A primer of calculus for numerical simulation of computer graphics

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Introduction

This document describes the fundamental calculus for numerical simulation of computer graphics. In 3D graphics, the photon emitted from a light source is affected, absorbed, and scattered by objects. We simulate the incident photon on the camera to generate images. It's called rendering and uses integration. Let's focus on the relationship between calculus and the simulation.

1. Sequences

For examples, a list of numbers ordered by a rule, such as $1, 2, 3, 4, \ldots$, is called a *sequence*. Each number in a sequence is called a *term*. In general, the sequence of numbers $a_1, a_2, a_3, \ldots, a_n$ (the first term is a_1) is written as:

$$\{a_n\} \tag{1.1}$$

Next, consider the sequence $1, 4, 7, 10, 13, \ldots$ This sequence starts at 1 and jumps 3 every term. Such that sequence is called an *arithmetic sequence*. In an arithmetic sequence, the difference between one term and the next is a constant. This constant is called the *common difference*. Therefore, the first term and the common difference are required to represent the arithmetic sequence. Letting a be the first term and d be the common difference, each term is given by

$$a_1 = a, \quad a_2 = a + d, \quad a_3 = a + 2d, \quad \cdots \tag{1.2}$$

The equation for each term of the sequence is called the *general term*, and that equation is

$$a_n = a + (n-1)d \tag{1.3}$$

In addition, the relationship between the *n*th term and the next is expressed as

$$a_{n+1} = a_n + d \tag{1.4}$$

This form is called a *recurrence relation* or a *recursive equation*.

Next, consider multiplying the first term by a constant the one after another. For example, if the first term is 5 and the constant is 2, we have

$$5, 10, 20, 40, 80, 160, \ldots \tag{1.5}$$

Such that sequence is called a *geometric sequence*, and that constant is called the *common ratio*. The general term of a geometric sequence is obtained by letting a be the first term and r be the common ratio

$$a_n = a \cdot r^{n-1} \tag{1.6}$$

And the recurrence relation of a geometric sequence is

$$a_{n+1} = r \cdot a_n. \tag{1.7}$$

Here, consider the common difference of an arithmetic sequence. Suppose the common difference d is $r \cdot a_n$, then the recurrence relation of an arithmetic sequence is

$$a_{n+1} = a_n + r \cdot a_n \tag{1.8}$$

Letting $R = 1 + r$, we have

$$a_{n+1} = R \cdot a_n \tag{1.9}$$

which is the recurrence relation of a geometric sequence. Thus, we can see the recurrence relation of a geometric sequence is the recurrence relation of an arithmetic sequence with varying the common difference.

Now consider the sequence with an infinite number of terms such as $a_1, a_2, a_3, \ldots, a_n, \ldots$. This is called an *infinite sequence*. If let $n \to \infty$ and the value a_n approach α, the sequence $\{a_n\}$ *converges* to α. That is expressed as

$$\lim_{n\to\infty} a_n = \alpha \tag{1.10}$$

where α is called the *limit value* of the sequence $\{a_n\}$. Some diverge to positive or negative infinity if they do not converge.

$$\lim_{n\to\infty} a_n = \infty, \qquad \lim_{n\to\infty} a_n = -\infty \tag{1.11}$$

Others might not converge or diverge, in this case, they are oscillated.

For example, An infinite geometric sequence $\{r^n\}$ diverge if $\{r^n\}$.

$$\lim_{n\to\infty} r^n = \infty \quad (r > 1) \tag{1.12}$$

Thus if $r = 1$ or $|r| < 1$, it converges.

$$\lim_{n\to\infty} r^n = 1 \quad (r = 1), \quad \lim_{n\to\infty} r^n = 0 \quad (|r| < 1) \tag{1.13}$$

If $r \leq 1$, it oscillates (limits do not exist).

If two converging sequences $\{a_n\}, \{b_n\}$ and an arbitrary sequence $\{c_n\}$ are related by $a_n \leq c_n \leq b_n$, then the following equation

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \alpha \tag{1.14}$$

is satisfied, we have

$$\lim_{n\to\infty} c_n = \alpha \tag{1.15}$$

which is called the *squeeze theorem*.

2. Function

A function, simply put, that the corresponding value y is uniquely determined when an arbitrary value x is given. That is expressed as

$$y = f(x) \tag{2.1}$$

A function can take some arbitrary values.

$$y = f(x_1, x_2, \ldots) \tag{2.2}$$

The important fact is that parameters $x_1, x_2, \ldots$ of a function can take real numbers such as 0.1 or -0.2 as opposed to positive integers such as the sequence $n = 1, 2, 3, 4, \ldots$. Therefore, values of a function is continuous (but not necessarily continuous). In addition, the limit of a function can be considered in the same way as sequences. For example, if x approaches a, the value of $f(x)$ is as close to α as possible to the value of $f(x)$, that is expressed as $x \to a$ and $f(x)$ is said to converge to α. Therefore,

$$\lim_{x \to a} f(x) = \alpha \tag{2.3}$$

Of course, divergence, oscillation, and the squeeze theorem holds for a function.

3. Differentiation

You may have often heard or been taught that the differential is finding the slope of a tangent line. Of course, that is correct. However, that is one way of looking at the differential. Thus, you can see it as a slope geometrically when a function is represented in a graph. The differential may be done many times. For example, in the case of the second differential, we are finding the slope of the slope. Can you imagine that intuitively? What could it be? I feel that if you assume the differential is a slope, you may be stuck in higher mathematics.

So I would like to look at the differentiation from a physical rather than geometric point of view. The simulation of photons described at the beginning of this document is in physics (optics). Let's look at it from a different perspective.

Mathematics deals with numbers, but physics deals with quantities. Here, the function $y = f(t)$ computes the distance and t is the time. This function compute the moved distance while a given time t. Suppose the change in t as Δt and the change in y corresponding to Δt as Δy. we have

$$\frac{\Delta y}{\Delta t} \tag{3.1}$$

which is the *rate of change*. Since Δy is the change in y, we have

$$\frac{\Delta y}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{3.2}$$

If the rate of change of the distance y represent as the function $g(f(t))$, we have

$$\frac{\Delta y}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t} = g(f(t)). \tag{3.3}$$

Letting $f(t) = u_n$ and $f(t + \Delta t) = u_{n+1}$, we obtain

$$\frac{u_{n+1} - u_n}{\Delta t} = g(u_n) \tag{3.4}$$

This can be rewritten as

$$u_{n+1} = u_n + \Delta t \cdot g(u_n) \tag{3.5}$$

Letting $d = \Delta t \cdot g(u_n)$, we have

$$u_{n+1} = u_n + d \tag{3.6}$$

which shows that it same as the recurrence relation of an arithmetic sequence. The common difference d is multiplying the change in Δt by the rate of change $g(u_n)$. What this is? u_{n+1} is $f(t + \Delta t)$, or the distance after Δt. Of course, u_n is the distance at t. This means that by computing d and adding it to the previous result, we can find the next distance. Repeating this, we perform the simulation.

Now consider the function $g(f(t))$ of the rate of change. Letting $\Delta t = 1$, we have

$$g(u_n) = \frac{u_{n+1} - u_n}{\Delta t} = u_{n+1} - u_n \tag{3.7}$$

which means it is the *difference*. Since the common difference d is also multiplied by Δt when computing, we have

$$d = \Delta t \cdot g(u_n) = \Delta t \cdot \frac{u_{n+1} - u_n}{\Delta t} = u_{n+1} - u_n \tag{3.8}$$

If we make this Δt as small as possible to close to 0, we can find the exact difference between $f(t)$ and $f(t + \Delta t)$. Then that is

$$g(f(t)) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{3.9}$$

Here, the limit of the rate of change of the function $f(t)$ is expressed as the following:

$$f'(t), \quad \frac{df}{dt}, \quad \frac{df(t)}{dt}, \quad \frac{d}{dt}f(t) \tag{3.10}$$

which all means the same, they are called a *derivative*. Hence, Eq. (3.9) has the form

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{3.11}$$

which shows that $g(f(t))$ is the *derivative*. Therefore, differentiating a function is finding that derivative. And the value obtained by passing a value to the derivative is called a *differential coefficient*. Substituting $f(t) = u_n$ into Eq. (3.8), we have

$$d = \Delta t \cdot g(f(t)) \tag{3.12}$$

where $g(f(t))$ is the differential coefficient.

In summary, in quantities of physics, the differential is the rate of change, or roughly speaking, that is finding the difference. We talked about the second differential earlier. I think it is better to assume that is the difference of the difference rather than the slope of slope. What do you think?

Note: ddx, ddy

There are 'ddx' and 'ddy' functions for shader programming. These are functions that compute the gradient, or slope, of a partial differential. In fact, it is the value computed the difference between the neighboring pixels. You can easily imagine that this is the difference if you assume $\Delta t = 1$.

4. Integration

Integration is as opposed to differentiation. Suppose that there is a derivative $f'(t)$ by differentiating a function $f(t)$. Integrating that derivative $f'(t)$ is the original function $f(t)$. The simulation is nothing more than computing the integral. And we have already come up with computing an integral. As you can probably guess, it is computing the recurrence relation of an arithmetic sequence.

The integrals is often described as an area, of course, it is also the way of looking at it geometrically. The function are continuous, but the differential obtain the difference. This difference is *discrete*. The integrals is to add these differences together. That is, you can seem that the differential is to discretize the function and the integrals is to bring the difference back to continuous.

5. Simulation

So, let's try to simulate. It's very simple. By using the built-in $\sin$ function:

```
x = Math.sin(time);
```

which get an oscillating motion. We will simulate this using calculus. First, we know that we obtain $\cos$ function by differentiating $\sin$ function and $-\sin$ function by differentiating $\cos$ function. On the contrary, we obtain $\cos$ function by integrating $-\sin$ function and $\sin$ function by integrating $\cos$ function. Needless to say, to integrating is compute the recurrence relation of an arithmetic sequence:

$$u_{n+1} = u_n + d$$

Using the derivative $g(f(t))$, we obtain

$$u_{n+1} = u_n + \Delta t \cdot g(f(t)).$$

First, to computing $\sin$ function integrates $\cos$ function.

```
c = Math.cos(time);
x = x + deltaTime * c;
```

Next, to computing $\cos$ function integrates $-\sin$ function.

```
s = -Math.sin(time);
c = c + deltaTime * s;
x = x + deltaTime * c;
```

where x is the computed $\sin$ function. Therefore, we can write as

```
c = c + deltaTime * -x;
x = x + deltaTime * c;
```

Now, we can simulate the $\sin$ function. Note that x should be initialized with an amplitude.

I made a sample of this simulation, refer to the following URL.

http://mebiusbox.github.io/contents/cg_calculus_sim/

In the sample, the red square are simulated using integration, the green square are computed using built-in trigonometric functions.

6. The relationship between Calculus and Units

In this section, Let try to look at the relationship between calculus in a slightly different way. As mentioned earlier, the integrals is as opposed to the differentiation likewise the relationship between multiplication and division and exponents and logarithms. Multiplication and division, exponents and logarithms are multiplied and divided by real numbers. So what about calculus? Couldn't calculus also be considered to be multiplied or divided by something? It is **units**.

Finding the average velocity v by letting d be the distance and t be the time, we have

$$v = \frac{d}{t}$$

Dividing the average velocity v by time obtains the instantaneous velocity, or acceleration, we have

$$a = \frac{v}{t}$$

Consider these changes. The distance is considered as the position of an object, the change in position by the motion of an object is called *displacement*. Velocity is the change in displacement per unit of time. Acceleration is the change in velocity per unit of time. The derivative of displacement is velocity and the derivative of the velocity is acceleration. On the contrary, integrating the acceleration is velocity and integrating the velocity is displacement.

Now, we focus on each of these units. Conforming to the International System Of Units (SI), The unit of displacement is m, the unit of velocity is m/s, and the unit of acceleration is m/s^2.

Consider the derivative as division. If we differentiate displacement (m), that is, divide it by the unit of time (s), we obtained m/s. That equals to the unit of velocity. Furthermore, differentiating velocity (m/s) with respect to time (s) is m/s^2. That equals to the unit of acceleration. The integrals can be considered in the same way. Integrating acceleration (m/s^2) with respect to time (s) is velocity (m/s), and integrating velocity (m/s) with respect to time (s) is displacement (m).

From this, we can consider that the following form is displacement dividing by the unit of time.

$$\frac{df}{dt}$$

So this notation may be easier to figure out. This equation expresses displacement per unit of time.

6.1. Optics

Only those who are interested should read this section. if you read it, pay attenuation to units.

The smallest unit of physical quantity of light is a photon. A collection of photons is called a *radiant energy*, which represent by Q. The unit is J (joule).

The radiant energy per unit of time is called a *radiant flux* Φ.

$$\Phi = \frac{dQ}{dt}$$

The unit is W (watt). It can also represent by J/s or $J \cdot s^{-1}$.

The radiant flux per unit of solid angle is called a *radiant intensity* I.

$$I = \frac{d\Phi}{d\omega}$$

The unit is W/sr (watt/steradian).

Next, the radiant flux per unit of area is called a *irradiance* E.

$$E = \frac{d\Phi}{dA}$$

The unit is W/m^2 (watt/square meter).

The radiant flux per unit of solid angle and per unit of projected area is called a *radiance* L.

$$L = \frac{1}{\cos\theta}\frac{d}{dA}\frac{d\Phi}{d\omega} = \frac{1}{\cos\theta}\frac{d^2\Phi}{dAd\omega}$$

This unit is $W/m^2 sr$.

Integrating the radiance with respect to the unit area is the radiant intensity.

$$I = \int_A L\cos\theta dA = \int_A \frac{1}{\cos\theta}\frac{d}{dA}\frac{d\Phi}{d\omega}\cos\theta dA = \frac{d\Phi}{d\omega}$$

The unit of radiance is $W/m^2 sr$. Since it was integrated with respect to the unit area m^2, the unit is W/sr.

Hemispheric integrating the radiance with respect to the unit solid angle is the irradiance B.

$$B = \int_\Omega L\cos\theta d\omega = \int_\Omega \frac{1}{\cos\theta}\frac{d}{dA}\frac{d\Phi}{d\omega}\cos\theta d\omega = \frac{d\Phi}{dA}$$

The unit of radiance is $W/m^2 sr$, which is integrated by the unit solid angle sr, so it is W/m^2.

As you can see, there are units in physics when dealing with quantities, so it is interesting to see the relationship between calculus and units.

In conclusion

I didn't use the squeeze theorem after all in this document, even though I wrote about it. There are some very useful formulas in calculus that are quick to learn. But I recommend you should derive them. The squeeze theorem is used to differentiate trigonometric functions. I was going to write that too, but I decided not to.
I hope this will be helpful to you.

References

- 尾崎　淳「微分方程式と差分方程式（漸化式）」, http://www.fbs.osaka-u.ac.jp/labs/skondo/ozaki/what%20is%20RD%204%20tips%20for%20simulation.htm