Non-monotone Submodular Maximization under Matroid and Knapsack Constraints [5]

Jon Lee, Vahab S. Mirrokni, Viswanath Nagarajan, Maxim Svirdenko

Merkouris Papamichail

IGP ALMA, AL1.20.0018

Fall 2021

Abstract

In this report we present a synopsis of Jon Lee's et al. paper on *Non-monotone Submodular Maximization under Matroid and Knapsack Constraints*. The authors provide two *approximate local search* methods for the problem of maximizing a submodular function under $k \geq 1$ matroid or knapsack constraints. They achieve an $\left(\frac{1}{k+2+\frac{1}{k}+\epsilon}\right)$–approximation for matroid constraints and an $\left(\frac{1}{5}-\epsilon\right)$–approximation for knapsack constraints.

1 Introduction

Jon Lee's et al. paper is rich in results regarding many varieties and special instances of the constraint submodular maximization problem. In this report we will only be considering the two general cases, of $k \geq 1$ matroid constraints and $k \geq 1$ knapsack constraints.

Let's introduce some elementary definitions. Let Ω be a finite set and $f\colon 2^{\Omega} \to \mathbb{R}_{\geq 0}$ be a function on the power set of Ω. The function f will be called *submodular* if, for every $X, Y \subseteq \Omega$, with $X \subseteq Y$ and every $x \in \Omega \setminus X$ the equation 1 holds.

$$f(X \cup x) - f(X) \geq f(Y \cup x) - f(Y) \tag{1}$$

In this report we will be considering *non–negative*, *non–monotone* submodular functions.

Regarding the optimization on submodular functions, there seems to be an asymmetry between the minimization and maximization. The *unconstraint* minimization problem is computable in polynomial time. Adding even a simple constraint makes the minimization problem NP-hard. On the other hand, the maximization problem is NP-hard, even in the *unconstraint* setting. Moreover, if we allow the submodular function f to take negative values, the problem becomes inapproximable [3].

In Section 2 we discuss the problem of submodular maximization under matroid constraints. There we give a local search method that achieves $\left(\frac{1}{k+2+\frac{1}{k}+\epsilon}\right)$ approximation ratio. In Section 3 we consider the variant of submodular maximization under knapsack constraints. There, we give a fractional relaxation of the problem and use a local search method to obtain a good approximation; then we use a simple rounding technique to achieve a $\left(\frac{1}{5}-\epsilon\right)$ approximation to the original (integer) problem. Lastly, in Section 4 we present some of the most recent results in the area, along with the remaining open problems.

2 Matroid Constraints

In this section we consider the problem of submodular maximization, under k matroid constraints. We begin with few definitions. Let E be a finite set and a collection $\mathscr{I} \subseteq 2^E$ of subsets of E, called *independent* sets. The pair $\mathcal{M} = (E, \mathscr{I})$ will be called a *matroid*, if the following conditions hold, the empty set is independent, every subset of an independent set is also independent, and the *Augmentation Property* is true. The Augmentation Property states that, if I_1, I_2 two independent sets, and $|I_1| < |I_2|$, then there is a $i \in I_2 \setminus I_1$, such that $I_1 \cup i$ is also independent.

A maximal independent set of a matroid is called *basis*. Each two bases B_1, B_2 of a matroid are equicardinal, i.e. $|B_1| = |B_2|$. For the bases of a matroid the *Exchange Property* is always true. The Exchange Property states that, if B_1, B_2 two bases, for every $b_1 \in B_1$, there is a $b_2 \in B_2$, such that $(B_2 \setminus b_2) \cup b_1$ is also a basis.

We can now state formally the problem we will be considering. Let E be a finite set, and $f\colon E \to \mathbb{R}_{\geq 0}$ a submodular, non-negative, non-monotone function. Also, let $\mathcal{M}_1, \ldots, \mathcal{M}_k$ a collection of k matroids on E. We are required to find some $S \subseteq E$, which maximizes the function f, granted that S is independent in all k matroids.

Let's consider the problem with a single matroid constraint first. Let S some independent set. Let, also $e \in E \setminus S$, some other arbitrary element. Then, either $S \cup e$ is an independent set, or there is a $s \in S$, such that $(S \setminus s) \cup e$ is an independent set. In the first case, we use the *Augmentation Property*, while in the second we use the *Exchange Property*. The authors devise a local search algorithm, where given a partial solution S, we may apply the operations of *Deletion*, $S' \leftarrow S \setminus e$, *Augmentation*, $S' \leftarrow S \cup e$ and *Exchange*, $S' \leftarrow (S \setminus \{d_1, \dots, d_\lambda\}) \cup e$, provided that the resulting S' is also independent in all k matroids. Note that for the Exchange Operation each matroid forces us to delete *at most* one element, in order to maintain feasibility, thus $\lambda \leq k$. This results in a neighborhood of size $n^{O(k)}$. In order to achieve a polynomial time algorithm (in n), the authors introduce a sensitivity parameter $\mathcal{E} = \left(1 + \frac{\epsilon}{n^4}\right)$, where we accept S' only if $f(S') > \mathcal{E} f(S)$. The associated time complexity is $n^{O(k)}$, while the algorithm achieves a $\left(\frac{1}{k+2+\frac{1}{k}+\epsilon}\right)$ approximation guarantee.

3 Knapsack Constraints

In this section we discuss the problem of submodular maximization, under knapsack constraints. Let E be a finite set, and $f\colon 2^E \to \mathbb{R}_{\geq 0}$ a submodular, non-negative, non-monotone function. Also, let $C_1, \dots C_k$ be the capacities of k knapsacks on E, and $\mathbf{w}^1, \dots, \mathbf{w}^k \in \mathbb{R}^n_{\geq 0}$ be the weight vectors for each knapsack. We are required to find some set $S \subseteq E$, which maximizes the function f and is feasible in every knapsack. Without loss of generality we assume that $C_i = 1$, for all knapsacks, and that the singletons $\{e\}$ are feasible.

The authors utilize the *Multilinear Extension* of a submodular function. Namely, let $f\colon E \to \mathbb{R}_{\geq 0}$ be a submodular function, the Multilinear Extension of f is a real function $F\colon [0,1]^n \to \mathbb{R}_{\geq 0}$, such that,

$$F(x) = \sum_{S \subseteq E} f(S) \prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i). \tag{2}$$

The function F is *multilinear*, that is linear for each variable separately. Also, F is polynomial in the variables $x_1, \dots, x_n$, and smooth in the sense that has continuous derivatives of every order. Moreover, F has the property of *Continuous Submodularity*, i.e. $\frac{\partial^2}{\partial x_i \partial x_j} F \leq 0$ [6].

We now consider the fractional relaxation of the problem. Let W be the matrix that has the weight vectors $\mathbf{w}^1, \dots, \mathbf{w}^k$ as *rows*, and $C = [C_1 \dots C_k]^t$ the column vector of the capacities. We are required to find some "fractional characteristic vector" $\mathbf{x} \in [0,1]^n$, that maximizes the function $F(\mathbf{x})$, while $W\mathbf{x} \leq C$ [1]. In order to solve the fractional relaxation, the authors presented a simple local search method. Let $\mathbf{y} \in [0,1]^n$ a feasible solution. In each step, we choose arbitrarily two *disjoint* sets $A, D \subseteq [n]$, with $|A| = |D| = k$. We decrease $\mathbf{y}(D)$ and increase $\mathbf{y}(A)$, while maintaining the feasibility. Moreover we allow $\mathbf{y}$ to take values only in $\mathcal{G} = \left\{p \cdot \zeta \mid p \in \mathbb{N},\ 0 \leq p \leq \frac{1}{\zeta}\right\}$ and introduce a sensitivity parameter $\mathcal{E} = 1 + \epsilon$. For $\zeta = \frac{1}{8n^4}$, the resulting neighborhood is of size $n^{O(k)}$, while the time complexity is $O\left(\frac{1}{\epsilon} \log n\right)$. The approximation guarantee is $\left(\frac{1}{4} - \delta\right)$.

In order to solve the original problem, we construct two solutions, the *heavy* and the *light* one, and return the optimal. We obtain the heavy solution by enumerating all the k/δ feasible solutions, where we consider *only* the elements e, with $w(e) > \delta$, for *any* knapsack. For the light elements, with $w(e) \leq \delta$ in every knapsack we use the above mentioned relaxation and round the fractional solution with a simple randiomized method. Combining the above solutions we obtain a $\left(\frac{1}{5} - \epsilon\right)$ approximation guarantee.

4 Conclusions & Future Work

Jon Lee's et al. paper had been published for the first time in 2009, since then has undergone numerous revisions. The version considered for this report was lastly updated in 2018. The submodular maximization is a rich area of interest, which attracted many researchers in the last decade. Many algorithmic techniques have been utilized to tackle the submodular maximization, such as local search, continuous optimization, sampling and others. The currently best known algorithm achieves an approximation ratio of 0.385 for both cardinality and matroid constraints [2]. In contrast, it is known that no polynomial algorithm can achieve approximation ratio of 0.497 for cardinality constraints, or 0.478 for matroid constraints [4]. It remains a long standing open problem to settle the approximability of the submodularity maximization [1].

[1] Note that the programme $F(\mathbf{x}^\star) = \max\{\mathbf{x} \in [0,1]^n \mid W\mathbf{x} \leq C\}$ is *not* linear, since the objective function F is *multilinear*.

References

[1] N. Alaluf, A. Ene, M. Feldman, H. L. Nguyen, and A. Suh, *Optimal streaming algorithms for submodular maximization with cardinality constraints*, arXiv preprint arXiv:1911.12959, (2019).

[2] N. Buchbinder and M. Feldman, *Constrained submodular maximization via a nonsymmetric technique*, Mathematics of Operations Research, 44 (2019), pp. 988–1005.

[3] U. Feige, V. S. Mirrokni, and J. Vondrak, *Maximizing non-monotone submodular functions*, SIAM Journal on Computing, 40 (2011), pp. 1133–1153.

[4] S. O. Gharan and J. Vondrák, *Submodular maximization by simulated annealing*, in Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete Algorithms, SIAM, 2011, pp. 1098–1116.

[5] J. Lee, V. S. Mirrokni, V. Nagarajan, and M. Sviridenko, *Non-monotone submodular maximization under matroid and knapsack constraints*, in Proceedings of the forty-first annual ACM symposium on Theory of computing, 2009, pp. 323–332.

[6] J. Vondrák, *Optimal approximation for the submodular welfare problem in the value oracle model*, in Proceedings of the fortieth annual ACM symposium on Theory of computing, 2008, pp. 67–74.