Spring 2015: Real Analysis Graduate Exam

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Problem 1. Prove that for almost all $x \in [0, 1]$, there are at more finitely many rational numbers with reduced form p/q such that $q \geq 2$ and $|x - p/q| < 1/(q \log q)^2$. (Hint: consider intervals of length $2/(qlogq)^2$ centered at rational points p/q.)

Proof. $\square$

Problem 2. Suppose that the real-valued function $f(x)$ is nondecreasing on the interval $[0, 1]$. Prove that there exists a sequence of continuous functions $f_n(x)$ such that $f_n \to f$ pointwise on this interval.

Proof. $\square$

Problem 3. Let (X, μ) be a finite measure space. Assume that a sequence of integrable functions f_n satisfies $f_n \to f$ in measure, where f is measurable. Assume that f_n satisfies the following property: For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) \leq \delta \Longrightarrow \int_E |f_n| d\mu \leq \varepsilon.$$

Prove that f is integrable and that

$$\lim_n \int_X |f_n - f| d\mu = 0.$$

Proof. □

Problem 4. Consider the following two statements about a function $f : [0, 1] \to \mathbb{R}$:

(i) f is continuous almost everywhere

(ii) f is equal to a continuous function g almost everywhere.

Does (i) imply (ii)? Prove or give a counterexample. Does (ii) imply (i)? Prove or give a counterexample.

Proof. $\square$