

Spring 2013: Real Analysis Graduate Exam

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Problem 1. Suppose that $\{f_n\}$ is a sequence of real valued continuously differentiable functions on $[0, 1]$ such that

$$\lim_{n \rightarrow \infty} \int_0^1 |f_n(x)| dx = 0 \text{ and } \lim_{n \rightarrow \infty} \int_0^1 |f'_n(x)| dx = 0.$$

Show that $\{f_n\}$ converges to 0 uniformly on $[0, 1]$.

Proof.

□

Problem 2. Investigate the convergence of $\sum_{n=0}^{\infty} a_n$, where

$$a_n = \int_0^1 \frac{x^n}{1-x} \sin(\pi x) dx$$

Proof. Because the integrand $\sin(\pi x)x^n/(1-x)$ is positive, Tonelli's theorem gives

$$\sum_{n=0}^{\infty} \int_0^1 \frac{x^n}{1-x} \sin(\pi x) dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^n}{1-x} \sin(\pi x) dx \quad (1)$$

$$= \int_0^1 \frac{\sin(\pi x)}{1-x} \sum_{n=0}^{\infty} x^n dx \quad (2)$$

$$= \int_0^1 \frac{\sin(\pi x)}{1-x} \cdot \frac{1}{1-x} dx \quad (3)$$

$$= \int_0^1 \frac{\sin(\pi x)}{(1-x)^2} dx. \quad (4)$$

Notice that (2) implies (3) because the bounds of the integral ensure that x is m -almost everywhere within the radius of converge of the sum.

Because the integrand is non-negative, by elementary calculus

$$\int_0^1 \frac{\sin(\pi x)}{(1-x)^2} dx \geq \int_{1/2}^1 \frac{\sin(\pi x)}{(1-x)^2} dx \geq \int_{1/2}^1 \frac{1-x}{(1-x)^2} dx = \int_{1/2}^1 (1-x)^{-1} dx = \infty.$$

Therefore $\sum_{n=0}^{\infty} a_n = \infty$. □

Problem 3. Let (X, \mathcal{M}, μ) be a measure space, $f_n, f \in L^1(\mu)$. Show that $\int_X |f_n - f| d\mu \rightarrow 0$ as $n \rightarrow \infty$ if and only if

$$\sup_{A \in \mathcal{M}} \left| \int_A f_n d\mu - \int_A f d\mu \right| \rightarrow 0$$

as $n \rightarrow \infty$

Proof. (\implies) Assume that $\int_X |f_n - f| d\mu \rightarrow 0$ as $n \rightarrow \infty$; in particular choose sufficiently large n so that $\int_X |f_n - f| d\mu < \varepsilon$. Then for any set $A \subset X$

$$\varepsilon = \int_X |f_n - f| d\mu = \int_A |f_n - f| d\mu + \int_{A^c} |f_n - f| d\mu \geq \int_A |f_n - f| d\mu \geq \left| \int_A f_n - f d\mu \right|.$$

Since this holds for all $A \in \mathcal{M}$, it also holds for the supremum. □

Problem 4. Let μ and ν be σ -finite positive measures, $\mu \geq \nu$ and assume that $\nu \ll \mu - \nu$ (ν is absolutely continuous with respect to $\mu - \nu$).

Prove that

$$\mu \left(\left\{ \frac{d\nu}{d\mu} = 1 \right\} \right) = 0.$$

Proof.

□