Applications of Mean Field Games in Economic Theory

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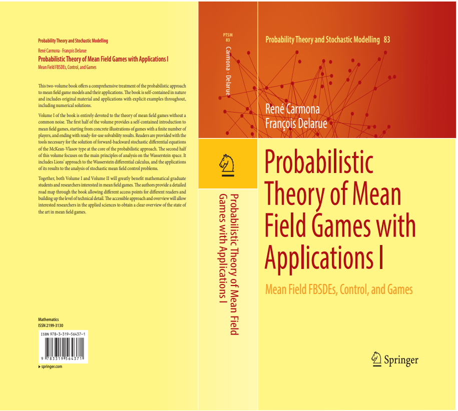
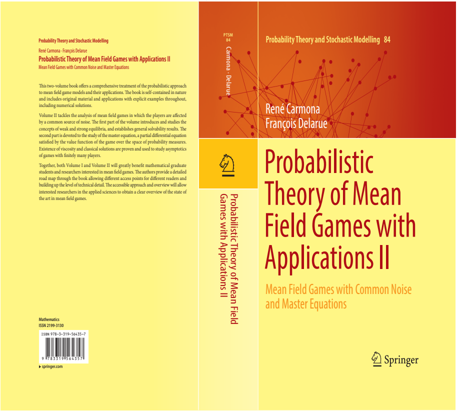
Outline of the Chapter

Shameless Advertisement

Several of the examples presented in this chapter are taken from





More details and specific references will be given for the other examples.

A Toy Model of Systemic Risk

R.C. - Fouque - Sun [12]

- $X_t^i, i = 1, \ldots, N$ log-monetary reserves of N banks
- $W_t^i, i = 0, 1, \ldots, N$ **independent Brownian motions**, $\sigma > 0$
- **Borrowing and lending** through the drifts:

$$dX_t^i = \left[a(\overline{X}_t - X_t^i) + \alpha_t^i\right] dt + \sigma\left(\sqrt{1-\rho^2}\,dW_t^i + \rho\, dW_t^0\right), \quad i = 1, \cdots, N$$

α^i **is the control of bank** i which tries to **minimize**

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left\{\int_0^T \left[\frac{1}{2}(\alpha_t^i)^2 - q\alpha_t^i(\overline{X}_t - X_t^i) + \frac{\epsilon}{2}(\overline{X}_t - X_t^i)^2\right] dt + \frac{\epsilon}{2}(\overline{X}_T - X_T^i)^2\right\}$$

Regulator chooses $q > 0$ to control the cost of borrowing and lending.

- If X_t^i is small (relative to the empirical mean $\overline{X}_t$) then bank i will want to borrow($\alpha_t^i > 0$)
- If X_t^i is large then bank i will want to lend ($\alpha_t^i < 0$)

Example of an N **- Player Game** with **Mean Field Interactions** and a **common noise**.

Explicit Solutions for each Finite N

Search for an **open loop** Nash equilibrium $(\boldsymbol{\alpha}_t)_{0\le t\le T}$ with $\boldsymbol{\alpha}_t = (\alpha_t^1, \cdots, \alpha_t^N)$

In equilibrium

$$\alpha_t^i = [q + (1 - \frac{1}{N})\eta_t](\overline{X}_t - X_t^i) \tag{1}$$

where $t \mapsto \eta_t$ solves a **Riccati equation** and

$$dX_t^i = [a + q + (1 - \frac{1}{N})\eta_t](\overline{X}_t - X_t^i)\,dt + \sigma\rho dW_t^0 + \sigma\sqrt{1-\rho^2}dW_t^i, \quad i = 1, \cdots, N, \tag{2}$$

NB: the optimal control is in **closed loop / feedback form** $\boldsymbol{\alpha}_t = \phi_t(\mathbf{X}_t)$!

Search for an **closed loop** Nash equilibrium $(\boldsymbol{\alpha}_t)_{0\le t\le T}$ with $\boldsymbol{\alpha}_t = \phi_t(\mathbf{X}_t)$

In equilibrium, **STILL** (1) and (2) but

η_t solves a different Riccati equation !

However, both solutions **coincide** in the limit $N \to \infty$

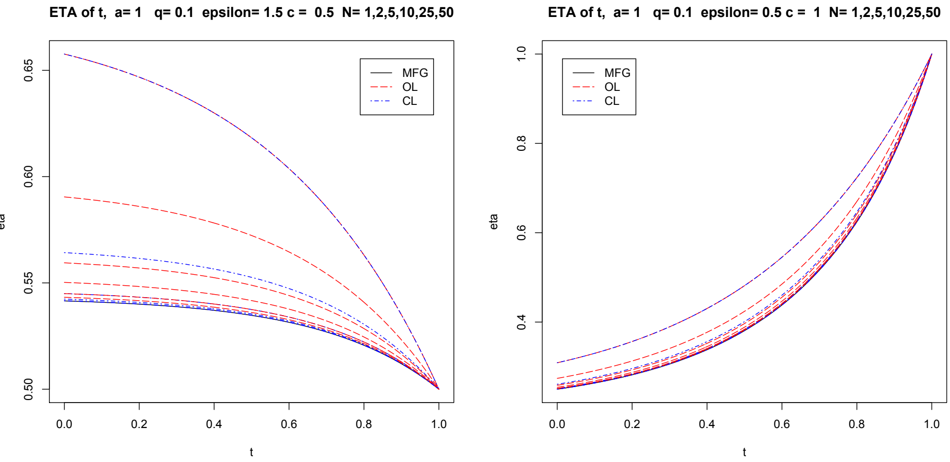


FIGURE: Plot of the function η_t giving the open loop Nash equilibrium for several values of the parameters and numbers of players N increasing from 1 to 50.

Relevance to Mean Field Games (MFGs)

Because of the **explicit** nature of the solutions of the finite player games

- We can take the limit as $N \to \infty$
  - In the dynamics of the states
  - In the Nash equilibrium controls (open and closed loop)
  - In the expected costs optimized by the players
- **Despite the presence of the common noise**
- We can read off the impact of the common noise in the limit

The limits $N \to \infty$

- Of the open and closed loop models coincide
- Identify the so-called Mean Field Game models
- Identify the **Master Equation** as a reduction of the system of the HJB systems

MFGs as Models for Systemic Risk

- Interesting features
  - Multi-period (continuous time) **dynamic equilibrium** model
  - **Explicitly solvable**
- Shortcomings
  - **Naive** model of bank lending and borrowing
  - Only a small jab at **stability** of the system
- Interesting challenges:
  - Introduction of **major** and **minor** players (mostly done)
  - Introduction of **time delays** (**Carmona-Fouque-Moussavi-Sun**[11])
  - Introduction of **graphical constraints** quantifying the levels of exchanges between the financial institutions

A Model for Price Impact

Carmona-Lacker [13]

Start with a model for N traders

- X_t^i number of shares owned at time t by player i,

$$dX_t^i = \alpha_t^i \, dt + \sigma^i dW_t^i$$

- α_t^i *rate of trading*
- $\mathbf{W}^i = (W_t^i)_{t\geq}$ independent Wiener processes for $i = 1, \cdots, N$
- σ^I idiosyncratic volatility
- K_t^i amount of cash held by trader i at time t

$$dK_t^i = -[\alpha_t^i S_t + c(\alpha_t^i)] \, dt,$$

- where S_t price of one share,
- $\alpha \to c(\alpha) \geq 0$ *cost for trading at rate* α

  c should be thought of as the Legendre transform of the shape of the order book, see **Carmona-Webster** [15], so for a flat order book

$$c(\alpha) = c\alpha^2$$

Price impact formula

Almgren-Chriss price impact [4]

$$dS_t = \frac{1}{N}\sum_{i=1}^{N} h(\alpha_t^i)\, dt + \sigma_0 dW_t^0$$

- V_t^i is the wealth of trader i at time t

$$V_t^i = K_t^i + X_t^i S_t.$$

- Using the standard self-financing condition

$$\begin{aligned} dV_t^i &= dK_t^i + X_t^i\, dS_t + S_t\, dX_t^i \\ &= \Big[-c(\alpha_t^i) + X_t^i \frac{1}{N}\sum_{j=1}^{N} h(\alpha_t^j)\Big] dt + \sigma S_t dW_t^i + \sigma_0 X_t^i dW_t^0. \end{aligned} \tag{3}$$

- so if player i minimizes their **expected trading costs**

$$J^i(\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^N) = \mathbb{E}\Big[\int_0^T c_X(X_t^i)dt + g(X_T^i) - V_T^i\Big]$$

NB: *The common noise* $\mathbf{W}^0$ *disappears from the optimization problem because of the* *risk neutrality* *of the traders !*

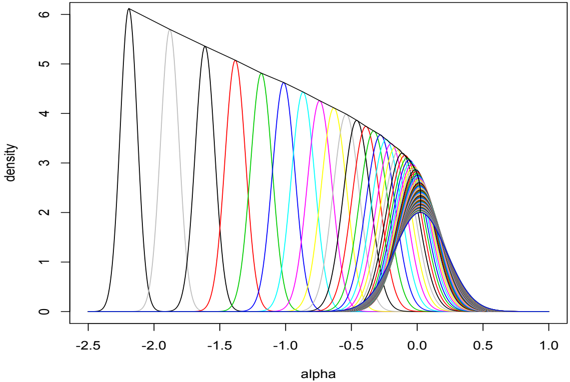
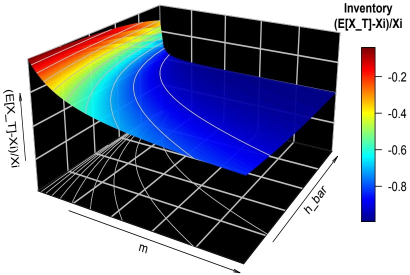


FIGURE: Time evolution (from t ranging from 0.06 to $T = 1$) of the marginal density of the optimal rate of trading $\hat{\alpha}_t$ for a representative trader.

Terminal Inventory of a Typical Trader



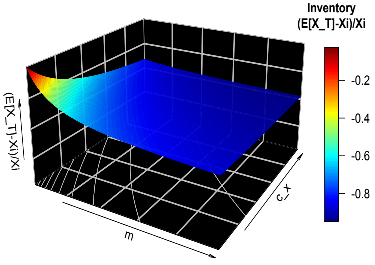
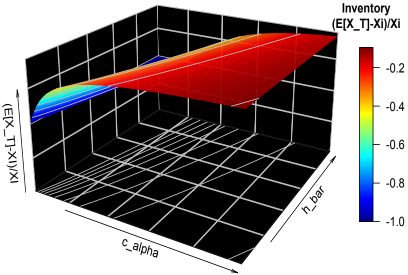


Figure: Expected terminal inventory as a function of m and c_X (left) , and as a function of m and $\overline{h}$ (right).

Terminal Inventory of a Typical Trader



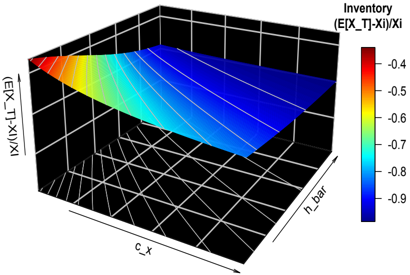


FIGURE: Expected terminal inventory as a function of c_α and $\overline{h}$ (left), and as a function of c_X and $\overline{h}$ (right).

Optimal Execution

- For a single trader/broker **Almgren** [3]
- Game Models for **Predatory Trading** **Brunnermeier-Pedersen**[6], **Carlin-Lobo-Viswanathan**[8], **CarmonaYang** [16]
- MFGs and Optimal Execution
  - **Extended MFGs**: interaction through the controls
  - in the **weak formulation**: **Carmona-Lacker** [13],
  - in the Linear-Quadratic (**LQ**) case: **Carmona-Delarue** [9],
  - in slightly more general form than LQ but still with explicit solutions: **Cartea-Jaimungal** [17],
  - with extensions to **fictitious strategies**: **Cardaliaguet-Lehalle** [7]

Motivation:

lectures / talks at Systemic Risk Summer School / Conference (Vancouver July 2014)

Rochet-Vives, **Fong-Gossner-Hörner-Sannikov**

As **economists**, use a **continuum of players** (atomless measure space)
Morris-Shin,**He-Xiong**,

Try to understand models for **finitely many players** in the Mean Field framework

Gossner's talk

- N depositors
- Amount of each individual (initial & final) deposit $D_0^i = 1/N$
- Current interest rate r
- Depositors promised return $\bar{r} > r$
- Y_t = value of the assets of the bank at time t,
- Y_t Itô process, $Y_0 \geq 1$
- $L(y)$ liquidation value of bank assets if $Y = y$
- Bank has a credit line of size $L(Y_t)$ at time t at rate $\bar{r}$
- Bank uses credit line each time a depositor runs (withdraws his deposit)

Bank Run Model (cont.)

- Assets mature at time T, no transaction after that
- If $Y_T \geq 1$ every one is paid in full
- If $Y_T < 1$ **exogenous default**
- **Endogenous default** at time t if depositors try to withdraw **more** than $L(Y_t)$

Bank Run Model (cont.)

Each depositor $i \in \{1, \cdots, N\}$

- has access to a **private signal** X_t^i at time t

$$dX_t^i = dY_t + \sigma dW_t^i, \qquad i = 1, \cdots, N$$

- **chooses a time** τ^i at which to **ATTEMPT** to withdraw his/her deposit
- collects **return** $\bar{r}$ until time τ^i
- tries to **maximize**

$$J^i(\tau^1, \cdots, \tau^N) = \mathbb{E}\Big[g(\tau^i, Y_{\tau^i}, \tau^{-i})\Big]$$

where for example:

  - $g(t, Y_t, \tau^1, \cdots, \tau^N) = e^{(\bar{r}-r)t\wedge\tau} + e^{-rt\wedge\tau}(L(Y_t) - N_t/N)^+ \wedge \frac{1}{N}$
  - $N_t = \#\{i;\ \tau^i \le t\}$ number of withdrawals before t
  - $\tau = \inf\{t;\ L(Y_t) < N_t/N\}$

Bank Run Model: case of full information

Again **Gossner**'s talk

Assume

- $\sigma = 0$, i.e. Y_t is **public knowledge** !
- the function $y \hookrightarrow L(y)$ is also public knowledge

In **ANY** equilibrium

$$\tau^i = \inf\{t;\ L(Y_t) \leq 1\}$$

- Depositors withdraw at the **same time** (**run on the bank**)
- Each depositor gets his deposit back (**no one gets hurt!**)

Highly Unrealistic

Depositors should **wait longer** because of **noisy private signals**

GAMES OF TIMING

Geoffrey Zhu (PhD unpublished) N players with individual states $X_t^{N,i}$ at time t

$$dX_t^{N,i} = b(t, X_t^{N,i}, \overline{\nu}_t^N)dt + \sigma(t, X_t^{N,i})dW_t^i, \qquad i = 1, \cdots, N$$

interacting through the empirical distribution

$$\overline{\nu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}}.$$

Each player chooses a $\mathcal{F}^{X^i}$ stopping time τ^i and try to maximize

$$J^i(\tau^1, \cdots, \tau^N) = \mathbb{E}\Big[g(\tau^i, X_{\tau^i}, \overline{\mu}^N([0, \tau^i])\Big]$$

where

- $\overline{\mu} = \frac{1}{N}\sum_{i=1}^N \delta_{\tau^i}$ empirical distribution of the τ^i's
- $g(t, x, p)$ is the reward to a player for
    - exercising his timing decision at time t when
    - his private signal is $X_t^I = x$,
    - the proportion of players who already exercised their right is p.

MFG Formulation

$N \to \infty$ and follow a **representative player**
e.g. $b(t, x, \nu) = b(t, x)$ states do not interact (for simplicity)

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

- $\mathbb{F}^X = (\mathcal{F}_t^X)_{0 \le t \le T}$ information available to agent at time t
- $\mathcal{S}^X$ set of $\mathbb{F}^X$-stopping times

MFG of Timing

1. **Best Response Optimization**: for each fixed environment $\boldsymbol{\mu} \in \mathcal{P}([0, T])$ solve

$$\hat{\theta} \in \arg \sup_{\theta \in \mathcal{S}^X, \theta \le T} \mathbb{E}[g(\theta, X_\theta, \mu([0, \theta]))]$$

2. **Fixed Point Step**: find $\boldsymbol{\mu}$ so that

$$\mu([0, t]) = \mathbb{P}[\hat{\theta} \le t]$$

Solution with Randomized Stopping Times

RC-G. Zhu

Recall that even if

$$\begin{cases} \lim_{n\to\infty}(X, Y_n) = (X, Y) \quad \text{in law} \\ Y_n \ \text{ is a function of } \ X \end{cases}$$

Y is not necessarily a function of X, i.e. $Y \in \sigma\{X\}$ may not hold.

Assume the reward $g : [0, T] \times \mathbb{R} \times \mathcal{P}([0, T]) \ni ((t, x, \mu) \mapsto g(t, x, \mu)\mathbb{R}$ is

- bounded
- continuous in (t, x) for μ fixed
- Lipschitz continuous in μ for (t, x) fixed

Then

$$\Pi : \mathcal{P}([0, T]) \times \mathcal{P}(C([0, T] \times [0, T])) \mapsto \mathbb{R}$$

$$(\mu, \xi) \hookrightarrow \Pi(\nu, \xi) = \int_{C([0,T])\times[0,T]} g(t, x_t, \mu)\xi(dx, dt)$$

is continuous

NB: 3rd assumption **NOT SATISFIED** for functions $t \hookrightarrow \mu([0, t])$,
unless $t \in \mathbb{T} \subset [0, T]$ with $\mathbb{T}$ finite!

Existence Proof

- Space $\tilde{\mathcal{S}}$ of randomized stopping times is compact (Baxter-Chacon)
- **Berge's maximum theorem** implies

$$\mathcal{P}([0,T]) \ni \nu \hookrightarrow \arg\sup_{\xi \in \tilde{\mathcal{S}}} \Pi(\nu, \xi)$$

  is upper hemicontinuous and compact-valued
- Followed by the projection on the first marginal, it is still upper hemicontinuous and compact-valued
- **Kakutani's fixed point theorem** implies existence

Existence with usual stopping times

Work directly on the **space of stopping times** (**complete lattice**) and use **Tarski's fixed point theorem**

Assume (for example):

- **Time increments of g are monotone in ν**

so we can check

$$\tau \hookrightarrow \arg\sup_{\tau' \in \mathcal{S}} \mathbb{E}[g(\tau', X_{\tau'}, F_\tau(\tau'))]$$

is monotone. Here $F_\tau(t) = \mathbb{P}[\tau \leq t]$ is the c.d.f. of τ.

Unfortunately, NOT THE CASE for "Bank Run" models !!!

Solution in the general case (including common noise)
R.C.-Delarue-Lacker [10]

Some References

MFG Models for Oil Production

- Plain model for a large population of oil producers **Giraud,Guéant, Lasry, Lions** [22, 26, 25]
- Game models for oil production **Ludkovski-Sircar** [31]
- Inclusion of the exhaustibility constraint **Chan-Sircar** [18], **Bensoussan-Graber**[24]
- Inclusion of Major-Minor MFG models **Achdou-Giraud-Lasry-Lions**[1]

MFG Models for the Electric Grid

- **Djehiche-Barreiro Gomez-Tembine** [20],
- **Alasseur-Ben Tahar-Matoussi** [2]

MFG Models for Negotiations on the Environment

- **Bahn-Haurie-Malhame** [5],

Moral Hazard & Contract Theory

- How should a government control a flu outbreak by encouraging citizens to vaccinate?
- How should taxes be levied to influence people's consumption, saving and investment decisions?
- How should an employer incentivize and compensate its employees in order to boost productivity?

Economic lingo we shall try to elucidate

Agency Problem, Contract Theory, Moral Hazard,
Asymmetric Information

Early Contributors

- **Static** One Period **Models**.Mirrlees [32], Holmstrom [27, 28],
- **Dynamic models**: Holmstrom - Milgrom [29, 30], Y. Sannikov [33, 34]
- **Stochastic Control & Game Models**: Sannikov and Financial Mathematicians Cvitanic-Zhang [19] using Backward Stochastic Differential Equations (BSDEs)

Bengt Holmström & Oliver Hart awarded Nobel Memorial Prize in Economic Sciences in 2016 for their work on Contract Theory

Standard Model Data

Present in all these scenarios are two parties:

1. the ***principal*** who devises a *contract*, according to which **incentives** are given to, and/or **penalties** are imposed on,
2. the ***agent*** who may accept the contract and work for the principal.

Major assumptions:

- We assume that all the agents are *rational*
  - they behave optimally to maximize their utilities,
  - tradeoff between the rewards/penalties they received and the efforts they put in.
- The principal designs a **contract**
- The agent may walk away (**reservation utility**)
- The principal **partially observes** the agent actions!

Asymmetric information, Moral hazard

The fact that the principal de facto does not see the agent behavior forces a **special formulation** of the optimization problems, different from the usual mathematical formulation of stochastic games and stochastic control !!! **Sannikov**

One Principal vs MANY MANY MANY Agents

New situation:

1. ***ONE principal*** who devises *one single contract*, according to which **incentives** are given to, and/or **penalties** are imposed on,
2. ***MULTIPLE agents*** may accept the contract and work for the principal.

Major assumptions:

- We assume that all the agents are *rational* and statistically identical in their rationality
  - they behave **selfishly** maximize their utilities and reach a **NASH equilibrium**,
- The principal designs a contract that maximizes the principal's own utility, anticipating that the agents will reach a Nash equilibrium.
- The agent may walk away (reservation utility)

Asymmetric information, Moral hazard

ONE PRINCIPAL vs a MEAN FIELD OF AGENTS
Continuous case: **Elie-Mastrolia-Possamai** [21]

Discrete case: **Carmona-Wang** [14]

WEAK FORMULATION FOR CONTROL & TWO-PLAYER GAMES

Stochastic Optimal Control

- **State** X_t of system at time t
- One single controller takes **actions** / **controls**
- Actions affect system and **cost** / **reward** of the controller

Stochastic Games

- **Several** / **Many** controllers (players, agents) take actions
- Each one has a cost / reward to worry about
- System affected by individual actions and their **interactions**

Weak Formulation

- Also called **Martingale Approach** (**Davis, Kurtz**)
- $\omega \hookrightarrow X_t(\omega)$ NOT AFFECTED by actions!
- The **distribution** of the process $\mathbf{X} = (X_t)_{0\le t\le T}$ affected by controls
- Roughly speaking:
  the choice of a control is the choice of a law for the state process

Weak Formulation because of Moral Hazard

Agent controls / influences their effort

- **effort** α_t controls of the state at time t
- Agent sees the state X_t
- Agent optimizes **cost** / **reward** given the contract

Principal chooses the remuneration of the agent

- without observing directly the effort α_t of the agent
- observing, only partially, the state X_t
- seeing the impact of the agent's effort only through the expected returns
- de facto through the values of expected quantities

Weak Formulation

- Roughly speaking:
  the principal guesses the return through the distribution of the output of the agent

Continuous State Case. I

Canonical Representation

- Ω space of continuous functions from $[0, T]$ to E (typically $E = \mathbb{R}$ or $E = \mathbb{R}^d$)
- $W_t(\omega) = \omega(t)$ for $t \geq 0$
- $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ natural filtration generated by $\mathbf{W}$
- μ_0 fixed probability on E (initial distribution of the state, i.e. $X_0 \sim \mu_0$)
- $\mathcal{F} := \mathcal{F}_T$
- $\mathbb{P}$ Wiener measure on $(\Omega, \mathbb{F}, \mathcal{F})$ so $\mathbf{W}$ is a Wiener process
- $X_t = \xi_0 + \int_0^t \sigma_s(X_\cdot)dW_s$ for some σ bounded from above and below away from 0

$$dX_t = \sigma_t(X_\cdot)dW_t, \qquad \text{under } \mathbb{P}$$

independently of what the agent does.

Continuous State Case. II

We want to control

- $\mathbb{A}$ space of admissible control (agent effort) processes $\boldsymbol{\alpha} = (\alpha_t)_{0 \leq t \leq T}$
- Bounded progressive drift (only thing controlled by the agent)

$$(t, x, \alpha) \mapsto b_t(x_\cdot, \alpha) \in \mathbb{R}^d$$

- $\mathbb{P}^{\boldsymbol{\alpha}}$, state distribution if teh effort of the agent is $\boldsymbol{\alpha}$

$$\frac{d\mathbb{P}^{\boldsymbol{\alpha}}}{d\mathbb{P}} = \mathcal{E}\Big[\int_0^T \sigma_t(X_\cdot) b_t(X_\cdot, \alpha_t) dW_t\Big]$$

where $\mathcal{E}(\mathbf{M}) = \exp[M_t - \frac{1}{2} < M, M >_t]$ is the Doleans exponential of the continuous square integrable martingale $\mathbf{M} = (M_t)_{0 \leq t \leq T}$

- Girsanov Theorem says

$$dX_t = b_t(X_\cdot, \alpha_t)dt + \sigma_t(X_\cdot)dW_t^{\boldsymbol{\alpha}}, \qquad \text{under } \mathbb{P}^{\boldsymbol{\alpha}}$$

where

$$W_t^{\boldsymbol{\alpha}} = W_t - \int_0^t \sigma_s(X_\cdot) b_s(X_\cdot, \alpha_s) ds$$

is a Brownian motion under the measure $\mathbb{P}^{\boldsymbol{\alpha}}$.

Continuous State Case. III

Final Formulation

- The principal offers a contract $(\boldsymbol{r}, \xi)$ where
  - $\boldsymbol{r} = (r_t)_{0 \le t \le T}$ is a payment stream
  - ξ is a terminal payment
- the agent decides whether or not to accept the contract and work for the principal
- chooses an effort level $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$ to maximize

$$J^{\boldsymbol{r},\xi}(\boldsymbol{\alpha}, \mu_0) = \mathbb{E}^{\mathbb{P}^{\boldsymbol{\alpha}}}\left[U_A(\xi) + \int_0^T [u_A(r_t) - c_t(X_\cdot, \alpha_t)]dt\right]$$

  where
  - u_A is the agent running utility
  - U_A is the agent terminal utility
  - $c_t(x_\cdot, \alpha)$ is the cost for the effort level α at time t when the history of the state is $x_{[0,t]}$

Continuous State Case. IV

Principal Optimization Problem

- For each contract $(\boldsymbol{r}, \xi)$
- Assuming knowledge of the utility and cost functions of the agent
- Assuming that the agent is rational
- the Principal computes an **optimal** effort level $\boldsymbol{\alpha}^* = (\alpha^*_t)_{0 \le t \le T}$

$$\boldsymbol{\alpha}^* \in \arg \inf_{\boldsymbol{\alpha} \in \mathbb{A}} J^{\boldsymbol{r}, \xi}(\boldsymbol{\alpha}, \mu_0)$$

and then search for an optimal contract $(\boldsymbol{r}^*, \xi^*)$

$$(\boldsymbol{r}^*, \xi^*) \in \arg \inf_{(\boldsymbol{r}, \xi)} \mathbb{E}^{\mathbb{P}^{\boldsymbol{\alpha}^*}} \left[U_P \left(X_T - \xi - \int_0^T r_t dt \right) \right]$$

where U_P is the (terminal) utility of the principal.

Stackelberg game between the principal **going first** and the agent.

What Changes for a Large Number of Agents

- Statistically speaking the agents behave similarly (**symmetry** assumption)
- Because of their large number, their individual influences are **negligeable**
- The agents **compete** with each other

Principal Optimization Problem

- For each contract $(\boldsymbol{r}, \xi)$
- Assuming knowledge of the utility and cost functions of the agents
- Assuming that the agents settle in a Mean Field Nash Equilibrium
- the Principal
  - solves the MFG of the agents
  - computes an **Nash Equilibrium** effort level $\boldsymbol{\alpha}^* = (\alpha_t^*)_{0 \le t \le T}$
  - then search for an optimal contract $(\boldsymbol{r}^*, \xi^*)$

$$(\boldsymbol{r}^*, \xi^*) \in \arg\inf_{(\boldsymbol{r},\xi)} \mathbb{E}^{\mathbb{P}^{\boldsymbol{\alpha}^*}} \left[U_P\Big(X_T - \xi - \int_0^T r_t dt \Big) \right]$$

where U_P is the (terminal) utility of the principal.

Stackelberg game between the principal **going first** and the field of the agents, but now

$$b_t(X_t, \alpha_t, \mu_t) \qquad \text{and} \qquad c_t(X_t, \alpha_t, \mu_t)$$

where μ_t is the distribution at time t of the state X_t under $\mathbb{P}^{\boldsymbol{\alpha}}$.

The Canonical Process for the Discrete Case

We want to control

- $\mathbf{X} = (X_t)_{0 \le t \le T}$ a continuous-time Markov chain with m states
- State space $E = \{e_1, \ldots, e_m\}$, e_i's basis vectors in $\mathbb{R}^m$
- $t \to X_t$ are *càdlàg*, i.e. right continuous with left limits, and continuous at T (i.e. $X_{T-} = X_T$)

Canonical Representation

- Ω space of càdlàg functions from $[0, T]$ to E, continuous at T
- $X_t(\omega) := \omega_t$
- $\mathbb{F} := (\mathcal{F}_t)_{t \in [0,T]}$ natural filtration generated by $\mathbf{X}$
- $\mathbf{p}^\circ$ fixed probability on E (initial distribution of the process $\mathbf{X}$)
- $\mathcal{F} := \mathcal{F}_T$
- $\mathbb{P}$ probability on $(\Omega, \mathbb{F}, \mathcal{F})$ for which $\mathbf{X}$
  - is a continuous-time Markov chain
  - has initial distribution $\mathbf{p}^\circ$
  - has transition rates between any two different states equal to 1

So if $i \neq j$ and $\Delta t > 0$,

$$\mathbb{P}[X_{t+\Delta t} = e_j | \mathcal{F}_t] = \mathbb{P}[X_{t+\Delta t} = e_j | X_t] \quad \text{and} \quad \mathbb{P}[X_{t+\Delta t} = e_j | X_t = e_i] = \Delta t + o(\Delta t)$$

SDE Representation

Disclaimer

"Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different."

Johann Wolfgang von Goethe

The process **X** has the representation (**Cohen-Elliott**) :

$$X_t = X_0 + \int_{(0,t]} Q^0 \cdot X_{t-} dt + \mathcal{M}_t,$$

- Q^0 is the square matrix with
  - $Q^0_{i,i} = -(m-1), \qquad i = 1, \cdots, m$
  - $Q^0_{i,j} = 1$ if $i \neq j$
- $\mathcal{M} = (\mathcal{M}_t)_{t \geq 0}$ is a $\mathbb{R}^m$-valued $\mathbb{P}$-martingale.
- $\cdot$ is matrix multiplication.

The predictable quadratic variation of the martingale $\mathcal{M}$ under $\mathbb{P}$ is given by the formula:

$$\langle \mathcal{M}, \mathcal{M} \rangle_t = \int_0^t \psi_t dt, \tag{4}$$

where ψ_t is given by:

$$\psi_t := diag(Q^0 \cdot X_{t-}) - Q^0 \cdot diag(X_{t-}) - diag(X_{t-}) \cdot Q^0. \tag{5}$$

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Players' Controls, I

Notation

- Probability measures on the state space

$$\mathcal{P}(E) = \mathcal{S} := \{p \in \mathbb{R}^m;\ \sum_{i=1}^m p_i = 1, p_i \geq 0\},$$

- $A \subset \mathbb{R}^k$ set of **actions** / **controls**
- Q-**matrices**

$$Q(t, \alpha, p, \nu) = [q(t, i, j, \alpha, p, \nu)]_{1 \leq i,j \leq m}$$

  where

$$[0, T] \times \{1, \dots, m\}^2 \times A \times \mathcal{S} \times \mathcal{P}(A) \to q(t, i, j, \alpha, p, \nu),$$

Major Hypotheses

(I) $Q(t, \alpha, p, \nu)$ is a Q-matrix.

(II) $0 < C_1 < q(t, i, j, \alpha, p, \nu) < C_2$.

(III) For all $(t, i, j) \in [0, T] \times E^2$, $\alpha, \alpha' \in A$, $p, p' \in \mathcal{S}$ and $\nu, \nu' \in \mathcal{P}(A)$, we have:

$$|q(t, i, j, \alpha, p, \nu) - q(t, i, j, \alpha', p', \nu')| \leq C(\|\alpha - \alpha'\| + \|p - p'\| + \mathcal{W}_1(\nu, \nu')).$$

where $\mathcal{W}_1$ is the 1-Wasserstein on $\mathcal{P}(A)$.

Players' Controls, II

The set $\mathbb{A}$ of player strategies is the set of A-valued, $\mathbb{F}$-predictable process $\boldsymbol{\alpha} = (\alpha_t)_{0\le t\le T}$

Given

- $\boldsymbol{\alpha} = (\alpha_t)_{0\le t\le T} \in \mathbb{A}$
- $\boldsymbol{p} = (p_t)_{0\le t\le T}$ a flow of probability measures on E
- $\boldsymbol{\nu} = (\nu_t)_{0\le t\le T}$ a flow of probability measures on A

Define the martingale $\mathbf{L}^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})} = (L_t^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})})_{0\le t\le T}$ by

$$L_t^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})} := \int_0^t X_{s-}^* \cdot (Q(s,\alpha_s,p_s,\nu_s) - Q^0) \cdot \psi_s^+ \cdot d\mathcal{M}_s.$$

Simple calculations show

$$\begin{aligned}\Delta L_t^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})} &= X_{t-}^* \cdot (Q(t,\alpha_t,p_t,\nu_t) - Q^0) \cdot \psi_t^+ \cdot \Delta X_t \\ &= q(t,i,j,\alpha_t,p_t,\nu_t) - 1,\end{aligned}$$

so that $\Delta L_t^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})} \ge -1$. Also $\mathcal{E}(\mathbf{L}^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})})$ is **uniformly integrable** so we can apply

Girsanov's Theorem.

Players' Controls, III

Define:

$$\frac{d\mathbb{Q}^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})}}{d\mathbb{P}} := \mathcal{E}(\mathbf{L}^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})})_T.$$

The process $\boldsymbol{\mathcal{M}}^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})} = (\boldsymbol{\mathcal{M}}_t^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})})_{0\le t\le T}$ defined as:

$$\mathcal{M}_t^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})} := \mathcal{M}_t - \int_0^t (Q^*(s,\alpha_s,p_s,\nu_s) - Q^0)\cdot X_{s-}\,ds, \tag{6}$$

is a $\mathbb{Q}^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})}$-martingale.

- The canonical decomposition of X reads:

$$X_t = X_0 + \int_0^t Q^*(s,\alpha_s,p_s,\nu_s)\cdot X_{s-}\,dt + \mathcal{M}_t^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})}. \tag{7}$$

- Under $\mathbb{Q}^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})}$, the stochastic intensity rate of $\mathbf{X}$ is $Q(t,\alpha_t,p_t,\nu_t)$
- X_0 has still distribution $\mathbf{p}^\circ$
- If $\alpha_t = \phi(t, X_{t-})$ for some measurable function ϕ, $\mathbf{X}$ is a **continuous-time Markov chain** with **jump rate intensity** $q(t,i,j,\phi(t,i),p_t,\nu_t)$ under the measure $\mathbb{Q}^{(\boldsymbol{\alpha},\boldsymbol{p},\boldsymbol{\nu})}$.

A Naive Model of Epidemy Containment

- Regulator tries to control the spread of a virus over a time period $[0, T]$
- Jurisdiction consists of two cities A and B
- Each individual is either infected (I) or healthy (H), lives in city A or B.
- State space is $E = \{AI, AH, BI, BH\}$
- $\pi_{AI}, \pi_{AH}, \pi_{BI}, \pi_{BH}$ proportions of individuals in each state

State Evolution

(1) Rate of contracting the virus depends on the proportion of infected individuals in the city so
  - transition rate from state AH to state AI is $\theta_A^-(\frac{\pi_{AI}}{\pi_{AI}+\pi_{AH}})$
  - transition rate from state BH to state BI is $\theta_B^-(\frac{\pi_{BI}}{\pi_{BI}+\pi_{BH}})$.

(2) Rate of recovery is a function of the proportion of healthy individuals in the city, so
  - transition rate from state AI to state AH is $\theta_A^+(\frac{\pi_{AH}}{\pi_{AI}+\pi_{AH}})$
  - transition rate from state BH to state BI is $\theta_B^+(\frac{\pi_{BH}}{\pi_{BI}+\pi_{BH}})$.

(3) Each individual can try to move to the other city: $\nu_I\alpha$ transition rates between the states AI and BI, and $\nu_H\alpha$ as the transition rates between the states AH and BH.

(4) Status of infection does not change when individual moves between cities.

θ_A^-, θ_B^-, θ_A^+ and θ_B^+ increasing differentiable functions from $[0, 1]$ to $\mathbb{R}^+$ characterize the quality of health care in cities A and B.

The Q-matrix of the Model

$$Q(t,\alpha,\pi) := \begin{array}{c} \\ \\ \end{array} \begin{array}{cccc} AI & AH & BI & BH \end{array}$$

$$Q(t,\alpha,\pi) := \begin{array}{cc} \begin{array}{cccc} AI & AH & BI & BH \end{array} & \\ \left[\begin{array}{cccc} \cdots & \theta_A^+\left(\frac{\pi_{AH}}{\pi_{AI}+\pi_{AH}}\right) & \nu_I\alpha & 0 \\ \theta_A^-\left(\frac{\pi_{AI}}{\pi_{AI}+\pi_{AH}}\right) & \cdots & 0 & \nu_H\alpha \\ \nu_I\alpha & 0 & \cdots & \theta_B^+\left(\frac{\pi_{BH}}{\pi_{BI}+\pi_{BH}}\right) \\ 0 & \nu_H\alpha & \theta_B^-\left(\frac{\pi_{BI}}{\pi_{BI}+\pi_{BH}}\right) & \cdots \end{array}\right] & \begin{array}{c} AI \\ AH \\ BI \\ BH \end{array} \end{array}$$

More on the Epidemic Containment Model

Agent cost functions:

$$c_1(t, AI, \pi) = c_1(t, AH, \pi) := \phi_A\left(\frac{\pi_{AI}}{\pi_{AI} + \pi_{AH}}\right), \tag{8}$$

$$c_1(t, BI, \pi) = c_1(t, BH, \pi) := \phi_B\left(\frac{\pi_{BI}}{\pi_{BI} + \pi_{BH}}\right), \tag{9}$$

$$\gamma_{AI} = \gamma_{BI} := \gamma_I, \quad \gamma_{AH} = \gamma_{BH} := \gamma_H, \tag{10}$$

where ϕ_A and ϕ_B are two increasing mappings on $\mathbb{R}$.

Authority (Principal) running and terminal costs:

$$c_0(t, \pi) = \exp(\sigma_A \pi_{AI} + \sigma_B \pi_{BI}), \tag{11}$$

$$C_0(\pi) = \sigma_P \cdot (\pi_{AI} + \pi_{AH} - \pi_A^0)^2, \tag{12}$$

where π_A^0 is the population of city A at time 0.

Objectives of the authority:

- Trade-off between the control of the epidemic and population planning
- Minimization of the infection rate of both cities
- σ_A, σ_B and σ_P reflects the relative importance the authority attributes to each of these objectives.

Equilibrium behavior of the individuals

- Tendancy to move away from the city with higher infection rate and poorer health care
- May result in overpopulation of the other city. Therefore, the authority also wishes to maintain the population of both cities at a steady level.

Infection Rates w/wo Intervention of the Regulator

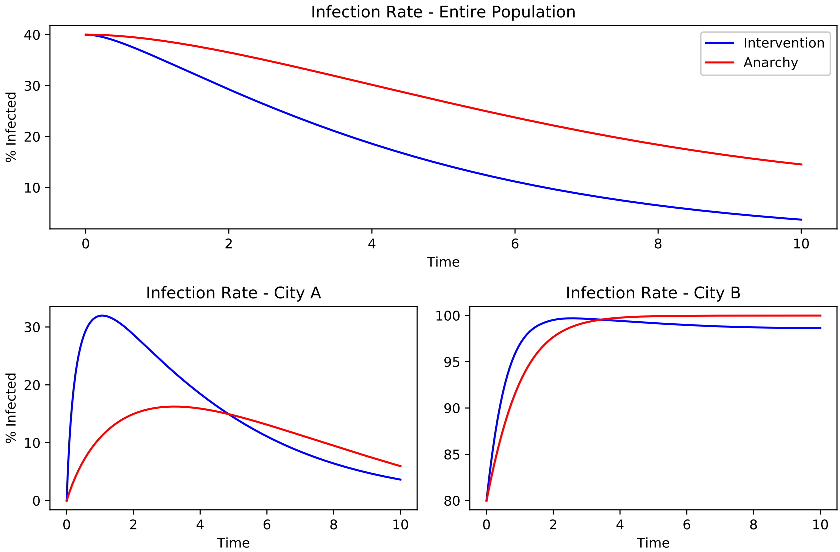

FIGURE: Evolution of infection rate with and without authority's intervention.

Optimal Effort of each Individual

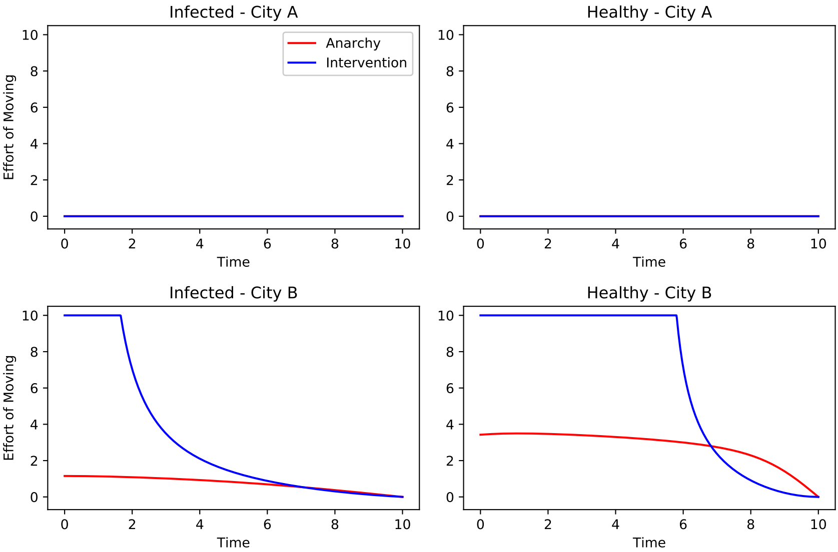

Figure: Optimal effort of moving for an individual in different states.

City Population Evolutions

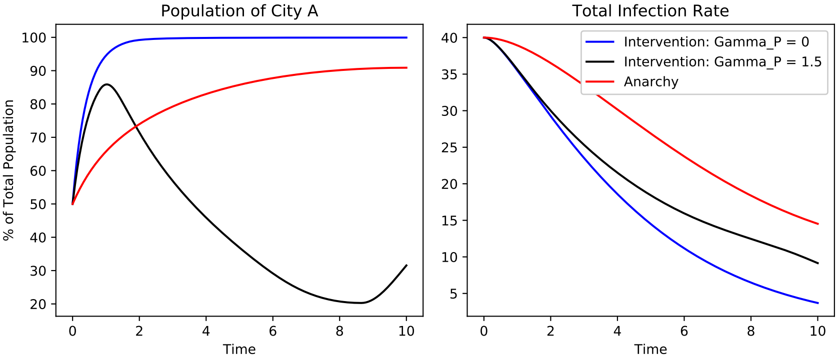

FIGURE: Evolution of population in city A and total infection rate: blue curve corresponds to intervention without population planning ($\sigma_P = 0$), black curve corresponds to intervention with population planning ($\sigma_P = 1.5$), red curve corresponds to the absence of intervention.

Optimal Effort w/wo Regulator Intervention

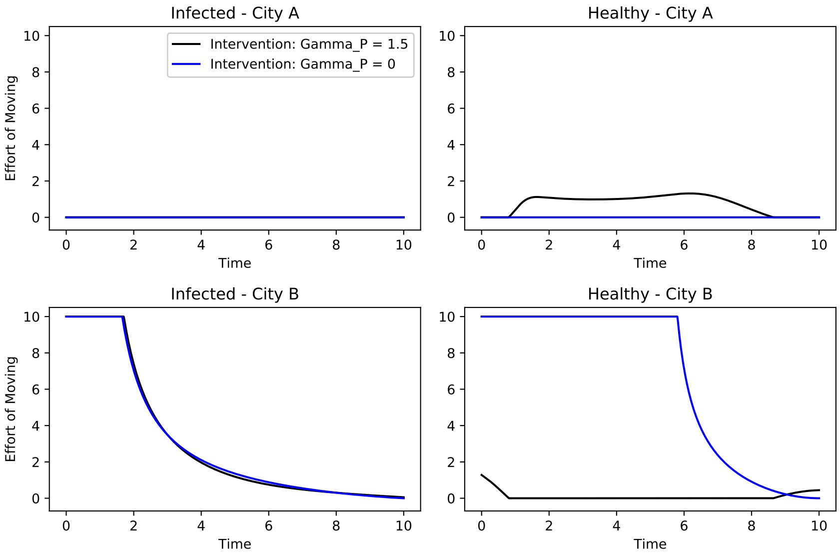

FIGURE: Individual's optimal effort of moving with and without population planning.

ENVIRONMENT ECONOMICS: EARLY RELATED WORKS

Externality and Taxes

- **Golosov-Hassler-Krusell-Tsyvinski** [23]

Negotiations and the Environment

- **Bahn-Haurie-Malhamé**, [5],

Proposal for a New MFG Model

R.C.-Dayanikli-Lauriere

- M Countries (populations) $\{1, \cdots, M\}$
  - Each country sets its own regulation (carbon tax)
  - A weighted graph $\mathbf{G} = [\mathbf{G}^{i,k}]_{i,k=1,\cdots,M}$ gives the strength of the economic exchanges between country i and country k
- N_i generators in country $i \in 1, 2, \ldots, M$.

Contract theory model *on steroids*

- Given the individual regulations and the graph of exchange strengths between the countries
  - In each country, the **large** populations of producers reach a **MFG Nash equilibrium**
  - The country regulators search for a **Nash equilibrium** in optimizing the cost/benefits produced by the production policies adopted by the producers in their MFG Nash equilibria.

Most likely, only numerical solutions are possible, and even then, ...

States & Controls

States:

- State at time t of **Generator** (Minor Player) j in Country (population) i:
  - $Q_t^{i,j} \in \mathbb{R}$: Total Energy Production
  - $E_t^{i,j} \in \mathbb{R}$: Cumulative Emission Level
- **Government/Regulator** (Major Player) i does not have state

Controls:

Minor player (generator) j in population i has the following controls:

- $I_t^{1,i,j} \in \mathbb{R}$: Investment at time t in Nonrenewable Energy Resources
- $I^{2,i,j} \in \mathbb{R}$: Fixed (initial) investment in Renewable Energy Resources

Major player (government) i control:

- $T^i \in \mathbb{R}$: Carbon tax level (Time-independent)

Generator Dynamics and Cost Function:

For Generator j in population i:
State Dynamics:

$$dQ_t^{i,j} = \kappa^{1,i} I_t^{1,i,j} dt + \kappa^{2,i} I^{2,i,j} dS_t^i + dW_t^{i,j}$$
$$dE_t^{i,j} = \delta^i I_t^{1,i,j} dt + d\widetilde{W}_t^{i,j}$$

- $\kappa^{1,i} > 0, \kappa^{2,i} > 0, \delta^i > 0$ country specific constants;
- $S_t^i \geq 0$ noise *common* to the entire population i.
- $W_t^{i,j}$ i.i.d. Brownian motions, *idiosyncratic* noises.

Expected Cost:

$$C^i\Big(I^{1,i,j}, I^{2,i,j}; (I^{1,i',j'}, I^{2,i',j'})_{(i',j')\neq(i,j)}, T^i\Big)$$
$$= \mathbb{E}\Bigg[\int_0^T \Big[\underbrace{c^{1,i}(I_t^{1,i,j})^2}_{\text{Term 1}} \underbrace{-p_t^i\Big(\sum_{k=1}^M \mathbf{G}^{k,i}\bar{Q}_t^k\Big) Q_t^{i,j}}_{\text{Term 2}} \underbrace{+c^i\Big|\bar{Q}_t^i - D_t^i\Big|^2}_{\text{Term 3}}\Big] dt \underbrace{+T^i(E_T^{i,j})^2}_{\text{Term 4}} \underbrace{+c^{2,i}(I^{2,i,j})^2}_{\text{Term 5}}\Bigg]$$

Generator Expected Costs

$$
\begin{aligned}
&C^i\left(I^{1,i,j}, I^{2,i,j}; (I^{1,i',j'}, I^{2,i',j'})_{(i',j')\neq(i,j)}, T^i\right)\\
&= \mathbb{E}\left[\int_0^T \Big[\underbrace{c^{1,i}(I_t^{1,i,j})^2}_{\text{Term 1}} \underbrace{-p_t^i\Big(\sum_{k=1}^{M}\mathbf{G}^{k,i}\bar{Q}_t^k\Big)Q_t^{i,j}}_{\text{Term 2}} \underbrace{+c^i\left|\bar{Q}_t^i - D_t^i\right|^2}_{\text{Term 3}}\Big]dt \underbrace{+T^i(E_T^{i,j})^2}_{\text{Term 4}} \underbrace{+c^{2,i}(I^{2,i,j})^2}_{\text{Term 5}}\right]
\end{aligned}
$$

- **Term 1:** *Variable cost of nonrenewable energy sources.*
- **Term 2:** *Revenue from energy production.*
  - $p_t^i(\cdot)$ decreasing (price) function. For example

$$p_t^i\Big(\sum_{k=1}^{M}\mathbf{G}^{k,i}\bar{Q}_t^k\Big) = p_0^i - p_1^i\sum_{k=1}^{M}\mathbf{G}^{k,i}\bar{Q}_t^k$$

  - $\mathbf{G}^{k,i}$ denotes the strength of the exchanges between countries k and i.

  NB: It includes the **mean-field interactions** within the populations and **network connections** among the countries.
- **Term 3:** *Penalization for not satisfying the demand.* Here D_t^i exogenous (stochastic) demand in country i and $c^i > 0$. Includes **mean-field interactions** within the populations.
- **Term 4:** *Cost of Carbon Tax.* The generators are paying a carbon tax that is proportional to the square of the cumulative emission level. Includes the **interaction with the major player**.
- **Term 5:** *Fixed cost of renewable energy investment.*

Major Player Expected Costs

Country i chooses the tax level T^i to minimize the following expected cost
Expected Cost:

$$J^i\left(T^i; T^{-i}, (I^{1,i,j}, I^{2,i,j})_{(i,j)}\right) = \mathbb{E}\left[\underbrace{\alpha^i \left|\bar{E}_T^i - O_T^i\right|^2}_{\text{Term 1}} \underbrace{- T^i(\bar{E}_T^i)^2}_{\text{Term 2}} + \underbrace{\sum_{k=1}^{M} \mathbf{G}^{k,i} \left|T^i - T^k\right|^2}_{\text{Term 3}}\right]$$

- **Term 1:** *Cost for missing the (average) emission target.* Here O_T^i gives country i's average carbon emission level target and $\alpha^i > 0$ is a country-specific constant. The government i is penalized if it exceeds the target level (because of the excess emission levels) or if they are below the target level (this is an opportunity cost for not using the given right). This term includes the **interaction with the mean-field of the minor players** in the country.
- **Term 2:** *Revenue from Tax Collection.* Includes the **mean-field interaction among the minor players** in the country.
- **Term 3:** *Tax Calibration Cost.* Country i is penalized for differing from the tax levels of the other countries that they are connected to. Again, this term includes the **interaction with the other major players**.

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