Fall 2012: Algebra Graduate Exam

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Problem 1. Use Sylow's theorems directly to find, up to isomorphism, all possible structures of groups of order $5 \cdot 7 \cdot 23$.

Proof. Sylow's theorems tell us that any group G must have

$$\begin{aligned} r_5 &\text{ Sylow 5-subgroups,} \\ r_7 &\text{ Sylow 7-subgroups, and} \\ r_{23} &\text{ Sylow 23-subgroups} \end{aligned}$$

where r_5, r_7, and r_{23} divide $5 \cdot 7 \cdot 23$, and $r_p \equiv 1 \bmod p$.

$$r_p = 1, 5, 7, 5 \cdot 7, 23, 5 \cdot 23, 7 \cdot 23, \text{ or } 5 \cdot 7 \cdot 23$$

considering the restriction on modulus, $r_5 \in \{1, 7 \cdot 23\}$, $r_7 = 1$, and $r_{23} = 1$. Let P and Q be the unique Sylow 23-subgroup and Sylow 7-subgroup respectively. Since $P \cap Q = 1$, $PQ \cong P \times Q$. Let R be a Sylow 5-subgroup.
Since $R \trianglelefteq G$ (why?), and R has a complement $P \times Q$, G is a semidirect product of R by $P \times Q$, that is $G = R \ltimes (P \times Q)$.
By Rotman Lemma 7.21, there is a homomorphism

$$\theta \colon \underbrace{R \to \operatorname{Aut}(P \times Q)}_{\mathbb{Z}_5 \to \mathbb{Z}_{22} \times \mathbb{Z}_6}.$$

But since $\gcd(5, 22) = \gcd(5, 6) = 1$, the only homomorphism is trivial. Therefore there is only one group of order $5 \cdot 7 \cdot 23$, the abelian group

$$G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{23}.$$

□

Problem 2. Let A, B, and C be finitely generated $F[x] = R$ modules for F a field with C torsion free. Show that $A \otimes_R C \cong B \otimes_R C$ implies that $A \cong B$. Show by example that this conclusion can fail when C is not torsion free.

Proof. (From Nicolle)
R is a PID since F is a field, so by the *structure theorem for finitely generated modules over a PID*,

$$\begin{aligned} A &\cong T(A) \oplus R^n \\ B &\cong T(B) \oplus R^m \\ C &\cong R^t, \end{aligned}$$

were $T(M)$ denotes the torsion submodule of M. Since $A \otimes_R C \cong B \otimes_R C$, it follows that

$$\begin{gathered} (T(A) \oplus R^n) \otimes_R R^t \cong (T(B) \oplus R^m) \otimes_R R^t \\ (T(A) \otimes_R R^t) \oplus (R^n \otimes_R R^t) \cong (T(B) \otimes_R R^t) \oplus (R^m \otimes_R R^t) \end{gathered}$$

Thus the free part of $A \otimes_R C$ is isomorphic to the free part of $B \otimes_R C$:

$$R^n \otimes_R R^t \cong R^m \otimes_R R^t,$$

so $n = m$. Similarly, the torsion submodules of $A \otimes_R C$ and $B \otimes_R C$ are isomorphic:

$$(T(A) \otimes_R R^t) \cong (T(B) \otimes_R R^t),$$

so $T(A) = T(B)$. Therefore,

$$A \cong T(A) \oplus R^n \cong T(B) \oplus R^m \cong B,$$

as desired.

As a counterexample, consider $A = B \oplus \mathrm{Ann}(C)$. Then

$$A \otimes_R C \cong B \otimes_R C \oplus \underbrace{\mathrm{Ann}(C) \otimes_R C}_{0} \cong B \otimes_R C,$$

but $A \ncong B$. $\square$

Problem 3. Working in the polynomial ring $\mathbb{C}[x, y]$, show that some power of $(x + y)(x^2 + y^4 - 2)$ is in $(x^3 + y^2, y^3 + xy)$.

Proof. $\square$

Problem 4. For integers $n, m > 1$, let $A \subseteq M_n(\mathbb{Z}_m)$ be a subring with the property that if $x \in A$ with $x^2 = 0$ then $x = 0$. Show that A is commutative. Is the converse true?

Proof. $\square$

Problem 5. Let F be the splitting field of $f(x) = x^6 - 2$ over $\mathbb{Q}$. Show that $\mathrm{Gal}(F/\mathbb{Q})$ is isomorphic to the dihedral group of order 12.

Proof. $\square$

Problem 6. Given that all groups of order 12 are solvable, show that any group of order $2^2 \cdot 3 \cdot 7^2$ is solvable.

Proof. $\square$