The Economics of Interest Rate Models in Decentralised Lending Protocols

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Abstract

We characterise the equilibrium between borrowers and lenders in a decentralised lending protocol (DLP). Using this equilibrium characterisation, we develop a methodology for a DLP to update the interest rate model (IRM) in order to move the proportion of liquidity being borrowed towards a desired level of utilization. Furthermore, we show how DLPs can estimate the supply and demand curves as borrowers and lenders interact with the pool of liquidity and compare our methodology with the well-known PID controller approach. Finally, we discuss possible exploits where lenders alter the quantities being borrowed and lent to benefit from the IRM, and we go on to study these exploits when there is response from borrowers and dynamic updating.

1 Introduction

Lending and borrowing of assets is one of the core activities of market participants in a financial ecosystem. Decentralised lending protocols (DLPs) replicate these activities on public blockchains by allowing users to lend and borrow assets without having to rely on a central trusted entity such as traditional banks [8, 15, 17, 9]. Assets in DLPs are pooled together, and the interest paid by borrowers is, typically, equally distributed among all lenders, in proportion to the quantities of assets supplied. Unlike traditional finance, variable interest rates are endogenously determined by the protocol governance through a predefined interest rate model (IRM). A common design choice is to use an IRM which sets interest rates as a function of the fraction of liquidity that is being borrowed; this percentage is called the utilization of the pool. For example, the IRM used by AAVE [1] and Compound [7], currently the two largest DLPs in terms of total value locked, are piecewise linear functions of the utilization of the pool.

Key objectives when designing an IRM are to ensure that the demand for borrowing and the supply of lending can reach an equilibrium and that this equilibrium is achieved at some desired level of utilization. Determining utilization is particularly important when rehypothecation of the collateral is allowed (that is, no distinction is made between assets supplied for lending and assets supplied as collateral against borrowing); in this setting, a high buffer of liquid (unborrowed) assets is required to allow withdrawals and repayments to occur without substantial risk. One of the key objectives of this work is to understand the equilibrium level of utilization based on the supply and demand for assets, which can be estimated from data, and hence to give methods to choose an IRM with the desired equilibrium utilization. In particular, we develop a framework for market clearing in DLPs and we describe how equilibrium interest rates are determined. From here, we derive update rules for how the protocol can change the IRM to move towards a desired utilization. We also study how to make the problem robust to model misspecification by allowing the protocol to estimate the market clearing mechanism and update the IRM simultaneously. In addition, we inspect how additional goals can be incorporated into the dynamic updates of the IRM. Finally, we consider a potential flaw in this market design, where the mechanism can be exploited by lenders who increase utilization artificially, leading to supercompetitive profits.

Literature. There is growing literature on understanding market efficiency and equilibrium in the context of lending protocols. An economic approach to interest rate setting mechanisms and the associated welfare implications are given in [14]; this is conceptually similar to the fixed point argument which we present in Section 2. Similarly, [5] studies an equilibrium model focusing on asymmetric information between borrowers and lenders. For empirical work on IRM see [3]. In contrast, our work focuses on how to adjust the IRM dynamically in line with changing market conditions. Since DLPs are mimicking the functionality of a margin trading account, one can study how the supply and demand for lendable assets relate to trading on margin at alternative venues, see e.g [3, 11, 2].

In recent work, [4] presents an exploit on dynamic IRMs. This is conceptually similar to the analysis that we present in Section 4.

While the focus of this work is on IRMs, other key elements of the design of DLPs are loan-to-value ratios at loan origination, liquidation thresholds, and bonuses, which control the acceptable level of risk for the protocol and its users, and also collateralisation rules, which in turn control the exposure of the protocol to a particular asset class. We refer the reader to [6] for a systematic analysis of these design choices. There are also a number of empirical works that shed light on risk and reward trade-offs in DLPs [13, 12, 10].

Organization of the paper. Section 2 derives an equilibrium level of lending and interest rates in DLPs from first principles, based on a classical model of supply and demand. This section builds intuition for how equilibria are formed and introduces the dynamics of market participants (borrowers and lenders) together with notation that is used throughout the paper. Section 3 derives equations for how the IRM should be updated if the protocol aims to achieve a desired level of utilization in the pool. In particular, this section shows how protocols can estimate the supply and demand curves as borrowers and lenders interact with the pool of liquidity, and compares our methodology with the well-known proportional-integral-derivative controller approach. Section 4 discusses exploits and presents the concept of a 'recursive trade' that lenders can use to alter utilization for their benefit, and we go on to study how these exploits unfold when there is response from the borrowers and dynamic updating.

2 An equilibrium interest rate model

We wish to determine the equilibrium interest rate and quantity of borrowing in a DLP. This is made somewhat complicated due to the endogenous dependence of the interest rate on the level of utilization, which we shall see typically prevents an equilibrium forming at an interest rate where supply equals demand (which would correspond to full utilization).

We begin our model by specifying the supply and demand functions $\mathbf{Q}^L$ and $\mathbf{Q}^B$, which give the amounts which lenders and borrowers (respectively) wish to lend/borrow, as a function of the interest rate which they obtain. We plot an indicative example of $\mathbf{Q}^L, \mathbf{Q}^B : \mathbb{R}^+ \to \mathbb{R}^+$ in Figure 1.

Assumption 2.1. *We assume that both $\mathbf{Q}^L$ and $\mathbf{Q}^B$ are positive, continuous, $\mathbf{Q}^L$ is strictly increasing, and $\mathbf{Q}^B$ is decreasing. We also assume that the map $r \mapsto r\mathbf{Q}^B(r)$ is strictly quasiconcave – that is, there exists r_M such that it is strictly increasing for $r < r_M$ and strictly decreasing for $r > r_M$.*

The assumption that $r \mapsto r\,\mathbf{Q}^B(r)$ is increasing for small interest rates is essentially a marginal substitution effect for borrowing, where increases in interest rates result in increased total interest payments, even though quantities borrowed may decrease. Allowing the map to become decreasing above a point r_M encodes our intuition that it may not be in lenders' interests to make the interest rate arbitrarily large, if it forces borrowers out of the market.[1]

From Figure 1, we immediately observe that there is a minimal interest rate $r_{\min}$ below which it is not possible for the market to clear – there are not enough assets supplied by lenders to satisfy the demands of borrowers.

However, for interest rates above $r_{\min}$, as not all assets offered by lenders will be borrowed, it is not the case that lenders will receive the nominal rate which is paid by borrowers. We define the utilization fraction $\mathcal{U}$ as

$$\mathcal{U} = \frac{Q^B}{Q^L}\,, \tag{1}$$

[1] As an example, if $\mathbf{Q}^B$ is linear, then $r\,\mathbf{Q}^B(r)$ is a parabola that opens downwards and r_M is the interest rate that characterises the vertex.

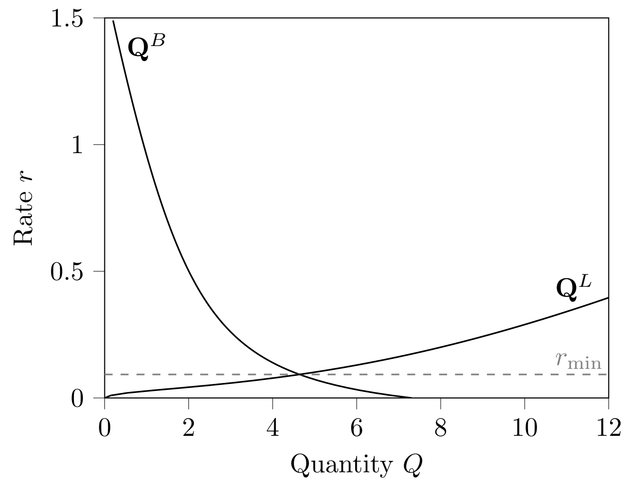

Figure 1: Supply and demand for borrowing in the market. Here and elsewhere $\mathbf{Q}^B$ and $\mathbf{Q}^L$ are plotted with their axes reversed, to conform with the classic convention for supply and demand curves with quantities on the horizontal axis.

where Q^B and Q^L are the amounts being borrowed and lent in the DLP. If the nominal interest rate is r, assuming borrowing is in equilibrium, when providing $q > 0$ assets for lending, lenders will only receive interest at the rate $\rho\, r\, \mathcal{U} = \rho\, r\, \mathbf{Q}^B(r)/q$, where $\rho \in [0, 1]$ is a coefficient introducing a multiplicative spread between borrowers and lenders. Therefore, lenders face a fixed point problem, because for a given nominal rate r, the amount q which they wish to post on the market should satisfy

$$q = \mathbf{Q}^L\Big(\rho\, r\, \frac{\mathbf{Q}^B(r)}{q}\Big). \tag{2}$$

Proposition 2.1. *Under Assumption 2.1, there is a unique function* $\tilde{\mathbf{Q}}^L : \mathbb{R}^+ \to (0, \infty)$ *such that for* $r > 0$ *we have that*

$$\tilde{\mathbf{Q}}^L(r) = \mathbf{Q}^L\Big(\rho\, r\, \frac{\mathbf{Q}^B(r)}{\tilde{\mathbf{Q}}^L(r)}\Big). \tag{3}$$

Proof. As $\mathbf{Q}^L$ is strictly increasing, it is invertible, and its inverse is also strictly increasing. Together with positivity, this implies that the map $M : q \mapsto q \times (\mathbf{Q}^L)^{-1}(q)$ is also strictly increasing, and hence is also invertible. It is then easy to see that the (unique) fixed point in (2) is given by

$$\tilde{\mathbf{Q}}^L(r) := q = M^{-1}(\rho\, r\, \mathbf{Q}^B(r)).$$

□

We call $\tilde{\mathbf{Q}}^L$ the effective supply, as it represents how much lenders are willing to offer at a given nominal interest rate r, accounting for the interest they actually receive. Given our assumptions, we know that $r \mapsto r\, \mathbf{Q}^B(r)$, and hence $\tilde{\mathbf{Q}}^L$, is a quasiconcave function of r, and $\tilde{\mathbf{Q}}^L(r) < \mathbf{Q}^L(r)$ for all $r > r_{\min}$. We also observe that $\tilde{\mathbf{Q}}^L$ may have a maximum value, which occurs at the point $r_M = \arg\max_r\{r\, \mathbf{Q}^B(r)\}$, at which lenders are earning a maximal interest payment from borrowers.

Adding this function to the plot in Figure 2 (left), we sketch the effective supply $\tilde{\mathbf{Q}}^L$ and demand $\mathbf{Q}^B$ curves for borrowing. We note that the intercept of these curves is the point where utilization $\mathcal{U} = 1$, that is, all quantities offered for loan are borrowed.

Based on these effective supply $\tilde{\mathbf{Q}}^L$ and demand $\mathbf{Q}^B$ curves, we next observe that there is a relationship between the interest rate and the level of utilization in the market. In particular, we can sketch out the curve $\mathcal{U} = \mathbf{Q}^B(r)/\tilde{\mathbf{Q}}^L(r)$ for $r > r_{\min}$. For values of $r < r_M$, this will be a strictly decreasing function, as lower interest rates correspond to higher utilization (higher quantities borrowed and lower quantities offered for loan). We thus obtain a function

$$r_{\mathrm{MKT}} : u \mapsto \inf \left\{ r > 0 : u = \mathbf{Q}^B(r)/\tilde{\mathbf{Q}}^L(r) \right\}$$

which defines the (minimal) interest rate which would need to be offered to the market in order to obtain a specified level of utilization. Given $\tilde{\mathbf{Q}}^L$ is not monotone, it is not clear that such an interest rate exists for all levels of utilization. However, this is certainly the case for all $u \geq \mathbf{Q}^B(r_M)/\tilde{\mathbf{Q}}^L(r_M)$, as we know $\tilde{\mathbf{Q}}^L$ is increasing for $r < r_M$; for these values, utilization is guaranteed to be decreasing in r.

At this point, we introduce the interest rate model specified by the protocol. In a protocol like AAVE, this is a map $r_{\mathrm{IRM}} : [0, 1] \to \mathbb{R}^+$ which defines the interest rate as a function of utilization. In order to prevent manipulation of the interest rate, it is necessary that this is an increasing function, and we will assume that it is also continuous. We plot these curves in Figure 2 (right).

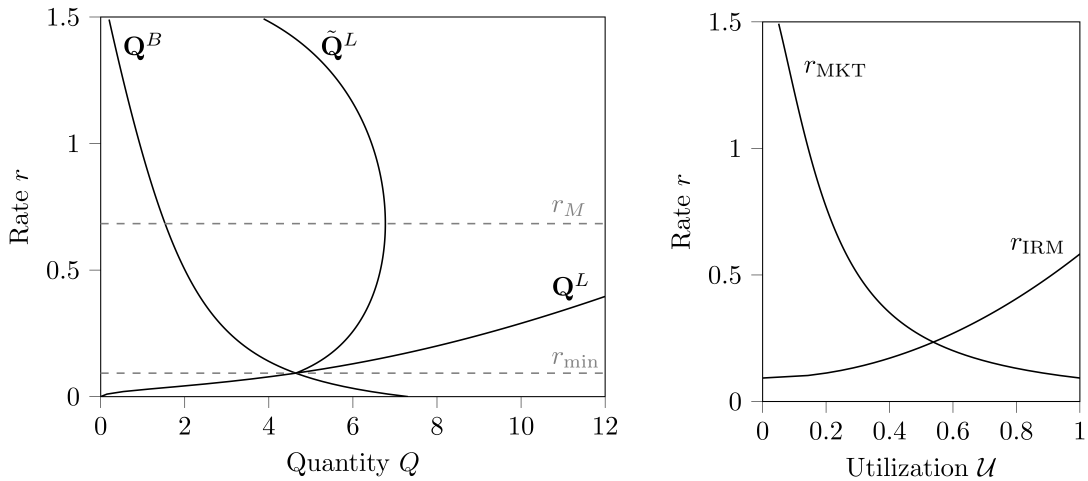

Figure 2: Left: Supply and demand curves, together with effective supply. Right: The market and IRM interest rates (in bold), as a function of utilization

In order for the market to be at equilibrium, we look for a point $(r^*, \mathcal{U}^*)$ where

$$r^* = r_{\mathrm{IRM}}(\mathcal{U}^*) = r_{\mathrm{MKT}}(\mathcal{U}^*).$$

Assuming $r_{\mathrm{IRM}}(0) \leq r_{\mathrm{MKT}}(0)$ and $r_{\mathrm{IRM}}(1) \geq r_{\mathrm{MKT}}(1) = r_{\min}$, by the intermediate value theorem there exists at least one equilibrium point. As $\tilde{\mathbf{Q}}^L$ is increasing for $r < r_M$, we observe that there is at most one equilibrium with $r^* \leq r_M$, and this is usually the value of economic interest (because above r_M, all market participants have an interest in reducing the interest rate). We can obtain the corresponding equilibrium quantities of borrowing $B^* = \mathbf{Q}^B(r^*)$ and lending $L^* = \tilde{\mathbf{Q}}^L(r^*)$, as shown in Figure 3.

2.1 Convergence to equilibrium

Let us study an example in more detail. Suppose borrowers are borrowing $\tilde{B} < B^*$ and lenders are supplying $\tilde{L} > L^*$, in the setting where $r^* < r_M$. It then follows that the utilization in the lending protocol is at $\tilde{\mathcal{U}} = \tilde{B}/\tilde{L}$ which is less than $\mathcal{U}^*$. The IRM implies that the interest rate in the market is given by $\tilde{r} = r_{\mathrm{IRM}}(\tilde{\mathcal{U}}) < r^*$. This implies that borrowers should be willing to borrow $\mathbf{Q}^B(\tilde{r}) > B^*$ and similarly, lenders should be willing to lend $\mathbf{Q}^L(\tilde{r}) < L^*$. Both market participants would then adjust their quantities in the direction of B^* and L^* respectively. Other cases are similar, with nuances around one of the quantities moving in the opposite direction but returning to equilibrium after some updates. We come back to this point below when we run simulations on the convergence to the equilibrium $(r^*, \mathcal{U}^*)$.

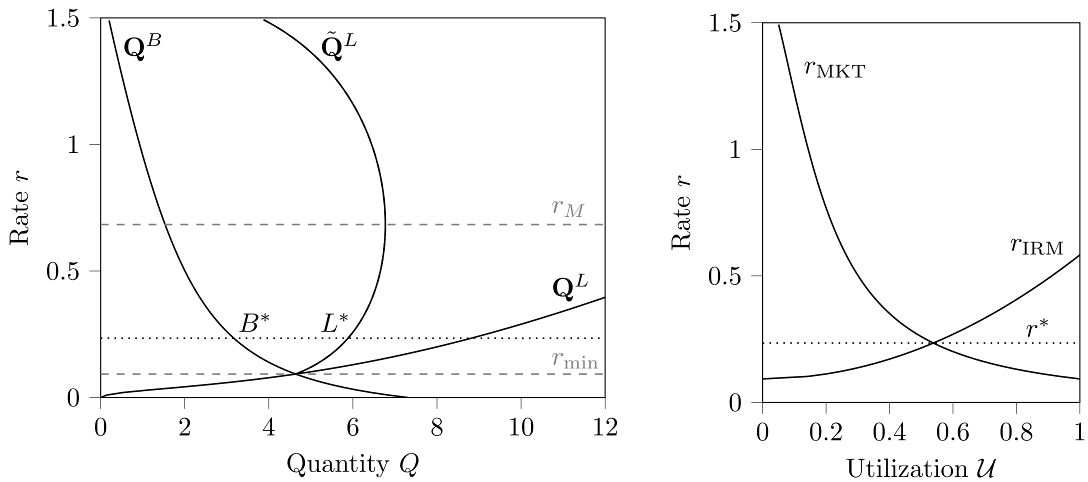

Figure 3: Equilibrium interest rates, utilization and quantities borrowed and offered for loan.

2.2 The linear case

For the sake of explicit formulae, we now consider the case where $\mathbf{Q}^B$ and $\mathbf{Q}^L$ are linear, as is the IRM curve. In particular,

$$r_{\mathrm{IRM}}(\mathcal{U}) = a\,\mathcal{U}\,, \quad \mathbf{Q}^B(r) = B_0 - r\,\bar{B}\,, \quad \text{and} \quad \mathbf{Q}^L(r) = L_0 + r\,\bar{L}\,, \tag{4}$$

for positive parameters $a, B_0, \bar{B}, L_0$, and $\bar{L}$. We also fix $\rho = 1$.

A short calculation using (2) and the functions $\mathbf{Q}^L(r)$ and $\mathbf{Q}^B(r)$ implies that $\tilde{\mathbf{Q}}^L(r)$ is given by

$$\tilde{\mathbf{Q}}^L(r) = \frac{L_0 + \sqrt{L_0^2 + 4\,r\,\bar{L}\,(B_0 - r\,\bar{B})}}{2} = \frac{L_0 + \sqrt{L_0^2 + 4\,r\,\bar{L}\,Q_B(r)}}{2}\,. \tag{5}$$

We observe that $r_M = B_0/(2\bar{B})$ in this setting. The utilization, as a function of r, is given by

$$\mathcal{V}(r) = \frac{\mathbf{Q}^B(r)}{\tilde{\mathbf{Q}}^L(r)} = \frac{2\,(B_0 - r\,\bar{B})}{L_0 + \sqrt{L_0^2 - 4\,r\,\bar{L}\,(r\,\bar{B} - B_0)}}\,, \tag{6}$$

and thus

$$\begin{aligned} r_{\mathrm{MKT}}(\mathcal{U}) &= \mathcal{V}^{-1}(\mathcal{U}) \\ &= -\frac{\mathcal{U}\,L_0\,\bar{B} - \mathcal{U}^2\,\bar{L}\,B_0 - 2\,B_0\,\bar{B}}{2(\mathcal{U}^2\,\bar{L}\,\bar{B} + B^2)} \\ &\qquad + \frac{\sqrt{(\mathcal{U}\,L_0\,\bar{B} - \mathcal{U}^2\,\bar{L}\,B_0 - 2\,B_0\,\bar{B})^2 - 4\,(\mathcal{U}^2\,\bar{L}\,\bar{B} + B^2)(B_0^2 - \mathcal{U}\,L_0\,B_0)}}{2(\mathcal{U}^2\,\bar{L}\,\bar{B} + B^2)}\,. \end{aligned}$$

We obtain the equilibrium by finding $\mathcal{U}^*$ such that

$$\begin{aligned} a\,\mathcal{U}^* = &-\frac{\mathcal{U}^*\,L_0\,\bar{B} - (\mathcal{U}^*)^2\,\bar{L}\,B_0 - 2\,B_0\,\bar{B}}{2((\mathcal{U}^*)^2\,\bar{L}\,\bar{B} + B^2)} \\ &\qquad + \frac{\sqrt{(\mathcal{U}^*\,L_0\,\bar{B} - (\mathcal{U}^*)^2\,\bar{L}\,B_0 - 2\,B_0\,\bar{B})^2 - 4\,((\mathcal{U}^*)^2\,\bar{L}\,\bar{B} + B^2)(B_0^2 - \mathcal{U}^*\,L_0\,B_0)}}{2((\mathcal{U}^*)^2\,\bar{L}\,\bar{B} + B^2)}\,, \end{aligned}$$

and letting $r^* = r_{\mathrm{IRM}}(\mathcal{U}^*)$. Unfortunately, while this point can be found efficiently numerically, the formula does not simplify.

2.3 Dynamic models and convergence

We now consider a dynamic version of the above model, where we no longer assume that lenders and borrowers instantaneously reach equilibrium. We base this on the linear model, and assume that borrowers and lenders gradually adapt their positions towards their optimal asset levels, ignoring their effect on the interest rate.

We use the the following model parameter values:

(i) $a = 0.15$, that is $r_{\text{IRM}}(\mathcal{U}) = 0.15\mathcal{U}$, and $\rho = 1$,

(ii) $B_0 = 100$, $\bar{B} = 100$, so the Buyer desired quantity is $\mathbf{Q}^B(r) = 100 - 100r$,

(iii) $L_0 = 75$, $\bar{L} = 100$, so the lender desired quantity is $\mathbf{Q}^L(r) = 75 + 100r$.

For a given pair of initial quantities Q_0^B and Q_0^L with $Q_0^B < Q_0^L$ and the subscript denoting time, the utilization is $\mathcal{U}_0 = Q_0^B/Q_0^L$ and the interest rate in the market is $r_0 = r_{\text{IRM}}(\mathcal{U}_0)$.

We suppose that borrowers and lenders will not instantly converge to equilibrium, but will do so in a sequential way. At time t, the current interest rate is given by

$$r_t = r_{\text{IRM}}(\mathcal{U}_t) = r_{\text{IRM}}(Q_t^B/Q_t^L),$$

so borrowers wish to borrow $Q_B(r_t)$. We suppose that they update using a simple learning algorithm, based on the current interest rate, that is,

$$Q_{t+1}^B = \min\left\{Q_t^B + \alpha_B \left(\mathbf{Q}^B(r_t) - Q_t^B\right), \quad Q_t^L\right\}$$

for a learning rate $\alpha_B > 0$. We limit borrowers to not borrow more assets than have already been posted by lenders.

Similarly, lenders desire to supply $\tilde{Q}_L(r_t)$ and thus we let

$$Q_{t+1}^L = \max\left\{Q_t^L + \alpha_L \left(\tilde{\mathbf{Q}}^L(r_t) - Q_t^L\right), \quad Q_{t+1}^B\right\}$$

for a learning rate $\alpha_L > 0$. We note that we have included the constraint that borrowers can adjust their positions first and 'lock' the lenders' liquidity.

In the plots below, the equilibrium is given by $(r^*, \mathcal{U}^*) = (0.268, 0.765)$, and $(B^*, L^*) = (73.18, 95.54)$. Figure 4 shows the trajectories of $(Q_t^B)_{t\in[0,100]}$, $(Q_t^L)_{t\in[0,100]}$, and $(\mathcal{U}_t)_{t\in[0,100]}$ for four starting values that different from (B^*, L^*); we take $\alpha_B = \alpha_L = 0.1$.

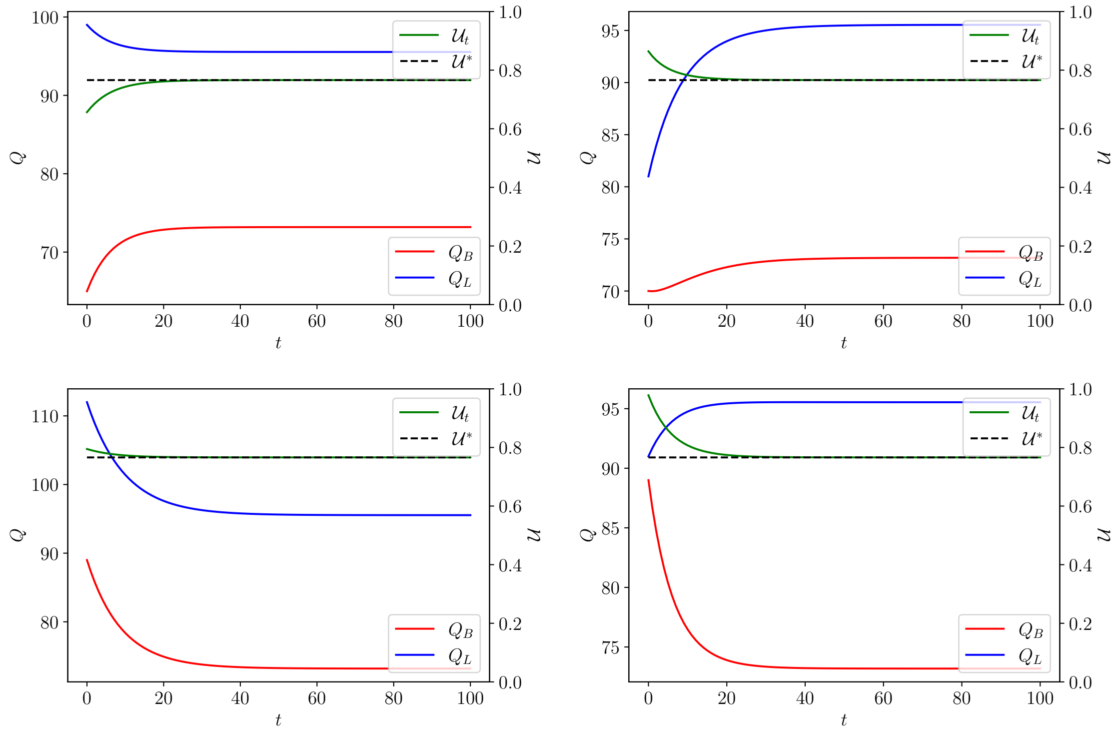

Figure 4: Convergence of $Q_t^B \to Q_B^*$, $Q_t^L \to Q_L^*$, and $\mathcal{U}_t \to \mathcal{U}^*$ as $t \to \infty$.

In practice, the quantities posted by borrowers and lenders will be much more noisy than the smooth paths we have given above. We adjust this modelling in Figure 5, which shows the trajectories depicted above when we add noise to the update equations, that is, for $t \in \{1, 2, 3, \dots\}$ the borrowers adjust their quantity using

$$Q_{t+1}^B = \max\left\{ \min\left\{ Q_t^B + \alpha_B \left(\mathbf{Q}^B(r_t) - Q_t^B \right) + \sigma^B \, Z_t^B, \quad Q_t^L \right\}, 0 \right\}$$

and lenders use

$$Q_{t+1}^L = \max\left\{ Q_t^L + \alpha_L \left(\tilde{\mathbf{Q}}^L(r_t) - Q_t^L \right) + \sigma^L \, Z_t^L, \quad Q_{t+1}^B \right\}$$

where Z_t^L, Z_t^B are a collection of independent standard normal random variables and $\sigma^B, \sigma^L \in \mathbb{R}^+$. For these plots, we take $\sigma^L = \sigma^B = 0.5$. Given these noise terms can be arbitrarily large, the effect of borrowers locking liquidity should be included here, and we also modify the dynamics to ensure that our randomness does not lead to negative borrowing quantities.

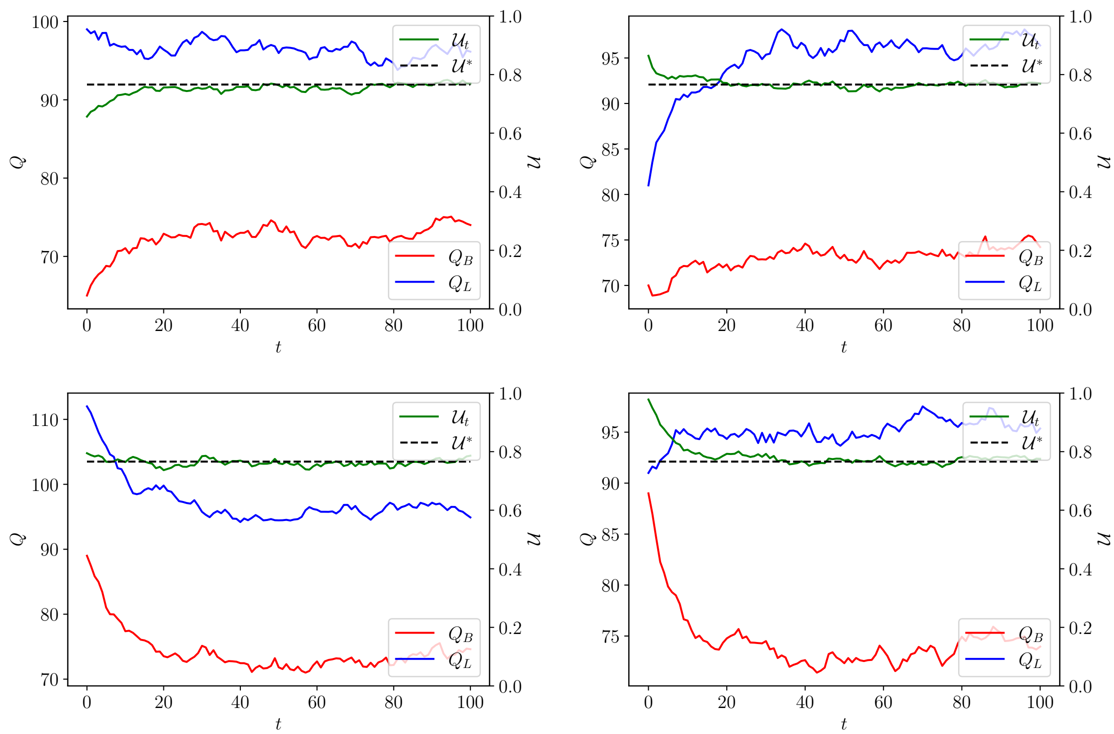

Figure 5: Convergence of $Q_t^B \to B^*$, $Q_t^L \to L^*$, and $\mathcal{U}_t \to \mathcal{U}^*$ as $t \to \infty$.

3 Moving towards a desired utilization

Let $\mathcal{U}_{\text{goal}}$ be a desired utilization rate that the protocol wishes to reach. In this section we study how to change $r_{\text{IRM}}(\cdot)$ so that the market equilibrium utilization $\mathcal{U}^*$ moves towards the desired $\mathcal{U}_{\text{goal}}$. Of course there is not a unique r_{IRM} giving a desired level of utilization – the only requirement is that $r_{\text{IRM}}(\mathcal{U}^*) = r_{\text{MKT}}(\mathcal{U}^*)$. Therefore, we will restrict our attention to parametric families of r_{IRM}, and consider how to optimize the parameters to achieve these goals.

In this section, we initially assume that the functions $\mathbf{Q}^B$ and $\mathbf{Q}^L$ are known by the protocol, and are differentiable. In principle, this means that it would be possible to simply jump to a good choice of r_{IRM}. However, we will look at designing iterative methods to proceed towards the desired utilization, which will give us insights for the case where $\mathbf{Q}^B$ and $\mathbf{Q}^L$ are not fully known.

Let $r_{\text{IRM}}(\cdot)$ be a parametric family of curves, and let Θ be the parameters. In the example above, $r_{\text{IRM}}(\mathcal{U}) = a\mathcal{U}$ so we have that $\Theta = (a)$.

We wish to understand how the equilibrium utilization $\mathcal{U}^*$ changes as we change the IRM parameters Θ. Thus, we employ comparative statics using the equilibrium utilization equation

$$\mathcal{U}^* = \frac{\mathbf{Q}^B(r_{\text{IRM}}(\mathcal{U}^*))}{\tilde{\mathbf{Q}}^L(r_{\text{IRM}}(\mathcal{U}^*))}\,, \tag{7}$$

and observe that

$$\nabla_\Theta \mathcal{U}^* = \frac{\frac{\mathrm{d}}{\mathrm{d}r}\mathbf{Q}^B(r_{\text{IRM}}(\mathcal{U}^*))\,\nabla_\Theta r_{\text{IRM}}(\mathcal{U}^*) - \mathcal{U}^*\,\frac{\mathrm{d}}{\mathrm{d}r}\tilde{\mathbf{Q}}^L(r_{\text{IRM}}(\mathcal{U}^*))\,\nabla_\Theta r_{\text{IRM}}(\mathcal{U}^*)}{\tilde{\mathbf{Q}}^L(r_{\text{IRM}}(\mathcal{U}^*))}\,. \tag{8}$$

If the market is at equilibrium utilization $\mathcal{U}^*$ (for the current choice of r_{IRM}) and the goal utilization is $\mathcal{U}_{\text{goal}}$, then we can update parameter Θ using a gradient step[2] :

$$\Theta_{\text{NEW}} = \Theta_{\text{OLD}} + \alpha_\Theta\,(\mathcal{U}_{\text{goal}} - \mathcal{U}^*)\,\nabla_\Theta \mathcal{U}^*\,, \tag{9}$$

[2]This corresponds to one step of a first-order gradient descent minimising $\frac{1}{2}|\mathcal{U}_{\text{goal}} - \mathcal{U}^*|^2$.

for a given learning rate $\alpha_\Theta > 0$. For the linear case defined in (4), we have that

$$\frac{\mathrm{d}\mathcal{U}^*}{\mathrm{d}a} = \frac{-\bar{B}\,\mathcal{U}^* - (\mathcal{U}^*)^2\,\left(\bar{L}\,(B_0 - r\,\bar{B}) - r\,\bar{L}\,\bar{B}\right)\,\left(L_0^2 + 4\,r\,\bar{L}(B_0 - r\,\bar{B})\right)^{-1/2}}{\tilde{\mathbf{Q}}^L(a\,\mathcal{U}^*)}\,. \tag{10}$$

Figure 6 shows how the above methodology would work in the linear case. For this, we take $\mathcal{U}_{\text{goal}} = 0.9$, that is, the target utilization is 90%. We use $Q_0^B = 65$ and $Q_0^L = 99$ which is the case shown in the top-left panel of Figures 4 and 5. The remaining values for the initial conditions are: $r_{\text{IRM}}(\mathcal{U}) = 0.35\mathcal{U}$ at time zero, $\alpha_\Theta = \alpha_B = \alpha_L = 0.1$ and the rest of the model parameters are the same as above. For a given r_{IRM} we employ 20 update steps for Q_t^B and Q_t^L so they converge to the equilibrium level as shown in Figure 4. Then, we perform an update step for $r_{\text{IRM}}(\cdot)$ which means that we update the parameter a using (9) and (10). We repeat this procedure 50 times to obtain a total of $20 \times 50 = 1000$ time steps.

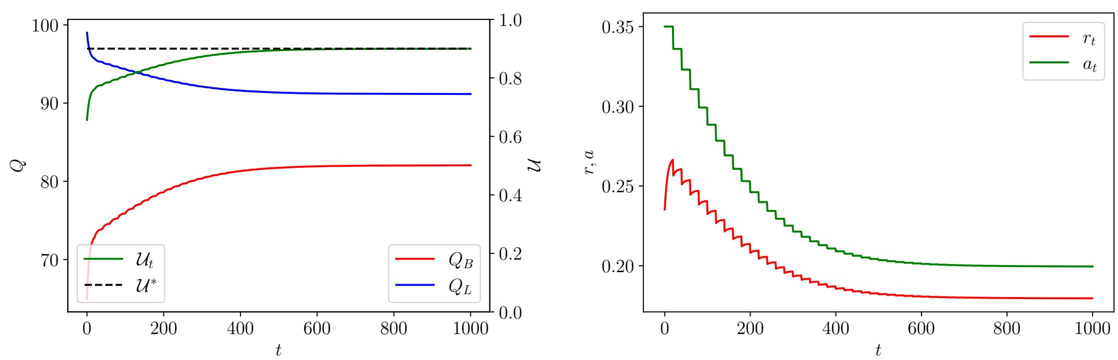

Figure 6: Convergence of $\mathcal{U}^*$ using update equation (10) as $t \to \infty$ without noise.

We see that the equilibrium is effectively achieved shortly after the 30th update of r_{IRM}, which corresponds to $t = 600$. The final utilization is $\mathcal{U}^* = \mathcal{U}_{\text{goal}} = 0.9$ and $r_{\text{IRM}}(\mathcal{U}) = 0.2\mathcal{U}$.

We can also consider the setting with noise in the dynamics of Q_t^B and Q_t^L. The case for $\sigma^L = \sigma^B = 0.5$ is shown in Figure 7. We see that this method still leads to convergence of the utilization to the desired level, with some additional noise in the final utilization and interest rate.

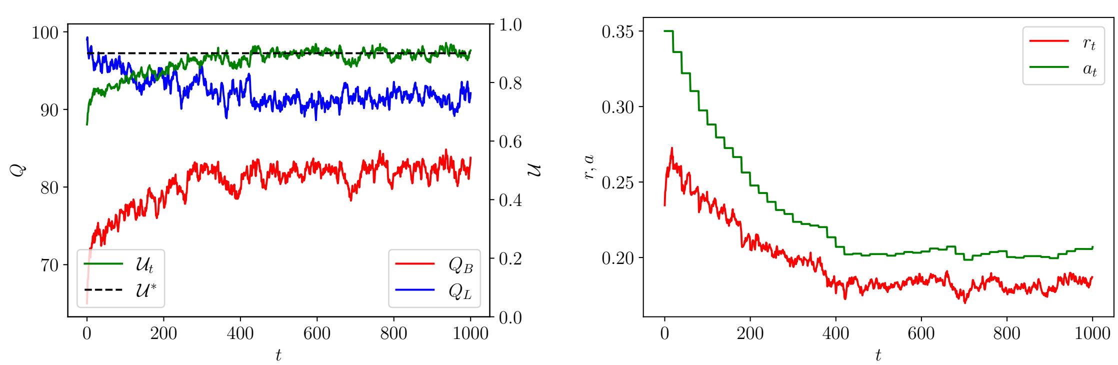

Figure 7: Convergence of $\mathcal{U}^*$ using update equation (10) as $t \to \infty$ with noise.

3.1 The incomplete information case

In real markets the protocol does not know the underlying functions $\mathbf{Q}^B(\cdot)$ and $\tilde{\mathbf{Q}}^L(\cdot)$. In this section we present a simple algorithm to update beliefs about these functions while changing the IRM to guide the market towards an equilibrium utilization $\mathcal{U}_{\text{goal}}$. For simplicity we derive the dynamics under the assumption that both $\mathbf{Q}^B$ and $\tilde{\mathbf{Q}}^L$ are linear functions and constant through time; however, the algorithm we propose performs well in practice when $\mathbf{Q}^B$ and $\tilde{\mathbf{Q}}^L$ are differentiable (so approximately linear for small changes of r) and vary only slowly in time.

The protocol approximates the functions $\mathbf{Q}^B(\cdot)$ and $\tilde{\mathbf{Q}}^L(\cdot)$ as

$$\begin{aligned}\mathbf{Q}^B(r) &\approx \mathfrak{Q}^B(r) = \mathfrak{a} - \mathfrak{b}\, r,\\ \tilde{\mathbf{Q}}^L(r) &\approx \tilde{\mathfrak{Q}}^L(r) = \mathfrak{c} + \mathfrak{d}\, r,\end{aligned}$$

and we let $(\mathfrak{a}_0, \mathfrak{b}_0, \mathfrak{c}_0, \mathfrak{d}_0)$ be initial estimates of the values of the parameters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$. The updating works as follows:

Before updating the IRM, the protocol collects noisy observations of the realized supply and demand, Q_t^L and Q_t^B, together with observations of r_t and $\mathcal{U}_t = Q_t^B/Q_t^L$. To account for randomness, the protocol averages these observations, to store the data point $(\bar{r}_0, \bar{\mathcal{U}}_0, \bar{Q}_0^B, \bar{Q}_0^L)$. We then begin the first epoch, with $t = 1$, and follow the following steps:

(i) The protocol computes an estimate of the sensitivity of utilization to the parameters of the IRM, at the current estimate of equilibrium utilization and interest rate:

$$\frac{\mathrm{d}\mathcal{U}_{t-1}^*}{\mathrm{d}a} = \frac{-\mathfrak{b}_{t-1}\, r_{t-1} - \left(\mathcal{U}_{t-1}^*\right)^2\, \mathfrak{d}_{t-1}}{\mathfrak{c}_{t-1} + \mathfrak{d}_{t-1}\, r_{t-1}}\,. \tag{11}$$

Using this estimate, the protocol changes the IRM following (9), that is, to the function $r_{\mathrm{IRM}}(\mathcal{U}) = a_t\, \mathcal{U}$, with

$$a_t = a_{t-1} + \alpha_\Theta\, (\mathcal{U}_{\mathrm{goal}} - \mathcal{U}_{t-1}^*)\, \frac{\mathrm{d}\mathcal{U}_{t-1}^*}{\mathrm{d}a}\,. \tag{12}$$

(ii) Using the newly updated IRM, borrowers and lenders adjust their quantities, and after some time they will converge to a new equilibrium, around which they produce (noisy) observations. More precisely, we assume that the protocol collects noisy observations of $(r_i, \mathcal{U}_i, Q_i^B, Q_i^L)_{i=1}^N$ and stores the data point $(\bar{r}_1, \bar{\mathcal{U}}_1, \bar{Q}_1^B, \bar{Q}_1^L)$ given by the average of the second half of these observations (assuming that it took the borrowers and lenders approximately $N/2$ observations to get to the new equilibrium), that is

$$(\bar{r}_t, \bar{\mathcal{U}}_t, \bar{Q}_t^B, \bar{Q}_t^L) = \frac{1}{N/2} \sum_{i=N/2+1}^{N} (r_i, \mathcal{U}_i, Q_i^B, Q_i^L)\,. \tag{13}$$

(iii) The protocol uses their stored data points to update their current beliefs $\mathfrak{a}_{t-1}, \mathfrak{b}_{t-1}, \mathfrak{c}_{t-1}, \mathfrak{d}_{t-1}$ by fitting a line through the points $(\bar{r}_i, \bar{Q}_i^B)_{i=0}^1$ and a line through the points $(\bar{r}_i, \bar{Q}_i^L)_{i=0}^1$, in particular, we obtain

$$\hat{\mathfrak{b}}_t = -\frac{\bar{Q}_t^B - \bar{Q}_{t-1}^B}{\bar{r}_t - \bar{r}_{t-1}}, \qquad \hat{\mathfrak{a}}_t = \bar{Q}_{t-1}^B + \hat{\mathfrak{b}}_t \bar{r}_{t-1} \tag{14}$$

$$\hat{\mathfrak{d}}_t = \frac{\bar{Q}_t^L - \bar{Q}_{t-1}^L}{\bar{r}_t - \bar{r}_{t-1}}, \qquad \hat{\mathfrak{c}}_t = \bar{Q}_{t-1}^L + \hat{\mathfrak{d}}_t \bar{r}_{t-1}. \tag{15}$$

These are crude estimates of the current slope and intercept of the corresponding supply and demand curves. For a given learning rate $\alpha_{\mathfrak{Q}} \in (0, 1)$ the protocol does a weighted average between their previous beliefs $\mathfrak{a}_0, \mathfrak{b}_0, \mathfrak{c}_0, \mathfrak{d}_0$ and the slopes and intercepts of the two lines, that is,

$$\mathfrak{a}_1 = (1 - \alpha_{\mathfrak{Q}})\, \mathfrak{a}_0 + \alpha_{\mathfrak{Q}}\, \hat{\mathfrak{a}}_1 \tag{16}$$

$$\mathfrak{b}_1 = (1 - \alpha_{\mathfrak{Q}})\, \mathfrak{b}_0 + \alpha_{\mathfrak{Q}}\, \hat{\mathfrak{b}}_1 \tag{17}$$

$$\mathfrak{c}_1 = (1 - \alpha_{\mathfrak{Q}})\, \mathfrak{c}_0 + \alpha_{\mathfrak{Q}}\, \hat{\mathfrak{c}}_1 \tag{18}$$

$$\mathfrak{d}_1 = (1 - \alpha_{\mathfrak{Q}})\, \mathfrak{d}_0 + \alpha_{\mathfrak{Q}}\, \hat{\mathfrak{d}}_1\,. \tag{19}$$

Given these three steps, the protocol iterates the above procedure indefinitely, building a sequence of IRM curves which, we expect, will yield a utilization process converging to $\mathcal{U}_{\mathrm{goal}}$.

In the figures below, we assume the market is characterized by $\mathbf{Q}^B(r) = 100 - 100\,r$ and $\mathbf{Q}^L(r) = 75 + 100\,r$ which implies that $\tilde{\mathbf{Q}}^L(r) = (75 + \sqrt{75^2 - 4\,r\,100\,(100\,r - 100)})/2$. The protocol does not know these functions but has an initial belief[3] $\mathfrak{Q}^B(r) = \mathfrak{a}_0 - \mathfrak{b}_0\,r$ and $\tilde{\mathfrak{Q}}^L(r) = \mathfrak{c}_0 + \mathfrak{d}_0\,r$, for $\mathfrak{a}_0 = 105, \mathfrak{b}_0 = 95, \mathfrak{c}_0 = 80, \mathfrak{d}_0 = 95$. The initial IRM is $r_{\text{IRM}}(\mathcal{U}) = a_0\,\mathcal{U}$ for $a_0 = 0.35$.

The goal is to get to 90% utilization, that is, $\mathcal{U}_{\text{goal}} = 0.9$, and the learning rates are taken to be

$$\alpha_B = \alpha_L = \alpha_\Theta = \alpha_{\mathfrak{Q}} = 0.1.$$

For any given r_{IRM} we observe $N = 100$ data points from borrowers and lenders together with the utilization and the interest rate in the market $(Q^B, Q^L, \mathcal{U}, r)$ and we average the last 50 to obtain $(\bar{r}, \bar{\mathcal{U}}, \bar{Q}^B, \bar{Q}^L)$. We deploy 100 update steps for $r_{\text{IRM}}(\cdot)$ and beliefs $\mathfrak{a}_0, \mathfrak{b}_0, \mathfrak{c}_0, \mathfrak{d}_0$. Figure 8 shows the above procedure implemented for the linear case when the noise is zero, i.e., $\sigma_B = \sigma_L = 0$.

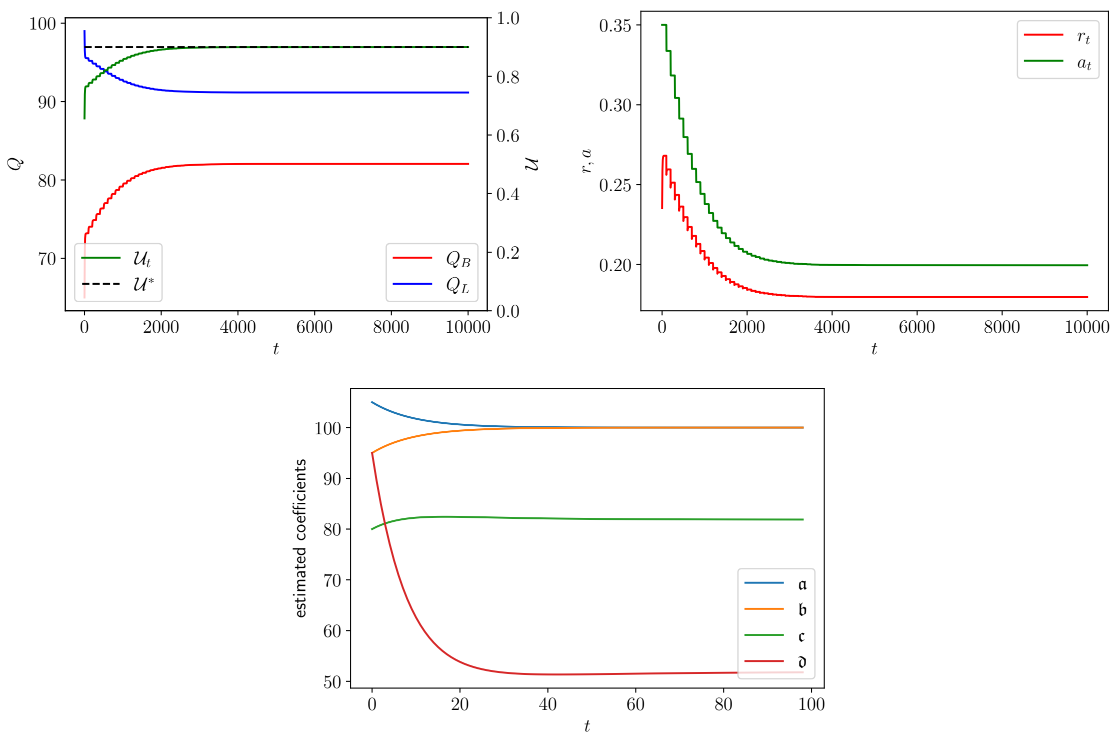

Figure 8: Convergence of $\mathcal{U}^*$ using update equation (10) as $t \to \infty$ while updating beliefs $\mathfrak{Q}^B(\cdot)$ and $\tilde{\mathfrak{Q}}^L(\cdot)$ without noise.

Of course, with zero noise, the only difficulty is that the estimate of the parameters of $\mathfrak{Q}^B$ and $\mathfrak{Q}^L$ need to converge to good approximations of $\mathbf{Q}^B$ and $\tilde{\mathbf{Q}}^L$, but there is no additional estimation noise. Conversely, Figure 9 shows the above procedure implemented for the linear case when the noise is $\sigma_B = \sigma_L = 0.5$. We see that there is some initial instability in this method, but the system nevertheless successfully calibrates to the desired utilization rate quickly.

Figure 10 shows the first two updates for borrowers (left panel) and lenders (right panel) when there is noise. The points in light blue are not used because the market is moving towards the equilibrium. The points in orange (first IRM) and pink (second IRM) are used to compute $\bar{r}_0, \bar{Q}^B_0, \bar{Q}^L_0$ and $\bar{r}_1, \bar{Q}^B_1, \bar{Q}^L_1$ respectively. The intercepts and slopes of the dotted lines are then used to update beliefs as in the equations above.

[3]Note that, not only are the parameters mismatched (e.g., compare $\mathbf{Q}^B(r)$ with $\mathfrak{Q}^B(r)$), but the functional form of the lender's effective demand is incorrect (compare $\tilde{\mathbf{Q}}^L(r)$ with $\tilde{\mathfrak{Q}}^L(r)$).

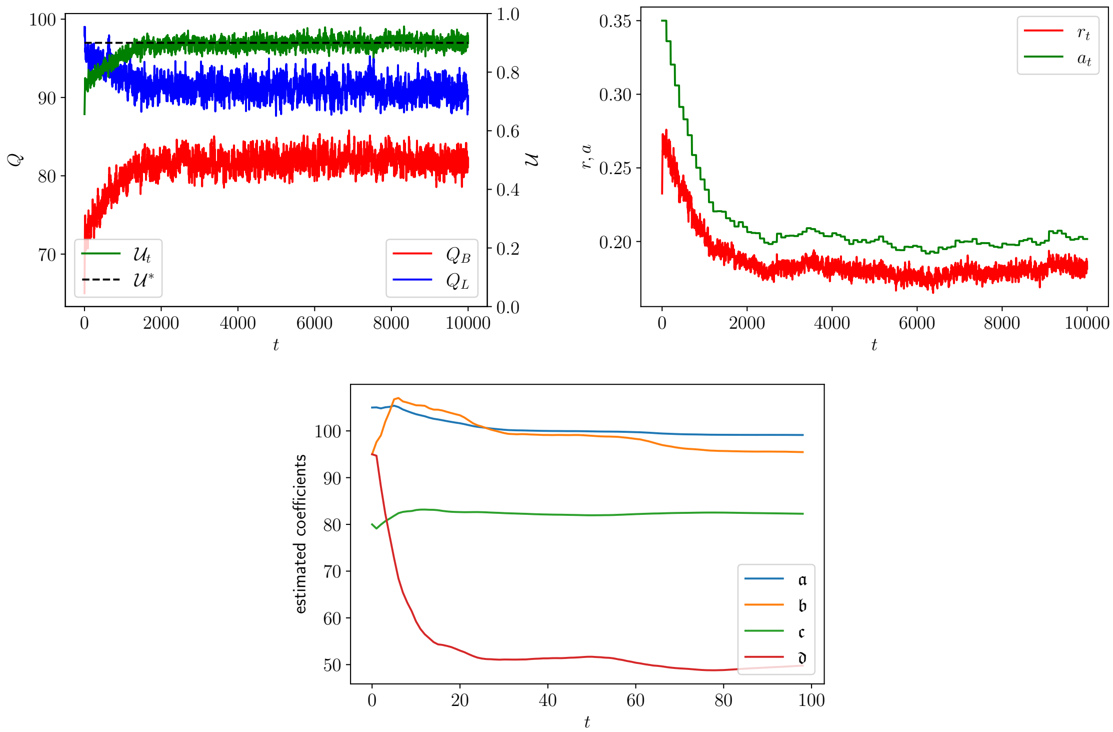

Figure 9: Convergence of $\mathcal{U}^*$ using update equation (10) as $t \to \infty$ while updating beliefs $\mathfrak{Q}^B(\cdot)$ and $\tilde{\mathfrak{Q}}^L(\cdot)$ with noise.

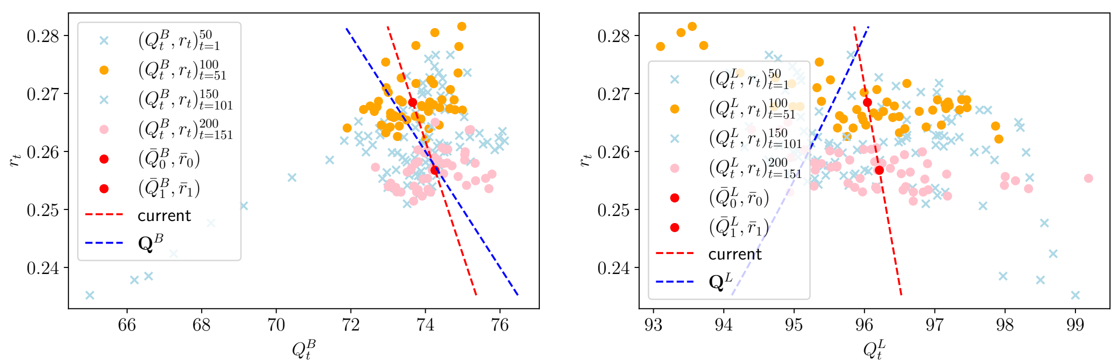

Figure 10: Calculation of $(\bar{r}_i, \bar{Q}_i^B)_{i=0}^1$ and $(\bar{r}_i, \bar{Q}_i^L)_{i=0}^1$. Points marked with $\times$ in light blue are discarded in the calculation of the averages. Average data points and fitted line are in red.

The right panel of Figure 10 illustrates why one should have small learning rates, as the current estimate of the supply curve is a bad approximation with a wrong sign for the slope. It also suggests that improvements may be possible, by restricting the regression to plausible values of the slope/intercept. Below, in Section 3.2 we present an alternative formulation that makes the estimation process robust to these issues.

In practice, the various parameters of this algorithm (in particular the learning rates α_Θ, $\alpha_{\mathfrak{Q}}$, and the number of data points taken to allow convergence to equilibrium and for estimation) would need to be calibrated to typical market dynamics. We note that the learning rates α_B, α_L do not appear in the protocol's algorithm, but indirectly affect the choice of the protocol's learning parameters.

3.2 Robust estimation of demand-supply beliefs

Instead of updating an intercept and slope, the protocol may choose to keep track of a reference point and the angle made between the approximated line and the horizontal axis. This has the effect of improving the stability of the estimation, as a near-vertical curve corresponds to an angle $\theta \approx \pi/2$, rather than an infinitely large linear coefficient; this is a particular concern when supply or demand is locally inelastic in response to the interest rate. For the borrower curve, the protocol models their demand line with the reference point $(\mathfrak{r}, \mathfrak{q}^B) \in (0,\infty) \times (0,\infty)$ and the angle $\theta^B \in (\frac{3}{2}\pi, 2\pi)$. Similarly, the protocol models the supply curve of lenders with the point $(\mathfrak{r}, \mathfrak{q}^L) \in (0,\infty) \times (\mathfrak{q}^B,\infty)$ and the angle $\theta^L \in (0, \frac{1}{2}\pi)$. Then, as in the linear case, the protocol models the functions $\mathbf{Q}^B(\cdot)$ and $\tilde{\mathbf{Q}}^L(\cdot)$ as

$$\mathfrak{Q}^B(r) = \mathfrak{q}^B + (r - \mathfrak{r})\, \tan(\theta^B)\,,$$
$$\mathfrak{Q}^L(r) = \mathfrak{q}^L + (r - \mathfrak{r})\, \tan(\theta^L)\,,$$

and equation (8) becomes

$$\frac{\mathrm{d}\mathcal{U}^*}{\mathrm{d}a} = \frac{\tan(\theta^B)\,\mathcal{U}^* - (\mathcal{U}^*)^2\,\tan(\theta^L)}{\mathfrak{Q}^L(r_{\mathrm{IRM}}(\mathcal{U}^*))}\,. \tag{20}$$

Given current beliefs $\mathfrak{r}_t$, $\mathfrak{q}^B_t$, $\mathfrak{q}^L_t$, θ^B_t, and θ^L_t, for $t \in \{0, 1, \dots\}$, the protocol updates their beliefs as follows

$$\mathfrak{r}_{t+1} = (1 - \alpha_{\mathfrak{Q}})\,\mathfrak{r}_t + \alpha_{\mathfrak{Q}}\,\bar{r}_t\,, \tag{21}$$
$$\mathfrak{q}^B_{t+1} = (1 - \alpha_{\mathfrak{Q}})\,\mathfrak{q}^B_t + \alpha_{\mathfrak{Q}}\,\bar{Q}^B_t\,, \tag{22}$$
$$\mathfrak{q}^L_{t+1} = (1 - \alpha_{\mathfrak{Q}})\,\mathfrak{q}^L_t + \alpha_{\mathfrak{Q}}\,\bar{Q}^L_t\,, \tag{23}$$
$$\theta^B_{t+1} = (1 - \alpha_{\mathfrak{Q}})\,\theta^B_t + \alpha_{\mathfrak{Q}}\,\bar{\theta}^B_t\,, \tag{24}$$
$$\theta^L_{t+1} = (1 - \alpha_{\mathfrak{Q}})\,\theta^L_t + \alpha_{\mathfrak{Q}}\,\bar{\theta}^L_t\,, \tag{25}$$

where $\bar{\theta}^L_t$ and $\bar{\theta}^L_t$ are the angles from the lines through $(\bar{r}_i, \bar{Q}^L_i)^t_{i=0}$ and $(\bar{r}_i, \bar{Q}^B_i)^t_{i=0}$ respectively. If desired, we can also enforce the requirement that $\theta^B_t \in (\frac{3}{2}\pi, 2\pi)$ and $\theta^L_t \in (0, \frac{1}{2}\pi)$, by appropriately truncating $\bar{\theta}^L_t$ and $\bar{\theta}^B_t$.

In the experiments below we use $\mathfrak{r}_0 = 0.05$, $\mathfrak{q}^B_0 = 87$, $\mathfrak{q}^L_0 = 83$, $\theta^B_0 = \frac{3}{2}\pi + 0.004$, and $\theta^L_0 = 0.95\,\frac{\pi}{2}$. The rest of the parameters and learning rates are taken to be as above and we work with noisy observations. The left panel of Figure 11 shows the initial beliefs using $\mathfrak{r}_0, \mathfrak{q}^B_0, \mathfrak{q}^L_0, \theta^B_0, \theta^L_0$ and the unknown market supply and demand curves $\tilde{\mathbf{Q}}^L$ and $\mathbf{Q}^B$. The right panel shows last beliefs after convergence has been obtained.

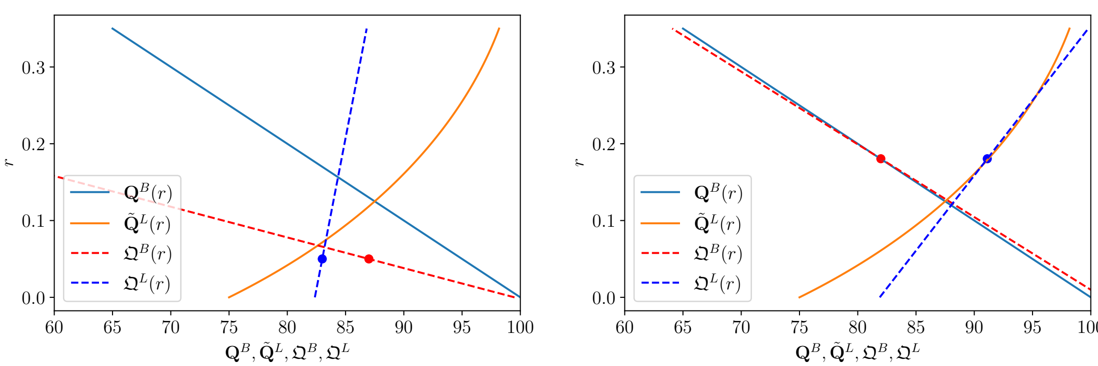

Figure 11: Market supply $\tilde{\mathbf{Q}}^L$ and demand $\mathbf{Q}^B$ with beliefs $\mathfrak{Q}^B$ and $\mathfrak{Q}^L$ at the beginning of training (left panel) and the end of training (right panel).

We see that the beliefs converge to the best linear approximation of $\tilde{\mathbf{Q}}^L$ and $\mathbf{Q}^B$ around the quantities where the equilibrium is attained. Figure 12 shows the analogous of Figure 9 for the top two panels – in particular, we observe similar convergence and behaviour. The bottom two panels of Figure 12 are the values of the beliefs $(\mathfrak{q}^B_t, \mathfrak{q}^L_t, \theta^B_t, \theta^L_t)^{100}_{t=0}$. The dotted lines in the right hand panel of Figure 11 is done with values of $\mathfrak{q}^B_{100}, \mathfrak{q}^L_{100}, \theta^B_{100}, \theta^L_{100}$. We see that this algorithm retains the qualitative performance of our earlier model – given we have not chosen inelastic demand or supply functions, this is as expected.

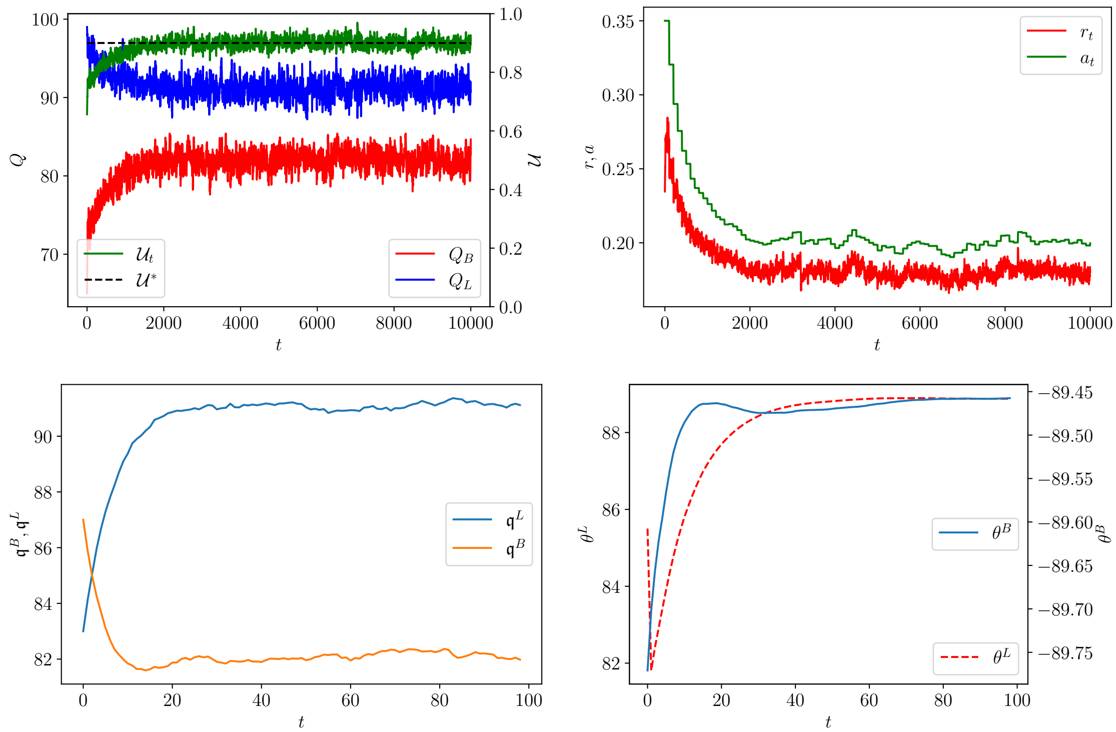

Figure 12: Convergence of $\mathcal{U}^*$ using update equation (20) as $t \to \infty$ while updating beliefs $\mathfrak{Q}^B(\cdot)$ and $\tilde{\mathfrak{Q}}^L(\cdot)$ with noise.

3.3 Additional objectives

The above analysis studies how to move towards a desired utilization $\mathcal{U}_{\text{goal}}$. Additionally, the protocol might want to remain competitive and 'stay close' to the IRM of another venue. Let r_{ABC} be the IRM in the other venue. The venue defines the metric it wishes to use for measuring distance between IRMs, for example,

$$d^\Theta = -\int_0^1 \left(r_{\text{IRM}}(u) - r_{\text{ABC}}(u)\right)^2 \mathrm{d}u\,, \tag{26}$$

and then it takes learning steps to maximize d^Θ. That is, the update equation (9) becomes

$$\Theta_{\text{NEW}} = \Theta_{\text{OLD}} + \alpha_\Theta \left(\mathcal{U}_{\text{goal}} - \mathcal{U}^*\right) \nabla_\Theta \mathcal{U}^* + \alpha_d \nabla_\Theta d^\Theta\,, \tag{27}$$

where α_d is the new learning rate. In general, d^Θ can be any quantity we wish to maximize or the negative of a quantity we wish to minimize. In the above example, the quantity $\nabla_\Theta d^\Theta$ is easy to compute because d^Θ only uses Θ through $r_{\text{IRM}}(\cdot)$.

3.4 Comparing to a proportional-integral-derivative (PID) controller

In the update equation (9), one sees that the sign of $\nabla_\Theta \mathcal{U}^*$ is the same regardless of market conditions. For example, if we unpack the derivative term (8) we see that the denominator $(\tilde{\mathbf{Q}}^L(r_{\text{IRM}}(\mathcal{U}^*)))$ is positive and

$$\tfrac{\mathrm{d}}{\mathrm{d}r}\mathbf{Q}^B(r_{\text{IRM}}(\mathcal{U}^*))\,\nabla_\Theta r_{\text{IRM}}(\mathcal{U}^*) < 0 \tag{28}$$

$$-\mathcal{U}^*\,\tfrac{\mathrm{d}}{\mathrm{d}r}\tilde{\mathbf{Q}}^L(r_{\text{IRM}}(\mathcal{U}^*))\,\nabla_\Theta r_{\text{IRM}}(\mathcal{U}^*) < 0\,. \tag{29}$$

When quantities Q^L and Q^B are close to equilibrium, these gradients vary only slightly (assuming sufficient smoothness). If we approximate them with a constant, or choose an appropriate time-varying learning rate, the simplified update equation (12) becomes a PID (proportional-integral-derivative) controller where the venue employs the 'P' (proportional) part of the PID for the update of the parameters. That is, (12) is equivalent to, for some constant $\alpha_P > 0$ and $\Delta t > 0$,

$$a_{t+1} = a_t - \alpha_P \left(\mathcal{U}_{\text{goal}} - \mathcal{U}^*\right) \Delta t \,. \tag{30}$$

The above argument lends support to the usage of 'P' controller for updating IRMs. In particular, as the choice of α_Θ in (9) can be determined freely (provided $\alpha_\Theta \in (0,1)$), the coefficient α_P in (30) can be chosen to suit general market conditions, rather than being based on the update we discussed in Section 3.1. See [4] where this approach is studied in more detail.

One can consider generalizing this approach to the class of PID controllers, where the update equation can be written in the form

$$a_{t+1} = a_t - \left[\alpha_P \underbrace{\left(\mathcal{U}_{\text{goal}} - \mathcal{U}_t^*\right)}_{\text{Proportional}} + \alpha_I \underbrace{\left(\frac{1}{t}\sum_{s=0}^{t}(\mathcal{U}_{\text{goal}} - \mathcal{U}_s^*)\right)}_{\text{Integral}} + \alpha_D \underbrace{\left(\frac{\mathcal{U}_t^* - \mathcal{U}_{t-1}^*}{\Delta t}\right)}_{\text{Derivative}}\right] \Delta t \,, \tag{31}$$

for positive constants α_P, α_I, and α_D. Effectively, the introduction of an 'Integral' term in the controller is equivalent to a different choice of optimization algorithm to the simple gradient update which we used in (9).

The 'Derivative' term of the controller is more delicate. Two particular issues arise, if the time-increment Δt is taken to be moderately small.[4] At a theoretical level, if we assume that the underlying supply and demand include stochastic components, it is inevitable that this stochasticity will be inherited by the utilization. This means that it is difficult to estimate the derivative of the utilization with respect to time, leading to a noisy term. We can see the effect of this in Figure 13 below, where introducing dependence on a one-step derivative estimate leads to additional instability in the convergence to equilibrium. This is unsurprising, given in classical implementations of PD controllers one often needs to implement a low-pass filter before computing the derivative, to avoid excessive noise in the estimates.

Figure 13 explores the PD controller approach further. We consider the update equation in (30) and add a term to account for the derivative of the utilization, approximated using the change in utilization over the final period of the epoch.

We see this effect displayed further by considering the volatility in the value of a_t (over the final 300 time-steps of our simulation) as a function of α_D, as shown in Figure 14.

A further concern is that the use of a D term allows easier manipulation of the utilization. For example, consider a net-borrower who has an interest in keeping the utilization curve low. This agent will then typically make the utilization appear to be rising at the time of the update, that is in the final time-step of the epoch. By borrowing assets for this single period, the agent will only be exposed to a small cost, but can modify the gradient of the utilization significantly – this is particularly the case if the time-steps used to compute the derivative are small.

4 Exploiting the IRM

In this final section we will investigate a potential issue with utilization-based interest rate models. This issue is due to recursive borrowing: recursive borrowers are lending protocol users who borrow and supply the same token simultaneously. This has been a popular yield farming strategy, particularly in the periods when lending protocols incentives users by supplying lenders and borrowers with additional rewards.[5] Here,

[4] The classic 'derivative' control is typically used in a setting where the time-step is taken to be small, as a way of attempting to smooth out undesirable fluctuations of the path. However, if the time increment $\Delta t = 1$, and $\alpha_D < \alpha_P$, then one can see that this system corresponds, in some sense, to a proportional update based on a combination of the last two states. If the derivative is estimated after a low-pass filter has been applied, this essentially yields an average based on a larger number of steps, in some sense corresponding to the use of separate epochs in our algorithm.

[5] See, for example, [16].

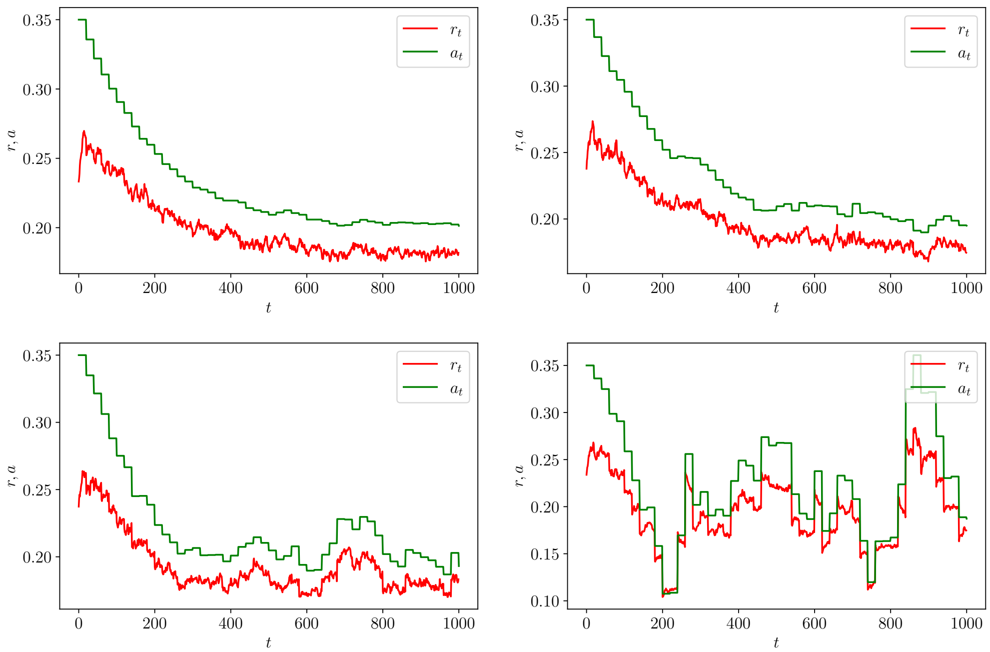

Figure 13: Trajectory of r_t and a_t for various values of the derivative gain α_D in the PD controller case. The value of α_D is 0.1 in the top left panel, 0.5 in the top right, 1 in the bottom left panel, and 5 in the bottom right panel. We see that, due to the presence of noise, even a small derivative gain only results in an increase in interest rate volatility.

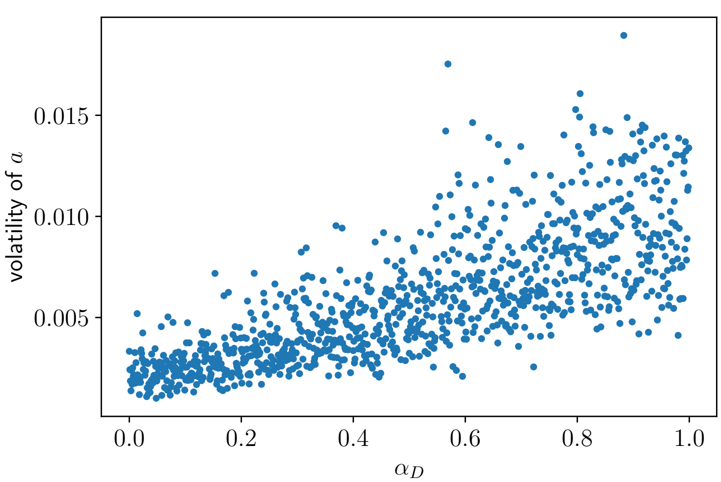

Figure 14: Standard deviation of the last 300 values of a_t as a function of α_D.

we will see that recursive borrowing gives lenders the ability to manipulate the level of utilization in the market for their own profit, which raises the question of stability of these systems. This is similar in spirit to the exploit considered in [4].

Consider a linear interest model $r_{\mathrm{IRM}}(\mathcal{U}) = a\,\mathcal{U}$ with $a > 0$, and as before let Q^L, Q^B be the amounts of supplied and borrowed assets. We see that since $\mathcal{U} = Q^B/Q^L$ then:

(i) Borrowing from the pool ($Q^B \nearrow$) increases the utilization ($\mathcal{U} \nearrow$), and consequently increases interests rates. Repaying the loan has an opposite effect.

(ii) Providing liquidity to the pool ($Q^L \nearrow$) decreases the utilization ($\mathcal{U} \searrow$), and consequently decreases interest rates. Withdrawing liquidity from the pool has an opposite effect.

However, due to the asymmetry of borrowing and lending, we can see that simultaneously providing liquidity and borrowing the same amount (which leaves an agent's position largely unchanged, even after accounting for collateral requirements) will increase utilization and hence interest rates. This provides an opportunity for a lender to independently increase the interest rate.

Hence, we see that an agent – by simultaneously altering the amount of borrowed and supplied asset – can increase, decrease, or keep constant the utilization, ceteris paribus. The lemma below makes this statement precise.

Lemma 4.1. *Let $Q^L, Q^B > 0$ and $\Delta Q^L > -Q^L$, $\Delta Q^B > -Q^B$ and $\Delta Q^B < Q^L - Q^B$ be the amounts by which an agent alters the pools. Let $\mathcal{U} = \frac{Q^B}{Q^L}$. We have*

(i) $\Delta Q^B = \mathcal{U}\Delta Q^L$ if and only if $\frac{Q^B+\Delta Q^B}{Q^L+\Delta Q^L} = \frac{Q^B}{Q^L}$.

(ii) Similarly $\Delta Q^B > \mathcal{U}\Delta Q^L$ ($\Delta Q^L B < \mathcal{U}\Delta Q^L$) if and only if $\frac{Q^B+\Delta Q^B}{Q^L+\Delta Q^L} > \frac{Q^B}{Q^L}$ ($\frac{Q^B+\Delta Q^B}{Q^L+\Delta Q^L} < \frac{Q^B}{Q^L}$).

Proof. To prove (i) we observe that

$$\frac{Q^B + \Delta Q^B}{Q^L + \Delta Q^L} = \frac{Q^B}{Q^L} \iff Q^L \Delta Q^B = Q^B \Delta Q^L \iff \Delta Q^B = \mathcal{U}\,\Delta Q^L\,.$$

Changing equalities for inequalities proves the second statement. □

A standard approach to risk management in these markets is to require all loans to be over-collateralized. In particular, the market specifies a constant $\theta^0 \in (0,1]$ as a haircut coefficient, meaning that a user can take a loan of value $\theta^0 \Delta Q^L$ if they have provided ΔQ^L collateral. Typically this is done with collateral being provided in a different currency, however for simplicity we will only consider the single currency setting[6] where we suppose that it is technically possible to provide collateral in the same currency as is being borrowed (and as we shall see, there may be incentives to do so).

Suppose that an agent is a lender in the market, and wishes to manipulate the interest rate. They can do so by repurposing y units of the liquidity they have provided as follows:

Definition 4.2 (Recursive Trade)**.** *Let Q^L, Q^B be the amounts of supplied and borrowed assets. Suppose the agent holds $\beta\, Q^L$ of the amount being supplied with $\beta \in (0,1]$. The steps taken by the agent in a recursive trade of size $y \in [0, \beta\, Q^L]$ are:*

(i) Take y units out from the lending position.

(ii) Take a flash-loan for $y\,\theta_0/(1-\theta_0)$ units.

(iii) Use (i) the y units taken out from the lending position and (ii) the $y\,\theta_0/(1-\theta_0)$ units from the flash-loan and deposit the total $y/(1-\theta_0)$ units as collateral in the lending pool.

[6] A multiple currency version of our construction is possible, however would often involve an additional model for the exchange market between currencies. A more complicated version of our strategy would involve simultaneously borrowing and lending in a pair of currencies, taking both trades in both directions. This would ultimately provide the same result as we present here, but with a significant increase in notational complexity.

(iv) Borrow $y\,\theta_0/(1-\theta_0)$ against the collateral deposited in the previous step (iii).

(v) Use borrowed amount to pay back the flash-loan.

When the agent executes a recursive trade of size $y \in [0, \beta\, Q^L]$, the utilization in the pool becomes

$$\mathcal{U}(y) = \frac{Q^B + \frac{\theta_0\, y}{1-\theta_0}}{Q^L - y + \frac{y}{1-\theta_0}} = \frac{Q^B + \frac{\theta_0\, y}{1-\theta_0}}{Q^L + \frac{\theta_0\, y}{1-\theta_0}} = \frac{(1-\theta_0)\, Q^B + \theta_0\, y}{(1-\theta_0)\, Q^L + \theta_0\, y}\,. \tag{32}$$

Note that, for the recursive trade, $\Delta Q^B = \Delta Q^L = \frac{\theta_0\, y}{1-\theta_0}$ using the notation of Lemma 4.1, and hence utilization is increasing as long as $1 > \mathcal{U}$.

Theorem 4.3. *In the absence of fees, for a linear IRM, assuming interest rates for borrowing and lending are the same, interest is paid on collateral, and assuming that no borrowers will change their position, a lender will maximize their profit by using their entire position to recursively lend in the market.*

Proof. The nominal interest rate after the recursive trade is carried out is $a\,\mathcal{U}(y)$, with $y \mapsto \mathcal{U}(y)$ defined in (32), and the profit/loss incurred after a time $t > 0$ in the future (assuming that other market participants do not adjust their quantities borrowed and supplied) is given by

$$\underbrace{\left(\beta\, Q^L - y\right)\, \exp\left(\rho\, a\, t\, \mathcal{U}(y)\right)}_{\text{unaltered liquidity}} + \underbrace{\left(\frac{y}{1-\theta_0} - \frac{y\theta_0}{1-\theta_0}\right) \exp\left(\rho\, a\, t\, \mathcal{U}(y)\right)}_{\text{liquidity in recursive trade}} = \left(\beta\, Q^L\right)\, \exp\left(\rho\, a\, t\, \mathcal{U}(y)\right)\,. \tag{33}$$

Next, for a given $t > 0$, we are interested in finding the maximum of the function

$$y \to \beta\, Q^L\, \exp\left(\rho\, a\, t\, \mathcal{U}(y)\right)\,, \qquad y \in [0, \beta\, Q^L]\,. \tag{34}$$

The first derivative is

$$\beta\, Q^L\, \exp\left(\rho\, a\, t\, \mathcal{U}(y)\right)\, \rho\, a\, t\, \frac{\mathrm{d}\mathcal{U}(y)}{\mathrm{d}y} = \beta\, Q^L\, \exp\left(\rho\, a\, t\, \mathcal{U}(y)\right)\, \rho\, a\, t\, \frac{\theta_0\, \left((1-\theta_0)\, Q^L + \theta_0\, y\right) - \theta_0\, \left((1-\theta_0)\, Q^B + \theta_0\, y\right)}{\left((1-\theta_0)\, Q^L + \theta_0\, y\right)^2} \tag{35}$$

$$= \rho\, a\, t\, \beta\, Q^L\, \exp\left(\rho\, a\, t\, \mathcal{U}(y)\right)\, \frac{\theta_0\, (1-\theta_0)\, \left(Q^L - Q^B\right)}{\left((1-\theta_0)\, Q^L + \theta_0\, y\right)^2} \geq 0\,. \tag{36}$$

Thus, it is optimal for the agent to use all their position ($y = \beta\, Q^L$) to execute the recursive trade and increase the interest rates.

□

From this calculation, we see that, ignoring the changes of other agents, a lender will naturally manipulate the utilization of the protocol through recursive trades, in order to increase the utilization level and hence the interest rate. This suggests that these markets will be open to manipulation by lenders. Here we have assumed that borrowing and lending interest rates are the same – if this is not the case, or fees are substantial, appropriate adjustments will be needed to this calculation. Similarly, if one departs from the single-currency setting studied above, then one should account for liquidations.

Remark 4.4. *We observe at this point that the assumption that collateral is paid interest (which essentially means that there is no distinction between assets offered for loan by lenders and borrowers' collateral) is important. If collateral did not earn interest (which will typically be the case if collateral is kept separate from assets available for lending, which may be desirable for the purposes of risk management), then the recursive trade described above results in a lender losing the interest they would earn on the quantity y they withdraw from the lending pool. For typical IRM parameters, it can be checked that this is a larger loss than the gains to lenders resulting from the higher interest rate, particularly as they also have to pay interest on their borrowed amount (without receiving this interest back as interest on their own collateral).*

4.1 Exploitation with borrower response

If we follow our initial analysis based on supply and demand, our conclusion is slightly altered. We observed, from (32), that recursive lending gives liquidity providers the ability to raise utilization, towards a maximum value $\theta_0+(1-\theta_0)\frac{Q^B}{Q^L}$ (which is achieved when $y = Q^L$, when all available liquidity has been used in a recursive trade[7]). This effectively gives lenders the ability to act as a monopolist in the lending market – in particular as the interests of lenders are all aligned towards increasing utilization towards a profit-maximization point. Accounting for the changing behaviour of borrowers gives the following variation of the previous result.

Over a short period, the total return to lenders is given by $\rho\, r\, \mathbf{Q}^B(r)\mathrm{d}t$. We recall that $r \mapsto \rho\, r\, \mathbf{Q}^B(r)$ has a unique maximizing value

$$r_M := \arg\max_r \left\{\rho\, r\, \mathbf{Q}^B(r)\right\}. \tag{37}$$

For example, for the function $\mathbf{Q}^B$ in the above experiments $\rho = 1$ and $r\,\mathbf{Q}^B(r) = 100\,r - 100\,r^2$, which implies that $r_M = 0.5$.

From (32), and employing $r_{\mathrm{IRM}}(\mathcal{U}) = a\mathcal{U}$, we know that a recursive trade of size y results in the nominal interest rate

$$a\,\frac{(1-\theta_0)Q^B+\theta_0 y}{(1-\theta_0)Q^L+\theta_0 y}\,. \tag{38}$$

In order to achieve the maximizing rate r_M, we therefore seek y such that $\mathcal{U}(y) = \frac{r_M}{a}$. Assuming $r_M < a$,

$$y = Q^L\left(\frac{1-\theta_0}{\theta_0}\right)\left(\frac{r_M}{a}-\frac{Q^B}{Q^L}\right)\left(1-\frac{r_M}{a}\right)^{-1}. \tag{39}$$

It follows that there are three possibilities for the optimal trade (observing that $Q^B/Q^L < 1$):

(i) If $\frac{Q_B}{Q_L} \geq \frac{r_M}{a}$, so utilization is already above the monopolist optimizing level, there is no incentive for lenders to trade recursively (and, in fact, lenders have a desire to reduce the interest rate, in order to improve their profits).

(ii) If $\frac{r_M}{a} > (1-\theta_0)\frac{Q^B}{Q^L}+\theta_0$, then lenders have incentive to enter into a maximal recursive trade, of size $y = Q^L$, which increases utilization to $(1-\theta_0)\frac{Q^B}{Q^L}+\theta_0$.

(iii) If $\frac{Q_B}{Q_L} < \frac{r_M}{a} \leq (1-\theta_0)\frac{Q^B}{Q^L}+\theta_0$, then lenders have incentive to enter into a recursive trade with $y = Q^L\left(\frac{1-\theta_0}{\theta_0}\right)\left(\frac{r_M}{a}-\frac{Q^B}{Q^L}\right)\left(1-\frac{r_M}{a}\right)^{-1}$, in order to increase utilization to r_M/a.

In the numerical examples above, where $a = 0.5$, $\rho = 1$, $\mathbf{Q}^B(r) = 100 - 100\,r$, and thus $r_M = 0.5$, we have that lenders fall into case (ii) and thus have incentives to enter into a maximal recursive trade.

4.2 Exploitation with dynamic updating

One final remark is that this exploit only considers manipulation of the utilization rate in a static setting, however the effect is heightened when we consider dynamic IRM settings, as considered in Section 3.

Consider the simple setting where the IRM is updated following a 'P controller', as in (30), and the desired utilization level $\mathcal{U}_{\mathrm{goal}} < \theta_0$. Suppose a lender has performed recursive trades, which have led to the utilization $\mathcal{U}_t > \mathcal{U}_{\mathrm{goal}}$. The response of the protocol will be to increase the value of a, and hence the interest rate, in hope that this will lead to a decrease in the level of utilization.

However, it is immediately apparent that this simply increases the benefit to lenders, as they benefit from increased interest rates at every level of utilization. In particular, if they are in a setting where they are taking the maximal recursive trade, lenders will continue to do so, resulting in persistently high utilization, and persistent increases in the IRM.

[7] We are here ignoring the collateral that borrowers initially post; one could think that net-borrowers are posting liquidity in a different currency, and only net-lenders are entering into recursive trades. Therefore, only the net-lenders' positions are determining the level of supply used when calculating the utilization.

Eventually, however, the IRM will be sufficiently high that lenders will begin targeting the monopolist rate, r_M, defined in (37), rather than simply maximizing utilization. Therefore, the additional manipulation of the system will not increase, leading to the potential for a new equilibrium. This new equilibrium will take place at the level $(\mathcal{U}_{\text{goal}}, r_M)$, with IRM $\overline{r_{\text{IRM}}}(\mathcal{U}) := (r_M/\mathcal{U}_{\text{goal}})\mathcal{U}$, and corresponds to the situation where lenders simply fix the monopolist interest rate r_M, then enter into a sufficiently large recursive trade to ensure the utilization level is $\mathcal{U}_{\text{goal}}$. Observing that $r_M/a = \mathcal{U}_{\text{goal}}$ for this choice of IRM, this equilibrium is achievable provided

$$\frac{\mathbf{Q}^B(r_M)}{\mathbf{Q}^L(r_M)} < \mathcal{U}_{\text{goal}} < (1-\theta_0)\frac{\mathbf{Q}^B(r_M)}{\mathbf{Q}^L(r_M)} + \theta_0,$$

which is often the case in practical examples, for example where $\theta_0 \approx 1$, $\mathcal{U}_{\text{goal}} \approx 0.5$.

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References

[1] Aave. Aave Protocol Whitepaper. `https://github.com/aave/aave-protocol/blob/master/docs/Aave_Protocol_Whitepaper_v1_0.pdf`, 2020.

[2] Sirio Aramonte, Sebastian Doerr, Wenqian Huang, and Andreas Schrimpf. Defi lending: intermediation without information? Technical report, Bank for International Settlements, 2022.

[3] Amit Chaudhary, Roman Kozhan, and Ganesh Viswanath-Natraj. Interest rate rules in decentralized finance: Evidence from compound. In *4th International Conference on Blockchain Economics, Security and Protocols (Tokenomics 2022)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2023.

[4] Tarun Chitra, Peteris Erins, and Kshitij Kulkarni. Attacks on dynamic defi interest rate curves. *arXiv preprint arXiv:2307.13139*, 2023.

[5] Jonathan Chiu, Emre Ozdenoren, Kathy Yuan, and Shengxing Zhang. On the inherent fragility of defi lending, 2022.

[6] Samuel N Cohen, Marc Sabate-Vidales, Lukasz Szpruch, and Mathis Gontier Delaunay. The paradox of adversarial liquidation in decentralised lending. *Available at SSRN 4540333*, 2023.

[7] Compound. Compound: The Money Market Protocol. `https://github.com/compound-finance/compound-money-market/blob/master/docs/CompoundWhitepaper.pdf`, 2018.

[8] Campbell R Harvey, Ashwin Ramachandran, and Joey Santoro. *DeFi and the Future of Finance*. John Wiley & Sons, 2021.

[9] Kose John, Leonid Kogan, and Fahad Saleh. Smart contracts and decentralized finance. *Available at SSRN*, 2022.

[10] Alfred Lehar and Christine A Parlour. Systemic fragility in decentralized markets. *Available at SSRN*, 2022.

[11] Peter Mueller. Defi leveraged trading: Inequitable costs of decentralization. *Available at SSRN 4241356*, 2023.

[12] Daniel Perez, Sam M Werner, Jiahua Xu, and Benjamin Livshits. Liquidations: Defi on a knife-edge. In *Financial Cryptography and Data Security: 25th International Conference, FC 2021, Virtual Event, March 1–5, 2021, Revised Selected Papers, Part II 25*, pages 457–476. Springer, 2021.

[13] Kaihua Qin, Liyi Zhou, Pablo Gamito, Philipp Jovanovic, and Arthur Gervais. An empirical study of defi liquidations: Incentives, risks, and instabilities. In *Proceedings of the 21st ACM Internet Measurement Conference*, pages 336–350, 2021.

[14] Thomas J Rivera, Fahad Saleh, and Quentin Vandeweyer. Equilibrium in a defi lending market. *Available at SSRN 4389890*, 2023.

[15] Fabian Schär. Decentralized finance: On blockchain-and smart contract-based financial markets. *FRB of St. Louis Review*, 2021.

[16] DeFi Scientist. How to add 4% to your recursive lending returns on compound. `https://medium.com/@DeFiScientist/how-to-add-4-to-your-recursive-lending-returns-on-compound-3c576dfdd99d`, 2022.

[17] Jiahua Xu, Krzysztof Paruch, Simon Cousaert, and Yebo Feng. Sok: Decentralized exchanges (dex) with automated market maker (amm) protocols. *ACM Computing Surveys*, 55(11):1–50, 2023.