Spring 2013: Algebra Graduate Exam

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Problem 1. Let $p > 2$ be a prime. Describe, up to isomorphism, all groups of order $2p^2$.

Proof. Next, note that the number of Sylow p groups must divide the order of the group, and be congruent to $1 \bmod p$. Therefore there must be exactly one Sylow p group, and since it is unique it is normal. Call the Sylow p-subgroup N and the Sylow 2-subgroup K. Thus $G \cong N \rtimes_\varphi K$ where $\varphi \colon K \to \mathrm{Aut}(N)$ is a homomorphism.
Note that all groups of order p^2 are abelian, so in particular $N \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ or $N \cong \mathbb{Z}_{p^2}$.

Case 1. Assume $N \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, so that $\mathrm{Aut}(N) \cong GL_2(p)$, the general linear group over the field of integers modulo p. Then there are four homomorphisms which give three distinct groups up to isomorphism: the identity, the map $(x, y) \mapsto (x^{-1}, y)$, and the map $(x, y) \mapsto (x^{-1}, y^{-1})$. (Note: I'm not sure what these are the only homomorphisms)

(i) $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_2$,

(ii) $G \cong (\mathbb{Z}_p \oplus \mathbb{Z}_p) \times \mathbb{Z}_2$ with operation $((x_1, y_1), a) \cdot ((x_2, y_2), b) = \begin{cases} ((x_1x_2, y_1y_2), a + b) & a = 0 \\ ((x_1x_2^{-1}, y_1y_2), a + b) & a = 1 \end{cases}$, or

(iii) $G \cong (\mathbb{Z}_p \oplus \mathbb{Z}_p) \times \mathbb{Z}_2$ with operation $((x_1, y_1), a) \cdot ((x_2, y_2), b) = \begin{cases} ((x_1x_2, y_1y_2), a + b) & a = 0 \\ ((x_1x_2^{-1}, y_1y_2^{-1}), a + b) & a = 1 \end{cases}$.

Case 2. Assume $N \cong \mathbb{Z}_{p^2}$ so that $\mathrm{Aut}(N)$ is of order $\phi(p^2) = p(p - 1)$. Since p^2 is a power of a prime, $\mathrm{Aut}(N) \cong \mathbb{Z}_{p(p-1)}$. Since φ is a homomorphism, it must map $\overline{0} \mapsto \mathrm{id}$, and $\overline{1}$ to an automorphism of order 1 or 2. The only two such automorphisms are the identity and the map $1 \mapsto -1$.

(iv) $G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_2$, or

(v) $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_2$ with operation $(x_1, a) \cdot (x_2, b) = \begin{cases} (x_1x_2, a + b) & a = 0 \\ (x_1x_2^{-1}, a + b) & a = 1 \end{cases}$.

This is the dihedral group of order $2p^2$.

□

Problem 2. Let R be a commutative Noetherian ring with 1. Show that every proper ideal of R is the product of finitely many (not necessarily distinct) prime ideals of R.

Hint . Consider the set of ideals that are not products of finitely many prime ideals. Also note that if R is not a prime ring Then $IJ = (0)$ for some non-zero ideals I and J of R

Proof. $\square$

Problem 3. In the polynomial ring $R = \mathbb{C}[x, y, z]$ show that there is a positive integer m and polynomials $f, g, h \in R$ such that

$$(\underbrace{x^{16}y^{25}z^{81} - x^7z^{15} - yz^9 + x^5}_{p(x,y,z)})^m = (x-y)^3f + (y-z)^5g + (x+y+z-3)^7h.$$

Proof. Firstly, let

$$I = ((x-y)^3, (y-z)^5, (x+y+z-3)^7).$$

It is sufficient to show that $p(x, y, z)$ vanishes on $\mathrm{Var}(I)$; by Hilbert's Nullstellensatz, this implies that $p(x, y, z)^m \in I$ for some $m \in \mathbb{N}$.

By definition the variety of I is the points where all polynomials vanish:

$$\mathrm{Var}(I) = \{(x, y, z) : (x-y)^3 = (y-z)^5 = (x+y+z-3)^7 = 0\}$$

Ignoring multiplicity and looking the system of equations

$$\begin{aligned} x-y \quad &= 0 \\ y-z \quad &= 0 \\ x+y+z-3 &= 0 \end{aligned}$$

yields $x = y = z = 1$.

Evaluating $p(x, y, z)$ at $(1, 1, 1)$ yields

$$p(1,1,1) = \underbrace{1^{16}1^{25}1^{81}}_{1} \underbrace{-1^71^{15}}_{-1} \underbrace{-1 \cdot 1^9}_{-1} \underbrace{+1^5}_{+1} = 0,$$

so $p(x, y, z)$ vanishes on $\mathrm{Var}(I)$ and $p(x, y, z)^m \in I$ for some $m \in \mathbb{N}$ by Nullstellensatz. $\square$

Problem 4. Let $R \neq (0)$ be a finite ring such that for any element $x \in R$ there is $y \in R$ with $xyx = x$. Show that R contains an identity element and that for $a, b \in R$ if $ab = 1$ then $ba = 1$.

Proof. $\square$

Problem 5. Let $f(x) = x^{15} - 2$, and let L be the splitting field of $f(x)$ over $\mathbb{Q}$.

(a) What is $[L : \mathbb{Q}]$?

(b) Show there exists a subfield F of degree 8 that is Galois over $\mathbb{Q}$.

(c) What is $\mathrm{Gal}(F/\mathbb{Q})$

(d) Show that there is a subgroup of $\mathrm{Gal}(L/\mathbb{Q})$ that is isomorphic to $\mathrm{Gal}(F/Q)$.

Proof. $\square$

Problem 6. Let $F/\mathbb{Q}$ be a Galois extension of degree 60, and suppose F contains a primitive ninth root of unity. Show $\mathrm{Gal}(F/\mathbb{Q})$ is solvable.

Proof. $\square$

Problem 7. Let n be a positive integer. Show that $f(x, y) = x^n + y^n + 1$ is irreducible in $\mathbb{C}[x, y]$.

Proof. $\square$