Symmetry in Dynamical System Networks

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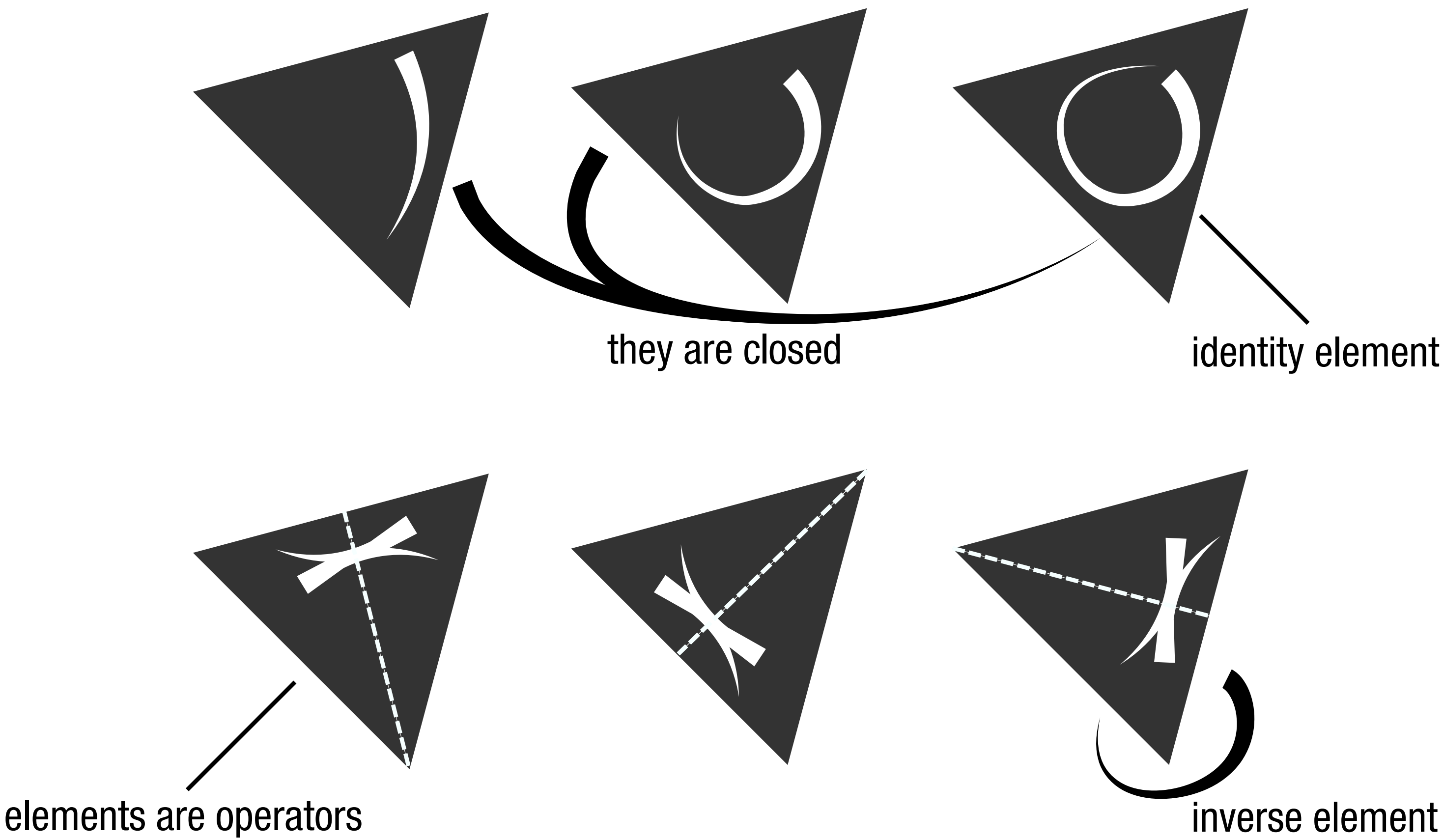
Symmetry is immunity to a possible change

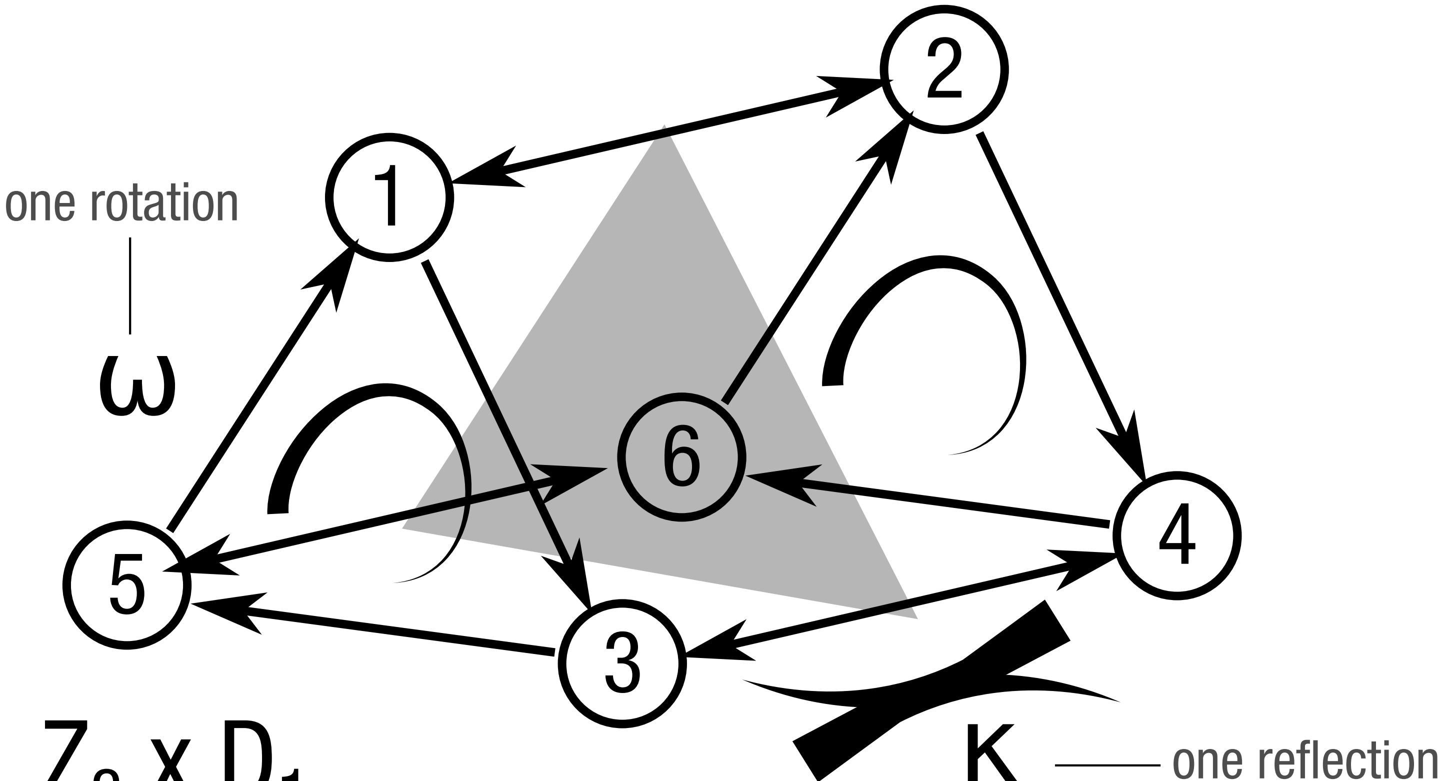
120° rotation   240° rotation   360° rotation

Cyclic ◄ Z_3

Symmetry operations form a group



Networks can have symmetries



$Z_3 \times D_1$

$G=\{e,\omega,\omega^2,\kappa,\kappa\omega,\kappa\omega^2\}$

Homorphism is a structure preserving mapping with a twist

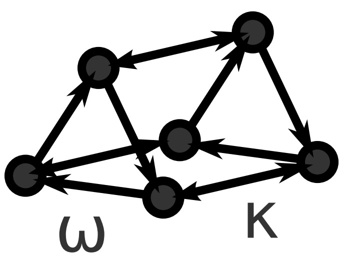
Ker(φ)

$$G = \{ \underline{e , \omega , \omega^2} , \underline{\kappa , \kappa\omega , \kappa\omega^2} \}$$

φ

$$G' = \{ \underline{e' , k'} , \ldots \}$$

Im(φ)



γ is a symmetry of a Dynamical System if it preserves all solutions to $\dot{x}(t) = f(x)$

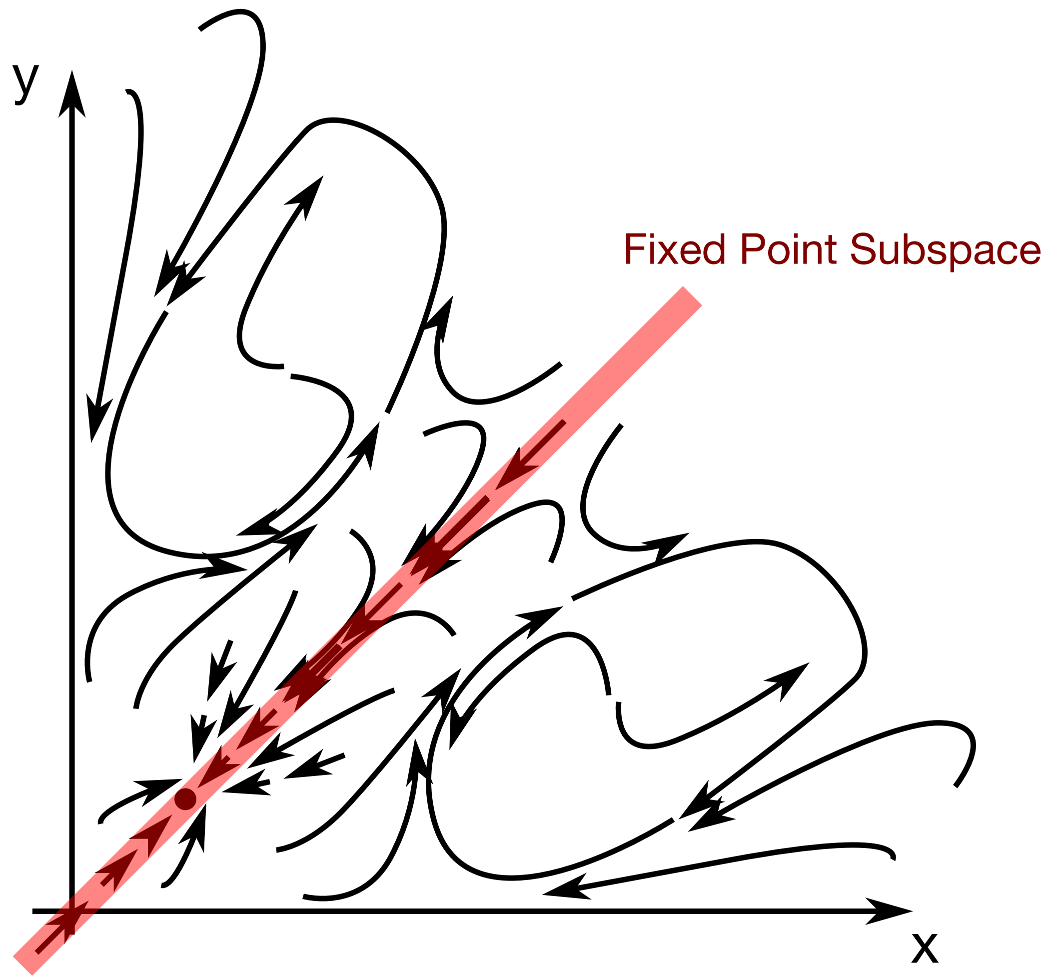
dynamical system

$$\dot{x}(t) = f(x)$$

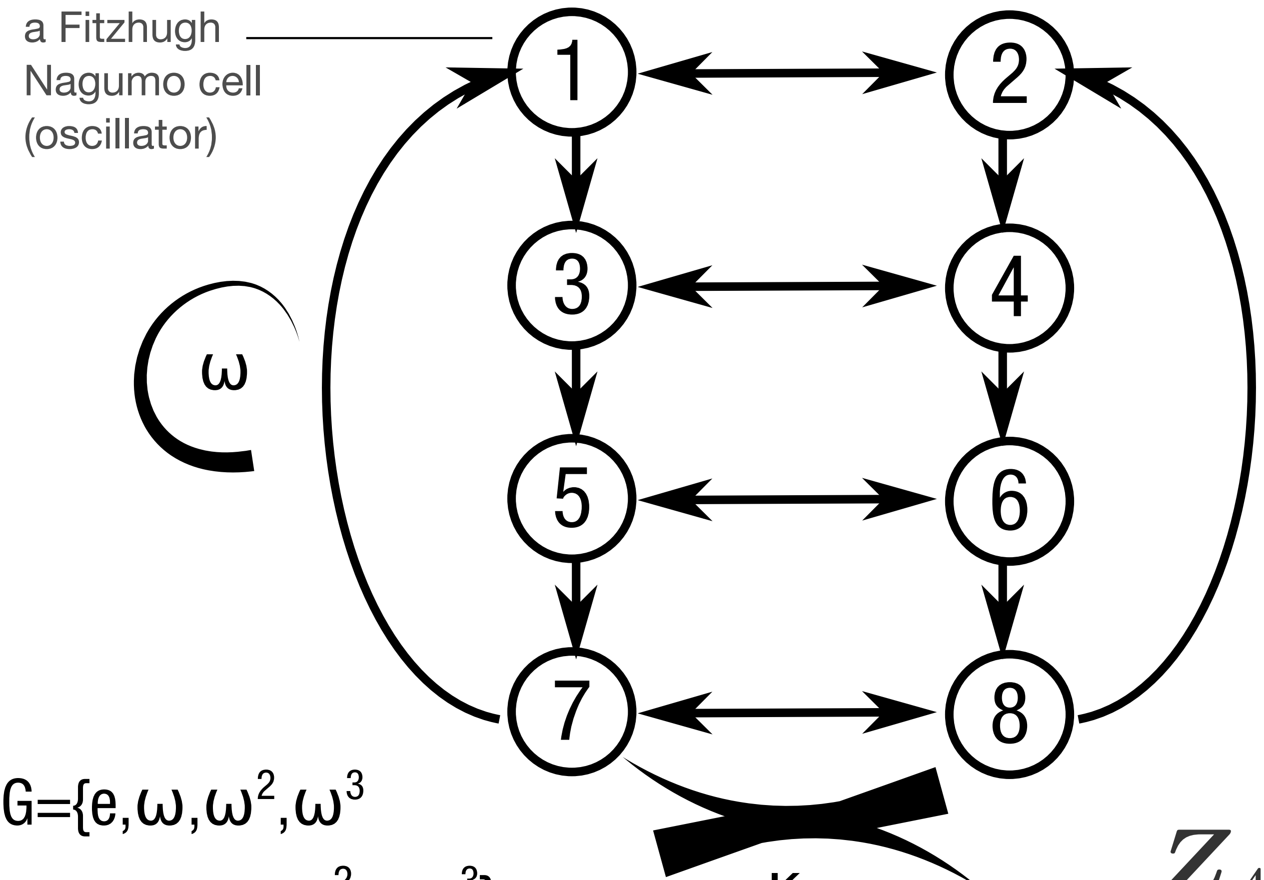
$$\gamma \frac{\partial}{\partial t} x(t) = \frac{\partial}{\partial t} \gamma x(t) = f(\gamma x)$$

equivariance condition

A symmetry group fixes a subspace of the state space of the system



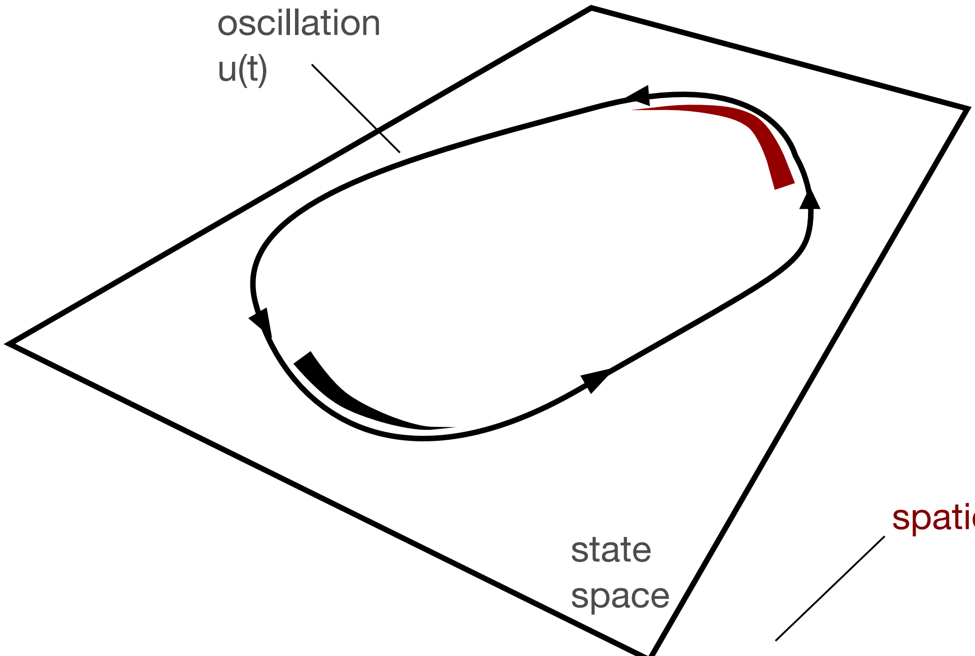
A neural network may have a symmetric coupling



$G = \{e, \omega, \omega^2, \omega^3$
$\kappa, \kappa\omega, \kappa\omega^2, \kappa\omega^3\}$

$$\boldsymbol{Z}_4 \times \boldsymbol{D}_1$$

A symmetric network of oscillators will have two kinds of symmetries



spatiotemporal symmetries

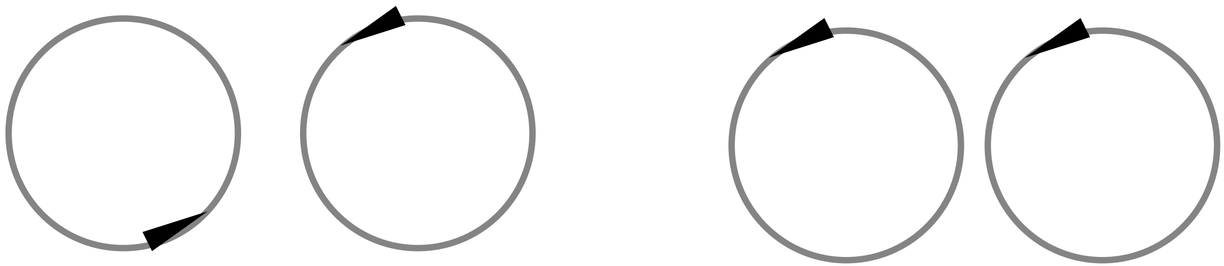
$$H = \{\gamma \in G : \gamma \cdot \{u(t)\} = \{u(t)\}\}$$

$$K = \{\gamma \in G : \gamma \cdot u(t) = u(t)\}$$

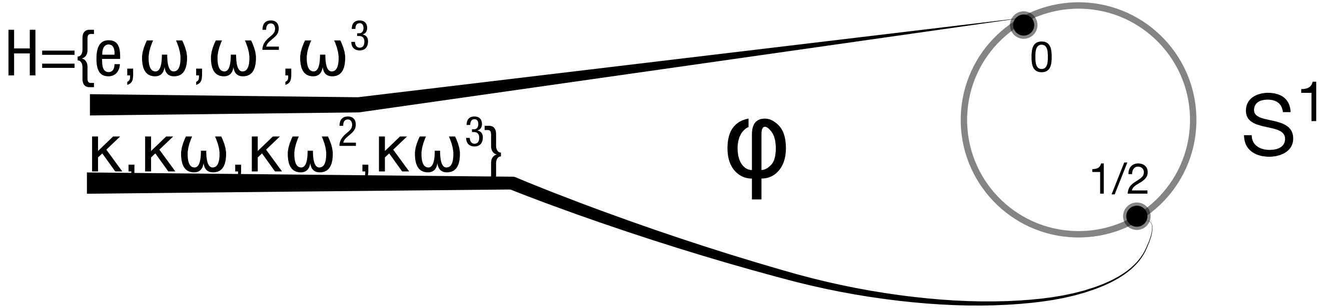
spatial symmetries

A phase difference between symetrically coupled oscillators arise from a homomorphism

$$H = \{\gamma \in G : \gamma \cdot \{u(t)\} = \{u(t)\}\}$$



$$K = \{\gamma \in G : \gamma \cdot u(t) = u(t)\}$$



First Isomorphism Theorem

The image of φ, Im(φ) is isomorphic to the closed quotient group H/K.

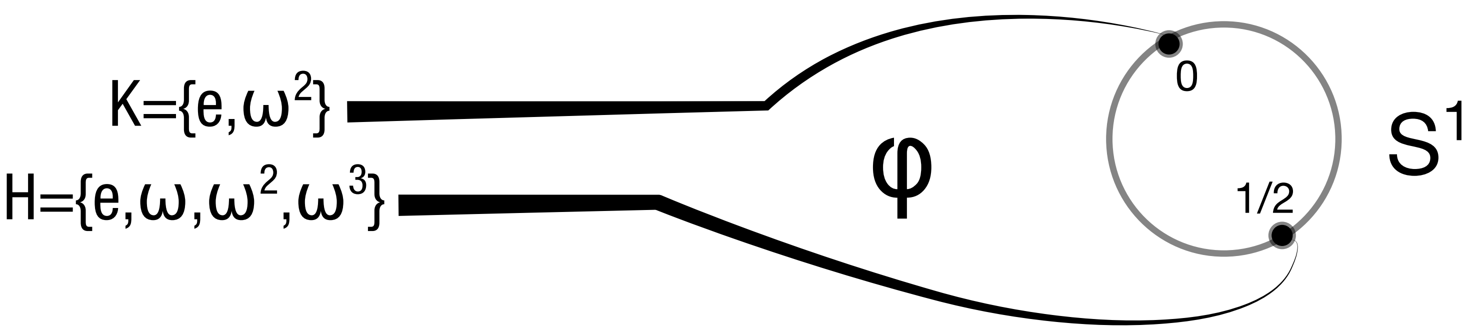
$$H=\{e,\omega,\omega^2,\omega^3\}$$

$$K=\{e,\omega^2\}$$

$$H / K=\{e.\{e,\omega^2\},\omega.\{e,\omega^2\},\omega^2.\{e,\omega^2\},\omega^3\{e,\omega^2\}\}$$

$$H / K=\{\{e,\omega^2\},\{\omega,\omega^3\},\{\omega^2,e\},\{\omega^3,\omega\}\}$$

$$=\{\{e,\omega 2\},\{\omega,\omega 3\}\} \cong \mathbf{Z}_2$$



Numerical simulations also confirm the presence of such solutions

$H=\{e,\omega,\omega^2,\omega^3\}$ $K=\{e,\omega^2\}$