Probability summary note ①

- model : Capture essense of problem (iteration)
- elements ① possible outcomes
  ② Subset of interest
  ③ probability (weights)

set theory op.

↑

- Framework (Ω, F, P) (1) Ω: Sample space (2) F: event: subset of Ω ⟷ sigma algebra of possible/feasible space

(3) P: probability defined on set $P(\Omega)=1$

- uniform dist : same probability
- $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
- Extension: Inclusion - Exclusion principle:

① $P(\cup E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{i=1}^{n}\sum_{j=1}^{n} P(E_i \cap E_j) + \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} P(E_i \cap E_j \cap E_k) + (-1)^{n+1} P(\cap_{i=1}^{n} E_i)$ induction

k, $\binom{k}{2}$, $(-1)^{n+1}\binom{k}{k}$

② $P(\cup_{i=1}^{n+1} E_i \cup E_{i+1}) = P(\cup_{i=1}^{n+1} E_i) + P(E_{i+1}) - P(\cup_{i=1}^{n+1} E_i \cap E_{i+1})$

$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B)P(B) = P(B|A)P(A) \Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B|A) \cdot P(A) + P(B|A') \cdot P(A')}$

Probability Conditioning: $\cup E_i = \Omega \Rightarrow P(F) = P(\cup(F \cap E_i)) = \sum P(F \cap E_i) = \sum P(F|E_i) \cdot P(E_i)$ Divide & Conquer

Simplify by simpler smaller pieces

Bays Formula: Back track: observe today → find out what happened yesterday

$P(E_i|F) = \frac{P(F|E_i) \cdot P(E_i)}{\sum P(F|E_i) \cdot P(E_i)}$ Reverse logic order

independance: $P(A|B) = P(B)$ ≡ sample space not reduced

$P(A \cap B) = P(A)P(B)$

r.v : random variable : real value func. defined on Sample space

how behaves?

Distribution: probability of lower or equal to something

Dist	pdf	median	cdf	Skewness	mean	Var	Ex kurtosis	MGF	CF
Bernouli	$\begin{cases} p & k=1 \\ 1-p & k=0 \end{cases}$	$\begin{cases} 0 & q>p \\ 0.5 & q=p \\ 1 & q<p \end{cases}$	$\begin{cases} 0 & k<0 \\ q & 0 \le k < 1 \\ 1 & k \ge 1 \end{cases}$	$\frac{q-p}{\sqrt{qp}}$	p	$p(1-p)$	$\frac{1-6pq}{pq}$	$q+pe^t$	$q+pe^{it}$
Binomial	$\binom{n}{k} p^k (1-p)^{n-k}$	$\lfloor np \rfloor$	$I_{1-p}(n-k, 1+k)$	$\frac{1-2p}{\sqrt{np(1-p)}}$	np	$np(1-p)$	$\frac{1-6p(1-p)}{np(1-p)}$	$(1-p+pe^t)^n$	$(1-p+pe^{it})^n$
Geometric	$(1-p)^{k-1} p$	$\lceil \frac{-1}{\log_2(1-p)} \rceil$	$1-(1-p)^k$	$\frac{2-p}{\sqrt{1-p}}$	$1/p$	$\frac{1-p}{p^2}$	$6+\frac{p^2}{1-p}$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{pe^{it}}{1-(1-p)e^{it}}$
Negative Binom	$\binom{k+r-1}{k}(1-p)^r p^k$	?	$1-I_p(k+1, r)$	$\frac{1+p}{\sqrt{pr}}$	$\frac{pr}{1-p}$	$\frac{pr}{(1-p)^2}$	$\frac{6}{r}+\frac{(1-p)^2}{pr}$	$\left(\frac{1-p}{1-pe^t}\right)^r$	$\left(\frac{1-p}{1-pe^{it}}\right)^r$
Poisson Dist	$\frac{\lambda^k}{k!} e^{-k}$	$\lambda + \frac{1}{3} - \frac{0.02}{\lambda}$	$\frac{\Gamma(\lfloor k+1 \rfloor, \lambda)}{\lfloor k \rfloor !}$	$\lambda^{-1/2}$	λ	λ	λ^{-1}	$e^{\lambda(e^t-1)}$	$e^{\lambda(e^{it}-1)}$
Uniform Dist	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{x-a}{b-a}$ $x \in [a, b)$	0	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$-6/5$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
Expon. Dist	$\lambda e^{-\lambda t}$	$\frac{\ln 2}{\lambda}$	$1-e^{-\lambda x}$	2	λ^{-1}	λ^{-2}	6	$(1-\frac{t}{\lambda})^{-1}$	$(1-\frac{it}{\lambda})^{-1}$
Erlang Dist.	$\frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$	?	$\frac{\gamma(k, \lambda x)}{(k-1)!}$	$\frac{2}{\sqrt{k}}$	k/λ	k/λ^2	$\frac{6}{k}$	$(1-\frac{t}{\lambda})^{-k}$	$(1-\frac{it}{\lambda})^{-k}$
Gamma Dist	$\frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$	?	$\frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}$	$\frac{2}{\sqrt{\alpha}}$	α/β	α/β^2	$6/\alpha$	$(1-\frac{t}{\beta})^{-\alpha}$	$(1-\frac{it}{\beta})^{-\alpha}$
Normal	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	$\Phi(\frac{x-\mu}{\sigma})$	0	μ	σ^2	0	$e^{\mu t+\frac{\sigma^2 t^2}{2}}$	$e^{i\mu t-\frac{\sigma^2 t^2}{2}}$
Beta Dist.	$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}$	$\frac{\alpha-1/3}{\alpha+\beta-2/3}$	$I_x(\alpha, \beta)$	$\frac{2(\beta-\alpha)\sqrt{\alpha+\beta+1}}{(\alpha+\beta+2)\sqrt{\alpha\beta}}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$		$1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$	${}_1F_1(\alpha; \alpha+\beta; it)$

②

Incomplete beta func: $I_x(a,b) = \frac{B(x|a,b)}{B(a,b)}$

$B(x|a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$

Incomplete gamma function: $\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t} dt$

$\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt$

Poisson Assumptions

(1) rate of interval Constant
(2) intervals should be disjoint and independent
(3) large n: $np = \lambda$ $x_n \sim Bin(n,p) \Rightarrow \lim_{n \to \infty} \{x_n = i\} = \frac{e^{-\lambda}\lambda^i}{i!}$

Exponential Dist: ① key: no memory ② life time dist.

Erlang/Gamma Dist.: $X = \sum_{i=1}^{n} X_i \sim Exp \Rightarrow X \sim Gamma(n, \lambda)$

e.g life of five tables

Normal Dist: AVG of large # of indep. rand var.

$E(x) = \begin{cases} \sum x f_x & \equiv E(x) = \sum_{i=0}^{\infty} P(X > i) \\ \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{\infty} x dF(x) = \int_0^\infty P(X > t) dt \equiv \int_0^\infty \bar{F}(t) dt \end{cases}$

only if x nonnegative

by Fubini theorem $x = \sum_{i=0}^{+\infty} I(x > i)$

$E(g(x)) = \begin{cases} \sum g(x) f_x(x) \\ \int_{-\infty}^{\infty} g(x) dF_x(x) \end{cases}$

$E(x^n) \equiv n$'th moment

mean dev risk: Ratio $= \frac{E(x) - r_f}{\sigma_x}$ ↑ reference

$F_{X,Y}(x,y) = Pr\{X \le x, Y \le y\} \Rightarrow F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y)$

independance: $F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$, $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

Tail, survival Function: $S = 1 - F \Rightarrow$ used in $x < min(x_1, x_2)$

e.g $X_1 \sim exp(\lambda_1)$, $X_2 \sim exp(\lambda_2)$, $X_3 \sim exp(\lambda_3)$ $\begin{cases} Y_1 = min(X_1, X_3) \\ Y_2 = min(X_2, X_3) \end{cases} \Rightarrow P(Y_1, Y_2) = P(Y_1 > u, Y_2 > v) = P(X_1 > u, X_3 > u, X_2 > v, X_3 > v)$

$= P\{X_1 > u, X_3 > max(u,v), X_2 > v\} = P\{X_1 > u\} P\{X_3 > max(u,v)\} P\{X_2 > v\}$

$= e^{-\lambda_1 u} e^{-\lambda_2 v} e^{-\lambda_3 max(u,v)}$

$P(Y_1 > u) = e^{-(\lambda_1+\lambda_3)u}$ $P(Y_2 > v) = e^{-(\lambda_2+\lambda_3)v}$

$\Rightarrow P(Y_1, Y_2) \neq P(Y_1) \cdot P(Y_2)$ ≡ not independent

① $E[g(x,y)] = \sum_x \sum_y g(x,y) f_{x,y}(x,y)$
$= \iint g(x,y) f_{x,y}(x,y) dx dy$

② $E(X+Y) = E(X) + E(Y)$

③ $Cov(X,Y) = E(x-\bar{x})(Y-\bar{Y})$, $\bar{x} = E(x)$, $\bar{Y} = E(y)$
$= E(XY) - E(X)E(Y)$

④ $E(XY) = \iint xy f_{x,y}(x,y) dx dy$

⑤ $Cov[cX, Y] = c \cdot Cov(X, Y)$
$Cov(X,X) = Var(X)$
if X indep. of $Y \Rightarrow Cov(X,Y) = 0$

⑥ If X, Y indep $\Rightarrow E[g(x)h(Y)] = E[g(x)]E(h(y))$

⑦ $Cov = 0 \Rightarrow$ indep. not hold, since Cov captures mean (cancel out)

⑧ $\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \rightarrow$ cancel out scale

⑨ $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$ ↑ reduce variability in finance by diversification = 0

⑩ $Var(\sum_{i=1}^{m} c_i X_i) = \sum_{i=1}^{m} c_i^2 Var(X_i) + \sum_{i \neq j} c_i c_j Cov(X_i, X_j)$

⑪ Convolution ① x, y indep $\Rightarrow P(X+Y \le t) = \int_{-\infty}^{\infty} F_Y(t-x) f_X(x) dx$

$\Rightarrow f_{X,Y}(t) = \int_{-\infty}^{\infty} f_Y(t-x) f_X(x) dx$

⑫ $X \sim Bin(n,p)$, $Y \sim Bin(m,p)$ $\Rightarrow P(X+Y=k) = \binom{m+n}{k} p^k (1-p)^{m+n-k}$

≡ Bernoli $\begin{cases} m+n & \text{trials} \\ p & \text{success probability} \end{cases}$

Probability summary note

$X \sim$ poisson λ_1, $Y \sim$ poisson λ_2, indep. X, Y $\Rightarrow P(X+Y=n) = e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!}$ with Poisson rate $(\lambda_1+\lambda_2)$

$X \sim exp(\lambda)$, $Y \sim exp(\lambda)$, X, Y: indep

$f_{X+Y}(t) = \int_0^\infty \lambda e^{-\lambda(t-x)} \lambda e^{-\lambda x} dx = \lambda^2 t e^{-\lambda t} \equiv$ Gamma $= e^{-\lambda t} \frac{(\lambda t)^{2-1}}{(2-1)!} \lambda$

MGF ① $\varphi_X(t) = E(e^{tX})$ ② $\frac{d\varphi_X(t)}{dt} = E(X)$ ③ $\frac{d^n}{dt^n}\varphi_X(t) = E(X^n)$

eg ① $X \sim Bin$ $\varphi_X(t) = \sum_{i=0}^n e^{it}\binom{n}{i}p^i(1-p)^{n-i} = (e^t p + 1-p)^n$

④ X, Y indep $\Rightarrow \varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$

eg: X, Y Binom $\Rightarrow \varphi_{X+Y}(t) = (e^t p + 1-p)^{m+n}$

② $X \sim Exp(\lambda) \Rightarrow E(e^{tX}) = \int_0^\infty \lambda e^{tx} e^{-\lambda x} dx = \frac{\lambda}{\lambda - t}$

② $X \sim Gamma(n, \lambda) \Rightarrow E(e^{tX}) = (\frac{\lambda}{\lambda-t})^n$

PGF probability Generating Function

① $E(Z^X)$ $x \geq 0 \in \mathbb{N}$ $0 \leq Z \leq 1$
$= \sum_{j=0}^\infty Z^j P\{X=j\}$

② $\lim_{Z\to 0} \frac{1}{n!}\frac{d^n}{dZ^n} E(Z^X) = P\{X=n\}$ $Z^X = e^{(\ln Z)X}$

③ laplace transformation $x \geq 0 \in \mathbb{N} \Rightarrow E(e^{-tX})$

eg ④ $X \sim poisson(\lambda) \Rightarrow E(Z^i) = \sum_{j=0}^\infty Z^j e^{-\lambda}\frac{\lambda^j}{j!} = e^{-\lambda(1-Z)}$

$Z_1, \dots, Z_m \sim iid\ N(0,1)$ $X = AZ + \mu$ $X \sim$ multivar normal

$x_1 = a_{11}Z_1 + a_{12}Z_2 + \dots + a_{1m}Z_m + \mu_1$
$x_2 = a_{21}Z_1 + a_{22}Z_2 + \dots + a_{2m}Z_m + \mu_2$
$\vdots$
$x_n = a_{n1}Z_1 + a_{n2}Z_2 + \dots + a_{nm}Z_m + \mu_n$

MGF $E(Z^{t^T X}) = e^{\sum_{i=1}^m t_i\mu_i + \frac{\sum_i\sum_j t_i t_j Cov(x_i, x_j)}{2}}$

$E(t^T X) = \sum_{i=1}^m t_i \mu_i$

$Var(t^T X) = \sum_i\sum_j Cov(t_i x_i, t_j x_j) = \sum_i\sum_j Cov(x_i, x_j) t_i t_j$

⊛ in multivariate zero Cov implies independance

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weak law of large number $E(\hat{X}(n)) = \mu$

$Var(\hat{X}(n)) = \frac{\sum_{i=1}^n \sigma^2}{n^2} = \frac{\sigma^2}{n}$

$\lim P\{|\hat{X}(n) - \mu| > \varepsilon\} = 0$

$X \geq 0 \Rightarrow P\{X > t\} \leq \frac{E(X)}{t}$ Markov

Proof: $E(X) = \int_0^\infty x dF(x) = \int_0^t x F(x) + \int_t^\infty t dF(x) \geq \int_t^\infty t F(x)$
$= tP\{X > t\}$

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weak law of large number Chebyshev

$P\{|X-\mu| > \varepsilon\} \leq \frac{Var(X)}{\varepsilon^2} \Leftrightarrow P\{|X-\mu|^2 \geq \varepsilon^2\} \leq \frac{Var(X)}{\varepsilon^2}$ $\equiv$ Probability of outside = Zero

proof: $\lim_{n\to\infty} P\{|\hat{X}(n) - \mu| > \varepsilon\} \leq \frac{Var(X)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$

SLLN Strong law of Large Number

$n \to \infty$ $\hat{X}(n) = \frac{\sum_{i=1}^n X_i}{n} \to \mu$ with Probability 1

cluster: $\neq$ mean, mean ☑; Not Converg

$\equiv$ would be incluster and not outside

CLT Central limit theorem — mean appears, but as you push $\Rightarrow$ see the distribution

$\lim_{n\to\infty} \{\frac{\hat{X}(n) - \mu}{\sigma/\sqrt{n}} \leq x\} \sim N(0,1)$ MGF $= e^{t^2/2}$

Stochastic Process a collection of rand. variable (e.g. Mkt Share)

$\{X(t), t \in T\}$ t: time or order $T \in [0, \infty)$ Cont. time; $T = 0, 1, 2, \dots$: discrete time

$\hookrightarrow$ state space (Cont. or discr.)

$X(t, \Omega) \mapsto$ sample space

$\to$ Specify joint dist. of $(X(t_1), X(t_2), \dots, X(t_k))$ $0 \leq t_1 < t_2 < \dots < t_k < \infty$, $k \geq 1$

Conditional distribution $P(E|F) = \frac{P(E \cap F)}{P(F)}$

① limiting state space
② sequential definition: $X_1 \sim F$, $X_2 \sim F_{X_2|X_1}$, $X_3 \sim F_{X_3|X_2, X_1}$

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eg $X_1, X_2 \sim$ independent poisson λ_1, λ_2

key $P\{X_1 = i | X_1 + X_2 = n\} = \frac{P\{X_1 = i, X_2 = n-i\}}{P\{X_1 + X_2 = n\}} = \frac{e^{-\lambda_1}\frac{\lambda_1^i}{i!} e^{-\lambda_2}\frac{\lambda_2^{n-i}}{(n-i)!}}{e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}} =$

$\binom{n}{i}(\frac{\lambda_1}{\lambda_1+\lambda_2})^i(\frac{\lambda_2}{\lambda_1+\lambda_2})^{n-i}$ Poisson split

$\sim Bin(n, \frac{\lambda_2}{\lambda_1+\lambda_2})$

$E(X|Y=y) = \begin{cases} \sum x f_{X|Y}(x|y) \\ \int x f_{X|Y}(x|y)dx \end{cases}$

e.g poisson $E(X_1 | X_1 + X_2 = n) = \sum_{i=1}^n i (\frac{\lambda_1}{\lambda_1+\lambda_2})^{n-i}(\frac{\lambda_2}{\lambda_1+\lambda_2})^i\binom{n}{i}$
$= \frac{n\lambda_2}{\lambda_1+\lambda_2}$

module ⊛ for $E(X|Y=y)$
① Calculate $f(x|y)$
② $\int x f(x|y)$

Calculate Expectation By Conditioning $E_Y[E(X|Y)] = E_Y[f(Y)] = E(X)$

$\equiv \sum \frac{i \cdot P(X=i, Y=j)}{P(Y=j)} \times P(Y=j) = \sum i\, P(X=i)$

trial & Error

Steps of Calculation $E(X) = E(E(X|Y))$ — Divide & Conquer
Step1 Compute $E(X|Y=y)\ \forall y$ Condition
Step2 Compute $\int E(X|Y=y)dF_Y(y)$ Uncondition

Conditioning Example $\begin{cases} E(X|I=1) = 1 & P \\ E(X|I=0) = E(X)+1 & 1-P \end{cases}$ Prob; $X \equiv$ # trial; info finished

$\Rightarrow E(X) = P + (1-P)(E(X)+1) \Rightarrow E(X) = \frac{1}{P}$

$E(X^2|I=0) = E(1+X)^2$

Prisoner Problem $\begin{cases} E(X|I=1) = 2 + E(X) \\ E(X|I=2) = 3 + E(X) \\ E(X|I=3) = 5 \end{cases}$

2 days Back ①, ③ 5 days free, ② 3 days Back

$\Rightarrow E(X) = \frac{2}{3} + \frac{1}{3}E(X) + 1 + \frac{1}{3}E(X) + \frac{5}{3}$
$\Rightarrow E(X) = 10$

Recursion Conditioning

$X = \sum X_i$ $E(X) = ?$ helpful Piece of info? what if I have Can calc?

$E(X|N=n) = nE(X)$

$\Rightarrow E(E(X_i|N)) = E[N \cdot E(X_i)] = E(N) \cdot E(X_i)$ This side of Eq known

$Cov(X, Y) = E_Z[Cov[X, Y|Z]] + Cov_Z[E(X|Z), E(Y|Z)]$
expected (Cov) + Cov (Expectation) — only worry about this side of Eq

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$Var(\sum_{i=1}^N X_i) = Cov(\sum_{i=1}^N X_i, \sum_{i=1}^N X_i) = E(N) \cdot Var(X_i) + E(X_i)^2 Var(N)$

$ECov(X, Y)$ $X = \begin{cases} Y & P \\ Z & 1-P \end{cases}$ $\Rightarrow$ define $I=1$ $X=Y$, $I=0$ $X=Z$

$\Rightarrow \begin{cases} Cov(X, Y|I=1) = Var(Y)P \\ Cov(X, Y|I=1) = 0\ (Cov(X,Y))(1-P) \end{cases}$ $\begin{cases} E(X|I=1) = E(Y) & E(Y|I=1) = E(Y) \\ E(X|I=0) = E(Z) & E(Y|I=0) = E(Y) \end{cases}$

First term: $PV(Y) =$ Constant $\Rightarrow Cov = 0$

Probability summary note

Using Dummy info $1_E = \begin{cases} 1 & E \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$

$E(1_E) = P(E) \Rightarrow E[E[1_E|Y]] = E[P(E|Y)] \Rightarrow P(E) = E[P(E|Y)]$

① $P\{X<Y\}$ (X, Y indep)
$= \int P\{X<Y|Y=y\} dF_Y(y) = \int F_X(y) dF_Y(y)$

② $P\{X<Y\}$ (X, Y indep) $= \int P\{X<Y|Y=y\} dF_Y(y) = \int F_X(y) dF_Y(y)$

e.g. $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$ $\Rightarrow P(X<Y) = \int_0^\infty (1-e^{-\lambda y}) \mu e^{-\mu y} dy = \frac{\lambda}{\lambda+\mu}$

③ Extension: $X_1, X_2, \cdots X_n \sim Exp(\lambda_i)$ — event which happen first

$P\{X_1 = \min(X_1, \dots, X_n)\} = \frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n}$

④ Convolution: $P\{X+Y \le t\} = \int \underbrace{P\{X+Y \le t | Y=y\}}_{P(X \le t-y)} dF_Y(y) = \int F_X(t-y) dF_Y(y)$
(independance)

⑤ **Telephone Blocking** $I_1, \dots, I_n, \dots$ iid seq Bernoulli(p)

$Y = \sum_{i=1}^{X} I_i$

$\Rightarrow P\{Y=k\} = \sum_{n=0}^\infty P\{Y=k|X=n\} P\{X=n\} = \sum_k^\infty \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} e^{-\lambda} \frac{\lambda^n}{n!}$

$= e^{-\lambda} \frac{\lambda p^k}{k!} \sum_k^\infty \frac{[\lambda(1-p)]^{n-k}}{(n-k)!} = e^{-\lambda} \frac{\lambda p^k}{k!} e^{\lambda(1-p)} \Rightarrow Y \sim \text{poisson}(\lambda p)$
$E(Y) = \lambda p$

⑥ **Stochastically larger** $X \ge^{st} Y$ if $P\{X>t\} \ge P\{Y>t\}$ — Larger tail

First order dominance $F_X(t) \le F_Y(t)$ $\forall t$ — compare two population

⑦ $P(X|X>Y) \ge^{st} Y$:

$P(X>t, X>Y) = \int P(X>t, X>Y|Y=y) dF_Y(y) = \int P\{X > \max(t,y)\} dF_Y(y)$

$\ge P\{X>t\} \int P\{X>Y|Y=y\} dF_Y(y) = P\{X>t\} \cdot P\{X>Y\}$

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① **Bonferroni inequality**: $P(EF) \ge P(E) + P(F) - 1$

② $\boxed{E \subset F \Rightarrow P(F) \ge P(E)}$: $P(F) = P(E) + P(FE^c) \ge P(E)$

③ $\cup_{i=1}^n E_i = E_1 \cup E_1^c E_2 \cup E_1^c E_2^c E_3 \cup \cdots \cup E_1^c \cdots E_{n-1}^c E_n$

$\Rightarrow \boxed{P(\cup_i E_i) \le \sum_{i=1}^n P(E_i)}$

④ two fair die sum $= i \Rightarrow \begin{cases} \frac{i-1}{36} & i = 2, \dots, 7 \\ \frac{13-i}{36} & i = 8, \dots, 12 \end{cases}$

⑤ experiment repeat until event E or F: $P(E < F) = \frac{P(E)}{P(E) + P(F)}$

$P\{E<F\} = P\{E<F | \text{current} = E\} \times P(E) + P\{E<F | \text{outcome} = F\} \times P(F) + P\{E<F | \text{outcome} \ne E, F\} \times [1 - P(E) - P(F)]$

nth time E appears: $P(E) \times (1-p)^{n-1}$ $p = P(E) + P(F)$

⑥ dice game: $\{7, 11\}$ win; $\{2, 3, 12\}$ loose; else $\to$ same number i or 7 loose twice

$\Rightarrow \begin{cases} 0 & i = 2, 12 \\ \frac{i-1}{5+i} & i = 3, \dots, 6 \\ 1 & i = 7, 11 \\ \frac{13-i}{19-i} & i = 8, \dots, 10 \end{cases}$

$E(\text{win}) = \sum_{i=2}^{12} P\{\text{win} | \text{throw } i\} P\{\text{throw } i\}$

④, ⑤ $\Rightarrow \frac{13-i}{13-i+6}$

⑦ E: 3 dice but same number on exactly two
D: all three numbers differed
S: all three numbers the same
$\Rightarrow P(E) = 1 - P(D) - P(S) = 1 - \frac{20}{36} - \frac{1}{36} = \frac{5}{12}$

⑧ throw hat center room & select random.

E_i: person i selects own hat

$P(\text{no own hat}) = 1 - P(E_1 \cup E_2 \cdots \cup E_n) = 1 - [\sum_i P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \cdots + (-1)^{n+1} P(E_1 E_2 E_n)]$

$\boxed{\sum_{i_1 < i_2 < \cdots < i_k} P(E_{i_1} \cdots E_{i_k}) = \binom{n}{k} \frac{(n-k)!}{n!} = \frac{1}{k!}}$ $\Rightarrow P(\text{no own hat}) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$

trick: i person took their own hat, but rest not
$P(\text{no one}) = 1 - P(\text{one right}) + P(\text{two right}) - \dots$

⑨ $\begin{cases} b: \text{black} \\ r: \text{red} \end{cases} \Rightarrow \begin{cases} (b+c, r) & \text{if } b \text{ draw} \\ (b, r+c) & \text{if } r \text{ draw} \end{cases}$

$\Rightarrow P(\text{first} = \text{black} | 2\text{nd} = \text{red}) = \frac{P(\text{First} = b \,\&\, 2\text{nd} = r)}{P(2\text{nd} = r)} = \frac{b}{b+r+c}$

since: $\frac{b}{b+r} \frac{r}{b+c+r}$

$\frac{b}{b+r} \frac{r}{b+c} + \frac{r}{b+c} \frac{r+c}{b+c+r}$

⑩ 3 prisoners $\to$ 1 Executed, 2 Freed
no harm divulge which free?

$P(A = Ex | X = B) = \frac{P(A = Ex, X = B)}{E(X = B)} = \frac{\frac{1}{3} \times \frac{1}{2}}{\frac{1}{2}} = \frac{1}{3}$ $A \cap X$ $(X = B | A = Ex)$

say B free

Similarly $P(A = Ex | X = A) = \frac{1}{3}$

⑪ Binom(n, p) $\nearrow \searrow$: $\frac{P\{X=k\}}{P\{X=k-1\}} = \frac{n-k+1}{k} \frac{p}{1-p}$

⑫ r possible outcome
ith outcome: p_i $i = 1, \dots, r$
$\sum_{i=1}^r p_i = 1$

$x_1, x_2, \dots, x_r$ — first outcome: # times; 2nd outcome: # times

$P(x_1, x_2, \dots, x_r) = \frac{n!}{x_1! x_2! \cdots x_r!} p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}$

$\sum x_i = n$

⑬ Prob mass func (pmf) of $X_1 + X_2 + \cdots + X_k$ in ⑫ $i = 1, \dots, r$:

$P\{X_1 + \cdots + X_k = m\} = \binom{n}{m} (p_1 + \cdots + p_k)^m (p_{k+1} + \cdots + p_r)^{n-m}$

⑭ negative binomial dist. ① p head ② until rth appear $n \ge r$

$P\{X=n\} = \binom{n-1}{r-1} p^r (1-p)^{n-r}$ $n \ge r$

⑮ Coin flip by A, B, equal head = total of k heads
(k) ($n-k$)

$P\{\text{same \# heads}\} = \sum_i P(A=i, B=i) = \sum_i \binom{k}{i} (\frac{1}{2})^k \binom{n-k}{i} (\frac{1}{2})^{n-k} = \binom{n}{k} (\frac{1}{2})^n$

⑯ Poisson $\frac{P\{X=i\}}{P\{X=i-1\}} = \frac{\lambda}{i}$ increase moneton in i for $P\{X=i\}$

⑰ $x_1, x_2, \dots, x_n \sim$ iid Unif(0,1)

$M = \max(x_1, x_2, \dots, x_n) \Rightarrow F_M(x) = x^n$ $0 \le x \le 1$

$F_M(x) = P(M<x) = P(\max(x_1, \dots, x_n) < x) = P(x_1 < x) P(x_2 < x) \cdots P(x_n < x) = x^n$

$f_M(x) = n x^{n-1}$

⑱ ① m different coupons ② iid draw

$E(N)$ N: trial to get at least one of each

$N = \sum_i^m X_i$ $X_i \sim$ Geometric mean $= \frac{1}{p}$

Trick: one stage: Got i different, draw until $i+1$ type

$p_i = \frac{n-i}{n}$ $i = 0, 1, \dots, n-1$ $\Rightarrow$ mean $\frac{n}{n-i}$ times

$\Rightarrow E(N) = \sum_{i=0}^m \frac{m}{m-i} = m \sum_{i=1}^n \frac{1}{i}$

intuition of conditioning Q: what you need to have to calculate result? ① assume you have them $\equiv$ condition on them and solve problem (no problem now that you put them right of bar; as noris said) ② Go forward and uncondition by mean of finding weight of each value assumed known and take weighted AVG (mean expectation)

⑲ $E(X^2) \ge E^2(X) \Rightarrow E(X^2) - E^2(X) = Var(X) \ge 0$ equal when $Var(X) = 0$

⑳ $X > 0$, $X \sim r.v.$, $g(0) = 0$

$\Rightarrow E[g(X)] = \int_0^\infty P(X>t) g'(t) dt$

Probability summary note

① n Coupons
② k iid draw p_i $i=1,\dots,n$
③ x: # distinct types

$\to E(X)=\sum_{i=0}^n E(x_i)=\sum_{i=0}^n \underbrace{(1-(1-p_i)^k)}$

prob of Existence = 1 − unexistance

Trick: Define indicator $x_i \begin{cases}1 & \text{if type } i \text{ occur}\\ 0 & \text{otherwise}\end{cases}$

$Var(X)=\sum var(x_i)+2\,Cov(x_i,x_j)=\sum_{i=1}^n (1-p_i)^k(1-(1-p_i)^k)$

$+2\,[\underbrace{1-[(1-p_i)^k+(1-p_j)^k-(1-p_i-p_j)^k]}_{P\{x_ix_j=1\}}\underbrace{-(1-p_i)^k(1-p_j)^k}_{-E(x_i)E(x_j)}]$

two com = none come + none the come

---

(21) $x_1,\dots,x_{10} \sim$ poison $\lambda=1$

① markov: $P\{x>t\}=\frac{E(x)}{t} \Rightarrow P\{x_1+\dots+x_{10}>15\}\le \frac{2}{3}=\frac{10}{15}$

② CLT: $P\{x_1+\dots+x_{10}>15\} \approx 1-\Phi\left[\frac{15-10}{\sqrt{10}}=\frac{10}{\sqrt{10}}\right]$

---

(22) $X_i \sim$ poison $\lambda=1 \Rightarrow P\{\sum_i^n X_i\le n\}=e^{-n}\sum_{k=0}^n \frac{n^k}{k!}$

CLT: $\sum_i^n x_i-n \sim N(0,1) \Rightarrow \lim_{n\to\infty} e^{-n}\sum_{k=0}^n \frac{n^k}{k!}=\frac{1}{2}$

---

(23) x: white ball from batch of k (urn of n white and m black)

ⓐ $P\{x=i\}=\frac{\binom{n}{i}\binom{m}{k-i}}{\binom{n+m}{k}}$ $i=0,1,\dots,\min(k,n)$

ⓑ $x_i\begin{cases}1 & \text{if } i\text{th ball selected is white}\\ 0 & \text{otherwise}\end{cases}$ $Y_j\begin{cases}1 & \text{if white ball } j \text{ is selected}\\ 0 & \text{otherwise}\end{cases}$

$\Rightarrow E(x)=E(\sum x_i)=\sum E(x_i)=\sum^k \frac{n}{m+n}=\frac{kn}{m+n}$

$\Rightarrow E(x)=E(\sum Y_i)=\sum E(Y_i)=\sum^n \frac{k}{m+n}=\frac{nk}{m+n}$

---

(24) $X \to$ # men select own hat

Trick Define $X=X_1+\dots+X_N$ $X_i=\begin{cases}1 & i\text{th man select own hat}\\ 0 & \text{otherwise}\end{cases}$

$Var(x)=\sum_{i=1}^N Var(X_i)+2\sum\sum_{i<j} Cov(x_i,x_j)$, $P(X_i=1)=\frac{1}{N}$ Bernoulli

① Symmetry ② Aggregation

$Var(x_i)=\frac{1}{N}(1-\frac{1}{N})=\frac{N-1}{N^2}$

$Cov(x_i,x_j)=E[x_ix_j]-E[x_i]E[x_j]$ $x_ix_j\begin{cases}1 & \text{if } i\text{th } j\text{th own hat}\\ 0 & \text{otherwise}\end{cases}$

$\Rightarrow E[x_ix_j]=P\{x_i=1,x_j=1\}=P\{x_i=1\}.P\{x_j=1|x_i=1\}=\frac{1}{N}\frac{1}{N-1}$

$\Rightarrow Cov(x_i,x_j)=\frac{1}{N(N-1)}-\left(\frac{1}{N}\right)^2=\frac{1}{N^2(N-1)}$

$Var(X)=\frac{N-1}{N}+2\binom{N}{2}\frac{1}{N^2(N-1)}=\frac{N-1}{N}+\frac{1}{N}=1$

---

(25) multinomial Dist. N_i: number of times outcome i occurs

ⓐ $E[N_i]=np_i$ like binomial (n,p_i)

ⓑ $Var[N_i]=np_i(1-p_i)$

ⓒ $Cov(N_i,N_j)=Cov(\sum_k X_k,\sum_k Y_k)=\sum_k\sum_\ell Cov(X_k,Y_\ell)$

$=\sum E(X_kY_k)-E(X_k)E(Y_k)=-np_ip_j$

ⓓ $E[\text{outcome not occure}]=E[\sum_i^r Y_i]=\sum_i^r E[Y_i]=\sum_i^r (1-p_i)^n$

---

(26) $x_1,x_2,\dots,x_n$ iid $\quad x_n>\max(x_1,\dots,x_{n-1})$ x_n is record

ⓐ $P\{\text{a record occurs at time } n\}=\frac{1}{n}$ ≡ equally likely & symmetry trick

ⓑ $E[\#\text{ records by time } n]=\sum_{i=1}^n \frac{1}{i}$

$I_j=\begin{cases}1 & \text{if a record occure at } j\\ 0 & \text{otherwise}\end{cases}$ $E[\sum_i^n I_j]=\sum_i^n E[I_j]=\sum_i^n \frac{1}{j}$

ⓒ $Var[\#\text{ records by time } n]=\sum_{i=1}^n \frac{(i-1)}{i^2}$

$Var\sum_i^n I_j=\sum_i^n Var(I_j)=\sum_{j=1}^n\left[\frac{1}{j}\right]\left[\frac{j-1}{j}\right]$ Trick: Sum of Indicator Bernoli

ⓓ $N=\min\{n: n>1, \text{ and a record occurs at time } n\} \to E[N]=\infty$

$P\{N>n\}=P\{x_1 \text{ largest } x_1\dots x_n\}=\frac{1}{n}$

$E[N]=\sum_{n=1}^\infty P\{N>n\}=\sum_{n=1}^\infty \frac{1}{n}=\infty$

---

(27) $a_1<a_2<\dots<a_n$ ≡ n numbers $\quad a_j<a_i,\ i<j \to$ permutation

$N_i=\#k, k<i$ a_i precede a_k in Permut

$N=\sum_{i=1}^n N_i$

ⓐ $N_1-N_n \sim$ indep: (reason1) $N_1,\dots N_n$ knowing ≡ know relative ordering

(reason2) a_1-a_i that follows a_{i+1} does not depend prob. on a_1-a_i's ordering.

ⓑ $P\{N_i=k\}=\frac{1}{i}$ $k=0,1,\dots,i-1$ Key: Symmetry

of $a_1,\dots,a_{i+1}$; a_{i+1} equally likely be 1st, 2nd, …, or $(i+1)$s

ⓒ $E(N_i)=\frac{1}{i}\sum_{k=0}^{i-1}k=\frac{i-1}{2}$

$E(N_i^2)=\frac{1}{i}\sum_{k=0}^{i-1}k^2=\frac{(i-1)(2i-1)}{6}$

$\Rightarrow Var(N_i)=\frac{(i-1)(2i-1)}{6}-\frac{(i-1)^2}{4}=\frac{i^2-1}{12}$

---

(28) $E(XY)=\mu_x\mu_y$

$E(XY)^2=(\mu_x^2+\sigma_x^2)(\mu_y^2+\sigma_y^2)$

$\Rightarrow Var(XY)=\sigma_x^2\sigma_y^2+\mu_y^2\sigma_x^2+\mu_x^2\sigma_y^2$

---

(29) ① $x_1,\dots,x_n \sim \phi(t_1,\dots,t_n)$ joint moment Gen func

$\Rightarrow \phi_{x_i}(t_i)=?$: $\phi_{x_i}(t_i)=\phi(0,0,\dots,0,1,0,\dots,0)$ $\hookrightarrow$ ith place

② if independent: $E[e^{\sum t_ix_i}]=\Pi_i[e^{t_ix_i}]$

$\Rightarrow \phi(t_1,\dots,t_n)=\phi_{x_1}(t_1)\cdots\phi_{x_n}(t_n)$

---

(30) $g_1(x,y)=x+y=u$ $\quad g_2(x,y)=(x-y)=v$

$J=\begin{bmatrix}\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2}\\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2}\end{bmatrix}=2$ $\quad f_{U,V}(u,v)=\frac{1}{2}f_{X,Y}\left[\frac{u+v}{2},\frac{u-v}{2}\right]$

$=\frac{2}{4\pi\sigma^2}\exp\left[-\frac{1}{2\sigma^2}\left[\left[\frac{u+v}{2}-\mu\right]^2+\left[\frac{u-v}{2}-\mu\right]^2\right]\right] \Rightarrow$ indep

---

(31) unbiased die $\quad X$: # roll to get 6 $\quad Y$: # roll to get 5

① $E(X)=\frac{1\times 1}{6}+(1+E(X))\times\frac{5}{6} \Rightarrow E(X)=6$

② $E(X|Y=1)=E(1+E(X))=7$

③ $E(X|Y=5)=1\left(\frac{1}{5}\right)+2\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)+3\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)+4\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)+6\left(\frac{4}{5}\right)^4\left(\frac{1}{6}\right)$ + …

---

(32) $X\sim \exp$ $\mu_x=\frac{1}{\lambda}$ $\quad E[X|X>1]=?$

$f_{X|X>1}(x)=\frac{f(x)}{P\{X>1\}}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda}}$ $x>1$

$E[X|X>1]=e^{\lambda}\int_1^\infty x\lambda e^{-\lambda x}dx=1+\frac{1}{\lambda}$: intuition mean of exp +1

---

(33) $X\sim unif(0,1)$ $\quad E(X|X<\frac{1}{2})$

$f(x|x<\frac{1}{2})=\frac{f(x)}{F(x<\frac{1}{2})}=\frac{f(x)}{\frac{1}{2}}=2 \Rightarrow \int_0^{1/2}2x\,dx=x^2\Big|_0^{1/2}=\frac{1}{4}$

Probability summary note

34 Polya urn r: red, b: blue $\Rightarrow \begin{cases}(m+r, b) & r \\ (r, m+b) & b\end{cases}$

X_k: # red balls drawn in the first k selection

① $E(X_1) = \frac{r}{r+b}$

② $E(X_2) = ?$ $Y_i \begin{cases}1 & i \text{ selection is red} \\ 0 & \text{otherwis}\end{cases}$ Trick: Dummy variable

$E(X_k) = \sum_{i=0}^{k} E(Y_i)$

$E(Y_2) = E[E[Y_2|X_1]] = E\left[\frac{r+mX_1}{r+b+m}\right] = \frac{r+m\frac{r}{r+b}}{r+b+m} = \frac{r}{r+b}$

$\Rightarrow E(X_2) = \frac{r}{r+b}$ red at stage → return with m red → probability of being red

③ $E(Y_k) = E[E[Y_k|X_{k-1}]] = E\left[\frac{r+mX_{k-1}}{r+b+(k-1)m}\right]$ = # of red balls

$= \frac{r+m(k-1)\frac{r}{r+b}}{r+b+(k-1)m} = \frac{r}{r+b}$ ≡ since random not past dependent

intuition: any selection could be any of $r+b$ types

$\Rightarrow E(X_k) = \sum_{i=0}^{k} E(Y_i) = k\frac{r}{r+b}$

---

35 ① Two player – ② player i hits: P_i $i=1,2$ ③ End shooting when 1, 2 consecutively hit

① $\begin{cases} h: \text{hit} \\ m: \text{miss} \\ q_i = 1-P_i\end{cases}$ M_i: mean # shots when player i shoot first

$M_1 = E[N|h]\cdot P_1 + E[N|m]\cdot q_1 = P_1P_2\, E[N|h,h] + P_1q_2\, E[N|h,m] + q_1(1+M_2) = 2P_1P_2 + (2+M_1)P_1q_2 + (1+M_2)q_1$

Trick: step Backward 2 step missed; one step missed, restart; 2 step

reset M_i as last two missed

intuition: one step at a time and reset M_i as last two missed

$\Rightarrow \begin{cases} M_1(1-P_1q_2) = 1+P_1+M_2q_1 \\ M_2(1-P_2q_1) = 1+P_2+M_1q_2\end{cases}$

② h_i: mean # times target is hit, player i shoots first → total hit & not by player i

$\begin{cases} h_1 = 2P_1P_2 + (1+h_2)P_1q_2 + h_2q_1 \\ h_2 = 2P_1P_2 + (1+h_1)P_2q_1 + h_1q_2\end{cases}$ # hit = # hit start from 2nd player

---

36 indep trials: success prob: p

① $E(\text{trial} \mid \text{at least } n \text{ success (both) } m \text{ failure})$ S: # trials needed for n success; F: # trials needed for m failure

at least ≡ $\min(S,F) = S+F-\max(S,F)$

Both: $\max(S,F) = E(T) = \sum_{i=0}^{n+m} E[T|N=i]\binom{n+m}{i}p^i(1-p)^{n+m-i}$

$n+m$ → S S F S F S at least one is met; condition on i

$E[T|N=i] = n+m+\frac{n-i}{p}$ $i \le n$ (worry only about success)

$E[T|N=i] = n+m+\frac{i-m}{1-p}$ $i \ge n$ (worry only about failure)

intuition: when two problem get rid of one by conditioning & use pegeon hole

---

37 n cards: $1\ldots n$, $n!$ possible m_n: mean # of cycles

① until 1 appears turn over

② pick lowest not yet appeared

③ cont face up until that card appear

Cont

① $m_n = \frac{1}{n}\sum_{i=1}^{n} E[N|X=i] = \frac{1}{n}\sum_{i=1}^{n}(1+m_{n-i}) = 1+\frac{1}{n}\sum_{j=1}^{n-1} m_j$

$m_1 = 1,\ m_2 = 1+\frac{1}{2} = \frac{3}{2}$ $m_3 = 1+\frac{1}{3}(1+\frac{3}{2}) = 1+\frac{1}{2}+\frac{1}{3} = \frac{11}{6}$

$\Rightarrow m_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} = \frac{1}{n}\left(n-1+\frac{(n-2)}{2}+\cdots+\frac{1}{n-1}\right)+1$

$= 1+\frac{1}{n}\left[n+\frac{n}{2}+\cdots+\frac{n}{n-1} - (n-1)\right] = 1+\frac{1}{2}+\cdots+\frac{1}{n}$

② # cycles $= N = \sum_{i=1}^{n} X_i$ $X_i = \begin{cases}1 & \text{cycle ends with } i \\ 0 & \text{otherwise}\end{cases}$ Trick Dummy

$i = 1,\ldots,n$

③ $m_n = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} P\{i \text{ is last of } 1\ldots i\} = \sum_{i=1}^{n}\frac{1}{i}$

intuition: to have i as end of cycle equal likely with previous cards

④ $X_1,\ldots,X_n$: indep? Yes, knowing $i+1$ is last of $1,\ldots,i+1$ tells nothing about last of $1\ldots i$

Trick: ① Backward ② Can you derive one info from the other?

⑤ $Var(N) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n}\left(\frac{1}{i}\right)\left(1-\frac{1}{i}\right)$ Key Backward Thinking

---

38 maze mouse

$\frac{1}{2}$ Right left $\frac{1}{2}$

$3+E(x)$ Back $\frac{1}{3}(2)$ Free $\frac{2}{3}[E(x)+5]$ Back

$\Rightarrow E(x) = \frac{1}{2}\left(\frac{2}{3}+\frac{2}{3}E(x)+\frac{10}{3}+\frac{3}{2}+\frac{1}{2}E(x)\right) \Rightarrow E(x) = 21$

---

39 x_{t-1} $\xrightarrow{dt}$ x_t $\Rightarrow d_t \sim (0, \beta x^2)$

① $E(X_n|X_{n-1}) = 0$ ② $Var(X_n|X_{n-1}) = \beta X_{n-1}^2$

③ $E(X_n) = 0$ only info needed is previous moment but given that we know we expect no change

④ $Var(X_n) = E(E[X_n^2|X_{n-1}]) = E[\beta X_{n-1}^2] = \beta E(X_{n-1}^2)$

$= \beta^2 E[X_{n-2}^2] = \cdots = \beta^n X_0^2$

intuition: Condition on what know, recursively and although mean would be 0, but the variance is really high, so it could be anywhere

mean captures average (cancell outs bias)

past dependance shows itself in variance

---

40 $Cov(X,Y) = Cov(X, E(Y|X))$

since ① $E[XY] = E[E[XY|X]] = E[XE[Y|X]]$

② $E[Y] = E[E[Y|X]]$

③ $Cov(X,Y)$? if $E(Y|X) = a+bX$

$= Cov(a+bX, X) = bVar(X) \Rightarrow b = \frac{Cov(X,Y)}{Var(X)}$

---

41 $X \sim \text{poiss } \lambda$, $\lambda \sim \exp(\mu=1)$ $\Rightarrow P(X=n) = \int_0^\infty \frac{e^{-\lambda}\lambda^n e^{-\lambda}}{n!}d\lambda = \left(\frac{1}{2}\right)^{n+1}$

---

42 $\begin{cases}\text{stage 1: select 1 out of 10 coin} \\ \text{stage 2: flip until head}\end{cases}$

$P\{N=k\} = \sum_{n=1}^{10}\left(\frac{10-n}{10}\right)^{k-1}\frac{n}{10}\frac{1}{10}$ Geometric if all coins the same

---

43 $X_i \sim iid\ U(0,1)$ $i \ge 1$ (no bound) $N = \min\{n: X_n < X_{n-1}\}$

① $f(x) = E(N) = \int_0^1 E[N|X_1=y]dy$ $\Rightarrow f(x) = 1+\int_x^1 f(y)dy$

$E[N|X_1=y] = \begin{cases}1 & \text{if } y<x \\ 1+f(x) & \text{if } y>x\end{cases}$

② $f'(x) = -f(x)$ $\Rightarrow$ ③ $f(x) = ce^{-x}$, $f(1)=1 \Rightarrow f(x) = e^{1-x}$

④ $P\{x<X_1<X_2\cdots<X_n\} = P\{N>n\} = \frac{(1-x)^n}{n!}$

$\Rightarrow E[N] = \sum_n \frac{(1-x)^n}{n!} = e^{1-x}$ integral = for loop

Trick: Condition on whatever info you have

probability Summary note

1. $f_X = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{other} \end{cases}$ Exponential dist $\begin{cases} E(X) = \frac{1}{\lambda} \\ V(X) = \frac{1}{\lambda^2} \\ \text{Coeff var} = 1 \end{cases}$

   Charact: (i) memoryless : $P\{X = s+t \mid X = s\} = P\{X = t\} \; \forall s,t \equiv X_s = P(X - s \mid X > s) = P(X)$ → residual
   (ii) Failure rate

2. $X_1 \sim exp(\lambda_1)$, $X_2 \sim exp(\lambda_2)$, $X_3 \sim exp(\lambda_3)$ $\quad P(\text{job } 3 = \text{last}) = \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{\lambda_1}{\lambda_1 + \lambda_3} + \frac{\lambda_1}{\lambda_3 + \lambda_1} \frac{\lambda_2}{\lambda_2 + \lambda_3}$ Condition on who finished first

3. if expon $X_1 - X_n$ $\quad P\{X_1 = \min(X_1, \dots, X_n)\} = \frac{\lambda_1}{\lambda_1 + \dots + \lambda_n}$

4. $P(\text{job } 2 \text{ last}) = \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\lambda_3}{\lambda_3 + \lambda_2}$ $\boxed{\text{memoryless property}}$

   $\boxed{\text{Geometric also memoryless for discrete}}$

5. Proof of memoryless : $g(s+t) = g(s) g(t) \Rightarrow g(t) = e^{-\lambda t}$

   $t = m/n \Rightarrow g(\frac{m}{n}) = g(\frac{1}{n})^m = g(1)^{m/n} = e^{(m/n) \ln(g(1))}$ $\quad -\ln(g(1)) = \lambda$

6. Failure rate func: $P\{t < x < t+s \mid x > t\}$ $\quad x \in [t, t+s]$

   $\Rightarrow r_X(t) = \frac{f(t)}{\bar{F}(t)}$ $\quad$ for exp: $r_X(t) = \lambda$ $\quad \bar{F}(t) = e^{-\int_0^t r(y) dy}$

7. Bass $\frac{f(t)}{\bar{F}(t)} = p + qF(t)$ hazard rate $\Rightarrow F(t) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}}$

   $F(t)$ ⟶ who has already hazard

8. $[N(t), t > 0]$ Counting/poisson Process (CP) (PP) $\quad N(t) = \#$ events by time t

   $N(0) = 0$ $\quad \rightarrow \equiv Cov(X_1 + X_2, X_2 + X_3) = Var(X_2)$

9. $\boxed{\text{independent increment}}$ (II) $\quad N_1 \perp N_2$ $\quad N(t_1) - N(s_1) = N_1 \quad (s_1, t_1)$

   $N(t_2) - N(s_2) = N_2 \quad (s_2, t_2)$

10. $\boxed{\text{stationary increment}}$ (SI) $\quad N(s_1 + t) - N(s_1) = N(s_2 + t) - N(s_2)$

11. $\boxed{\text{Definition 1}}$ $C.P(\lambda)$ (i) $N(0) = 0$ (ii) II, SI (iii) $P\{N(t) = n\} = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$

12. $\boxed{\text{Definition 2}}$ $C.P(\lambda)$ (i) $N(0) = 0$ (ii) II, SI (iii) $P\{N(h) = 1\} = \lambda h + o(h)$

    (iv) $P\{N(h) = 0\} = 1 - \lambda h + o(h)$ $\quad h$: infinitesimal

13. $f(x) \sim o(g(x))$ iff $\lim_{n \to 0} \frac{f(x)}{g(x)} = 0$

14. $\boxed{\text{Definition 3}}$ $C.P(\lambda)$ inter event times $\sim$ iid $exp(\lambda)$

15. $P_0(t+h) = P\{\underbrace{\overset{P_0(t)}{N(t) = 0}, N(t+h) - N(t) = 0}_{\text{independence}}\} = P_0(t)(1 - \lambda h + o(h))$

    $\Rightarrow \frac{\partial P_0(t)}{\partial t} = \frac{P_0(t+h) - P_0(t)}{h} = -\lambda h P_0(t)$

16. $P\{N(h) = 0\} = e^{-\lambda h} = 1 - \frac{\lambda h}{1!} + \frac{(\lambda h)^2}{2!} + \dots$

    $\Rightarrow P\{N(h) = 0\} = 1 - \frac{\lambda h}{1!} + o(h)$, $P\{N(h) = 1\} = \lambda h + o(h)$

17. $P\{X_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$ $\quad X_1 \sim Exp(\lambda)$ $\quad X_1, X_2$ = inter event

    $P\{X_1 + X_2 > t\} = P\{N(t) = 0 \text{ or } 1\} = \underbrace{P\{N(t) = 0\} + P\{N(t) = 1\}}_{\text{indep \& each could be true}} = e^{-\lambda t} + \lambda t e^{-\lambda t}$

    $\Rightarrow \frac{\partial P\{X_1 + X_2 > t\}}{\partial t} = -\lambda^2 e^{-\lambda t}$ $\boxed{\text{Erlang } (2, \lambda)}$

18. $Y_1, \dots, Y_n \sim$ iid dist $f(\cdot)$ $\quad Y_{[i]}$ = i'th smallest $\Rightarrow f_{Y_{[1]} \dots Y_{[n]}}(Y_1, Y_2, \dots, Y_n) = n! \prod_{i=1}^{N} f(y_i)$

    $\boxed{\text{intuition}}$ $n!$ different permutation of separate $Y_1 - Y_n$ that lead to left side is only considered once for drawing

19. train at time: T $\quad w_i \sim$ iid $Unif(0, T)$

    Passengers arrival: λ rate

    $\Rightarrow E(W) = E(E(\sum w_i \mid N(t))) = E[\sum_{i=1}^{N(t)} w_{[i]}] = \frac{N(T) T}{2} = \frac{\lambda T^2}{2}$

    $\boxed{\text{Trick : decompose \& Indicator}}$

20. $M/G/\infty$

    $M \sim$ expon, poiss(λ) $\quad N(t)$ = # in the system

    G : General $\quad P\{N(t) = k\} = \sum_{n=0}^{\infty} P\{N(t) = k \mid A(t) = n\}$

    ∞ : infinite servers

    $* \{A(t) = n\} = \binom{n}{k} [p(t)]^k [1 - p(t)]^{n-k}$

    probability of (entering × staying) $\quad G(x) \equiv$ service time lower than x

    $P(t) = \int_0^t \bar{G}(t-y) dy \times \frac{1}{t}$ → prob of entrance

    service time more than $t-y$

    $\Rightarrow N(t) \sim \text{poisson}(\lambda t p(t))$

    $[N(t) = k] \Rightarrow N(t) = \lambda t p(t) = \lambda \int_0^t \bar{G}(y) dy \Rightarrow L = \lambda W$

21. $A(t) - N(t)$ = # arrived & departed $[0, t] = D(t)$

    $N(t) \sim \text{poisson}(\lambda t p(t))$ $\boxed{\text{poisson Spliting Process}}$

    $\Rightarrow D(t) \sim \text{poisson}(\lambda t(1 - p(t)))$

    $D(t) \perp N(t)$

22. $I_1, I_2, \dots \sim$ iid Bernouli(P)

    $\boxed{N \sim \text{poisson}(\lambda)}$ $\Rightarrow N_1 = \sum_{i=1}^{N(t)} I_i \sim \text{poisson}; \lambda(P)$

    $N_2 = \sum_{i=1}^{N(t)} (1 - I_i) \sim \text{poisson}; \lambda(1-P)$

23. $N_1 \perp N_2$: $P\{N_1 = m, N_2 = n\} = P\{N_1 = m, N_2 = n \mid N = n+m\}$

    $\cdot \Pr\{N = n+m\} = \binom{m+n}{n} p^m (1-p)^n \frac{e^{-\lambda} \lambda^{n+m}}{(n+m)!} = \frac{e^{-\lambda p} (\lambda p)^m}{m!} \cdot \frac{e^{-\lambda(1-p)} [\lambda(1-p)]^n}{n!}$

    $= P\{N_1 = m\} P\{N_2 = n\}$ $\Rightarrow$ $\boxed{\text{intuition}}$: Bernouli split of poisson is two new poisson

$\boxed{\text{Names}}$ ① Decomposition of poisson process ② thinning process (Bernouli trial split)

24. $\{N_i(t), t \geq 0\}$ $i = 1, 2$ indep rate λ_i

    $N(t) = N_1(t) + N_2(t) \Rightarrow N(t)$ is poisson at rate $(\lambda_1 + \lambda_2)$

25. $\boxed{\text{Non homogeneous poisson Process}}$ $\{N(t), t \geq 0\}$

    intensity func: $\{\lambda(t), t \geq 0\} \Rightarrow$ ① $P(N(t) = k) = \frac{m(t)^k}{k!} e^{-m(t)}$

    ② $P(N(h) = 1) = \lambda(t) h$, ③ $m(t) = \int_0^t \lambda(y) dy$ $\quad$ e.g. Call Center

    ④ $P(N(h) = 0) = 1 - \lambda(t) h$ $\quad$ rate function of time

    ⑤ interevent $\neq$ iid

26. $(S_1, \dots, S_n \mid N(t) = n) \sim (Y_{[1]}, \dots, Y_{[n]})$ $\quad Y_i \sim$ iid $Unif(0, t)$

    ↳ intervals $\quad \sim pdf = \frac{\lambda(u) \cdot R}{m(t)}$ $\quad$ e.g. random order diff amnt

27. $\boxed{\text{Compound poisson Process}}$ with poisson process

    $X_1, X_2, \dots \sim$ iid Jumps arriving

    $Y(t) = \sum_{i=1}^{N(t)} X_i , t \geq 0$ $\quad \{Y(t), t \geq 0\}$ = Compound poisson process

28. $\boxed{\text{Birth \& Death Process}}$ one level up or time at time

    e.g. inventory, demand $\quad$ Backlog = event negative

    duration: random to change state (so journ in the state)

    $Min(X, Y)$ where $X \sim Exp(\lambda_i)$, $Y \sim Exp(\mu_i)$

    $P(\text{going up}) = \frac{\lambda_i}{\lambda_i + \mu_i}$ $\quad P(\text{going down}) = \frac{\mu_i}{\lambda_i + \mu_i}$

Probability summary note (13)

① $P_n(t+h) = P\{N(t)=n+1\}\{(\lambda_{n+1}+\mu_{n+1})h + o(h)\}\frac{\mu_{n+1}}{\lambda_{n+1}+\mu_{n+1}}$ ①

at least one event — that event is going down

③ $+P\{N(t)=n\}[1-(\lambda_n+\mu_n)h+o(h)]$ happens

no event happens

② $+P\{N(t)=n-1\}[(\lambda_{n-1}+\mu_{n-1})h+o(h)]\frac{\lambda_{n-1}}{\lambda_{n-1}+\mu_{n-1}} + o(h)$

at least one event happens — that event is going up

intuition: peogen whole thinking, hierarchical thinking

$\Rightarrow P_n(t+h)-P_n(t) = \overset{①}{P_{n+1}(t)\mu_{n+1}h} + \overset{②}{P_{n-1}(t)\lambda_{n-1}h} - \overset{③}{P_n(t)(\lambda_n+\mu_n)h}$

$\Rightarrow \frac{\partial P_n(t)}{\partial t} = P_{n+1}\mu_{n+1} + P_{n-1}\lambda_{n-1} - P_n(t)(\lambda_n+\mu_n)$

Convergence assumption: $\lim_{t\to\infty} P_n(t)=P_n \Rightarrow \frac{\partial P_n(t)}{\partial t}\to 0$

$\Rightarrow \boxed{P_n\times(\lambda_n+\mu_n) = P_{n+1}\times\mu_{n+1} + P_{n-1}\times\lambda_{n-1}}$ Balance Equation

Rate of leaving — rate of entering

$\boxed{P_n = \lim_{T\to\infty}\frac{1}{T}\int_0^T 1\{N(y)=n\}\,dy}$ time Average

② $\lambda_0P_0=\mu_1P_1 \Rightarrow P_1=\frac{\lambda_0}{\mu_1}P_0$

$(\lambda_1+\mu_1)P_1 = \underbrace{\lambda_0P_0}_{\lambda_0P_0}+\lambda_2P_2 \Rightarrow \lambda_1P_1=\mu_2P_2 \Rightarrow P_2=\frac{\lambda_1}{\mu_2}P_1$

$\Rightarrow \sum P_i=1 \Rightarrow \sum_{i=0}^\infty P_0\left(1+\frac{\lambda_0}{\mu_1}+\frac{\lambda_1\lambda_0}{\mu_2\mu_1}+\cdots\right)=1$

$\Rightarrow P_n = \frac{\lambda_{n-1}\cdots\lambda_0}{\mu_n\cdots\mu_1}\left(1+\frac{\lambda_0}{\mu_0}+\frac{\lambda_1\lambda_0}{\mu_2\mu_1}+\cdots\right)^{-1}$ $n\ge0$

Server utilization / Busy time / throughput

③ M/M/1 — λ: arrival, μ: service $\Rightarrow \boxed{\rho=\frac{\lambda}{\mu}}$ → input / output

$\Rightarrow P_0=(1+\rho+\rho^2\cdots)^{-1} = \left(\frac{1}{1-\rho}\right)^{-1}=1-\rho$ $\boxed{\rho<1}$ — input rate / serv free rate

$P_n=\rho^n(1-\rho)$

$L=\#$ in system $=\sum_{n=0}^\infty nP_n=\rho\sum_{n=0}^\infty n\rho^{n-1}(1-\rho) = \frac{\rho}{1-\rho}=\frac{\lambda}{\mu-\lambda}$

$W=$ AVG (waiting) $=\left(\left[\sum_{n=0}^\infty nP_n\right]+1\right)\frac1\mu = \frac1\mu\left(\frac{\lambda}{\mu-\lambda}+1\right)=\frac{1}{\mu-\lambda}$

People waiting to be served; under service; mean duration; Free rate inverse

PASTA: Poisson Arrivals See time Averag

Prob. of state as seen by an outside random observer is same as prob of the state seen by an arriving customer

④ Little Formula $L=\lambda W$ — intuition: Cost Structure

total amount collected $[0,T]\approx LT=\lambda TW$ — Pay 1\$ per minute; earned by server; entered; Customer AVG cost

time AVG of having busy server: $\frac\lambda\mu=\rho=1-P_0$

④ M/M/1/N — Truncate after state N (Capacity) $\lambda_N=0$

⑤ M/M/K — k server parallel $\lambda_i=\lambda$

$\mu_i=\begin{cases} i\mu & 1\le i\le k\\ k\mu & i>k\end{cases}$ → intuition higher rate of service

(14)

⑥ M/M/K/K: Erlang Loss model e.g telephone line

⑦ M/M/1 Batch Arrival: markov chain

$P_{ij}=\begin{cases}\frac{\lambda}{\lambda+\mu} & j=i+k\\ \frac{\mu}{\lambda+\mu} & j=i-1 \quad i\ge1\\ 0 & \text{otherwise}\end{cases}$

P(transition) only based on current state, and not previous state

next state j, given current state i

$\Rightarrow L=\sum_{n=0}^\infty nP_n$

$W=\sum_{n=0}^\infty P_n\frac1\mu\left(n+\frac{k+1}{2}\right)$

$\Rightarrow W=\frac1\mu\sum_{n=0}^\infty nP_n+\frac1\mu\left(\frac{k+1}{2}\right) = \frac{L}{\mu}+\frac{k+1}{2\mu}$ $\boxed{\sum_{j=1}^k j\frac1k}$ Batch of k expected position

$L=\Lambda W$, $\Lambda=\lambda k$ — server utilization: $\rho=\lambda k\times\frac1\mu<1$

$L_Q=L-\rho$ length of queue

$W_Q=W-\frac1\mu$ waiting in queue

⑧ Related method: $M/E_k/1$ — $E_k=$ Erlang(k,μ)

mean $=k/\mu$, sum of k iid — Exponential model every piece in term of Expon ver $\Rightarrow$ CTMC (Cont. time Markov chain)

⑨ State Definition — enough information so that: markovian = memoryless = sufficient info in state

e.g $N(t)=\#$ phases in entire system

k=5 phase remain, only when all phase done next enter $\Rightarrow$ 14 phase in system $\equiv$ 3 person in system, one only passed 1 phase

$\lceil\frac{N(t)}{k}\rceil\equiv\#$ Customers in system

$W_{M/E_k/1}=W_{M/M/1\text{ Batch }k}+\frac{k-1}{2}\frac1\mu$

⑩ Method of phases — $X=\sum_{i=1}^k X_i$, $X_i\sim$ iid Exp$(k\mu)$

$E(X)=k\frac{1}{k\mu}=\frac1\mu$ — $Var(X)=k\frac{1}{(k\mu)^2}=\frac1k\frac1{\mu^2}\xrightarrow{k\to\infty}0$

⑪ Exponential & Geometric

① any Constant can be approx. by rand. variable

$Y=\begin{cases}c_1 & p\\ c_2 & 1-p\end{cases}$ — $c_1\simeq\sum_{i=1}^{k_1}X_{1i}$, $X_{1i}\sim$ iid-Exp; $c_2\simeq\sum_{i=1}^{k_2}X_{2i}$, $X_{2i}\sim$ iid-Exp

② $Z_1\sim Exp(\lambda_1)$, $Z_2\sim Exp(\lambda_2)$, $\lambda_1>\lambda_2$ $\Rightarrow Z_2=\sum_{i=1}^N Z_{1i}$, $N\sim$ Geometric$\left(\frac{\lambda_2}{\lambda_1}\right)$ — very important

Proof: ① Conditioning ② MGF

- Regeneration of greater value by summing up smaller range (# discrete r.v) — Denseness Concept

whatever small interval, find rational # in

⑫ Denseness: any r.v $=\sum_{i=1}^n Z_i$, $Z_i\sim$ iid Expon. is called Phase-type

e.g Decompose GI/GI/1 (GI: General independent) to phase type dist. (Building Blocks)

⑬ curse of dimensionality — putting lots of info in state space

⑭ Barber shop: Markovian (Exponential time to next state)

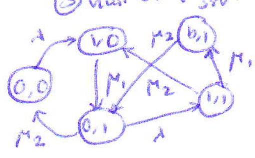
① no queue
② (haircut, waiting) $=(\mu_1,\mu_2)$
③ hair cut (rcv/srv or Block)

ⓐ Loss probability: $P_{10}+P_{11}+P_{b1}$

ⓑ Entrance rate: $\lambda[1-(P_{10}+P_{11}+P_{b1})]$

ⓒ $L=0\cdot P_{00}+1(P_{10}+P_{01})+2(P_{11}+P_{b1})$

(d) $W=L/\lambda_e$



probability Summary note

(14) waiting in barbeq Cont:

(ii) $W = 0\,(P_{10}+P_{11}+P_{b1}) + P_{00}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right) + P_{01}\left[\frac{\mu_1}{\mu_1+\mu_2}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_1}\right) + \frac{\mu_2}{\mu_1+\mu_2}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\right]$

- $0\,(P_{10}+P_{11}+P_{b1})$: Can't enter so no waiting
- $P_{00}(\ldots)$: not blocked so quickly moves to each chair & done
- $\frac{\mu_1}{\mu_1+\mu_2}(\ldots)$: ① finishes first, ② waits for other finish (Expo. memoryless) ③ waits himself finish washing

since washing chair finishes before him just has his own time

(iii) $A = \frac{1}{\mu_1} + \underbrace{\frac{\mu_2}{\mu_1+\mu_2}\cdot\frac{1}{\mu_2}} + \frac{\mu_1}{\mu_1+\mu_2}\left(\frac{1}{\mu_2}+\frac{1}{\mu_2}\right)$

① he finishes ② not blocked so just own waiting ③ Blocked so waits only for both himself and other

(iv) $A = \underbrace{\frac{1}{\mu_1}+\frac{1}{\mu_2}-\frac{1}{\mu_1+\mu_2}}_{\text{max time takes to move from first chair to 2nd}} + \underbrace{\frac{1}{\mu_2}}_{\text{time land from 2nd chair}}$

(d) $W_e = \frac{P_{00}}{P_{00}+P_{01}}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right) + \frac{P_{01}}{P_{00}+P_{10}}\cdot A$

server utilization: server 1: $P_{10}+P_{11}$
server 2: $P_{01}+P_{11}+P_{b1}$

server occupancy: server 1: $P_{10}+P_{11}+P_{b1}$
server 2: $P_{01}+P_{11}+P_{b1}$

(e) If service at 2nd chair Erlang ⇒ $(0,0),(1,0),(0,1),(0,1'),(1,1),(1,1'),(b,1),(b,1')$

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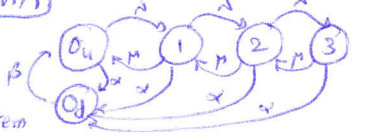
(15) Server Breakdown M/M/1

- α: failure rate (expon)
- β: Repair

1, 2, 3 : # people in system
one server



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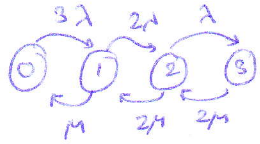
(16) Repairman Problem

3 machines - uptime ~ exp(λ)
2 repair - Repair time ~ exp(λ)
states: # down machines



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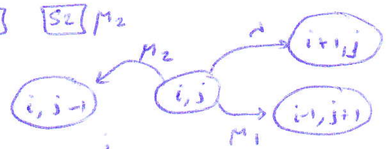
(17) Tandem Queue M/M/1 → M/1

{ 2× serial server
{ 2 × serial Queue

state $(S_1, S_2) = (i,j)$ ⇒



$\rho_1 = \frac{\lambda}{\mu_1}$ $\rho_2 = \frac{\lambda}{\mu_2}$ $P_{ij} = \rho_1^i(1-\rho_1)\,\rho_2^j(1-\rho_2)$ indep assump.

---

(18) Renewal process relax assump of equal # events = events f(time)

x_1, x_2, x_3, x_i ~ iid $F(\cdot)$ $m(t) = E[N(t)]$

in inhomog. Poiss rate funct. time → still Inter arrival ~ exp

not necessarily Exponential but positive ⇒ $\frac{N(t)}{t} \to \frac{1}{E(x_1)=\mu_F}$

$\frac{\sum_{i=1}^{N(t)} X_i}{N(t)} \le \frac{t}{N(t)} \le \frac{\sum_{i=1}^{N(t)+1} X_i}{N(t)+1}\cdot\frac{N(t)+1}{N(t)} \Rightarrow \mu_F \le \frac{t}{N(t)} \le \mu_F$

(19) Y_i = renewal recieved at the end of i'th ren. interval

(X_i, Y_i) $i \ge 1$ ~ iid r. vector — Renewal Reward process
X_i, Y_i ~ not necessarily indep

e.g. ① monkey randomtype type Macbeth
② long run benefits of different insurance policies

$R(t) = \sum_{i=1}^{N(t)} Y_i$

$\frac{R(t)}{t} \to \frac{E(Y_1)}{E(X_1)}$ with prob 1

CTMC but tim b/w not Exponential but general form

$\frac{N(t)}{t}\cdot\frac{\sum_{i=1}^{N(t)} Y_i}{N(t)} \le \frac{R(t)}{t} \le \frac{\sum_{i=1}^{N(t)+1} Y_i}{N(t)+1}\cdot\frac{N(t)+1}{t}$

(20) application: Train dispatching (Dispatch when # = K)

C: $\frac{\text{waitin Cost}}{\text{time}}$ for each cust. Long run AVG Cost = $\frac{E\left[\frac{\text{Cost}}{\text{Cycle}}\right]}{E(\text{cycle's time})}$

Q: long run AVG waiting time $= \frac{\left[\frac{1}{\lambda}+\frac{2}{\lambda}+\cdots+\frac{k-1}{\lambda}\right]}{\frac{k}{\lambda}}$

Trick: Calc $\frac{\text{reward}/\text{Cycle}}{\text{Cycle time}}$

i'th person will wait for $(k-i)$ person in each Com $\frac{1}{\lambda}$ minute
indep: $(k-1)\times\frac{1}{\lambda}$

- Reward during regen process per cycle

---

(21) Car replacement Life time: $F(\cdot)$

Replacement cost { C_1 if working
{ $C_1 + C_2$ if dead

① take T as cycle
② E(Cycle Time) = $E(\min(T, X))$ → use tail Prob
③ E[Cost in a cycle] = $C_1 + C_2\,P[X<T] = C_1 + C_2 F(T)$

---

① $E[X^2 \mid X>1] = E[(X+1)^2]$ memoryless

---

② (S) ① [ ] $\cdot\,\lambda_1$ ② [ ] $\cdot\,\lambda_2$ $P(S \text{ not last}) = \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2 + \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2$

---

③ X_1, X_2 ~ indep nonNeg Cont. r.v.

$P\{X_1 < X_2 \mid \min(X_1,X_2) = t\} = \frac{r_1(t)}{r_1(t)+r_2(t)}$ $r_i(t)$: failure rate func X_i

$= \frac{P\{X_1<X_2, \min(X_1,X_2)=t\}}{P\{\min(X_1,X_2)=t\}} = \frac{P\{X_1=t, X_2>t\}}{P\{X_1=t, X_2>t\} + P\{X_2=t, X_1>t\}}$

$= \frac{f_1(t)\bar{F}_2(t)}{f_1(t)\cdot\bar{F}_2(t) + f_2(t)\cdot\bar{F}_1(t)}$

then divide by $\bar{F}_1(t)\bar{F}_2(t)$

intuition: in Denomin when min of two is t, either first one is greater the other equal, or reverse

④ { $X \sim \exp(\lambda)$
{ $Y \sim \exp(\mu)$
{ X, Y ~ indep Exp
{ $M = \min(X, Y)$

(a) $E(MX \mid M=x) = E[M^2 \mid M=x] = \frac{2}{(\lambda+\mu)^2}$

$= \underbrace{\frac{1}{(\lambda+\mu)^2}}_{\mu^2} + \underbrace{\frac{1}{(\lambda+\mu)^2}}_{+\sigma^2}$

intuition: ① when X, Y Exp(λ, μ) min prob: $\frac{1}{\lambda+\mu}$
when two person work together they do faster in same amount of time but still Exponentially

② You can generalize when you have info mean take thing from right side bar

Probability summary note

independence

ⓑ $E[mx \mid m=Y] = E[m(m+x')] = E[m^2] + E[m]E[x']$

residual X on Y ← $= \dfrac{2}{(\lambda+\mu)^2} + \dfrac{1}{\lambda(\lambda+\mu)}$

$x' \sim$ Expon

intuition: residual of one Exp. variable is exp. variable with same rate due to memoryless

ⓒ $Cov(x,m) = E[mx] - E(m)\cdot E(x)$

$E[mx] = E[mx \mid m=x]\dfrac{\lambda}{\lambda+\mu} + E[mx \mid m=Y]\dfrac{\mu}{\lambda+\mu} = \dfrac{2\lambda+\mu}{\lambda(\lambda+\mu)^2}$

$\Rightarrow Cov(x,m) = \dfrac{\lambda}{\lambda(\lambda+\mu)^2}$

---

⑤ ⓐ $x_1, x_2, x_3 \sim exp(\lambda_1, \lambda_2, \lambda_3)$ → indep (lack of memory)

$\Rightarrow P\{x_1 < x_2 < x_3\} = P\{x_2 < x_3 \mid x_1 = \min(x_1,x_2,x_3)\} \cdot Pr\{x_1 = \min(x_1,x_2,x_3)\}$

$= \dfrac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \dfrac{\lambda_2}{\lambda_2+\lambda_3}$

intuition: always condition on lower values, since indep of lack of memory helps you not care about rightber

ⓑ $P\{x_2 < x_3 \mid x_1 = \max(x_1,x_2,x_3)\} = \dfrac{P\{x_2 < x_3 < x_1\}}{P\{x_2 < x_3 < x_1\} + P\{x_3 < x_2 < x_1\}}$ ①

$= \dfrac{\dfrac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} \dfrac{\lambda_3}{\lambda_1+\lambda_3}}{\dfrac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}\dfrac{\lambda_3}{\lambda_1+\lambda_3} + \dfrac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}\dfrac{\lambda_2}{\lambda_1+\lambda_2}}$

intuition: You don't need to use condition formula sometimes based on two info you can get since condition tells where in the intersection e.g. ① tree you are

$p(a|L)$ — $p(a|LL)$, $p(a|LR)$

a — $p(a|R)$

trick: ① to calculate $P\{x_1 < x_2 \mid \max(x_i) = x_3\}$ you need to find $P\{x_1 < x_2 < x_3\}$

ⓒ $E(\max x_i \mid x_1 < x_2 < x_3) = \underbrace{\dfrac{1}{\lambda_1+\lambda_2+\lambda_3}}_{\text{min of all}} + \underbrace{\dfrac{1}{\lambda_2+\lambda_3}}_{\text{min of second two}} + \underbrace{\dfrac{1}{\lambda_3}}_{\text{min of third}}$

intuition: when memoryless to get expectation of max, first get min (time) of all, then from memoryless second max is just from remaining and continue with tail method

$E(\max|\ldots) = E(\text{first min}) + E(\text{2nd min}) + E(\text{remain}\ldots)$ when memoryless

ⓓ $E[\max x_i] = \sum_{i \neq j \neq k} \dfrac{\lambda_i}{\lambda_1+\lambda_2+\lambda_3} \dfrac{\lambda_j}{\lambda_j+\lambda_k}\left[\dfrac{1}{\lambda_1+\lambda_2+\lambda_3} + \dfrac{1}{\lambda_j+\lambda_k} + \dfrac{1}{\lambda_k}\right]$

intuition: to calculate $E[\max x_i]$ condition on reality: $E(\max x_i \mid x_1 < x_2 < x_3)$

⑥ ⓐ $E[x \mid x<c] = \dfrac{1}{\lambda} - \dfrac{ce^{-\lambda c}}{1-e^{-\lambda c}}$

$f(x \mid x<c) = \dfrac{f(x)}{P\{x<c\}} = \dfrac{\lambda e^{-\lambda x}}{1-e^{-\lambda c}}$ $0 < x < c$

$\Rightarrow E[x \mid x<c] = \int_0^c \dfrac{x\lambda e^{-\lambda x}}{1-e^{-\lambda x}}dx$ ⇒ integration by parts

ⓑ $E(x) = E(x \mid x<c)P\{x<c\} + E[x \mid x>c]P\{x>c\}$

main rather than tail ← $1-e^{-\lambda c}$ $\quad (c+1/\lambda)\underbrace{e^{-\lambda c}}$ ↑ tail prob of c

---

⑦ $x_1, x_2 \sim$ ind. exp (μ)

ⓐ $x_{(1)} = \min(x_1, x_2)$ $\quad x_{(2)} = \max(x_1, x_2)$

$\Rightarrow E(x_{(1)}) = E(x_{(1)} \mid x_1 < x_2)\cdot P\{x_1<x_2\} + E\{x_2 \mid x_2 < x_1\}\cdot P\{x_2 < x_1\}$.

$E\{x_1 \mid x_1 < x_2\} = \int x_1 e^{-\lambda x}\lambda e^{-\lambda x}dx = \int x\lambda e^{-2\lambda x}dx = \dfrac{1}{2\lambda}$

$P\{x_1<x_2\} = \dfrac{\lambda}{2\lambda} \Rightarrow E[x_{(1)}] = 1/2\mu$

ⓑ $Var[x_{(1)}] = \dfrac{1}{4\mu^2}$ Var of Expon. is mean square

intuition: min of two exponential variable has an exponential distribution

ⓒ $E(x_{(2)}) = E(x_{(1)} + A) = \dfrac{1}{2\mu} + \dfrac{1}{\mu}$

intuition: Forward approach from memoryless allows simple summation

Backward approach: $\min + \max = x_1 + x_2$

$\Rightarrow E[\max] = E(x_1) + E(x_2) - E(\min)$

ⓓ $Var[x_{(2)}] = Var[\underbrace{x_{(1)} + A}_{\text{No Covariance}}] = \dfrac{1}{4\mu^2} + \dfrac{1}{\mu^2} = \dfrac{5}{4\mu^2}$

---

duality intuition

① sum and split of Poisson process is poisson process since sum and deduction of Exponential dist. is exponential dist. due to memoryless property

② server reset: in exponential you start from server that finishes sooner than others, and then at start of each you reset time

③ waiting in system $= \underbrace{\dfrac{1}{\mu_1} + \dfrac{1}{\mu_2} - \dfrac{1}{\mu_1+\mu_2}}_{\max(x_1,x_2) = x_1+x_2-\min(x_1,x_2)} + \overbrace{\dfrac{1}{\mu_2}}^{\text{service on 2nd chair}}$

① service when waiting ←

② service when NOT waiting ←

$\boxed{1} \rightarrow \boxed{2}$ **Barber shop model**

$\mu_1 \quad \mu_2$

④ Birth & Death process are proportional to # people ⇒ state i: $i\lambda$: Birth $\quad i\mu$: Death

probability summary note

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⑧ Satelite problem $M/G/\infty$ { launching rate λ ; life time F dist }

(a) $X(t)$: # satelite orbiting

$P(X(t)=k) = e^{-\lambda(t)} [\lambda(t)]^k / k!$

$\lambda(t) = \lambda \int_0^t (1-F(s))ds$ enter any time but survive until now

(b) system functioned if at least one satelite orbiting $E(T)$

renewal process: long run Avg cost $= \underbrace{\lim P\{X(t)=0\}}_{\text{no sateiite orbiting}} = \dfrac{1/\lambda}{1/\lambda + E(T)}$

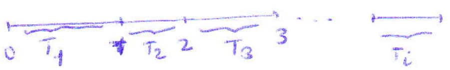
renewal process shock: recognizing none orbiting ⇒ launch one

$e^{-h(t)} = 1-F(t)$ ⟶ hazard function ⇒ $\mu = \int_0^\infty (1-F(s))d(s)$ mean time satelite orbits

≡ lives for ever when enters at s

$\underbrace{e^{-\lambda(t)}}_{\text{no satelite orbiting}} = \dfrac{1/\lambda}{1/\lambda + E[T]}$

⑨ Population problem — each person fertility λ (Birth only process) ⇒ rate: $n\lambda$



⇒ $T_i \sim exp(i\lambda)$ Forward approach

⇒ Backward approach $Prob(T_1 + \cdots + T_n < t) = prob(\max(x_i) < t)$

$\prod prob(x_i < t) = (1-e^{-\lambda t})^j$

intuition You look back word and all current people die until time zero ⇒ each has lifetime lower than t

intuition: Backword approach

$\underset{\text{move population from size i to j}}{P_{ij}} = \binom{j-1}{i-1} \underbrace{e^{-\lambda i t}}_{\text{Prob of longer life until } i} \underbrace{(1-e^{-\lambda t})^{j-i}}_{\text{probability of die (mean lower duration of life)}}$

⑩ T_i time of entrance in poisson is $\frac{1}{t}$ mem unit $(0,t)$

$E[\sum_{i=1}^n g(S_i) | N(t)=n] = E[\sum_{i=1}^n g(U_{[i]}) | N(t)=n] = E[\sum_{i=1}^{N(t)=n} g(u_i)]$

$\sum g(U_{[i]}) = \sum g(u_i)$ since random batch & sort ≡ sorted batch

⑪ Infection Ban model:

① N indiv ② Contact: poisson λ ③ $\binom{N}{2}$ Equally likely

④ pinfection: $\dfrac{i(n-i)}{\binom{n}{2}}$ intuition: poisson duality

⇒ $E[\text{time all infected}] = \sum_{i=1}^n \frac{1}{\lambda_i} = \frac{n(n-1)}{2\lambda} \sum_{i=1}^n \frac{1}{i(n-i)}$

intuition: Sort, and start from smallest: Expon. variable

intuition
① Equally likely $\frac{i(n-i)}{\binom{n}{2}}$
② poisson splitting ⇒ $p\lambda$
③ $\sum[Time] = \sum_{i=1}^n \frac{1}{\lambda_i}$ memoryless Exponential

important not on Quick Solving Problems:
① organized mind ② multiple reviews ③ Teaching
④ Conditioning ⑤ Previously solve Problems (all) & multiple reviews

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⑫ poisson non homogeneous: $m(t) = \int_0^t \lambda(t)dt$

≡ poisson : $\lambda h = \int_t^{t+h} \lambda(s)ds$ Break into smaller interval

$P\{t_i < S_i < t_i + h_i | N(t)=n\} = e^{-[m(t_i+h_i)-m(t_i)]} \times [m(t_i+h_i)-m(t_i)]$

Exponential poisson duality

⑬ customers ⟶ $exp(\lambda)$

N customer — summon

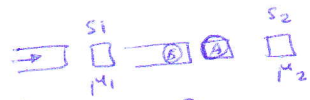
K arrival

long run cost $= \dfrac{E[\text{cycle cost}]}{E[\text{length of cycle}]} = \dfrac{ⓐ}{\frac{N}{\lambda}+K}$

ⓐ $E[\text{cycle cost}] = c[\underbrace{\frac{N-1}{\lambda}+K}_{\text{waiting for others by first person}} + \frac{N-2}{\lambda}+K + \cdots + K$

$+ \underbrace{\frac{\lambda K^2}{2}}]$ ⟶ { λK person ; $K/2$ mean of waiting } ≡ $\underbrace{\int_0^K \lambda k\, dk}_{\text{integral of waiting}} = \frac{\lambda k^2}{2}$

⑭



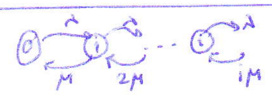
$E(T) = \underbrace{\frac{1}{\mu_1}}_{\text{done with first server}} + \underbrace{\frac{P_A}{\mu_2}}_{\text{A was there and waited to out}} + \underbrace{\frac{P_B}{\mu_2}}_{\text{B was there and wait to end service}} + \underbrace{\frac{1}{\mu_2}}_{\text{main person 2nd serve done}}$

$P_A = \dfrac{\mu_1}{\mu_1+\mu_2}$: first entrance enters Befor A goes out

$P_B = 1 - P_B(out) = 1 - (\frac{\mu_2}{\mu_1+\mu_2})(\frac{\mu_2}{\mu_1+\mu_2})$: Prob B in system when entrant moves to 2nd server

Intuition probability that person exists when move to next server to calculate Expectation of entrant's waiting time in system

⑮ λ → O μ
⋮
O $\infty\mu$

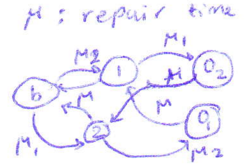


$E[\underline{\text{first departure time}}] = \sum_{i=1}^\infty P_i(t) \frac{1}{\lambda + i\mu}$

↳ $\frac{i\mu}{\lambda+i\mu} \times \frac{1}{i\mu} = \frac{1}{\lambda+i\mu}$

⑯ two machines μ_i $i=1,2$ time to detect $\sim exp$

μ : repair time $\sim exp$



$P_{b,1} = \dfrac{\mu_2}{\mu_1+\mu_2}$

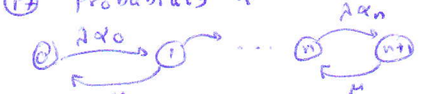
↳ prob moving from state b to state 1

1: 2nd server needs repair

2: 1st server needs repair

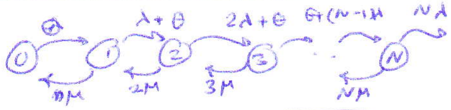
0_2: server 2 under repair, 1st serve down

⑰ Probability α_i of entrance at state i



Probability summary note

(21)

(18) 

$$\begin{cases} \lambda_n = n\lambda + \theta & n < N \\ \lambda_n = n\lambda & n \geq N \\ \mu_n = n\mu \end{cases}$$

Birth death with immigration

$$P_k = \frac{(k-1)\lambda}{k\mu} P_{k-1}$$

$$\Rightarrow P_k = \frac{(k-1)(k-2)\cdots(3)}{k(k-1)\cdots(4)} \left(\frac{\lambda}{\mu}\right)^{k-3} \Rightarrow P_k = \left(\frac{3}{k}\right)\left(\frac{\lambda}{\mu}\right)^{k-3} P_3 \quad N=3$$

$$\Rightarrow \sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3 P_3 \sum_{k=3}^{\infty} \frac{1}{k}\left(\frac{\lambda}{\mu}\right)^k$$

$$\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{\lambda}{\mu}\right)^k = \log\left[\frac{1}{1-\lambda/\mu}\right] = \log\left[\frac{\mu}{\mu-\lambda}\right] \text{ if } \frac{\lambda}{\mu} < 1$$

---

(19) $x_1 - x_n \sim$ iid $U(0,1)$

$$\Rightarrow P(y \leq x_1 - x_2 < z) = (z-y)^n$$

$$E[Z-Y] = \int_0^1 E(I(t))\,dt \qquad I(t) = \begin{cases} 1 & z < t < Y \\ 0 & \text{otherwise} \end{cases}$$

skimming

---

(20) $\lambda \rightarrow$ |||| , θ; S1 μ, S2 μ, S3 μ

Prob (empty → forth person leaves before service) $= \frac{\lambda}{\lambda+\mu} \frac{\lambda}{\lambda+2\mu} \frac{\lambda}{\lambda+3\mu} \frac{\theta}{\theta+3\mu}$

reason: $\frac{P(\text{failure})}{P(\text{success}) + P(\text{Failure})}$

Since these are only param of interest and next entrance won't matter

---

(21) Local train: 5 min
Express train: 15 min

$0 \leq y \leq 5$ train comes within y min $\Rightarrow \frac{y}{5}$ → uniform poisson duality

$0 \leq y \leq 15 \Rightarrow \frac{y}{15}$ → uniform poisson duality

PASTA: prob train to come from server point of view
mean(train) = prob. to come from customer point of view

---

(22) Shooting match: ① chang prob of win p to q = 1−p when A wins
② two consecutive win → match is end

$$\begin{cases} E[N_p] = (pq+q^2)2 + qp[2+E[N_q]] + p^2[2+E[N_p]] \\ E[N_q] = (pq+p^2)2 + qp[2+E(N_p)] + q^2[2+E[N_q]] \end{cases}$$

	A	B	C
A wins	p	q	
A loss	q	p	

---

(23) $Cov[L, K-L] = E[L(K-L)] - E[L]E[K-L]$

$E[L(K-L)] = E[LK] - E[L^2]$

$E[LK] = E[E[LK|K]] = E[K\,E[L|K]]$

---

(24) $\xrightarrow{\lambda}$ insurance | μ_1 | filing | μ_2 | payment $\Rightarrow$ not file until 1 month

① $\int_0^1 e^{-\mu_1(1-y)}\,dy$

Survive until min given entered at y

tail since we seek survival

② $L(\text{filing}) = \frac{\lambda}{\mu_1}$ # λ → rate, μ_1 → time

(22)

(25) Record ⇒ Number greater than before

$N^{min} = \{n \mid n > 1,\ n \text{ lowest record}\}$

⇓

Condition on the first record 1, 2, …
it could be any of remaining points

$E[N] = \sum_{i=1}^{n} \frac{1}{i} \quad \xrightarrow{n\to\infty} \infty$

5	6	7
1	2	3
3	8	8

→ N
1st record condition

---

(26) Cards turned over then find min not turned over at turn over to find → m_n: # cycles

$$m_n = \frac{1}{n}\sum_{i=1}^{n} 1 + m_{n-i}$$

↳ remaining card that lead to new cycle

it could break at any point

to say two things are independent ≡ knowing second does not reduce event space ≡ probability of first

(27) important key to solve problems: independence
(specially to separate exponential times)

e.g. $E[MX \mid M=Y] = E[MY + X' \mid M=Y]$

$= E[MY \mid M=Y] + \underbrace{E[M \mid M=Y] \times E[X']}_{\text{independence}}$

memoryless property explains independence

that's why you skim time with minimum until you skim whole duration to calculate

$T = \min(t_1 t_2 t_3) + \min(t_i, t_j) + \min(t_j)$

$t_i \sim exp(\mu) \quad i = 1,2,3$

$= \frac{1}{3\mu} + \frac{1}{2\mu} + \frac{1}{\mu}$

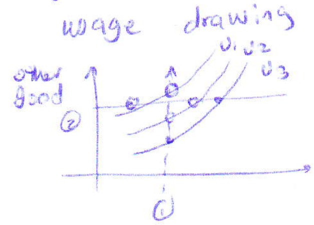
intuition: Ozlap's rule
Discover what promises by paper by look at equations (Sul, Ali Heitor)

General intuition − Conditioning is very important

① not just in Prob, but also in Micro Econ, since simplifies analysing, sometimes through independence, sometimes through logic applying

mind can not analyse confound

e.g: utility func work, and other good wage drawing



Conditioning ≡ ①, ②
⇓
$U_1 > U_2 > U_3$ step's order
analyse in one direction at a time

② order & organization & structure is not seen but creates the world & culture

e.g. ① theory of stat that tells us what bias is
② theory of Econometrics that tells how data could be used (e.g. price signal of opportunity cos Erdem

probability summery note

①

demand jamed: λR

1 sold	2 sold	...	i sold
$\frac{\lambda(\lambda R)^1 e^{-\lambda R}}{1!}$	$\frac{\lambda(\lambda R)^2 e^{-\lambda R}}{2!}$	...	$\frac{(\lambda R)^i e^{-\lambda R}}{i!}$

$\sum \frac{\text{this}}{\text{total}} \times$ total profit

$= \sum$ this profit

---

when poisson $e^{-m(t)}$

Count: $e^{-m(t)} \frac{(m(t))^n}{n!}$

b-y

$\int F(b-y) d\pi$

Count ≡ poisson counting

Exponential:

Yes

No

$\leq u$: past

Exhaust src

finished appointment any time indep

$(1-e^{-\lambda t})^2 + e^{-\lambda t} \int \lambda (\lambda t) e^{-\lambda t} dt$

---

not interval $= \int e^{m(t)} dt$

Before $= \frac{\lambda}{\lambda + r}$ $\frac{\lambda}{\lambda + 2\mu}$

Length of line: Little $L = \lambda w \rightarrow$ wait in system

---

① time proportion ⇒ uniform

② customer proportion: P_{ij}

Erlang $\frac{\lambda(\lambda x)^n}{n!} e^{-\lambda x}$

PASTA rule

Substitute θ: $\frac{1}{\mu_1+\mu_2+2\theta} + \frac{1}{\mu_1+\mu_2+\theta}$

ABandon ≠ Not enter

decide to stay = enter

Past:

$\frac{\lambda}{\lambda+\mu}$ $\frac{\lambda}{\lambda+2\mu}$ $\frac{\lambda}{\lambda+3\mu}$ $\frac{\theta}{\theta+\mu}$

past

$P(y>s, y<s+c) =$

$P(y<s+c) - P(y<s)$

Shopping checkout

No $\xleftarrow{\lambda e^{-\lambda t}}$ Yes

$N(t) = \lambda t$

$N(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$

⇒ $n \rightarrow n+1$ increase

$G(t)$ $H(t)$

$K(t) = G(t) H(t)$ ⇒ $\frac{\lambda t}{n+1}$ ⇒ $\lambda \int h(t) dt \sim$ poisson

---

Threshold Based:

$\frac{\lambda}{\lambda + s}$ s = fixed, be not expon. ⇒ split $e^{-\lambda s}$, $1-e^{-\lambda s}$

$E[\min[X,s]] = \int_0^s e^{-\lambda s} dy = \frac{1}{\lambda}(1-e^{-\lambda s})$

②

× Prob. of time of service expires

= out of service (complete)

Out of System: rate $= \frac{1}{\text{time}}$

For mean time when time Prob. available ⇒

tail prob $\rightsquigarrow$ integral

---

when waiting time from order ⇒ λ_e = order rate × batch size

---

mamoryless exponential

All entered go out

entered not served ⇐ $P = \int_0^1 \bar{G}(1-y) d(t) dt$

⇒ e^{-P} not exist

$G(t)$: finished service at t

$\bar{G}(t)$: Cont. service at t

$L = w\lambda$ ⇒ $L_2 = \frac{2\lambda\pi}{\mu}$

⇒ $E_{L_2} = \lambda_2$ ⇒ $\frac{e^{-\lambda_2 t}(\lambda_2 t)^i}{i!}$

$Var = \mu_w(60)(Var(\text{unit}) + E(\text{unit})^2)$

---

① Context

② tail

③ transformations

④ modular

⑤ Recursion

⑥ plug in (test case) of value 1, 2...

⑦ Conditioning (Divide & Conquer)

⑧ simplify

⑨ Complement

From 2012 Prelim

Questions on Advanced Managerial Economics

1. Suppose there are two commodities. Consider an indirect utility function defined by

$$v(p,w) = -\exp(-b\frac{p_1}{p_2})\left[\frac{w}{p_2} + \frac{1}{b}\left(a\frac{p_1}{p_2} + \frac{a}{b} + c\right)\right]$$

u

where w is the consumer's wealth, p_1 and p_2 are the price of commodity one and two, respectively, a, b, and c are parameters. Find the consumer's Walrasian demand function for commodity one, $x_1(p, w)$, the expenditure function, $e(p, u)$, and the Hicksian demand function for commodity one, $h_1(p, u)$.

$\frac{\partial h_i}{\partial p_k} > 0$ substitute

2. Consider a two-firm Cournot model with constant returns to scale but in which firms' costs may differ. Let c_j denote firm j's cost per unit of output produced, and assume that $c_1 > c_2$. Assume also that the inverse demand function is $p(q) = a - bq$, with $a > c$. Find the Nash equilibrium outcome (outputs and profits) of this model. Under what conditions does it involve only one firm producing and what conditions does it involve both firms producing?

$x_1 = -\frac{\partial v/\partial p_1}{\partial v/\partial w}$ Roy

1. e(p,u)

$u = -\exp(-b\frac{p_1}{p_2})\left[\frac{w}{p_2} + \frac{1}{b}(a\frac{p_1}{p_2} + \frac{a}{b} + c)\right] \Rightarrow$

$\frac{w}{p_2} = -u\exp(b\frac{p_1}{p_2}) - \frac{1}{b}(a\frac{p_1}{p_2} + \frac{a}{b} + c)$

$\Rightarrow w = e(p,u) = -u p_2 \exp(b\frac{p_1}{p_2}) - \frac{p_2}{b}(a\frac{p_1}{p_2} + \frac{a}{b} + c)$

$h_1(p,u) = \frac{\partial e}{\partial p_1} = $

2. Firm 1: $\max_{q_1} (a - b(q_1+q_2) - c_1)q_1$ F.O.C. $a - b(q_1+q_2) - c_1 - bq_1 \Rightarrow q_1 = \frac{a - bq_2 - c_1}{2b}$

Firm 2: $\max_{q_2} (a - b(q_1+q_2) - c_2)q_2$ F.O.C. $a - b(q_1+q_2) - c_2 - bq_2 \Rightarrow q_2 = \frac{a - bq_1 - c_2}{2b}$

$q_1 = \frac{a - b\left(\frac{a - bq_1 - c_2}{2b}\right) - c_1}{2b} \Rightarrow 4bq_1 = 2a - a + bq_1 + c_2 - 2c_1 \Rightarrow q_1 = \frac{a + c_2 - 2c_1}{3b}$

$q_2 = \frac{a - b\left(\frac{a - bq_2 - c_1}{2b}\right) - c_2}{2b} \Rightarrow 4bq_2 = 2a - a + bq_2 + c_1 - 2c_2 \Rightarrow \boxed{q_2 = \frac{a + c_1 - 2c_2}{3b}}$

$\boxed{a \not> 2c_1 - c_2 \Rightarrow q_1 = 0}$