Naveen Jindal School of Management

UT DALLAS

The University of Texas at Dallas

LBOE

Game Theory

Gary E. Bolton

Spring 2013

Bolton studies decision making

- Decision making
  - Individual
    Ex., Handling uncertainty
  - Interactive (Games)
    - Autonomous
      Ex., Trust & reciprocity
    - Joint
      Negotiating

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A gift for you!

Becker, DeGroot and Marschak (1964) trading mechanism

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What to write on your index card:

•Your name

•If you own a pencil, write down how much you would be willing to sell it for.

•If you don't own a pencil, write down how much you would be willing to pay.

•**DON"T SHOW ANYBODY YOUR ANSWER!**

Later, a random price will be determined using rand in Excel (price between $0.01 and $1.00). Each owner will be match with a non-owner who trade only if the later's price is higher than the former's.

•

You have a dominant strategy in this market

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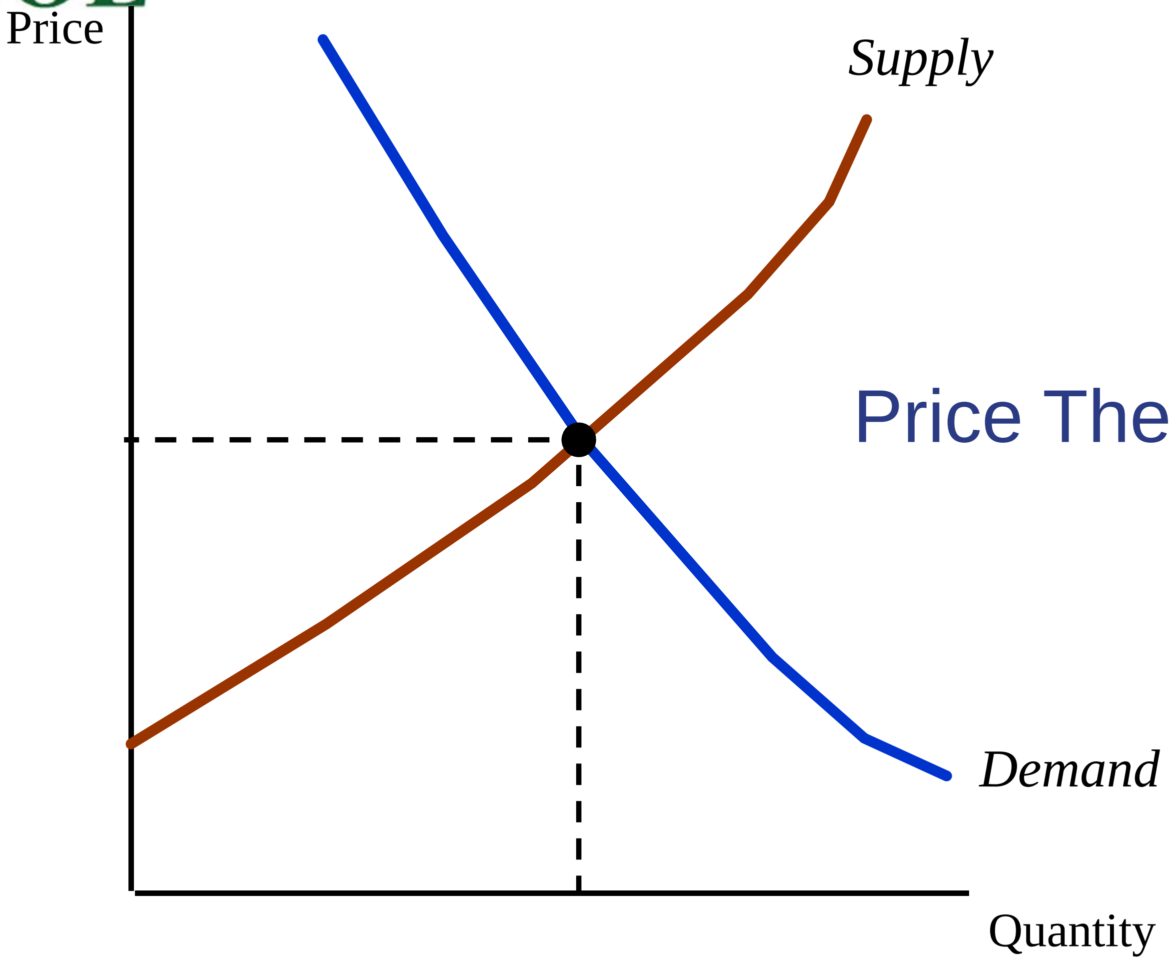
- Observe that the price is chosen randomly, fully independent of anyone's state willingness-to-pay or willingness-to sell.

  You should therefore write down the TRUE value you put on the pencil.

- We call this a dominant strategy because its optimality is independent what others in the market do.

- **Any Questions?**

Classical Economics



Price Theory

When do supply and demand curves predict the market outcome?

The standard price theory response:

- Many "small" buyers and sellers.
- Firms produce homogenous (identical) products.
- Traders have perfect information about price and quality offered by others.
- There are no transaction costs.
- There is free entry and exit.

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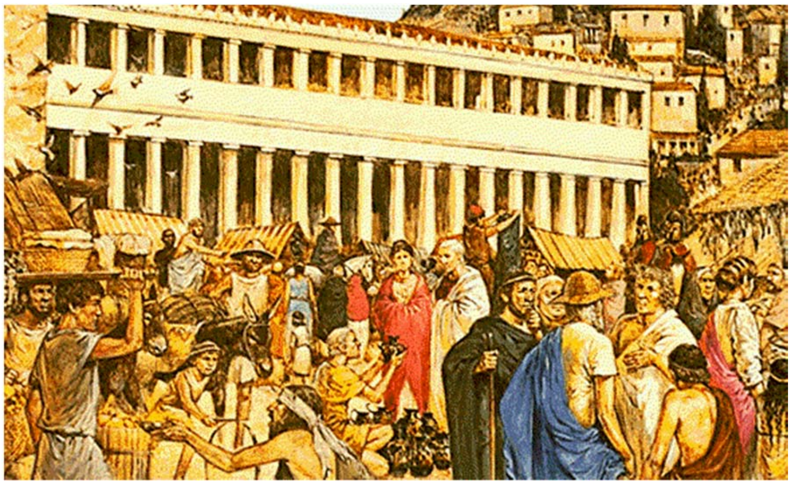
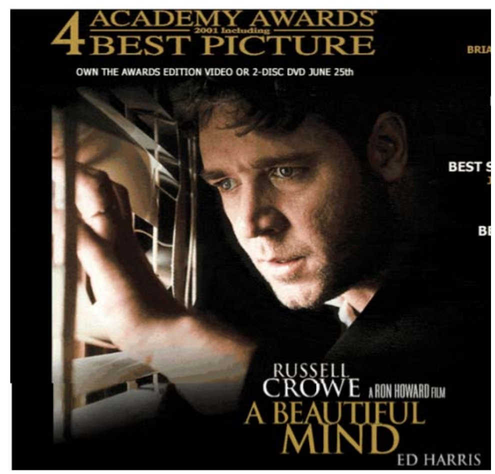
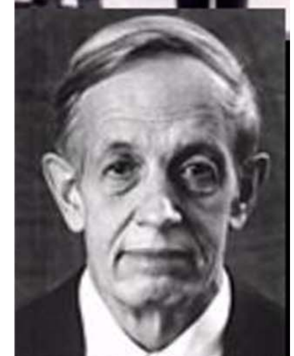
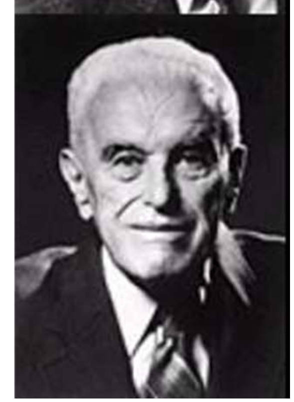
Strategic Interaction



Game
Theory







CHAPTER I

FORMULATION OF THE ECONOMIC PROBLEM

1. The Mathematical Method in Economics

1.1. Introductory Remarks

1.1.1. The purpose of this book is to present a discussion of some fundamental questions of economic theory which require a treatment different from that which they have found thus far in the literature. The analysis is concerned with some basic problems arising from a study of economic behavior which have been the center of attention of economists for a long time. They have their origin in the attempts to find an exact description of the endeavor of the individual to obtain a maximum of utility, or, in the case of the entrepreneur, a maximum of profit. It is well known what considerable—and in fact unsurmounted—difficulties this task involves given even a limited number of typical situations, as, for example, in the case of the exchange of goods, direct or indirect, between two or more persons, of bilateral monopoly, of duopoly, of oligopoly, and of free competition. It will be made clear that the structure of these problems, familiar to every student of economics, is in many respects quite different from the way in which they are conceived at the present time. It will appear, furthermore, that their exact positing and subsequent solution can only be achieved with the aid of mathematical methods which diverge considerably from the techniques applied by older or by contemporary mathematical economists.

1.1.2. Our considerations will lead to the application of the mathematical theory of "games of strategy" developed by one of us in several successive stages in 1928 and 1940–1941.[1] After the presentation of this theory, its application to economic problems in the sense indicated above will be undertaken. It will appear that it provides a new approach to a number of economic questions as yet unsettled.

We shall first have to find in which way this theory of games can be brought into relationship with economic theory, and what their common elements are. This can be done best by stating briefly the nature of some fundamental economic problems so that the common elements will be seen clearly. It will then become apparent that there is not only nothing artificial in establishing this relationship but that on the contrary this

[1] The first phases of this work were published: *J. von Neumann*, "Zur Theorie der Gesellschaftsspiele," Math. Annalen, vol. 100 (1928), pp. 295–320. The subsequent completion of the theory, as well as the more detailed elaboration of the considerations of loc. cit. above, are published here for the first time.

1

THEORY OF GAMES AND ECONOMIC BEHAVIOR

By JOHN VON NEUMANN, and

OSKAR MORGENSTERN

PRINCETON

PRINCETON UNIVERSITY PRESS

1953

on Neumann, Math. Ann., 100, 295-320,

uman and Oscar Morgenstern, 1944.

From the introduction of vN-M's book explaining their approach

- A need for greater clarity

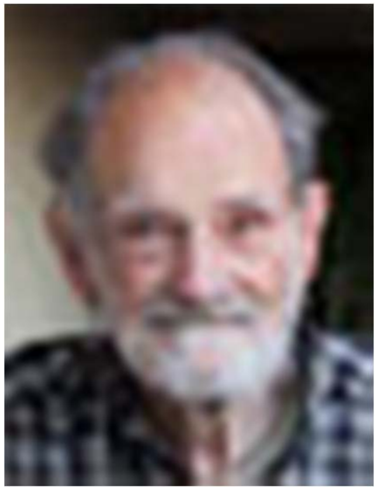
"[T]he economic problems were not formulated clearly and are often stated in such vague terms…"

- A need to start small and work your way up

"The free fall is a very trivial physical phenomenon, but it was the study of the exceedingly simple fact … which brought forth [Newton's] mechanics.

A recent Nobel in Economics went to two game theorists

The prize went for an idea that Shapley published in 1962 and Roth followed up on beginning in the early 1980's. (Picture at time they won: Shapley is 92, Roth is 60.)



Allocation Games

Lloyd S. Shapley

Professor Emeritus, UCLA

Game Theory

The Deferred Acceptance Algorithm for Stabilized Dating

Boys' Preferences			
Adam	**Bob**	**Charlie**	**Don**
Mary	Jane	Mary	Mary
Jane	Mary	Kate	Kate
Kate	Kate	Jane	Jane

Girls' Preferences		
Mary	**Jane**	**Kate**
Adam	Adam	Don
Bob	Charlie	Charlie
Charlie	Don	Bob
Don	Bob	Adam

	Day 1	Day 2	Day 3	Day 4	Day 5
Mary	**Adam** (Charlie & Don rejected)	**Adam** (no new proposal)	**Adam** (no new proposal)	**Adam** (Bob rejected)	**Adam** (no new proposal)
Kate	No proposal	**Don** (Charlie rejected)	**Don** (no new proposal)	**Don** (no new proposal)	**Don** (Bob rejected)
Jane	**Bob**	Bob (no new proposal)	**Charlie** (Bob rejected)	**Charlie** (no new proposal)	**Charlie** (no new proposal)

The process ends when all unattached boys have been rejected by all the girls.

The Deferred Acceptance Algorithm for Stabilized Dating

There are mathematical proofs of these theorems:

•The algorithm ends in a finite number of steps, resulting in a stable solution.

•The solution is the best possible outcome for the group proposing (that is, the boys). If the girls proposed, the result would be the best for the girls, but probably not the same result.

•This works for any different number of boys and girls.

The Theory and Practice of Market Design
(work in progress)

Nobel Lecture

December 8, 2012

Redesign of the resident match: Growing problems with couples, etc.

- Increasing percentage of women docs, starting in 1970’s
- Some defection of couples
  - Iron law of marriage: you can’t be happier than your spouse
- Various attempts made to deal with this, including finally allowing couples to state preferences over pairs of positions
- But stable matching with couples is still a hard problem: deferred acceptance algorithm won’t work, and a stable matching might not even exist
- Roth Peranson algorithm…’95 (‘99)
- Recent work on markets for doctors later in their career, e.g. gastroenterologists (Niederle, Proctor, Roth…)

Stable Clearinghouses (blue -> Roth Peranson Algorithm)

NRMP / SMS:

Medical Residencies in the U.S. (NRMP) (1952)
Abdominal Transplant Surgery (2005)
Child & Adolescent Psychiatry (1995)
Colon & Rectal Surgery (1984)
Combined Musculoskeletal Matching Program (CMMP)

- Hand Surgery (1990)

Medical Specialties Matching Program (MSMP)

- Cardiovascular Disease (1986)
- **Gastroenterology (1986-1999; rejoined in 2006)**
- Hematology (2006)
- Hematology/Oncology (2006)
- Infectious Disease (1986-1990; rejoined in 1994)
- Oncology (2006)
- Pulmonary and Critical Medicine (1986)
- Rheumatology (2005)

Minimally Invasive and Gastrointestinal Surgery (2003)
Obstetrics/Gynecology

- Reproductive Endocrinology (1991)
- Gynecologic Oncology (1993)
- Maternal-Fetal Medicine (1994)
- Female Pelvic Medicine & Reconstructive Surgery (2001)

Ophthalmic Plastic & Reconstructive Surgery (1991)
Pediatric Cardiology (1999)
Pediatric Critical Care Medicine (2000)
Pediatric Emergency Medicine (1994)
Pediatric Hematology/Oncology (2001)
Pediatric Rheumatology (2004)
Pediatric Surgery (1992)

Primary Care Sports Medicine (1994)
Radiology

- Interventional Radiology (2002)
- Neuroradiology (2001)
- Pediatric Radiology (2003)

Surgical Critical Care (2004)
Thoracic Surgery (1988)
Vascular Surgery (1988)

Postdoctoral Dental Residencies in the United States

- Oral and Maxillofacial Surgery (1985)
- General Practice Residency (1986)
- Advanced Education in General Dentistry (1986)
- Pediatric Dentistry (1989)
- Orthodontics (1996)

Psychology Internships in the U.S. and CA (1999)

Neuropsychology Residencies in the U.S. & CA (2001)
Osteopathic Internships in the U.S. (before 1995)
Pharmacy Practice Residencies in the U.S. (1994)
Articling Positions with Law Firms in Alberta, CA(1993)
Medical Residencies in CA (CaRMS) (before 1970)

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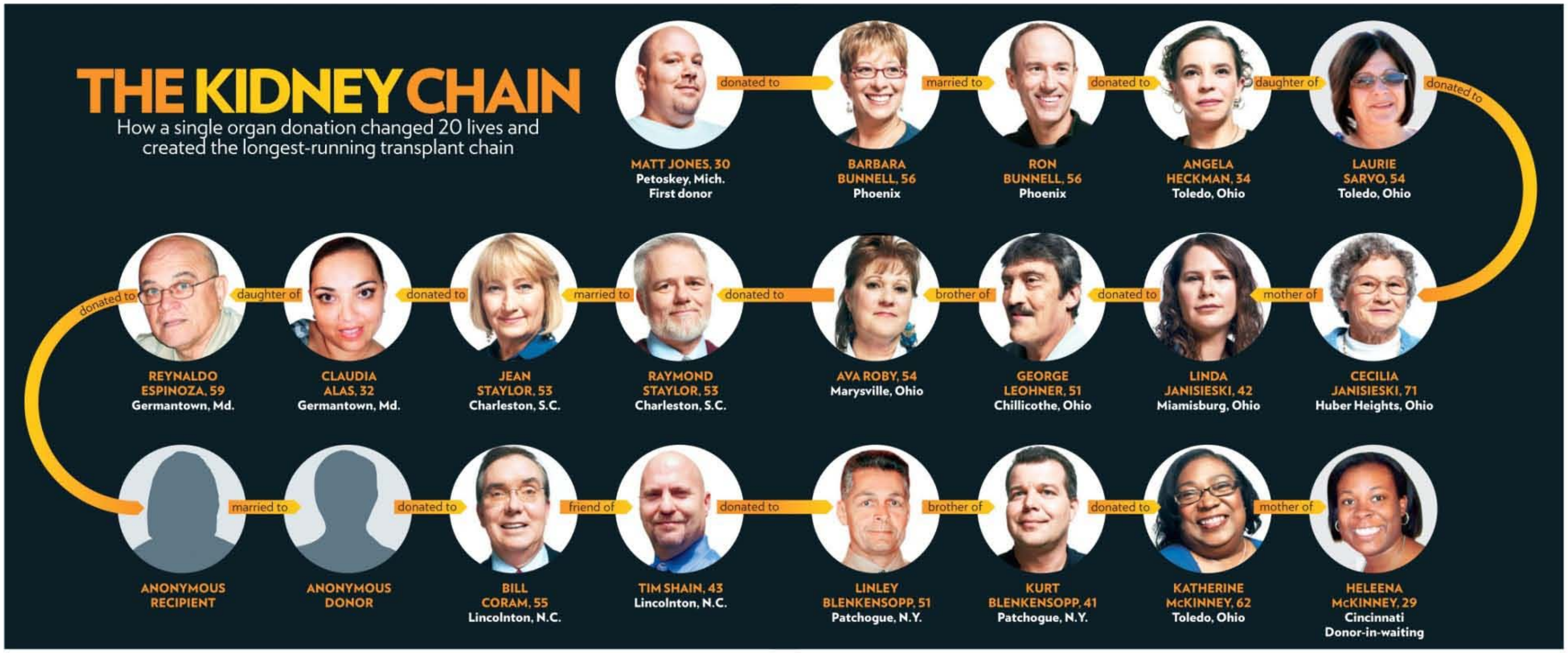
British (medical) house officer positions

- Edinburgh (1969)
- Cardiff (197x)

New York City High Schools (2003)
Boston Public Schools (2006)
Denver, Wasington DC (2012)

School choice

- Initially, NYC high schools (2003)
  - Abdulkadiroglu, Pathak and Roth
  - Two-sided matching—perhaps this is the application closest to what Gale and Shapley ‘62 might have imagined.
- Then Boston Public Schools (2004)
  - One sided allocation problem—schools aren’t strategic players (Abdulkadiroglu and Sonmez; Abdulkadiroglu, Pathak, Roth and Sonmez)
- Lately Denver and New Orleans (2012)
- (with Abdulkadiroglu, Pathak, Neil Dorosin and many other education professionals)
- Initially deferred acceptance
  - But with many indifferences, leading to lots of new questions and new theory
- Also top trading cycles—in New Orleans




Dr. Mike Rees (center, left) and his team perform a kidney transplant.

A recent Nobel in Economics went to two game theorists

This research trajectory, of starting with insights into a simple, abstract problem and then building up to solve complicated practical problems is exactly what von Neumann and Morganstern had in mind.

The wide range of applications for game theory

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Economics:
Oligopoly theory, strategic trade policy, control of money supply by central banks,...

Finance:
Design of tax systems, appropriation of public goods, IPO pricing ...

Marketing:
Sales force management, negotiation ...

Accounting:
Optimal contracts, auditing behavior...

Supply chain:
Auction procurement, supply chain coordinating contracts

Management decision making:
Strategic management, decision theory, personal leadership, . . .

Game theory is used in psychology, biology, philosophy, sociology, mathematics, law, theology, physics, . . .

So let's get started

- Some of these slides are mine.
- Some are Al Roth's
- Some are Axel Ockenfel's

It's all in the (academic) family

Game theory: What is it? Answer 1

"Classical" definition

Game theory is a set of analytical tools for studying strategic interactions between **rational** decision makers.

So about those pencils…

If decision makers are rational, how do we expect buyers' and sellers' stated willingness-to-pay and willingness-to-sell to differ?

That is, what's the rational model and what does it predict about behavior?

What constitute rational behavior

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A simple formulation:

Goal oriented, maximizing behavior in which choices can be represented by *preferences*.

Individual i's choices on sets of alternatives can be represented by a preference relation R_i (>) (with strict component P_i and indifference relation I_i) such that i's choice from a set A can always be represented by

$$[\ C_i(A) = \{x \text{ in } A \mid x\ R_i\ y \text{ for all } y \text{ in } A\}]$$

The **endowment effect**

- The tendency to appraise value more highly if you own the object.

- Endowment effects may have large pracitical consequences:
  - o Madrian and Shea (QJE 2001) found that employee participation in a large company's retirement plan went from 25% when employees needed to opt in to 61% when participation was the default option.

•

•

The **endowment effect**

- There is presently an effort to incorporate the endowment effect as a consequence of a more boundedly rational model of behavior:

- A specific case of a larger phenomenon known as Loss Aversion
  - The tendency people have to do more to prevent a loss than achieve a gain.
  - Koszegi and Rabin (2006,2007,2009) model this idea.

- We will say more about this approach next time.
-

Game theory: What is it? Answer 2

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Pehaps a more up-to-date definition

Game theory is a set of analytical tools for studying strategic interactions between decision makers.

The investigation commonly begins with the assumption that the decision makers are fully rational. As evidence comes in, the models are updated or new game theory models are formulated.

Different models assume different levels of rationality

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Varieties of rationality we will study

- Goal oriented, maximizing behavior,
- **Ordinal utility maximization**
- **Expected utility maximization, (and its important special case, expected value maximization)**
- Subjective expected utility maximization (various levels of Bayesian belief formation).

Ordinal utility maximization

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It is analytically convenient to be able to express a decision makers preferences as a utility function:

Ordinal utility maximization: individual i's choices can be represented by a real valued utility function u_i

$$[\ C_i(A) = \{x \text{ in } A \mid u_i(x) > u_i(y) \text{ for all } y \text{ in } A\}]$$

Not every binary relation on a set A can be *represented* by a utility function [u(x) > u(y) iff xRy]. Two necessary conditions :

•1. complete: for each x,y in A either xRy or yRx (since either u(x) > u(y) or u(y) > u(x)).

•2. transitive: xRyRz implies xRz
(since u(x) >u(y) > u(z) implies u(x) > u(z)).

An intransitivity demo:

You like your tea with two spoons of sugar; you are presented with 101 cups of tea ordered from zero to two spoons of sugar in increments too small to taste the difference. So, you are indifferent between any two adjacent cups. Transitivity implies you should be indifferent between any two cups, but you are not…☹

Ordinal utility maximization

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Theorem: If A is finite or countable, then completeness and transitivity are *sufficient* as well as necessary conditions for a preference to be representable by a utility function.

Lexicographic preference question

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Consider lexicographic preferences over bundles of butter (B) and margarine (M):

(B_1,M_1) P (B_2,M_2) if and only if

$$B_1 > B_2$$

or $\quad B_1 = B_2$ and $M_1 > M_2$

Restrict attention to bundles in which the smallest unit of measure is 1 ounce.

Utility function?

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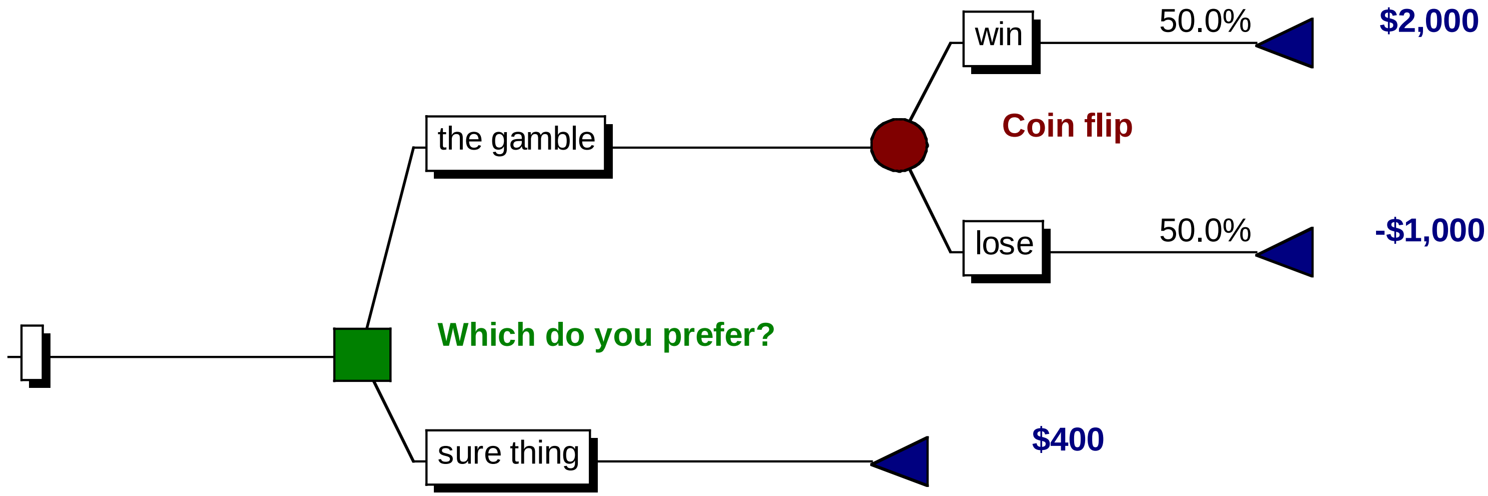
Utility functions – so far

- Represent rational (complete and transitive) preferences
- Exist for a wide class of rational preferences but not for all
- Are ordinal, so that if there is one utility representation there are an infinite number
- By themselves are limited in an important way: Do not admit preferences over gambles (risk)

Expect Utility Theory

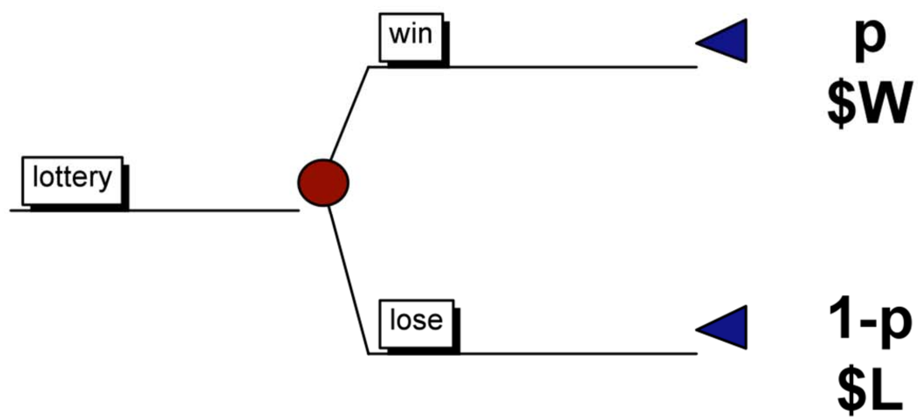
Survey: Suppose you have two choices…



Would you take the sure thing or the gamble?

Expected Value Definition

Consider an uncertain event with monetary consequences (i.e., a *lottery or gamble*)



Define

Expected Value (EV) = pW + (1 - p)L

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- For a long time in decision theory, the expected payoff was used as the value for an uncertain payoff. But Bernoulli noted the following problems with this practice:

- *St. Peterburg Paradox* (Bernoulli 1728)

  A fair coin is tossed until a heads comes up. You get $1 when it lands on heads the first time, $2 when it lands on heads the second time, $4 when it takes three tosses, $8 when it takes four tosses. Name the greatest certain amount that you pay to play this game once.

- The expected value of this bet is (1/2)1 + (1/4)*2 + (1/8)*4 + ..., ad infinitum.

- Bernoulli proposed a "utility function" with diminishing marginal utility so that the sums converge.

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- However, for a long time the St. Petersburg Paradox did not greatly change theories on decision making ander uncertainty.
- Von Neumann and Morgenstern axiomatized expected utility in 1944.
- It should not be forgotten that the maximization of expected payoffs can be a good *approximation* for many decision situations.

"The objection could be raised that it is not necessary to go into all these intricate details concerning the measurability of utility, since evidently the common individual, whose behavior one wants to describe, does not measure his utilities exactly but rather conducts his economic activities in a sphere of considerable haziness. The same is true, of course, for much of his conduct regarding light, heat, muscular effort, etc. But in order to build a science of physics these phenomena had to be measured. …"

Von Neumann & Morgenstern

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If a preference relation R is defined on M (with strict preference P and indifference I), then an *expected utility function* u:M$\rightarrow$R is a function u such that

(i) u(a) $\geq$ u(b) if and only if aRb; and

(ii) u[pa;(1-p)b] = pu(a)+(1-p)u(b)

von Neumann-Morgenstern ‘s (1944) Theorem

Axioms: Let x, y and z be lotteries. (to assign certain probabilities to certain payoffs). The following conditions apply to the preferences for the lottery:

- *Order*
  The preferences are exhaustive (either xPy or yPx or xIy) and transitive (if xPy and yPz, then xPz).
- *Archimedian*
  For all xPyPz there exists a definite probability p so that [px;(1-p)z]Iy.
- *Independence*
  If xPy, then [px;(1-p)z]P[py;(1-p)z] for all values of z and all p when $0 < p < 1$.

Both independence and consistency axioms have behavioral content

Many people's intuition is that the independence condition is natural, and probably describes their behavior pretty well, but that the archimedian condition might be hard to take: e.g. suppose

- x = finding a $20 bill as you leave class
- y = leaving class today as you expected
- z = getting boiled in oil.

The Allais "paradox" (Allais, Econometrica, 1953)

Choice 1: choose one of the two lotteries P or Q, where Q gives you $500,000 for certain, and P is the lottery [.89$500,000; .10$2,500,000; .01$0]

Choice 2: choose one of the two lotteries R or S, where R is the lottery [.11$500,000; .89$0] and S is the lottery [.10$2,500,000; .90$0].

Allais observed that many people prefer Q to P in choice 1, and S to R in choice 2.

However it is easy to confirm that no expected utility maximizer can simultaneously hold both opinions. That is preferring Q to P implies u($500,000) > .89u($500,000) + .1u($2,500,000) + .01u($0), which reduces to .11u($500,000) > .1u($2,500,000) + .89u($0), while preferring S to R implies the reverse inequality.

Some of the modifications of utility theory designed to account for this kind of phenomenon focus on the fact that it can be viewed as a violation of the independence condition which we assumed was satisfied by an individual's preferences. That is, if we define the lottery

T = [(10/11)$2,500,000; (1/11)$0], then

Q= $500,000 = [.11Q; .89Q]

P = [.89$500,000; .10$2,500,000; .01$0] = [.11T; .89Q]

R = [.11$500,000; .89$0] = [.11Q; .89$0]

S = [.10$2,500,000; .90$0] = [.11T; .89$0].

So if an individual whose preferences obey the independence condition prefers Q to P then he prefers Q to T, which implies he must prefer R to S.

Neumann and Morgenstern's (1944) Theorem

- The three axioms imply that the preferences can be represented in a numerical utility function in which the utility of a lottery is the expected utility (EU) of the possible results.
- The utility function is unique up to linear affine transformations.
- For a discrete lottery with result x, each having a p,- probability, the utility of the lottery can be represented with an „appropriate“ function *u* through $EU = \Sigma\, p_i\, u(x_i\,)$.
- The expected utility is generally not identical to the expected payoff $\Sigma\, p_i\, x_i$.

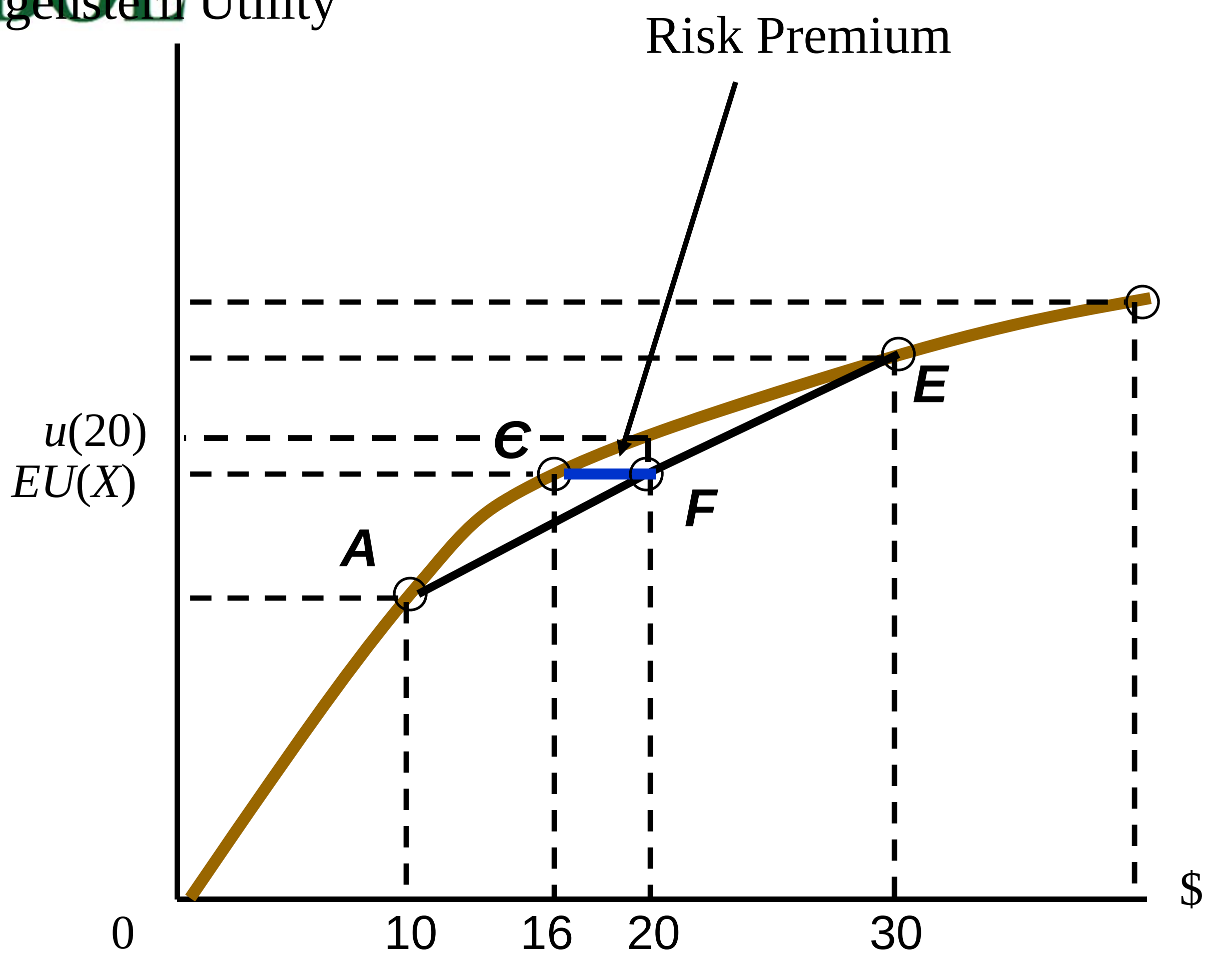
Risk postures

- If u() is concave, we call this *risk aversion.*
  - If u() is convex, risk seeking; if u() linear, risk neutral
- The amount a person truly values a gamble at is called the *certainty equivalent* (CE).

  EMV minus CE = the *risk premium.*
- “Risk seeking”, preferring the gamble to the EMV implies a negative risk premiums.

Example

Neumann-
Morgenstern Utility



Lottery X, which pays off $30 or $10 with a probability of ½, has an expected value of $20.

Because the utility of the expected value of u(20) is greater than the expected value of the lottery EU(X)=1/2u(10) + 1/2*u*(30), the person is risk averse.

The person would pay an sum of $4 in advance to avoid the risk of the lottery (risk premium).

Example.Constant absolute risk aversion

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- $U(x) = 1 - e^{-x/R}$ where R is a constant

- For R=5000

50-50 Gamble		Expected Value (EV)		Certainty Equivalent (CE)		Risk Premium (EV-CE)	Relative Prem. 1-(CE/EV)
(\$2500) 5000		\$1,250		\$0		\$1,250	1.00
0 7500		3750		2500		1250	0.33
2500 10000		6250		5000		1250	0.20
5000 12500		8750		7500		1250	0.14

Example. Decreasing absolute risk aversion

- U(x) = ln(x + A) for x > -A
- For A=15000

50-50 Gamble	Expected Value (EV)	Certainty Equivalent (CE)	Risk Premium (EV-CE)	Relative Prem. 1-(CE/EV)
($10000) 0	($2,500)	($6,339)	$1,339	-1.54
0 10000	5000	4365	635	0.13
10000 20000	15000	14580	420	0.03
20000 30000	25000	24686	314	0.01

Holt and Laury (2002)

of its functional form. We are able to compare behavior under real and hypothetical incentives, for lotteries that range from several dollars up to several hundred dollars. The wide range of payoffs allows us to specify and estimate a hybrid utility function that permits both the type of increasing relative risk aversion reported by Binswanger and decreasing absolute risk aversion needed to avoid "absurd" predictions for the high-payoff treatments. The procedures are ex-

Holt and Laury (2002)

TABLE 1—THE TEN PAIRED LOTTERY-CHOICE DECISIONS WITH LOW PAYOFFS

ion A	Option B	Expected p differen
of $2.00, 9/10 of $1.60	1/10 of $3.85, 9/10 of $0.10	$1.17
of $2.00, 8/10 of $1.60	2/10 of $3.85, 8/10 of $0.10	$0.83
of $2.00, 7/10 of $1.60	3/10 of $3.85, 7/10 of $0.10	$0.50
of $2.00, 6/10 of $1.60	4/10 of $3.85, 6/10 of $0.10	$0.16
of $2.00, 5/10 of $1.60	5/10 of $3.85, 5/10 of $0.10	−$0.18
of $2.00, 4/10 of $1.60	6/10 of $3.85, 4/10 of $0.10	−$0.51
of $2.00, 3/10 of $1.60	7/10 of $3.85, 3/10 of $0.10	−$0.85
of $2.00, 2/10 of $1.60	8/10 of $3.85, 2/10 of $0.10	−$1.18
of $2.00, 1/10 of $1.60	9/10 of $3.85, 1/10 of $0.10	−$1.52
0 of $2.00, 0/10 of $1.60	10/10 of $3.85, 0/10 of $0.10	−$1.85

Holt and Laury (2002) show evidence for a hybrid expected utility function

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aversion model. This can be done with a hybrid "power-expo" function (Atanu Saha, 1993) that includes constant relative risk aversion and constant absolute risk aversion as special cases:

$$(2) \qquad U(x) = \frac{1 - \exp(-\alpha x^{1-r})}{\alpha},$$

Actuarially fair insurance

- $u(\,)$ is strictly concave
- w is income without a loss
- Loss L occurs with probability π
- \$1 of insurance costs p

How much insurance will this person purchase?

Insuring a loss L for \$$A$ costs pA.

Expected payout on loss is πA, so actuarially fair insurance costs $p = \pi$.

Comments

- Neumann and Morgenstern ʻs Expected Utility Theory is the dominant decision theory of economics.
- The axioms „define“ rationality.
- Early on, this theory received a good deal of criticism. Next we ʻll look at a classic example often cited by critics of this theory. Then, I ʻll introduce a few new problems with the EU theory.
- This area of study is very broad (and actually deserves its own course on „limited rationality“), so only brief insight can be given here.
- More on this topic can be found in the article by Colin Camerer in the *Handbook of Experimental Economics*.

Game Theory

Gary Bolton
Spring 2013)

1

Intro to non-cooperative game theory

Princess Bride

2

1. Static Games with Complete Information

Definition 1.1 [Normal form]

For every player $i = 1, \ldots , n$ of an n-person game,the normal form of that game specifies

• The strategy set S_i

• As well as the payoff function $u_i(s)$, $s = (s_1, \ldots, s_n)$, $s_i \in S_i$ for all values of i.

The game is designated as $G = \{S;u\}$, where $S = (S_1, \ldots, S_n)$ and $u = (u_1, \ldots, u_n)$.

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1. Static Games with Complete Information

• Static: ‘simultaneous‘ actions, no repetitions or sequential structure.

• Complete Information: strategy sets and payoff functions are *common knowledge*.

•Payoffs = von Neumann-Morgenstern utility

• Looking for: Solution concept

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1. Static Games with Complete Information

• The ‘repeated elimination of strictly dominated strategies’ leads to a simple pair of strategies: (*o*,*m*).

• Assumption: Rationality is ‘common knowledge‘: All players are rational; all players know that all players are rational; all players know that all players know that all players are rational... (Aumann 1976).

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1. Static Games with Complete Information

Digression: Common Knowledge

Something is common knowledge when not only

You and I know it, but also

I know that you know it and you know that I know it, and

I know that you know that I know it and you know that I know that you know it, and . . .

Every sentence in this form, however long, is true.

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1. Static Games with Complete Information

Nobel Prize 2005 for Robert Aumann.

Excerpt from the Statement:

Another of Aumann's fundamental contributions concerns the cognitive foundations of game theory, i.e., the implications of the parties' knowledge about the various aspects of the game, including "knowledge about each others' knowledge". In the early days of game theory, analysis was often simplified by assuming that the parties know everything about all aspects of the game, in analogy, e.g., to physics, where friction or air resistance are sometimes disregarded. Knowledge that another party is rational can affect one's own behavior, as will knowledge about someone else's knowledge about one's own rationality, and so on. Aumann's formalization of the concept of common knowledge allowed for systematic analysis of the relation between the knowledge of the parties and the outcome of the game.

.

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1. Static Games with Complete Information

Informally: Something is common knowledge, when it can be seen publicly by everyone - I.e. I see it, you see it, I see that you see it, etc. ad infinitum.

Common knowledge does not apply to every situation.

Nonetheless, it is sometimes the case that a bit of information that is common knowledge has completely different implications than a bit of information that everyone simply knows.

Example: Suppose 100 people are sitting in a circle, and every person wears either a red or a blue hat. No one knows the color of his own hat – but in fact all of the hats are red.

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1. Static Games with Complete Information

The master of ceremonies stands in the middle of the circle and loudly announces: “Every sixty seconds, I will ring a bell. At that point, if you know that you wear a red hat, please leave the room.”

It should be clear that no one leaves the room, because the bell ringing does not reveal any information.

But suppose that the master of ceremonies now says before his announcement that at least one person is wearing a red hat.

This is not new information, since everyone can see that at least 99 hats must be red. However, this announcement changes the situation...

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1. Static Games with Complete Information

After the bell has been rung 100 times, all should leave the room!

Why? Suppose that there are only two people in the room. You and I see that the other wears a red hat, but we can‘t see our own hat color. We know then that there must be at least one red hat. But only the announcement of the master of ceremonies makes this *common knowledge*.

This means that now we both know that we both know that the other knows.

So because you don‘t leave the room after the bell is rung once (which would happen if I wore a blue hat), I know that I wear a red hat. The same principle applies to you, so that on the second ringing we both leave the room.

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1. Static Games with Complete Information

In a 3 person situation, you and I see that the third person wears a red hat. We both also know that the third person knows that at least one person wears a red hat. But until the announcement, this is not common knowledge, i.e. you didn 't know that i know that he knows.

The first ring eliminates the possibility that there are two blue hats in the room, otherwise someone would have stood up. The question then becomes whether there is one blue hat in the room. If there is, then someone will stand up on the second ring since they can see the person wearing the blue hat. No one stands up on the second ring and so on the third ring everyone leaves the room.

... Induction for every group of *n* people is possible.

QED

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Common knowledge assumptions do not always apply
Sebenius, "Negotiation analysis," *Management Science* 1992

[T]he full set of actual and potential players, interests, beliefs, issues, alternatives to agreement, rules, and agreements, are often only imperfectly known, and even the character of what is known by one party is not known to others. Indeed, purposive action by involved or excluded parties can often change the set of involved players, bring in or exclude issues, raise or lower the salience of different interests, alter the "rules" of the interaction, or take other actions to change the collective perception of the "game's" configuration.

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1. Static Games with Complete Information

1.2 Dominated and Maximin Strategies

Definition 1.2 [Dominated Strategies]

Given: normal form game $G = \{S;u\}$.

The strategy $s'_i \in S_i$ is said to be strictly dominated, if $s''_i \in S_i$ exists so that $u_i(s'_i, s_{-i}) < u_i(s''_i, s_{-i})$ for all $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in S_{-i}$.

The strategy $s'_i \in S_i$ is said to be weakly dominated if $s''_i \in S_i$ exists so that $u_i(s'_i, s_{-i}) \leq u_i(s''_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

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1. Static Games with Complete Information

Example 1.1 [Prisoner's Dilemma]:

		Player 2: cooperate	Player 2: defect
Player 1	cooperate	3,3	1,4
Player 1	defect	4,1	2,2

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Many people think of negotiation as a game of split-or-steal

Split or steal?

Split-or-Steal is a dilemma game

		Player 2	
		Split	*Steal*
Player 1	*Split*	£ 50,000 / £ 50,000	£ - / £ 100,000
	Steal	£ 100,000 / £ -	£ - / £ -

Both sides have a nearly dominant strategy to Steal – and a big incentive to lie about it before the fact. But both playing steal is lose-lose.

1. Static Games with Complete Information

Example 1.1 [Prisoner's Dilemma]:

Player 1 \ Player 2	cooperate	defect
cooperate	3,3	1,4
defect	4,1	2,2

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1. Static Games with Complete Information

- Cooperation is a dominated strategy for both players.

- Rational players do not play strictly dominated strategies (dominance principle) because there is no belief about the behavior of the other player that would make the strategy optimal.

- Individually rational behavior can lead to Pareto inferior results (prisoner ‘s dilemma structure).

- Cooperaation does not occur, even though it would be advantageous for all parties involved!

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1. Static Games with Complete Information

Example 1.2 [Linear Public Goods]:

Payoff Function: $u_i(x) = w - x_i + m\Sigma^n_{j=1} x_j$
(Non-rivalry in consumption; no exclusion from consumption).

Strategy $x_i \in S_i = [0,w]$ (individual contribution to public good).

Number of players = n, each with endowment = w, $\Sigma^n_{j=1} x_j$ = public good, m = 'social' marginal utility of private contributions.

For $m < 1$ all strategies $x_i > 0$ are dominated, and for $m > 1/n$, rational behavior ($x_i = 0$) leads to a deficiency of the public good.

Examples: environmental goods, national defense, oligopolistic price fixing, advertisement blocking devices on TVs (TiVo, etc.)...

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1. Static Games with Complete Information

1.1 Normal Form Games

Experiment 1.1 [Number Selection Game]:

All ($n > 1$) players simultaneously select a (real) number z_i from an interval $[0,k]$.

The winner of the selection game is the player whose number comes closest to the goal value λz, where $0 < \lambda < 1$ and
$z = \Sigma^n_{j=1} z_j/n$.

The winner receives a fixed payoff u; all others receive nothing.

If there is more than one winner, the payoff will be divided equally between the winners.

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1. Static Games with Complete Information

Example 1.3 [Repeated Elimination Strictly Dominated Strategies]:

	l	*m*	*r*
o	**1,0**	**1,2**	**0,1**
u	**0,3**	**0,1**	**2,0**

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1. Static Games with Complete Information

Example 1.3 [Repeated Elimination of Strictly Dominated Strategies]:

	l	*m*	*r*
o	**1,0**	**1,2**	**0,1**
u	**0,3**	**0,1**	**2,0**

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1. Static Games with Complete Information

Example 1.3 [Repeated Elimination of Strictly Dominated Strategies]:

	l	*m*	~~*r*~~
o	1,0	1,2	~~0,1~~
~~*u*~~	~~0,3~~	~~0,1~~	~~2,0~~

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1. Static Games with Complete Information

Example 1.3 [Repeated Elimination of Strictly Dominated Strategies]:

	~~*l*~~	*m*	~~*r*~~
o	~~1,0~~	1,2	~~0,1~~
~~*u*~~	~~0,3~~	~~0,1~~	~~2,0~~

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1. Static Games with Complete Information

Example 1.4 [Pick a Number Example]:

- See Experiment 1.1.

- After (infinitely) repeated elimination of (also weakly) dominated strategies, one gets the definite solution: $z_i \equiv 0$.

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1. Static Games with Complete Information

Pick a Number Game, Beauty Contests, and Investment

„ . . . professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view. It is not a case of choosing those which, to the best of one's judgment, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practise the fourth, fifth and higher degrees."

[Keynes, 1936]

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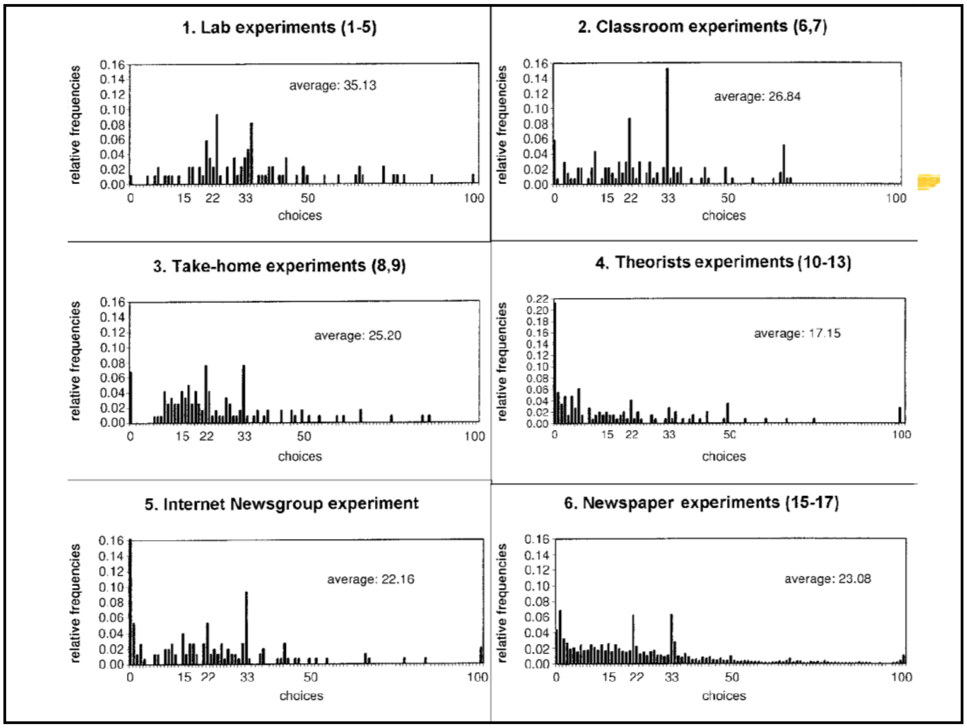

1. Static Games with Complete Information

Problem with the Solution Concept of "Eliminating Dominated Strategies"

• Many games cannot be solved through the dominance priciple.

• Rationality must be common knowledge through elimination.

• Caution with the (repeated) elimination of *weakly* dominated strategies. It is not necessarily irrational to play weakly dominated strategies (compare to the examples below).

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1. Static Games with Complete Information

Example 1.5 [Normal Form Game Without Dominated Strategies]:

(a)

	l	*m*	*r*
o	0,4	4,0	5,3
v	4,0	0,4	5,3
u	3,5	3,5	6,6

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1. Static Games with Complete Information

Example 1.5 [Normal Form Game Without Dominated Strategies]:

(b)

	l	*m*	*r*
o	0,0	6,6	2,2
v	6,6	1,1	0,2
u	2,2	2,0	8,8

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1. Static Games with Complete Information

For (a) and (b) no dominated strategies exist. Accordingly, the dominance principle does not generate any predictions for this game. The players must think about the behavior of the other players (strategic uncertainty).

The maximum solution implies such expectations in a very special way.

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1. Static Games with Complete Information

Definition 1.3 [Maximin Strategy von Neumann, and Morgenstern 1944]

The strategy that maximizes ones own minimum payoff.

The strategy $s'_i \in S_i$ is said to be the maximin strategy of Player i, if:

$\min u_i(s'_i, s_{-i}) \geq \min u_i(s_i, s_{-i})$ for all values of $s_i \in S_i$,

where above $s_{-i} \in S_{-i}$ is minimized, or:

s'_i solves: $\max [\min u_i(s_i, s_{-i})]$,

where max is over $s_i \in S_i$ min is over all s_{-i}.

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1. Static Games with Complete Information

Example 1.4a [Zero Sum Game]:

Player 1 \ Player 2	left	right
up	3,-3	1,-1
down	-2,2	0,0

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1. Static Games with Complete Information

Example 1.5 [Normal Form Game Without Dominated Strategies]:

(a)	*l*	*m*	*r*
o	0,4	4,0	5,3
v	4,0	0,4	5,3
u	3,5	3,5	6,6

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1. Static Games with Complete Information

Example 1.5 [Normal Form Game Without Dominated Strategies]:

(a)

	l	*m*	*r*
o	#0,4	4,0#	5,3#
v	4,0#	#0,4	5,3#
u	#3,5	#3,5	6,6

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1. Static Games with Complete Information

Example 1.5 [Normal Form Game Without Dominated Strategies]:

(b)

	l	*m*	*r*
o	0,0	6,6	2,2
v	6,6	8,8	0,2
u	2,2	2,0	1,1

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1. Static Games with Complete Information

Example 1.5 [Normal Form Game Without Dominated Strategies]:

(b)	*l*	*m*	*r*
o	${}_{\#}0,0^{\#}$	6,6	2,2
v	6,6	8,8	${}_{\#}0,2$
u	2,2	$2,0^{\#}$	${}_{\#}1,1^{\#}$

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1. Static Games with Complete Information

For Player 1 (2) the strategy *u* (*r*) is the maximin strategy in the examples 1.5(a) and (b).

Maximin strategies are not often based on rational expectations. For example, in Example 1.5(b), Player 1 knows that *r* can never be an optimal answer for a rational Player 2 who simply follows his/her own self interest.

Maximin strategies make sense primarily in zero-sum games. In a two-person zero-sum game: $u_i(s) = - u_j(s)$ (s. Bsp. 1.11). In these games, maximin strategies (taking mixed strategies into account) are also always a Nash equilibrium. However, many economic problems are not zero-sum games.

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1. Static Games with Complete Information

Excerpt from the statement granting the 2005 Nobel Prize to Schelling:

Thomas Schelling's book The Strategy of Conflict (1960) launched his vision of game theory as a unifying framework for the social sciences. Turning attention away from zero-sum games, such as chess, where players have diametrically opposed interests, he emphasized the fact that almost all multi-person decision problems contain a mixture of conflicting and common interests, and that the interplay between the two concerns could be effectively analyzed by means of non-cooperative game theory.

…

Bargaining always entails some conflict of interest in that each party usually seeks an agreement that is as favorable as possible. Yet, any agreement is better for both parties than no agreement at all. Each player has to balance the quest for a large "share of the pie" against the concern for agreement.

1. Static Games with Complete Information

For a long time, it was believed that money didn't just go unused, i.e. that rational agents are always able to maximize the „cake." The strategic problem therefore is a distribution problem –a zero-sum game (or more accurately, a constant sum game).

Cournot had shown that this is not so. But only later did game theory show how dramatically false this belief was in regards to asymmetrical information, cooperation problems, or external effects.

The concept of Nash equilibrium considers „rational" instead of „pessimistic" expectations. For example, (m,v) in Example 1.5(b) is a plausible (Nash) equilibrium.

1. Static Games with Complete Information

Example 1.5 [Normal Form Game Without Dominated Strategies]:

(b)	*l*	*m*	*r*
o	0,0	6,6*	*2,2
v	*6,6	*8,8*	0,2
u	2,2*	2,0	1,1

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1. Static Games with Complete Information

1.3 Nash Equilibrium

In Nash equilibrium, every player's strategy is a best response to the other players' strategies.

Definition 1.4 [Nash Equilibrium in Pure Strategies]

In the *n*-Person normal form game $G = \{S;u\}$ the strategy profile s^* constitutes a Nash equilibrium, if for every player *i* the Strategy s_i^* is the best response to the other player 's strategy s_{-i}^*, i.e. if:

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \text{ for all values of } s_i \in S_i \text{ and } i = 1, \ldots, n$$

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1. Static Games with Complete Information

- The Nash equilibrium describes “simultaneous best responses,” and in this sense a stable situation.

- When the Nash equilibrium is unique, it is the logical consequence if the game G and rationality are common knowledge.

- It can also sometimes be interpreted as a stable point of rest in a dynamic or evolutionary adjustment process in which only comparatively limited rationality assumptions are necessary.

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1. Static Games with Complete Information

- If the goal is to obtain a recommendation rather than a prediction, then a non-Nash equilibrium recommendation for the actions of all players will be falsified.

- In contrast, a Nash equilibrium passes the „announcement test:“ if all players simultaneously announce their equilibrium strategies, no one will change his/her strategy.

- A Nash equilibrium is finally a self-enforcing arrangement (implicitly or explicitly). It requires no external agent to enforce it (cooperative vs. Non-cooperative game theory).

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1. Static Games with Complete Information

Proposition 1.1 [Dominated and Maximin Strategies]

- If the repeated elimination of strictly dominated strategies in an *n*-person normal form game leads to the definite solution s^*, then s^* is the unique Nash equilibrium of the game.

- Conversely: if s^* is the Nash equilibrium of an *n*-person normal form game, then s^* is also a solution after repeated elimination of strictly dominated strategies. (i.e. The Nash equilibrium is a stronger solution concept than the „repeated elimination of strictly dominated strategies."

- In a 2-person zero sum game, maximin strategies also always constitute a Nash equilibrium (taking mixed strategies into account).

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1. Static Games with Complete Information

Proof for second part of Proposition 1.1

Proof through contradiction. Suppose that this statement is *not* true. In that case, a Nash equilibrium s^* from $G = \{S, u\}$, would exist that would not „survive" the repeated elimination of strictly dominated strategies.

Let s_i^* be the strategy of the equilibrium that would be eliminated first. Then:

There exists an s_i' in S_i, that still has not been eliminated so that:

$$u_i(s_i', s_{-i}) > u_i(s_i^*, s_{-i}) \text{ for all values of } s_{-i}\text{, that are not yet eliminated.}$$

Since s_i^* is the first equilibrium strategy eliminated, it must be true that:

$$u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*).$$

This is, however, a contradiction to the assumption that s^* is a Nash equilibrium.

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1. Static Games with Complete Information

Example 1.6 [Nash equilibria in the previous examples]:

• Equilibrium in dominant strategies: *(defect, defect)* is the unique Nash equilibrium in the normal form game from Example 1.1; $x_i \equiv 0$ is the unique Nash equilibrium in the normal form game from Example 1.2.

• Equilibrium after repeated elimination of dominant strategies: (o,m) is the unique Nash equilibrium in the normal form game from Example 1.3; $z_i \equiv 0$ is the unique Nash equilibrium in the normal form game from Example 1.4.

• (u,r) and. (v,m), respectively, represent the unique Nash equilibria in the normal form game from Example 1.5(a) and (c), respectively.

• Side note: uniqueness also applies when considering mixed strategies.

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3.3 Breakdowns in Coordination

3.3.1 Minimum Games

Smallest Value of X

		7	6	5	4	3	2	1
	7	1.30	1.10	0.90	0.70	0.50	0.30	0.10
	6	-	1.20	1.00	0.80	0.60	0.40	0.20
Your	5	-	-	1.10	0.90	0.70	0.50	0.30
Choice	4	-	-	-	1.00	0.80	0.60	0.40
of	3	-	-	-	-	0.90	0.70	0.50
X	2	-	-	-	-	-	0.80	0.60
	1	-	-	-	-	-	-	0.70

What is your number?

Sources: Van Huyck, Battalio, and Beil 1990.

3.3.1 Minimum Games

The Simple Theory of Minimum Games

- **In a minimum game, one person ‛s own optimal „effort“ is positively dependent on the efforts of others (also called „strategic complementarity“).**
- **In formulas: $y(x_i,m) = \$0.2m - \$0.1x_i + \$0.6$**
  - y = payoff
  - x_i = selected number of i
  - m = smallest selected number of all players
  - Deviations from the minimum will be penalized.

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3.3.1 Minimum Games

The Simple Theory of Minimum Games

- **Many equilibria exist.**
  - In the „payoff dominated“ equilibrium all players pick 7 (with a payoff of $1.30).
  - In the „secure“ equilibrium, all players select 1 *with a payoff of $0.70).
  - A secure action, therefore, is an action whose smallest payoff is at least as large as the smallest payoff of every other strategy („maximin strategy“).
- **Which equilibrium will be selected? The problem of equilibrium selection:**
  - Deductive Principles: Pareto dominance or risk dominance (maximin)?
  - Inductive Principles: History, learning?

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1. Static Games with Complete Information

Example 1.7: [Cournot-Duopoly]

- 2 firms produce a homogeneous good.
- Cost structure: no fixed costs; identical, constant marginal costs *c*.
- Every firm *i* simultanelously determines their quantity of the good $q_i \in [0, \infty)$
- (Inverse) demand function: $P(Q) = a - bQ$ with $a > c > 0$; $b > 0$ and $Q = q_1 + q_2$

How can this problem be transferred into a normal form game?

Players: Firms $i = 1, 2$

Strategy range of the players: $S_i = [0, \infty)$, with $q_i \in S_i$.

Player payoffs:

$u_i(q_i, q_{-i}) = \pi_i(q_i, q_{-i}) = P(q_1 + q_2)q_i - cq_i = q_i(a - b(q_1 + q_2) - c)$

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1. Static Games with Complete Information

In Nash equilibrium : s_i^* solves

$$\max_{s_i \in S_i} u_i(s_i, s_{-i}^*)$$

For $n = 2$ (q_1^*, q_2^*) is also a Nash equilibrium, if

$$q_1^* \text{ solves } \max_{0 \le q_1 < \infty} u_1(q_1, q_2^*) = q_1\left[a - b\left(q_1 + q_2^*\right) - c\right]$$

and

$$q_2^* \text{ solves } \max_{0 \le q_2 < \infty} u_2(q_1^*, q_2) = q_2\left[a - b\left(q_1^* + q_2\right) - c\right]$$

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1. Static Games with Complete Information

First Order of Conditions:

$$\frac{\partial u_1}{\partial q_1} = a - b(q_1 + q_2^*) - c - bq_1 = 0 \Leftrightarrow \boxed{q_1^*(q_2^*) = \frac{1}{2b}(a - c - bq_2^*)}$$

und

$$\frac{\partial u_2}{\partial q_2} = a - b(q_1^* + q_2) - c - bq_2 = 0 \Leftrightarrow \boxed{q_2^*(q_1^*) = \frac{1}{2b}(a - c - bq_1^*)}$$

Strategies in Nash equilibrium must be mutual best responses, i.e. both equations must be fulfilled simultaneously.

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1. Static Games with Complete Information

First Order of Conditions :

$$\frac{\partial u_1}{\partial q_1} = a - b(q_1 + q_2^*) - c - bq_1 = 0 \Leftrightarrow \boxed{q_1^*(q_2^*) = \frac{1}{2b}(a - c - bq_2^*)}$$

and

$$\frac{\partial u_2}{\partial q_2} = a - b(q_1^* + q_2) - c - bq_2 = 0 \Leftrightarrow \boxed{q_2^*(q_1^*) = \frac{1}{2b}(a - c - bq_1^*)}$$

In Nash equilibrium: $q_1^* = q_2^* = \frac{1}{3b}(a-c)$

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1. Static Games with Complete Information

Condtion 1. Order:

$$\frac{\partial u_1}{\partial q_1} = a - b\left(q_1 + q_2^*\right) - c - bq_1 = 0 \Leftrightarrow \boxed{q_1^*\left(q_2^*\right) = \frac{1}{2b}\left(a - c - bq_2^*\right)}$$

and

$$\frac{\partial u_2}{\partial q_2} = a - b\left(q_1^* + q_2\right) - c - bq_2 = 0 \Leftrightarrow \boxed{q_2^*\left(q_1^*\right) = \frac{1}{2b}\left(a - c - bq_1^*\right)}$$

Reaction Functions (Best Response Functions)

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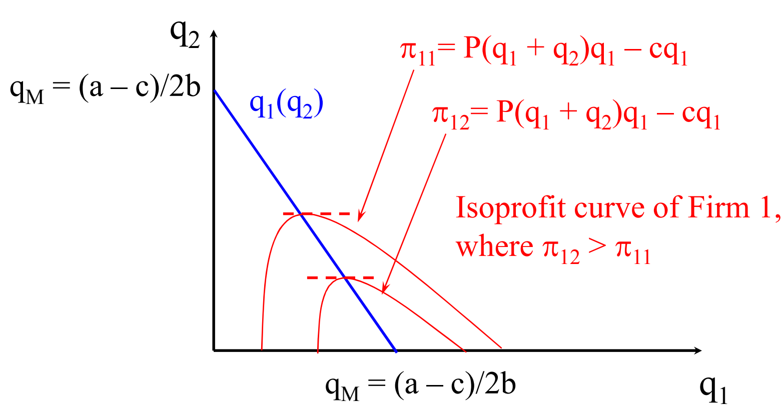
1. Static Games with Complete Information



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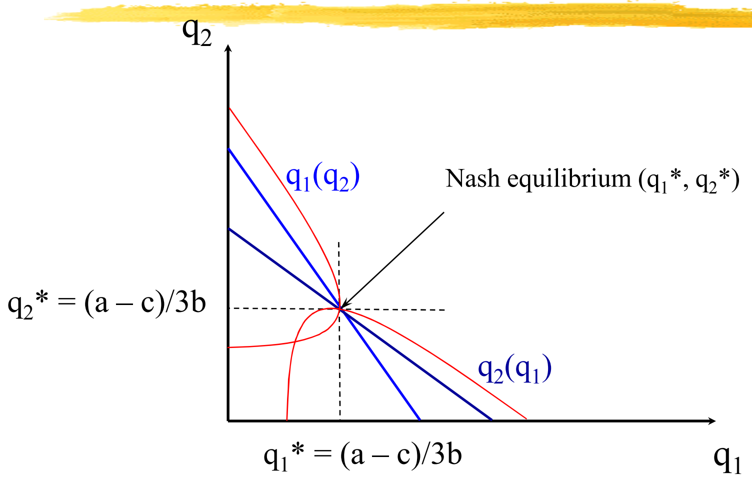
1. Static Games with Complete Information



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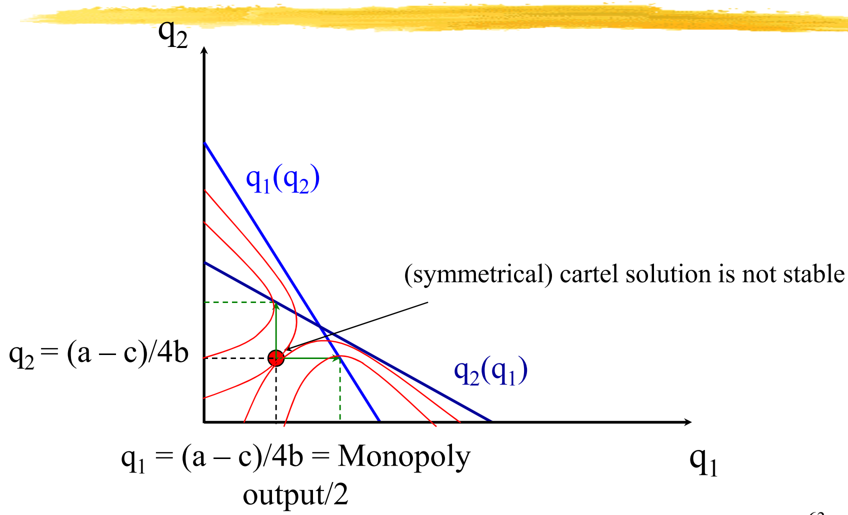
1. Static Games with Complete Information



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1. Static Games with Complete Information

Example 1.8: [Bertrand Duopoly]

- 2 firms produce a homogenous good.
- Cost structure: no fixed costs; identical, constant marginal costs *c;*
  cost function for firm *i*: $C(q_i) = cq_i$
- Both firms simutaneously set their prices $p_1, p_2 \in [0, \infty)$
- Quantity q_i market demands from Firm *i*:

$$q_i\left(p_i, p_j\right) = \begin{cases} q\left(p_i\right) & \text{when } p_i < p_j \\ \frac{1}{2} q\left(p_i\right) & \text{when } p_i = p_j \\ 0 & \text{when } p_i > p_j \end{cases}$$

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1. Static Games with Complete Information

How can this problem be converted to a normal form game?

Players: Firms $i = 1, 2$

Strategy range of players: $S_i = [0, \infty)$

Player payoffs: $u_i(p_i, p_j) = \pi_i(p_i, p_j) = (p_i - c)q_i(p_i, p_j)$

$\Rightarrow$ $\boxed{\text{In Nash equilibrium: } p_1^* = p_2^* = c}$

Price competition between just two firms here leads to competition price (“Bertrand-Paradox”)

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1. Static Games with Complete Information

Side note:

Bertrand himself, the „Inventor of Price Competition“, erred in his analysis. He wrote:

> “... one of the proprietors will reduce his price to attract buyers to him, and [the] other will in turn reduce his price even more to attract buyers back to him. … Whatever the common price adopted, if one of the owners, alone, reduced his price, he [would], ignoring any minor exceptions, attract all of the buyers, and thus double his revenue if his rivals let him do so. …There is no solution under this assumption, in that there is no limit to the downward movement [in prices].”
>
> [Translation: Chevaillier (1992)].

Cournot ‘s interpretation was also not one of a Nash equilibrium. Only after Nash was it understood that the concepts shared a close methodical relation.

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1. Static Games with Complete Information

1.4.2 Unambiguousness

Example 1.9 [Ambiguity: Battle of the Sexes]:

	Opera	*Boxing*
Opera	*2,1*	0,0
Boxing	0,0	*1,2*

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1. Static Games with Complete Information

- Nash equilibrium are sometimes not definite. In that case, how should an equilibrium be chosen?
  - Based on ‚focal points ‘, (culture specific) norms, ...?
  - Or by using sophisticated solution concepts, such as the elimination of weakly dominated strategies, trembling hand equilibrium, selection according to payoff principles or risk principles, evolutionary stability, etc . . . ?
- The following examples deal with the problem of equilibrium selection.

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1. Static Games with Complete Information

Example 1.10(a) [Equilibrium Selection: Focal Points (Schelling 1960)]:

(„Population Experiment):

- Pick heads or tails.
- Pick heads or tails.
- Circle one of the following numbers: 7, 100, 13, 261, 99, 666.
- Circle one of the following numbers : 14, 15, 16, 17, 18, 100.
- Divide $100. If the alloted shares add to more than 100%, no one wins anything.
- Drive right or left.
- Meeting Sunday afternoon in Boston. But where? And in New York?

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1. Static Games with Complete Information

Example 1.10 (d) [Experiment and communication; see *Cooper et al.* 1992]

	1	2
1	800, 800	800, 0
2	0, 800	1000, 1000

	Strategy 1	Strategy 2
Without communication	98,5%	1,5%
One-sided comm.	31,2%	68,8%
Two-sided comm.	4,5%	95,5%

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1. Static Games with Complete Information

2005 Nobel Prize for Thomas Schelling.
Excerpt from the statement:

Schelling was also concerned with the ability of individuals to coordinate their behavior in situations without any strong conflict of interest, but where unsuccessful coordination would give rise to high costs for all parties. In his research, including classroom experiments with his students, Schelling found that coordinative solutions – which he called focal points – could be arrived at more often than predicted by theory. The ability to coordinate appears to be related to the parties' common frames of reference. Social conventions and norms are integral parts of this common ground. Schelling's work in this area inspired the philosopher David Lewis to specify the idea that language originated as a means of coordination.

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1. Static Games with Complete Information

1.4.3 Existence

Example 1.11 [Non-existence in pure strategy: matching pennies]:

	Heads	*Tails*
Heads	-1,1	1,-1
Tails	1,-1	-1,1

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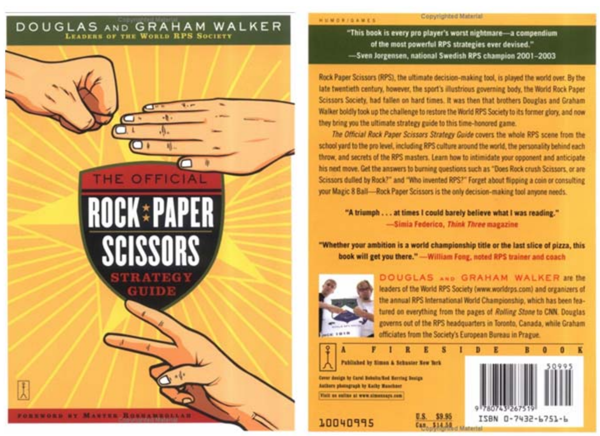
1. Static Games with Complete Information

1.4.3 Existence

Example 1.11 [Non-existence in pure strategy: rock, paper, scissors]



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1. Static Games with Complete Information

1.4.3 Existence

Example 1.12 [Non-existence in pure strategy: rock, paper, scissors]

	Rock	*Paper*	*Scissors*
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

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1. Static Games with Complete Information

• Examples: Penalties, Poker, Jäger-Beute models

• No equilibrium in pure strategies

• But when we also accept „probabilistic“ strategies („mixed strategies“), existence is generally assured.

• First we will learn how to recognize mixed strategies and how they are evaluated, and then we will come back to the problem of existence.

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1. Static Games with Complete Information

Definition 1.5 [Mixed Strategy]

If the strategy range of an *n*-person normal form game eventually comes to $G = \{S;u\}$, then let the mixed strategy of Player *i* be defined with

$S_i = \{s_{il}, ..., s_{iK}\}$ through a distribution function $p_i = (p_{il}, ..., p_{iK})$, where $0 \leq p_{ik} \leq 1$ for $k = 1, ..., K$ and $\sum^{K}_{k=1} p_{ik} = 1$.

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1. Static Games with Complete Information

- Interpretation: the mixed strategy is a formalization of the strategic uncertainty about the behavior of the other players.

- Special case: the pure Strategy s_{ij} can be described through a distribution function

$$p_i = (p_{il}, ..., p_{ij-1}, p_{ij}, p_{ij+1}, ..., p_{iK}) = (0, ...,0,1, 0, ..., 0)$$

.

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1. Static Games with Complete Information

Definition 1.6 [Nash equilibrium in mixed strategies]

A mixed strategy profile $p^* = (p_1^*, ..., p_n^*)$ is a Nash equilibrium in an n-person normal form game $G = \{S;u\}$, if for all players i the mixed Strategy p_i^* the best response to the other players' strategy p_{-i}^*, i.e. if

$$v_i(p_i^*, p_{-i}^*) \geq v_i(p_i, p_{-i}^*) \text{ for all } p_i \text{ and } i = 1, ..., n$$

where v_i the expected payoff describes the von Neumann-Morgenstern utility in regards to risk-neutral players and monetary payoffs.

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1. Static Games with Complete Information

Example 1.13 [The Nash equilibrium of matching pennies]:

	Heads	*Tails*
Heads	**-1,1**	**1,-1**
Tails	**1,-1**	**-1,1**

p_1 Probability that Player 1 plays "Heads" (mixed $0 \leq p_1 \leq 1$)

$1-p_1$ Probability that Player 1 plays "Tails"

p_2 Probability that Player 2 plays "Heads" (mixed $0 \leq p_2 \leq 1$)

$1-p_2$ Probability that Player 2 plays "Tails"

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1. Static Games with Complete Information

Suppose Player 1 chooses "Heads". His/her expected payoff is then:

$$v_1\left(K,\left(p_2,1-p_2\right)\right)=p_2\cdot\left(-1\right)+\left(1-p_2\right)\cdot 1=1-2p_2$$

Suppose Player 1 chooses "Tails". Then:

$$v_1\left(Z,\left(p_2,1-p_2\right)\right)=p_2\cdot 1+\left(1-p_2\right)\cdot\left(-1\right)=2p_2-1$$

From $v_1(K, (p_2,1-p_2)) = v_1(Z, (p_2,1-p_2))$ we can determine that: Player 1 is indifferent between "Heads" and "Tails" when $p_2 = 1/2$.

From $v_2(K, (p_1,1-p_1)) = v_2(Z, (p_1,1-p_1))$, we can determine that: Player 2 is indifferent between "Heads" and "Tails" when $p_1 = 1/2$.

$p_1 = p_2 = 1/2$ is a mutual best response.

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1. Static Games with Complete Information

Proposition 1.2: Let $S_i^+ \subset S_i$ the set of all pure strategies that Player 1 plays with positive probability in the mixed strategy profile $p = (p_1, ..., p_n)$.

The mixed strategy profile $p = (p_1, ..., p_n)$ is then a Nash equilibrium in the normal form game G when for all players $i = 1, ..., n$ the following applies:

(i) $v_i(s_i, p_{-i}) = v_i(s_i', p_{-i})$ for all $s_i, s_i' \in S_i^+$

(ii) $v_i(s_i, p_{-i}) \geq v_i(s_i', p_{-i})$ for all $s_i \in S_i^+$ and all $s_i' \notin S_i^+$

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1. Static Games with Complete Information

„Sufficient“:

‘trivial’

„ Necessary”:

If (i) or (ii) is not true for a player i, then it is understood that two strategies $s_i \in S_i^+$ and $s_i' \in S_i$ exist so that $v_i(s_i, p_{-i}) < v_i(s_i', p_{-i})$. But player i can then raise his payoffs by *always* choosing strategy s_i' instead of strategy s_i. This contradicts the assumption that a Nash equilibrium exists.

1. Static Games with Complete Information

Example 1.14 [Symmetrical Solution to Battle of the Sexes]:

For the Battle of the Sexes, $p_1 = (2/3, 1/3)$ and $p_2 = (1/3, 2/3)$ are Nash equilibria in a mixed strategy:

$p_1 = 2/3$ makes Player 2 indifferent ($1p_1 = 2(1 - p_1) \Rightarrow p_1 = 2/3$)

$p_2 = 1/3$ makes Player 1 indifferent ($2p_2 = 1(1 - p_2) \Rightarrow p_2 = 1/3$)

While this equilibrium is symmetrical, it is payoff dominated by both other equilibria. The expected payoff for Player 1 (2) is:

$p_1 p_2 2 + (1 - p_2)(1 - p_1)1 = 2/3 < 1.$

	Opera	Boxing
Opera	2,1	0,0
Boxing	0,0	1,2

1. Static Games with Complete Information

Side Note:

- The payoff for a player does not play a role in his/her combination ratio in a completely mixed equilibrium.
- In equilibrium, a player makes his opponent indifferent; therefore, only the payoffs of the opponent are relevant.
- For example in the example at right, $p_2 = 1/3$ remains Player 2 's combination in mixed equilibrium.
- However, this implication can typically not be confirmed in lab testing.

	Opera	Boxing
Opera	2,10	0,0
Boxing	0,0	1,2

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O'Neill's zero sum card game

		Column Player			
		1	2	3	J
Row Player	1	−	+	+	−
	2	+	−	+	−
	3	+	+	−	−
	J	−	−	−	+

- **Column player payoffs: + denotes win and – denotes loss**
- **Unique mixed strategy equilibrium:**
  Each plays (J,1,2,3) = (.4, .2, .2, .2)
  Row player wins 40% of the time.

Description of the game to subjects

1. Each player has four cards—ace, two, three, and a joker.

2. Each player will start with $2.50 in nickels for the series of hands.

3. When I say "ready" each of you will select a card from your hand and place it face down on the table. When I say "turn," turn your card face up and determine the winner. (I will be recording the cards as played.)

4. The winner should announce, "I win" and collect 5¢ from the other player.

5. Then return the card to your hand.

Now to determine the winner . . . [Subjects were shown a placard giving these rules, which were read aloud to them.]

[One subject's name] wins if there is a match of jokers (two jokers played) or a mismatch of number cards (two, three, for example).

[Other subject's name] wins if there is a match of number cards (three, three, for example) or a mismatch of a joker (one joker, one number card).

Given 6 conditions, the game is the simplest of its kind.

These ideas are formalized in the following requirements:

Condition 1: The game is in normal (matrix) form.

Condition 2: There are exactly two levels of payoff for each player.

Condition 3: It is not true that a player has two identical strategies.

Condition 4: Neither player has a dominated strategy.

Condition 5: The game is not completely symmetrical in strategies.

Condition 6: Any other game satisfying *Conditions 1–5* has at least as many strategies for each player.

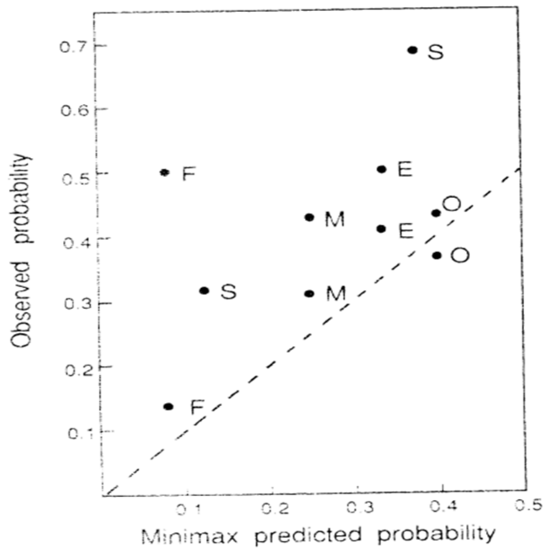

FIG. 1. From 2 × 2 zerosum game experiments, the predicted probability of the move with lower predicted probability is plotted versus observed probability. The values of Frenkel were reversed in the original publication due to a typographical error. E. Estes from figure 7 in ref. 6; S. Suppes and Atkinson from group E in ref. 7; M. Malcolm and Lieberman from ref. 8; F. Frenkel from game 75 in ref. [illegible] O, B.O'N. from the present study.

O'Neill argues his experiment is closer to theory because of a more careful design

- **Simplicity of game**
- **Higher subject motivation**
- **Careful control of expected utility considerations**

O'Neill's evidence

- **The aggregate data is in line with the theory**
  - Correct proportion of strategies used
  - Correct proportion of wins
- **Individual data is less confirming**
  - Serial dependence of moves
  - High individual variance of Joker play

Thus the theory seemed validated by the large-scale statistics but not the finer ones. This is puzzling. How could the overall proportions have followed the theory when the individual moves that generated them did not? One theoretical explanation, which seems unlikely given people's psychological limitations, is that the subjects regarded the 105 plays as one large supergame and randomized over all possible strategies for this game, a set that includes some with nonindependent moves and greater than predicted variance in number of jokers. A more plausible explanation is that players were constrained to follow the minimax in its gross statistics because these were relatively observable by the opponent. However, at each move players felt free to invent patterns, follow hunches, or do other things that introduced dependencies and variance into the sequence of plays. They could do this without significant danger because the opponent had a limited ability to calculate all the relevant probabilities especially when only a small sample of moves was available. But a large deviation from the overall minimax proportion was easier to notice so players avoided the risk of loss by sticking close to the minimax proportions.

Brown and Rosenthal: Point predictions are accurate but the standard deviations are not.

TABLE I

RELATIVE FREQUENCIES OF CARD CHOICES IN O'NEILL'S EXPERIMENT[a]

		Column Player Choice 1	2	3	J	Marginal Frequencies For Row Player:
Row Player Choice	1	.044 (.040) [.004]	.043 (.040) [.004]	.043 (.040) [.004]	.091 (.080) [.005]	.221 (.200) [.008]
	2	.046 (.040) [.004]	.038 (.040) [.004]	.038 (.040) [.004]	.092 (.080) [.005]	.215 (.200) [.008]
	3	.049 (.040) [.004]	.032 (.040) [.004]	.037 (.040) [.004]	.085 (.080) [.005]	.203 (.200) [.008]
	J	.086 (.080) [.005]	.065 (.080) [.005]	.051 (.080) [.005]	.158 (.160) [.007]	.362 (.400) [.010]
Marginal Frequencies for Column Player:		.226 (.200) [.008]	.179 (.200) [.008]	.169 (.200) [.008]	.426 (.400) [.010]	

[a] Numbers in parentheses represent minimax predicted relative frequencies. Numbers in brackets represent standard deviations for observed relative frequencies under the minimax hypothesis.

I.I.D. hypothesis is violated

TABLE III

OBSERVED AND EXPECTED WINNING PERCENTAGES UNDER THE INDEPENDENCE HYPOTHESIS[a]

Pair #	Observed Row Win %	Expected Row Win %	Mixture Effect	Correlated Play Effect
1	.391	.418	.018	−.027
2	.295	.410	.010	−.115
3	.390	.437	.037	−.046
4	.419	.366	−.034	.054
5	.343	.406	.006	−.063
6	.419	.397	−.003	.022
7	.476	.408	.008	.069
8	.467	.414	.014	.053
9	.362	.413	.013	−.051
10	.390	.400	.000	−.010
11	.390	.403	.003	−.013
12	.543	.401	.001	.142
13	.410	.368	−.032	.042
14	.467	.427	.027	.040
15	.324	.395	−.005	−.071
16	.323	.396	−.004	−.053
17	.362	.400	.000	−.038
18	.486	.398	−.002	.087
19	.390	.386	−.014	.004
20	.438	.402	.002	.037
21	.476	.405	.005	.072
22	.400	.393	−.007	.007
23	.448	.402	.002	.046
24	.495	.407	.007	.088
25	.333	.377	−.023	−.043
Mean Absolute Value:	—	—	.011	.052
Chi-Square Statistic:	—	—	2.655	40.428
Chi-Square Prob. Value:	—	—	.999	.026

[a] Expected winning percentages assume independent play with mixtures observed over all 105 games for each pair. Mixture effects are defined as the difference between values in column (2) and .40. Correlation effects are defined as the difference between values in columns (1) and (2).

O'Neill's response (*Econometrica* 1991)

I believe the heart of our difference is that their tests answer the question, "Is minimax exactly correct?", while my focus was, "How close is minimax to the truth?" To me the latter question is more important. There is no doubt that minimax is wrong. All precise theories of human action are wrong. However if minimax is close to real behavior, we should try to build on it. Addressing the "how close" question guides a research program, whereas the "exactly correct" question produces a foregone answer that points nowhere.

A reader might overlook the difference between these two questions. When Brown and Rosenthal report "strong evidence of a serial correlation," that is different from "evidence of a strong serial correlation." When their tests "reject the minimax model quite clearly," they are not demonstrating that minimax is *very* wrong. To say that a discrepancy is statistically detectable is not to say it is important.

Idea for another example

- Aumann and Schelling with acceptance, Nobel Prize statement, etc. www.nobel.org
- World Cup Final 74 (W. Germany vs. Holland): This game could not be matched for its drama and hectic pace. In just the second minute of the game, there was a penalty kick for Holland, which Johann Neeskens cooly scored on. He shot in the middle and discovered a design mechanism that increased the probability of success from ½ to 2/3!! Later, Paul Breitner shot to the left.

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1. Static Games with Complete Information

Example 1.16 [Bertrand Paradox revisited]:

It is often said that the definite equilibrium in Bertrand games produces marginal costs. However, at least with inelastic demand, this is false (it applies similarly to elastic demand as long as the demand is positive at all prices, similar to the case of isoelastic demand).

In a Bertrand duopoly, the identical marginal costs are standardized at zero and the inelastic demand is standardized at 1. The following strategy then describes an equilibrium in mixed strategy:

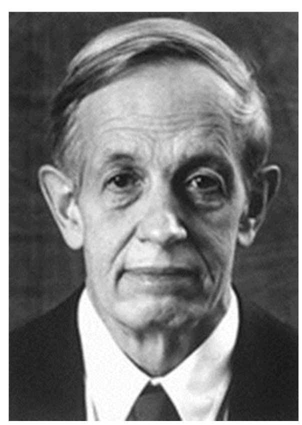
A value of $k \geq 0$ is given. The probability of a price p ($p \geq k$) or greater is k/p; the probability of a price less than k is zero.

The standard Bertrand equilibrium comes from values of k around 0. Equilibria in mixed strategies come about from values of $k > 0$.

1. Static Games with Complete Information

Proposition 1.3 [Existence of Nash equilibria]:

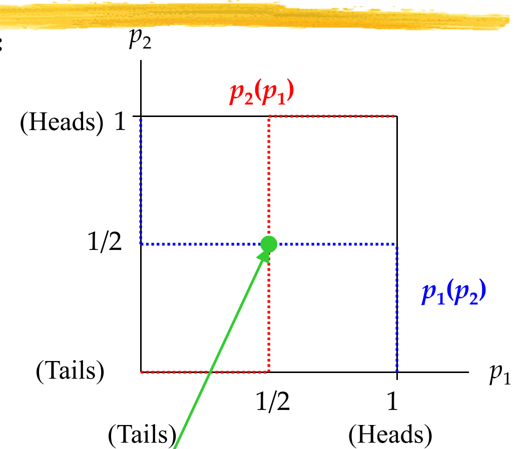
In an n-person normal form game $G = \{S;u\}$ with finite n and S_i for all i, there exists at least one Nash equilibrium (possibly in a mixed strategy).



1. Static Games with Complete Information

Example 1.15 [Matching Pennies: Graphic Representation]:

	Heads (p_2)	Tails ($1 - p_2$)
Heads (p_1)	-1, 1	1, -1
Tails ($1 - p_1$)	1, -1	-1, 1



$p_1 = p_2 = 1/2$ is the unique Nash equilibrium

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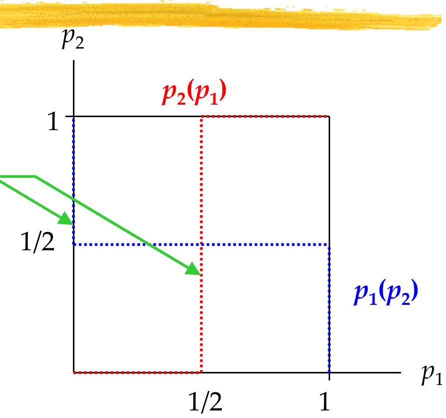
1. Static Games with Complete Information

best response Correspondence



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1. Static Games with Complete Information

The general existence theory is based on a fixed point theorem:

Breakdown of the Argument (fairly advanced ☺)

Brouwer Fixed Point Theorem (1910)

Let $X \subset \Re^N$ be a non-empty, compact* and convex set, and f: $X \rightarrow X$ be a continuous function from X to X. f then has a fixed point, i.e. There exists an $x \in X$, so that $f(x) = x$.

* A compact set must be bounded and closed.

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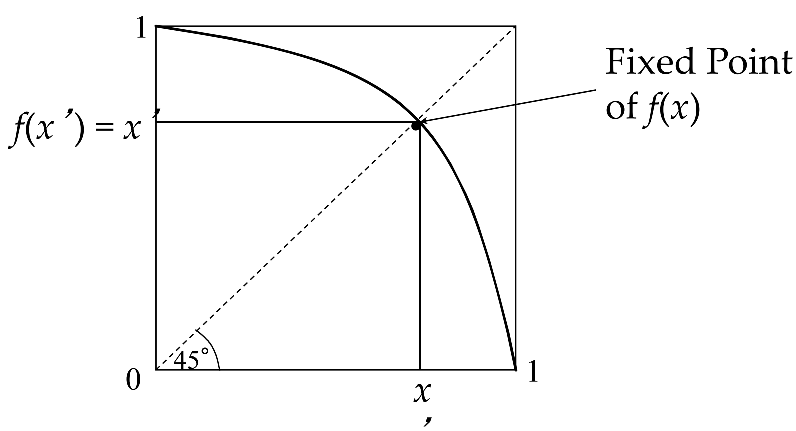
1. Static Games with Complete Information

Example: $X = [0, 1]$ and $f(x) = 1 - x^2$



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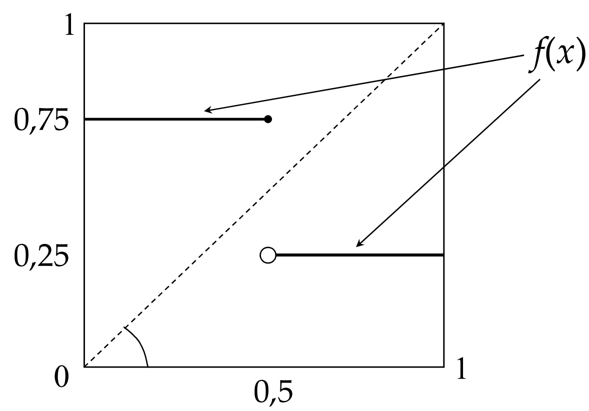
1. Static Games with Complete Information

Example: $X = [0, 1]$, $f(x)=\frac{3}{4}$, if $x \leq \frac{1}{2}$, and $f(x)=\frac{1}{4}$, if $x > \frac{1}{2}$



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1. Static Games with Complete Information

→ The proof plays out in two steps:

1) Show that every fixed point of a given (best response) correspondence is a Nash equilibrium.
2) Use an applicable fixed point theorem to show that a fixed point exists for this (best response) *correspondence*.

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1. Static Games with Complete Information

"*n*-Person Best Response Correspondence":

Let p_i be a mixed strategy for Player i and P_i the set of all mixed strategies for Player i that can be derived from S_i.

The Best Response Correspondence for Player i, $B_i : P_{-i} \rightarrow P_i$, is defined as:

$$B_i(p_{-i}) = \arg\max_{p_i} u_i(p_i, p_{-i})$$

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1. Static Games with Complete Information

The *n-Person Best Response Correspondence* $B(p) : P \rightarrow P$ is defined as:

$$B(p) = B(p_1, p_2, \ldots, p_n) = \begin{cases} B_1(p_{-1}) \\ B_2(p_{-2}) \\ \ldots \\ B_n(p_{-n}) \end{cases}$$

This results in every possible strategy profile p being assigned the profile of the best response to that strategy profile.

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1. Static Games with Complete Information

For a fixed point p^* of $B(p)$ the following applies: $B(p^*) = p^*$. I.e. p_i^* *is* the best response to p_{-i}^* for all Players *i*.

$$B(p^*) = p^*$$

This means:

$$B_1(p^*_{-1}) = p^*_1$$

$$B_2(p^*_{-2}) = p^*_2$$

...

$$B_n(p^*_{-n}) = p^*_n$$

In other words, every fixed point p^* of $B(p)$ is a Nash equilibrium.

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1. Static Games with Complete Information

Is there an applicable fixed point theorem for *correspondences*? (Brouwer 's theorem only applies to functions.)

Kakutani 's Fixed Point Theorem (1941)

Let $P \subset \Re^N$ be a non-empty, compact, and convex set.

And B: $P \to P$ an upper semi-continuous correspondence such that the set $B(p) \subset$ X is a non-empty and convex set for every $p \in P$.

Then B has a fixed point, i.e. there exists a value p so that $p \in B(p)$.

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EQUILIBRIUM POINTS IN N-PERSON GAMES

By John F. Nash, Jr.*

Princeton University

Communicated by S. Lefschetz, November 16, 1949

One may define a concept of an n-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.

Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the $n - 1$ strategies of the other players in the countered n-tuple. A self-countering n-tuple is called an equilibrium point.

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The correspondence of each n-tuple with its set of countering n-tuples gives a one-to-many mapping of the product space into itself. From the definition of countering we see that the set of countering points of a point is convex. By using the continuity of the pay-off functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: if $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots, Q_n, \ldots$ are sequences of points in the product space where $Q_n \rightarrow Q$, $P_n \rightarrow P$ and Q_n counters P_n then Q counters P.

Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem[1] that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person zero-sum case the "main theorem"[2] and the existence of an equilibrium point are equivalent. In this case any two equilibrium points lead to the same expectations for the players, but this need not occur in general.

* The author is indebted to Dr. David Gale for suggesting the use of Kakutani's theorem to simplify the proof and to the A. E. C. for financial support.

[1] Kakutani, S., *Duke Math. J.*, 8, 457–459 (1941).

[2] Von Neumann, J., and Morgenstern, O., *The Theory of Games and Economic Behaviour*, Chap. 3, Princeton University Press, Princeton, 1947.

John F. Nash (1950), *Proceedings of the National Academy of Sciences of the USA.*

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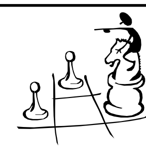
1. Static Games with Complete Information



Conclusion: Nash equilibria are not always efficient or definite, but they exist in the broad catetory of „finite“ games.

But there also exists a Nash equilibrium in the Cournot Model, although that model assumes infinite strategy.

The conditions listed are sufficient, but not necessary conditions for the existence of a Nash equilibrium!

Example: *Dasgupta/Maskin* (1986): Even if the payoff functions are not continuous, Nash equilibria exist in many cases.

Example for non-existence: Pick a Number game with strategy set $(0,k)$, i.e., the chosen real number must be *greater* than zero … (why?)

The main problem is not existence, but rather ambiguity.

3. Static Games with Incomplete Information

3.1 Completion of Incomplete Information

Incomplete information: at least one player does not know the payoffs of one of the other players.

In such a situation, how should equilibrium be calculated?

Idea: The game with incomplete information is "completed" so that only imperfect information remains - which can be handled using standard methods.

3. Static Games with Incomplete Information

Example 3.1 [Oligopoly theory with asymmetrical information]:

Company 1, which is serving the market alone, is deciding on whether or not to build another factory. Company 2 is simultaneously deciding on whether or not to enter the market. Company 2 is unsure whether Company 1's startup cost for a factory is 1.5 or 3, whereas company 1 has private information about the cost.

3. Static Games with Incomplete Information

Example 3.1 [Oligopoly theory with asymmetrical information]:

Payoff functions, in the case that costs for company 1 are high:

	enter	*don't enter*
build	0, - 1	2, 0*
don't build	*2, 1*	*3, 0

3. Static Games with Incomplete Information

Example 3.1 [Oligopoly theory with asymmetrical information]:

Payoff functions, in the case the costs for company 1 are low:

	enter	*don't enter*
build	1.5, - 1	*3.5, 0*
don't build	*2, 1*	3, 0

3. Static Games with Incomplete Information

Example 3.1 [Oligopoly theory with asymmetrical information]:

Company 1:

• In the case of high costs 'don't build' is the dominant strategy for company 1. In the case of low costs, the decision to build depends on whether or not company 1 thinks company 2 is going to enter the market.

• That means: When company 1 assumes an entrance probability of y < ½, they will build when costs are low:

u(build) = y(1.5) + (1 – y)(3.5)

> u(don't build) = y(2) + (1 – y)(3),

if y < 1/2.

	enter	*don't enter*
build	1.5, - 1	*3.5, 0*
don't build	*2, 1*	3, 0

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3. Static Games with Incomplete Information

Example 3.1 [Oligopoly theory with asymmetrical information]:

Company 2:

• The payoffs for company 2 depend on company 1's decision to build or not: 'Enter' is profitable when company 1 does not build.

• For every possibility (high and low costs) equilibriums can be calculated. But which situation are we in?

• Idea: Transform the game with incomplete information into a game with imperfect but complete information. (Harsanyi, 1967-68):

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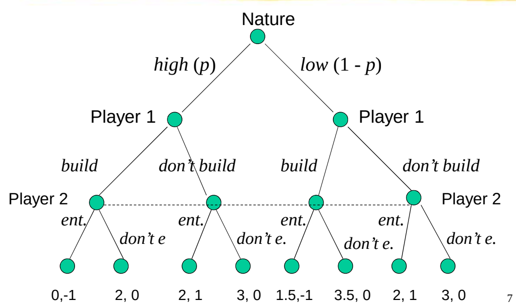
3. Static Games with Incomplete Information



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3. Static Games with Incomplete Information

Trick

• Assume a random move by 'nature' to determine type (high cost or low).

• All players have the same expectations, p (common prior).

• Response functions can be calculated through the maximization of *expected* payoffs.

• A *Bayesian-Nash equilibrium* with incomplete information is identical to the analogous Nash equilibrium with imperfect but complete information.

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3. Static Games with Incomplete Information

Solutions

• For every p, (‘don’t build, regardless if costs are high or low’, ‘enter') is an equilibrium.

• (‘don’t build in the case of high costs, and build when the costs are low’, ‘don’t enter’) is the equilibrium when $p \leq 1/2$:

u(enter) = $p(1) + (1 – p)(– 1)$

$\leq u$(don’t enter) = $p(0) + (1 – p)(0)$,

if $p \leq 1/2$.

3. Static Games with Incomplete Information

Structure of a static Bayesian game (Harsanyi, 1967):

• Nature chooses the type vector $t = (t_1, ..., t_n)$, $t_i \in T_i$ for all i.

• The type t_i is only known to player i (private Information).

• Every player i simultaneously chooses an action $a_i \in A_i$.

• Every player i gets his payoff $u_i(a,t_i)$.

3. Static Games with Incomplete Information

Definition 3.2 [Strategy]

In a static game with imperfect information $G = \{A,T,p,u\}$, player i’s strategy is a function $s_i(t_i)$.

$s_i(t_i)$ specifies that for every type $t_i \in T_i$, an action, $a_i \in A_i$, is chosen.

(An action must also be specified for every type that will not be used.)

3. Static Games with Incomplete Information

Definition 3.3 [Bayesian Nash Equilibrium]

In a static game with incomplete information $G = \{A,T,p,u\}$, the strategies $s^* = (s_1^*, ..., s_n^*)$ describe a Bayesian Nash equilibrium (in pure strategies) if for every player i the strategy s_i^* solves the following optimization problem:

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} | t_i)\, u_i(s_1^*(t_1), ..., s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), ..., s_n^*(t_n); t_i)$$

for all $t_i \in T_i$.

That means that for every possible type t_i, player i maximizes his expected payoffs through his choice of an action a_i given s_{-i}^*. The result is $s_i^*(t_i)$.

3. Static Games with Incomplete Information

Existence

Every static game with incomplete information and a finite number of players, actions and types contains a Bayesian Nash equilibrium (possibly with mixed strategies).

The proof follows a path similar to the previous existence proofs …

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3. Static Games with Incomplete Information

Example 3.2 [Cournot Duopoly with incomplete information]:

- Inverse Demand: $P(Q) = a - bQ$ with $Q = q_1 + q_2$.
- $C_1(q_1) = cq_1$
- $C_2(q_2) = \begin{cases} c_H\, q_2 & \text{with probability } p \\ c_L\, q_2 & \text{with probability } 1 - p \end{cases}$
- $c_H > c_L$
- Company 1 does not know the implementation of C_2, but the probability p.
- Payoff: $u_i(q_i, q_j) = \pi_i(q_i, q_j) = q_i(P - c_i) = q_i\left[a - b\left(q_i + q_j\right) - c_i\right]$

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3. Static Games with Incomplete Information

It can be said that in equilibrium:

(1) $q_2^*(c_H)$ solves $\max\limits_{0 \le q_2 < \infty} \pi_2(q_1^*, q_2) = q_2\left[a - b\left(q_1^* + q_2\right) - c_H\right]$

and

(2) $q_2^*(c_L)$ solves $\max\limits_{0 \le q_2 < \infty} \pi_2(q_1^*, q_2) = q_2\left[a - b\left(q_1^* + q_2\right) - c_L\right]$

(3) $q_1^*(c)$ solves $\max\limits_{0 \le q_1 < \infty} \pi_1(q_1, q_2^*) = p \cdot q_1\left[a - b\left(q_1 + q_2^*(c_H)\right) - c\right]$

$$+ (1 - p) \cdot q_1\left[a - b\left(q_1 + q_2^*(c_L)\right) - c\right]$$

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3. Static Games with Incomplete Information

First Order Conditions:

$$\frac{\partial \pi_2}{\partial q_2(c_H)} = a - bq_1^* - c_H - 2bq_2(c_H) = 0 \;\Leftrightarrow\; q_2^*(c_H) = \frac{1}{2b}\left(a - c_H - bq_1^*\right)$$

$$\frac{\partial \pi_2}{\partial q_2(c_L)} = a - bq_1^* - c_L - 2bq_2(c_L) = 0 \;\Leftrightarrow\; q_2^*(c_L) = \frac{1}{2b}\left(a - c_L - bq_1^*\right)$$

$$\frac{\partial \pi_1}{\partial q_1} = p \cdot \left(a - bq_2^*(c_H) - c\right) + (1 - p) \cdot \left(a - bq_2^*(c_L) - c\right) - 2bq_1 = 0$$

$$\Leftrightarrow\; q_1^* = \frac{p \cdot \left(a - bq_2^*(c_H) - c\right) + (1 - p) \cdot \left(a - bq_2^*(c_L) - c\right)}{2b}$$

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3. Static Games with Incomplete Information

Solution:

$$q_2^*(c_H) = \frac{a - 2c_H + c}{3b} + \frac{1-p}{6b}(c_H - c_L)$$

$$q_2^*(c_L) = \frac{a - 2c_L + c}{3b} - \frac{p}{6b}(c_H - c_L)$$

$$q_1^* = \frac{a - 2c + (1-p)\cdot c_L + pc_H}{3b}$$

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3. Static Games with Incomplete Information

• Company 1's decision is similar to the Cournot solution with complete information ($q_i^* = (a - 2c_i + c_j)/(3b)$), if the true costs of company 2 are the same as the expected value.

• In a high cost scenario, company 2 is better positioned then when it has complete information because it can produce more. The size of the advantage depends on the degree of uncertainty of company 1. If $p = 1$ or $c_L = c_H$, there would be no uncertainty and no advantage.

• In a low cost scenario, company 2 is in a worse position then when it has complete information. The disadvantage depends again on the degree of uncertainty of company 1.

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3. Static Games with Incomplete Information

3.2 Auction theory

- An auction is a market institution with explicit rules that govern the prices and allocations of objects based on bids from participants in the market.
- Example: The worth of an item to be auctioned is v (and zero for the seller). This is known to all (common knowledge). Which allocations and price should one expect? Which allocation should one expect in the case that the bidders have different valuations? Are the resulting allocations efficient?
- What consequence does asymmetric information have on pricing, on the resulting allocations and on the efficiency of players that interact strategically?

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3. Static Games with Incomplete Information

- The individual seller's uncertainty about valuations creates a positive expected payoff for the bidder. Even if though the seller cannot observe the bidder's willingness to pay in the case of incomplete information, he can leverage the competition between bidders to disclose relevant information.
- Auctions can maximize profits (within limits), create efficiency and uncover private information.
- This is one of the reasons why auctions have a special empirical relevance and a special relevance to economic policy …

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3. Static Games with Incomplete Information

- Some of the best known examples regarding the usage of advanced auction theory are the Spectrum or UMTS auctions in the USA and Europe, which were essentially designed by game theorists and lead to astonishing revenues.
- There are also other auctions such as traditional auctions of government securities and recently more and more auctions for energy.
- In the last few years, the number of uses has ‘exploded’ in all areas, and this is not just because of modern communication technologies.

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3. Static Games with Incomplete Information

Common Auction Formats

English Auction: The price raises until only one bidder remains who obtains the good at the last price bid.

Dutch Auction: The price decreases until the first bidder accepts the price.

First-Price Auction: All bidders simultaneously make a bid. The object goes to the bidder with the highest bid at a price equal to that bid.

Second-Price Auction: The highest bidder wins at a price equal to the second highest bid.

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3. Static Games with Incomplete Information

Questions:

- Do open auctions, such as the English auction, lead to more aggressive bidding and higher revenues than a closed auction (ex., sealed bids)?
- Does an auction in which the winner pays a lower price than his bid lead to lower revenues?
- Which form of auction is strategically simpler than the others?
- ...

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3. Static Games with Incomplete Information

Further References

- Vijay Krishna: “Auction Theory”, Academic Press 2002.
- Paul Milgrom: “Putting Auction Theory to Work”, Cambridge University Press, 2004.
- Paul Klemperer: “Auctions: Theory and Practice”, Princeton University Press 2004.
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3. Static Games with Incomplete Information

3.2.1 Independent and Private Valuations

Given

- The seller can set the rules of the auction.
- Bidding is on a single, inseparable object.
- n potential buyers.
- All market participants are risk neutral.
- The bidders' valuations (willingness to pay) v_i are private information (only known to each bidder) and stochastically independent from each other ("independent private values").
- The bidders are symmetrical, meaning that the valuations are taken from the same known distribution : $F_i(v_i) \equiv F(v_i)$.

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3. Static Games with Incomplete Information

Bayesian Nash Equilibrium

- A strategy is a function $b_i(v_i)$, in which every possible valuation v_i is assigned a bid b_i.
- A Bayesian Nash Equilibrium is a profile made up of bid functions $b_i(v_i)$, $i = 1,2,\ldots n$, so that every b_i is the best response to the bid function b_j of all the other bidders j.

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3. Static Games with Incomplete Information

English Auction

- (weakly dominant) equilibrium strategy: bid until v_i.
- The equilibrium price is equal to the second highest valuation. In the following, the highest valuation amongst the n bidders is $v_{(1)}$, the second highest is $v_{(2)}$, etc.
- The bidder with the highest valuation wins the auction. Hence, the English Auction leads to *efficiency*.

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3. Static Games with Incomplete Information

Vickreys Second-price Auction

- Lets look at the second-price auction that at first glance seems a little strange.
- Different than the (open) English Auction, the second-price auction is a closed auction in which bidders 'simultaneously' give their bids.
- And unlike the English Auction, the price is equal to the *second highest* bid.
- Nevertheless, it turns out that both auctions (under our assumptions) lead to exactly the same revenues and allocations.

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3. Static Games with Incomplete Information

- Payoff Function:

$$\pi_i = \begin{cases} v_i - \max\limits_{j \neq i} b_j, & \textbf{if } b_i > \max\limits_{j \neq i} b_j \\ 0, & \textbf{if } b_i < \max\limits_{j \neq i} b_j \end{cases}$$

- If more than one bidder makes an identical high bid, every bidder has the same probability of obtaining the object.
- *Equilibrium*: In the second-price auction, it is a (weakly) dominant strategy to bid ones valuation: $b_i(v_i) = v_i$.

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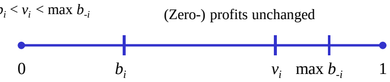
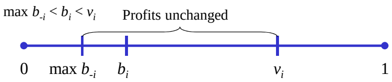
3. Static Games with Incomplete Information

Is is worth bidding less?

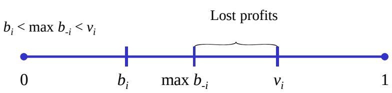
1) $b_i < v_i < \max b_{-i}$



2) $\max b_{-i} < b_i < v_i$



3) $b_i < \max b_{-i} < v_i$



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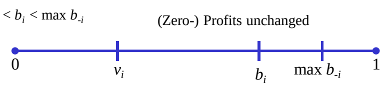
3. Static Games with Incomplete Information

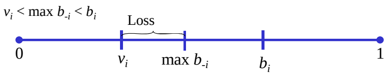
Is it worth bidding more?

1) $v_i < b_i < \max b_{-i}$



2) $\max b_{-i} < v_i < b_i$

3) $v_i < \max b_{-i} < b_i$



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3. Static Games with Incomplete Information

- Bidding less or more than the valuation is never worth it and sometimes hurts!
- The essential property of both the English and second-price auction is that the price the winner must pay is determined from the offers of the other bidders.
- In this sense, the bidders are "price takers", who should only accept prices up until the amount they are willing to pay. The auctions are thus incentive aligned.
- Because of this, both auctions have the same resulting equilibrium (under the assumption that the bidders never play weakly dominated strategies; otherwise other equilibriums apply.)

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3. Static Games with Incomplete Information

Thus:

- The auctions lead to the efficient result: the bidder with the highest valuation wins.
- The respective bid functions are $b_i(v_i) = v_i$.
- The respective price is $v_{(2)}$.
- The winner's payoff is $v_{(1)} - v_{(2)}$. All others go home empty handed.

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3. Static Games with Incomplete Information

If the n bidders' valuations v_i are independently and uniformly distributed over [0,1] than it is true that:

- Bidder i's expected payoff is: v_i^n/n.
  - This holds true for the distribution function $F(v_i) = v_i$. Thus, v_i^{n-1} is the probability that the $n - 1$ other valuations v_j are smaller than the valuation v_i; in this case, i wins.
  - v_i/n is the expected 'space' between one's own bid and the next highest bid in the case of winning (in other words the expected profit in the case of winning).
  - Because the payoff is zero in all other cases, $v_i^{n-1} v_i/n$ is the expected payoff.
  - (The payoff goes from v_i for $n = 1$ to 0 as $n \to \infty$.)

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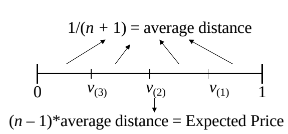
3. Static Games with Incomplete Information

- The expected price is $E\{v_{(2)}\} = (n - 1)/(n + 1)$.
- Intuition:



- Using our assumptions, $E\{v_{(r)}\} = (n + 1 - r)/(n + 1)$.
- The price goes from 0 for $n = 1$ to 1 as $n \to \infty$.

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3. Static Games with Incomplete Information

First-price Auction

- If the bidders were to give their 'true' valuation in a first-price auction, they could never make a profit.
- Thus unlike the English Auction, there are incentives to bid 'false' strategically motivated valuations (the auction is thus not incentive aligned).
- In addition, strategic uncertainty plays a deciding role. Nobody wants to bid too much or too little – which again depends, for example, on the bids and the number of other bidders as well as the degree of risk aversion etc.

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3. Static Games with Incomplete Information

An Example

- Assuming the valuations v_i of n bidders are independently normally distributed over [0,1]. That means $F(v_i) = v_i$.
- Strategies, or bid functions $b_i(v_i)$ for all i are searched for, so that $(b_1(v_1), b_2(v_2),\ldots, b_n(v_n))$ is a Nash equilibrium.
- Solution: $b_i(v_i) = (n-1)v_i/n$ for all i.
- That means that if there is one bidder he will bid zero, if there are two bidders, each bidder will bid half of his valuation, and if there are many bidders, the bid approaches the valuation.

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3. Static Games with Incomplete Information

Proof for Two Bidders:

- The profit function is given by:

$$\pi_i(b_i,b_j) = \begin{cases} v_i - b_i, & \textbf{if } b_i > b_j \\ (v_i - b_i)/2 & \textbf{if } b_i = b_j \\ 0, & \textbf{if } b_i < b_j \end{cases}$$

- Given bid function $b_j(v_j)$, i maximizes his expected payoff:

$$(v_i - b_i)\Pr\left[b_i > b_j\left(v_j\right)\right] + \frac{1}{2}(v_i - b_i)\Pr\left[b_i = b_j\left(v_j\right)\right]$$

- In the following we assume that the bidders possess strictly monotonically increasing bid functions (one can show that this must be the case in all equilibriums), such that the probability of equal bids is zero.

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3. Static Games with Incomplete Information

- Because $\Pr[b_i = b_j(v_j)] = 0$, only the first term is important:

$$\max_{b_i}(v_i - b_i)\Pr\left[b_i > b_j\left(v_j\right)\right]$$

- For $b_j(v_j) = v_j/2$ it follows that

$$\max_{b_i}(v_i - b_i)\Pr\left[b_i > v_j/2\right]$$

- That means (because $F(v) = v$):
  $\Pr[b_i > v_j/2] = \Pr[2b_i > v_j] = 2b_i$
- Resulting in:

$$\max_{b_i}(v_i - b_i)2b_i$$

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3. Static Games with Incomplete Information

- Thus:

$$\max_{b_i}(v_i - b_i)2b_i$$

$$2(v_i - 2b_i) = 0 \Leftrightarrow b_i = v_i/2.$$

- With that we have shown that $b_i(v_i) = v_i/2$ is the best response to $b_j(v_j) = v_j/2$. Because we are dealing with a symmetrical problem, it follows that $b_j(v_j) = v_j/2$ is an equilibrium bid strategy.
- In the case of n bidders the equilibrium bid function is:

$$b_i(v_i) = \frac{n-1}{n}v_i.$$

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3. Static Games with Incomplete Information

- That means that the profits a bidder makes decreases with the number of bidders. The bid function reflects balance 'the higher the bid, the higher the probability of winning' versus 'the payoff decrease as the winning bid increases'.

- The expected price is
  $E\{b(v_{(1)})\} = E\{(n-1)\, v_{(1)}/n\}$
  $= (n-1)\, E\{v_{(1)}\}/n = (n-1)[n/(n+1)]/n$
  $= (n-1)/(n+1)$.

- This price is equal to the expected price in the English and in the second price auction!

- Intuition:

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3. Static Games with Incomplete Information

- In equilibrium, the bidder with the highest willingness to pay wins. Thus, first-price auctions always lead to efficiency.

- Therefore, in his bidding strategy, a bidder can ignore the possibility that he doesn't have the highest valuation (in this case he cannot win or lose).

- In equilibrium every bidder *i* bids the expected worth of the second highest bid, under the assumption that his valuation is the highest!

- Because this assumption is correct for the winner, the expected value of the price is equal to the second highest valuation.

- In the English and second-price auction, the price is also equal to the second highest valuation (not only the expected value).

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3. Static Games with Incomplete Information

Dutch Auction

- The Dutch auction is strategically equivalent to the first-price auction.

- In both cases, the bidder must consider how much he wants to bid/when he wants to accept, without having information about how much the others are bidding/accepting.

- Therefore, the strategy spaces and the payoff functions and consequently the equilibrium are equivalent.

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3. Static Games with Incomplete Information

Revenue Equivalence Theorem

- We have proven that the four different types of auctions lead to identical solutions with regards to expected values.

- Revenue equivalence is actually much more general:

  In all types of auctions that lead to an efficient allocation as well as to zero payoffs for the bidder with a valuation of zero ($v = 0$), the auction profit is identical.

- The bidders adapt their strategies so that the seller, in the case of efficient allocation, cannot extract additional profits even when using the most clever market rules.

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3. Static Games with Incomplete Information

- Similar is true for the revenue equivalence theorem.
- Today, auction research concentrates on whether and how the assumptions of the theorem are violated in reality, and which influence this can have on the conflicting goals between efficiency and revenue.
- Examples:
  - If the seller is risk averse, he should prefer auctions that implement the results in dominant strategies.
  - If the bidder valuations are statistically correlated, he should prefer the English auction.
  - If the bidders are asymmetric, the efficiency of the first-price auction is no longer secure…

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3. Static Games with Incomplete Information

Experiment 3.2 [Auctioning a Jar full of Coins]

• I am going to pass a jar filled with pennies around. Everyone can take a good look at it, but you are not allowed to open it.

• Please pass the jar to your neighbor and then write down a bid (in dollars) for the jar, without talking to your neighbor about it or showing your neighbor your bid.

• In the case that you submit the highest bid, you win the auction. That means you get the amount of the money in the jar in dollars, but you must pay me the amount of your bid.

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3. Static Games with Incomplete Information

3.2.2 Interdependent Valuations

• In this chapter we will look at the situation when the valuations are not independently chosen but are interdependent.

• An example of this is the jar that we auctioned. The dollar value of the jar is the same for everyone (which is why such an auction is called 'common-value' auction).

• A typical effect that appears in such auctions is the "winner's curse".

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3. Static Games with Incomplete Information

• Assume: the true and for everyone identical valuation is v.

• Every bidder has private information about v.

• Assume that bidder i's information about v can be represented by a number x_i, so that a larger x_i implies a larger v. Than it is true that:

$$E(v \mid x_i) \geq E(v \mid x_i, x_i > x_j \ \forall\, j \neq i).$$

• That means that a bidder who naively conditions his bid on his signal and who does not consider that the auction mechanism selects the one who has the most optimistic estimate as the winner, suffers the winner's curse.

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3. Static Games with Incomplete Information

Example 3.4 [A Standard Model without Proof]

- The common value v is randomly drawn from a normal distribution of the interval $(\underline{v}, \bar{v})$
- Before submitting their bids, every bidder receives a private signal s_i, that is randomly drawn from the interval $[v - \varepsilon, v + \varepsilon]$.
- The unknown valuation v determines the support (the interval from which the signal is drawn) and ε describes the precision of the signal.
- Auction mechanism: First –price auction

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3. Static Games with Incomplete Information

- The expected value of the object given the signal s_i is

$$E[v|s_i] = s_i.$$

- In symmetrical equilibriums, the auction will be won by the one who has the highest signal s_i.
- The bidders should therefore anticipate that the winner of the auction received the highest signal of all the other bidders.
- The valuation should accordingly be conditioned to this information:

$$E[v|s_{max} = s_i] \text{ , where } s_{max} := \max\{s_1, \ldots, s_n\}$$

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3. Static Games with Incomplete Information

• Stated (without proof):

$$E[v|s_{\max} = s_i] = s_i - \varepsilon\frac{n-1}{n+1}$$
$$< E[v|s_i]$$

• The smaller the precision of the signal (i.e.the larger ε) the greater the deduction.

•The more bidders, the greater the deduction.

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3. Static Games with Incomplete Information

Equilibrium (without proof):

Symmetrical Bayesian Nash equilibrium strategies in a common value auction: For all *i:*

$$b(s_i) = s_i - \varepsilon + \phi(s_i),$$

where $\phi(s_i) := \frac{2\varepsilon}{n+1} e^{-\frac{n}{2\varepsilon}[s_i - (\underline{v}+\varepsilon)]}$

Note that s_i is quickly approaching zero and therefore can be neglected for our purposes.

The bidder's equilibrium payoff is $2\varepsilon/(n + 1)$, thus positive, increasing in ε and decreasing in n.

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3. Static Games with Incomplete Information

Example 3.5 [Real World Examples]:

Auctions of oil fields, book manuscripts, UMTS auctions, baseball players, managers, IPOs, online auctions, experimental auctions, . . .

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An Early Example of the "Winner's Curse" in an Auction

After murdering the Roman Emperor Pertinax (March 28, A.D.193), the praetorian guards had an empire to dispose of:

> Then ensued a most disgraceful business and one unworthy of Rome. For, just as if it had been in some market or auction-room, both the City and its entire empire were auctioned off. The sellers were the ones who had slain their emperor, and the would-be buyers were Sulpicianus and Julianus, who vied to outbid each other, one from the inside, the other from the outside [of the praetorian camp]. They gradually raised their bids up to twenty thousand sesterces per soldier. Some of the soldiers would carry word to Julianus, "Sulpicianus offers so much; how much more do you make it?" And to Sulpicianus in turn, "Julianus promises so much; how much do you raise him?" Sulpicianus would have won the day, being inside and being prefect of the city and also the first to name the figure twenty thousand, had not Julianus raised his bid no longer by a small amount but by five thousand at one time, both shouting it in a loud voice and also indicating the amount with his fingers. So the soldiers, captivated by this excessive bid and at the same time fearing that Sulpicianus might avenge Pertinax (an idea that Julianus put into their heads), received Julianus inside and declared him emperor. [*Dio's Roman History*, book LXXIV]

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Auction theorists will note the success of the final "jump" bid in what we would now call an English auction. Assuming Julianus committed to pay all of the more than 10,000 praetorians, the total bid corresponded to well over a billion dollars today. (We approximated the current value by reference to the salaries of elite soldiers. The winning bid corresponded to about five years' wages per praetorian.) We know of many smaller auctions in the early Roman Empire, a loan of seven million of sesterces (when prices were somewhat lower than in 193), and imperial revenues that must have been far larger, but this bid appears enormous even for the fully monetized market economy of the early Empire. Julianus appears to have defaulted on his bid in any case, and the soldiers' resulting anger may have contributed to his short tenure in office.

Two months later, Septimius Severus marched an army to Rome, the praetorians deserted Julianus, and he was murdered after a reign of only 66 days. In Gibbon's words, Julianus was "beheaded as a common criminal, after having purchased, with an immense treasure, an anxious and precarious reign of only sixty-six days." Severus then disbanded the troublesome praetorians.

[*Dio's Roman History*, book LXXIV, translated by Earnest Cary (Cambridge, Mass.: Harvard Univ. Press, 1960; Edward Gibbon, *The Decline and Fall of the Roman Empire* (1776), vol. 1, chap. V.]

Suggested by Paul Klemperer and Peter Temin

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2. Dynamic Games with Complete Information

2.1 Extensive Form Games

• Dynamic: sequence of actions

• Assumptions: Rationality, game rules, and preferences are common knowledge; ‘perfect recall‘

• Dynamic games can be represented in extensive form (for more exact definitions compare Fudenberg/Tirole p. 77ff.)

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2. Dynamic Games with Complete Information

Definition 2.1 [Extensive Form]

The extensive form of an *n*-person game specifies:

(1) The order (when each player „makes a move“)

(2) Which actions are possible for the player who is making a move,

(3) Which knowledge the move-making player has (perfect or imperfect information), and

(4) The payoff for each player for each combination of moves.

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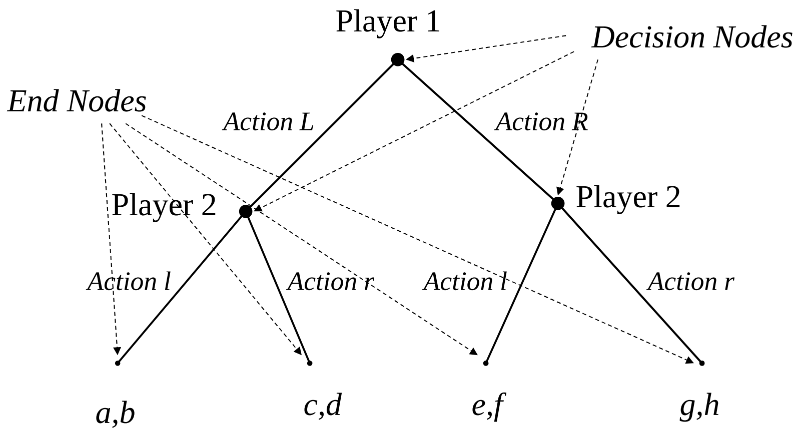
2. Dynamic Games with Complete Information

Game Tree:



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2. Dynamic Games with Complete Information

With certain strategy definitions, a game in extensive form can also be represented in (reduced) normal form. (von Neumann and Morgenstern, 1947).

Definition 2.2 [Strategy]

A (pure) strategy of a player is a complete plan of action. It specifies an action for the player at every decision node where the player makes a move. With imperfect information, it specifies an action for every set of information on which the player makes a move.

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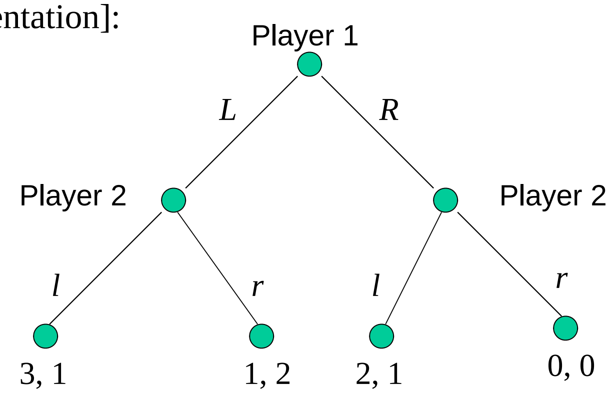
Example 2.1 [Extensive Form and Normal Form Representation]:



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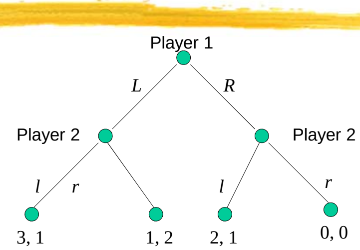
2. Dynamic Games with Complete Information

Normal Form Representation



	(*l*, *l*)	(*l*, *r*)	(*r*, *l*)	(*r*, *r*)
L	**3,1**	**3,1**	**1,2**	**1,2**
R	**2,1**	**0,0**	**2,1**	**0,0**

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2. Dynamic Games with Complete Information

• The action range of Player 1 is identical to the strategy range (without mixed strategy).

• Player 2 must decide between two actions, but four strategies.

• The normal form is a reasonable and simple description of a dynamic game if all players can already determine their strategies at the beginning of the game.

• *However*: The solution concepts for games in extensive form can (as we will see) depend on the chronological progression of the actions taken in the game. The normal form abstracts from the time dimension.

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2. Dynamic Games with Complete Information

2.2 Perfect and Imperfect Information

Definition 2.3 [Information Sets]

The information set of a player is a combination of decision nodes for which the following applies:

(1) The player makes a move at every node in the information set.

(2) If the game reaches the information set, the player does not know which decision node in the information set will be reached.

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2. Dynamic Games with Complete Information

• It follows that the set of possible actions at every decision node within the information set must be identical.

• If all information sets in a game (with complete information) are singletons, then it is said to be a game with perfect information. Otherwise, it is said to be a game with imperfect information.

• In a game with perfect information, every player making a move knows how the entire game has played out until that point.

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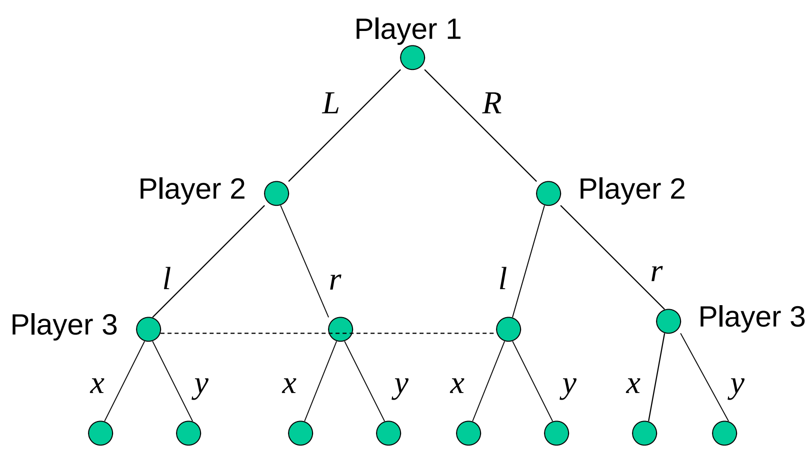
2. Dynamic Games with Complete Information

Example 2.2 [Game with Imperfect Information]:



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2. Dynamic Games with Complete Information

Example 2.2 [Game with Imperfect Information]:

• Player 3 can only watch whether or not the course of the game was (R, r).

• The dotted line indicates the information set.

• Every normal form game can be represented as a game in extensive form. To do this, multi-element information sets are generally required.

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2. Dynamic Games with Complete Information

Example 2.3 [Normal Form and Extensive Form Representation]:

	c	*d*
C	3,3	1,4
D	4,1	2,2

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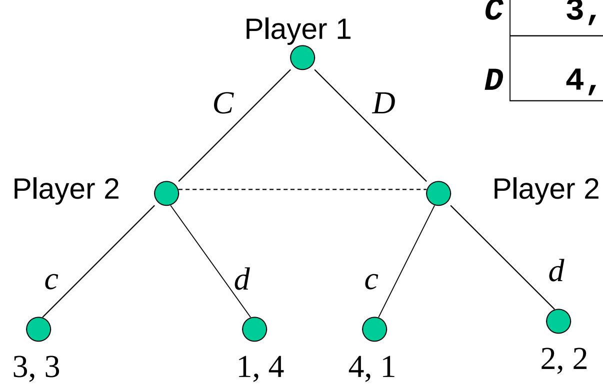
2. Dynamic Games with Complete Information

Extensive Form Representation

	c	d
C	3,3	1,4
D	4,1	2,2



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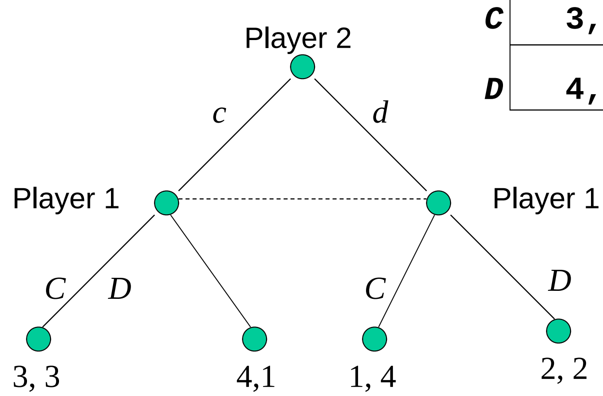
2. Dynamic Games with Complete Information

Alternative Extensive Form Representation

	c	d
C	3,3	1,4
D	4,1	2,2



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2. Dynamic Games with Complete Information

Example 2.4 [Imperfect Information and Moral Hazard]:

• The principal possesses imperfect information about the actions of the other players: 'hidden action ‘.

• Strategic exploitation of private information ('hidden action' or 'hidden knowledge') often leads to Pareto-inferior results: 'moral hazard’.

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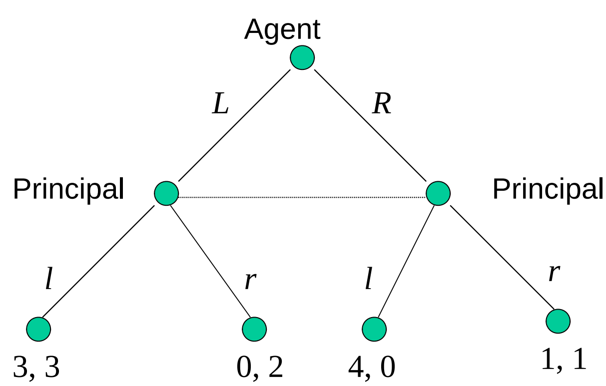
2. Dynamic Games with Complete Information

Example 2.4 [Imperfect Information and Moral Hazard]:



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2. Dynamic Games with Complete Information

Example 2.4 [Imperfect Information and Moral Hazard]:

• *Hidden action*: The principal does not know what the agent is doing. Accordingly, he cannot condition his actions off of the behavior of the agent.

• The agent has the dominant strategy *R*. Accordingly, in Nash equilibrium the principal chooses *r*, resulting in a payoff of (1, 1).

• *Moral hazard*: If the principal could make his decision dependent on the decision of the agant (perfect information), then the agent would select *L*. With perfect information, the payoff (3,3) can therefore be reached in a Nash equilibrium.

•In other words, inefficiency results from asymmetric information!

• The standard application for moral hazard with hidden action is the insurance market.

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2. Dynamic Games with Complete Information

Experiment 2.1 [Ultimatum Game]:

In the ultimatum game (Güth, Schmittberger, Schwarze, 1982), the „distributor“ suggests a division of a sum of \$10 between himself and the „receiver.“

The receiver has veto power. This means that he can accept the suggested distribution, so that the money is divided as suggested, or he can decline, meaning that neither player receives anything.

Imagine that you are playing the ultimatum game. As the proposer, how much would you offer the receiver ($0 \leq x \leq 10$)? What minimum offer would you accept as the receiver ($0 \leq m \leq 10$)?

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2. Dynamic Games with Complete Information

2.3 Backwards Induction

- How do we solve games in extensive form?
- The Nash equilibrium sometimes suggests implausible solutions for extensive form games since it does not exclude „empty“ threats.

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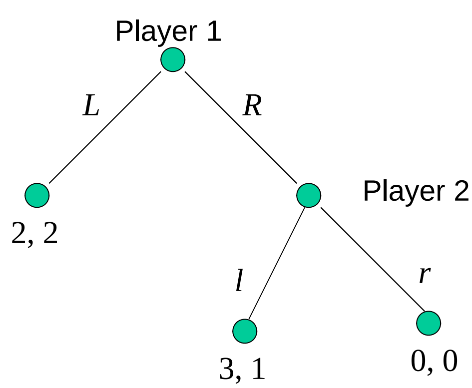
2. Dynamic Games with Complete Information

Example 2.5 [Incredible Threat; Selten 1965]:



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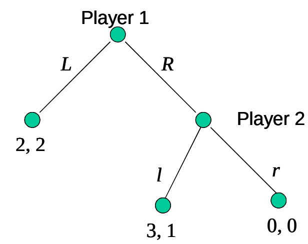
2. Dynamic Games with Complete Information



	l	r
L	2,2	2,2
R	3,1	0,0

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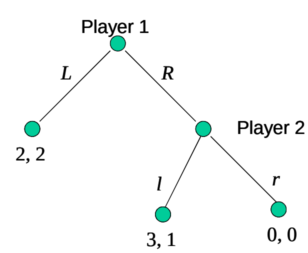
2. Dynamic Games with Complete Information



	l	r
L	2,2*	*2,2*
R	*3,1*	0,0

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2. Dynamic Games with Complete Information

• Two Nash equilibria: (R, l) and (L, r).

• Analysis of the sequential structure shows that (L, r) depends on an incredible threat (even when the corresponding subgame in equilibrium is not reached) and is therefore is implausible. It is not „perfect."

• Backwards induction is an alternative solution concept that can be applied to extensive form games with perfect information and singles out equilibria with empty threats.

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2. Dynamic Games with Complete Information

- Idea of Backwards Induction:
- First, the optimal action for every possible course of the game is determined for the player who makes the last move of the game.
- Then, the player who makes the next-to-last move anticipates this optimal action for the player making the last move; considering this optimal action, the next-to-last player decides on his/her actions for every possible course of the game.
- In this way, a player works through the game all the way back to the first move of the game and comes up with game plan that is said to be the „result of backwards induction."

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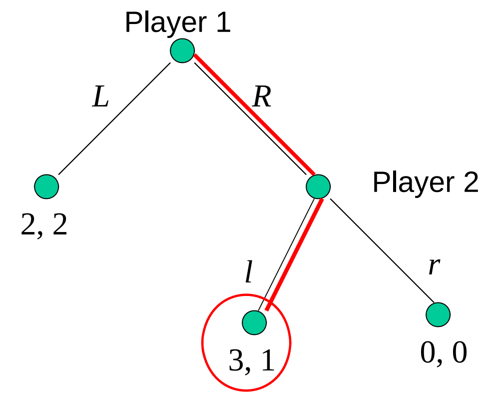
2. Dynamic Games with Complete Information

Example 2.6 [Incredible Threat; Selten 1965]: Solution by Backwards Induction



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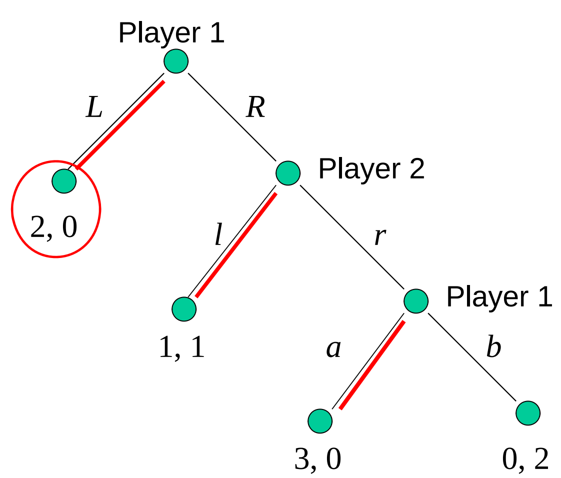
2. Dynamic Games with Complete Information

Example 2.6 [Backwards Induction]:



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2. Dynamic Games with Complete Information

• Player 1 should choose L since he anticipates that Player 2 would choose l since he in turn anticipates that Player 1 would choose a.

• What would happen if the equilibrium path were *not* followed is critical to the solution of the game.

• (However, such hypothetical considerations not always simple, as we will see.)

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2. Dynamic Games with Complete Information

Example 2.7 [General Two-Stage Game]:

Stage 1: Player 1 chooses an action $a_1 \in A_1$.

State 2: Player 2 observes a_1 (Perfect Information) and chooses $a_2 \in A_2$.

Payoff functions: $u_1(a_1, a_2)$, $u_2(a_1, a_2)$.

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2. Dynamic Games with Complete Information

Example 2.7 [General Two-Stage Game]: *(Continued)*

Backwards Induction:

• Player 2 chooses a_2 so that a_2 solves the optimization problem $\max_{a2 \in A2} u_2(a_1, a_2)$ for every possible course a_1 . If the optimization problem has a definite solution, Player 2 ‘s best response is given by $R_2(a_1)$.

• Player 1 anticipates Player 2 ‘s reaction $R_2(a_1)$. As such, Player 1 ‘s problem is: $\max_{a1 \in A1} u_1(a_1, R_2(a_1))$.

• $(a_1^*, R_2(a_1^*))$ is the result of backwards induction.

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2. Dynamic Games with Complete Information

Example 2.8 [Stackelberg Model]:

Stage 1: Firm 1 (Stackelberg Leader) chooses a quantity $q_1 \geq 0$

Stage 2: Firm 2 (Stackelberg Follower) observes q_1 and chooses a quantity $q_2 \geq 0$

Cost Structure: no fixed costs; identical, constant marginal costs *c;*

(inverse) Demand Function: $P(Q) = a - bQ$ with $Q = q_1 + q_2$

Payoff for Firm *i*:

$$u_i(q_i, q_j) = \pi_i(q_i, q_j) = q_i[P(Q) - c] = q_i[a - bQ - c]$$

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2. Dynamic Games with Complete Information

Solution:

1) Determine $R_2(q_1)$:

$$\max_{q_2 \geq 0} \pi_2(q_1, q_2) = q_2[a - b(q_1 + q_2) - c]$$

First Order of Conditions:

$$a - b(q_1 + q_2) - bq_2 = c$$

$$\Rightarrow R_2(q_1) = \frac{1}{2b}(a - c - bq_1)$$

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2. Dynamic Games with Complete Information

2) Determine q_1*:

$$\max_{q_1 \geq 0} \pi_1(q_1, R_2(q_1)) = q_1[a - b(q_1 + R_2(q_1)) - c]$$

$$= q_1\left[a - b\left(q_1 + \frac{1}{2b}(a - c - bq_1)\right) - c\right]$$

First Order of Conditions :

$$\frac{1}{2}(a - c) - bq_1^* = 0$$

$$\Rightarrow q_1^* = \frac{1}{2b}(a - c)$$

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2. Dynamic Games with Complete Information

3) Result of Backwards Induction:

$$q_1^* = \frac{1}{2b}(a-c) \text{ und } R_2(q_1^*) = \frac{1}{4b}(a-c)$$

Total Quantity**:**

$$Q = \frac{3}{4b}(a-c)$$

Market Price**:**

$$P(Q) = \frac{1}{4}(a+3c)$$

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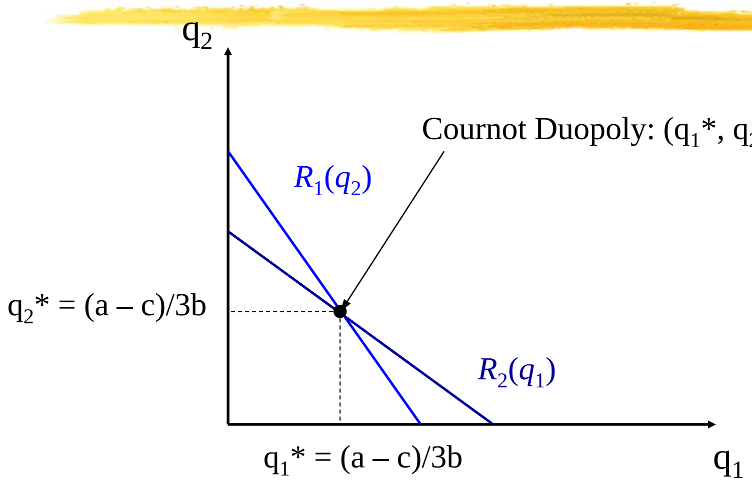
2. Dynamic Games with Complete Information



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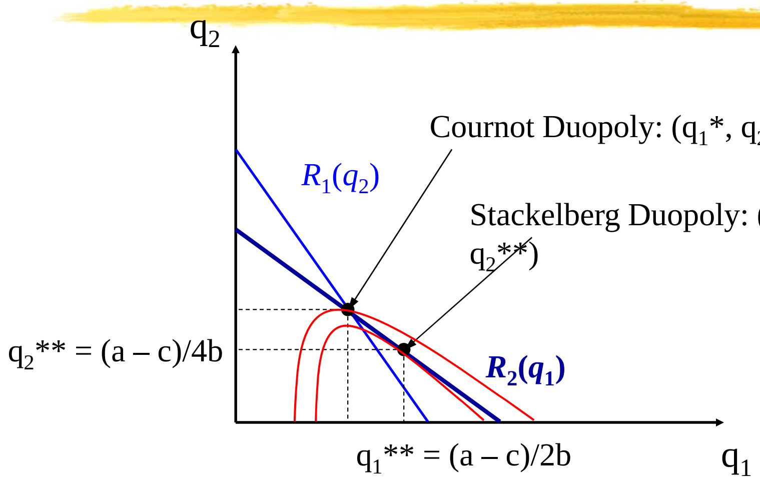
2. Dynamic Games with Complete Information



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2. Dynamic Games with Complete Information

Was is the solution of negotiations?

- *„Rather than solve the two-person bargaining game by analyzing the bargaining process, one can attack the problem axiomatically by stating general properties that "any reasonable solution" should possess. By specifying enough such properties one excludes all but one solution.“*
  John Nash. (1953) Two-person Cooperative Games, Econometrica.

Axioms of Nash ‘s Bargaining Solution:

- Efficiency and Rationality
- Symmetry
- Invariance
- Independence of irrelevant alternatives

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2. Dynamic Games with Complete Information

Example 2.9 [Ultimatum Game]:

Stage 1: In divvying up \$10, Player 1 offers a share x to Player 2, where $0 \leq x \leq 10$.

Stage 2: Player 2 observes x either accepts the offer (j) or declines (n).

Player Payoffs:

$$u_1(x, j) = 10 - x, \quad u_2(x, j) = x, \quad u_1(x, n) = u_2(x, n) = 0$$

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2. Dynamic Games with Complete Information

Solution by Backwards Induction:

1) Player 2 will accept every offer $x > 0$. When $x = 0$, Player 2 is indifferent between accepting and declining. (This means that declining is weakly dominated.)

2) Player 1 will offer Player 2 as little as possible and keep the rest for him/herself.

3) Result of Backwards Induction: (0, j).
(Is the solution definite? What is your answer if offers must be made in whole-cent amounts?)

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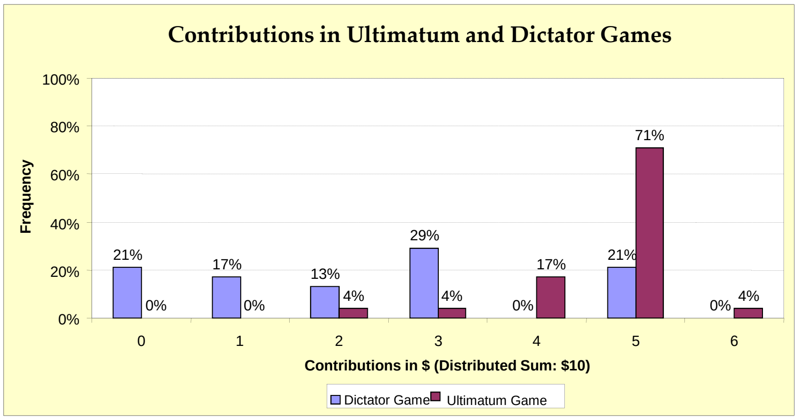
2. Dynamic Games with Complete Information

Observations from *Forsythe et al.* 1994:



2. Dynamic Games with Complete Information

Example 2.10 [Sequential Negotiation Game in 3 Stages, 1982]:

Period 1:

Stage 1: Player 1 offers a share of $1: s_1 for him/herself and $(1 - s_1)$ for Player 2.

Stage 2: Player 2 observes s_1 and accepts (*j*, game is over) or declines (*n*, game continues).

Period 2:

Stage 1: Player 2 offers a share: s_2 for Player 1 and $(1 - s_2)$ for him/herself.

Stage 2: Player 1 observes s_2 and accepts (*j*, game is over) or declines (*n*, game continues).

Player 1 gets his/her share *s* and Player 2 his/her share $(1 - s)$. 40

2. Dynamic Games with Complete Information

s_t is Player 1 ‘s share in the t-th round.

The players are impatient: every period, they negotiate with the factor δ, where $0 < \delta < 1$, payoffs that they will receive only later.

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2. Dynamic Games with Complete Information

Solution:

1) Determine s_2^*:

Player 1 will only accept Player 2 ‘s offer if $s_2 \geq \delta s$.

Player 2 must therefore decide between a share of $(1 - \delta s)$ in Period 2 and $(1 - s)$ in Period 3, which is only worth $\delta(1 - s)$ in Period 2.

Because $(1 - \delta s) > \delta(1 - s)$, the optimal offer is $s_2^* = \delta s$.

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2. Dynamic Games with Complete Information

2) Determine s_1*:

Player 2 will only accept Player 1 ‘s offer if $(1 - s_1) \geq \delta(1 - s_2^*)$ or alternately if $s_1 \leq 1 - \delta(1 - s_2^*) = 1 - \delta(1 - \delta s)$.

Player 1 must therefore decide between a share of $1 - \delta(1 - \delta s)$ in Period 1 and $s_2^* = \delta s$ in Period 2, which is worth $\delta(\delta s) = \delta^2 s$ in Period 1.

Because $1 - \delta(1 - \delta s) > \delta^2 s$, the optimal offer is $s_1^* = 1 - \delta(1 - \delta s)$.

3) Result of Backwards Induction:
Player 1 offers $s_1^* = 1 - \delta(1 - \delta s)$ and Player 2 accepts.

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2. Dynamic Games with Complete Information

- The larger δ *is*, the smaller the advantage of Player 1 (what happens when $\delta = 0$, when $\delta = 1$?).
- Rubinstein negotiations lead to efficient allocations (no time delay, no declinations).
- This concept also applies to infinite negotiations (see Example 2.28).
- In finite negotiations, the last “offerer” can siphon off the remainder of the sum to be divided. The bargaining power of the other players comes from the opportunity to delay negotiations.
- However, this is not true with incomplete information (Myerson-Satterthwaite Theorem) or with “continuous” negotiations with an indefinite deadline.

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2. Dynamic Games with Complete Information

2.5.1 Finitely Repeated Games

Definition 2.5 [Finitely Repeated Game]

$G(T)$ describes a finitely repeated game in which the stage game G is played exactly T times ($T < \infty$).

The results of the previous round are observed before the next round begins.

The payoff functions of $G(T)$ result from the payoffs of the stage game T.

(The stage game G can be a normal or extensive form game.)

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2. Dynamic Games with Complete Information

• Repeated games, as we will study them here, have a stationary, time-invariant structure : $u_t(s_t) = u(s_t)$.

• However, the strategies can have an indirect influence on payoffs over time, since s_{it} is a function of the complete course of the game.

The strategies thereby become very large very fast.

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2. Dynamic Games with Complete Information

Example 2.20 [Twice Repeated Prisoner's Dilemma]:

stage game *G*:

	c	*d*
C	4,4	0,5
D	5,0	1,1

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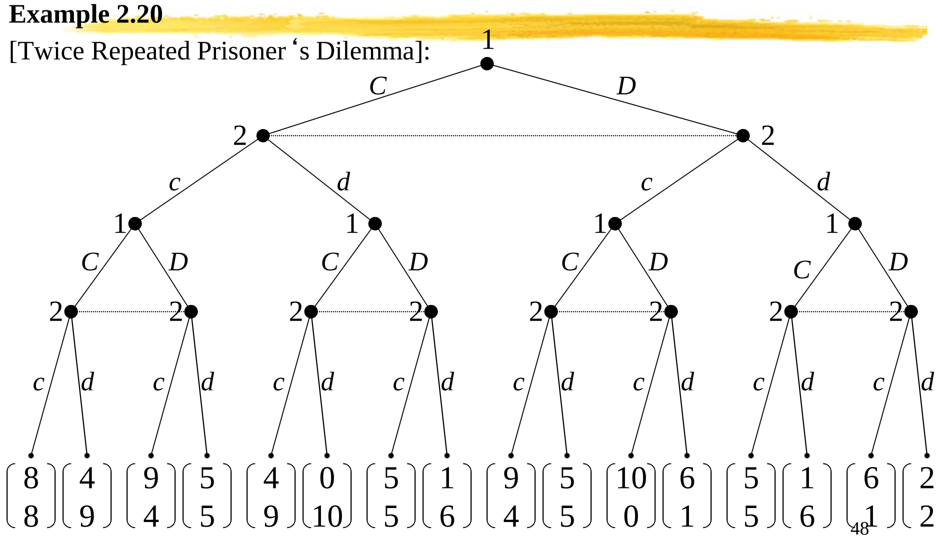
2. Dynamic Games with Complete Information

	c	d
C	4,4	0,5
D	5,0	1,1

Example 2.20

[Twice Repeated Prisoner's Dilemma]:



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2. Dynamic Games with Complete Information

Example 2.20 [Twice Repeated Prisoner ‘s Dilemma]:

Subgame Perfect Nash Equilibrium:

((D, (D|(D,d), D|(D,c), D|(C,d), D|(C,c))),
(d, (d|(D,d), d|(D,c), d|(C,d), d|(C,c))))

How many strategies per player are there in this game?

Player 1 must decide between 2 actions 5 different times.

Player 2 must decide between 2 actions 5 different times.
Therefore, every player possesses $2^5 = 32$ strategies in the game.

In general, a player in a Prisoner ‘s Dilemma repeated T times has
$2^{4^0} \times 2^{4^1} \times \ldots \times 2^{4^{T-1}} = 2^{(4^T - 1)/3}$ pure strategies
(for $T = 10$, the number is a 105,218-digit number).

2. Dynamic Games with Complete Information

Definition 2.6 [Subgame in finitely repeated game]

In a finitely repeated game $G(T)$, let there be a subgame that begins in $t + 1$ and a repeated game in which G is played for exactly $T - t$ rounds, so that $G(T - t)$.

For every possible course of the game until round t, there exists a subgame that begins in $t + 1$.

2. Dynamic Games with Complete Information

Proposition 2.2 [Equilibrium with Definite Equilibrium of the Stage Game]

If the stage game *G* has a definite equilibrium, then *G*(*T*) possesses for all $T < \infty$ a definite subgame perfect equilibrium in which the Nash equilibrium of G will be played in every round.

Evidence: through backward induction!

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2. Dynamic Games with Complete Information

Example 2.21 [Finitely Repeated Prisoner ‘s Dilemma]:

Stage Game *G*:

	c	*d*
C	4,4	0,5
D	5,0	1,1

• Subgame Perfect Equilibrium for all *G*(*T*) with $T < \infty$: Defection of both players in all rounds.

• This means that threats and promises are incredible from the beginning. This is evidence through backward induction!

• This is also a definite Nash equilibrium. Evidence through the repeated eliminataion of strictly dominated strategies.

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2. Dynamic Games with Complete Information

But:

If the equilibrium of the stage game is not definite, then threats and promises can be credible.

This means that in a subgame perfect equilibrium of $G(T)$, results can come about during the stage $t < T$ that cannot be reached through equilibria of G.

This is illustrated in the following example:

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2. Dynamic Games with Complete Information

Example 2.22 [Twice repeated game with credible threats]:

Stage Game G:

	c	*d*	*r*
C	4,4	0,5*	0,0
D	*5,0	*1,1*	0,0
R	0,0	0,0	*3,3*

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2. Dynamic Games with Complete Information

Example 2.22 [Twice repeated game with credible threats]:

- Nash equilibria of stage games: (*D*, *d*) and (*R*, *r*).
- Claim: There exists a subgame perfect equilibrium for *G*(2) so that (*C*, *c*) will be played in the first round.
- Suppose the players anticipate that (*R*, *r*) will be played in the second round if (*C*, c) was played in the first round, and otherwise (*D*, *d*).
- These trigger strategies create the following reduced representation of the entire game (in that the equilibrium payoffs of the second round subgames are considered.)

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2. Dynamic Games with Complete Information

Example 2.22 [Twice repeated game with credible threats]:

	c	*d*	*r*
C	4,4	0,5*	0,0
D	5,0	1,1*	0,0
R	0,0	0,0	3,3*

	c	*d*	*r*
C	7,7	1,6	1,1
D	6,1	2,2	1,1
R	1,1	1,1	4,4

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2. Dynamic Games with Complete Information

Example 2.22 [Twice repeated game with credible threats]:

	c	d	r
C	4,4	0,5*	0,0
D	*5,0	*1,1*	0,0
R	0,0	0,0	*3,3*

	c	d	r
C	*7,7*	1,6	1,1
D	6,1	*2,2*	1,1
R	1,1	1,1	*4,4*

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2. Dynamic Games with Complete Information

Example 2.22 [Twice repeated game with credible threats]:

- An equilibrium in this normal form game corresponds with a subgame perfect equilibrium in the dynamic game, since only equilibria will be played in the second stage.
- (C, c) is part of an equilibrium strategy in the first round.
- (Credible) threats and promises can therefore influence present behavior in a subgame perfect equilibrium.
- Problem: renegotiations are likely, since punishment contains Pareto-inferior equilibrium. However, if the Pareto-efficient equilibrium is always played, then the incentive to play (K, k) in the first round is lost.
- But, this problem does not come up in the following stage game.

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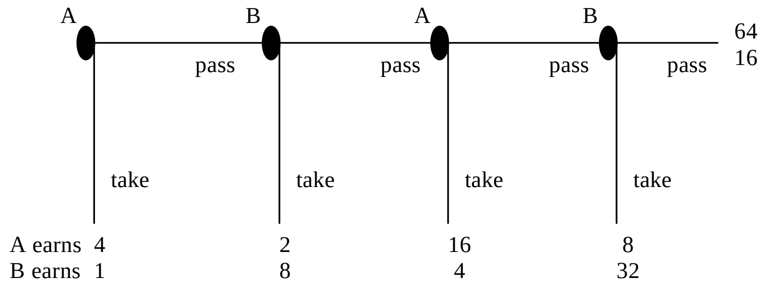
Centipede game

- **Players A and B take turns passing or taking. Passing doubles the size of the payoff pie. Taking ends the game. The rules and possible payoffs to the game are known, common knowledge.**



Do for next time: Solve this game by backward induction. Are there other Nash equilibria? Is the game iterative dominance solvable?

4. Dynamic Games with Incomplete Information

4.1 Motivation

In dynamic games with incomplete information, types can be ‘signaled’ or information can be ‘communicated’ and reputations can be ‘built’.

Accordingly, the most important application areas are:

• Signaling Games (in which the different types can signal their type based on the different costs of signaling)

• Cheap talk Games (in which signaling does not cost anything and hence the degree of information transmission depends on the degree to which the players have shared interests)

• Reputation Games (in which the player’s can build a reputation for being a certain type)

4. Dynamic Games with Incomplete Information

The problem presents itself in dynamic games with incomplete information as in games with complete information, to eliminate equilibria that are based on implausible threats.

Subgame perfection serves no purpose with incomplete information. Because the players do not know their opponent’s type, there is no longer a singleton information set after nature has chosen the types. There is no subgame.

(Note: The slides in this chapter partially are from a lecture of Prof. Klaus Schmidt from Munich.)

4. Dynamic Games with Incomplete Information

In this chapter we will generalize the idea of subgame perfection by examining continuation games.

A continuation game can also begin with a multiple element information set. However the player in this type of set, who has the move, must have a belief (i.e. a probability distribution) about which node of the set he finds himself in.

When we've defined continuation games, we can analyze the sequential rational behavior, i.e. behavior that is optimal in all continuation games created from the original game, and also those that are out-of-equilibrium.

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4. Dynamic Games with Incomplete Information

This brings us to a new equilibrium concept: *Perfect Bayesian Equilibrium*.

To the extent that the game we are analyzing becomes more complicated, we must set additional requirements to rule out the implausible equilibria.

You must however consider that not all the equilibrium concepts are arbitrary, rather they build on each other.

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4. Dynamic Games with Incomplete Information

We will see that perfect Bayesian equilibria are consistent with the following:

→ subgame perfect equilibria, when they are dealing with a dynamic game with complete information,

→ Bayesian Nash equilibria, when they are dealing with a static game with incomplete information,

→ Nash Equilibria, when they are dealing with a static game with complete information.

First, we will apply the idea of sequential rationality to games with complete but imperfect information. The jump to playing with incomplete information is then no longer large.

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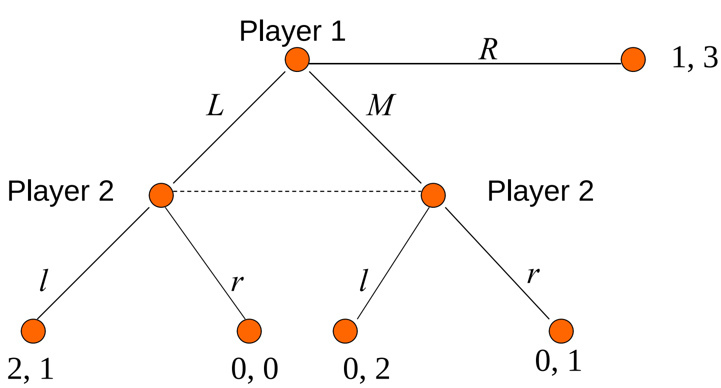
4. Dynamic Games with Incomplete Information

Example 4.1 [A variation of Selten's Horse, Selten, 1975]:



How should player 2 behave?

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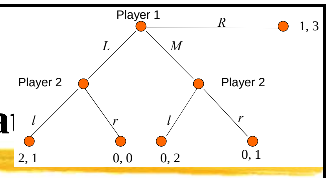
4. Dynamic Games with Incomplete Informa



When player 2 has the move, he has a dominant strategy: *l* . Then it is optimal for player 1 to also play *L*.

Are there other equilibria in this game?

1 \ 2	*l*	*r*
L	*2 , 1*	0 , 0
M	0 , 2*	0 , 1
R	1 , 3*	*1 , 3*

Normal form of Selten's Horse.

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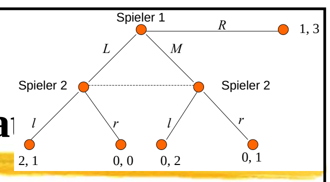
4. Dynamic Games with Incomplete Informa



Analysis of the normal form of the game shows that there is a second Nash equilibrium: (*R*, *r*). Is this equilibrium (*R*, *r*) subgame perfect?

Yes! Otherwise there would not be a subgame. Because (*R*, *r*) is a Nash equilibrium during the entire game, it is therefore also subgame perfect. However, it is certainly not sequentially rational.

How can we rule out the implausible equilibrium?

Consider the continuation game that begins when player 2 has the move. This continuation game begins in an information set that is not singleton and is therefore not a subgame. In spite of that, we also want the strategies to prescribe optimal behavior in such continuation games.

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4. Dynamic Games with Incomplete Information

The optimal behavior in a continuation game generally depends on the probability that a player assigns to the different nodes in his/her information set.

Additional Requirements on an Equilibrium:

Condition 4.1: In every information set the player who has the move must have a *belief* about which node he is on. A belief is a probability distribution of the possible nodes. A system of beliefs of all information sets is μ.

Condition 4.2: Given his beliefs, the behavior of every player must be sequentially rational, i.e., given his beliefs, his strategy in every continuation game must be the best response to the strategy of his opponent.

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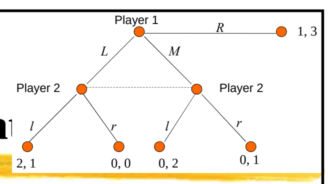
4. Dynamic Games with Incomplete Information



These conditions rule out the equilibrium (R, r). Regardless of which beliefs player 2 has in his information set, it is always better to play l than r.

It is not the rule that the beliefs of the other player do not play a role. In general, the optimal action of a player depends on his beliefs. Therefore, we cannot allow random beliefs for rational behavior. Instead, we must insure that the beliefs are consistent with the player's strategies and the history of the game up till that point.

Condition 4.3: A player's beliefs in every information set (in equilibrium and out-of-equilibrium) result from the equilibrium strategy of the player and from Bayes' rule, when this rule can be applied.

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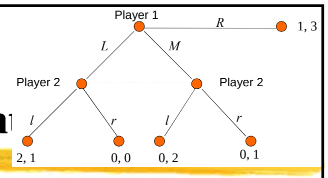
4. Dynamic Games with Incomplete Informa



In example 4.1 and in the equilibrium (*L, l*), player 2 must believe that when he reaches his information set, he finds himself on the left node with a probability of 1.

Let's assume that in this game, these is an equilibrium with mixed strategies in which player 1 moves with a probability of p, q and $1 - p - q$ to *L, M, R* respectively.

If player 2's information set is then reached with the node *L* and *M,* player 2 must believe, based on Bayes' Rule, that he finds himself on node *L* with a probability of $p/(p + q)$ and on node *M* with the probability of $q/(p + q)$.

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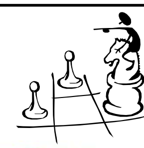
4. Dynamic Games with Incomplete Information



4.2 Perfect and Sequential Equilibria

Definition 4.1: A *perfect Bayesian equilibrium (PBE)* is a strategy profile and a system of beliefs (s, μ), so that the conditions 4.1-4.3 hold true, i.e. so that

→ The strategies of all players are sequentially rational given μ;

→ The beliefs (in equilibrium and out-of-equilibrium) are derived from the player's equilibrium strategies with help from Bayes' Rule, whenever this rule is applicable.

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4. Dynamic Games with Incomplete Information

Remarks

When the players use complete mixed strategies, all the nodes would be reached with a positive probability. Then we can always use Bayes' Rule to "update" the beliefs of the players.

If the strategies are not complete mixed strategies, certain strategies are reached with the probability of zero. In such information sets, it is possible that Bayes' rule is not applicable. Then the concept of PBE allows random beliefs, so that often many results can be supported as PBEa. Therefore we need to further refine the equilibrium concept.

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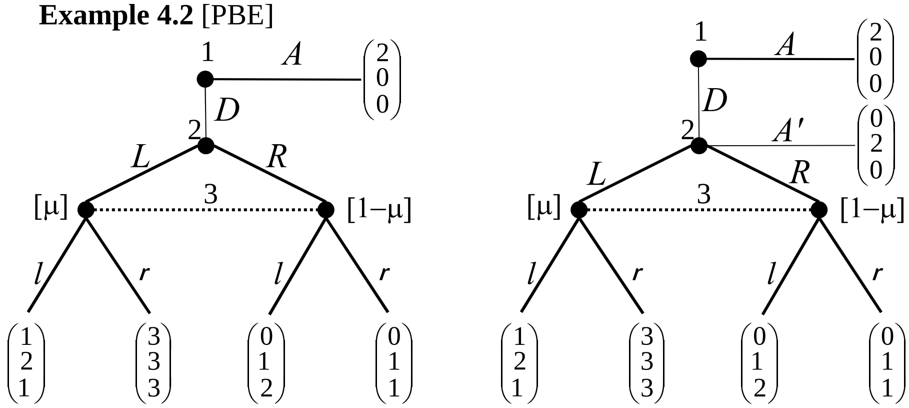
4. Dynamic Games with Incomplete Information

Example 4.2 [PBE]



(a) Bayes' Rule applicable

(b) Bayes' Rule not applicable

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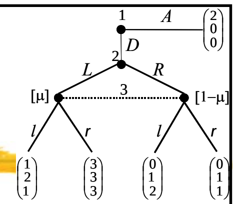
4. Dynamic Games with Incomplete Information



Example (a) has a true subgame with a clear equilibrium: (*L,r*). Therefore, the clear PBE is: (*D,L,r*), $\mu = 1$. This is also the only subgame perfect equilibrium.

There are other Nash equilibria, for example (*A,L,l*). In this equilibrium, player 3's information set is reached with a probability of zero.

If we only require that Bayes' rule be used in equilibrium, than player 3's beliefs are random. Particularly $\mu = 0$ is possible. With $\mu = 0$ it is actually optimal for player 3 to choose *l.*

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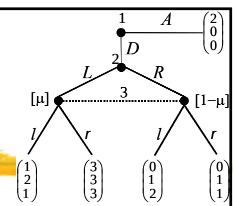
4. Dynamic Games with Incomplete Information



We additionally require that whenever possible Bayes' rule should also be applied out-of-equilibrium.

Here it is possible. Given player 2's strategy, *L*, Bayes' rule specifies: $\mu = 1$. Therefore *l* is not optimal for player 3 and (*A,L,l*) is not a PBE.

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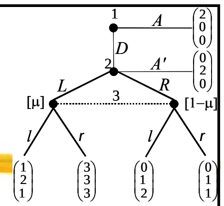
4. Dynamic Games with Incomplete Information



In *example (b)* player 2 has the option to end the game (A'). The old equilibrium (D,L,r) with $\mu = 1$ is still a PBE. Now, however, there is a second PBE: (A,A',l) and $\mu \leq 1/3$.

When player 2 has the move in this equilibrium, he chooses either L or R with the probability of zero. Bayes' Rule therefore is not applicable.

- → The beliefs of player 3 are random; particularly, $\mu \leq 1/3$ is possible.
- → For 3 it is actually optimal to play l.
- → For 2 it is actually optimal to play A'.
- → For 1 it is actually optimal to play A.

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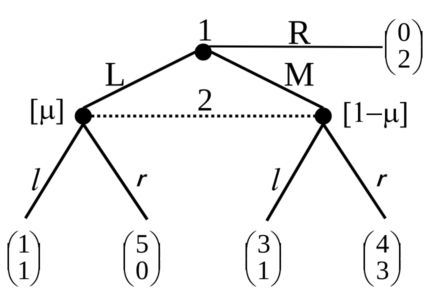
4. Dynamic Games with Incomplete Information

Example 4.3 [PBEs in mixed strategies]



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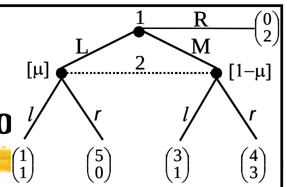
4. Dynamic Games with Incomplete Informatio



→ When player 2's information set is reached, *l* is optimal if and only if $1 \geq 3(1 - \mu) \Leftrightarrow \mu \geq 2/3$.

→ Assume, $\mu > 2/3$ is a subset of a PBE. Then player 2 plays *l* and player 1 plays *M*. That would imply $\mu = 0$, which is a contradiction.

→ Assume, $\mu < 2/3$ is a subset of PBEs. Then player 2 would play *r* and player 1 would play *L*. That would imply $\mu = 1$, which is again a contradiction.

→ Therefore, $\mu = 2/3$. Now player 2 is indifferent and must play in equilibrium *l* with a probability *p,* so that player 1 is also indifferent between *L* and *M*. This is exactly the case when $p + 5(1 - p) = 3p + 4(1 - p) \Leftrightarrow p = 1/3$.

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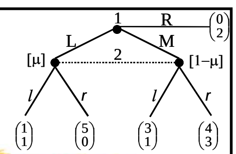
4. Dynamic Games with Incomplete Information



→ Therefore the unequivocal PBEs in this game:
- Strategy of player 1: (2/3, 1/3, 0)
- Strategy of player 2: (1/3, 2/3)
- Beliefs of player 2: $\mu = 2/3$.

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4. Dynamic Games with Incomplete Information

4.3 Signaling Games

An important class of dynamic games with incomplete information is the so called signaling game that has the following structure.

There are two players, a sender and a receiver.

1. Nature chooses the type t out of a set $T = \{t_1, \ldots, t_I\}$ according to the probability distribution $\mu(t)$.
2. The sender learns his type and chooses a message m $M = \{m_1, \ldots, m_J\}$.
3. The receiver observes the message (but not t_i) and than chooses an action a out of the set of possible actions $A \in \{a_1, \ldots, a_K\}$.
4. The payoffs are $U_S (t, m, a)$ and $U_E (t, m, a)$.

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4. Dynamic Games with Incomplete Information

Examples

Job market signaling: The sender is a bidder, whose abilities are private information. He chooses a level of education. The employer observes the education, but not the ability and decides upon the wage.

Initial public offering: The sender is an entrepreneur who knows the value of his firm. He chooses a portion of his firm that he wants to sell on the stock market. The market observes this portion, but not the worth of the firm, and decides upon the worth of the stock.

Limit pricing: The sender is a monopolist who knows his marginal costs. He chooses the amount of his output in the first quarter. The receiver is a potential market entrant. He observes the chosen amount of output, but not the marginal costs of the monopolist. He then decides upon his market entrance.

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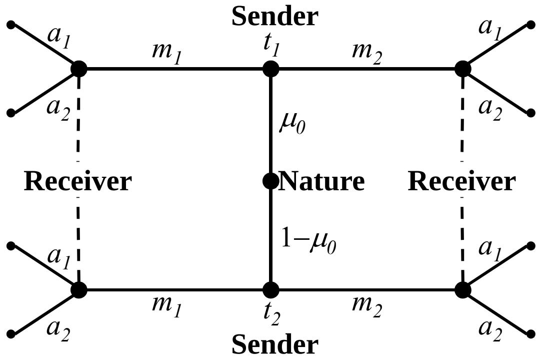
4. Dynamic Games with Incomplete Information

Example 4.4 [Signaling Game's Structure]

We will initially examine an abstract signaling game with two types, two sender's messages, and two receiver's actions:



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4. Dynamic Games with Incomplete Information

In this game, every player has four possible *strategies*. Be aware that a strategy for each information set in which player *i* has the move must announce how player *i* should behave.

→ A possible strategy of the sender is (m_2, m_1): "Play m_2, if you are type t_1, and m_1, if you are type t_2."

→ A possible strategy for the receiver is (a_2, a_1): "Play a_2, if the sender sent m_1, and a_1, if the sender sent m_2."

Furthermore, following both messages, the receiver must state a *belief* with regards to the node at which he resides. Since there are two possible information sets for the receiver, there must also be two *beliefs* (μ_1, μ_2).

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4. Dynamic Games with Incomplete Information

There are two possible types of equilibria in pure strategies.

Separating equilibria: Different types of senders choose different messages.

Pooling-equilibria: Both types of senders choose the same message.

If there are more than two types there can also be semi-separating equilibria: Certain messages are only chosen from some types and not from others; However, the separation is not perfect.

In mixed strategies, hybrid equilibria are also possible: One type sends a message with a probability of 1 and the other type randomizes between both messages.

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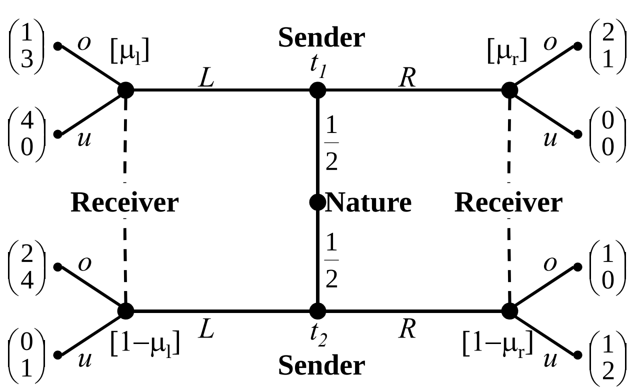
4. Dynamic Games with Incomplete Information

Example 4.5 [A Signaling Game]



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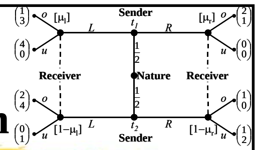
4. Dynamic Games with Incomplete Information



1. *Pooling on L*: Assume there is a PBE in which both types select L
   → $\mu_1 = {}^1/_2$.
   → The receiver reacts with o.

   Is it actually optimal for both types to choose L? Certainly for t_2; However, for t_1,it is only optimal to choose L when the receiver reacts with u if the sender plays R.

   What would the receiver do if he observed R? For him, u is optimal if $\mu_r \leq {}^2/_3$. Note that the Bayes' rule cannot be applied here to determine μ_r.

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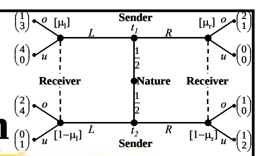
4. Dynamic Games with Incomplete Information



Thus, the following strategies and beliefs are PBEs:

(L, L)	(Strategy of the sender)
(o, u)	(Strategy of the receiver)
$\mu_1 = {}^1/_2$	(Belief of the receiver after L)
$\mu_r \leq {}^2/_3$	(Belief of the receiver after R)

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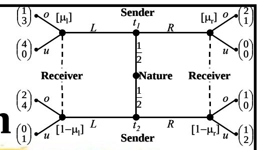
4. Dynamic Games with Incomplete Information



2. *Pooling on R:*
   - → $\mu_r = {}^1/_2$.
   - → Receiver reacts with *u*.
   - → Sender of type 1 receives 0.

   This cannot be an equilibrium since t_1 is guaranteed a payoff of at least 1 if he plays *L*.

3. *Separation (L, R):*
   - → $\mu_1 = 1$, $\mu_r = 0$.
   - → The receiver reacts to *L* with *o* and to *R* with *u*.
   - → Both types of sender receive payoff 1.

   This cannot be an equilibrium: Type t_2 would be better off if he chose *L*. If he did this, the receiver would react with *o* and hence the sender would have 2 instead of 1.

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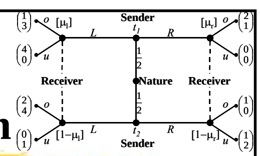
4. Dynamic Games with Incomplete Information



4. *Separation (R, L):*
   - → $\mu_1 = 0$, $\mu_r = 1$.
   - → Receiver reacts to *L* with *o* and to *R* with *o*.
   - → Both types of sender receive payoff 2.

   No type can better his position by changing his strategy.

   Thus, the following strategies and beliefs are a PBE:

(*R*, *L*)	(Strategy of the sender)
(*o*, *o*)	(Strategy of the receiver)
$\mu_1 = 0$	(Belief of the receiver after *L*)
$\mu_r = 1$	(Belief of the receiver after *R*)

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4. Dynamic Games with Incomplete Information

Spence - Nobel Prize 2001

Example 4.6 [Education as Signal; Spence 1974]
(see Osborne 2003, 340-342)

- Education as signal for potential Employers (E), such that employee (EE) is highly skilled.
- The same education would not be beneficial for someone who is less skilled, since this person would face higher costs to obtain an equivalent education.
- EE: Skill level $K = \begin{cases} H \\ L \end{cases}$, *where* $L < H$
- The EE knows his skill level K, the two potential Es do not.

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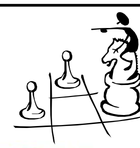
4. Dynamic Games with Incomplete Information



- EE chooses a certain level e of education.
- Then, both Es who observe e, offer a wage of w_1 and w_2 simultaneously .
- Lastly, EE chooses one of the two offers.
- An EE with skill level K needs to pay e/K to obtain e units of education.

Payoffs:

- EE with skill level K: Payoff $= w - e/K$.
- The firm who employs EE with skill level K:
  Payoff = $K - w$.

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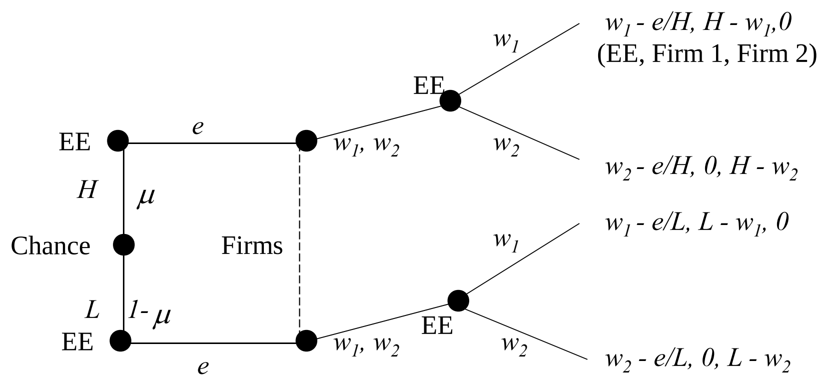
4. Dynamic Games with Incomplete Information

Signaling Game for a possible education level *e*:



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Separating Equilibrium, in which an EE with a high skill level chooses a positive amount of education:

- **EE's Strategy:** Type H chooses $e = e^*$ and type L chooses $e = 0$.
  After observing the offers of the Es, both types choose the highest wage. In the case where the offers are the same, both types choose firm 1.

- **Firm's Belief:** Each firm believes that the EE is of type H if he chooses e^*, otherwise they believe that the E is of type L.

- **Firm's Strategy:** Each firm offers an EE who chooses e^* a wage of H. If the EE chooses any other value for *e*, the firm offers a wage of L.

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4. Dynamic Games with Incomplete Information

Conditions for the parameter under which the player's strategy is sequentially rational:

- **EE:**

Type H: $e = e^*$: Payoff = $H - e^*/H$

$e \neq e^*$: Payoff = $L - e/H$ (Type H would choose $e = 0$ here)

$\longrightarrow H - e^*/H \geq L$

$e^* \leq H(H - L)$

Type L: $e = e^*$: Payoff = $H - e^*/L$

$e = 0$: Payoff = L (Type L would never choose an education level $e \neq e^*$ that is greater than 0 since his payoff would only decrease as a result thereof)

$\longrightarrow L \geq H - e^*/L$

$e^* \geq (H - L)L$

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4. Dynamic Games with Incomplete Information

- **Firms:** $e = e^*$: Payoff: $K - w = H - H = 0$

$e = 0$: Payoff : $L - L = 0$

Departing from this strategy does not make sense for the firms. For all other offers, their payoff is always 0 or negative.

For there to exist a separating equilibrium, e^* must fulfill the following condition:

$$L(H\text{-}L) \leq e^* \leq H(H\text{-}L).$$

Since $H > L$, the condition for all L, H is fulfilled.

(However, there is also a pooling equilibrium.)

Education has no influence on the productivity of EE. Rather, it serves as a signal to potential Es that one is highly skilled.

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4. Dynamic Games with Incomplete Information

4.5 Reputation Models

Example 4.11 [Finitely Repeated Prisoner’s Dilemma]:
(Gibbons, 224-232)

• In the finitely repeated prisoner’s dilemma with complete information, the defection of both players in all rounds is the clear subgame perfect equilibrium.

• However this is contrary to intuition (keyword: Chainstore paradox) and to experimental evidence that shows that often successful cooperation occurs shortly before the end of the game.

• The ‘gang of four’ (Kreps, Milgrom, Roberts and Wilson, 1982) show that in games with incomplete information, plausible equilibria are ‘probable’.

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4. Dynamic Games with Incomplete Information

• For the sake of simplicity, we now assume that there is no incomplete information regarding the payoff, rather there is incomplete information regarding the possible strategies.

• Specifically, we assume that player 1 with a probability p can only play the ‘tit for tat’ game.

• Empirically, this is not an implausible assumption, because there are actually tit for tat players (see also Axelrod’s tournament).

• The gang of four show that even when p is very small, incomplete information can have a large effect. Because for a positive p an upper limit on rounds exists in which one can defect. This upper limit depends on p but not on the number of rounds.

• i.e., if the total number of rounds T is large enough, cooperation predominantly occurs, and cooperation ends primarily at the end of the game.

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4. Dynamic Games with Incomplete Information

- In equilibrium it holds that:

  1. In the case that player 1 ever deviates from the tit for tat (tft) strategy, it is common knowledge that he is not a tft player. Thus in all following rounds no defecting occurs.
  2. Thus even 'rational' player 1s have an incentive to imitate a tft player.
  3. The best response of player 2 to a tft player (for our prisoner's dilemma; see below) is to cooperate until the last round and then defect in the last round.

- We will not assume here like the gang of four that p is very small, rather we will show that for adequately large p's, all players (regardless if they are tft players or not) cooperate up until the next to last round.

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4. Dynamic Games with Incomplete Information

Basic Game:

	k	***d***
K	**1,1**	**b,a**
D	**a,b**	**0,0**

$a > 1$ and $b < 0$

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	k	d
K	1,1	b,a
D	a,b	0,0

- We will start with the case of two rounds. The game structure is:
  1. Chance chooses the type of player 1 (tft with the probability p or rational with the probability $1 - p$). Player 2 is rational. Player 1 knows both types, but player 2 does not know the type of player 1.
  2. The prisoner's dilemma is played two times.
  3. The payoffs are the respective sums of the payoffs from both rounds.

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	k	d
K	1,1	b,a
D	a,b	0,0

- In the last round, all rational players would defect.
- Hence a rational player 1 would defect in the first round.
- A tft player 1 would cooperate in the first round and then do what player 2 choose to do in the first round as his second round move.
- If the rational player 2 cooperates in the first round, he receives $p1 + (1 - p)b + pa$.
- If he defects, he receives pa.
- Thus he cooperates when $p + (1 - p)b \geq 0$, which we assume in the following case.

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4. Dynamic Games with Incomplete Informatio

	k	d
K	1,1	b,a
D	a,b	0,0

• What happens in three rounds? We will show that if $p + (1 - p)b \geq 0$, $1 + pa \geq a$ and $a + b \leq 1$, than the following table describes the behavior in a perfect Bayesian (reputations) equilibrium:

	$t = 1$	$t = 2$	$t = 3$
Player 1 = tit for tat	*K*	*K*	*K*
Player 1 = rational	*K*	*D*	*D*
Player 2	*k*	*k*	*d*

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4. Dynamic Games with Incomplete Informatio

	k	d
K	1,1	b,a
D	a,b	0,0

- In this equilibrium, the payoff for the rational player 1 is equal to $1 + a$, and the payoff for player 2 is equal to $1 + p + (1 - p)b + pa$.
- If the rational player 1 defects in the first round, than only defecting occurs in all following rounds. Thus, his payoff would be a, which is less than the equilibrium payoff.
- If player 2 defects in the first and second rounds (see table 1), his payoff is a, which is less that the equilibrium payoff $1 + p + (1 - p)b + pa \geq a$, or equivalent: $1 + pa \geq a$.
- If player 2 cooperates in the first round and defects in the second round (see table 2), his payoff is $a + b + pa$, which is less than the equilibrium payoff, if $1 + p + (1 - p)b + pa \geq a + b + pa$, or equivalent: $1 + pa \geq a$.

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4. Dynamic Games with Incomplete Informati

	k	d
K	1,1	b,a
D	a,b	0,0

Table 1: Player 2 defects in the first and second rounds

	t = 1	*t = 2*	*t = 3*
Player 1 = tit for tat	*K*	*D*	*D*
Player 1 = rational	*K*	*D*	*D*
Player 2	*d*	*d*	*d*

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4. Dynamic Games with Incomplete Informati

	k	d
K	1,1	b,a
D	a,b	0,0

Table 2: Player 2 cooperates in the first round and defects in the second round

	t = 1	*t = 2*	*t = 3*
Player 1 = tit for tat	*K*	*D*	*K*
Player 1 = rational	*K*	*D*	*D*
Player 2	*d*	*k*	*d*

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4. Dynamic Games with Incomplete Information

	k	d
K	1,1	b,a
D	a,b	0,0

- Since $p + (1 - p)b \geq 0$, the following is a sufficient condition for the fact that this deviation is not worthwhile: $1 \geq a + b$.
- Thus, we have derived sufficient conditions for the fact that all players will cooperate in the first round.
- One can now show that for all $T > 3$ perfect Bayesian equilibria exist in which all player types cooperate until period $T - 2$.
- ...

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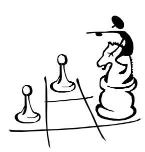

End

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Cooperative Game Theory

Much of the intro to the theory is taken from Serrano's (2007), "Cooperative games", a very nice primer on the subject.

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Cooperative Game Theory

- Analyzes games without detailing the moves of the game.
- Origin of cooperative/non-cooperative labels (Harsanyi): Possible or profitable to agree on a joint strategy?

The pair (N, V) is called a *cooperative game*

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- N = {1, ...,n} is a finite set of players. Each non-empty subset of N is called a *coalition*. The set N is referred to as the *grand coalition*.
- For each coalition S, the *characteristic function* V (S) is the set of |S|-dimensional payoff vectors that are feasible for coalition S.
- These are sometimes called transferable utility games.

The reduced form approach

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- Note how a reduced form approach is taken because one does not explain what strategic choices are behind each of the payoff vectors in V(S).
- It is implicitly assumed that the actions taken by the complement coalition (those players in N\S) cannot prevent S from achieving each of the payoff vectors in V(S).

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How Communication Links Influence Coalition Bargaining

Gary E Bolton, Kalyan Chatterjee & Kathleen L. Valley, *Management Science* 1996

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The game in the new experiment

- V(*SCT*) = 100
- V(*SC*) = 90
- V(*ST*) = 70
- V(*CT*) = 40
- V(*S*) = V(*C*) = V(*T*) = 0

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Two of the most common cooperative game solution concepts.

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The core

- The core of the game (N, V) is the set of payoff vectors
- $C(N, V) = \{x \in V(N)$: there exists no $S \subseteq N$ such that x is pareto dominated in $V(S)\}$.
- In words, it is the set of feasible payoff vectors for the grand coalition that no coalition can upset. If such a coalition S exists, we shall say that S can improve upon or block x, and x is deemed unstable.

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The core

- The set of core outcomes for a game can be
- non-singular
- empty

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The core

- **Theorem** (Debreu and Scarf, 1963): Consider an exchange economy. Then,
- (i) The set of competitive equilibrium allocations is contained in the core.
- (ii) There exists a sufficiently large replica of the economy for which the replica of any non-competitive core allocation of the original economy is blocked.

The Shapley Value

LBOE

The Shapley value is the function that assigns to each player i the payoff

$$Sh_i(N,v) = \sum_{S \subset N} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S-i)]$$

where n = | N| , s = | S|

The Shapley Value

LBOE

- To understand the intuition, suppose the players enter a room in some order, all orders equally likely. The probability that the ith player enters after all other members in the S coalition is (s-1)!(n-s)!/n!

- Then the Shapley value awards to each player the average of his marginal contributions to each coalition. In taking this average, all orders of the players are considered to be equally likely.

Axioms. Shapley (1953)

LBOE

- (i) Efficiency: The payoffs must add up to v(N), which means that all the grand coalition surplus is allocated.
- (ii) Symmetry: If two players are substitutes because they contribute the same to each coalition, the solution should treat them equally.
- (iii) Additivity: The solution to the sum of two TU games must be the sum of what it awards to each of the two games.
- (iv) Dummy player: If a player contributes nothing to every coalition, the solution should pay him nothing.

Axioms. Shapley (1953)

LBOE

- **Theorem** (Shapley, 1953): There is a unique single-valued solution to transferable utility games satisfying efficiency, symmetry, additivity and dummy.
- Connection with competitive equilibrium: Similar result as for core but only holds for transferable utility.

Solutions to the game

LBOE

The core is unique:

(S, C, T) = (60, 30, 10)

- **The Shapley value:**

(S, C, T) = (46.67, 31.67, 21.66)

Communication Links

LBOE

A *communication link* exists between two bargainers if it is possible for them to exchange information directly with one another.

Bilateral Only one link configuration possible.

Multilateral (Link depend on structure)

- Trade treaty negotiations typically allow all links.
- Labor union contract negotiations, the links are often constrained.
- Communication links are also an issue in thin markets.

How do link configurations influence multilateral negotiation?

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The Myerson value

- Axiomatized by Myerson (1977)
- The Myerson value is the Shapley value for the characteristic function game modified so that the value of any coalition that does not include the controlling bargainer is zero.
- Example, in the *S*-controls condition, the modified characteristic function, V', is the same as V with the exception that V'(*CT*) = 0.

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Predicted Myerson value allocations

condition	payoff		
	S	*C*	*T*
complete and public	46.67	31.67	21.66
S-controls	60	25	15
C-controls	35	55	10
T-controls	31.67	16.67	51.66

Modified core

LBOE

Table 2. Modified core predictions

configuration/treatment	payoffs *S*	*C*	*T*
Unconstrained & Public	60	30	10
S-controls	≥ 60	≤ 30	≤ 10
C-controls	≤ 60	≥ 30	≤ 10
T-controls	≤ 60	≤ 30	≥ 10

New Experiment

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Measures influence the communication configuration has on a simple 3-person coalition bargaining game.

Two features distinguish our work.

1. Focus on the influence of communication configuration, and systematically manipulate the feasible communication links.

2. Compare the data with models that focus on the influence of the communication configuration.

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- Unconstrained: any bargainer can send a message or proposal to either one or both of the other bargainers.
- Public: all communication from one bargainer must go to the other two bargainers.
- *S*-controls: all communication must pass through *S*.
- *C*-controls: all communication must pass through *C*.
- *T*-controls: all communication must pass through *T*.

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Subject instructions very similar to those used by Raiffa and Kohlberg (Raiffa, 1982).

Scandinavian Cement (*SC*) the Cement Corporation (*CC*) and Thor Cement (*Thor*), are contemplating a merger.

Merging Parties	Total Profit of Merger (thalers)
SC, *CC*, and *Thor*	100
SC and *CC*	90
SC and *Thor*	70
CC and *Thor*	40

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Laboratory protocol

- 99 subjects from 5 Boston universities.
- Each subject participated in exactly one session.
- Sessions run at Harvard Business School during the summer of 1996.
- Free form negotiations by e-mail. Eight minute limit.
- All negotiations were carried out anonymously.
- Round robin design: roles fixed, partners rotated.
- All parties included in the agreed-upon merger must fill out a contract form.
- Average earnings = $21.40, standard deviation of $10.49 (figures include a $10 show-up fee).

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The coalition with the highest per capita value is modal in all treatments.

Table 4. Observed coalition frequency

treatment	n†	coalition frequency *SCT*	*SC*	*ST*	*CT*	*impasse*
Unconstrained	30	.10	.50	.23	.03	.13
Public	30	.30	.53	.03	.03	.10
S-controls	30	.17	.33	.27	–	.23
C-controls	40	.35	.45	–	.03	.18
T-controls	35	.80	–	.20	–	–
Average	33	.36	.36	.14	.02	.13

†n = 5 rounds x games per round. Games per round = # of subjects/3. There were 18 subjects in Unconstrained, Public, *S*-controls, 24 in *C*-controls, and 21 in *T*-controls.

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Table 6. Average payoffs within coalitions

treatment	coalition freq.†	payoffs (std. error)†† S	C	T
Unconstrained	.58 (15)	46.7 (1.16)	43.3 (1.16)	
	.27 (7)	51.9 (1.65)		18.1 (1.65)
	.04 (1)		30 (-)	10 (-)
	.12 (3)	45 (2.89)	38.3 (4.41)	16.7 (6.67)
Public	.59 (16)	54.1 (1.89)	35.9 (1.89)	
	.04 (1)	35 (–)		35 (–)
	.04 (1)		20 (–)	20 (–)
	.33 (9)	44.5 (3.53)	36.6 (2.00)	18.9 (3.79)

treatment	coalition freq.†	payoffs (std. error)†† S	C	T
S-controls	.43 (10)	45.5 (0.50)	44.5 (0.50)	
	.35 (8)	52.5 (3.00)		17.5 (3.00)
	.22 (5)	50 (0)	40 (0)	10 (0)
C-controls	.55 (18)	43.5 (0.64)	46.5 (0.64)	
	.03 (1)		30 (-)	10 (-)
	.42 (14)	44.9 (1.21)	47.2 (1.06)	7.9 (0.79)
T-controls	.20 (7)	33.6 (0.92)		36.4 (0.92)
	.80 (28)	32.9 (0.61)	30.3 (0.61)	36.9 (0.64)

†Measured as a proportion of coalitions formed (actual number in parentheses.

††A blank indicates that the bargainer is not a member of the coalition.

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- The coalition with the highest per capita value tends to split equally while other coalitions, when they form, appear to be competitive responses to the equal high per capita split.
- Exception: The public communication treatment, where all coalitions look quite competitively formed.

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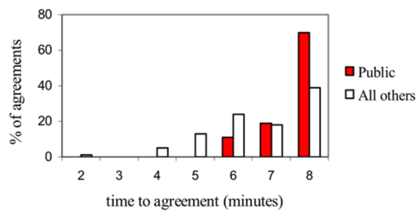
Figure 4. Time to agreement



	average (minutes)
Unconstrained	6.8
Public	7.6
S-controls	6.4
C-controls	6.8
T-controls	6.7

The “deadline effect” is more pronounced in the public than in the other treatments.

Evidence that the detailed strategies bargainers are following differs across treatments – and that this matters.

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The alternating offer extensive-form models negotiation as a sequence of proposals and counterproposals.

A proposal consists of a coalition and a division of the coalition payoff. Members of the named coalition respond sequentially, indicating either accept or reject. The first rejecter gets to propose the next coalition. Payoffs are discounted at a common rate of d between a rejection and a counter-proposal. Utility is transferable. The game is analyzed using stationary subgame perfect equilibrium; stationary that bargainers make proposals and responses that are independent of play in past rounds.

Suppose…

1. Myerson modification

2. First proposer is chosen at random

We work out the stationary perfect equilibrium allocations of the infinite horizon version of the extensive-form, for the limiting case when $\delta \rightarrow 1$.

Table 3. Alternating offer predictions

condition	coalition frequency	payoff allocations† *S*	*C*	*T*
complete, public	2/3	45	45	
& *S*-controls	1/3	45		25
C-controls	2/3	45	45	
	1/3	45	45	10
T-controls	2/3	35		35
	1/3	35	30	35

†a blank space indicates the bargainer is not a member of the coalition

Table 9. Modified alternating offer model predictions vs. the data
Italicized numbers are predictions , regular font is data (average payoffs).

treatment	coalition frequency		payoff allocations † S		C		T	
Unconstrained	*.44*	.58	*45*	46.7	*45*	43.3		
	.33	.27	*45*	51.9			*25*	18.1
	.22	.12	*45*	45.0	*40*	38.3	*15*	16.7
Public	*.44*	.59	*45*	54.1	*45*	35.9		
	.33	.04	*45*	35.0			*25*	35.0
	.22	.33	*45*	44.5	*40*	36.6	*15*	18.9
S-controls	*.44*	.43	*45*	45.5	*45*	44.5		
	.33	.35	*45*	52.5			*25*	17.5
	.22	.22	*45*	50.0	*40*	40.0	*15*	10.0
C-controls	*.44*	.55	*45*	43.5	*45*	46.5		
	.56	.42	*45*	44.9	*45*	47.2	*10*	7.9
T-controls	*.44*	.20	*35*	33.6			*35*	36.4
	.56	.80	*35*	32.9	*30*	30.3	*35*	36.9

†a blank space indicates the bargainer is not a member of the coalition

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Algorithm for finding CDRS solutions

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- Identify the coalition with highest average value.
- The theory predicts that each member of this coalition, if chosen proposer, will name this coalition together with the equal division allocation.
- If a bargainer excluded from this coalition is chosen proposer, she picks the coalition that gives her the largest payoff given that she must offer her partners the payoff they would receive if one of them were chosen proposer.

Summary

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The above example illustrates how cooperative game theory, with its extremely reduced form approach, won't be adequate for all strategic problems we're interested in.

But it is extremely useful for an important class of markets, which is out next subject:

Matching markets

3. Static Games with Incomplete Information

3.2 Matching Theory

(see amongst others Wolfstetter, Topics in Microeconomics, Chapter 7)

- Workers must be allocated to companies, students to universities, organ donors to patients, authors to publishers, etc. These types of allocation problems are analyzed under the concept “matching”.
- Matching Theory belongs, along with auction theory, to the most active areas of applied game theory.
- Are there stable pairings, or can the pairings be changed through individual bargaining? Which procedures and institutions lead to stable pairings? Are these manipulable?

1

3. Static Games with Incomplete Information

Example 3.3 [The Marriage Problem]:

Notation and Basic Assumptions:

- We assume that the members of two different groups should be paired with each other:

1. The group of women: $W = \{w_1, \ldots, w_n\}$
2. The group of men: $M = \{m_1, \ldots, m_n\}$

- Every man possesses a complete and transitive order of preference concerning *W;* in the same manner, every woman has such an order of preference concerning *M*.

2

3. Static Games with Incomplete Information

- These preferences are portrayed in a preference matrix.
- In the present case, the matrix portrays the preferences of 3 men and 3 women: The first number of every pair in the matrix is the ranking that the men gave to the women. The second number is the women's ranking for the men.
- Man 1 (m_1) ranked woman 1 (w_1) the highest, m_2 favored w_2 and m_3 likes w_3 the best. For the women, the preferences are the exact opposite.

3

3. Static Games with Incomplete Information

	w_1	w_2	w_3
m_1	1 , 3	2 , 2	3 , 1
m_2	3 , 1	1 , 3	2 , 2
m_3	2 , 2	3 , 1	1 , 3

- How many pairings are possible?

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3. Static Games with Incomplete Information

- If M and W each have n elements, than there are $n!$ possible pairings. One can clarify this by portraying every pairing through an n-dimensional vector:

$$\left(w_{i_1}, w_{i_2}, \ldots, w_{i_n}\right)$$

- That means: m_1 is paired with w_{i1}, m_2 w_{i2} etc. and m_n w_{in} .
- Thus the number of possible pairings is equal to the permutation of a group with n elements, thus $n!$.
- The current example therefore has in total six possible pairings.

3. Static Games with Incomplete Information

Definition 3.4 [Instable Matching]

A matching is considered instable, if there are two men and two women who are paired in the form (m_1,w_1) and (m_2,w_2), although m_1 would prefer w_2 and w_2 would prefer m_1:

Formal: $w_2 \succ_{m_1} w_1$ and $m_1 \succ_{w_2} m_2$

A matching that is not instable is called stable.

3. Static Games with Incomplete Inforr

	w_1	w_2	w_3
m_1	1 , 3	2 , 2	3 , 1
m_2	3 , 1	1 , 3	2 , 2
m_3	2 , 2	3 , 1	1 , 3

- In the current example, there are six possible matchings, of them the following three are stable:

1. Every man is paired with his first choice. In this case a woman cannot convince a man to get a divorce.
2. Analogous to case 1, every woman is paired with her first choice.
3. Every man and every woman is matched with his or her second choice.

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3. Static Games with Incomplete Inforr

	w_1	w_2	w_3
m_1	1 , 3	2 , 2	3 , 1
m_2	3 , 1	1 , 3	2 , 2
m_3	2 , 2	3 , 1	1 , 3

- An instable matching is the pairing of (m_1,w_1), (m_2,w_3), (m_3,w_2): m_3 and w_1 could improve themselves by divorcing their current partners and marrying each other.
- One could believe that it is always possible to reach stable matches by pairing either the men or the women with their first choice. However, this is not the case.
- Example: Two or more members of a group give their highest ranking to the same member of the other group.

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3. Static Games with Incomplete Information

- Another Example: The Roommate Problem

	s_1	s_2	s_3	s_4
s_1		1		4
s_2			1	4
s_3	1			4
s_4				

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3. Static Games with Incomplete Inform

	s_1	s_2	s_3	s_4
s_1		1		4
s_2			1	4
s_3	1			4
s_4				

- Not every matching problem has a stable matching.
- Four students $\{s_1, s_2, s_3, s_4\}$ should be ordered into pairs of roommates that will each share a room in the dorm.
- No one wants to share a room with s_4. Hence, it follows that regardless of who is paired with s_4 he/she can always find someone he/she would rather room with. Hence, there does not exist a stable matching.

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3. Static Games with Incomplete Information

- However, for the marriage problem the following holds:

  Proposition 3.1

  The marriage problem *always* has, independent from the preferences of the group, at least one stable solution.

- In this case, the deciding factor is that the members are matched between two *different* groups (as opposed to the roommate problem in which the members are paired with other members of the same group).

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3. Static Games with Incomplete Information

Proof: Procedure "The man proposes"

Step 1:

a) Every man proposes to his most treasured woman.

b) Every woman rejects all proposals except the one from the man she likes the best.

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3. Static Games with Incomplete Information

Step k:

a) Every man, who was rejected in the last step, proposes to the woman whom he likes the best *out of all the women he hasn't proposed to yet*.

b) Every woman accepts the one whom she likes *the best out of all the men who propose* and rejects the others.

3. Static Games with Incomplete Information

- This procedure ends when every woman has received at least one proposal. At this point, every woman has a partner.
- The solution exists because the procedure ends after at the most n^2 steps, because every man can at most propose to each woman.
- The solution is stable because every woman who would prefer a man more than her husband, would have already rejected this man in preference for her current partner in the steps before.

3. Static Games with Incomplete Information

Which matching procedure is useful to whom?

- The matching illustrated in the proof is only one of many possible matching procedures. When more than one stable matching exists, the “man proposes” procedure creates a matching that benefits the men.
- If instead the women propose, one has a matching that benefits the women.
- As soon as more than one stable matching exists, a conflict of interest is created between the group of men and women.

3. Static Games with Incomplete Information

Proposition 3.2

Amongst all the stable matchings from the marriage problem, there is one that is slightly preferred by the men and one that is slightly preferred by the women. Thus, there is a “male-optimal” and a “female-optimal” matching as long as there does not exist an unequivocal stable matching.

3. Static Games with Incomplete Information

Proof:

- One can show: There is no stable matching situation in which the man can place better than in the stable matching that was created through the “man proposes” procedure.
- Because the problem is symmetrical in nature, this proof works just as well for the group of women when they do the proposing.

3. Static Games with Incomplete Information

- For every man m_i, a woman w_j should be named *attainable, if a stable matching exists that pairs the two with each other.*
- Assume that up until step $k - 1$ of the “man proposes” procedure, no man has been rejected by a woman who is attainable for him.
- Assume again, in step k, m_i is rejected by w_j: thus, what remains to be shown is that w_j cannot be attainable for this man.

3. Static Games with Incomplete Information

- Since a man is never rejected by a woman who is attainable for him, it follows that a man is never rejected from his *most treasured attainable* woman.
- Assuming this is not true: w_j would be attainable for m_i, although she rejected him in step k. At the same time, in step k, m_r is her best possible partner.
- Hence: $m_r \succ_{w_j} m_i$ (w_j prefers m_r over m_i)

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3. Static Games with Incomplete Information

- Furthermore: $w_j \succ_{m_r} w_k$ (m_r prefers w_j over all the other attainable women w_k for him)
- Thus, the matching of m_i to w_j and of m_r to any other attainable woman cannot be stable. Both w_j as well as m_r could improve their choice by divorcing their partner and marrying each other.
- Thus, w_j cannot be attainable for m_i.

End of Example 3.3.

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3. Static Games with Incomplete Information

Strategic Bargaining

- Problems can arise when certain matching *mechanisms* are used to solve matching problems.
- A mechanism is a procedure that generates a stable matching from the preferences.
- If all the preferences are public information, the matching can be given over to a neutral third party (e.g. computer software).

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3. Static Games with Incomplete Information

Strategic Bargaining

- In the following, only *direct* mechanisms are considered: A mechanism is direct when every man and every woman initially must name their preference. These preferences are accepted as true and implemented with the given procedure.
- Such a direct mechanism leads to a game under *imperfect* information (no one knows which preferences the others specified) and *incomplete* information (no one knows the true preferences of the others).

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3. Static Games with Incomplete Information

Strategic Bargaining

- If the agents only know their own preference and not the preferences of the others, than strategic bargaining would be possible.
- One must consider the question, whether a stable matching procedure can be developed that induces the revelation of true preferences.
- The literature in this subject is still in a developmental phase. The following will introduce a few of the most important results.

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3. Static Games with Incomplete Information

Two Impossibility Results

[Proposition 3.3]

No stable matching procedure exists in which the revelation of true preferences is a dominant strategy for all preferences.

- To prove proposition 3.3, it is sufficient to find a matching problem in which the revelation of true preferences is not a dominant strategy in any of the stable matching procedures.

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3. Static Games with Incomplete Information

Example 3.5 [Marriage Problem II]

	w_1	w_2	w_3
m_1	2 , 1	1 , 2	3 , 1
m_2	1 , 3	2 , 3	3 , 2
m_3	1 , 2	2 , 1	3 , 3

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3. Static Games with Incomplete Information

Example 3.5 [Marriage Problem II]

	w_1	w_2	w_3
m_1	2 , 1	1 , 2	3 , 1
m_2	1 , 3	2 , 3	3 , 2
m_3	1 , 2	2 , 1	3 , 3

- This marriage problem has exactly two stable matchings: The "female-optimal (1) and the "Male-optimal" (2):

  (1) (w_1, w_3, w_2)

  (2) (w_2, w_3, w_1)

- Every stable matching procedure selects either (1) or (2).
- Assume, the male-optimal matching is selected: Than w_1 can influence the pairings to her benefit by giving false preferences.

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3. Static Games with Incomplete Information

Example 3.5 [Marriage Problem II]

	w_1	w_2	w_3
m_1	2,1	1,2	3,1
m_2	1,3	2,3	3,2
m_3	1,2	2,1	3,3

- If w_1 gives the following *false* preferences: $m_1 \succ_{w_1} m_2 \succ_{w_1} m_3$, than the matching (2) (w_2, w_3, w_1) is no longer stable. w_1 could than provoke m_2 to leave w_3.
- At this same moment, the female-optimal matching becomes the only possible stable matching.
- Thus, w_1 can cause the matching mechanism to select the female-optimal matching through the appropriate portrayal of false preferences.

3. Static Games with Incomplete Information

Example 3.5 [Marriage Problem II]

	w_1	w_2	w_3
m_1	2,1	1,2	3,1
m_2	1,3	2,3	3,2
m_3	1,2	2,1	3,3

- Analogously, it can be shown that men can benefit from concealing their true preferences, if the matching mechanism selected matching (1).
- Thus, there does not exist a procedure that makes the revelation of the truth a dominant strategy.

Two Impossibility Results

- This result does not mean that the revelation of true preferences cannot be reached. It only means that stating the truth cannot be a dominant strategy in *all* matching problems.
- As Roth showed, this conclusion can be even further supported.

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Two Impossibility Results

[Proposition 3.4]

There is no stable matching procedure in which a Nash equilibrium for all preferences is such that all players state the truth.

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3. Static Games with Incomplete Information

Stable Matchings?

- There are mechanisms that are stable in regards to both stated and true preferences. Among them are the before described processes ‘the man proposes’ and ‘the woman proposes’.
- In the procedure ‘the man proposes’, it is a dominant strategy for all the men to tell the truth, even when there is incomplete information about all the preferences (as Roth and Sotomayor showed).

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3. Static Games with Incomplete Information

Stable Matchings?

[Proposition 3.5]

The “man proposes” mechanism makes it a dominant strategy for every man to state his true preferences.

(Analogously, the “woman proposes” mechanism makes it a dominant strategy for every woman to express her true preferences.)

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3. Static Games with Incomplete Information

Stable Matchings?

- Although truthfulness is the optimal behavior for the men in the mechanism “the man proposes”, women can improve their positions if they strategically bargain (see example 3.5).
- Remember: There exists an equilibrium in which all women respond optimally to the preferences that the men specify (see slide 18 and following).
- Does a stable matching in which all express their true preferences occur from this equilibrium? Roth and Sotomayor say yes.

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3. Static Games with Incomplete Information

Stable Matchings?

[Proposition 3.6]

Assume, women choose preferences that are the best answer to the true preferences of the men in the “man proposes” mechanism. Then the resulting matching is stable based on stated and true preferences.

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3. Static Games with Incomplete Information

Stable Matchings?

- These results show that one can find a simple mechanism that can lead to a stable matching even with incomplete information.
- However, there is a tendency towards manipulation through the declaration of false preferences when using such mechanisms.
- Therefore, the matching that results from such procedures is stable but in general not pareto-optimal.

3. Static Games with Incomplete Information

Matching Problem Literature

- Up until now the matching problem has been used for the apartment market, college admissions, and job placements for entry level positions.
- The basic analyses are the works of Roth (1984 and 1991) about the placement of medical students as interns in hospitals.
- Roth showed that up until 1952 chaotic circumstances ruled in the American market. Students would change their position at the last minute although they had already agreed to go elsewhere. In 1952 a new matching algorithm was introduced, which lead to stable pairings.

3. Static Games with Incomplete Information

Matching Problem Literature

- In England, the same problem lead to the introduction of a centralized regional matching procedure, of which, however, a few were instable.
- Interestingly, all the *successful* matching procedures are variations of the introduced the “man proposes” or the “woman proposes” method.
- In the meanwhile, new challenges have arisen since more and more couples are looking for internships at the same location. However, this problem is not yet solved.

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3. Static Games with Incomplete Information

Matching Problem Literature

- An application of two sided matching problems is job markets. Shapley and Shubik (1972) showed that the properties of stable matchings can be generalized to models such that the pairing of the work force and the determination of the salary can be combined with each other.
- A few new publications examine the timing of transactions. Amongst others, Roth and Xing (1994) and Li and Rosen (1998) explain the tendency that transactions always occur earlier and earlier. This is a phenomenon that can be observed in some markets for highly qualified workers.

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Economic Design of the Entry Level Job Market

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1. The Entry Level Job Market for American Doctors *Introduction to the Case Study*

The material for this case study is from Roth (2001) and the works cited within it.

- Young doctors must do an internship in a hospital after graduation.
- The choice of hospitals can have a significant influence on careers and the choice of intern has a significant influence on the efficiency of the hospital.
- The competition for positions and doctors lead to a market failure in the 40s.

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1. The Entry Level Job Market for American Doctors
Introduction to the Case Study

- These market failures were remedied in the 50s through the introduction of a central matching mechanism. (National Resident Matching Program = NRMP)
- In the years after the needs of the system changed which lead to the reworking of the matching mechanism in the 90s.
- The present Roth - Peranson Mechanism matches 20,000 interns to hospitals every year, and is also utilized by many other markets with similar problems.

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1. The Entry Level Job Market for American Doctors
The Problem: Prisoners Dilemma

- In 1900 the market was decentralized, thus the students applied for the positions themselves.
- There were far more positions than students, which lead to a strong competition for the students among the hospitals.
- The competitions lead to agreements being made earlier and earlier, despite different attempts to stop this process.

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1. The Entry Level Job Market for American Doctors
The Problem: Prisoners Dilemma

- Around 1940, the agreements were already being made a full two years before graduation. That meant that the hospitals knew comparatively little about the qualities of the students that they hired and that the students had to choose before they could build preferences about their career (and used a lot of time during their studies to look for a position). The result: market failure.

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1. The Entry Level Job Market for American Doctors
A Solution Attempt by the Hospitals

- "For a number of years attempts have been made to defer the appointment of hospital interns until towards the close of the fourth year. … The difficulty has been in persuading someone to take the lead. This is to inform you that it has been decided to defer the appointments of interns at the Presbyterian Hospital in the City of New York until some time in April. It is earnestly hoped that other hospitals and schools will be able to act in a similar manner." (Darrach, 1927)

- "… for some unknown reason the Presbyterian Hospital soon abandoned its stand …" (Fitz, 1939)

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1. The Entry Level Job Market for American Doctors
A Solution Attempt by the Universities

- To solve the prisoners dilemma structure for their students, the universities decided in 1945 that no information or recommendations can be given out before a set date.
- As a matter of fact the agreements did move further back but new problems arose: Incentives to delay making agreements as long as possible lead to the introduction of deadlines for accepting the offer.

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1. The Entry Level Job Market for American Doctors
A Solution Attempt by the Universities

- Students, who were offered a position at a hospital but knew that they were on a waiting list at a more preferred hospital, waited with the acceptance as long as possible.
- Waiting lists were only slowly dismantled, and there were a whole bunch of hectic decisions being made in the last minute. These often were regretted or even ended up no longer being possible, or students already forced themselves into an early decision which they later regretted.
- The result was a chaotic market with unacceptable results for many.

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1. The Entry Level Job Market for American Doctors
A Central Solution

- After a period of chaotic conditions (with all kinds of futile attempts to regulate the market), a decision was made to implement a central solution.
- Students and hospital administrators should, as before, have decentralized interviews, however afterwards they should submit their rankings.
- An algorithm then recommends an allocation of students to hospitals based on the rankings. But what should the algorithm look like?

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2. Simple Market Theory
Matching

- There are firms, $F = \{f_1,..., f_n\}$, and workers, $W = \{w_1,..., w_p\}$, that want be matched with eachother based on their preferences.
- Every worker is looking for one job and every firm f_i is looking for (up to) q_i workers.

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2. Simple Market Theory
Matching

- A matching result is a subset of FxW, i.e. a set of pairs, such that every worker is in at least one pair, and every company f_i is in no more than q_i pairs.
- A matching result μ is identified with a correspondence μ: $F \cup W \rightarrow F \cup W$, so that $\mu(w) = f$ and $w \in \mu(f)$ if and only if (f,w) is a pair in μ .
- In the case that no pair in μ contains worker w, than $\mu(w) = w$ (i.e. the worker is matched with himself).

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2. Simple Market Theory
Worker's Preference

- Every worker has complete and transitive preferences about the firms.
- The preferences of w_i would be, for example, $P(w_i) = f_2, f_4, \ldots w_i \ldots$, and they show that the worker prefers f_2 over f_4 [$f_2 >_{wi} f_4$,], etc., and that *acceptable* position fi [$f_i >_{wi} w_i$] would be preferred to the 'secondary job market'.
- A firm f with wi >wi f is called unacceptable.
- The respective other group receives a preference list of acceptable agents.

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2. Simple Market Theory
Firm's Preferences

- Since companies have a group of workers distributed to them, they have complete and transitive preferences about the group of acceptable workers. (A preference list in the form $P(f_i) = w_i, w_j, \ldots w_k$ does not suffice.)

- For the simple model, it is enough to assume that the preferences are *responsive* in regards to the preferences about individual workers, i.e.
  - For every set of workers $S \subset W$ with $|S| < q_i$, and for all workers w and w' in W/S, $S \cup w >_{f_i} S \cup w'$ if and only if $w >_{f_i} w'$,
  - and $S \cup w >_{f_i} S$ if and only if w for f_i is acceptable.

- In words: if a firm prefers w to the worker w', than this firm is always going to choose w instead of w' to add to any group of other workers, and it would always prefer an acceptable worker instead of leaving the position open.

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2. Simple Market Theory
Stability

- A matching μ is *blocked through an individual k* if $\mu(k)$ is unacceptable for k.

- A matching μ is blocked by a *pair (f,w) of agents*, if they themselves prefer the matching with μ, i.e. if
  [$w >f\ w'$ for a w' in $\mu(f)$ or
  w is acceptable for f l and $|\mu(f)| < qf$]
  and $f >w\ \mu(w)$

- A matching is called *stable* if it is not blocked.

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2. Simple Market Theory

- The deferred acceptance algorithm (DAA) (Gale und Shapley, 1962) shows that there are always stable matchings.
- Experience also shows how rank order lists can be worked centrally but be written *as if* the worker is going through a decentralized process.

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2. Simple Market Theory
Deferred Acceptance Algorithm: 'Worker optimal'

1 a. Every worker applies at his/her first choice firm.

b. Every firm f (with q_f positions) rejects all unacceptable offers and in the case it receives more the q_f acceptable offers, it *keeps* the q_f most preferred offers and rejects the rest.

…

…

k a. Every worker whose application was rejected during step k-1 applies to another company which he/she prefers the most and from which he/she has not yet been rejected (i.e. where he/she has not yet applied).

b. Every firm f keeps the (up to) q_f most preferred acceptable received offers and rejects the rest.

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2. Simple Market Theory
Deferred Acceptance Algorithm: ‘Worker-optimal’

Stopping Rule: As soon as there are no new applications, every firm is then allocated the workers that they keep.

- The resulting matching μW is stable.
  - Unacceptable pairs are never kept so that the only reason for instability is a blocking pair of agents.
  - Assume w prefers f to the result $\mu W(w)$. In this case he must have applied to f and been rejected. As a consequence, f does not prefer worker w to any of its workers. Thus, (w,f) cannot be a blocking pair.

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2. Simple Market Theory
Stability Theory

Theorem 1: The set of stable matchings is never empty.

Theorem 2: The DAA, in which the workers (firms) make the offers creates a worker- (firm-) optimal matching. The optimal stable matching of the one is the least preferred matching of the other.

Theorem 3: In every stable matching, the same workers are distributed and the same positions filled.

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2. Simple Market Theory
Incentive Theorem

Theorem 4:

- There is no stable matching method in which all participating agents have a dominant strategy, in which they reveal their true preferences.
- However, if the workers make the offers (in the worker-optimal matching method) it is a dominant strategy for every worker to reveal his/her true preferences.
- If the firms make the offers, a worker can simply improve his standings through manipulation of the preferences that would obtain a different matching when used in the worker optimal procedure.
- There is a Nash equilibrium rank order list for the DAA in which the matching procedure is stable in regards to the submitted as well as the actual preferences.

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3. Why is stability so important: Empiricism

- Theoretically, one could find a pair of agents in an instability that tries to elude the matching.
- It is often quite simple to uncover instabilities even in large markets. If for example a students is paired with his third choice, he only needs to make two telephone calls to find out if he is part of a blocking pair.
- The empiricism supports the intuition.

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3.1 A Natural Experiment (Roth, 1991)

- The DAA in the USA has emerged as successful. The (volunteer!) participation rate has been around 95% for years. Does this lead back to the stability property?
- Other markets have experienced similar problems as the American market, but have chosen other central matching systems which allows for the possibility of a “natural experiment”.
- For example, the following hypothesis can be tested: because a centralized mechanism reduces the high transaction costs of a chaotic decentralized process, *every* central mechanism (independent of whether or not it’s stable) can count on a high participation rate.

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3.1 A Natural Experiment (Roth, 1991)
Example: Priority Matching

- Priority Matching means that the rank order lists of the workers and firms are combined and every pair is given a priority.
- The pairs are then matched based on priority. First, the pairs with a priority 1 are matched, than the pairs with priority 2 etc.

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3.1 A Natural Experiment (Roth, 1991)
Birmingham (1966) and Newcastle (1967)

- Priority = Product of Ranks.
- If a firm and a worker both prefer each other the most (a (1,1)-match) than they have a priority of 1.
- A (1,2)-match has a priority of 2, etc.
- Cases of the priority is processed as followed:
  - Birmingham: Firms are favored, i.e. a (1,2)-match is given preference to a (2,1)-match.
  - Newcastle: the reverse.

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3.1 A Natural Experiment (Roth, 1991)
Edingburgh (1967)

- Priorities are given lexicographically in regards to the preferences of the firms. (1,1) has the highest priority, (1,2) the second highest etc.
- After all the (1,x)-matches are checked, the (2,x)-matches are consulted, etc.

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3.1 A Natural Experiment (Roth, 1991)
Stability Problems for Priority Matching

Proposition: Priority Matching can lead to an instable result.

Proof

- 6 firms with one position each and 6 workers.
- Preferences:

f_1: w_1, ...	w_1: f_1, …
f_2: w_1, w_3, w_2, w_4, w_5, w_6	w_2: f_2, f_1, f_3, f_4, f_5, f_6
f_3: w_3, w_4, …	w_3: f_4, f_3, …
f_4: w_4, w_3, …	w_4: f_3, f_4, …
f_5: w_1, w_2, w_5, w_3, w_4, w_6	w_5: f_1, f_2, f_5, f_3, f_4, f_6
f_6: w_2, w_5, …	w_6: f_5, f_6, …

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3.1 A Natural Experiment (Roth, 1991)
Stability Problems with Priority Matching

The resulting matchings are:

- Birmingham: f_1w_1 (1,1), f_3w_3 (1,2), f_4w_4 (1,2), f_2w_2 (3,1), f_5w_6 (6,1), f_6w_5 (2,6)
  - instable with regards to f_5 and w_5 (3,3).
- Newcastle: f_1w_1 (1,1), f_3w_4 (2,1), f_4w_3 (2,1), f_2w_2 (3,1), f_5w_6 (6,1), f_6w_5 (2,6)
  - instable with regards to f_5 and w_5 (3,3).
- Edingburgh: f_1w_1 (1,1), f_3w_3 (1,2), f_4w_4 (1,2), f_6w_2 (1,6), f_5w_5 (3,3), f_2w_6 (6,2)
  - instable with regards to f_2 and w_2 (3,1).
- Intuition: Two agents that each end up in the fourth position in the priority after a (1,15)-match.

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3.1 A Natural Experiment (Roth, 1991)
Incentive Problems with Priority Matching

"… shortly before the scheme was discarded we found that in up to 80% of cases students and consultants only used the computer to indicate a first preference … The main reason for the abandonment of the scheme, therefore, was that there were problems in getting students and consultants to participate in an orderly way, and this led to those who rigidly observed the requirements of the scheme to be penalized." (Dean at Newcastle, 1987; see Roth, 1991).

- In many reports it became clear that after an introduction of priority matching, an increasing number of placements were negotiated outside of the system (at the expense of those that followed the central matching procedure).
- See above example:
  - f_5 in Birmingham is unhappy with w_6 – even more so when f_5 hears that w_5 is also unhappy. If both of them had given each other the highest priority, they would both have done better.

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3.1 A Natural Experiment (Roth, 1991)
Incentive Problems with Priority Matching

- The more agents that succumb to this temptation, the greater the incentive for the left over agents to withdraw from the mechanism.
  - Assume f_5 and other firms learn from experience and arrange (1,1)-matches.
  - For example, w_1, w_2, w_3 and f_3, f_4, f_5 have agreed on a (1,1) match for the next year. The (true) rank order list of f_2 would be w_1, w_2, w_3, w_4, w_5, ... And the (true) rank order list of w_4 would be f_3, f_4, f_5, f_2, ...
  - W_4 is the most preferred and attainable worker for $f_{2,}$, and vice versa. If true preferences are submitted, the priority is only 16, thus f_2 must also be satisfied with its 15th choice.

Proposition. It is not a dominant strategy for any agent to say the truth in a priority matching system.

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3.1 A Natural Experiment (Roth, 1991)

- An extensive analysis of other matching procedures (stable and instable) can be found in Roth (1991) and in Roth and Sotomayor (1990).
- Results of the natural experiments:

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3.1 A Natural Experiment (Roth, 1991)
Stable and Instable Centralized Mechanisms

Market	Stable	In Use
American Medical Market		
NRMP	yes	yes (new Design '98)
Medical Specialties	yes	yes (approx.. 30 Markets)

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3.1 A Natural Experiment (Roth, 1991)
Stable and Instable Centralized Mechanisms

Market	Stable	In Use
British Regional Medical Markets		
Edinburgh ('69)	*yes*	*yes*
Cardiff	*yes*	*yes*
Birmingham (Priority)	*no*	*no*
Edinburgh (‘67) (Priority)	*no*	*no*
Newcastle (Priority)	*no*	*no*
Sheffield	no	no
Cambridge	no	yes
London Hospital	no	yes

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3.1 A Natural Experiment (Roth, 1991)
Stable and Instable Centralized Mechanisms

Market	Stable	In Use
Other Markets		
Dental Residencies	yes	yes
Osteopaths (< ‘94)	no	no
Osteopaths (< ‘94)	yes	yes
Pharmacists	yes	yes
Other Markets and Matching Procedures		
Canadian Lawyers	yes	yes (with the exception of British Columbia since 1996)
Sororities	yes	yes

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3.2 A Laboratory Experiment (Kagel and Roth, 2000)
Idea

- Laboratory experiments can test the different procedures that are used in Newcastle and in the USA under controlled conditions (ceteris paribus).
- The experiment allows the subjects to gain experience with a decentralized market in the simplest form possible. In this market there are sufficient incentives to make arrangements earlier and earlier as a result of competition and shortage.
- Early arrangements cost money and mean loss of efficiency.

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3.2 A Laboratory Experiment (Kagel and Roth, 2000)
Idea

- As soon as the subjects have adjusted to the markets, one of two central matching systems will be offered to the subjects that have not yet made an early arrangement.
- The single difference between the two experiment designs is the selection of the matching procedure.
  - In one design, priority matching as it is used in Newcastle is chosen.
  - In the other design a variation of the DAA is used.
- The idea was not to reproduce the real markets, but rather to investigate if the different outcomes of the procedures in a simple environment can be clearly attributed to the procedure itself.

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3.2 A Laboratory Experiment (Kagel and Roth, 2000)
Experimental Design

- Every market contains 12 subjects: 6 firms and 6 workers.
- The amount of the payment depends on with whom and when the subjects made the agreement (between $4 and $16).
- Every market consists of three periods, -2, -1, and 0, whose names indicate that the payment is reduced by $2 when the arrangement is made during period –2, etc. An agent without a match earns nothing.
- In every market, a firm can hire one worker and every worker can accept one job.

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3.2 A Laboratory Experiment (Kagel and Roth, 2000)
Experimental Design

- Firms can give one offer per period. Workers can receive many offers but can only accept at the most one.
- All agreements are binding.
- Every session begins with 10 successive exchanges with decentralized procedures. After the 10th exchange it is announced that in period 0 a (volunteer) central procedure will be arranged.

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3.2 A Laboratory Experiment (Kagel and Roth, 2000)
Experimental Design

- The subjects know that if they are not matched before period 0, they can submit a rank order list concerning the possible matching, and that "the centralized matching mechanism (which is a computer program) will determine the final matching, based on the submitted rank order lists."
- Here upon 15 more markets are played including the central process in period 0.
- Half of the sessions will be played with the DAA, the other half with priority matching.

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3.2 A Laboratory Experiment (Kagel and Roth, 2000)
Results

- The following graph shows the evolution of the average cost that is carried by all subjects due to the early agreements. (If all 12 subjects made their agreements in period 2, than the cost is $24, in period –1, $12, and in period 0, $0.)
- The subjects learned in the first phase to make arrangements earlier and earlier.
- Directly after introducing the central procedure, the costs do not differ greatly. After that the costs decrease in the stable process but do not decrease in the instable process.
- The experiment can therefore replicate the key field observations. The procedure here, however, is clearly the cause for the effects of efficiency. This makes the assumption more plausible that also in the field, the stability of the procedure causes the success or failure of the procedure.

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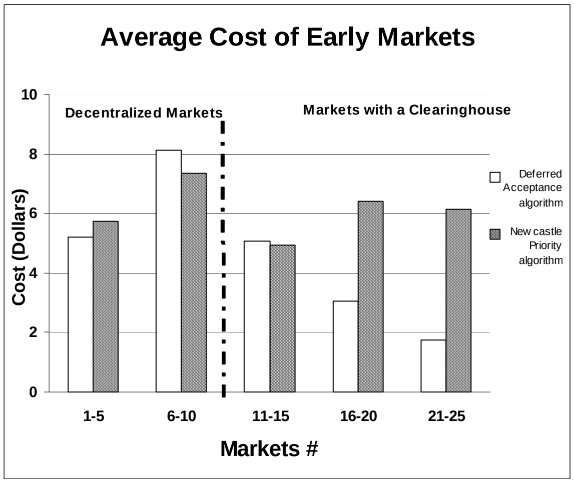
3.2 A Laboratory Experiment (Kagel and Roth, 2000)
Results



Average amounts that are paid for early matching.
(Kagel and Roth, 2000)

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