Spring 2013: Complex Analysis Graduate Exam

Peter Kagey

July 25, 2018

Problem 1. Evaluate

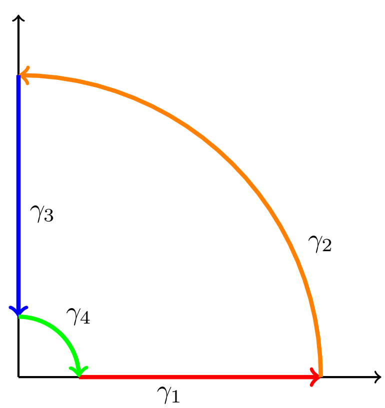
$$\int_0^\infty \frac{x^{1/3}}{1+x^4}\,dx$$

being careful to justify your answer.

Proof. For ease of notation, name the integrand f; that is,

$$f(z) = \frac{z^{1/3}}{1+z^4}.$$

We will compute the integral by using the Residue Theorem together with (the limit of) the following contour:



$$\gamma_1 = \{t + 0i \mid t \in [\varepsilon, R]\} \tag{1}$$
$$\gamma_2 = \{Re^{it} \mid t \in [0, \pi/2]\} \tag{2}$$
$$\gamma_3 = \{0 + ti \mid t \in [\varepsilon, R]\} \tag{3}$$
$$\gamma_4 = \{\varepsilon e^{it} \mid t \in [0, \pi/2]\}. \tag{4}$$

For sufficiently small ϵ and large R, this contour encloses one singularity of f, namely $z_0 = e^{\pi i/4}$.

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz = 2\pi i \operatorname{Res}_{z_0}(f).$$

In the limit, both arcs (γ_2 and γ_4) vanish.

$$\begin{aligned}
\left|\int_{\gamma_2} f(z)\,dz\right| &= \left|\int_0^{\pi/2} \frac{(Re^{it})^{1/3}}{1+(Re^{it})^4} iRe^{it}\,dt\right| \\
&\leq \int_0^{\pi/2} \left|\frac{(Re^{it})^{1/3}}{1+(Re^{it})^4} iRe^{it}\,dt\right| \\
&= \int_0^{\pi/2} \left| iR^{4/3} \frac{e^{4it/3}}{1+R^4e^{4it}}\,dt\right| \\
&\leq \int_0^{\pi/2} \left| iR^{4/3} \frac{e^{4it/3}}{R^4e^{4it}}\,dt\right| = \frac{\pi}{2}R^{-8/3}
\end{aligned}$$

which vanishes as $R \to \infty$. Similarly,

$$
\begin{aligned}
\left| \int_{\gamma_4} f(z)\, dz \right| &= \left| \int_0^{\pi/2} \frac{(\varepsilon e^{it})^{1/3}}{1+(\varepsilon e^{it})^4} i\varepsilon e^{it}\, dt \right| \\
&\le \int_0^{\pi/2} \left| \frac{(\varepsilon e^{it})^{1/3}}{1+(\varepsilon e^{it})^4} i\varepsilon e^{it}\, dt \right| \\
&= \int_0^{\pi/2} \left| i\varepsilon^{4/3} \frac{e^{4it/3}}{1+\varepsilon^4 e^{4it}}\, dt \right| \\
&\le \int_0^{\pi/2} \left| i\varepsilon^{4/3} \frac{e^{4it/3}}{1}\, dt \right| = \frac{\pi}{2}\varepsilon^{4/3}
\end{aligned}
$$

which also vanishes as $\varepsilon \to 0$. This means that our equation simplifies in the limit to

$$
\int_{\gamma_1} f(z)\, dz + \int_{\gamma_3} f(z)\, dz = 2\pi i \operatorname{Res}_{z_0}(f).
$$

And the right hand side further simplifies to

$$
\begin{aligned}
\int_\varepsilon^R \frac{z^{1/3}}{1+z^4}\, dz + \int_R^\varepsilon \frac{(iz)^{1/3}}{1+(iz)^4} i\, dz &= \int_\varepsilon^R \frac{z^{1/3}}{1+z^4}\, dz - i^{4/3} \int_\varepsilon^R \frac{z^{1/3}}{1+z^4}\, dz \\
&= (1 - i^{4/3}) \int_\varepsilon^R \frac{z^{1/3}}{1+z^4}\, dz.
\end{aligned}
$$

So by the Residue Theorem, the integral evaluates to

$$
\int_\varepsilon^R \frac{z^{1/3}}{1+z^4}\, dz = \frac{2\pi i \operatorname{Res}_{z_0}(f)}{1 - i^{4/3}} = \frac{2\pi i}{1 - e^{2\pi i/3}} \operatorname{Res}_{z_0}(f),
$$

and it is enough to compute the residue:

$$
\operatorname{Res}_{z_0}(f) = \frac{z_0^{1/3}}{(z_0^2 + i)(z_0 + z_0)} = \frac{e^{\pi i/12}}{(2e^{\pi i/2})(2e^{\pi i/4})} = \frac{1}{4} e^{-2\pi i/3}.
$$

Therefore

$$
\begin{aligned}
\int_\varepsilon^R \frac{z^{1/3}}{1+z^4}\, dz &= \frac{2\pi i}{1 - e^{2\pi i/3}} \cdot \frac{1}{4} e^{-2\pi i/3} \\
&= \frac{2\pi i}{1 - (-1/2 + \sqrt{3}i/2)} \cdot \frac{1}{4} e^{-2\pi i/3} \\
&= \frac{4\pi i}{3 - \sqrt{3}i} \cdot \frac{1}{4} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) \\
&= \frac{\pi}{3 - \sqrt{3}i} \cdot i \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) \\
&= \frac{\pi}{2} \left(\frac{\sqrt{3} - i}{3 - \sqrt{3}i} \right) \\
&= \frac{\pi}{2\sqrt{3}}.
\end{aligned}
$$

$\square$

Problem 2. Assume that f is an entire function such that

$$|f(z)| \geq \frac{1}{1+|z|} \text{ for all } z \in \mathbb{C}.$$

Prove that f is a constant function.

Proof. $\square$

Problem 3. Let $f_n,\ n \geq 1$, be a sequence of holomorphic functions on an open connected set D such that $|f_n(z)| \leq 1$ for all $z \in D,\ n \geq 1$. Let $A \subseteq D$ be the set of all $z \in D$ for which the limit $\lim_n f_n(z)$ exists.

Show that if A has an accumulation point in D, then there exists a holomorphic function f on D such that $f_n \to f$ uniformly on every compact set of D as $n \to \infty$.

Proof. $\square$

Problem 4. Let $f(z)$ be meromorphic on $\mathbb{C}$, holomorphic for $\operatorname{Re} z > 0$ and such that $f(z+1) = zf(z)$ in its domain with $f(1) = 1$.

Show that f has the first order poles at $0, -1, -2, \ldots$, and find the residues of f at these points.

Proof. Notice first that inductively, we can write

$$f(z) = \frac{f(z+1)}{z} = \frac{\left(\frac{f(z+2)}{z+1}\right)}{z} = \frac{f(z+2)}{z(z+1)} = \ldots = \frac{f(z+k+1)}{z(z+1)\cdots(z+k)}.$$

Now, for each $k \in \mathbb{N}$, checking the limit

$$\begin{aligned}
\lim_{z\to -k}(z+k)f(z) &= \lim_{z\to -k}(z+k)\frac{f(z+k+1)}{z(z+1)\cdots(z+k)} \\
&= \lim_{z\to -k}\frac{f(z+k+1)}{z(z+1)\cdots(z+k-1)} \\
&= \frac{f(1)}{(-k)(-k+1)\cdots(-1)} \\
&= \frac{(-1)^k}{k!}.
\end{aligned}$$

Since this limit exists and is finite for all $k \in \mathbb{N}$, f has first order poles at $-k$ for all k, and the residue of each pole is the value computed above:

$$\operatorname{Res}_{-k}(f) = \frac{(-1)^k}{k!}.$$

$\square$