Spring 2012: Geometry/Topology Graduate Exam

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Problem 1. Prove that a compact manifold of dimension n cannot be immersed in $\mathbb{R}^n$.

Proof. $\square$

Problem 2. *(Topology)*

Proof. $\square$

Problem 3. Let S be an oriented embedded surface in $\mathbb{R}^3$ and ω be an area form on S which satisfies $\omega(p)(e_1, e_2) = 1$ for all $p \in S$ and any orthonormal basis (e_1, e_2) of T_pS with respect to the standard Euclidean metric on $\mathbb{R}^3$. If (n_1, n_2, n_3) is the unit normal vector field of S, then prove that

$$\omega = n_1 dy \wedge dz - n_2 dx \wedge dz + n_1 dx \wedge dy,$$

where (x, y, z) are the standard Euclidean coordinates on $\mathbb{R}^3$.

Proof. $\square$

Problem 4. *(Topology)*

Proof. $\square$

Problem 5. *(Topology)*

Proof. □

Problem 6. Let $f\colon M \to N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N respectively, and suppose that $f_* X = Y$.
Prove that $f^*(\mathcal{L}_Y \omega) = \mathcal{L}_X(f^*\omega)$, where ω is a 1-form on N. Here $\mathcal{L}$ denotes the Lie derivative.

Proof. $\square$

Problem 7. *(Geometry)* Consider the linearly independent vector fields on $\mathbb{R}^4 - \{0\}$ given by:

$$X(x_1, x_2, x_3, x_4) = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}$$
$$Y(x_1, x_2, x_3, x_4) = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}.$$

Is the rank 2 distribution orthogonal to these two vector fields integrable? Here orthogonality is measured with respect to the standard Euclidean metric on $\mathbb{R}^4$.

Proof. $\square$