Fall 2012: Complex Analysis Graduate Exam

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Problem 1. Evaluate the integral

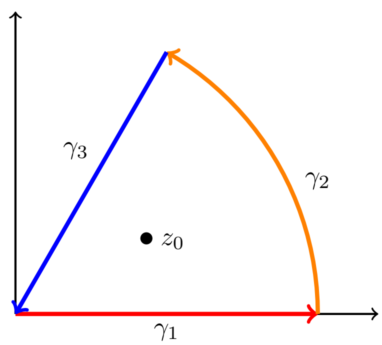
$$\int_0^\infty \frac{dx}{1+x^n}\,dx$$

being careful to justify your methods.

Proof.
First notice that the integrand $f(z) = (1+x^n)^{-1}$ has poles at

$$\begin{aligned}
1 + x^n &= 0 \\
x^n &= e^{\pi i + 2\pi i k} \\
z_k &= e^{(2k+1)\pi i/n} \text{ where } 0 \leq k < n.
\end{aligned}$$

The idea is to draw a contour around the first pole $z_0 = e^{\pi i/n}$ along an n-th root of unity, and then compute the integral via the Residue Theorem. In particular, we will use the contour given by:



$$\gamma_1 = \{t + 0i \mid x \in [0, R]\} \tag{1}$$
$$\gamma_2 = \{Re^{it} \mid t \in [0, 2\pi/n]\} \tag{2}$$
$$\gamma_3 = \{te^{2\pi i/n} \mid t \in [0, R]\} \tag{3}$$

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = 2\pi i \operatorname{Res}_{z_0}(f).$$

In the limit, the integral over γ_2 vanishes.

$$\begin{aligned}
\left|\int_{\gamma_2} f(z)\,dz\right| &= \left|\int_0^{2\pi/n} \frac{dt}{1+(Re^{it})^n} iRe^{it}\right| \\
&\leq \int_0^{2\pi/n} \left|\frac{iRe^{it}}{1+R^n e^{tni}}\right| dt \\
&\leq \int_0^{2\pi/n} \left|\frac{iRe^{it}}{R^n e^{tni}}\right| dt \\
&= \frac{1}{R^{n-1}} \int_0^{2\pi/n} dt \\
&= \frac{2\pi}{nR^{n-1}}
\end{aligned}$$

which vanishes as $R \to \infty$. This means that our equation simplifies in the limit to

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = 2\pi i \operatorname{Res}_{z_0}(f).$$

Also the integral over γ_3 is a multiple of the integral over γ_1,

$$\begin{aligned}\int_R^0 \frac{1}{1+(te^{2\pi i/n})^n} e^{2\pi i/n}\,dt &= -e^{2\pi i/n} \int_0^R \frac{dt}{1+t^n} \\ &= -e^{2\pi i/n} \int_{\gamma_1} f(z)\,dz,\end{aligned}$$

so the equation further simplifies to

$$\int_{\gamma_1} f(z)\,dz - e^{2\pi i/n} \int_{\gamma_1} f(z)\,dz = 2\pi i \operatorname{Res}_{z_0}(f).$$

So by the Residue Theorem, the integral evaluates to

$$\int_{\gamma_1} f(z)\,dz = \frac{2\pi i \operatorname{Res}_{z_0}(f)}{1 - e^{2\pi i/n}},$$

and it is enough to compute the residue:

$$\operatorname{Res}_{z_0}(f) = \lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \frac{1}{\left(\dfrac{1+z^n}{z - z_0}\right)} = \frac{1}{\dfrac{d}{dz}\left[1+z^n\right]_{z=z_0}} = \frac{1}{n z_0^{n-1}}$$

Therefore

$$\begin{aligned}\int_0^\infty \frac{dx}{1+x^n} &= \frac{2\pi i}{n z_0^{n-1}(1 - e^{2\pi i/n})} \\ &= \frac{2\pi i/n}{e^{\pi i(n-1)/n}(1 - e^{2\pi i/n})} \\ &= \frac{2\pi i/n}{\underbrace{e^{\pi i}}_{=-1} e^{-\pi i/n}(1 - e^{2\pi i/n})} \\ &= \frac{2\pi i/n}{-e^{-\pi i/n} + e^{\pi i/n}} \\ &= \frac{\pi}{n} \cdot \left(\frac{e^{\pi i/n} - e^{-\pi i/n}}{2i}\right)^{-1} \\ &= \frac{\pi}{n \sin(\pi/n)}\end{aligned}$$

$\square$

Problem 2. Find the Laurent series expansion for

$$\frac{1}{z(z+1)}$$

valid in $\{1 < |z-1| < 2\}$.

Proof. It is easiest to find the expansion centered at 0, so we will instead substitute $z-1=w$ and find the Laurent series expansion of

$$\hat{f}(w) = \frac{1}{(w+1)(w+2)}$$

valid when $1 < |w| < 2$.
Notice that by partial fraction decomposition,

$$\frac{1}{(w+1)(w+2)} = \frac{A}{w+1} + \frac{B}{w+2}$$

where A and B satisfy the system of equations

$$A + B = 0$$
$$2A + B = 1$$

and so $A = 1$ and $B = -1$.
Now we can add the Laurent series expansion of the two summands to give the Laurent series of $\hat{f}$.

Laurent series of $(w+1)^{-1}$ for $|w| > 1$.
Notice that

$$\frac{1}{w+1} = \frac{1}{w} \cdot \frac{1}{1-(-\frac{1}{w})} = \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{-1}{w}\right)^n = -\sum_{n=1}^{\infty}(-1)^{-n}w^{-n}$$

for $|1/w| < 1$ (or equivalently $|w| > 1$).

Laurent series of $-(w+2)^{-1}$ for $|w| < 2$.
Similarly

$$\frac{1}{w+2} = \frac{1}{2} \cdot \frac{1}{1-(-\frac{w}{2})} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-w}{2}\right)^n = -\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1} w^n$$

for $|w/2| < 1$ (or equivalently $|w| < 2$).

Thus the Laurent series of $\hat{f}$ is

$$\hat{f}(w) = -\sum_{n=1}^{\infty}(-1)^{-n}w^{-n} - \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1} w^n,$$

in $\{1 < |w| < 2\}$, so the Laurent series of f is

$$f(z) = -\sum_{n=1}^{\infty}(-1)^{-n}(z-1)^{-n} - \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1}(z-1)^n$$

in $\{1 < |z-1| < 2\}$. □

Problem 3. Suppose that f is an entire function and that there is a bounded sequence of distinct real numbers $a_1, a_2, a_3, \ldots$ such that $f(a_k)$ is real for each k. Show that $f(x)$ is real for all real x.

Proof. Because the sequence of real numbers is bounded, there must be a (real) accumulation point by the Bolzano-Weierstrass Theorem; call it z_0.

Now, since f is entire, we can do a Taylor Series expansion about z_0. All derivatives are real, because we can always find real z arbitrarily close to z_0 and both the numerator and denominator in the definition of the derivative are real,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

so the derivative itself is real.
This same idea works for higher order derivatives, so the Taylor expansion about z_0 has all real coefficients. Therefore, $f(x)$ is real-valued for all real x. □

Problem 4. Suppose

$$f_n(z) = \sum_{k=0}^{n} \frac{1}{k!z^k}, \;\; z \neq 0$$

and let $\varepsilon > 0$. Show that for large enough n all the zeros of f_n are in the disk $D(0, \varepsilon)$ with center 0 and radius ε

Proof. The plan here is to show that the function $\hat{f}_n(x) = f_n(1/x)$ has no zeros inside the disk of radius $R = 1/\varepsilon$ for any large R and sufficiently large n.

Since $\hat{f}_n(z)$ is just a partial sum of the Taylor series of e^z, which converges □