The following is a formal, computer-checked proof in Isabelle/HOL. This differs only slightly from the usual mathematical notation. Most noticeably, function application is written $f\ x$, not $f(x)$. *Suc n* is simply $n + 1$, but in a form that works better for automation.

theory *PoTD-15*
  imports
    HOL−Analysis.Analysis
    HOL−Library.Quadratic-Discriminant
begin

context
  fixes $a :: real$
  assumes $a > 0$
begin

We will define the sequence as a function $x : \mathbb{N} \to \mathbb{R}$.

fun $x :: nat \Rightarrow real$ **where**
  $x\ 0 = 0\ |$
  $x\ (Suc\ n) = a + (x\ n)^2$

lemma *x-nonneg*: $x\ n \geq 0$
  using $\langle a > 0\rangle$ **by** (*induction n*, *auto*)

lemma *x-incseq*: $x\ n \leq x\ (Suc\ n)$
proof (*induction n*)
  case *0*
  from $\langle a > 0\rangle$ **show** $x\ 0 \leq x\ (Suc\ 0)$ **by** *simp*
next
  case (*Suc k*)
  from $\langle x\ k \leq x\ (Suc\ k)\rangle$
  have $(x\ k)^2 \leq (x\ (Suc\ k))^2$
    using *x-nonneg* **by** (*smt* (*z3*) *power2-le-imp-le*)
  hence $a + (x\ k)^2 \leq a + (x\ (Suc\ k))^2$ **by** *auto*
  thus $x\ (Suc\ k) \leq x\ (Suc\ (Suc\ k))$ **by** *auto*
qed

theorem *convergent x* $\longleftrightarrow a \leq 1/4$
proof
  assume *convergent x*
  then obtain L **where** $x \longrightarrow L$
    using *convergent-def* **by** *auto*
  define f **where** $f\ u = a + u^2$ **for** $u :: real$
  have *continuous* (*at u*) *f* **for** $u :: real$
    unfolding *f-def* **by** *auto*
  with $\langle x \longrightarrow L\rangle$ **have** $(f \circ x) \longrightarrow f\ L$
    using *continuous-at-sequentially* **by** *auto*
      — *continuous-at-sequentially* is the theorem usually known as sequential continuity.
  moreover have $(f \circ x) \longrightarrow L$
  proof −
    have $f \circ x = (\lambda n.\ x\ (Suc\ n))$
      using *f-def* **by** *auto*
        — i.e. $f \circ x$ is the same sequence as x, but without the first element.
    thus $(f \circ x) \longrightarrow L$
      using $\langle x \longrightarrow L\rangle$ **and** *LIMSEQ-Suc* **by** *fastforce*
  qed
  ultimately have $f\ L = L$
    by (*rule LIMSEQ-unique*)
  hence $L^2 - L + a = 0$
    unfolding *f-def* **by** *simp*
  hence *discrim 1* (-1) $a \geq 0$
    using *discriminant-negative*[*of* $1\ {-1}\ a$] **by** *fastforce*
  thus $a \leq 1/4$

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    unfolding discrim-def by simp
next
  assume a ≤ 1/4
  have x n ≤ 1/2 for n
  proof (induction n)
    case 0
    then show x 0 ≤ 1/2 by simp
  next
    case (Suc k)
    from ‹x k ≤ 1/2›
    have (x k)² ≤ (1/2)²
      using x-nonneg by (smt (z3) power2-le-imp-le)
    hence a + (x k)² ≤ 1/2
      using ‹a ≤ 1/4› by (simp add: power2-eq-square)
    then show x (Suc k) ≤ 1/2 by simp
  qed
  with x-incseq obtain L where x ⟶ L and ∀ n. x n ≤ L
    using incseq-convergent by (blast intro!: incseq-SucI)
  thus convergent x by (auto simp add: convergent-def)
qed

end
end
```