Accurate 3D periodic orbits for the three body problem

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1 Introduction

The three-body problem concerns the motion of three bodies attracted to each other according to Newton's universal law of gravitation. While the two-body problem is exactly solvable and is a major triumph of theoretical physics, the three-body problem cannot be solved by a closed expression. Instead, the equations must be solved numerically. We will investigate the equal mass case $m_1 = m_2 = m_3 = m$ with units chosen such that $m = G = 1$. The equations of motion for the positions $\mathbf{r}_i$ and momenta $\mathbf{p}_i$ are

$$\dot{\mathbf{r}}_i = \mathbf{p}_i, \qquad \dot{\mathbf{p}}_i = -\sum_{i \neq j} \frac{\mathbf{r}_{ij}}{r_{ij}^3},$$
$$\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j, \qquad r_{ij} = \|\mathbf{r}_{ij}\|_2. \tag{1}$$

Here $i = 1, 2, 3$ is the particle index, boldface letters denote 3D vectors and $\dot{\mathbf{r}}$ is the time derivative of $\mathbf{r}$. The full state space is described by 18 degrees of freedom, but many of these are trivial. The total momentum $\mathbf{p} \equiv \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$ is a constant of motion due to the antisymmetry of the force, and the total position $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3$ evolves trivially $\dot{\mathbf{r}} = \mathbf{p}$.

$$\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{p}(0)t, \qquad \mathbf{p}(t) = \mathbf{p}(0). \tag{2}$$

The Galilean and translation symmetry o the problem can always be used to map the state such that $\mathbf{r} = \mathbf{p} = 0$. We will assume from this point on that all states have this property, which reduces the dimension of the problem from 18 dimensions to 12 dimensions. Only the position and momenta of the first two particles will be tracked, since $\mathbf{r}_3 = -\mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{p}_3 = -\mathbf{p}_1 - \mathbf{p}_2$.

We will be concerned with finding periodic solutions to equations (1). Note that periodic solutions can only exist for negative energy, where energy is defined by the Hamiltonian H.

$$T = \frac{1}{2} \sum_i \mathbf{p}_i \cdot \mathbf{p}_i, \tag{3}$$

$$H = T - \frac{1}{2} \sum_{i \neq j} \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j\|_2}. \tag{4}$$

The proof that periodic orbits only exist for negative energy follows from the time derivative of the virial function U.

$$U = \sum_i \mathbf{r}_i \cdot \mathbf{p}_i, \tag{5}$$

$$\dot{U} = T + H. \tag{6}$$

Since kinetic energy is positive definite $T \geq 0$, only negative energy will allow for a non-monotonic (possibly periodic) U. Furthermore, we fix the natural scaling symmetry of the problem

$$(\mathbf{r}_i', \mathbf{p}_i', t') = (\lambda^{-2}\mathbf{r}_i, \lambda\mathbf{p}_i, \lambda^{-3}t) \tag{7}$$

by enforcing $H' = \lambda^2 H = -1$. This removes another dimension of state space, but we will not in practice use fewer than 12 dimensions numerically.

2 Periodic Orbit Hunting

The C code in main.c uses a symplectic integration scheme to solve the equations of motion. To search for periodic orbits, we randomly sample the components of $\mathbf{r}_i$ and $\mathbf{p}_i$ from the interval $[-1, 1]$. If a bound state with $H < 0$ is selected, the state is rescaled to have $H = -1$. A period T is randomly selected from $[0, 30]$. Let $\mathbf{Z} \equiv (\mathbf{r}_1, \mathbf{r}_2, \mathbf{p}_1, \mathbf{p}_2, T)$ denote the 13 dimensional vector of initial condition and period. We will hunt for zeros of the forward shooting objective function $\mathbf{F}(\mathbf{Z})$.

$$\mathbf{F}(\mathbf{Z}) = \mathbf{f}^T(\mathbf{z}) - \mathbf{z}, \tag{8}$$

where $\mathbf{f}^T$ denotes numerical evolution by time T and $\mathbf{z}$ denotes the 12 dimensional subvector $\mathbf{z} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{p}_1, \mathbf{p}_2)$. Numerical integration is performed with the fourth order symplectic scheme of Forest and Ruth [1] assuming a constant number of timesteps $N = 1024$. The Jacobian of $\mathbf{F}$ is numerically evaluated exactly, except for $\partial_T \mathbf{F}$. This last column of the Jacobian is assumed to be its analytic value $\mathbf{v}(T)$ as if the time integration is done exactly.

Newton-Raphson iteration is used to converge to a zero of $\mathbf{F}$. Note that the rotational symmetry of the problem ensures the Jacobian will be rank deficient. To get around this, as well as numerically small singular values, we use an iterative solver: the LSQR algorithm [2]. Newton steps are weighted by 0.01 since we are not starting with good initial guesses. Each guess is given 2048 iterations to fall below $\|\mathbf{F}\|_2 < 10^{-3}$. If at any point in integration, a close pass of $r_{ij} < 0.01$ occurs, convergence is exited with a failure code. Such close pass solutions are not of interest in this work.

After running for many hours, the directory states/ is filled with 7,348 forward shooting solutions.

2.1 sorting solutions

Of the thousands of solutions found, only a fraction are unique. To sort solutions, two time averaged quantities will be considered: a rank-2 tensor $\mathbf{Q}$ and the mean angular momentum $\mathbf{l}$.

$$\mathbf{Q} = \frac{1}{T}\int_0^T dt \sum_{i=1}^{3} \mathbf{r}_i \otimes \mathbf{r}_i, \tag{9}$$

$$\mathbf{l} = \frac{1}{T}\int_0^T dt \sum_{i=1}^{3} \mathbf{r}_i \times \mathbf{p}_i. \tag{10}$$

Since $\mathbf{Q}$ is symmetric, there are rotation matrices $\mathbf{R}$ such that $\mathbf{Q} = \mathbf{R}\mathbf{D}\mathbf{R}^T$ such that $\mathbf{D}$ is diagonal. If the values of $\mathbf{D}$ are distinct, there is a unique rotation of our solution up to reflections. Solutions are considered identical if the eigenvalues of $\mathbf{Q}$ and absolute value of $\mathbf{l}$ are below a threshold. Unique solutions are stored in unique_solutions/.

3 spectral methods

To obtain accurate solutions to the equations of motion, we will solve for the trajectories globally. Local solutions suffer from the accumulation of numerical error and numerical stability constraints. Take $\mathbf{x}$ to be a vector of size 12×1024, where it captures a trajectory in all 12 degrees of freedom at 1024 equally spaced point in time. Let $\mathbf{z} = \mathbf{x} \oplus T$, where T is the period of the solution. Our objective function $\mathbf{f}$ is

$$\mathbf{f}(\mathbf{z}) = \frac{2\pi}{T}\left[\mathbf{x} - \langle \mathbf{x} \rangle_t\right] - \mathcal{F}^{-1}\left[\frac{1}{\delta_{0,\omega} + i\omega}\mathcal{F}(\mathbf{x})\right] = 0. \tag{11}$$

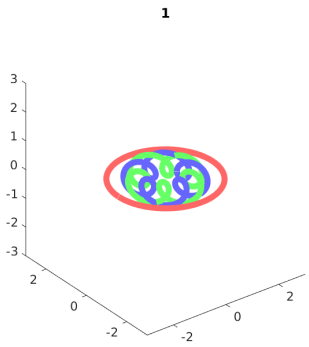
Here $\mathcal{F}$ is the Fast Fourier Transform (FFT) and $\omega \in \mathbb{Z}$ are the discrete frequencies. This global objective function. In practive to make the Jacobian square and well-suited for power-iteration, I add the mean Hamiltonian constraint $\langle H \rangle_t + 1 = 0$ as the final element. The action of the Jacobian is estimated with a finite difference, and a relatively small Krylov subspace is used. Orbits are loaded in using the symplectic forward shooting trajectory, and O(100) Newton-Raphson-Krylov steps are taken.

Note the most recent code also supports finding RPOs, in which an SO(3) drift velocity is appended to $\mathbf{z}$ as well.

References

[1] Etienne Forest and Ronald D Ruth. "Fourth-order symplectic integration". In: *Physica D: Nonlinear Phenomena* 43.1 (1990), pp. 105–117.

[2] Christopher C Paige and Michael A Saunders. "LSQR: An algorithm for sparse linear equations and sparse least squares". In: *ACM Transactions on Mathematical Software (TOMS)* 8.1 (1982), pp. 43–71.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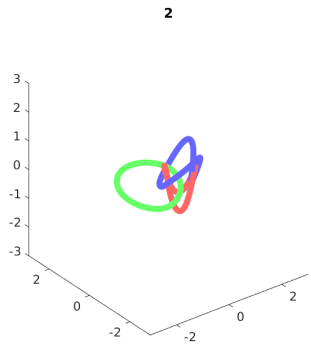
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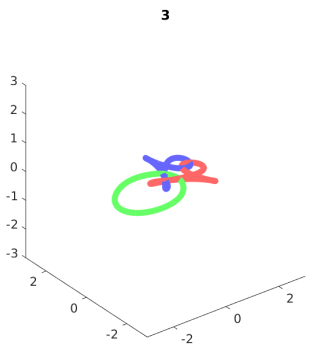
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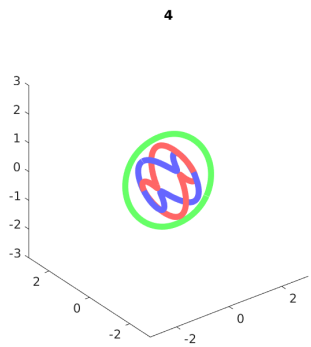
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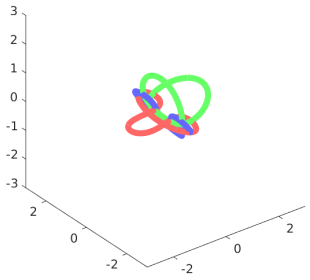

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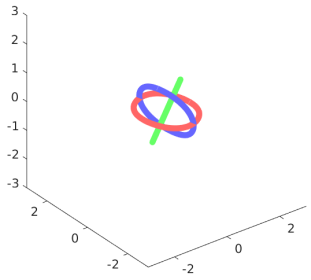

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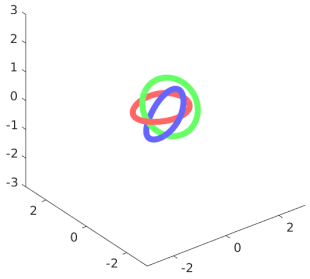

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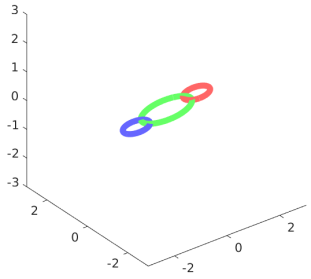

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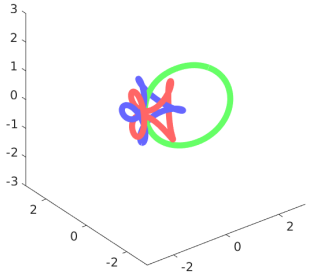

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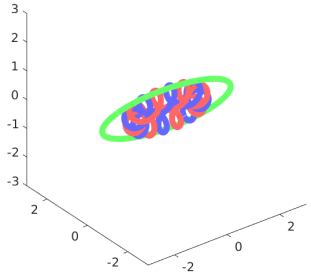

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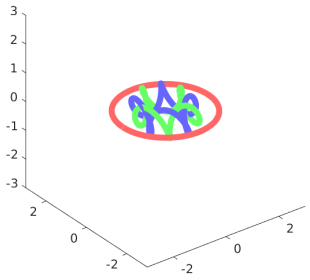

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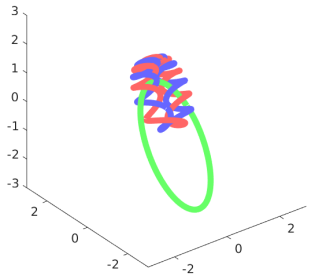

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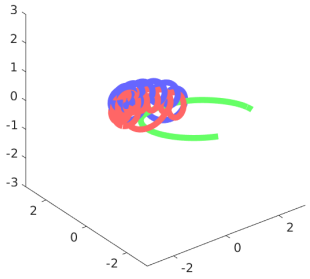

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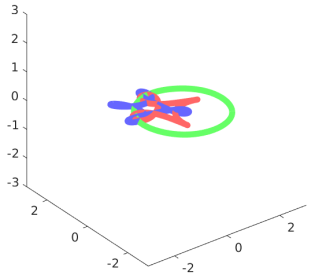

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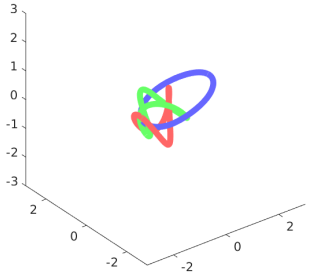

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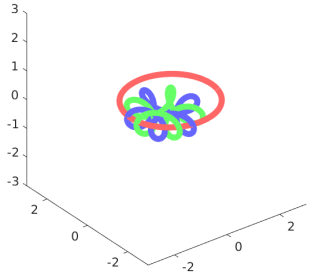

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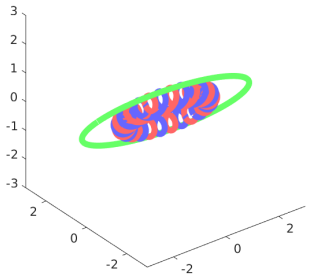

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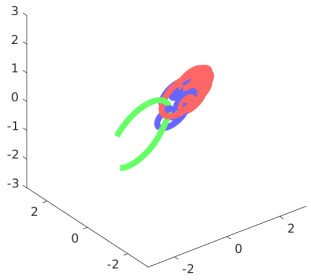

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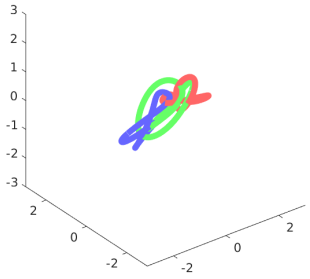

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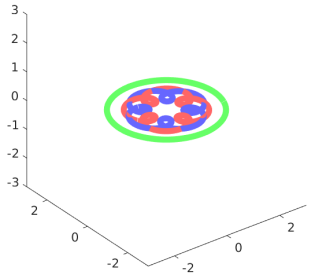

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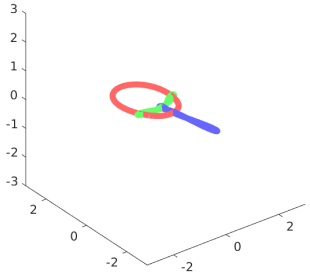

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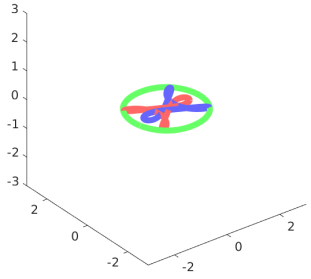

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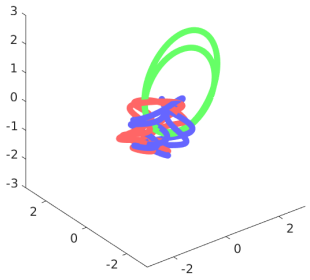

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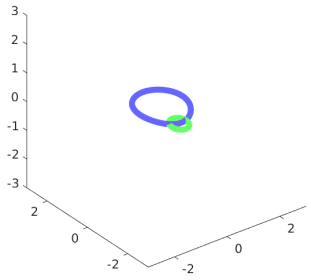

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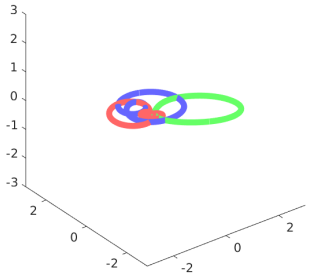

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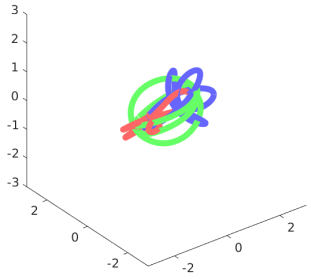

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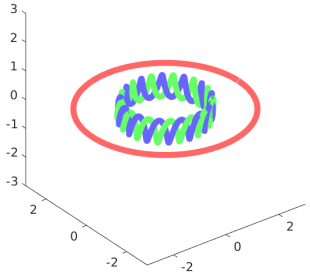

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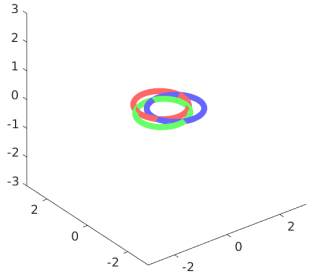

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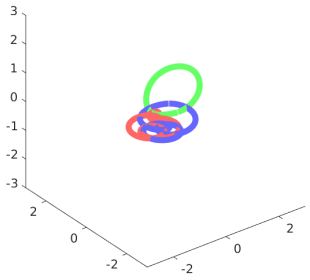

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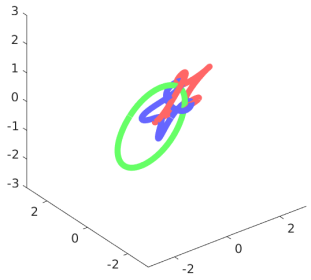

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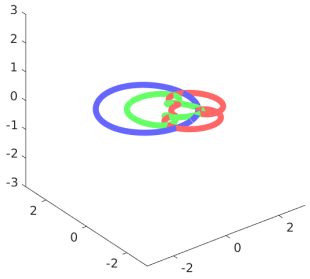

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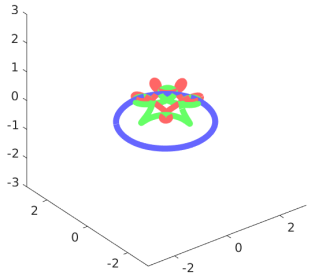

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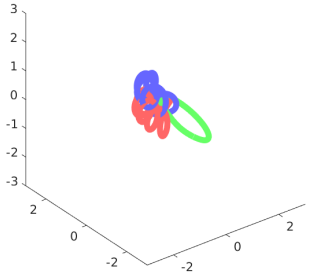

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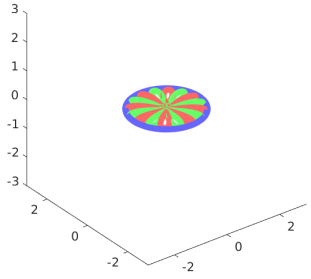

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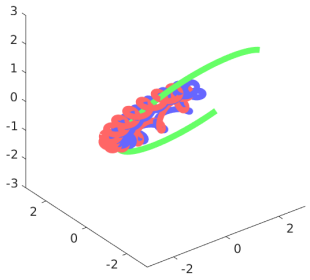

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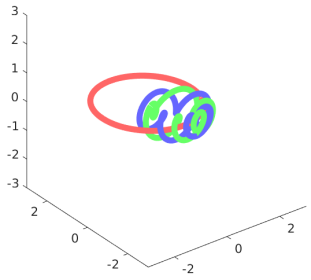

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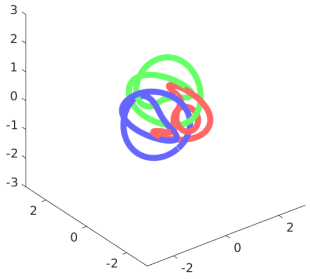

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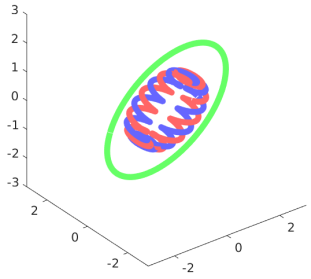

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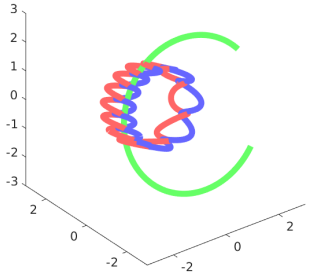

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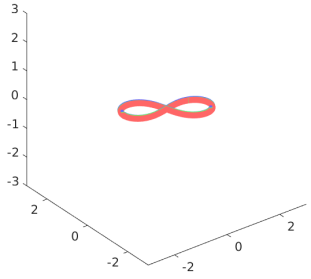

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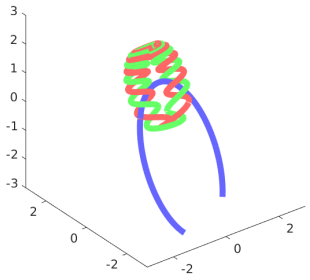

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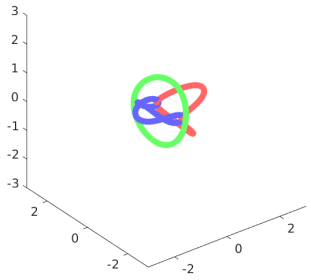

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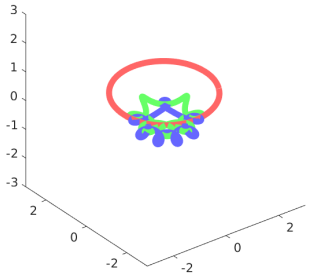

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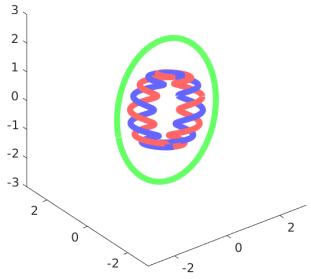

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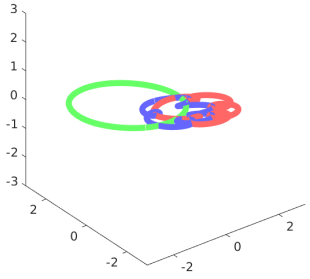

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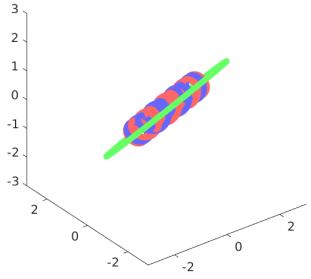

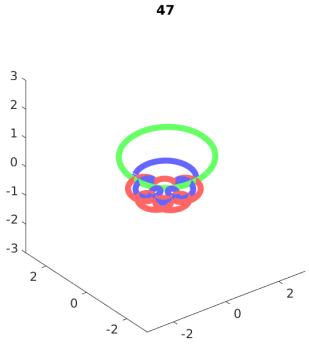
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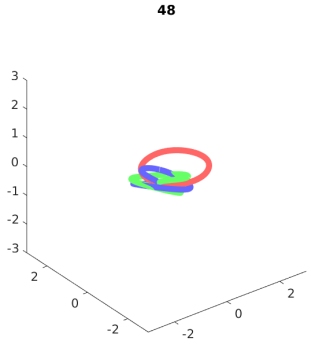
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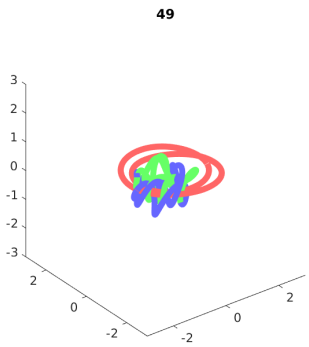
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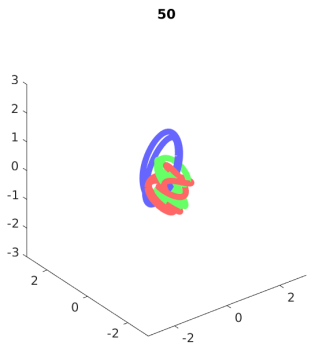
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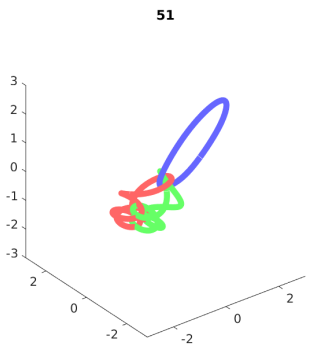
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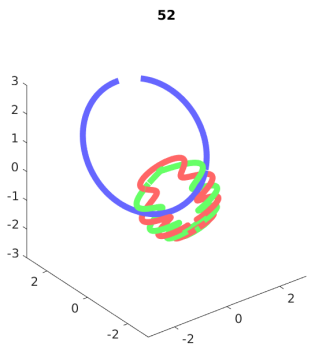
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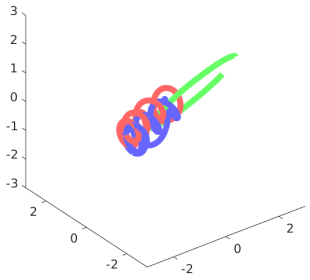

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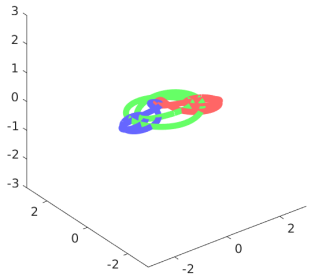

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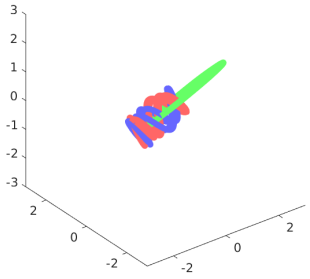

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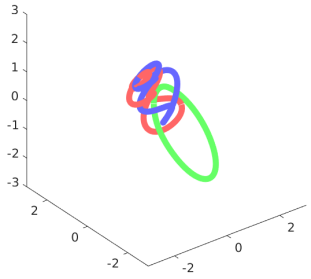

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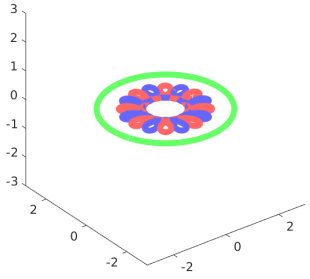

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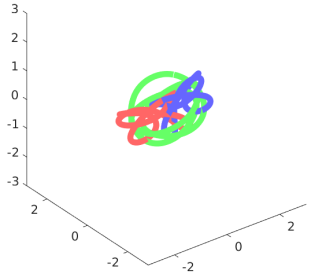

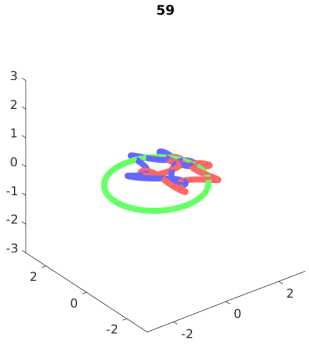
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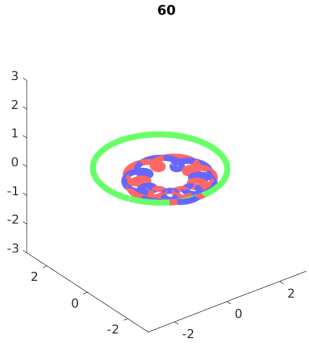
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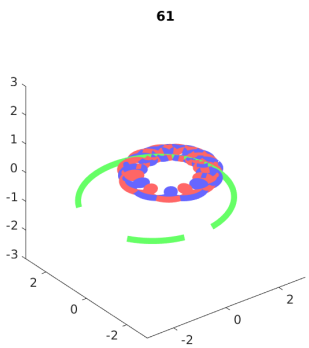
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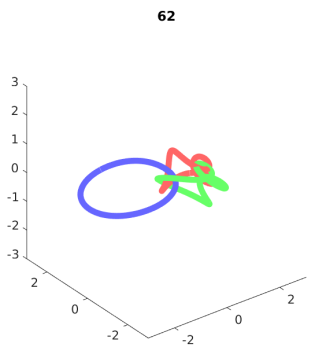
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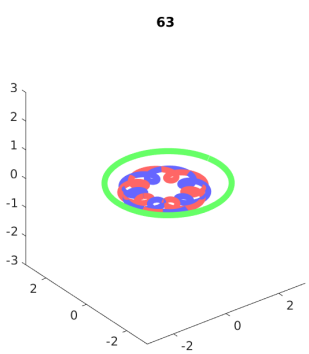
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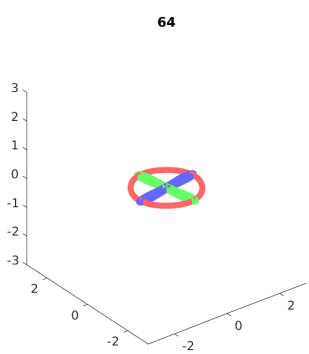
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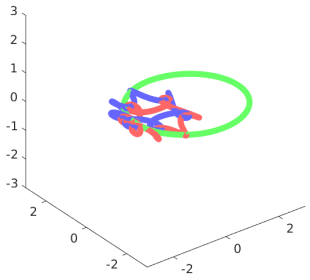

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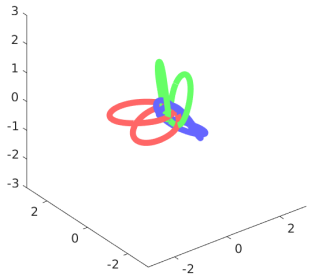

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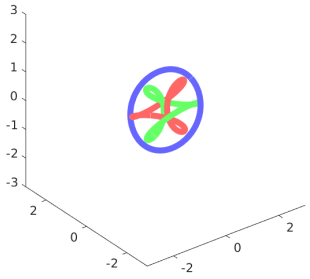

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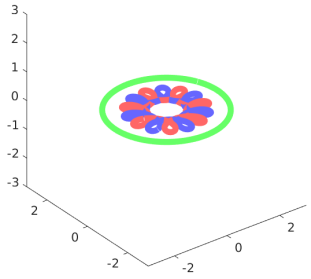

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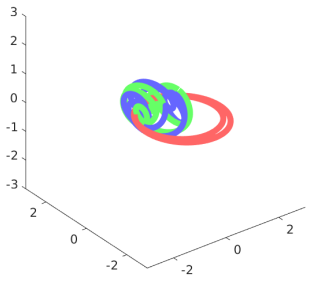

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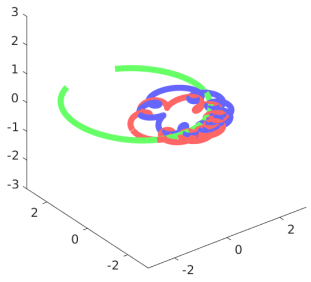

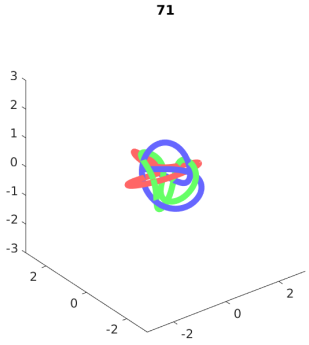
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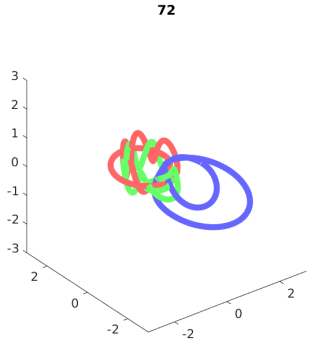
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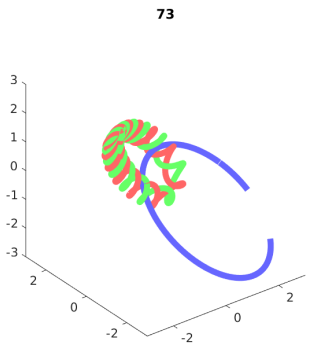
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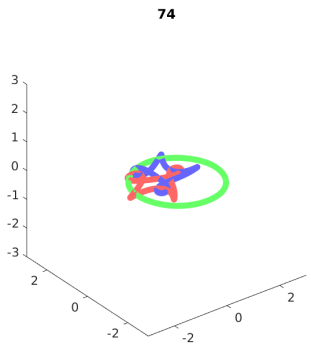
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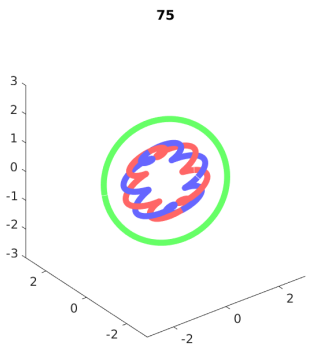
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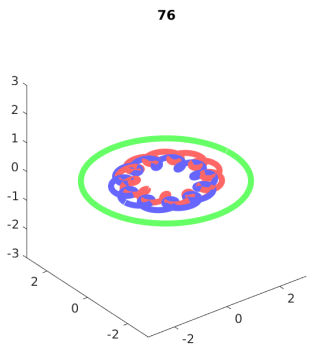
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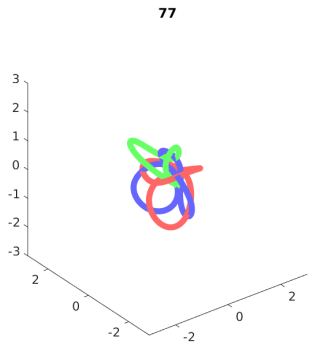
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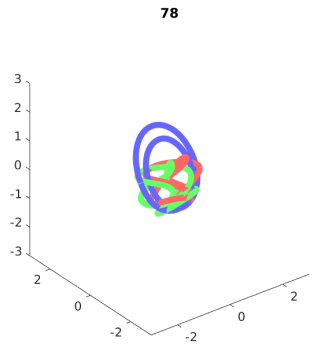
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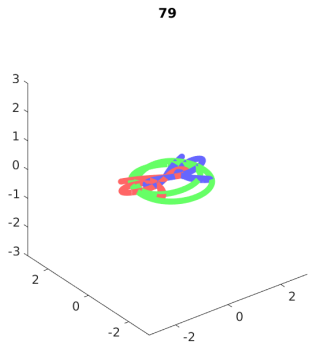
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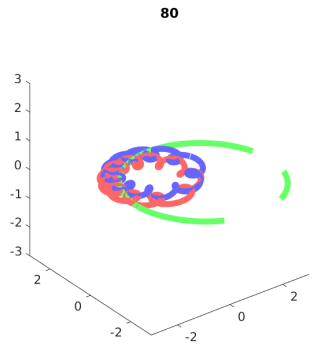
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