

Background

RNNs:

- Sequence-to-sequence mappings of the form:

$$h^{(k)} = \phi(W h^{(k-1)} + F x^{(k)} + b),$$

$$y^{(k)} = C h^{(k)}, \quad h^{(-1)} = h_{-1}.$$

- Parameters: $\Theta = (W, F, B, C, h_{-1})$
- Input-output mapping: $y = G(x, \Theta)$

Equivalence of RNNs:

- Given Θ_1 and Θ_2 :
 $G(x, \Theta_1) = G(x, \Theta_2)$ for all $x = (x^{(0)}, \dots, x^{(T-1)})$
- Internal states may be different
- Does not imply that parameters are identical
- Example: invertible T , identity activation
- $W \rightarrow TWT^{-1}, \quad C \rightarrow CT^{-1}, \quad F \rightarrow TF, \quad h_{-1} \rightarrow Th_{-1}$

Contractive RNNs:

- $\|W\| := \max_{h \neq 0} \frac{\|Wh\|_2}{\|h\|_2}$
- Contractive: $\|W\| < 1$, non-expansive: $\|W\| \leq 1$.
- Non expansive activation function:
 $\|\phi(x) - \phi(y)\| \leq \|x - y\|$ for all x, y

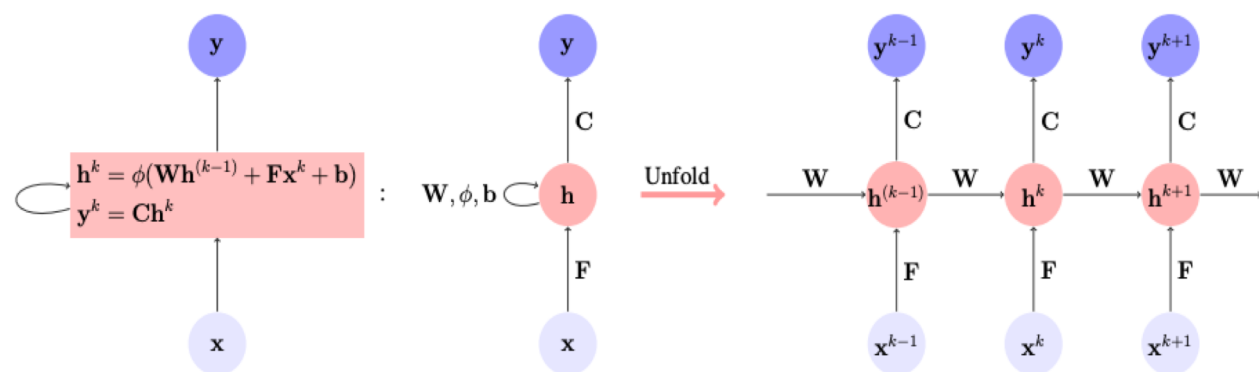


Figure 1: Recurrent neural network (RNN) model

Unitary RNNs (URNN):

- $W^H W = W W^H = I$
- Overcome the vanishing/exploding gradient problem
- Improve the stability of the network

Main Results

This work:

- Characterizes how restrictive the unitary constraint is on an RNN.
- Compares input-output mappings achievable by URNNs and RNNs

Equivalence Results for RNNs with ReLU Activations:

Theorem 1: Given any contractive RNN with n hidden states, bounded input, and ReLU activations, there exists a URNN with at most $2n$ hidden states with the identical input-output mapping.

- No loss in modeling with URNNs compared to RNNs
- Cost: two-fold increase in state dimension

Proof idea:

- Construct a URNN with $2n$ states
- Match the first n states with the original RNN
- Last n states are zero

Theorem 2: For every positive n , there exists a contractive RNN with ReLU activations and state dimension n such that every equivalent URNN has at least $2n$ states.

- Converse result for Theorem 1
- $2n$ achievability is tight

Equivalence Results for RNNs with Sigmoid Activations:

Theorem 3: There exists a contractive RNN with sigmoid activations such that there is no URNN with any finite number of states that exactly matches the input-output mapping.

- Difference in equivalence for smooth and non-smooth activations
- No exact equivalence even with arbitrary number of states

Synthetic Data Generation

- Multiple instances of a synthetic RNN with 4 hidden units
- F, C, b matrices \sim iid Gaussian
- Contractive $W_g = I - \epsilon A^T A / \|A\|^2$, A : Gaussian iid
- $\epsilon = 0.01 \rightarrow$ slow varying system \rightarrow long term dependencies
- Biases adjusted to ensure hidden states are on 60% of the time
- Additive output noise with SNR= 15, 20 dB
- Each trial: $T= 1000$ i.e. $(10 \times \text{time constant}(\frac{1}{\epsilon}))$, test ratio = 0.3

Numerical Simulation

Learning the system:

- Standard RNNs, URNNs, LSTMs with [2,4,6,8,10,12,14] hidden units
- MSE loss, batch-size =10, learning-rate = 0.01
- Averaged over 30 realization of original contractive system
- Unitary constraint: projection on unitary space using SVD

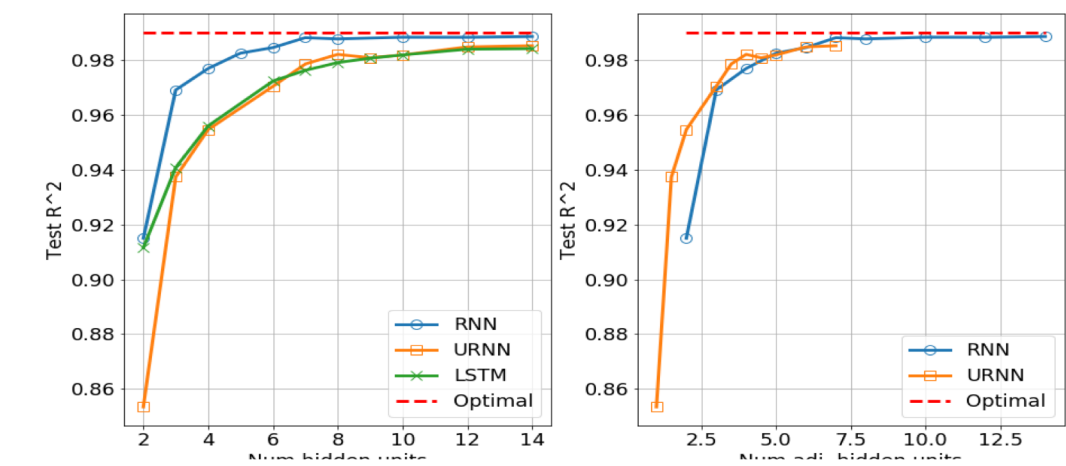


Figure 2: Test R^2 on synthetic data for a Gaussian i.i.d. input and output SNR=20 dB.

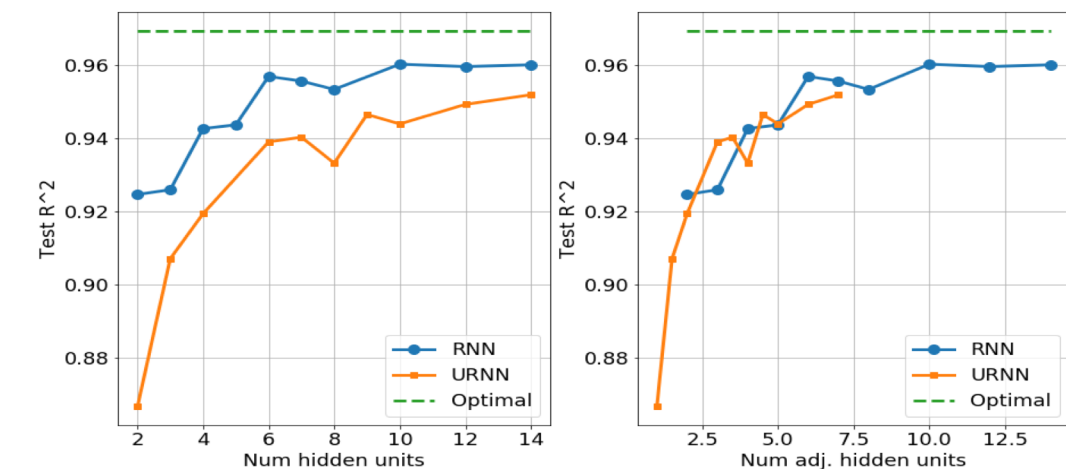


Figure 3: Test R^2 on synthetic data for a Gaussian i.i.d. input and output SNR=15 dB.

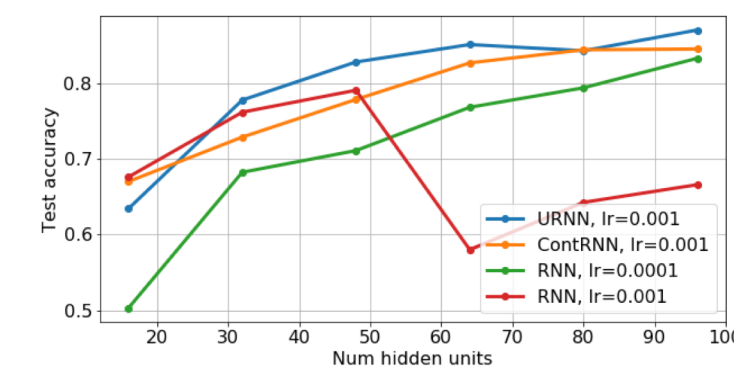


Figure 4: Accuracy on Permuted MNIST task for various models trained with RMSProp, validation-based early termination.

References:

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- [3] Scott Wisdom, Thomas Powers, John Hershey, Jonathan Le Roux, and Les Atlas. Full-capacity unitary recurrent neural networks. In NIPS, 2016.

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