

Fall 2013: Algebra Graduate Exam

Problem 1.

Let H be a subgroup of the symmetric group S_5 . Can the order of H be 15, 20 or 30?

Proof.

□

Problem 2.

Let R be a PID and M a finitely generated torsion module of R . Show that M is a cyclic R -module if and only if for any prime \mathfrak{p} of R , either $\mathfrak{p}M = M$ or $M/\mathfrak{p}M$ is a cyclic R -module.

Proof.

□

Problem 3.

Let $R = \mathbb{C}[x_1, \dots, x_n]$ and suppose I is a proper non-zero ideal of R . The coefficients of a matrix $A \in M_n(R)$ are polynomials in x_1, \dots, x_n and can be evaluated at $\beta \in \mathbb{C}^n$; write $A(\beta) \in M_n(\mathbb{C})$ for the matrix so obtained. If for some $A \in M_n(R)$ and all $\alpha \in \text{Var}(I)$, $A(\alpha) = 0_{n \times n}$, show that for some integer m , $A^m \in M_n(I)$.

Proof.

□

Problem 4.

If R is a noetherian unital ring, show that the power series ring $R[[x]]$ is also a noetherian unital ring.

Proof.

□

Problem 5.

Let p be a prime. Prove that $f(x) = x^p - x - 1$ is irreducible over $\mathbb{Z}/p\mathbb{Z}$. What is the Galois group?

Hint. Observe that if α is a root of $f(x)$, then so is $\alpha + i$ for $i \in \mathbb{Z}/p\mathbb{Z}$.

Proof.

□

Problem 6.

Let $K \subset \mathbb{C}$ be the field obtained by adjoining all roots of unity in \mathbb{C} to \mathbb{Q} . Suppose $p_1 < p_2$ are primes, $a \in \mathbb{C} \setminus K$, and write L for a splitting field of

$$g(x) = (x^{p_1} - a)(x^{p_2} - a)$$

over K . Assuming each factor of $g(x)$ is irreducible, determine the order and the structure of $\text{Gal}(L/K)$.

Proof.

□