theory *PoTD-15*
imports
HOL−Analysis.Analysis
HOL−Library.Quadratic-Discriminant
begin

Let a be a positive real number. Define a sequence x_n by

$x_0 = 0,\ x_{n+1} = a + {x_n}^2,\ n \geq 0.$

Find a necessary and sufficient condition on a for a finite limit $\lim_{n\to\infty} x_n$ existing.

context
fixes $a :: real$
assumes $a > 0$
begin

We will define the sequence as a function $x : \mathbb{N} \to \mathbb{R}$.

fun $x :: nat \Rightarrow real\ (x_{-})$ **where**
$x_0 = 0\ |$
$x_{Suc\ n} = a + {x_n}^2$

lemma *x-nonneg*: $x_n \geq 0$
using $\langle a > 0\rangle$ **by** (*induction n*, *auto*)

lemma *x-incseq*: $x_n \leq x_{Suc\ n}$
proof (*induction n*)
case *0*
from $\langle a > 0\rangle$ **show** $x_0 \leq x_{Suc\ 0}$ **by** *simp*
next
case (*Suc k*)
from $\langle x_k \leq x_{Suc\ k}\rangle$
have ${x_k}^2 \leq {x_{Suc\ k}}^2$
using *x-nonneg* **by** (*smt* (*z3*) *power2-le-imp-le*)
hence $a + {x_k}^2 \leq a + {x_{Suc\ k}}^2$ **by** *auto*
thus $x_{Suc\ k} \leq x_{Suc\ (Suc\ k)}$ **by** *auto*
qed

theorem *convergent* $x \longleftrightarrow a \leq 1/4$
proof
assume *convergent* x
then obtain L **where** $x \longrightarrow L$
using *convergent-def* **by** *auto*
define f **where** $f\ u = a + u^2$ **for** $u :: real$
have *continuous* (*at u*) f **for** $u :: real$
unfolding *f-def* **by** *auto*
with $\langle x \longrightarrow L\rangle$ **have** $(f \circ x) \longrightarrow f\ L$
using *continuous-at-sequentially* **by** *auto*
— *continuous-at-sequentially* is the theorem usually known as sequential continuity.
moreover have $(f \circ x) \longrightarrow L$
proof −
have $f \circ x = (\lambda n.\ x_{Suc\ n})$
using *f-def* **by** *auto*
— i.e. $f \circ x$ is the same sequence as x, but without the first element.
thus $(f \circ x) \longrightarrow L$
using $\langle x \longrightarrow L\rangle$ **and** *LIMSEQ-Suc* **by** *fastforce*
qed
ultimately have $f\ L = L$
by (*rule LIMSEQ-unique*)
hence $L^2 - L + a = 0$
unfolding *f-def* **by** *simp*
hence *discrim* $1\ (-1)\ a \geq 0$
using *discriminant-negative*[*of* $1\ {-1}\ a$] **by** *fastforce*
thus $a \leq 1/4$
unfolding *discrim-def* **by** *simp*

next
  assume $a \le 1/4$
  have $x_n \le 1/2$ **for** n
  proof (*induction n*)
    case *0*
    then show $x_0 \le 1/2$ **by** *simp*
  next
    case (*Suc k*)
    from ‹$x\ k \le 1/2$›
    have $x_k{}^2 \le (1/2)^2$
      using *x-nonneg* **by** (*smt* (*z3*) *power2-le-imp-le*)
    hence $a + x_k{}^2 \le 1/2$
      using ‹$a \le 1/4$› **by** (*simp add*: *power2-eq-square*)
    then show $x_{Suc\ k} \le 1/2$ **by** *simp*
  qed
  with *x-incseq* **obtain** L **where** $x \longrightarrow L$ **and** $\forall n.\ x_n \le L$
    using *incseq-convergent* **by** (*blast intro*!: *incseq-SucI*)
  thus *convergent x* **by** (*auto simp add*: *convergent-def*)
qed

end
end