Proportionality for Rankings

GERDUS BENADÈ, CHRISTOPHER DONNAY, MOON DUCHIN, AND THOMAS WEIGHILL

In this paper, we offer a framework to measure the degree of proportionality of representation to voter preference under a wide class of election systems, even when elections are conducted without party labels. To do this, we introduce definitions of *blocs* of voters and their *slates* of preferred candidates, which need not be known to voters but could be implicit in their voting patterns. Next, we build out a list of generative models for creating synthetic ranked preference profiles, with an emphasis on flexibility and realism; in particular, we define models that efficiently generate polarized elections with properties motivated by naturalistically observed examples. Putting these together will allow researchers to compare systems of election in terms of their tendency to produce proportional outcomes; we illustrate this by giving both simulated and theoretical results for single transferable vote (STV) elections. This work brings a statistical modeling toolkit to the questions around ranked choice voting and proportionality.

Contents

Manuscript submitted for review to the 25th ACM Conference on Economics & Computation (EC'24).

1 INTRODUCTION

In this paper, we give what we believe to be the first definition of proportionality of votes to seats that is general enough for use with ranked choice elections. We construct novel generative models of ranking that are inspired by polarized elections in messy, real-world settings. We then pair these to run voting systems on real and synthetic preference profiles, which gives information—some provable and analytic and some qualitative and simulation-based—on whether roughly proportional outcomes do indeed tend to result from so-called semi-proportional systems.

Generative models of voting use inputs like historical voting patterns, demographics, and polarization parameters to build a probability distribution from which ballots are sampled and elections can be simulated. In this paper we build and test generative models. These are the first generative systems for producing ranked ballots that incorporate parameters for bloc and crossover voting.

Proportionality of representation for a subgroup of voters could have a very simple interpretation as demographic proportionality (the group's seat share is in line with its share of the electorate). However, this fails to account for any complexity in the voting patterns of that group and the complementary voters. We will define a framework that replaces demographic proportionality with *support proportionality*, where a group's seat share is in line with the combined support for its preferred candidates. We note that this kind of proportional representation is broader than that of PR systems such as party list voting, which secure party-based support proportionality by construction. The finding of proportional outcomes is vacuous in that setting.

Our main interest will be in ranked choice election systems, and we will use *single transferable vote* (STV) as a test case for the work in this paper. Ranked choice voting is a family of systems in which voters rank candidates in order of preference; STV is a smaller family within ranked choice. In STV elections, there is a threshold level of support needed to be elected, depending on the number of seats to be filled—typically the threshold is about $1/(k+1)$ of the first-place votes, where k is the number of seats. The election is conducted in rounds. As candidates are either elected (by passing the threshold) or eliminated from contention, the (surplus) votes supporting those candidates are transferred to the next options on their respective ballots.[1]

Though STV is the basis for the examples in the present paper, the express goal of the work is to set up a framework suitable for the comparative study of systems of election in the future.

Important note: links to data and code used to produce the examples and data visualization in this paper have been suppressed for anonymization purposes, and will be provided in a full replication repo after the paper is reviewed.

1.1 Related work

Statistical ranking models, or models that assign a probability to permutations on a set of elements, have been studied at least since the early 20th century, going back to Thurstone [1927]. Subsequent models include those introduced by Bradley and Terry [1952] and Plackett [1975], which form the basis for the BT and PL models in this paper, respectively. Benter [2008] introduced a variation of the Plackett model with a dampening parameter to account for less careful deliberation of lower-ranked objects. Johnson et al. [2002] proposed a model to combine rankings that were determined by

[1]Specific mechanics vary; in this paper we have implemented the vote-tallying mechanism used by Cambridge, MA for its City Council elections, except as noted below. Since we want to explore the potential of ranked choice voting to secure proportional representation, we will focus on multi-member districts—that is, on STV rather than IRV or *instant runoff voting*, a system that redistributes votes by a nearly identical mechanism until a single winner is selected. IRV is now in wide use around the country, including a recent high-profile introduction in New York City, and it has significant reform momentum. However, the benefits claimed by advocates for IRV are quite different—mitigation of vote splitting and spoiler effects, creation of incentives for positive campaigning, ease of entry for third parties, and so on.

several different sources—which could have used different methods and criteria—into an aggregate, or meta, ranking scheme.

Ranking models have been used in a variety of applications in the broader social science literature. Stern [1990] apply the methods to horse races, where the marginal probability of each horse finishing first is known in advance. Bradlow and Fader [2001] apply time series models to Billboard "Hot 100" list, to show how song rankings change over time. Graves et al. [2003] apply a combination of ranking models to racecar competition outcomes. In the area of election analysis, Upton and Brook [1975] fit a Plackett model to ranked ballots in London elections to determine the effect of candidate name ordering on the ballots, also known as positional bias. Gormley and Murphy [2008] fit a combination of Plackett-Luce and Benter models to polling data from Irish elections in 1997 and 2002. In particular, they find the models to be effective in identifying voting blocs (groups of voters with similar ranked preferences) within the electorate. These analyses are descriptive, based on historical data. In a recent EC paper, Garg et al. [2022] model outcomes of elections in multi-member Congressional districts under a solid coalition assumption, which means that the ballots are effectively unranked (and do not differentiate candidates within each coalition).

Our work is related in several respects to the computational social choice literature. There is a large body of work on the axiomatic properties of voting rules in various settings, including notions of proportionality for representation. For example, defining (extended) justified representation [Aziz et al., 2017] allows proportionality-like guarantees for cohesive blocs of voters in approval-based committee voting; refer to Lackner and Skowron [2022] for a more thorough discussion. Numerical experiments in this literature traditionally rely on assumptions of *impartial culture* [Pritchard and Wilson, 2009], under which voters are independent and every permutation of candidates is equally likely, *impartial anonymous culture*, in which Lebesgue measure is used to set relative preferences, or use *spatial* or distance-based models [Elkind et al., 2017, Tideman and Plassmann, 2010]. Refer to Szufa et al. [2022, 2020] for a comparison of common statistical cultures and recent discussion of how to sample approval elections.

In particular, spatial models Enelow and Hinich [1984], which represent voters (and candidates) as ideal points in a metric space—in other words, using a space with a distance function as the latent space for voter preferences—are common across fields. Voters are presumed to vote either deterministically for their closest representatives or probabilistically (upweighting closer candidates) [Burden, 1997]. Two commonly used methods for estimating ideal points (typically from Congressional roll-call data) are NOMINATE [Poole and Rosenthal, 1985] and IDEAL [Clinton et al., 2004]. Ranked choice voting models can be built from spatial models. For example, Gormley and Murphy [2007] combine a spatial and Plackett-Luce model to analyze Irish STV elections (discussed further in §5), and Kilgour et al. [2020] use a spatial model (where voters rank by proximity) to measure the effect of ballot truncation on single-winner ranked choice outcomes. Garg et al. [2022] also use a spatial model in one section, with voter ideal points extracted from ideology ratings in a commercial voter file, to relate the "diversity" of elected officials to the sizes of multimember districts.

Spatial models are favored by the mathematically inclined because they lend themselves to provable theoretical properties of voting rules. For example, under the implicit utilitarian voting framework, ordinal votes are proxies for underlying utilities and the *distortion* of a voting rule captures its worst-case loss compared to having full information [Procaccia and Rosenschein, 2006]. Anshelevich et al. [2018] study the distortion of STV under metric preferences, and Gkatzelis et al. [2020] recently settled a well-known conjecture on the optimal metric distortion when aggregating rankings to elect a single winner.

Our goal is to strike out in a new direction by formulating novel models and definitions that enable different kinds of questions to surface.

2 BLOCS, SLATES, AND PROPORTIONALITY

2.1 Defining blocs, slates, and notions of preference

The concept of blocs and slates is straightforward: *blocs* are disjoint groups of voters, such that each has a corresponding *slate* of candidates; as a group, each bloc prefers its own slate to the other slates.

To make this precise, we must delineate what it means for the preference profile consisting of ranked votes from a group of voters to display an overall preference for one group of candidates over another. We begin by defining notions of preference that are suitable for measuring in a given observed profile—that is, these are measurements that can be made on any cast vote record that has been minimally cleaned so that each ballot is a partial ranking (a permutation of a subset of the candidates).

Definition 2.1. Suppose an election is conducted with bloc structure $(A, \mathcal{A}, B, \mathcal{B})$ consisting of sets of voters A, B and corresponding slates of candidates $\mathcal{A} = \{A_1, \ldots, A_r\}$ and $\mathcal{B} = \{B_1, \ldots, B_s\}$. Suppose voters are allowed to rank $n \leq r + s$ candidates on their ballots.

- Bloc B prefers slate $\mathcal{B}$ with *first-place preference* p_B if the share of first-place votes in the profile for $\mathcal{B}$ candidates is p_B.
- Bloc B prefers slate $\mathcal{B}$ with *positional preference* $P_B = (p_1, p_2, \ldots, p_n)$ if the share of ballots placing an $\mathcal{B}$ candidate in position i (among those for which a vote is cast and neither slate was exhausted in the higher positions) is p_i. In particular, the special case of *consistent positional preference* p_B corresponds to $P_B = (p_B, p_B, \ldots, p_B)$.
- Given a positional scoring rule with weights $(w_1, w_2, \ldots, w_n)$, we say that B prefers slate $\mathcal{B}$ with *score preference* p_B if the share of their score for $\mathcal{B}$ candidates is p_B. A default option will be to use standard Borda weights $(n, n-1, \ldots, 1)$ and refer to *Borda preference.*

Preferences for the A bloc are defined exactly similarly, and the only difficulty in extending to more than two blocs is one of cumbersome notation.

We will interpret each of these preference parameters as an indication of how *cohesive* bloc B is, with larger preference parameters indicating more strongly aligned blocs.

Example 2.2. Suppose an election has been conducted with $r = 3, s = 2, n = 5$ (i.e., complete rankings are allowed), and suppose the voters are labeled as A voters or B voters. Suppose that the summarized preference profile for the B bloc is given by

×2	×3	×8	×1	×5	×3	×5
B_1	B_1	B_1	A_1	B_2	B_2	B_1
B_2	A_2	B_2	B_1	B_1	A_3	B_2
A_1	B_2	A_2	B_2	A_1	A_1	
A_2	A_3	A_1		A_3	B_2	
A_3	A_2			A_2	A_2	

(by name)

i.e.,

×7	×3	×8	×1	×3	×5
B	B	B	A	B	B
B	A	B	B	A	B
A	B	A	B	A	
A	A	A		B	
A	A			A	

(by slate)

Then the first-place preference for the B bloc is 26/27, the positional preference is $(\frac{26}{27}, \frac{21}{27}, \frac{4}{7}, \frac{3}{3}, -)$, and the (standard) Borda preference is 232/364.

2.2 Defining proportionality

In a situation where two blocs partition the whole electorate, we have a simple heuristic for a proportional outcome of an election. If π_B is any preference parameter for bloc B towards its candidates and likewise π_A for bloc A, then the target would be to have seat share S_B for the $\mathcal{B}$

slate satisfy

$$S_B \approx N_B \cdot \pi_B + (1 - N_B)(1 - \pi_A),$$

where N_B is the share of voters from the B bloc. That is, the combined support for $\mathcal{B}$ candidates is the size of the B bloc times its level of cohesion plus the size of the complementary bloc times its level of crossover.

This enables us to say, for instance, whether a particular election outcome was near-proportional in a given bloc structure with respect to first-place preferences, or to Borda preferences, or any other notion of preference. Proportionality is not a foregone conclusion for ranked choice voting even in the extremely simple case where π is first-place preference and the blocs are also defined by first-place preference. (This is because results depend on lower-ranked choices as well, which may or may not track closely with first-place preference.)

Example 2.3. We use a sample of real-world Scottish local government STV elections to illustrate the definition in Table 1. We will use the simplified bloc structure where three Scottish parties—Green, Liberal Democrat, and Labour—are defined as a "left-leaning" slate $\mathcal{B}$, and all other parties are combined as slate $\mathcal{A}$. A voter is assigned to bloc A or bloc B by their first-place vote, so by definition the cohesion is 1 using first-place preference, and the proportionality target with respect to first-place preference reduces to $N_B \cdot k$, the share of B voters times the number of seats.

election	(r, s, k)	N_B	proportionality target	STV outcome
North Ayrshire 2022 Ward 1	$(8, 4, 5)$	0.17	0.87 seats	0 seats
Angus 2012 Ward 8	$(4, 2, 4)$	0.24	0.96 seats	1 seat
Clackmannanshire 2012 Ward 2	$(5, 3, 4)$	0.32	1.27 seats	1 seat
Aberdeen 2022 Ward 12	$(7, 3, 4)$	0.31	1.26 seats	1 seat
Aberdeen 2017 Ward 12	$(6, 4, 4)$	0.33	1.33 seats	1 seat
Falkirk 2017 Ward 6	$(3, 3, 4)$	0.34	1.35 seats	2 seats
Renfrewshire 2017 Ward 1	$(4, 4, 4)$	0.37	1.49 seats	1 seat
Fife 2022 Ward 21	$(4, 4, 4)$	0.46	1.86 seats	2 seats
Glasgow 2012 Ward 16	$(7, 5, 4)$	0.60	2.40 seats	3 seats

Table 1. Here, s is the number of left-leaning candidates (defined by membership in the Green, Liberal Democrat, and Labour parties), r is the number of candidates from all other parties, and k is the number of seats to be filled in the election.

3 GENERATIVE MODELS

3.1 Constructing the models

In this section, we set up generative models of election, including several variants derived from classical statistical ranking literature in the style of Plackett-Luce and Bradley-Terry models.[2] Though at first the by-name and by-slate versions may seem extremely similar, we find that Slate-PL and Slate-BT have several desirable properties compared to Name-PL and Name-BT. Together with the model called the Cambridge sampler (Slate-CS), these will form the list of generative models explored in the empirical work in this paper.

[2]Earlier versions of the Name-PL, Name-BT, and Slate-CS models have been discussed in unpublished work by an overlapping collection of authors. References are suppressed here for anonymization purposes.

Definition 3.1. For all of the models below, assume a fixed bloc structure $(A, \mathcal{A}, B, \mathcal{B})$ with $\mathcal{A} = (A_1, \ldots, A_r)$ and $\mathcal{B} = (B_1, \ldots, B_s)$.

We will use the term *ballot* for a partial or complete ranking of the $r + s$ candidates and a *ballot type* for a permutation of the symbols $A^r B^s$, i.e., a simplified ballot that treats the candidates of each slate as indistinguishable from each other.

The models below will use the following parameters to generate a profile for bloc B:

Cohesion Cohesion or preference lean of the bloc between the slates, given as a parameter $\pi_B \leq 1$ (typically required to be $> 1/2$).

Strength Strength of the overall preference of bloc B within slates. This consists of probability vectors $I_{BA} = (a_1, \ldots, a_r)$ and $I_{BB} = (b_1, \ldots, b_s)$.

We can combine this data into a single probability vector

$$I_B = \big((1 - \pi_B)a_1, \ldots, (1 - \pi_B)a_r, \quad (\pi_B)b_1, \ldots, (\pi_B)b_s\big).$$

Using these specs, we can define five generative models as follows. The first two work directly with ballots, while the latter three first construct ballot types. These are analogous to the profile by name and the profile by slate in Example 2.2.

Name-PL Plackett-Luce by name: A B-bloc voter chooses candidate i to be ranked first with probability $I_B(i)$. They continue to select candidates for the lower-ranking positions by choosing from the remaining j with probability proportional to $I_B(j)$ among the remaining candidates. In other words, each voter samples without replacement from all candidates proportional to their weighting in I_B.

Name-BT Bradley-Terry by name: The probability that a B voter casts a ballot σ is proportional to

$$\prod_{i <_\sigma j} \frac{I_B(i)}{I_B(i) + I_B(j)},$$

where $i <_\sigma j$ means that i is ranked before (i.e., higher than) j in σ. In other words, for each pairwise comparison of candidates, we introduce a term for the likelihood of ranking one before the other coming from the relative weights in I_B.

Slate-PL Plackett-Luce by slate: A B-bloc voter chooses between the symbol A and B in the ith position with probability π_B of choosing B, as long as both $\mathcal{A}$ candidates and $\mathcal{B}$ candidates remain available. Once one slate runs out, the rest of the complete ranking is filled in with the available symbol.

Slate-BT Bradley-Terry by slate: Suppose a ballot type σ is a permutation of $A^r B^s$, that is, an ordered list containing r A symbols and s B symbols. Suppose that out of the rs comparisons of the instances of A with the instances of B, the A occurs earlier than the B a total of $0 \leq i \leq rs$ times. The probability that a B voter casts this ballot is proportional to $(1 - \pi_B)^i (\pi_B)^{rs-1}$.

Slate-CS Cambridge sampler: We draw from a dataset consisting of ten years of ranked votes from city council elections in Cambridge, MA. Candidates have been labeled as white (W) or as people of color (C) (with help from local organizers). To use this model, we make a choice to designate bloc $\mathcal{B}$ as corresponding to voters who put a W candidate first, or who put a C candidate first. We use the cohesion parameter π_B to decide probabilistically whether the voter chooses their own slate or the other slate in the first position. Then we complete the ballot type by drawing proportional to frequency from the cast ballots with that header.

In all three Slate models, we must then assign candidate names to the symbols A and B. We do so by drawing without replacement (Plackett-Luce style) from I_{BA} and I_{BB} separately to order $\mathcal{A}$ and $\mathcal{B}$, then fill in names accordingly.

Remark 1 (Names versus slates). *It turns out to be an important distinction to work directly with the names or to create a type first, then add names. The reason for the divergence is that the Slate models handle I_{BA} and I_{BB} separately; concatenating them into I_B before making length comparisons yields unintended results, such as a highly cohesive bloc whose voters tend to put their strong candidate first and then immediately cross over to supporting the opposite slate. These effects can be explored in the supplementary plots (§A) which compare all five models.*

Remark 2 (About the Cambridge data). *Cambridge, Massachusetts uses STV for its city council and school board elections and has done so continuously since 1941. Our source of Cambridge historical data is city council elections to elect $k = 9$ seats by STV from 2009 to 2017; there are frequently 20 or more candidates who run in each contest. To explain the method a bit more fully: if a ballot type is selected from the historical frequency histogram that has more candidates from a given slate than the (r, s) chosen for a given simulation run, then we ignore further instances. For instance, a ballot type of AAABB in an election where $r = s = 2$ will be read as AABB.*

One valuable aspect of our use of Cambridge historical data in the present study is that it lets us incorporate realistic short-ballot voting behavior without a proliferation of extra parameters. For instance, Cambridge voters cast "bullet votes" (listing only one candidate and leaving other positions blank) 7501 times out of 87,914 ballots cast in our data set. One serious limitation is that we have coded the candidates by race, while Cambridge city council politics are demonstrably more polarized by other candidate features—for instance, an explicit slate of affordable housing candidates is routinely advertised before election day and is highly salient to voter behavior. Nevertheless race is a candidate feature that is often well known to voters and allows us to observe naturalistic patterns of alternation in voting.

These give new generative models to study. As noted in the literature review (§1.1), many authors have considered only solid bloc voting, in which every A voter casts a ballot of type $AA \dots ABB \dots B$, and numerous others have considered only extremely stylized assumptions like Impartial Culture, Impartial Anonymous Culture, and spatial voting. Thus the new models greatly expand on the generative models in the literature.

Remark 3 (PL preferences). *Slate-PL with $(A, \mathcal{A}, B, \mathcal{B})$ and any cohesion and candidate strength parameters produces blocs with consistent positional preference π_B (respectively π_A) for their own slates, and therefore with first-place preference π_B (or π_A) as well.*

Remark 4 (Mixture models). *The definitions above are in terms of specified blocs of voters with different voting preferences. However, there is a strong connection to mixture models suggested by the structure here. In a mixture model, each voter is assigned independently to a class, and then randomly submits a ballot based on the parameters for that class. More precisely, if N_1 and N_2 are the weights for two different classes of voter with $N_1 + N_2 = 1$, and μ_1 and μ_2 are two distributions on ballots corresponding to the two classes, the probability of a ballot σ is*

$$\mu(\sigma) = N_1\mu_1(\sigma) + N_2\mu_2(\sigma).$$

As the number of voters increases, the fraction of voters assigned to each class converges to N_1 and N_2 respectively; for large numbers of voters we can therefore consider the size of each class to be predetermined and treat voters as if they belong to two blocs of fixed size.

3.2 Visualization

3.2.1 MDS plot of profiles. One difficulty in studying ranked choice elections is that, unlike oversimplified Example 2.2, real-world elections frequently have too many kinds of ballot to effectively see the full preference profile. For instance, an election with six candidates can be thought of as having 1236 possible ballots to cast—there are 6! complete rankings and a roughly equal number of partial rankings.[3] Thinking of profiles as distributions over possible ballots allows us to define natural notions of distance between profiles; for instance, the L^1 distance between profiles is the sum over possible ballots of the absolute value of the difference between the share in one profile and the share in the other. (Up to a constant factor, this is the same as the total variation distance of distributions.) With this notion we can provide an understandable visualization that shows how much our generative models tend to differ as we vary parameters.

To illustrate the importance of candidate strength, we introduce four out of the infinitely many variations on I_B concerning the preferences of B-bloc voters.

- **U** (uniform-uniform): preferences are uniform over $\mathcal{A}$ candidates and uniform over $\mathcal{B}$ candidates, so within each slate any two candidates have a $1:1$ ratio of support.
- **S** (strong-strong): preferences are strong over both slates, namely with one candidate receiving 10 times the support of all the others, who are equal.
- **X** (uniform-strong): uniform support for $\mathcal{A}$ candidates and 10:1:1 support for $\mathcal{B}$ candidates;
- **Y** (strong-uniform): the reverse.

In the multi-dimensional scaling (MDS) plot in Figure 1, the first-place preference for $\mathcal{B}$ candidates is $p_B = .7$; Supplemental Figure 37 shows how the outputs vary in p. In this plot, we can see some systematic differences and similarities.[4] For instance, strength scenarios Y and X interpolate between U and S, as we might have expected. Also, BT profiles resemble both kinds of Cambridge outputs more than PL profiles do, though the reason for this is far less clear.

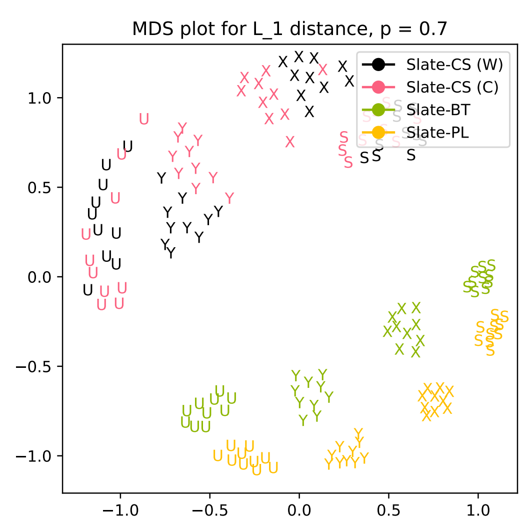

Fig. 1. Multi-dimensional scaling (MDS) plots for one-bloc profiles with $r = s = 3$ (3 candidates per slate), under a variety of generative models and candidate strength scenarios. Each model is designated by a different color, and the candidate strength scenarios are denoted U, S, X, Y, as described above. The pairwise distances between profiles are computed with L^1 distance on the distributions. Each preference profile has 1000 ballots, and we have generated 10 profiles by each of the 16 model/strength pairs.

[3]Here, we identify a ballot of length 5 with a complete ranking of length 6, since the last-place candidate is implicit.

[4]The reader should recall that MDS plots are simply low-distortion planar embeddings, which depend on a choice of random seed. The x and y axes have no meaning; only the relative pairwise distances are meaningful with respect to the data. We have verified that the structure of the plots stays the same for a few choices of random seed.

3.2.2 Election outcomes for two blocs. We begin our consideration of electoral outcomes with an analytical observation that the asymptotics of two-bloc elections interpolate between solid coalitions and unpolarized voting in an intuitive way. (See Figure 2.)

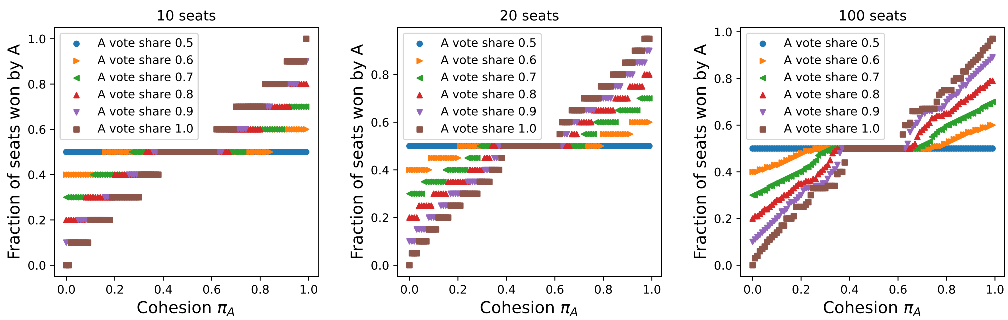

Fig. 2. Exact asymptotics (as the number of voters gets large) showing the share of seats won by the A bloc as their vote share and cohesion varies. The elections have $m = 10, 20, 100$ seats, with an inexhaustible supply of candidates. We use the Slate-PL model, suppose both blocs use the same fixed ordering over $\mathcal{A}$ and $\mathcal{B}$ and apply the one-by-one election variant of STV defined in §4.

One interesting (and real) artifact visible in these plots is that the outcome with seat share of 50% is a plateau that occurs for a range of cohesion values. To get an idea of the reason for this, note that since this plot assumes both blocs use a fixed candidate order $A_1, A_2, \dots$ and $B_1, B_2, \dots,$ the first candidate elected with $\pi_A, N_A > .5$ will always be A_1. For large numbers of seats, so that the election threshold is close to zero, there is a phase transition when $\pi^2 = (1-\pi) + \pi(1-\pi)$, occurring at $\pi = 1/\sqrt{2} \approx .707$, that determines whether the first transfer will result in the election of A_2 following A_1. For smaller π, enough support will transfer to B_1 that they are next to be elected. (Similar polynomial thresholds determine how many A candidates are elected between each B.) For π approaching $1/2$, the order of election will alternate $ABABAB\dots$, giving $1/2$ seat share to each side.

Next, we turn to simulation studies for the more complex cases that are not readily accessible to exact analysis. We have selected four candidate strength scenarios of the infinitely many that are possible for two blocs; these are chosen to give a small window on how candidate strength can interact with other factors.

- **UU** both blocs have uniform preference across both slates;
- **UX**: I_{BB} has a strong (10:1:1) candidate while others are uniform;
- **XX-same**: A and B blocs strongly prefer the same $\mathcal{B}$ candidate;
- **XX-diff**: A and B blocs strongly prefer different $\mathcal{B}$ candidates.

We vary N_B over $\{.1, .2, .3, .4\}$ and we vary both π_A and π_B. Figures 3 and 4 show a great deal of information about the tendency to proportionality, including an intriguing "winner's bonus" and some interesting asymmetries. In the present paper they serve to illustrate the richness of this approach for understanding interactive effects for STV under realistically complex conditions.

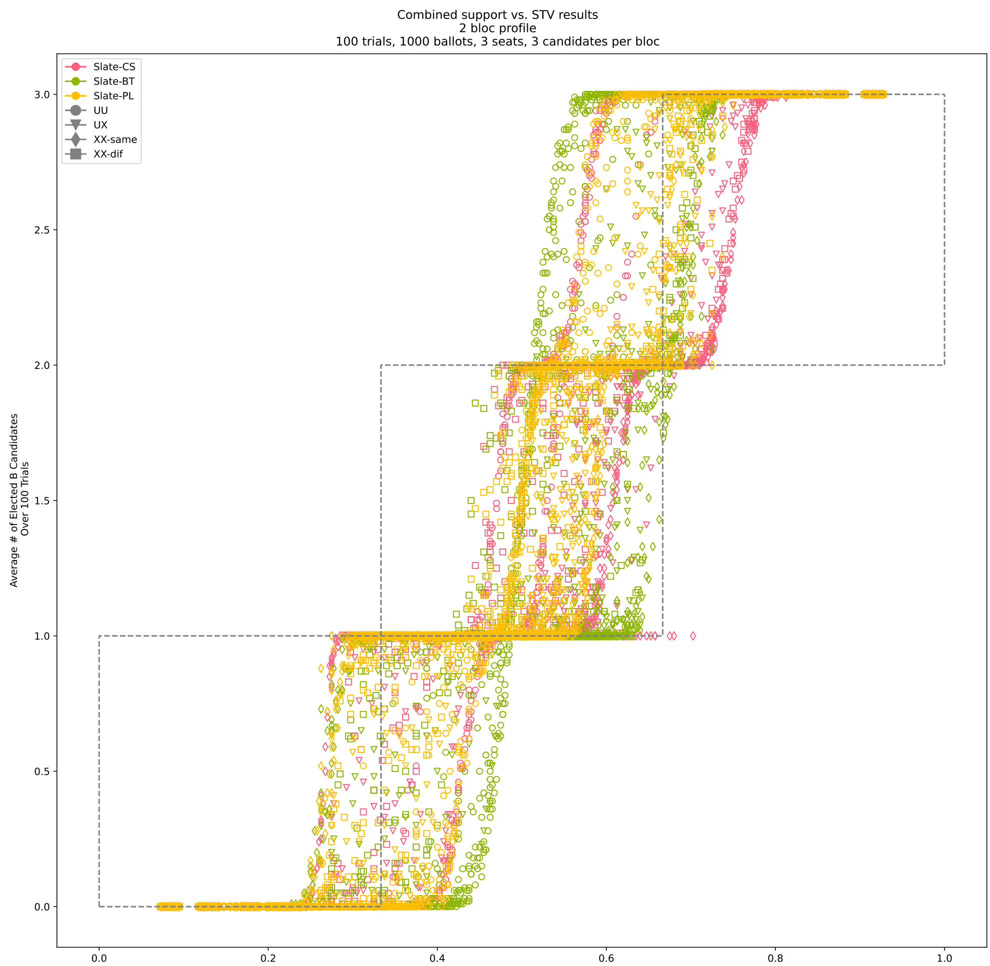

Fig. 3. Setting $(r, s, k) = (3, 3, 3)$, we independently vary the B proportion of the electorate, the generative model, the A and B cohesion, and the candidate strength settings. In this visualization, we have run 100 trials for each parameter tuple, recording the number of $\mathcal{B}$ candidates elected for each simulated profile. The x axis position is the combined support for $\mathcal{B}$ (with respect to first-place votes) and the y-axis position records the average number of seats over the trials with each tuple of parameters. The dashed lines show the proportionality target rounded up and down to the nearest whole number of seats.

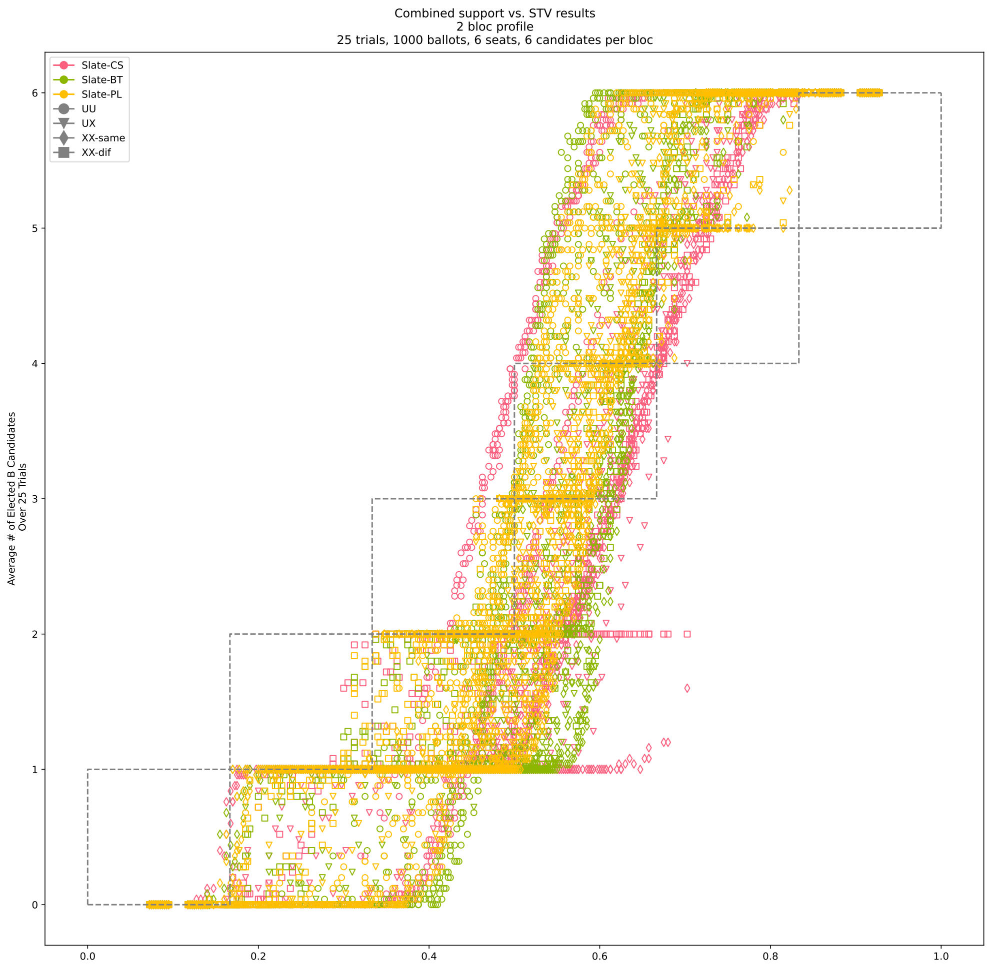

Fig. 4. This time $(r, s, k) = (6, 6, 6)$. We again independently vary the B proportion of the electorate, the generative model, the A and B cohesion, and the candidate strength settings. In this visualization, we have run 25 trials for each parameter tuple, recording the number of $\mathcal{B}$ candidates elected for each simulated profile. The x axis position is the combined support for $\mathcal{B}$ (with respect to first-place votes) and the y-axis position records the average number of seats over the trials with each tuple of parameters. The dashed lines show the proportionality target rounded up and down to the nearest whole number of seats.

4 PROVABLE PROPERTIES

In this section, we give proof of concept that the framework presented here is robust enough to admit provable statements about STV, a system of election for which theorems have so far been elusive.[5]

Within the context of the Slate-PL and Name-PL models, we can prove theoretical results that offer a kind of asymptotic generalization of the well-known Proportionality for Solid Coalitions (PSC). (We give asymptotics as the number of voters goes to infinity, since our models are probabilistic.) In this section, we focus on the case of one bloc of voters; the case for two competing blocs is much more difficult to describe analytically, so we rely on simulation experiments for our main results in that regime.

We start by giving bounds on the outcomes for a bloc voting under Slate-PL model. The results reveal that the choice of precise method for tallying votes has a profound impact on the expected outcomes. With that in mind, we define two different methods for deciding which candidates are elected in each round of an STV vote tallying process.

- **Simultaneous election:** if multiple candidates exceed the threshold for election in a certain round, they are all elected and their excess votes transfer down to the remaining candidates before the next round.
- **One-by-one election:** if multiple candidates exceed the threshold for election in a certain round, the one with the most votes is elected and their excess votes are transferred. The tallying process then proceeds to the next round.

Based on the way that election results are reported by the city of Cambridge, we conclude that Cambridge follows the simultaneous election method.[6]

Proposition 4.1 (Slate-PL with simultaneous election). *Consider an election for k open seats, a single bloc of N voters, and two slates of candidates $\mathcal{A}$ and $\mathcal{B}$. Suppose that the voters vote according to a Slate-PL model with cohesion parameter $0.5 < \pi_A \leq 1$, and all voters rank the candidates within each slate in the same order. Suppose also that the number of candidates in each slate is more than k, and the votes are tallied using simultaneous election.*

(a) *For all $\varepsilon > 0$, the number of candidates elected from slate $\mathcal{B}$ is bounded below by $\lfloor (1-\pi_A)(k+1) - \varepsilon \rfloor$ and above by $\lfloor k/2 \rfloor$ asymptotically almost surely as $N \to \infty$.*
(b) *Suppose $\pi_A < 1$. As $k \to \infty$, the fraction of elected candidates which are from slate $\mathcal{B}$ (asymptotically almost surely as $N \to \infty$) tends to $1/2$.*

Note that the lower bound in (a) is precisely the number of thresholds exceeded by the first-place votes for slate $\mathcal{B}$. See Figure 5 for an empirical demonstration of these results.

Proof. We first derive the lower bound in (a). At any stage during the vote tallying process, let ω_A (resp. ω_B) denote the fraction of the original N ballots which have both not been discarded yet, and whose top vote is from $\mathcal{A}$ (resp. $\mathcal{B}$). Since we are concerned with results as $N \to \infty$, we may assume that these fractions are, up to an arbitrarily small error, deterministic quantities at each stage of the vote tallying process.

Note that the top candidate of any ballot is determined by its slate, since the ranking of all candidates within a slate is the same across ballots. This means that only one candidate from $\mathcal{A}$ and one candidate from $\mathcal{B}$ receive first-place votes at a time. Let $t = 1/(k+1)$ denote the Droop quota as a fraction of total votes. If $\omega_A > t$ and $\omega_B > t$, then one candidate is elected from each slate.

[5]The assumption of solid coalitions, in particular, assumes away any role for transfer between blocs.
[6]See for instance https://www.cambridgema.gov/Election2023/Official/Council%20Round.htm

After discarding votes and fractional transfers, the fractions ω_A and ω_B are updated as follows

$$\omega'_A = (\omega_A + \omega_B - 2t)\pi_A$$
$$\omega'_B = (\omega_A + \omega_B - 2t)(1 - \pi_A)$$

Thus, the ratio ω_A/ω_B returns to $\pi_A/(1-\pi_A)$ immediately after two candidates (one from each slate) are elected in a singe round. If $(1-\pi_A) \leq t = 1/(k+1)$, then the bound holds trivially, so suppose $(1-\pi_A) > t$. Two candidates are elected in round i if $(1-\pi_A)(1-2t(i-1)) > t$. Setting $i = (1-\pi_A)(k+1) + \varepsilon$, we obtain

$$\begin{aligned}(1-\pi_A)(1-2t(i-1)) &= (1-\pi_A) - 2t(i-1)(1-\pi_A) \\ &\geq (1-\pi_A) - t(i-1) \qquad (\text{since } \pi_A \geq 0.5) \\ &> t\end{aligned}$$

so at least $\lfloor(1-\pi_A)(k+1)\rfloor$ candidates from slate $\mathcal{B}$ are elected. This proves the lower bound.

For the upper bound, note that if we start with $\omega_A > \omega_B$, then this inequality is maintained except for after a round where only an $\mathcal{A}$ candidate is elected. Following such a round, a single $\mathcal{B}$ candidate can be elected, after which the inequality $\omega_A > \omega_B$ is restored. Thus the only rounds in which a single $\mathcal{B}$ candidate can be elected are directly after rounds where a single $\mathcal{A}$ candidate is elected. It follows that at least as many $\mathcal{A}$ candidates as $\mathcal{B}$ candidates are elected.

To prove (b), note that the first election of single $\mathcal{A}$ candidate (rather than the simultaneous election of and $\mathcal{A}$ and $\mathcal{B}$ candidate) takes place when $(1-\pi_A)(\omega_A + \omega_B) < t$. Thus the number of candidates which can still be elected is at most

$$\begin{aligned}(\omega_A + \omega_B)/t &= (1-\pi_A)\left(1 + \frac{\pi_A}{1-\pi_A}\right)(\omega_A + \omega_B)/t \\ &< 1 + \frac{\pi_A}{1-\pi_A}.\end{aligned}$$

As $t \longrightarrow \infty$, the ratio of this quantity to k goes to zero, which gives the required result. □

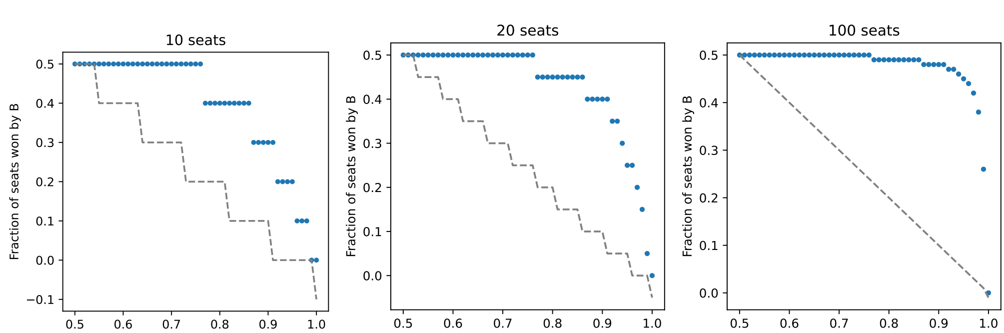

Fig. 5. A visualization of the lower bound and limit behavior described in Proposition 4.1. Results are asymptotic as $N \longrightarrow \infty$ for one bloc of voters voting with cohesion parameter π_A.

It is somewhat surprising that, as $k \longrightarrow \infty$, $\mathcal{A}$ and $\mathcal{B}$ are equally represented even though all voters are in bloc A. Proposition 4.1 assumes simultaneous election transfers—this, together with the fact that there are fixed rankings over $\mathcal{A}, \mathcal{B}$, creates a situation where in nearly every round all first-place votes land on the top remaining $\mathcal{A}$ and $\mathcal{B}$ candidates, and both are elected.

We now consider the one-by-one vote tallying method. A practical difference between the simultaneous and one-by-one elections is that one-by-one election may exhibit a kind of leap-frogging, where a candidate who is over the threshold in round 1 may nonetheless be elected after a candidate who was below the threshold in round 1. This can not happen in simultaneous elections.

Proposition 4.2 (Slate-PL with one-by-one election). *Consider an election for k open seats, a single bloc of N voters, and two slates of candidates $\mathcal{A}$ and $\mathcal{B}$. Suppose that the voters vote according to a Slate-PL model with cohesion parameter $0.5 < \pi_A \leq 1$, and all voters rank the candidates within each slate in the same order. Suppose also that the number of candidates in each slate is more than k, and the votes are tallied using one-by-one election.*

Then, as $k \to \infty$, the fraction of candidates elected from $\mathcal{A}$ is lower bounded by

$$1 - 1/\lceil \log_{\pi_A}(1/2)) \rceil$$

and upper bounded by

$$1 - 1/(1 + \lceil \log_{\pi_A}(1/2)) \rceil)$$

asymptotically almost surely as $N \to \infty$. See Figure 6 for an visualization of this result.

Proof. Let ω_A^z (ω_B^z) denote the fraction of ballots which have not been discarded, whose top vote is from $\mathcal{A}$ ($\mathcal{B}$, respectively) at the start of round z.

Since $\pi_A > 0.5$, $\omega_A^1 = \pi_A > \omega_B^1 = 1 - \pi_A$ and an $\mathcal{A}$ candidate is elected in the first round. We have the following update:

$$\omega_A^2 = (\omega_A^1 - t)\pi_A = \pi_A^2 - \pi_A t$$
$$\omega_B^2 = \omega_B^1 + (\omega_A^1 - t)(1 - \pi_A)$$

If an $\mathcal{A}$ candidate is elected again in the second round, then

$$\omega_A^3 = (\omega_A^2 - t)\pi_A = ((\pi_A^2 - \pi_A t) - t)\pi_A = \pi_A^3 - \pi_A^2 t - \pi_A t$$
$$\omega_B^3 = \omega_B^2 + (\omega_A^2 - t)(1 - \pi_A)$$

Starting from the first round of the election, suppose $s - 1$ $\mathcal{A}$ candidates were elected thusfar. Now an $\mathcal{A}$ candidate is elected in seat s if $\pi_A^s \geq 1 - \pi_A^s - (s-1)t$. Letting $k \to \infty$ (so $t \to 0$), an $\mathcal{A}$ candidate is elected for the s-th seat if $\pi_A^s \geq 1 - \pi_A^s$, or $\pi_A^s \geq 1/2$, otherwise a $\mathcal{B}$ candidate is elected. It follows that $\lfloor s^* \rfloor$ candidates from $\mathcal{A}$ are elected consecutively at the start of the election, with $s^* = \log(1/2)/\log(\pi_A)$ satisfying $\pi_A^{s^*} = 1 - \pi_A^{s^*}$, followed by the election of the first $\mathcal{B}$ candidate. At this stage the fraction of $\mathcal{A}$ candidates elected is $1 - 1/(\lfloor s^* \rfloor + 1)$.

Let a *sequence* of rounds consist of the time between the elections of $\mathcal{B}$ candidates. The first sequence starts at the beginning of the election process and ends with the election of the first $\mathcal{B}$ candidate in seat $\lfloor s^* \rfloor + 1$. The second sequence starts from seat $\lfloor s^* \rfloor + 2$ and ends when a $\mathcal{B}$ candidate is next elected, etc.

Notice that at the start of the first sequence $\mathcal{A}$'s fraction of the overall first-place votes were π_A. At the start of any subsequent sequence the most recent update from electing a $\mathcal{B}$ candidate is

$$\omega_A' = \omega_A + (\omega_B - t)\pi_A$$
$$\omega_B' = (\omega_B - t)(1 - \pi_A)$$

from which it follows that $\mathcal{A}$'s share of the first-place votes is at least π_A. As a result, each sequence will elect at least as many $\mathcal{A}$ candidates as the first, from which we conclude that $\mathcal{A}$'s fraction of seats is at least $1 - 1/(\lfloor s^* \rfloor + 1)$.

For the upper bound, observe that if initially $\omega_A^0 = 1$, then after electing the first $\mathcal{A}$ candidate and updating, $\omega_A^1 = \pi_A$ as in our starting condition. Other words, when a sequence starts with $\omega_A = \pi_A$

it elects $\lfloor s^* \rfloor$ $\mathcal{A}$ candidates before the first $\mathcal{B}$ candidate, and if it starts with $\omega_A = 1$ it elects $\lfloor s^* \rfloor + 1$. At the start of every sequence, after electing a $\mathcal{B}$ candidate, $\pi_A \leq \omega_A \leq 1$, from which we conclude that at most $\lfloor s^* \rfloor + 1$ $\mathcal{A}$ candidates are elected for every $\mathcal{B}$ candidate. The bound follows. □

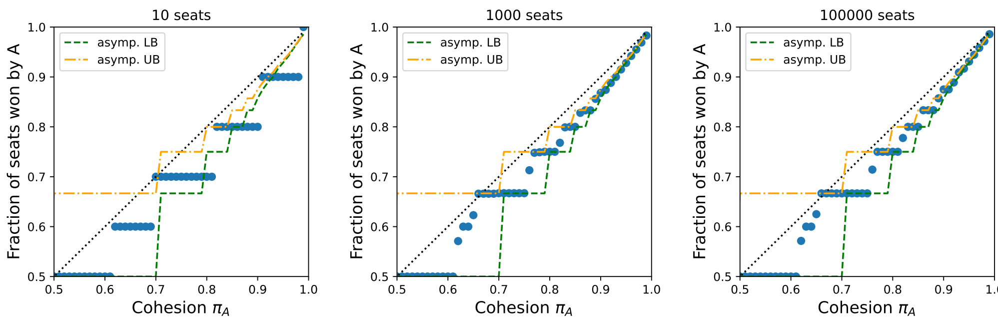

Fig. 6. Visualizations demonstrating the limit behavior described in Proposition 4.2 for $k = 10, 1000, 100000$. The theoretical bounds hold in the limit as $k \to \infty$ for $\pi_A > 0.5$. The dotted line is $y = \pi_A$, which is also A's combined share since there is no bloc B.

Proposition 4.3 (Name-PL with one-by-one election). *When one bloc votes by Name-PL, asymptotic results are easy to describe for extreme candidate strength scenarios, assuming there are more candidates in each slate than seats open, and equal numbers of candidates in each slate. For ballots generated by a Plackett-Luce model, the STV winners are (a.a.s.) the top candidates by support value. Thus we obtain the following results a.a.s. as $N \to \infty$.*

(a) *If $a_1 \gg a_2 \gg \ldots$ and $b_1 \gg b_2 \gg \ldots$, then equal numbers of candidates are elected from both slates if there are an even number of seats open. If there are an odd number of seats open and $\pi_A > 0.5$, then one more $\mathcal{A}$ candidate is elected than $\mathcal{B}$ candidates.*
(b) *If the support is uniform and $\pi_A > 0.5$, then only $\mathcal{A}$ candidates are elected.*

5 CONCLUSION AND FUTURE WORK

A natural next step would be to develop methods to fit our models to historical election data, first by choosing a model that best describes observed behavior and then by obtaining parameter estimates. In [Gormley and Murphy, 2008] the authors fit mixtures of Plackett-Luce models to cast vote records from Irish elections, with the main goal of identifying blocs within the electorate. This roughly corresponds to fitting a Name-PL model (see Remark 4) with unknown group sizes and no slate structure. That is, their method is designed to learn preferences for all candidates by each of two blocs. Fitting a mixture model in this way does not produce a canonical division of candidates into slates, and fitting slate models may prove challenging.[7]

Our particular aim is to lay the groundwork to systematically study the tendency of systems to deliver more or less proportional outcomes for voters. Crucially, the framework allows but does not require party labels, so that we can also consider emergent blocs with similar voting behavior. Besides opening up rich directions for mathematical study, applications will include better-informed decisionmaking by democracy stakeholders when considering practical electoral reform.

[7]It may also be necessary in some cases to estimate cohesion parameters for models where the bloc sizes are fixed, either at the level of the entire electorate or precinct-by-precinct. For plurality voting, the standard technique for estimating cohesion levels is known as ecological inference (or EI) [King, 2013]. A generalization might be required for ranking models.

REFERENCES

Elliot Anshelevich, Onkar Bhardwaj, Edith Elkind, John Postl, and Piotr Skowron. 2018. Approximating optimal social choice under metric preferences. *Artificial Intelligence* 264 (2018), 27–51.

Haris Aziz, Markus Brill, Vincent Conitzer, Edith Elkind, Rupert Freeman, and Toby Walsh. 2017. Justified representation in approval-based committee voting. *Social Choice and Welfare* 48, 2 (2017), 461–485.

William Benter. 2008. Computer based horse race handicapping and wagering systems: a report. In *Efficiency of racetrack betting markets*. World Scientific, 183–198.

Ralph Allan Bradley and Milton E. Terry. 1952. Rank Analysis of Incomplete Block Designs: I. The Method of Paired Comparisons. *Biometrika* 39, 3/4 (1952), 324–345. http://www.jstor.org/stable/2334029

Eric T Bradlow and Peter S Fader. 2001. A Bayesian Lifetime Model for the "Hot 100" Billboard Songs. *J. Amer. Statist. Assoc.* 96, 454 (2001), 368–381. https://doi.org/10.1198/016214501753168091

Barry C. Burden. 1997. Deterministic and Probabilistic Voting Models. *American Journal of Political Science* 41, 4 (1997), 1150–1169. http://www.jstor.org/stable/2960485

Joshua Clinton, Simon Jackman, and Douglas Rivers. 2004. The statistical analysis of roll call data. *American Political Science Review* (2004), 355–370.

Edith Elkind, Piotr Faliszewski, Jean-François Laslier, Piotr Skowron, Arkadii Slinko, and Nimrod Talmon. 2017. What do multiwinner voting rules do? An experiment over the two-dimensional euclidean domain. In *Proceedings of the AAAI Conference on Artificial Intelligence*, Vol. 31.

James M Enelow and Melvin J Hinich. 1984. *The spatial theory of voting: An introduction*. CUP Archive.

Nikhil Garg, Wes Gurnee, David Rothschild, and David Shmoys. 2022. Combatting gerrymandering with social choice: The design of multi-member districts. In *Proceedings of the 23rd ACM Conference on Economics and Computation*. 560–561.

Vasilis Gkatzelis, Daniel Halpern, and Nisarg Shah. 2020. Resolving the optimal metric distortion conjecture. In *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE, 1427–1438.

Isobel Claire Gormley and Thomas Brendan Murphy. 2007. A Latent Space Model for Rank Data. In *Statistical Network Analysis: Models, Issues, and New Directions*, Edoardo Airoldi, David M. Blei, Stephen E. Fienberg, Anna Goldenberg, Eric P. Xing, and Alice X. Zheng (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 90–102.

Isobel Claire Gormley and Thomas Brendan Murphy. 2008. Exploring voting blocs within the Irish electorate: A mixture modeling approach. *J. Amer. Statist. Assoc.* 103, 483 (2008), 1014–1027.

Todd Graves, C Shane Reese, and Mark Fitzgerald. 2003. Hierarchical models for permutations: Analysis of auto racing results. *J. Amer. Statist. Assoc.* 98, 462 (2003), 282–291.

Valen E Johnson, Robert O Deaner, and Carel P Van Schaik. 2002. Bayesian analysis of rank data with application to primate intelligence experiments. *J. Amer. Statist. Assoc.* 97, 457 (2002), 8–17.

D Marc Kilgour, Jean-Charles Grégoire, and Angèle M Foley. 2020. The prevalence and consequences of ballot truncation in ranked-choice elections. *Public Choice* 184 (2020), 197–218.

Gary King. 2013. *A solution to the ecological inference problem: Reconstructing individual behavior from aggregate data*. Princeton University Press.

Martin Lackner and Piotr Skowron. 2022. Approval-based committee voting. In *Multi-Winner Voting with Approval Preferences*. Springer.

Robin L Plackett. 1975. The analysis of permutations. *Journal of the Royal Statistical Society: Series C (Applied Statistics)* 24, 2 (1975), 193–202.

Keith T Poole and Howard Rosenthal. 1985. A spatial model for legislative roll call analysis. *American Journal of Political Science* (1985), 357–384.

Geoffrey Pritchard and Mark C Wilson. 2009. Asymptotics of the minimum manipulating coalition size for positional voting rules under impartial culture behaviour. *Mathematical Social Sciences* 58, 1 (2009), 35–57.

Ariel D Procaccia and Jeffrey S Rosenschein. 2006. The distortion of cardinal preferences in voting. In *International Workshop on Cooperative Information Agents*. Springer, 317–331.

Hal Stern. 1990. Models for distributions on permutations. *J. Amer. Statist. Assoc.* 85, 410 (1990), 558–564.

Stanisław Szufa, Piotr Faliszewski, Łukasz Janeczko, Martin Lackner, Arkadii Slinko, Krzysztof Sornat, and Nimrod Talmon. 2022. How to Sample Approval Elections? *arXiv preprint arXiv:2207.01140* (2022).

Stanisław Szufa, Piotr Faliszewski, Piotr Skowron, Arkadii Slinko, and Nimrod Talmon. 2020. Drawing a map of elections in the space of statistical cultures. In *Proceedings of the 19th International Conference on Autonomous Agents and Multiagent Systems*. 1341–1349.

Louis L Thurstone. 1927. A law of comparative judgment. *Psychological review* 34, 4 (1927), 273.

T Nicolaus Tideman and Florenz Plassmann. 2010. The structure of the election-generating universe. (2010). Manuscript.

GJG Upton and D Brook. 1975. The determination of the optimum position on a ballot paper. *Journal of the Royal Statistical Society: Series C (Applied Statistics)* 24, 3 (1975), 279–287.

A ONE-BLOC PROFILES

A.1 Attributes of profile, split by candidate pool and strength scenario

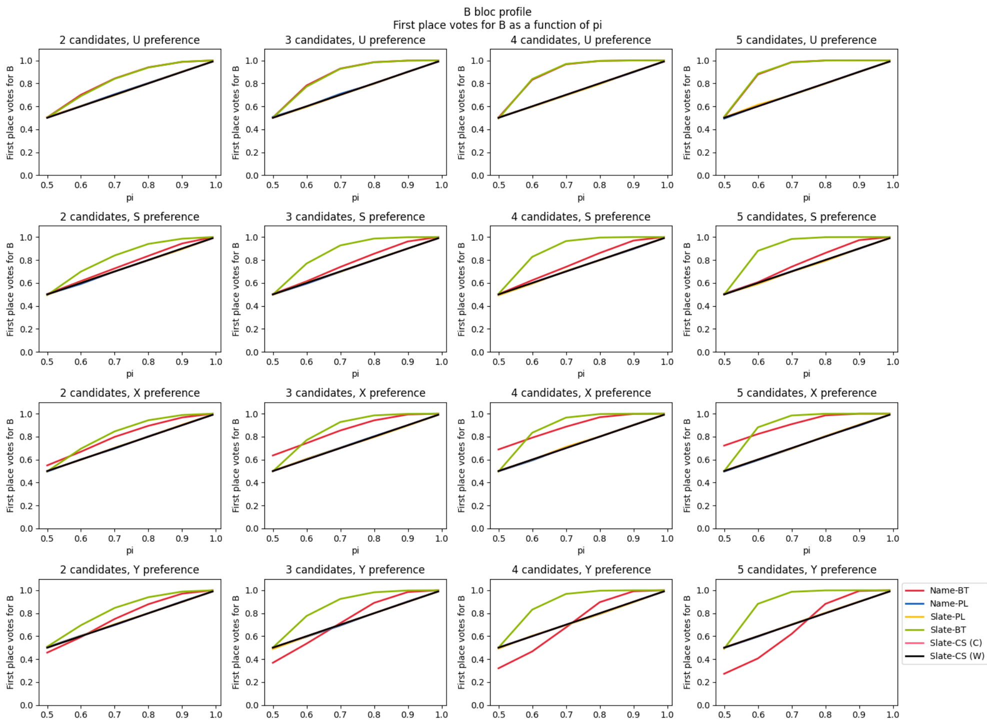

Fig. 7. The proportion of first-place votes for $\mathcal{B}$ candidates. Shown across different generative models, numbers of candidates, and strength scenarios. Notice that Slate-BT and Name-BT are the only two models for which first-place support may differ in expectation from the model parameter π.

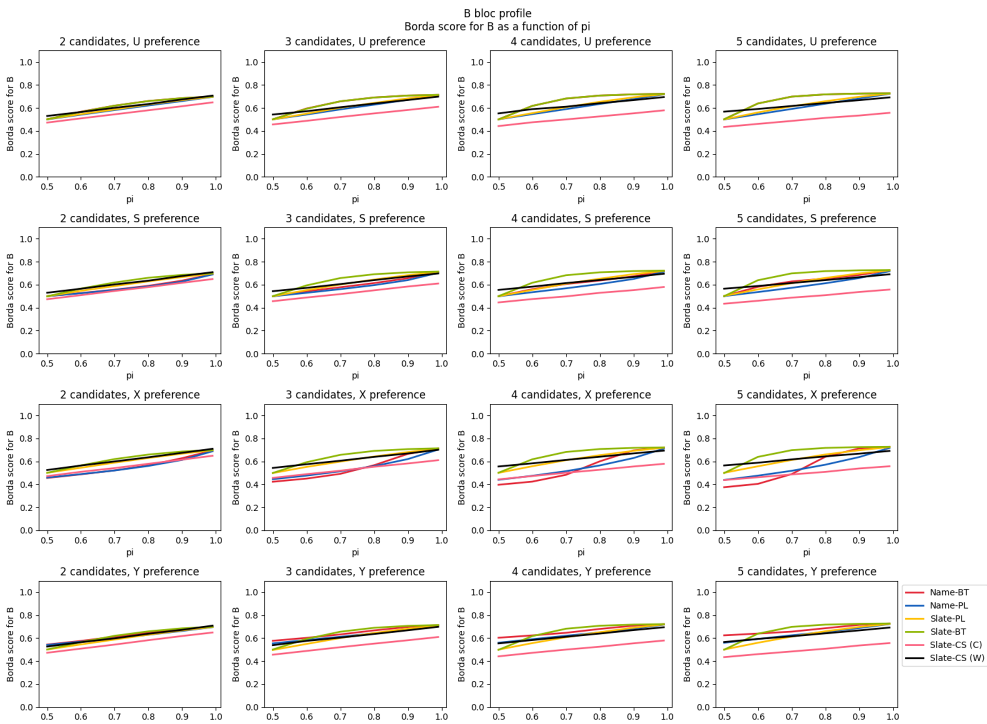

Fig. 8. The proportion of Borda points for $\mathcal{B}$ candidates. Shown across different generative models, numbers of candidates, and strength scenarios.

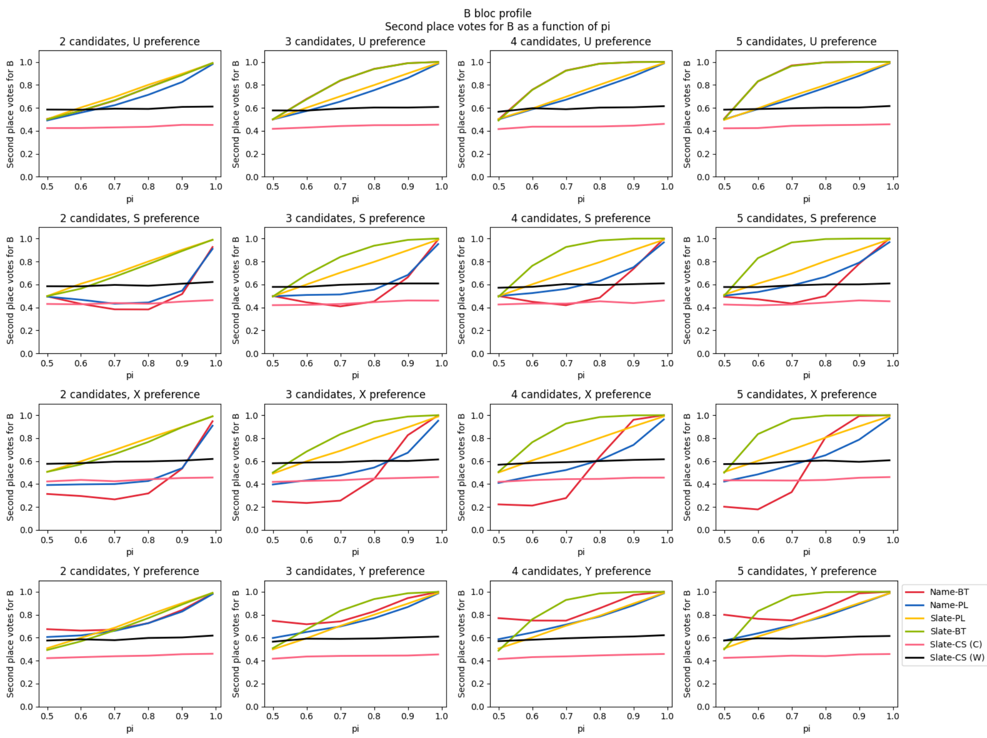

Fig. 9. The proportion of second-place votes for $\mathcal{B}$ candidates. Shown across different generative models, numbers of candidates, and strength scenarios. Notice that in the by-name models, the probability of ranking your own bloc's candidate second can actually be less than 50%, even in cases of high cohesion, if your slate has a strong candidate. (We regard this as evidence that the Slate models are more realistic, but others may hold different views.)

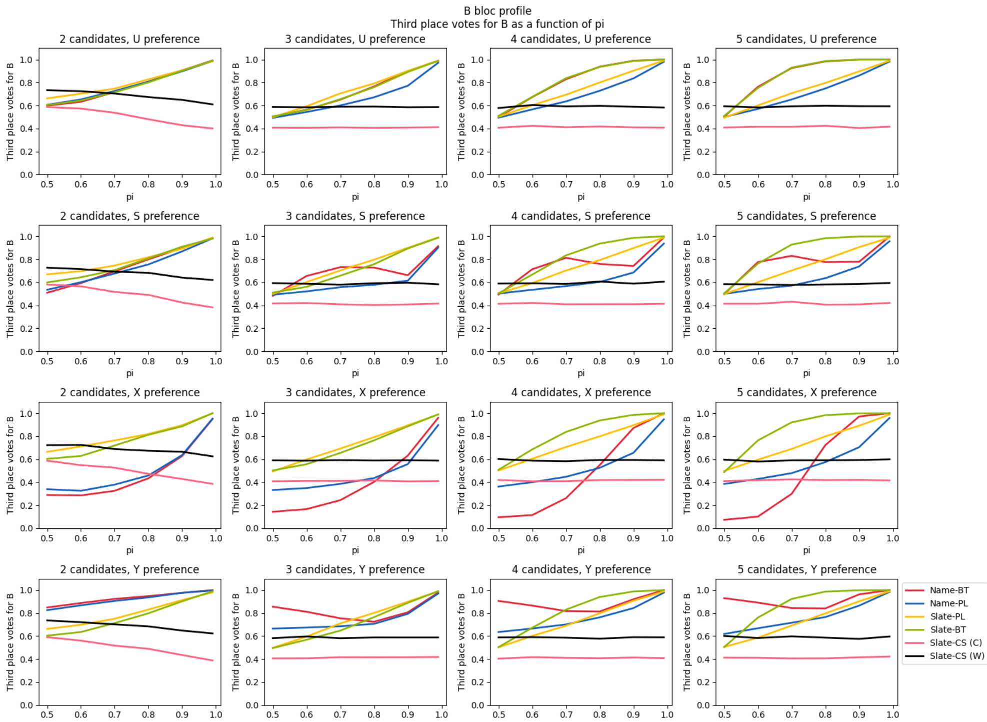

Fig. 10. The proportion of third-place votes for $\mathcal{B}$ candidates. Shown across different generative models, numbers of candidates, and strength scenarios.

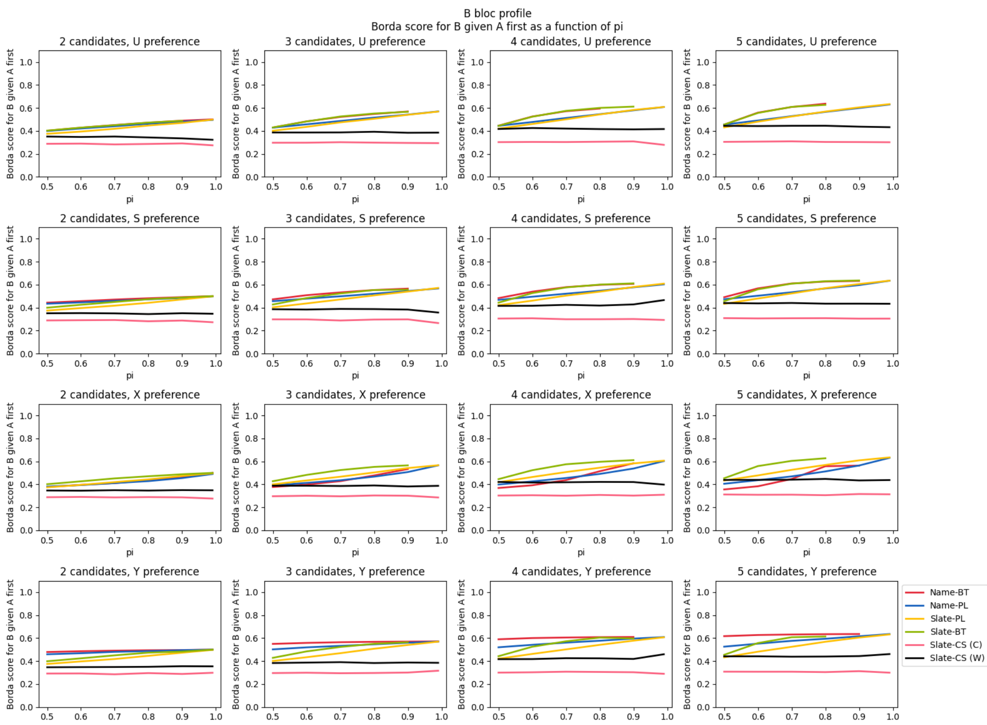

Fig. 11. The proportion of Borda points for $\mathcal{B}$ candidates given that a ballot started with an $\mathcal{A}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

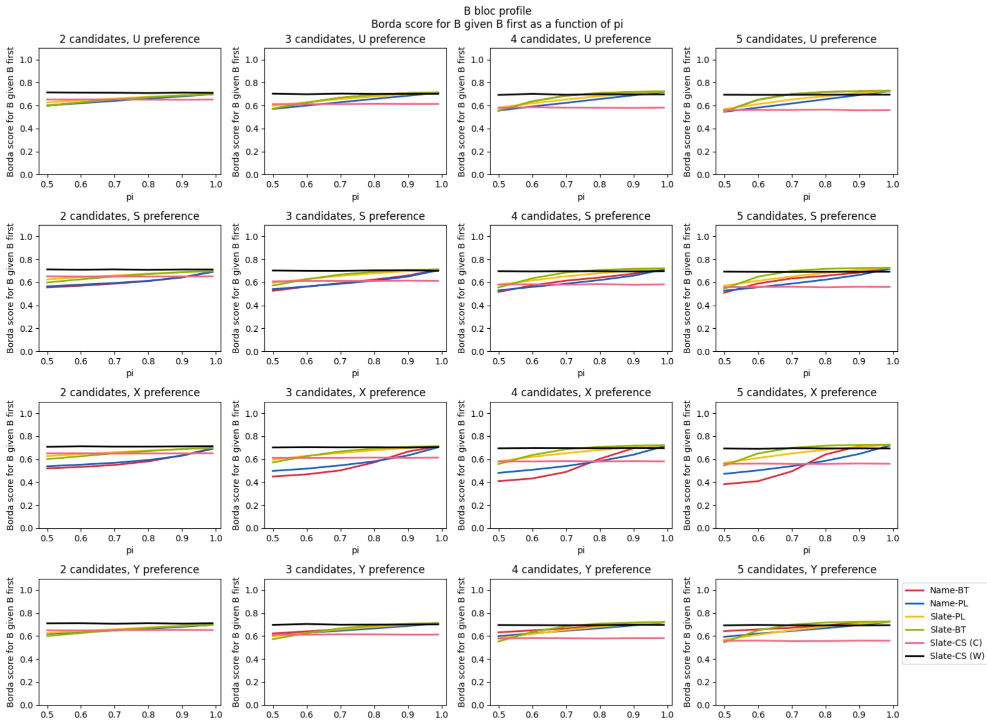

Fig. 12. The proportion of Borda points for $\mathcal{B}$ candidates given that a ballot started with a $\mathcal{B}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

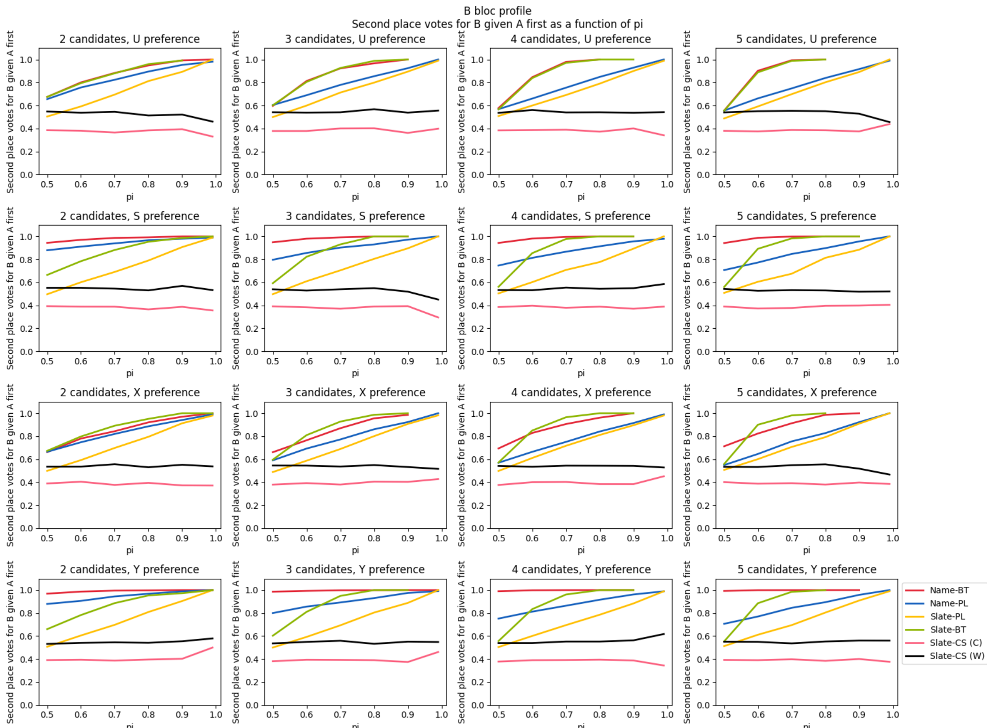

Fig. 13. The proportion of second-place votes for $\mathcal{B}$ candidates given that a ballot started with an $\mathcal{A}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

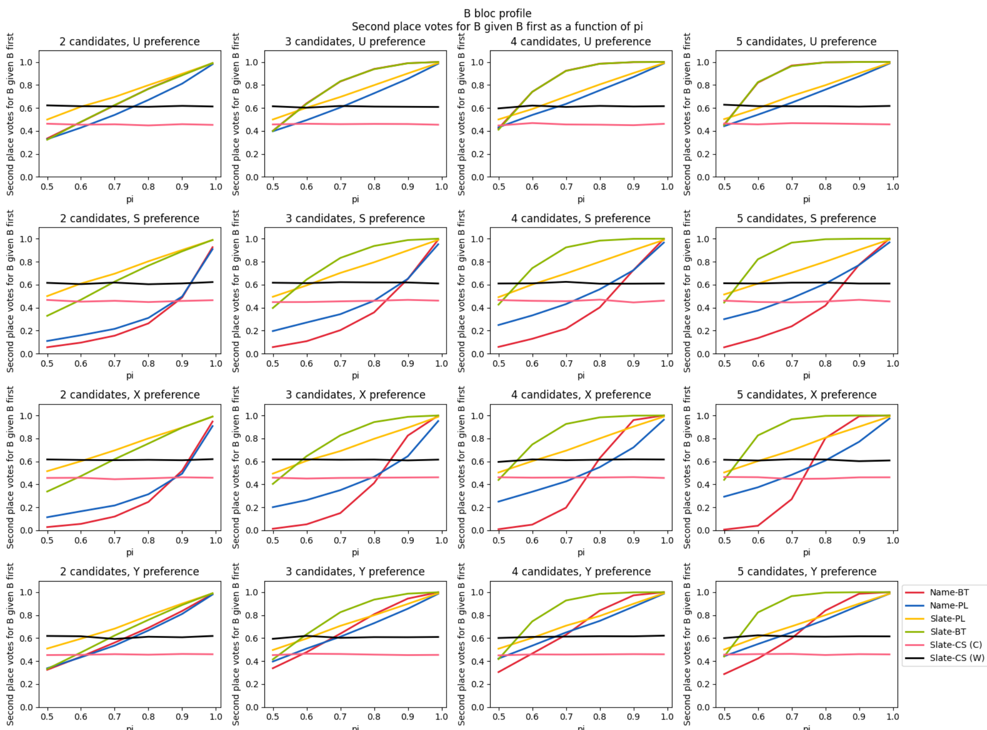

Fig. 14. The proportion of second-place votes for $\mathcal{B}$ candidates given that a ballot started with a $\mathcal{B}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

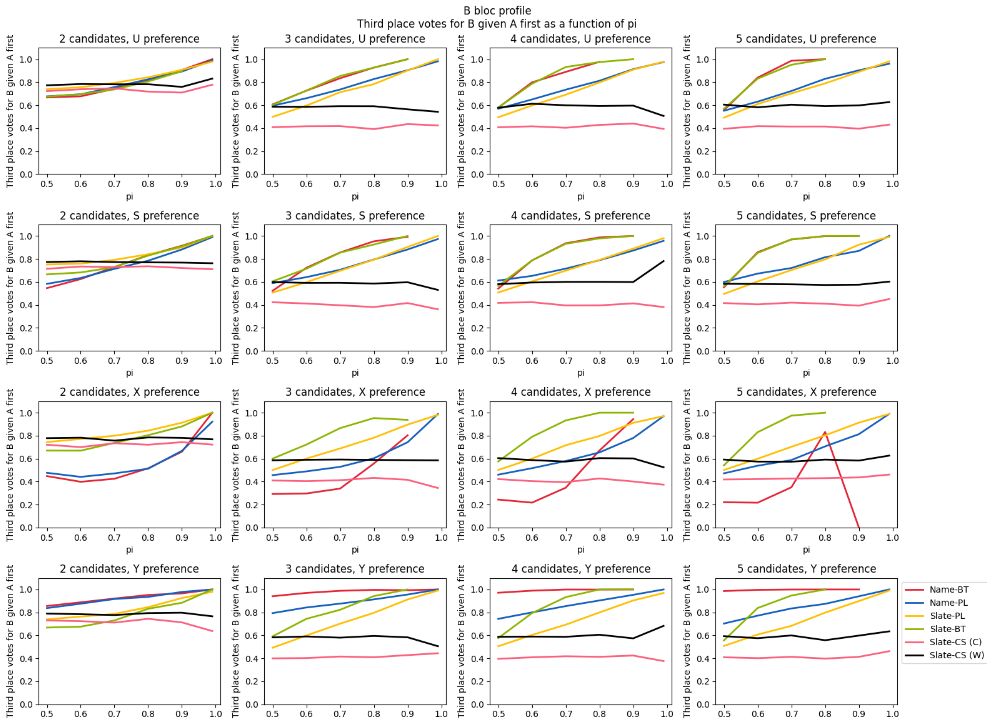

Fig. 15. The proportion of third-place votes for $\mathcal{B}$ candidates given that a ballot started with an $\mathcal{A}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

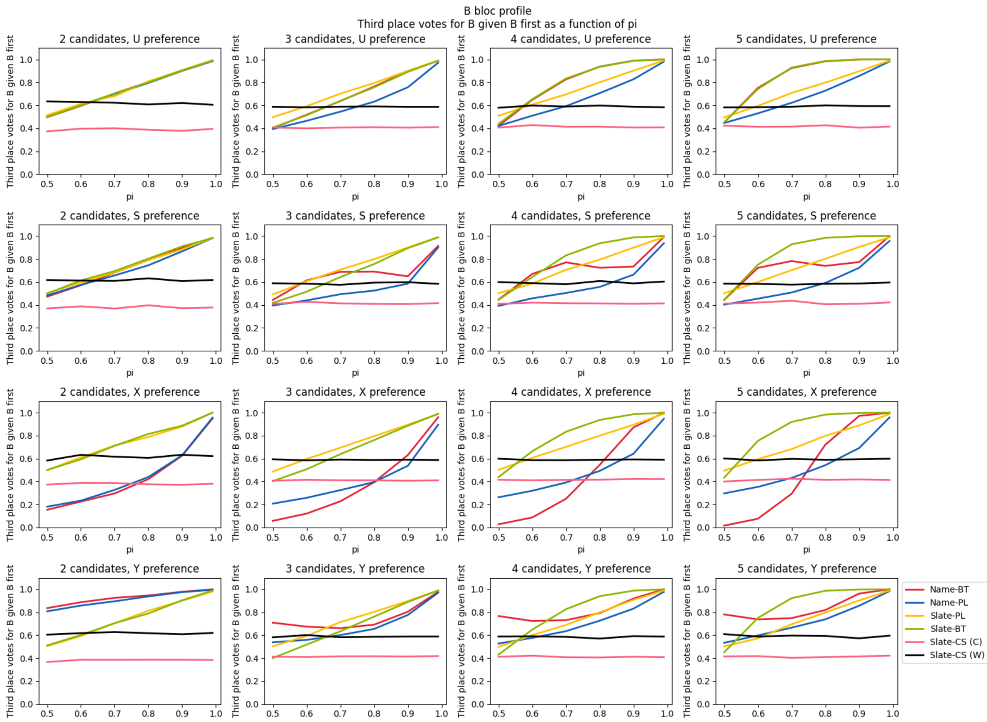

Fig. 16. The proportion of third-place votes for $\mathcal{B}$ candidates given that a ballot started with a $\mathcal{B}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

A.2 Attributes of profile, split by candidate pool and generative model

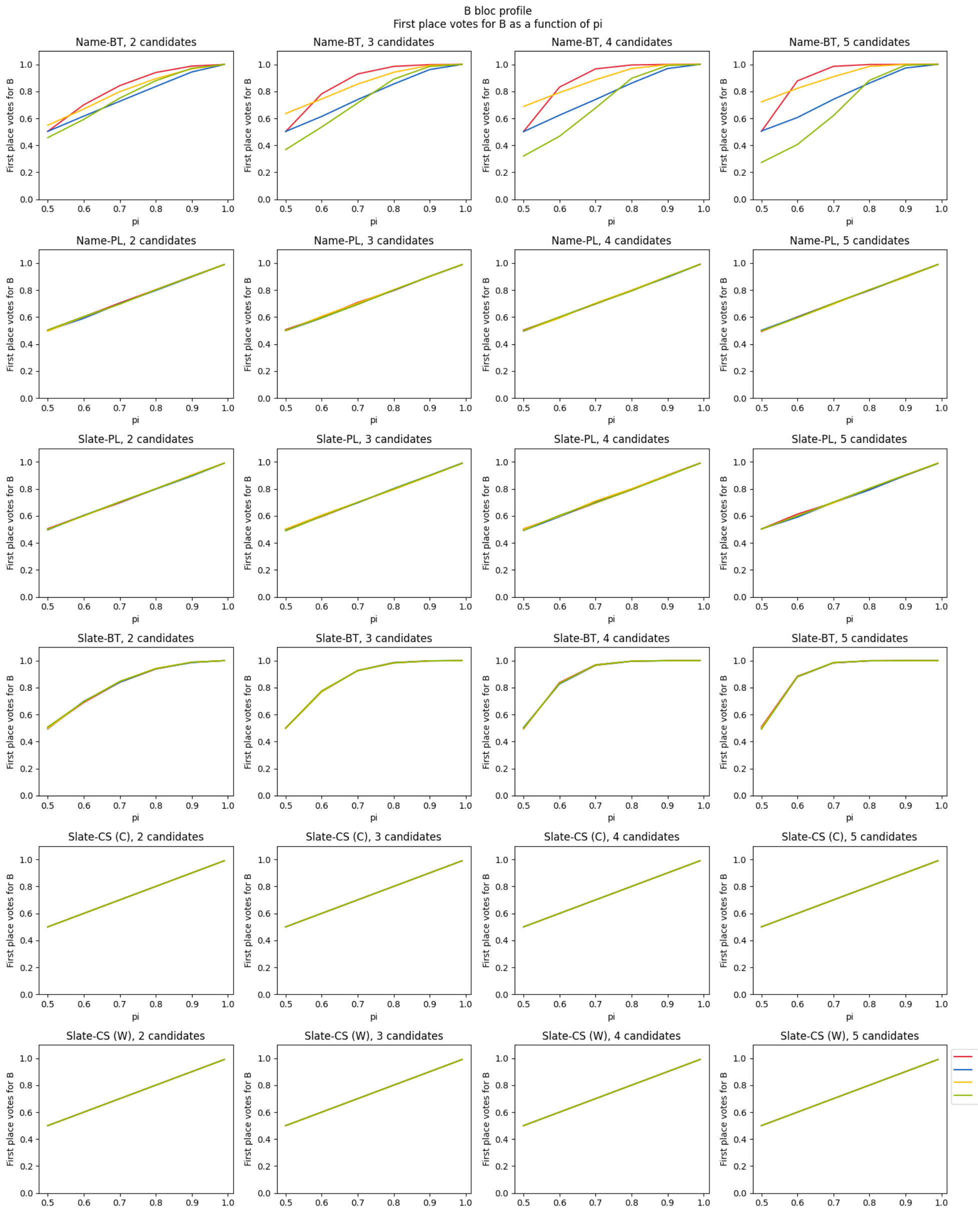

Fig. 17. The proportion of first-place votes for $\mathcal{B}$ candidates. Shown across different generative models, numbers of candidates, and strength scenarios.

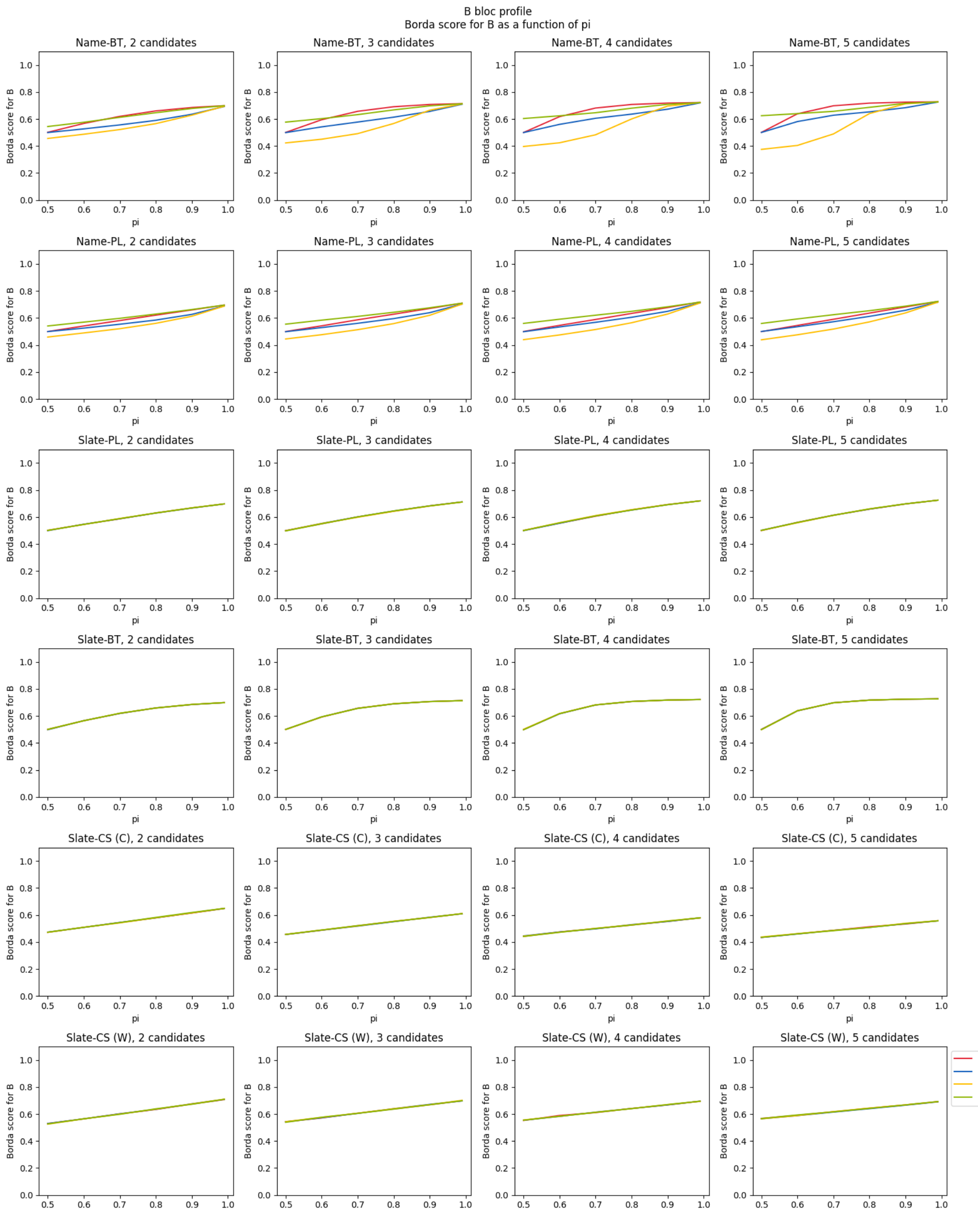

Fig. 18. The proportion of Borda points for $\mathcal{B}$ candidates. Shown across different generative models, numbers of candidates, and strength scenarios.

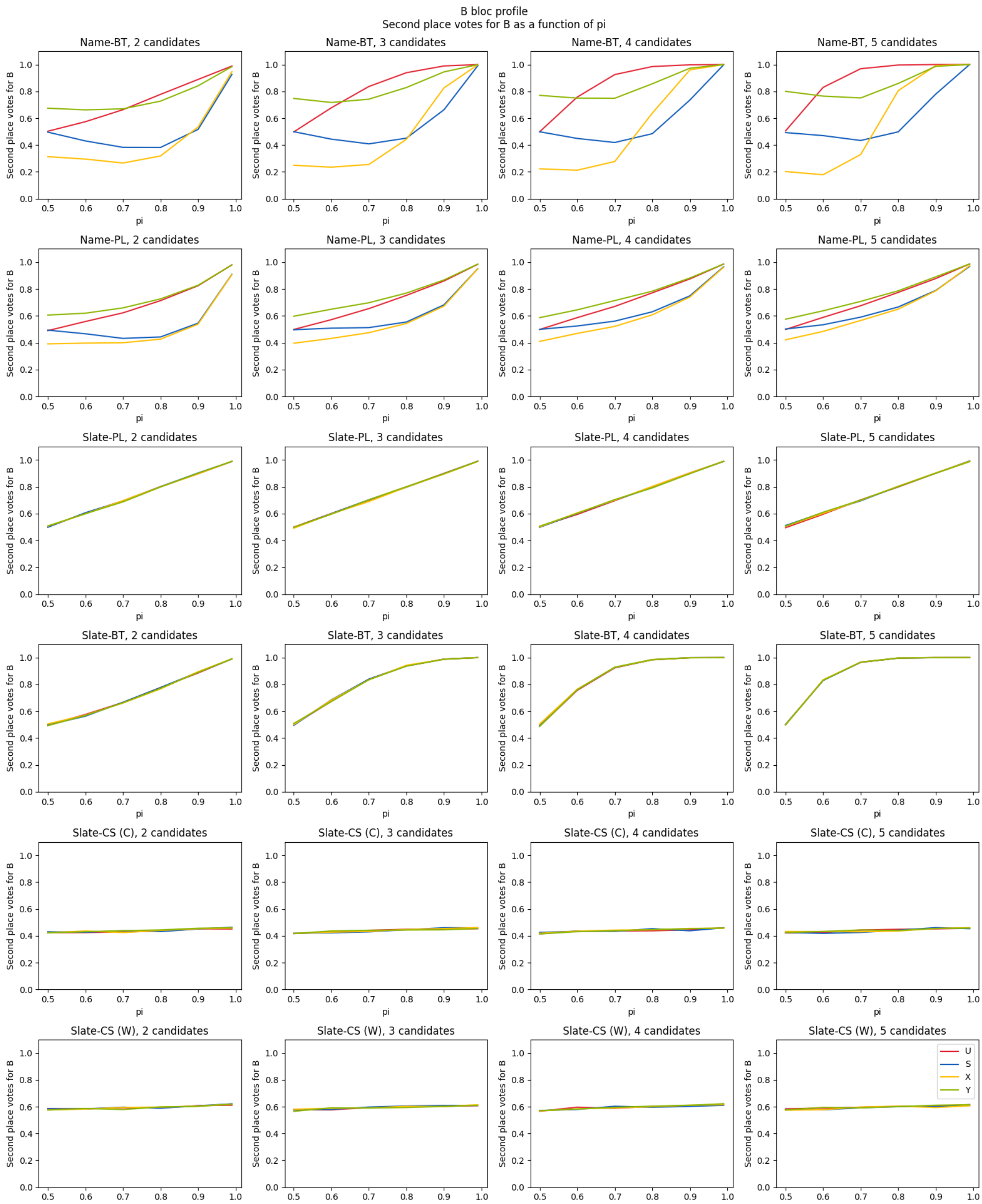

Fig. 19. The proportion of second-place votes for $\mathcal{B}$ candidates. Shown across different generative models, numbers of candidates, and strength scenarios. Notice that in the by-name models, the probability of ranking your own bloc's candidate second can actually be less than 50%, even in cases of high cohesion, if your slate has a strong candidate. (We regard this as evidence that the Slate models are more realistic, but others may hold different views.)

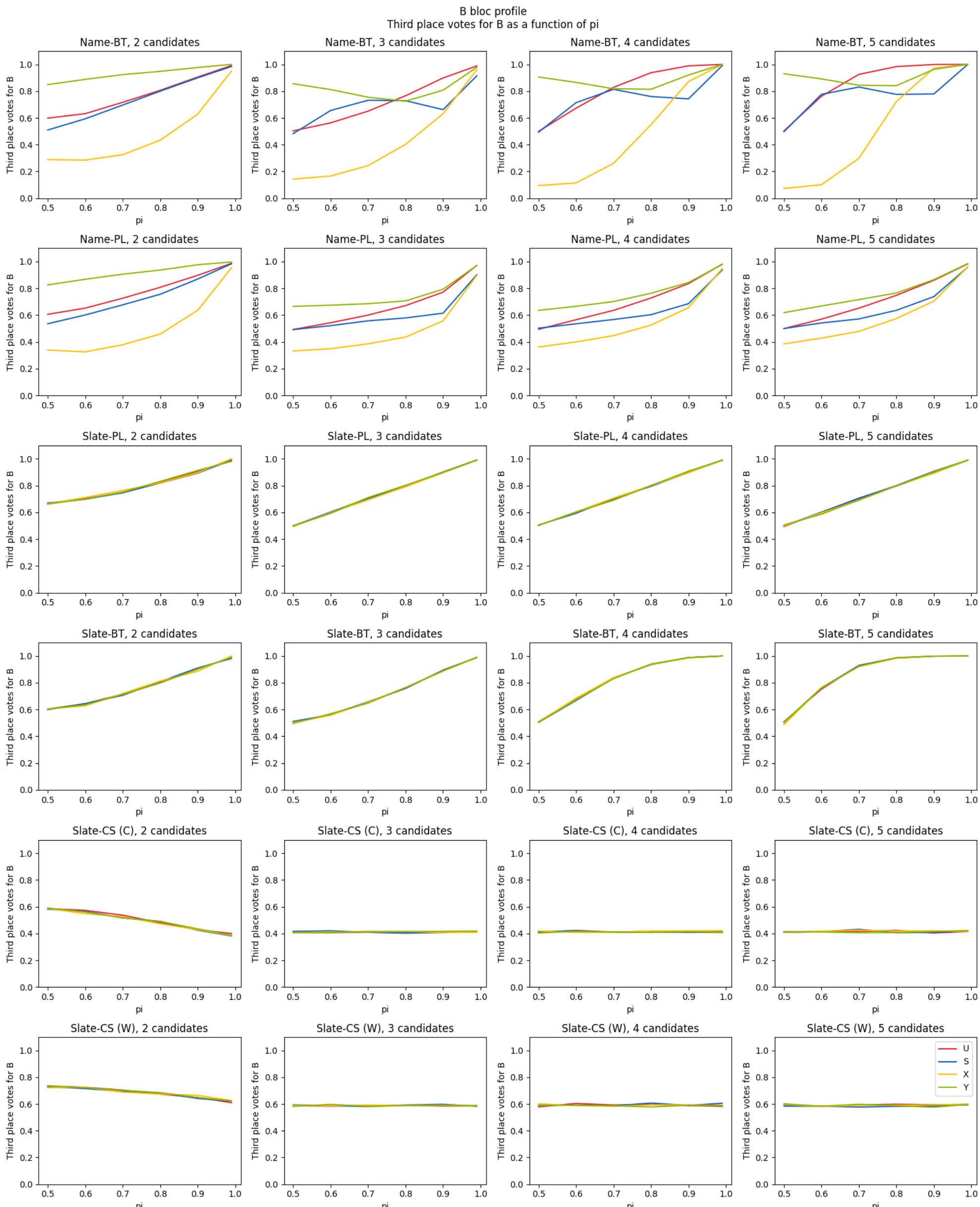

Fig. 20. The proportion of third-place votes for $\mathcal{B}$ candidates. Shown across different generative models, numbers of candidates, and strength scenarios.

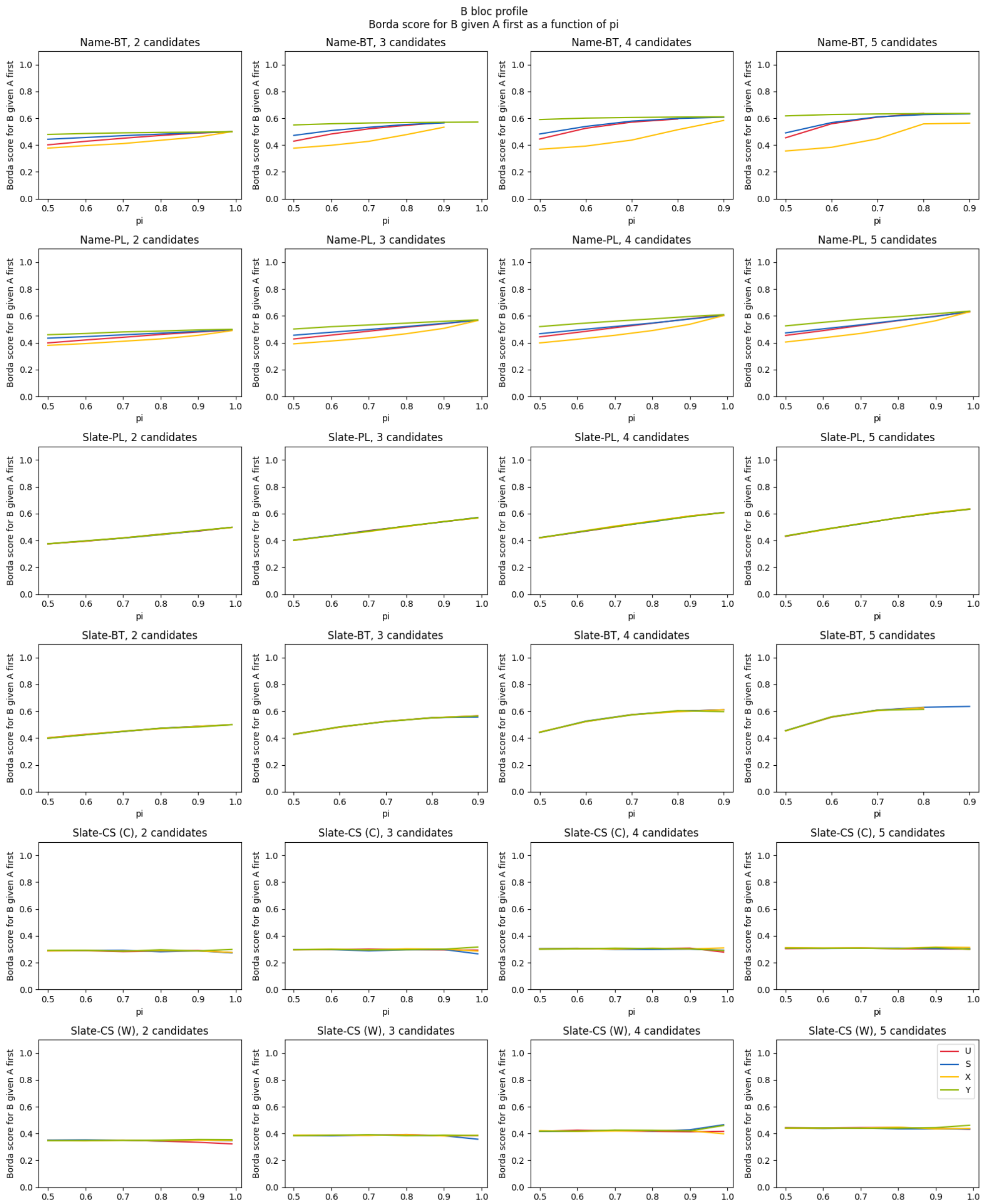

Fig. 21. The proportion of Borda points for $\mathcal{B}$ candidates. given that a ballot started with an $\mathcal{A}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

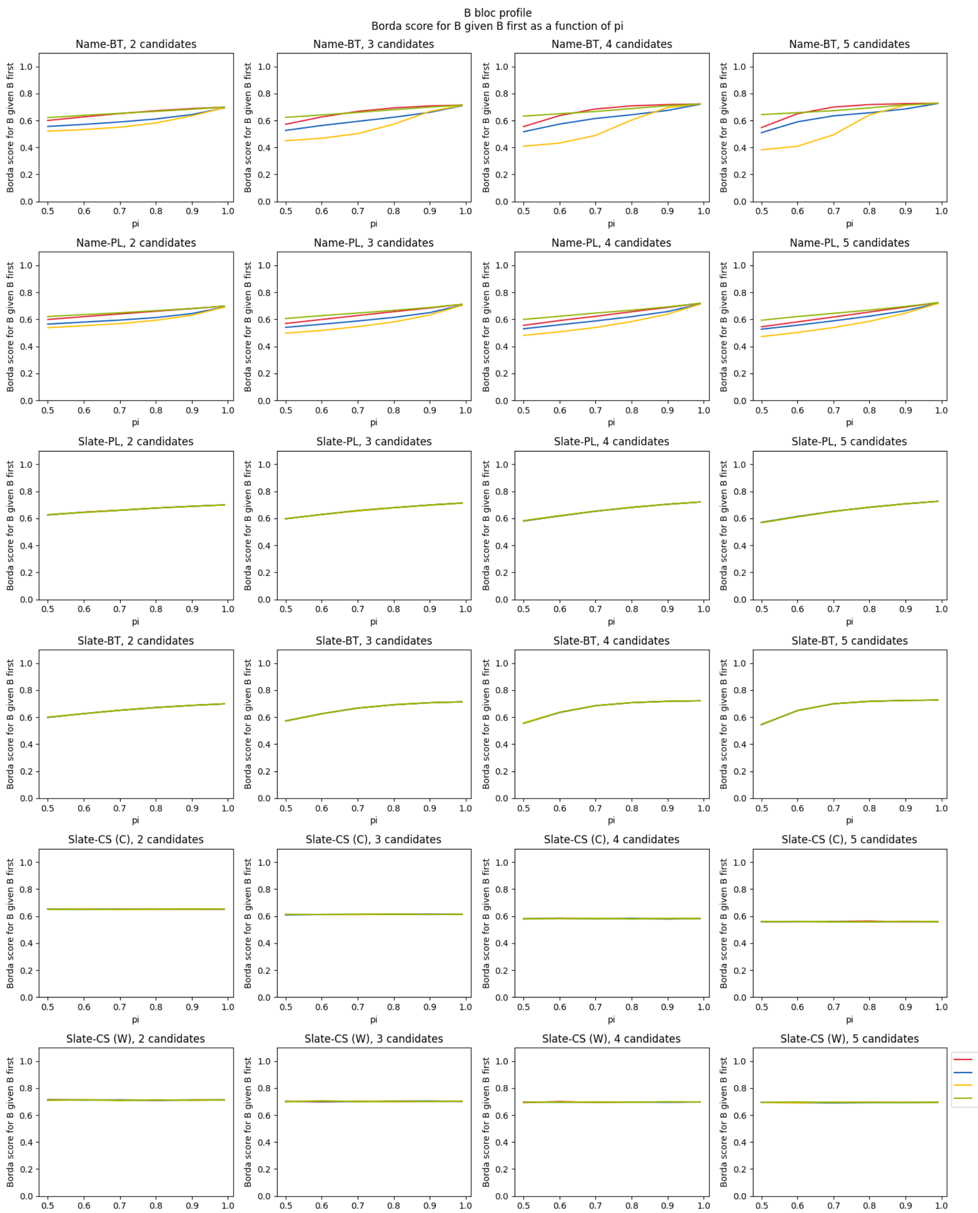

Fig. 22. The proportion of Borda points for $\mathcal{B}$ candidates. given that a ballot started with a $\mathcal{B}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

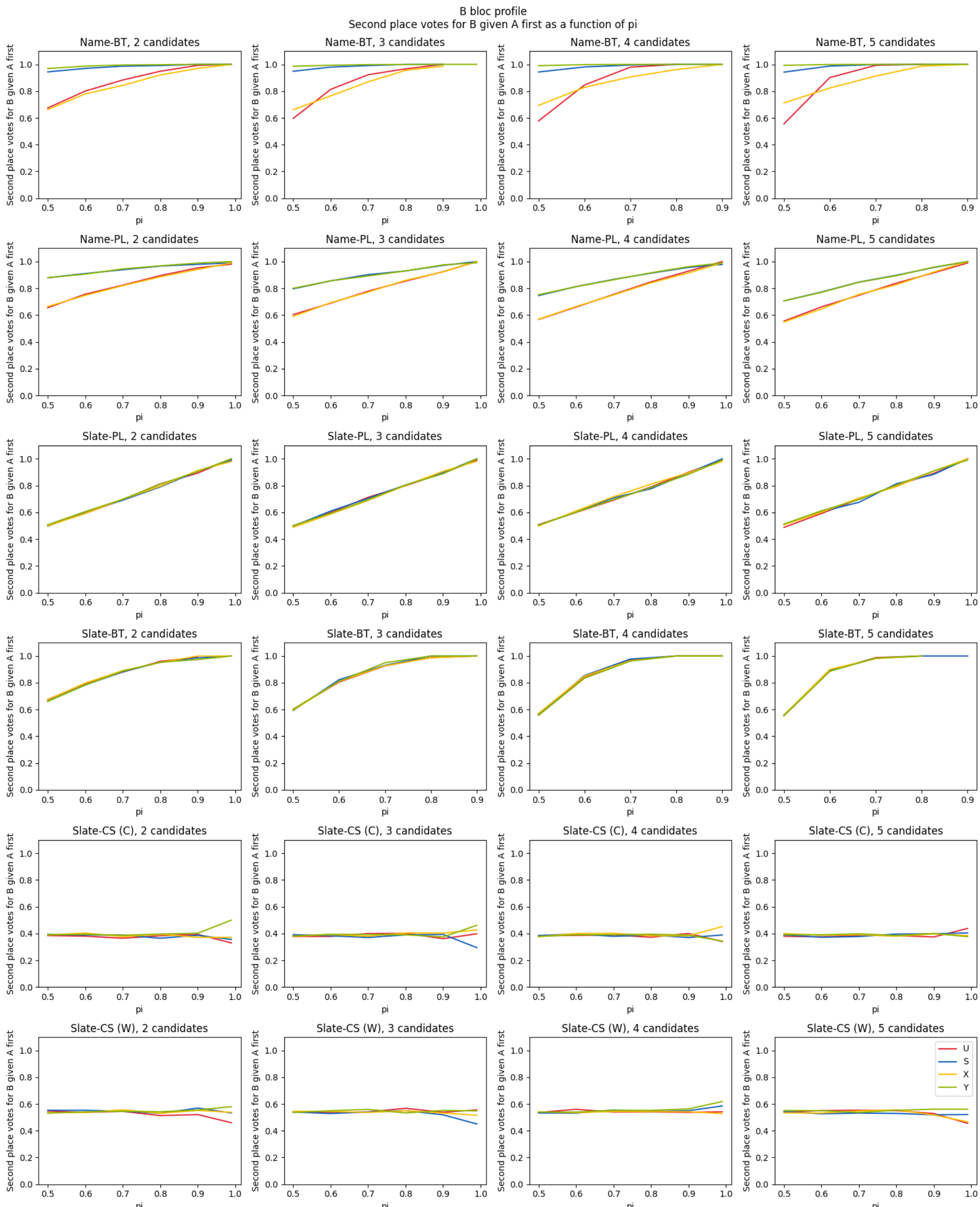

Fig. 23. The proportion of second-place votes for $\mathscr{B}$ candidates. given that a ballot started with a $\mathscr{A}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

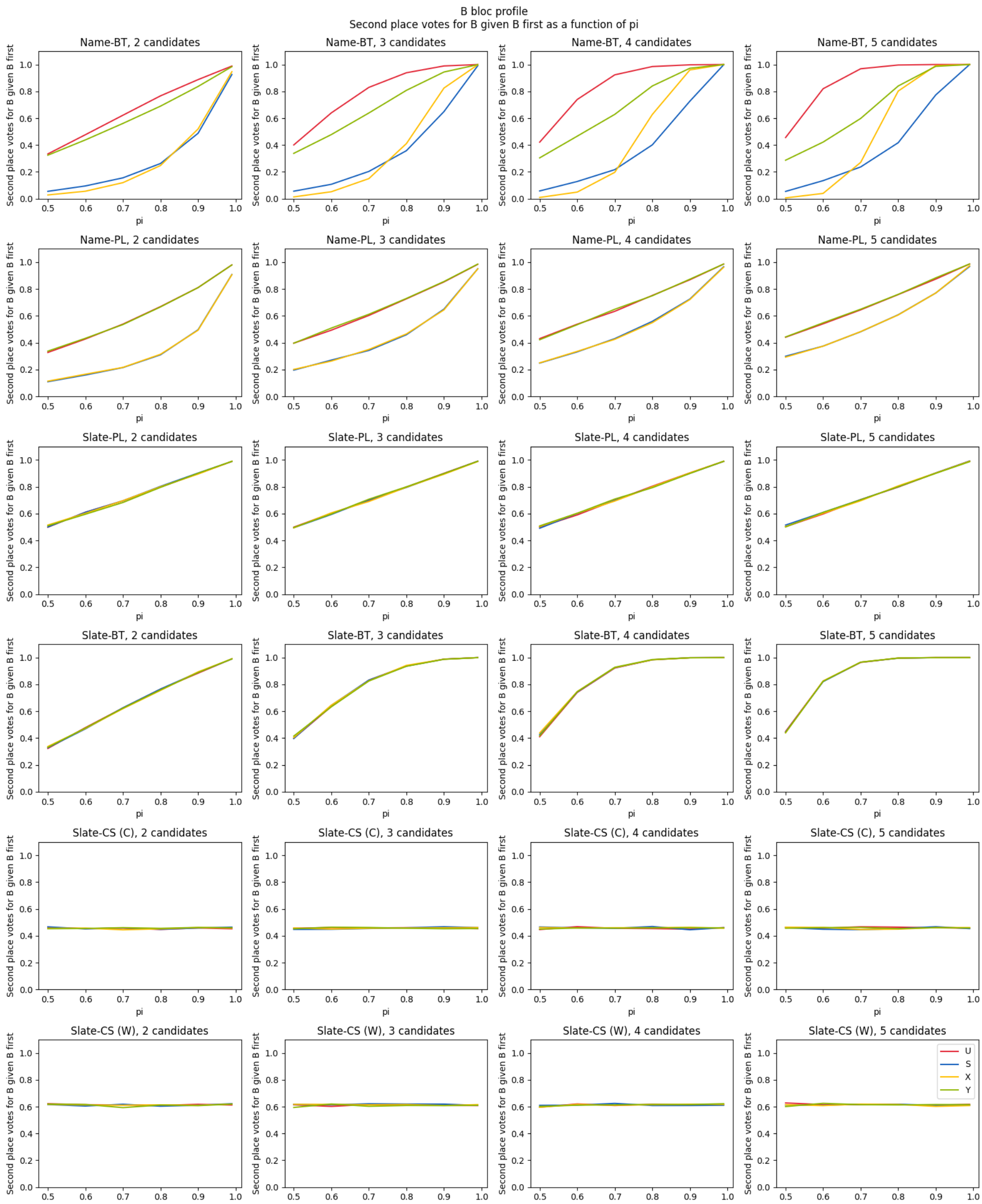

Fig. 24. The proportion of second-place votes for $\mathcal{B}$ candidates given that a ballot started with a $\mathcal{B}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

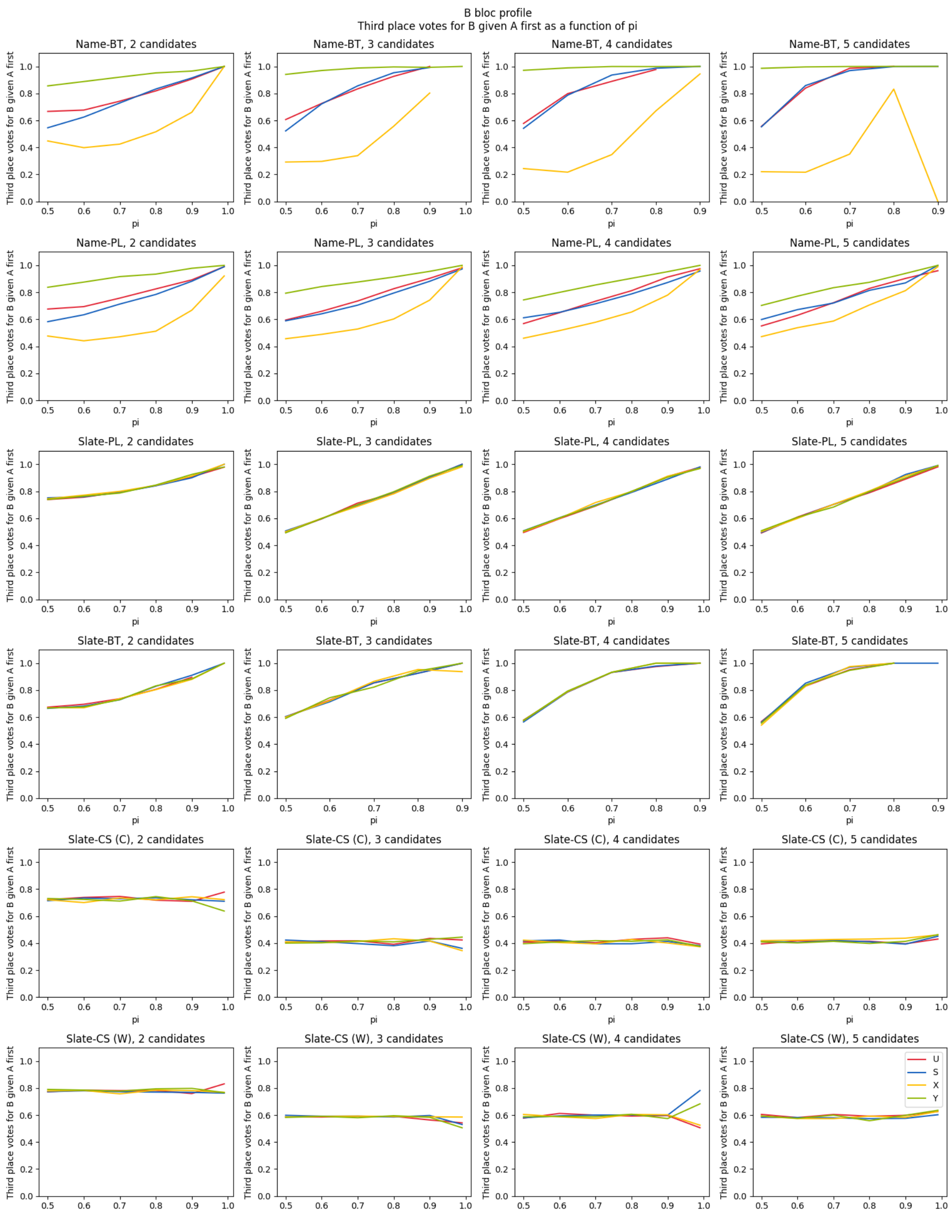

Fig. 25. The proportion of third-place votes for $\mathcal{B}$ candidates given that a ballot started with an $\mathcal{A}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

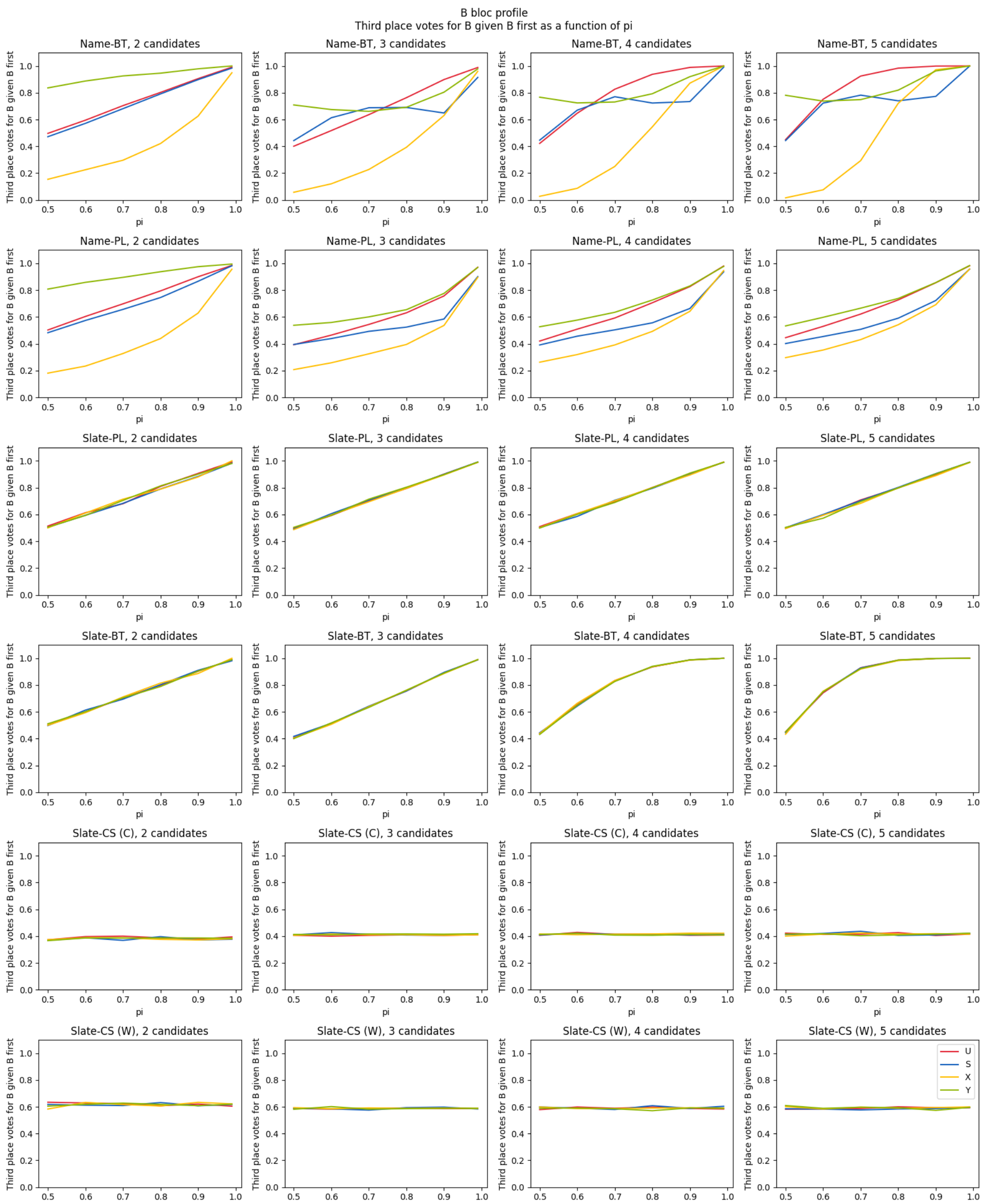

Fig. 26. The proportion of third-place votes for $\mathcal{B}$ candidates given that a ballot started with a $\mathcal{B}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

A.3 Attributes of profile, split by strength scenario and generative model

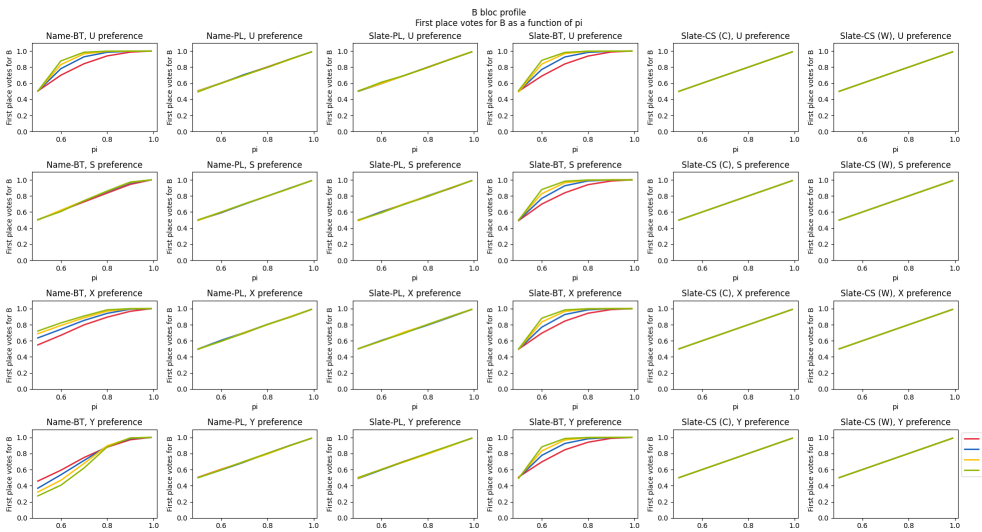

Fig. 27. The proportion of first-place votes for $\mathcal{B}$ candidates across different generative models, numbers of candidates, and strength scenarios.

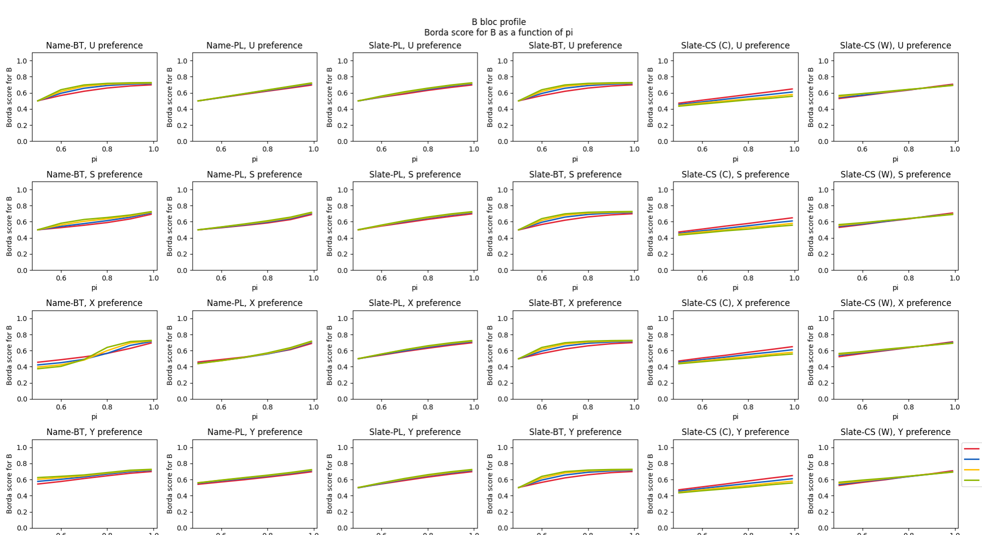

Fig. 28. The proportion of Borda points for $\mathcal{B}$ candidates across different generative models, numbers of candidates, and strength scenarios.

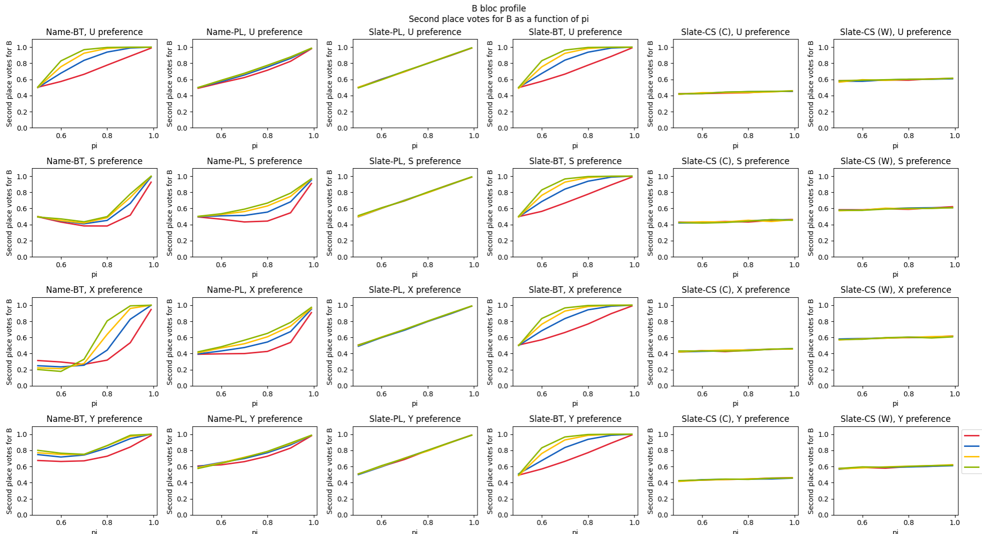

Fig. 29. The proportion of second-place votes for $\mathcal{B}$ candidates across different generative models, numbers of candidates, and strength scenarios. Notice that in the name models, the probability of ranking your own bloc second can actually be less than 50%, even in cases of high cohesion, given particular strength scenarios.

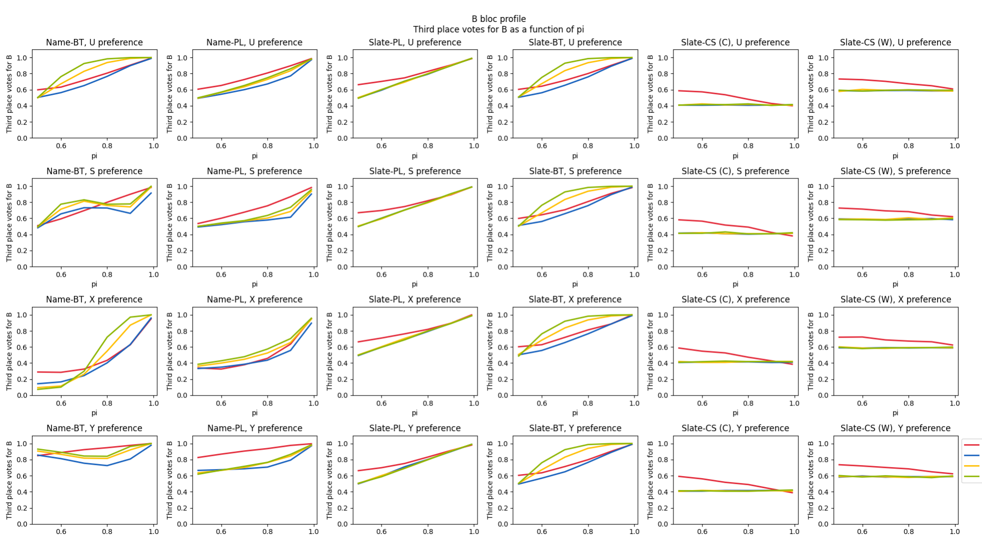

Fig. 30. The proportion of third-place votes for $\mathcal{B}$ candidates across different generative models, numbers of candidates, and strength scenarios.

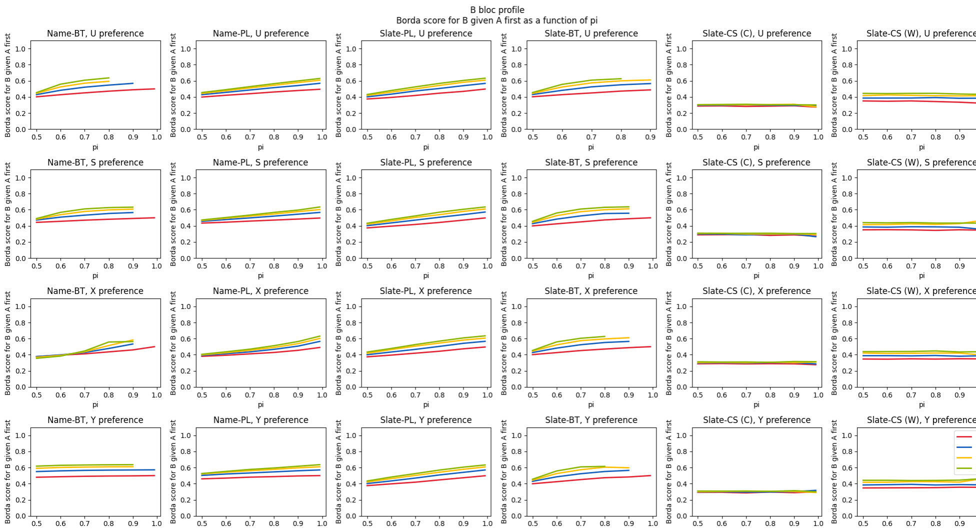

Fig. 31. The proportion of Borda points for $\mathcal{B}$ candidates given that a ballot started with an $\mathcal{A}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

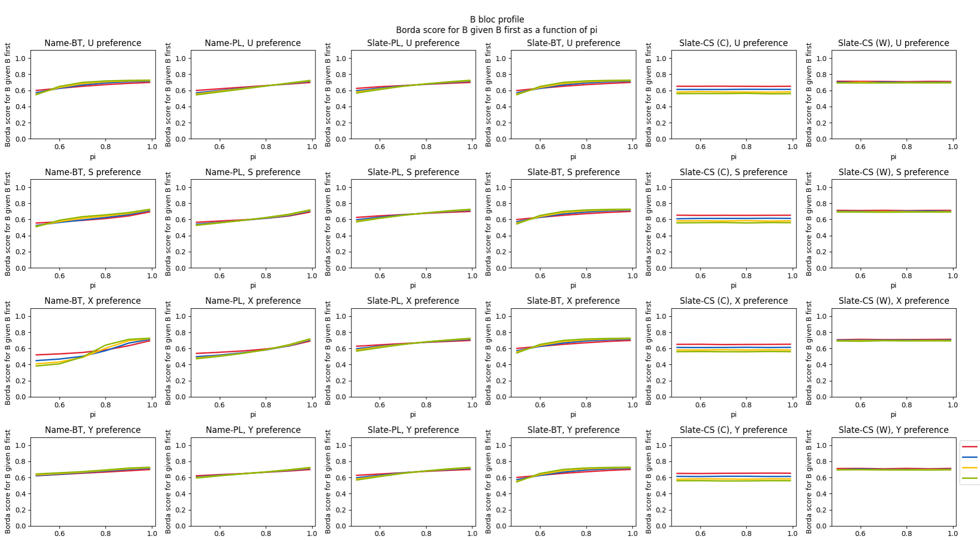

Fig. 32. The proportion of Borda points for $\mathcal{B}$ candidates given that a ballot started with a $\mathcal{B}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

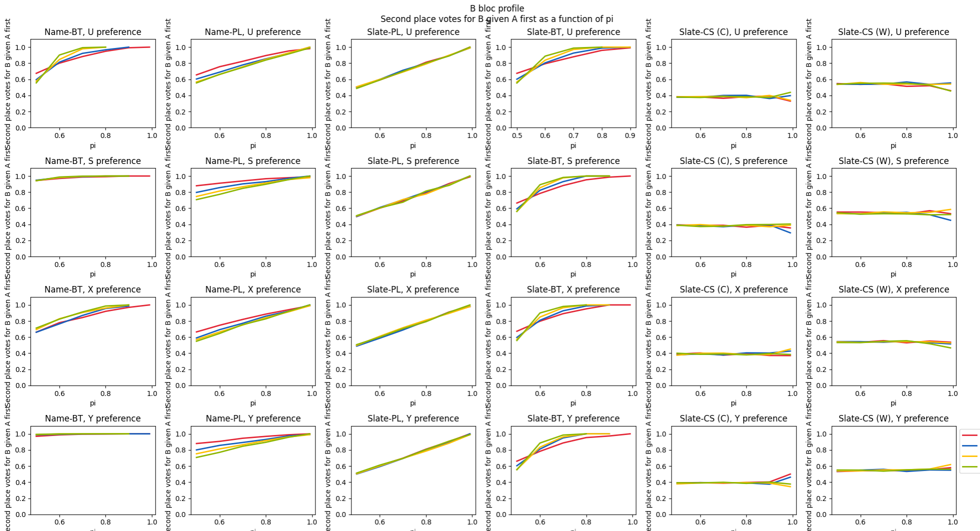

Fig. 33. The proportion of second-place votes for $\mathcal{B}$ candidates given that a ballot started with a $\mathcal{A}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

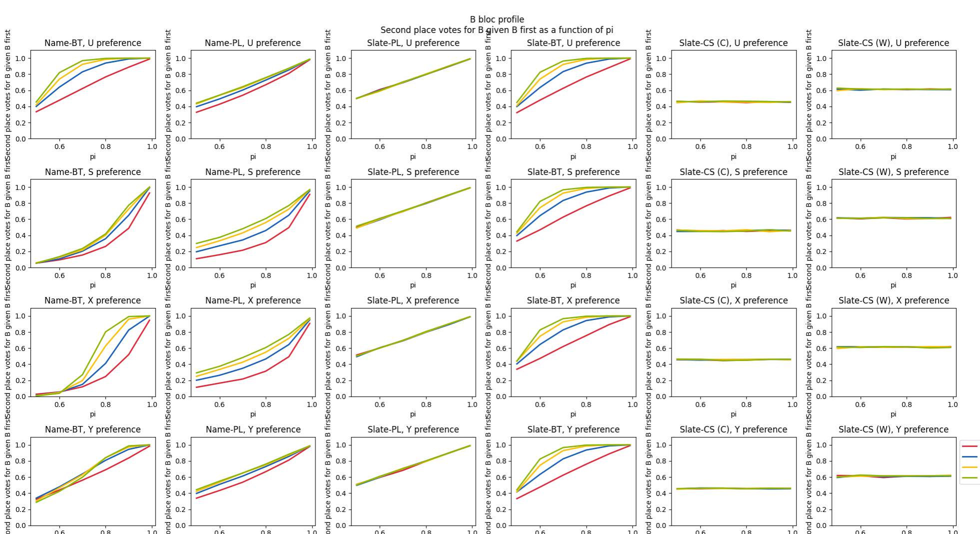

Fig. 34. The proportion of second-place votes for $\mathcal{B}$ candidates given that a ballot started with a $\mathcal{B}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

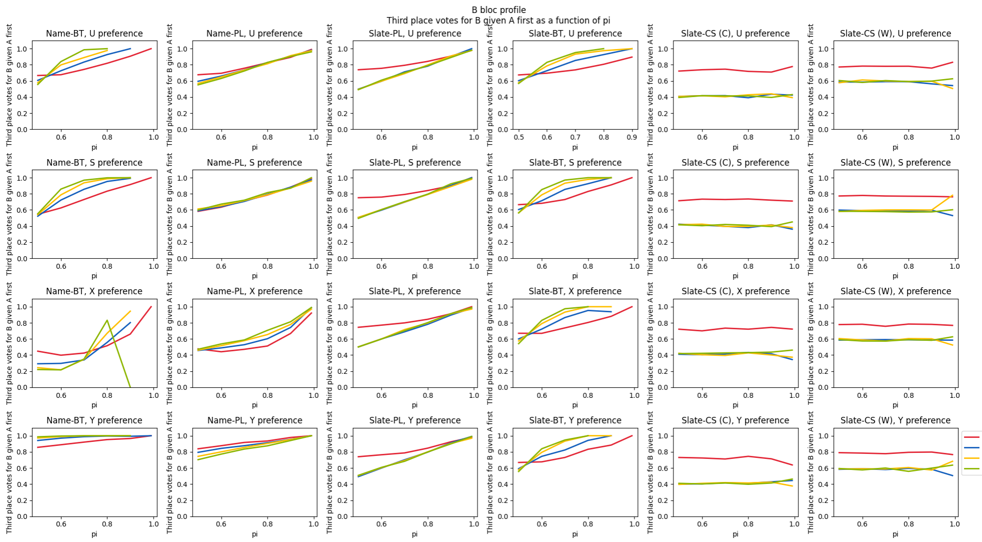

Fig. 35. The proportion of third-place votes for $\mathcal{B}$ candidates given that a ballot started with an $\mathcal{A}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

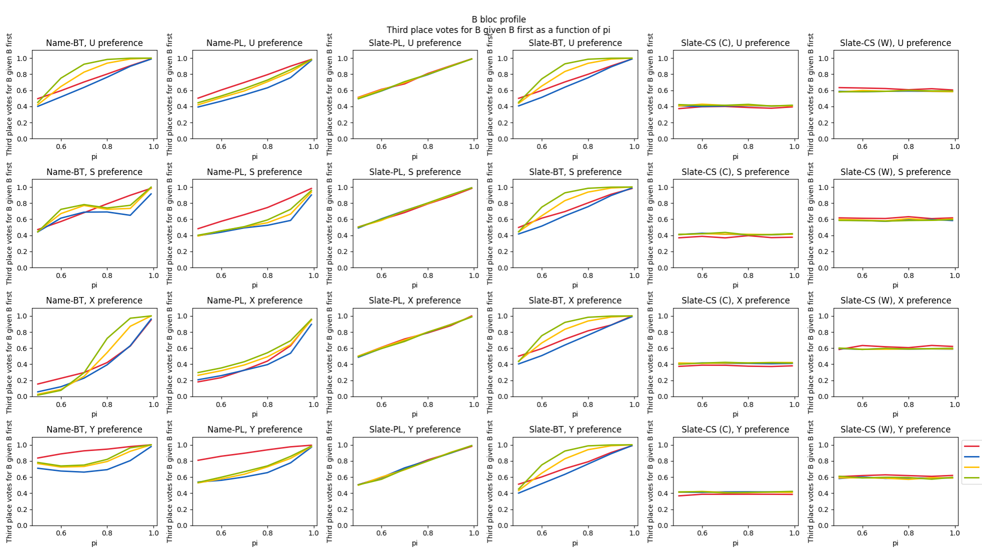

Fig. 36. The proportion of third-place votes for $\mathcal{B}$ candidates given that a ballot started with a $\mathcal{B}$ candidate. Shown across different generative models, numbers of candidates, and strength scenarios.

B MORE MDS PLOTS

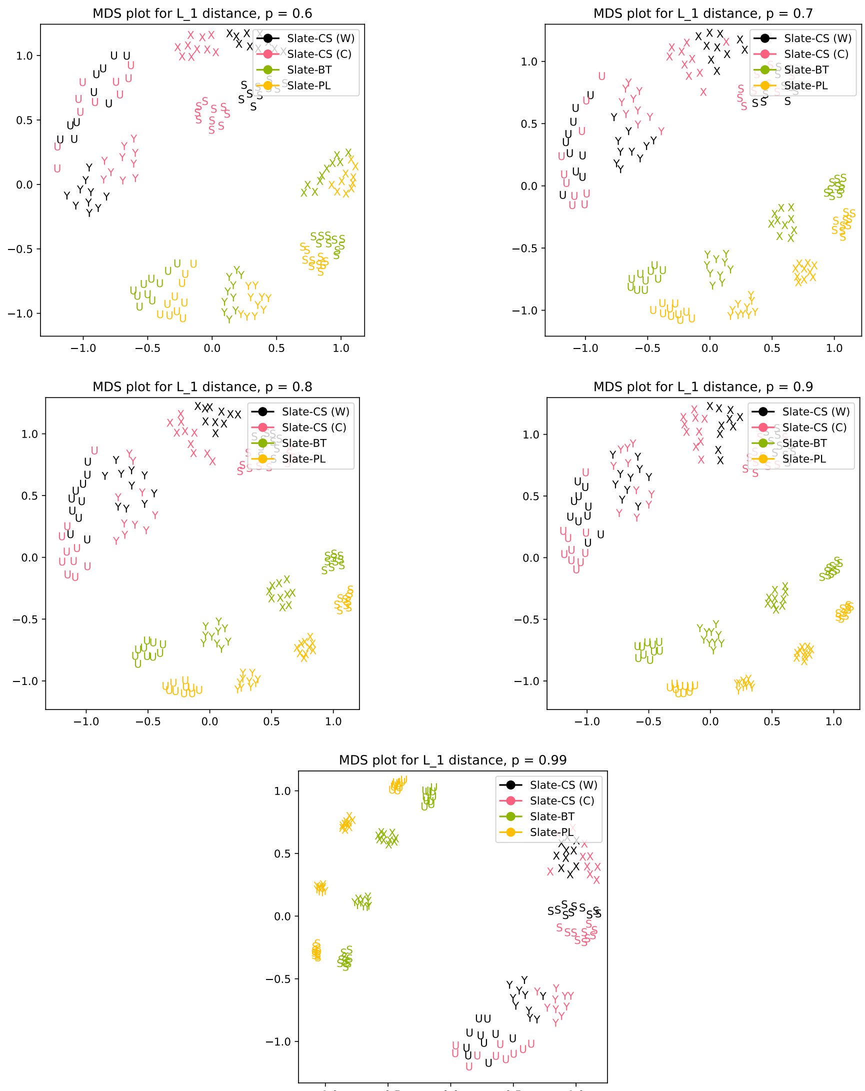

Fig. 37. Multi-dimensional scaling (MDS) plots for profiles with $r = s = 3$ (3 candidates per bloc), under a variety of generative models and candidate strength scenarios. The preference parameters π in each model are chosen to produce an expectation of p first-place votes for one's own slate (which means $\pi = p$ except for BT models, which require calibration). Each model is designated by a different color, and the candidate strength scenarios are denoted U, S, X, Y, as described above. The pairwise distances between profiles are computed with L^1 distance on the profiles. Each preference profile has 1000 ballots, and we have generated 10 profiles by each of the 16 model/strength pairs. As $p \to 1$, the main difference appearing in the models is that the BT and PL profiles become tightly clustered for each candidate strength scenario, while the CS profiles remain more variable.