Convergence of the variational inversion of the cyclogeostrophic balance

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Contents

The goal of this document is to open questions on some optimization results obtained when estimating cyclogeostrophic currents using a variational approach.

First, we briefly present the oceanographic background in Sections *Geostrophic approximation* and *Cyclogeostrophic balance*. In Section *Cyclogeostrophic balance*, we also introduce our numerical resolution setting (*Variational approach*), and an aspect of our problem of potential interest (*Solution existence*). Finally, Section *Optimization related questions* reveals some results, and our related questions.

1 Geostrophic approximation

Sea Surface Currents (SSC) can be easily approximated from satellite altimetry observations of the Sea Surface Height (SSH) using the geostrophic balance. Geostrophy describes the balance between the pressure gradient force (indirectly observed via SSH), and the Coriolis force. Geostrophic currents satisfy this equilibrium:

$$f\left(\vec{k} \times \vec{u}_g\right) = -g\nabla\eta, \tag{1.1}$$

where f is the Coriolis parameter, $\vec{k}$ the vertical unit vector, $\vec{u}_g$ the geostrophic velocity, g the gravity, and η the SSH.

2 Cyclogeostrophic balance

However, geostrophy alone is not always sufficient to accurately estimate SSC [Penven *et al.*, 2014], and the advective term $\vec{u} \cdot \nabla \vec{u}$ should be added back to the balance to take into consideration the centrifugial acceleration. Skipping some rearrangements, we can express the cyclogeostrophic balance as:

$$\vec{u}_c - \frac{\vec{k}}{f} \times (\vec{u}_c \cdot \nabla \vec{u}_c) = \vec{u}_g, \tag{2.1}$$

where $\vec{u}_c$ is the cyclogeostrophic velocity.

2.1 Solution existence

We know that, under some physically unstable conditions, Equation (2.1) does not hold a mathematical solution [Knox and Ohmann, 2006]. It can be exhibited in the idealized scenario of a gaussian eddy.

In that setting, the nonlinear term $\vec{u}_c \cdot \nabla \vec{u}_c$ simplifies to $-\frac{V_{gr}^2}{r} \vec{e}_r$, with V_{gr} the azimuthal component of the velocity, r the radial distance to the eddy center, and $\vec{e}_r$ the outward-directed radial unit vector. Consequently, Equation (2.1) simplifies to the gradient-wind equation [Knox and Ohmann, 2006]:

$$V_{gr} + \frac{V_{gr}^2}{fr} = V_g, \tag{2.2}$$

where V_g is the azimuthal geostrophic velocity. From Equation (2.2), we obtain the "normal" physical solution:

$$V_{gr} = \frac{2V_g}{1 + \sqrt{1 + 4V_g/(fr)}} \tag{2.3}$$

By convention, in the northern hemisphere, r is positive for cyclonic eddies, and negative for anticyclonic ones. Therefore, because of the square root in the denominator, there are no real solutions of V_{gr} for small r in anticyclonic eddies [Knox and Ohmann, 2006].

2.2 Variational approach

Because of the advective term $\vec{u}_c \cdot \nabla \vec{u}_c$, Equation (2.1) is nonlinear, and solving it analytically is conceivable only in idealized scenarios (such as the gaussian one). We propose to solve it numerically by formulating the cyclogeostrophy as the variational problem:

$$J(\vec{u}_c) = \left\| \vec{u}_c - \frac{\vec{k}}{f} \times (\vec{u}_c \cdot \nabla \vec{u}_c) - \vec{u}_g \right\|^2, \tag{2.4}$$

where $\|.\|$ is the discrete L^2 norm.

jaxparrow

Our Python package `jaxparrow` implements this variational approach, leveraging JAX [Bradbury *et al.*, 2021]. Thanks to JAX automatic differentiation capabilities, ∇J is numerically available, and the cyclogeostrophic currents are estimated by minimizing Equation (2.4) using a gradient-based optimizer, with $\vec{u}_c^{(0)} = \vec{u}_g$ as initial guess.

3 Optimization related questions

Our interrogations are based on some unexpected (to us) results we obtained when using `jaxparrow` with different optimizers for estimating cyclogeostrophic currents in the Alboran sea (a region of the Mediterranean sea).

3.1 Alboran sea experiments

The Alboran sea is an highly energetic (currents are strong) area where cyclogeostrophy is expected to better represent the ocean dynamic than geostrophy. We use plausible and realistic (but not real) data from the state-of-the-art NEMO ocean circulation model [Madec *et al.*, 2022] as reference.

More precisely, we have access to simulated SSH and SSC velocities, and we derive SSC normalized vorticities (ξ/f) from the velocities (see Fig. 3.1). Highly negative (smaller than -2) vorticities are typical unstable conditions for which the cyclogeostrophic balance does not hold a mathematical solution (because cyclogeostrophy is no longer a physically valid approximation).

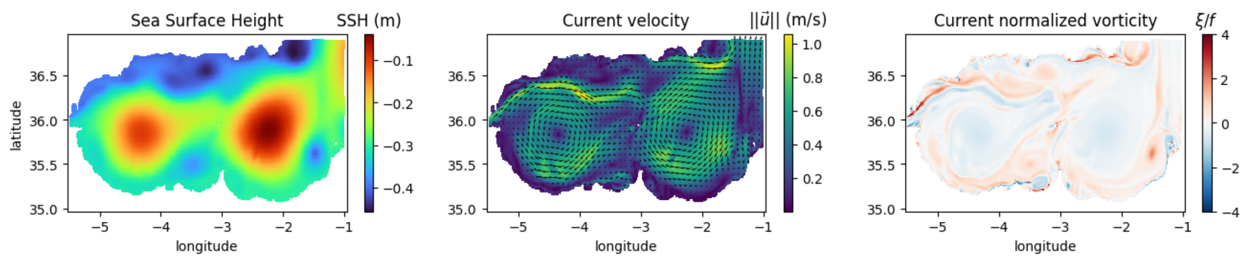

Fig. 3.1: NEMO data

`jaxparrow` first estimates the geostrophic velocities (following Equation (1.1)), from which we can computes the geostrophic vorticities (see Fig. 3.2).

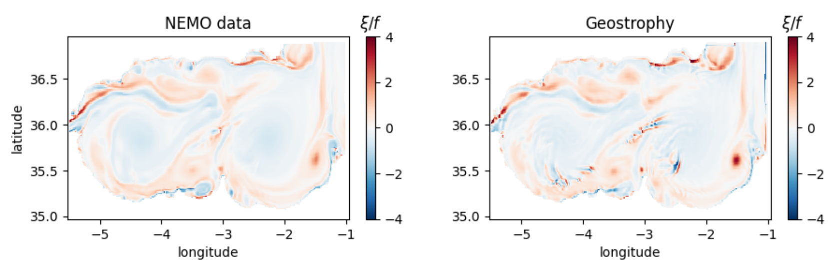

Fig. 3.2: NEMO (left) and geostrophic (right) vorticities

Next, `jaxparrow` estimates the cyclogeostrophic velocities by minimizing Equation (2.4), with an optimizer chosen by the user.

Classical gradient descent

Using gradient descent, we obtain very good qualitative results (see Fig. 3.3), with little sensitivity to the learning rate (not shown here).

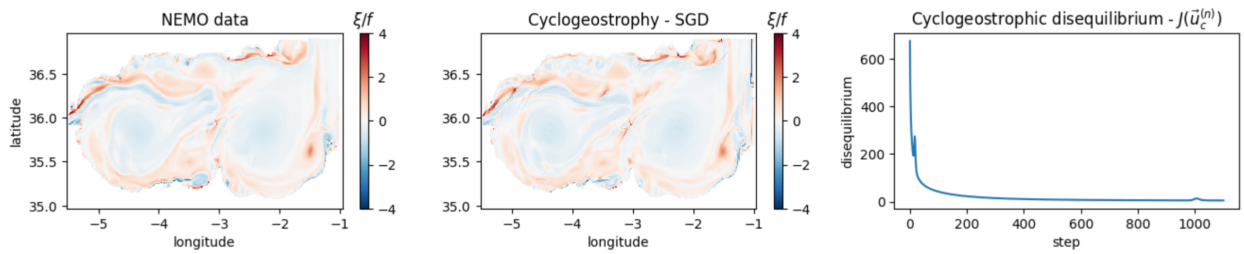

Fig. 3.3: NEMO (left) and SGD (right) vorticities

Adam variation

When using Adam, the loss curve suggests that we are still converging, but clearly, towards a qualitatively worse solution (see Fig. 3.4).

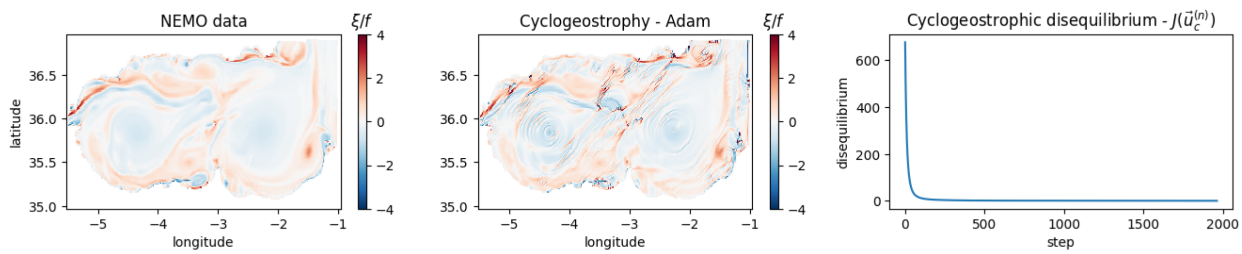

Fig. 3.4: NEMO (left) and Adam (right) vorticities

3.2 Open questions

These first observations lead us to believe that the optimization problem is probably not "smooth", and that several solutions exist. We think that it could be related to physical unstable conditions resulting in the absence of mathematical solutions to Equation (2.1) (as exhibited in Equation (2.3)). Our questions at this point could be formulated as:

- are there optimization strategies that could account for these irregularities?
  - optimizing on subparts of the domain, masking areas with highly negative geostrophic vorticities, …
- are there theoretical properties of the optimizers explaining our observations?
- how sensitive the solution is to the initial guess (that is geostrophic velocities)?

References

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