Spring 2015: Real Analysis Graduate Exam

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Problem 1. Consider the sequence

$$f_n(x) = \left(1 + \frac{x}{n}\right)^{-n} cos\left(\frac{x}{n}\right), \quad n = 1, 2, \dots .$$

Evaluate

$$\lim_n \int_0^\infty f_n(x)\ dx,$$

being careful to justify your answer.

Proof. The boundedness of cosine and non-negativity of $1 + \frac{x}{n}$ imply that for $n > 2$ and $x > 0$

$$|f_n(x)| \leq \left(1 + \frac{x}{n}\right)^{-n} \leq \left(1 + \frac{x}{2}\right)^{-2} = g(x).$$

The function g is integrable on $[0, \infty)$ because by the p-test

$$\int_0^\infty g(x)\ dx = \int_0^\infty \frac{dx}{1 + x + \frac{x^2}{4}} \leq 4 \int_0^\infty \frac{dx}{x^2} < \infty$$

Thus the Dominated Convergence Theorem with g allows the limit to be moved inside the integral:

$$\lim_n \int_0^\infty f_n(x)\ dx = \int_0^\infty \lim_n f_n(x)\ dx = \int_0^\infty \frac{\lim_n \cos\left(x/n\right)}{\lim_n \left(1 + x/n\right)^n} dx = \int_0^\infty \frac{\cos(0)}{e^x} dx = \int_0^\infty e^{-x} dx = 1$$

$\square$

Problem 2. Suppose that $f : [0, \infty) \to \mathbb{R}$ is Lebesgue integrable.

(i) Show that there exists a sequence $x_n \to \infty$ such that $f(x_n) \to 0$.

(ii) Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or counterexample.

(iii) Suppose additionally that f is differentiable and $f'(x) \to 0$ as $x \to \infty$. Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or counterexample.

Proof. (i) Suppose that there was no point x_ε such that $|f(x_\varepsilon)| < \varepsilon$. Then $|f| \geq \varepsilon$ for all $\varepsilon > 0$. This means that

$$\int_0^\infty |f| \, dm \geq \int_0^\infty \varepsilon \, dm = \epsilon m((0, \infty)) = \infty,$$

which is a contradiction, because f is integrable by hypothesis. Therefore for each $\varepsilon > 0$, there exists some x_ε such that $f(x_\varepsilon) < \varepsilon$. Take a sequence such that $|f(x_n)| < 1/n$ for all n. This sequence converges to 0.

(ii) Let $f = \mathbb{1}_\mathbb{Q}$ be the indicator function for the rational numbers. Then f does not converge to 0, but it is integrable:

$$\int_0^\infty f \, dm = m(\mathbb{Q}) = 0$$

(iii) The idea is that if $f \not\to 0$, then there exists some ε such that for any M, there exists $x_0 > M$ such that $f(x_0) > \varepsilon$. Choose M large enough that $|f'(x)| < 2$, then the area from x_0 to the next x-intercept must be greater than ε. But this means that the integral $\int_0^\infty |f| \, dm = \sum_{i=1}^\infty \varepsilon = \infty$.

$\square$

Problem 3. Define $f_n(x) = a\exp(-nax) - b\exp(-nbx)$ where $0 < a < b$.

(i) Show that

$$\sum_{n=1}^{\infty}\int_0^{\infty} f_n(x)\ dx = 0$$

and

$$\int_0^{\infty}\sum_{n=1}^{\infty} f_n(x)\ dx = \log(b/a).$$

(ii) What can you deduce about the value of

$$\int_0^{\infty}\sum_{n=1}^{\infty}|f_n(x)|\,dx?$$

Proof. (i) Notice that

$$\int_0^{\infty} f_n(x)\ dx = \frac{\exp(-nbx)}{n} - \frac{\exp(-nax)}{n}\Big|_0^{\infty} = 0 - \left(\frac{-1}{n} + \frac{1}{n}\right) = 0$$

because $exp(-kx) \to 0$ as $x \to \infty$ for $k > 0$, and $\exp(0) = 1$.

(idea from Daniel Douglas's solution)
By the identity

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$$

we can see that

$$\sum_{n=1}^{\infty} a\exp(-nax) - b\exp(-nbx) = a\sum_{n=1}^{\infty}\exp(-ax)^n - b\sum_{n=1}^{\infty}\exp(-bx)^n = \frac{a}{1-e^{-ax}} - \frac{b}{1-e^{-bx}}$$

so integrating this value gives

$$\int_0^{\infty}\sum_{n=1}^{\infty} f_n(x)dx = \int_0^{\infty}\frac{ae^{-ax}}{1-e^{-ax}} - \frac{be^{-bx}}{1-e^{-bx}}\ dx$$

doing substition with $u = 1 - e^{-ax}$ and $v = 1 - e^{-bx}$ gives

$$\int_0^1 \frac{dv}{v} - \int_0^1 \frac{du}{u} = \log(v) - \log(u)\Big|_0^1 = \log(1-e^{-a}) - \log(1-e^{-b}) = \log\left(\frac{1-e^{-a}}{1-e^{-b}}\right)$$

(ii) Let $g_n = \sum_{k=1}^{n} f_k(x)$. Fubini's theorem says that when g_n is integrable,

$$\lim_{n\to\infty}\int_0^{\infty} g_n dm = \int_0^{\infty}\lim_{n\to\infty} g_n dm$$

Thus g_n is not integrable and by the triangle inequality

$$\int_0^{\infty}\sum_{n=1}^{\infty}|f_n(x)|\,dx \geq \int_0^{\infty}\left|\sum_{n=1}^{\infty} f_n(x)\right| dx = \int_0^{\infty}\left|\lim_{n\to\infty} g_n\right| dm = \infty.$$

$\square$

Problem 4. Assume that f is integrable on $[0,1]$ with respect to the Lebesgue measure m, and let $F(x) = \int_0^x f(t)dt$. Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz, i.e., there exists a constant $C \geq 0$ such that

$$\phi(x_1) - \phi(x_2) \leq C(x_1 - x_2), \quad x_1, x_2 \in \mathbb{R}.$$

Prove that there exists a function g which is integrable on $[0,1]$ such that $\phi(F(x)) = \int_0^x g(t)\, dt$ for $x \in [0,1]$.

Note. This is similar to problem # from the Spring 20## Real Analysis Exam.

Proof. Replace this text with the details of your proof or solution. $\square$