```
theory Warmup-Problem-D
  imports
    Complex-Main
    Common.Future-Library
    HOL−Analysis.Analysis
begin
```

0.1 Warmup problem D

There is a set of n points on a plane with the property that, in each triplet of points, there's a pair with distance at most 1. Prove that the set can be covered with two circles of radius 1.

There's nothing special about the case of points on a plane, the theorem can be proved without additional difficulties for any metric space:

```
theorem warmup4-generic:
  fixes S :: 'a::metric-space set
  assumes finite S
  assumes property: ⋀T. T ⊆ S ∧ card T = 3 ⟹ ∃p∈T. ∃q∈T. p ≠ q ∧ dist p q ≤ 1
  obtains O1 O2 where S ⊆ cball O1 1 ∪ cball O2 1
proof
  let ?pairs = S × S
  let ?dist = λ(a, b). dist a b
  define widest-pair where widest-pair = arg-max-on ?dist ?pairs
  let ?O1 = (fst widest-pair)
  let ?O2 = (snd widest-pair)
  show S ⊆ cball ?O1 1 ∪ cball ?O2 1
  proof
    fix x
    assume x ∈ S

    from ‹finite S› and ‹x ∈ S›
    have finite ?pairs and ?pairs ≠ {} by auto
    hence OinS: widest-pair ∈ ?pairs
      unfolding widest-pair-def by (simp add: arg-max-if-finite)

    have ∀(P,Q)∈?pairs. dist ?O1 ?O2 ≥ dist P Q
      unfolding widest-pair-def
      using ‹finite ?pairs› and ‹?pairs ≠ {}›
      by (metis (mono-tags, lifting) arg-max-greatest prod.case-eq-if)
    hence greatest: dist P Q ≤ dist ?O1 ?O2 if P ∈ S and Q ∈ S for P Q
      using that by blast

    let ?T = {?O1, ?O2, x}
    have TinS: ?T ⊆ S using OinS and ‹x ∈ S› by auto

    have card ?T = 3 if ?O1 ≠ ?O2 and x ∉ {?O1, ?O2} using that by auto
    then consider
      (primary) card ?T = 3 |
      (limit) x ∈ {?O1, ?O2} |
      (degenerate) ?O1 = ?O2 by blast
    thus x ∈ cball ?O1 1 ∪ cball ?O2 1
    proof cases
      case primary
      obtain p and q where p ≠ q and dist p q ≤ 1 and p ∈ ?T and q ∈ ?T
        using property [of ?T] and ‹card ?T = 3› TinS
        by auto
      then have
        dist ?O1 ?O2 ≤ 1 ∨ dist ?O1 x ≤ 1 ∨ dist ?O2 x ≤ 1
        by (metis dist-commute insertE singletonD)
      thus x ∈ cball ?O1 1 ∪ cball ?O2 1
        using greatest and TinS
```

by *fastforce*
next
case *limit*
then have $\mathit{dist}\ x\ ?O_1 = 0 \lor \mathit{dist}\ x\ ?O_2 = 0$ **by** *auto*
thus *?thesis* **by** *auto*
next
case *degenerate*
with *greatest* **and** *TinS* **have** $\mathit{dist}\ ?O_1\ x = 0$ **by** *auto*
thus *?thesis* **by** *auto*
qed
qed
qed

Let's make sure that the particular case of points on a plane also works out:

corollary *warmup4*:
fixes $S :: (\mathit{real}\ \widehat{}\ 2)\ \mathit{set}$
assumes *finite S*
assumes *property*: $\bigwedge T.\ T \subseteq S \land \mathit{card}\ T = 3 \Longrightarrow \exists p{\in}T.\ \exists q{\in}T.\ p \neq q \land \mathit{dist}\ p\ q \leq 1$
obtains $O_1\ O_2$ **where** $S \subseteq \mathit{cball}\ O_1\ 1 \cup \mathit{cball}\ O_2\ 1$
using *warmup4-generic* **and** *assms* **by** *auto*

end