Spring 2012: Geometry/Topology Graduate Exam

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Problem 1. *(Topology)*

Proof. □

Problem 2. *(Topology)*

Proof. $\square$

Problem 3. *(Topology)*

Proof. $\square$

Problem 4. Does there exist a smooth embedding of the projective plane $\mathbb{R}P^2$ into $\mathbb{R}^2$? Justify your answer.

Proof. $\square$

Problem 5. Let M be a manifold and let $C^\infty(M)$ be the algebra of C^∞ functions $M \to \mathbb{R}$. Explain the relationship between vector fields on M and $C^\infty(M)$. If we consider vector fields as maps $C^\infty(M) \to C^\infty(M)$ is the composition map XY also a vector field? What about $[X, Y] = XY - YX$?

Proof. $\square$

Problem 6. Let S be the unit sphere defined by $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbb{R}^4$. Compute $\int_S \omega$ where

$$\omega = (w + w^2)\, dx \wedge dy \wedge dz$$

Proof. By Stokes' Theorem, knowing that $\partial B^4 = S^3$, we can rewrite the integral as

$$\int_S \omega = \int_{\partial B^4} \omega = \int_{B^4} d\omega.$$

Then

$$\begin{aligned} d\omega &= d((w + w^2)\, dx \wedge dy \wedge dz) \\ &= (1 + 2w)\, dw \wedge dx \wedge dy \wedge dz \\ &= -(1 + 2w)\, dx \wedge dy \wedge dz \wedge dw, \end{aligned}$$

where $dx \wedge dy \wedge dz \wedge dw$ is the usual volume form.

By linearity,

$$\begin{aligned} \int_{B^4} d\omega &= -\underbrace{\int_{B^4} dx \wedge dy \wedge dz \wedge dw}_{\text{volume}(B^4)} - \underbrace{\int_{B^4} 2w\, dx \wedge dy \wedge dz \wedge dw}_{=0 \text{ by symmetry}} \\ &= -\operatorname{volume}(B^4). \end{aligned}$$

$\square$

Problem 7. Does the equation $x^2 = y^3$ define a smooth submanifold in $\mathbb{R}^3$?

Proof. Consider the map $f\colon \mathbb{R}^3 \to \mathbb{R}$ which sends $(x, y, z) \overset{f}{\mapsto} x^2 - y^3$. The function f is not submersive at $f^{-1}(0)$ because

$$df_{(x,y,z)} = \begin{bmatrix} 2x & -3y^2 & 0 \end{bmatrix}$$

has rank 0 when $x = y = 0$. Thus $0 \in \mathbb{R}$ is a critical value, and $f^{-1}(0)$ cannot be given the structure of a smooth manifold. $\square$