

Fall 2012: Algebra Graduate Exam

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Problem 1. Use Sylow's theorems directly to find, up to isomorphism, all possible structures of groups of order $5 \cdot 7 \cdot 23$.

Proof. Sylow's theorems tell us that any group G must have

r_5 Sylow 5-subgroups,
 r_7 Sylow 7-subgroups, and
 r_{23} Sylow 23-subgroups

where r_5, r_7 , and r_{23} divide $5 \cdot 7 \cdot 23$, and $r_p \equiv 1 \pmod{p}$.

$$r_p = 1, 5, 7, 5 \cdot 7, 23, 5 \cdot 23, 7 \cdot 23, \text{ or } 5 \cdot 7 \cdot 23$$

considering the restriction on modulus, $r_5 \in \{1, 7 \cdot 23\}$, $r_7 = 1$, and $r_{23} = 1$. Let P and Q be the unique Sylow 23-subgroup and Sylow 7-subgroup respectively. Since $P \cap Q = 1$, $PQ \cong P \times Q$. Let R be a Sylow 5-subgroup.

Since $R \trianglelefteq G$ (why?), and R has a complement $P \times Q$, G is a semidirect product of R by $P \times Q$, that is $G = R \rtimes (P \times Q)$.

By Rotman Lemma 7.21, there is a homomorphism

$$\theta: R \rightarrow \underbrace{\text{Aut}(P \times Q)}_{\mathbb{Z}_5 \rightarrow \mathbb{Z}_{22} \times \mathbb{Z}_6}.$$

But since $\gcd(5, 22) = \gcd(5, 6) = 1$, the only homomorphism is trivial. Therefore there is only one group of order $5 \cdot 7 \cdot 23$, the abelian group

$$G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{23}.$$

□

Problem 2. Let A , B , and C be finitely generated $F[x] = R$ modules for F a field with C torsion free. Show that $A \otimes_R C \cong B \otimes_R C$ implies that $A \cong B$. Show by example that this conclusion can fail when C is not torsion free.

Proof.

□

Problem 3. Working in the polynomial ring $\mathbb{C}[x, y]$, show that some power of $(x + y)(x^2 + y^4 - 2)$ is in $(x^3 + y^2, y^3 + xy)$.

Proof.

□

Problem 4. For integers $n, m > 1$, let $A \subseteq M_n(\mathbb{Z}_m)$ be a subring with the property that if $x \in A$ with $x^2 = 0$ then $x = 0$. Show that A is commutative. Is the converse true?

Proof.

□

Problem 5. Let F be the splitting field of $f(x) = x^6 - 2$ over \mathbb{Q} . Show that $\text{Gal}(F/\mathbb{Q})$ is isomorphic to the dihedral group of order 12.

Proof.

□

Problem 6. Given that all groups of order 12 are solvable, show that any group of order $2^2 \cdot 3 \cdot 7^2$ is solvable.

Proof.

□