Spring 2014: Complex Analysis Graduate Exam

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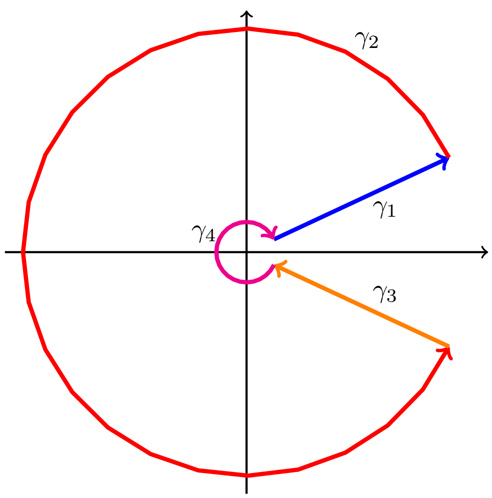
Problem 1. For $a > 0$, evaluate the integral

$$\int_0^\infty \frac{\log x}{(a+x)^3}\,dx$$

being careful to justify your methods.

Proof. First, call this integral S, and denote the integrand by $f(x)$. We will evaluate this integral using the usual trick: integrating $g(z) = f(z)\log(z)$ around a keyhole contour, with the branch cut of the logarithm along the positive real axis, so that the contour contains the pole at $z = -a$.



$$\gamma_1 = \{te^{i\rho} \mid t \in [\varepsilon, R]\} \tag{1}$$
$$\gamma_2 = \{Re^{it} \mid t \in [\rho, 2\pi - \rho]\} \tag{2}$$
$$\gamma_3 = \{-te^{(2\pi-\rho)i} \mid t \in [-R, -\varepsilon]\} \tag{3}$$
$$\gamma_4 = \{\varepsilon e^{-it} \mid t \in [-2\pi + \rho, -\rho]\}. \tag{4}$$

Then by the residue theorem,

$$\int_{\gamma_1} g(z)dz + \int_{\gamma_2} g(z)dz + \int_{\gamma_3} g(z)dz + \int_{\gamma_4} g(z)dz = \operatorname{Res}_{-a}(g).$$

As expected, the integrals along the arcs vanish in the limit.
Firstly, the large arc is bounded by

$$\begin{aligned}\left|\int_{\gamma_2} g(z)dz\right| &\leq \int_0^{2\pi} \left|\frac{\log^2(Re^{i\theta})}{(a+Re^{i\theta})^3} Rie^{i\theta}\right| d\theta \\ s &\leq \int_0^{2\pi} \left|\frac{\log^2(Re^{i\theta})}{R^3e^{3i\theta}} R\right| d\theta \\ &\leq \int_0^{2\pi} \left|\frac{\log^2 R + 2i\theta \log R - \theta}{R^2}\right| d\theta,\end{aligned}$$

which vanishes as $R \to \infty$ by the ML inequality. Next, the small arc is bounded by

$$\begin{aligned}
\left| \int_{\gamma_4} g(z) dz \right| &\leq \int_0^{2\pi} \left| \frac{\log^2(\rho e^{i\theta})}{(a + \rho e^{i\theta})^3} \rho i e^{i\theta} \right| d\theta \\
&\leq \int_0^{2\pi} \left| \frac{\log^2(\rho e^{i\theta})}{(a/2)^3} \rho \right| d\theta \\
&\leq \frac{8}{a^3} \int_0^{2\pi} \left| \rho(\log^2 \rho + 2i\theta \log \rho - \theta) \right| d\theta,
\end{aligned}$$

which vanishes as $\rho \to 0$ with the ML inequality together with two applications of L'Hôpital's rule

$$\lim_{\rho \to 0} \rho \log^2 \rho = \lim_{\rho \to 0} \frac{\log^2 \rho}{\rho^{-1}} = \lim_{\rho \to 0} \frac{2\rho^{-1} \log \rho}{-\rho^{-2}} = \lim_{\rho \to 0} \frac{2 \log \rho}{-\rho^{-1}} = \lim_{\rho \to 0} \frac{2\rho^{-1}}{\rho^{-2}} = \lim_{\rho \to 0} 2\rho = 0.$$

Now the integral has been reduced to

$$\int_{\gamma_1} g(z) dz + \int_{\gamma_3} g(z) dz = \operatorname{Res}_{-a}(g).$$

Evaluating the remaining integrals yields

$$\begin{aligned}
\int_{\gamma_1} g(z) dz + \int_{\gamma_3} g(z) &= \lim_{\rho \to 0} \int_{\varepsilon}^{R} \frac{\log^2(te^{i\rho})}{(a + te^{i\rho})^3} e^{i\rho} dt + \int_{R}^{\varepsilon} \frac{\log^2(te^{i(2\pi - \rho)})}{(a + te^{i(2\pi-\rho)})^3} e^{i(2\pi - \rho)} dt \\
&= \int_{\varepsilon}^{R} \frac{\log^2(t)}{(a+t)^3} dt + \int_{R}^{\varepsilon} \frac{(\log(t) + 2\pi i)^2}{(a+t)^3} dt \\
&= \int_{\varepsilon}^{R} \frac{4\pi^2 - 4\pi i \log(t)}{(a+t)^3} dt \\
&= \int_{\varepsilon}^{R} \frac{4\pi^2}{(a+t)^3} dt - 4\pi i \int_{\varepsilon}^{R} \frac{\log(t)}{(a+t)^3} dt.
\end{aligned}$$

Using ordinary techniques of integration, we can evaluate the integral

$$\int_0^{\infty} \frac{4\pi^2}{(a+t)^3} dt = \left[\frac{4\pi^2}{-2(a-t)^2} \right]_0^{\infty} = \frac{2\pi^2}{a^2}.$$

Computing the residue of g at $z = -a$ just requires computing the Taylor series of $\log^2 z$ centered at $z = -a$ to a second order term, which requires computing the second derivative of $\log^2 z$.

$$\begin{aligned}
\frac{d}{dz}[\log^2 z] &= 2 \log(z) z^{-1} \\
\frac{d^2}{dz^2}[\log^2 z] &= 2(-\log(z) z^{-2} + z^{-2})
\end{aligned}$$

Thus

$$\frac{\log^2 z}{(z+a)^3} = \frac{1}{(z+a)^3} = \left[\log^2(-a) + \frac{2\log(-a)}{-a}(z+a) + \frac{2(1 - \log(-a))}{2a^2}(z+a)^2 + \dots \right].$$

So the residue at $z = -a$ is

$$\operatorname{Res}_{-a}(g) = \frac{1 - \log(-a)}{a^2} = \frac{1 - \log(a) - \log(-1)}{a^2} = \frac{1 - \log(a) - \pi i}{a^2}.$$

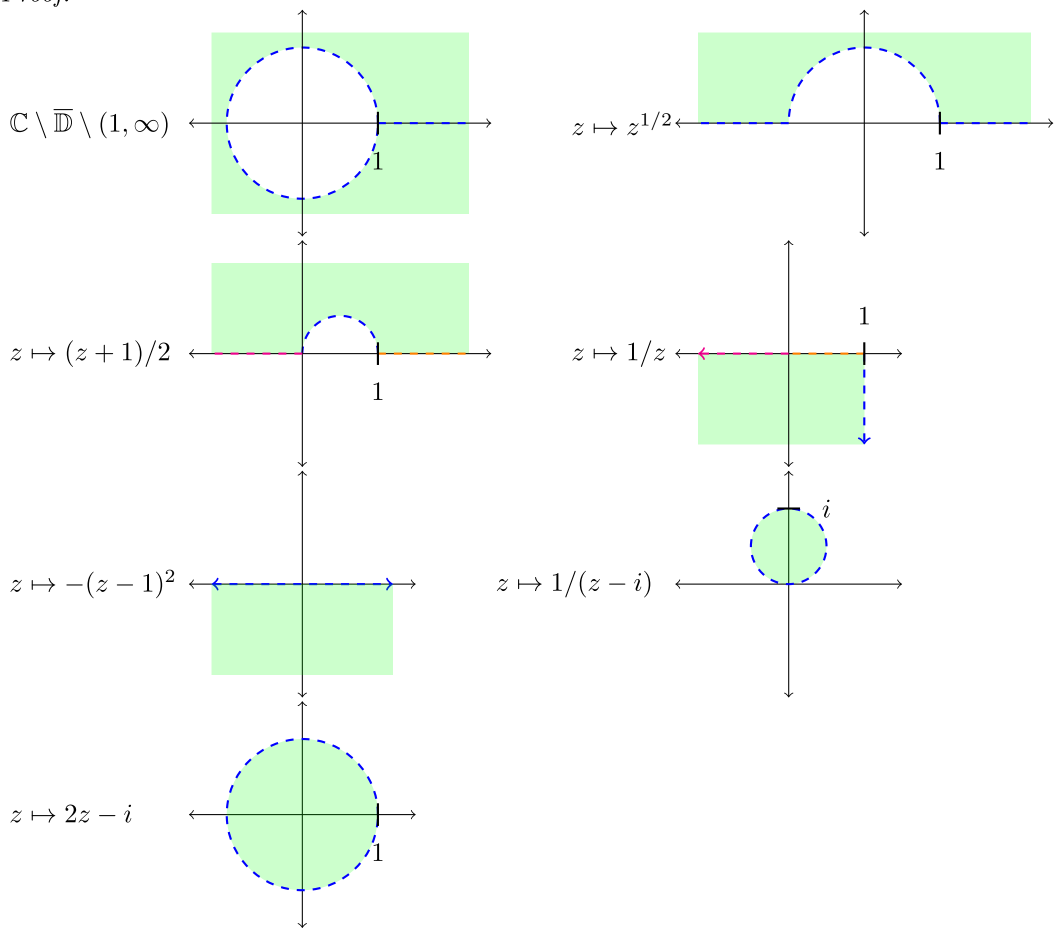
Now we have all of the ingredients to compute the integral:

$$\begin{aligned}
\frac{2\pi^2}{a^2} - 4\pi i \int_0^{\infty} \frac{\log(t)}{(a+t)^3} dt &= 2\pi i \left(\frac{1 - \log(a) - \pi i}{a^2} \right) \\
\int_0^{\infty} \frac{\log(t)}{(a+t)^3} dt &= \frac{1}{4\pi i} \left(\frac{2\pi^2}{a^2} - \frac{2\pi i(1 - \log(a) - \pi i)}{a^2} \right) = \frac{\log(a) - 1}{2a^2}.
\end{aligned}$$

$\square$

Problem 2. Find a conformal mapping of the region $\{z : |z| > 1\} \setminus (1, \infty)$ onto the open unit disk $\{z : |z| < 1\}$.

Proof.



□

Problem 3. Suppose that f_n are analytic functions on a connected open set $U \subset \mathbb{C}$ and that $f_n \to f$ uniformly on compact subsets of U. In each case indicate the main steps in the proofs of the following standard results.

(i) f is analytic in U;

(ii) $f_n' \to f'$ uniformly on compact subsets of U;

(iii) if $f_n(z) \neq 0$ for all n and all $z \in U$, then either $f(z) \neq 0$ for all $z \in U$ or else $f \equiv 0$.

Proof. $\square$

Problem 4.

(a) Suppose that f is analytic on the open unit disk $\{z : |z| < 1\}$ and that there exists a constant M such that $|f^k(0)| \leq k^4 M^k$ for all $k \geq 0$. Show that f can be extended to be analytic on $\mathbb{C}$.

(b) Suppose that f is analytic on the open unit disk and that there exists a constant $M > 1$ such that $|f(1/k)| \leq M^{-k}$ for all $k \geq 1$. Show that f is identically zero.

Proof. $\square$