5.4 Problems

1. Cournot Oligopoly. Suppose there are 2 firms producing an identical product. Let q_i be the quantity produced by by firm i so the total sold on the market is given by $Q = q_1 + q_2$. Both firms have the same costs of production given by $C_i(q_i) = 5q_i$. Demand for the product given the market price of P is given by:

$$Q = 70 - .5P.$$

   (a) Suppose both firms simultaneously select the quantity they produce. Identify the Nash equilibrium. Also, identify each firm's profit.

   (b) Suppose the two firms merge (or collude) in order to form a monopoly for this product. Identify the price and quantity produced and total profit.

   (c) If the two firms are colluding, then their agreement cannot be enforced by law. Suppose that both firms are simultaneously deciding to collude as in (b) above (assume they split production equally) or to produce as in (a). Do the firms have a dominant strategy? What is the Nash equilibrium? Is this game similar to any other game we have seen in class?

   (d) Suppose there are n firms that are identical to the two described earlier. What is the Nash equilibrium if each firm decides on a quantity to produce simultaneously? What happens as n goes to infinity?

2. Tollway Pizza and Moon Face Pizza are the only two competing pizza restaurants in So Cold Pennsylvania. They are simultaneously selecting a price to charge for a pizza. The following table describes the profit for each pizza sold at three different price levels (assumed to be the same for each firm):

Price	Profit (per pizza)
High	\$12
Medium	\$10
Low	\$5

   Both pizza places also face similar demand. They both can count on 3,000 pizzas of loyal demand a week (people who go to their pizza place regardless of price). However, an additional 4,000 pizzas per week in demand are floating, going to the pizza parlor with the lowest price. If the pizza places charge the same price, they split the floating demand (2,000 pizzas each).

   (a) Show the normal form representation of this game.

   (b) Are there any strategies for either player that are dominated?

   (c) Identify the pure strategy Nash equilibrium.

3. Linear public goods. There are n players, each with an endowment of cash w, who must decide simultaneously on their individual contribution $x_i \in [0, w]$ to the public good which

is given by sum of everyone's contribution $\sum_{j=1}^{n} x_j$. The payoff function for all agents is identical and given by

$$u_i(x) = w - x_i + m \sum_{j=1}^{n} x_j$$

where m is the 'social' marginal utility of private contributions.

(a) If $m < 1$, do the players' have a dominant strategy? If so, what?

(b) Under what conditions will this rational (dominant strategy) behavior result in non-Pareto optimal outcomes?

4. Consider the following payoff table:

		Firm 2	
		Sell Online	Don't Sell Online
Firm 1	Sell Online	10,10	20,6
	Don't Sell Online	6,20	15,15

(a) Does either player have a dominant strategy?

(b) What will be the outcome of this game?

5. Consider the following payoff table:

		Firm 2	
		Sell Online	Don't Sell Online
Firm 1	Sell Online	50,60	20,30
	Don't Sell Online	40,20	60,40

(a) Does either player have a dominant strategy?

(b) What will be the Nash equilibria?

6. Many economists specializing in international trade argue that, at least in some industries, the world market as a whole has room for only one profitable entrant because of large economies of scale. The following example is due to MIT economist Paul Krugman. The passenger aircraft industry may be one such example. Two firms (Boeing and Airbus) are capable of producing a 150-seat aircraft. Boeing is located in America and Airbus is located in Europe. Each firm has the option of producing or not. If they both produce, they both lose money (or at least make economic losses) to the tune of -\$50 million. On the other hand, if only one firm produces, then that firm makes \$1 billion in economic profit. Of course, if any firm does not produce, it makes no profits or losses.

(a) Construct a payoff table for the game described above.

(b) Identify the pure strategy Nash equilibria of the game. Briefly explain why they are Nash equilibria.

(c) Suppose that the European Community commits itself to a policy of paying Airbus a subsidy of $100 million regardless of whether Boeing produces or not. Construct a new payoff table and identify the Nash equilibria. Explain.

7. The Battle of the Sexes. Tony and Tina would like to spend Saturday night together but like different types of entertainment. Tina would prefer to go to the movies, but Tony wants to go to the casino. While they would most prefer to do their favorite activity, both would be very unhappy if they ended up separated. The following payoff table depicts the game.

		Tina	
		Casino	Movies
Tony	Casino	2,1	0,0
	Movies	0,0	1,2

(a) Identify the pure strategy Nash equilibria of the game.

(b) Identify the mixed strategy Nash equilibrium of the game.

8. Asahi and Kirin are the two largest sellers of beer in Japan. These two firms compete head-to-head in the dry beer category in Japan. The following table shows the profit (in millions of yen) that each firm earns when it charges different prices for its beer.

		Kirin			
		¥630	¥660	¥690	¥720
Asahi	¥630	180,180	184,178	185,175	186,173
	¥660	178,184	183,183	192,182	194,180
	¥690	175,185	182,192	191,191	198,190
	¥720	173,186	180,194	190,198	196,196

(a) Does either firm have a dominant strategy?

(b) Both Asahi and Kirin have a dominated strategy. Find it and identify it.

(c) Find the Nash equilibrium to the game.

9. Two new electronics stores are choosing where to locate along a 5 mile stretch of highway within the town of Johnsonville. Johnsonville is a one street town and nearly all the town's 40,000 residents live directly off the 5 mile stretch of highway in town. It is also reasonable to assume that the 40,000 residents are spread evenly along the highway. There are no other competing stores along this stretch of highway. While the two stores are independently owned and operated, pricing agreements with the manufacturers mean that they will charge essentially the same prices (except for occasional sales). Customers, therefore, choose where to shop solely on the basis of location; they will always buy from the store that is closest. If the stores are the same distance, then the customers are indifferent and flip a coin about which store to go to. The expected demand in any given month is 1,000 customers spending $120 each on average (assume that this demand is generated randomly from the town's population).

(a) Identify the Nash equilibrium of this game. Explain why it is the only Nash equilibrium.

THE PENNSYLVANIA STATE UNIVERSITY

BA 597D
Advanced Microeconomic Analysis

Kwasnica
Spring 2008

Section 5 Solutions

1. Cournot Oligopoly. Suppose there are 2 firms producing an identical product. Let q_i be the quantity produced by by firm i so the total sold on the market is given by $Q = q_1 + q_2$. Both firms have the same costs of production given by $C_i(q_i) = 5q_i$. Demand for the product given the market price of P is given by:
$$Q = 70 - .5P.$$

   (a) Suppose both firms simultaneously select the quantity they produce. Identify the Nash equilibrium. Also, identify each firm's profit. The maximization problem for firm 1 is to select a quantity q_1 to maximize profits given that firm 2 is producing q_2 units, or

$$\begin{aligned} \max_{q_1} \pi_1(q_1, q_2) &= Pq_1 - C_1(q_1) & (1) \\ &= (140 - 2(q_1 + q_2))q_1 - 5q_1 & (2) \end{aligned}$$

where $P(Q) = 140 - 2Q$ is the *inverse demand* function. We have the following first order condition:

$$140 - 4q_1 - 2q_2 - 5 = 0. \quad (3)$$

Solving for q_1 as a function of q_2, we get firm 1's reaction curve:

$$\begin{aligned} 4q_1 &= 135 - 2q_2 & (4) \\ q_1 &= 33.75 - .5q_2. & (5) \end{aligned}$$

Given the symmetry of the problem, it is obvious that firm 2's reaction curve is similarly defined:

$$q_2 = 33.75 - .5q_1. \quad (6)$$

Thus, a Nash equilibrium will be a point q_1^*, q_2^* satisfying both equations 5 and 6. Substituting equation 6 into equaiton 5, we obtain:

$$\begin{aligned} q_1^* &= 33.75 - .5(33.75 - .5q_1^*) & (7) \\ q_1^* &= 16.875 + .25q_1^* & (8) \\ .75q_1^* &= 16.875 & (9) \\ q_1^* &= 22.5 & (10) \end{aligned}$$

and

$$\begin{aligned} q_2^* &= 33.75 - .5(q_1^*) & (11) \\ &= 33.75 - .5(22.5) & (12) \\ &= 22.5. & (13) \end{aligned}$$

Thus, the Nash equilibrium is for both firms to produce 22.5 units. Putting this quantity back into each firm's profit function we obtain their profits:

$$\begin{aligned} \pi_i(q_1, q_2) &= Pq_i - C_i(q_i) & (14)\\ &= (140 - 2(45))22.5 - 5(22.5) & (15)\\ &= 1012.5 & (16) \end{aligned}$$

(b) Suppose the two firms merge (or collude) in order to form a monopoly for this product. Identify the price and quantity produced and total profit.

This is simply the monopolists problem, or

$$\begin{aligned} \max_Q \pi(Q) &= P(Q) - C(Q) & (17)\\ &= (140 - 2Q)Q - 5Q & (18)\\ &= 135Q - 2Q^2 & (19) \end{aligned}$$

which yields the following first order condition:

$$\begin{aligned} 135 - 4Q &= 0 & (20)\\ 4Q &= 135 & (21)\\ Q &= 33.75. & (22) \end{aligned}$$

This monopolist quantity $Q = 33.75$ would result in a price of $P = 140 - 2(33.75) = 72.5$ and profits of 2,278.125.

(c) If the two firms are colluding, then their agreement cannot be enforced by law. Suppose that both firms are simultaneously deciding to collude as in (b) above (assume they split production equally) or to produce as in (a). Do the firms have a dominant strategy? What is the Nash equilibrium? Is this game similar to any other game we have seen in class?

If the firms are colluding by splitting the production at the monopolist level then $q_1 = q_2 =$ 16.875 and profit for each firm is half of the monopoly profit, or 1,139.06. When firm 1 'defects' by producing the Cournot quantity but firm 2 continues to collude and produce 16.875, then we have $q_1 + q_2 = 22.5 + 16.875 = 39.375$ and profits of:

$$\begin{aligned} \pi_1(q_1, q_2) &= (140 - 2(39.375))22.5 - 5(22.5) & (23)\\ &\approx 1,265.63 & (24)\\ \pi_2(q_1, q_2) &= (140 - 2(39.375))16.875 - 5(16.875) & (25)\\ &\approx 949.22. & (26) \end{aligned}$$

Therefore, this can be setup of as the following 2 by 2 game:

		Firm 2	
		Collude	Cournot
Firm 1	Collude	1,139.06, 1,139.06	949.22, 1,265.63
	Cournot	1,265.63, 949.22	1,012.50, 1,012.50

It is plain to see that this game looks a lot like the Prisoner's Dilemma game discussed in class; each firm has a dominant strategy to play 'Cournot' so that the only Nash equilibrium is the Cournot outcome as described above.

(d) Suppose there are n firms that are identical to the two described earlier. What is the Nash equilibrium if each firm decides on a quantity to produce simultaneously? What happens as n goes to infinity?

Let q_{-i} be the amount produced by the $n-1$ firms other than firm i. Then, firm i will select q_i to maximize profits given the production of the other firms, or

$$\begin{aligned} \max_{q_i} \pi_i(q_i, q_{-i}) &= Pq_i - C_i(q_i) && (27) \\ &= (140 - 2(q_i + q_{-i}))q_i - 5q_i && (28) \end{aligned}$$

which yields the following first order condition:

$$140 - 4q_i - 2q_{-i} - 5 = 0. \quad (29)$$

Solving for q_i as a function of q_{-i}, we get firm i's reaction curve:

$$\begin{aligned} 4q_i &= 135 - 2q_{-i} && (30) \\ q_i &= \frac{135 - 2q_{-i}}{4}. && (31) \end{aligned}$$

Now, since all firms are symmetric, we know that in equilibrium $q_i = q_j$ for all i, j which implies that $q_{-i} = (n-1)q_i$. Substituting this into equation 31 above, we can solve for q_i:

$$\begin{aligned} q_i &= \frac{135 - 2(n-1)q_i}{4} && (32) \\ q_i\left(\frac{1}{2}(n+1)\right) &= \frac{135}{4} && (33) \\ q_i &= \frac{135}{2(n+1)}. && (34) \end{aligned}$$

This tells us that as n gets large (goes to infinity) the amount that each firm produces goes to zero. But, the amount produced in the whole market is given by:

$$\begin{aligned} Q &= \sum_{i=1}^{n} q_i && (35) \\ &= nq_i && (36) \\ &= \left(\frac{135}{2}\right)\left(\frac{n}{n+1}\right) && (37) \end{aligned}$$

which goes to $135/2 = 67.5$ as n goes to infinity. At this quantity, the market price is $P = 5$ or we are at the perfectly competitive equilibrium of price equals marginal cost.

2. Tollway Pizza and Moon Face Pizza are the only two competing pizza restaurants in So Cold Pennsylvania. They are simultaneously selecting a price to charge for a pizza. The following table describes the profit for each pizza sold at three different price levels (assumed to be the same for each firm):

Price	Profit (per pizza)
High	\$12
Medium	\$10
Low	\$5

Both pizza places also face similar demand. They both can count on 3,000 pizzas of loyal demand a week (people who go to their pizza place regardless of price). However, an additional 4,000 pizzas per week in demand are floating, going to the pizza parlor with the lowest price. If the pizza places charge the same price, they split the floating demand (2,000 pizzas each).

(a) Show the normal form representation of this game. The following is the normal form representation of the game where all units are in thousands of dollars.

		Moonface		
		High	Medium	Low
	High	60,60	36,70	36,35
Tollway	Medium	70,36	50,50	30,35
	Low	35,36	35,30	25,25

(b) Are there any strategies for either player that are dominated?

Low Price is dominated for both pizza places.

(c) Identify the pure strategy Nash equilibrium.

Solve by iteratively deleting dominated strategies. First, Low is dominated for both pizza places. After eliminating the Low price, High price is dominated by Medium price for both pizza places. Thus, all that is left is **Medium price for both pizza places**. It is easy to check that indeed this is a Nash equilibrium.

3. Linear public goods. There are n players, each with an endowment of cash w, who must decide simultaneously on their individual contribution $x_i \in [0, w]$ to the public good which is given by sum of everyone's contribution $\sum_{j=1}^{n} x_j$. The payoff function for all agents is identical and given by

$$u_i(x) = w - x_i + m \sum_{j=1}^{n} x_j$$

where m is the 'social' marginal utility of private contributions.

(a) If $m < 1$, do the players' have a dominant strategy? If so, what?

Yes. They have a dominant strategy. To see this look at result of agent i's payoff maximizing choice of x_i.

$$\max_{x_i} w - x_i + m \sum_{j=1}^{n} x_j \tag{38}$$

subject to

$$0 \le x_i \le w \tag{39}$$

The necessary first order condition is given by

$$-1 + m + \lambda_1 - \lambda_2 = 0 \tag{40}$$

where $\lambda_1 > 0$ if $x_i = 0$ and $\lambda_2 > 0$ if $x_i = w$. Since $m < 1$ by assumption it must be that $\lambda_1 > 0$ and $x_i = 0$. So making no contribution $x_i = 0$ is a dominant strategy. It is a dominant strategy because $x_i = 0$ is optimal no matter what the choice of x_j for $j \neq i$.

(b) Under what conditions will this rational (dominant strategy) behavior result in non-Pareto optimal outcomes?

Notice that contributions any individual x_i provide a marginal benefit of m to all n players. So that the total (social) marginal benefit is nm where the marginal cost is 1 for player i.

So as long as $mn > 1$ full contributions are Pareto optimal $x_i = w$ for all i. So as long as $m > 1/n$ the dominant strategy will be non-Pareto optimal.

4. Consider the following payoff table:

		Firm 2: Sell Online	Firm 2: Don't Sell Online
Firm 1	Sell Online	10,10	20,6
	Don't Sell Online	6,20	15,15

   (a) Does either player have a dominant strategy?

   Both firms have a dominant strategy to sell online since the payoff from selling online is always greater than not (10 versus 6 and 20 versus 15). Therefore, the outcome will be both firms selling online.

   (b) What will be the outcome of this game?

   Therefore, the outcome will be both firms selling online.

5. Consider the following payoff table:

		Firm 2: Sell Online	Firm 2: Don't Sell Online
Firm 1	Sell Online	50,60	20,30
	Don't Sell Online	40,20	60,40

   (a) Does either player have a dominant strategy? Neither firm has a dominant strategy now. They both want to sell online when the other firm is selling online and don't sell online when the other firm is doing likewise.

   (b) What will be the pure-strategy Nash equilibria?

   This leads to two pure strategy equilibria where the firms either both sell online or both don't sell online.

6. Many economists specializing in international trade argue that, at least in some industries, the world market as a whole has room for only one profitable entrant because of large economies of scale. The following example is due to MIT economist Paul Krugman. The passenger aircraft industry may be one such example. Two firms (Boeing and Airbus) are capable of producing a 150-seat aircraft. Boeing is located in America and Airbus is located in Europe. Each firm has the option of producing or not. If they both produce, they both lose money (or at least make economic losses) to the tune of -$50 million. On the other hand, if only one firm produces, then that firm makes $1 billion in economic profit. Of course, if any firm does not produce, it makes no profits or losses.

   (a) Construct a payoff table for the game described above.

   The payoff table should look like the following (profits in millions of dollars).

		Boeing: Produce	Boeing: Don't
Airbus	Produce	-50,-50	1000,0
	Don't	0,1000	0,0

(b) Identify the pure strategy Nash equilibria of the game. Briefly explain why they are Nash equilibria.

The two pure strategy Nash equilibria involve one firm producing the plane and the other not. These are Nash equilibria because, if one firm is producing the plane, the other firm does not want to switch to production because -50 million is less than 0.

(c) Suppose that the European Community commits itself to a policy of paying Airbus a subsidy of $100 million regardless of whether Boeing produces or not. Construct a new payoff table and identify the Nash equilibria. Explain.

The subsidy simply increase Airbus' payoff when the produce by 100 million. Therefore, the new game looks like the following.

		Boeing	
		Produce	Don't
Airbus	Produce	50,-50	1100,0
	Don't	0,1000	0,0

Now, Airbus has a dominant strategy to produce the plane so the only Nash equilibrium is where Airbus produces and Boeing does not.

7. The Battle of the Sexes. Tony and Tina would like to spend Saturday night together but like different types of entertainment. Tina would prefer to go to the movies, but Tony wants to go to the casino. While they would most prefer to do their favorite activity, both would be very unhappy if they ended up separated. The following payoff table depicts the game.

		Tina	
		Casino	Movies
Tony	Casino	2,1	0,0
	Movies	0,0	1,2

(a) Identify the pure strategy Nash equilibria of the game.

The two pure strategy Nash equilibria involve either them both going to the casino or both going to the movies.

(b) Identify the mixed strategy Nash equilibrium of the game.

Assume that Tony goes to the casino with probability p (and to the movies with probability $1-p$), and Tina goes to the casino with probability q (and to the movies with probability $1-q$). We can find the mixed strategy equilibrium by having each player select their probabilities in order to make the other player indifferent between selecting the two different strategies. For Tony, he want to make Tina get the same expected payoff from picking the casino or going to the movies. Her expected payoff from going to the casino is $p1+(1-p)0$ (Tina's payoff when she plays casino and Tony is mixing with probability p). This simplifies to p. Tina's expected payoff from the movies is $p0+(1-p)2 = 2-2p$. We can now solve for p by setting these two payoffs equal to each other (Tina must be indifferent), or $p = 2-2p$. Solving for p we get $p = \frac{2}{3}$. Use the same logic to find that $q = \frac{1}{3}$ for Tina (Tina must make Tony indifferent).

8. Asahi and Kirin are the two largest sellers of beer in Japan. These two firms compete head-to-head in the dry beer category in Japan. The following table shows the profit (in millions of yen) that each firm earns when it charges different prices for its beer.

		Kirin			
		¥630	¥660	¥690	¥720
Asahi	¥630	180,180	184,178	185,175	186,173
	¥660	178,184	183,183	192,182	194,180
	¥690	175,185	182,192	191,191	198,190
	¥720	173,186	180,194	190,198	196,196

(a) Does either firm have a dominant strategy?

Neither firm has a dominant strategy since they want to charge different prices given the different actions of their competitor.

(b) Both Asahi and Kirin have a dominated strategy. Find it and identify it.

The high price (¥720) is dominated for both firms since their is always a price that gives them a higher payoff given the strategy of their opponent.

(c) Find the Nash equilibrium to the game.

The high price (¥720) is dominated for both firms since their is always a price that gives them a higher payoff given the strategy of their opponent.

9. Two new electronics stores are choosing where to locate along a 5 mile stretch of highway within the town of Johnsonville. Johnsonville is a one street town and nearly all the town's 40,000 residents live directly off the 5 mile stretch of highway in town. It is also reasonable to assume that the 40,000 residents are spread evenly along the highway. There are no other competing stores along this stretch of highway. While the two stores are independently owned and operated, pricing agreements with the manufacturers mean that they will charge essentially the same prices (except for occasional sales). Customers, therefore, choose where to shop solely on the basis of location; they will always buy from the store that is closest. If the stores are the same distance, then the customers are indifferent and flip a coin about which store to go to. The expected demand in any given month is 1,000 customers spending $120 each on average (assume that this demand is generated randomly from the town's population).

(a) Identify the Nash equilibrium of this game. Explain why it is the only Nash equilibrium.

The only Nash equilibrium is for the two store to locate next to each other in the middle of the highway. If not, the other firm can move to a location where they get more customers.

(b) Suppose instead that customers will not travel more than 1.5 miles to visit any store. Assuming everything else is the same, will this change the Nash equilibrium? Identify at least one feasible Nash equilibrium.

If the firms have to be within 1.5 miles of their customer, then we can see some separation in Nash equilibrium. One equilibrium would be for each firm to place themselves 1.5 miles from opposite ends of the road.

10. Two candidates, a Democrat and a Republican are competing for office in a district which is divided into K wards. In the ith ward there are n_i potential voters, with $n = \sum_{i=1}^{K} n_i$ being the total number of potential voters in the district. The Democrat has total campaign resources of D which he can divide up between the wards, allocating d_i to the ith ward. Thus, $\sum_{i=1}^{K} d_i = D$ with $d_i > 0$ for all i. Similarly, the Republican has total resources R which he can divide up, allocating r_i to each ward, so $\sum_{i=1}^{K} r_i = R$ with $r_i > 0$ for all i. For any given ward the vote going to the Democrat and Republican is proportional to their respective shares of the total expenditures in that ward. Thus,

$$v_i^D = \frac{n_i d_i}{d_i + r_i}, v_i^R = \frac{n_i r_i}{d_i + r_i}.$$

The total vote for each candidate, then, for a given choice of strategies $d = (d_1, d_2, \ldots, d_K)$ and $r = (r_1, r_2, \ldots, r_K)$ is

$$v^D = \sum_{i=1}^{K} v_i^D = \sum_{i=1}^{K} \frac{n_i d_i}{d_i + r_i}, v^R = \sum_{i=1}^{K} v_i^R = \sum_{i=1}^{K} \frac{n_i r_i}{d_i + r_i}.$$

Assuming each candidate tries to maximize their total vote, find a Nash equilibrium for the above game.

In a Nash equilibrium each candidates (Democrat and Republican) must be maximizing given the choices of his opponent. Setup the following Lagrangians:

$$\mathcal{L}^D = \sum_{i=1}^{K} \frac{n_i d_i}{d_i + r_i} - \lambda \left(\sum_{i=1}^{K} d_i - D \right) \tag{41}$$

$$\mathcal{L}^R = \sum_{i=1}^{K} \frac{n_i r_i}{d_i + r_i} - \mu \left(\sum_{i=1}^{K} r_i - R \right) \tag{42}$$

which yields the following necessary first order conditions for all $i = 1, \ldots, K$

$$\frac{\partial \mathcal{L}^D}{\partial d_i} = \frac{(d_i + r_i)n_i - n_i d_i}{(d_i + r_i)^2} - \lambda = 0 \tag{43}$$

$$\frac{\partial \mathcal{L}^R}{\partial r_i} = \frac{(d_i + r_i)n_i - n_i r_i}{(d_i + r_i)^2} - \mu = 0 \tag{44}$$

which yields the following:

$$\mu = \frac{d_i n_i}{(d_i + r_i)^2} \tag{45}$$

$$\lambda = \frac{r_i n_i}{(d_i + r_i)^2} \tag{46}$$

for all $i = 1, \ldots, K$. Dividing (46) by (45) results in:

$$\frac{\lambda}{\mu} = \frac{d_i n_i}{r_i n_i} \tag{47}$$

$$= \frac{d_i}{r_i} \tag{48}$$

$$r_i = \frac{\lambda}{\mu} d_i \tag{49}$$

$$\tag{50}$$

for all i. Summing up over all $i = 1, \ldots, K$ yields the following:

$$\sum_{i=1}^{K} r_i = \frac{\lambda}{\mu} \sum_{i=1}^{K} d_i \tag{51}$$

$$R = \frac{\lambda}{\mu} D \tag{52}$$

$$\frac{\lambda}{\mu} = \frac{D}{R} \tag{53}$$

which obviously implies that

$$d_i = \frac{D}{R} r_i. \tag{54}$$

We want to express the equilibrium in terms of parameters (D, R, N, n_i) rather than strategies of the other candidates. In order to accomplish this, substitute (54) into (46) to obtain:

$$\begin{aligned}
\lambda &= \frac{r_i n_i}{\left(\frac{D}{R} r_i + r_i\right)^2} && (55)\\
&= \frac{r_i n_i}{r_i^2 \left(\frac{D}{R} + 1\right)^2} && (56)\\
&= \frac{n_i}{r_i \left(\frac{D}{R} + 1\right)^2}. && (57)
\end{aligned}$$

Solving for n_i we have the following:

$$n_i = \lambda r_i \left(\frac{D}{R} + 1\right)^2. \tag{58}$$

Then summing up for all $i = 1, \ldots, K$ we have that:

$$\begin{aligned}
\sum_{i=1}^{K} n_i &= \lambda \left(\frac{D}{R} + 1\right)^2 \sum_{i=1}^{K} r_i && (59)\\
N &= R\lambda \left(\frac{D}{R} + 1\right)^2 && (60)\\
\lambda &= \frac{N}{R \left(\frac{D}{R} + 1\right)^2}. && (61)
\end{aligned}$$

The substituting (61) into (57) we obtain:

$$\begin{aligned}
\frac{n_i}{r_i \left(\frac{D}{R} + 1\right)^2} &= \frac{N}{R \left(\frac{D}{R} + 1\right)^2} && (62)\\
\frac{n_i}{N} &= \frac{r_i}{R} && (63)\\
r_i &= \frac{n_i}{N} R. && (64)
\end{aligned}$$

Using the same logic for the Democratic candidate (e.g. with μ), we find that

$$r_i = \frac{n_i}{N} R, d_i = \frac{n_i}{N} D \tag{65}$$

for all $i = 1, \ldots, K$ is a Nash equilibrium to this game.

11. Consider a population of voters uniformly distributed along the ideological spectrum from left ($x = 0$) to right ($x = 1$). Each of the candidates for office simultaneously choose a campaign platform (i.e., a point on the line between $x = 0$ and $x = 1$). The voters observe the candidates' platforms and then each voter votes for the candidate whose platform is closest to the voters' position on the spectrum. For example, if there are two candidates and they choose platforms $x_1 = .3$ and $x_2 = .6$, then all voters to the left of $x = .45$ vote for candidate 1 and all those to the right of $x = .45$ vote for candidate 2. Thus, candidate 2 wins the election with 55% of the vote. Assume the candidates care only about getting elected and that any candidates who select the same platform ($x_i = x_j$) equally split the votes cast for that platform and that ties among the leading vote getters are resolved by a coin flip.

(a) Suppose there are two candidates. What is the pure strategy Nash equilibriun? Explain.

The pure strategy Nash equilibrium is for each candidate to select the point $x_1 = x_2 = .5$. To see that this is a Nash equilibrium fix the position of candidate 2 ($x_2 = .5$) and see that for any other point for x_1 candidate 1's vote share decreases (he losses the election). Let $S_i(x_1, x_2)$ be the interval of voters who vote for candidate i given the platforms. Without loss of generality let $x_1 \leq x_2$, then we have the following:

$$S_1(x_1, x_2) = \begin{cases} x_1 + \frac{x_2 - x_1}{2} & x_1 \neq x_2 \\ .5 & x_1 = x_2 \end{cases} \tag{66}$$

$$S_2(x_1, x_2) = \begin{cases} (1 - x_2) + \frac{x_2 - x_1}{2} & x_1 \neq x_2 \\ .5 & x_1 = x_2 \end{cases} \tag{67}$$

Let $x_2 = .5$ and $x_1 \neq .5$. Then we have:

$$S_1(x_1, .5) = x_1 + .25 - .5x_1 \tag{68}$$
$$= .5x_1 + .25. \tag{69}$$
$$S_2(x_1, .5) = .5 + .25 - .5x_1 \tag{70}$$
$$= .75 + .5x_1 \tag{71}$$

Since $x_1 < .5$, we have that $S_1 < .5$ and $S_2 > .5$ or 1 losses the election. The same logic follows for a move to the left of .5 or a move by candidate 2. It is clear to see that this is the only Nash equilibrium by considering any selection of points such other that $x_1 = x_2 = .5$. Consider any other potential equilibrium such that $S_2 \geq S_1$. Then, given a position for candidate 2 (x_2), candidate 1 can win for sure, or $S_1 > S_2$, by selecting either $x_1 = x_2 + \epsilon$ if $x_2 < .5$ and $x_1 = x_2 - \epsilon$ if $x_2 > .5$. Of course, by the same logic, candidate 2 would want to deviate from any point were candidate 1 is winning with a similar strategy.

(b) Suppose there are three candidates. Find a pure strategy Nash equilibrium. Explain.

There are many pure strategy Nash equilibria to the three candidate game when all candidates want to do is 'win' the election. One example is to set $(x_1, x_2, x_3) = (.3, .4, .7)$. Under this arrangement each candidate's vote share is .35, .20, and .45 respectively so candidate 3 wins the election with a plurality. However, neither candidate 1 or 2 can select an alternative platform (given the fixed position of the other two candidates) such that they garner greater than .45 of the vote. Note: the distinction between 'winning' and maximizing 'vote share' is somewhat subtle. If the three candidates tried to maximize the percentage of the vote they got (regardless of whether they won or not) then there would be no pure-strategy Nash equilibrium.

(c) Suppose that voters are distributed along the line $[0, 1]$ something other than uniformly (but the distribution is known). If there are two candidates, can you describe what the Nash equilibrium will be? (A formal proof is not needed here.)

It will be at the *median* of the distribution.

12. Answer problem 1.13 in Gibbons (pg. 51). Let $(1/2)w_1 < w_2 < 2w_1$. The normal form representation of the game is given as follows:

		Worker 2: Apply to Firm 1	Worker 2: Apply to Firm 2
Worker 1	Apply to Firm 1	$\frac{1}{2}w_1, \frac{1}{2}w_1$	w_1, w_2
	Apply to Firm 2	w_2, w_1	$\frac{1}{2}w_2, \frac{1}{2}w_2$

Note that the relationship between w_1 and w_2 stated also implies that $w_1 > \frac{1}{2}w_2$ which automatically gives us that two pure-strategy Nash equilibria exist where each worker selects a different firm to apply to. Formally, if $s = (s_1, s_2)$ is the strategy pair for workers 1 and 2 respectively, then (Apply to Firm 1, Apply to Firm 2) and (Apply to Firm 2, Apply to Firm 1) are both Nash equilibria.

Let p_1 be the probability that worker 1 plays 'Apply to Firm 1' and p_2 be the probability that worker 2 plays 'Apply to Firm 1'. The expected value for player 1 for each pure strategy given the mixed strategy of worker 2 is given by:

$$v_1(\text{Apply to Firm 1}, (p_2, 1-p_2)) = p_2(\frac{1}{2}w_1) + (1-p_2)w_1 \tag{72}$$

$$v_1(\text{Apply to Firm 2}, (p_2, 1-p_2)) = p_2 w_2 + (1-p_2)(\frac{1}{2}w_2) \tag{73}$$

with nearly identical functions for worker 2. Then for a Nash equilibrium in mixed strategies to exist here is must be that $v_1(\text{Apply to Firm 1}, (p_2, 1-p_2)) = v_1(\text{Apply to Firm 2}, (p_2, 1-p_2))$ which yields:

$$p_2(\frac{1}{2}w_1) + (1-p_2)w_1 = p_2 w_2 + (1-p_2)(\frac{1}{2}w_2) \tag{74}$$

$$p_2(\frac{1}{2}w_1) + w_1 - p_2 w_1 = p_2 w_2 + \frac{1}{2}w_2 - p_2\frac{1}{2}w_2 \tag{75}$$

$$w_1 - \frac{1}{2}w_1 p_1 = \frac{1}{2}w_2 + \frac{1}{2}w_2 p_2 \tag{76}$$

$$2w_1 - w_1 p_2 = w_2 + w_2 p_2 \tag{77}$$

$$2w_1 - w_2 = p_2 w_1 + p_2 w_2 \tag{78}$$

$$p_2 = \frac{2w_1 - w_2}{w_1 + w_2}. \tag{79}$$

Doing the same calculation for worker 2 also yields:

$$p_1 = \frac{2w_1 - w_2}{w_1 + w_2}. \tag{80}$$

Thus, p_1 and p_2 as described above constitutes a mixed-strategy Nash equilibrium.

6.7 Problems

1. Problem 2.5 in Gibbons (pg. 132).
2. Problem 2.11 in Gibbons (pg. 135).
3. Consider a voting game in which three players 1, 2, and 3, are deciding among three alternatives, A, B, and C. Alternative B is the "status quo" and alternatives A and C are the "challengers." At the first stage, players choose which of the two challengers should be considered by casting votes for either A or C, with the majority choice being the winner and abstentions not allowed. At the second stage, players vote between the status quo B and whichever alternative was victorious in the first round, with majority rule again determining the winner. Players vote simultaneously in each round. The players only care about the alternative that is finally selected, and are indifferent as to the sequence of votes that leads to a given selection. The payoff functions are $u_1(A) = 2, u_1(B) = 0, u_1(C) = 1; u_2(A) = 1, u_2(B) = 2, u_2(C) = 0; u_3(A) = 0, u_3(B) = 1, u_3(C) = 2$.
   (a) Draw an extensive form representation of the game.
   (b) What would happen if at each stage the players voted for the alternative they would most prefer as the final outcome?
   (c) Find the subgame perfect Nash equilibrium that satisfies the additional condition that no strategy can be eliminated by iterated weak dominance. Indicate what happens if dominated strategies are allowed.
   (d) Discuss whether different "agendas" for arriving at a final decision by voting between two alternatives at a time would lead to a different equilibrium outcome.
4. Consider an infinitely repeated Cournot duopoly model with discount factor $\delta < 1$. Two firms, 1 and 2, simultaneously choose the quantities they will sell on the market q_1 and q_2. The price each receives for each unit given these quantities is $P(q_1+q_2) = a-b(q_1+q_2)$. Each firm has a constant marginal cost $c > 0$ with $a > c$ and $b > 0$. Under what conditions can the symmetric joint monopoly outputs $(q_1, q_2) = (q_m/2, q_m/2)$ be sustained with strategies that call for $(q_m/2, q_m/2)$ to be played if no one has yet deviated and for the single-period Nash equilibrium output to be played otherwise?

THE PENNSYLVANIA STATE UNIVERSITY

BA 597D
Advanced Microeconomic Analysis

Kwasnica
Spring 2008

Section 6 Solutions

1. Problem 2.5 in Gibbons (pg. 132). There are three stages to consider in this game.

   (a) The *firm* makes wage offers (w_D, w_E) where w_D is the wage in the difficult job and w_E is the wage in the easy job.

   (b) Having observed the wage offer, the *employee* decides to invents in a firm specific skill S or not at a cost C.

   (c) Having observed whether the employee invested in S or not, the *firm* decides to promote the employee to D or not. If they are promoted, the employee earns the wage w_D and if not they earn w_E.

   In order to find the subgame perfect Nash equilibrium, we solve by backward induction. In the third stage, the firm knows w_D and w_E and whehter the employee invested in S or not. For any (w_D, w_E), if the firm observes S, then they will prefer to promote the employee only if $y_{DS} - w_D \geq y_{ES} - w_E$ or $y_{DS} - y_{ES} \geq w_D - w_E$. Otherwise, they will not promote the employee. If the firm observers not S, then they will prefer to promote the employee only if $y_{D0} - w_D \geq y_{E0} - w_E$ or $y_{D0} - y_{E0} \geq w_D - w_E$.

   Note that $y_{DS} - y_{ES} > 0 > y_{D0} - y_{E0}$ by assumption.

   Moving to the second stage, the employee must decide whether to invest in S or not. Consider three cases. First, if $w_D - w_E > y_{DS} - y_{ES}$, then the firm will not promote the employee in any case so the investment choices for the employee is a choice between w_E (if not S) and $w_E - C$ (if S). Obviously, the employee will pick not S. Second, if $y_{DS} - y_{ES} \geq w_D - w_E > y_{D0} - y_{E0}$, then the firm will promote the employee if S and not promote them if not S. Thus the employee's choice is between w_E (not S) and $w_D - C$ (S) so the employee will invest in S only if $w_D - C \geq w_E$ or $w_D - w_E \geq C$. Third, if $y_{D0} - y_{E0} \geq w_D - w_E$ then the firm will promote regardless of employee investment. Thus, the choice for the employee is between w_D (not S) and $w_D - C$ (S) so they will obviously select not S.

   In the first stage, the firm can select the wages. The outcome of the wages choice (w_D, w_E) depends crucially upon $w_D - w_E$. Consider the following cases:

   (a) If $w_D - w_E \leq y_{D0} - y_{E0}$, then the employee will play not S but the firm will still promote so the payoff to the firm is $y_{D0} - w_D$.

   (b) If $C > w_D - w_E > y_{D0} - y_{E0}$, then employee will play not S and the firm will not promote so the payoff to the firm is $y_{E0} - w_E$.

   (c) If $y_{DS} - y_{ES} \geq w_D - w_E \geq C$, then the employee will invest in S and the firm will promote so the payoff to the firm is $y_{DS} - w_D$.

   (d) If $w_D - w_E > y_{DS} - y_{ES}$, then the employee will play not S and the firm will not promote so the payoff to the firm is $y_{E0} - w_E$

In order to find the profit maximizing wage combination for the firm we can find the maximal combination satisfying each case and the additional given assumption that $w_D \geq 0$ and $w_E \geq 0$. The maximal profit in each case is given by:

(a) setting $w_D = 0$ and $w_E \geq y_{E0} - y_{D0}$ yielding y_{D0}.
(b) setting $w_E = 0$ and $C > w_D \geq 0$ yielding y_{E0}.
(c) setting $w_E = 0$ and $w_D = C$ yielding $y_{DS} - C$.
(d) setting $w_E = 0$ and $w_D > y_{DS} - y_{ES}$ yielding y_{E0}.

In order to determine the profit maximizing case, remember that by assumption $y_{DS} - y_{E0} > C$ so (if possible) the third case represents the profit maximizing choice. However, it is important to recall that we are not guaranteed that $y_{DS} - y_{ES} \geq C$ as is required by the third case. So, we really have two possible outcomes depending upon this.

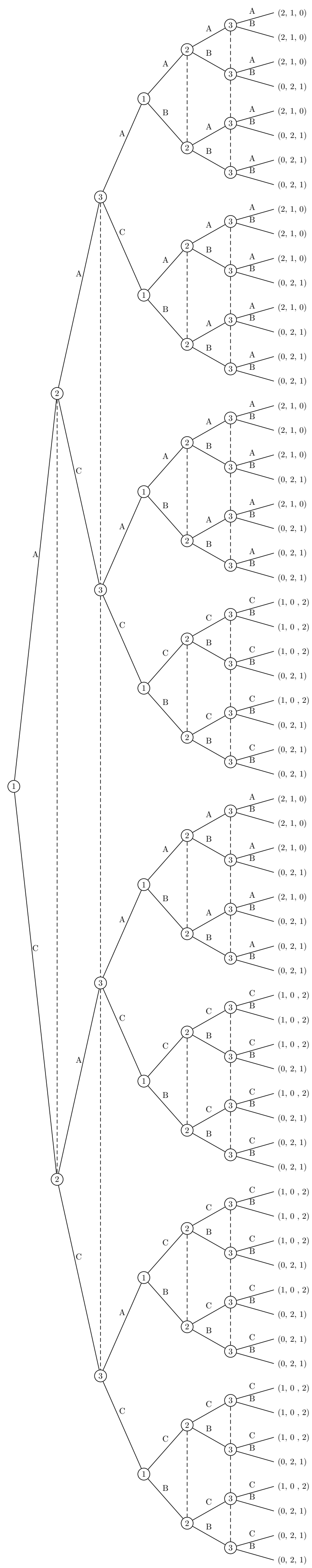
(a) If $y_{DS} - y_{ES} \geq C$, then the firm offers $w_D = C$ and $w_E = 0$ and the employee invests in S and the firm promotes. This results in payoffs of $y_{DS} - C$ for the firm and 0 for the employee.
(b) If $y_{DS} - y_{ES} < C$, then the firm offers $w_E = 0$ and any $w_D \geq 0$ and the employee does not invest in S and the firm does not promote. This results in payoffs of y_{E0} for the firm and 0 for the employee.

2. Problem 2.11 in Gibbons (pg. 135). It is easy to see that the two pure strategy Nash equilibria of this game are (T, L) and (M, C) where the players earn $(3, 1)$ and $(1, 2)$ respectively. Now, consider the twice repeated game. In order for a set of strategies to be a pure-strategy subgame perfect equilibrium is must be that it is a pure-strategy Nash equilibrium of every subgame. Since the second time playing the game for every history of first period play is a subgame, we know that either (T, L) or (M, C) must be played in those subgames. Otherwise, they would not be a Nash equilibrium of the subgame. Consider the following second stage strategies. If in the first-stage, player 1 plays T or M then (M, C) is played in stage 2. If player 1 played B in the first stage, then (T, L) is played in stage 2. Now consider whether playing (B, R) in the first stage (given the actions in the second stage) is a Nash equilibrium. Clearly, it is for player 2 since (B, R) achieves his maximal payoff. On the other hand, 1 would increase her payoff to 5 by playing T, but in the second stage (M, C) would be played. This would result in a payoff of $5 + 1 = 6$ which is less than the payoff from playing B of $4 + 3 = 7$. The same is obviously also true of playing M for player 1. So (B, R) for payoffs of $(4, 4)$ can be supported in the first-stage as part of a subgame perfect equilibrium.

3. Consider a voting game in which three players 1, 2, and 3, are deciding among three alternatives, A, B, and C. Alternative B is the "status quo" and alternatives A and C are the "challengers." At the first stage, players choose which of the two challengers should be considered by casting votes for either A or C, with the majority choice being the winner and abstentions not allowed. At the second stage, players vote between the status quo B and whichever alternative was victorious in the first round, with majority rule again determining the winner. Players vote simultaneously in each round. The players only care about the alternative that is finally selected, and are indifferent as to the sequence of votes that leads to a given selection. The payoff functions are $u_1(A) = 2, u_1(B) = 0, u_1(C) = 1; u_2(A) = 1, u_2(B) = 2, u_2(C) = 0; u_3(A) = 0, u_3(B) = 1, u_3(C) = 2$.

(a) Draw an extensive form representation of the game.

The following is the extensive form of this game. Denote the strategy for each player as the alternative that they vote for in each stage. The payoffs are listed in the terminal node.



(b) What would happen if at each stage the players voted for the alternative they would most prefer as the final outcome?

In the first stage 1 and 2 would vote for A over C and 3 would vote for C. Then, in a vote between A and B 2 and 3 would vote for B and 1 would vote for A so the final outcome would be B with payoffs of $(0, 2, 1)$.

(c) Find the subgame perfect Nash equilibrium that satisfies the additional condition that no strategy can be eliminated by iterated weak dominance. Indicate what happens if dominated strategies are allowed.

In order to find the subgame perfect Nash equilibrium first determine what happens in the two final voting subgames. First, in the vote between A and B, 1 will vote for A and 2 and 3 will vote for B and B will be the outcome. Second, in the vote between C and B, 1 and 3 will vote for C and 2 will vote for A and C will be the outcome. That means that the vote in the first subgame is not a vote between the outcomes of A versus C but a vote between B and C. In other words, a vote for A in the first subgame is really the same as a vote for B. Thus, in this subgame 1 and 3 will vote for C and 2 will vote for A. Formally, the subgame perfect Nash equilibrium in undominated strategies is given by $\{(C, A, C), (A, B, B)(C, B, C)\}$ where each 3-tuple is the votes (in numerical order) in each of the three subgames.

If weakly dominated strategies were allowed, then anything is possible since and strategy where everybody votes for the same outcome is a Nash equilibrium (no one person can change the outcome). By eliminating weakly dominated strategies, you are ensuring 'honest' voting in the final subgames.

(d) Discuss whether different "agendas" for arriving at a final decision by voting between two alternatives at a time would lead to a different equilibrium outcome.

It fairly easy to see that other voting schemes or agendas will result in different outcomes. For example, a first vote between A and B with the winner to face C will result in A winning.

4. Consider an infinitely repeated Cournot duopoly model with discount factor $\delta < 1$. Two firms, 1 and 2, simultaneously choose the quantities they will sell on the market q_1 and q_2. The price each receives for each unit given these quantities is $P(q_1 + q_2) = a - b(q_1 + q_2)$. Each firm has a constant marginal cost $c > 0$ with $a > c$ and $b > 0$. Under what conditions can the symmetric joint monopoly outputs $(q_1, q_2) = (q_m/2, q_m/2)$ be sustained with strategies that call for $(q_m/2, q_m/2)$ to be played if no one has yet deviated and for the single-period Nash equilibrium output to be played otherwise?

The book provides the solution to the static Nash equilibrium of this game. The Cournot equilibrium results in each firm making a profit of $\frac{(a-c)^2}{9b}$. The maximum gain from deviation can be obtained by playing the best response to $\frac{q_m}{2} = \frac{(a-c)}{4b}$ which is $\frac{3(a-c)}{8b}$. The profit from this deviation is $\frac{9(a-c)^2}{64b}$. Monopoly profit for each firm is given by $\frac{(a-c)^2}{8b}$. The payoff from deviating is given by:

$$\frac{9(a-c)^2}{64b} + \sum_{t=1}^{\infty} \delta^t \frac{(a-c)^2}{9b} = \frac{9(a-c)^2}{64b} + \frac{\delta}{1-\delta}\frac{(a-c)^2}{9b} \tag{1}$$

and the payoff from not deviating is given by:

$$\sum_{t=0}^{\infty} \delta^t \frac{(a-c)^2}{8b} = \frac{1}{1-\delta}\frac{(a-c)^2}{8b}. \tag{2}$$

Combining Equations 1 and 2 we find that deviation is not profitable if and only if

$$\begin{aligned}
\frac{1}{1-\delta}\frac{(a-c)^2}{8b} &\geq \frac{9(a-c)^2}{64b}+\frac{\delta}{1-\delta}\frac{(a-c)^2}{9b} && (3)\\
\frac{1}{1-\delta}\frac{1}{8} &\geq \frac{9}{64}+\frac{\delta}{1-\delta}\frac{1}{9} && (4)\\
\frac{1}{8} &\geq (1-\delta)\frac{9}{64}+\delta\frac{1}{9} && (5)\\
\delta &\geq \frac{9}{17}. && (6)
\end{aligned}$$

5. Consider the (single-period) Cournot duopoly model in which two firms, 1 and 2, simultaneously choose the quantities they will sell on the market q_1 and q_2. The price each receives for each unit given these quantities is $P(q_1+q_2)=a-b(q_1+q_2)$ with $b>0$. Each firm has a constant marginal cost. However, now each firm's marginal cost can be c_L or c_H (with $a>c_H>c_L>0$) with probability μ and $1-\mu$ respectively. Each firm knows its own marginal cost when making a quantity decision but not the cost of the other firm. Solve for the Bayes Nash equilibrium.

   Let q_{iH} be the quantity produced by firm i when they have marginal cost c_H and q_{iL} be the quantity produced by firm i when they have marginal cost c_L. Each firm, i, will maximize its expected profit taken as a given that the other firm will supply q_{jH} or q_{jL}. A type $k\in\{H,L\}$ firm 1 will maximize the following:

$$\max_{q_{1k}}(1-\mu)[(a-b(q_{1k}+q_{2H})-c_k)q_{1k}]+\mu[(a-b(q_{1k}+q_{2L})-c_k)q_{1k}] \quad (7)$$

   which yields the following first order condition:

$$(1-\mu)[a-b(2q_{1k}+q_{2H})-c_k]+\mu[a-b(2q_{1k}+q_{2L})-c_k]=0 \quad (8)$$

   for each $k\in\{H,L\}$. In a symmetric Bayesian Nash equilibrium we have that $q_{1H}=q_{2H}=q_H$ and $q_{1L}+q_{2L}=q_L$. Substituting this into the first order conditions results in the following two equations:

$$\begin{aligned}
(1-\mu)[a-3bq_H-c_H]+\mu[a-b(2q_H+q_L)-c_H] &= 0 && (9)\\
(1-\mu)[a-b(q_H+2q_L)-c_L]+\mu[a-3bq_L-c_L] &= 0. && (10)
\end{aligned}$$

   Solving for q_H and q_L, we obtain that

$$\begin{aligned}
q_H &= \left[a-c_H+\frac{\mu}{2}(c_L-c_H)\right]\frac{1}{3b} && (11)\\
q_L &= \left[a-c_L+\frac{1-\mu}{2}(c_H-c_L)\right]\frac{1}{3b}. && (12)
\end{aligned}$$

OPRE 6311 Problem Set 3

1. 3.3 in Gibbons (pg. 169)

2. Consider two people who can combine their labor inputs e_1 and e_2 respectively to produce some public good according to the production function

$$f(e_1, e_2) = 2\min\{e_1, e_2\}.$$

Each agent has a disutility cost of providing labor given by $c(e_1, e_2) = \frac{e_1+e_2}{2}$. For simplicity, just two levels of effort 10 and 20 are assumed available. The net payoff for player i is simply $m(e_1, e_2) = f(e_1, e_2) - c(e_1, e_2)$.

   (a) Depict the normal form representation of this game.

   (b) Identify all the Nash equilibria of the game.

   (c) Assume that the players are not completely certain of their opponents' payoffs in the event of *unsuccessful* coordination on the high effort equilibrium $(20, 20)$. In particular assume that player 1 gets and added payoff of t_1 in the event that they selected effort of 20 but player 2 selected effort of 10, and player 2 receives an added payoff of t_2 in the that they selected effort of 20 but player 1 did not. These added payoff parameters t_1 and t_2 are private information for each respective player. Assume that it is common knowledge that t_1 and t_2 are independently and uniformly distributed on the interval $[0, \epsilon]$. Depict this new game in the normal form.

   (d) Identify the Bayes Nash equilibrium of this game.

3. Consider the following two-player game:

		2: L	2: R
1	U	x, 0	0, 1
	D	0, 1	1, 0

where $x > 0$. Let p be the probability that player 1 chooses D and q be the probability that player 2 chooses R.

   (a) Find any Nash equilibria of the game as a function of x and show that p is independent of x.

Now assume that player 1 receives and additional payoff of γ_1 for playing U, and that player 2 receives an additional payoff of γ_2 for playing L. Assume that γ_i is private information to player i and that γ_i is uniformly distributed on $[0, \epsilon]$, where $0 \leq \epsilon \leq 1$. The distribution of γ_i is assumed to be common knowledge.

(b) For a fixed ϵ, find a Bayesian equilibrium to the resulting game.

(c) Show that as ϵ goes to zero, the Bayesian equilibrium approaches the equilibrium of the full information game.

(d) Show that in the Bayesian equilibrium, for any $\epsilon > 0$, that the expected probability that player 1 plays U increases with x.

4. Consider n bidders competing to purchase a single, indivisible object. Each bidder has a private value v_i for the object being sold where values are drawn independently from the uniform distribution on $[0, 1]$. Each bidder's valuation is their own private information. Each bidder has the same Bernoulli utility function given by $u_i(x) = x^\alpha$ where $0 < \alpha < 1$ which they use to evaluate auction outcomes.

   (a) Are these bidders risk averse, risk neutral, or risk seeking? Explain why.

   (b) Suppose bidders participate in a first-price sealed bid auction to purchase the object. Each bidder submits a bid b_i and if her bid is maximal she wins the objects and pays the price b_i for the object for a profit of $v_i - b_i$. If the bidder's bid is not maximal, then she has profits of 0. Identify the Bayes Nash equilibrium of this auction under the assumption that bidders have the Bernoulli utility function identified earlier.

(c) Suppose bidders participate in a second-price sealed bid auction to the purchase the object. Each bidders submits a bid b_i and if her bid is maximal she wins the object *but* now she pays a price equal to the highest losing bid (or the second highest bid) for a profit of $v_i - \overline{b}$ where $\overline{b}$ is the second highest bid. If the bidder's bid is not maximal, then she has profits of 0. Identify the Bayes Nash equilibrium of this auction. Does the equilibrium depend upon α?

(d) Suppose the person selling the object is risk neutral. Given that bidders have these preferences, what auction format (first or second price auction) will he prefer? Explain.

Problem Set 3 Solutions

1. 3.3 in Gibbons (pg. 169) The action space for each firm is a price p_i to charge for its product where $p_i \in A_i = [0, \infty)$. The type space for each firm is the value of the b_i coefficient which measure responsiveness of demand to prices, or $T_i = \{b_H, b_L\}$ with probabilities of each type given by $\pi_i(b_H) = \theta$ and $\pi_i(b_L) = 1 - \theta$. Each firm's ex post payoffs or utility is given by

$$\begin{aligned} u_i(p_i, p_j; b_i, b_j) &= q_i(p_i, p_j) \cdot p_i && (1) \\ &= (a - p_i - b_i p_j) \cdot p_i && (2) \\ &= a p_i - p_i^2 - b_i p_i p_j. && (3) \end{aligned}$$

A strategy for each player is an action (price to charge) given their observation of b_i or $p_i : T_i \to A_i$. In a Bayes Nash equilibrium, each players strategy must maximize their expected utility given the strategy of the other firm where each firm's expected utility is given by:

$$\begin{aligned} Eu(p_i; b_i) &= \left[a p_i - p_i^2 - b_i p_i p_j(b_H)\right] \theta + \left[a p_i - p_i^2 - b_i p_i p_j(b_L)\right](1 - \theta) && (4) \\ &= a p_i - p_i^2 - b_i p_i \left[p_j(b_H)\theta + p_j(b_L)(1 - \theta)\right] && (5) \end{aligned}$$

The necessary first order condition is therefore given by:

$$a - 2p_i - b_i \left[p_j(b_H)\theta + p_j(b_L)(1 - \theta)\right] = 0 \qquad (6)$$

Since the players are ex ante identical, lets look for a symmetric equilibrium where $p_i(b_i) = p_j(b_i)$. If $b_i = b_H$ then we obtain the following from the first order condition:

$$\begin{aligned} a - 2p(b_H) - b_H \left[p(b_H)\theta + p(b_L)(1 - \theta)\right] &= 0 && (7) \\ p(b_H) &= \frac{a - b_H(1 - \theta)p(b_L)}{2 + b_H\theta} && (8) \end{aligned}$$

If $b_i = b_L$ then we obtain:

$$\begin{aligned} a - 2p(b_L) - b_L \left[p(b_H)\theta + p(b_L)(1 - \theta)\right] &= 0 && (9) \\ p(b_H) &= \frac{a - p(b_L)\left[2 + b_L(1 - \theta)\right]}{b_L\theta}. && (10) \end{aligned}$$

Combining (8) and (10) we obtain:

$$\begin{aligned} \frac{a - p(b_L)\left[2 + b_L(1 - \theta)\right]}{b_L\theta} &= \frac{a - b_H(1 - \theta)p(b_L)}{2 + b_H\theta} && (11) \\ p(b_L)(2 + b_H\theta)(2 + b_L(1 - \theta)) - p(b_L)(b_L\theta)(b_H(1 - \theta)) &= (2 + b_H\theta)a - b_L\theta a && (12) \\ p(b_L) &= \frac{a}{2}\left[\frac{2 + \theta(b_H - b_L)}{2 + \theta(b_H - b_L) + b_L}\right]. && (13) \end{aligned}$$

We can find $p(b_H)$ in a similar fashion:

$$p(b_H) = \frac{a}{2}\left[\frac{2 - (1 - \theta)(b_H - b_L)}{2 + \theta(b_H - b_L) + b_L}\right]. \qquad (14)$$

2. Consider two people who can combine their labor inputs e_1 and e_2 respectively to produce some public good according to the production function

$$f(e_1, e_2) = 2\min\{e_1, e_2\}.$$

Each agent has a disutility cost of providing labor given by $c(e_1, e_2) = \frac{e_1+e_2}{2}$. For simplicity, just two levels of effort 10 and 20 are assumed available. The net payoff for player i is simply $m(e_1, e_2) = f(e_1, e_2) - c(e_1, e_2)$.

(a) Depict the normal form representation of this game.

The following is the normal form representation:

		Player 2: 10	Player 2: 20
Player 1	10	10, 10	5, 5
	20	5, 5	20, 20

(b) Identify all the Nash equilibria of the game.

It easy to check that there are two pure strategy equilibria at (10, 10) and (20, 20) and one mixed strategy equilibrium where each player plays 10 with probability .75.

(c) Assume that the players are not completely certain of their opponents' payoffs in the event of *unsuccessful* coordination on the high effort equilibrium $(20, 20)$. In particular assume that player 1 gets and added payoff of t_1 in the event that they selected effort of 20 but player 2 selected effort of 10, and player 2 receives an added payoff of t_2 in the that they selected effort of 20 but player 1 did not. These added payoff parameters t_1 and t_2 are private information for each respective player. Assume that it is common knowledge that t_1 and t_2 are independently and uniformly distributed on the interval $[0, \epsilon]$. Depict this new game in the normal form.

The following is the normal form representation of the new game:

		Player 2: 10	Player 2: 20
Player 1	10	10, 10	5, 5+ t_2
	20	5+t_1, 5	20, 20

(d) Identify the Bayes Nash equilibrium of this game.

First, notice that the game is completely symmetric with the exception of the type drawn. Since these types are independent drawn from the same interval, the same type-contingent strategies will be optimal. Also, notice that the type-contingent strategy of each player's opponent will induce a "mixed strategy" since for some types the opponent will play 10 and for some other types they will play 20. Let q_2 be the probability that player 2 plays 10 and $1 - q_2$ the probability that they play 20. Player 1 will prefer to play 10 if (given his type draw) his expected value from playing 10 is greater than that from playing 20, or if $q_2 10 + (1 - q_2)5 > q_2(5 + t_1) + (1 - q_2)20$. Solving for t_1, it is easy to see that an optimal strategy for player 1 requires the following strategy (denoted by s_1) given an t_1 and q_2:

$$s_1(q_2, t_1) = \begin{cases} 10 & t_1 < 20 - \frac{15}{q_2} \\ 20 & t_1 > 20 - \frac{15}{q_2} \end{cases} \tag{15}$$

Player 1 is indifferent between the actions when $t_1 = 20 - \frac{15}{q_2}$, but, since this is a probability zero event for the uniform distribution, we do not have to worry about this special case.

The optimal strategy of player 2 is similarly defined given the "mixed strategy" of player 1 (denoted by q_1 and $(1 - q_1)$).

$$s_2(q_1, t_2) = \begin{cases} 10 & t_2 < 20 - \frac{15}{q_1} \\ 20 & t_2 > 20 - \frac{15}{q_1} \end{cases} \tag{16}$$

Since t_1 and t_2 are drawn from the uniform distribution on $[0, \epsilon]$, the probability that nature will draw a type smaller than x is given by $\frac{x}{\epsilon}$. Thus, the "mixed strategy" induced by the type-contingent strategy s_1 and the uniform distribution on t_1 is given by

$$\begin{aligned} q_1 &= \Pr\{\text{player 1 plays 10}\} && (17) \\ &= \Pr\{t_1 < (20 - \frac{15}{q_2})\} && (18) \\ &= \frac{1}{\epsilon}(20 - \frac{15}{q_2}). && (19) \end{aligned}$$

Using the same logic for player 2 we get the following:

$$q_2 = \frac{1}{\epsilon}(20 - \frac{15}{q_1}) \tag{20}$$

In a symmetric equilibrium it must be that $q_1 = q_2 = q$, solving for q yields a quadratic equation. Only one of the two roots makes sense:

$$q = \frac{20 - \sqrt{400 - 60\epsilon}}{2\epsilon}. \tag{21}$$

We find the Bayes Nash equilibrium strategy by substituting it back into s_1 and s_2 in order to obtain:

$$\begin{aligned} s_1(t_1) &= \begin{cases} 10 & t_1 < 20 - \frac{30\epsilon}{20-\sqrt{400-60\epsilon}} \\ 20 & t_1 > 20 - \frac{30\epsilon}{20-\sqrt{400-60\epsilon}} \end{cases} && (22) \\ s_2(t_2) &= \begin{cases} 10 & t_2 < 20 - \frac{30\epsilon}{20-\sqrt{400-60\epsilon}} \\ 20 & t_2 > 20 - \frac{30\epsilon}{20-\sqrt{400-60\epsilon}} \end{cases} && (23) \\ & && (24) \end{aligned}$$

3. Consider the following two-player game:

		2: L	R
1	U	x, 0	0, 1
	D	0, 1	1, 0

where $x > 0$. Let p be the probability that player 1 chooses D and q be the probability that player 2 chooses R.

(a) Find any Nash equilibria of the game as a function of x and show that p is independent of x.

Note: Original problem failed to stipulated that $x > 0$.

It is easy to see that there are no pure strategy Nash equilibria of this game when $x > 0$ ((D, L) is a Nash equilibrium if $x \leq 0$). Thus, the only possible Nash equilibrium must be in mixed strategies. For a given q by player 2, player 1 must be indifferent between U and D, or

$$\begin{aligned} Eu_1(U, (1-q, q)) &= Eu_1(D, (1-q, q)) \quad (25) \\ (1-q)x + q0 &= (1-q)0 + q1 \quad (26) \\ (1-q)x &= q \quad (27) \\ q &= \frac{x}{1+x}. \quad (28) \end{aligned}$$

Player 2 must be indifferent between L and R given p by player 1, or

$$\begin{aligned} Eu_2(L, (1-p, p)) &= Eu_2(R, (1-p, p)) \quad (29) \\ (1-p)0 + p1 &= (1-p)1 + p0 \quad (30) \\ p &= 1 - p \quad (31) \\ p &= \frac{1}{2}. \quad (32) \end{aligned}$$

So the mixed strategy Nash equilibrium is given by $p^* = 1/2$ and $q^* = x/1+x$ for all $x > 0$. Clearly, p does not depend on x.

Now assume that player 1 receives and additional payoff of γ_1 for playing U, and that player 2 receives an additional payoff of γ_2 for playing L. Assume that γ_i is private information to player i and that γ_i is uniformly distributed on $[0, \epsilon]$, where $0 \leq \epsilon \leq 1$. The distribution of γ_i is assumed to be common knowledge.

(b) For a fixed ϵ, find a Bayesian equilibrium to the resulting game. In order to simplify the presentation and mathematics let $\gamma_i = \epsilon t_i$ where t_i is distributed uniformly on the $[0, 1]$ interval. The following is the depiction of the normal form game induced by these added payoffs

		2: L	2: R
1	U	$x + \epsilon t_1$, ϵt_2	ϵt_1, 1
1	D	0, $1 + \epsilon t_2$	1, 0

It is easy to see that the expected utility of player 1 for playing U is increasing in t_1 for any beliefs about player 2, and, likewise, the expected utility of player 2 for playing L is increasing in t_2 for any beliefs about player 1. Thus, we can look for a cutoff strategy or a $(\bar{t}_1, \bar{t}_2)$ such that if $t_1 > \bar{t}_1$ then player 1 plays U and plays D otherwise, and if $t_2 > \bar{t}_2$ then player 2 plays L and plays R otherwise. Given the strategy of player 2, the probability that 2 plays L is given by $1 - \bar{t}_2$ and the probability that 2 plays R is given by $\bar{t}_2$ so the expected utility for player 1 for the various actions is given by:

$$\begin{aligned} Eu(U; t_1) &= (x + \epsilon t_1)(1 - \bar{t}_2) + \epsilon t_1 \bar{t}_2 \quad (33) \\ &= (1 - \bar{t}_2)x + \epsilon t_1 \quad (34) \\ Eu(D; t_1) &= 0(1 - \bar{t}_2) + 1\bar{t}_2 \quad (35) \\ &= \bar{t}_2. \quad (36) \end{aligned}$$

Lets find the the t_1 that makes player 1 indifferent between U and D, or

$$\begin{aligned}
Eu(U;t_1) &= Eu(D;t_1) && (37)\\
(1-\bar{t}_2)x+\epsilon t_1 &= \bar{t}_2 && (38)\\
t_1 &= \frac{\bar{t}_2(1+x)-x}{\epsilon}. && (39)
\end{aligned}$$

Given the strategy of player 1, the probability the 1 plays U is given by $1-\bar{t}_1$ and the probability that 1 plays D is given by $\bar{t}_1$ so the expected utility for player 2 for the various actions is given by:

$$\begin{aligned}
Eu(L;t_2) &= \epsilon t_2(1-\bar{t}_1)+(1+\epsilon t_2)\bar{t}_1 && (40)\\
&= \epsilon t_2+\bar{t}_1 && (41)\\
Eu(R;t_2) &= 1(1-\bar{t}_1)+0\bar{t}_1 && (42)\\
&= 1-\bar{t}_1. && (43)
\end{aligned}$$

Player 2 will be indifferent between L and R if:

$$\begin{aligned}
Eu(L;t_2) &= Eu(R;t_2) && (44)\\
\epsilon t_2+\bar{t}_1 &= 1-\bar{t}_1 && (45)\\
\bar{t}_1 &= \frac{1-\epsilon t_2}{2}. && (46)
\end{aligned}$$

In a Bayes Nash equilibrium, the cutoffs $(\bar{t}_1,\bar{t}_2)$ must jointly satisfy (39) and (46), or

$$\begin{aligned}
\frac{1-\epsilon\bar{t}_2}{2} &= \frac{\bar{t}_2(1+x)-x}{\epsilon} && (47)\\
\bar{t}_2 &= \frac{\epsilon+2x}{2+2x+\epsilon^2} && (48)
\end{aligned}$$

Similarly, using (39) and (46) to find $\bar{t}_1$, we obtain:

$$\begin{aligned}
\frac{x+\epsilon\bar{t}_1}{1+x} &= \frac{1-2\bar{t}_1}{\epsilon} && (49)\\
\bar{t}_1 &= \frac{1+x(1-\epsilon)}{2+2x+\epsilon^2}. && (50)
\end{aligned}$$

So the cutoff strategy defined by (48) and (50) is a Bayes Nash equilibrium of this game for a fixed ϵ.

(c) Show that as ϵ goes to zero, the Bayesian equilibrium approaches the equilibrium of the full information game. Remember that $\bar{t}_1=p$ the probability that 1 plays D and that $\bar{t}_2=q$ the probability that player 2 plays R. Take the limit of $\bar{t}_1$ and $\bar{t}_2$ as ϵ goes to 0 to find:

$$\begin{aligned}
\lim_{\epsilon\longrightarrow 0}\bar{t}_1 &= \frac{1+x}{2+2x} && (51)\\
&= \frac{1+x}{2(1+x)} && (52)\\
&= \frac{1}{2} && (53)\\
\lim_{\epsilon\longrightarrow 0}\bar{t}_2 &= \frac{2x}{2+2x} && (54)\\
&= \frac{x}{1+x} && (55)
\end{aligned}$$

which are precisely the probabilities identified in part (a).

(d) Show that in the Bayesian equilibrium, for any $\epsilon > 0$, that the expected probability that player 1 plays U increases with x.

It is sufficient to show that $\bar{t}_1$ is decreasing in x since that obviously implies that $1-\bar{t}_1$ is increasing as x increases.

$$\frac{d\bar{t}_1}{dx} = \frac{(1-\epsilon)\left(2+2x+\epsilon^2\right) - 2\left(1+x(1-\epsilon)\right)}{\left[2+2x+\epsilon^2\right]^2} \tag{56}$$

$$= \frac{\epsilon\left(-2+\epsilon-\epsilon^2\right)}{\left[2+2x+\epsilon^2\right]^2} \tag{57}$$

Since the term in the denominator is positive, ϵ is positive, and the term inside the parenthesis in the numerator is negative, then this derivative is negative.

4. Consider n bidders competing to purchase a single, indivisible object. Each bidder has a private value v_i for the object being sold where values are drawn independently from the uniform distribution on $[0, 1]$. Each bidder's valuation is their own private information. Each bidder has the same Bernoulli utility function given by $u_i(x) = x^\alpha$ where $0 < \alpha < 1$ which they use to evaluate auction outcomes.

   (a) Are these bidders risk averse, risk neutral, or risk seeking? Explain why.

   These bidders are *risk averse*. They exhibit constant relative risk aversion or CRRA. To see why note that $u''(x) = (\alpha-1)\alpha x^{\alpha-2}$ which is negative since $\alpha < 1$.

   (b) Suppose bidders participate in a first-price sealed bid auction to purchase the object. Each bidder submits a bid b_i and if her bid is maximal she wins the objects and pays the price b_i for the object for a profit of $v_i - b_i$. If the bidder's bid is not maximal, then she has profits of 0. Identify the Bayes Nash equilibrium of this auction under the assumption that bidders have the Bernoulli utility function identified earlier.

   Let us assume that bidders are following the symmetric equilibrium strategy given by $\beta : [0, 1] \to \mathbb{R}$ such that $\beta(0) = 0$ and β is increasing and differentiable. Consider an arbitrary bidder i and assume that all $n-1$ are following the equilibrium bid strategy β. It is obvious that the bidder would never bid above $\beta(1)$ since that would be equivalent to throwing money away. For any value of v_i the bidder's problem is to select a report r_i and a bid amount $\beta(r_i)$ such that expected utility is maximized, or

$$\max_{r_i \in [0,1]} G(r_i)\left(v_i - \beta(r_i)\right)^\alpha \tag{58}$$

   where $G(r_i) = F(r_i)^{n-1}$ or the probability that r_i is greater than the $n-1$ values of the other bidders. The necessary first-order condition is therefore given by:

$$g(r_i)\left(v_i - \beta(r_i)\right)^\alpha - \alpha G(r_i)\beta'(r_i)\left(v_i - \beta(r_i)\right)^{\alpha-1} = 0. \tag{59}$$

   If this is a symmetric equilibrium, it must be expected utility maximizing to select $r_i = v_i$, and rearranging terms we obtain that:

$$\beta'(v_i)G(v_i) + \frac{1}{\alpha}\beta(v_i)g(v_i) = \frac{1}{\alpha}v_i g(v_i). \tag{60}$$

   Multiply both sides by $G(v_i)^{(1/\alpha)-1}$ to obtain:

$$\beta'(v_i)G(v_i)^{1/\alpha} + \frac{1}{\alpha}\beta(v_i)g(v_i)G(v_i)^{(1/\alpha)-1} = \frac{1}{\alpha}v_i g(v_i)G(v_i)^{(1/\alpha)-1} \tag{61}$$

which is equivalent to

$$\frac{d}{dv_i}\Big(G(v_i)^{1/\alpha}\beta(v_i)\Big) = \frac{1}{\alpha}v_i g(v_i)G(v_i)^{(1/\alpha)-1} \tag{62}$$

and since we are assuming $\beta(0) = 0$, we have that

$$\begin{aligned} G(v_i)^{1/\alpha}\beta(v_i) &= \int_0^{v_i} \frac{1}{\alpha} y g(y) G(y)^{(1/\alpha)-1} dy && (63)\\ \beta(v_i) &= \frac{1}{G(v_i)^{1/\alpha}} \int_0^{v_i} \frac{1}{\alpha} y g(y) G(y)^{(1/\alpha)-1} dy. && (64)\end{aligned}$$

With uniform values on $[0,1]$ we have that $F(v) = v$ and $f(v) = 1$ so $G(v) = v^{n-1}$ and $g(v) = (n-1)v^{n-2}$. Substituting into (64) above we obtain:

$$\begin{aligned} \beta(v_i) &= \frac{n-1}{\alpha v_i^{\frac{n-1}{\alpha}}} \int_0^{v_i} y^{n-1} y^{\frac{(n-1)(1-\alpha)}{\alpha}} dy && (65)\\ &= \frac{n-1}{\alpha v_i^{\frac{n-1}{\alpha}}} \int_0^{v_i} y^{\frac{n-1}{\alpha}} dy && (66)\\ &= \left(\frac{\alpha}{n-1+\alpha}\right) \frac{n-1}{\alpha v_i^{\frac{n-1}{\alpha}}} v_i^{\frac{n-1}{\alpha}+1} && (67)\\ &= \frac{n-1}{n-1+\alpha} v_i. && (68)\end{aligned}$$

(c) Suppose bidders participate in a second-price sealed bid auction to the purchase the object. Each bidders submits a bid b_i and if her bid is maximal she wins the object *but* now she pays a price equal to the highest losing bid (or the second highest bid) for a profit of $v_i - \overline{b}$ where $\overline{b}$ is the second highest bid. If the bidder's bid is not maximal, then she has profits of 0. Identify the Bayes Nash equilibrium of this auction. Does the equilibrium depend upon α?

As we discussed in class, it is a weakly dominant strategy for a bidder to bid his or her value in a second-price sealed bid auction. Thus, it follows that for any α is also a weakly dominant strategy to bid $\beta(v_i) = v_i$.

(d) Suppose the person selling the object is risk neutral. Given that bidders have these preferences, what auction format (first or second price auction) will he prefer? Explain.

The seller will prefer the first-price sealed bid auction because it generate higher expected revenues when the bidders are risk averse. To see this note that the auctioneer can except to collect $\beta(v^{(1)})$ where $v^{(1)}$ is the maximum of n value draws from the uniform distribution where the distribution is given by $F(v)^n$ The expected value is given as follows.

$$\begin{aligned}
E(\beta(v^{(n)} &= \int_0^1 \beta(v^{(1)})f(v)^n dv && (69)\\
&= \left(\frac{n-1}{n-1+\alpha}\right)\int_0^1 v^{(1)}f(v)^n dv && (70)\\
&= \left(\frac{n-1}{n-1+\alpha}\right)\int_0^1 (n)v^n dv && (71)\\
&= \left(\frac{n-1}{n-1+\alpha}\right)\left(\frac{n}{n+1}\right) && (72)\\
&= \left(\frac{n-1}{n+1}\right)\left(\frac{n}{n-1+\alpha}\right) && (73)
\end{aligned}$$

The expected revenue in the second-price sealed bid auction is simply the second highest value out of n values or

$$\begin{aligned}
E(v^{(2)}) &= \int_0^1 v f_{(2)}(v) dv && (74)\\
&= \int_0^1 n(n-1)v^{n-1}(1-v)dv && (75)\\
&= \frac{n-1}{n+1} && (76)
\end{aligned}$$

Notice that the first term in (73) is the same as (76) but the second term is greater than 1 since $\alpha < 1$ so the expected revenue from the first-price auction is greater.