# Algebra Definitions

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# 1 Groups

## 1.1 Notation and definitions

### 1.1.1 Basic definitions

**Definition 1.1.1.1** (Normal subgroup)**.** Let $G$ be a group and $K$ be a subgroup of $G$. If $gkg^{-1} \in K$ for all $k \in K$ and $g \in G$, then $K$ is called a normal subgroup of $G$ and is denoted $K \trianglelefteq G$.

**Definition 1.1.1.2** (Simple group)**.** A group $G$ is called a simple group is a group whose only normal subgroups are $\{e\}$ and $G$.

**Definition 1.1.1.3** (Semidirect product)**.** Let $K \trianglelefteq G$ and $Q \leq G$. A group $G$ is a semidirect product of $K$ by $Q$ (denoted $G = K \ltimes Q$) if there exists $Q_1 \cong Q$ such that $Q_1$ is a complement of $K$ in $G$, that is $K \cap Q_1 = 1$ and $KQ_1 = G$.

### 1.1.2 Galois Theory

**Definition 1.1.2.1** (Normal series)**.** A normal series of a group $G$ is a sequence of subgroups

$$G = G_0 \geq G_1 \geq \ldots \geq G_n = 1$$

in which $G_{i+1} \trianglelefteq G_i$ for all $i$.

**Definition 1.1.2.2** (Factor groups)**.** The factor groups of a normal series are the groups $G_i/G_{i+1}$ for $i = 0, 1, \ldots, n-1$.

**Definition 1.1.2.3** (Length)**.** The length of a a normal series is the number of nontrivial factor groups.

**Definition 1.1.2.4** (Solvable group)**.** A finite group is solvable if it has a normal series whose factor groups are cyclic of prime order.

### 1.1.3 Centralizer/Normalizer

**Definition 1.1.3.1** (Center)**.** The center of a group $G$, denoted by $Z(G)$, is the set of all $a \in G$ that commute with every element of $G$.

**Definition 1.1.3.2** (Centralizer)**.** The centralizer of a subset $S$ of a group $G$ is defined to be

$$C_G(S) = \{g \in G \mid gs = sg \text{ for all } s \in S\}.$$

**Definition 1.1.3.3** (Normalizer)**.** The centralizer of a subset $S$ in the group $G$ is defined to be

$$N_G(S) = \{g \in G \mid gS = Sg\}.$$

**Definition 1.1.3.4** (Commutator)**.** If $a, b \in G$, the commutator of $a$ and $b$, denoted $[a, b]$, is

$$[a, b] = aba^{-1}b^{-1},$$

and the commutator subgroup of $G$, denoted $G'$, is the subgroup of $G$ generated by all of the commutators.

**Definition 1.1.3.5** (Class equation)**.** Partition $G$ into its conjugacy classes, with $x_i$ the representative of the $i$th conjugacy class. The class equation of the finite group $G$ is

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)].$$

### 1.1.4 Group Actions

**Definition 1.1.4.1** (Group action)**.** Let $G$ be a group and $X$ be a set. Then a group action on $X$ is a function $\varphi \colon G \times X \to X$ denoted $\varphi(g, x) = g \cdot x$ and satisfying

(i) Identity: group action by the identity is trivial for all $x \in X$: $1 \cdot x = x$.

(ii) Compatibility: $(gh) \cdot x = g \cdot (h \cdot x)$.

And $X$ is called a $G$-set.

**Definition 1.1.4.2** (Orbit)**.** The orbit of an element $x \in X$ is denoted by

$$G \cdot x = \{g \cdot x \mid g \in G\}$$

**Definition 1.1.4.3** (Stabilizer subgroup)**.** The stabilizer subgroup of $G$ with respect to $x \in X$ is denoted

$$G_x = \{g \in G \mid g \cdot x = x\}$$

**Definition 1.1.4.4** (Transitive)**.** A group action is called transitive is for each $x, y \in X$ there exists some $g \in G$ such that $g \cdot x = y$.

## 1.2 Theorems

**Theorem 1.2.1** (First isomorphism theorem)**.** If $\varphi\colon G \to H$ is a group homomorphism then $\ker(\varphi) \trianglelefteq G$ and $G/\ker(\varphi) \cong \varphi(G)$.

**Theorem 1.2.2** (Second isomorphism theorem)**.** Let $G$ be a group with $S \leq G$ and $N \trianglelefteq G$. Then

1. $SN \leq G$
2. $S \cap N \trianglelefteq S$, and
3. $(SN)/N \cong S/(S \cap N)$.

Strictly speaking, $N$ does not have to be a normal subgroup as long as $S$ is a subgroup of the normalizer of $N$, $S \leq N_G(N)$.

**Theorem 1.2.3** (Third isomorphism theorem)**.** Let $G$ be a group with normal subgroup $N \trianglelefteq G$. Then

1. If $K \leq G$ (resp. $K \trianglelefteq G$) such that $N \subseteq K \subseteq G$, then $K/N \leq G/N$ (resp. $K/N \trianglelefteq G/N$).
2. Every subgroup (resp. normal subgroup) of $G/N$ is of the form $K/N$, for some subgroup (resp. normal subgroup) $K \subset G$ such that $N \subseteq K \subseteq G$.
3. If $K \trianglelefteq G$ such that $N \subseteq K \subseteq G$, then $(G/N)/(K/N) \cong G/K$.

**Theorem 1.2.4** (Simplicity of the $A_n$)**.** $A_n$ is simple for all $n \geq 5$.

**Theorem 1.2.5** (Sylow's theorem)**.**

(i) If $P$ is a Sylow $p$-subgroup of a finite group $G$, then all Sylow $p$-subgroups of $G$ are conjugate to $P$.

(ii) If there are $r$ Sylow $p$-subgroups, then $r$ divides $|G|$ and $r \equiv 1 \bmod p$.

**Theorem 1.2.6** (Fundamental Theorem of Abelian Groups)**.** If $G$ and $H$ are finite abelian groups, then $G \cong H$ if and only if, for all primes $p$, they have the same elementary divisors.