) \tag{3a}$$

$$g = \Delta t - \frac{1}{\sqrt{\mu}}\chi^3 S(\alpha\chi^2) \tag{3b}$$

$$\dot{f} = \frac{\sqrt{\mu}}{rr_0}[\alpha\chi^3 S(\alpha\chi^2) - \chi] \tag{3c}$$

$$\dot{g} = 1 - \frac{\chi^2}{r}C(\alpha\chi^2) \tag{3d}$$

where α is the inverse of the semi-major axis, μ is the gravitational parameter of the primary body, χ is the universal anomaly at time t, and the functions C and S are the Stumpff functions defined in [Kaplan]. Symbols without vector notation denote the magnitudes of those vectors.

If we can determine the universal anomaly at time t, then we can determine the position and velocity. To solve for χ, we use Kepler's Equation (4a), modified for use in numerical solvers:

$$f(\chi) = \frac{r_0 v_r)_0}{\sqrt{\mu}}\chi^2 C(z) + (1 - \alpha r_0)\chi^3 S(z) + r_0\chi - \sqrt{\mu}\Delta t \tag{4a}$$

$$f'(\chi) = \frac{r_0 v_r)_0}{\sqrt{\mu}}\chi[1 - \alpha\chi^2 S(z)] + (1 - \alpha r_0)\chi^2 C(z) + r_0 \tag{4b}$$

$$\begin{aligned} f''(\chi) = \frac{r_0 v_r)_0}{\sqrt{\mu}}\Big(1 - 3zS(z) + z\big(C(z) - 3S(z)\big)\Big) \\ + \chi\big(1 - zS(z)\big)(1 - \alpha \mathrm{r}_0) \end{aligned} \tag{4c}$$

where $z = \alpha\chi^2$, and $v_r)_0$ denotes the radial velocity at t_0.

We describe two numerical methods: Newton's Method [Weisstein] and Laguerre's Method [Conway]. We compare the two methods over a range of orbital conditions. We also describe some practical considerations for software implementations of either of these methods in a spacecraft.

2. Newton's Method

Using Newton's Method, we begin with some initial estimate χ_0 and apply the iteration equation

$$\chi_{i+1} = \chi_i - \frac{f(\chi)}{f'(\chi)} \tag{5}$$

to solve for χ within some desired tolerance. [Chobotov] recommends $\chi_0 = \sqrt{\mu}|\alpha|\Delta t$.

3. Laguerre's Method

[Conway] applies Laguerre's numerical method for solving polynomials to the Kepler Equation. The Kepler equation is obviously not a polynomial, but the method seems to work well in testing. We begin with some initial estimate of χ_0 and apply the iteration equation

$$\chi_{i+1} = \chi_i - \frac{nf(\chi_i)}{f'(\chi_i) \pm \sqrt{(n-1)^2\left(f'(x_i)\right)^2 - n(n-1)f(x_i)f''(x_i)}} \tag{6}$$

where n is nominally the order of the polynomial. Conway notes that n must not be 0 or 1, but that the speed of convergence is fairly insensitive to the value of n—they use $n = 5$, since that value gives consistently rapid convergence, although Conway notes that other values of n yield faster convergence under certain conditions.

4. Empirical Testing of Each Method

4.1. Experimental Conditions

We model a spacecraft orbiting Earth ($\mu = 398{,}600$ km^3/s^2) with an initial periapsis position of $r_0 = 10000$ km. Note that since the spacecraft is at periapsis at time t_0, the value of $v_r)_0 = 0$ km/s. We test each method over the full range of elliptical orbits, $e = [0.0, 1.0)$, and for a range of hyperbolic orbits, $e = (1.0, 5.0]$. For each eccentricity, we test the method over a time difference between 1 second and 24 hours (the period of geosynchronous satellites). We consider a value to be converged when it reaches 7 decimal places of precision, per the [IEEE] 754 standard for single precision floating points. For χ_0, we use the initial value of $\sqrt{\mu}|\alpha|\Delta t$ recommended by [Chobotov]. For the value of n in Eq. (6), we use 5, as recommended by [Conway].

4.2. Results

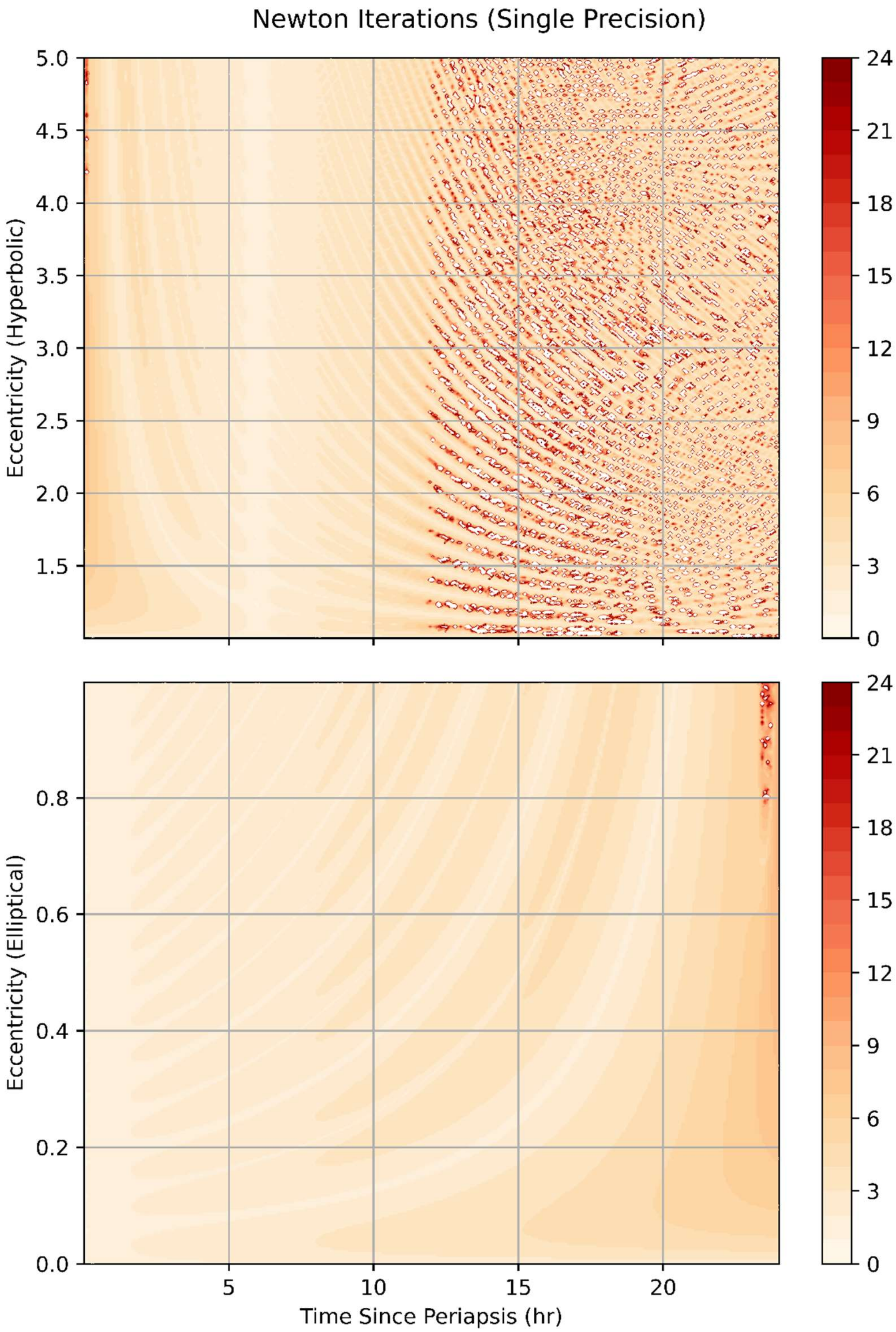

Figure (1) Number of iterations required for Newton's Method for convergence. Darker regions correspond to more iterations required. White regions correspond to a failure to converge within 25 iterations.

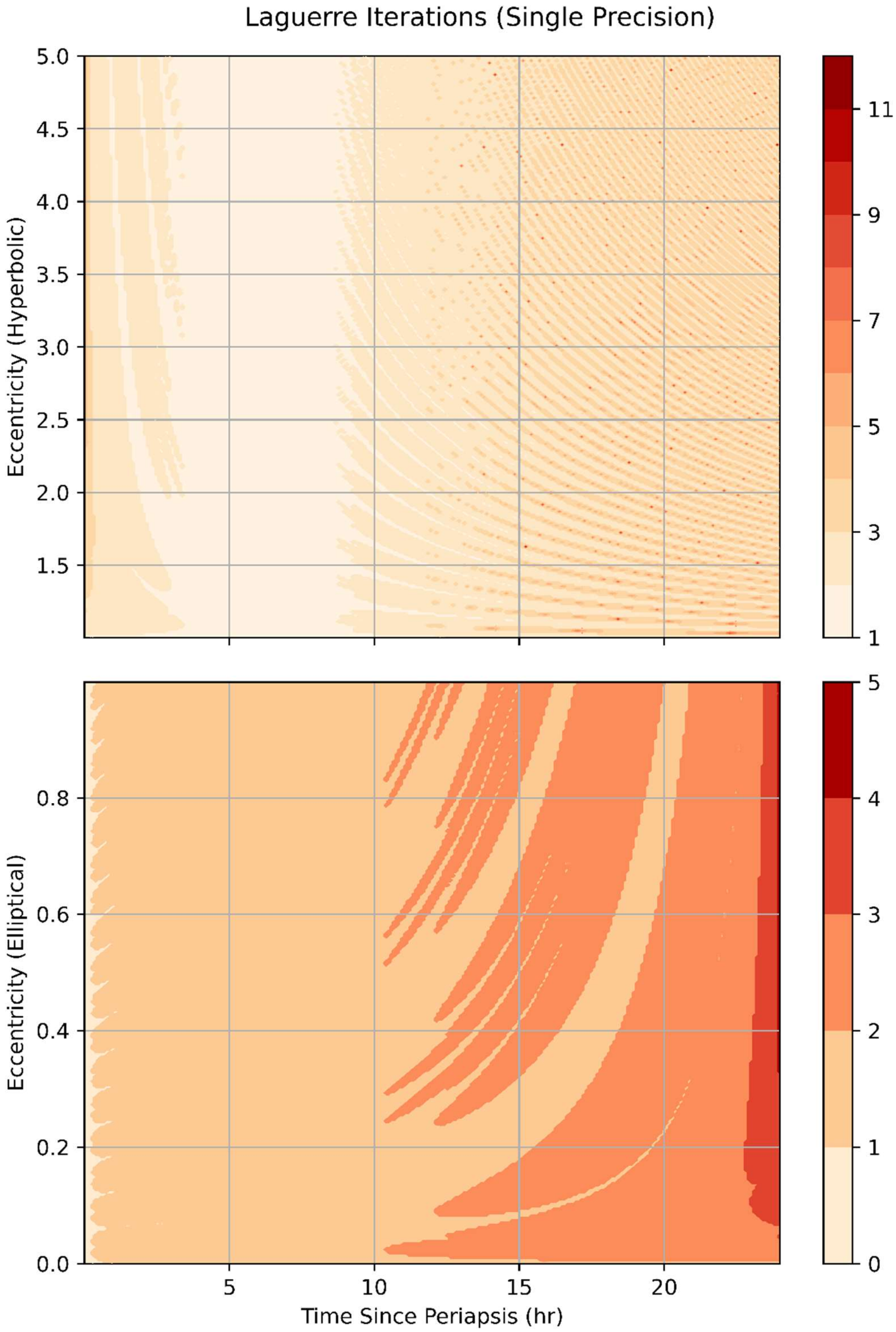

Figure (2) Number of iterations required for Laguerre's Method for convergence. Darker regions correspond to more iterations required. Laguerre's Method never failed to converge in any of these cases.

4.3. Discussion

In the two figures shown, we plot time since periapsis, in hours, on the abscissa, and eccentricity on the ordinate. In each figure, we separate the elliptical eccentricities from the hyperbolic by means of a horizontal separation, and have distinct color mappings for the lower and upper plots.

In Figure (1), we have the number of iterations required for Newton's Method for the range of conditions described in Section 4.1. In Figure (2), we have the number of iterations required for Laguerre's Method. In Figures (1) and (2), the darker shade of red a region is, the more iterations required to converge.

We note that Laguerre's Method converged in fewer than 12 iterations in all cases. On the contrary, Newton's Method took *many* more iterations on average than Laguerre's Method. In Figure (1), white regions of the plots denote cases in which Newton's Method failed to converge within 25 iterations—twice as many as Laguerre's took at maximum. In those regions the method took anywhere from 25 iterations to at least 100, if it converged at all.

5. Practical Considerations

When optimizing for number of iterations, Laguerre's Method is clearly superior. If a spacecraft's orbit is predicted using ground-based computers, Laguerre's Method offers the same precision as Newton's Method with fewer iterations and does not converge to parasitic solutions [Conway], unlike Newton's Method.

We observe that Newton's Method requires fewer multiplications than Laguerre's Method; this implies that each iteration of Newton's Method *should* take less time than each iteration of Laguerre's Method. However, upon optimizing the software implementation of each method, we found that Newton's Method yielded more iterations per second such that it converged faster than Laguerre's Method on average, although *only* in the elliptical region. For hyperbolic cases, Laguerre's Method yielded more iterations per second, always converged, and always converged faster than Newton's Method for cases where Newton's Method did converge. We were unable to determine why Laguerre iterations took less time than Newton iterations in the Hyperbolic region.

If a spacecraft's orbit were to be determined from a ground computer with high processing power availability, Laguerre's Method would clearly be superior in all cases; Laguerre's Method would be especially better for objects on interplanetary trajectories while still in Earth's sphere of influence, when eccentricity is highly hyperbolic.

However, we also consider that processing power is limited on a spacecraft. If a spacecraft's orbit were to be determined using on-board flight computers for elliptical orbits, such as Earth satellites, Newton's Method would be worth considering; it offers a faster convergence than Laguerre's Method, requiring less computing resources—although the potential cost of convergence to a parasitic solution might make Newton's Method too unstable for use.

6. Conclusion

We have observed that Laguerre's Method is superior to Newton's Method in terms of iterations required for convergence; this led us to the conclusion that Laguerre's Method should be implemented in ground-based orbit determination computers. We did, however, discuss practical scenarios in which Newton's Method might be deemed suitable.

This analysis was limited to examining the number of iterations required for convergence to single-precision floating point values. It would be worthwhile, in future analyses, to examine the number of iterations required for double-precision floating point values, which are more likely to be used in modern spacecraft computers.

We were also unable to determine why Laguerre's Method iterations were computationally faster, on average, than Newton's Method for hyperbolic orbits. We recommend examining why this is the case in future analyses.

In terms of applications, Laguerre's Method is not only limited to solving Kepler's Equation, or even to problems in astrodynamics. It would be worthwhile to examine the suitability of Laguerre's Method to other astrodynamics problems, such as attitude determination, or trajectory estimation during entry, descent, and landing on a celestial body with an atmosphere.

References

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