PHYS-101 Projectile Motion Lab Example
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In this lab, we fired a projectile ten times from a spring gun onto a landing site covered in carbon paper to mark the landing locations on impact. The range of the projectile recorded from the carbon paper marks after each trial is recorded in the table below:

Trial Number	Range (m)	Trial Number	Range (m)
1	$1.923 \pm .0005$	6	$1.937 \pm .0005$
2	$1.918 \pm .0005$	7	$1.940 \pm .0005$
3	$1.941 \pm .0005$	8	$1.929 \pm .0005$
4	$1.950 \pm .0005$	9	$1.944 \pm .0005$
5	$1.932 \pm .0005$	10	$1.935 \pm .0005$

Table 1: Range of projectile fired from spring gun in each of ten trials. Error reflects the ability to measure distance only to the nearest millimeter using the two-meter stick.

1. The average range of the projectile calculated from the above data is $\overline{R} = 1.935$ m. The measurement uncertainty is of order less than .0005 m while the statistical fluctuations in the range is of order .01 m or more. Since this is a difference of about two orders in magnitude, we are justified in neglecting the measurement error and taking only the statistical spread in the range as our error on $\overline{R}$. This statistical uncertainty is $\Delta\overline{R} = .016$ m. The measured range is therefore:

$$\overline{R} \pm \Delta\overline{R} = (1.935 \pm .016) \text{ m}.$$

2. Using the formula for the range in terms of the initial velocity and launch angle, we can calculate the average velocity from the average range:

$$R = \frac{v_0^2 \sin 2\theta}{g} \implies \overline{v}_0 = \sqrt{\frac{g\overline{R}}{\sin 2\theta}} \tag{1}$$

Similarly, we are given the formula for the ucertainty in the average velocity in terms of the uncertainty in the average range:

$$\Delta\overline{v}_0 = \frac{\Delta\overline{R}}{2\sqrt{\overline{R}}}\sqrt{\frac{g}{\sin 2\theta}} \tag{2}$$

Using Equations (1) and (2) and the fact that the spring gun was set at an angle of 64°, we calculate the average velocity and uncertainty on the velocity as:

$$\overline{v}_0 \pm \Delta\overline{v}_0 = (4.91 \pm .02) \text{ m/s} \tag{3}$$

It is instructive to also examine the degree to which the uncertainty in the angle of the spring gun plays a role. The measurement uncertainty in this angle is $\Delta\theta = 0.5°$ since the resolution of the protractor measuring the angle is to the nearest degree. We are given the more general formula for the uncertainty in the average velocity including this angular uncertainty as:

$$\Delta\overline{v}_0 = \sqrt{\frac{g}{4\overline{R}\sin 2\theta}(\Delta\overline{R})^2 + \frac{g\overline{R}\cos^2 2\theta}{\sin 2\theta}(\Delta\theta)^2} \tag{4}$$

Note that we must convert $\Delta\theta$ to radians in the above formula so that the units work out correctly, since radians are technically unitless. Plugging in the quantities calculated above, we find the revised result for the uncertainty:

$$\Delta\overline{v}_0 = .03 \text{ m/s} \tag{5}$$

Remarkably, we find that the angular uncertainty is in fact an important correction which modifies the uncertainty in average velocity by a factor of about 1.5.

A last comment on this part is that it is useful to check whether or not this velocity is physically reasonable. We note that the average time the projectile was in the air was about a second each time it was fired. To find the average velocity, we need the length of the trajectory. This is nontrivial for a parabola, but to get order of magnitude let's approximate the parabola by a semicircle. The diameter of this semicircle is about 2 m since this is the order of the average range, so the arc length is π m. We should thus expect a velocity on the order of π m/s which is consistent with our computation above to order of magnitude.

3. Now, we have derived in the pre-lab that the trajectory of the projectile $(x(t), y(t))$ is given by:

$$x(t) = (v_0 \cos\theta)\, t, \qquad y(t) = (v_0 \sin\theta)\, t - \frac{1}{2} g t^2 \tag{6}$$

when the origin is measured from center of the ball at the moment of launch and g is taken to be positive. Plugging in our measured values of $\overline{v}_0$ and θ to this formula we obtain the trajectories numerically:

$$x(t) = (2.15 \text{ m/s})\, t, \qquad y(t) = (4.41 \text{ m/s})\, t - (4.90 \text{ m/s}^2)\, t^2 \tag{7}$$

4. Yes, given our values for v_0 and R, the value $\theta = 90^\circ - 64^\circ = 26^\circ$ would also allow us to hit the target. Conceptually this is because at the larger angle, the projectile has a lower horizontal velocity but spends more time in the air. At the smaller angle, the projectile spends less time in the air but has a compensating larger horizontal velocity. Mathematically, note that Equation (1) which gives the range as a function of initial velocity and angle is directly proportional to $\sin 2\theta$. Since $\sin x$ is symmetric about $x = 90^\circ$, the range is symmetric about $2\theta = 90^\circ \implies \theta = 45^\circ$. Thus θ and $90^\circ - \theta$ yield the same range.

It is interesting to consider whether air resistance would cause the trajectory at smaller or larger θ to have a larger range. We are given the the force due to air resistance scales as the velocity squared and acts opposite to the direction of motion. For the smaller theta this force acts primarily to reduce the horizontal velocity and vice versa. So for small theta the acceleration due to air resistance goes as v_x^2, the average change in velocity scales as $v_x^2 t$ and the change in range scales as $v_x^2 t^2$. For large theta, the acceleration due to air resistance scales as v_y^2, the average change in y velocity scales as $v_y^2 t$. Note that this tends to slow down the projectile faster on the way up *and* the way down, so the effects roughly cancel and the time of flight is roughly the same. Thus we should expect that the air resistance has a more significant effect at small θ, where the change in horizontal velocity affects the range more than the negligible change in time-of-flight at large θ.

5. For the second part of the lab we set two rings at vertical positions so that the trajectory of the projectile would pass through the rings. In order to set the vertical positions, we first obtained the vertical height of the trajectory as a function of the horizontal distance away from the launcher. Using Equation (6), we can solve for t given x from the first equation as $t = \frac{x}{v_0 \cos\theta}$. Substituting into $y(t)$, we find the trajectory:

$$y(x) = x \tan\theta - \frac{g x^2}{2 v_0^2 \cos^2\theta} \tag{8}$$

Notice that this is in fact a parabola i.e quadratic in x as expected.

6. Given this formula for the trajectory, we found where to set the vertical height of the rings given the horizontal positions of each ring. The rings were at distances $x_1 = .343$ m and $x_2 = 1.142$ m. Equation (8) thus prescribes the corresponding vertical positions $y_1 = .579$ m and $y_2 = .961$ m.

7. If the rings were oriented perpendicularly to the table, then these vertical positions need only to be accurate to within the radius of the ring minus the radius of the ball $= .05$ m$-.01$ m $= .04$ m. However, we tilted the rings by an angle ϕ to the horizontal. This changes that error Δy as shown in Figure 1 From Figure 1, the new vertical error is:

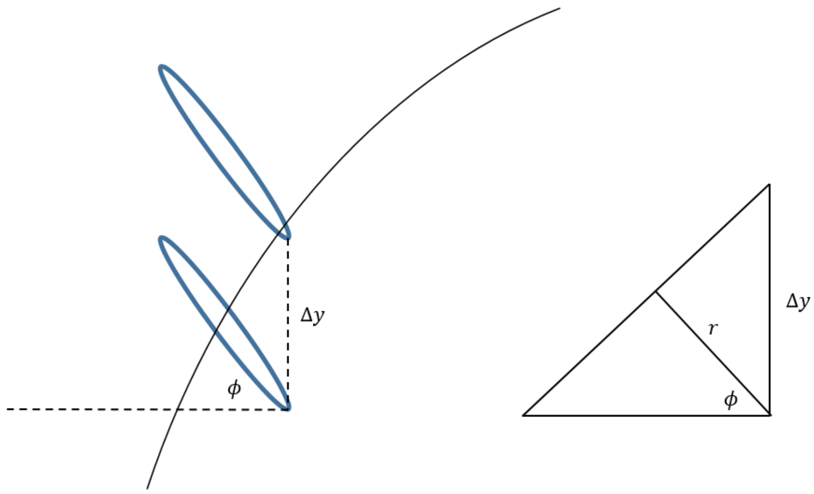

Figure 1: Left: the distance Δy at which the rings may be shifted vertically while still containing the trajectory of the ball depends on the angle ϕ at which the rings are tilted. Right: the geometrical setup of the scenario on the left.

$$\Delta y = \frac{r}{\cos(90° - \phi)} - r_{ball} \tag{9}$$

where $r = .05$ m is the radius of the ring. We estimated by measuring the horizontal and vertical distances to the center of the rings that the rings were oriented at an angle of $\phi = 50°$ to the horizontal. Therefore, the permissible vertical shift is:

$$\Delta y = \frac{.05 \text{ m}}{\cos 40°} - .01 \text{ m} = .056 \text{ m} \tag{10}$$

We obtain an extra margin of error due to the fact that the trajectory is headed upwards; the angular shift of the ring lets the back end of the ring travel further upwards to meet the higher trajectory. Note that this estimate is only true where the trajectory has small curvature i.e. away from the top of the trajectory, as you can see from the triangle in Figure 1 which approximates the trajectory as linear.

8. No, we did not make both rings on the first try. The ball clipped the right side of the second ring and bounced off. However, the ball was perfectly aligned along the other two axes.

   One obvious culprit seems like air resistance. We can put a general bound on the magnitude of the air resistance force using the fact given to us that drag force is approximately $F = \frac{1}{2}\rho A v^2$, where ρ is the density of air, A is the cross-sectional area of the ball, and v is the velocity. We can approximate the velocity of the ball as $\frac{\overline{v}_0}{2} = 2.46$ m/s. We are given that the density of air is about 1.23 kg/m^3, and since the measured diameter of the ball is $d = .02$ m, $A = \pi(.01^2 \text{ m}^2)$. The average magnitude of the force is thus about $\overline{F} = 1.2 \times 10^{-3}$ N. Since we are given that the mass of the ball is about .02 kg, the average magnitude of the acceleration is: $\overline{a} = .06$ m/s^2. Over the roughly second-long duration of flight, this air resistance force thus decreases the velocity by about $v = .06$ m/s. This is close to the scale of the uncertainty on the average velocity and so is probably not a major effect. At the location of the second ring, which we'd expect the ball to reach at $t = \frac{1.142\text{ m}}{4.85\text{ m/s }\cos 64^\circ} = .54$ s, the height would then be:

$$y = (4.85\text{ m/s})\sin 64^\circ(.54\text{ s}) - (4.90\text{ m/s}^2)\,(.54\text{ s})^2 = 0.925\text{ m} \tag{11}$$

This is change of $\Delta y_2 = .036$ m from the estimate without air resistance. Since the radius of the ring is .05 m, this should not be a sufficient error as to cause the ball to miss in the vertical direction! Note, however, that the error is on the same order of magnitude as the radius of the ring, so we should be more careful with exact numerical factors to determine precisely if the air resistance is enough to cause the ball to miss.

Regardless, since the air resistance doesn't cause any shift of the ball in the transverse direction, it's unlikely that this was the dominant source of uncertainty in our calculations. More likely to have caused problems was the uncertainty in the transverse angle of the spring gun. We did not have a separate protractor to measure this angle; however, we can estimate that the spring gun was perfectly straight to within an angle $\Delta\phi = 3^\circ$. To see how large a shift in the transverse direction this could cause at the location of the second ring, consult Figure 2: We see that the slight angle to the spring

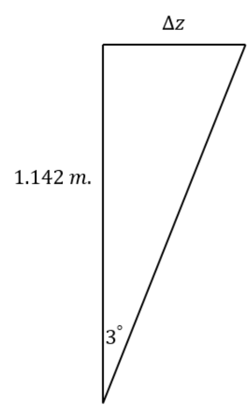

Figure 2: Possible transverse shift in ball trajectory due to angular uncertainty of 3° of the alignment of the spring gun in the transverse direction

gun could cause a transverse shift of:

$$\Delta z = (1.142\text{ m})(\tan 3^\circ) = .06\text{ m} \tag{12}$$

This is actually larger than the radius of the ring: therefore, this slight angular displacement of the spring gun would be sufficient to cause the ball to clip the side of the ring as occurred on our first try.