Fall 2012: Algebra Graduate Exam

Peter Kagey

May 31, 2019

Problem 1. Use Sylow's theorems directly to find, up to isomorphism, all possible structures of groups of order $5 \cdot 7 \cdot 23$.

Proof. Sylow's theorems tell us that any group G must have

$$\begin{aligned} r_5 &\text{ Sylow 5-subgroups,} \\ r_7 &\text{ Sylow 7-subgroups, and} \\ r_{23} &\text{ Sylow 23-subgroups} \end{aligned}$$

where r_5, r_7, and r_{23} divide $5 \cdot 7 \cdot 23$, and $r_p \equiv 1 \bmod p$.

$$r_p = 1, 5, 7, 5 \cdot 7, 23, 5 \cdot 23, 7 \cdot 23, \text{ or } 5 \cdot 7 \cdot 23$$

considering the restriction on modulus, $r_5 \in \{1, 7 \cdot 23\}$, $r_7 = 1$, and $r_{23} = 1$. $\square$

Let P and Q be the unique Sylow 23-subgroup and Sylow 7-subgroup respectively. Since $P \cap Q = 1$, $PQ \cong P \times Q$. Let R be a Sylow 5-subgroup.
Since $R \trianglelefteq G$ (why?), and R has a complement $P \times Q$, G is a semidirect product of R by $P \times Q$, that is $G = R \ltimes (P \times Q)$.
By Rotman Lemma 7.21, there is a homomorphism

$$\theta \colon \underbrace{R \to \mathrm{Aut}(P \times Q)}_{\mathbb{Z}_5 \to \mathbb{Z}_{22} \times \mathbb{Z}_6}.$$

But since $\gcd(5, 22) = \gcd(5, 6) = 1$, the only homomorphism is trivial. Therefore there is only one group of order $5 \cdot 7 \cdot 23$, the abelian group

$$G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{23}.$$

Problem 2. Let A, B, and C be finitely generated $F[x] = R$ modules for F a field with C torsion free. Show that $A \otimes_R C \cong B \otimes_R C$ implies that $A \cong B$. Show by example that this conclusion can fail when C is not torsion free.

Proof. $\square$