**Advance managerial economics**

- Check sheet or formula Sheet standard size on both side is fine (in exam memorizing is not important but analysis)
- Theory and foundation course and everything would be related to things you will learn in future
- Models and how important it is to conceptually develop the way of thinking will be discuss
- Here we try to abstract from reality
- Two important things of the class:
  - You have to have appreciation for theory, and understand it. All papers have some theory behind things.
  - Second many things we do is relevant to what we do in research.
- Natural subsequent classes for this: game theory, and industrial organizations
- Copy of things of other students is forbidden ; solution should be in your own words
- Late assignments is not allowed since solution would be available on the same day (turn the assignment in the class or by classmate is fine)
- Office hours: after the class from 13:30 to 14:30
- The assignment due is next day of assigning

**Theory of consumer**

**Preferences and choices**

- Let X be the set of possible alternatives for which an individual must choose (all goods could be subsumed here).
- Two sets of allowance:
  - Preference based
  - Choice base

**Preference base**

Basic notations:

1. "$\succsim$" means "at least as good as"
   X $\succsim y \Rightarrow$ x is at least as good as y
2. ">" strictly preferred
   X> y => x is strictly preferred by y ⇔ x $\succsim y$ $but\, not\, y \succsim x$
3. "~" indifference
   X~y ⇔ x $\succsim y$ $and\, y \succsim x$

Definition: "rational preferences"

The preference relation $\succeq$ is rational if it possesses the following properties:

1. Completeness:
   For $x, y \in X$ we have x $\geq y$ or y $\geq x$
   It means "don't know" is a problem and should not exist in answering them, since on that case we would encounter "incompleteness"

2. Transitivity For $x, y \in X$ we have x $\geq y$ and y $\geq z$ then x $\geq z$

   Violation of transitivity case:

   - if you have preferences that do not have perceptible differences, and you became indifferences then you will violate the second one. (e.g. colors from two side of continuum but then come to the point from two side that the difference faded, since at the middle also we should have strict preference but we have indifference)
   - Change of preference can also violate it (e.g. you change from abstemious to smoking)
   - Let say we have three person in the family (Dad, Mom, Child) between (Honda, Toyota, VW)
     Dad's preference $V >$ H $> T$
     Mom's preference H $\succ T \succ V$
     Child's preference T$\succ V \succ H$
     Rule Majority in voting is showing that V$\succ H \succ T \succ V$
     We are running a cycle so transitivity is violated

Rationality is needed in order to be able to take the utility function

**Utility function**:

Definition: A function U that will e defined over X, and maps them to real numbers X→ R is a utility function representing preference relation "at least as good as" ("$\succeq$ ") if for all $x, y \in X$

$x \succeq y \Leftrightarrow$ U(x) >= U(y)

The first notation on above is "at least as good as", but the second one is "great or equal to" since it is numerical value (since the preferences are not of value, and are not the same type to be compared, that is why we use utility function to map it to values; moreover variables such as x, and y could be represented as bundles)

Not all preferences can be presented by utility function, but only rational ones could be represented

**Any increasing transformation would represent the underlying preferences**

- As a result you can use any of the utility function that is increasing transformation that helps you optimization

- If x~y then U(x)=U(y)
- Utility considers satisfaction, but price considers what actually will become physical act
- Utility function maps the effect of all attributes such as functionality and brand name (in the combined form) to a numerical value
- Different individuals have different utility functions
- Utility functions are not unique
- If there is no resource limitation then utility function does not make any sense, and I would take everything; as a result we are considering resource scarcity

Prep: a preference relation $\succsim$ *can be represented by a utility funciton only if it is rational*

Proof: Let U(.) be utility function representation $\succsim$

(1) Completeness
Define on X, it must be true that for any x,y$\in X$ it is going to be either U(x) >= U(y), or U(y)>=U(x)
=> each of the cases mean in order either x$\succsim y$ or y$\succsim x$
This means you can always rank them so it is "complete"
(2) Transitivity
x$\succsim y \succsim z$ and U is defined on this and this implies U(x)>=U(y)>=U(z) and this means U(x)>=U(z) that suggests transitivity

**Choice based approach (Choice rules)**

Choice structure has two components (B, C(.))

(1) "B" is a family of non empty sets a subset of X. Every elements of B is a set B that is subset of X
Budget sets a list of al <u>feasible</u> choice experiment
(2) C(.) is a choice rule that assigns a non empty set of chosen elements which means C(B) would be subset of B for every budget set "B"

Example:

suppose X={ x, y,z}, B1={x,y}, B2={x, y,z}

"B"={B1, B2}: only feasible ones

C1{B1}={x}

C2{B2}={x}

Another example:

C1(B1)={x}

C2(B2)={x,y}

Definition: Weak axiom of reveal preference (WARP)

The choice structure ("B", C(.)) satisfies the weak axiom of reveal preference if the following property holds:

If for some B part of "B" with x,y$\in B$, we have x$\in C(B)$ then for any B' part of family of all budget sets "B", $x, y \in B'$ and y $\in C(B')$

Then we must have

x $\in C(B')$

Means: if there's ever situation which both alternatives available and x is part of that set, then there would be no other situation where y is chosen while x is not. The choice has to be consistent.

The first example was WARP but the second one not, because "y" should have been in the first set

The following example:

C1{B1}={x}

C2{B2}={z}

Is not violating consistence since you chose "z" over that.

For the following example:

C1{B1}={x}

C2{B2}={x,z}

Does not violate consistence.

Definition: Reveal preference relation $\succsim^*$

Given a choice structure ("B", C(.)), the reveal preference relation $\succsim^*$ is defined by $x \succsim^* y$ ⇔ there is some B part of "B" (budget set) such that both x, and y are part of B, and x is part of C(B).

If we use strictly reveal >* then it means x is in there and y is not in there.

Two questions arise from different perspective:

1. Is the choice of rational preference of customer will satisfy weak axiom of reveal preference? Yes.
2. Is the satisfying weak axiom of revel preference of choice satisfying the rational preference? Maybe

**The relationship between preference relation and choice rules**

(1) Suppose you have rational preference => choice rule that satisfying WARP? Yes
(2) Suppose you have rule that satisfies WARP=> rational preference as the choice rule ? Maybe

**Definition**: Choice rule for rational preference at least as good as ≥

C*(B, ≥)={ x$\in B | x \geq y \; for \; y \in B\}$

(“B”, C*(., $\geq)\}$

Proposition: suppose that $\geq is \; rational$, then the choice structure generated by the preference relation ≥ in this case, (“B”, C*(.,$\geq))$ satisfies the WARP

**Proof**: suppose that for some B element of family of budget sets “B”, we have x,y elements of B, and x x∈ C*(“B”, ≥)

By definition of chosen set C*(B, $\geq), x \in C*(B, \geq) => x \geq y$

Suppose that we have B’ $\in "B"$ and $x, y \in B'$ and we have $y \in C*(B, \geq)$

That means y≥ $z$ for all $z \in B'$, that means due to the transitive x also satisfies x≥ $z$ so we will have $x \in C*(B', \geq)$

3. Choice rule C(.) satisfies the WARP => rational preference ≥ generate C(.)

Suppose that you have X={x,y,z}

“B”={{x,y},{y,z}, {x,z}}

C({x,y})={x}, C({y,z})={y}, C({x,z})={z}

They satisfy all Weak Axiom of Reveal Preference.

This **violates transitivity**, and as a result is not rational preference.

If budget set if consists of all subsets with three elements will eliminates cycles and conditions that violate WARP. So if you put {x,y,z} in then it would result in the correct statement.

- On e-learning is the first assignment and you should work on it for the next session
- Last time:
    - Introduce two different approaches to analyze decision maker approaches
        - Preference based
            - Restriction on preferences
                - Completeness
                - Transitivity
            - Utility function
        - Choice based
            - Some restrictions
                - Weak axiom of revealed preference
            - Equivalence between preference approach and choice based approach
            - Rational preference lead to weak axiom of revealed preference, but reverse is not
- Research paper
    - You start with setting and model
    - You put restrictions
    - You introduce restriction to get solution
    - Think whether restriction are too restrictive that let us assume the result which is not true
    - The assumptions should be technically important but not important economically

Today:

Preposition: if (B, C(.)) is a choice structure such that

- Weak axiom of revealed preference is satisfied
- B includes all subsets of X up to 3 elements

Then there exist a $\succeq$ such that C(B)=C*(B, $\succeq$) (choice rule that is generated based on rational preference) for all $B \in \mathcal{B}$

What we try to find is that If we start with choice structure can we find rational preference choice rule that the decision would be the same as choice rule.

1. $\succeq^*$ :revealed preference relation
2. C* (B, $\succeq^*$)= C(B)

Proof: (1) We need to show that revealed preference relation is rational

Completeness: based on (ii), we know that {x,y} going to be in the B (family of budget sets)

The choice rule will tell us either x, or y, or both will be in C({x,y}) if x is in it means that x$\succsim^* y$

if y is in that implies $y \succsim^* x$

if x and y are in it, it means y$\succsim^* x$ *and* $x \succsim^* y$

this proves completes

Transitivity: let x
$\succsim^* y$ *and* $y \succsim^* z$ *and we need to show for the transittivity we need to prove* $x \succsim^* z$

We start with {x,y,z} part of the Budget set (B)

For C({x,y,z}) completeness implies that it could not be empty

Suppose x is in it implies x$\succsim^* y$ *and* $x \succsim^* z$

Suppose y is part of C({x,y,z}) this implies that by WARP there is budget set in which x, y are in and this implies that based on WARP x should be in choice set C({x,y,z}) and this means x$\succsim^* z$ and transitivity is approved

If z is in the chosen set, the same reasoning will be applied to y, and
y$\succsim^* z$ *and y has to be in it, and now we are back to previous argument and x should be in it*

Now we need to reveal that C(B)=C*(B, $\succsim^*$)

Proof: C(B)=C*(B, $\succsim^*$) *for all B part of* budget set (B)

First show C(B) is subset of C*(B, $\succsim^*$)

Suppose x is part of C(B) as a result x$\succsim^* y$ for all elements that are in B (and this is precise definition of construction C*(B, $\succsim^*$) as a result x is part of C*(B, $\succsim^*$)

We are going to prove the reverse side now:

C* (B, $\succsim^*$) is going to be subset of C(B)

Suppose x is part of C*(B, $\succsim^*$) : this will imply that x $\succsim^*$ y for all y that is actually in B, that means this choice only applies to B, and then there would be no budget set that x, and y are available and if x is available y needs to be in it due to WARP. Consequently, for each y part of B (budget set), there must exist some set B y which is also one possible budget set in family of budget set, and this B y has both x, and y in it. In the mean time, x is in the chosen set of that B y mean x is member of C(B). y is arbitrary element, and that means what y is you can always identify with x, and y in it. Means x should be in B, for all family of budget set, so x should be an element in C(B).

This means that by implementing some constraint over two models, choice model which is observable, and preference based model that is so restrictive and unobservable if we apply WARP on the choice model we will have the same result of rationality that we have in preference approach.

We need to choose less restrictive assumption and that is all subset of 3 present and not more (i.e. 4 or 5)

Consumer choice:

2.1. Commodities: there are versions of goods and services (e.g. consumption goods, clean air or anything that they care about)that are available for purchase in the market place. Also we assume that number of them are finite, so indexed by l: 1, 2, …. . a commodity vector or bundle is presented by x=(x1, x2, …, xl) that would be in the L space (R^L). When something is bad (e.g. pollution) it would be negative (you can redefine this in positive form).

2.2. Consumption set: is the subset of commodity space, denoted by X is subset of R^L, whose elements are the consumption bundles that individual can conceivably consume, given a physical environment (physical constraint)
X=R^L = {x part of R^l: x>=0 for all l=1,2, …}

Consumption set is convex:
If x1, x2 both are in X then αx+ (1-α)y is part of X, for all 0<=α<=1. (it means all point in the middle is included)
Note: this does not mean that the person will choose all points in the middle since the preference function is concave, and we need to find optimal solution.

2.3. Competitive budget set
A consumer faces economic constraints.
(i) All l commodities are traded at price p=(p1,p2,…,pl) and we assume p>>0 means Pl >0 for all l: 1,2,…
(ii) Price takers
The affordability constraint mean total expenditure is lower than wealth. Sigma(i=1;l; xi.pi)<=w (wealth)

Definition: Walrasian budget set is defined as Bp,w ={x is an element in consumption set, and x.p<=w: affordable}

If x, x' were in B p,w then a combination of these two would be in there as well.
Proof: If x is there it means p.x <=w, and p.x'<=w, then x<αx+ (1-α)x' <x' so (αx+ (1-α)y).p<w

2.4. Demand functions and comparative statistics.
Consumers Walrasion demand consumption
x(p,w)--- assigns a set of chosen consumption bundles for each (p,w). If x(p,w) is single value, we refer to it as demand function. It is actually: x(p,aw)=C({B p,w})

if multiple consumption bundle exists it would be consumption correspondence but if it is one it would be called demand function

Definition: The Walrasian Demand Correspondance (W.D.C) is homogeneous of degree zero if $x(\alpha p, \alpha w)=x(p,w)$ for all p,w, and $\alpha >=0$.
In general this does not hold, since people have different expectation in reality we call it illusion, you see people have different perception to that. They say I am actually wealthier and never realize that the price is more, and they are going to increase their consumption.

$B\ \alpha p,\alpha w=B\ p,w$ as well

The consumption we are talking about now is not only about today, but about tomorrow, a day after that and so on.

Definition: Walras law says W.D.C satisfies the walras law if $p.x=w$ for all x is part of X p,w.
Means the decision maker is going to fully spend his wealth. (if you are talking about charity, then one of those x would be charity)

Example:
Suppose: L=2
$X1(p,w)=p2/(p1+p2)*w/p1=\alpha p2/\alpha(p1+p2)*\alpha w/\alpha p1$
$X2(p,w)=\beta p1/(p1+p2).w/p$
Are homogeneous.

For walrasian law
$P1x1+p2x2=p1p2/(p1+p2)*w/p1+p2\beta p1/(p1+p2).w/p2=p2/(p1+p2)*w+\beta p1/(p1+p2).w=(p2+\beta p1)/(p1+p2).w$
Whether this is equal to w or not is the matter of β.

Competitive statistics

Wealth effect—changing w
Hold the price fix and then see how w will react
Engel function $=\{x\ (p\^{}, w): w>0\}$
Knowing the Engel curve on wealth expansion path
If wealth is decreased the demand curve will shift down, and if wealth increases the demand curve will shift up
If you connect the corresponding point on the demand curves that shift for wealth changes it will be called expansion path

$\partial x\ (p\^{}, w)/\partial w>= p$ it would be normal good
$\partial x\ (p\^{}, w)/\partial w< p$ it would be called inferior good (wealthy family hamburger, and beef stake)

In general
$D_w X_l(p,w)=(\partial x_1(p,w)/\partial w, \partial x_2(p,w)/\partial w, \ldots, \partial x_l(p,w)/\partial w)$ would be part of $R^L$

Price effect

e.g. Two commodities: $x(p_1, p_2, w)$
we would have offer curve $p_1=p^\wedge$ , $w=w^\wedge$

$\partial x_l(p,w)/\partial p_k$ is most of the case negative
If it is positive we will call it Giffen: the most common example is potato in Irish era

The price effect Matrix:

$D_p x(p, w)= ((\partial x_1/\partial p_1, \partial x_1/\partial p_2, \ldots, \partial x_1/\partial p_l), ((\partial x_2/\partial p_1, \partial x_2/\partial p_2, \ldots, \partial x_2/\partial p_l), \ldots.((\partial x_l/\partial p_1, \partial x_l/\partial p_2, \ldots, \partial x_l/\partial p_l))$

Implication of homogeneity and walrase law on price and wealth effect:

Proposition: if $x_1(p, w)$ is W.D.F., then for all P and w, $\Sigma$ ($\partial x_l/\partial p_k .p_k.w$, $k=1; L$) + $\partial x_l/\partial w =0$ for all $k=1,2,\ldots, l$
$D_p\ x(p,2)p+D_w\ x(p,w)w=0$

Proof:
$X(\alpha p, \alpha w)= x(p,w)$
Differentiate both sides with respect to $\alpha$
$\partial x_1/\partial(\alpha p_1). P_1+\partial x_1/\partial(\alpha p_2). P_2+\ldots+\partial x_1/\partial(\alpha p_l). P_l- \alpha/\partial w .w=0$

Assuming that $\alpha=1$ then
$\partial x_1/\partial p_1. P_1+\partial x_1/\partial p_2.P_2+\ldots++\partial x_l/\partial p_l.P_l=\partial \alpha/\partial w .w=0$

The same will happen for other x's as well.

Elasticity means percentage change of one variable when the percentage change in anther variable.

$\partial x_1/x_1/ \partial w/w=\partial x_1/\partial w.w/x_1$

The term of above sum of elasticity of each product with respect to good I, if we divided everyone with x1. means if there is one percentage change is price, and one percent change in wealth the demand will not change.

The implication of Walras Law: $p.x(p,w)=w$

1. Differentiate $p.x(p,w)=w$ w.r.t. P
   Prop: if $x(p,w)$ satisfies Walras law then for every p and w we will have
   $\Sigma(p_l.\partial x_l/\partial p_k, l=1; l)+x_k (p,w)=0$ for all $k=1,\ldots, l$

$P. D_p x(p,w) + x(p,w)^T = 0$

Proof: $\Sigma(p_k . x_k(p,w), k=1; l) = w$

$P_1x_1(p,w)+p_2x_2(p,w)+\ldots+p_lx_l(p,w)=0$

Differentiate with respect to $p_k$:

$P_1. \partial x_1/\partial p_k + P_2. \partial x_2/\partial p_k + \ldots + x_k(p,w) + P_k. \partial x_k/\partial p_k + \ldots + P_l. \partial x_l/\partial p_k = 0$

$\Sigma(P_l. \partial x_l/\partial p_k; l=1; L) + x_k(p,w) = 0$

Implication: If the wealth remain the same if you change the prices the consumer will make the change into his consumption so that total expansion would not be changed.

2. Differentiate p.x(p,w)=w w.r.t (with respect to) w

   Proof: if x(p,w) satisfies then for the Walras law then for all pa and w:

   $\Sigma(p_l.\partial x_1/\partial w, l=1; L)=1$ (it is some of expenditure)

   When demand is going to change of all commodities, the total expenditure would be totally equal to the wealth change.

$\prec$

- Demand function and the relationship with weak axiom of revealed preference
- Link b/w WARP and utility function
- Use utility function and use in optimization to find optimal position

Last time we discussed:

- the relationship b/w choice structure (WARP), and rational decision , that all budget sets until three elements should have been supported
- the we talked about Walrasian demand function
- two important properties, which was homogeneity of degree zero and Walras law, and convexity
- then we did comparative statistics

Today:

The relationship b/w WARP and its implication for demand function

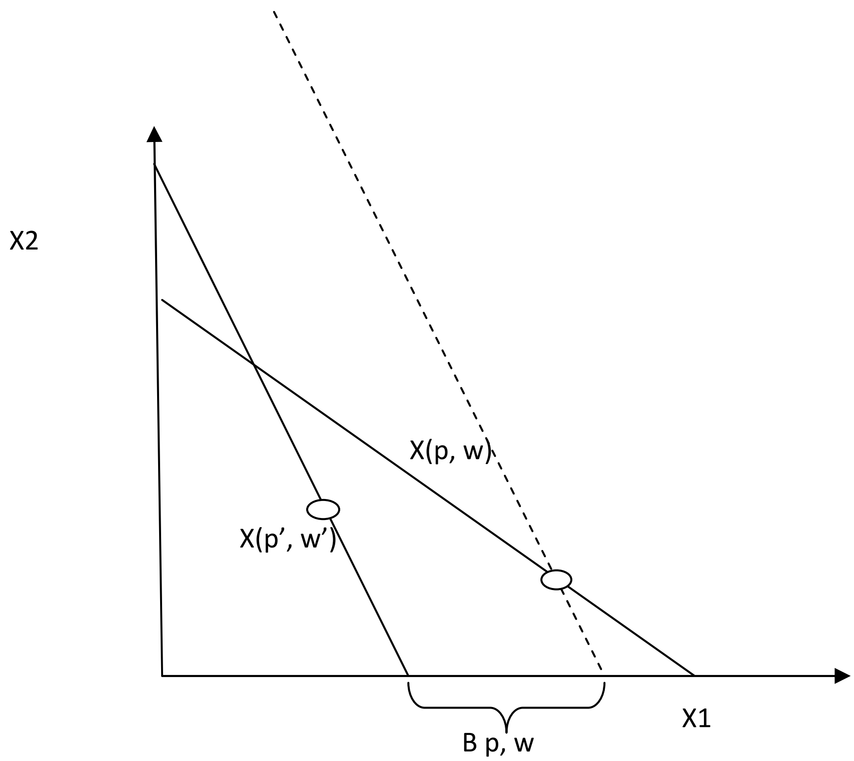
2.5. The WARP and the law of demand.

Definition: The Walrasian Demand Function (W.D.F) X(p,w) satisfies the WARP if for any two price and wealth situations (p,w), and (p',w'), we have if p.x(p',w')< w, and x(p',w') $\neq$ x(p,w) then p'.x(p,w)>w'

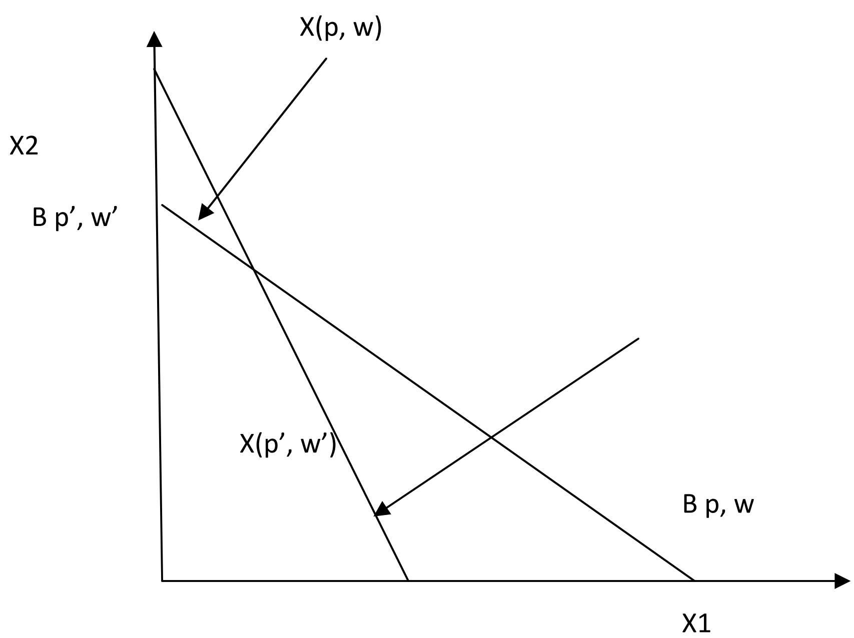
We introduce two budget sets B p,w, B p',w'

We are going to check the availability (it means it is lower than wealth (w))

L=2




Following does not satisfied the weak axiom of revealed preference:




Definition: **Sluthkey wealth compensation**: we start with price wealth pair (p,w) -> x(p,w)

Then for Walras law p.x(p,w)=w

X is consumption bundle (x1 (p1,w), x2(p2,w),…, xl (pl, wl))

Price is changed from p→ p'

How much we need to adjust person wealth so that the original consumption would be affordable?

P'.x(p,w) this is the money that you need to give him.

So the amount of wealth adjustment: P'.x(p,w)-w=p'x(p,w)-p.x(p,w)=(p-p').x(p,w)= $\Delta$ p.x(p,w)

( $\Delta$ p.x(p,w)) is Slutskey wealth compensation. This is giving them money or taking money from them.

You cannot say that increase the wealth with the same percentage, since different prices change differently **(Since the consumption is a vector)**

**Definition**: Compensated price change: refers to the price changes that are accompanied by Slutskey wealth compensation.

$\Delta$ p= (p'-p)

$\Delta$ w=(p'-p).x(p, w)

Change of them at the same time is called compensated price change.

**Proposition**: The Walrasian Demand Function is Homogeneity of Degree zero and satisfies Walras Law, then the demand function x(p, w) satisfies the weak axiom of revealed preference if and only if the following is satisfied:

For any compensated price change from (p,w) to (p',w') =(p', p'.x(p,w))

(*) (p'-p).[(x(p',w')-x(p,w)]<=0

And <0 if x(p',w') $\neq$ x(p,w)

Proof: (1) The WARP →

(p'-p).[(x(p',w')-x(p,w)]= p'(x(p',w')-x(p,w))- p(x(p',w')-x(p,w))= w'-w'- p(x(p',w')-x(p,w))= - p(x(p',w')-x(p,w))

Since this is compensated price change it means the original consumption bundle would be available so p'.x(p,w) would be available as well.

The weak axiom of revealed preference implies that x'(p',w') should have been affordable but not chosen, as a result x(p',w') should not have been affordable, and that means p.x(p',w') > p.x(p,w), and this means p(x(p',w')-x(p,w)) >0

Now we are going after the reverse, we assume that (p'-p).[(x(p',w')-x(p,w)]<=0 satisfies, but the weak axiom of revealed preference is not satisfied.

Look at compensated price change from (p',w') to (p,w) such that $x(p',w') \neq x(p,w)$

$P.x(p',w') = w$ and $P'.x(p,w) \leq w'$

This means the second one is revealed preference to the first one, but according to definition it <u>should not have been affordable</u>.

$W'=p'.x(p',w') \Rightarrow p.x(p',w') \leq p'.x(p',w') \Rightarrow p.x(p',w') - p'.x(p',w') \leq 0 \Rightarrow p'(x(',w')-x(p,w)) >= 0$

On the other hand we had $p(x(p',w')-x(p,w))=0 \Rightarrow (p'-p)(x(p',w')-x(p,w))>=0$

Compensated Law of demand: $\Delta p.\ \Delta x<=0$

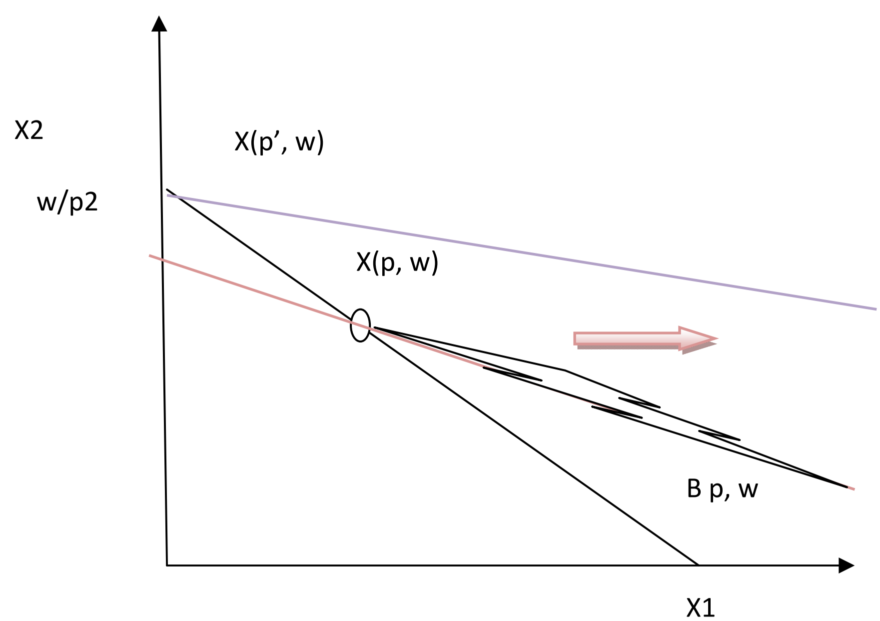
$\Delta p=p'-p$

$\Delta x=x(p',w')-x(p,w)$

$\Delta p=(0,0,\ldots., \Delta pl , 0,0,\ldots,0)$

$\Delta x=(\Delta x1,\ \Delta x2,\ \ldots.,\ \Delta x\ l,\ ,\ldots,\ \Delta xl)$

$\Rightarrow \Delta p.\ \Delta x= \Delta p\ l.x\ l <0$




- If x(p,w) is differentiable:
  Start with (p,w) let dp be the compensated price change. dw= dp. X(p,w)= X T (p,w).dp
  The compensated law of demand => dp .dx <=0

dx= Dp X(p,w).dp + D w X(p,w).dw = [ D p X (p,w).dp+ D w x T(p,w).X (p,w).dp]= [ D p x (p,w).D+ D w x T(p,w)]dp

dp. dx= dp [ D P x(p,w)+ D w X T Op,w)]dp

dp= [ d p1, d p2,…, d p l] part of R L

the term [ D P x(p,w)+ D w X T Op,w)] is **Slutsky substitute matrix**. This matrix is **Negative Semidefinite Matrix**.

for compensated price change it should be zero.

$$S(p,w)=\begin{bmatrix} s11 & s12 & S1l \\ S21 & S22 & S2l \\ \\ SL1 & SL2 & SLL \end{bmatrix}$$

S lk= $\partial$ x l (p,w)/ $\partial$ p k + $\partial$ x2(p,w)/ $\partial$ w . x l (p,w)

1.3. classical demand theory

3.1. preference relations

"$\succ$=" X c R L +

Φ rational preference

**Desirability**: more is preferred to less

**Convexity**: concerns the trade off that the consumer is willing to make among different goods

Y >> x => $y_l > x_l$ for all l=1,2, …, l

y $\geq$ x => No $y_l < x_l$ but some $y_l > x_l$

Definition: $\succ$= is monotone if x, y$\in$X implies that y$\succ$ x if y$\succ\succ$x

It is *<u>strongly monotone</u>* if x,y$\in$X implies that y$\succ$x if y$\succ$=x

- With theory model with *no restriction* you can draw no result, and then you go to put restriction, but putting restriction will result in *trivial* answer that does not work, so we *relax* some of them to derive the result

Def: Local non satiation

$\succ$= on X is *<u>locally nonsatiation</u>* if for any x$\in$X every $\varepsilon$>0 there is y$\in$X so that ||y-x||<$\varepsilon$ and y $\succ$x.

For any x$\in$X we define indifference set: {x$\in$X: y ~x}

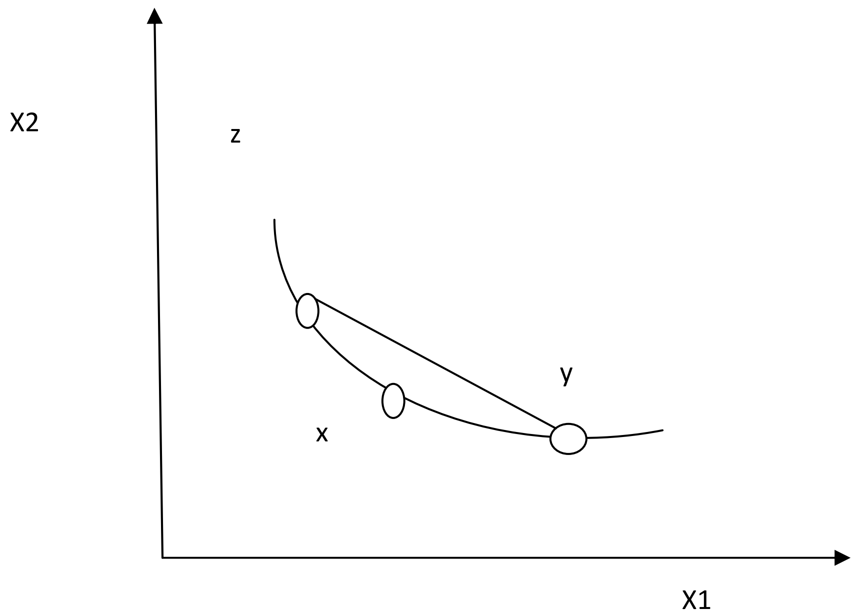
*<u>Upper contour set</u>*: {y$\in$X, y $\succ$=x}

*<u>Lower contour set</u>*: {y$\in$X: x$\succ$=y}

Convexity:

Definition: $\succ$= on X is convex if for every x$\in$X the upper contour set {y$\in$X: y$\succ$=x} is convex.

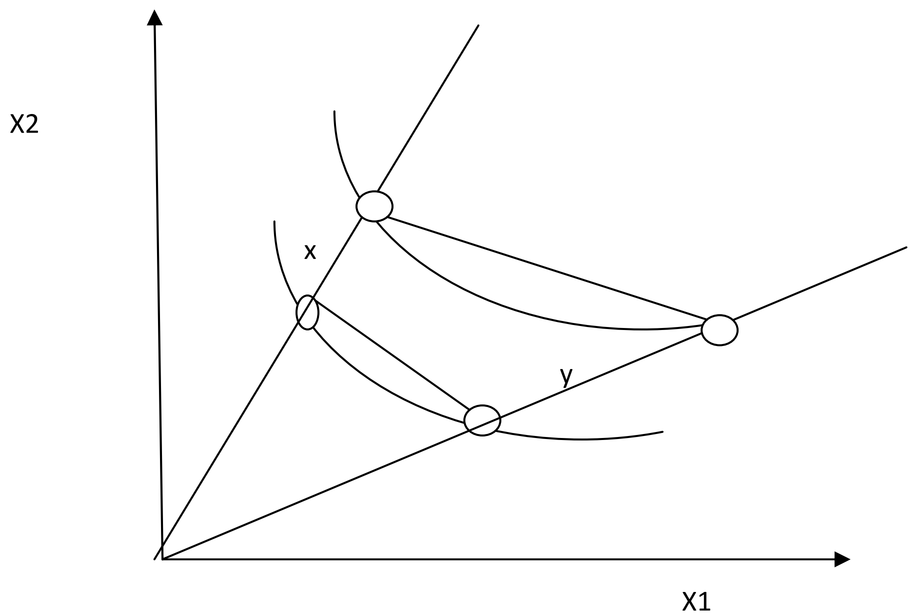
That is, $y \succ= x$ and $z \succ= x$ implies $\alpha y + (1-\alpha) z \succ= x$ for any $\alpha \in [0,1]$




Strictly convex: is the state that any combination is strictly preferred to the x.

Definition: *Homothetic* preference.

Suppose that we have $\succ=$ on X is homothetic if all indifference sets are related by *proportional expansion* along rays that is , if $x \sim y$, then $\alpha x \sim \alpha y$ for all $\alpha >= 0$




Quasi Linear properties:

Definition: $\succ$ = on x= (-∞, +∞) * R Y l is **quasilinear** w.r.t to y and l if:

(1) All indifference sets are *parallel* (displacement) of each other along the axis of commodity 1. That is x~y, then x+ $\alpha\varepsilon_1 \sim y + \alpha\varepsilon_1$
$\varepsilon$=(1,0,…,0), $\alpha \in R$
(2) Good is desirable. That is x+ $\alpha\ \varepsilon_1 \succ x$ for $\alpha$>0

*Means more money is better.*

Assignment will be posed.

$\leq$

Review last time:

- Started with implication of weak axiom of revealed preference for demand function
- Desirability of utility function was also defined
- Convexity: what trade of combination exists between commodities

Today:

- Introduce another assumption which is important
- Utility function actually was going to be continuous
- Last time preference was not continuous, and we were not able to define utility function which was not continuous

---

3.2. preference and utility function:

Definition: $\succ=$ is continuous if it presents

For any sequence of pairs {(x$^n$, y$^{n)}$ }, n=1; ∞

With x$^n$ $\succ=$ y$^n$ for all n, x=$\lim\limits_{n->\infty} x^n$ and y=$\lim\limits_{n->\infty} y^n$, we have x x=$\succ=$ y

Alternative definition: for all α, the upper contour set { x=$y \in X; y \succ= x\}$ and the lower contour set$\{y \in X; x \succ= y\}$ are both closed.

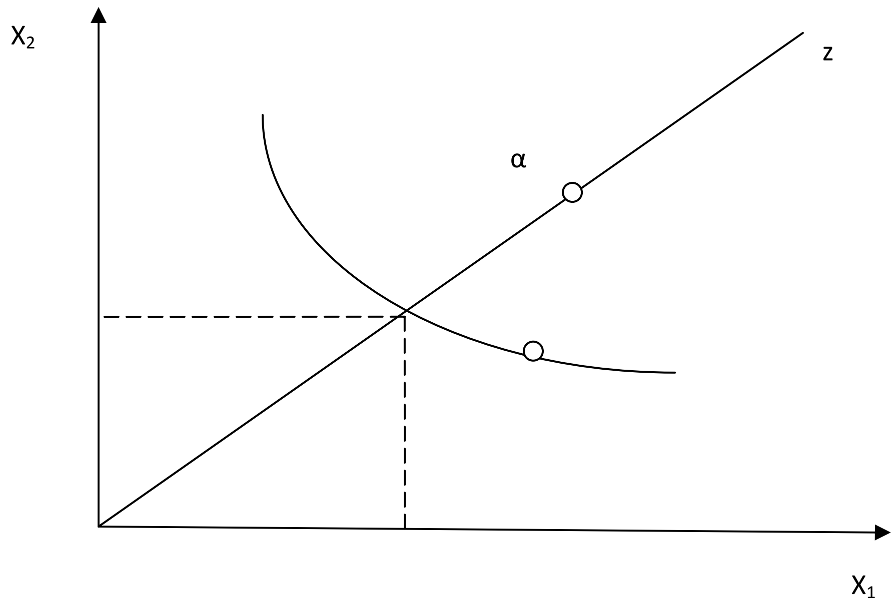
Proposition: suppose $\succ=$ on X is continuous, then there exists a continuous utility function u(x) that represented $\succ=$

Proof: denote a diagonal ray in $R^L{}_t$ by z for e=(1,1,…,1) αe$\in$Z

For any consumption bundle $x \in X$ monotonicity implies that x$\succ=$0 (having something is better than nothing)

Also not that for any $\hat{\alpha}$ including $\hat{\alpha}$ $\succ\succ$ x, then e$\hat{\alpha}$ $\hat{\alpha}$ e $\succ=$ x

$\hat{\alpha}$ e $\succ=$ x $\succ=$ 0




By monotonicity and continuity we can find α* such that α*e~x => u(x)=αx

- Previously at 90s economist tried to use the most sophisticated tools, however the trend is changed now and now behavior economic and empirical analysis is hot and they only try to be users and not invertors that compete with mathematician

Next, need to prove for any $x, y \in X$ if and only if u(x) $\succ=$ u(y)⇔ $x \succ= y$

1. u(x) $\succ=$ u(y)=> $x \succ= y$
   u(x)=α(x)
   u(y)=α(y)
   α(x)e α(y)e
   u(x)>=u(y) => α(x) $\succ=$ α(y)
   α(x)e $\succ=$ α(y)e
   x $\succ=$ y (transitivity)
2. x $\succ=$ y => u(x)>=u(y)
   x~α(x)e
   y~α(y)e
   α(x) $\succ=$ α(y)

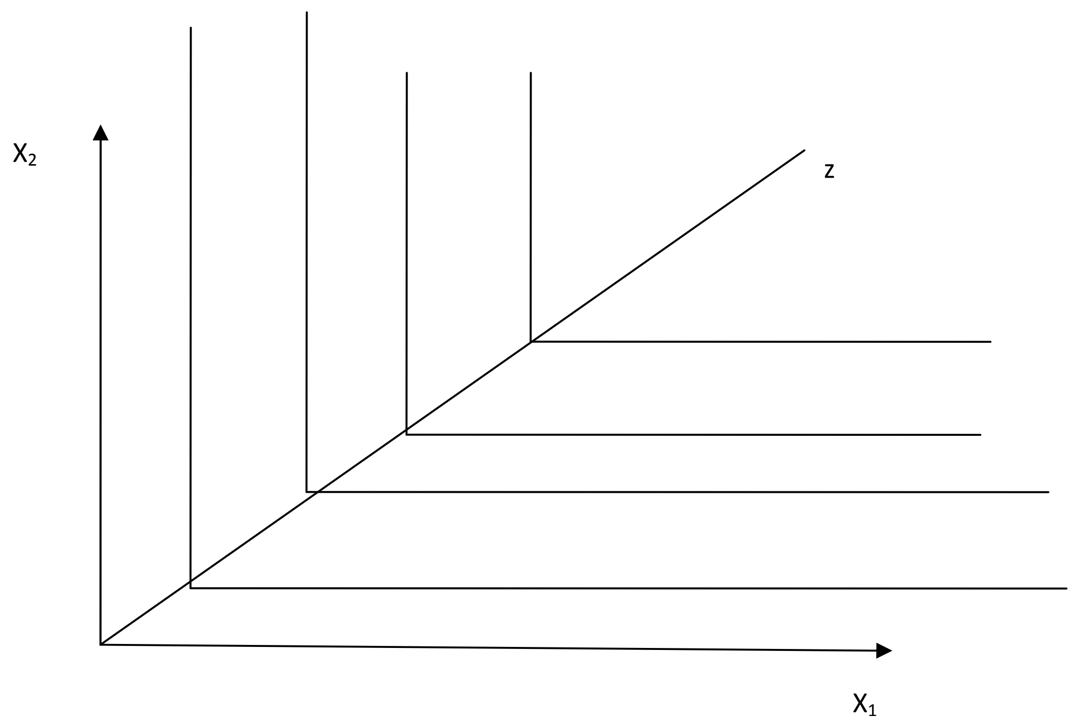
⇨ u(x) $\succ=$ u(y)

u(x) is assumed to be differentiable in most of the cases but there are exceptions.

Leonteif preference (one exception)

L=2

U(x1,x2)= min {x1,x2} // two commodities and utility will be determined by whichever value is //smaller



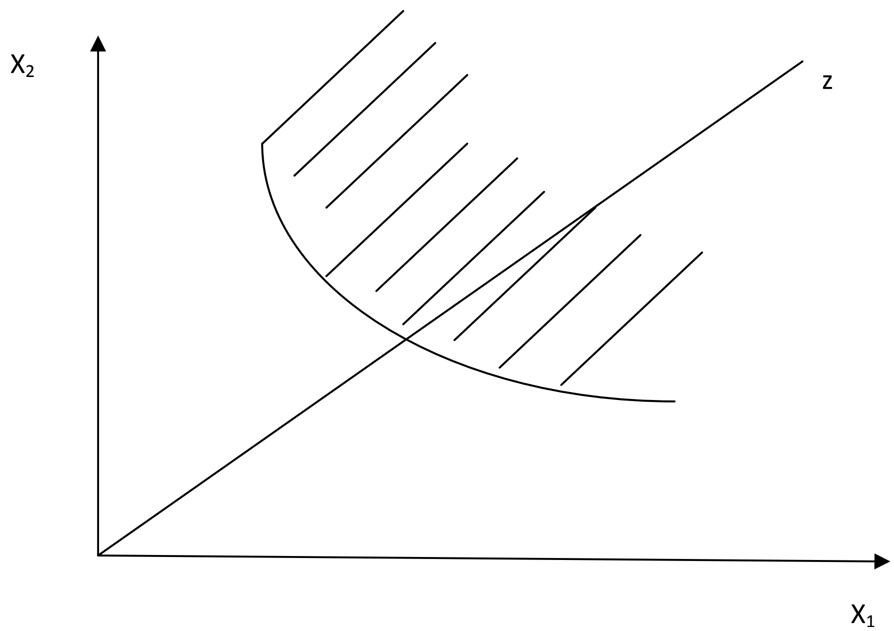

Concavity of the utility function.

Definition: Convexity of preferences implies that u(.) is quasi concave. That is the set $\{y \in R^l, u(y) >= u(x)\}$ is convext for all x or of u(αx+(1-α)y) >= min (u(x),u(y) )

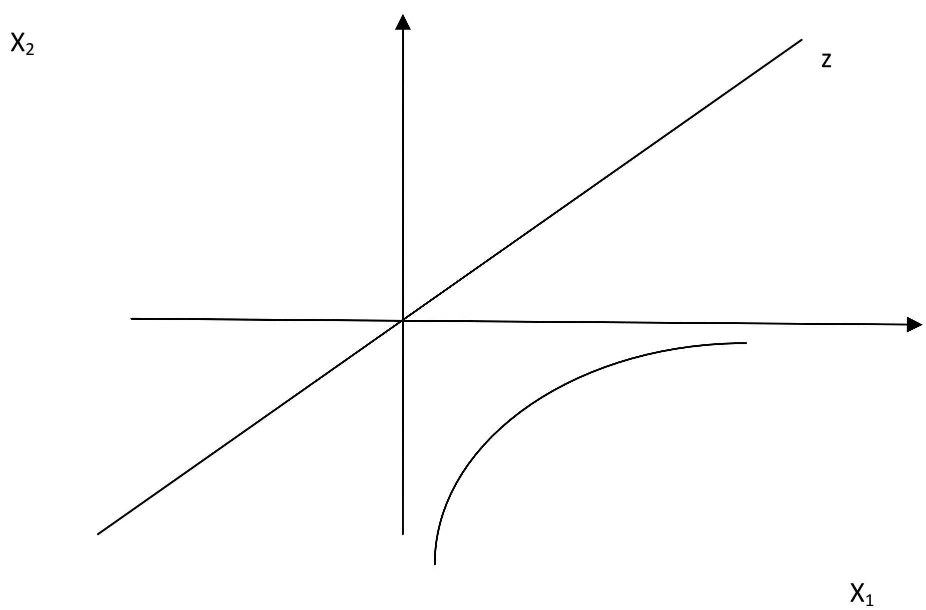
For any x,y, and $\alpha \in [0,1]$

Strictly quasi concave: if u(αx+(1-α)y) > min (u(x),u(y) )

The previous set was upper contour set.




Following is the case of strictly quasi concave:




3.3 The utility Maximization problem (UMP)

The consumer's problem would be:

$\max_{x>=0} \quad u(x)$ (this is monotonicity condition)

s. t. p. x <=w

(price vector ) . consumption bundle <= wealth

proposition: if D>>0 and u(.) is continuous then the U.M.P has the solution

Proof: Given that u(.) is continuous we need to show that the budget set is compact.

(compact set is 1. bounded and 2. closed)

We know that the budget set is ***closed*** because it includes the boundary, but we need to show that it is ***bounded***. The maximum amount of commodity of good l can be purchased is w/p(l), l=1,2,…, l

As a result the budget set is bounded, so since it is bounded and closed then it is compact.

⇨ The budget set is compact

There is a solution to the U.M.P.

The solution to the U.M.P in X(p,w) W.D.C

Properties of Walrasian Demand Correspondence.

Proof: suppose that x(p,w) is the solution to the problem of u.m.p. it has following properties

(i) Homogeneity of degree zero in (p,w)

X(p,w)=x(αp,αw) for α>0

(ii) Walrasian law: p.x=w for all x $\in x(p,w)$

(iii) Convexity and uniqueness:

If the preference is convex such that utility function u(.) is *quasi concave* then W.D.C. is convex set. If $\succ=$ is strictly convex then x(p,w) is single valued.

Proof: First one is easy to prove since it is like the previous proposition and proof.

(ii) if p.x<w for some $x \in X(p,w)$

Then according to local *non-satiation*, there must exist a bundle y very close to x such that p.y<w and y $\succ$ x.

Implies that u(y) > u(x) contradicts that x is the solution to the utility maximization problem. It has to be true that p.x=w

Proof: consider two bundles x,x’ $\in x(p,w)$ then u(x)=u(x’)=u* (which is maximized utility level)

We have to show that any combination of x, x’ would be in that as well.

Consider x'': αx+(1-α)x' for $\alpha \in [0,1]$. By quasiconcavity of u(.) we have (u(x'') = u(αx+(1-α)x')>= min (u(x), u(x'))

U(x'')>u*

Now we need to check whether it is affordable point.

Px''= p(αx+(1-α)x')=αw+(1-α)w=w

⇨ X'' $\in x(p, w)$

Strictly quasicancavity $\alpha \in (0,1)$

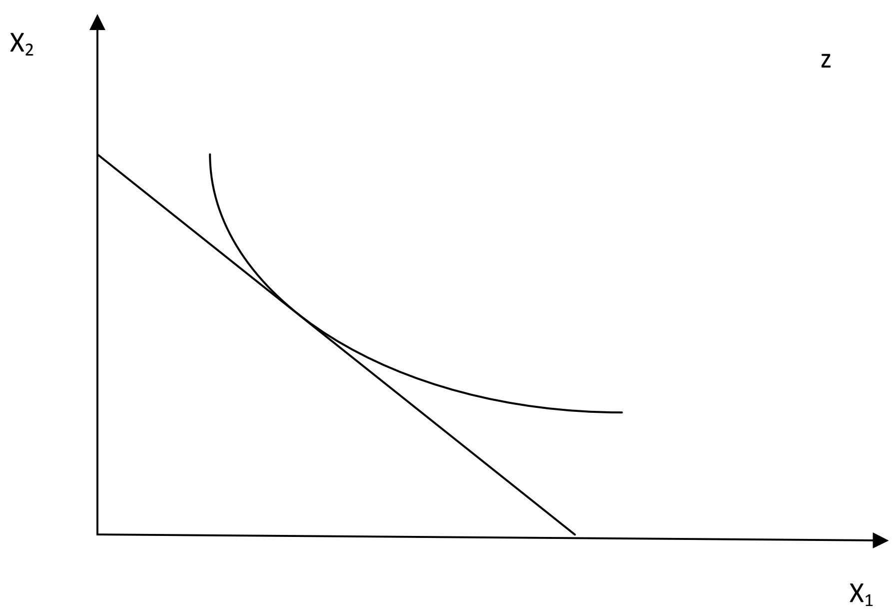
(u(x'') = u(αx+(1-α)x')> min (u(x), u(x'))

U(x'')>u*

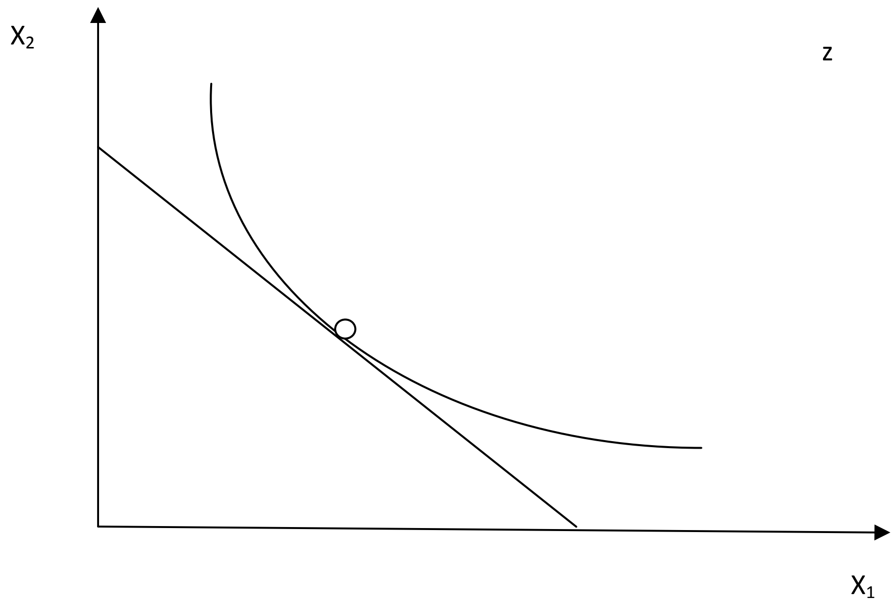
Px''=w=> contradiction

This means the original points x, x' did not satisfy Walrasian law, and that means contradiction, and the only state would be x=x'=x''

In the following budget set and indifference curve is more than one point, and therefore it would be Walrasian demand correspondence (in contrast to set)




The following is the Walrasian function due to one point of solution:




If there were two points it means at one point you violated Walrasian law and you need to push it upward to get one point of intersection.

To find the solution:

L=u(x)+λ(w-p.x)

$\frac{\partial L}{\partial x_l}$ for l=1,2,…, l let x* be the solution

$\frac{\partial u(x^*)}{\partial x_l}$ <=λp $_l$ = λ $p_l$ if $x^*_l$ >0

The marginal utility vector: $\nabla u(x) = [\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, \ldots, \frac{\partial u(x)}{\partial x_1}]$

X*. ( $\nabla u(x^*) - \delta p$ ) =0 Kuln-Fuckels condition

If $x_l$*>0 ; $\frac{\partial u(x^*)}{\partial x_l}$ =λ$p_l$ , l=1,2,…, l

Interpretation of λ:

λ is marginal utility of the wealth

$\nabla u(x(p,w))$ $D_w$ x(p,w)

$\nabla u(x(p,w)) = \lambda p$

$\nabla u(x(p,w))$ $D_w$ x(p,w) = λ

Example: coup-Douglas utility function :

$u(x_1, x_2) = k\, x_1^{\alpha} x_2^{1-\alpha}$

$\ln u(x_1, x_2) = \ln k + \alpha . \operatorname{Ln}(x_1) + (1-\alpha) . \operatorname{Ln}(x_2)$

$$\frac{\partial \ln(u)}{\partial x_1} = \frac{\alpha}{x_1} = \lambda p_1$$

$$\frac{\partial \ln(u)}{\partial x_2} = \frac{1-\alpha}{x_2} = \lambda p_2$$

$X_1(p,w)=\alpha w/p_1$

$X_2(p,w)=(1-\alpha)w/p_2$

X(p,w) a solution the U.M.P

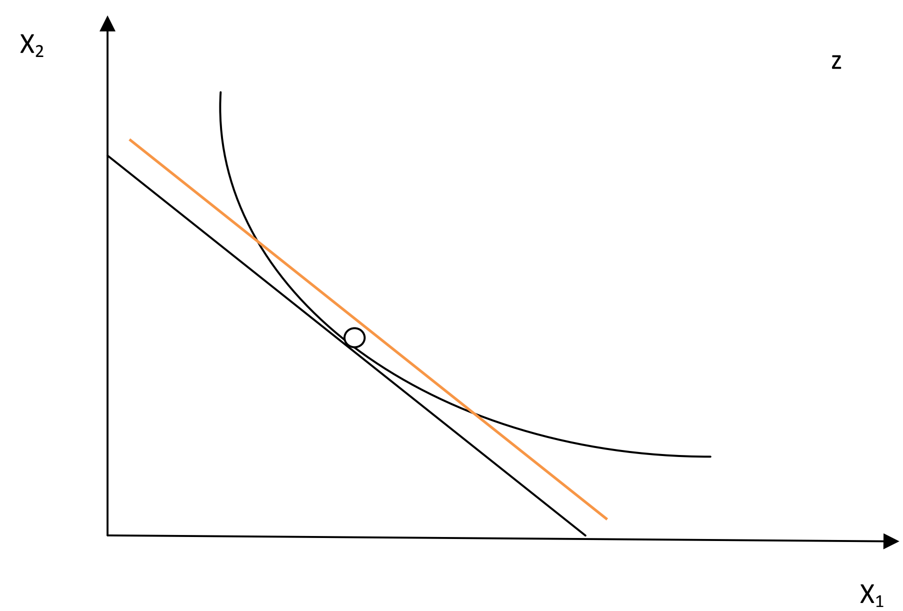
➔ U(x)

V(p,w)=U(x(p,w)) is called indirect utility function (I.U.F)

The properties of Indirect Utility function (IUF), v(p,w):

(i) Homogeneity of degree zero in (p,w)
(ii) Strictly increasing in w and non-increasing in $p_l$ for all l=1,2, …, l
(iii) Quasiconvex: that is the set $\{(p,w): V(p,w) <= \bar{v}\}$ is convex set for any $\bar{v}$
(iv) Continuous in p and w

Proof:



(iii) suppose that v(p,w)<= $\bar{v}$ and v(p',w') <= $\bar{v}$

Let p''=αp+(1-α)p' and w''= αw+(1-α)w' for any α $\in$ [0,1]

We need to show that v(p'',w'')<= $\bar{v}$

Consider any consumption bundle in (p'',w'')

If $x \in B_{p'',w''}$ then p''.x<=w''

αp.x+(1-α)p'.x<=αw+(1-α)w'h

- ⇨ X is either in B $_{p,w}$ or B $_{p',w'}$ or bot
- ⇨ V(p'',w'')<= $\bar{v}$
- ⇨ (p'',w'') $\in$ {(p,w): V(p,w)<= $\bar{v}$}

Find the I.U.F for coup-Douglas Utility function.

$X_1(p,w)=\alpha w/p_1$

$X_2(p,w)=(1-\alpha)/p_2$

V(p,w)=u(x(p,w))=log k + αlog $x_1$(p,w)+ (1-α)log $x_2$(p,w)= ln k + [α ln α + (1-α) ln (1-α)]+ ln w – α ln p1 – (1-α) ln p2

$\leq$

- Last time we discussed utility maximization problem, that told us given the budget constraint, then he is gonna pick the consumption bundle that will maximize his utility
- From that two relevant objects were discussed, that quasilinear function if multiple solution in it
- If there was one it would be walrasian demand function
- Whether it is one or two, depends on preferences
- If it is concave there would be multiple solutions
- If it would be strictly concave then we would have unique walrasian demand function
- The second object that we obtained from utility maximization was the objective function itself, which is actually optimized is called indirect utility function (the utility function hat is optimized by given price and wealth)
- Propertieis of indirect utility function preoperties, and the properties to the solution of utility function were also discussed
- What is that the decision maker doing here, the decision maker first looks for the budget set that is available and then tries to basically identify the consumption bundle that gives him until the satisfaction level; we do have the budget set, and then we try to push out the level of satisfaction until we see that this is the highest level of satisfaction that we can take
- Different perspective: suppose I know what I want to take. I am asking a different question: what kind of wealth I need to achieve that level of satisfaction or standard of living that helps me to reach there with the lowest cost
- Given the wealth level what would be the highest satisfaction level was the first question, now we have the satisfaction level and we are looking for minimizing the cost ; these are the same questions, and the second one is called expenditure minimization problem.

---

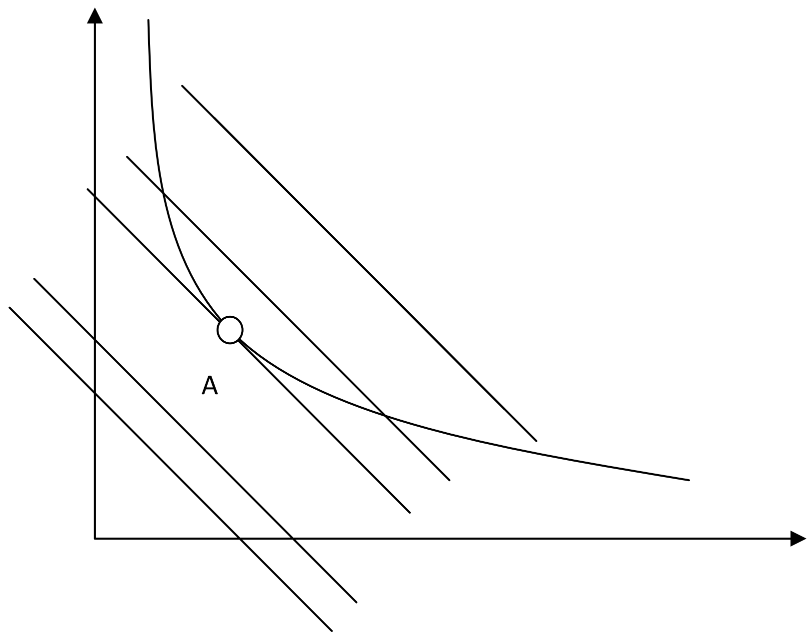
**The expenditure minimization problem (EMP)**

Min p.x = $\sum_{l=1}^{L} x_l p_l$

We need to satisfy the constrain that $u(x) \geq u$

Minimize the expenditure is one thing, and then solution to this problem which is called Hicksian demand correspondence/function.

Indirect utility function maximizes utility level, and we have hexian demand function for solution to this problem.




Apparently the only one that allows you to achieve that is the point A, which is corresponds to the minimum level.

We have different approaches from different perspective, and we call it duality (there is relationship between them)

Relationship between the u.m.p and the e.m.p:

Proposition: Suppose that u(.) is continuous and satisfies all the conditions. P>>0 (means every single price is positive)

(i) If x* is optimal in the u.m.p for w>0 then the same x* is optimal in the E.M.P when the required utility is u(x*), and the minimize expenditure is p.x*=w

(ii) If x* is optimal in the expenditure minimization problem when the required utility level is u > u(0) (greater than consuming nothing) then x* is also optimal in the utility minimization problem when wealth is p.x* and the maximized utility is u. i.e. v(p,p.x*)=u.

Proof:

(i) Suppose that x* is not optimal in the expenditure minimization problem (it is optimal in the utility maximization problem) with utility level u(x*) then there exists x' such that u(x')>=u(x*), and p.x'<p.x*. By local nonsatiation, there exists x'', very close to x'such that u(x'')> u(x') and p.x'' < p.x* (still affordable). Since u(x')>= u(x*), it leads x''>x*, and this is a problem since x* were optimal (this is contradiction). By Walras Law, p.x*=w

(ii) Suppose that x* is not optimal in the utility maximization problem (u.m.p) when wealth is p.x* then there exists x' such that u(x') > u(x*) and p.x' <= p.x*. Consider x'' =αx' for α$\in (0,1)$, by continuity u(x'') >u (x*) and p.x''< p.x* suggesting contradiction since we found a solution is better than optimal which is not possible.

Expenditure function is: e(p,w)=p.x*= $\sum_{l=1}^{L} x^*_l \; p_l$

The expenditure function properties:

Proposition: suppose that u(.) is continuous. The expenditure function e(p,u) satisfies:

(i) Homogeneous of degree 1 in p. e(αp,w)=αe(p,u)

(ii) Strictly increasing in u and non-decreasing in $p_l$ for l=1,…, L

(iii) Concave in p

(iv) Continuous in p and u.

Proof:

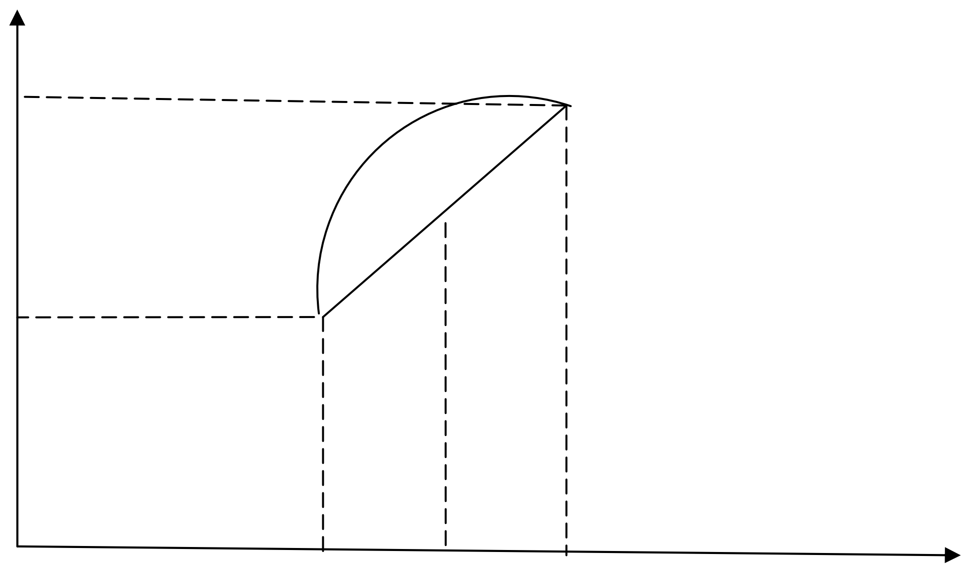
(i) Min αp.x = $\sum_{l=1}^{L} \alpha x_l p_l$ ; e(αp,w)=αp.x*=αe(p,u)

(ii) Suppose that e (p, u) is not strictly increasing in u. let x' and x'' denote optimal consumption bundles for required utility levels u' and u'', and u''> u', and p.x''<=p.x' (we need to prove that px''>px'). Let $\tilde{x} = \alpha x'', \alpha \in (0,1)$, and α is close to 1. By continuity, $u(\tilde{x}) > x'$ and p. $\tilde{x}$ <p. x'. here we had the problem since x' should have been the optimal solution, and here we found another solution that costs less and provides higher utility. This would suggest that x' is not optimal under then E.M.P. and this is contradiction.

Suppose that p'', and p' such that $p''_l > p'_l$ and for the others $p''_k = p'_k$ for $k \neq l$.

Let x'' be the optimal for the expenditure minimization problem for price p''. Then e.(p'', u)=p''.x'' $\geq$ p'.x'' $\geq$ e(p',u) => e(p'',u) $\geq$ e(p',u): this will happen in the case that the customer is on the corner solution.

Definition: a function is concave if for x', and x'', and f(αx'+ (1-α) x'') $\geq$ αf(x')+(1-α)f(x'') for $\alpha \in [0,1]$ then f (.) is concave.



αx'+ (1-α) x''

Proof: (iii) fix $\bar{u}$, let p''=αp+ (1-α)p' for $\alpha \in [0,1]$, suppose that x' is optimal in the E.M.P for p'' then e(p'', $\bar{u}$)=p''.x''=αp.x''+(1-α)p'.x'' ; the first term is $\geq$ e(p, $\bar{u}$), since it is not optimal on p (it is on p''), the same happens for the second term and it is $\geq$ e(p', $\bar{u}$). Therefore: e(p'', $\bar{u}$)= p''.x''=αp.x''+(1-α)p'.x''> α e(p, $\bar{u}$)+(1-α) e(p', $\bar{u}$).

**Hicksian demand correspondence**

h (p,u) is the solution to E.M.P

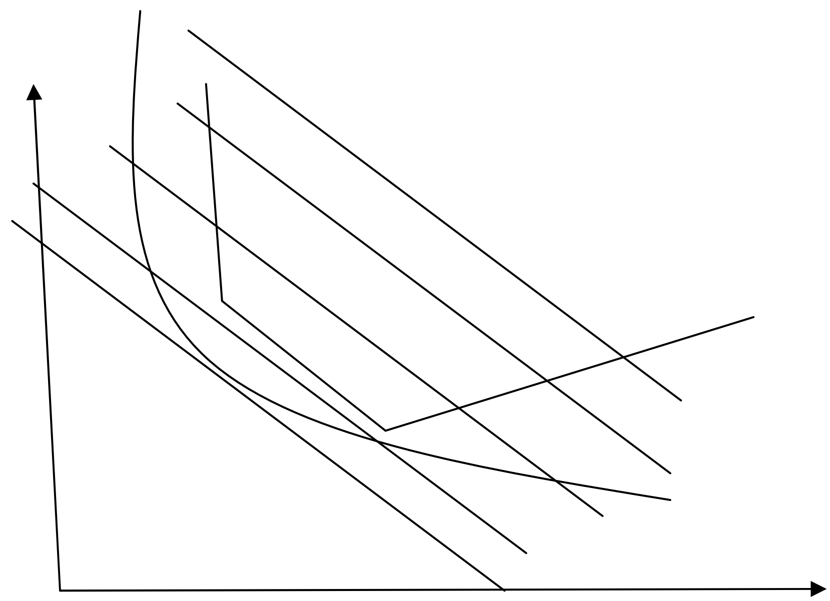
proposition: h(p,u) has two following properties:

(i) Homogeneity of degree zero in p.

h(αp,u)=h(p,u) for α>0 and any p, u

(ii) No excess utility. $x \in h(p,u), u(x) = u$.

(iii) Convexity/ uniqueness: if $\succ$ is convex, then h(p,u) is also convex set, and if $\succ$ is strictly convex, h(p,u) is single valued/unique.



Proof: (ii) suppose there exists $x \in h(p,u)$ such that u(x)>u. [The condition was solving u=p.x, so that u(x)>=u. ] Consider x' = αx, $\alpha \in (0,1)$, by continuity, for α close to 1, u(x')>u and p.x' < p.x= e(p,u) and this is contradiction. Which means our assumption could not have happen.

The E.M.P

Find p.x

So that u(x)>=u

L=p.x +λ (u(x)-u)

First condition: p>=λ $\lambda \nabla u(x^*) = \lambda \nabla u(x^*)$ if x*>0

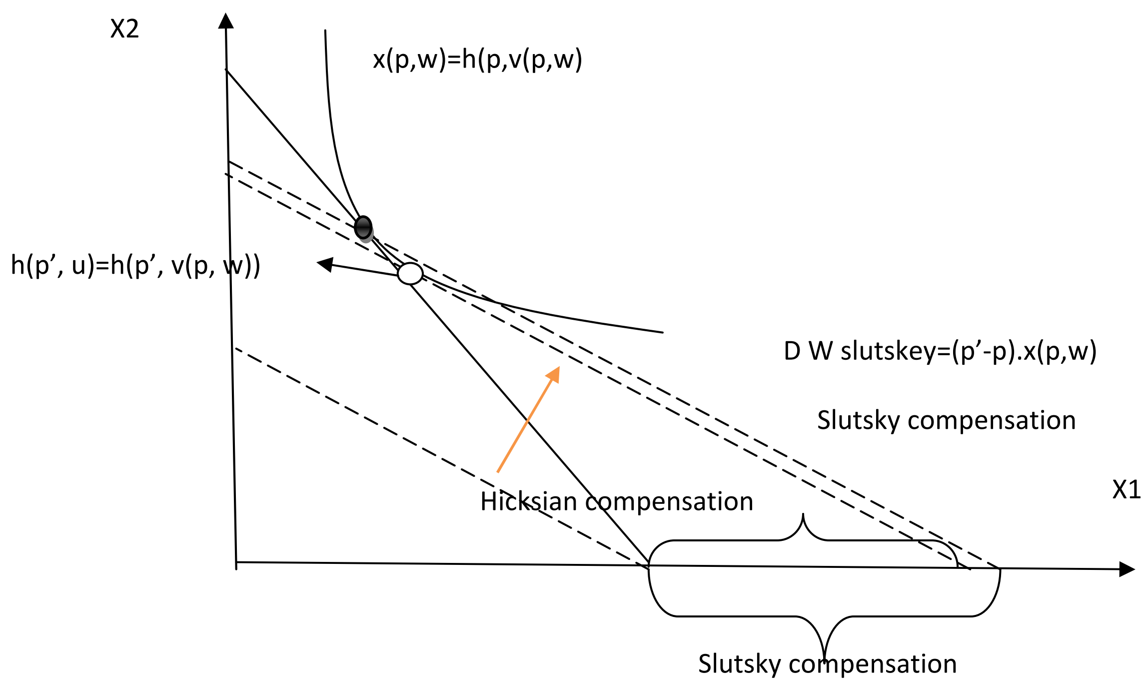
X*[p- $\lambda \nabla u(x^*)$ ]=0

$$\nabla u(u^*) = \begin{bmatrix} \partial u(u^*)/\partial x_1 \\ \\ \\ \partial u(u^*)/\partial x_l \end{bmatrix}$$

Correspondence between walrasian demand law and Hisdican demand correspondence

> h(p,u) = x(p,e(p,u)): solution to utility maximization the minimum wealth that allows to achieve the utility level u
>
> x(p,w)=h(p,v(p,w)): utility level that you have to achieve the maximum utility, and find solution on expenditure problem, then the solution would be the solution to utility maximization problem

h(p,u) = x(p,e(p,u)): in contrast to slutsky substitution matrix, iw now Hicksian compensation




These compensations have implication in adjusting consumer price index (CPI). Since, customers usually configure his consumption bundle based on change in prices, Hicksian compensation suits better. The money that government will give can be hicksian compensation or Slutskey compensation.

The relationship between Hicksian demand function and compensated law of demand.

Proposition: The Hicksian demand function h(p,u) satisfies the compensated law of demand.

For all p', and p'':

(P''-p').[h(p'',u)-h(p',u)]<=0 (u is the same for both h (.,.) and this means the compensation since the utility level is preserved

Proof: for every p>>0, h(p,u) is optimal in the E.M.P

p''.h(p'', u)<= p". h(p', u)

p'.h(p'', u) >= p'. h(p' ,u)

if we do the subtraction from both side we will have:

(p''-p').h(p'',u)<= (p''-p').h(p',u)

(p''-p').(h(p'',u)-h(p',u))<=0

This satisfies the compensated law of demand for Hicksian function.

$\Delta p = [0,0,...,\Delta p_l, 0,0,0,0]$ => $\Delta p_l . \Delta h_l \leq 0$ not always true for walrasian

Example: Coup Douglas utility function :

$U(x_1,x_2)= x_1^{\alpha} x_2^{1-\alpha}$, $\alpha \in (0,1)$

H(p,u) and e(p,u) find h(p,u) & e(p,u)

$\underset{x1,x2}{Min} \quad p.x = p_1 x_1 + p_2 x_2$ so that $x_1^{\alpha} x_2^{1-\alpha} >= u$

$P_1 = \lambda \frac{\partial u}{\partial x_1}$

$P_2 = \lambda \frac{\partial u}{\partial x_2}$

$$\frac{P_1}{P_2} = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}} = \frac{\alpha}{1-\alpha}\frac{x_2}{x_1}$$

$$x_2 = \frac{1-\alpha}{\alpha}\frac{P_2}{P_1} x_1$$

$$u = x_1^{\alpha} x_2^{1-\alpha}$$

$$x_1^{\alpha}\left(\frac{1-\alpha}{\alpha}\frac{P_1}{P_2} x_1\right)^{1-\alpha} = u$$

$$h_1(p,u) = \left[\frac{\alpha}{1-\alpha}\frac{p_2}{p_1}\right]^{1-\alpha} u$$

$$h_2(p,u) = \left[\frac{\alpha}{1-\alpha}\frac{p_1}{p_2}\right]^{-\alpha} u$$

Expenditure function=$p_1x_1+p_2x_2$=$p_1h_1(p,u)+p_2h_2(p,u)$= $\alpha^{-\alpha}(1-\alpha)^{1-\alpha} p_1^{\alpha} p_2^{1-\alpha} u$

$\leq$

- Focus of the exam would be on how to solve problems, proving axioms would not be asked, but it would be fair game to ask you find the demand function, indirect utility function and so on ( you need to understand theory papers as well)

- Risk says that I know the distribution, but I don't know the realization. Uncertainty says that you don't know what distribution is that exposes difficulty in decision. If you know the distribution you will make sure that expected values would be maximized. You will look at the worst case scenario that is mean return would be really low, that works against you. You going to get a very different outcome in comparison to the case that you get the average from other. Short sell vs. long sell decision. You will take the position no matter what your risk aversion is this. No matter how risky it is you will take a position. What is the implication of *preference* on uncertainty would be done in economic theory paper. Preference is not *everywhere differentiable.*

Brief review of what we had last time:

- Expenditure  minimization and relationship with utility maximization

- Slutskey substitution (original consumption affordable) vs. Hicksian substitution

- Indirect utility function and Walrasian demand function, and Hicksian demand function and the relationship between them

- Question is what is the relationship b/w Hicksian demand function, and Walrasian demand function and indirect utility function

Relationship of all those important concepts:

3.5 Relationship between demand indirect utility function and the expenditure function (demand, IUF, EF)

1. H.D.F (Hexian function) and E.F

2. H.D.F and W.D.F

3. W.D.F and I.U.F (indirect utility function)

Expenditure function define as minimum

$$e(p,u) = p.x^* = p.h(p,u)$$

Proposition: suppose that u(.) is a continuous utility function for all price (p) and u, we have: $h(p,u) = \nabla_p e(p,u)$ or

$$h_l(p,u) = \frac{\partial e(p,u)}{\partial p_l}, l = 1,\ldots,L$$

Proof: $\nabla_p e(p,u) = h(p,u) + [p.D_p h(p,u)]$

Hicksian demand function was homogeneity of degree zero for the price

So, from H.D.Z. for h(p, u) in p, we have

Differentiate w.r. to α and evaluate at α=1; we do this at point 1 so that we make $.D_p h(p,u)$ to zero.

h(αp,u)=h(p,u) holds for all α>0 => $p.D_p h(p,u)$=0 (Due to chain rule)

as a result we will have: $\nabla_p e(p,u) = h(p,u)$ which proves it

Properties of Hicksian demand function (HDf):

1. $D_p h(p,u) = D^2{}_p e(p,u)$
2. $D_p h(p,u)$ is a negative semi-definite Matrix; since expenditure function is concave in price p. That means the second derivative would be ***negative semi-definite***. Means: v. $D_p h(p,u)$ .v<=0 for any v (vector of $R^L$)
3. $D_p h(p,u)$ is symmetric, since it is second derivative.
4. $D_p h(p,u)$ .p=0

The ***compensated law*** of Demand (C.L.D) suggest that dp. dh(p,u)<=0

dp = [0,0,….,$dp_l$,0,0,..,0]

dp.dh(p,u)=$dp_l$.$dh_l$(p,u)<=0 (in general is not true since it holds only for compensation)

Suppose $p_k$ is changed now we are interested $\frac{\partial h_l(p,u)}{\partial p_k} > 0$: it is substitution of two goods

$\frac{\partial h_l(p,u)}{\partial p_k} < 0$: l and k are complements in this case

This is the different way of looking at this which would be cross price effect

Since $D_p h(p,u)$ is symmetric we can not have l have specific effect on k, and k have different effect on l

Proof for relationship b/w W.D.F and H.D.F

For all (p,w) and u=v(p,w) (i.e. maximum), we have

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,u)}{\partial p_k} + \frac{\partial x_l(p,u)}{\partial w} . x_k(p,w) \text{ for all l, k}$$

$$D_p h(p,u) = D_p x(p,u) + D_w x(p,u) x(p,w)^T$$

Proof: consider consumer facing ( $\overline{p}, \overline{w}$) and attaining $\overline{u}$ , when level is $\overline{w} = e(\overline{p}, \overline{w})$

For all (p, w), we have $h_l(p,u) = x_l(p, e(p,w))$

Differentiate w.r.t. $p_k$ .

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,e(p,w))}{\partial p_k} + \frac{\partial x_l(p,e(p,w))}{\partial w} \frac{e(\overline{p},\overline{w})}{\partial p_k}$$ using H.D.F and E.F relationship and duality

$$\frac{\partial h_l(\overline{p},\overline{u})}{\partial p_k} = \frac{\partial x_l(\overline{p},e(\overline{p},\overline{w}))}{\partial p_k} + \frac{\partial x_l(\overline{p},e(\overline{p},\overline{w}))}{\partial w} x_k(\overline{p},\overline{w})$$

$$\frac{\partial x_l(p,e(p,w))}{\partial p_k} = \frac{\partial h_l(p,u)}{\partial p_k} + [-\frac{\partial x_l(p,e(p,w))}{\partial w} x_k(\overline{p},\overline{w})]$$

$$\frac{\partial x_l(p,e(p,w))}{\partial p_k} = \frac{\partial h_l(p,u)}{\partial p_k} + [\frac{\partial x_l(p,e(p,w))}{\partial w} (-x_k(\overline{p},\overline{w}))]$$

Price effect Substitution effect wealth effect

Left side: price effect of Walrasian demand function. Would be equal to the first part which is substitution effect Plus the Walrasian demand multiply the total wealth change (the negative term)) and sum of two is called income effect.

Proof of the last one W.D.F and I.U.F:

(Roy's identity) I.U.F is differentiable. $(\overline{p},\overline{u})$ >>0 we have x $(\overline{p},\overline{u})$ = $\frac{1}{D_w v(\overline{p},\overline{w})} D_p v(\overline{p},\overline{w})$ or

$$x_l(\overline{p},\overline{u}) = \frac{\dfrac{\partial v(\overline{p},\overline{w})}{\partial p_l}}{\dfrac{\partial v(\overline{p},\overline{w})}{\partial w}} \text{ for all } l=1,\dots, L$$

Proof: For $\overline{u} = v(\overline{p},\overline{w})$; by duality, $v(p,\overset{\overline{w}}{e(p,\overline{u})}) = \overline{u}$ for all the prices p

Differentiate the above w.r.t p and evaluate at $\overline{p}$

$$D_p v(\overline{p},e(p,\overline{u})) + \frac{\partial v(\overline{p},e(p,\overline{u}))}{\partial w}\overset{h(\overline{p},\overline{u})}{D_p e(\overline{p},\overline{u})} = 0$$ ; using duality and the first E.F and H.D.F

$$D_p v(\overline{p},e(p,\overline{u})) + \frac{\partial v(\overline{p},e(p,\overline{u}))}{\partial w} x(\overline{p},\overline{u}) = 0$$

$$x(\overline{p},\overline{u}) = \frac{1}{\dfrac{\partial v(\overline{p},\overline{w})}{\partial w}} D_p v(\overline{p},e(p,\overline{u}))$$

In practice you will get the sense of utility function from indirect utility function, and in other word we go backward.

Those three are important and make sure you understand them

3.7. Welfare evaluation of the Economic changes. (Focus on price changes)

Starting point for each consumer (p,w)

V(p,w) solution to the problem is Walrasian problem, however objective function is taken from this indirect utility function that maximizes utility

Welfare change is change in the indirect utility function corresponding to price change

= $v(p^1,w)-v(p^l,w)$

Welfare here is your standard of living that would be taken from indirect utility function that maximizes utility

Relation b/w welfare change to expenditure function

Start with v(p,w) choose price vector $\overline{p}$ >>0, Consider $e(\overline{p}, v(p,w))$: since expenditure function will tell you for given price factor what would be min expenditure to reach that utility level

Expenditure function was strictly increasing in utility level (as we proved that e(p,u) is strictly increasing in u) => e( $\overline{p}$ , v(p,w)) is strictly increasing in v(p,w) => ***e( $\overline{p}$ , v(p,w)) is by itself and Indirect Utility Function (I.U.F)***: *Since this function is strictly increasing, and shows the same underlying preference*

e( $p^1$ ,v($p^1$,w))-e( $p^0$ ,v( $\overline{p}$ ,w)): this is in term of money and it is also indirect utility function, and now we have interpretation for the welfare change

We call above function mean e( $\overline{p}$ , v(p, w)) ***money metric indirect utility function***

***Equivalent variation (E.V.)****:* let $\overline{p}$ to change from $p^0$ to $p^1$

$$u^0 = v(p^0, w),\ u^1 = v(p^0, w)$$

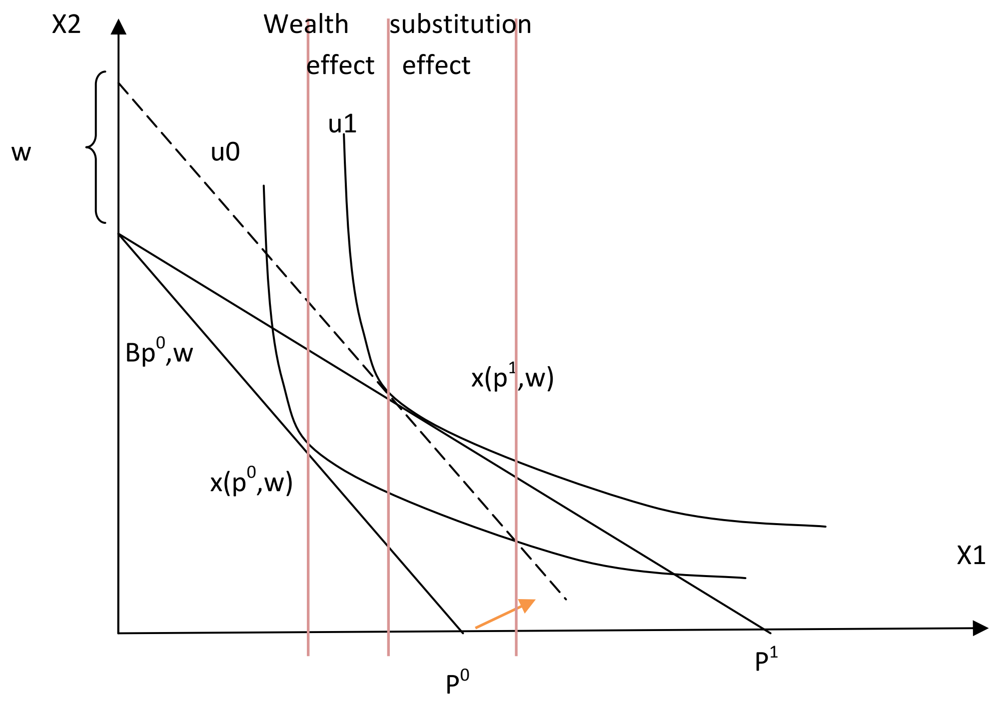
E.V. = $e(p^0, u^1)$ -w (the price is not change, but the utility is changed)

You change the price, and as a result your standard of living is change; however, we decide to change the wealth instead of changing the price, so that this new utility level be met.

**Compensating Variation (C.V.)**: the amount of wealth that after price change leads the consumer's standard of living (***utility level***) would be **<u>maintained</u>**.

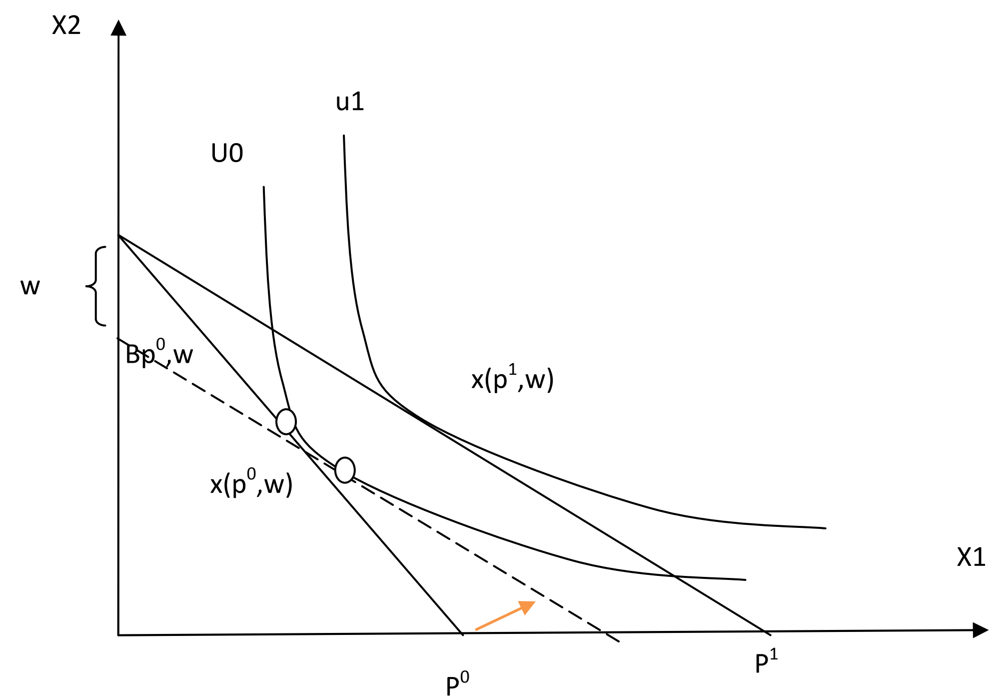
= w- $e(p^1, u^0)$

Equivalent variation (E. V.):




Here price did not changed (relative), but wealth only changed.

Compensating Variation: Maintain the original utility level



This is exactly how the index program works over CPI (Consumer price index), tries to maintain the utility levels.

The aggregation of demand, and the question is how the distribution is changed, for example from 15 years ago the wealth increased however, rich people are better off, so the distribution is important

I.4. Aggregation Demand

When can aggregated demand be expected as a function of aggregate wealth?

4.1. A.D. and A.W.

I consumers. i=1,2,…, I

$X_i(p,2)$

A.D

$$\sum_{i=1}^{I} x_i(p,u_o)$$

$$x\ (p,w_1,w_2,...,w_I) = x(p,\sum_{i=1}^{I} w_i)$$

Let ( $w_1,w_2,...,w_I$) be a wealth distribution where $(dw_1,2w_2,...,dw_I)$ is part of $\in R^L$ such that $\sum_{i=1}^{I} dw_i = 0$ as a wealth redistribution $\sum_{i=1}^{I} \frac{\partial x_l}{\partial w_i} dw_i = 0$ => $\frac{\partial x_l}{\partial w_i} = \frac{\partial x_l}{\partial w_j}$ , $i \neq j$ for all l=1,.., L

We need this in order to hold the $x\ (p,w_1,w_2,...,w_I) = x(p,\sum_{i=1}^{I} w_i)$ as true, but this means that in case we ***give one dollar to poor and rich and everyone*** the ***demand change would be the same***.

***Efficiency vs. fairness*** is an issue usually in economics. The marginal utility of dollars is different.

Proposition: A **necessary and sufficient** condition for the set of consumers to exhibit **parallel straight wealth expansion** path at all p >>0 is that preferences admit I.U.F (**indirect utility function** of ) Gorman form:

$v_i(p,w_i) = a_i(p) + b(p)w_i$ ; it means the utility function would be ***homothetic***

Proof: $x_{li} = \frac{\partial v / \partial p_l}{\partial v / \partial w_i} = \frac{\partial a_i / \partial p_l + \partial b / \partial p_l \, w_i}{b(p)}$; the second term of nominator has nothing to do with the customer: $\partial x_{li} / \partial w_i = \partial b / \partial p_l$

- Consumer’s theory was finished on the previous time
- And the purpose of going to those consumer theory was to derive the aggregate demand
- We moved from individual demand to aggregate demand
- There were sort of complication on whether the aggregate demand relates to aggregate level
- There were special cases which was true, was that all those consumer had homothetic preferences, and the utility function could have been represented by Gourman form
- In general we do not expect that
- The individual demand will affect aggregate level, but will be dependent on the distribution
- On marketing and finance, therefore, you work over heterogeneous consumer model, and there you need to overcome this, and there locus work (Nobel price laureate), in his paper he works over agents, which works over represent agent economy
- On which level you do aggregation depend on complete market assumption.
- Complete market needs to be hold to have the agent model
- On that case you would not need to do aggregate decision, and we analyze the individual level and by assuming the same decision rule (not the same decision), will help to do the analysis
- On that case you would be able to work on heterogeneous agent
- We use represent agent economy to overcome the challenge that each person has different decision, which is very important
- Result of that assumption many economic model that were not tractable, now we have solution to them
- There are two people at that time that mad important contribution to rational expectation theory which basically says, decision makers are there, they have rational expectation; they use all the information; they can be consistent. First person was tom locus, and second is tom surgent, and people wonder why the second one who is laureate on NYU, and was not bitter, and he said I made important contribution; because after locus’s presentation in Stanford, I take it to him
- We will now work over firm decision, and it would be much easier.
- The objective is now to find the aggregate supply
- Just as aggregate demand is aggregate of individual demand, aggregate firm supply would be sum of aggregate firm supply

- Now we work over what it is. They must have some logic for their decision
- There are not many firms there to maximize profit, and many of them emphasis on social responsibility; although it is kind of contradict with profit maximization, in the theory apparatus we use this assumption, and profit maximization and cost minimization will be scrutinized here

The theory of the firm

II. 1. Production set

Consider the firm with L commodities,  Let $y = (y_1, y_2, \ldots, y_l) = R^L$

Be the ***net output*** of L commodity for a production process

We allow some of the elements to have negative value. We say if some of net output is negative we will have input. As a result we define:

$Y_i > 0$: net output

$Y_i < 0$ would be net input

Y is called production plan (or production vector)

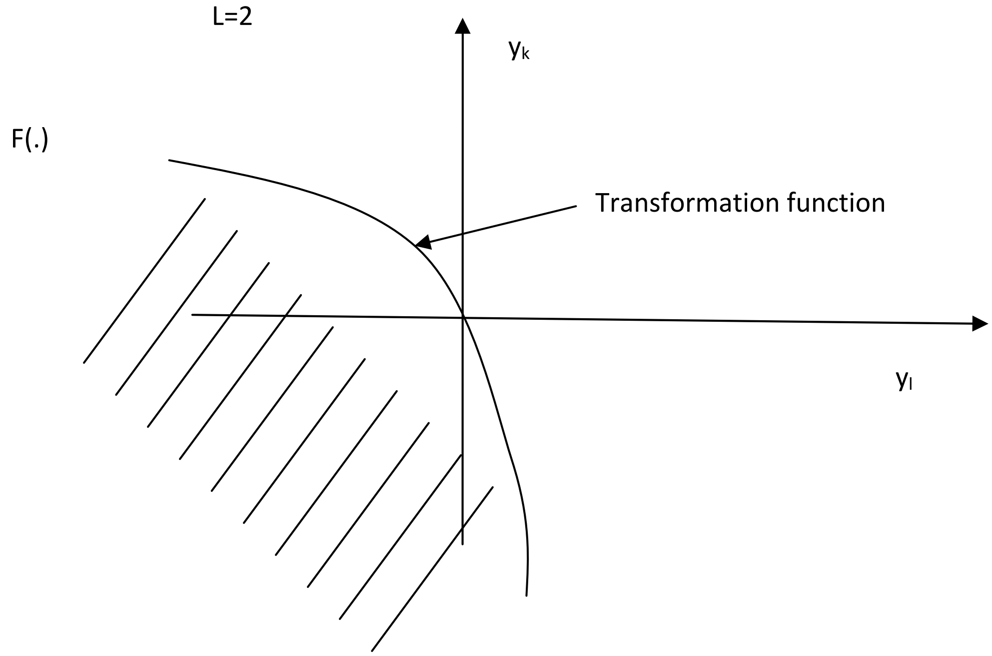
We define feasible production plan (refers to all those production plans, combination of output or input that physically, and technologically they would be able to do it: e.g. trip to moon possible, and mars not feasible):

Transformation function: F(.): is a function that if you evaluate the function on different production plan would not be greater than zero:

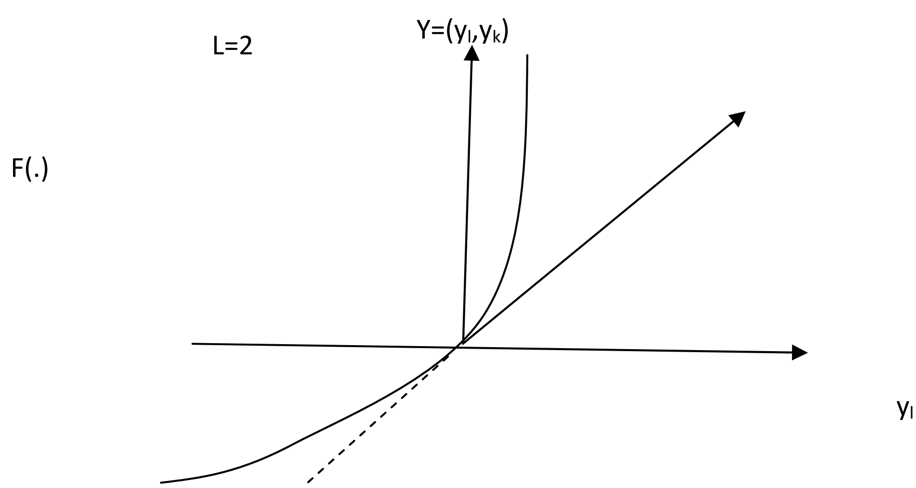
$$Y = \{y \in R^L : F(y) \leq 0\}$$

**Transformation frontier**: $Y = \{y \in R^L : F(y) = 0\}$ : we also call that **boundary of the production set**

Intuition is that you need to consume one thing to produce the other thing. If you were in 3rd quadrant you are wasting everything.



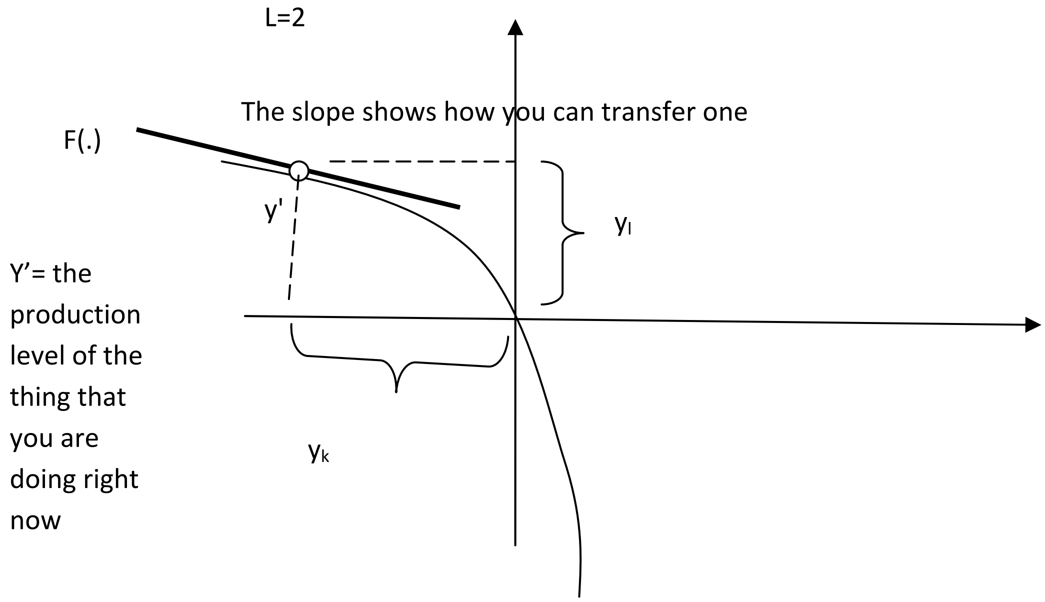

The shaded part is feasible, and frontier is the boundary



The value would become clear when we introduce technology. This is general situation when we have many outputs and multiple inputs.

MRT: Marginal rate of transformation

$$MRT_{lk} = \frac{\dfrac{\partial F(\bar{y})}{\partial y_l}}{\dfrac{\partial F(\bar{y})}{\partial y_k}} = \frac{\partial y_k}{\partial y_l}$$




Transformation function represents technology

Example could be that on one part of the word labor is not expensive so you use less capital and in other place you have expensive labor, and so you use more capital; it has that meaning

This is convex but not necessarily

$q = (q_1, q_2, .., q_m) \in R_+^{\ m}$ : it would be outputs

$z = (z_1, z_2, .., z_{l-m}) \in R_+^{\ L-M}$ : would be inputs

$Y = \{(z, q) : F(-z, q) \leq 0 \; for z \in R_+^{\ L-M} \ \& \ q \in R_+^{\ L}\}$

For a single output case,  we have **production function f(z)** (would not be any function but) gives us ***maximum*** output for the given input z.

$$Y = \{(z_1, z_2,.., z_{l-1}, q) : q - f(z_1, z_2,.., z_{l-1}) \leq 0 \, for z_1, z_2,.., z_{l-1} \geq 0\}$$

$$F(-z, q \mid f = f(z)) \leq 0$$

Transformation function evaluates production plan by treating inputs as negative, and outputs as positive, but production function says given all inputs, what is the maximum level of production

Production function will depend on technology

$MRT_{lk} = \dfrac{\dfrac{\partial F(\overline{z})}{\partial z_l}}{\dfrac{\partial F(\overline{z})}{\partial z_k}}, \overline{q} = f(\overline{z})$ : Marginal rate of technical substitution

Substitution will depend on contribution of each factor to the output. If the factor contributes a lot, then you need to replace a lot. If it is not very productive you will not need to put many of other material to substitute

Example:

$$F(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$$

$$\frac{\partial f}{\partial z_1} = \alpha z_1^{\alpha-1} z_2^{\beta}$$

$$\frac{\partial f}{\partial z_2} = \beta . z_1^{\alpha} z_2^{\beta-1}$$

$$MRT_{12} = \frac{\alpha z_2}{\beta z_1}$$

It shows that marginal substitution would be inversely related to the rates. As you use more of same input the contribution of that input would be declining, means the substitutability would be lower. Means other are more productive. Assuming every other things fixed. [**Diminishing Product productivity**]

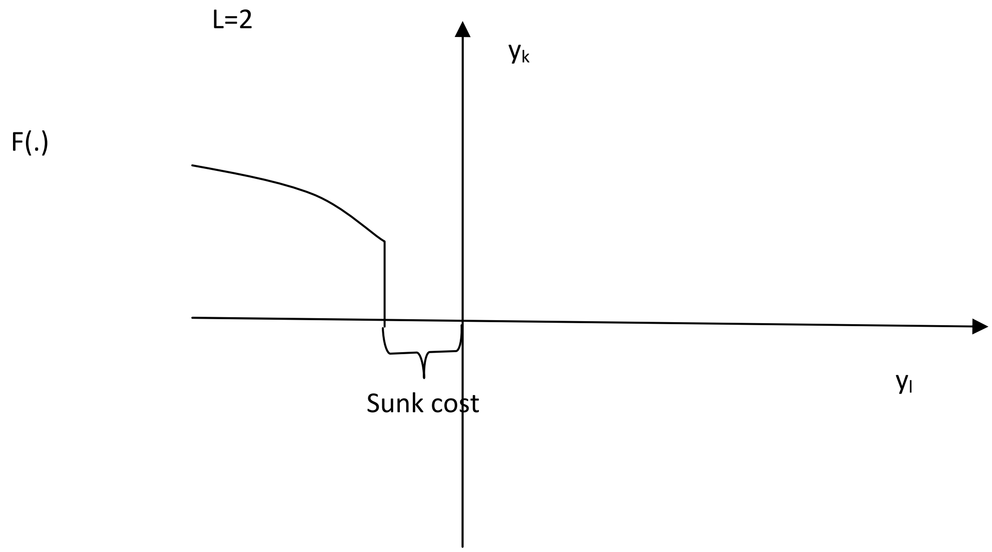
Properties of production function:

1. Y is non-empty: there is always production function
2. Y is closed: Y includes its boundary

3. No free lunch: means if $y \in Y, y \geq 0$ then y=0: there is production function but you may choose to not produce anything

4. The possibility of inaction $0 \in Y$

Sunk cost: you have invested and you can't back up. Sunk cost means 0 is not feasible.



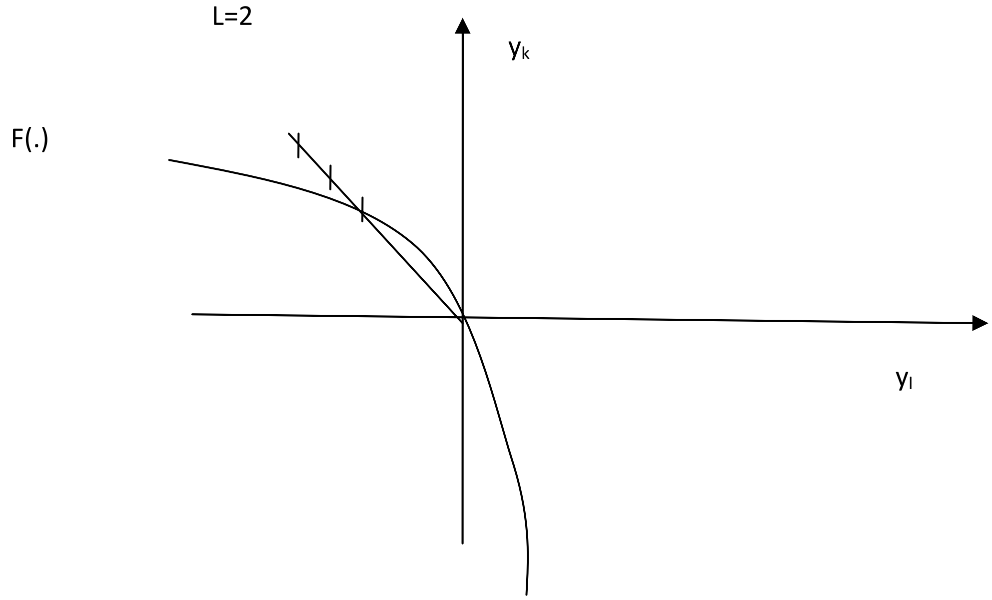

5. Free disposal: if $y \in Y, y' \leq y \rightarrow y' \in Y$ : using more input to produce the same output.

6. Irreversibility

   If $y \in Y, y \neq 0$ then $-y \notin Y$

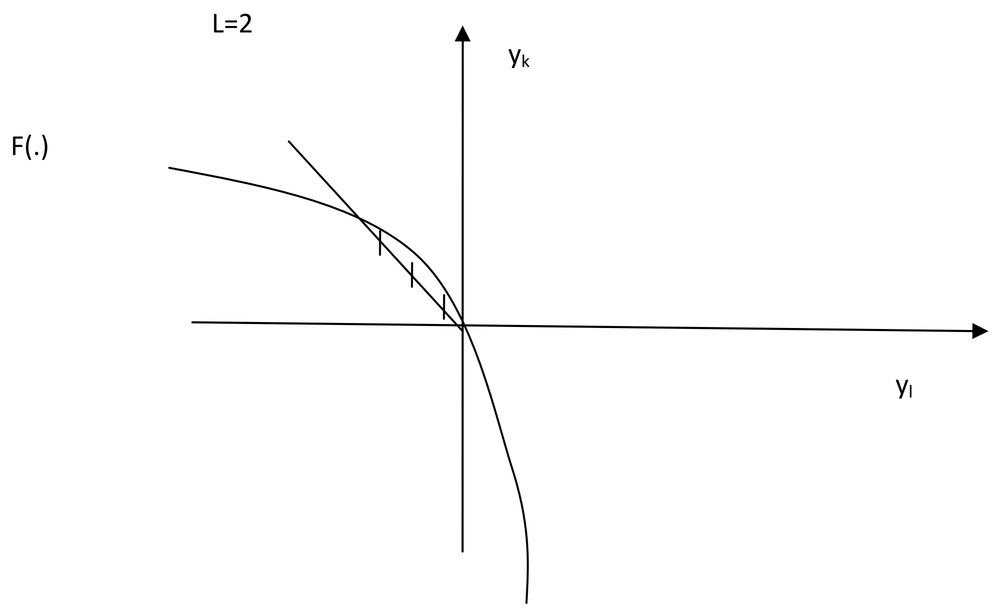
7. Non increasing return on scale

   If $y \in Y$ we have $\alpha y \in Y$ for all $\alpha \in [0,1]$



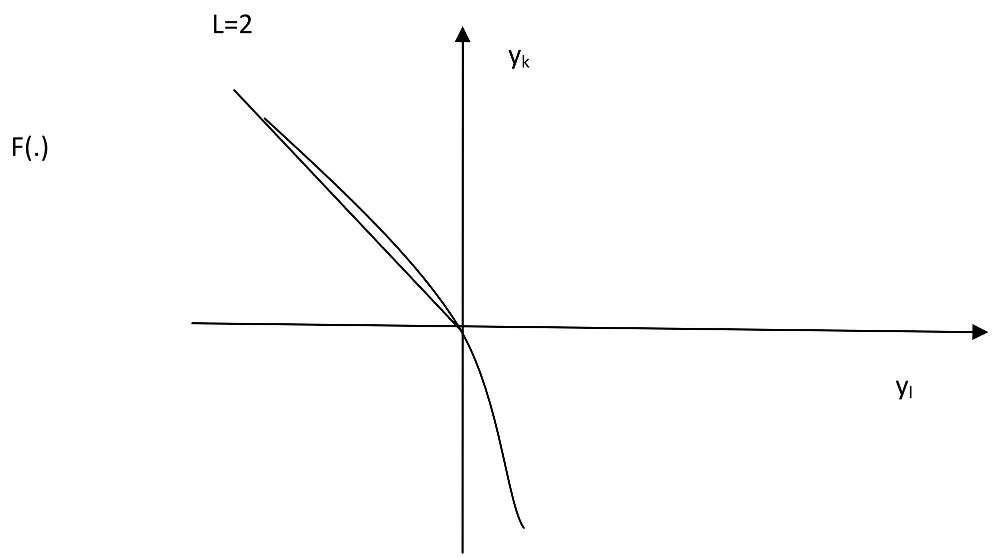

8. Non-decreasing return to scale

If for any $y \in Y$ we have $\alpha y \in Y$ for all $\alpha \geq 1$



9. constant return to scale

   If for any $y \in Y$ we have $\alpha y \in Y$ for all $\alpha \geq 0$



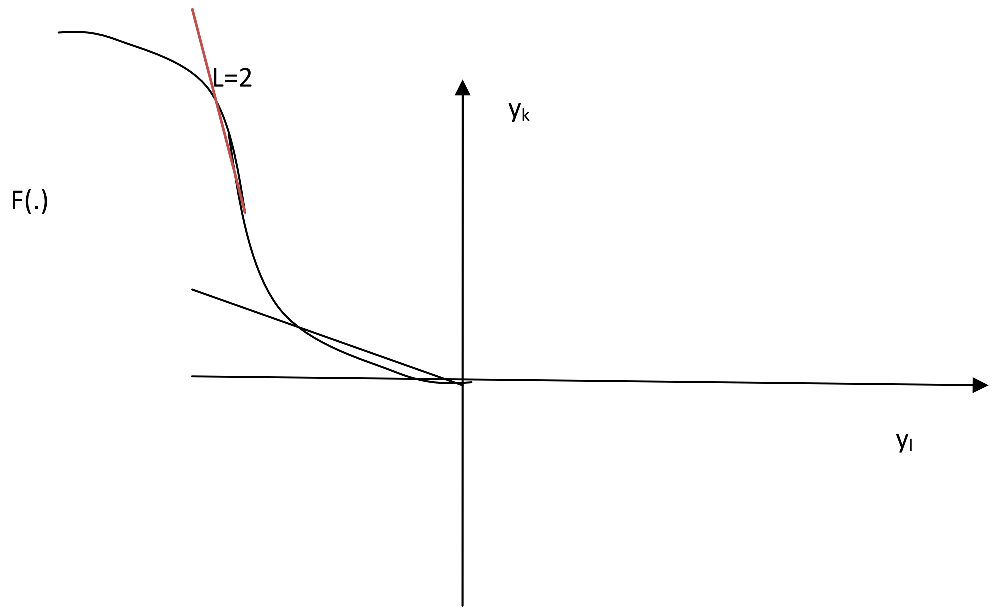

10. additively (or free entry)

    If $y \in Y$ and $y' \in Y$ then $y + y' \in Y$

    ⇨ $ky \in Y$ for any positive integer (mean they are feasible, although you may not do it)

At different time the return to scale would be different:

1. First stage is positive return to scale
2. Second stage: maturity: constant return to scale: most of companies such as Microsoft are in this. They first invest, they find out they got problem, and then they divest.
3. Third stage: would be when the firm will have negative return to scale




Production function is saying for given input what would be maximum output, so the boundary of this transformation function is production function.

Some of the properties may not be satisfied by production functions.

11. Production function is convex cone

    If for any $y \in Y$ and $y' \in Y$ for α, β>0 then we have $\alpha y + \beta y' \in Y$

II.2. profit maximization and cost minimization: (PMP problem)

Price vector $(p_1, p_2, \ldots, p_l) >> 0$

Firms are price takers

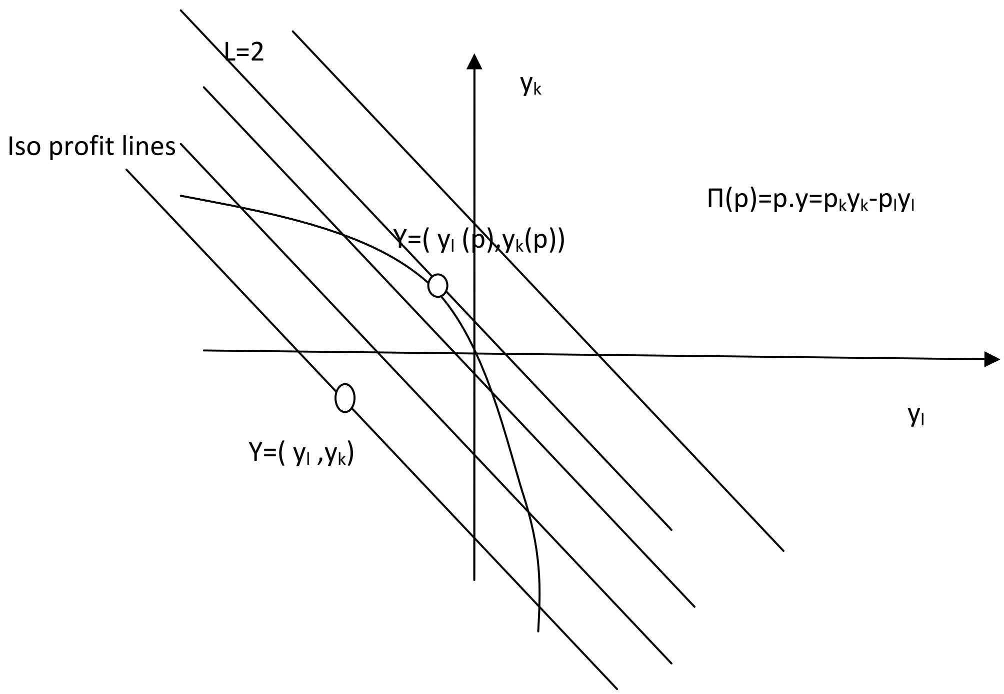
Firm's objective is to

$$\max_y p.y = \sum_{l=1}^{L} p_l y_l$$

s.t. $y \in Y$

if we solve this we will get the solution to supply correspondence (if single value it would be function) so, The solution to the PMP y(p) will be called supply correspondence

The maximized value or objective value in this case called $\pi(p)$ profit function. $\pi(p)=p.y(p)$




$$y(p)=\{y\in Y: p.y=\pi(p)\}$$

Example:

1 unit l to 1 unit of k

$p_k>p_l$; $(p_k-p_l)>0$=> no solution for $(p_k-p_l)y_k$

The assumption was that the firm was price takers. Not having solution means in long run we will not have the long run equilibrium. Therefore this is more complex problem.

If F(.) is differentiable (we will set the Lagrange equation)

L=p.y-λF(y)

If $y\in y^*(p)$, we have $p_l=\lambda\dfrac{\partial F(y^*)}{\partial y_l}$, l=1,2,…,L

$$p_k=\lambda\frac{\partial F(y^*)}{\partial y_k}$$

Therefore: $MRT_{lk} = \dfrac{\dfrac{\partial F(y^*)}{\partial y_l}}{\dfrac{\partial F(y^*)}{\partial y_k}} = \dfrac{p_l}{p_k}$

$$\frac{\dfrac{\partial F(y^*)}{\partial y_l}}{p_l} = \frac{\dfrac{\partial F(y^*)}{\partial y_k}}{p_k}$$

F(y)<=0

In the case of a single output f(z):

p- output price

w- input price vector

profit = p.f(z)-w.z

max p.f(z)-w.z (production technology identifies the production contour)

$\dfrac{\partial f(y^*)}{\partial z_l} p - w_l \leq 0$ l=1,2,.., L

For interior solution, we have $\dfrac{\partial f(y^*)}{\partial z_l} p = w_l$ , l=1,2,…, l-1

$$MRT_{lk} = \frac{\dfrac{\partial f(y^*)}{\partial z_l}}{\dfrac{\partial f(y^*)}{\partial z_k}} = \frac{w_l}{w_k}$$

$\dfrac{\dfrac{\partial f(y^*)}{\partial z_l}}{\dfrac{\partial f(y^*)}{\partial z_k}} = \dfrac{w_l}{w_k}$ : every dollar you put on each input the amount of the additional output would be?

$$\frac{\dfrac{\partial f(y^*)}{\partial z_l}}{w_l} = \frac{\dfrac{\partial f(y^*)}{\partial z_k}}{w_k}$$

- Last time we discussed that return to scale (constant, increasing, decreasing) could exist one at a time
- Today first we start with proposition that addresses the profit function and supply correspondence
- We are not going to prove every properties but we are going to single out those that are specific to the firm function
- For others it is the same as the one that we had for consumer's

Basic properties of profit function and supply correspondence

Prop: π(.) is a the profit function and y(.) is the supply correspondence,. They have the following properties:

(i) π(.) is H.D. One in p.

(ii) π(p) is convex in p.

(iii) if the production function (Y) is convex, then $Y = \{y \in R^L, p.y \le \pi(p)\ for \quad all \quad p >> 0\}$

(iv) y(p) H.D.Z in p

(v) If Y is convex, then y(p) is convex set

If Y is strictly context then y(p) is single value

(vi) (Hotelling's Lemma) if y(p) is single value then the production function π(p) is differentiable and $\nabla$ π($(\overline{p})$)=y$(\overline{y})$

(vii) If y(p) id differentiable, then $D_y$($(\overline{y})$=$D^2$π($(\overline{p})$ is symmetric and positive semidefinite with $D_y(\overline{p}).\overline{p} = 0$

P.M.P (Profit maximization problem)

f(αx+(1-α)y)<=αf(x)+(1-α)f(y): convexity

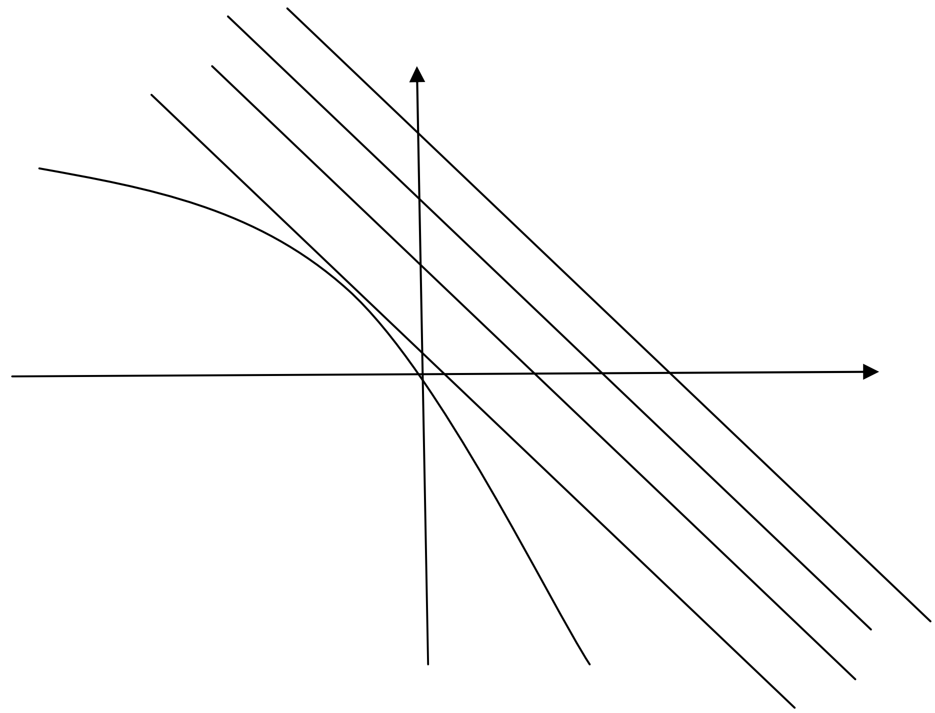
p''=αp+(1-α)p'

π(p'')= π(αp+(1-α)p')=p''.y''=> y''(p'')=αp.y''+(1-α)p'y''

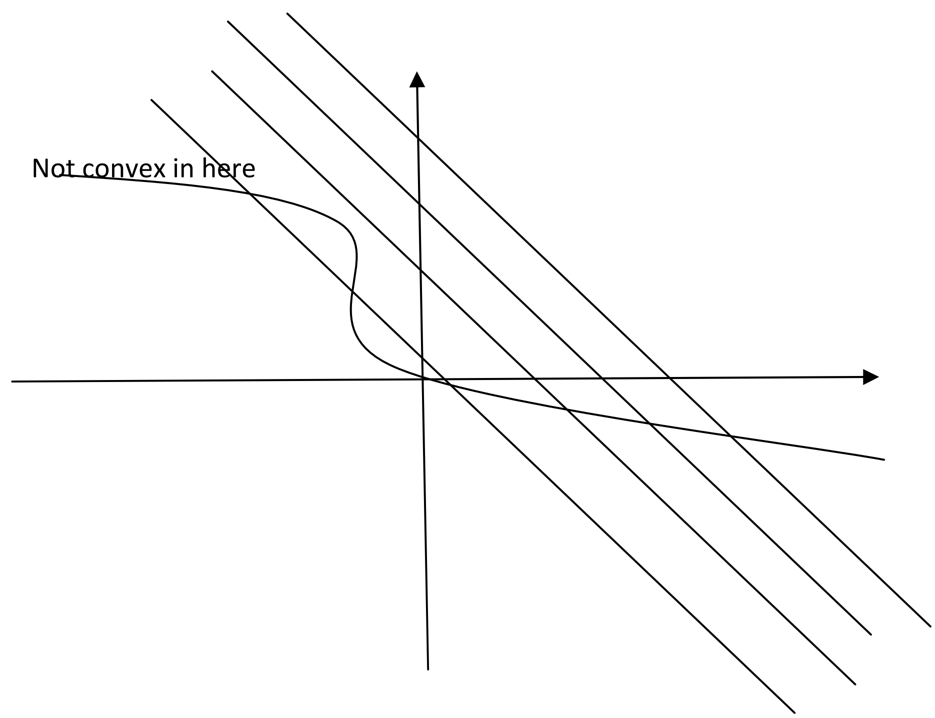
this is not optimal profit and optimal profit would be =απ(p)+(1-α)π(p') > αp.y''+(1-α)p'y''

so it is convex

The profit function maximizes profit:



In reality we usually have following curve:




In the developing country labor is not expensive, but capital is, so they use more labor, but in developed country it is reverse.

The fourth property one is similar to the Walrasian demand function and you can go to that for the proof of this.

π(p)=p.y(p)

$$\nabla \pi(p) = y(p) + p.Dy(p)$$

$$HDZ \quad in\ p : y(\alpha p) = y(p)$$

Differentiate with respect to α and evaluate when α=1

Now last property:

$Dy(\overline{p}) = D^2\pi(\overline{p})$ positive semidefinite

$$dpD^2\pi(\overline{p})dp \geq 0$$

(p'-p)(y(p')-y(p))>=0

(p'-p)(y(p')-y(p))=p'.y(p')-p.y(p')+p.y(p)-p'.y(p)= (p'.y(p') -p'.y(p))+ (p.y(p) -p.y(p'))>=0

The first term is greater than zero since the first term of it is maximized profit but second one is not

On the second term it is also greater than zero since the first term is maximized profit but second one is not

Here it was much easier since it was not wealth effect, and substitution.

**Cost minimization problem (C.M.P.)**

Consider a single output q

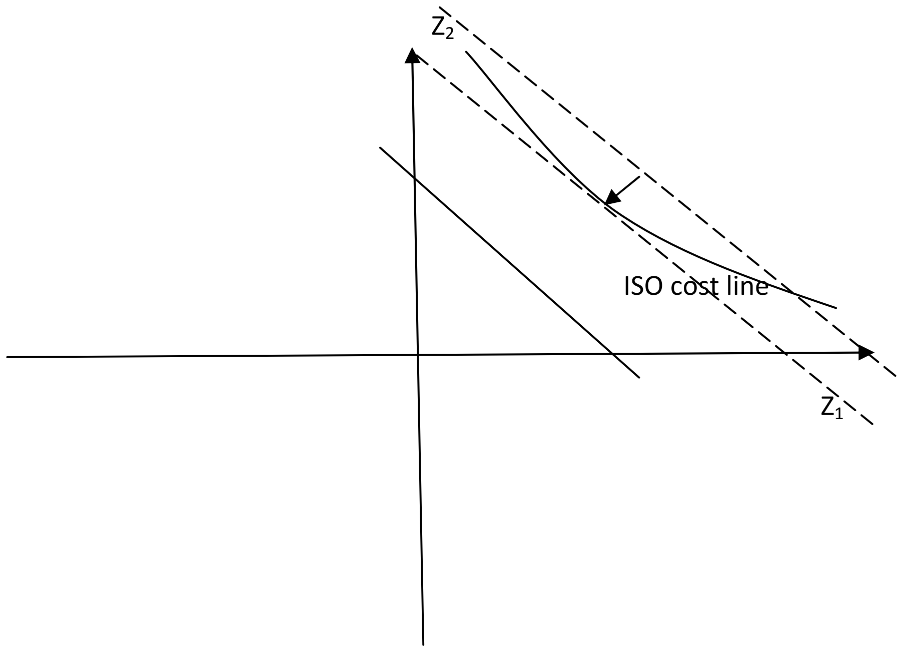
Production function f(z) where z- input vector and w- input price vector

$$\begin{aligned} &\underset{z}{\text{Min}}\ \ \text{w.z} \\ &\text{s.t. } f(z) >= q \end{aligned}$$

Cost function c(w,q)

Conditional factor demand z(w,q)

The amount of input is going to change with the output level and that is why we call it conditional factor demand




$$L = w.z - \lambda(f(z) - q)$$

Diff w.r.t.z.

$$w_1 \geq \lambda \frac{\partial f(p^*)}{\partial z_l}, R = 1,2,\ldots,l-1$$

$$w_l = \lambda \frac{\partial f(z^*)}{\partial z_l}$$

$$w_k = \lambda \frac{\partial f(z'')}{\partial z_k}$$

$$if \ z'' > 0$$

$$(w - \lambda \nabla f(z^*))z^* = 0$$

$$\frac{w_1}{w_k} = \frac{\frac{\partial f(z'')}{\partial z_l}}{\frac{\partial f(z'')}{\partial z_k}}$$

λ:marginal cost of output

$$w.\frac{\partial z(w,q)}{\partial q} = \frac{\partial c(w,q)}{\partial q}$$

since: f(z(w,q))=q, $\nabla f(z(w,q))\frac{\partial z(w,q)}{\partial q}$ we have: $\frac{\partial c(w,q)}{\partial q} = \lambda$

Prop: suppose c(w,q) is the cost function and z(w,q) is the conditional factor demand correspondence

(i) C(w,q) is H.D.one in w and nondecreasing in q

(ii) C(w,q) is concave function of w:

Similar with the prove that we made before for consumer case for expansion function

(iii) If the set $\{z \geq 0 : f(z) \geq q\} are\ convex$ for every q, then

$$Y = \{(-z,q) : wz \geq C(w,q)\ for\ all \quad w >> 0\}$$

It is upper contour set, which means minimum cost will be given by the cost function, any other will cost you more

(iv) Z(w,q) is H.D.Z in w

Solution is the same as profit maximization

(v) If the set $\{z \geq 0 : f(z) \geq q\}$ is convex then z(w,q) is convex set if the $\{z \geq 0 : f(z) \geq q\}$ is strictly convex

Z(w,q) is single value

Similar to Hicksian demand function and Walrasian demand function

(vi) (Shepend's Lemma) if $z(\overline{w},q)$ is single valued then c(.) is differentiable and $\nabla_w C(\overline{w},q) = z(\overline{w},q)$

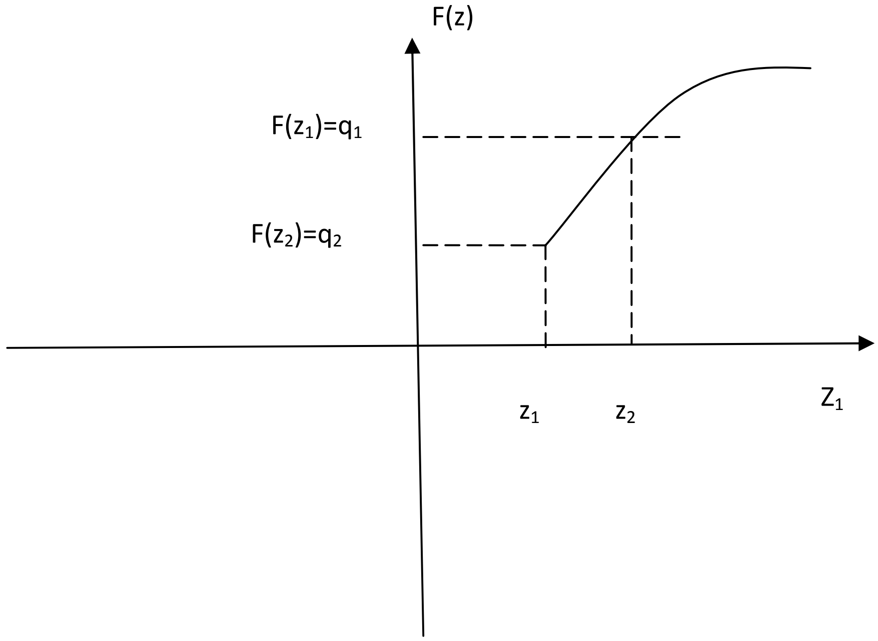
Cost function is C(w,q)=w.z(w,q), and you differentiate w.r.t. w, and then multiply α, and the second term would be equal to zero (very straight forward)

(vii) If z(.) is differentiable at $\overline{w}$, then $D*_m\ z(\overline{w},q)$ is a symmetric negative semidefinite

(viii) If f(.) is H.D.one and $z(\overline{w},q)$ then c(.) and z(.) are both are H.D.one in q

Constant return to scale means if you double the input, then you would produce twice as much which means you will encounter twice as much cost. Everything will be scaled down or up.

(ix) If f(.) is concave then the cost function is convex




$Z' = \alpha z_1 + (1-\alpha) z_2$

$F(z') = f(\alpha z_1 + (1-\alpha) z_2) >= \alpha f(z_1) + (1-\alpha) f(z_2)$

$c(w, f(z')) = w.z' >= C(\alpha f(z_1) + (1-\alpha) f(z_2)) <= \alpha w z_1 + (1-\alpha) w z_2 = \alpha C(w, z_1) + (1-\alpha) c(w.z_2)$

---

P price of output, q- input price

$$\underset{q}{\mathrm{Max}}\ pq - c(w,q)$$

$$p - \frac{\partial c(w,q^*)}{\partial q} \leq 0$$

$$=0 \text{ if } q^*>0$$

$$(p - \frac{\partial c(w,q^*)}{\partial q}) q^* = 0$$

$$p - \frac{\partial c(w,q^*)}{\partial q} = 0$$

$p = \dfrac{\partial c(w,q^*)}{\partial q}$ : p means marginal revenue (assuming that firms are price takers), and the right hand side of equation is marginal cost

If p would be greater it means that you are producing the unit of output that offsets more than cost, and this means that you have not reached the maximum profit, and if it would be lower it means that the revenue would be more than the revenue you lose for the cost, so it would be better for you that you cut that output, and this means that the only condition is that these two be equal.

Example: Coup Douglas production function

$F(z_1,z_2)=z_1^{\alpha}z_2^{\beta}$

$MP_1/w_1= MP_2/w_2$

$\alpha z_1^{\alpha}z_2^{\beta}/w_1= \beta z_1^{\alpha}z_2^{\beta}/w_2$

$q= z_1^{\alpha}z_2^{\beta}$

$z_2/z_1=\beta/\alpha.w_1/w_2$

$z_1(w_1,w_2,q)=q^{1/\alpha+\beta}(\alpha w_2/\beta w_1)^{\beta/(\alpha+\beta)}$

$z_1(w_1,w_2,q)=q^{1/\alpha+\beta}(\beta w1/\alpha w2)^{\alpha/(\alpha+\beta)}$

$Cost=wz(w,q)=w_1z_1(w,q)+ w_2z_2(w,q)= q^{1/\alpha+\beta} ((\alpha/\beta)^{\beta/(\alpha+\beta)}+ (\alpha/\beta)^{-\alpha/(\alpha+\beta)})w_1^{\alpha/(\alpha+\beta)}w_2^{\beta/(\alpha+\beta)}=\Theta\varphi(w_1,w_2)\ q^{1/(\alpha+\beta)}$

$MR=p$

$MC= =1/(\alpha+\beta)\ \Theta\varphi(w_1,w_2)\ q^{(1-\alpha-\beta)/(\alpha+\beta)}$

(i) $\alpha+\beta<1$: $p=1/(\alpha+\beta)\ \Theta\varphi(w_1,w_2)\ q^{(1-\alpha-\beta)/(\alpha+\beta)}$

$q^*=(\alpha+\beta)(P/ \Theta\varphi(w_1,w_2))^{\alpha+\beta /(1-\alpha-\beta)}$

(ii) $\alpha+\beta=1$:

if $p>MC$, no finite price maximizing solution

if $P<MC$, $q=0$

(iii) $\alpha+\beta>1$: no finite price maximizing solution

zero profit is not bad thing, it means that you are earning at the same level of the lowest profit in the market.

III. Cost curves.

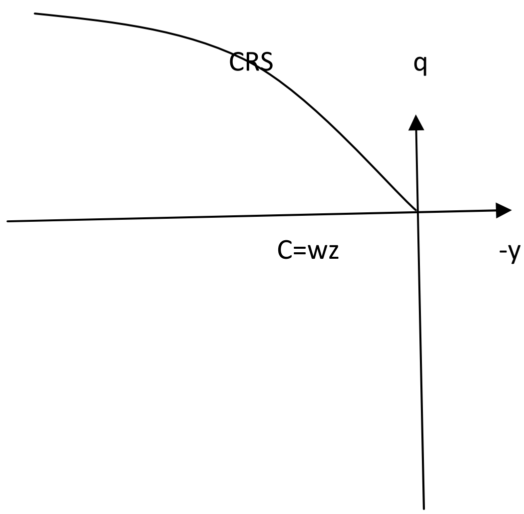
$C(\overline{w},q)$ : q: output

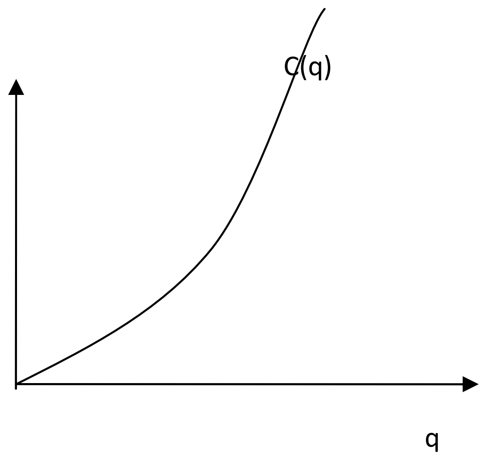
Average cost (AC)= $C(\overline{w},q)$ /q

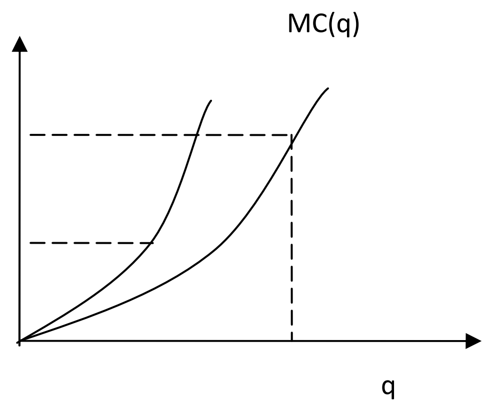
Marginal cost (MC) = $\dfrac{\partial \mathrm{c}(\overline{\mathrm{w}}, \mathrm{q})}{\partial q}$

$$\frac{\partial \mathrm{A}c}{\partial q} = \frac{q\dfrac{\partial c(\overline{w}, q)}{\partial q} - c(\overline{w}, q)}{q^2} = 0 \rightarrow q\frac{\partial c(\overline{w}, q)}{\partial q} - c(\overline{w}, q) = 0 \rightarrow \frac{\partial c(\overline{w}, q)}{\partial q} = \frac{c(\overline{w}, q)}{q} = AC$$

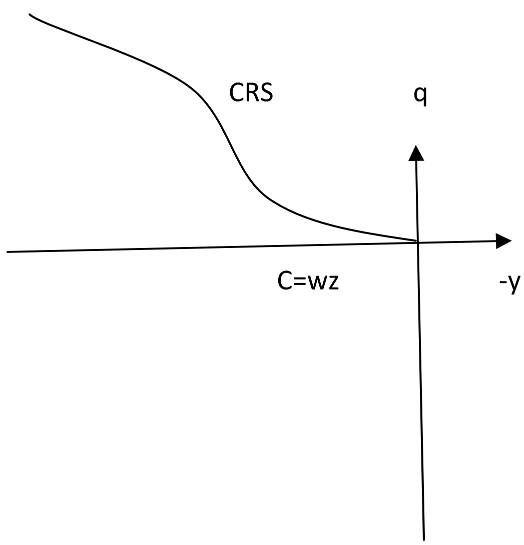
Means marginal cost equal to average cost at that point.

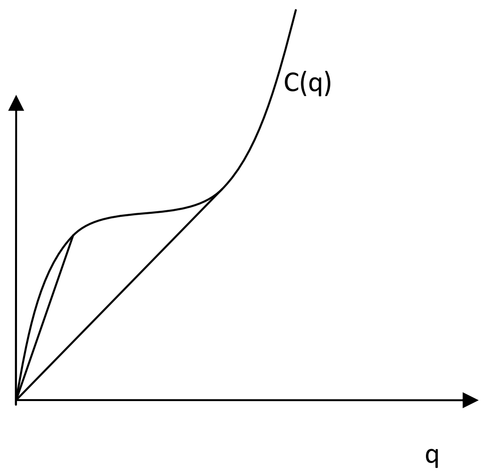


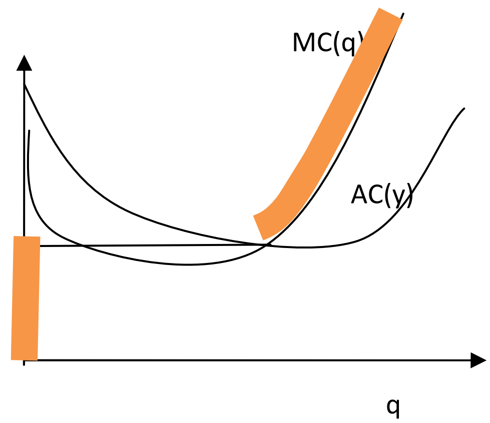




In reality we will have this:



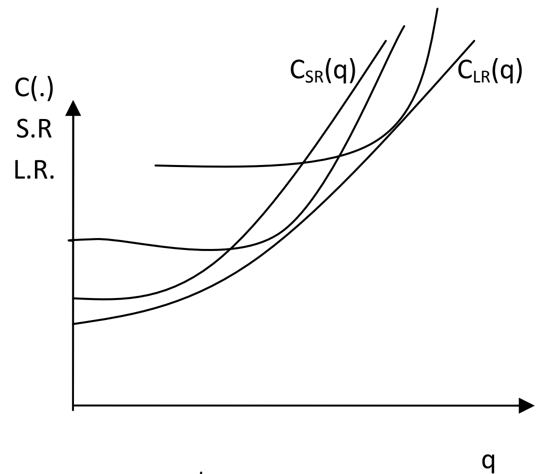

Production cost would be the orange one, since until the point of AC=MC you are losing money

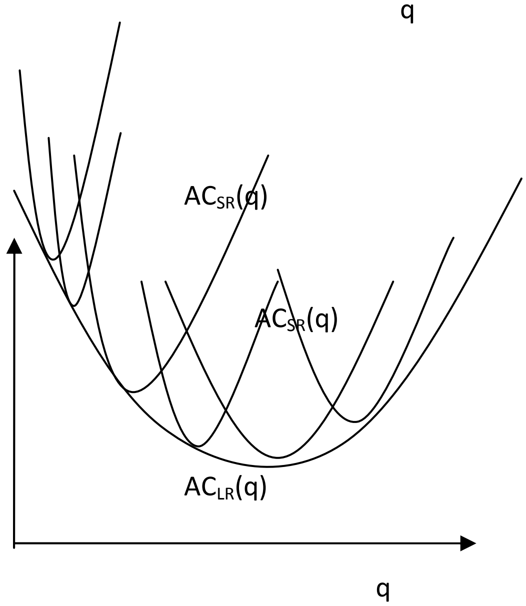
If the average cost is declining, I produce more, since my operating profit is increasing, so I will never ever stop at this point.

There is short run and long run. $z_1$, $z_2$ are the inputs, and in the short term $\bar{z}_2$ is fixed, although you can add shifts

$C_{SR}(w_1,w_2,q|z_2=\bar{z}_2) >= C_{LR}(w_1,w_2,q)$

Long run would be envelope of all the short runs. Since in long run you reshuffling everything while in short run something is always held fixed.




Aggregate supply for all firms is very straight forward, and simply when you change the price you aggregate all supplies of firms.

Efficient production and profit maximization

Wrap up:

- Today we started with the discussion of profit function and cost minimization
- We worked over conditional effect of demand correspondence (since it depended on output level)
- Even though conceptually firm problem are easier to work with, but we have another level of complication which is market size

- Sometimes there is infinite amount of solution for profit maximization and we do not have
- Next time we will talk about relationship between profit maximization and cost minimization

- Main topic for the day is market, up until the point we finished consumer decision and the result was demand curve which was the aggregated demand side
- Supply side were also talked which was aggregation of the supply side
- We now bring them together to see how equilibrium quantity are aggregated
- How the market is being structured will be discussed
- Most basic one which is competitive market
- Then we will talk about the extreme which is monopoly and then we will talk about the one in between these two extremes and will see what would be the effect of each firms' decision on the other

Last week we talked about properties of the supply correspondence and function and then we talked about the cost minimization which was the duality of profit maximization problem

The second thing was conditional factor demand correspondence which is the input that occurs for the given input prices

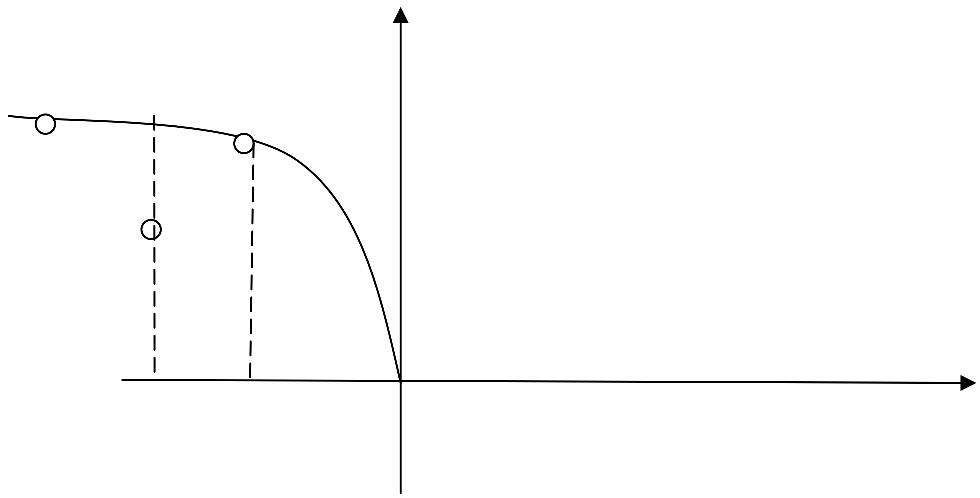
Then we introduced the concept of the cost curve which was dependent upon the production technology; for example if input and output change with the same proportion then the average cost and marginal would be the same (constant return to scale), and then on the case of decreasing return to scale discussed

Firm usually experience different conditions, that at first it starts with increasing return to scale, then constant return to scale and finally decreasing return to scale, and this happens on different phase of life cycle

When firms become mature and are facing decreasing return to scales the firm will choose the financial techniques such as divesting which was split the firm to different sub firms so that each of them either is increasing return to scale or constant return to scale. Using this we try to correct the situation. On this case we also would be able to focus on the things that we do better.

**Efficient production**

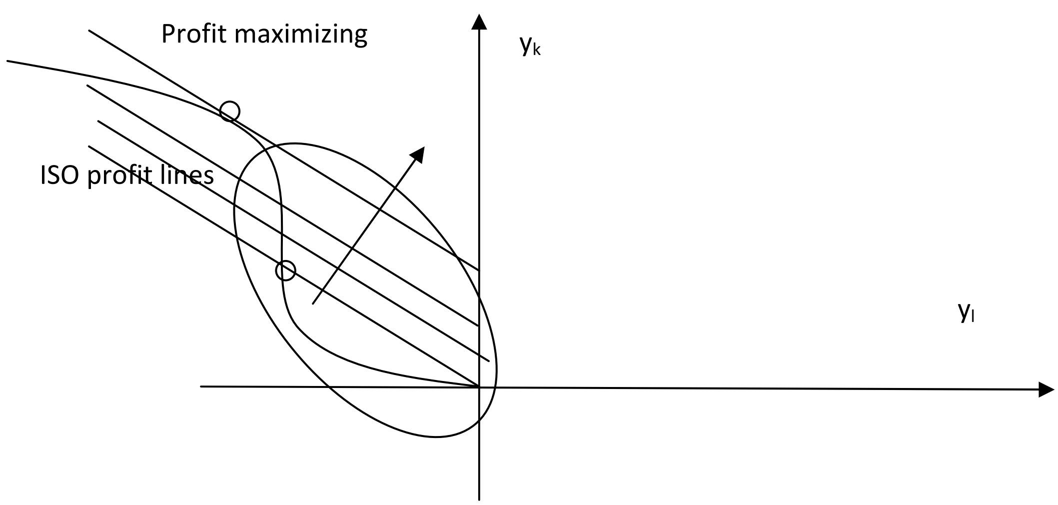
Definition: Production plan y is efficient if there is no alternative production plan, $y' \in Y$ such that $y' \geq y$ and $y' \neq y$



Proposition: if $y \in Y$ is profit maximizing for some p>>0, then y is efficient.

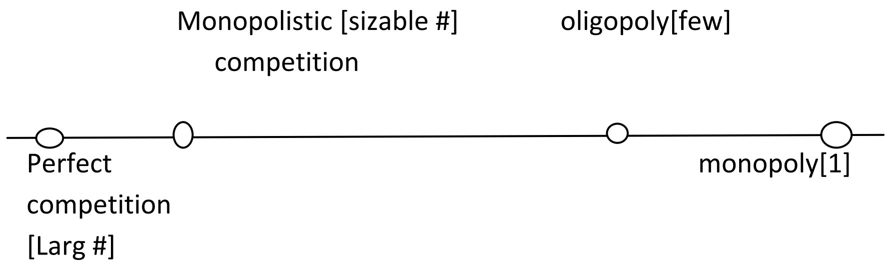
Suppose that y is not efficient there exists an alternative production plan $y' \geq y, y' \neq y$

$p.y' \geq p.y$ => contradicts that y is profit maximizing => you cannot find y'.




Proposition: suppose that y is **convex** then every ***efficient production*** $y \in Y$ is ***a profit maximizing*** production plan for some price vector $p \geq 0$

III Theory of the Market




On the buy side we have [Consumers, monosuming] spectrum

There is no clear cut between oligopoly and monopolistic competition and it is gray area.

The monopolistic discussion will be discussed in the Industrial Organization Class.

Competitive equilibrium:

I. Competitive markets:

Pareto optimality (efficiency) and competitive equilibrium

Consider a market consisting of:

I: consumers i=1,…, I

J- firms, j=1,….,J

l-goods l=1,2,…, L

$U_i$ – consumer utility

$X_i$ $(x_{1i},\dots,x_{si}) \in R^L$ consumer I's consumption bundle

$Y_j=(y_{1ji},\dots,y_{lj}) \in Y_j$ production plan of firm j

$(y'_1,y'_2,\dots, y'_j) \in R^{Lj}$ production plan of all J firms

$W_l \geq 0$, l=1,2,…, L

Initial endowment of good l

$w_l + \sum_{j=1}^{J} y_{lj}$ : total amount of good l

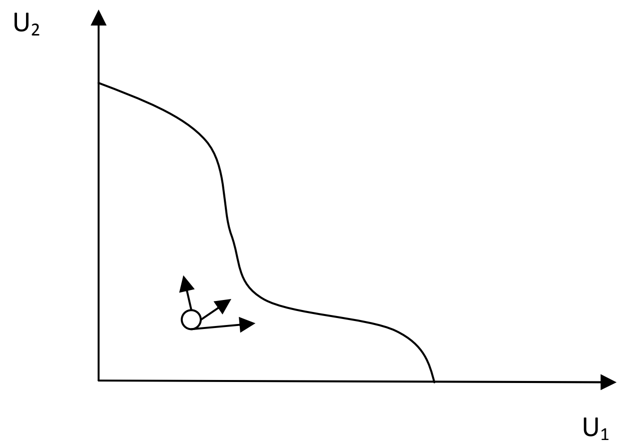
Def: feasible allocation.

A: economic allocation ($x_1, x_2,.., x_I, y_1, y_2,..., y_J$) is a specification of consumption vector $x_i \in X_i$ for each consumer i=1,2,.., I and a production vector $y_j \in Y_j$ for each term j=1,2,..,J. The allocation ($x_1, x_2,.., x_I, y_1, y_2,..., y_J$) is feasible if $\sum_{u-1}^{I} x_{li} \le w_l + \sum_{j=1}^{J} y_{lj}$ ,l=1,2,.., L

Pareto optimality:

Def: A feasible allocation $(x_1, x_2,.., x_I, y_1, y_2,..., y_J)$ is Pareto optimal if there is no other feasible allocation $(x'_1, x'_2,.., x'_I, y'_1, y'_2,..., y'_J)$ such that $u_i(x'_i) \ge u_i(x_i)$ for all i=1,…, I and $u_i(x'_i) > u_i(x_i)$ for some i.

Pareto optimality has nothing to do with fairness.




Competitive Equilibrium:

$w_{li}$ consumer i's initial endowment of good l

$$w_l = \sum_{i=1}^{I} w_{lj}$$

$(w_{1i}, w_{2i},.., w_{li})$ Endowment vector of i

$\theta_{ij}$ share of firm j consumed by consumer i

Profit of firm j $(p.y_j)$

$\theta_{ij}(p.y_j)$ share of profit to consumer I from firm j

Def: A competitive equilibrium:

The allocation $(x*_1, x*_2,.., x*_I, y*_1, y*_2,.., y*_J)$ and price vector p* constitute a competitive equilibrium if:

(i) Profit maximization: for each firm $y*_j$ solves,

Max $p*.y_i$

$$y_j \in Y_j$$

(ii) Utility maximization for each consumer I, x* solves

Max $u_i(x_i)$

$x_i \in X_i$ so that. $P*.x_i \leq p*w_i + \sum_{j=1}^{J} \theta_{ij}(p*.y*_j)$

(iii) Market clearing for each good l=1,…,L

$$\sum_{i=1}^{I} x*_{lj} = w_l + \sum_{j=1}^{J} y*_{lj}$$

Partial equilibrium competitive analysis:

Two commodities: good l and all other goods [we use term numerator for all other goods]

$X_i$ consumer I's consumption of good l

$M_i$ consumer I's consumption numerator

Each consumer I has a quasilinear utility

$U_i(x_i, m_i) = m_i + \Phi_i(x_i)$

$\Phi_i(x_i) > 0$ for all $x_i>0$

$\Phi_i(x_i) < 0$

$\Phi_i(0) = 0$

Firm Each firm has a cost function $c_j(q_j)$

Suppose the price for good l is p*, and the price for the numerator good is normalized to 1.

$W_l=0$, $\sum_{i=1}^{I} w_{mi} = w_m > 0$

Firms objective:

$$\max_{q_j} p*q_j - c_j(q_j)$$

$p^* \leq c'_j(q^*_j)$ if $q^*_j > 0$ then $p^* = c_j'(q^*_j)$

Consumer max $m_i + \phi_i(x_i)$

$M_i, x_i$

so that $m_i + p^* x_i \leq w_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q^*_j - c_j(q^*_j))$ [total wealth]

$$m_i = w_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q^*_j - c_j(q^*_j)) - p^* x_i$$

$$\max_{x_i} \phi_i(x_1) + w_{mi} + \sum_{j=1}^{J} \theta_{ij}(p^* q^*_j - c_j(q^*_j)) - p^* x_i$$

F.O.C (First Order Condition): $\phi_i'(x_1^*) \leq p^*$

$\phi_i'(x_1^*) = p^*$ if $x_i^* > 0$

Marginal utility of good l for consumer i.

$(x^*_1, x^*_2, .., x^*_I, q^*_1, q^*_2, .., q^*_J)$ and p* constitutes the competitive equilibrium if

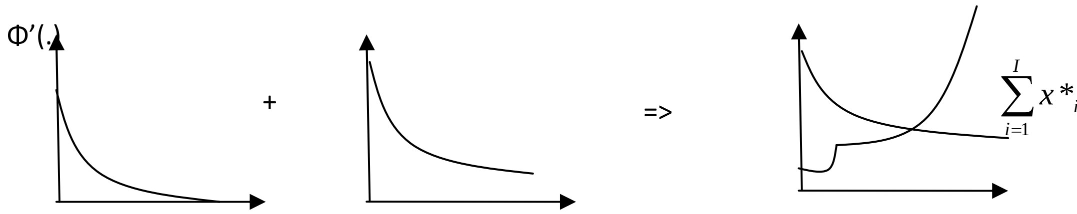
$p^* \leq c'_j(q_j^*)$ for all j $\phi_i(x^*_i) \leq p^*$ for all I, $\sum_{i=1}^{I} x^*_i = \sum_{j=1}^{J} q_j^*$

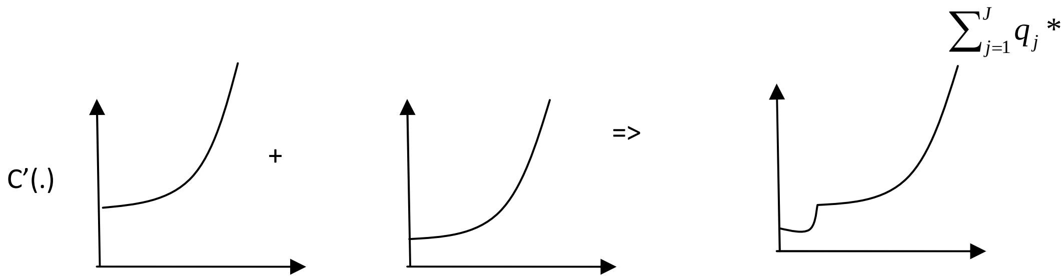
$$p^* = c'_j(q^*_j)$$

If $q^*_j > 0$ and $x^*_i > 0$ then $= p^* = \phi'_i(x^*_i)$

$$\sum_{i=1}^{I} x^*_I = \sum_{j=1}^{J} q_j^*$$

$$\max_u \phi'_i(x^*_i) > \min_j c'_j(0)$$




Comparative statistics

Utility function $\phi_i(.) \rightarrow \phi_i(.,\alpha) \rightarrow$ α preference parameters

Cost function $c_i(.) \rightarrow c_i(.,\beta) \rightarrow$ β technology shock

$$p_i \rightarrow \hat{p}_i(p,t)$$
$$p_j \rightarrow \hat{p}_j(p,t)$$

Unit tax $p_i = p + t$ gas tax

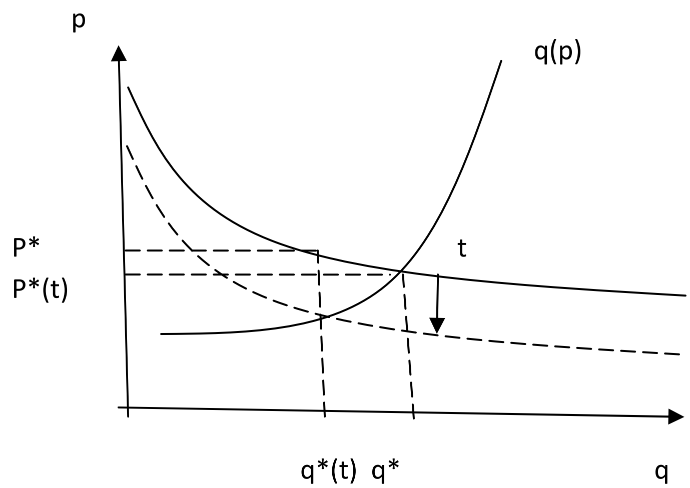
Percentage tax $p_i = p(1+t)$ sale tax

x(p) – demand without tax

q(p) – supply without tax

suppose paid by consumer $p_i = p + t$

$$x(p(t)+t)$$
$$q(p(t))$$




M equilibrium

X(p(t)+t)=q(p(t))

x'(p(t)+t)(p'(t)+1)=q'(p(t))p'(t)

$$P'(t)=\frac{x'(p(t)+t)}{x'(p(t)+t)-q'(p(t)}\Rightarrow\frac{P(t)'+1}{1}=\frac{x'(p(t)+t)}{x'(p'(t)+t)-q'(P(t))}-1=\frac{q'(p(t))}{x'(p'(t)+t)-q'(P(t))}$$

# Microeconomics Course of Professor Harold Zhang @ UTD: Market & Equlibrium

Meisam Hejazinia

11/7/2012

Last time we discussed competitive equlibrium and said that there are three main conditions:

1. Firms maximize the profit
2. Consumers maximize their utility
3. Market Clearning of goods

We looked at what are the conditions for competitor equilibrium, and we studied one good, as partial equilibrium and we focused over one market; however, in the real market substantive equlibrium of multiple equilibrium is much complex that the partial equlibrium.

We also discussed the government policy and taxation, two are somehow similar in term of calculation.

This week we will start with continuing the discussion of taxation.

$x(p)$ would be aggregate demand
$w(p)$ would be aggregate supply
$x(p*) = Q(p*)$
$p*$ equlibrium
$x(p*) = \alpha(p*)$

If there would be government intervention.

Unit Tax $p + t$
We could also have percentage tax which is in the form of $p(1 + t)$

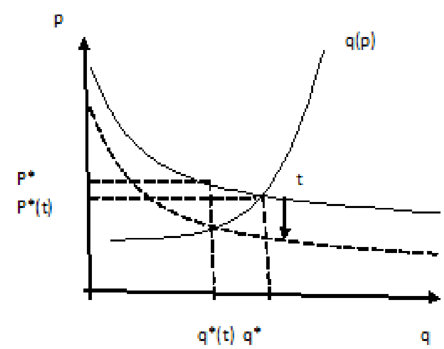



Figure 1: Tax policy

Demand and supply equlation would be in the form of $x[p(t) + t] = Q(p(t))$

We showed last week that $\frac{dP'(t)}{dt} < 0$
$P(t)$ Price recieved by producer $P(t)+t$ price paid by consumer
$P(t) \downarrow t \uparrow$

$\frac{dP(t)+t}{dt} > 0$

$P(t) + t \uparrow$ as $t \uparrow$

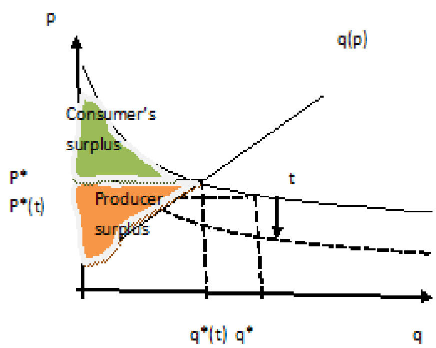



Figure 2: Consumer Producer Surplus

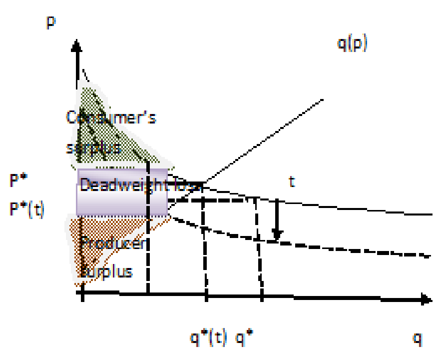



Figure 3: Deadweight Loss

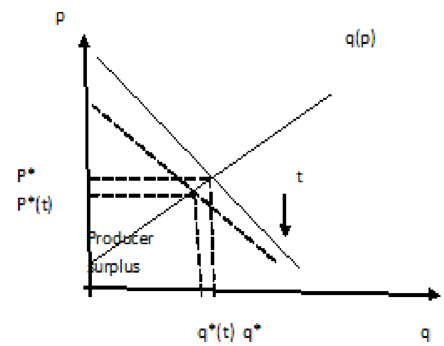



Figure 4: Percentage tax effect

For percentage tax we have the following:
$X(P(t) + tP(t)) = Q(P(t))$
$X'(P(t) + tP(t)).(P'(t) + P(t) + tP'(t)) = Q'(P(t))P'(t)$

We skipped the welfare part and from the class perspective and time constraint they are less important.

**1.E. Free- Entry and Long-run competition equlibrium**

When we say profit is zero in economy we just mean that the investment above normal return could not happen, and it means investment will move from industry to find a better place.

We assume that there is a production technology that is available for all firms and we capture it in the production function $C(q)$.
$C(0) = 0$

Aggregate demand $X(p)$

The Inverse demand function is P(.)

We want to find the long run competitive equlibrium.

This competitive equlibrium will tell us $(p, q, J)$ means price, output quantity for number of firms.

**Definition:** Given aggregate demand $x(p)$ and the cost function $c(q)$ a triple $(p^*, q^*, J^*)$ is a long run competitive equilibrium if:

- $q^*$ solves $Max(p^*q - c(q)), q \geq 0$: Firms profit maximization
- $x(p^*J^*q^*)$: Market clearning
- $p^*q^* - c(q^*) = 0$ Means free entry. If you invest in this industry you can not expect any return more than normal return, since if it would be higher the product would be increased and the profit margin will decrease, and firms will go out of market. You still need to pay wages and input, so the return would be lower, so those capitals for this industry will go to other markets.

In this case input price is fixed, since it is competitive market.

In reality this does not hold; for examle Pharmaceutical industry the cost of producing is very small, but most of the cost have been done prior to that for research, and firms can not enter due to legal protection for specific period (Patent Protection).

Let $\pi(p)$ is individual's firm's profit function.

$Q(P)$ would be aggregate supply

$$Q(p) = \begin{cases} \infty, & \text{if} \pi(p) > 0 \\ Q \geq 0, Q = Jq, q \in q(p) & \text{if} \pi(p) = 0 \end{cases}$$

For the firms solving the maximization problem is much easier than consumer, but it does not mean that you can go forward and set up lagrange multiplier, and you need to think about whether it is constant/increasing/decreasing to scale.

$P^*$ long run competitive equlibrium price.
$x(p^*) = Q(p^*)$

Case 1: $c(q) = cq, c > 0, x(c) > 0$

$p^* = c$ Marginal cost = Marginal Revenue

$Q(p^*) = J^*q^* = X(c)$

$J^*$ equlibrium number of firms
$q^*$ output per firm

$p^*q^* - cq^* = (p^* - c)q^* = 0$ You will have indeterminancy here as a result.

Case 2: C(.) increasing and convex (decreaasing return to scale)
$c'(.)$ is marginal cost
$x(c'(0)) > 0$
If $p > c'(0)$, no finite equlibrium, since $\pi(p) > 0$
If $p \leq c'(0), q = 0$
$p < c'(0)$ means you are loosing money, so $q = 0$
$p = c'(0)$ also means in the long run $c'(0) < c'(a)$ we must have $q = 0$

Case 3: c(.) has a unique efficient scale $\bar{q} > 0$. The corresponding minimize average cost would be $\bar{c} = \frac{c(\bar{q})}{\bar{q}} > 0$
$x(\bar{c}) > 0$

$p^* > \bar{c}, \pi(P^*) > 0$, No finite equlibrium
$p^* > \bar{c}, \pi(P^*) < 0$, No equlibrium exists
$p^* = \bar{c}, \pi(p^*) = 0$
$J^*\bar{q} = x(\bar{c})$
$J^* = \frac{x(\bar{c})}{\bar{q}}$ $(\bar{c}, \bar{q}, \frac{x(\bar{c})}{\bar{q}})$ : Long run equilibrium

This case is stable long term equilibrium and each firm will operate at the level corresponding to minimum average cost. Total demand will also work for price equal to minimum average cost.

If you look at the different between short run and long run it would be different between the technology change.

If we look at the technology part, the difference between short run and long run would be in the cost function.

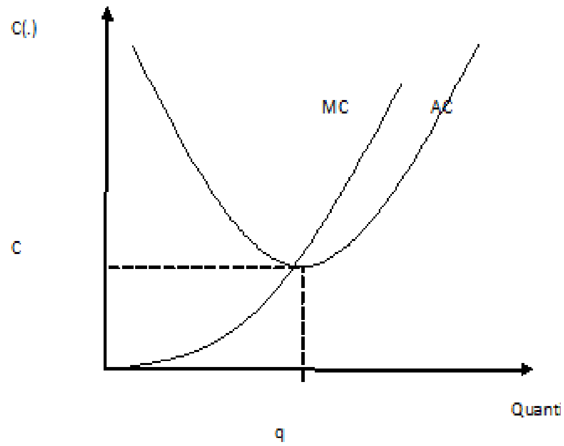



Figure 5: Average cost and Marginal cost

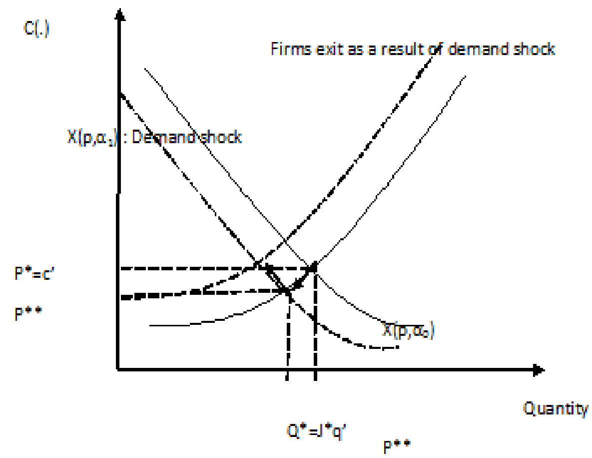



Figure 6: Demand shock and effect on supply curve [firm exit]

In long run everything is variable, and fix cost does not exist, since you will go to the board and go for new things.

**Differentation cost function**
in long run

$c(p) = \begin{cases} +\psi q, \; q > 0 \\ 0, \; q = 0 \end{cases}$

In the short run:$c(q) = k + \psi(q), q \leq 0$

**Market Power**
2.A Monopoly pricing
$x(p)$ : aggregate demand strictly decreasing as $p \uparrow$
The monopolist's objective is to max profit:
$Max_{p.q} = px(p) - c(q)$
$q = x(p)$ market clearning condition
$\Rightarrow Max_{p.q} pq - c(q)$
First order condition (.O.C):
$P'(q^m)q^m + p(q^m) \leq c'(q^n) = c'(q^m)\text{if} q^m > 0.$
$p'(q^m)q^m + p(q^m) = c'(q^m)$: MR=MC

Socially optimal outcome:Choose $q^0$ such that $p(q^0) = c'(q^0)$

$q^m < q^0$
$p(q^m) > p(q^o)$

E.g. Lineare inverse function.

$p(q) = a - bq$
cost function $c(q) = cq$

$a > c > 0.$

$max_q p(q)q - c(q)$
$max_q (a - bq)q - cq$

First order condition:

$a - 2bq - c = 0$

$q^m = \frac{a-c}{2b}$

$p(q^m) = a - b\frac{a-c}{2b} = \frac{a+c}{2}$
optimal therefore would be $(\frac{a+c}{2}, \frac{a-c}{2b})$
For socially optimal profit we will have:
$p(q^o) = c$

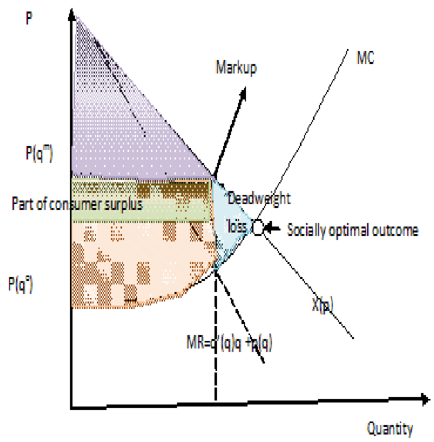



Figure 7: Monopoly equlibrium and social optimum outcome

$a - bq^o = c \rightarrow q^o = \frac{a-c}{b}$
$p(q^o) = c$
$\Rightarrow (c, \frac{a-c}{b})$

Static Models of oligopoly:

The Bertrand model of price competition. (Two firms so it would be duopoly)

Two profit maximizing firms 1,2, (duopoly).

The aggregate demand x(p).

$MC = C$: marginal cost by both firms

both firms will put their price simulteneously (it is important since otherwise it would be two stage game).

Name prices $P_1$ and $P_2$

sales for firm j and k:

$$x_j(p_j, p_k) = \begin{cases} x(p_j), & \text{if } p_j < p_k \\ 1/2x(p_j) & \text{if } p_j = p_k \\ 0 & \text{if } p_j > p_k \end{cases}$$

Firm j's profit $(p_j - c)X_J(p_j, p_k)$

A Nash equilibrium is a strategy profile $S = (S_1, S_2, ..., S_I)$ of a game this would be Nash game represented by: $N = [I, \{S_i\}, \{u_i, (.)\}]$ such that for every $i = 1, 2, ..., I$
$S_i$ strategy of other players.
$u_i(S_i, S_{-i}) \geq u_i(S_i', S_{-i})$ for all $S_i' \in S_i$

Players know their strategy, and it is one shot game.

If it is nash equilibrium the strategy that you pick conditional on the strategies that other people pick would be of greater payoff than any other strategy that you pick.

Next time we will discuss about Bertrand market condition and will prove that it is going to be Nash Equlibrium.

# Microeconomics of Professor Harold Zhang @ UTD: Bertrand Model

Meisam Hejazinia

15 November, 2012

**Static Model of Oligopoly Chapter 11 & 12**

**The Bertrand Model**

two profit Maximizing firms

c= MC= AC $> 0$ (Equal to marginal cost, and Average cost, regarding production technology

Demand $x(p) > 0$ downward sloping

$\frac{\partial x(p)}{\partial p} < 0$ , $x(c) > 0$

At the same time each firm will price $p_1$ and $p_2$

firm j's sales, j $k, j = 1, 2,\ k \neq j$

$$\mathrm{x}_j(p_j, p_k) = \begin{cases} 0 & \text{if } P_j > P_k \\ \frac{1}{2}x(p_j) & \text{if } p_j = p_k \\ x(p_j) & \text{if } P_j < P_k \end{cases}$$

Proposition: There is a Nash equilibrium $(P_1^*, P_2^*)$ in the Bertrand model. Both firms set their prices equal to marginal cost. $P_1^* = P_2^* = c$

Proof: Both firms make zero economic profit. Niether firm would benefit from raising its price above c, or lowering its price below c, so this is an equilibrium.

First. $\min\{p_1^*, P_2^*\} < c$, suppose that firm j, $p_j^* = \min\{p_1^*, p_2^*\} < c$, as a result $\pi_j < 0$, not an equilibrium.

Second. $p_J = C$ and $p_k > c$. Then firm $j$ will price its product at $p_j = c + \frac{p_k - c}{2}$ and will capture the entire market. Firm $j$ makes positive profit. Incentive to deviate; therefore it would not be equlibrium.

Third. $p_j > c$ and $p_k > c$
assume $p_j \leq p_k$. The maximum profit form firm k is $\frac{1}{2}(P_j - c)x(p_j)$
Try $p_k = p_j - \epsilon\ \epsilon > 0$ and small. It's profit is $(p_j - \epsilon - c)x(p_j - \epsilon)$. If $\epsilon > 0$ is small, $(p_j - \epsilon - c)x(p_j - \epsilon) > \frac{1}{2}(p_j - c)x(p_j)$ therefore here also there is incentive to deviate, so there would not be an equlibrium.

The bottom line is that in Bertrand condition there is Nash equilibrium, but it would be diffiduclt to find if price would be greater than marginal cost.

In airline industry the product is homogeneous, although their timeline is different, most of customers view them as substitute. Any time one airline decreases their price, other airline do the same. All will not set their price high, because if all of them increase their prices, then one will think that if I decrease the price subtly I will capture more market; moreover, the antitrust law also does not allow these companies to do that. If there would be evidance of private collaboration, it would be illigal, called price fixing.

**Quantity competition (The Cournet Model)**

Two firms maximize their profit, and they decide how much to produce.

$q_1$ and $q_2$ would be their quantity. The price will be adjusted to the level such that the quantity demanded would be equal to quantity supply.

$x(p(q_1+q_2)) = q_1 + q_2$

Inverse demand functuion $p(q_1+q_2)$

$MC = AC = c > 0$

$P(0) > c$

$p(q^0) = c$

$q^0 > 0$ socially optimal output is positive.

Consider firm j's decision, $j, k = 1, 2, j \neq k$. The decision would be:

$cd : q_j \geq 0$
$max_{cd} p(q_j + \bar{q_k})q_j - cq_j$

There would be conjecture of each player about what the other player is going to do, and vice versa and when two are equal we would have equlibrium.

First order condition:
$p'(q_j + \bar{q_k})q_j + p(q_j + \bar{q_k}) \leq c$

$= c$ if $q_j^* > 0$

This first order condition will give us best response function of firm j to firm k.

$b_j(\bar{q_k}), j, k = 1, 2, j \neq k$

**Definition** : A pair of quantity choices $(q_1^*, q_2^*)$ is a Nash equilibium, if and only if $q_j^* \in b_j(q_k^*)$ for $j, k = 1, 2$ and $j \neq k$.

Thus, $(q_1^*, q_2^*)$ satisfies $p'(q_1^*+q_2^*)q_1^*+p(q_1^*+q_2^*) \leq c$ $= c$ if $q_1^* > 0$

$p'(q_1^* + q_2^*)q_2^* + p(q_1^* + q_2^*) \leq c$
$= c$ if $q_2^* > 0$ .

First show $q_1^* > 0$ and $q_2^* > 0$

Proof. Suppose that $q_1^* = 0 \Rightarrow p(q_2^*) \leq c$ (**)

$p'(q_2^*)q_2^* + p(q_2^*) \leq c.$

if $q_2^* > 0$

$p'(q_2^*)q_2^* + p(q_2^*) = c$ (1)

From (**) we have $p'(q_2^*)q_2^* + p(q_2^*) < c$ (2)

(1) and (2) have contradition.

If $q_2^* = 0$ $p(0) \leq c$ contradicts $p(0) > c$

$q_1^* > 0$

$q_2^* > 0$

So following equation summerises the first order condition of both cases:

$p'(q_1^* + q_2^*)\frac{q_1^*+q_2^*}{2} + p(q_1^* + q_2^*) = c$

Proposition: In the Nash equlibrium of the Cournet model, with $c > 0$ per unit of output and an inverse demand fucntion p(.) with $p'(.) < 0$ for all $q \geq 0$ and $p(0) > c$, the market price is greater than $c$ (the competitive price), and less than the monopoly price.

$c < p(q_1^* + q_2^*) < p(q^m)$ where $q^m$ is monopoly output

Proof: $p'(q_1^* + q_2^*)\frac{q_1^*+q_2^*}{2} + p(q_1^* + q_2^*) = c \Rightarrow p(q_1^* + q_2^*) > c.$

To show $p(q_1^* + q_2^*) < p(q^m)$ we show $q_1^* + q_2^* > q^m$.

Suppose $q^m > q_1^* + q_2^*$

Consider $\hat{q_j} = q^m - q_k^*$

Total profit increases, and profit of firm x decreases $\Rightarrow$ firm j has incentive to deviate $\Rightarrow$ Contradiction, mean $q^m > q_1^* + q_2^*$ is not true.

Still need to show that $q^m \neq q_1^* + q_2^*$

Suppose $q^m = q_1^* + q_2^*$

Recall that first order condition for monopoly is $p'(q^m)q^m + p(q^m) = c$.

$p'(q^m)\frac{q^m}{2} + p(q^m) = c$ which is contradiction.

E.g. Inverse demand function, $p(q) = a - bq$ $a > 0$ and $b > 0$

$AC = MC = c > 0$ $a > c$

Socially optimal solution would be $p(q^0) = c$ & $a - bq^0 = c \Rightarrow q^0 = \frac{a-c}{b}$ & $p(q^0) = c$

Monoply situation: $q^m = \frac{a-c}{2b}$ & $p^m = \frac{a+c}{2} > c$

Cournot model: $max_{q_j}(a - bq_j - b\bar{q_k})q_j - cq_j$

First order condition would be: $-bq_j + (a - bq_j - bq_k) = c$

best response function would be $q_j = \frac{a-c-bq_k}{2b}$

as symmetry: $q_j = q_k$ & $\frac{a-c-bq_j}{2b} = q_j$

$q_j* = q_k^* = \frac{1}{3}\frac{a-c}{b}$

$p(q_1^* + q_2^*) = \frac{1}{3}(a + 2c) = c + \frac{1}{3}(a - c)$

$q_1^* + q_2^* = \frac{2}{3}\frac{a-c}{b}$

$p'(q_1^* + q_2^*)\frac{q_1^*+q_2^*}{2} + p(q_1^* + q_2^*) = c$

$p(Q*)\frac{Q*}{2} + P(Q^*) = c$

For n number of firms we will have $p(Q*)\frac{Q*}{n} + P(Q^*) = c$ and as $n \to \infty$ it would become $P(Q^*) = c$ which would be competitive case, and for $n = 1$ it would become monpoly; therefore this is a useful equation. Figure 1 shows the capacity constraint and this equation implication.

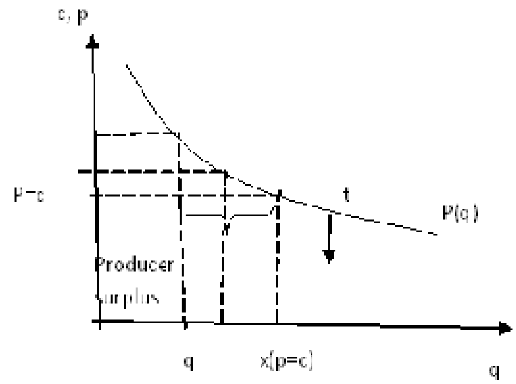



Figure 1: Capacity Constraint

**Production Differentiation**

A Linear city Model

Consider a city represented by a line segment

M Consumers lying uniformly on this line

Two firms on either end of the city. Figure 2 shows the case.

The cost of production is $c > 0$ for both firms, and the product is the same.

Each consumer buys one unit of the product

The cost of buying the product would be

$p_j + \frac{t}{2}(2d) = p_j + td$

$2d$ is for round trip, so $\frac{t}{2}$ is travel cost per distance

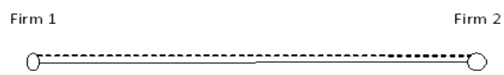



Figure 2: Linear City

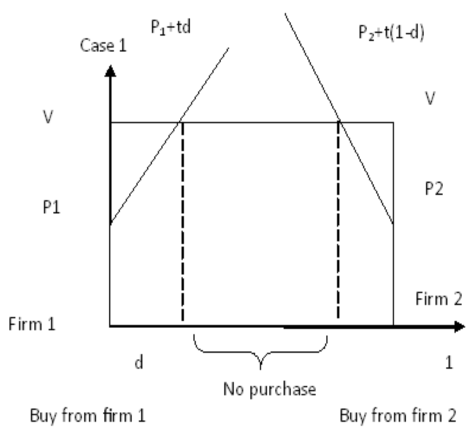



Figure 3: City Model Case 1

To Find $\hat{z}$

$p_1 + t\hat{z} = p_2 + t(1-\hat{z})$

$p_1 + t\hat{z} = p_2 + t - t\hat{z}$

$\hat{z} = \frac{t+p_2-p_1}{2t}$

Figure 3 and 4 show the two possible cases that we first analyze case 2.

Firm 1's demand.

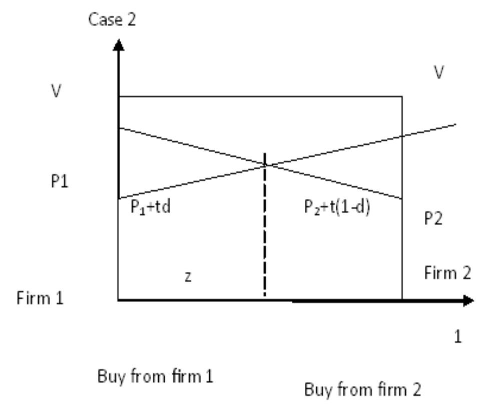



Figure 4: City Model Case 2

$$\mathrm{x}_1(p_1,p_2) = \begin{cases} 0 & , \hat{z} < 0 \\ \hat{z}m & \hat{z} \in [0,1] \\ M & \hat{z} > 1 \end{cases}$$

$$\mathrm{x}_1(p_1,p_2) = \begin{cases} 0 & , \text{if } p_1 > p_2 + t \\ \frac{(t+p_2-p_1)m}{2t} & \text{if } p_1 \in [p_2 - t, p_2 + t] \\ M & \text{if } p_2 > p_1 + t \Rightarrow P_1 < P_2 - t \end{cases}$$

For firm $j, j = 1, 2$, If $P_j > P_{-j} + t$ firm's profit j would be zero.

If $P_j < P_{-j} - t$, firm j makes Less profit than setting $P_j = P_{-j} - t$

Firm j's best response:

$max_{p_j}(p_j - c)(t + \bar{p}_{-j} - p_j)\frac{m}{2t}$

so that $p_j \in [\bar{p_j} - t, \bar{p}_{-j} + t]$

First order condition

$$\mathrm{t}+\bar{p}_{-j}+c-2p_j \begin{cases} \leq 0 & , \text{if } p_j > \bar{p}_{-j} - t \\ = 0, & \text{if } p_1 \in [p_2 - t, p_2 + t] \\ \geq & \text{if } pj > \bar{p}_{-j} + t \Rightarrow P_1 < P_2 - t \end{cases}$$

$t + \bar{p}_{-j} + c - 2(\bar{P}_{-j} + t) = 0 \Rightarrow \bar{P}_{-j} = c - t$

$t + \bar{p}_{-j} + c - 2(\bar{p}_{-j} - t) = 0$

$p_{-j}^{-} = c + 3t$

$$\text{b}(p_{-j}^{-}) \begin{cases} p_{-j}^{-} + t & \text{, if } p_j > p_{-j}^{-} \leq c - t \\ \frac{t + p_{-j}^{-} + c}{2} & \text{if } p_{-j}^{-} \in [c - t, c + 3t] \\ p_{-j}^{-} - t \text{ if} & p_{-j}^{-} \geq c + 3t) \end{cases}$$

$P_j = \frac{t + p_j + c}{2}$

$p^* = p_j = p_{-j} = \frac{t + p* + c}{2} \Rightarrow p^* = c + t$

This simple model has interesting implication for for gas station as you can see if they are close to eachother they have the same price, but if they are not the price would be different.

# Microeconomics of Professor Harold Zhang @ UTD: Bertrand Model

Meisam Hejazinia

29 November, 2012

Two cheat sheets mean four pages could be brought for the exam on next week. The review questions has been posted; the exam would be on chapter 5, 6, 10, 12, focused on the second half.

Last time we discussed product defferentiation, and we completed our discussion on oligopoly. In the specific example that we discussed which was travel distance, that particular example could be generalized to anything, such as product quality. As a result you can think of many variations that are extension of product quality. The discussion was based on the assumption that we know everything. The reality is not so simple, and there are uncertainties that occurs wherever we make the decision. Consumer's decision are mostly in the investment form, putting money in charging or saving account, or you say I want to take a risk and invest in mutual fund, and then your target may not be actualize. From firm point of view, also we have the same situation, for example pharmaceutical companies that they invest millions of dollars in RD, yet at the FDA said not approved and you are done.

Today, we start from consumer decision and then the same logic exactly could be applied to firm's decision. As a result our discussion will be focused on consumer's decision, since it would be difficult to deal with.

**II. Choice under uncertainty**
**Expected utility theory**

C: the set of all prossible outcome. C: firm finite & index by $n = 1, 2, ..., N$

simple lottary:

**Definition** A single lottery is a list $L = (p_1, p_2, .., p_N)$ with $p_n \geq 0$ for all n and $\sum_n P_n = 1$ where $p_n$ is implemented with the probability of n occuring.

In reality you have risks attached to set of outcomes, and different assets are exposed to different risks. In those cases we look at the portfolio as a whole, what would be probability on set of outcome. In other word we try to reduce the compound lottery to simple lotteries.

**Definition of Compound lottary** Give $k$ simple lotteries $l_k = (p_1^k, p_2^k, \ldots, p_N^k)$, $k = 1, 2, \ldots, k$, and probability $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, the compound lottery $(L_1, L_2, \ldots, L_k; \alpha_1, \alpha_2, \ldots, \alpha_k)$ is the riskey alternative that yields the simple lottery $L_k$ with probability $\alpha_k$ for $k = 1, 2, \ldots, k$.

The reduced simple lottery $L = (p_1, p_2, .., p_N)$ would be $p_n = \alpha_1 p_n^1 + \alpha_2 p_n^2 + \ldots + \alpha_k p_n^k$, $n = 1, 2, \ldots, N$.

Here is an example:

$c = \{1, 2, 3\}$

$$\begin{cases} \frac{1}{3} & L_1 = (1, 0, 0) \\ \frac{1}{3} & L_2 = (\frac{1}{4}, \frac{1}{8}, \frac{5}{8}) \\ \frac{1}{3} & L_3 = (\frac{1}{4}, \frac{3}{8}, \frac{3}{8}) \end{cases}$$

$L = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$

$$\left\{ \begin{array}{ll} \frac{1}{2} & L_4 = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6}) \\ \frac{1}{2} & L_5 = (\frac{1}{2}, 0, \frac{1}{2}) \end{array} \right.$$

$L = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$

$L$- the set of all simple lotteries over the set of outcomes C.

Rational preference $\succ$ on $L$

**Continuity Axiom and independence Axiom** $I.A.$

Definition: The preference relation $\succ$ on $L$ is contineous if for any $L$, $L'$, and $L'' \in L$ the set $s$ $\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succ L''\} \subset [0,1]$ and $\{\alpha \in [0,1] : L'' \succ \alpha L + (1-\alpha)L' \succ L''\} \subset [0,1]$ are closed.

The contineuity axiom implies the eistance of utility function representing $\succ$ , $U : L \to R$ such that $L, L' \in L$

$L \succ L'$ if $U(L) \geq U(L')$

Definition: The preference relation $\succ$ satisfies $I.A.$ if for all $L, L', L'' \in L$ and $\alpha \in (0,1)$ we have $L \succ L'$ if and only if $\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''$

If adding the third one does not change your decision mean they are independent, yet if not, they would be dependent.

Definition: The utility function $\cup : L \to R$ has an expected utility form if there is an assignment of numbers $(u_1, u_2, \ldots, u_N)$ to the $N$ outcomes such that for many simple lottery $L = (p_1, p_2, \ldots, p_N) \in L$ we have $\cup(L) = p_1 u_1 + p_2 u_2 + \ldots + p_N u_N = \sum_{n=1}^{N} p_n u_n$. This form is called Von Neuman-Morgenstern expected utility.

**Proposition** A utility funciton $\cup : L \to R$ has an expected utility form if and only if it is linear, that is $\cup(\sum_{k=1}^{k} \alpha_k L_k) = \sum_{k=1}^{k} \alpha_k \cup (L_k)$ $(*)$ for any k lotteries $L_k \in L$, $k = 1, 2, \ldots, K$, and probability $\alpha_1, \ldots, \alpha_k) \geq 0$, $\sum_k \alpha_k = 1$.

Proof. Suppose $\cup(.)$ satisfies $(*)$ write $L = \sum_n P_n L^n$ where $L''$ corresponds to lottery that yields outcome $n$ with probability 1.

Mean we will degenerate it. $\cup(L^n) = u_n$. This, $\cup(L) = \cup(\sum_n P_n L^n) = \sum_{n=1}^{N} P_n \cup (L^n) = \sum_{n=1}^{N} P_N U_N$ This is the expected utility.

Suppose that $\cup(.)$ has the expected utility form. Consider $(L_1, L_2, \ldots, L_k; \alpha_1, \ldots, \alpha_k)$ where $L_k = (p_1^*, p_2^*, \ldots, p_N^*)$ its reduced lottery is $L' = \sum_k \alpha_k L_k$

$\cup(\sum_k \alpha_k L_k) = \sum_n \cup_n (\sum_k \alpha_k p_n^k) = \sum_k \alpha_k (\sum_n \cup_n P_n^k) = \sum_k \alpha_k \cup (L_k)$

**Proposition: The expected utility theorem**

Suppose that the rational prefere $\succ$ on $L$, satisfies Contineuity axiom and Independent Axiom then $\succ$ admits a utility representation of the expected utility form, that is, we can assign, a number $U_n$ to each outcome $n = 1, 2, 3, \ldots, N$ in such a way that for any two lotteries $L = (p_1, p_2, \ldots, p_N)$ and $L' = (p_1', p_2', ..p_N')$, we have $L \succ L'$ if and only if $\cup(L) \geq \cup(L')$ mean $\sum_n p_n u_n \geq \sum_n p_n' u_n$.

**IV. Money Lotteries and Risk aversion**

$x$ amount of money

Cumulative distribution function (CDF)- F(x)

$F(x) = \int_{-\infty}^{x} \alpha F$

$(L_1, L_2, \ldots, L_k; \alpha_1, \alpha_2, \ldots, \alpha_k)$

where $L_i = F_i(\alpha)$

$F(x) = \sum_{k=1}^{K} \alpha_k F_k(x)$

$u_1, u_2, \ldots, u_N$

u(x) - utility associated with amount of money x

$u'(x) > 0$, $u''(x) < 0$

$u(x)$ is Bernouli utility

The expected utility:

$\cup(F) = \int u(x)dF$ which is from Von-Neumann Morgenston expected utility.

**Risk aversion and its measurement**
A decision maker is Risk averse if for any lottery $F(.)$ the degneerate lottery that yeields the amount $\int xdF(\alpha)$ with certainty is at least as good as the lottery if the decision maker is always indifferent between these two, he is risk neutral, and further, we say that he is strictly risk averse if the indifference holds only when $F(.)$ is degenerated.

What does it mean? $u(\int xdF(x))$

The uncertainty is $\int u(x)dF(x)$

If $u(\int xdF(x)) \equiv \int u(x)dF(x)$ you are risk neutral, and if $u(\int xdF(x)) \geq \int u(x)dF(x)$ means you are risk averse and $u(\int xdF(x)) > \int u(x)dF(x)$ mean you are strictly risk averse.

Risk aversion $\iff$ Utility function is concave. For this you can find the figure of this online, and get understanding from cartesian representation.

**Definition: Certainty Equivalent**

Given the bernouli utility function u(.), we define Centrainty Equivalent $F(.)$ as $C(F, u)$ as the amount of money for which the decision maker is indifferent between the lottery $F(.)$ and the certain amount $C(F, u)$ that is $U(C(F, u)) = \int u(x)dF(x)$.

For a risk averse decision maker you know that certain equivalent is definitely less than average pay off $c(F, n) \leq \int xdF(x) \equiv$ Utility function would be concave.

Example: Insurance.

Initial wealth is $w$

Possible loss is $D$

Probability of loss is $\pi$

$q$ cost of one unit of insurance paying \$1 in case of loss.

$\Phi$ insurance premium

$\alpha$ coverage (how many insurance to buy)
we loss. $w - \alpha q$ , and the probability would be $1 - \pi$

Loss $w - \alpha q - D + \alpha$, $\pi$

$\max_{\alpha \geq 0}(1-\pi)u(w - \alpha q) + \pi u(w - \alpha q - D + \alpha)$

First order condition at $\alpha^*$: $q(1-\pi)u'(w - \alpha^* q) + \pi(1-q)u'(w - \alpha * q - D + \alpha*) = 0$

Acturielly fair $\Rightarrow q = \pi \times \$1 + (1-\pi) \times \$0 = \pi$

$\Rightarrow u'(w - D + \alpha^*(1-\pi)) = u'(w - \alpha^* \pi) \Rightarrow D + \alpha^*(1-\pi) = \alpha^* \Rightarrow \alpha^* = D$

**The measurements of risk aversion**

Absolute risk averion $r_A$

Relative risk aversion $r_R$

**Definition of absolute risk aversion**

Given a Bernoulli utility function $u(.)$ the arrow-ratt coefficient of absolute risk aversion at $x$ is given by $r_A(x) = -\frac{u''(x)}{u'(x)}$

For hedge funds, there is less regulation since absolute risk aversion is higher for them, since they have money and they can handle it.

Example: exponential utility function $u(x) = -e^{-ax}$, so $u'(x) = ae^{-ax}$ and $u''(x) = -a^2 e^{-ax}$

$r_A(x) = \frac{a^2 e^{-ax}}{ae^{-ax}} = a$

**Definition: (Relative Risk aversion)**

Given $u(.)$, the relative risk aversion coefficient at x is $r_R(x) = -\frac{xu''(x)}{u'(x)} = xr_A(\alpha)$

example: Constant relative Risk averion utility function: $u(x) = \frac{x^\gamma - 1}{1-\gamma} = \frac{1-x^\gamma}{1-\gamma}$ and $u'(x) = x^{-\gamma}$, $u''(x) = -\gamma x^{-\gamma-1}$, and you will get $r_R(x) = -\frac{x(-\gamma x^{-\gamma-1})}{x^{-\gamma}} = \frac{\gamma x^{-\gamma}}{x^{-\gamma}} = \gamma$