**theory** *Problem-789*
  **imports** *Complex-Main*
**begin**

Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) + y) = 4yf(x) + f(x^2 - y).$$

**theorem**
  **fixes** $f :: real \Rightarrow real$
  **shows** $(\forall x\ y.\ f\ (f\ x + y) = 4*y*f\ x + f\ (x^2 - y))$
    $\longleftrightarrow (\forall x.\ f\ x = x^2) \vee (\forall x.\ f\ x = 0)$
  (**is** $(\forall x\ y.\ ?eqn\ x\ y) \longleftrightarrow$ -)
**proof**
  **assume** $\forall x\ y.\ ?eqn\ x\ y$
  **then have** *eqn*: $?eqn\ x\ y$ **for** $x\ y$ **by** *auto*
  **have** $[simp]$: $f\ (x^2 + f\ x) = 4*x^2*f\ x + f\ 0$ **for** $x$
    **using** *eqn*[**where** $y=x^2$ **and** $x=x$]
    **unfolding** *power2-eq-square* **by** *smt*
  **have** *opts*: $f\ x = 0 \vee f\ x = x^2$ **for** $x$
    **using** *eqn*[**where** $y=-f\ x$ **and** $x=x$, *simplified*]
    **by** *auto*
  {
    **fix** $a$
    **presume** $f\ a \neq a^2$
    **then have** $a \neq 0$ **and** $[simp]$: $f\ a = 0$
      **using** *opts* **by** *fastforce+*
    **fix** $y$
    **have** $*$: $f\ y = f\ (a^2 - y)$
      **using** *eqn*[**where** $x=a$ **and** $y=y$] **by** *simp*
    **presume** $f\ y \neq 0$
    **hence** $f\ y = y^2$ **using** *opts* **by** *auto*
    **moreover from** $*$ **and** $\langle f\ y \neq 0 \rangle$ **have** $f\ (a^2 - y) = (a^2 - y)^2$
      **using** *opts* **by** *auto*
    **ultimately have** $y^2 = (a^2 - y)^2$
      **using** $*$ **by** *auto*
    **with** $\langle a \neq 0 \rangle$ **have** $a^2 = 2*y$
      **by** (*simp add*: *power2-eq-square*) *algebra*
  }
  **hence** $**$: $f\ a \neq a^2 \implies 2*y \neq a^2 \implies f\ y = 0$ **for** $a\ y$
    **by** *fastforce*

  {
    **fix** $a$ **and** $y$
    **assume** $f\ a \neq a^2$ **and** $2*y = a^2$
    **moreover obtain** $b$ **where** $2*y \neq b^2$ **and** $b \neq 0$ **and** $b \neq y$
      **by** (*smt four-x-squared one-power2*)
    **ultimately have** $f\ b \neq b^2$ **using** $**$ [**where** $y=b$ **and** $a=a$]
      **by** *simp*
    **with** $**$ [**where** $a=b$ **and** $y=y$] **and** $\langle 2*y \neq b^2 \rangle$
    **have** $f\ y = 0$ **by** *simp*
  }
  **with** $**$ **have** $f\ a \neq a^2 \implies f\ y = 0$ **for** $a\ y$
    **by** *blast*
  **thus** $(\forall x.\ f\ x = x^2) \vee (\forall x.\ f\ x = 0)$
    **using** *opts* **by** *blast*
**next**
  **assume** $(\forall x.\ f\ x = x^2) \vee (\forall x.\ f\ x = 0)$
  **then show** $\forall x\ y.\ ?eqn\ x\ y$
    **apply** (*auto simp add*: *power2-eq-square*)
    **apply** (*thin-tac* $\forall x.\ f\ x = x * x$)
    **by** *algebra*
**qed**

1

**end**