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theory Problem-1
  imports Complex-Main
begin

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## 0.1 Problem 1

Solve the equation in the integers:

**theorem** *problem1*:

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  fixes x y :: int
  assumes x ≠ 0 and y ≠ 0
  shows 1 / x2 + 1 / (x*y) + 1 / y2 = 1
    ⟷ x = 1 ∧ y = -1 ∨ x = -1 ∧ y = 1
  (is ?eqn ⟷ ?sols)

```

**proof**

— Unfortunately, removing the conversions between int and real takes a few lines

let ?x = real-of-int x and ?y = real-of-int y

assume ?eqn

then have 1 / ?x<sup>2</sup> + 1 / (?x\*?y) + 1 / ?y<sup>2</sup> = 1 by auto

hence ?x<sup>2</sup>\*?y<sup>2</sup> / ?x<sup>2</sup> + ?x<sup>2</sup>\*?y<sup>2</sup> / (?x\*?y) + ?x<sup>2</sup>\*?y<sup>2</sup> / ?y<sup>2</sup> = ?x<sup>2</sup>\*?y<sup>2</sup>

by algebra

hence ?x<sup>2</sup> + ?x\*?y + ?y<sup>2</sup> = ?x<sup>2</sup> \* ?y<sup>2</sup> using ⟨x ≠ 0⟩ ⟨y ≠ 0⟩

by (simp add: power2-eq-square)

hence inteq: x<sup>2</sup> + x\*y + y<sup>2</sup> = x<sup>2</sup> \* y<sup>2</sup>

using of-int-eq-iff by fastforce

define g where g = gcd x y

then have g ≠ 0 and g > 0 using ⟨x ≠ 0⟩ ⟨y ≠ 0⟩ by auto

define x' y' where x' = x div g and y' = y div g

then have x' \* g = x and y' \* g = y using g-def by auto

from inteq and this have g<sup>2</sup> \* (x'<sup>2</sup> + x' \* y' + y'<sup>2</sup>) = x'<sup>2</sup> \* y'<sup>2</sup> \* g<sup>4</sup>

by algebra

hence reduced: x'<sup>2</sup> + x' \* y' + y'<sup>2</sup> = x'<sup>2</sup> \* y'<sup>2</sup> \* g<sup>2</sup> using ⟨g ≠ 0⟩ by algebra

hence x' dvd y'<sup>2</sup> and y' dvd x'<sup>2</sup>

by algebra+

moreover have coprime x' (y'<sup>2</sup>) coprime (x'<sup>2</sup>) y'

unfolding x'-def y'-def g-def

using assms div-gcd-coprime by auto

ultimately have is-unit x' is-unit y'

unfolding coprime-def by auto

hence abs1: |x'| = 1 ∧ |y'| = 1 using assms by auto

then consider (same-sign) x' = y' | (diff-sign) x' = -y' by fastforce

thus ?sols

**proof** cases

case same-sign

then have x' \* y' = 1

using abs1 and zmult-eq-1-iff by fastforce

hence g<sup>2</sup> = 3

using abs1 same-sign and reduced by algebra

hence 1<sup>2</sup> < g<sup>2</sup> and g<sup>2</sup> < 2<sup>2</sup> by auto

hence 1 < g and g < 2

using ⟨g > 0⟩ and power2-less-imp-less by fastforce+

hence False by auto

thus ?sols by auto

**next**

case diff-sign

then have x' \* y' = -1

using abs1

by (smt mult-cancel-left2 mult-cancel-right2)

hence g<sup>2</sup> = 1

using abs1 diff-sign and reduced by algebra

hence g = 1 using ⟨g > 0⟩

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      by (smt power2-eq-1-iff)
    hence  $x = x'$  and  $y = y'$ 
      unfolding  $x'$ -def and  $y'$ -def by auto
      thus ?sols using abs1 and diff-sign by auto
    qed
  next
    assume ?sols
    then show ?eqn by auto
  qed
end

```