Spring 2012: Algebra Graduate Exam

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Problem 1. Let I be an ideal of $R = \mathbb{C}[x_1, \dots, x_n]$. Show that $\dim_{\mathbb{C}}(R/I)$ is finite if and only if I is contained in only finitely many maximal ideals of R.

Proof. $\square$

Problem 2. If G is a group with $|G| = 7^2 \cdot 11^2 \cdot 19$, show that G must be abelian and describe the possible structures of G.

Proof. We'll start by using Sylow's theorems. Firstly, let r_p denote the number of Sylow p-subgroups. Since p divides $|G|$,

$$\begin{aligned} r_{19} &\in \{1, 7, 7^2, 11, 11 \cdot 7, 11 \cdot 7^2, 11^2, 11^2 \cdot 7, 11^2 \cdot 7^2\}, \\ r_{11} &\in \{1, 7, 7^2, 19, 19 \cdot 7, 19 \cdot 7^2\}, \\ r_7 &\in \{1, 11, 11^2, 19, 19 \cdot 11, 19 \cdot 11^2\}. \end{aligned}$$

Since $r_p \cong 1 \bmod p$, we can further refine this to

$$\begin{aligned} r_{19} &= 1, \\ r_{11} &\in \{1, 19 \cdot 7\}, \\ r_7 &= 1. \end{aligned}$$

This means that we have unique subgroups H_{19} and H_7 of orders 19 and 7 respectively. Since H_7 and H_{19} are unique and thus normal, the product of H_7 and H_{19} forms a normal subgroup, call it N. Since $H_7 \cap H_{19} = \{e\}$, $H_7 H_{19} \cong H_7 \times H_{19}$, where H_{19} is abelian because it is cyclic, and H_7 is abelian because all groups of order p^2 are abelian. Thus $N \cong \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{19}$ or $N \cong \mathbb{Z}_{49} \times \mathbb{Z}_{19}$.

Since N and H_{11} are complementary, that is $N \cap H_{11} = \{e\}$ and $|N||H_{11}| = |G|$, G can be realized as the semidirect product of N and H_{11}

$$G = N \rtimes H_{11}.$$

Thus it is enough to consider the possible structures of the semidirect product.

Case 1. Assume $N \cong \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{19}$. Consider homomorphisms $\varphi \colon H_{11} \to \operatorname{Aut}(N)$, noting that

$$\operatorname{Aut}(N) \cong \operatorname{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{19}) \cong \operatorname{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_7) \times \operatorname{Aut}(\mathbb{Z}_{19}) \cong \underbrace{\operatorname{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_7)}_{\text{order } 48 \cdot 42} \times \mathbb{Z}_{18}.$$

Since $\gcd(11, 48 \cdot 42 \cdot 18) = 1$, the only homomorphism is trivial. So the semidirect product is direct

$$G \cong \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{19} \times H_{11}$$

Case 2. Assume $N \cong \mathbb{Z}_{49} \times \mathbb{Z}_{19}$. Consider homomorphisms $\varphi \colon H_{11} \to \operatorname{Aut}(N)$, noting that

$$\operatorname{Aut}(N) \cong \operatorname{Aut}(\mathbb{Z}_{49} \times \mathbb{Z}_{19}) \cong \operatorname{Aut}(\mathbb{Z}_{49}) \times \operatorname{Aut}(\mathbb{Z}_{19}) \cong \underbrace{\operatorname{Aut}(\mathbb{Z}_7 \times \mathbb{Z}_7)}_{\text{order } 7 \cdot 6} \times \mathbb{Z}_{18}.$$

Since $\gcd(11, 7 \cdot 6 \cdot 18) = 1$, the only homomorphism is trivial. So the semidirect product is direct

$$G \cong \mathbb{Z}_{49} \times \mathbb{Z}_{19} \times H_{11}$$

Since $|H_{11}| = 11^2$, it is abelian, so by the fundamental theorem of abelian groups, G is isomorphic to

$$\begin{aligned} &\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{11} \times \mathbb{Z}_{11} \times \mathbb{Z}_{19}, \\ &\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{121} \times \mathbb{Z}_{19}, \\ &\mathbb{Z}_{49} \times \mathbb{Z}_{11} \times \mathbb{Z}_{11} \times \mathbb{Z}_{19}, \qquad \text{or} \\ &\mathbb{Z}_{49} \times \mathbb{Z}_{121} \times \mathbb{Z}_{19}. \end{aligned}$$

□

Problem 3. Let F be a finite field and G a finite group with $\gcd\{\operatorname{char} F, |G|\} = 1$. The group algebra $F[G]$ is an algebra over F with G as an F-basis, elements $\alpha = \sum_G a_g g$ for $g \in F$, and multiplication that extends $ag \cdot bh = ab \cdot gh$. Show that any $x \in F[G]$ that is not a zero left divisor must be invertible in $F[G]$.

Note: Since x is not a zero left divisor, if $xy = 0$ for $y \in F[G]$ then $y = 0$.

Proof. $\square$

Problem 4. If $p(x) = x^8 + 2x^6 + 3x^4 + 2x^2 + 1 \in \mathbb{Q}[x]$ and if $\mathbb{Q} \subseteq M \subseteq \mathbb{C}$ is a splitting field for $p(x)$ over $\mathbb{Q}$, argue that $\mathrm{Gal}(M/\mathbb{Q})$ is solvable.

Proof. $\square$

Problem 5. Let R be a commutative ring with 1 and let $x_1, \ldots, x_n \in R$ so that $x_1 y_1 + \ldots + x_n y_n = 1$ for some $y_j \in R$. Let $A = \{(r_1, r_2, \ldots, r_n) \in R^n \mid x_1 r_1 + \ldots + x_n r_n = 0\}$. Show that

(i) $R^n \cong_R A \oplus R$,

(ii) A has n generators, and

(iii) when $R = F[x]$ for F a field, then A_R is free of rank $n-1$.

Proof. $\square$

Problem 6. For p a prime, let F_p be the field of p elements and K and extension field of F_p of dimension 72.

(i) Describe the possible structures of $\mathrm{Gal}(K/F_p)$.

(ii) If $g(x) \in F_p[x]$ is irreducible of degree 72, argue that K is a splitting field of $g(x)$ over F_p

(iii) Which integers $d > 0$ have irreducibles in $F_p[x]$ of degree d that split in K?

Proof. □