Proportionality for ranked voting, in theory and practice

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Classical social choice theory includes a long list of criteria, or fairness axioms, for elections where individuals rank their preferences. Famous impossibility theorems from the 1970s concern the properties of voting rules to convert profiles of ranked preferences to winner sets. But though public perceptions of fairness are strongly keyed to proportional representation, notions of proportionality are strikingly missing from the standard roster of fairness axioms. We design a framework to measure *the degree of proportionality of seats to voter preference* under a wide class of systems for electing legislative bodies, even when elections are conducted without party labels. Given an election with a bloc structure—dividing voters into groups with a degree of preference for each of several slates of candidates—the proportionality target for a given set of candidates is their level of combined support from all the voter groups. Then the disproportionality is defined as the deviation of their seat share from this target.

We illustrate this proportionality framework by giving theoretical results relating seats to votes for single transferable vote (STV) elections. We show that changes in the STV mechanism, or comparison to other voting rules, can have a large impact on the degree of proportionality. We then conduct empirical exploration both with actual historical elections and with new generative models for creating synthetic ranked preference profiles. The new generative models are constructed with an emphasis on flexibility and realism; in particular, we can efficiently generate polarized elections with properties motivated by the extensive body of work on racially polarized voting in the United States.

This work brings a statistical modeling toolkit to the questions around ranked choice voting and proportionality. At the same time, it builds a much-needed bridge from computational social choice theory to political science, where degrees of proportionality have been intensely studied for well over a century, and to the work of practitioners in current reform efforts around voting rights and democracy.

ACM Reference Format:
Gerdus Benadè, Christopher Donnay, Moon Duchin, and Thomas Weighill. 2025. Proportionality for ranked voting, in theory and practice. 1, 1 (September 2025), 46 pages. https://doi.org/10.1145/nnnnnnn.nnnnnnn

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ACM XXXX-XXXX/2025/9-ART
https://doi.org/10.1145/nnnnnnn.nnnnnnn

1 Introduction

In this paper, we give what we believe to be the first definition of *the degree of proportionality of votes to seats* that is general enough for use with ranked preferences and with any structure of districts and voting rules that fills a legislative body.[1] This fills a gap in the classical social choice literature. Ken Arrow's foundational work studied social choice functions that combine multiple input rankings into one output ranking; following this, a series of important results were conjectured and proved from the 1960s to the 1990s concerning the use of rankings to output winner *sets* rather than rankings. Impossibility theorems of Müller–Satterthwaite, Gibbard–Satterthwaite, and Duggan–Schwartz rule out the viability for single-winner or multi-winner elections of simultaneously securing multiple axioms of fairness (see, for instance, [Taylor, 2002]). Examples of fairness axioms from early social choice theory include strategy-proofness, monotonicity, and the Condorcet criterion. However, in the public discourse around democracy, these kinds of considerations fall far behind proportionality as commonly articulated expressions of fairness.

Another area of study in the computational social choice literature is defining generative models of election that create synthetic preference profiles. There, several authors have noted the lack of realism and the distance from real-world electoral dynamics. In this paper, we introduce novel generative models of ranking that are inspired by polarized elections in real-world settings; in particular, we draw on domain knowledge from voting rights law in the United States, where definitions of voting blocs and their degrees of cohesiveness have been used for decades. (The term "generative model" is often associated with large language models as paradigms of artificial intelligence; here, what is being generated is realistic voting rather than realistic language.) With these models and access to observed electoral data, we can test voting rules on both real and synthetic preference profiles, yielding information—some provable and analytic and some qualitative and simulation-based—on whether roughly proportional outcomes do indeed tend to result from so-called "semi-proportional" systems.

1.1 Contributions

Rethinking proportionality. The proportionality of representation for a subgroup of voters could have a very simple interpretation in demographic terms (the group's seat share is in line with its share of the electorate). However, this fails to account for any complexity in the voting patterns of that group and the complementary voters. We define a framework that replaces demographic proportionality for a bloc of voters with *support proportionality* for a slate of candidates: the slate's seat share should be in line with the combined support for its candidates. We note that this kind of proportional representation is broader than that of so-called "PR systems" such as party list voting, which secure support proportionality—on the basis of party only—by construction. (This makes it vacuous to measure party proportionality for such systems.) Here, voters might not even be aware of which candidates constitute a slate; slates can be identified after the fact on the basis of trends in voter behavior. This kind of proportionality can be measured not only on the basis of party but with respect to any other cohesive preference.

Furthermore, an impossibility result in §3.1 will make it clear that exact proportionality is not achievable as a global yes/no property of a voting rule. This motivates our re-framing that focuses on measuring the observed or expected *degree* of proportionality for profiles P drawn from some distribution on profiles μ.

[1] In particular, both multi-winner elections and the use of districts conducting single-winner elections are covered in our framework. All other notions we are aware of work by recourse to approval ballots, as we describe further below.

New generative models. To get interesting distributions on profiles, we build and test new generative models of voting using parameters and data—in our case, historical voting patterns, demographics, cohesion parameters, and candidate strength—from which ballots are sampled and elections can be simulated. By allowing the user to specify sizes and behavior by bloc, these allow a study of the impact of polarization. We will offer some validation that our models comport better with real-world ranking data than previous simpler models (solid coalitions, IC, IAC), which builds our confidence in using them to analyze voting rules.

Incorporating domain knowledge. This project engages domain knowledge in voting rights law and practice in multiple ways. First, we shift the definition of voter cohesion to better line up with the ordinary and legal use of the term. In the previous social choice literature, definitions of *cohesive* groups of voters tend to revolve around overlapping approval ballots: for instance, Sánchez-Fernández et al. [2017] call a group of voters ℓ−cohesive, where n candidates are running for k seats, if they comprise at least $\ell n/k$ people and their preferences overlap in at least ℓ candidates. Then various axioms of fairness might revolve around guarantees that some of those voters approve of at least ℓ winners.[2] By contrast, this paper introduces notions of cohesiveness keyed to the rate at which members of a group tend to support candidates from a certain slate. Compare this to, for instance, the landmark *Thornburg v. Gingles* decision of the U.S. Supreme Court, requiring Voting Rights Act plaintiffs to ascertain "whether members of a minority group constitute a politically cohesive unit" by measuring whether "a significant number of minority group members usually vote for the same candidates."[3] Expert work supporting a finding of cohesiveness revolves around "statistical evidence of voting patterns" using past elections, and polarization is typically summarized by using standard inference techniques to estimate the share of support for slates of candidates by blocs of voters [Hebert et al., 2010]. This paper draws on just this kind of experience in voting rights expert work.[4]

Secondly, definitions related to justified representation are far removed from the notions of proportionality dominant in the political science literature (and the popular discourse): the seat share for a group should be in line with its vote share. The relationship of seat share to vote share has been intensely studied at least since the late 19th century, and measurement of deviation from ideal seats/votes curves has generated a significant literature in the last fifty years especially in the work of Tufte, King, Grofman, and many more.

Finally, our use of ranked ballots rather than approval ballots is aligned with practice (and reform momentum) in the United States and internationally. Several U.S. states have recently debated adoption of ranked choice elections: Maine and Alaska now use ranked voting for statewide elections, and Nevada got midway through the process of transitioning before the momentum faltered in the 2024 elections. Dozens of cities from San Francisco to Minneapolis use ranked choice for municipal elections, and New York City recently switched to ranked choice to elect city councillors and the mayor. Outside of the U.S., ranked choice voting is used for local or legislative elections in much of the Anglophone world—including Scotland, Ireland, New Zealand, and Australia—as well as for parliamentary elections in Malta and Papua New Guinea. As ranked voting is considered more broadly, stakeholders are increasingly asking about its properties, and one claim in common circulation is that they deliver more proportional outcomes for minority

[2]Definitions in this direction are variants of an original definition called JR, or *justified representation*, in which "cohesion" requires only a non-empty overlap in approvals.

[3]*Thornburg v. Gingles* (1986), https://www.oyez.org/cases/1985/83-1968.

[4]For instance, consider recent expert work in Texas: minority racial groups, taken together, were estimated to collectively support Democratic candidates in general elections from 2012–2020 at rates of 85-92%, while White voters supported Republican candidates at rates of 75-85% in the same contests. Expert report of REDACTED, *TX NAACP et al. v. Abbott*, Case No. 1:21-CV-00943-RP-JES-JVB.

voting groups than could be expected from first-past-the-post systems. We seek to investigate these claims.

Illustrating with STV. While our notion of proportionality and the generative models we propose do not assume use of any specific voting rule, we will use *single transferable vote* (STV) as a test case. STV is a family of voting rules within ranked choice voting, using a transfer mechanism for selection of multiple winners, where the number of seats to be filled in a single contest is called the *magnitude*. In STV elections, there is a threshold level of support needed to be elected—typically the threshold is about $1/(k+1)$ of the first-place votes, where k is the magnitude. The election is conducted in rounds. As candidates are either elected (by passing the threshold) or eliminated from contention, the (surplus) votes supporting those candidates are transferred to the next options on their respective ballots.[5] We note that *instant runoff voting* or IRV, an extremely popular alternative in practice, is the same voting rule as STV in the special case $k = 1$.

Though STV is the basis for many of the examples in this paper, the express goal of the work is to set up a framework suitable for the comparative study of any voting rules that fill a representative body. We will begin to do this below, both by comparing different voting rules within the STV family and by comparing STV to the Borda count. We emphasize that the definition of proportionality here works equally well in the multi-winner setting (also known as *committee voting*) or in districted elections with single-winner rules like first-past-the-post or IRV. All that is needed is a classification of candidates across districts—such as members of a party or a racial group in U.S. Congress—and then we can compare the combined vote support to the seat share in exactly the same manner.

1.2 Related work

Statistical ranking models, or models that assign a probability to permutations on a set of elements, have been studied at least since the early 20th century, going back to Thurstone [1927]. Subsequent models include those introduced by Bradley and Terry [1952], Plackett [1975], and Luce [1959], which form the basis for the BT and PL models in this paper, respectively. Benter [2008] introduced a variation of the Plackett model with a dampening parameter to account for less careful deliberation of lower-ranked items. Johnson et al. [2002] proposed a model to combine rankings that were determined by several different sources—which could have used different methods and criteria—into an aggregate, or meta, ranking scheme.

Ranking models have been used in a variety of applications in the broader social science literature. Stern [1990] applies the methods to horse races, where the marginal probability of each horse finishing first is known in advance. Bradlow and Fader [2001] apply time series models to Billboard "Hot 100" list, to show how song rankings change over time. Graves et al. [2003] apply a combination of ranking models to racecar competition outcomes. In the area of election analysis, Upton and Brook [1975] fit a Plackett model to ranked ballots in London elections to determine the effect of candidate name ordering on the ballots, also known as positional bias. Gormley and Murphy [2008] fit a combination of Plackett-Luce and Benter models to polling data from Irish elections in 1997 and 2002. In particular, they find the models to be effective in identifying voting blocs (groups of voters with similar ranked preferences) within the electorate. In the same paper, the authors fit mixtures of Plackett-Luce models to cast vote records from Irish elections, with the main goal of

[5] Specific mechanics vary; in this paper we have implemented the vote-tallying mechanism used by Cambridge, MA for its City Council elections, except as noted below.

identifying blocs within the electorate.[6] These analyses are descriptive, based on historical data. In a recent paper, Garg et al. [2022] model outcomes of elections in multi-member Congressional districts under a solid coalition assumption, which means that the ballots are effectively unranked (and do not differentiate candidates within each coalition).

Our work is related in several respects to the existing computational social choice literature. There is a large body of work on the axiomatic properties of voting rules in various settings, including notions with a family resemblance to proportionality. The classical fairness axiom called Proportionality for Solid Coalitions (PSC), introduced by Dummett [1984], has been widely noted to be inadequate because it only applies with perfectly solid voting blocs, which never occurs in practice. The chief examples of axioms improving on PSC are those of (extended) justified representation (JR/EJR) [Aziz et al., 2017], which are structured as guarantees under approval-based multi-winner voting: sufficiently large groups whose approvals have non-trivial overlap can't be shut out of the winner set. Refer to Lackner and Skowron [2022] for a more thorough discussion. Various papers have used proportionality language for functions that map approval ballots to ranked outcomes [Skowron et al., 2017] and, quite recently, for functions that carry ranked ballots to sets of approval ballots, and from there map to multi-winner outcomes [Brill and Peters, 2023]. While similar in spirit, it would be difficult to compare ideas invoking justified representation to ours directly because the JR family of axioms relies on a fundamentally different definition of cohesion. Furthermore, like PSC, these axioms are binary: a winner set satisfies rank-PJR+, for instance, or it does not. These definitions are not keyed to giving degrees of proportionality.

In terms of generative models of election, numerical experiments in this literature traditionally rely on assumptions of *impartial culture* [Pritchard and Wilson, 2009], under which voters are independent and every permutation of candidates is equally likely, *impartial anonymous culture*, in which Lebesgue measure is used to set relative preferences, or use *spatial* or distance-based models [Elkind et al., 2017, Tideman and Plassmann, 2010]. See Szufa et al. [2022, 2020] for a comparison of common generative models (called "statistical cultures") and a recent discussion of how to sample approval elections.

Spatial models [Enelow and Hinich, 1984] represent voters (and candidates) as ideal points in a metric space—in other words, using a space with a distance function as the latent space for voter preferences—and are common across fields. Voters are presumed to vote either deterministically for their closest representatives or probabilistically (upweighting closer candidates) [Burden, 1997]. Two commonly used methods for estimating ideal points (typically from Congressional roll-call data) are NOMINATE [Poole and Rosenthal, 1985] and IDEAL [Clinton et al., 2004]. Ranked choice voting models can be built from spatial models. For example, Gormley and Murphy [2007] combine a spatial and Plackett-Luce model to analyze Irish STV elections (discussed further in §5), and Kilgour et al. [2020] use a spatial model (where voters rank by proximity) to measure the effect of ballot truncation on single-winner ranked choice outcomes. Garg et al. [2022] also use a spatial model in one section, with voter ideal points extracted from ideology ratings in a commercial voter file, to relate the "diversity" of elected officials to the sizes of multimember districts.

Spatial models on one hand, and approval votes on the other, are favored by the mathematically inclined because they lend themselves to provable theoretical properties of voting rules. For example, under the implicit utilitarian voting framework, ordinal votes are proxies for underlying utilities and the *distortion* of a voting rule captures its worst-case loss compared to having full information [Procaccia and Rosenschein, 2006]. Anshelevich et al. [2018] study the distortion of STV under

[6] In the language that will be introduced below, this roughly corresponds to fitting a Name-PL model (see Remark 3) with unknown group sizes and no slate structure. That is, their method is designed to learn preferences for all candidates by each of two blocs. Fitting a mixture model in this way does not produce a partition of candidates into slates so it is not clear how it might fit with a notion of proportionality.

metric preferences, and Gkatzelis et al. [2020] recently settled a well-known conjecture on the optimal metric distortion when aggregating rankings to elect a single winner.

Our goal is to strike out in a new direction, with definitions that enable new questions to surface.

2 Blocs, slates, and proportionality

2.1 Defining blocs, slates, and notions of preference

The concept of blocs and slates is straightforward: *slates* are disjoint sets of candidates, such that voter support for the various slates can be measured. *Blocs* are groups of voters. The idea that voters exhibit a preference among slates makes sense whether we consider the electorate as a single bloc (undivided into types of voters, but with a probabilistic tendency to prefer one slate or the other) or split out into two or more disjoint blocs who exhibit distinct voting behavior.

To make this precise, we need ways of measuring what it means for a group of voters to display an overall preference for one group of candidates over another. There are several natural notions of preference that can be measured in an observed ranking profile—that is, they can be treated as descriptive statistics that can be read off of a cast vote record where each ballot is a partial ranking (i.e., a permutation of a subset of the candidates), as long as the profile is labeled so that the candidates are identified by slate. In this section, we develop notation and definitions for this descriptive setting. (In §4.1, we will give a notion of preference that is used as a parameter in a generative model that creates a vote profile, and we will discuss how these parameters might be estimated from a set of historical elections.)

Definition 2.1 (Descriptive statistics of group preference). Suppose an election is conducted with bloc structure $(A, \mathcal{A}, B, \mathcal{B})$ consisting of sets of voters A, B and corresponding slates of candidates $\mathcal{A} = \{A_1, \dots, A_r\}$ and $\mathcal{B} = \{B_1, \dots, B_s\}$. This accommodates the case of one bloc (where we will adopt the convention $A = \emptyset$) or the case of two blocs, and this definition is easily expanded to more than two blocs. Often, we will use A for the majority bloc and B for the minority, when those are clear. The preference definitions are given below for bloc B.

Suppose voters are allowed to rank up to $n \le r + s$ candidates on their ballots—that is, ballots may be incomplete rankings of varying length, up to some maximum.

- Bloc B prefers slate $\mathcal{B}$ with *first-place preference* p_B if the share of first-place votes in the profile for $\mathcal{B}$ candidates is p_B.
- Bloc B prefers slate $\mathcal{B}$ with *positional preferences* $P_B = (p_1, p_2, \dots, p_n)$ if the share of ballots placing a $\mathcal{B}$ candidate in position i (among those for which a vote is cast and neither slate was exhausted in the higher positions) is p_i. Then a single preference parameter can be defined by any linear combination of those values.
- Given a positional scoring rule with weights $(w_1, w_2, \dots, w_n)$, we say that B prefers slate $\mathcal{B}$ with *score preference* p_B if the share of their score for $\mathcal{B}$ candidates is p_B. The default option will be to give standard Borda weights to the top k ranks via the score vector $(k, k-1, \dots, 1, 0, \dots, 0)$ in a magnitude-k election; we will refer to this as (top-k) *Borda preference.*[7]

Preferences for the A bloc are defined analogously; the only difficulty in extending to *more* than two blocs is one of cumbersome notation.

We will interpret each of these preference parameters as an indication of how *cohesive* bloc B is, with higher preference parameters (closer to 1) indicating more strongly aligned blocs.

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[7]For a discussion of how to extend to partial rankings, see §C.

Example 2.2. Suppose an election has been conducted with $r = 3, s = 2, n = 5$ (i.e., complete rankings are allowed), and suppose the voters are labeled as A voters or B voters. Suppose that the summarized preference profile for the B bloc is given by

$\times 2$	$\times 3$	$\times 8$	$\times 1$	$\times 5$	$\times 3$	$\times 5$
B_1	B_1	B_1	A_1	B_2	B_2	B_1
B_2	A_2	B_2	B_1	B_1	A_3	B_2
A_1	B_2	A_2	B_2	A_1	A_1	
A_2	A_3	A_1		A_3	B_2	
A_3	A_2			A_2	A_2	

(by name)

i.e.,

$\times 7$	$\times 3$	$\times 8$	$\times 1$	$\times 3$	$\times 5$
B	B	B	A	B	B
B	A	B	B	A	B
A	B	A	B	A	
A	A	A		B	
A	A			A	

(by slate)

Then the first-place preference of the B bloc for $\mathcal{B}$ candidates is $26/27$, the positional preferences are $(\frac{26}{27}, \frac{21}{27}, \frac{4}{7}, \frac{3}{3}, -)$, the Borda preference to all five places is $232/405$ with ballot completion by averaging, and the top-2 Borda preference is $73/81$. Note that the last few positional scores are $4/7$, $3/3$, and undefined—rather than $4/22$, $3/21$, and 0—because we only consider ballots where the B candidates are not exhausted.

2.2 Defining proportionality

First let's consider the case that the electorate is undivided ($A = \emptyset$). We may have a preference profile where voters were observed to support slate $\mathcal{B}$ at the level p_B. Likewise, if an election is generated by a random process with a parameter π_B for the tendency to prefer slate $\mathcal{B}$, this can also be interpreted as the level of support. In either case, the proportionality ideal is extraordinarily simple: seat share equals vote support share, i.e.,

$$S_B = p_B \qquad \text{or} \qquad S_B = \pi_B.$$

When voters only select a single candidate, this is exactly the everyday (plain English) notion of proportionality.

When there are two distinct blocs with different voting behavior that partition the whole electorate, this extends by convex combination to a natural proportionality target. If p_B is the observed preference for bloc B towards its candidates (or π_B is a parameter for preference in a generative model), and likewise p_A (respectively, π_A) for bloc A, then the combined support for slate $\mathcal{B}$ is

$$C_B := N_B \cdot p_B + (1 - N_B)(1 - p_A) \qquad \text{or} \qquad C_B := N_B \cdot \pi_B + (1 - N_B)(1 - \pi_A)$$

where N_B is the share of voters from the B bloc. The proportionality target sets seat share to this level, namely $S_B = C_B$. That is, proportionality pins the representation to combined support for $\mathcal{B}$ candidates: the size of the B bloc times its level of cohesion (the propensity to vote for $\mathcal{B}$ candidates) plus the size of the complementary bloc times its level of crossover voting (again, the propensity to vote for $\mathcal{B}$ candidates).[8]

This enables us to say, for instance, whether a particular slate-labeled profile P had near-proportional outcomes under voting rule f with respect to first-place preferences, or to Borda score, or a Bradley-Terry model parameter, or any other notion of how to measure or generate support.[9]

[8]One could consider alternative definitions of proportionality, for example, based on a weighted combination of the number of seats a slate wins in each of the hypothetical elections in which only one of the blocs participates. However, this requires fixing a voting rule. We deliberately propose a notion of proportionality that is agnostic to the choice of voting rule.

[9]Notably, if a descriptive statistic like first-place support or Borda share is used, its linearity means that combined support can be read directly off the profile by just knowing the slate labeling of candidates but without identifying the voter blocs. (For instance, first-place support for Democrats can be measured without knowing which blocs of voters cast those votes at

Example 2.3. We use a large repository of real-world Scottish local government STV elections to illustrate how to use the definition in practice. We illustrate with first-place preference for a random sample of 30 Scottish contests, shown in Table 1.

In the table, we employ simple slate labels where two Scottish parties—SNP (Scottish National Party) and the Green Party—are defined as a slate $\mathcal{B}$, and the complementary slate $\mathcal{A}$ combines all other parties.[10] We measure the level of proportionality with respect to first-place preference to define p_B, the observed propensity of voters to support slate $\mathcal{B}$. This means that the number of seats needed to achieve (first-place) proportionality is $p_B \cdot k$, the target seat share times the number of seats. We then compute disprop as the defect from proportionality, or $S - C$.

2.3 Degree of disproportionality

In the previous section, where $S = C$ was presented as a target or ideal for proportionality, we defined disproportionality as the defect $S - C$. Over a distribution on elections, this allows us to define disprop as the mean signed difference (sometimes called MSD in the statistics literature) or the mean absolute difference/mean absolute error (MD or MAE).

Definition 2.4 (Degree of disproportionality). If P is a labeled profile (with blocs and/or slates identified), let $C = C(P)$ be the combined support for slate $\mathcal{B}$. Let f be a voting rule that gives seat share $S = S(f, P)$ for slate $\mathcal{B}$, so that $0 \le C, S \le 1$. Then

$$\mathrm{disprop}(f, P) := S - C, \qquad \widehat{\mathrm{disprop}}(f, P) := |S - C|.$$

If μ is a distribution on labeled profiles, then

$$\mathrm{disprop}(f, \mu) := \int S - C \, d\mu(P), \qquad \widehat{\mathrm{disprop}}(f, \mu) := \int |S - C| \, d\mu(P),$$

for $S = S(f, P)$, $C = C(P)$.

For disprop, a positive sign means that a group is overrepresented relative to the $S = C$ target. We note that this is defined by integrating $d\mu$ rather than integrating against Lebesgue measure on the combined support C, though the latter would be more in line with measures from some political science papers. Given μ with full support, we could in principle estimate the change of variables $d\mu/dC$, but this formulation is both more straightforward and arguably more interpretable as the expectation of disproportionality over a probabilistic distribution on profiles.

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what rate.) But incorporating a viewpoint of distinct groups and their cohesion is still important for assessments of overall fairness, whether theoretical or practical.

[10]The other parties include Conservatives, Labour, Liberal Democrats, multiple parties defined by their stance on independence from the UK, far-right parties like the National Front, and some farther-left socialist parties. STV can be tested with respect to any slate, but this choice to set $\mathcal{B} = \{\text{SNP, Green}\}$ was suggested by research by an overlapping group of authors in [redacted], in which these parties were found to be most clustered in voter preference across all possible groupings.

election	(r, s, k)	p_B (first-place pref.)	proportionality (first-place pref.)	STV outcome	disprop
North Ayrshire 2022 Arran	(4, 2, 1)	0.36	0.36	0	−0.36
Orkney 2022 Ward 3	(5, 1, 3)	0.14	0.42	0	−0.14
Highland 2012 Ward 12	(7, 1, 3)	0.15	0.45	0	−0.15
Dumgal 2022 Ward 12	(5, 2, 3)	0.17	0.51	0	−0.17
Inverclyde 2017 Ward 5	(6, 1, 3)	0.24	0.72	1	0.09
Edinburgh 2017 Ward 8	(4, 2, 3)	0.25	0.75	0	−0.25
Eilean Siar 2012 Ward 4	(5, 2, 3)	0.32	0.96	1	0.01
Fife 2012 Ward 12	(4, 2, 3)	0.32	0.96	1	0.01
North Lanarkshire 2012 Ward 19	(4, 2, 4)	0.26	1.04	1	−0.01
Clackmannanshire 2022 Ward 5	(5, 2, 3)	0.37	1.11	1	−0.04
Inverclyde 2017 Ward 7	(6, 2, 3)	0.38	1.14	1	−0.05
Stirling 2017 Ward 5	(3, 3, 3)	0.38	1.14	1	−0.05
East Renfrewshire 2022 Ward 5	(7, 2, 4)	0.29	1.16	1	−0.04
North Lanarkshire 2017 Ward 8	(5, 3, 4)	0.30	1.20	1	−0.05
Clackmannanshire 2012 Ward 5	(2, 2, 3)	0.40	1.20	1	−0.07
Stirling 2012 Ward 1	(4, 2, 3)	0.45	1.35	2	0.22
North Lanarkshire 2022 Ward 5	(4, 3, 3)	0.47	1.41	2	0.19
Clackmannanshire 2022 Ward 3	(4, 3, 3)	0.48	1.44	2	0.19
North Lanarkshire 2022 Ward 8	(6, 2, 4)	0.36	1.44	2	0.14
Highland 2012 Ward 19	(6, 2, 4)	0.39	1.56	2	0.11
Glasgow 2007 Hillhead	(8, 2, 4)	0.40	1.60	2	0.10
Renfrewshire 2022 Ward 8	(4, 2, 4)	0.40	1.60	2	0.10
East Ayrshire 2017 Ward 3	(3, 3, 4)	0.44	1.76	2	0.06
North Lanarkshire 2022 Ward 18	(4, 2, 4)	0.47	1.88	2	0.03
Renfrewshire 2017 Ward 4	(7, 3, 4)	0.47	1.88	2	0.03
Glasgow 2012 Ward 12	(10, 3, 4)	0.48	1.92	3	0.27
Glasgow 2022 Ward 6	(7, 3, 4)	0.53	2.12	3	0.22
Glasgow 2022 Ward 10	(7, 3, 4)	0.59	2.36	3	0.16
Glasgow 2022 Ward 7	(6, 3, 4)	0.60	2.40	3	0.15
North Lanarkshire 2022 Ward 4	(4, 4, 4)	0.63	2.52	3	0.12

Table 1. Here, s is the number of $\mathcal{B}$ candidates (defined by membership in the Scottish National Party and the Greens), r is the number of candidates from all other parties, and k is the number of seats to be filled in the election. We measure p_B as the level of first-place support for the $\mathcal{B}$ slate. The rows are listed in order of proportionality target, and the dashed lines show the divisions when rounding the target to the nearest integer.

3 Theoretical properties

In this section, we begin by proving an impossibility result, namely that without any control over voter behavior it is impossible to guarantee proportional representation to every conceivable slate of candidates in a given election.

We then move on to some positive results, which serve as proof of concept that the framework presented here is robust enough to admit provable statements about STV, a system of election for which theorems have so far been elusive. The standard assumption of solid coalitions, in particular, has every voter rank all candidates from one slate above all candidates from the other. This assumes away any role for transfer between slates. Therefore, although the assumptions below are strong, they are hugely more flexible than what has existed in the literature so far. In particular, the results of §3.2 fix the candidate order within slates, but allow probabilistic crossover between slates.

The first aim of the results in this section is to show that the generative models let us formulate and prove nontrivial quantitative results; beyond that, we can change one feature of the voting system at a time and see how results differ, as in Proposition 3.2 vs. Proposition 3.3, where a small change in how STV is tabulated can make a large difference. Corollary 3.4 is a surprisingly strong numerical bound on the seats-to-votes ratio for large elections.

3.1 Impossibility result

Proposition 3.1. *Consider an election to fill $k \geq 3$ seats with n candidates, where $n \geq k+2$. Assume a completely uniform election in which every possible ballot appears equally often in the profile. Then no election outcome adheres to proportionality relative to first-place votes (or Borda score) for every possible slate of candidates.*

Proof. Let $\mathcal{W}$ be the set of winners and let $\mathcal{L}$ be the set of losers in the election. Since the share of first-place votes (or Borda score) awarded to $\mathcal{L}$ is at least $\frac{n-k}{n} \geq \frac{2}{k+2} \geq \frac{1}{k}$, the proportionality target for $\mathcal{L}$ is at least $1/k$, i.e. at least one seat, but they receive none. □

In fact, Proposition 3.1 holds more generally for any election in which no set of k candidates achieves a share of first-place votes (or score) greater than $(k-1)/k$. In this setting, the "loser bloc" falls short of proportionality.

This is reminiscent of the classic single-winner impossibility theorems, where tied profiles are also a fundamental edge case. In response to the impossibility paradigm, some authors have moved to consider restricted domains, where certain kinds of edge cases are not allowed. Another extreme is to only consider stylized voting profiles, such as those with solid bloc voting. Our approach adopts a perspective shift to probabilistic models that balance realism and flexibility, and that allow proportionality to be a matter of degree.

3.2 Single bloc asymptotics

In this section, we focus on the case of one bloc of voters and two slates of candidates. Note that even with a single bloc the fact that we have two slates means any lack of cohesion immediately leads to the richer types of crossover ballots that motivated our generative models.

The first results in this section are stated for a particular probabilistic model of voting (which will be used later for empirical experiments) that we call Slate-Plackett-Luce or Slate-PL. We briefly describe the 1-bloc version of the model here, with details to come in §4.1 and §A. Under Slate-PL, each ballot is generated by drawing without replacement from the list of candidates, starting with the voter's first choice. Given two slates of candidates, $\mathcal{A}$ and $\mathcal{B}$, the (marginal) probability of drawing an $\mathcal{A}$ (resp. $\mathcal{B}$) candidate at each draw is fixed by a parameter α (resp. β), unless one of two slates has been exhausted, in which case the choice of slate becomes deterministic. Note that this defines only the slate represented at each rank; filling the ranks with candidates (e.g. $A_1, A_2, \ldots$) can be done in a number of ways (see §4.1 and §A), but in the theorems below we assume a fixed order of candidates within each slate. Note that the expected share of first-place votes (which we will treat as the proportionality target) for $\mathcal{B}$ candidates is β.

Our results reveal that the choice of precise method for tallying votes has a profound impact on the expected outcomes. With that in mind, we define two different methods for deciding which candidates are elected in each round of an STV vote tallying process.

- **Simultaneous election:** if multiple candidates exceed the threshold for election in a certain round, they are all elected and their excess votes transfer down to the remaining candidates before the next round.
- **One-by-one election:** if multiple candidates exceed the threshold for election in a certain round, the one with the most votes is elected and their excess votes are transferred. The tallying process then proceeds to the next round.

Based on the way that election results are reported by the city of Cambridge, it appears that Cambridge follows the simultaneous election method.[11]

Proposition 3.2 (STV with simultaneous election). *Consider an STV contest with simultaneous election for k open seats, a single bloc of N voters, and two slates of candidates $\mathcal{A}$ and $\mathcal{B}$. Suppose that the voters vote according to a Slate-PL model and all voters rank the candidates within each slate in a fixed order, $A_1 > A_2 > \cdots > A_k$ and $B_1 > B_2 > \cdots > B_k$. (Further candidates would therefore be irrelevant.) Write α and β for the tendency to support $\mathcal{A}$ and $\mathcal{B}$ candidates, respectively, and assume $\frac{1}{2} < \alpha < 1$. Then the share of $\mathcal{B}$ candidates elected satisfies*

$$\frac{1}{2} - \frac{1}{2k}\left(\frac{\alpha}{\beta} - 1\right) \leq S_B \leq \frac{1}{2} \qquad \text{a.a.s. as } N \to \infty.$$

Thus, the share of $\mathcal{B}$ candidates elected approaches $1/2$ with high probability as k, N get large, even if their support β is very small.

A proof of Proposition 3.2 can be found in §B; Figure 1 gives a visualization of the result. To obtain the exact $N \to \infty$ asymptotics plotted in the figure, we allow a fractional number of ballots of each kind, and assume that the number of ballots of each kind is exactly equal to the expectation under the model. We also assume that vote transfers are fractional and deterministic.

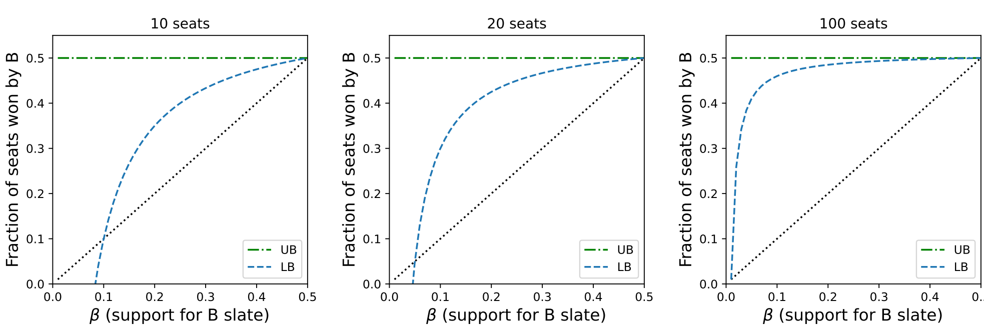

Fig. 1. A visualization of the lower bound and upper bounds in Proposition B.1 for various values of β.

It is somewhat surprising that, as $k \to \infty$, $\mathcal{A}$ and $\mathcal{B}$ are equally represented even though all voters prefer $\mathcal{A}$. Proposition 3.2 assumes simultaneous election transfers—this, together with the fact that there are fixed rankings over $\mathcal{A}, \mathcal{B}$, creates a situation where in nearly every round all first-place votes land on the top remaining $\mathcal{A}$ and $\mathcal{B}$ candidates, and both are elected.

[11] See for instance https://www.cambridgema.gov/Election2023/Official/Council%20Round.htm

We now consider the one-by-one vote tallying method. A practical difference between the simultaneous and one-by-one elections is that one-by-one election may exhibit a kind of leap-frogging, where a candidate who is over the threshold in round 1 may nonetheless be elected after a candidate who was below the threshold in round 1. This does not happen in simultaneous elections.

Proposition 3.3 (STV with one-by-one election). *Take the same setup assumptions as in Proposition 3.2 except for using one-by-one election rather than simultaneous election. Assume that* $\log_\alpha(1/2)$ *is not an integer. Let* $\gamma = \lfloor \log_\alpha(1/2) \rfloor$, *so that* $\alpha^\gamma \geq 1/2$ *but* $\alpha^{\gamma+1} < 1/2$. *Then seat share* S_B *for B candidates satisfies*

$$\frac{\frac{k+2}{\gamma+2} - 1}{k} \leq S_B \leq \frac{1}{2} - \frac{\delta(k+1)}{k}\left(\frac{\left\lceil \frac{2\alpha-1}{\frac{t}{1-\ell t} - 2\alpha \ln \alpha} \right\rceil}{1 + \left\lceil \frac{2\alpha-1}{\frac{t}{1-\ell t} - 2\alpha \ln \alpha} \right\rceil} - \frac{1}{2}\right) \quad \text{a.a.s. as } N \to \infty.$$

for any $\delta \in [0, 1]$. *By setting* $\delta = \frac{\sqrt{k}-1}{\sqrt{k}}$ *we obtain that as* $k \to \infty$, *the value* $\hat{S}_B$ *to which* S_B *tends a.a.s. satisfies*

$$\frac{1}{\gamma + 2} \leq \hat{S}_B \leq \frac{1}{1 + \left\lceil \frac{2\alpha-1}{-2\alpha \ln \alpha} \right\rceil}$$

Figure 2 contains a visualization of Proposition 3.3 using the same method to compute exact asymptotics as in Figure 1.

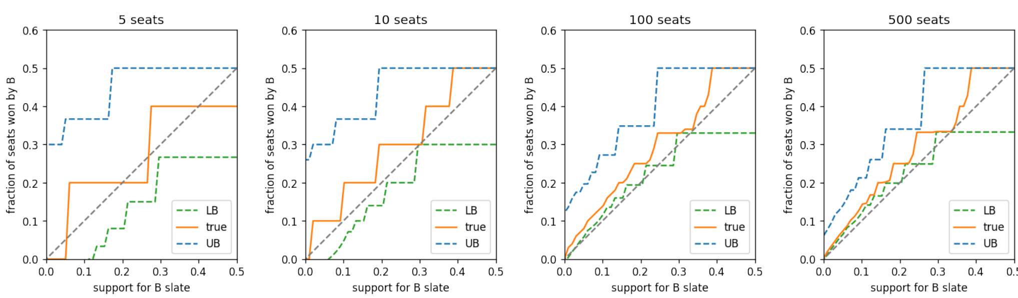

Fig. 2. Visualizations of the lower and upper bounds given by in Proposition 3.3 for $k = 5, 10, 100, 500$ and $\delta = (\sqrt{k} - 1)/\sqrt{k}$. The grey line indicates proportionality.

The preceding propositions let us assess the extent to which STV is likely to yield proportional representation in large elections, if all voters adhere to a rigid ordering of candidates within a slate.

Corollary 3.4 (Bounding disproportionality for STV with fixed candidate orders). *Consider STV under the same conditions as above (Slate-PL, fixed candidate order, sufficiently large* k*). Under simultaneous election, disproportionality can get arbitrarily severe as the election gets large. However, under one-at-a-time election, the asymptotic ratio of seats to votes for the minority party satisfies*

$$\frac{2}{3} \leq \frac{S_B}{\beta} \leq 2,$$

where β *is the support for* $\mathcal{B}$ *candidates. Moreover, for the range* $\beta \in [0, 0.2]$, *we have the lower-bound* $\frac{S_B}{\beta} \geq 1$, *i.e. the* $\mathcal{B}$ *slate always achieves proportionality or better.*

Proof. Simultaneous election means that $\mathcal{B}$ candidates will tend toward 1/2 seat share, no matter their level of support from voters.

Working with the asymptotic value for $N \to \infty$ for one-by-one election, we have, as $k \to \infty$,

$$S_B/\beta \geq \frac{1}{(1-\alpha)\left(\frac{\ln(1/2)}{\ln \alpha}+2\right)}$$

which is an increasing function of $\alpha \in [0.5, 1)$ and at $\alpha = 0.5$ achieves a minimum of $2/3$. For the upper bound, we have

$$S_B/\beta \leq \frac{1}{(1-\alpha)\left(1+\frac{2\alpha-1}{-2\alpha\ln\alpha}\right)}$$

which has a supremum of 2 for $\alpha \in [0.5, 1)$.

For the final claim, we use the tighter bound

$$\beta/S_B \leq (1-\alpha)\left(\lfloor \log_\alpha(1/2)\rfloor + 2\right).$$

Note that the right hand side equals 1 for $\alpha = 0.8$ and $\alpha \to 1$. We separate the intervening values into two cases: $\alpha \in [0.8, 2^{-1/4}]$ and $\alpha \in [2^{-1/4}, 1]$. Over $[0.8, 2^{-1/4}]$, $\lfloor \log_\alpha(1/2)\rfloor$ remains unchanged, but $1-\alpha$ is decreasing, which gives that $\beta/S_B \leq 1$. One can check that $(\log_\alpha \frac{1}{2}+2)(1-\alpha)$ is less than one and has negative first derivative at $\alpha = 2^{-1/4}$, and that it has negative second derivative for all $\alpha \leq 1$. This gives $\beta/S_B \leq 1$ for $\alpha \in [2^{-1/4}, 1]$, completing the proof. □

For completeness, we also consider voters voting under the classical Plackett-Luce model, which generates rankings of candidates outright without an intervening slate-selection step. With candidates $A_1, \ldots, A_r$ and $B_1, \ldots, B_s$ divided into two slates, voters vote by drawing without replacement from the set of candidates where the probability of drawing candidate A_i (resp. B_j) is given by a_i (resp. b_j). The preference for the $\mathcal{A}$ slate is therefore measured by $\alpha = \sum_i a_i$. In §A we outline a generative model, Name-PL, built on this framework.

Proposition 3.5. *Suppose that a single bloc of voters votes according to a Plackett-Luce model with parameters $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s$ corresponding to candidates $A_1, \ldots, A_r, B_1, \ldots, B_s$. In particular, $(\sum_i a_i) + (\sum_i b_i) = 1$ and $\alpha = \sum_i a_i$ denotes the overall preference for the $\mathcal{A}$ slate, so that we assume $\alpha \geq 1/2$ without loss of generality. Then the STV winners (with either vote tallying process) are (a.a.s.) the top candidates by support value, up to a choice about how to break ties between equally supported candidates. Thus we obtain the following results a.a.s. as $N \to \infty$.*

(a) *If there are strong preferences within slates, so that $a_1 > b_1 > a_2 > b_2 > \cdots$, then the number of $\mathcal{A}$ and $\mathcal{B}$ candidates elected is within ± 1, no matter the value of α.*
(b) *On the other hand, if the support is divided uniformly within slates and $\alpha > 1/2$, then only A candidates are elected, assuming there are enough to fill all the seats.*

Proof. Using the definition of the Plackett-Luce model one easily checks that the candidates with the most first-place votes are (a.a.s.) those with the highest support values. Moreover, after vote transfers, the candidates with the most first places will be (a.a.s.) those remaining candidates (not yet elected or eliminated) with the highest support values. This proves the main statement, and (a) and (b) follow. □

3.3 Two-bloc asymptotics with fixed candidate order

We conclude our consideration of electoral outcomes with an observation that the asymptotics of two-bloc elections for the one-by-one variant of STV interpolate between solid coalitions and unpolarized voting in an intuitive way. We consider two blocs A and B, each of which votes according to a Slate-PL model where their tendency to support their own slate is given by π_A and π_B respectively. In Figure 3 we plot exact asymptotic seat outcomes (in the sense of Figure 2) for various numbers of seats and A bloc sizes. Values on the far right and left of each plot represent

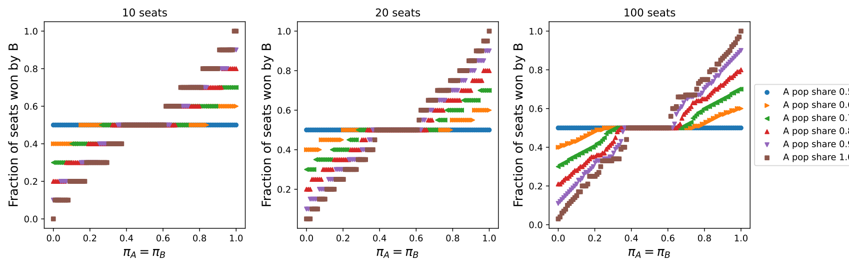

Fig. 3. Exact asymptotics (as the number of voters gets large) showing the share of seats won by the A bloc as their vote share and cohesion varies. The elections have $m = 10, 20, 100$ seats, with an inexhaustible supply of candidates. We use the Slate-PL model, suppose both blocs use the same fixed ordering over $\mathcal{A}$ and $\mathcal{B}$ and apply the one-by-one election variant of STV defined in §3.

solid coalition behavior (perfect cohesion), and thus are predicted by proportionality for solid coalitions. Values in the middle of the x-axis represent elections where voters do not strongly prefer one slate over another.

One interesting (and real) artifact visible in these plots is that the outcome with seat share of 50% is a plateau that occurs for a range of cohesion values. To get an idea of the reason for this, note that since this plot assumes both blocs use a fixed candidate order $A_1, A_2, \dots$ and $B_1, B_2, \dots$, the first candidate elected with $\pi_A, N_A > .5$ will always be A_1. For large numbers of seats, where the election threshold is close to zero, there is a phase transition when $\pi^2 = (1 - \pi) + \pi(1 - \pi)$, occurring at $\pi = 1/\sqrt{2} \approx .707$, that determines whether the first transfer results in the election of A_2. For smaller π, enough support will transfer to B_1 that they are next to be elected. Similar polynomial thresholds determine how many A candidates are elected between successive B candidates. For π approaching $1/2$, the order of election will alternate $ABABAB\dots$, giving $1/2$ seat share to each side.

4 Generative models

4.1 Constructing the models

In this section, we describe three generative models of election, which we call the **Plackett-Luce** (PL), **Bradley-Terry** (BT), and **Cambridge Sampler** (CS) models of preference. The PL and BT models are derived from classical statistical ranking literature, dating back to work the 1950s (with even earlier antecedents). Both PL and BT have parameters that correspond to the probabilistic preference of each kind of voter between the slates. For example, suppose voters in a certain bloc have probability p of preferring an A to a B. Then the PL probability that a ballot starts AAB is $p^2(1 - p)$. On the other hand, the BT model considers the pairwise comparisons, so the probability that a ballot starts AAB is proportional to p^2 because the ballot lists an A candidate above a B candidate in two different ways. The high-level descriptions above are of the Slate type; the models are fully explained in §A.[12] The Cambridge sampler is a randomized model that uses the parameter p to decide if a voter is likely to rank an A or a B first, then uses a labeled dataset from city council elections in Cambridge, MA in which candidates have been labeled as either White (W) or people of color (C). The user decides whether to map $A \mapsto W, B \mapsto C$ or vice versa, and then a ballot type is chosen proportional to the actual historical frequency from among the ones with that first

[12]To be more precise, each of these models has a Name and a Slate variant; the mathematics is similar, but the Slate variant works with ballot types like AAB while the Name variant works with detailed ballots like $A_3A_2B_1$. Versions of the Name-PL, Name-BT, and Slate-CS models have been discussed in unpublished work by an overlapping collection of authors. References are suppressed here for anonymization purposes.

choice type. By construction, the PL and BT models always generate complete rankings, while the CS model includes partial rankings at a rate derived from historical voter behavior in Cambridge. Together, these make up the new generative models explored in the empirical work in this paper.

We focus on settings with 1-2 blocs for notational convenience but the framework immediately expands to more groups. In all models, the key parameter for each bloc is the strength of support for their own slate. In the two bloc case, blocs A and B support their own slates with strengths π_A and π_B respectively. Typically, these parameters are at least 0.5, and setting them equal to 1 recovers solid coalition behavior for the PL and BT models (but not the CS model); see §3.3 above for some numerical asymptotics for Slate-PL in this setting. In the one bloc case, we treat the voters as being in bloc B by convention, leaving only π_B as the main parameter. Theoretical results for Slate-PL with this setup (with the additional assumption of fixed candidate ordering) were established in §3.2, where we used the notation $\alpha = 1 - \pi_B$ and $\beta = \pi_B$.

The models have first-principles descriptions as capturing different kinds of voting behavior. Since a PL voter can be said to fill in their ballot from top to bottom according to prior preferences, we can think of this as modeling an "impulsive" voter. By contrast, a BT voter makes comparisons of every two entries on their ballot and weighs that ballot against one with some reversals, modeling a "deliberative" voter. These give new generative models to study, greatly expanding on what is already present in the COMSOC literature. Furthermore, they build on a framework that comports well with U.S. voting rights law; the standard legal polarization analysis produces cohesion estimates for majority and minority groups suitable for use as the π_A, π_B parameters. We will give a brief validation showing model performance in matching a few observed elections in §4.3.

Remark 1 (First-place vote distribution). *Slate-PL with $(A, \mathcal{A}, B, \mathcal{B})$ and any cohesion and candidate strength parameters is expected to produce blocs with consistent positional preference π_B (respectively π_A) for their own slates, and therefore with first-place preference π_B (or π_A) as well. Likewise, the Cambridge sampler produces expected first-place preference π_B by construction. However, Slate-BT has first-place support related to π_B by a monotone increasing, but non-identity function.*

Remark 2 (About the Cambridge data). *Cambridge, Massachusetts uses STV for its city council and school board elections and has done so since 1941. Our source of Cambridge historical data is city council elections to fill $k = 9$ seats by STV from 2009 to 2017, coded by candidate race as described above; there are frequently 20 or more candidates who run in each contest. If a ballot type is selected from the historical frequency histogram that has more candidates from a given slate than the (r, s) chosen for a given simulation run allows, then we ignore further instances. For instance, a ballot type of AAABB in an election where $r = s = 2$ will be read as AABB.*

One valuable aspect of our use of Cambridge historical data in the present study is that it lets us incorporate realistic short-ballot voting behavior without a proliferation of extra parameters. For instance, Cambridge voters cast "bullet votes" (listing only one candidate and leaving other positions blank) 7501 times out of 87,914 ballots cast in our data set, and this will be reflected in the ballots generated by the CS model. However, a serious limitation is that we have coded the candidates by race, while Cambridge city council politics are likely more polarized by other candidate features—for instance, an explicit slate of affordable housing candidates is routinely advertised before election day and is highly salient to voter behavior. Nevertheless, race is a candidate feature often apparent to voters which allows us to observe naturalistic patterns of alternation in voting.

Remark 3 (Mixture models). *The definitions above are in terms of specified blocs of voters with different voting preferences. However, there is a strong connection to mixture models suggested by the structure here. In a mixture model, each voter is assigned independently to a class, and then randomly submits a ballot based on the parameters for that class. More precisely, if N_1 and N_2 are the weights* *for two different classes of voter with $N_1 + N_2 = 1$, and μ_1 and μ_2 are two distributions on ballots corresponding to the two classes, the probability of a ballot σ is $\mu(\sigma) = N_1\mu_1(\sigma) + N_2\mu_2(\sigma)$.*

As the number of voters increases, the fraction of voters assigned to each class converges to N_1 and N_2 respectively; for large numbers of voters we can therefore consider the size of each class to be predetermined and treat voters as if they belong to two blocs of fixed size. In particular, since it considers pairwise probabilities, the BT model with two blocs resembles a mixture of Mallows models. It differs in weighing swaps by preference between slates rather than by position in the ranking.

4.2 Visualization

One difficulty in studying ranked choice elections is that, unlike oversimplified Example 2.2, real-world elections frequently have too many valid ballots possible to effectively see the full preference profile. For instance, an election with six candidates can be thought of as having 1236 possible ballots to cast—there are 6! complete rankings and a roughly equal number of partial rankings.[13] Thinking of profiles as distributions over valid ballots allows us to define natural notions of distance between profiles, such as the L^1 distance between profiles given by the sum over possible ballots of the absolute value of the difference of shares for that ballot. (Up to a constant factor, this is the same as the total variation distance of distributions.) With this notion we can visualize differences between the generative models as we vary parameters. Below, U, S, X, and Y are four different *candidate strength* settings describing whether there is agreement in how voters rank their preferences *within* each slate. See §A.1 for details.

In the multi-dimensional scaling (MDS) plot in Figure 4, the first-place preference for $\mathcal{B}$ candidates is $\pi_B = .75$; Supplemental Figure 11 shows how the outputs change as π_B varies. In this plot, we can see some systematic differences and similarities.[14]

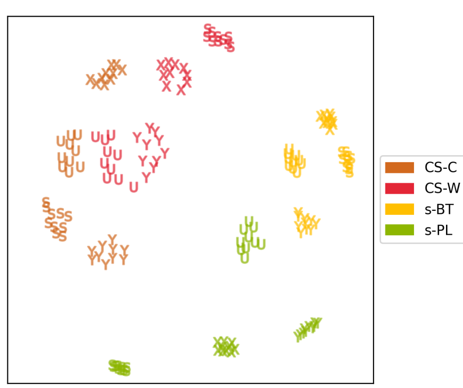

Fig. 4. Multi-dimensional scaling (MDS) plot for one-bloc profiles with $r = s = 3$ (3 candidates per slate) and $\pi_B = .75$, under several generative models (CS, BT, PL) and candidate strength scenarios (U, S, X, Y). The pairwise distances between profiles are computed with L^1 distance on the distributions. Each preference profile has 1000 ballots; we generate 10 profiles by each of the 16 model/strength pairs. Note: it is not surprising that CS profiles, where bloc B is identified with W-led or C-led ballots, fall far from PL and BT profiles, since PL and BT always generate complete rankings, while CS uses real historical data that includes many partial rankings. This observation can be used to give a sense of scale for the distances in the plot.

[13]Here, we identify a ballot of length 5 with a complete ranking of length 6, since the last-place candidate is implicit.
[14]The reader should recall that MDS plots are simply low-distortion planar embeddings, which depend on a choice of random seed. The x and y axes have no meaning; only the relative pairwise distances are meaningful with respect to the data. We have verified that the structure of the plots stays the same for a few choices of random seed.

It is notable that the strength settings can generate differences in the profiles that are as large as the overall model choice. Also, BT profiles resemble both kinds of Cambridge outputs more than PL profiles do, though the reason for this is far less clear. (Compare Supplemental Figures 14–22, which bear this out from another point of view.)

4.3 Validation on Scottish elections

A benefit of using parameterized generative models is the possibility of fitting to real-world elections. Though we leave a full-bore fitting effort to future work, this section shows the potential of this approach to match the observed non-solidity of coalitions.

To this end, we define a *swap distance* between two ballot types, partial or complete. For complete ballots, this counts the smallest number of swaps of adjacent symbols necessary to transform one ballot type into the other; for instance, $\mathrm{dist}(AABBB, ABBAB) = 2$. See §C for a discussion of efficiently measuring this distance, including an extension to partial or weakly ranked ballots.

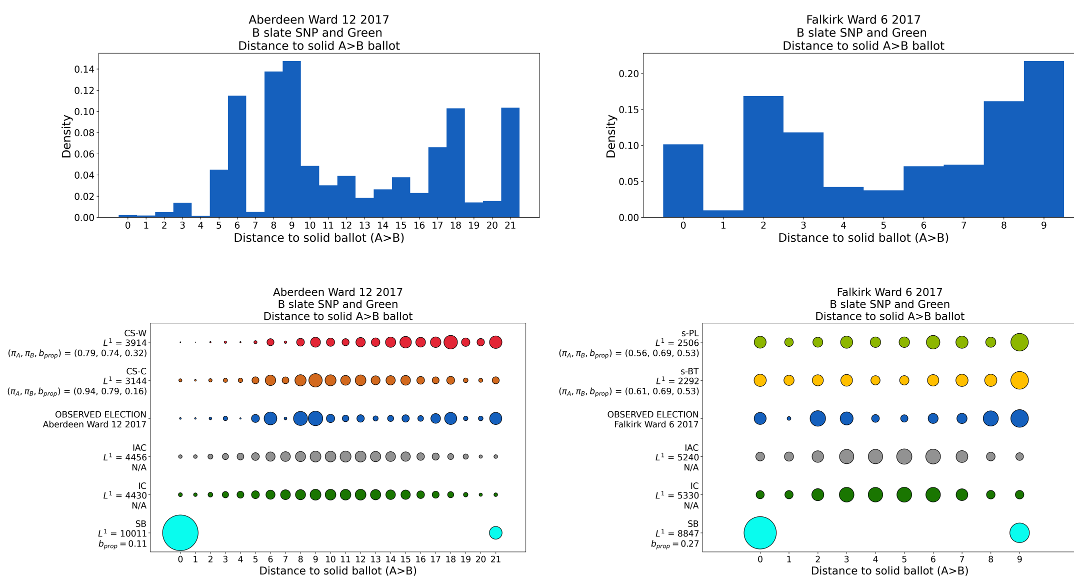

Fig. 5. Top: Histograms showing the distribution of swap distances to solid A-over-B type in Aberdeen Ward 12 and Falkirk Ward 6, 2017. Bottom: Bubble plots showing the distribution of swap distances, where the area of each circle is proportional to frequency. The top two colored rows show outputs from models introduced in this paper, with parameters optimized to match the observed election. The third row in dark blue is the observed election, for which the data exactly repeats the conventional histograms. The bottom three rows show the best fit when voters are constrained to solid ballots (SB), followed by samples under IC (impartial culture) and IAC (impartial anonymous culture) models already popular in the social choice literature.

Using swap distance, we can investigate the extent to which vote profiles deviate from the solid coalition assumption. Let us return to the Scottish elections and the slate $\mathcal{B}$ discussed in Example 2.3. For every ballot cast in the election, we can compute its distance to the solid A-over-B ballot type A^sB^r. (A solid vote of the opposite kind looks like B^rA^s, lying at distance rs.) For the Aberdeen Ward 12 and Falkirk Ward 6 elections from 2017, these distances are summarized in the histograms of Figure 5.

Next, we can attempt to generate profiles that are the best match for these histograms using the models in §4.1. We choose our cohesion parameter by optimizing π_B to minimize the L^1 distance to the observed election. The resulting distance distributions are visualized in the bubble plots of Figure 5 (and see §E for a full range of outputs). The traditional assumption of solid coalitions produces distributions that are point masses at distances 0 and rs, which clearly have little in common with the real-world ballot distributions. Both visually and in terms of measured L^1 distance, the models do well at matching observed patterns of non-solidity of coalitions.

4.4 Comparing voting rules

Next, we leverage the generative models in combination with a voting rule to produce simulations that highlight complex interactive effects between model parameters. We use this to examine the degree of proportionality of STV and Borda style elections in Figure 6.

We vary N_B over $\{.05, .15, \dots, .95\}$ and we vary both π_A and π_B over $\{.55, .65, .75, .85, .95\}$. We have selected four candidate strength scenarios for two blocs (as in the one-bloc scenarios in §4.2); these are chosen to give a small window on how powerfully candidate strength can interact with other factors. See §A.1 for details.

In effect, we must make five choices for each batch of runs: model, strength scenario, population share, cohesion for A voters, and cohesion for B voters. We then generate a batch of 100 profiles from each 5-tuple to place each symbol on the plot. The x-axis position is the combined support level for $\mathcal{B}$ candidates observed in the profiles, given by $C_B = N_B \cdot \pi_B + (1 - N_B)(1 - \pi_A)$ as above—so a given support level can be achieved in many different ways. The y-axis position is the average number of seats won by $\mathcal{B}$ candidates when the batch of profiles is run through a voting rule.

If the proportionality ideal were hit exactly, the symbols would all fall on the main diagonal. The proportionality target rounded up and down to whole numbers of seats is shown with dotted lines. Each plot uses the same preference profiles, so the difference in the degree of proportionality is entirely attributed to the election rule. And the difference is striking, not only visually, but in terms of measured disproportionality. Under STV, the dataset has $\widehat{\text{disprop}} = .08$, while the same set of profiles under a Borda voting rule gives $\widehat{\text{disprop}} = .14$, or almost twice the mean absolute error.

5 Conclusion and future work

Our goal in this paper is to lay the groundwork to systematically study the tendency of systems to deliver more or less proportional outcomes for voters. Crucially, the framework we propose allows but does not require party labels, so that we can also identify emergent blocs with similar voting behavior after an election has been conducted. Finally, the new generative models outlined here can be theoretically explored, opening up rich directions for mathematical study, but can also give decision-makers a powerful toolkit for practical electoral reform.

In §4.3 we make first steps toward fitting models and parameters to realistic elections, with immediate payoff in a starkly improved correspondence to Scottish ranked elections than solid coalitions could offer. A more comprehensive fitting effort along these lines—simultaneously learning optimal blocs and slates from observed elections—was begun in a current paper with overlapping authors [citation redacted]. This can also point the way to new methods of measuring the degree of polarization, which can feed back usefully into voting rights law.

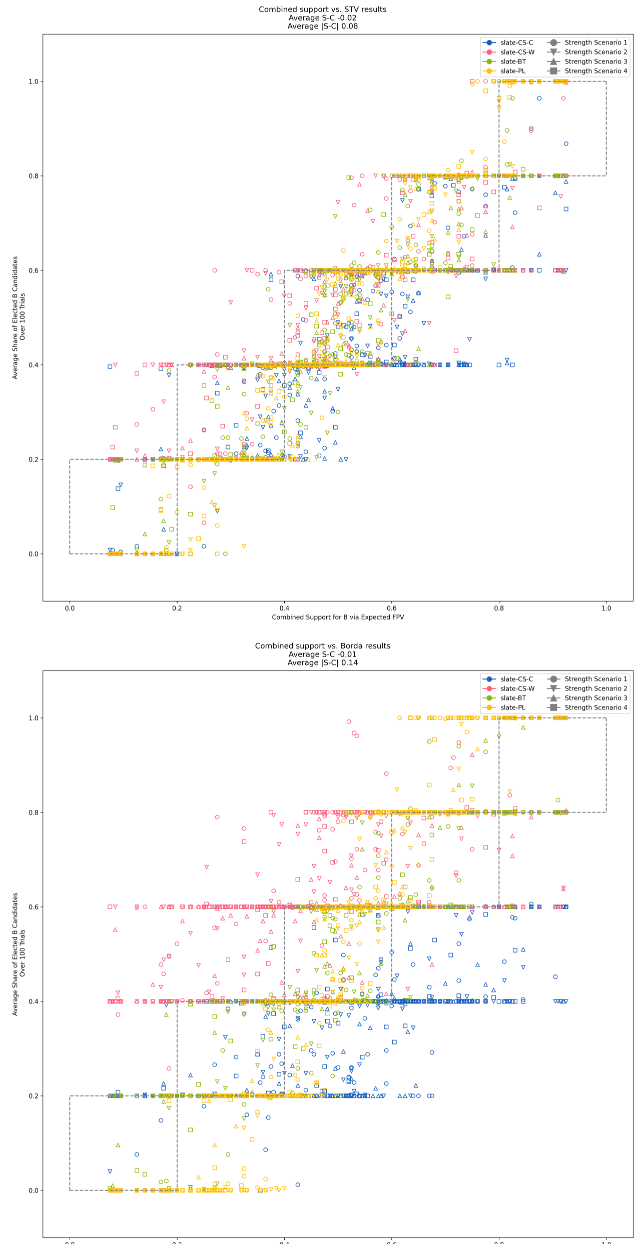

Fig. 6. Setting $(r, s, k) = (5, 5, 5)$, we independently vary the B proportion of the electorate, the generative model, the A and B cohesion, and the candidate strength settings, for a total of 4000 parameter tuples. We generate 100 profiles for each tuple. We then run the same set of profiles through the **STV** voting rule (top) and the **Borda** voting rule (bottom). The x axis position is the combined support for $\mathcal{B}$, which is the same in both plots, and the y-axis position is the average number of seats given by the voting rule to $\mathcal{B}$ candidates, over the 100 trials. The dashed lines show the proportionality target rounded up and down to the nearest whole number of seats. Since the profiles are the same, the difference in the degree of proportionality is entirely attributable to the voting rule.

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A More details on generative models

The statistical ranking models will be introduced with what we call Slate vs. Name versions: the Slate versions begin by constructing an abstract ballot type (such as $AABBA$) before filling in candidate names (producing, for instance, $A_2A_1B_3B_1A_3$), while the Name versions work directly with candidate names. Though at first the by-name and by-slate versions may seem extremely similar, we find that Slate-PL and Slate-BT have several desirable properties compared to Name-PL and Name-BT.

Definition A.1. For all of the models below, assume a fixed bloc structure $(A, \mathcal{A}, B, \mathcal{B})$ with $\mathcal{A} = (A_1, \ldots, A_r)$ and $\mathcal{B} = (B_1, \ldots, B_s)$, allowing the possibility that $A = \emptyset$ as before.

A *ballot* is a partial or complete ranking of the $r + s$ candidates and a *ballot type* is a partial or complete permutation of the symbols $A^r B^s$, i.e., a simplified ballot that treats the candidates of each slate as indistinguishable from each other.

The models below will use the following parameters to generate a profile for bloc B:

Cohesion Tendency of the bloc to support slate $\mathcal{B}$, given as a parameter $\pi_B \leq 1$ (typically required to be at least 1/2 in the multi-bloc case).

Strength Tendency of bloc B to agree on preferred candidates *within* each slate. This consists of probability vectors $I_{BA} = (a_1, \ldots, a_r)$ and $I_{BB} = (b_1, \ldots, b_s)$; i.e., the entries are non-negative and sum to one. For instance, if $I_{BA} = (.1, .8, .1)$, then typical B voters strongly prefer candidate A_2 to A_1 or A_3.

We can combine the cohesion and strength data into a single probability vector

$$I_B = \big((1 - \pi_B)a_1, \ldots, (1 - \pi_B)a_r, \quad (\pi_B)b_1, \ldots, (\pi_B)b_s\big).$$

Using these components, we can define five generative models as follows. The first two work directly with ballots, while the latter three first construct ballot types. These are analogous to the profile by name and the profile by slate in Example 2.2.

Name-PL Plackett-Luce by name: Each B-bloc voter chooses candidate i to be ranked first with probability $I_B(i)$. They continue to select candidates for lower-ranked positions in order, at each stage selecting candidate j with probability proportional to $I_B(j)$. In other words, each voter samples their ballot without replacement from all candidates proportional to their weighting in I_B.

Name-BT Bradley-Terry by name: The probability that a B voter casts a ballot σ is proportional to

$$\prod_{i <_\sigma j} \frac{I_B(i)}{I_B(i) + I_B(j)},$$

where $i <_\sigma j$ means that i is ranked before (i.e., higher than) j in σ. In other words, for each pairwise comparison of candidates, we introduce a term for the likelihood of ranking one before the other coming from the relative weights in I_B.

Slate-PL Plackett-Luce by slate: Each B-bloc voter chooses between the symbol A and B in the ith position with probability π_B of choosing B, as long as both $\mathcal{A}$ candidates and $\mathcal{B}$ candidates remain available. Once a slate is exhausted, the rest of the complete ranking is filled in with the remaining symbol.

Slate-BT Bradley-Terry by slate: Suppose a ballot type σ is a permutation of $A^r B^s$, that is, an ordered list containing r A symbols and s B symbols. Suppose that out of the rs comparisons of the instances of A with the instances of B, the A occurs earlier than the B a total of $0 \le i \le rs$ times. The probability that a B voter casts this ballot is proportional to $(1-\pi_B)^i(\pi_B)^{rs-i}$.

Slate-CS Cambridge sampler: We draw from a dataset consisting of ten years of ranked votes from city council elections in Cambridge, MA. Historical candidates have been labeled as white (W) or as people of color (C), with help from local organizers. To use this model, we make a choice to designate bloc B as corresponding to voters who put a W candidate first ($B = W$), or who put a C candidate first ($B = C$). We use the cohesion parameter π_B to decide probabilistically whether the voter chooses their own slate or the other slate in the first position. Then we complete the ballot type by drawing with weight proportional to frequency from the cast ballots with that header.

In all three Slate models, we must then assign candidate names to the symbols A and B. We do so by drawing without replacement (Plackett-Luce style) from I_{BA} and I_{BB} separately to order $\mathcal{A}$ and $\mathcal{B}$, then fill in names accordingly.

Remark 4 (Names versus slates). *It turns out to be an important distinction to work directly with the names or to create a type first, then add names. The reason for the divergence is that the Slate models handle I_{BA} and I_{BB} separately; concatenating them into I_B before making length comparisons yields unintended results, such as a highly cohesive bloc whose voters tend to put their strong candidate first and then immediately cross over to supporting the opposite slate.*

A.1 Strength parameters

To illustrate the importance of candidate strength, we introduce a few out of the infinitely many variations.

For the one-bloc profiles, we specify voter preferences with respect to the $\mathcal{A}$ and $\mathcal{B}$ slates.

- **U** (uniform-uniform): preferences are uniform over $\mathcal{A}$ candidates and uniform over $\mathcal{B}$ candidates.
- **S** (strong-strong): preferences are strong over both slates, namely with some candidates receiving more support.
- **X** (uniform-strong): uniform support for $\mathcal{A}$ candidates and strong support for some $\mathcal{B}$ candidates;
- **Y** (strong-uniform): the reverse.

For the two-bloc profiles, we again base four strength scenarios on similar considerations.

- **Scenario 1**: both blocs have uniformly random preference order over each slate;
- **Scenario 2**: I_{BB} has a strong candidate while others are uniform;
- **Scenario 3**: A and B blocs strongly prefer the same $\mathcal{B}$ candidate and are otherwise uniform;
- **Scenario 4**: A and B blocs strongly prefer different $\mathcal{B}$ candidates and are otherwise uniform.

A.2 More parameter interactions

While the plots in Figure 6 show all 4,000 of the model parameter tuples at once, we can also split these out by candidate strength scenario which reveals interesting effects of candidate strength on the degree of proportionality.

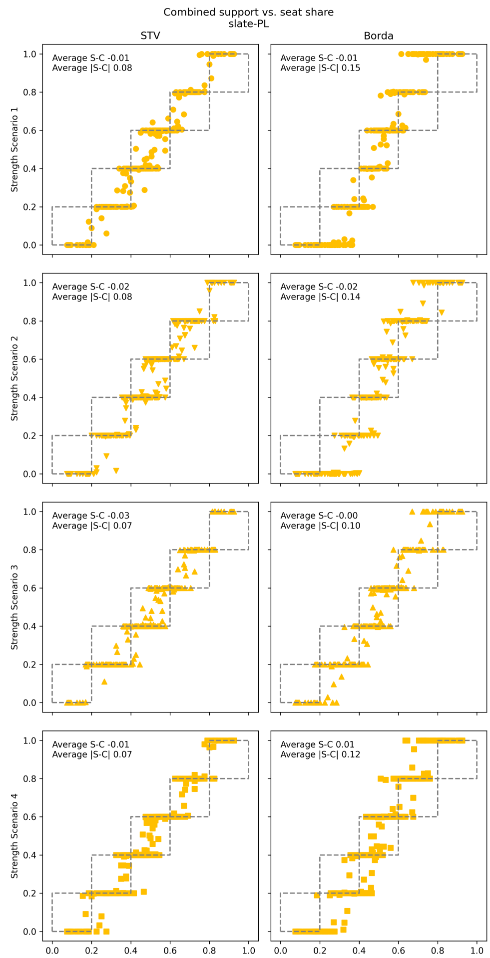

Fig. 7. This grid of plots shows how the choice of election rule and candidate strength scenario can impact the degree of proportionality for profiles generated by the slate-PL model. The left column shows STV results, and the right column shows Borda results. Each row uses the same preference profiles, so the difference in the degree of proportionality between the two plots in each row is entirely due to the election rule.

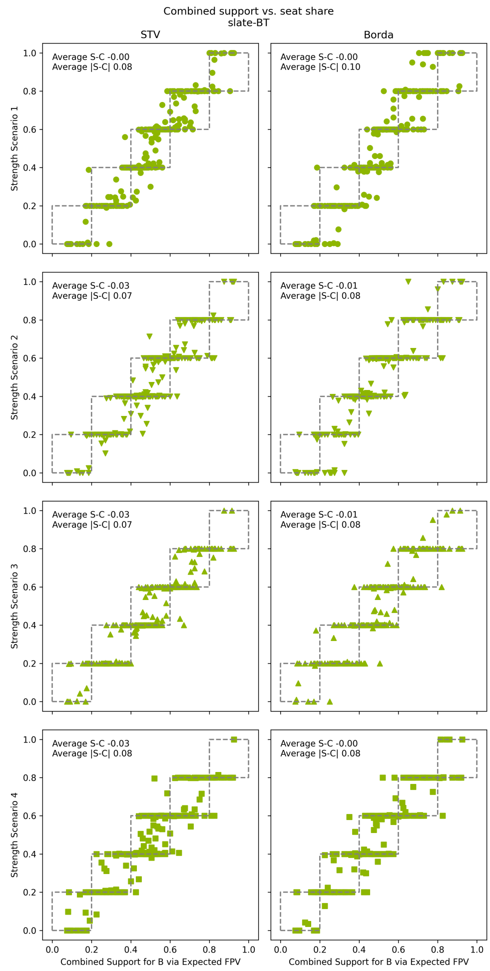

Fig. 8. This grid of plots shows how the choice of election rule and candidate strength scenario can impact the degree of proportionality for profiles generated by the slate-BT model. The left column shows STV results, and the right column shows Borda results. Each row uses the same preference profiles, so the difference in the degree of proportionality between the two plots in each row is entirely due to the election rule.

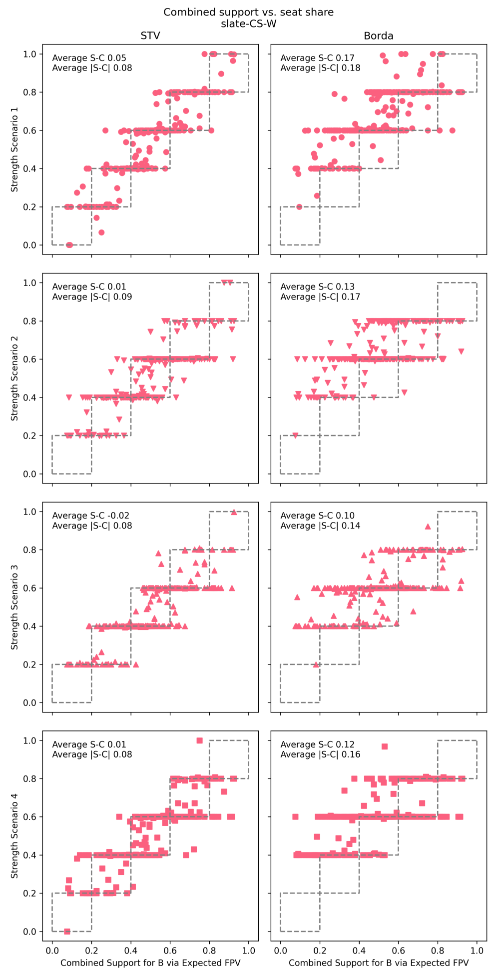

Fig. 9. This grid of plots shows how the choice of election rule and candidate strength scenario can impact the degree of proportionality for profiles generated by the slate-CS-W model. The left column shows STV results, and the right column shows Borda results. Each row uses the same preference profiles, so the difference in the degree of proportionality between the two plots in each row is entirely due to the election rule.

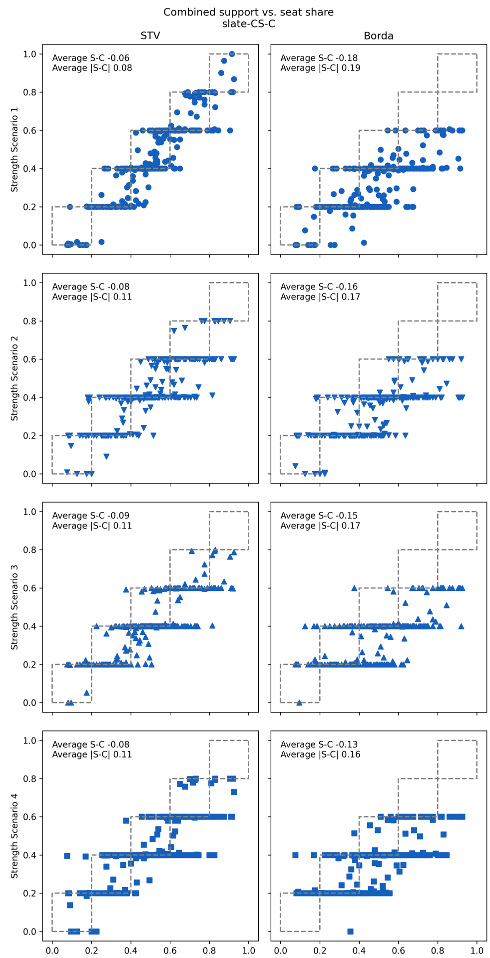

Fig. 10. This grid of plots shows how the choice of election rule and candidate strength scenario can impact the degree of proportionality for profiles generated by the slate-CS-C model. The left column shows STV results, and the right column shows Borda results. Each row uses the same preference profiles, so the difference in the degree of proportionality between the two plots in each row is entirely due to the election rule.

B Proofs

Proposition B.1 (STV with simultaneous election). *Consider an STV contest with simultaneous election for k open seats, a single bloc of N voters, and two slates of candidates $\mathcal{A}$ and $\mathcal{B}$. Suppose that the voters vote according to a Slate-PL model and all voters rank the candidates within each slate in a fixed order, $A_1 > A_2 > \cdots > A_k$ and $B_1 > B_2 > \cdots > B_k$. (Further candidates would therefore be irrelevant.) Write α and β for the tendency to support $\mathcal{A}$ and $\mathcal{B}$ candidates, respectively, and assume $\frac{1}{2} < \alpha < 1$. Then the share of $\mathcal{B}$ candidates elected satisfies*

$$\frac{1}{2} - \frac{1}{2k}\left(\frac{\alpha}{\beta} - 1\right) \leq S_B \leq \frac{1}{2} \qquad \text{a.a.s. as } N \to \infty.$$

Thus, the share of $\mathcal{B}$ candidates elected approaches $1/2$ with high probability as k, N get large, even if their support β is very small.

Proof. In this proof, we will treat the ballot shares as deterministically equaling their expectations, so that any strict inequalities we derive stay true with high probability as $N \to \infty$. First, observe that the fixed order means that in a given round, only one A candidate and one B candidate has any first-place votes. The use of the Droop quota means that $N/(k+1)$ votes is the threshold of election; when a candidate reaches that threshold, they are elected and their excess votes are transferred, leading to a reduction of the mass of total votes by $N/(k+1)$; ballots can not be exhausted because Slate-PL produces complete rankings. Until the full k seats are elected, this means that there is always enough vote mass to elect at least one candidate per round, and since only two have first-place support, either one candidate is elected, or one A and one B are elected simultaneously. Suppose that one candidate from each slate is elected in rounds $1, \ldots, \ell$, but that only one candidate is elected in round $\ell + 1$; this must be an $\mathcal{A}$ candidate with high probability, because $\alpha > \beta$. So in round $\ell + 1$, the support for $\mathcal{B}$ has dropped below threshold. Solving for ℓ, we have

$$\left(1 - \frac{2\ell}{k+1}\right)\beta \geq \frac{1}{k+1}; \quad \left(1 - \frac{2(\ell+1)}{k+1}\right)\beta < \frac{1}{k+1},$$

and this gives

$$(k - 2\ell - 1)\beta < 1 \leq (k - 2\ell + 1)\beta \implies k - \frac{\alpha}{\beta} - 1 \leq 2\ell < k - \frac{\alpha}{\beta}.$$

Thus there are at least $\frac{k}{2} - \frac{\alpha}{2\beta} - \frac{1}{2}$ candidates from $\mathcal{B}$ elected, as claimed.

For the upper bound, note that the only round in which a single B can be elected immediately follows a round in which a single A was elected. This is because after one of each is elected (whether this occurs simultaneously or in successive rounds), all first-place votes have been cleared and the full mass of remaining ballots now favors the next A candidate with a share close to α. □

The following technical lemma will be required in the next proof.

Lemma B.2. *Let $\alpha \in [0.5, 1)$ and $s \in \mathbb{N}$.*

(a) $2\alpha^s \geq 2\alpha s \ln \alpha + 2(\alpha - \alpha \ln \alpha)$.

(b) *If* $1 \leq s \leq \log_\alpha(1/2) + 1$, *then*

$$\frac{\alpha(1 - \alpha^{s-1})}{1 - \alpha} \leq s - 1 \leq \frac{2\alpha(1 - \alpha^{s-1})}{1 - \alpha}$$

Proof. (a) Follows by considering the tangent line to $f(s) = 2\alpha^s$ at $s = 1$. (b) For the left hand bound, notice that since $\alpha < 1$,

$$\frac{\alpha(1 - \alpha^{s-1})}{1 - \alpha} < (1 + \alpha + \ldots + \alpha^{s-2}) \leq (s - 1)$$

For the right hand bound, we have

$$\frac{2\alpha(1-\alpha^{s-1})}{1-\alpha}-(s-1)\geq\frac{2\alpha-\alpha^{s}}{1-\alpha}-\log_{\alpha}(1/2)\geq\frac{\alpha}{1-\alpha}-\log_{\alpha}(1/2)$$

The last expression is zero for $\alpha = 1/2$ and a derivative test shows that it is positive for all $\alpha \in (0.5, 1)$. □

Proposition B.3 (STV with one-by-one election). *Take the same setup assumptions as in Proposition B.1 except for using one-by-one election rather than simultaneous election. Assume that* $\log_\alpha(1/2)$ *is not an integer. Let* $\gamma = \lfloor \log_\alpha(1/2)) \rfloor$*, so that* $\alpha^\gamma \geq 1/2$ *but* $\alpha^{\gamma+1} < 1/2$*. Then seat share* S_B *for B candidates satisfies*

$$\frac{\frac{k+2}{\gamma+2}-1}{k}\leq S_B\leq\frac{1}{2}-\frac{\delta(k+1)}{k}\left(\frac{\left\lceil\frac{2\alpha-1}{\frac{t}{1-\ell t}-2\alpha\ln\alpha}\right\rceil}{1+\left\lceil\frac{2\alpha-1}{\frac{t}{1-\ell t}-2\alpha\ln\alpha}\right\rceil}-\frac{1}{2}\right)\quad\text{a.a.s. as } N\to\infty.$$

for any $\delta \in [0, 1]$*. By setting* $\delta = \frac{\sqrt{k}-1}{\sqrt{k}}$ *we obtain that as* $k \to \infty$*, the value* $\hat{S}_B$ *to which* S_B *tends a.a.s. satisfies*

$$\frac{1}{\gamma+2}\leq\hat{S}_B\leq\frac{1}{1+\left\lceil\frac{2\alpha-1}{-2\alpha\ln\alpha}\right\rceil}$$

Proof. Consider the tally after ℓ rounds. Let the share of all ballots (live or not) currently headed by an A be $\hat{\alpha} \leq 1-\ell t$. In the next round, the share of ballots headed by an A is $(\hat{\alpha}-t)\alpha$. After round $\ell+s-1$, assuming only A candidates have been elected since round ℓ, the share of ballots headed by an A is $\hat{\alpha}\alpha^{s-1}-t\alpha^{s-1}-\cdots-t\alpha$. Since the total mass of ballots left at that stage is $1-(s-1+\ell)t$, it follows that the difference between the share headed by an A and the share headed by a B is given by

$$\begin{aligned}\Delta(s) &= 2\hat{\alpha}\alpha^{s-1}-1-t(2\alpha^{s-1}+\cdots+2\alpha)+t(s-1+\ell) &(1)\\ &= 2\hat{\alpha}\alpha^{s-1}-(1-\ell t)-t\left(\frac{2\alpha(1-\alpha^{s-1})}{1-\alpha}-(s-1)\right) &(2)\end{aligned}$$

In particular, an A will be elected next if and only if $\Delta(s)$ is positive.

We define a *sequence* as a set of consecutive rounds consisting of electing A candidates, followed by a B candidate. If $\ell = 0$, then $\hat{\alpha} = \alpha$. By Lemma B.2, $\Delta(s) \leq 2\alpha^s - 1$ for all $s \leq \gamma$, so $\Delta(s)$ is negative for $s = \gamma + 1$. Thus the first sequence has length at most γ.

If round ℓ was the first round of some later sequence, then all ballots headed by an A transferred last round, so $\hat{\alpha} \leq \alpha(1-\ell t)$. Thus

$$\Delta(s)\leq(1-\ell t)(2\alpha^{s}-1)-t\left(\frac{2\alpha(1-\alpha^{s-1})}{1-\alpha}-(s-1)\right)$$

and using Lemma B.2 again, we have that this is negative for $s = \gamma + 1$. Thus this sequence has length at most $\gamma + 1$, since we allow for round ℓ to be the first round of the sequence.

Suppose there are r sequences, followed possibly by electing a final set of A candidates. The best case for A candidates is if this final set consists of $\gamma + 1$ A candidates. In that case there are $\gamma + (r-1)(\gamma+1) + \gamma + 1 = r(\gamma+1) + \gamma$ A candidates elected and r B candidates elected. Since $k = r(\gamma+1) + \gamma + r = (r+1)(\gamma+2) - 2$, we obtain

$$S_A\leq\frac{k-r}{k}=\frac{k-\frac{k+2}{\gamma+2}+1}{k}$$

We now turn our attention to the lower bound. Suppose that we are in round ℓ at the start of a sequence. By Lemma B.2 and the fact that $\hat{\alpha} \geq (1-\ell t)\alpha$, we have

$$\begin{aligned}\Delta(s) &\geq 2(1-\ell t)\alpha^s - (1-\ell t) - t(2(s-1)-(s-1))\\ &\geq 2(1-\ell t)\left(\alpha s\ln(\alpha) + (\alpha - \alpha\ln\alpha)\right) - (1-\ell t) - t(s-1).\end{aligned}$$

This last expression is decreasing in s. The root of the expression is given by

$$s = 1 + \frac{2\alpha - 1}{\frac{t}{1-\ell t} - 2\alpha\ln\alpha}$$

Defining

$$\lambda(\ell) := \left\lceil \frac{2\alpha - 1}{\frac{t}{1-\ell t} - 2\alpha\ln\alpha} \right\rceil$$

we have that $\lambda(\ell)$ is a lower bound on the number of A candidates elected, starting at round ℓ, before a B is elected.

Fix $\delta \in [0,1]$. Since a candidate is elected every round, there are k rounds total. Suppose that m is the number of rounds contained in sequences whose first round is before or equal to round $\lfloor \delta(k+1) \rfloor$, so that in particular, $m \geq \lceil \delta(k+1) \rceil$. The fraction of these m rounds in which an A is elected is at least $\lambda(\delta(k+1))/(\lambda(\delta(k+1))+1)$. Since an A is elected in round m, an A is elected in at least half of the remaining $k-m$ rounds. Thus the seat share for A satisfies

$$\begin{aligned}S_A &\geq \frac{1}{k}\left(m \cdot \frac{\lceil \frac{2\alpha-1}{\frac{t}{1-\ell t}-2\alpha\ln\alpha} \rceil}{1+\lceil \frac{2\alpha-1}{\frac{t}{1-\ell t}-2\alpha\ln\alpha} \rceil} + \frac{k-m}{2}\right)\\ &\geq \frac{1}{k}\left(\delta(k+1) \cdot \frac{\lceil \frac{2\alpha-1}{\frac{t}{1-\ell t}-2\alpha\ln\alpha} \rceil}{1+\lceil \frac{2\alpha-1}{\frac{t}{1-\ell t}-2\alpha\ln\alpha} \rceil} + \frac{k-\delta(k+1)}{2}\right)\\ &= \frac{1}{2} + \frac{\delta(k+1)}{k}\left(\frac{\lceil \frac{2\alpha-1}{\frac{t}{1-\ell t}-2\alpha\ln\alpha} \rceil}{1+\lceil \frac{2\alpha-1}{\frac{t}{1-\ell t}-2\alpha\ln\alpha} \rceil} - \frac{1}{2}\right)\end{aligned}$$

□

C Swap distance and ballot completion

The distance between two (complete) ballots measures the complexity of swaps to turn one ballot into the other. We will generalize this to ballot types (where candidates within each slate are indistinguishable from one another). We will allow swaps of individual neighboring candidates at unit cost as a special case of general individual swaps, whose cost is the difference in their positions. For instance, though ballot ABC could be transformed to CBA with three neighbor-swaps, its total cost will be just 2 because A and C can be exchanged directly, leaving B in place. To calculate this efficiently, we adapt a lemma from a preprint of Duchin and Tapp [2024]. Given an ordering of candidates, let the score vector sc of a ballot be defined as the vector of Borda scores earned by each candidate, so for the candidate order A, B, C we have

$$\mathrm{sc}(ABC) = (3,2,1); \quad \mathrm{sc}(CBA) = (1,2,3).$$

This admits a natural generalization to score vectors for incomplete ballots and weak orders over candidates; unmentioned candidates are regarded as being tied at the end of the ballot, and ties are handled as averages. There is also a merge/unmerge move for ballots with ties: merging or unmerging two sets in neighboring ballot positions costs one-half of the product of the set sizes. (Neighbor swaps are a special case realized by one merge and one unmerge.)

If the types are $\mathcal{A}$ and $\mathcal{B}$, let the score vector by type, denoted $\mathrm{sc}^{A|B}$, report each candidate's score as the average over its type, so that for an election with $(r,s) = (2,3)$ (i.e., $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{B} = \{B_1, B_2, B_3\}$), we have

$$\mathrm{sc}^{A|B}(AABBB) = (9/2, 9/2 \mid 2,2,2), \quad \mathrm{sc}^{A|B}(\{A,A\},\{B,B,B\}) = (9/2, 9/2 \mid 2,2,2),$$

$$\mathrm{sc}^{A|B}(ABBAB) = (7/2, 7/2 \mid 8/3, 8/3, 8/3), \quad \mathrm{sc}^{A|B}(AB\{A,B,B\}) = (7/2, 7/2 \mid 8/3, 8/3, 8/3).$$

Here, the first two are sorted with $\mathcal{A}$ candidates before $\mathcal{B}$ candidates, and either of the second two can be sorted by moves incurring swap distance 2.

Lemma C.1 (Duchin–Tapp). *Swap distance on ballots can be calculated as an L^1 vector difference, as can the distance of a ballot type to being sorted. For ballots b_1, b_2,*

$$dist(b_1, b_2) = \frac{1}{2} \left\| sc(b_1) - sc(b_2) \right\|_1;$$

for a ballot type σ,

$$dist(\sigma, (\mathcal{A}, \mathcal{B})) = \frac{1}{2} \left\| sc^{A|B}(\sigma_1) - sc^{A|B}(\mathcal{A}, \mathcal{B}) \right\|_1.$$

This is the distance to sorted that was used as a summary statistic of elections in §4.3.

D More MDS plots

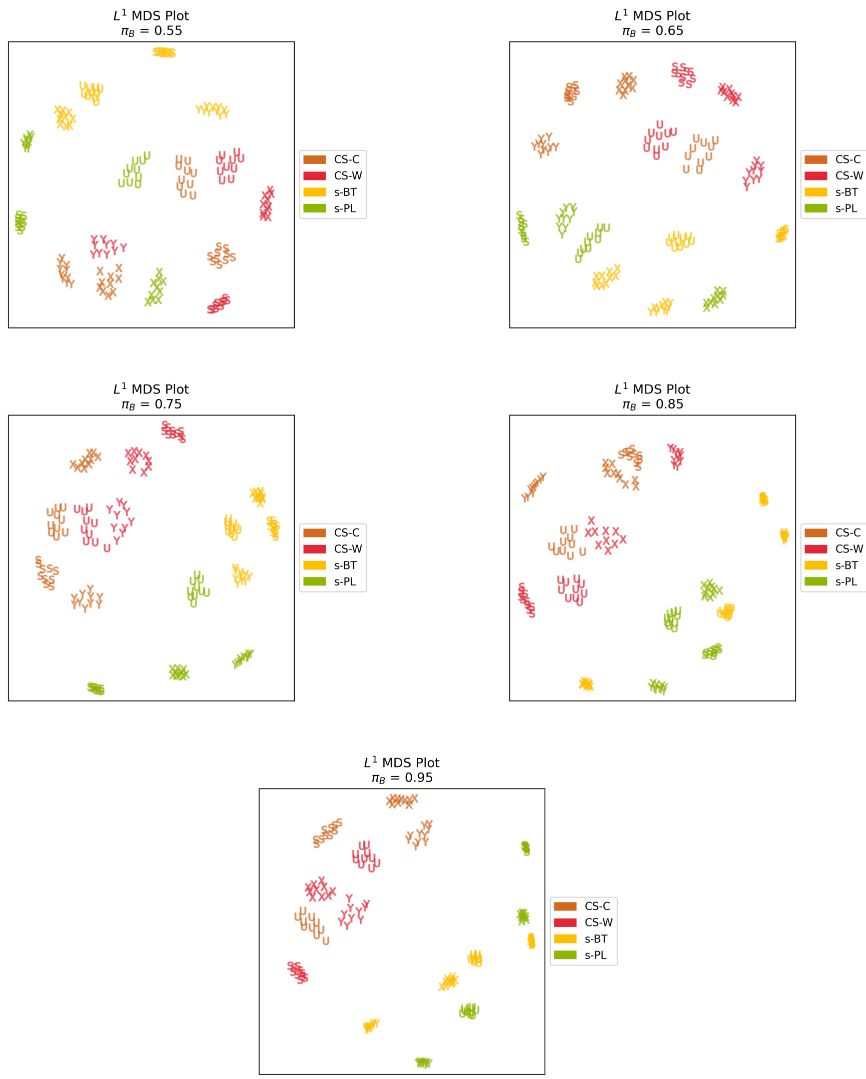

Fig. 11. Multi-dimensional scaling (MDS) plots for profiles with $r = s = 3$ (3 candidates per bloc), under a variety of generative models and candidate strength scenarios. Each model is designated by a different color, and the candidate strength scenarios are denoted U, S, X, Y, as described in §A.1. The pairwise distances between profiles are computed with L^1 distance on the profiles. Each preference profile has 1000 ballots, and we have generated 10 profiles by each of the 16 model/strength pairs. As $\pi_B \to 1$, the main difference appearing in the models is that the BT and PL profiles become tightly clustered for each candidate strength scenario, while the CS profiles remain more variable.

E More Scottish elections

Here, we include further exploration of the disproportionality of the Scottish elections. First is another version of Table 1 using top-k Borda share instead of first-place preference to measure p_B. Then we also include a set of tables that looks at the disproportionality of a different random sample of Scottish elections, this time split by the number of seats up for election.

election	(r, s, k)	top-k Borda p_B	top-k Borda proportionality	STV outcome	disprop top-k Borda
Orkney 2022 Ward 3	(5, 1, 3)	0.11	0.33	0	−0.11
North Ayrshire 2022 Arran	(4, 2, 1)	0.36	0.36	0	−0.36
Highland 2012 Ward 12	(7, 1, 3)	0.12	0.36	0	−0.12
Inverclyde 2017 Ward 5	(6, 1, 3)	0.17	0.51	1	0.16
Dumgal 2022 Ward 12	(5, 2, 3)	0.17	0.51	0	−0.17
Edinburgh 2017 Ward 8	(4, 2, 3)	0.24	0.72	0	−0.24
Eilean Siar 2012 Ward 4	(5, 2, 3)	0.31	0.93	1	0.02
Clackmannanshire 2022 Ward 5	(5, 2, 3)	0.33	0.99	1	0.01
Fife 2012 Ward 12	(4, 2, 3)	0.34	1.02	1	−0.00
East Renfrewshire 2022 Ward 5	(7, 2, 4)	0.25	1.00	1	−0.00
North Lanarkshire 2012 Ward 19	(4, 2, 4)	0.29	1.16	1	−0.04
Inverclyde 2017 Ward 7	(6, 2, 3)	0.40	1.20	1	−0.06
Glasgow 2007 Hillhead	(8, 2, 4)	0.33	1.32	2	0.17
Stirling 2017 Ward 5	(3, 3, 3)	0.45	1.35	1	−0.11
North Lanarkshire 2017 Ward 8	(5, 3, 4)	0.34	1.36	1	−0.09
North Lanarkshire 2022 Ward 8	(6, 2, 4)	0.35	1.40	2	0.15
Stirling 2012 Ward 1	(4, 2, 3)	0.47	1.41	2	0.20
Clackmannanshire 2022 Ward 3	(4, 3, 3)	0.48	1.44	2	0.19
Highland 2012 Ward 19	(6, 2, 4)	0.36	1.44	2	0.14
North Lanarkshire 2022 Ward 5	(4, 3, 3)	0.48	1.44	2	0.18
Renfrewshire 2022 Ward 8	(4, 2, 4)	0.37	1.48	2	0.13
Clackmannanshire 2012 Ward 5	(2, 2, 3)	0.50	1.50	1	−0.16
North Lanarkshire 2022 Ward 18	(4, 2, 4)	0.44	1.76	2	0.06
Glasgow 2012 Ward 12	(10, 3, 4)	0.47	1.88	3	0.28
Renfrewshire 2017 Ward 4	(7, 3, 4)	0.47	1.88	2	0.03
Glasgow 2022 Ward 6	(7, 3, 4)	0.50	2.00	3	0.25
East Ayrshire 2017 Ward 3	(3, 3, 4)	0.54	2.16	2	−0.04
Glasgow 2022 Ward 7	(6, 3, 4)	0.56	2.24	3	0.19
Glasgow 2022 Ward 10	(7, 3, 4)	0.56	2.24	3	0.19
North Lanarkshire 2022 Ward 4	(4, 4, 4)	0.66	2.64	3	0.09

Table 2. Here, s is the number of $\mathcal{B}$ candidates (defined by membership in the Scottish National Party and the Greens), r is the number of candidates from all other parties, and k is the number of seats to be filled in the election. We treat the electorate as a single bloc (undivided) and measure p_B as the level of top-k Borda share for the $\mathcal{B}$ slate. The dividing lines indicate where the proportionality target should be rounded to the nearest integer. These are the same elections as Table 1.

election	(r, s, k)	first-place pref. p_B	first-place pref. proportionality	STV outcome	disprop first-place pref.
Sc Borders 2017 Ward 10	(4, 2, 3)	0.15	0.45	1	0.19
Aberdeen 2017 Ward 9	(6, 2, 3)	0.18	0.54	0	−0.18
Sc Borders 2012 Ward 4	(5, 1, 3)	0.20	0.60	0	−0.20
Inverclyde 2012 Ward 2	(5, 1, 3)	0.22	0.66	1	0.11
Sc Borders 2017 Ward 7	(5, 2, 3)	0.23	0.69	1	0.10
South Ayrshire 2022 Ward 8	(6, 1, 3)	0.25	0.75	1	0.08
Sc Borders 2017 Ward 6	(4, 2, 3)	0.25	0.75	1	0.08
Aberdeenshire 2017 Ward 1	(3, 1, 3)	0.25	0.75	1	0.08
Highland 2017 Ward 15	(5, 1, 3)	0.26	0.78	1	0.07
Perth Kinross 2017 Ward 6	(5, 3, 3)	0.28	0.84	1	0.05
Highland 2012 Ward 14	(6, 2, 3)	0.30	0.90	1	0.03
South Lanarkshire 2022 Ward 9	(6, 2, 3)	0.31	0.93	1	0.02
Aberdeen 2022 Ward 3	(4, 3, 3)	0.36	1.08	1	−0.02
Renfrewshire 2017 Ward 3	(6, 3, 3)	0.40	1.20	1	−0.06
Fife 2022 Ward 1	(7, 3, 3)	0.40	1.20	1	−0.07
East Dunbartonshire 2022 Ward 6	(4, 2, 3)	0.40	1.20	1	−0.07
Argyll Bute 2022 Ward 6	(5, 2, 3)	0.42	1.26	1	−0.09
North Ayrshire 2017 Ward 10	(5, 2, 3)	0.42	1.26	1	−0.09
Midlothian 2012 Ward 5	(5, 2, 3)	0.43	1.29	1	−0.10
South Lanarkshire 2022 Ward 14	(5, 3, 3)	0.46	1.38	2	0.20
Eilean Siar 2012 Ward 9	(5, 1, 4)	0.20	0.80	1	0.05
Highland 2017 Ward 18	(8, 2, 4)	0.23	0.92	1	0.02
Highland 2022 Wick And East Caithness	(5, 1, 4)	0.25	1.00	1	−0.00
North Lanarkshire 2012 Ward 19	(4, 2, 4)	0.26	1.04	1	−0.01
Angus 2017 Ward 3	(6, 2, 4)	0.28	1.12	1	−0.03
Fife 2022 Ward 18	(4, 2, 4)	0.29	1.16	1	−0.04
Renfrewshire 2012 Ward 9	(5, 2, 4)	0.32	1.28	2	0.18
Glasgow 2012 Ward 19	(7, 3, 4)	0.32	1.28	1	−0.07
Highland 2017 Ward 20	(6, 2, 4)	0.33	1.32	2	0.17
Eilean Siar 2022 Ward 10	(5, 2, 4)	0.34	1.36	2	0.16
Glasgow 2007 Canal	(9, 2, 4)	0.37	1.48	2	0.13
South Ayrshire 2012 Ward 3	(3, 2, 4)	0.38	1.52	1	−0.13
West Dunbartonshire 2022 Ward 2	(5, 3, 4)	0.40	1.60	1	−0.15
Aberdeenshire 2012 Ward 4	(6, 2, 4)	0.41	1.64	2	0.09
Falkirk 2017 Ward 4	(4, 3, 4)	0.45	1.80	2	0.05
Fife 2022 Ward 21	(5, 3, 4)	0.45	1.80	2	0.05
Aberdeen 2022 Ward 12	(7, 3, 4)	0.48	1.92	2	0.02
Clackmannanshire 2022 Ward 1	(4, 3, 4)	0.48	1.92	2	0.02
West Lothian 2022 Ward 3	(5, 3, 4)	0.53	2.12	2	−0.03
North Lanarkshire 2022 Ward 4	(4, 4, 4)	0.63	2.52	3	0.12
North Ayrshire 2022 Garnock Valley	(9, 2, 5)	0.31	1.55	2	0.09
North Ayrshire 2022 North Coast	(9, 3, 5)	0.35	1.75	2	0.05
North Ayrshire 2022 Saltcoats And Stevenston	(6, 2, 5)	0.40	2.00	2	0.00

Table 3. Here, s is the number of $\mathcal{B}$ candidates (defined by membership in the Scottish National Party and the Greens), r is the number of candidates from all other parties, and k is the number of seats to be filled in the election. We treat the electorate as a single bloc (undivided) and measure p_B as the level of first-place support for the $\mathcal{B}$ slate. The dividing lines indicate the number of seats up for election, k.

election	(r, s, k)	top-k Borda p_B	top-k Borda proportionality	STV outcome	disprop top-k Borda
Sc Borders 2012 Ward 4	$(5, 1, 3)$	0.17	0.51	0	−0.17
Inverclyde 2012 Ward 2	$(5, 1, 3)$	0.17	0.51	1	0.16
Aberdeen 2017 Ward 9	$(6, 2, 3)$	0.18	0.54	0	−0.18
South Ayrshire 2022 Ward 8	$(6, 1, 3)$	0.19	0.57	1	0.14
Highland 2017 Ward 15	$(5, 1, 3)$	0.19	0.57	1	0.14
Sc Borders 2017 Ward 10	$(4, 2, 3)$	0.19	0.57	1	0.14
Aberdeenshire 2017 Ward 1	$(3, 1, 3)$	0.20	0.60	1	0.13
Sc Borders 2017 Ward 7	$(5, 2, 3)$	0.21	0.63	1	0.12
Sc Borders 2017 Ward 6	$(4, 2, 3)$	0.24	0.72	1	0.09
Perth Kinross 2017 Ward 6	$(5, 3, 3)$	0.30	0.90	1	0.03
South Lanarkshire 2022 Ward 9	$(6, 2, 3)$	0.33	0.99	1	−0.00
Highland 2012 Ward 14	$(6, 2, 3)$	0.34	1.02	1	−0.01
East Dunbartonshire 2022 Ward 6	$(4, 2, 3)$	0.38	1.14	1	−0.05
Argyll Bute 2022 Ward 6	$(5, 2, 3)$	0.40	1.20	1	−0.07
North Ayrshire 2017 Ward 10	$(5, 2, 3)$	0.40	1.20	1	−0.07
Fife 2022 Ward 1	$(7, 3, 3)$	0.41	1.23	1	−0.08
Renfrewshire 2017 Ward 3	$(6, 3, 3)$	0.41	1.23	1	−0.08
Midlothian 2012 Ward 5	$(5, 2, 3)$	0.42	1.26	1	−0.09
Aberdeen 2022 Ward 3	$(4, 3, 3)$	0.43	1.29	1	−0.09
South Lanarkshire 2022 Ward 14	$(5, 3, 3)$	0.47	1.41	2	0.20
Eilean Siar 2012 Ward 9	$(5, 1, 4)$	0.17	0.68	1	0.08
Highland 2022 Wick And East Caithness	$(5, 1, 4)$	0.17	0.68	1	0.08
Highland 2017 Ward 18	$(8, 2, 4)$	0.22	0.88	1	0.03
Fife 2022 Ward 18	$(4, 2, 4)$	0.26	1.04	1	−0.01
Angus 2017 Ward 3	$(6, 2, 4)$	0.26	1.04	1	−0.01
Glasgow 2007 Canal	$(9, 2, 4)$	0.28	1.12	2	0.22
North Lanarkshire 2012 Ward 19	$(4, 2, 4)$	0.29	1.16	1	−0.04
Highland 2017 Ward 20	$(6, 2, 4)$	0.29	1.16	2	0.21
Glasgow 2012 Ward 19	$(7, 3, 4)$	0.30	1.20	1	−0.05
Eilean Siar 2022 Ward 10	$(5, 2, 4)$	0.31	1.24	2	0.19
Renfrewshire 2012 Ward 9	$(5, 2, 4)$	0.32	1.28	2	0.18
Aberdeenshire 2012 Ward 4	$(6, 2, 4)$	0.39	1.56	2	0.11
South Ayrshire 2012 Ward 3	$(3, 2, 4)$	0.40	1.60	1	−0.15
West Dunbartonshire 2022 Ward 2	$(5, 3, 4)$	0.42	1.68	1	−0.17
Fife 2022 Ward 21	$(5, 3, 4)$	0.45	1.80	2	0.05
Falkirk 2017 Ward 4	$(4, 3, 4)$	0.46	1.84	2	0.04
Aberdeen 2022 Ward 12	$(7, 3, 4)$	0.48	1.92	2	0.02
Clackmannanshire 2022 Ward 1	$(4, 3, 4)$	0.48	1.92	2	0.02
West Lothian 2022 Ward 3	$(5, 3, 4)$	0.51	2.04	2	−0.01
North Lanarkshire 2022 Ward 4	$(4, 4, 4)$	0.66	2.64	3	0.09
North Ayrshire 2022 Garnock Valley	$(9, 2, 5)$	0.30	1.50	2	0.10
North Ayrshire 2022 Saltcoats And Stevenston	$(6, 2, 5)$	0.35	1.75	2	0.05
North Ayrshire 2022 North Coast	$(9, 3, 5)$	0.35	1.75	2	0.05

Table 4. Here, s is the number of $\mathcal{B}$ candidates (defined by membership in the Scottish National Party and the Greens), r is the number of candidates from all other parties, and k is the number of seats to be filled in the election. We treat the electorate as a single bloc (undivided) and measure p_B as the level of top-k Borda support for the $\mathcal{B}$ slate. The dividing lines indicate the number of seats up for election, k.

We can also plot the observed first-place preference and STV seat share of all Scottish elections, much like Figure 6.

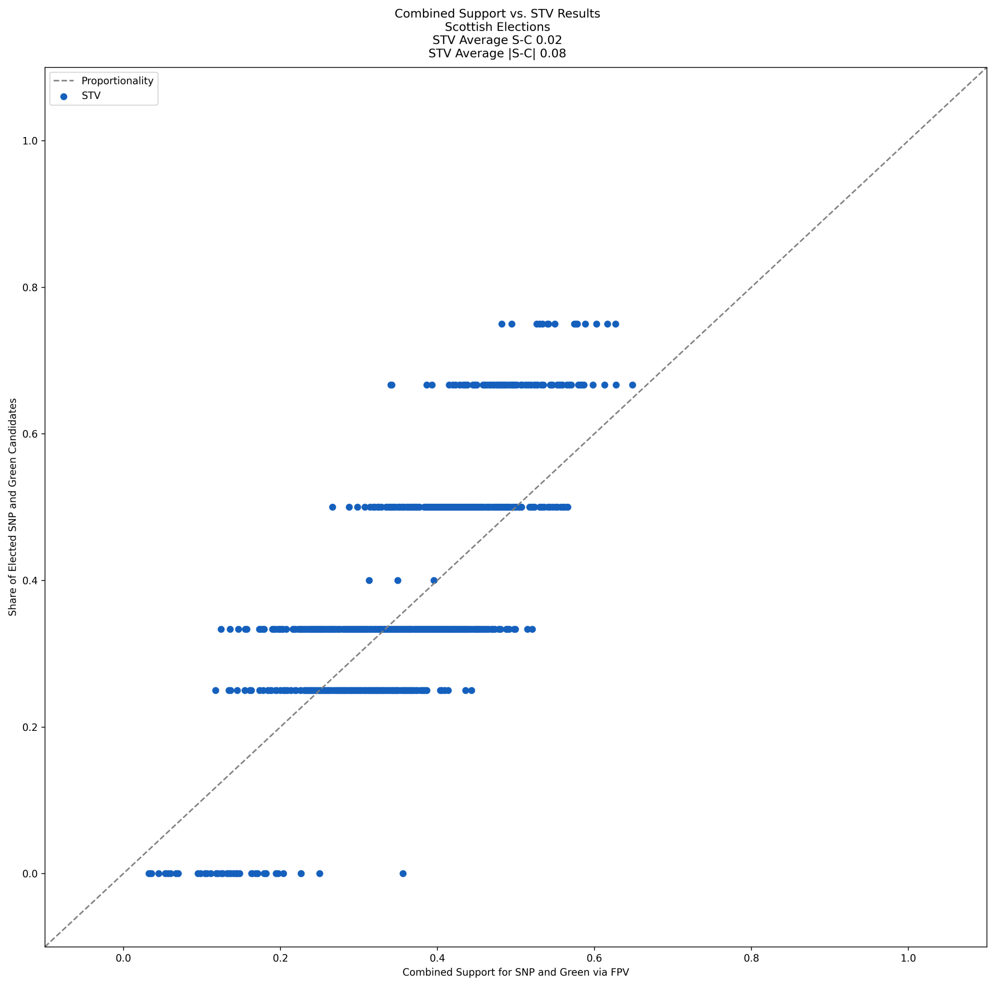

Fig. 12. A plot of first-place support for the Scottish National Party and the Greens against their STV seat share for all Scottish elections we had data for.

To conclude, we provide a full sweep of fitting outputs across nine elections and various models. All simulations use the same number of ballots as in the observed election. Plots for all elections and models follow.

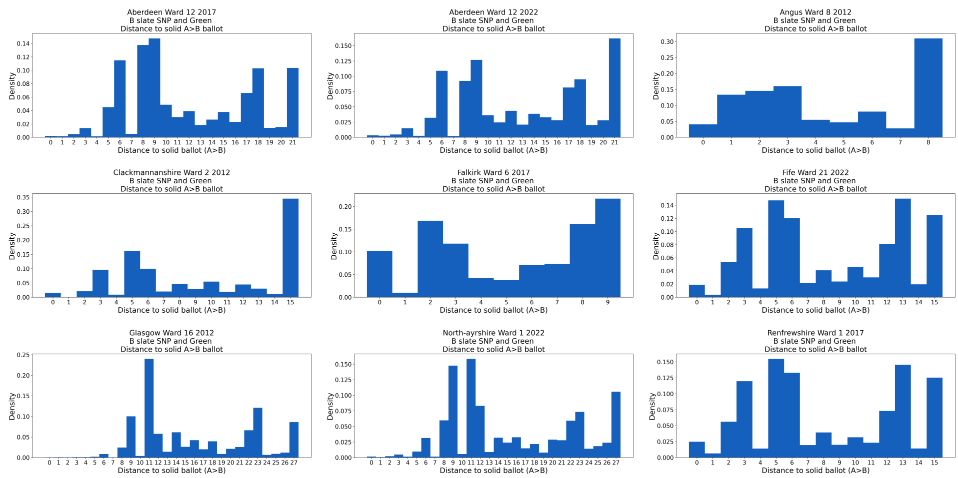

Fig. 13. Histograms showing the distribution of swap distances to solid A-over-B ballots in nine Scottish elections.

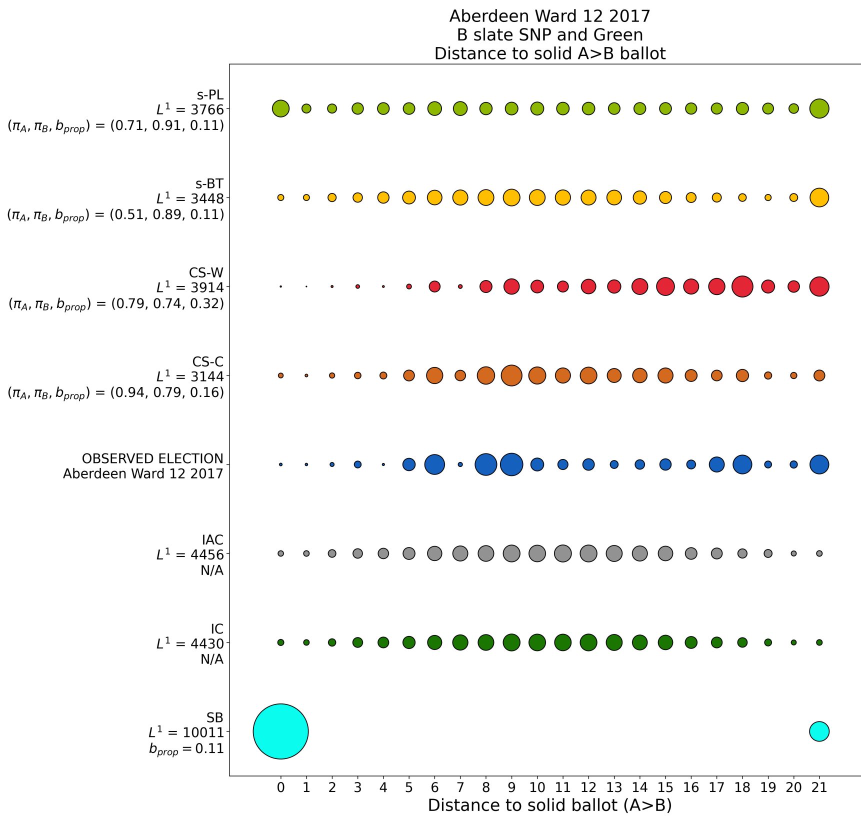

Fig. 14. Bubble plots showing the distribution of swap distances from our generative models, classical models, and a real election to A-over-B ballots. Both the generative models and solid-bloc election are optimized via a grid search to choose a value for π_B that minimizes L^1 to the real Aberdeen Ward 12 2017 election.

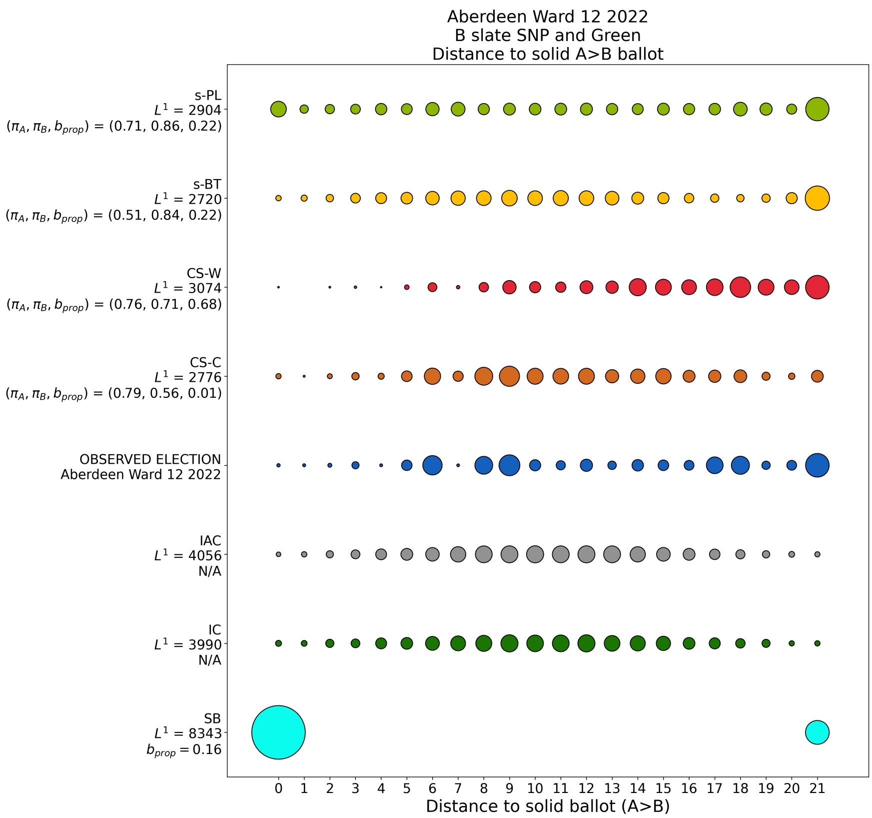

Fig. 15. Bubble plots showing the distribution of swap distances from our generative models, classical models, and a real election to A-over-B ballots. Both the generative models and solid-bloc election are optimized via a grid search to choose a value for π_B that minimizes L^1 to the real Aberdeen Ward 12 2022 election.

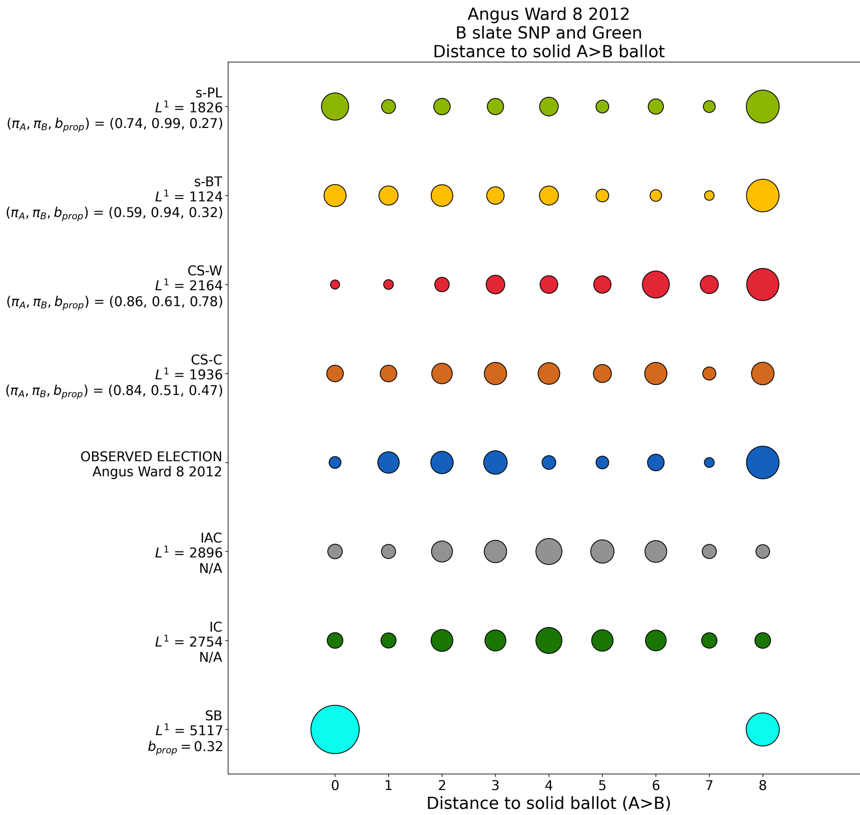

Fig. 16. Bubble plots showing the distribution of swap distances from our generative models, classical models, and a real election to A-over-B ballots. Both the generative models and solid-bloc election are optimized via a grid search to choose a value for π_B that minimizes L^1 to the real Angus Ward 8 2012 election.

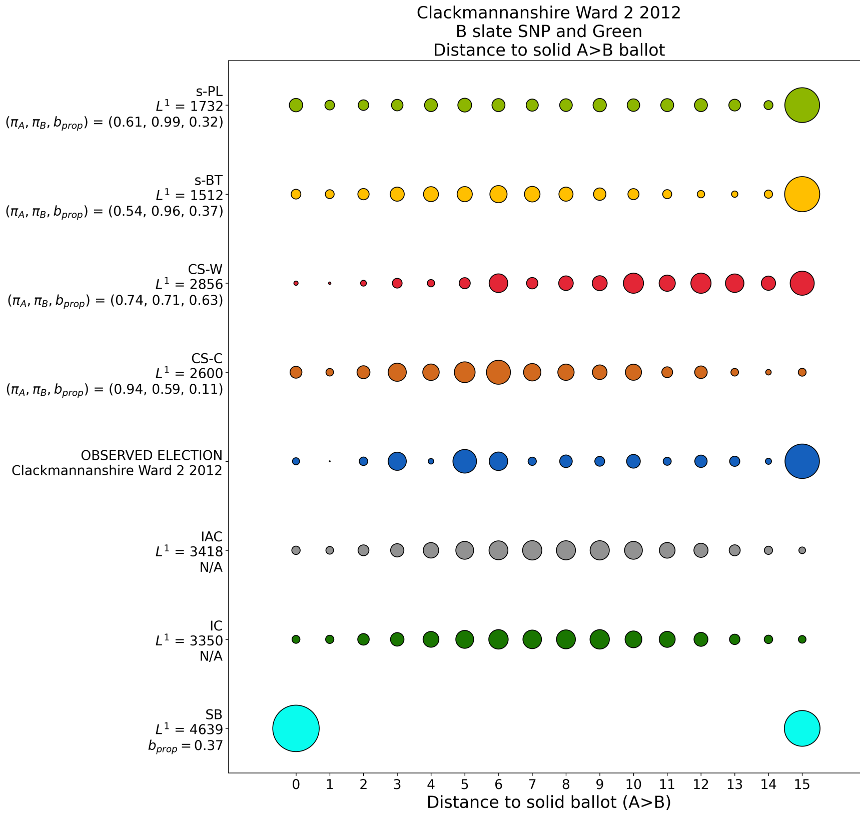

Fig. 17. Bubble plots showing the distribution of swap distances from our generative models, classical models, and a real election to A-over-B ballots. Both the generative models and solid-bloc election are optimized via a grid search to choose a value for π_B that minimizes L^1 to the real Clackmannanshire Ward 2 2012 election.

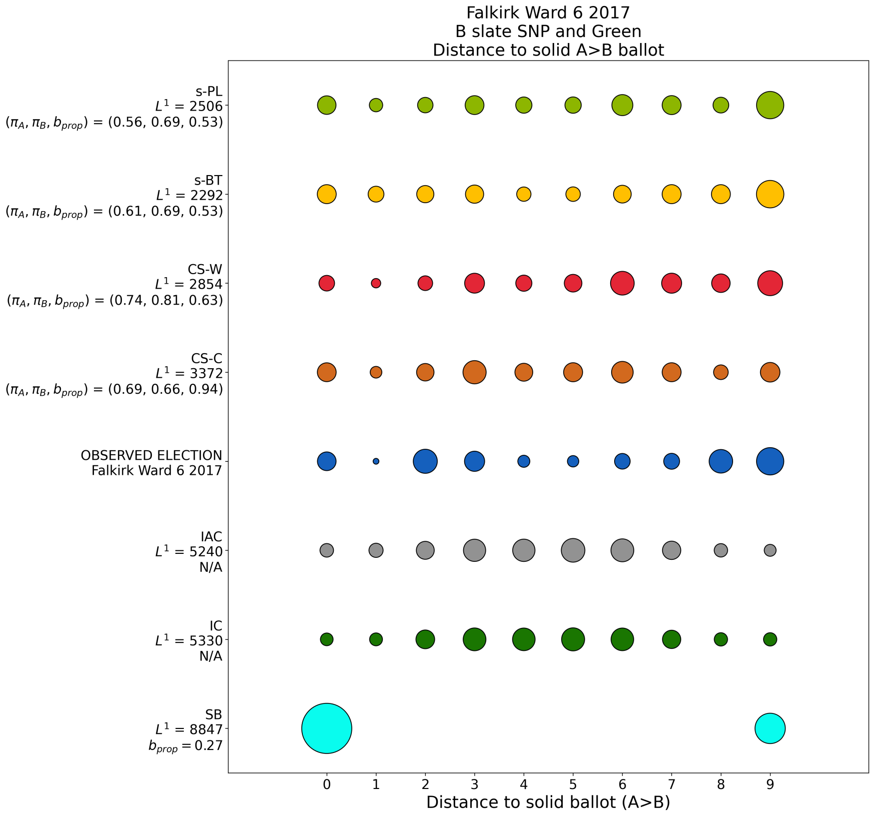

Fig. 18. Bubble plots showing the distribution of swap distances from our generative models, classical models, and a real election to A-over-B ballots. Both the generative models and solid-bloc election are optimized via a grid search to choose a value for π_B that minimizes L^1 to the real Falkirk Ward 6 2017 election.

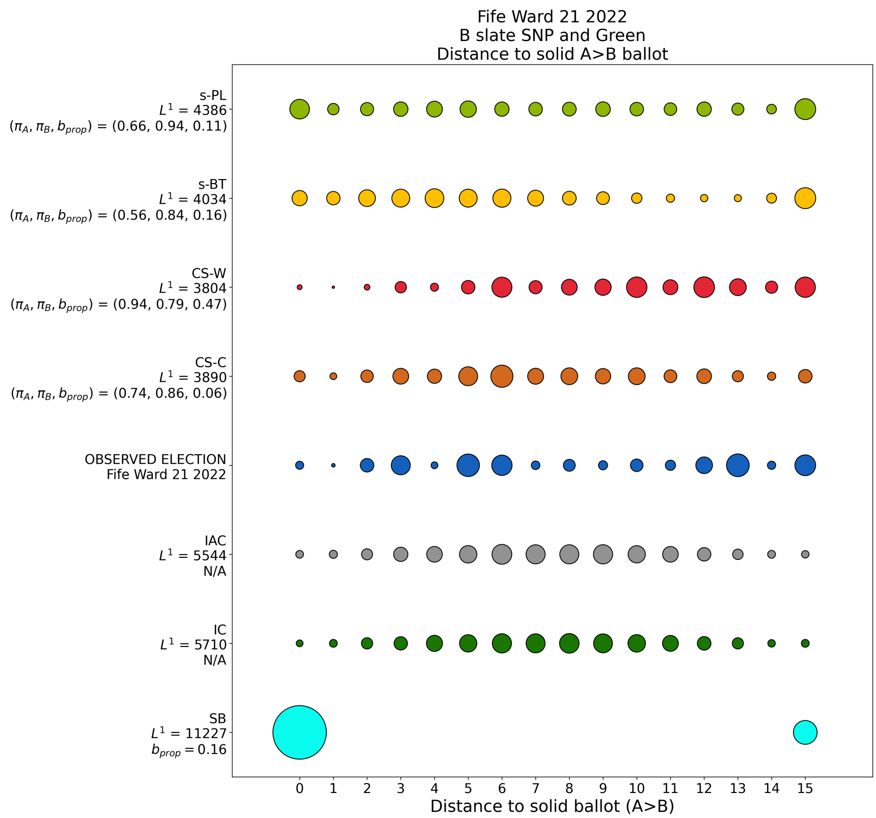

Fig. 19. Bubble plots showing the distribution of swap distances from our generative models, classical models, and a real election to A-over-B ballots. Both the generative models and solid-bloc election are optimized via a grid search to choose a value for π_B that minimizes L^1 to the real Fife Ward 21 2022 election.

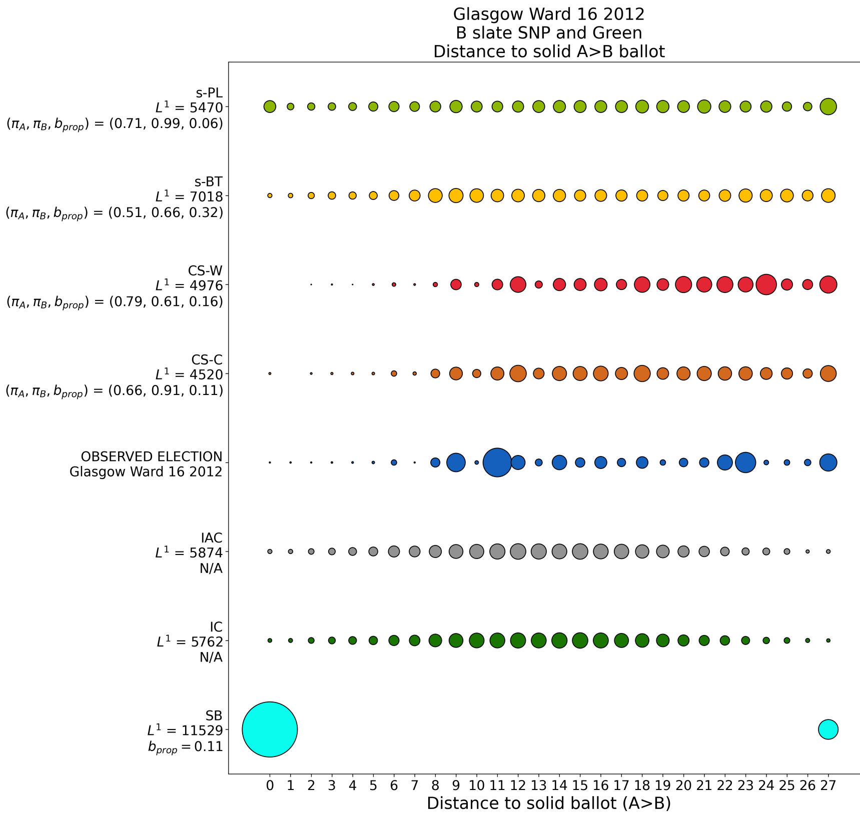

Fig. 20. Bubble plots showing the distribution of swap distances from our generative models, classical models, and a real election to A-over-B ballots. Both the generative models and solid-bloc election are optimized via a grid search to choose a value for π_B that minimizes L^1 to the real Glasgow Ward 16 2012 election.

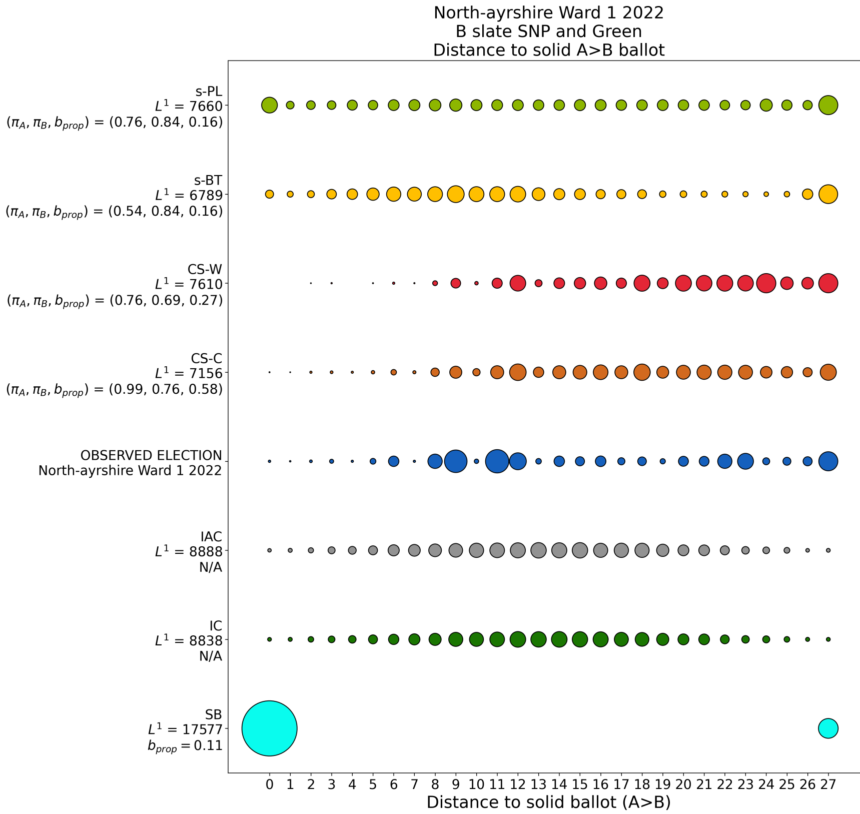

Fig. 21. Bubble plots showing the distribution of swap distances from our generative models, classical models, and a real election to A-over-B ballots. Both the generative models and solid-bloc election are optimized via a grid search to choose a value for π_B that minimizes L^1 to the real North Ayrshire Ward 1 2022 election.

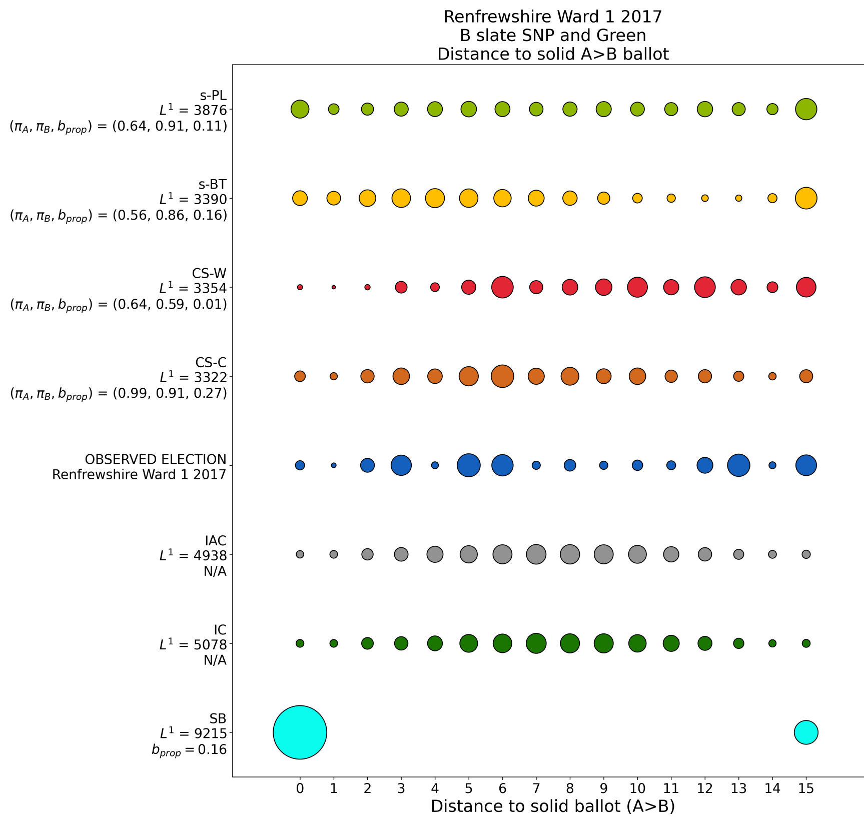

Fig. 22. Bubble plots showing the distribution of swap distances from our generative models, classical models, and a real election to A-over-B ballots. Both the generative models and solid-bloc election are optimized via a grid search to choose a value for π_B that minimizes L^1 to the real Renfrewshire Ward 1 2017 election.