Simultaneous Localization and Mapping (SLAM)

Lecture 01

Introduction

SLAM Objective

- Place a robot in an unknown location in an unknown environment and have the robot incrementally build a map of this environment while simultaneously using this map to compute vehicle location

- SLAM began with seminal paper by R. Smith, M. Self, and P. Cheeseman in 1990

- A solution to SLAM has been seen as the “Holy Grail”
  - Would enable robots to operate in an environment without *a priori* knowledge of obstacle locations

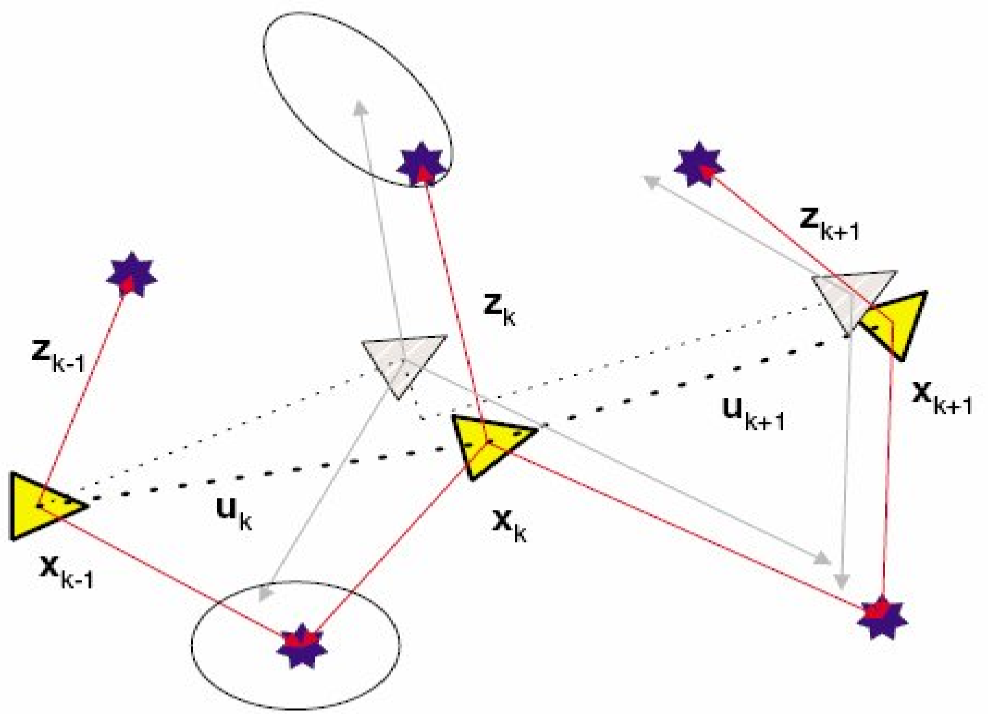
- Research over the last decade has shown that a solution is possible!!

The Localization Problem

Defined

- A map m of landmark locations is known a priori
- Take measurements of landmark location z_k (i.e. distance and bearing)
- Determine vehicle location x_k based on z_k
  - *Need filter if sensor is noisy!*

- x_k: location of vehicle at time k
- u_k: a control vector applied at k-1 to drive the vehicle from x_{k-1} to x_k
- z_k: observation of a landmark taken at time k
- X^k: history of states $\{x_1, x_2, x_3, \ldots, x_k\}$
- U^k: history of control inputs $\{u_1, u_2, u_3, \ldots, u_k\}$
- m: set of all landmarks

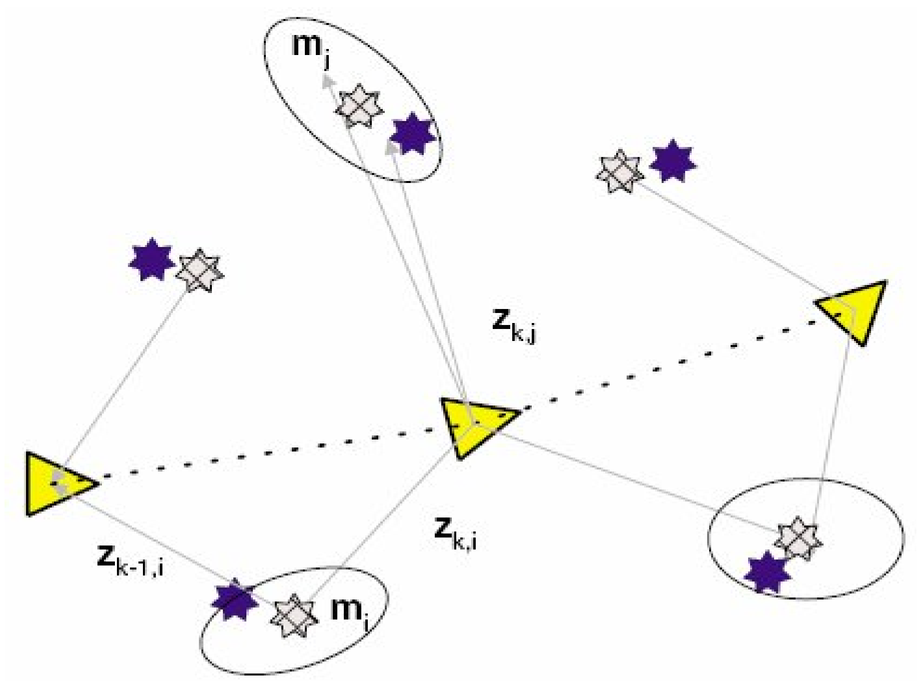


The Mapping Problem

Defined

- The vehicle locations X^k are provided
- Take measurement of landmark location z_k (i.e. distance and bearing)
- Build map m based on on z_k
  - *Need filter if sensor is noisy!*

- X^k: history of states $\{x_1, x_2, x_3, \ldots, x_k\}$
- z_k: observation of a landmark taken at time k
- m_i: true location of the i^{th} landmark
- m: set of all landmarks



Simultaneous Localization and Mapping

Defined

- From knowledge of observations Z^k
  - Determine vehicle locations X^k
  - Build map m of landmark locations

- x_k: location of vehicle at time k
- u_k: a control vector applied at k-1 to drive the vehicle from x_{k-1} to x_k
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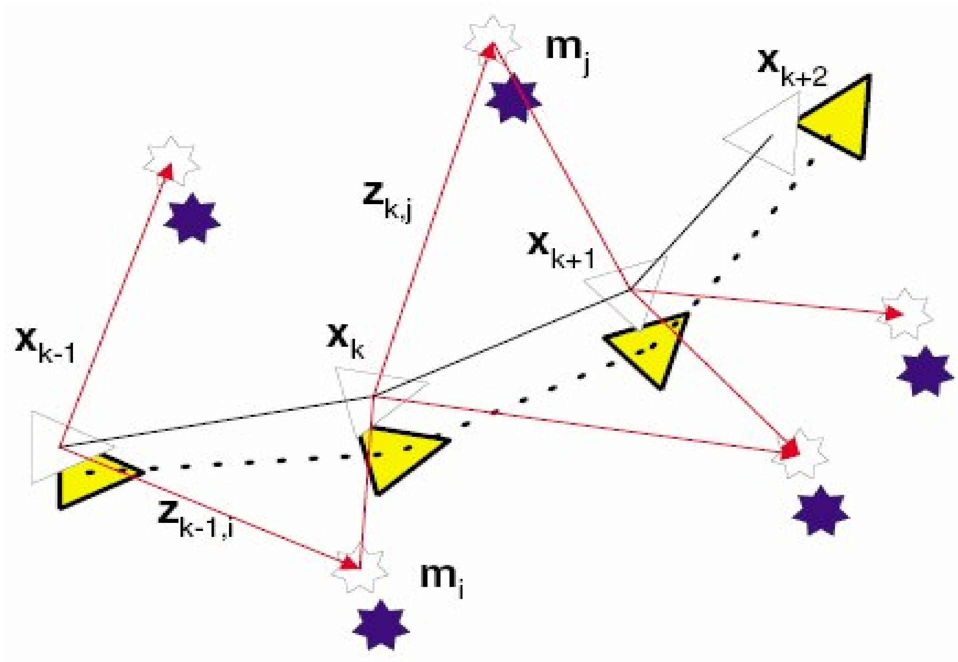
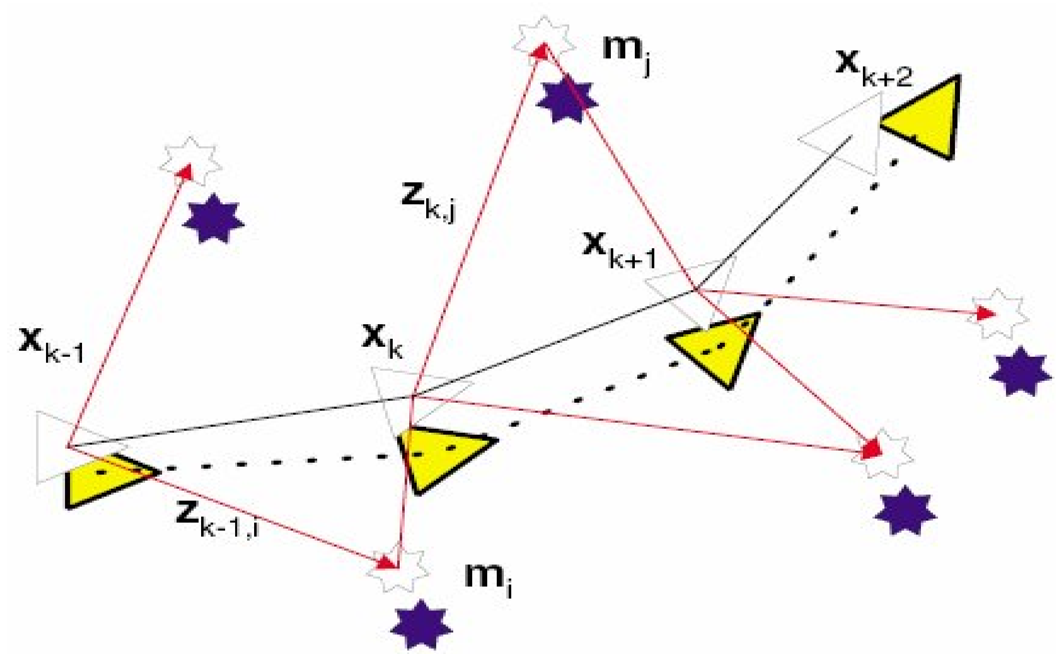


H. Durrant-Whyte, D. Rye, E. Nebot, “Localisation of Automatic Guided Vehicles”, ISRR 1995

Simultaneous Localization and Mapping

Characteristics

- Localization and mapping are coupled problems
  - Two quantities are to be inferred from a single measurement
- A solution can only be obtained if the localization and mapping processes are considered together

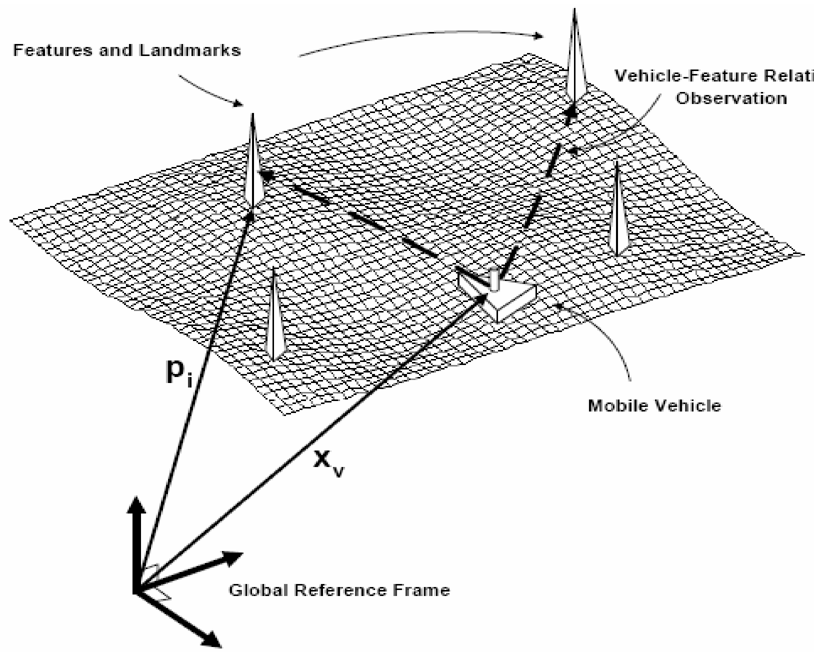


H. Durrant-Whyte, D. Rye, E. Nebot, "Localisation of Automatic Guided Vehicles", Robotics Research: The 7th International Symposium (ISRR 1995)

SLAM Fundamentals

Setting

- A vehicle with a known kinematic model moving through an environment containing a population of landmarks (process model)
- The vehicle is equipped with a sensor that can take measurements of the relative location between any individual landmark and the vehicle itself (observation model)



SLAM Fundamentals

Process Model

- For better understanding, a linear model of the vehicle is assumed
- If the state of the vehicle is given as $x_v(k)$ then the vehicle model is

$$x_v(k+1) = F_v(k)x_v(k) + u_v(k+1) + w_v(k+1)$$

where

  - $F_v(k)$ is the state transition matrix
  - $u_v(k)$ is a vector of control inputs
  - $w_v(k)$ is a vector of uncorrelated process noise errors with zero mean and covariance $Q_v(k)$

- The state transition equation for the i^{th} landmark is

$$p_i(k+1) = p_i(k) = p_i$$

SLAM considers all landmarks stationary!

SLAM Fundamentals

Process Model

- The augmented state vector containing both the state of the vehicle and the state of all landmark locations is

$$x(k) = \left[x_v^T(k) \quad p_1^T \quad \dots \quad p_N^T \right]^T$$

- The state transition model for the complete system is now

$$\begin{bmatrix} x_v(k+1) \\ p_1 \\ \vdots \\ p_N \end{bmatrix} = \begin{bmatrix} F_v(k) & 0 & \dots & 0 \\ 0 & I_{p_1} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & I_{p_N} \end{bmatrix} \begin{bmatrix} x_v(k) \\ p_1 \\ \vdots \\ p_N \end{bmatrix} + \begin{bmatrix} u_v(k+1) \\ 0_{p_1} \\ \vdots \\ 0_{p_N} \end{bmatrix} + \begin{bmatrix} w_v(k+1) \\ 0_{p_1} \\ \vdots \\ 0_{p_N} \end{bmatrix}$$

where

- I_{pi} is the dim(p_i) x dim(p_i) identity matrix
- 0_{pi} is the dim(p_i) null vector

SLAM Fundamentals

Observation Model

- Assuming the observation to be linear, the observation model for the i^{th} landmark is given as

$$z(k) = H_i x(k) + v_i(k)$$

where

  – $v_i(k)$ is a vector of uncorrelated observation errors with zero mean and variance $R_i(k)$

  – H_i is the observation matrix that relates the sensor output $z_i(k)$ to the state vector $x(k)$ when observing the i^{th} landmark and is written as

$$H_i = \left[-H_v, 0 \ldots 0, H_{p_i}, 0 \ldots 0 \right]$$

  – Re-expressing the observation model

$$z(k) = H_{p_i} p - H_v x_v(k) + v_i(k)$$

Estimation Process

Objective

- The state of our discrete-time process x_k needs to be estimated based on our measurement z_k
- This is the exact definition of the Kalman filter!!

Kalman Filter

- Recursively computes estimates of state x(k) which is evolving according to the process and observation models
- The filter proceeds in three stages
  - Prediction
  - Observation
  - Update

Estimation Process

Prediction

- After initializing the filter (i.e. setting values for $\hat{x}(k)$ and P(k)), a prediction is generated for
  - The *a priori* state estimate

$$\hat{x}(k+1|k) = F(k)\hat{x}(k|k) + u(k)$$

  - The *a priori* observation relative to the ith landmark

$$\hat{z}_i(k+1|k) = H_i(k)\hat{x}(k+1|k)$$

  - The *a priori* state covariance (e.g. a measure of how uncertain the states computed by the process model are)

$$P(k+1|k) = F(k)P(k|k)F^T(k) + Q(k)$$

Estimation Process

Observation

- Following the prediction, an observation $z_i(k+1)$ of the i^{th} landmark is made using the observation model
- An innovation and innovation covariance matrix are calculated
  - Innovation is the discrepancy between the actual measurement z_k and the predicted measurement $\hat{z}_k$

$$v_i(k+1) = z_i(k+1) - \hat{z}_i(k+1 \mid k)$$

$$S_i(k+1) = H_i(k)P(k+1 \mid k)H_i^T(k) + R_i(k+1)$$

Estimation Process

Update

- The state estimate and corresponding state estimate covariance are then updated according to

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + W_i(k+1)w_i(k+1)$$

$$P(k+1|k+1) = P(k+1|k) - W_i(k+1)S(k+1)W_i^T(k+1)$$

where the gain matrix $W_i(k+1)$ is given by

$$W_i(k+1) = P(k+1|k)H_i^T(k)S_i^{-1}(k+1)$$

Kalman Filter

A Closer Look…

Kalman Filter

Background

- Developed by Rudolph E. Kalman in 1960
- A set of mathematical equations that provides an efficient computational (recursive) means to estimate the state of a process
- It supports estimations of
  - Past states
  - Present states
  - Future states

  and can do so when the nature of the modeled system is unknown!

Discrete Kalman Filter

Process Model

- Assumes true state at time k evolves from state (k-1) according to

$$x(k) = Fx(k-1) + Gu(k-1) + w(k)$$

where

- F is the state transition model (A matrix)
- G is the control input matrix (B matrix)
- w(k) is the process noise which is assumed to be white and have a normal probability distribution

$$p(w) \sim N(0, Q)$$

covariance

Discrete Kalman Filter

Observation Model

- At time k, a measurement z(k) of the true state x(k) is made according to

$$z(k) = Hx(k) + v(k)$$

where

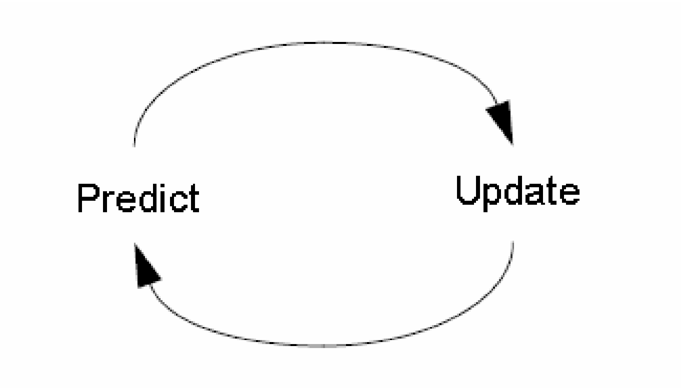
- H is the observation matrix and relates the measurement z(k) to the state vector x(k)
- v(k) is the observation noise which is assumed to be white and have a normal probability distribution

$$p(v) \sim N(0, \underbrace{R}_{\text{covariance}})$$

Discrete Kalman Filter

Algorithm

- It’s recursive!
  - Only the estimated state from the previous time step and the current measurement are needed to compute the estimate for the current state
- The state of the filter is represented by two variables
  - x(k): estimate of the state at time k
  - P(k|k): error covariance matrix (a measure of the estimated accuracy of the state estimate)
- The filter has two distinct stages
  - Predict (and observe)
  - Update



Discrete Kalman Filter (Notation 1)

Prediction

- Predicted state $\boxed{\hat{x}(k|k-1)} = F(k)\hat{x}(k-1|k-1) + B(k)u(k-1)$
- Predicted covariance $\boxed{P(k|k-1)} = F(k)P(k-1|k-1)F(k)^T + Q(k)$

Observation

- Innovation $\tilde{y}(k) = z(k) - H(k)\hat{x}(k|k-1)$
- Innovation covariance $S(k) = H(k)P(k|k-1)H(k)^T + R(k)$

Update

Not the same variable!!

- Optimal Kalman gain $K(k) = P(k|k-1)H(k)^T S(k)^{-1}$
- Updated state $\boxed{\hat{x}(k|k)} = \hat{x}(k|k-1) + K(k)\tilde{y}(k)$
- Updated covariance $\boxed{P(k|k)} = (I - K(k)H(k))P(k|k-1)$

Not the same variable!!

Discrete Kalman Filter (Notation 2)

Prediction

- Predicted state $\hat{x}(k)^- = F(k)\hat{x}(k-1) + Bu(k-1)$
- Predicted estimate covariance $P(k)^- = FP(k-1)F^T + Q$

Observation

- Innovation $\tilde{y}(k) = z(k) - H\hat{x}(k)^-$
- Innovation covariance $S(k) = HP(k)^- H^T + R$

Update

- Optimal Kalman gain $K(k) = P(k)^- HS(k)^{-1}$
- Updated state estimate $\hat{x}(k) = \hat{x}(k)^- + K(k)\tilde{y}(k)$
- Updated estimate covariance $P(k) = (I - K(k)H)P(k)^-$

Discrete Kalman Filter

Prediction

(1) Project the state ahead

$$\hat{x}(k)^{-} = F(k)\hat{x}(k-1) + Bu(k-1)$$

(2) Project the error covariance ahead

$$P(k)^{-} = FP(k-1)F^{T} + Q$$

Observation and Update

(1) Compute the Kalman gain

$$K(k) = P(k)^{-}H^{T}(HP(k)^{-}H^{T} + R)^{-1}$$

(2) Update estimate with measurement z(k)

$$\hat{x}(k) = \hat{x}(k)^{-} + K(k)[z(k) - H\hat{x}(k)^{-}]$$

(3) Update error covariance

$$P(k) = (I - K(k)H)P(k)^{-}$$

Initial estimates for

$\hat{x}(k-1)$ & $P(k-1)$

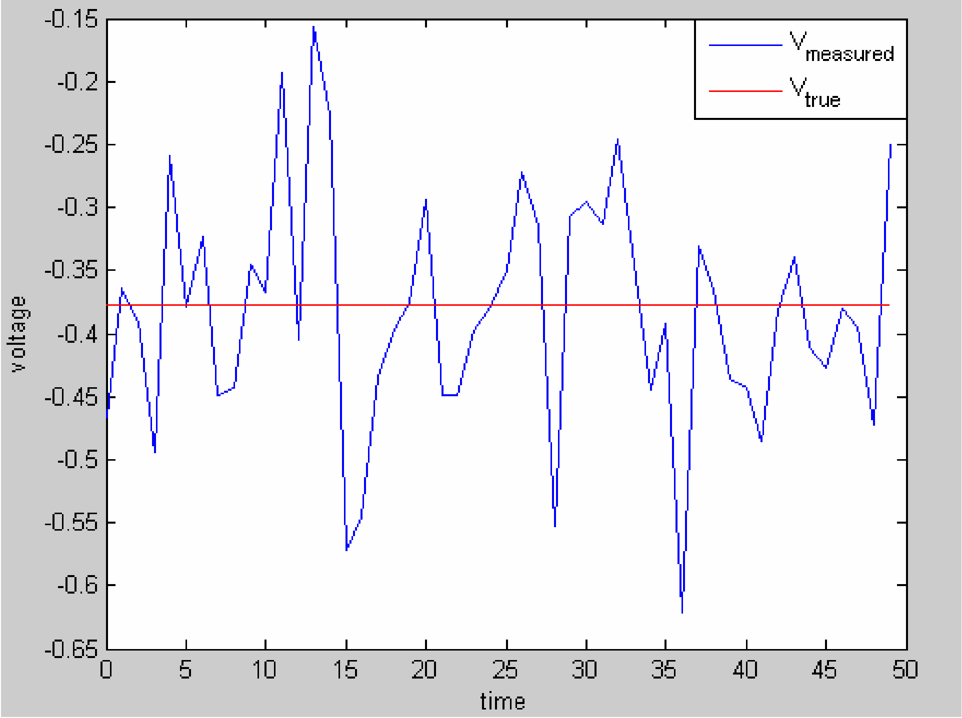
A Kalman Filter in Action

An Example…

Kalman Filter Example

Process Model

- Estimate a scalar random constant (e.g. voltage)
  - Measurements are corrupted by 0.1 volt RMS white noise



Kalman Filter Example

Process Model

- Governed by the linear difference equation

$$x(k) = Fx(k-1) + Gu(k-1) + w(k)$$

$$x(k) = x(k-1) + w(k)$$

State doesn’t change (F=0)
No control input (u=0)

with a measurement

$$z(k) = Hx(k) + v(k)$$

$$z(k) = x(k) + v(k)$$

Measurement is of state directly (H=1)

Kalman Filter Example

Output