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OPTIMAL CONTROL

Analysis of an Optimization Method for Solving the Problem of Complex Heat Transfer with Cauchy Boundary Conditions

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Abstract—An optimization method is proposed for solving a boundary value problem with Cauchy conditions for the equations of radiative-conductive heat transfer in the P_1-approximation of the radiative transfer equation. Theoretical analysis of the corresponding problem of boundary optimal control is carried out. It is shown that a sequence of solutions of extremal problems converges to the solution of the boundary value problem with the Cauchy conditions for temperature. The results of theoretical analysis are illustrated with numerical examples.

Keywords: equations of radiative-conductive heat transfer, diffusion approximation, optimal control problem, Cauchy conditions

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1. INTRODUCTION

Stationary radiative and diffusive heat transfer in a bounded region $\Omega \subset \mathbb{R}^3$ with a boundary $\Gamma = \partial\Omega$ is simulated in the P_1-approximation for the equation of radiative transfer by the following system of elliptic equations [1–3]:

$$-a\Delta\theta + b\kappa_a(|\theta|\theta^3 - \varphi) = 0, \quad -\alpha\Delta\varphi + \kappa_a(\varphi - |\theta|\theta^3) = 0, \quad x \in \Omega. \tag{1}$$

Here, θ is the normalized temperature and φ is the normalized radiation intensity averaged over all directions. Positive physical parameters a, b, κ_a, and α, describing the properties of the medium, are defined in a standard way [3]. A detailed theoretical and numerical analysis of various formulations of boundary and inverse problems, as well as control problems for the equations of radiative heat transfer in the P_1-approximation for the equation of radiative transfer, is presented in [1–21]. It is also worth noting a serious analysis of interesting boundary value problems related to radiative heat transfer, presented in [22–27].

We will assume that, at the boundary $\Gamma = \partial\Omega$, we have a known temperature field

$$\theta = \theta_b. \tag{2}$$

To impose the boundary condition for the radiation intensity, it is required to know the function describing the reflective properties of the boundary [4]. If this function is unknown, it is natural, instead the boundary condition for the radiation intensity, to specify the heat fluxes at the boundary:

$$\partial_n\theta = q_b. \tag{3}$$

Here, ∂_n denotes the derivative in the direction of the outward normal $\mathbf{n}$.

The nonlocal solvability of nonstationary and stationary boundary value problems for the equations of complex heat transfer without boundary conditions on the radiation intensity and with conditions (2) and (3) for temperature was proved in [20, 21].

This article is devoted to the analysis of an optimization method proposed for solving boundary value problem (1)–(3) with Cauchy conditions for temperature. This method consists in considering the optimal boundary control problem for system (1) with "artificial" boundary conditions

$$a(\partial_n\theta + \theta) = r, \quad \alpha(\partial_n\varphi + \varphi) = u \quad \text{on} \quad \Gamma. \tag{4}$$

The function $r(x)$, $x \in \Gamma$, is given, and the unknown function $u(x)$, $x \in \Gamma$, plays the role of a control. The extremal problem consists in finding a triple $\{\theta_\lambda, \varphi_\lambda, u_\lambda\}$ such that

$$J_\lambda(\theta, u) = \frac{1}{2}\int_\Gamma (\theta - \theta_b)^2 d\Gamma + \frac{\lambda}{2}\int_\Gamma u^2 d\Gamma \to \inf \tag{5}$$

on the solutions of boundary value problem (1) and (4). The function $\theta_b(x)$, $x \in \Gamma$, and the regularization parameter $\lambda > 0$ are specified.

As will be shown below, the optimal control problem (1), (4), and (5) with $r := a(\theta_b + q_b)$, where q_b is a function defined on Γ, is an approximation of boundary value problem (1)–(3) for small λ.

The article is organized as follows. In Section 2, the necessary spaces and operators are introduced and the formalization of the optimal control problem is given. A priori estimates for the solution of problem (1) and (4), on the basis of which the solvability of this boundary value problem and the optimal control problem of (1), (4), and (5) were proved, are obtained in Section 3. In Section 4, the optimality system is deduced. In Section 5, it is shown that the sequence $\{\theta_\lambda, \varphi_\lambda\}$ corresponding to the solutions of the extremal problem converges, as $\lambda \to +0$, to the solution of boundary value problem (1)–(3) with Cauchy conditions for the temperature. Finally, Section 6 presents an algorithm for solving the control problem, illustrated by numerical examples.

2. FORMALIZATION OF THE CONTROL PROBLEM

In what follows, we assume that $\Omega \subset \mathbb{R}^3$ is a bounded strictly Lipschitz domain, whose boundary Γ consists of a finite number of smooth pieces. By L^p, $1 \le p \le \infty$, we denote the Lebesgue space and, by H^s, the Sobolev space W_2^s. Let $H = L^2(\Omega)$, $V = H^1(\Omega)$. By V' we denote the space dual to the space V. We identify the space H with the space H', so that $V \subset H = H' \subset V'$. We denote by $\|\cdot\|$ the standard norm in H and, by (f, v), the value of the functional $f \in V'$ on the element $v \in V$, which coincides with the scalar product in H, if $f \in H$. By U we denote the space $L^2(\Gamma)$ with the norm $\|u\|_\Gamma = \left(\int_\Gamma u^2 d\Gamma\right)^{1/2}$.

We will assume that

(i) $a, b, \alpha, \kappa_a, \lambda = \text{Const} > 0$,

(ii) $\theta_b, q_b \in U$, $r = a(\theta_b + q_b)$.

We define operators $A : V \to V'$ and $B : U \to V'$, using the following equalities, which are valid for any $y, z \in V$ and $w \in U$:

$$(Ay, z) = (\nabla y, \nabla z) + \int_\Gamma yz d\Gamma, \quad (Bw, z) = \int_\Gamma wz d\Gamma.$$

The bilinear form (Ay, z) defines the scalar product in the space V, and the corresponding norm $\|z\|_V = \sqrt{(Az, z)}$ is equivalent to the standard norm in V. Therefore, a continuous inverse operator $A^{-1} : V' \mapsto V$ is defined. Note that, for any $v \in V$, $w \in U$, and $g \in V'$, we have the inequalities

$$\|v\|^2 \le C_0\|v\|_V^2, \quad \|v\|_{V'} \le C_0\|v\|_V, \quad \|Bw\|_{V'} \le \|w\|_\Gamma, \quad \left\|A^{-1}g\right\|_V \le \|g\|_{V'}. \tag{6}$$

Here, the constant $C_0 > 0$ depends only on the domain Ω.

In what follows, we use the notation: $[h]^s := |h|^s \operatorname{sgn} h$, $s > 0$, $h \in \mathbb{R}$, for a monotonic power function.

Definition. A pair $\theta, \varphi \in V$ is called a weak solution of problem (1) and (4) if

$$aA\theta + b\kappa_a([\theta]^4 - \varphi) = Br, \quad \alpha A\varphi + \kappa_a(\varphi - [\theta]^4) = Bu. \tag{7}$$

To formulate the optimal control problem, we define the constraint operator $F(\theta,\varphi,u): V\times V\times U\to V'\times V'$:

$$F(\theta,\varphi,u)=\{aA\theta+b\kappa_a([\theta]^4-\varphi)-Br,\ \alpha A\varphi+\kappa_a(\varphi-[\theta]^4)-Bu\}.$$

Problem (CP). Find a triple $\{\theta,\varphi,u\}\in V\times V\times U$ such that

$$J_\lambda(\theta,u)\equiv\frac{1}{2}\|\theta-\theta_b\|_\Gamma^2+\frac{\lambda}{2}\|u\|_\Gamma^2\to\inf,\quad F(\theta,\varphi,u)=0. \tag{8}$$

3. SOLVABILITY OF PROBLEM (CP)

Let us first prove the unique solvability of boundary value problem (1) and (4).

Lemma 1. *Let conditions* (i) *and* (ii), $u\in U$, *be satisfied. Then, there is a unique weak solution of problem* (1) *and* (4), *such that*

$$a\|\theta\|_V\le\|r\|_\Gamma+\frac{C_0\kappa_a}{\alpha}\|r+bu\|_\Gamma,$$
$$\alpha b\|\varphi\|_V\le\|r\|_\Gamma+\left(\frac{C_0\kappa_a}{\alpha}+1\right)\|r+bu\|_\Gamma. \tag{9}$$

Proof. Multiplying the second equation in (7) by b and adding the result to the first equation, we obtain the equalities

$$A(a\theta+\alpha b\varphi)=B(r+bu),\quad a\theta+\alpha b\varphi=A^{-1}B(r+bu),\quad \varphi=\frac{1}{\alpha b}(A^{-1}B(r+bu)-a\theta).$$

Therefore, $\theta\in V$ is a solution of the following equation:

$$aA\theta+\frac{\kappa_a}{\alpha}\theta+b\kappa_a[\theta]^4=g. \tag{10}$$

Here,

$$g=Br+\frac{\kappa_a}{\alpha}A^{-1}B(r+bu)\in V'.$$

The unique solvability of Eq. (10) with monotonic nonlinearity is well known (see, e.g., [28]). Therefore, problem (7) is uniquely solvable.

To obtain estimates (9), we find the scalar product of (10) by $\theta\in V$ and discard the non-negative terms on the left-hand side. Then,

$$a\|\theta\|_V^2\le(g,\theta)\le\|g\|_{V'}\|\theta\|_V,\quad a\|\theta\|_V\le\|g\|_{V'}.$$

Inequality (6) allows us to estimate $\|g\|_{V'}$ and $\|\varphi\|_V$:

$$\|g\|_{V'}\le\|r\|_\Gamma+\frac{C_0\kappa_a}{\alpha}\|r+bu\|_\Gamma,\quad \|\varphi\|_V\le\frac{1}{\alpha b}\|r+bu\|_\Gamma+\frac{a}{\alpha b}\|\theta\|_V.$$

As a result, we obtain estimates (9).

The estimates obtained for the solution of the controlled system make it possible to prove the solvability of the optimal control problem.

Theorem 1. *Let conditions* (i) *and* (ii) *be satisfied. Then, there is a solution of problem* (CP).

Proof. Let $j_\lambda=\inf J_\lambda$ on the set $u\in U$ and $F(\theta,\varphi,u)=0$. We choose a minimizing sequence $u_m\in U$, $\theta_m\in V$, $\varphi_m\in V$:

$$J_\lambda(\theta_m,u_m)\to j_\lambda,$$
$$aA\theta_m+b\kappa_a([\theta_m]^4-\varphi_m)=Br,\quad \alpha A\varphi_m+\kappa_a(\varphi_m-[\theta_m]^4)=Bu_m. \tag{11}$$

The boundedness of the sequence u_m in the space U implies, based on Lemma 1, the estimates

$$\|\theta_m\|_V\le C,\quad \|\varphi_m\|_V\le C,\quad \|\theta_m\|_{L^6(\Omega)}\le C.$$

Here, $C > 0$ denotes the largest of the constants that bound the corresponding norms and do not depend on m. Passing, if necessary, to subsequences, we conclude that there is a triple $\{\hat{u}, \hat{\theta}, \hat{\varphi}\} \in U \times V \times V$ such that

$$u_m \to \hat{u} \quad \text{weakly in} \quad U, \qquad \theta_m, \varphi_m \to \hat{\theta}, \hat{\varphi} \quad \text{weakly in} \quad V, \qquad \text{strongly in} \quad L^4(\Omega). \tag{12}$$

Note also that $\forall v \in V$ we have

$$\left|([\theta_m]^4 - [\hat{\theta}]^4, v)\right| \le 2\left\|\theta_m - \hat{\theta}\right\|_{L^4(\Omega)} \|v\|_{L^4(\Omega)} \left(\|\theta_m\|^3_{L_6(\Omega)} + \left\|\hat{\theta}\right\|^3_{L_6(\Omega)}\right). \tag{13}$$

The results on the convergence of (12) and (13) allow the passage to the limit in (11). Therefore,

$$aA\hat{\theta} + b\kappa_a([\hat{\theta}]^4 - \hat{\varphi}) = Br, \quad \alpha A\hat{\varphi} + \kappa_a(\hat{\varphi} - [\hat{\theta}]^4) = B\hat{u},$$

where $j_\lambda \le J_\lambda(\hat{\theta}, \hat{u}) \le J_\lambda(\theta_m, u_m) = j_\lambda$. Therefore, the triple $\{\hat{\theta}, \hat{\varphi}, \hat{u}\}$ is a solution of problem (CP).

4. OPTIMALITY CONDITIONS

To obtain an optimality system, it suffices to use the Lagrange principle for smooth-convex extremal problems [29, 30]. Let us check the validity of the key condition that the image of the derivative of the constraint operator $F(y,u)$, where $y = \{\theta, \varphi\} \in V \times V$, coincides with the space $V' \times V'$. It is this condition that guarantees the nondegeneracy of the optimality conditions. Recall that

$$F(y,u) = \{aA\theta + b\kappa_a([\theta]^4 - \varphi) - Br, \alpha A\varphi + \kappa_a(\varphi - [\theta]^4) - Bu\}.$$

Lemma 2. *Let conditions* (i) *and* (ii) *be satisfied. For any pair* $\hat{y} \in V \times V$, $\hat{u} \in U$, *we have the equality*

$$\operatorname{Im} F_y'(y,u) = V' \times V'.$$

Proof. It suffices to check that the problem

$$aA\xi + b\kappa_a(4|\hat{\theta}|^3\xi - \eta) = f_1, \quad \alpha A\eta + \kappa_a(\eta - 4|\hat{\theta}|^3\xi) = f_2$$

is solvable for all $f_{1,2} \in V'$. This problem is equivalent to the system

$$aA\xi + \kappa_a\left(4b|\theta|^3 + \frac{a}{\alpha}\right)\xi = f_1 + \frac{\kappa_a}{\alpha} f_3, \quad \eta = \frac{1}{\alpha b}(f_3 - a\xi).$$

Here, $f_3 = A^{-1}(f_1 + bf_2) \in V$. The solvability of the first equation of this system obviously follows from the Lax–Milgram lemma.

According to Lemma 2, the Lagrangian of problem (CP) has the form

$$L(\theta, \varphi, u, p_1, p_2) = J_\lambda(\theta, u) + (aA\theta + b\kappa_a([\theta]^4 - \varphi) - Br, p_1) + (\alpha A\varphi + \kappa_a(\varphi - [\theta]^4) - Bu, p_2).$$

Here, $p = \{p_1, p_2\} \in V \times V$ is the conjugate state. If $\{\hat{\theta}, \hat{\varphi}, \hat{u}\}$ is a solution of problem (CP), then, according to the Lagrange principle [29, Theorem 1.5], the variational equalities hold $\forall v \in V$, $w \in U$, and we have

$$(\hat{\theta} - \theta_b, v)_\Gamma + (aAv + 4b\kappa_a|\hat{\theta}|^3 v, p_1) - \kappa_a(4|\hat{\theta}|^3 v, p_2) = 0, \quad b\kappa_a(v, p_1) + (\alpha Av + \kappa_a v, p_2) = 0, \tag{14}$$

$$\lambda(\hat{u}, w)_\Gamma - (Bw, p_2) = 0. \tag{15}$$

Thus, from conditions (14) and (15), we obtain the following result, which, together with Eqs. (7) for the optimal triple, determines the optimality system for problem (CP).

Theorem 2. *Let conditions* (i) *and* (ii) *be satisfied. If* $\{\hat{\theta}, \hat{\varphi}, \hat{u}\}$ *is a solution of problem* (CP)*, then there is a unique pair* $\{p_1, p_2\} \in V \times V$ *such that*

$$aAp_1 + 4|\hat{\theta}|^3\kappa_a(bp_1 - p_2) = B(\theta_b - \hat{\theta}), \quad \alpha Ap_2 + \kappa_a(p_2 - bp_1) = 0. \tag{16}$$

In this case, $\lambda\hat{u} = p_2$.

5. APPROXIMATION OF THE PROBLEM WITH CAUCHY CONDITIONS

Consider boundary value problem (1)–(3) for the equations of complex heat transfer without boundary conditions on the radiation intensity. The existence of $\theta, \varphi \in H^2(\Omega)$ satisfying (1)–(3) for sufficiently smooth θ_b, q_b and sufficient conditions for the uniqueness of the solution were proved in [21]. Let us show that the solutions of problem (CP) at $\lambda \to +0$ approximate the solution of problem (1)–(3).

Theorem 3. *Suppose that conditions* (i) *and* (ii) *are satisfied and there exists a solution of problem* (1)–(3). *If* $\{\theta_\lambda, \varphi_\lambda, u_\lambda\}$ *is a solution of problem* (CP) *for* $\lambda > 0$, *then there is a sequence* $\lambda \to +0$ *such that*

$$\theta_\lambda \to \theta_*, \quad \varphi_\lambda \to \varphi_* \quad \textit{weakly in } V, \quad \textit{strongly in } H,$$

where θ_*, φ_* *is a solution of problem* (1)–(3).

Proof. Let $\theta, \varphi \in H^2(\Omega)$ be a solution of problem (1)–(3), $u = \alpha(\partial_n \varphi + \varphi) \in U$. Then,

$$aA\theta + b\kappa_a([\theta]^4 - \varphi) = Br, \quad \alpha A\varphi + \kappa_a(\varphi - [\theta]^4) = Bu,$$

where $r := a(\theta_b + q_b)$. Therefore, taking into account that $\theta|_\Gamma = \theta_b$, we get

$$J_\lambda(\theta_\lambda, u_\lambda) = \frac{1}{2}\|\theta_\lambda - \theta_b\|_\Gamma^2 + \frac{\lambda}{2}\|u_\lambda\|_\Gamma^2 \le J_\lambda(\theta, u) = \frac{\lambda}{2}\|u\|_\Gamma^2.$$

Therefore,

$$\|u_\lambda\|_\Gamma^2 \le C, \quad \|\theta_\lambda - \theta_b\|_\Gamma^2 \to 0, \quad \lambda \to +0.$$

Hereinafter, $C > 0$ does not depend on λ. The boundedness of the sequence u_λ in the space U implies, based on Lemma 1, the estimates

$$\|\theta_\lambda\|_V \le C, \quad \|\varphi\|_\lambda \le C.$$

Therefore, we can choose a sequence $\lambda \to +0$ such that

$$u_\lambda \to u_* \quad \text{weakly in } U, \quad \theta_\lambda, \varphi_\lambda \to \theta_*, \varphi_* \quad \text{weakly in } V, \quad \text{strongly in } L^4(\Omega). \tag{17}$$

The results (17) allow passage to the limit as $\lambda \to +0$ in the equations for $\theta_\lambda, \varphi_\lambda, u_\lambda$ and, then,

$$aA\theta_* + b\kappa_a([\theta_*]^4 - \varphi_*) = Br, \quad \alpha A\varphi_* + \kappa_a(\varphi_* - [\theta_*]^4) = Bu_*. \tag{18}$$

In this case, $\theta_*|_\Gamma = \theta_b$. From the first equation in (18), taking into account that $r = a(\theta_b + q_b)$, we deduce

$$-a\Delta\theta_* + b\kappa_a([\theta_*]^4 - \varphi_*) = 0 \quad \text{a.e. in } \Omega, \quad \theta_* = \theta_b, \quad \partial_n\theta = q_b \quad \text{a.e. in } \Gamma.$$

The second equation in (18) implies that $-\alpha\Delta\varphi + \kappa_a(\varphi - [\theta]^4) = 0$ almost everywhere in Ω. Thus, the pair θ_*, φ_* is a solution of problem (1)–(3).

Remark. The boundedness of the sequence u_λ in the space U implies its weak relative compactness and the existence of a sequence (possibly not unique) $\lambda \to +0$ such that $u_\lambda \to u_*$ weakly in U. For the practical solution of problem (1)–(3), it is important that, for any sequence $\lambda \to +0$, the estimate $\|\theta_\lambda - \theta_b\|_\Gamma^2 \le C\lambda$ holds and, since $\partial_n\theta_\lambda = \theta_b + q_b - \theta_\lambda$, we have $\|\partial_n\theta_\lambda - q_b\|_\Gamma^2 \le C\lambda$. These inequalities guarantee that the boundary values $\theta_\lambda, \partial_n\theta_\lambda$ at small λ approximate the boundary conditions of problem (1)–(3).

6. NUMERICAL SIMULATION

Let us present an iterative algorithm for solving the optimal control problem. Let $\tilde{J}_\lambda(u) = J_\lambda(\theta(u), u)$, where $\theta(u)$ is the component of the solution of problem (1) and (3) corresponding to the control $u \in U$.

In accordance with (16), the gradient of the functional $\tilde{J}_\lambda(u)$ is

$$\tilde{J}'_\lambda(u) = \lambda u - p_2.$$

Here, p_2 is the corresponding component of the conjugate state from system (16), where $\hat{\theta} := \theta(u)$.

Gradient descent algorithm

Chose the value of the gradient step ε.
Choose the number of iterations N.
Choose an initial approximation for the control, $u_0 \in U$.
for $k \leftarrow 0, 1, 2, \ldots, N$ **do**
    For the given u_k, calculate the state $y_k = \{\theta_k, \varphi_k\}$: the solution of problem (1) and (2).
    Calculate the value of the objective functional $J_\lambda(\theta_k, u_k)$.
    Calculate the conjugate state $p_k = \{p_{1k}, p_{2k}\}$ from Eqs. (14), where $\hat{\theta} := \theta_k$, $\hat{u} = u_k$.
    Recalculate the control $u_{k+1} = u_k - \varepsilon(\lambda u_k - p_2)$.

The value of the parameter ε is chosen empirically so that the value $\varepsilon(\lambda u_k - p_2)$ be a significant correction for u_{k+1}. The number of iterations N is chosen to be sufficient to satisfy the condition $J_\lambda(\theta_k, u_k) - J_\lambda(\theta_{k+1}, u_{k+1}) < \delta$, where $\delta > 0$ determines the accuracy of the calculations.

The examples considered below illustrate the performance of the proposed algorithm for small (which is important) values of the regularization parameter $\lambda \le 10^{-12}$. In the first example, test calculations for a cube are performed. The second example compares the calculations by the proposed algorithm with the results of [20].

Note that, for the numerical solution of the direct problem with a given control, a simple iteration method was used to linearize the problem and solve it by the finite element method. Solving the conjugate system, which is linear at a given temperature, is not difficult. For numerical simulation, the FEniCS solver was used [31, 32].

The source code of the experiments can be found in [33].

Example 1. Consider a cube $\Omega = (x, y, z)$, $0 \le x, y, z \le l$. Assume that $l = 1$ cm, $a = 0.006$ [cm^2/s], $b = 0.025$ [cm/s], $\kappa_a = 1$ [cm^{-1}], and $\alpha = 0.(3)$ [cm]. These parameters correspond to glass [11]. The regularization parameter is $\lambda = 10^{-12}$.

Let the boundary data r and u in (2) have the form:

$$r = 0.7, \quad u = \hat{u} = 0.5.$$

Next, we calculate the state θ and φ as a solution of problem (1) and (2) and choose as θ_b the boundary value of the function θ on Γ. The values of the normal derivative $\partial_n\theta$ on Γ must correspond to the values $q_b = r/a - \theta_b$. Applying the proposed algorithm with an initial approximation $u_0 = 0.1$, we find an approximate solution $\{\theta_\lambda, \varphi_\lambda, u_\lambda\}$ of problem (CP). To demonstrate that the algorithm finds an approximate solution of the problem with Cauchy data for temperature, it is important to compare the values of $\partial_n\theta_\lambda$ on Γ with q_b.

Figures 1a and 1b show the absolute value of the relative deviation of $\partial_n\theta_\lambda$ from q_b on the edge of the cube in the plane $z = l$, where $\partial_n\theta_\lambda = \partial\theta_\lambda/\partial z$ and the dynamics of the objective functional, which determines the norm of the difference $\|\theta_\lambda - \theta_b\|^2_\Gamma$. On the other faces of the cube, the relative deviation has the same order of smallness.

Example 2. Let us compare the performance of the proposed algorithm with the results of article [20], for which one of the authors of this article was a coauthor. The problem is considered in the domain $\Omega \times (-L, L)$, where $\Omega = \{x = (x_1, x_2)\colon 0 < x_{1,2} < d\}$ and, at large L, it is reduced to a two-dimensional problem with a computational domain Ω. The following values of the problem parameters were selected: $d = 1$ (m), $a = 0.9210^{-4}$ (m^2/s), $b = 0.19$ (m/s), $\alpha = 0.0333$ (m), and $\kappa_a = 1$ (m^{-1}). The parameters correspond to air at normal atmospheric pressure and a temperature of 400°C.

The functions θ_b and q_b in boundary condition (3) are specified as follows: $\theta_b = \hat{\theta}\big|_\Gamma$ and $q_b = \partial_n\hat{\theta}\big|_\Gamma$, where $\hat{\theta} = (x_1 - 0.5)^2 - 0.5x_2 + 0.75$.

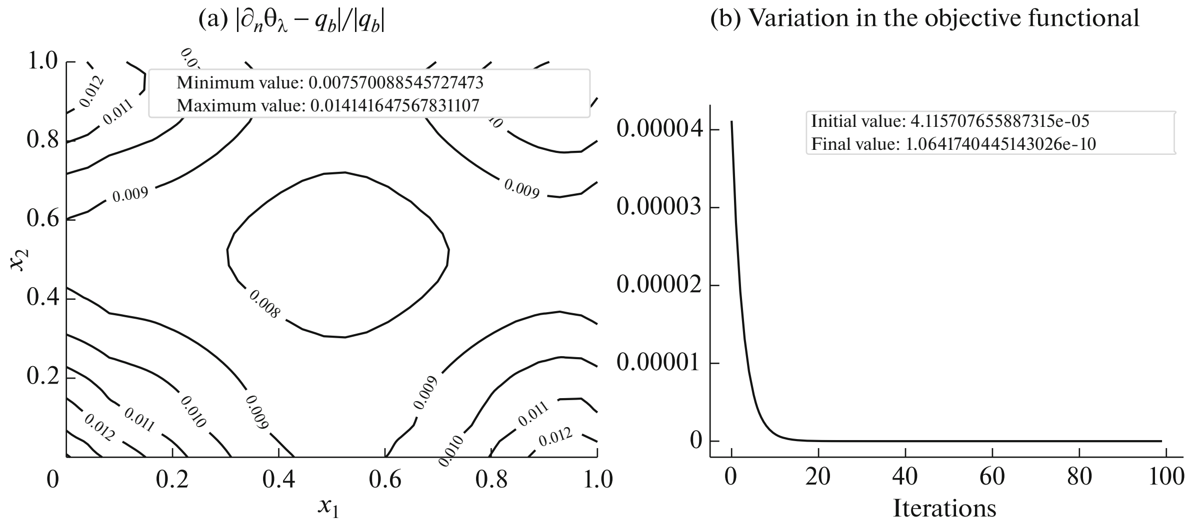

Fig. 1. Example 1.

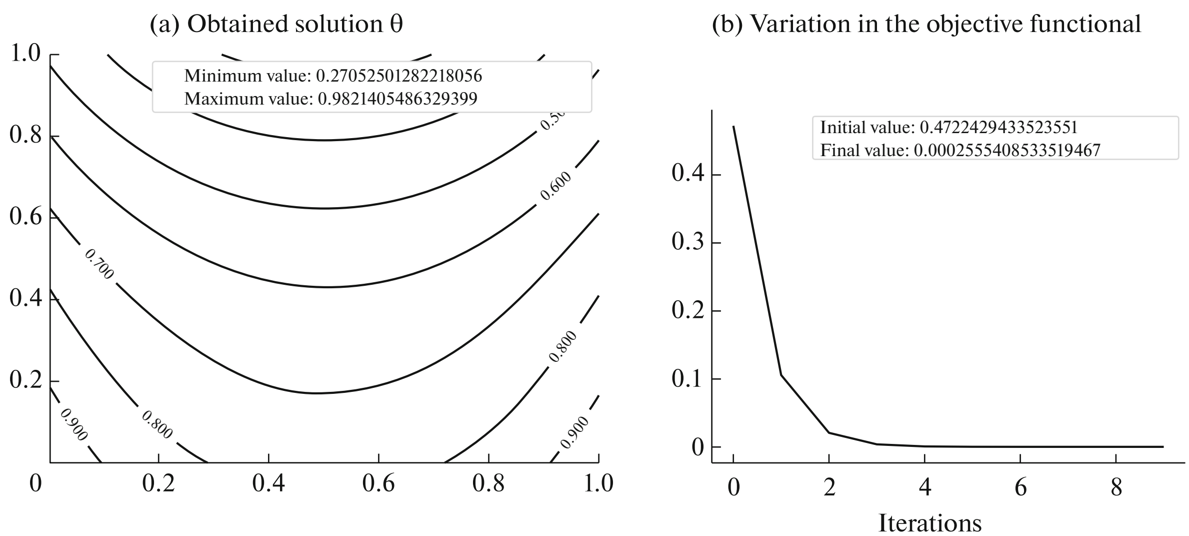

Fig. 2. Example 2.

An approximate solution of the problem with the Cauchy data presented in [20] was obtained by solving a fourth-order elliptic problem for temperature by the method of relaxation in time. The Bogner–Fox–Schmidt H^2 conformal finite elements and the FeliCs solver, developed at the Technical University of Munich, were used. The solution stabilized after 120 s, but the calculations at each time step had rather significant costs [20].

Figure 2a shows the temperature field obtained by the method proposed in this article, which quite accurately coincides with the result in [20]. The value of $\|\partial_n\theta_\lambda - q_b\|_{L^2(\Gamma)} / \|q_b\|_{L^2(\Gamma)}$ is equal to 0.000567. The value of the objective functional, which determines the norm of the difference $\|\theta_\lambda - \theta_b\|_\Gamma^2$, is equal to 0.000255 and stabilizes after 10 iterations, as shown in Fig. 2b.

The numerical examples presented show that the algorithm proposed successfully copes with finding a numerical solution of problem (1)–(3).

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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