Fall 2015: Complex Analysis Graduate Exam

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Problem 1. Evaluate the integral

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx,$$

being careful to justify your answer.

Proof. $\square$

Problem 2. Determine the number of roots of $f(z) = z^9 + z^6 + z^5 + 8z^3 + 1$ inside the annulus $1 < |z| < 2$.

Proof. By Rouché's Theorem, if g is analytic and $|f - g| < |f|$ along a simple curve, then both f and g have the same number of zeros inside this region.

Case 1: $|z| = 2$ First, consider the curve $|z| = 2$, and the function $g = z^9$. Now along this curve, by the triangle inequality

$$\begin{aligned} |f(z) - g(z)| &= |z^6 + z^5 + 8z^3 + 1| \\ &\leq |z^6| + |z^5| + |8z^3| + |1| \\ &= 2^6 + 2^5 + 8(2^3) + 1 \\ &= 161. \end{aligned}$$

Similarly by the triangle inequality

$$\begin{aligned} |f(z)| &= |z^9 + z^6 + z^5 + 8z^3 + 1| \\ &\geq |z^9| - |z^6| - |z^5| - |8z^3| - |1| \\ &= 2^9 - 161 \\ &= 351. \end{aligned}$$

Thus $|f - g| \leq 161 < 351 \leq |f|$, and f has the same number of zeros as g when $|z| < 2$. Clearly g has all nine zeros inside this region at $z = 0$, so f has nine zeros in $|z| < 2$

Case 2: $|z| = 1$ Now, consider the curve $|z| = 1$, and the function $g = 8z^3$. Along this curve, by the triangle inequality

$$\begin{aligned} |f(z) - g(z)| &= |z^9 + z^6 + z^5 + 1| \\ &\leq |z^9| + |z^6| + |z^5| + |1| \\ &= 4. \end{aligned}$$

Similarly by the triangle inequality

$$\begin{aligned} |f(z)| &= |z^9 + z^6 + z^5 + 8z^3 + 1| \\ &> |8z^3| - |z^9| - |z^6| - |z^5| - |1| \\ &= 4. \end{aligned}$$

(* I'm not sure of a nice way to show that this inequality is strict.)
Thus $|f - g| \leq 4 < |f|$, and f has the same number of zeros as g when $|z| < 1$. Clearly g has three zeros inside this region at $z = 0$, so f has three zeros in $|z| < 1$, and by the inclusion-exclusion principle, f has $9 - 3 = 6$ zeros inside the annulus $1 < |z| < 2$. ☐

Problem 3. Suppose that f is holomorphic on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and suppose that for $z \in \mathbb{D}$, one has $\Re(f(z)) > 0$ and $f(0) = 1$. Prove that

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}$$

for all $z \in \mathbb{D}$.

Proof. $\square$

Problem 4. For $a_n = 1 - \frac{1}{n^2}$, let

$$f(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - a_n z}.$$

(a) Show that f defines a holomorphic function on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

(b) Prove that f does not have an analytic continuation to any larger disk $\{z \in \mathbb{C} : |z| < r\}$ for some $r > 1$.

Proof.

(a)

(b) Suppose that f did have an analytic continuation on $B_{1+\varepsilon}(0)$. Then by continuity,

$$f(1) = \lim_{z \to 1} f(z) = \lim_{n \to \infty} f(1 - 1/n^2) = 0,$$

because f has zeros at a_n. However, f has poles at $n^2/(n^2-1)$, so taking the limit from the right along the real axis cannot equal zero because there are arbitrarily large values to the right of 1. This is a contradition, so f cannot have an analytic continuation on $B_{1+\varepsilon}(0)$.

$\square$