Istanbul Technical University ★ Electrical and Electronics Engineering

SYMMETRIES OF DYNAMICAL SYSTEM NETWORKS

B.Sc. Thesis by
Mehmet Ali Anıl

Department: Electronics and Communication Engineering
Programme: Electronics Engineering

Supervisor : Assoc. Prof. Dr. Neslihan Serap Şengör

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Summary

This work investigates the link between symmetry of a dynamical system network, and the solutions regarding this dynamical system. A network was selected to demonstrate the impact of symmetry in the connections of the system, by selecting Fitzhugh-Nagumo cells as nodes of dynamical systems. These cells are connected to each other with that certain symmetry and this symmetry is investigated. A numerical simulation was presented at the end of this work to show that constraints on the solutions.

Nomenclature

(123)	A cycle that maps 1 to 2, 2 to 3 and 3 back to 1
$\cdot$	Law of composition for a group.
$\dot{x}$	time derivative of the variable x
$\equiv$	Equivalence
$\forall$	For all conditions or elements that follow
Γ	The symmetry group
$\langle a, b, c \rangle$	Space spanned by bases a, b and c
$\langle g \rangle$	Group generated by element g
$\mathbb{C}$	Complex space of all possible complex numbers
$\mathbb{Q}$	Set of all rational numbers
$\mathbb{R}$	Set of all real numbers.
$\mathbf{D}_n$	dihedral group of order n.
$\mathbf{S}^n$	Lie group of a n dimensional hypersphere
$\mathbf{S}_n$	Permutation group of all permutations for an n length sequence
$\mathbf{Z_n}$	Cyclic group of order n
$\mathcal{O}(G)$	Order of a group G
ϕ	Homomorphism
$\rightarrow$	Implies that
Fix(A)	Fixed point subspace of the group A
Im(ϕ)	Image of a group G
Ker(ϕ)	Kernel of a group homomorphism ϕ
$\times$	Cartesian product for groups and spaces
$a \cdot B$	Left coset of a and group B
$a \mapsto b$	Indication of an element a is mapped to b.

$A \to B$ A mapping from spaces or sets A to B.

$A \triangleleft B$ A is a normal subgroup of B

A/B A quotient B

e Identity element for a specified group.

$G \cong G'$ G is isomorphic to G'

$g_i \in G$ g_i is an element of group or set G

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Chapter 1

Introduction

Symmetry is mostly known to be an observational notion in daily life. People by far more observe it or use it in an aesthetic manner rather than put it into use. Thats simply because the cause and effect relationship between symmetry and engineering is still vague and the tools of trade when an engineer advances in her design is inadequate fro the symmetry perspective.

Despite of its lack in the world of engineering, the world of physics is using symmetries for more than a decade, to understand the laws of nature. Assuming that there **are** laws of nature, in order to investigate them, symmetry of a phenomenon in nature is one of the first to check. Though the world of science tries to discover the inner workings of nature, by making measurements and by linking phenomena, they have little idea what the **nature** of these fundamental laws that one unfolds those little by little. At first, metaphysical nature of the law of the nature was popular amongst the philosophers, but it is hardly mentioned with notable credit since the mid 19th century. [1] The new way to look at the fundamental laws changed in form when people realized that the symmetry language, with the help of the group theory, was very helpful in explaining the then-unexplored areas of particle physics. Symmetry groups of Lie type were so successful in interpreting the results of the experiments, symmetry in the space-time as we know it started to be taken for granted. For every different symmetry group that was proposed, a set of particles were generated on paper, then spins, colors and other fundamental properties were investigated. As the chart of subatomic particles populated in the mid 1900's, the symmetry groups started to dictate how many of them must there be out there, or which families these particles must be present in. Right now, the language of sub-atomic physics is dominated with the language of symmetry, since its ability to link the observation to the reality is still unsurpassed. [2]

There are other branches of science that use symmetry arguments in order to explain other macroscopic phenomena. One of them is crystallography, which is in close contact to electrical engineering via the semiconductor. The crystals in nature, their atomic arrangements are ideally applicable to symmetry groups acting on the three dimensional space. They exhibit spatial symmetries, some of them are rotational symmetries, and many of them are translational symmetries. When a perfect crystal is translated in space a length of its lattice constant, will remain unchanged. Assuming that it is large enough, this means that the problem that is to be solved is exactly the same of the previous one (the problem in the untranslated coordinates one had). Though other symmetries might have caused a different constraints, translational symmetry dictates the crystal to have periodic solutions to the Schrödinger wave equation. This constraints affects the energies of the fermions (electrons in this case). Introduction of the symmetry to the problem affects the energy distribution of one electron under such circumstances,in such a way that some bands of energy turn out to be forbidden for an electron to propagate with. This is not the case for a free-electron, since its kinetic energy is a quadratic function of its momentum, and an electron can have any momentum within a broad range. The symmetry in Silicon for example, is the reason behind its well celebrated bandgap. This bandgap gives the engineers to have a control over the behavior of electrons, and this control is achieved by breaking the symmetry of the crystal for a desired amount. [3, 4]

Though the language of symmetry is seldom spoken behind the walls of the room of crystallography, its applicability does not degrade over time. The set of tools acquired when one understands group theory and uses it in order to understand symmetries of systems of real life, is totally valid for any symmetry that any type of system can have. This work will investigate symmetries of dynamical system networks, networks coupled to themselves in a symmetric manner, and will try to shed light on how symmetry in capable of transforming an arbitrary problem to a problem with constraints.

If engineering is the act of using mathematical and physical reality in order to create systems that have exhibit some sort of control, an approach to the problem with the knowledge of symmetries and their implications on such systems, one can use this set of tools to mold the system into ones liking. For example, as the reader may see in the following chapters, symmetries can be used in order to dictate limits on phase portraits of nonlinear dynamical systems of high dimensions. This may have applications in nonlinear control theory, or theory of oscillators, bot more importantly than its areas of direct applicability, is the vision it introduces into observations an engineer would make.

The first chapter after this introduction, introduces group theory, to an extent that is required to follow and understand the flow on this work. The second chapter creates the link between the symmetry groups and the object of interest (which may be a different one depending on what is investigated) , namely, an arbitrary dynamical system network. Theorems are shown and explained in this chapter, that will be useful in other sections. In the fourth chapter, an example is given for a network of dynamical systems, and its properties is investigated from the perspective that is delineated. Also, the results are simulated with a dynamical system simulator named XPPAUT, and the results obtained from the symmetry perspective is backed up.

Chapter 2

Group Theory

2.1 Group

In this section, the mathematical background that will give us the ability to work on the symmetries of a system will be presented. In order to work on the symmetries of a system qualitatively, one often uses the fact the symmetry operations form a **group**, and that they show characteristics that can be explained by **group theory**. The properties of symmetry groups will be covered afterwards, in this section fundamentals that will be used in order to have a thorough understanding of symmetry groups will be given. This chapter only includes the background that is needed in the subsequent chapters. For a more rigorous and complete approach, one can consult **(author?)** [5, 6].

Definition 2.1.1 (Group)
A **group** is a set of elements, **G** representing a group, related with a law of composition,

$$\mathbf{G} \times \mathbf{G} \rightarrow \mathbf{G}$$

satisfying the following requirements:

(Let us denote our group with G, and its law of composition with $\cdot$.)

Closure: For all g_i and g_j that satisfies $g_i, g_j \in \mathbf{G}$

$$g_i \cdot g_j, g_j \cdot g_i \in \mathbf{G}$$

Associativity: For all g_i, g_j and g_k that satisfies $g_i, g_j, g_k \in G$ such that:

$$g_i \cdot (g_j \cdot g_k) = (g_i \cdot g_j) \cdot g_k$$

Existence of Identity: For all $g_i \in G$, there is one and only one $e \in G$ such that:

$$g_i \cdot e = e \cdot g_i = g_i \ \ , \ \forall g_i \in \mathbf{G}$$

Existence of Inverses In a group with identity element, for each $g_i \in \mathbf{G}$, there is an inverse element $g_i^{-1} \in \mathbf{G}$ such that:

$$g_i^{-1} \cdot g_i = g_i \cdot g_i^{-1} = e$$

These requirements probably are familiar to those who are dealing with vector spaces. It may be favorable here to note that group encompasses vector spaces in definition, it is an object that has less restrictions than requirements for being a vector space, thus a vector space is also a group. In abstract algebra, a group is an object that is derived from **monoids** , in which inverse element is not mandatory.
The group concept is best comprehended by examining examples of it. First, we will demonstrate some trivial examples.

Example 2.1.1
Integers, under the law of composition of addition form a group.

$$\mathbf{G} = (\{\dots, -2, -1, 0, 1, 2, \dots\}, +)$$

The of G have associativity and closure properties under summation.
For this group, the identity element is 0, whereas the inverse element of each element g_i is the negative of it, $g_i^{-1} = -g_i$

Example 2.1.2
The set of all rational numbers *do not* form a group under multiplication.
This arises from the fact that though the set is closed and associative under multiplication, and has the identity element 1, the inverse of the element zero in that set is undefined. Since every single element is required to have an inverse, it fails to satisfy all group criteria. But the set of elements $\mathbb{Q} - \{0\}$ does form a group under multiplication,

$$\mathbf{G} = (\mathbb{Q} - \{0\}, \cdot)$$

since all the requirements are met with this modification.

Example 2.1.3
Real numbers form a group under summation.

$$\mathbf{G} = (\mathbb{R}, +)$$

It can be shown that the set of real numbers is closed and associative under summation. The identity element is 0 in this case, whereas for every real number a an inverse can be found: $a^{-1} = -a$

Though examples that are given are helpful in getting used to groups, they are not the most instructive examples for our intentions. So, we will try to focus on other possible sets that will satisfy the group criteria.

Operations, or transformations can also form a group, which is actually a more suitable interpretation of a group in our case, than the examples given above. A group of operators is simply an ensemble of operators that are generally related to each other with the law of composition of consecutive application. † Here, the elements of the group are operators themselves, and certainly not the objects that they act on.

Example 2.1.4
Rotation Group of $\mathbf{Z}_4$: ‡ Let R_ϕ be the rotation operation that acts upon vectors in $\mathbb{R}^2$, rotating them ϕ degrees clockwise. Let G be the group that has 0°, 90°, 180°, 270° rotation operators:

$$G_{90^\circ} = \{R_{0^\circ}, R_{90^\circ}, R_{180^\circ}, R_{270^\circ}\}$$

As mentioned, the law of composition will be consecutive rotations, in which we will omit the symbol unless needed, for example when more than one laws of composition is present. So $R_{90^\circ}R_{180^\circ}$ is a 180°rotation followed by 90°rotation, resulting in a 270°rotation.

As it is evident, any combination of consecutive rotations applied on a vector space is also achievable with one of the rotation operator in the group. Since the results are indistinguishable, these two operations should be the same, so closure holds with this set. As an example,

$$R_{90^\circ}R_{180^\circ} = R_{270^\circ} \in G_{90^\circ}$$

The order of composition is insignificant on the final operation. The reader can check that associativity also holds with this set and law of composition. The identity element

†Other laws can also be introduced, for example commutation can be the law of composition, as it is the case with Pauli matrices.

‡Group theory books, like **(author?)** [5] may call the rotation (cyclic) group as $\mathbf{C}_i$, we are going to use $\mathbf{Z}_i$ for a cyclic group of order i, as our primary source, **(author?)** [7] follows that notation.

is R_{0° which does not rotate the vector that it acts on. The inverse elements are self evident and are as follows:

$$R_{90^\circ}^{-1} = R_{270^\circ},\ R_{180^\circ}^{-1} = R_{180^\circ},\ R_{270^\circ}^{-1} = R_{90^\circ},\ R_{0^\circ}^{-1} = R_{0^\circ}$$

So, we have a system that group theory will be applicable upon seamlessly, since it satisfies all criteria.

Example 2.1.5
A different group could be the following matrices:

$$\begin{gathered}\left\{\begin{pmatrix} cos(\frac{\pi}{2}) & sin(\frac{\pi}{2}) \\ -sin(\frac{\pi}{2}) & cos(\frac{\pi}{2}) \end{pmatrix}, \begin{pmatrix} cos(\pi) & sin(\pi) \\ -sin(\pi) & cos(\pi) \end{pmatrix},\right. \\ \left.\begin{pmatrix} cos(\frac{3\pi}{2}) & sin(\frac{3\pi}{2}) \\ -sin(\frac{3\pi}{2}) & cos(\frac{3\pi}{2}) \end{pmatrix}, \begin{pmatrix} cos(0) & sin(0) \\ -sin(0) & cos(0) \end{pmatrix}\right\}\end{gathered} \tag{2.1}$$

which is:

$$\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \tag{2.2}$$

Which are, 90 ° rotation matrices in $\mathbb{R}^2$. These matrices, are closed under matrix multiplication. They satisfy all other criteria, thus they form a group.

Example 2.1.6
Another analogous group could be a set of irrational numbers, which is related by the law of composition of multiplication:

$$G_i = \{1, i, -1, -i\} = \{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3\pi}{2}}\}$$

in which we are not acting on vectors in $\mathbb{R}^2$ anymore but instead on complex numbers $\gamma = \alpha + \beta i \in \mathbb{C}$. Also, it is notable that our law of composition changed from consecutive rotation and matrix product to multiplication defined in the set of complex numbers.

The resemblance of the three interpretations given in Examples 2.1.4 , 2.1.5 and 2.1.6 is evident, and actually they are also related to each other with a mathematical relationship. These three different groups are denoted as the *realizations of an* **abstract group**, which is labeled as $\mathbf{Z}_4$ in which the notation will be introduced in the following sections.

One should also note that examples that can be given for a group, is also not limited to operators or transformations, but for the sake of the argument, we will focus on groups that are in this nature. A more rigorous treatment for the fundamentals and a bigger variety of examples, one can consult algebra textbooks such as **(author?)** [8]

2.2 Finite Abstract Groups and Realizations

An **abstract group** is a group that is characterized with its abstract properties. As formerly mentioned, an abstract group is a platonic entity that is without any direct physical resemblance, is a group that one can relate to any group that has the same structure. These groups are often noted with an uppercase letter and a subscript, such as $\mathbf{D_4}$ or $\mathbf{C_3}$. The letter is meant to distinguish the structure, whereas the subscript is an integer characteristic to the group, able to differentiate groups of the same structure. A **realization** of an abstract group is a group that have the same structure of it. [5]In this section, we will introduce some well known finite abstract groups with examples for different representations. In our case the symmetry is a discrete symmetry, which implies that our number of elements in our symmetry groups is finite (our groups have a finite order). There is a set of other abstract groups that came out to be very useful while interpreting physical reality that are called Lie Groups, named after Norwegian mathematician Sophus Lie, that represent continuous symmetries.

Definition 2.2.1 (Order of a group)
The **order** of a group is the number of elements in the group. If a group is composed of infinite elements, it is considered to be of infinite order. Order of a group will be denoted with $\mathcal{O}(G)$. [5]

2.2.1 Permutation Groups

A permutation is a rearrangement of a sequence of elements. When a sequence of elements is permuted, the sequence is mapped onto another one that the position of every element redetermined. [5] For example, $1234 \mapsto 4213$ is a permutation, which is customary to show as $\begin{pmatrix} 1234 \\ 4213 \end{pmatrix}$. The set of possible permutations forms a permutation group.

Example 2.2.1
An instructive example is the permutation group of order 6, noted as $\mathbf{S}_3$ since it is formed by permutations of a sequence of three elements. The elements of this group

are:

$$\mathbf{S}_3 = \left\{ \begin{pmatrix}123\\123\end{pmatrix}, \begin{pmatrix}123\\312\end{pmatrix}, \begin{pmatrix}123\\231\end{pmatrix}, \begin{pmatrix}123\\132\end{pmatrix}, \begin{pmatrix}123\\213\end{pmatrix}, \begin{pmatrix}123\\321\end{pmatrix} \right\} \tag{2.3}$$

Two consecutive permutations will always result in another one, such as,

$$\begin{pmatrix}123\\132\end{pmatrix} \begin{pmatrix}123\\321\end{pmatrix} = \begin{pmatrix}123\\312\end{pmatrix} \tag{2.4}$$

Readers may check that $\mathbf{S}_3$ has an identity element $\begin{pmatrix}123\\123\end{pmatrix}$, and every element has an inverse.

Other groups either have too many elements to list, or they are trivially small, since the number of elements for permutations of an n-element sequence is $n!$.

Permutations can also be expressed by a series of other operations, some well known ones are **transpositions** and **cycles** . A switch of two elements is called a transposition. A transposition is denoted with two of the elements within parentheses:

$$(12) = \begin{pmatrix}12345678\\21345678\end{pmatrix} \tag{2.5}$$

as it is the case here, a transposition for a sequence of eight elements.

A cycle is a permutation that a group of elements switch places in a chainlike fashion. For example, a cycle of (1234) within a sequence of six elements would be:

$$(1234) = \begin{pmatrix}123456\\412356\end{pmatrix} \tag{2.6}$$

where 1 moves into 2's position, 2 moves into 3's position, 3 moves into 4's position, and 4 moves into 1's position in the sequence.

Note that cycles are elements of an encompassing permutation group, whereas transpositions are cycles of two elements. Any permutation can be expressed by a series of cycles.

$$\begin{pmatrix}123456789\\412356978\end{pmatrix} = (789)(1234) \tag{2.7}$$

this is one of many possible ways to decompose a permutation.

2.2.2 Cyclic Groups

A cyclic group is a group that all elements of it can be generated from one "generator element". A useful analogy is rotation, in which consecutive rotations generate other rotations in the group. A formal definition may be given as: [5]

Definition 2.2.2 (Cyclic Group)
A group is **cyclic** if:

$$\text{There exists a } g \in \mathbf{G} : G = \langle g \rangle \tag{2.8}$$

Order 1

There is only one abstract group that has an order one, which is denoted as $\mathbf{Z}_1$. It has only one element, and evidently it is the identity element, e, which is the inverse of itself.

Order 2

Being another trivial abstract group, the group $\mathbf{Z}_2$ is the only group with order 2. The group consists of one element that is its own inverse, since it must satisfy closure, thus

$$\mathbf{Z}_2 = \{a, e\}, \quad a \cdot a = e \tag{2.9}$$

A realization of this group is:

$$\mathbf{G} = (\{1, -1\}, +) \tag{2.10}$$

Another notable realization is the set of transformations

$$\mathbf{G} = (\{\kappa, e\}) \tag{2.11}$$

In which κ is the reflection operator, and e is the identity operation, which maps what it acts upon to itself.

Order 3

The cyclic group of order 3 is denoted as $\mathbf{Z}_3$. It is instructive to show the groups structure with tables from now on, since once the number of elements get larger, it gets harder to show the structure with multiplications. Also it provides a useful way to visualize subgroups. Here, each element is represented with a unique letter, which for every possible multiplication of two elements, the resultant group element is the element in the intersection of the corresponding rows and columns.

$\mathbf{Z}_3$

e	a	b
a	b	e
b	e	a

$\mathbf{Z}_3$ can be represented with a group of coordinate transformations for a two dimensional vector space of 120 degree, 240 degree and 360 degree rotations in one direction.

Higher Orders

The properties of this group can be guessed, since it requires merely a generalization of the notion of a cyclic group.

$\mathbf{Z}_4$

e	a	b	c
a	b	c	e
b	c	e	a
c	e	a	b

$\mathbf{Z}_5$

e	a	b	c	d
a	b	c	d	e
b	c	d	e	a
c	d	e	a	b
d	e	a	b	c

$\mathbf{Z}_6$

e	a	b	c	d	f
a	b	c	d	f	e
b	c	d	f	e	a
c	d	f	e	a	b
d	f	e	a	b	c
f	e	a	b	c	d

2.2.3 Dihedral Groups

Dihedral groups are the abstract group of rotations and reflections that preserve a regular m-gon. [5]

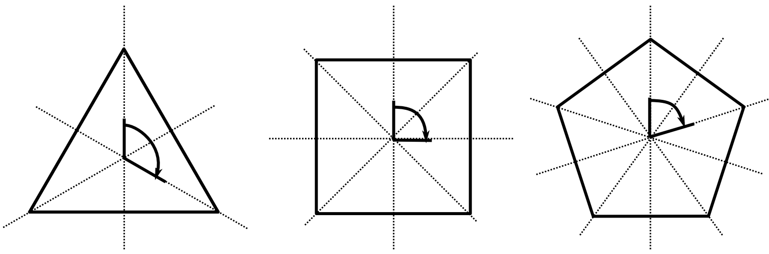

Figure 2.1: Geometrical representation of the finite dihedral groups, the dashed lines represent the reflection symmetries, and curved arrows represent the generator of the rotational symmetries.

Order 4

The dihedral group of order 4 is denoted as $\mathbf{D}_2$, and it is the group of reflections and rotations that preserve a line. (Maybe a rectangle is more pleasing visually.) The group elements act on a geometric object by rotating it by 180 °, taking the reflection of the object on a specified axis, rotating and reflecting consecutively, or preserving the orientation as it is. On a group table we can show it as:

D$_2$

e	a	b	c
a	e	c	a
b	c	e	a
c	b	a	e

Here, every single operation is the inverse of itself. Since the number of elements is 4, $\mathcal{O}(\mathbf{D}_2) = 4$. It should be noted that this group is not Abelian, since $a \cdot c \neq c \cdot a$

Order 6

The dihedral group of order 6 is **D**$_3$, and it consists of all rotations and reflections that preserve an equilateral triangle, as denoted in Figure 2.1. There are three reflection operations, one for each axis and three rotations, 0°, 120°, 240 ° respectively.

D$_3$

e	a	b	c	d	f
a	b	e	f	c	d
b	e	a	d	f	c
c	d	f	e	a	b
d	f	c	b	e	a
f	c	d	a	b	e

Dihedral groups of higher order may be constructed by the regular n-gon analogy that is presented above.

2.3 Isomorphism

Let's consider a one-to-one mapping ϕ from a group G to a group G'. These two groups are isomorphic to each other under this mapping if the *structure of the group is preserved*. In other words, under this map from G to G', every equality such as $a \cdot b = c$ preserves its structure as $\phi(a) \cdot \phi(b) = \phi(c)$. A more formal definition may be given as: [5, 6]

Definition 2.3.1 (Isomorphism)
Two groups G and G' are **isomorphic** if there is a bijection (a one-to-one mapping)[6]

$$\phi : G \longleftrightarrow G'$$

that satisfies

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b), \quad \forall a, b \in G.$$

The bijection ϕ itself is called the **isomorphism** between G and G'. The isomorphism of the groups is shown as $G \cong G'$.

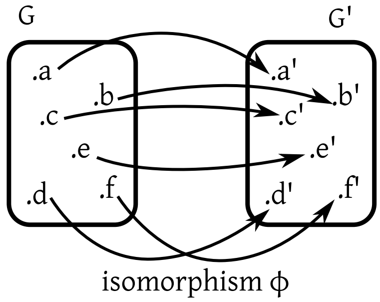

Figure 2.2: Schematical representation of isomorphism

2.4 Homomorphism

Example 2.4.1
$\mathbf{S}_3$ is the permutation group of sequences with three elements. We have shown in Example 2.2.1 that $\mathbf{S}_3$ can be shown as:

$$\begin{array}{cccccccc} \mathbf{S}_3 = \{ & \begin{pmatrix}123\\123\end{pmatrix}, & \begin{pmatrix}123\\312\end{pmatrix}, & \begin{pmatrix}123\\231\end{pmatrix}, & \begin{pmatrix}123\\132\end{pmatrix}, & \begin{pmatrix}123\\213\end{pmatrix}, & \begin{pmatrix}123\\321\end{pmatrix} & \} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbf{D}_3 = \{ & e & a & b & c & d & f & \} \end{array} \tag{2.12}$$

Homomorphism is a more general case of isomorphism, where the requirement of being a bijection is removed. The mapping that we will call as a homomorphism may be a many-to-one mapping.[6]

Definition 2.4.1 (Homomorphism)
Let G and G' be groups, and a and b be elements of two different subgroups of G; A and B respectively. A mapping $\phi : G \mapsto G'$ is a **homomorphism** if, [5, 6]

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b), \quad \forall a \in A \text{ and } b \in B$$

Some useful tools and definitions that we are going to use are derived from homomorphism, thus it is important to point this function concerning groups. The fact that ϕ may not be a bijection does not alter some properties attributed to isomorphisms, those rest only on the fact that the mapping $\phi(\cdot)$ is structure preserving are still valid.

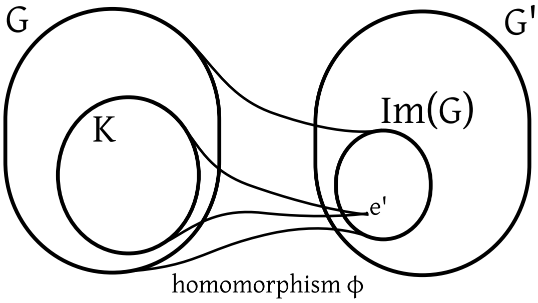

Figure 2.3: Schematic representation of homomorphism.

Definition 2.4.2 (Kernel of a homomorphism)
The **kernel** of a homomorphism ϕ, denoted by Ker(ϕ) is the set of the elements of G that are mapped to the identity element of G'. [5]

$$\text{Ker}(\phi) = \{g \in G \mid \phi(g) = e' \}$$

Definition 2.4.3 (Image of a homomorphism)
The **image** of a homomorphism ϕ, denoted by Im(ϕ) is the subgroup of G' onto which ϕ maps the whole group G. [5]

$$\text{Im}(\phi) = \{g' \in G' \mid \phi(g) = g', \quad \forall g \in G\}$$

Definition 2.4.4 (Conjugation)
Conjugation is the operation $C_g : a \mapsto g^{-1} \cdot a \cdot g$, g and a being in G.
Two elements $a, b \in G$ are called **conjugate** if $g \cdot a \cdot g^{-1} = b$. Conjugacy between two elements is denoted as $a \equiv b$. [5]

There are some properties that may deserve some attention. First of all, every element $g \in G$ is *conjugate with itself*, since $e^{-1} \cdot a \cdot e = e \cdot a \cdot e = a$. Conjugacy is *symmetric*, since for every element u that satisfies $u^{-1} \cdot a \cdot u = b$, one could always do as following:

$$u^{-1} \cdot a \cdot u = b \tag{2.13}$$
$$u \cdot u^{-1} \cdot a \cdot u \cdot u,^{-1} = u \cdot b \cdot u^{-1} \tag{2.14}$$
$$a = u \cdot b \cdot u^{-1} \tag{2.15}$$
$$v^{-1} \cdot b \cdot v = a \quad , v = u^{-1} \in \mathbf{G} \tag{2.16}$$

thus prove that (since for every element u there must exist an inverse, v), one can always find a group element that make b conjugate with a as a is conjugate with b. In an Abelian (commuting) group, every element is *only conjugate with itself*. An example might solidify the concept.

Example 2.4.2

One of the few groups that we can populate all conjugacies without falling into a trivial situation (like the cases when the group is Abelian) is $\mathbf{D}_3$, dihedral group of order 6. The group table was presented in Section 2.2.3. The following conjugations can be taken: [5]

$$\begin{aligned} b^{-1}ab &= b^{-1}e = a \\ c^{-1}ac &= c^{-1}f = b \\ d^{-1}ad &= d^{-1}c = dc = b \\ f^{-1}af &= f^{-1}d = fd = b \end{aligned} \tag{2.17}$$

$$\begin{aligned} a^{-1}ba &= a^{-1}e = b \\ c^{-1}bc &= c^{-1}d = a \\ d^{-1}bd &= d^{-1}f = df = a \\ f^{-1}bf &= f^{-1}c = fc = a \end{aligned} \tag{2.18}$$

$$\begin{aligned} a^{-1}ca &= bca = bf = c \\ b^{-1}cb &= adb = ac = f \\ d^{-1}cd &= cdc = cb = f \\ f^{-1}cf &= fdf = fa = c \end{aligned} \tag{2.19}$$

$$\begin{aligned} a^{-1}da &= bda = bf = c \\ b^{-1}db &= adb = ac = f \\ c^{-1}dc &= cdc = cb = f \\ f^{-1}df &= fdf = fa = c \end{aligned} \tag{2.20}$$

$$\begin{aligned} a^{-1}fa &= bfa = bc = d \\ b^{-1}fb &= afb = ad = c \\ c^{-1}fd &= cfc = ca = d \\ d^{-1}fc &= dfd = fb = d \end{aligned} \tag{2.21}$$

As it can be seen, different subsets of mutual conjugacy has formed. The identity element, e is conjugate with itself, a and b are conjugate with themselves and each other, c,d and f are conjugate within themselves. Since these subsets are all closed under conjugation, they form subgroups of $\mathbf{D}_3$. These groups are called **conjugacy classes**.

$$\{e\} \;\; \{a,b\} \;\; \{c,d,f\}$$

It is instructive to note that the symmetries of the equilateral triangle are divided into conjugacy classes as rotations, reflections and the identity element.

Definition 2.4.5 (Normal Subgroup)

K is a **normal subgroup** of H if and only if K is a subgroup of G such that elements of it remain in K under conjugation. [6] Let $h \in H$ and $k \in K \subset H$. Then K is a normal subgroup of H if:

$$h \cdot k \cdot h^{-1} \in K \qquad \forall k, h \tag{2.22}$$

(The law of composition is the same for both groups, and is shown with $\cdot$.) This relation is denoted as:

$$K \lhd H \tag{2.23}$$

Definition 2.4.6 (Quotient Group)
Let H and K be two different groups. The quotient group denoted as H/K is the group of all left cosets of K in H: [5, 6]

$$H/K = h \cdot K : \forall h \in H \tag{2.24}$$

Let us demonstrate it with an example: Let us take

$$H = \{h_1, h_2, h_3, h_4, h_5, h_6\} \tag{2.25}$$

$$K = \{k_1, k_2, k_3\} \tag{2.26}$$

Then H/K can be shown as:

$$\begin{gathered} H/K = \{h_1 \cdot \{k_1, k_2, k_3\}, h_2 \cdot \{k_1, k_2, k_3\}, h_3 \cdot \{k_1, k_2, k_3\}, \\ h_4 \cdot \{k_1, k_2, k_3\}, h_5 \cdot \{k_1, k_2, k_3\}, h_6 \cdot \{k_1, k_2, k_3\}\} \end{gathered} \tag{2.27}$$

Theorem 2.4.1
First Isomorphism Theorem [5, 6] Let G and G' be groups, and $\phi : G \mapsto G'$ be a homomorphism. Then,
1.The kernel of ϕ , Ker(ϕ), is a normal subgroup of G.

2.The image of ϕ, Im(ϕ) is a subgroup of G'

3.The image of ϕ, Im(ϕ) is isomorphic to the quotient group G/K.

Chapter 3

Symmetry in Dynamical Systems

Having introduced groups, one might go on with explaining dynamical systems with the toolbox obtained. First, definition of a dynamical system is given , for the sake of being complete. Details about dynamical systems will be omitted, the curious reader may consult to related textbooks.

Definition 3.0.7 (Dynamical System)
A **dynamical system** is a notion that consists of:

a state space: or a phase space that we will denote as $\mathbb{X}$ in which all possible solutions to it will reside within

a time set: which we will denote as T which will contain the parameters $t \in T$ that our dynamical system will evolve within, namely, a set of times.

evolution operators: which is a set $\{\phi^t\}$ which satisfies the following:

$$\phi_t : \mathbb{X} \mapsto \mathbb{X}$$

This is an all encompassing definition for dynamical systems, including discrete and continuous time dynamical systems. Systems with boolean functions, with discrete time definitions are also included in this broad definition. We would like to focus our interest on dynamical systems with continuous dynamical systems, thus we will narrow our definition down.

Definition 3.0.8 (Flow)
A **flow** is a set of mappings $\{\phi_t : X \times \mathbb{R} \to X\}, \quad t \in \mathbb{R}$
With the following restrictions:

1. The flow must map any element $x \in X$ to itself if the time $t \in \mathbb{R}$ is equal to zero, in other words for $t = 0$, it must be the identity map.

$$\phi_0 = e \tag{3.1}$$

2. And two consecutive flows are applied to x with two different time variables $t_1, t_2 \in \mathbb{R}$ and $t_1 \neq t_2$:

$$\phi_{t_1} \cdot \phi_{t_2} = \phi_{t_1+t_2} \tag{3.2}$$

such that the flow is a linear mapping in time.

3.1 Symmetry in Dynamical Systems

A symmetry surely resembles an invariance under an operation, but we must point out what actually is remaining invariant to be able to talk about symmetry. The element that will remain invariant might be ambiguous, so we will solidify it.
A flow with a phase space defined in $\mathbb{R}^n$ can be represented with a n differential equations:

$$\dot{x} = F(x) \tag{3.3}$$

where $x \in \mathbb{R}^n$, and $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a smooth map. This paper will exemplify symmetries of dynamical systems of such structure, those can be represented with a set of first order differential equations. These systems will have a continuous time and phase space, with a smooth map.

Definition 3.1.1 (Symmetry of a set of differential equations)
Let γ be an operator defined in the same state space,

$$\gamma : \mathbb{R}^n \mapsto \mathbb{R}^n \tag{3.4}$$

γ is a **symmetry operator** of Equation 3.3 when $\gamma x(t)$ is a solution for that dynamical system for all $x(t)$ that are also solutions.

There can be an ensemble of these symmetry operators for an arbitrary dynamical system. One of them is the identity operator whose action on the n dimensional phase space is $e : x \mapsto x, x \in \mathbb{R}^n$. Thus the identity operator is a symmetry operator for the set of dynamical systems in the form of Equation 3.3.
This ensemble of symmetry operations form a group themselves.

Definition 3.1.2 (Symmetry Group of a set of differential equations)
A symmetry group of Equation 3.3 is a group of all symmetry operator of Equation 3.3. Let $x_1(t)$ and $x_2(t)$ be functions of time. Let C the the set of all possible solutions to Equation 3.3.

$$\Gamma = \{\gamma : \gamma \cdot x_1(t) \in C \forall x(t) \in C\} \tag{3.5}$$

Though it is readily defined, another common way to represent a symmetry operator is through the *equivariance condition*:

Proposition 3.1.1:
γ is a symmetry operator if and only if

$$\gamma \cdot f(x) = f(\gamma \cdot x) \tag{3.6}$$

is satisfied.

From a different point of view, an operator γ is a symmetry of a dynamical system $\dot{x} = f(x)$ if it can commute with f on the field that they are defined in.
We can show that equivariance condition holds for all symmetry operations defined in Equation 3.4. Let $x(t)$ be a solution of the system $\dot{x}(t) = f(x(t))$. Then, if γ is a time-independent symmetry of the system,

$$\gamma \cdot \dot{x}(t) = \gamma \cdot f(x(t)) \tag{3.7}$$

Since γ commutes with time derivative (it does not have an obvious time dependence):

$$\gamma \cdot \frac{\partial}{\partial t} x(t) = \frac{\partial}{\partial t} \gamma \cdot x(t) \tag{3.8}$$

Since $\gamma \cdot x(t)$ is also a solution to f(x):

$$\frac{\partial}{\partial t} \gamma \cdot x(t) = f(\gamma \cdot x) \tag{3.9}$$

thus,

$$\gamma \cdot \dot{x}(t) = \gamma \cdot f(x) = f(\gamma \cdot x). \tag{3.10}$$

Symmetries of differential equations induce constraints to the solutions of the differential equations. For example, if a problem is invariant (meaning that the differential equation is invariant) under any translations on x-y plane of the Cartesian coordinates, then solutions should also be invariant to these translations, since any of these translations fail to introduce a new problem.

Symmetries that this paper is concerned with, also introduce constraints to the solutions of the system, and the nature of these constraints should be mentioned.

Definition 3.1.3 (Fixed Point Subspace)

Fixed Point Subspace: Let $G \subseteq \Gamma$ be a subgroup of the symmetry group of a set of differential equations. The fixed point sub space of G is defined as:

$$Fix(G) = \{U \in \mathbb{R}^n : g \cdot U = U, \forall g \in G\} \tag{3.11}$$

It is the space that is populated by phase space points that remain invariant in any group action of G.

An instructive example is the fixed point subspace of the action of the Lie group $\mathbf{S}^2$ in three dimensional Cartesian coordinates.

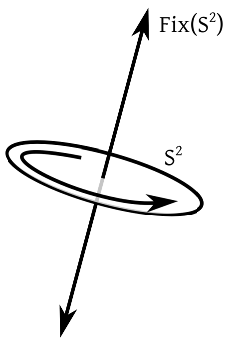

Figure 3.1: Fixed point subspace of $\mathbf{S}^1$. The circle represents all two dimensional rotations in the space that it lies, and all the points that remain invariant under operations of $\mathbf{S}^1$ is the subspace perpendicular to this space.

As shown in Figure 3.1, the action of the Lie group on $\mathbb{R}^3$ fixes a one dimensional subspace of $\mathbb{R}^3$. As fixed point subspace may be defined with the help of a symmetry group, a group may also be defined from the solutions that it will fix:

Definition 3.1.4 (Isotropy Subgroup)

Let G be a group that acts on a space $\mathbb{X}$. An isotropy subgroup of G is a subgroup of G that fixes a point in $\mathbb{X}$.[7, 9] Namely,

$$\Sigma_x = \{g \in G : g \cdot x = x\}, \quad x \in \mathbb{X} \tag{3.12}$$

One might use the term for a **set** of points, trajectories:

$$\Sigma_{u(t)} = \{g \in G : g \cdot u(t) = u(t)\}, \quad u(t) \in \mathbb{X} \tag{3.13}$$

in which u(t) is the trajectory in question.

This definition leads to an important theorem:

Theorem 3.1.1

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Γ equivariant mapping, and let $\Sigma \subseteq \Gamma$ be a subgroup.[7] Then,

$$f(\text{Fix}(\Sigma)) \subseteq \text{Fix}(\Sigma) \tag{3.14}$$

Proof 3.1.1

Let $v \in \mathbb{R}^n$ be a point in our phase space and $\sigma \in \Sigma$ be a symmetry operation. Then since equivarience condition, Equation 3.1.1 holds,

$$\sigma \cdot f(v) = f(\sigma \cdot v). \tag{3.15}$$

If v is in the fixed point subspace of Σ, then $\sigma \cdot v = v$ implies:

$$f(\sigma \cdot v) = f(v) \quad \forall v \in \text{Fix}(\Sigma) \tag{3.16}$$

$$f(\text{Fix}(\Sigma)) \subset \text{Fix}(\Sigma) \tag{3.17}$$

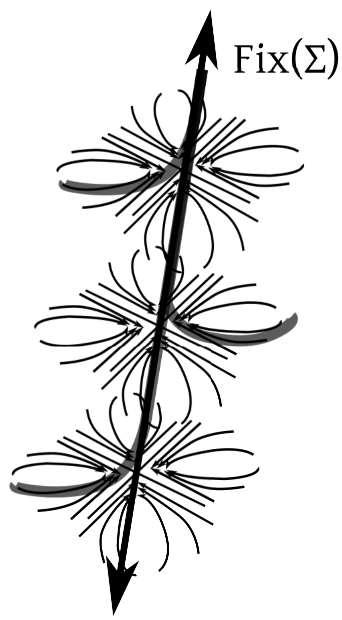

Figure 3.2: Flow invariance under a discrete group Σ with one dimensional Fix(Σ). Bold lines represent the solutions for different initial conditions.

Theorem 3.1.1 has an important consequence. If the flow f is Γ equivariant, any point in its fixed point subspace will be evolved in time into some point in the same subspace. For example, if a flow in $\mathbb{R}^3$ is $\mathbf{S}^1$ equivariant, any point in its Fix($\mathbf{S}^1$) will be mapped onto itself. If the points in this subspace are asymptotically stable equilibria, then any initial conditions within its attractive basin will end up in the fixed point subspace and remain there.

3.1.1 Networks of Dynamical Systems

For a network of dynamical systems many examples can be given. One is a network of neurons, in which a single neuron is modeled as a dynamical system. Another one can be a set of pendula, which are coupled to each other by forces, in which the state of a single pendulum is represented as a point in the state space of its corresponding mechanical system.

Example 3.1.1
Let us consider a particle moving under the influence of a force. Its state in three dimensional space can be represented by orbits in its state space, most generally being the space of its generalized coordinates, q_i and $\dot{q}_i$. But for simplicity, and also without losing any generality, we can use Cartesian coordinates x,y,z. It is clear that the state space, which will represent the state of our system is in, will have information about the position of the particle, thus it must be able to represent have a position vector $(x, y, z) \in \mathbb{R}^3$. Also, this particle would have a mass and a velocity, which should also be represented by our state space, so it must also hold a momentum vector $(p_x, p_y, p_z) \in \mathbb{R}^3$. Since a state of our system of one particle, can be in *any* combination of the two, our state space will be the Cartesian product of these two spaces,

$$\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6 \tag{3.18}$$

The state of two distinguishable particles will require an additional six dimensional space. Even if these two particles interact with each other, the dimension of the state space will remain as it is, requiring that the interaction force is not a function of speed or its higher time derivatives. This is because solution of a mechanical system is identified with equations:

$$\dot{\mathbf{p}} = F(\mathbf{x}) \tag{3.19}$$

$$\dot{\mathbf{x}} = \frac{\mathbf{p}}{m} \tag{3.20}$$

So we can also consider these particles interacting with a very small force, it can be for example, a gravitational attraction. This attraction will be added to the mechanical equations as an additional term, resulting these two dynamical systems governed by differential equations **to be coupled** to each other. With this system of two single particles which all individually have six differential equations and thus six generalized coordinates, can be represented fully with one dynamical system with double the number of differential equations and double the state space dimension.
Here, the nature of a system composed of numerous subsystems, may be too general for

our use. Subsystems may be dynamical systems of different nature, with no restriction on the nature of the coupling. This may lead to an arbitrary set of equations that for most of the cases will be very hard to interpret in simple terms. Thus, in order to be lead to a satisfactory interpretation, simplifications are made in such ways that the phenomena that is to be interpreted will be in dominance. Here, we will take all systems similar in nature, and take inter-system couplings as weak and linear.

Therefore, all networks of dynamical systems can be represented with another dynamical system with a new set of differential equations and state space, which makes the symmetry arguments that can be made for a dynamical system equally applicable to networks of systems. Given that micro-systems have the same inner dynamics, permutational symmetries of a dynamical system network, can always be interpreted as symmetries of the dynamical system as a whole.

3.2 Examples

In order to create an understanding of symmetric dynamical system networks, it is instructive to give a simple example.
Let us take two dynamical systems, that have the same inner dynamics, consequently, have the same set of differential equations,

$$\begin{aligned} \dot{x_1} &= f(x_1, x_2) \qquad &, x_1 \in \mathbb{R}^k \\ \dot{x_2} &= f(x_2, x_1) \qquad &, x_2 \in \mathbb{R}^k \end{aligned} \tag{3.21}$$

When mentioning dynamical systems ① and ②, and their corresponding variables x_1 and x_2, one must consider a **set** of equations such as:

$$\frac{d}{dt}\begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1k} \end{pmatrix} = F\left(\begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1k} \end{pmatrix}, \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2k} \end{pmatrix}\right) \tag{3.22}$$

$$\frac{d}{dt}\begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2k} \end{pmatrix} = F\left(\begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2k} \end{pmatrix}, \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1k} \end{pmatrix}\right) \tag{3.23}$$

in the most general form, since any variable of each system can be a coupling parameter for the other system. In most of our examples, we will omit cross-coupling between variables of different type, though the statement above is still valid, $f(\cdot)$ can be constructed such that all variables are multiplied with zero but one which is the coupling parameter.

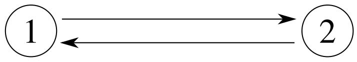

Figure 3.3: Graphical representation of two coupled systems, ① and ②

Without the coupling, every single node in the graph is a k dimensional dynamical system within itself. The coupling that is introduced with the arrows on the graph join these two separate systems, thus form a 2k-dimensional system.

If we look at the graph we can easily point out the symmetry of the system. When systems ① and ② are interchanged, the graph, and evidently the whole dynamical system remains invariant. We will call such an operation a permutation operation $\sigma(x_1, x_2) = (x_2, x_1)$ and show it as (1,2) in some context (this is the well practiced way to show permutations). When the Equation 3.21 undergoes permutation operation $\sigma(x_1, x_2)$, the coordinates x_1 of system ① is exchanged with the coordinates x_2 of system,

$$\dot{x_1} = f(x_1, x_2) \rightarrow \dot{x_2} = f(x_2, x_1) \tag{3.24}$$

$$\dot{x_2} = f(x_2, x_1) \rightarrow \dot{x_1} = f(x_1, x_2) \tag{3.25}$$

Let us create such a system with two identical Fitzhugh-Nagumo neuron models. Fitzhugh-Nagumo model is a two dimensional model of a neuron exhibiting spiking, suitable for our needs.

Neurons, exhibiting a large variety of behavior, are modeled as electrically excitable dynamical systems. Generally, a set of relations between membrane potential and current of ions are taken into account, the potential and currents relate to each other via differential equations of transport and voltage - current relationship of membrane capacitance.

Fitzhugh-Nagumo model of neural dynamics is a stripped down nonlinear dynamical system. It is a good and simple replacement of more complex models such as Hodgkin-Huxley model in some cases, since it is a simpler one.

The behavior of a group of neurons is complex enough to lose cause and effect relationship between form and result. When the numbers of neurons get bigger, an arbitrary arrangement of neurons cease to be comprehensible, but a symmetric arrangement depending on the symmetry, can result in a variety of collective behavior. Understanding of symmetries of a bigger system might give clues to the neuroscientist, give chances to distinguish integrated blocks of neural arrangements, in a non-familiar way. Since neuroscience is a thriving area of interest, this thesis will use the Fitzhugh-Nagumo model in simulations. [10] The main research in this area is also dominated by neural applications, which is obvious from the primary sources that are cited.

Model 3.2.1
Fitzhugh-Nagumo Model:

$$\dot{v} = v - \frac{v^3}{3} - w + i \tag{3.26}$$

$$\dot{w} = 0.08(v + 0.7 - 0.8w) \tag{3.27}$$

where v and w are variables of the system, v models the membrane potential, and w provides the intrinsic feedback mechanism. The parameter i is the stimulus current, a variable that is open to outer interference, models the stimuli.

Using this model for systems ① and ②, we can probe further down in the dynamics.

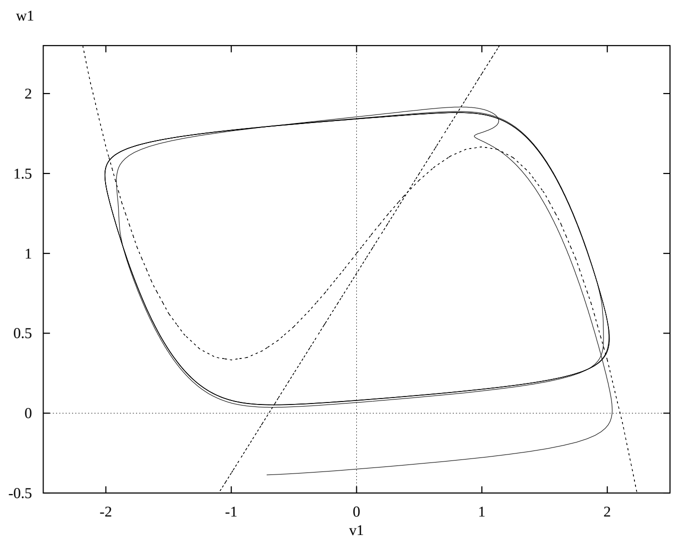

Figure 3.4: Projection of the phase space representation of the coupled system onto w_1-v_1 axes. This is the phase space plot of system ① only. The dotted curves are the nullclines, the solid line is a solution. $i = 1$, $\gamma = 0.1$

This is a characteristic phase space of a Fitzhugh-Nagumo cell, where the nullclines

$$v - \frac{v^3}{3} - w + i = 0 \tag{3.28}$$

$$v + 0.7 - 0.8w = 0 \tag{3.29}$$

are clearly visible and the intersection is an unstable equilibrium point.

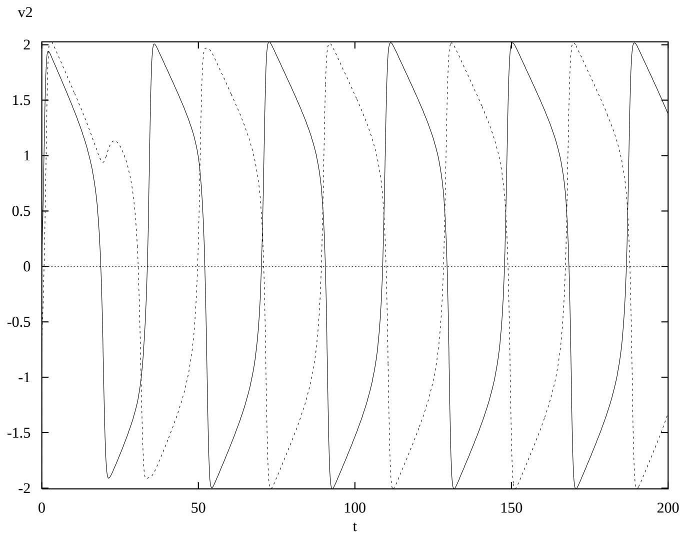

Figure 3.5: Evolution of v_1 (dashed curve) and v_2 (solid curve) with time

If we look from a symmetry perspective, these two systems exhibit a permutation symmetry, as stated. So, under any possible arrangements of these systems, the encapsulating system remains invariant, which leads to the fact that the symmetry group of the system is $\mathbf{S_2}$. It is one of the main points that will be made clear that this synchrony is the result of the symmetry that this dynamical system exhibits in its structure, and can be explained via the introduction of symmetry groups.

3.3 H/K Theorem

The H/K Theorem lies at the heart of our interpretation of symmetries in dynamical system networks. The main motivation behind analyzing dynamical system networks and symmetry from a mathematically intensive perspective was the ability to point out general properties of to solutions to dynamical system networks that have symmetry. Now, we are capable of interpreting symmetries of any system with its symmetry group, we also know that the symmetry of a graph correspond to a symmetry in the dynamical system independent of the system concerned. The link between the symmetry properties of the network and the solutions to the individual subsystems is still missing. The H/K theorem gives us the opportunity to make generalizations on the way the individual systems behave. First, some definitions are necessary. [7]

Definition 3.3.1 (Spatial and spatiotemporal symmetries)
Let us define the symmetry groups of a system as H,K and Γ:

$$K \triangleleft H \triangleleft \Gamma \tag{3.30}$$

and let $x(t)$ be solutions to this system. the spatiotemporal symmetries of the system form a subgroup satisfying:

$$H = \{\gamma \in \Gamma : \gamma\{x(t)\} = \{x(t)\}\} \tag{3.31}$$

and the spatial symmetries of the system also form a subgroup satisfying:

$$K = \{\gamma \in \Gamma : \gamma x(t) = x(t),\ \forall t\} \tag{3.32}$$

An explanation is needed in order to unveil the definitions from the notation. The spatiotemporal symmetries of the system, when act upon the system, each solution of the system is mapped to a solution of the same system, but strictly the same solution, same point in the phase space in the same time. This state will be noted as synchrony between systems. The spatial symmetries of the system, when they act on the system, they preserve the trajectories of the solutions, but points that form the solutions on the phase space are not mapped onto the same point for a particular time, a solution might trace the same trajectory, but in a different time, namely, with a delay.
Let $u(t)$ be a periodic solution to a Γ equivariant Equation 3.3. Let us define groups H and K as introduced in Equations 3.32 and 3.31. Let $u(0)$ be an initial condition for the solution $u(t)$ and γ be in the isotropy subgroup of $u(0)$. Then, γ will map $u(0)$ to $u(0)$, and $\gamma \cdot u(t)$ will be another solution to the differential equation with the same initial condition, $u(0)$. Uniqueness of solutions asserts $\gamma \cdot u(t)$ and $u(t)$ ought to be the same solution, since they are governed with the same flow with the same initial conditions. In that case, $\gamma \in K$ for all $\gamma \in \Sigma_{u(0)}$.
Let us define a homomorphism $\Theta : H \to \mathbf{S}^1$ where $\theta = \Theta(h_i)$. For every element of H, there is a corresponding θ, which denotes the phase difference of a periodic solution $u(t+\theta)$ with a reference solution $u(t)$. Elements of K are mapped onto $\theta = 0$, since K is the kernel of this homomorphism. Since $K = \text{Ker}(\theta)$, K is a normal subgroup, dictated by Theorem 2.4.1 , First Isomorphism Theorem. This theorem also states that the image of θ, Im(θ) is isomorphic to the quotient group H/K. The image of θ is sure to be a subgroup of $\mathbf{S}^1$. That implies $H/K \cong \mathbf{Z}_m$ or $H/K \cong \mathbf{S}^1$. If we take the former isomorphism as granted, the form of H/K will determine the phase differences in the dynamical system network, and since H/K is a cyclic group, the solutions to the dynamical system will exhibit phase differences $\theta = T/m$,where m is the order of H/K and T is one period of u(t), which is in Fix(K).
For now, it was assumed that the solution u(t) was a periodic solution of Equation 3.3. In order for a differential equation to exhibit a stable oscillation, the dimension of its phase space must be at least two. The symmetry of the system will force u(t) to be in $\text{Fix}(\Sigma_{u(t)}) = \text{Fix}(K)$, so $\dim(\text{Fix}(K)) \geq 2$. [7]
These conditions are all put together in a form of a Theorem as the H/K theorem: [7]

Theorem 3.3.1
(H/K Theorem) [7, 11]
Let γ be a symmetry group of coupled cell network of cells of at least two dimensional state space. Let

$$K \triangleleft H \triangleleft \Gamma \tag{3.33}$$

be normal subgroups of Γ.
There exists spatial symmetries K and spatiotemporal symmetries H if and only if H/K is a cyclic quotient group, and K is a isotropy subgroup.

Chapter 4

Eight Cell Network with Symmetry Group $\mathbf{Z}_4 \times \mathbf{Z}_2$

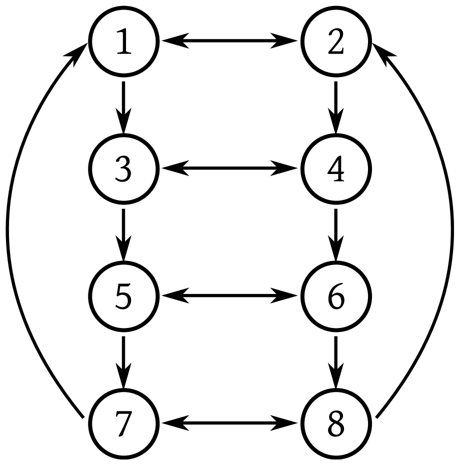

Figure 4.1: An eight cell network with $\mathbf{Z}_2 \times \mathbf{Z}_4$ symmetry

Let us give an example to shed light on the implications of the H/K Theorem.

Here is a network of 8 similar cells, each of them being a dynamical system of order two or more, so that they can exhibit oscillatory behavior.

The graph has several symmetries. One of them is an operator that maps ① $\longrightarrow$ ③,③ $\longrightarrow$ ⑤,⑤ $\longrightarrow$ ⑦ and ⑦ $\longrightarrow$ ① while it also maps ② $\longrightarrow$ ④,④ $\longrightarrow$ ⑥,⑥ $\longrightarrow$ ⑧ and ⑧ $\longrightarrow$ ②. This group operator can be denoted with the cycle notation as (1357)(2468). One can see that rotating one of the sides does not preserve the structure. If only ① $\longrightarrow$ ③,③ $\longrightarrow$ ⑤,⑤ $\longrightarrow$ ⑦ and ⑦ $\longrightarrow$ ① is applied, the branches that couple odd numbered systems to even numbered systems would not remain in the same combination.

Another symmetry operation would be the one that would map ① $\longleftrightarrow$ ② , ③ $\longleftrightarrow$ ④, ⑤ $\longleftrightarrow$ ⑥, ⑦ $\longleftrightarrow$ ⑧, or (12)(34)(56)(78). This is equivalent to taking a mirror image, if a visual aid may be practical.

The set of operators generated by (1357)(2468) are closed within that set, and form a group. Let us call this group $\mathbf{Z}_4$, being a cyclic group of order 4,

$$\mathbf{Z}_4 = \{e, \rho, \rho^2, \rho^3\} \tag{4.1}$$

ρ^i being the ith rotation, e being the unit operation.
The second family of operations would be the reflection group, generated by (12)(34)(56)(78), which could be shown as,

$$\mathbf{D}_1 = \{e, \sigma\} \tag{4.2}$$

It is evident that having both of these symmetry groups, any combination of these symmetry operation still also be a symmetry operation, in other words, when applied, the system will remain as is. Therefore, the Cartesian product of these two groups,

$$\mathbf{Z}_4 \times \mathbf{D}_1 = \{1, \rho, \rho^2, \rho^3, \sigma\rho^1, \sigma\rho^2, \sigma\rho^3\} \tag{4.3}$$

will also form a symmetry group (Reader can check whether it meets definition 2.1.1, the group criteria.)
This also forms our main symmetry group,

$$\gamma = \mathbf{Z}_4 \times \mathbf{D}_1 = \{1, \rho, \rho^2, \rho^3, \sigma\rho^1, \sigma\rho^2, \sigma\rho^3\} \tag{4.4}$$

Our system exhibiting symmetry of $\mathbf{Z}_4 \times \mathbf{D}_1$, we can proceed applying the H/K Theorem.

4.1 Solutions for $H \cong Z_4(\rho)$ and $K \cong Z_2(\rho^2)$

Let us select H and K, both of them must be normal subgroups of Γ, H/K must form a cyclic quotient group, and K must be an isotropy subgroup.

$$H = \{i, \rho^1, \rho^2, \rho^3\} \tag{4.5}$$
$$K = \{i, \rho^2\} \tag{4.6}$$

There are different ways to show that both of these groups are normal subgroups of γ. First of all the symmetry group Γ is Abelian (in other words, its elements commute).

And every subgroup of an Abelian group is a normal subgroup of that group. Normally, of the symmetry group Γ is not Abelian, one would have to check that

$$\gamma_j h_i \gamma_j^{-1} \in H; \quad \forall \gamma_j \in \Gamma, \quad \forall h_i \in H \tag{4.7}$$

holds.

We should check whether K is an isotropy subgroup of Γ.

that way, the quotient group H/K will be populated by left cosets of H multiplied with K,

$$H/K = \{i \cdot \{i, \rho^2\}, \rho^1 \cdot \{i, \rho^2\}, \rho^2 \cdot \{i, \rho^2\}, \rho^3 \cdot \{i, \rho^2\}\} \tag{4.8}$$

When multiplications are executed, some of the elements of the set turn out to be analogous to other elements as their action:

$$H/K = \{\{i, \rho^2\}, \{\rho^1, \rho^3\}\} \tag{4.9}$$

which **is** cyclic, since every element of the group can be generated by $\{\rho^1, \rho^3\}$ by multiplication. Multiplication is defined as follows. When two cosets are multiplied, the resultant is a coset, and it consists of individual elements of the cosets multiplied (law of composition of the group) with each other. So, if we multiplied $\{\rho^1, \rho^3\}$ with itself, we would have:

$$\{\rho^1, \rho^3\} \cdot \{\rho^1, \rho^3\} = \{\rho^1\rho^1, \rho^3\rho^3, \rho^3\rho^1, \rho^1\rho^3\} = \{\rho^2, i\} \tag{4.10}$$

which gives the other element of the H/K multiplication.

Since all requirements are satisfied we can conclude that the system will have spatial symmetries K and spatiotemporal symmetries H in its solutions.

Lets take the solution for ① as $x_1(t)$. If we apply the generator for K, namely, ρ^2 to it, it is stated by the H/K theorem that it will have a spatiotemporal symmetry, meaning that $x_5(t)$ will follow $x_1(t)$ for all t. The same applies to $x_2(t)$, which is followed by $x_6(t)$. Algebraically, the existence of our generator for the cyclic symmetry group K †

† It is instructive to note that this symmetry group is also in the form of $\mathbf{Z_2}$. We named it as $\mathbf{D_1}$.

implies that:

$$\begin{aligned} x_1(t) &= x_5(t) \\ x_2(t) &= x_6(t) \\ x_3(t) &= x_7(t) \\ x_4(t) &= x_8(t) \end{aligned} \quad , \forall t \in \mathbb{T} \tag{4.11}$$

Depending on the H/K Theorem, we also can predict that solutions that have a phase shift of $T/2$, since the generator of the H is the operator ρ^1. The phase shift is determined by the fact that the operation ρ^2 is within spatio-temporal symmetries of the system, thus must be invariant for all the solutions, since two consecutive operations of ρ^1 must result in a spatially and temporally symmetric solutions.

This example that we have gone through is the model that is proposed by **(author?)** [12] in order to explain how more complex walking rhythms are observed in four legged animals within the context of symmetry. In order to maintain all the observed gaits in four legged animals. In addition to a coupled four cell network each one representing one leg, another layer of cells that are coupled in such a way that the system presents the symmetry of $\mathbf{Z_4} \times \mathbf{Z_2}$, from which all primary gaits for all four legged animals can derived. Every different way to select a H/K group (in other words, every different way to select a pair of H and K, satisfying the conditions stated) a different gait is produced. A more thorough treatment is available in **(author?)** [7, 11].

Since we have spotted out the symmetries of the system, we could go on showing other solutions also exist. Let us introduce a notation and then investigate more. In order to point out a particular symmetry of the graph with a particular generator, lets write $\Gamma(\gamma)$, Γ being the symmetry group and γ being the generator associated. So, $\mathbf{Z_2}(\sigma\rho^2)$ represents a cyclic group of order 2, its generator being one mirror and two consecutive rotation operations.

4.2 Solutions for $H \cong \Gamma$ and $K \cong Z_2(\sigma\rho^2)$

Having sorted notation out, lets select two different H and K groups:

$$H = \{i, \rho^1, \rho^2, \rho^3, \sigma, \sigma\rho^1, \sigma\rho^2, \sigma\rho^3\} \tag{4.12}$$

$$K = \{i, \sigma\rho^2\} \tag{4.13}$$

with the new notation, we can show them as:

$$H = \gamma \quad K = \mathbf{Z_2}(\sigma\rho^2) \tag{4.14}$$

building H/K,

$$H/K = \{\{i, \sigma\rho^2\}, \rho^1 \cdot \{i, \sigma\rho^2\}, \rho^2 \cdot \{i, \sigma\rho^2\}, \rho^3 \cdot \{i, \sigma\rho^2\}, \tag{4.15}$$
$$\sigma \cdot \{i, \sigma\rho^2\}, \sigma\rho^1 \cdot \{i, \sigma\rho^2\}, \sigma\rho^2 \cdot \{i, \sigma\rho^2\}, \sigma\rho^3 \cdot \{i, \sigma\rho^2\}\} \tag{4.16}$$

if we apply the operators to the elements of the cosets and eliminate repetition,

$$H/K = \{\{i, \sigma\rho^2\}, \{\rho^1, \sigma\rho^3\}, \{\rho^2, \sigma\}, \{\rho^3, \sigma\rho^1\}, \} \tag{4.17}$$

H/K is cyclic, and its generator is $\{\rho, \sigma\rho^3\}$:

$$\{\rho, \sigma\rho^3\} \cdot \{\rho, \sigma\rho^3\} = \{\rho^2, \rho^2, \sigma\rho^4, \sigma\rho^4\} = \{\rho^2, \sigma\} \tag{4.18}$$
$$\{\rho, \sigma\rho^3\} \cdot \{\rho^2, \sigma\} = \{\sigma\rho, \rho^3, \sigma\rho, \rho^3\} = \{\sigma\rho, \sigma\rho^3\} \tag{4.19}$$
$$\{\rho, \sigma\rho^3\} \cdot \{\sigma\rho, \sigma\rho^3\} = \{\sigma\rho^2, \sigma\rho^6, \rho^4, \rho^4\sigma^2\} = \{\sigma\rho^2, i\} \tag{4.20}$$

Hence we have shown that we can populate every element of H/K with the generator $\{\rho, \sigma\rho^3\}$.

With the help of the H/K Theorem, we can conclude that we have the spatiotemporal symmetries of H and spatial symmetries of K.

4.3 Simulations

For all possible legal combinations of H and K, we will have a resultant limit cycle $U(t)$ that will have the corresponding symmetries. Here is a table for some possible combinations of H and K, and the corresponding solutions:

H	K	$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8)$
Γ	Γ	$(0, 0, 0, 0, 0, 0, 0, 0)$
Γ	$\mathbf{Z}_4(\rho)$	(0,½,0,½,0,½,0,½)
Γ	$\mathbf{Z}_4(\sigma\rho)$	(0,½,½,0,0,½,½,0)
Γ	$\mathbf{Z}_2(\sigma)$	(0,0,¼,¼,½,½,¾,¾)
Γ	$\mathbf{Z}_2(\sigma\rho^2)$	(0,½,¼,¾,½,0,¾,¼)

Let us verify this by finding the corresponding solutions with a numerical aid. These solutions are obtained by using a software named XPPAUT.[13]

When we connect cells of FitzHugh-Nagumo cells to each other by coupling every variable to the corresponding variable of the cell that is linked to the original one as:

$$
\begin{aligned}
v_1' =& v_1 - (v_1^3)/3 - w_1 + i + a \cdot v_7 + c \cdot v_2 \\
w_1' =& 0.08 \cdot (v_1 + 0.7 - 0.8 \cdot w_1) + a \cdot w_7 + c \cdot w_2 \\
v_2' =& v_2 - (v_2^3)/3 - w_2 + i + a \cdot v_8 + c \cdot v_1 \\
w_2' =& 0.08 \cdot (v_2 + 0.7 - 0.8 \cdot w_2) + a \cdot w_8 + c \cdot w_1 \\
v_3' =& v_3 - (v_3^3)/3 - w_3 + i + a \cdot v_1 + c \cdot v_4 \\
w_3' =& 0.08 \cdot (v_3 + 0.7 - 0.8 \cdot w_3) + a \cdot w_1 + c \cdot w_4 \\
v_4' =& v_4 - (v_4^3)/3 - w_4 + i + a \cdot v_2 + c \cdot v_3 \\
w_4' =& 0.08 \cdot (v_4 + 0.7 - 0.8 \cdot w_4) + a \cdot w_2 + c \cdot w_3 \\
v_5' =& v_5 - (v_5^3)/3 - w_5 + i + a \cdot v_3 + c \cdot v_6 \\
w_5' =& 0.08 \cdot (v_5 + 0.7 - 0.8 \cdot w_5) + a \cdot w_3 + c \cdot w_6 \\
v_6' =& v_6 - (v_6^3)/3 - w_6 + i + a \cdot v_4 + c \cdot v_5 \\
w_6' =& 0.08 \cdot (v_6 + 0.7 - 0.8 \cdot w_6) + a \cdot w_4 + c \cdot w_5 \\
v_7' =& v_7 - (v_7^3)/3 - w_7 + i + a \cdot v_5 + c \cdot v_8 \\
w_7' =& 0.08 \cdot (v_7 + 0.7 - 0.8 \cdot w_7) + a \cdot w_5 + c \cdot w_8 \\
v_8' =& v_8 - (v_8^3)/3 - w_8 + i + a \cdot v_6 + c \cdot v_7 \\
w_8' =& 0.08 \cdot (v_8 + 0.7 - 0.8 \cdot w_8) + a \cdot w_6 + c \cdot w_7
\end{aligned}
\tag{4.21}
$$

As it can be seen from the set of equations, each cell is coupled to its cyclic neighbor (the neighbor that it is connected to with the $\mathbf{Z_4}$) symmetry, with the a coupling parameter, whereas it is connected to its lateral neighbor with the parameter c. The coupling parameters taken for v and w may have been chosen to be adjusted independently, but it further complicates an readily complex problem, thus we chose not to. There is a global parameter i, which is the current that excites a single Fitz-Hugh Nagumo neuron. It is fixed as 0.5, since we want all of our systems to be in their oscillatory regime, all the time.

Let is check the results. We can see one of our proposed solutions to our 16 dimensional coupled system in Figure 4.3. When parameters are chosen as $a = -0.08$ and 0.06, this solution where we have 180°phase difference between consecutive cells having odd and even tags. We will use a shorthand notation (0,0,½,½,0,0,½,½) for this mode of operation, the fractions showing the phase difference of each cell with the first cell.

Since we know the steady state solution to this system specifically to these parameters, we can point out which symmetries were dominant. Let us name our periodic solution as $u(t) \in \mathbb{R}^{16}$. We know that our trajectory satisfies conditions,

$$\begin{aligned}
v_1(t) &= v_2(t) \qquad & w_1(t) &= w_2(t) \\
v_3(t) &= v_4(t) \qquad & w_3(t) &= w_4(t) \\
v_5(t) &= v_6(t) \qquad & w_5(t) &= w_6(t) \\
v_7(t) &= v_8(t) \qquad & w_7(t) &= w_8(t) \\
v_1(t) &= v_5(t) \qquad & w_1(t) &= w_5(t) \\
v_2(t) &= v_6(t) \qquad & w_2(t) &= w_6(t)
\end{aligned} \tag{4.22}$$

These conditions form a subspace, just like a condition $x = y, x, y$ for points $(x, y) \in \mathbb{R}^2$ forms a line, a one dimensional subspace $\{(x, x)|x \in \mathbb{R}\}$ of $\mathbb{R}^2$. Since any point in our phase space can be represented by variables v_1, v_3, w_1, w_3 but not less, we can deduce that the subspace our solution lies on is a four dimensional real space. One can also reach a similar argument by applying a series of projections for the variables that are equal to each other. The isotropy subgroup associated with u(t) is

$$\Sigma_{u(t)} = \{e, \kappa, \omega^2, \kappa \cdot \omega^2\} \cong \mathbf{Z}_2(\kappa) \times \mathbf{Z}_2(\omega^2) \cong \mathbf{D}_2 = \langle \kappa, \omega^2 \rangle \tag{4.23}$$

Here, $\langle, \rangle$ is used in order to denote that the group is generated by those elements within the brackets. Same notation will be used to denote the spaces that are spanned by the elements within the brackets. The fixed point subspace of this isotropy group (the set of points that form a space that are invariant for every single element of a group) is:

$$\text{Fix}\left(\Sigma_{u(t)}\right) = \langle v_1, v_3, w_1, w_3 \rangle \tag{4.24}$$

$$\dim\left(\text{Fix}\left(\Sigma_{u(t)}\right)\right) = 4 \tag{4.25}$$

As seen in Figure 4.3, such a solution that has the isotropy group of is $\mathbf{D}_2$, the dimension of the space that the action of it fixes is at least 2, we can take the isotropy subgroup of the solution as K, whereas we will select the group denoted as H in Theorem 3.3.1 as Γ. We could have taken H something else, since there is no contradiction to do so, but one could note that selecting H as a different subgroup of Γ, such as $\mathbf{D}_2$, we lead to a situation as following:

When we select the group H as $\mathbf{D}_2$, the spatiotemporal symmetries will be:

$$\Delta = (\gamma, \theta) \in \mathbf{D}_2 \times \mathbf{S}^1$$

in which for all γ, a trajectory will be invariant to the action of the group element in the phase space. But because $H \neq \Gamma$, the solutions will have two independent spa-

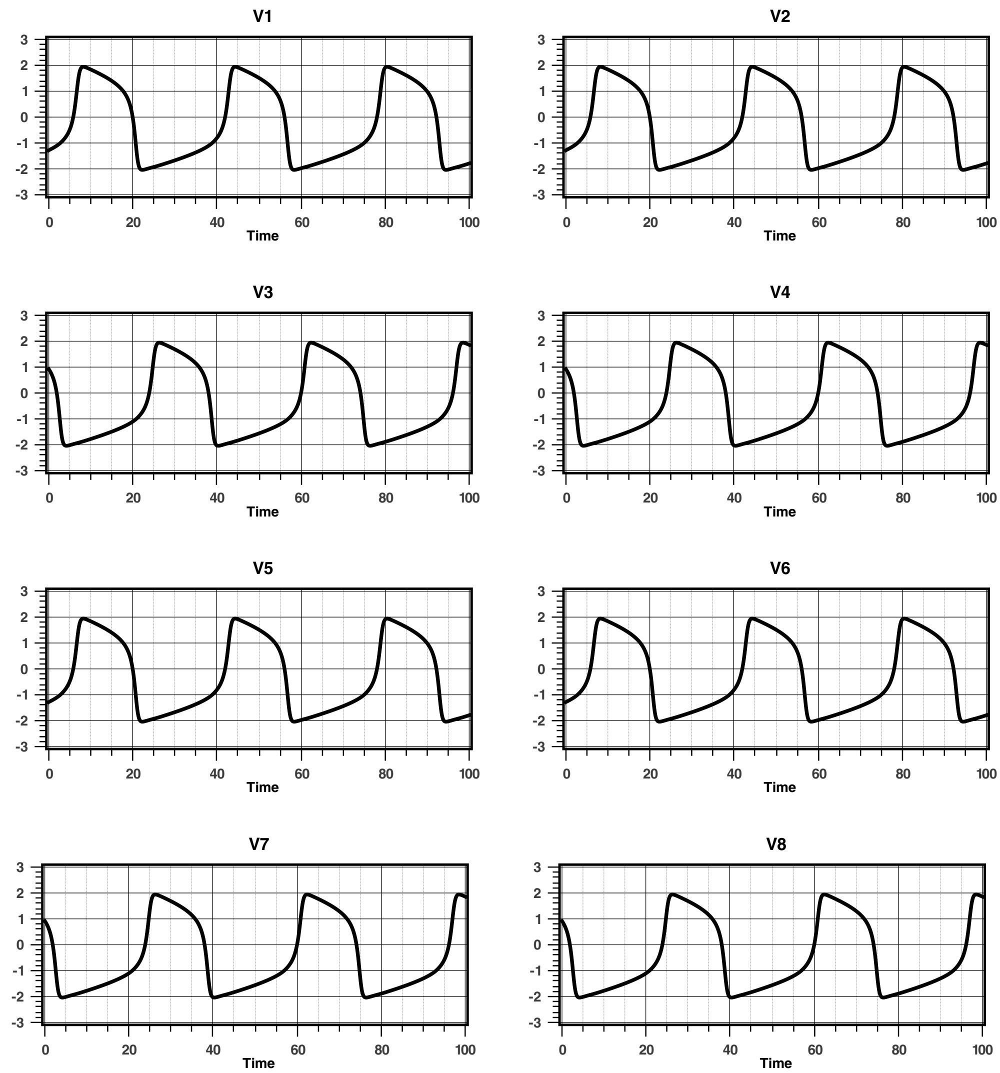

Figure 4.2: Steady state solution for the system, $a = -0.08$ and $c = 0.06$

tiotemporal symmetries, one concerning $v_1, v_2, v_5, v_6, w_1, w_2, w_5, w_6$ and one concerning $v_3, v_4, v_7, v_8, w_3, w_4, w_7, w_8$. That means the solutions for the first set of variables may be uncorrelated with the second one. Since it is not our case, we can proceed with Γ. With intuition, one can say that these cases arise where there is no coupling between some nodes, the case when some are transparent to other. The case when $H \cong Z_4(\omega)$ would lead to two independent systems uncoupled, the right hand side visualized in Figure 4 would be independent of the left side.

The solutions corresponding to the groups $H \cong \mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$ and $K \cong \mathbf{D}_2$ are found to be in existence, and that they are stable.

Other solutions that correspond to different coupling parameters are as follows. All correspond to a different symmetry group H/K.

When the parameters are chosen as $a = 0.08$ and $b = -0.02$, the system exhibits a rhythm (0,½,0,½,0,½,0,½).

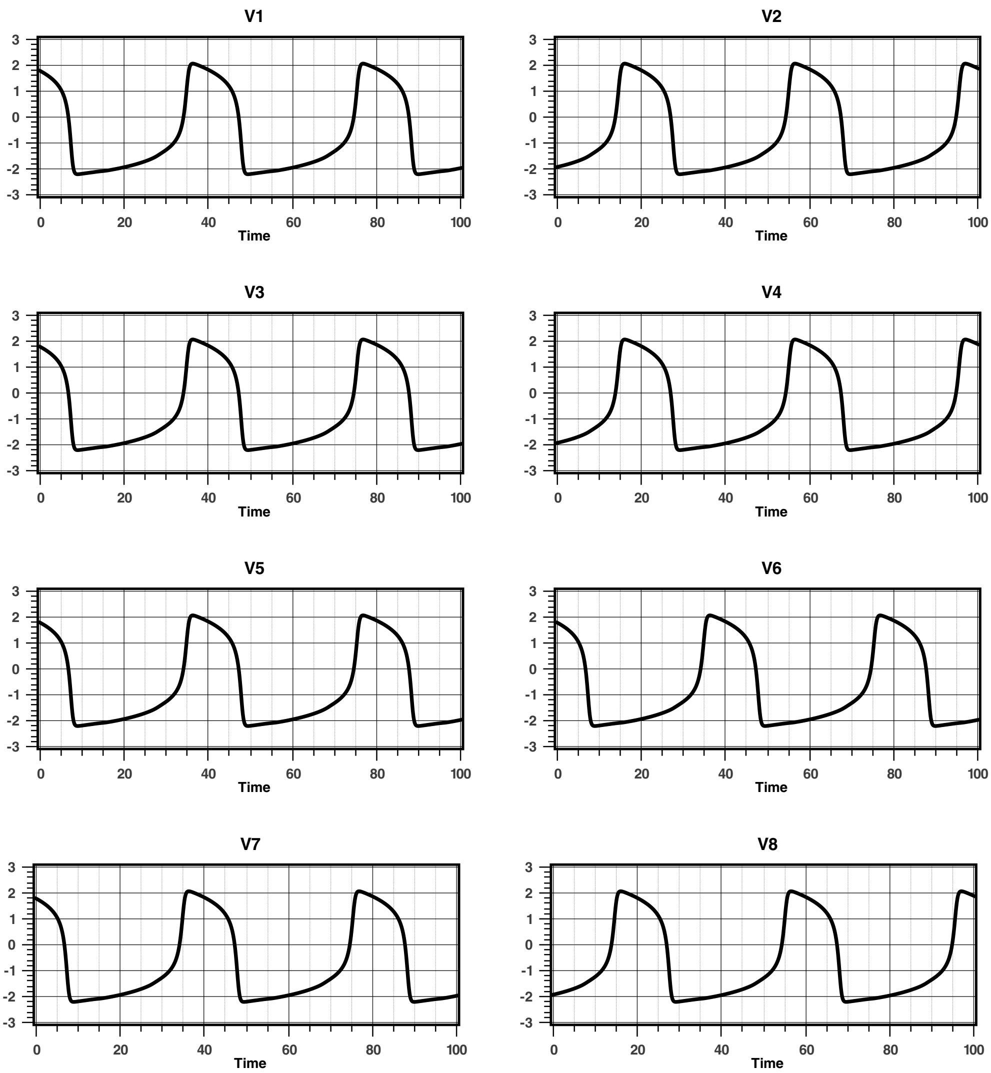

Figure 4.3: Steady state solution for the case when coupling parameters are chosen as $a = 0.08$ and $b = -0.02$.

When the parameters are chosen as $a = -0.0056$ and -0.0018, the system exhibits a rhythm (0,½,¼,¾,½,0,¾,¼).

When the parameters are chosen as $a = -0.004$ and -0.002, the system exhibits a rhythm (0,0,¾,¾,½,½,¼,¼). But in this case, if the coupling constant for v_i is a , coupling constant for w_i is $b = -a$

When the parameters are chosen as $a = 0.004$ and -0.002, the system exhibits a rhythm (0,0,¼,¼,½,½,¾,¾). $b = -a$ holds for this case also.

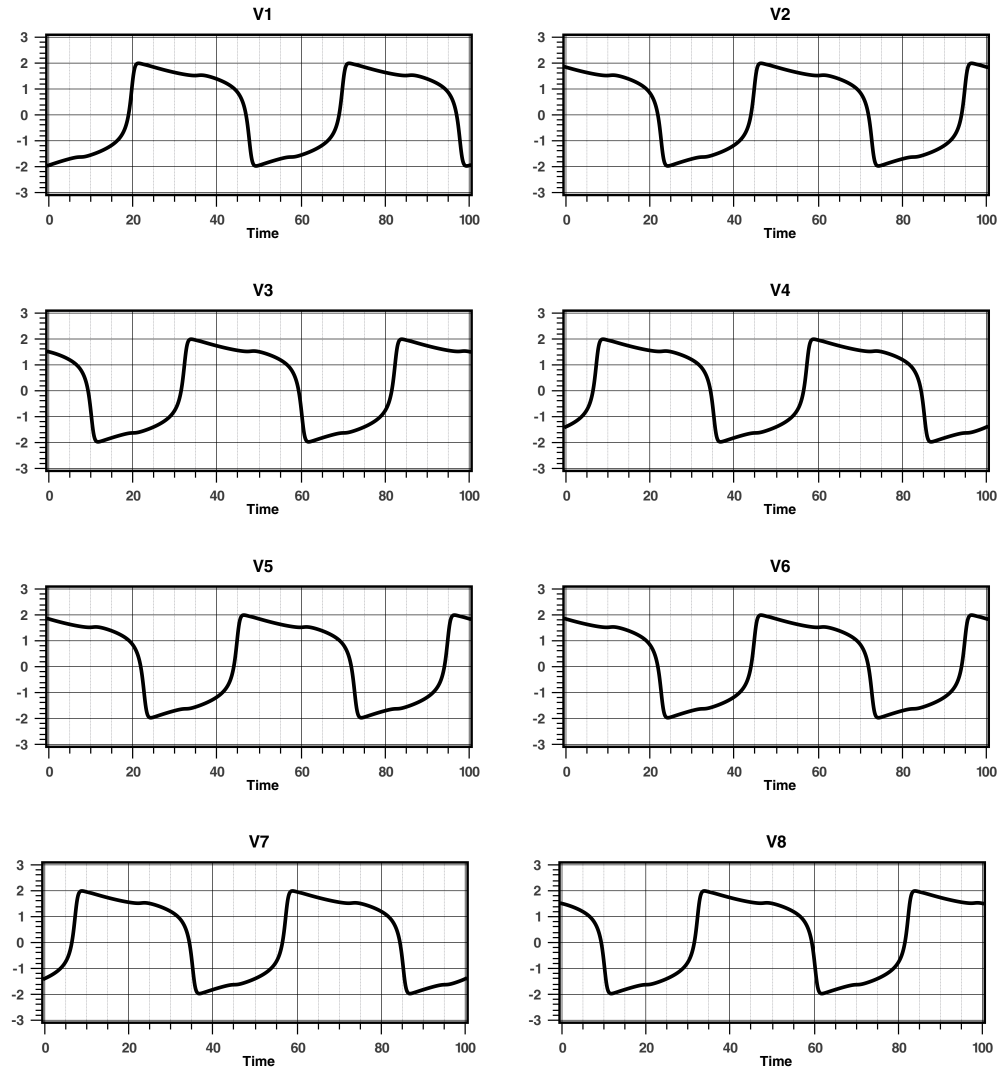

Figure 4.4: Steady state solution for the case when coupling parameters are chosen as $a = -0.0056$ and -0.0018

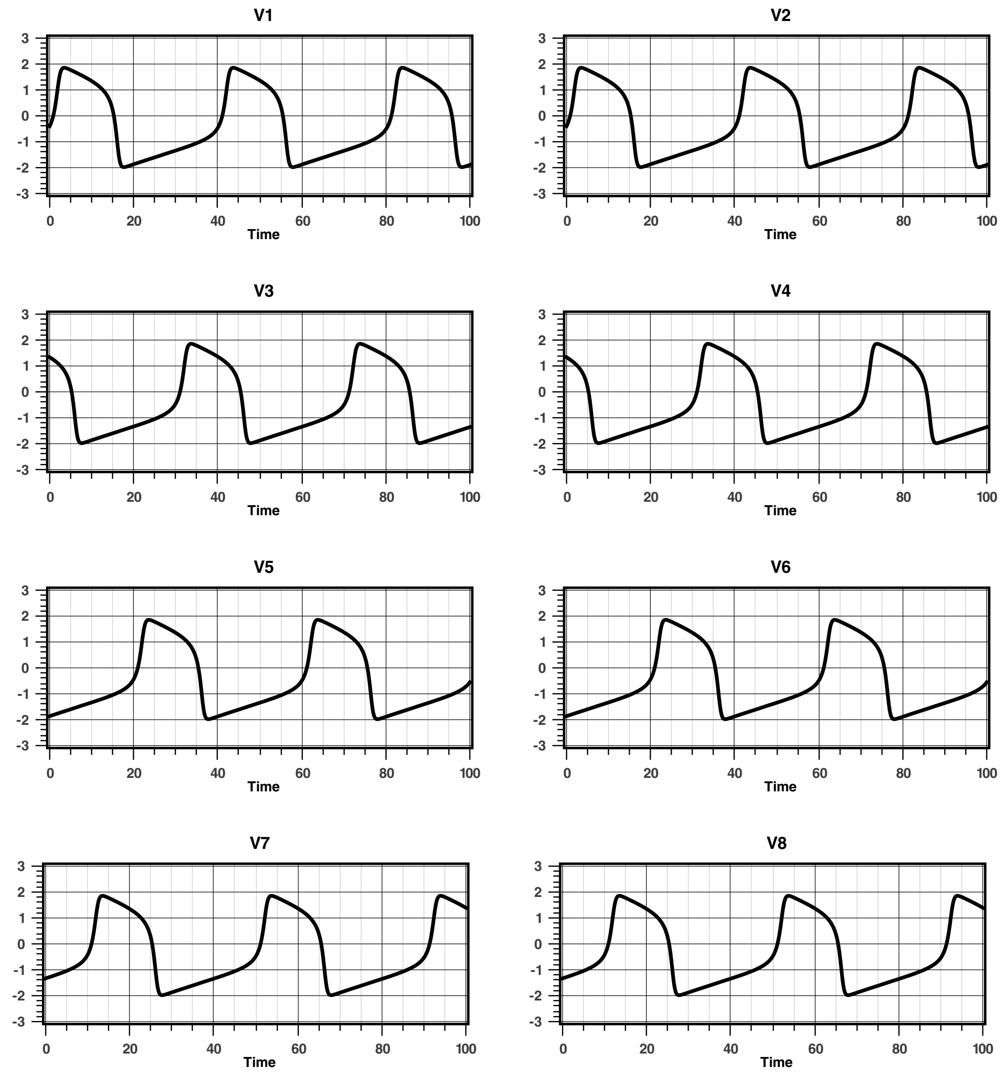

Figure 4.5: Steady state solution for the case when coupling parameters are chosen as $a = -0.004$ and -0.002, with antisymmetrical coupling.

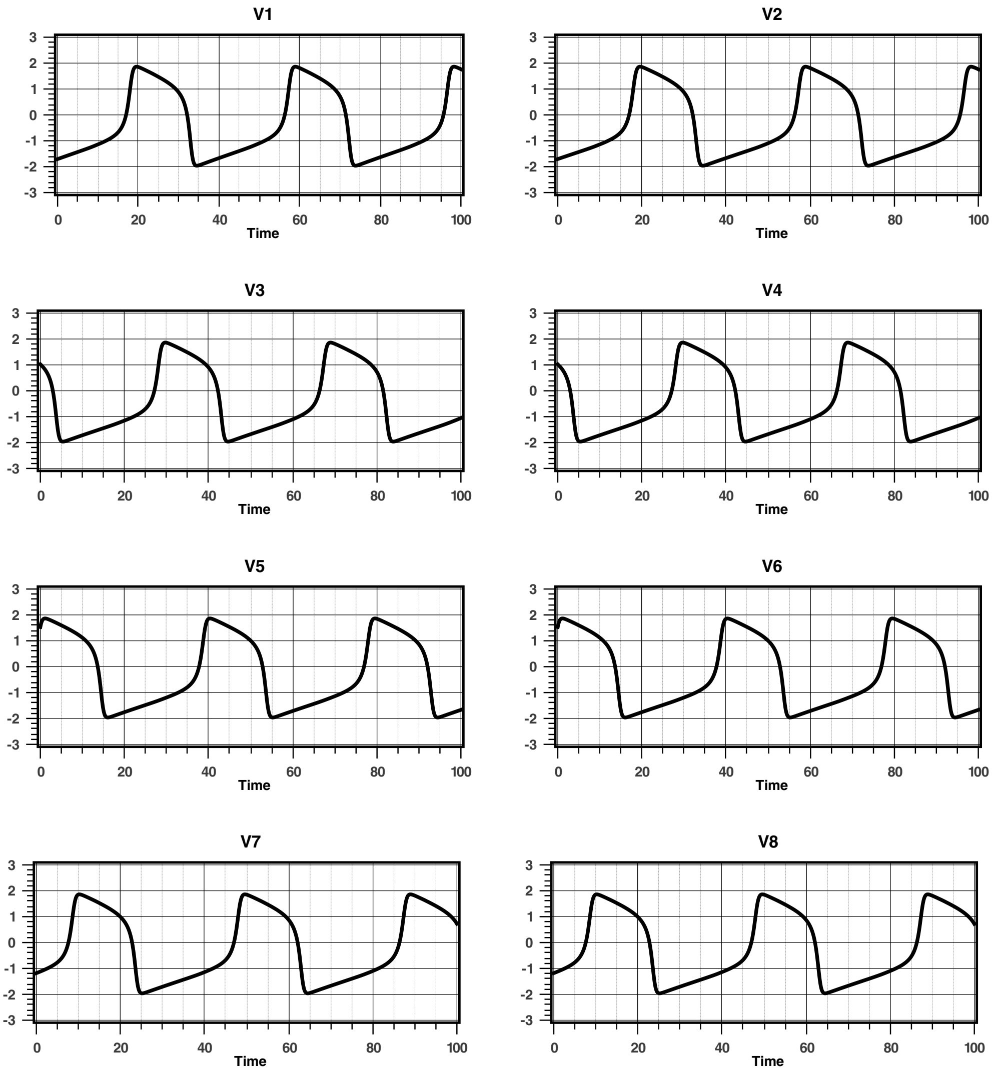

Figure 4.6: Steady state solution for the case when coupling parameters are chosen as $a = 0.004$ and -0.002

Chapter 5

Conclusion

In Chapter 4, it is demonstrated that symmetries of dynamical system networks put notable constraints on the observable solutions that might be generated. With the help of the First Isomorphism Theorem and the flow invariant subspaces, H/K theorem was introduced, with one is capable of extracting possible rhythms of oscillations from a dynamical system network. In this paper it was shown that these solutions exist and are stable, backing the resultant theorem with numerical data. But it still remains as a question how can one stable solution can be triggered into another one. This question gains importance when one speculates how animals change their gait, as they accelerate. In order to answer this question, and speculate on how network of systems with symmetry change "mode" of operation, one should go beyond the point it was reached in this work, and investigate the Equivariant Branching Lemma [7, 14].

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