Spring 2012: Complex Analysis Graduate Exam

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Problem 1. Suppose $a > 0$. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)}\, dx$$

being careful to justify your methods.

Proof. After a transformation, this integral can be computed by using the *Cauchy principal value* of the integral. In particular, the given integral can be rewritten as

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)}\, dx = \int_{-\infty}^{\infty} R(x)e^{ix}\, dx$$

where $R(x)$ is a rational function. First, by the substitution $u = ax$, the integral can be rewritten as

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)}\, dx = \frac{1}{a}\int_{-\infty}^{\infty} \frac{\sin(u)}{\frac{u}{a}((\frac{u}{a})^2+1)}\, du. = \int_{-\infty}^{\infty} \sin(u)\frac{a^2}{u(u^2+a^2)}\, du,$$

Then by the identity $\sin(u) = -i(e^{iu} - \cos(u))$, this can be further rewritten as

$$-ia^2\int_{-\infty}^{\infty} \frac{e^{iu}-\cos(u)}{u(u^2+a^2)}\, du = -ia^2\int_{-\infty}^{\infty} \frac{e^{iu}}{u(u^2+a^2)}\, du + \underbrace{ia^2\int_{-\infty}^{\infty} \frac{\cos(u)}{u(u^2+a^2)}\, du}_{=0\text{x, odd integrand}} = \int_{-\infty}^{\infty} R(u)e^{iu}\, du.$$

where

$$R(u) = \frac{-ia^2}{u(u^2+a^2)}.$$

Now it is enough to compute some poles and residues. In particular, the integrand $g(z) = R(z)e^{iz}$ has poles at $z = 0$, $z = ai$, and $z = -ai$. The residue $\operatorname{Res}_0(g)$ at $z = 0$ can be determined from the Taylor expansion about 0:

$$g(z) = \frac{-ia^2}{z(z^2+a^2)}e^{iz} = \frac{1}{z}\left(\frac{-ia^2e^{iz}}{z^2+a^2}\right) = \frac{1}{z}\left[\frac{-ia^2e^0}{0^2+a^2} + \dots\right].$$

Thus $\operatorname{Res}_0(g) = -i$. Next, the residue $\operatorname{Res}_{ai}(g)$ can be determined similarly:

$$g(z) = \frac{-ia^2}{z(z^2+a^2)}e^{iz} = \frac{1}{z-ai}\left(\frac{-ia^2e^{iz}}{z(z+ai)}\right) = \frac{1}{z-ai}\left[\frac{-ia^2e^{-a}}{ai(ai+ai)} + \dots\right].$$

so $\operatorname{Res}_{ai}(g) = \frac{i}{2}e^{-a}$. Therefore

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)}\, dx = 2\pi i\left[\operatorname{Res}_{ai}(g) + \frac{1}{2}\operatorname{Res}_0(g)\right] = 2\pi i\left[\frac{i}{2}e^{-a} + \frac{1}{2}(-i)\right] = \pi(1-e^{-a}).$$

$\square$

Problem 2. Let $f(z)$ be analytic for $0 < |z| < 1$. Assume there are $C > 0$ and $m \geq 1$ such that

$$|f^{(m)}(z)| \leq \frac{C}{|z|^m}, \quad 0 < |z| < 1.$$

Show that f has a removable singularity at $z = 0$.

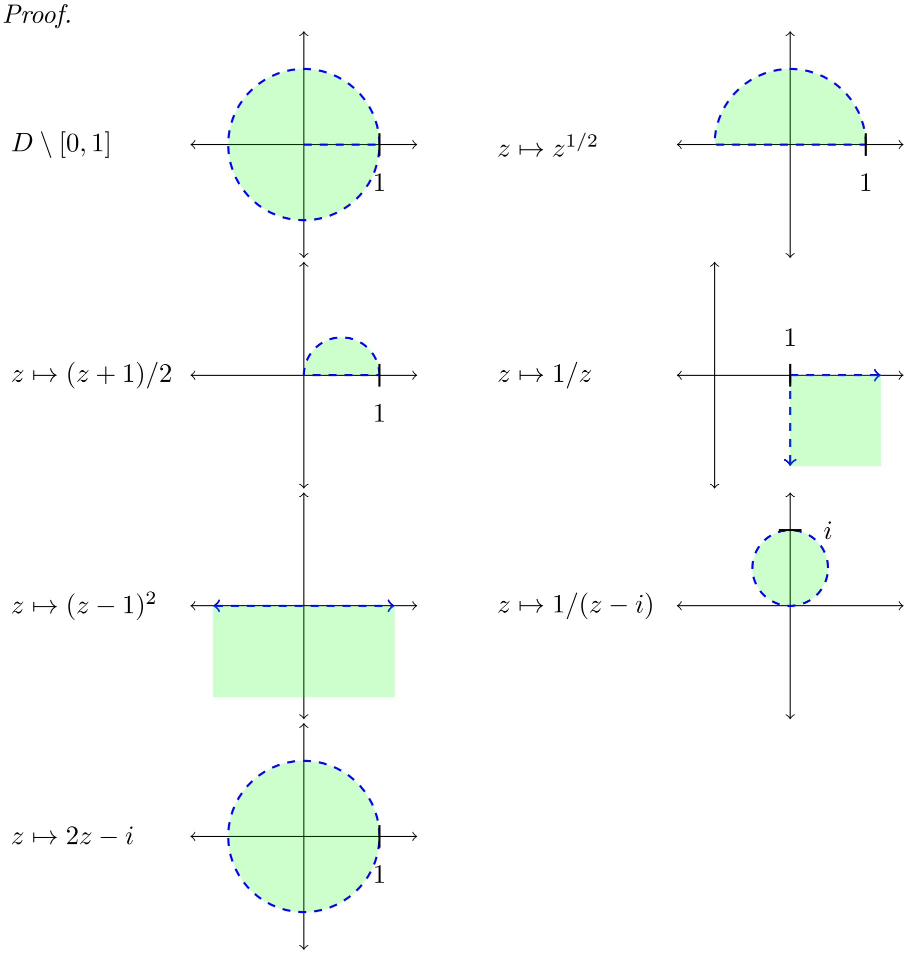
Proof. $\square$

Problem 3. Let $D \subseteq \mathbb{C}$ be a connected open subset and let (u_n) be a sequence of harmonic functions $u_n\colon D \to (0, \infty)$. Show that if $u_n(z_0) \to 0$ for some $z_0 \in D$, then $u_n \to 0$ uniformly on compact subsets of D.

Proof. $\square$

Problem 4. Let D be the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane, and define $\Omega = D \setminus [0,1]$. Find a conformal mapping of Ω onto D. You may give your answer as the composition of several mappings, so long as each mapping is precisely described.

Proof.



$\square$