Fall 2012: Algebra Graduate Exam

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Problem 1. Use Sylow's theorems directly to find, up to isomorphism, all possible structures of groups of order $5 \cdot 7 \cdot 23$.

Proof. Sylow's theorems tell us that any group G must have

$$\begin{aligned} r_5 &\text{ Sylow 5-subgroups,} \\ r_7 &\text{ Sylow 7-subgroups, and} \\ r_{23} &\text{ Sylow 23-subgroups} \end{aligned}$$

where r_5, r_7, and r_{23} divide $5 \cdot 7 \cdot 23$, and $r_p \equiv 1 \bmod p$.

$$r_p = 1, 5, 7, 5 \cdot 7, 23, 5 \cdot 23, 7 \cdot 23, \text{ or } 5 \cdot 7 \cdot 23$$

considering the restriction on modulus, $r_5 \in \{1, 7 \cdot 23\}$, $r_7 = 1$, and $r_{23} = 1$. $\square$

Let P and Q be the unique Sylow 23-subgroup and Sylow 7-subgroup respectively. Since $P \cap Q = 1$, $PQ \cong P \times Q$. Let R be a Sylow 5-subgroup.
Since $R \trianglelefteq G$ (why?), and R has a complement $P \times Q$, G is a semidirect product of R by $P \times Q$, that is $G = R \ltimes (P \times Q)$.
By Rotman Lemma 7.21, there is a homomorphism

$$\theta\colon \underbrace{R \to \operatorname{Aut}(P \times Q)}_{\mathbb{Z}_5 \to \mathbb{Z}_{22} \times \mathbb{Z}_6}.$$

But since $\gcd(5, 22) = \gcd(5, 6) = 1$, the only homomorphism is trivial. Therefore there is only one group of order $5 \cdot 7 \cdot 23$, the abelian group

$$G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{23}.$$

Problem 2. Let A, B, and C be finitely generated $F[x] = R$ modules for F a field with C torsion free. Show that $A \otimes_R C \cong B \otimes_R C$ implies that $A \cong B$. Show by example that this conclusion can fail when C is not torsion free.

Proof. $\square$

Problem 3. Let F be a finite field and G a finite group with $\gcd\{\operatorname{char} F, |G|\} = 1$. The group algebra $F[G]$ is an algebra over F with G as an F-basis, elements $\alpha = \sum_G a_g g$ for $g \in F$, and multiplication that extends $ag \cdot bh = ab \cdot gh$. Show that any $x \in F[G]$ that is not a zero left divisor must be invertible in $F[G]$.

Note: Since x is not a zero left divisor, if $xy = 0$ for $y \in F[G]$ then $y = 0$.

Proof. $\square$

Problem 4. If $p(x) = x^8 + 2x^6 + 3x^4 + 2x^2 + 1 \in \mathbb{Q}[x]$ and if $\mathbb{Q} \subseteq M \subseteq \mathbb{C}$ is a splitting field for $p(x)$ over $\mathbb{Q}$, argue that $\mathrm{Gal}(M/\mathbb{Q})$ is solvable.

Proof. $\square$

Problem 5. Let R be a commutative ring with 1 and let $x_1, \ldots, x_n \in R$ so that $x_1y_1 + \ldots + x_ny_n = 1$ for some $y_j \in R$. Let $A = \{(r_1, r_2, \ldots, r_n) \in R^n \mid x_1r_1 + \ldots + x_nr_n = 0\}$. Show that

(i) $R^n \cong_R A \oplus R$,

(ii) A has n generators, and

(iii) when $R = F[x]$ for F a field, then A_R is free of rank $n - 1$.

Proof. □

Problem 6. For p a prime, let F_p be the field of p elements and K and extension field of F_p of dimension 72.

(i) Describe the possible structures of $\mathrm{Gal}(K/F_p)$.

(ii) If $g(x) \in F_p[x]$ is irreducible of degree 72, argue that K is a splitting field of $g(x)$ over F_p

(iii) Which integers $d > 0$ have irreducibles in $F_p[x]$ of degree d that split in K?

Proof. $\square$