# Algebra Definitions

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# 1 Groups

## 1.1 Notation and definitions

### 1.1.1 Basic definitions

**Definition 1.1.1.1** (Normal subgroup)**.** Let $G$ be a group and $K$ be a subgroup of $G$. If $gkg^{-1} \in K$ for all $k \in K$ and $g \in G$, then $K$ is called a normal subgroup of $G$ and is denoted $K \trianglelefteq G$.

**Definition 1.1.1.2** (Simple group)**.** A group $G$ is called a simple group is a group whose only normal subgroups are $\{e\}$ and $G$.

**Definition 1.1.1.3** (Semidirect product)**.** Let $K \trianglelefteq G$ and $Q \leq G$. A group $G$ is a semidirect product of $K$ by $Q$ (denoted $G = K \rtimes Q$) if there exists $Q_1 \cong Q$ such that $Q_1$ is a complement of $K$ in $G$, that is $K \cap Q_1 = 1$ and $KQ_1 = G$.

### 1.1.2 Galois Theory

**Definition 1.1.2.1** (Normal series)**.** A normal series of a group $G$ is a sequence of subgroups

$$G = G_0 \geq G_1 \geq \ldots \geq G_n = 1$$

in which $G_{i+1} \trianglelefteq G_i$ for all $i$.

**Definition 1.1.2.2** (Factor groups)**.** The factor groups of a normal series are the groups $G_i/G_{i+1}$ for $i = 0, 1, \ldots, n-1$.

**Definition 1.1.2.3** (Length)**.** The length of a a normal series is the number of nontrivial factor groups.

**Definition 1.1.2.4** (Solvable group)**.** A finite group is solvable if it has a normal series whose factor groups are cyclic of prime order.

### 1.1.3 Centralizer/Normalizer

**Definition 1.1.3.1** (Center)**.** The center of a group $G$, denoted by $Z(G)$, is the set of all $a \in G$ that commute with every element of $G$.

**Definition 1.1.3.2** (Centralizer)**.** The centralizer of a subset $S$ of a group $G$ is defined to be

$$C_G(S) = \{g \in G \mid gs = sg \text{ for all } s \in S\}.$$

**Definition 1.1.3.3** (Normalizer)**.** The centralizer of a subset $S$ in the group $G$ is defined to be

$$N_G(S) = \{g \in G \mid gS = Sg\}.$$

**Definition 1.1.3.4** (Commutator)**.** If $a, b \in G$, the commutator of $a$ and $b$, denoted $[a, b]$, is

$$[a, b] = aba^{-1}b^{-1},$$

and the commutator subgroup of $G$, denoted $G'$, is the subgroup of $G$ generated by all of the commutators.

**Definition 1.1.3.5** (Class equation)**.** Partition $G$ into its conjugacy classes, with $x_i$ the representative of the $i$th conjugacy class. The class equation of the finite group $G$ is

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)].$$

### 1.1.4 Group Actions

**Definition 1.1.4.1** (Group action)**.** Let $G$ be a group and $X$ be a set. Then a group action on $X$ is a function $\varphi\colon G \times X \to X$ denoted $\varphi(g, x) = g \cdot x$ and satisfying

(i) Identity: group action by the identity is trivial for all $x \in X$: $1 \cdot x = x$.

(ii) Compatibility: $(gh) \cdot x = g \cdot (h \cdot x)$.

And $X$ is called a $G$-set.

**Definition 1.1.4.2** (Orbit)**.** The orbit of an element $x \in X$ is denoted by

$$G \cdot x = \{g \cdot x \mid g \in G\}$$

**Definition 1.1.4.3** (Stabilizer subgroup)**.** The stabilizer subgroup of $G$ with respect to $x \in X$ is denoted

$$G_x = \{g \in G \mid g \cdot x = x\}$$

**Definition 1.1.4.4** (Transitive)**.** A group action is called transitive is for each $x, y \in X$ there exists some $g \in G$ such that $g \cdot x = y$.

## 1.2 Theorems

**Theorem 1.2.1** (First isomorphism theorem)**.** If $\varphi\colon G \to H$ is a group homomorphism then $\ker(\varphi) \trianglelefteq G$ and $G/\ker(\varphi) \cong \varphi(G)$.

**Theorem 1.2.2** (Second isomorphism theorem)**.** Let $G$ be a group with $S \leq G$ and $N \trianglelefteq G$. Then

1. $SN \leq G$
2. $S \cap N \trianglelefteq S$, and
3. $(SN)/N \cong S/(S \cap N)$.

Strictly speaking, $N$ does not have to be a normal subgroup as long as $S$ is a subgroup of the normalizer of $N$, $S \leq N_G(N)$.

**Theorem 1.2.3** (Third isomorphism theorem)**.** Let $G$ be a group with normal subgroup $N \trianglelefteq G$. Then

1. If $K \leq G$ (resp. $K \trianglelefteq G$) such that $N \subseteq K \subseteq G$, then $K/N \leq G/N$ (resp. $K/N \trianglelefteq G/N$).
2. Every subgroup (resp. normal subgroup) of $G/N$ is of the form $K/N$, for some subgroup (resp. normal subgroup) $K \subset G$ such that $N \subseteq K \subseteq G$.
3. If $K \trianglelefteq G$ such that $N \subseteq K \subseteq G$, then $(G/N)/(K/N) \cong G/K$.

**Theorem 1.2.4** (Simplicity of the $A_n$)**.** $A_n$ is simple for all $n \geq 5$.

**Theorem 1.2.5** (Sylow's theorem)**.**

(i) If $P$ is a Sylow $p$-subgroup of a finite group $G$, then all Sylow $p$-subgroups of $G$ are conjugate to $P$.

(ii) If there are $r$ Sylow $p$-subgroups, then $r$ divides $|G|$ and $r \equiv 1 \bmod p$.

**Theorem 1.2.6** (Fundamental Theorem of Abelian Groups)**.** If $G$ and $H$ are finite abelian groups, then $G \cong H$ if and only if, for all primes $p$, they have the same elementary divisors.

**Theorem 1.2.7.** Let $G$ be a finite group and $p$ be the least prime divisor of $|G|$. Then if $H$ is a subgroup of $G$ such that $[G : H] = p$, then $H \trianglelefteq G$.

# 2 Fields

## 2.1 Notation and definitions

### 2.1.1 Basic definitions

**Definition 2.1.1.1** (Degree of a field extension)**.** Suppose that $E/k$ is a field extension. Then $E$ may be considered as a vector space over $k$. The dimension of this vector space is called the degree of the field extension and is denoted by $[E : k]$.

**Definition 2.1.1.2** (Field automorphism)**.** A field automorphism of a field $K$ is an isomorphism $\phi\colon K \to K$. In particular,

$$\phi(a + b) = \phi(a) + \phi(b) \text{ and}$$
$$\phi(ab) = \phi(a)\phi(b).$$

**Definition 2.1.1.3** (Splitting field)**.** A splitting field of a polynomial $p$ over a field $K$ is a field extension $L \supseteq K$ over which $p$ factors into linear factors.

**Definition 2.1.1.4** (Separable polynomial)**.** A polynomial $p$ is called separable if if factors into distinct linear factors in its splitting field.

**Definition 2.1.1.5** (Separable extension)**.** A separable extension is an field extension $E \supseteq F$ such that for every $\alpha \in E$, the minimal polynomial of $\alpha$ over $F$ is a separable polynomial.

**Definition 2.1.1.6** (Normal extension)**.** A normal extension $K \supseteq L$ is one for which every polynomial that is irreducible over $K$ either has no root in $L$ or splits into linear factors in $L$.

**Definition 2.1.1.7** (Galois extension)**.** A Galois extension is an algebraic field extension $E/F$ that is normal and separable.

**Definition 2.1.1.8** (Galois group)**.** Let $E \supseteq F$ be a field extension. The Galois group $\mathrm{Gal}(E/F)$ is the set of automorphisms of $E$ that fix $F$ under function composition.

**Definition 2.1.1.9** (Galois correspondence)**.** Let $E \supseteq F$ be a finite, Galois extension. The Galois correspondence is the bijection between intermediate fields $F \supseteq K \supset E$ and subgroups of the Galois group $E/F$.

**Definition 2.1.1.10** (Trace)**.** ???

**Definition 2.1.1.11** (Norm)**.** ???

**Definition 2.1.1.12** (Radical extension)**.** A radical extension of a field $K$ is an extension that is obtained by adjoining a sequence of $n$th roots of elements of $K$.

**Definition 2.1.1.13** (Finite field)**.** A finite field is a field with a finite number of elements. Note: any finite field has $p^k$ elements for some prime $p$ and $k \in \mathbb{N}$.

**Definition 2.1.1.14** (Cyclotomic extension)**.** A cyclotomic extension $\mathbb{Q}(\xi_n)$ of $\mathbb{Q}$ is an extension formed by adjoining a primitive $n$th root of unity.

**Definition 2.1.1.15** (Algebraic closure)**.** An algebraic closure of a field $K$ is an algebraic extension $F/K$ such that $F$ contains a root for every non-constant polynomial in $F[x]$.

## 2.2 Theorems

**Theorem 2.2.1** (Isomorphism extension theorem)**.** Let $F$ be a field and $\phi\colon F \to F'$ an isomorphism. Then if $E$ is an extension field of $F$, $\phi$ can be extended into an isomorphism $\tau\colon E \to E'$.

**Theorem 2.2.2** (Fundamental theorem of Galois theory)**.** Let $E/k$ be a finite Galois extension with Galois group $G = \mathrm{Gal}(E/k)$. The function

$$\gamma\colon \mathrm{Sub}(\mathrm{Gal}(E/k)) \to \mathrm{Int}(E/k),$$

defined by $H \mapsto E^H$, is an order reversing bijection whose inverse maps $B \mapsto \mathrm{Gal}(E/B)$.

**Theorem 2.2.3** (Primitive element theorem)**.** Finite separable extensions are simple.

# 3 Commutative Algebra

## 3.1 Notation and definitions

### 3.1.1 Basic definitions

**Definition 3.1.1.1** (Maximal ideal)**.** An ideal $\mathfrak{m}$ of $R$ is maximal if $\mathfrak{m} \subsetneq R$ and there is no ideal $K$ of $R$ such that $\mathfrak{m} \subsetneq K \subsetneq R$.

**Definition 3.1.1.2** (Localization)**.** ???

*Note.* Localization is a formal way to introduce the "denominators" to a given ring or module.

**Definition 3.1.1.3** (Integral element)**.** Let $B$ be a ring and $A \subset B$ a subring. Then an element $b \in B$ is called integral over $A$ if for some $n \geq 1$, there exist $a_j \in A$ such that $b^n = a_{n-1}b^{n-1} + \cdots + a_1 b + a_0$.

**Definition 3.1.1.4** (Integral extension)**.** A ring $B$ is called an integral extension of $A \subset B$ if every element of $B$ is integral over $A$.

*Note.* The set of elements of $B$ that are integral over $A$ is called the integral closure of $A$ in $B$.

**Definition 3.1.1.5** (Unique factorization domain)**.** A unique factorization domain is an integral domain in which every non-zero non-unit element can be written as the product of prime elements uniquely.

*Note.* In general every prime is irreducible, but in a UFD, the converse is true.

**Definition 3.1.1.6** (Principal ideal domain)**.** A principal ideal domain is one in which every ideal is generated by a single element. That is, if $R$ is a ring and $I \subseteq R$ is an ideal of $R$, then $I = (a)$ for some element $a \in R$.

**Definition 3.1.1.7** (Noetherian ring)**.** A Noetherian ring is a ring that satisfies the ascending chain condition on ideals, that is given any chain

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq I_{k+1} \subseteq \cdots$$

there exists some $n$ after which $I_n = I_{n+1} = I_{n+2} = \cdots$.

**Definition 3.1.1.8** (Variety)**.** Let $k$ be an algebraically closed field, and $F$ a subset of $k[x_1, \ldots, x_n]$. Then the variety defined by $F$ is

$$\mathrm{Var}(F) = \{a \in k^n | f(a) = 0 \text{ for all } f \in F\}$$

**Definition 3.1.1.9** (Zariski topology)**.** The Zariski topology is a topology on algebraic varieties where the closed sets on $k^n$ are $\mathrm{Var}(F)$ for some $F \subset k[x_1, \ldots, x_n]$.

*Note.* For any two ideals of polynomials $I$ and $J$,

1. $V(I) \cup V(J) = V(IJ) = \mathrm{Var}(I \cap J)$, and
2. $V(I) \cap V(J) = V(I + J)$.

## 3.2 Theorems

**Theorem 3.2.1** (Eisenstein criterion)**.** Let $D$ be an integral domain and $f(x) = a_n x^n + \ldots + a_1 x + a_0$ where $a_i \in D$, and so $f(x) \in D[x]$. Then if there exists a prime ideal $\mathfrak{p}$ of $D$ such that

1. $a_i \in \mathfrak{p}$ for each $i < n$,
2. $a_n \notin \mathfrak{p}$, and
3. $a_0 \notin \mathfrak{p}^2$,

then $f$ cannot be written as the product of two non-constant polynomials in $D[x]$.

**Theorem 3.2.2** (Hilbert basis theorem)**.** A polynomial ring $R[x]$ over a Noetherian ring $R$ is Noetherian.

**Theorem 3.2.3** (Hilbert's Nullstellensatz)**.** *F12.3, S13.3.*
Let $k$ be a field and $\overline{k}$ be an algebraically closed field extension. Consider the polynomial ring $k[x_1, \ldots, x_n]$, and let $I$ be an ideal in this ring. Hilbert's Nullstellensatz states that if $p \in k[x_1, \ldots, x_n]$ vanishes on $\mathrm{Var}(I)$, then $p^r \in I$ for some $r \in \mathbb{N}$.

# 4 Modules

## 4.1 Notation and definitions

### 4.1.1 Basic definitions

**Definition 4.1.1.1** (Irreducible module)**.** An irreducible module or a simple module over a ring $R$ are nonzero modules whose only submodules are the module itself and the zero module.

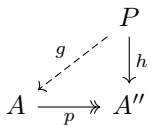
**Definition 4.1.1.2** (Torsion element)**.** An element $m$ of a module $M$ over a ring $R$ is called a torsion element of the module if there exists (a non zero divisor) $r \in R$ such that $rm = 0$.

**Definition 4.1.1.3** (Torsion module)**.** A module $M$ over a ring $R$ is called a torsion module if all of its elements are torsion elements.

**Definition 4.1.1.4** (Free module)**.** A free module is a module that has a basis, $E$. That is

1. $E$ is a generating set for $M$, and
2. $E$ is linearly independent: $r_1e_1 + \ldots + r_ne_n = 0_M$ implies $r_1 = \ldots = r_n = 0_R$.

**Definition 4.1.1.5** (Projective module)**.** A left $R$-module $P$ is projective if whenever $p\colon A \to A''$ is a surjective module homomorphism and $h\colon P \to A''$ is any module homomorphism, then there exists a homomorphism $g\colon P \to A$ with $h = p \circ g$.




*Note.* A left $R$-module $P$ is projective if and only if every short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} P \longrightarrow 0$$

is split, that is, $B \cong A \oplus P$.

**Definition 4.1.1.6** (Modules over PIDs)**.** ???

**Definition 4.1.1.7** (Chain conditions)**.** ???

**Definition 4.1.1.8** (Tensor products)**.** Let $A$ and $B$ be modules over a commutative ring $R$. Then $A \otimes_R B$ is an $R$ module where $\phi\colon A \times B \to A \otimes_R B$ defined by $(a, b) \overset{\phi}{\mapsto} a \otimes b$ is a middle linear map:

1. $\phi(a + a', b) = \phi(a, b) + \phi(a', b)$
2. $\phi(a, b + b') = \phi(a, b) + \phi(a, b')$

3. $\phi(ar, b) = \phi(a, rb)$

Where any bilinear map $h\colon A \times B \to Z$ can be written as $h = \widetilde{h} \circ \phi$ for some unique $\widetilde{h}$.

*Note.* The tensor product is the freest bilinear operation.

**Definition 4.1.1.9** (Exact sequences)**.** An exact sequence is a sequence of objects and morphisms between them

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} G_n$$

such that $\mathrm{Im}(f_k) = \ker(f_{k+1})$.

## 4.2 Theorems

**Theorem 4.2.1** (Structure theorem for finitely generated modules over a PID)**.** For every finitely generated module $M$ over a PID $R$, there is a unique decreasing sequence of proper ideals

$$(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$$

such that $M$ is isomorphic to the sum of cyclic modules:

$$M \cong \bigoplus_{i=1}^{n} R/(d_i) = R/(d_1) \oplus R/(d_2) \oplus \cdots \oplus R/(d_n).$$

*Note.* The number of $d_i$s that are equal to zero is the dimension of the free part of $M$.

# 5 Noncommutative Rings

## 5.1 Notation and definitions

### 5.1.1 Basic definitions

**Definition 5.1.1.1** (Artinian Rings)**.** A ring is called left artinian if it has descending chain condition on left ideals.

**Definition 5.1.1.2** (Jacobson radical)**.** The Jacobson radical of $R$, denoted $J(R)$, is the intersection of all of the maximal left ideals in $R$.

**Definition 5.1.1.3** (Jacobson semisimple)**.** A ring $R$ is called Jacobson semisimple if $J(R) = (0)$.

**Definition 5.1.1.4** (Division rings)**.** A division ring is a "possibly noncommutative field"; that is $D$ is a ring in which $1 \neq 0$ and ever nonzero element $a \in D$ has a multiplicative inverse.

## 5.2 Theorems

**Theorem 5.2.1** (Artin-Wedderburn theorem)**.** *Spring 2012, problem 3.*
Every semisimple ring $R$ is a direct product,

$$R \cong \mathrm{Mat}_{n_1}(\Delta_1) \times \cdots \times \mathrm{Mat}_{n_m}(\Delta_m)$$

**Theorem 5.2.2** (Skolem-Noether theorem)**.** Let $A$ be a central simple $k$-algebra over a field $k$ and let $B$ and $B'$ be isomorphic simple $k$-subalgebras of $A$. If $\psi\colon B \to B'$ is an isomorphism, then there exists a unit $u \in A$ with $\psi(b) = ubu^{-1}$ for all $b \in B$.

**Theorem 5.2.3** (Wedderburn's theorem on finite division rings)**.** Every finite division ring $D$ is a field.

**Theorem 5.2.4** (Maschke's theorem)**.** *Spring 2012, problem 3.*
Let $G$ be a finite group and K a field whose characteristic does not divide the order of $G$. Then $K[G]$, the group algebra of $G$, is semisimple.