

# Algebra Qualifying Exam Solutions

MGSA

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## Fall 2012: Algebra Graduate Exam

### Problem 1.

Use Sylow's theorems directly to find, up to isomorphism, all possible structures of groups of order  $5 \cdot 7 \cdot 23$ .

*Proof.* Sylow's theorems tell us that any group  $G$  must have

$$\begin{aligned} r_5 & \text{ Sylow 5-subgroups,} \\ r_7 & \text{ Sylow 7-subgroups, and} \\ r_{23} & \text{ Sylow 23-subgroups} \end{aligned}$$

where  $r_5, r_7$ , and  $r_{23}$  divide  $5 \cdot 7 \cdot 23$ , and  $r_p \equiv 1 \pmod{p}$ .

$$r_p = 1, 5, 7, 5 \cdot 7, 23, 5 \cdot 23, 7 \cdot 23, \text{ or } 5 \cdot 7 \cdot 23$$

considering the restriction on modulus,  $r_5 \in \{1, 7 \cdot 23\}$ ,  $r_7 = 1$ , and  $r_{23} = 1$ . Let  $P$  and  $Q$  be the unique Sylow 23-subgroup and Sylow 7-subgroup respectively. Since  $P \cap Q = 1$ ,  $PQ \cong P \times Q$ . Let  $R$  be a Sylow 5-subgroup.

Since  $R \trianglelefteq G$  (why?), and  $R$  has a complement  $P \times Q$ ,  $G$  is a semidirect product of  $R$  by  $P \times Q$ , that is  $G = R \ltimes (P \times Q)$ .

By Rotman Lemma 7.21, there is a homomorphism

$$\theta: \underbrace{R \rightarrow \text{Aut}(P \times Q)}_{\mathbb{Z}_5 \rightarrow \mathbb{Z}_{22} \times \mathbb{Z}_6}.$$

But since  $\gcd(5, 22) = \gcd(5, 6) = 1$ , the only homomorphism is trivial. Therefore there is only one group of order  $5 \cdot 7 \cdot 23$ , the abelian group

$$G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{23}.$$

□

**Problem 2.**

Let  $A$ ,  $B$ , and  $C$  be finitely generated  $F[x] = R$  modules for  $F$  a field with  $C$  torsion free. Show that  $A \otimes_R C \cong B \otimes_R C$  implies that  $A \cong B$ . Show by example that this conclusion can fail when  $C$  is not torsion free.

*Proof.* (From Nicolle)

$R$  is a PID since  $F$  is a field, so by the *structure theorem for finitely generated modules over a PID*,

$$\begin{aligned} A &\cong T(A) \oplus R^n \\ B &\cong T(B) \oplus R^m \\ C &\cong R^t, \end{aligned}$$

where  $T(M)$  denotes the torsion submodule of  $M$ . Since  $A \otimes_R C \cong B \otimes_R C$ , it follows that

$$\begin{aligned} (T(A) \oplus R^n) \otimes_R R^t &\cong (T(B) \oplus R^m) \otimes_R R^t \\ (T(A) \otimes_R R^t) \oplus (R^n \otimes_R R^t) &\cong (T(B) \otimes_R R^t) \oplus (R^m \otimes_R R^t) \end{aligned}$$

Thus the free part of  $A \otimes_R C$  is isomorphic to the free part of  $B \otimes_R C$ :

$$R^n \otimes_R R^t \cong R^m \otimes_R R^t,$$

so  $n = m$ . Similarly, the torsion submodules of  $A \otimes_R C$  and  $B \otimes_R C$  are isomorphic:

$$(T(A) \otimes_R R^t) \cong (T(B) \otimes_R R^t),$$

so  $T(A) = T(B)$ . Therefore,

$$A \cong T(A) \oplus R^n \cong T(B) \oplus R^m \cong B,$$

as desired.

As a counterexample, consider  $A = B \oplus \text{Ann}(C)$ . Then

$$A \otimes_R C \cong B \otimes_R C \oplus \underbrace{\text{Ann}(C) \otimes_R C}_0 \cong B \otimes_R C,$$

but  $A \not\cong B$ . □

**Problem 3.**

Working in the polynomial ring  $\mathbb{C}[x, y]$ , show that some power of  $f(x, y) = (x + y)(x^2 + y^4 - 2)$  is in  $I = (x^3 + y^2, y^3 + xy)$ .

**Note.** This is identical to the Problem 5 in the 2014 fall exam.

*Proof.* It is sufficient to show that  $f(x, y)$  vanishes on  $\text{Var}(I)$ ; by Hilbert's Nullstellensatz, this implies that  $f(x, y)^m \in I$  for some  $m \in \mathbb{N}$ .

First note that  $y^3 + xy = y(y^2 + x)$  vanishes when  $y = 0$  or  $x = -y^2$ .

**Case 1.** Assume  $y = 0$ . Then  $x^3 + y^2$  vanishes at  $(0, 0)$ .

**Case 2.** Assume  $x = -y^2$ . Substituting this yields  $(-y^2)^3 + y^2 = y^2(-y^4 + 1)$ , so the polynomial vanishes at  $(0, 0), (-1, 1), (-1, -1), (1, i), (1, -i)$ . Checking these:

$$\begin{aligned} 0^3 + 0^2 &= 0^3 + 0 \cdot 0 &= 0 \\ (-1)^3 + 1^2 &= 1^3 + (-1) \cdot 1 &= 0 \\ (-1)^3 + (-1)^2 &= (-1)^3 + (-1)(-1) &= 0 \\ 1^3 + i^2 &= i^3 + 1 \cdot i &= 0 \\ 1^3 + (-i)^2 &= (-i)^3 + 1(-i) &= 0. \end{aligned}$$

Now it is enough to check that  $f(x, y)$  vanishes on  $\text{Var}(I) = \{(0, 0), (-1, 1), (-1, -1), (1, i), (1, -i)\}$ :

$$\begin{aligned} f(0, 0) &= \underbrace{(0 + 0)}_0 (0^2 + 0^4 - 2) = 0 \\ f(-1, 1) &= \underbrace{(-1 + 1)}_0 ((-1)^2 + 1^4 - 2) \\ f(-1, -1) &= (-1 + (-1)) \underbrace{((-1)^2 + (-1)^4 - 2)}_0 \\ f(1, i) &= (1 + i) \underbrace{(1^2 + i^4 - 2)}_0 \\ f(1, -i) &= (1 + (-i)) \underbrace{(1^2 + (-i)^4 - 2)}_0. \end{aligned}$$

Thus by Hilbert's Nullstellensatz, since  $f$  vanishes on  $\text{Var}(I)$ , a power of  $f$  is in  $I$ . □

**Problem 4.**

For integers  $n, m > 1$ , let  $A \subseteq M_n(\mathbb{Z}_m)$  be a subring with the property that if  $x \in A$  with  $x^2 = 0$  then  $x = 0$ . Show that  $A$  is commutative. Is the converse true?

*Proof.* The idea here it to show that  $A$  is semisimple, and so by Artin-Wedderburn can be written as

$$A \cong M_{n_1}(\Delta_1) \times \dots \times M_{n_m}(\Delta_m)$$

where  $\Delta_i$  is a field because it is finite and  $n_i = 1$ .

The converse is false. Let  $A$  be the ring generated by a single element with  $n = m = 2$ :

$$A = \left\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Then  $A$  is commutative, but  $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  while  $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . □

**Problem 5.**

Let  $F$  be the splitting field of  $f(x) = x^6 - 2$  over  $\mathbb{Q}$ . Show that  $\text{Gal}(F/\mathbb{Q})$  is isomorphic to the dihedral group of order 12.

*Proof.* Firstly,  $F = \mathbb{Q}[\sqrt[3]{2}, \omega]$  where  $\omega$  is a sixth root of unity. Then

$$\begin{aligned} [\mathbb{Q}[\sqrt[3]{2}] : \mathbb{Q}] &= 6, \text{ and} \\ [F : \mathbb{Q}[\sqrt[3]{2}]] &= \varphi(6) = 2, \end{aligned}$$

so  $[F : \mathbb{Q}] = [F : \mathbb{Q}[\sqrt[3]{2}]] \cdot [\mathbb{Q}[\sqrt[3]{2}] : \mathbb{Q}] = 12$  and  $\text{Gal}(F/\mathbb{Q}) = 12$ . Now consider the automorphisms

$$\tau : \begin{cases} \omega \mapsto \bar{\omega} \\ \sqrt[3]{2} \mapsto \sqrt[3]{2} \end{cases} \quad \text{and} \quad \sigma : \begin{cases} \omega \mapsto \omega \\ \sqrt[3]{2} \mapsto \omega \sqrt[3]{2} \end{cases}.$$

Now  $\tau$  is of order 2 and  $\sigma$  is of order 6, and the dihedral relation is satisfied:

$$\begin{aligned} \sigma\tau\sigma\tau(\omega) &= \sigma\tau\sigma(\bar{\omega}) = \sigma\tau(\bar{\omega}) = \sigma(\omega) = \omega \\ \sigma\tau\sigma\tau(\sqrt[3]{2}) &= \sigma\tau\sigma(\sqrt[3]{2}) = \sigma\tau(\omega\sqrt[3]{2}) = \sigma(\bar{\omega}\sqrt[3]{2}) = \underbrace{\bar{\omega}\omega}_1 \sqrt[3]{2} = \sqrt[3]{2}. \end{aligned}$$

□

**Problem 6.**

Given that all groups of order 12 are solvable, show that any group of order  $2^2 \cdot 3 \cdot 7^2$  is solvable.

*Proof.* Let  $r_p$  denote the number of Sylow  $p$ -subgroups of  $G$ . Sylows theorems state that  $r_p$  divides  $2^2 \cdot 3 \cdot 7^2$ , so

$$\begin{aligned} r_2 &\in \{1, 3, 7, 3 \cdot 7, 7^2, 3 \cdot 7^2\} \\ r_3 &\in \{1, 2, 2^2, 7, 2 \cdot 7, 2^2 \cdot 7, 7^2, 2 \cdot 7^2, 2^2 \cdot 7^2\} \\ r_7 &\in \{1, 2, 2^2, 3, 2 \cdot 3, 2^2 \cdot 3\} \end{aligned}$$

also  $r_p \equiv 1 \pmod{p}$ , so

$$\begin{aligned} r_2 &\in \{1, 3, 7, 3 \cdot 7, 7^2, 3 \cdot 7^2\} \\ r_3 &\in \{1, 2^2, 7, 2^2 \cdot 7, 7^2, 2^2 \cdot 7^2\} \\ r_7 &= 1 \end{aligned}$$

This means that there is a unique—and thus normal—Sylow 7-subgroup, call it  $N \cong \mathbb{Z}_7$ . Therefore  $G \cong N \rtimes K$  where  $K$  is a subgroup of order 12.

Now a group is solvable if it has a normal series whose factor groups are cyclic of prime order. Since  $K$  is solvable, it has a normal series

$$K = K_0 \leq K_1 \leq K_2 \leq \dots \leq K_n = 1.$$

where  $K_i/K_{i+1}$  is a cyclic group of prime order. Moreover, since  $N$  is normal,  $NK_{i+1}$  is a subgroup of  $NK_i$ . Thus

$$G = NK_0 \leq NK_1 \leq NK_2 \leq \dots \leq \underbrace{NK_n}_N \leq 1$$

is a normal series of  $G$  where  $NK_i/NK_{i+1} \cong K_i/K_{i+1}$  is a cyclic group of prime order for  $i \in \{0, 1, \dots, n-1\}$ , and  $N/1 \cong N \cong \mathbb{Z}_7$  is a cyclic group of prime order. Therefore  $G$  is solvable.  $\square$

## Spring 2012: Algebra Graduate Exam

### Problem 1.

Let  $I$  be an ideal of  $R = \mathbb{C}[x_1, \dots, x_n]$ . Show that  $\dim_{\mathbb{C}}(R/I)$  is finite if and only if  $I$  is contained in only finitely many maximal ideals of  $R$ .

*Proof.*

□



**Problem 2.**

If  $G$  is a group with  $|G| = 7^2 \cdot 11^2 \cdot 19$ , show that  $G$  must be abelian and describe the possible structures of  $G$ .

*Proof.* We'll start by using Sylow's theorems. Firstly, let  $r_p$  denote the number of Sylow  $p$ -subgroups. Since  $p$  divides  $|G|$ ,

$$\begin{aligned} r_{19} &\in \{1, 7, 7^2, 11, 11 \cdot 7, 11 \cdot 7^2, 11^2, 11^2 \cdot 7, 11^2 \cdot 7^2\}, \\ r_{11} &\in \{1, 7, 7^2, 19, 19 \cdot 7, 19 \cdot 7^2\}, \\ r_7 &\in \{1, 11, 11^2, 19, 19 \cdot 11, 19 \cdot 11^2\}. \end{aligned}$$

Since  $r_p \equiv 1 \pmod{p}$ , we can further refine this to

$$\begin{aligned} r_{19} &= 1, \\ r_{11} &\in \{1, 19 \cdot 7\}, \\ r_7 &= 1. \end{aligned}$$

This means that we have unique subgroups  $H_{19}$  and  $H_7$  of orders 19 and 7 respectively. Since  $H_7$  and  $H_{19}$  are unique and thus normal, the product of  $H_7$  and  $H_{19}$  forms a normal subgroup, call it  $N$ . Since  $H_7 \cap H_{19} = \{e\}$ ,  $H_7 H_{19} \cong H_7 \times H_{19}$ , where  $H_{19}$  is abelian because it is cyclic, and  $H_7$  is abelian because all groups of order  $p^2$  are abelian. Thus  $N \cong \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{19}$  or  $N \cong \mathbb{Z}_{49} \times \mathbb{Z}_{19}$ .

Since  $N$  and  $H_{11}$  are complementary, that is  $N \cap H_{11} = \{e\}$  and  $|N||H_{11}| = |G|$ ,  $G$  can be realized as the semidirect product of  $N$  and  $H_{11}$

$$G = N \rtimes H_{11}.$$

Thus it is enough to consider the possible structures of the semidirect product.

**Case 1.** Assume  $N \cong \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{19}$ . Consider homomorphisms  $\varphi: H_{11} \rightarrow \text{Aut}(N)$ , noting that

$$\text{Aut}(N) \cong \text{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{19}) \cong \text{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_7) \times \text{Aut}(\mathbb{Z}_{19}) \cong \underbrace{\text{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_7)}_{\text{order } 48 \cdot 42} \times \mathbb{Z}_{18}.$$

Since  $\gcd(11, 48 \cdot 42 \cdot 18) = 1$ , the only homomorphism is trivial. So the semidirect product is direct

$$G \cong \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{19} \times H_{11}$$

**Case 2.** Assume  $N \cong \mathbb{Z}_{49} \times \mathbb{Z}_{19}$ . Consider homomorphisms  $\varphi: H_{11} \rightarrow \text{Aut}(N)$ , noting that

$$\text{Aut}(N) \cong \text{Aut}(\mathbb{Z}_{49} \times \mathbb{Z}_{19}) \cong \text{Aut}(\mathbb{Z}_{49}) \times \text{Aut}(\mathbb{Z}_{19}) \cong \underbrace{\text{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_7)}_{\text{order } 7 \cdot 6} \times \mathbb{Z}_{18}.$$

Since  $\gcd(11, 7 \cdot 6 \cdot 18) = 1$ , the only homomorphism is trivial. So the semidirect product is direct

$$G \cong \mathbb{Z}_{49} \times \mathbb{Z}_{19} \times H_{11}$$

Since  $|H_{11}| = 11^2$ , it is abelian, so by the fundamental theorem of abelian groups,  $G$  is isomorphic to

$$\begin{aligned} &\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{11} \times \mathbb{Z}_{11} \times \mathbb{Z}_{19}, \\ &\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{121} \times \mathbb{Z}_{19}, \\ &\mathbb{Z}_{49} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11} \times \mathbb{Z}_{19}, \quad \text{or} \\ &\mathbb{Z}_{49} \times \mathbb{Z}_{121} \times \mathbb{Z}_{19}. \end{aligned}$$

□

**Problem 3.**

Let  $F$  be a finite field and  $G$  a finite group with  $\gcd\{\text{char } F, |G|\} = 1$ . The group algebra  $F[G]$  is an algebra over  $F$  with  $G$  as an  $F$ -basis, elements  $\alpha = \sum_G a_g g$  for  $g \in G$ , and multiplication that extends  $ag \cdot bh = ab \cdot gh$ . Show that any  $x \in F[G]$  that is not a zero left divisor must be invertible in  $F[G]$ .

**Note:** Since  $x$  is not a zero left divisor, if  $xy = 0$  for  $y \in F[G]$  then  $y = 0$ .

*Proof.* Since  $\text{char } F$  does not divide  $|G|$ , by Maschke's Theorem,  $F[G]$  is semisimple, so by the Artin-Wedderburn theorem,

$$F[G] \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \dots \times M_{n_k}(D_k)$$

where  $M_{n_i}(D_i)$  is an  $n_i$ -by- $n_i$  matrix ring over a division ring  $D_i$ .

Thus any  $\alpha = \sum_{g \in G} a_g g \in F[G]$  maps under the isomorphism to

$$\varphi(\alpha) = (a_1, a_2, \dots, a_k) \in M_{n_1}(D_1) \times \dots \times M_{n_k}(D_k).$$

Now suppose for the sake of contradiction that some  $a_i$  is not invertible for some  $i$ ; without loss of generality, say that  $i = 1$ . Then there exists some  $b \neq 0 \in M_{n_1}(D_1)$  such that  $a_1 b = 0$  (why?), and

$$(a_1, a_2, \dots, a_k) \cdot (b, 0, 0, \dots, 0) = (\underbrace{a_1 b}_0, 0, 0, \dots, 0).$$

Therefore  $\varphi^{-1}(a_1, a_2, \dots, a_k) = x$  is a left divisor.

Thus in order for  $x$  not to be a left divisor, all  $a_i$  must be invertible. Thus  $x^{-1} = \varphi^{-1}(a_1^{-1}, a_2^{-1}, \dots, a_k^{-1})$ .  $\square$

**Problem 4.**

If  $p(x) = x^8 + 2x^6 + 3x^4 + 2x^2 + 1 \in \mathbb{Q}[x]$  and if  $\mathbb{Q} \subseteq M \subseteq \mathbb{C}$  is a splitting field for  $p(x)$  over  $\mathbb{Q}$ , argue that  $\text{Gal}(M/\mathbb{Q})$  is solvable.

*Proof.* Let  $q(y) = y^4 + 2y^3 + 3y^2 + 2y + 1$  so that  $q(x^2) = p(x)$ . Since  $\deg(q) = 4$ ,  $q$  is solvable by radicals with roots  $\{a_1, a_2, a_3, a_4\}$  expressible as radicals. Thus  $p$  is also solvable by radicals with roots  $\{\pm\sqrt{a_1}, \pm\sqrt{a_2}, \pm\sqrt{a_3}, \pm\sqrt{a_4}\}$ .  $\square$

**Problem 5.**

Let  $R$  be a commutative ring with 1 and let  $x_1, \dots, x_n \in R$  so that  $x_1y_1 + \dots + x_ny_n = 1$  for some  $y_j \in R$ . Let  $A = \{(r_1, r_2, \dots, r_n) \in R^n \mid x_1r_1 + \dots + x_nr_n = 0\}$ . Show that

- (i)  $R^n \cong_R A \oplus R$ ,
- (ii)  $A$  has  $n$  generators, and
- (iii) when  $R = F[x]$  for  $F$  a field, then  $A_R$  is free of rank  $n - 1$ .

*Proof.* First consider the map  $\varphi: R^n \rightarrow R$  that sends  $(r_1, \dots, r_n) \mapsto x_1r_1 + \dots + x_nr_n$  so that  $\varphi(y_1, \dots, y_n) = 1$  and thus is surjective. Notice also that  $\ker(\varphi) = A$ . So the short exact sequence splits:

$$0 \rightarrow A \hookrightarrow R^n \xrightarrow{\varphi} R \rightarrow 0$$

- (i) Since  $R$ , as a module over itself, is free and thus projective, so  $R^n \cong_R A \oplus R$ .
- (ii) (?)
- (iii) If  $R = F[x]$ , then  $R$  is a PID. Thus by the structure theorem for finitely generated modules over a PID,

$$A \cong T(A) \oplus R^k$$

and since  $R^n \cong A \oplus R = T(A) \oplus R^{k+1}$ ,  $T(A) \cong 0$  and  $k = n - 1$ , so  $\text{rank}(A) = \text{rank}(R^{n-1}) = n - 1$ .

□

**Problem 6.**

For  $p$  a prime, let  $F_p$  be the field of  $p$  elements and  $K$  an extension field of  $F_p$  of dimension 72.

- (i) Describe the possible structures of  $\text{Gal}(K/F_p)$ .
- (ii) If  $g(x) \in F_p[x]$  is irreducible of degree 72, argue that  $K$  is a splitting field of  $g(x)$  over  $F_p$ .
- (iii) Which integers  $d > 0$  have irreducibles in  $F_p[x]$  of degree  $d$  that split in  $K$ ?

*Proof.*

□

## Fall 2013: Algebra Graduate Exam

### Problem 1.

Let  $H$  be a subgroup of the symmetric group  $S_5$ . Can the order of  $H$  be 15, 20 or 30?

*Proof.* First note that the only normal subgroup of  $S_5$  is  $A_5$ , which has order 60.

**Case 1.** Assume  $|H| = 15$ . Then  $H$  must have both a Sylow 5-subgroup and a Sylow 3-subgroup, and thus  $H$  must contain a 5-cycle and a 3-cycle. Since neither of the subgroups generated by these elements is normal in  $S_5$ , their product is not a subgroup. Therefore any subgroup of  $S_5$  containing a 5-cycle and a 3-cycle has more than 15 elements.

**Case 2.** Assume  $|H| = 20$ . Of course,  $H$  must have both a Sylow 5-subgroup and a Sylow 2-subgroup, so

**Case 2a.** Assume the Sylow 2-subgroup contains a transposition. Then a 5-cycle and a transposition generates  $S_5$ , so  $|H| = 120$ , a contradiction.

**Case 2b.** If the Sylow 2-subgroup does not contain a transposition, it must contain element of the form  $(s_1 s_2)(s_3 s_4)$ . (...?)

**Case 3.** Assume  $|H| = 30$ . Thus  $H$  has a Sylow 5-subgroup, a Sylow 3-subgroup, and a Sylow 2-subgroup. Based on Case 2a, if  $H$  has a Sylow 2-subgroup, it must be of the form  $(s_1 s_2)(s_3 s_4)$  (...?)

□

**Problem 2.**

Let  $R$  be a PID and  $M$  a finitely generated torsion module of  $R$ . Show that  $M$  is a cyclic  $R$ -module if and only if for any prime  $\mathfrak{p}$  of  $R$ , either  $\mathfrak{p}M = M$  or  $M/\mathfrak{p}M$  is a cyclic  $R$ -module.

*Proof.*

□

**Problem 3.**

Let  $R = \mathbb{C}[x_1, \dots, x_n]$  and suppose  $I$  is a proper non-zero ideal of  $R$ . The coefficients of a matrix  $A \in M_n(R)$  are polynomials in  $x_1, \dots, x_n$  and can be evaluated at  $\beta \in \mathbb{C}^n$ ; write  $A(\beta) \in M_n(\mathbb{C})$  for the matrix so obtained. If for some  $A \in M_n(R)$  and all  $\alpha \in \text{Var}(I)$ ,  $A(\alpha) = 0_{n \times n}$ , show that for some integer  $m$ ,  $A^m \in M_n(I)$ .

*Proof.*

□



**Problem 4.**

If  $R$  is a noetherian unital ring, show that the power series ring  $R[[x]]$  is also a noetherian unital ring.

*Proof.*

□

**Problem 5.**

Let  $p$  be a prime. Prove that  $f(x) = x^p - x - 1$  is irreducible over  $\mathbb{Z}/p\mathbb{Z}$ . What is the Galois group?

**Hint.** Observe that if  $\alpha$  is a root of  $f(x)$ , then so is  $\alpha + i$  for  $i \in \mathbb{Z}/p\mathbb{Z}$ .

*Proof.*

□

**Problem 6.**

Let  $K \subset \mathbb{C}$  be the field obtained by adjoining all roots of unity in  $\mathbb{C}$  to  $\mathbb{Q}$ . Suppose  $p_1 < p_2$  are primes,  $a \in \mathbb{C} \setminus K$ , and write  $L$  for a splitting field of

$$g(x) = (x^{p_1} - a)(x^{p_2} - a)$$

over  $K$ . Assuming each factor of  $g(x)$  is irreducible, determine the order and the structure of  $\text{Gal}(L/K)$ .

*Proof.*

□

## Spring 2013: Algebra Graduate Exam

### Problem 1.

Let  $p > 2$  be a prime. Describe, up to isomorphism, all groups of order  $2p^2$ .

*Proof.* Next, note that the number of Sylow  $p$  groups must divide the order of the group, and be congruent to 1 mod  $p$ . Therefore there must be exactly one Sylow  $p$  group, and since it is unique it is normal. Call the Sylow  $p$ -subgroup  $N$  and the Sylow 2-subgroup  $K$ . Thus  $G \cong N \rtimes_{\varphi} K$  where  $\varphi: K \rightarrow \text{Aut}(N)$  is a homomorphism.

Note that all groups of order  $p^2$  are abelian, so in particular  $N \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$  or  $N \cong \mathbb{Z}_{p^2}$ .

**Case 1.** Assume  $N \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ , so that  $\text{Aut}(N) \cong GL_2(p)$ , the general linear group over the field of integers modulo  $p$ . Then there are four homomorphisms which give three distinct groups up to isomorphism: the identity, the map  $(x, y) \mapsto (x^{-1}, y)$ , and the map  $(x, y) \mapsto (x^{-1}, y^{-1})$ . (Note: I'm not sure what these are the only homomorphisms)

(i)  $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_2$ ,

(ii)  $G \cong (\mathbb{Z}_p \oplus \mathbb{Z}_p) \times \mathbb{Z}_2$  with operation  $((x_1, y_1), a) \cdot ((x_2, y_2), b) = \begin{cases} ((x_1 x_2, y_1 y_2), a + b) & a = 0 \\ ((x_1 x_2^{-1}, y_1 y_2), a + b) & a = 1 \end{cases}$ , or

(iii)  $G \cong (\mathbb{Z}_p \oplus \mathbb{Z}_p) \times \mathbb{Z}_2$  with operation  $((x_1, y_1), a) \cdot ((x_2, y_2), b) = \begin{cases} ((x_1 x_2, y_1 y_2), a + b) & a = 0 \\ ((x_1 x_2^{-1}, y_1 y_2^{-1}), a + b) & a = 1 \end{cases}$ .

**Case 2.** Assume  $N \cong \mathbb{Z}_{p^2}$  so that  $\text{Aut}(N)$  is of order  $\phi(p^2) = p(p-1)$ . Since  $p^2$  is a power of a prime,  $\text{Aut}(N) \cong \mathbb{Z}_{p(p-1)}$ . Since  $\varphi$  is a homomorphism, it must map  $\bar{0} \mapsto \text{id}$ , and  $\bar{1}$  to an automorphism of order 1 or 2. The only two such automorphisms are the identity and the map  $1 \mapsto -1$ .

(iv)  $G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_2$ , or

(v)  $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_2$  with operation  $(x_1, a) \cdot (x_2, b) = \begin{cases} (x_1 x_2, a + b) & a = 0 \\ (x_1 x_2^{-1}, a + b) & a = 1 \end{cases}$ .

This is the dihedral group of order  $2p^2$ .

□

**Problem 2.**

Let  $R$  be a commutative Noetherian ring with 1. Show that every proper ideal of  $R$  is the product of finitely many (not necessarily distinct) prime ideals of  $R$ .

**Hint.** Consider the set of ideals that are not products of finitely many prime ideals. Also note that if  $R$  is not a prime ring Then  $IJ = (0)$  for some non-zero ideals  $I$  and  $J$  of  $R$

*Proof.*

□

**Problem 3.**

In the polynomial ring  $R = \mathbb{C}[x, y, z]$  show that there is a positive integer  $m$  and polynomials  $f, g, h \in R$  such that

$$\underbrace{(x^{16}y^{25}z^{81} - x^7z^{15} - yz^9 + x^5)}_{p(x,y,z)} = (x-y)^3f + (y-z)^5g + (x+y+z-3)^7h.$$

*Proof.* Firstly, let

$$I = ((x-y)^3, (y-z)^5, (x+y+z-3)^7).$$

It is sufficient to show that  $p(x, y, z)$  vanishes on  $\text{Var}(I)$ ; by Hilbert's Nullstellensatz, this implies that  $p(x, y, z)^m \in I$  for some  $m \in \mathbb{N}$ .

By definition the variety of  $I$  is the points where all polynomials vanish:

$$\text{Var}(I) = \{(x, y, z) : (x-y)^3 = (y-z)^5 = (x+y+z-3)^7 = 0\}$$

Ignoring multiplicity and looking the system of equations

$$\begin{aligned} x-y &= 0 \\ y-z &= 0 \\ x+y+z-3 &= 0 \end{aligned}$$

yields  $x = y = z = 1$ .

Evaluating  $p(x, y, z)$  at  $(1, 1, 1)$  yields

$$p(1, 1, 1) = \underbrace{1^{16}1^{25}1^{81}}_1 - \underbrace{1^71^{15}}_{-1} - \underbrace{1 \cdot 1^9}_{-1} + \underbrace{1^5}_{+1} = 0,$$

so  $p(x, y, z)$  vanishes on  $\text{Var}(I)$  and  $p(x, y, z)^m \in I$  for some  $m \in \mathbb{N}$  by Nullstellensatz. □

**Problem 4.**

Let  $R \neq (0)$  be a finite ring such that for any element  $x \in R$  there is  $y \in R$  with  $xyx = x$ . Show that  $R$  contains an identity element and that for  $a, b \in R$  if  $ab = 1$  then  $ba = 1$ .

*Proof.*

□

**Problem 5.**

Let  $f(x) = x^{15} - 2$ , and let  $L$  be the splitting field of  $f(x)$  over  $\mathbb{Q}$ .

- (a) What is  $[L : \mathbb{Q}]$ ?
- (b) Show there exists a subfield  $F$  of degree 8 that is Galois over  $\mathbb{Q}$ .
- (c) What is  $\text{Gal}(F/\mathbb{Q})$ ?
- (d) Show that there is a subgroup of  $\text{Gal}(L/\mathbb{Q})$  that is isomorphic to  $\text{Gal}(F/\mathbb{Q})$ .

*Proof.* Let  $\omega$  be a fifteenth root of unity. Then  $L = \mathbb{Q}[\omega, \sqrt[15]{2}]$ .

- (a) Since the extension of  $\mathbb{Q}[\sqrt[15]{2}]$  by a fifteenth root of unity is degree  $\phi(15) = 8$ ,

$$[L : \mathbb{Q}] = \underbrace{[L : \mathbb{Q}[\omega]]}_{15} \underbrace{[\mathbb{Q}[\omega] : \mathbb{Q}]}_{\phi(15)=8} = 8 \cdot 15 = 120.$$

- (b) Let  $F = \mathbb{Q}[\omega]$ . As shown above,  $[F : \mathbb{Q}] = \phi(15) = 8$ . Note that  $F$  is Galois because every extension of  $\mathbb{Q}$  by a root of unity is normal and thus Galois.
- (c) An automorphism of  $F$  which fixes  $\mathbb{Q}$  is of the form  $\omega \mapsto \omega^k$  where  $k \in \mathbb{Z}_{15}^\times$ , the multiplicative group of  $\mathbb{Z}_{15}$ , which as a set consists of  $\{\bar{1}, \bar{2}, \bar{4}, \bar{7}, \bar{8}, \bar{11}, \bar{13}, \bar{14}\}$ . and is isomorphic to  $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ .
- (d) This follows from the fundamental theorem of Galois theory. Since  $\mathbb{Q}[\sqrt[15]{2}]$  is an intermediate field ( $\mathbb{Q} \subset \mathbb{Q}[\sqrt[15]{2}] \subset L$ ), then there exists an (order reversing) bijection which sends intermediate fields to subgroups of  $\text{Gal}(L/\mathbb{Q})$ . In particular, this map sends  $\mathbb{Q}[\sqrt[15]{2}] \mapsto \text{Gal}(L/\mathbb{Q}[\sqrt[15]{2}])$ , the group of automorphisms of  $L$  that fix  $\mathbb{Q}[\sqrt[15]{2}]$ . This is isomorphic to  $\text{Gal}(F/\mathbb{Q})$ , the group of automorphisms of  $F$  that fix  $\mathbb{Q}$ .

□



**Problem 6.**

Let  $F/\mathbb{Q}$  be a Galois extension of degree 60, and suppose  $F$  contains a primitive ninth root of unity. Show  $\text{Gal}(F/\mathbb{Q})$  is solvable.

*Proof.* First, let  $\omega$  denote the ninth root of unity. Then

$$\underbrace{[F : \mathbb{Q}]}_{60} = [F : \mathbb{Q}[\omega]] \underbrace{[\mathbb{Q}[\omega] : \mathbb{Q}]}_{\varphi(9)=6},$$

so  $[F : \mathbb{Q}[\omega]] = 10$ .

Now the automorphism group of  $\mathbb{Q}[\omega]$  is isomorphic to the cyclic group of order 6 with generator  $\varphi: \omega \mapsto \omega^2$ . In particular,

$$\omega \xrightarrow{\varphi} \omega^2 \xrightarrow{\varphi} \omega^4 \xrightarrow{\varphi} \omega^8 \xrightarrow{\varphi} \omega^7 \xrightarrow{\varphi} \omega^5 \xrightarrow{\varphi} \omega.$$

□

**Problem 7.**

Let  $n$  be a positive integer. Show that  $f(x, y) = x^n + y^n + 1$  is irreducible in  $\mathbb{C}[x, y]$ .

*Proof.*

□