

Fall 2013: Complex Analysis Graduate Exam

Peter Kagey

July 20, 2018

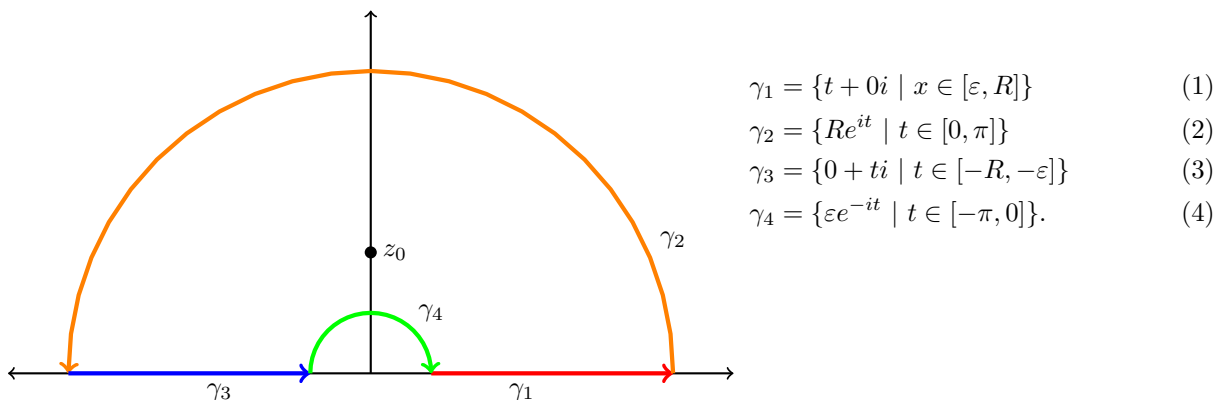
Problem 1. Compute

$$\int_0^\infty \frac{\log^2 x}{1+x^2} dx.$$

Proof. For ease of notation, name the integrand f ; that is,

$$f(z) = \frac{\log^2 z}{1+z^2}.$$

We will compute the integral by using the Residue Theorem together with (the limit of) a contour carefully designed to avoid the singularity at the origin, and including one of the simple poles of f :



For small ϵ and large R , this contour encloses a single simple pole of f , namely $z_0 = i$.

$$\int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz + \int_{\gamma_3} f(z) dz + \int_{\gamma_4} f(z) dz = 2\pi i \operatorname{Res}_i(f).$$

□

In the limit, the integrals over each arcs (γ_2 and γ_4) vanishes.

$$\begin{aligned} \left| \int_{\gamma_2} f(z) dz \right| &= \left| \int_0^\pi \frac{\log^2(Re^{it})}{1+R^2e^{2it}} iRe^{it} dt \right| \\ &\leq \int_0^\pi \left| \frac{\log^2(Re^{it})}{1+R^2e^{2it}} iRe^{it} \right| dt \\ &\leq \int_0^\pi \left| \frac{\log^2(Re^{it})}{R} \right| dt \\ &\leq \int_0^\pi \left| \frac{\log^2(R) + 2it \log(R) - t}{R} \right| dt \end{aligned}$$

which vanishes by the ML inequality as $R \rightarrow \infty$. Similarly,

$$\begin{aligned}
\left| \int_{\gamma_2} f(z) dz \right| &= \left| \int_0^\pi \frac{\log^2(\varepsilon e^{it})}{1 + \varepsilon^2 e^{2it}} i \varepsilon e^{it} dt \right| \\
&\leq \int_0^\pi \left| \frac{\log^2(\varepsilon e^{it})}{1 + \varepsilon^2 e^{2it}} i \varepsilon e^{it} \right| dt \\
&\leq \int_0^\pi \left| \frac{\log^2(\varepsilon e^{it})}{1} i \varepsilon e^{it} \right| dt \\
&\leq \int_0^\pi |\varepsilon \log^2(\varepsilon e^{it})| dt \\
&\leq \int_0^\pi |\varepsilon (\log^2(\varepsilon) + 2it \log(\varepsilon) + t)| dt,
\end{aligned}$$

which also vanishes as $\varepsilon \rightarrow 0$ by the ML inequality, as can be seen by two applications of L'Hôpital's rule:

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \varepsilon \log^2(\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \frac{\log^2(\varepsilon)}{\varepsilon^{-1}} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{2 \log(\varepsilon) \varepsilon^{-1}}{-\varepsilon^{-2}} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{2 \log(\varepsilon)}{-\varepsilon^{-1}} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{2\varepsilon^{-1}}{\varepsilon^{-2}} \\
&= \lim_{\varepsilon \rightarrow 0} 2\varepsilon \\
&= 0.
\end{aligned}$$

This means that our equation simplifies in the limit to

$$\int_{\gamma_1} f(z) dz + \int_{\gamma_3} f(z) dz = 2\pi i \operatorname{Res}_i(f).$$

And the left-hand side further simplifies to

$$\begin{aligned}
\int_\varepsilon^R \frac{\log^2 z}{1 + z^2} dz + (-1) \int_R^\varepsilon \frac{\log^2(-z)}{1 + (-z)^2} dz &= \int_\varepsilon^R \frac{\log^2 z + \log^2(-z)}{1 + z^2} dz \\
&= \int_\varepsilon^R \frac{\log^2 z + (\log(z) + \log(-1))^2}{1 + z^2} dz \\
&= 2 \int_{\gamma_1} f(z) dz + \int_\varepsilon^R \frac{2\pi i \log(z)}{1 + z^2} dz + \int_\varepsilon^R \frac{-\pi^2}{1 + z^2} dz
\end{aligned}$$

So by the Residue Theorem, the integral evaluates to

$$\int_0^\infty \frac{\log^2 z}{1 + z^2} dz = \pi i \operatorname{Res}_i(f) - \underbrace{\pi i \int_0^\infty \frac{\log(z)}{1 + z^2} dz}_{\text{purely imaginary}} - \frac{1}{2} \int_0^\infty \frac{-\pi^2}{1 + z^2} dz,$$

and by only considering the real part, it is enough to compute the residue and the last integral:

$$\operatorname{Res}_i(f) = \frac{\log^2(i)}{2i} = \frac{(\pi i/2)^2}{2i} = \frac{i\pi^2}{8},$$

and

$$-\pi^2 \int_\varepsilon^R \frac{1}{1 + z^2} dz = -\frac{\pi^3}{2}$$

Therefore

$$\begin{aligned}\int_0^\infty \frac{\log^2 z}{1+z^2} dz &= \pi i \left(\frac{i\pi^2}{8} \right) - \frac{1}{2} \left(-\frac{\pi^3}{2} \right) \\ &= -\frac{\pi^3}{8} + \frac{\pi^3}{4} \\ &= \frac{\pi^3}{8}.\end{aligned}$$

Problem 2. Find the number of *distinct* zeros of $f(z) = z^6 + (10 - i)z^4 + 1$ inside $(-1, 1) \times (-1, 1)$.

Proof. First, we will use Rouché's Theorem to establish a bound on the number of roots (with multiplicity) inside of the region $D = (-1, 1) \times (-1, 1)$.

For a lower bound, we will count the number of roots inside $|z| = 1$ and for an upper bound, we will count the number of roots inside $|z| = \sqrt{2}$. In both cases we will compare against the function $g(z) = (10 - i)z^4 + 1$.

(**Case 1:** $|z| = 1$) Notice that when $|z| = 1$,

$$\begin{aligned} |f - g| &= |z^6| = 1 \\ &< |(10 - i)z^4| - |z^6| - 1 = |10 - i| - 2 \\ &< |f|, \end{aligned}$$

by the triangle inequality. So f and g have the same number of roots inside the unit disk, and g has all four roots inside the unit disk:

$$\begin{aligned} g(z) &= (10 - i)z^4 + 1 = 0 \\ |z| &= \left| \frac{-1}{10 - i} \right|^{1/4} < 1. \end{aligned}$$

Thus f has at least four roots in D .

(**Case 2:** $|z| = \sqrt{2}$) When $|z| = \sqrt{2}$,

$$\begin{aligned} |f - g| &= |z^6| = 8 \\ &< |(10 - i)z^4| - |z^6| - 1 = 4|10 - i| - 8 - 1 \\ &< |f|, \end{aligned}$$

by the triangle inequality. And since g has all four roots inside the unit disk, it certainly has all roots inside the disk of radius $\sqrt{2}$.

Now that we have established that f has four roots inside D , it remains to check multiplicity, which can be done by comparing the roots of f and f' inside of D .

Notice that $f'(z) = 6z^5 + 4(10 - i)z^3$ factors as $f'(z) = z^3(6z^2 + 40 - 4i)$. Clearly f does not have any roots at $z = 0$, so it is enough to check the roots of $6z^2 + 40 - 4i$.

$$\begin{aligned} z^2 &= \frac{40 - 4i}{6} \\ |z| &= \left| \frac{40 - 4i}{6} \right|^{1/2} > \sqrt{6}. \end{aligned}$$

Therefore $f'(z)$ does not share any roots with $f(z)$ inside D , and so all roots inside D are distinct. Thus f has exactly four distinct roots inside D . \square

Problem 3. Suppose that f is holomorphic in a neighborhood U of $a \in \mathbb{C}$. Consider the following two statements.

- (i) There exist two sequences $\{z_k\}_{k=1}^{\infty}$ and $\{w_k\}_{k=1}^{\infty}$ in $U \setminus \{a\}$ converging to a such that $z_k \neq w_k$ and $f(z_k) = f(w_k)$ for all $k \in \mathbb{N}$.
- (ii) $f'(a) = 0$.

Proof.

Case 1: (i) $\not\Rightarrow$ (ii).

Let $a = 0$ and $f(z) = 1/z^2$. Clearly f is holomorphic in $\mathbb{C} \setminus \{0\}$.

Next, let $z_k = 1/k$ and $w_k = -1/k$, so that $f(z_k) = f(w_k) = k^2$ for all k . Lastly, see that $f'(z) = -2z^{-3}$ is not defined at 0, so (ii) does not follow.

Case 2: (ii) \Rightarrow (i).

□

Problem 4. Let f be analytic in an open set $U \subseteq \mathbb{C}$, and let $K \subseteq U$ be compact. Shw that there exists a constant C depending on U and K such that

$$|f(z)| \leq C \left(\int_U |f|^2 \right)^{1/2}$$

for all $z \in K$.

Proof.

□