Signals & Systems

EEE-223

Lab # 10

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LAB # 10

Trigonometric (Real) Fourier Series Representation and its Properties

Lab 10- Trigonometric (Real) Fourier Series Representation and its Properties

Pre-Lab Tasks

10.1 Trigonometric Fourier Series:

A second form of Fourier series is introduced in this section. Suppose that a signal $x(t)$ is defined in the time interval $[t_0, t_0 + T]$. Then $x(t)$, by using the trigonometric Fourier series, can be expressed in time interval $[t_0, t_0 + T]$ as a sum of sinusoidal signals, namely, sines and cosines, where each signal has frequency $k\Omega_0$ rad/s.

The mathematical expression (equation 10.1) is

$$x(t) = a_0 + \sum_{k=1}^{\infty} b_k \cos(k\Omega_0 t) + \sum_{k=1}^{\infty} c_k \sin(k\Omega_0 t)$$

The coefficients a_0, b_1, b_2, . . ., c_1, c_2, . . . of the trigonometric Fourier series are computed by

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt \qquad 10.2$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\Omega_0 t)\,dt, \quad n = 1, 2, 3, \ldots \qquad 10.3$$

$$c_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(n\Omega_0 t)\,dt, \quad n = 1, 2, 3, \ldots \qquad 10.4$$

Example:

The signal that will be expanded is the same signal used at the previous example. Thus, the problem is to expand in trigonometric Fourier series the signal $x(t) = e^{-t}, 0 \le t \le 3$.

First the trigonometric Fourier coefficients b_n, c_n and the dc component a_0 are computed according to equations 10.3, 10.4 and 10.2, respectively, for $n = 1, 2, ..., 200$. Next, $x(t)$ is approximated according to the relationship (equation 10.5)

$$x(t) = a_0 + \sum_{k=1}^{N} b_k \cos(k\Omega_0 t) + \sum_{k=1}^{N} c_k \sin(k\Omega_0 t)$$

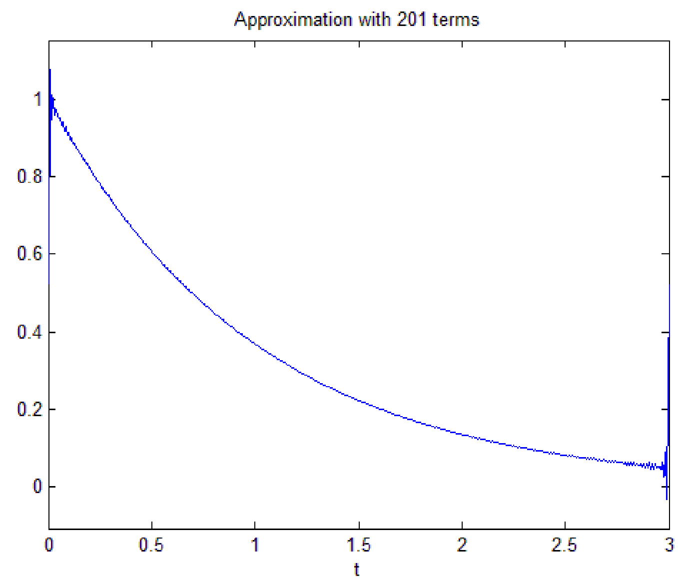
Commands	Results/Comments

`T=3;`
`t0=0;`
`w=2*pi/T;`
`syms t`
`x=exp(-t);`
	Definition of $t_0 = 0, T = 3, \Omega_0 = 2\pi/T$ and of the signal $x(t)$

`a0=(1/T)*int(x,t,t0,t0+T);`
`for n=1:200`
`b(n)=(2/T)*int(x*cos(n*w*t),t,t0,t0+T);`
`end`
`for n=1:200`
`c(n)=(2/T)*int(x*sin(n*w*t),t,t0,t0+T);`
`end`
	Computation of trigonometric coefficients according to equations 10.2, 10.3 and 10.4.

`k=1:200;`
`xx=a0+sum(b.*cos(k*w*t))+sum(c.*sin(k*w*t)`
`)`
	The signal $x(t)$ is approximated by equation 10.1 or more precisely from equation 10.5 with N=200.

`ezplot(xx, [t0 t0+T]);` `title('Approximation with 201 terms')`	Graph of the approximate signal $xx(t)$ that was computed by the terms of the trigonometric Fourier series.



The approximation with 201 terms of the original signal $x(t)$ is very good.

In order to understand the importance of the number terms used for the approximation of the original signal $x(t)$, the approximate signal $xx(t)$ is constructed for different values of n.

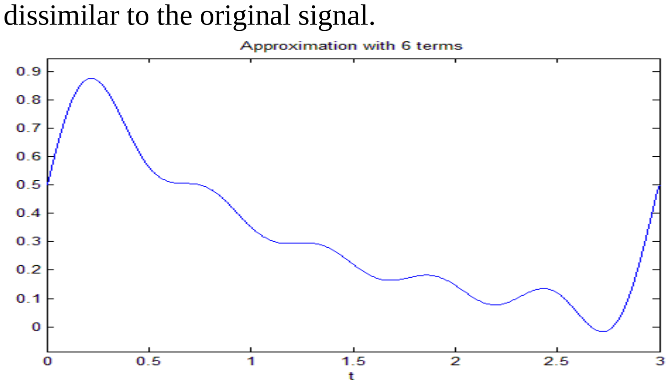
Approximation with five terms ($n = 1, ..., 5$)

Commands

```
clear b c
for n=1:5
b(n)=(2/T)*int(x*cos(n*w*t),t,t0,t0+T);
c(n)=(2/T)*int(x*sin(n*w*t),t,t0,t0+T);
end
k=1:5;
xx=a0+sum(b.*cos(k*w*t))+sum(c.*sin(k*w*t)
)
ezplot(xx, [t0 t0+T]);
title('Approximation with 6 terms')
```

Results/Comments

When the signal is approximated with 5 terms, it is pretty dissimilar to the original signal.



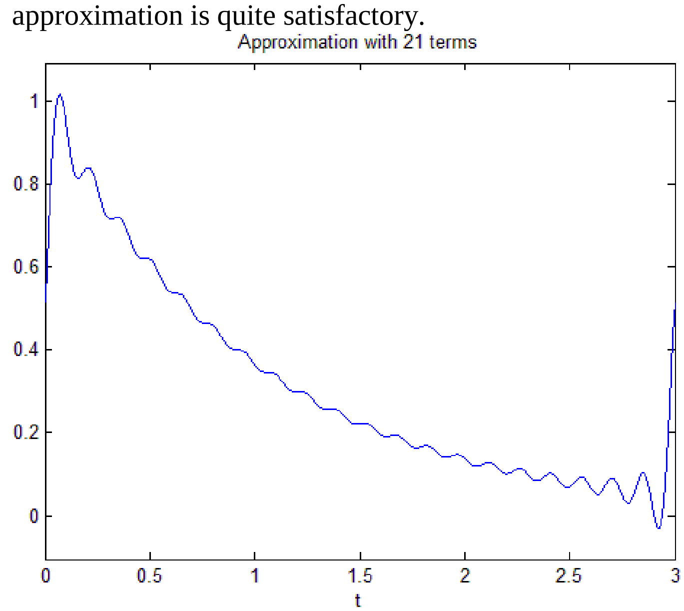
Approximation with 20 terms ($n = 1, ..., 20$)

Commands

```
for n=1:20
b(n)=(2/T)*int(x*cos(n*w*t),t,t0,t0+T);
c(n)=(2/T)*int(x*sin(n*w*t),t,t0,t0+T);
end
k=1:20;
xx=a0+sum(b.*cos(k*w*t))
+sum(c.*sin(k*w*t));
ezplot(xx, [t0 t0+T]);
title('Approximation with 21 terms')
```

Results/Comments

The approximated signal $xx(t)$ is now approximated by 20 terms and it is quite similar to the original signal. The approximation is quite satisfactory.



10.2 Properties of Fourier Series:

10.2.1 Linearity

Suppose that the complex exponential Fourier series coefficients of the periodic signals $x(t)$ and $y(t)$ are denoted by a_k and b_k, respectively, or In other words $x(t) \rightarrow a_k$ and $y(t) \rightarrow b_k$. Moreover, let z_1, z_2 denote two complex numbers. Then

$$z_1 x(t) + z_2 y(t) \leftrightarrow z_1 a_k + z_2 b_k$$

To verify the linearity property, we consider the periodic signals $x(t) = \cos(t)$, $y(t) = \sin(2t)$ and the scalars $z_1 = 3 + 2i$ and $z_2 = 2$.

Commands

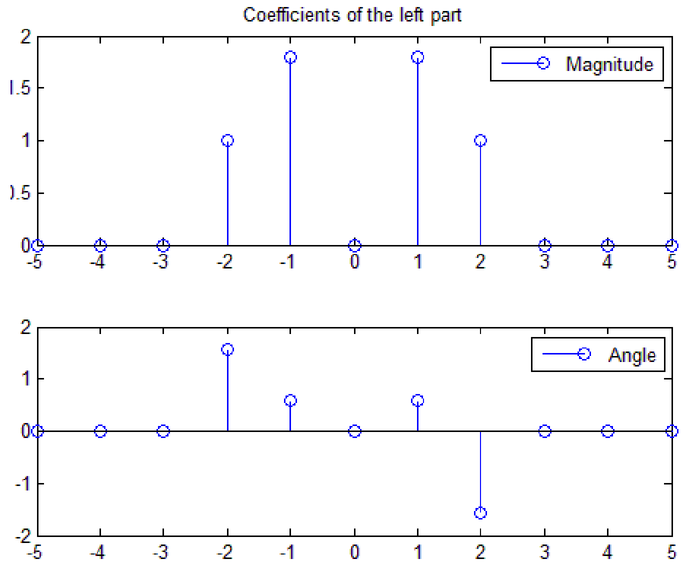
```
t0=0;
T=2*pi;
w=2*pi/T;
```

Results/Comments

First we determine the complex exponential Fourier series coefficients of the left part; that is, we compute the

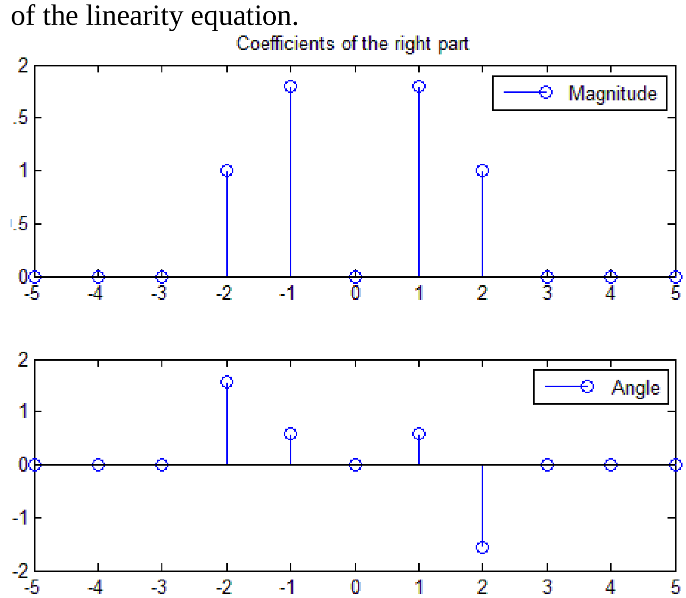
```
syms t
z1=3+2i; z2=2;
x=cos(t); y=sin(2*t);
f=z1*x+z2*y;
k=-5:5;
left=(1/T)*int(f*exp(-j*k*w*t),t,t0,t0+T);
left=eval(left);
subplot(211);
stem(k,abs(left));
legend('Magnitude');
title('Coefficients of the left part');
subplot(212);
stem(k,angle(left));
legend('Angle');
```

coefficients of the signal $f(t) = z_1 x(t) + z_2 y(t)$. The period of $f(t)$ is $T = 2\pi$




```
a=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);
b=(1/T)*int(y*exp(-j*k*w*t),t,t0,t0+T);
right=z1*a+z2*b;
subplot(211);
right=eval(right);
stem(k,abs(right));
legend('Magnitude');
title('Coefficients of the right part');
subplot(212);
stem(k,angle(right));
legend('Angle');
```

In order to derive the right part of the linearity equation, first coefficients are computed and formulate the right part of the linearity equation.



The two graphs are identical; thus the linearity property is verified.

10.2.2 Time Shifting

A shift in time of the periodic signal results on a phase change of the Fourier series coefficients. So, if $x(t) \to a_k$, the exact relationship is

$$x(t-t_1) \leftrightarrow e^{-jk\Omega_0 t_1} a_k$$

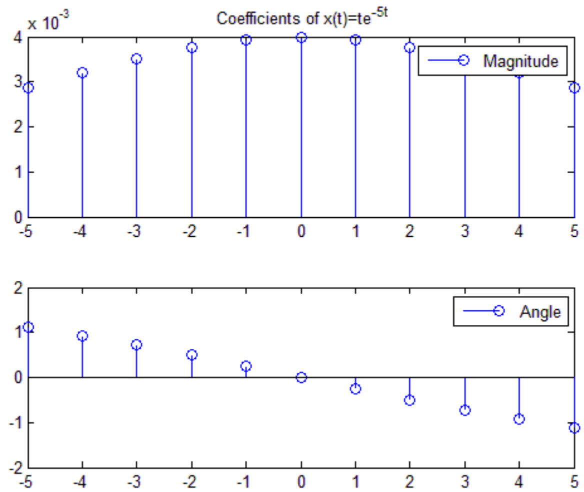
In order to verify time shifting property, we consider the periodic signal that in one period is given by $x(t) = te^{-5t}, 0 \le t \le 10$. Moreover, we set $t_1 = 3$. Consequently the signal $x(t-t_1)$ is given by $x(t-t_1) = x(t-3) = (t-3)e^{-5(t-3)}$.

Commands

```
t0=0;
T=10;
w=2*pi/T;
syms t
x=t*exp(-5*t)
k=-5:5;
a=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);
a1=eval(a);
subplot(211);
stem(k,abs(a1));
title(' Coefficients of x(t)=te^-^5^t');
legend('Magnitude');
subplot(212);
stem(k,angle(a1));
legend('Angle');
```

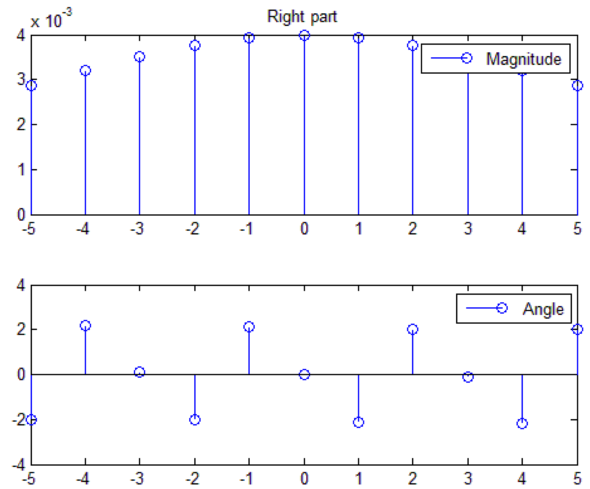
Results/Comments

First the Fourier series coefficients for the given signal are computed and plotted.



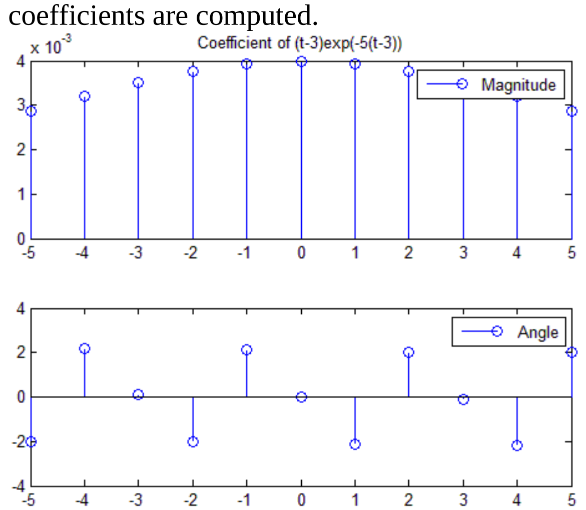

```
t1=3;
right= exp(-j*k*w*t1).*a;
right =eval(right);
subplot(211);
stem(k,abs(right));
legend('Magnitude');
title('Right part');
subplot(212);
stem(k,angle(right));
legend('Angle');
```

Next, the right part of the time shifting equation is computed.



Finally, the time shifted version of $x(t)$ is defined, i.e., $y(t)=(t-3)e^{-5(t-3)}$, and corresponding Fourier series coefficients are computed.

```
x=(t-t1).*exp(-5*(t-t1));
a=(1/T)*int(x*exp(-j*k*w*t),t,t0+t1,t0+T+t1);
coe=eval(a);
subplot(211);
stem(k,abs(coe));
legend('Magnitude');
title(' Coefficient of (t-3)exp(-5(t-3)) ');
subplot(212);
stem(k,angle(coe));
legend('Angle');
```



The two last graphs are identical; hence, the time shift property is confirmed. Comparing the two last graphs with the first one, we notice that indeed the magnitude does not change, but the phase is different.

10.2.3 Time Reversal

The Fourier series coefficients of the reflected version of a signal $x(t)$ are also a reflection of the coefficients of $x(t)$. So, if $x(t)\rightarrow a_k$, the mathematical expression is

$x(-t) \leftrightarrow a_{-k}$

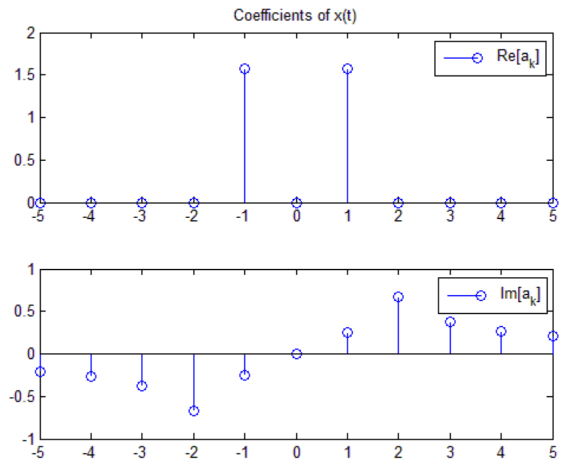
In order to validate the time reversal property, we consider the periodic signal that in one period is given by $x(t) = t\cos(t), 0 \le t \le 2\pi$.

Commands

```
t0=0;
T=2*pi;
w=2*pi/T;
syms t
x=t*cos(t) ;
k=-5:5;
a=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);
a1=eval(a);
subplot(211);
stem(k,real(a1));
legend('Re[a_k]');
title('Coefficients of x(t)');
subplot(212);
stem(k,imag (a1));
legend('Im[a_k]');
```

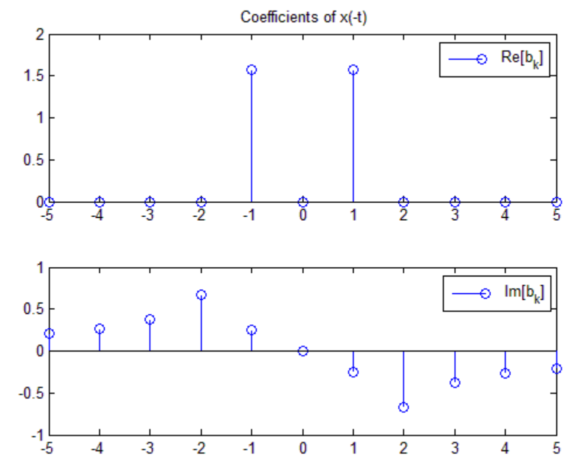
Results/Comments

Fourier series coefficients a_k are computed and plotted.




```
x_=-t*cos(-t) ;
b=(1/T)*int(x_*exp(-j*k*w*t),t,t0-T,t0);
b1=eval(b)
subplot(211);
stem(k,real(b1));
legend('Re[b_k]');
title(' Coefficients of x(-t)');
subplot(212);
stem(k,imag (b1));
legend('Im[b_k]');
```

Next the coefficients b_k are computed for the time reversed version $x(-t)$, and we notice that $b_k = a_{-k}$. Hence the time reversal property is confirmed.



10.2.4 Time Scaling

The Fourier series coefficients of a time scaled version $x(\lambda t)$ and $x(t)$ do not change. On the other hand, the fundamental period of the time scaled version becomes T/λ, and the fundamental frequency becomes $\lambda\Omega_0$. The mathematical expression is

$$x(\lambda t) \leftrightarrow a_k$$

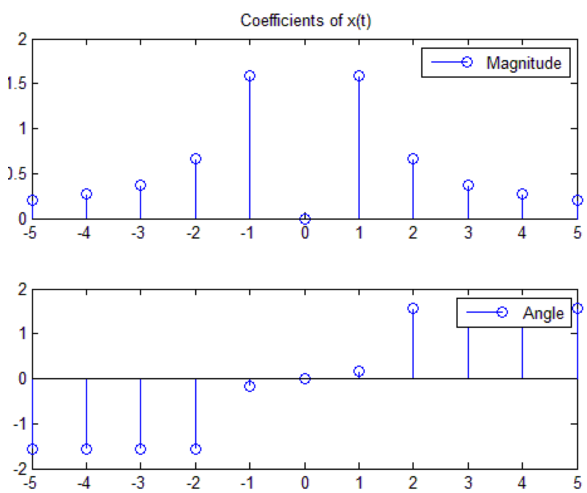
The time scaling property is confirmed by using the periodic signal that in one period is given by $x(t) = t\cos(t), 0 \le t \le 2\pi$.

Commands

```
syms t
t0=0;
T=2*pi;
w=2*pi/T;
x=t*cos(t) ;
k=-5:5;
a=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);
a1=eval(a)
subplot(211);
stem(k,abs(a1));
legend('Magnitude');
title(' Coefficients of x(t)');
subplot(212);
stem(k,angle(a1));
legend('Angle');
```

Results/Comments

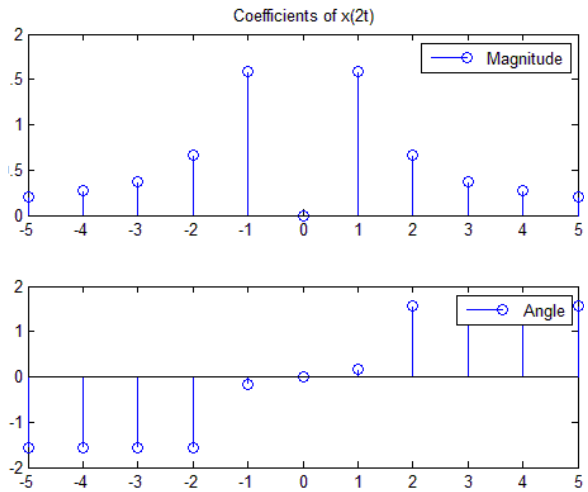
First the Fourier series exponential components $a_k, -5 \le k \le 5$ for the signal $x(t) = t\cos(t), 0 \le t \le 2\pi$ are computed and plotted.




```
lamda=2;
T=T/ lamda ;
w=2*pi/T;
x= lamda *t*cos(lamda *t) ;
k=-5:5;
a=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);
a1=eval(a)
subplot(211);
stem(k,abs(a1));
legend('Magnitude');
title(' Coefficients of x(2t)');
subplot(212);
stem(k,angle(a1));
```

Next the coefficients b_k of the time scaled signal $x(2t) = 2t\cos(2t), 0 \le t \le \pi$, and it is seen that $a_k = b_k$. Hence the time scaling property is confirmed.

legend('Angle');



10.2.5 Signal Multiplication

The Fourier series coefficient of the product of two signals equals the convolution of the Fourier series coefficients of each signal. Suppose that $x(t) \leftrightarrow a_k$ and $y(t) \leftrightarrow b_k$, we have 10. 6 as

$$x(t)y(t) \leftrightarrow a_k * b_k$$

Where * denotes discrete time convolution. To verify property 10.6, we consider the signals $x(t) = \cos(t)$ and $y(t) = \sin(t)$.

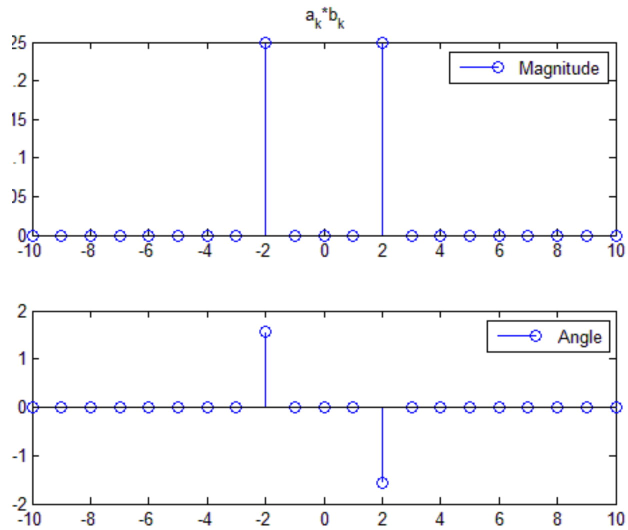
Commands

```
syms t
t0=0;
T=2*pi;
w=2*pi/T;
x=cos(t) ;
k=-5:5;
a=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);
a1=eval(a);
y=sin(t);
b=(1/T)*int(y*exp(-j*k*w*t),t,t0,t0+T);
b1=eval(b);
left=conv(a1,b1);
subplot(211);
stem(-10:10,abs(left));
legend('Magnitude');
title(' a_k*b_k');
```

Results/Comments

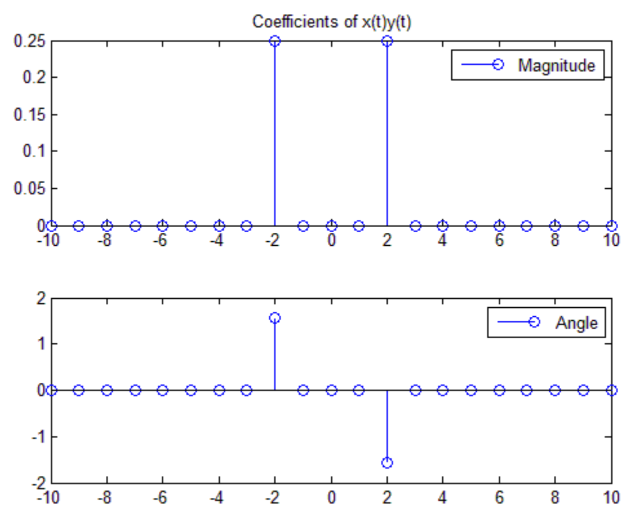
First, the exponential Fourier series coefficients $a_k, -5 \le k \le 5$ and $b_k, -5 \le k \le 5$ and their convolution is computed. Notice that the convolution $d_k = a_k * b_k$ is implemented between two complex valued sequences.

```
subplot(212);
stem(-10:10,angle(left));
legend('Angle');
```




```
z=x*y;
k=-10:10;
c=(1/T)*int(z*exp(-j*k*w*t),t,t0,t0+T);
c1=eval(c)
subplot(211);
stem(k,abs(c1));
legend('Magnitude');
title(' Coefficients of x(t)y(t)');
subplot(212);
stem(k,angle(c1));
legend('Angle');
```

Next, the Fourier series coefficients c_k are computed for the signal $z(t)=\cos(t)\sin(t), -10 \le k \le 10$.



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In-Lab Tasks

Task 01: The periodic signal $x(t)$ is defined in one period as $x(t) = te^{-t}, 0 \le t \le 6$. Plot approximate signal using 81 terms of trigonometric form of Fourier series.

```
T = 6;                    %Time Period
t0 = 0;
w = 2*pi/T;               %Angular Frequency

syms t                    %t as symbol declaration
x = t.*exp(-t);

a0 = (1/T)*int(x,t,t0,t0+T);
for n = 1:80
b(n)  = (2/T)*int(x*cos(n*w*t),t,t0,t0+T);
end
for n = 1:80
c(n)  = (2/T)*int(x*sin(n*w*t),t,t0,t0+T);
end
k = 1:80;
xx = a0+sum(b.*cos(k*w*t))+sum(c.*sin(k*w*t))
ezplot(xx, [t0 t0+T]);
title('Approximation with 81 terms')
```

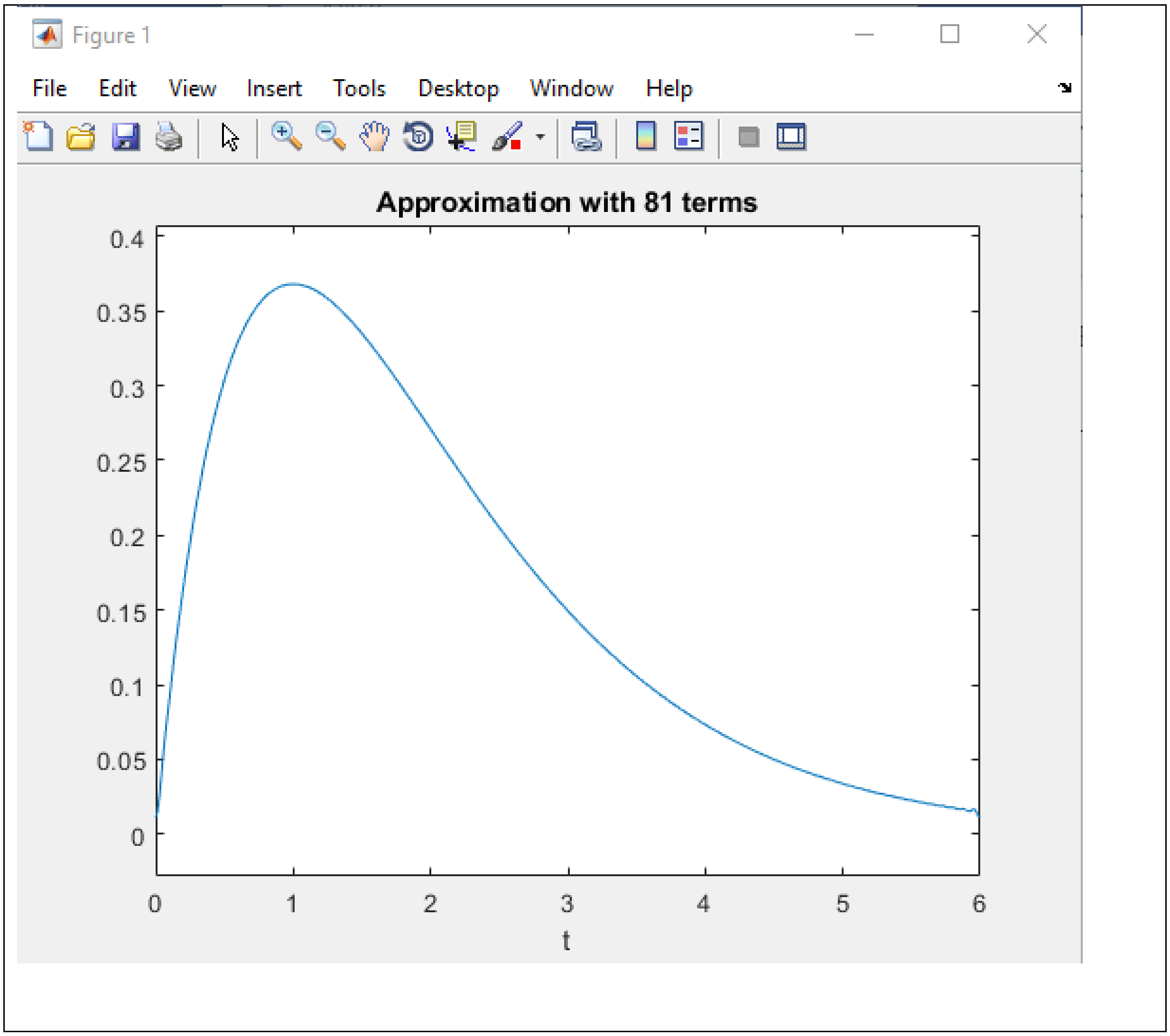


Figure 1
File Edit View Insert Tools Desktop Window Help
Approximation with 81 terms
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0 1 2 3 4 5 6
t

Task 02: Plot the coefficients of the trigonometric Fourier series for the periodic signal that in one period is defined by $x(t) = e^{-t^2}, -3 \le t \le 3$.

```
T = 6;                  %Time Period
t0 = -3;
w = 2*pi/T;             %Angular Frequency

syms t                  %t as symbol declaration
x = exp(-t.^2);
k = -3:3;
a0 = (1/T)*int(x,t,t0,t0+T);
b = (2/T)*int(x*cos(k*w*t),t,t0,t0+T);
c = (2/T)*int(x*sin(k*w*t),t,t0,t0+T);

figure
subplot(3,1,1)
stem(0,a0),legend('a0'),title('Trignometric Coefficients of x(t)');
subplot(3,1,2)
stem(k,b),legend('b_k');
subplot(3,1,3)
stem(k,c),legend('c_k');
```

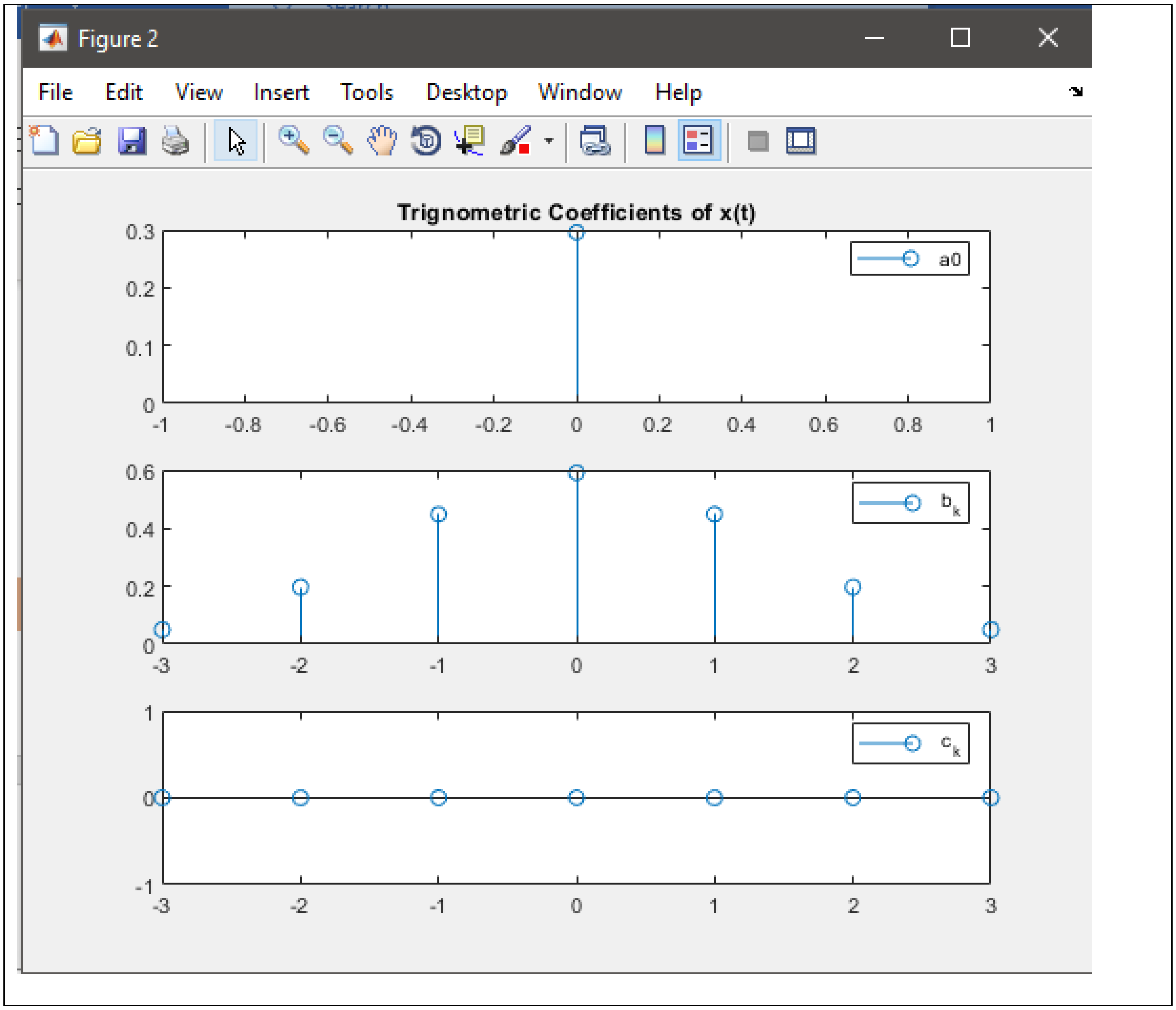


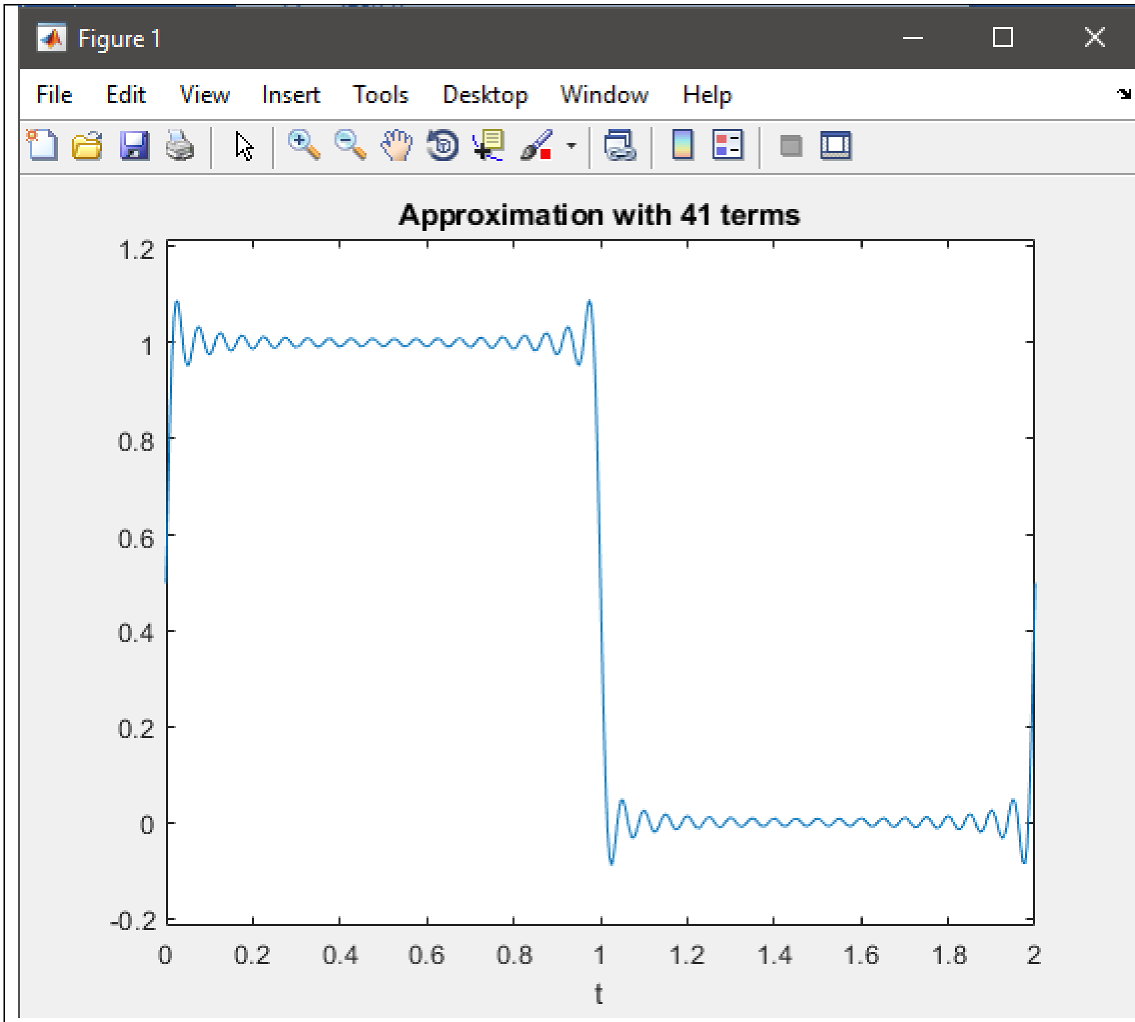
Figure 2
File Edit View Insert Tools Desktop Window Help
Trignometric Coefficients of x(t)
a0
b_k
c_k

Task 03: The periodic signal $x(t)$ in a period is given by

$$x(t)=\begin{cases}1, & 0\le t\le 1\\ 0, & 1\le t\le 2\end{cases}$$

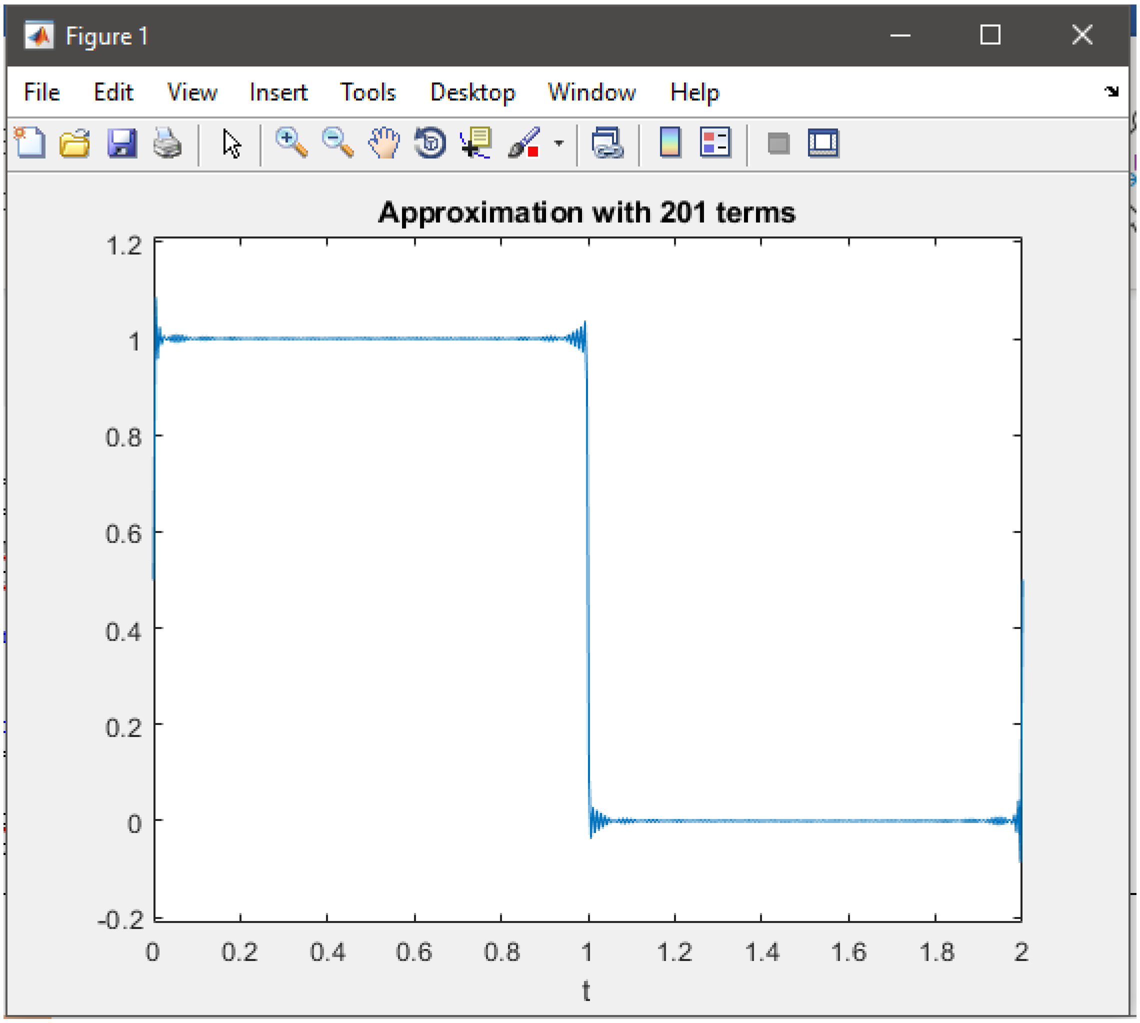
Plot in one period the approximate signals using 41 and 201 term of the trigonometric Fourier series. Furthermore, each time plot the complex exponential coefficients.

```
t0 = 0;
T = 2;
w = 2*pi/T;
syms t
x=heaviside(t)-heaviside(t-1);

a0 = (1/T)*int(x,t,t0,t0+T);
for n = 1:40      %Approximation using 41 terms
    b(n) = (2/T)*int(x*cos(n*w*t),t,t0,t0+T);
end
for n = 1:40      %Approximation using 41 terms
    c(n) = (2/T)*int(x*sin(n*w*t),t,t0,t0+T);
end

k = 1:40;         %Approximation using 41 terms
xx = a0 + sum(b.*cos(k*w*t)) + sum(c.*sin(k*w*t));
ezplot(xx, [t0 t0+T]);
title('Approximation with 41 terms')
```




```
t0 = 0;
T = 2;
w = 2*pi/T;
syms t
x=heaviside(t)-heaviside(t-1);
a0 = (1/T)*int(x,t,t0,t0+T);
for n = 1:200
    b(n) = (2/T)*int(x*cos(n*w*t),t,t0,t0+T);
    c(n) = (2/T)*int(x*sin(n*w*t),t,t0,t0+T);
end
k = 1:200;
xx = a0 + sum(b.*cos(k*w*t)) + sum(c.*sin(k*w*t));
ezplot(xx, [t0 t0+T]);
```

```
title('Approximation with 201 terms')
```

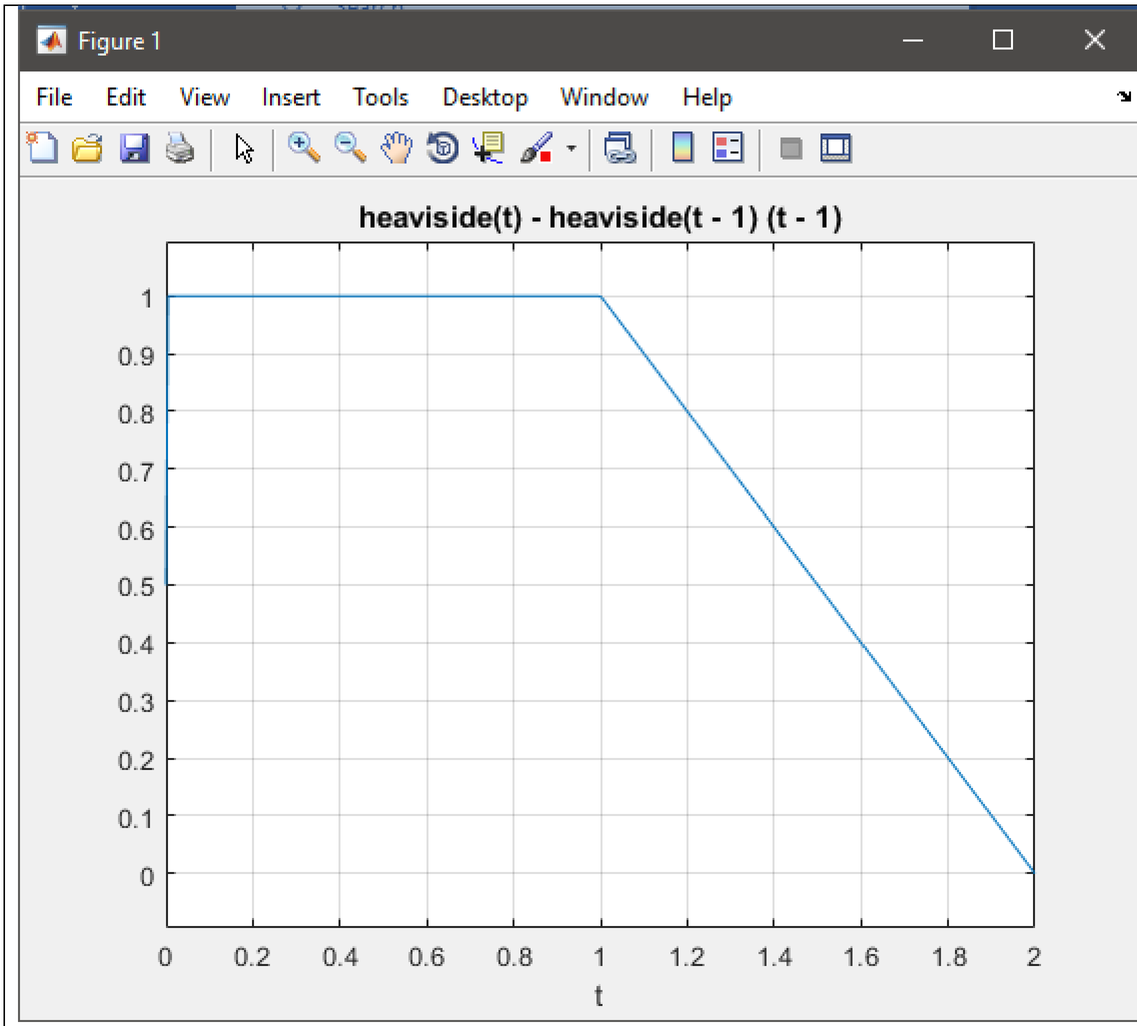


Task 04: The periodic signal $x(t)$ in a period is given by

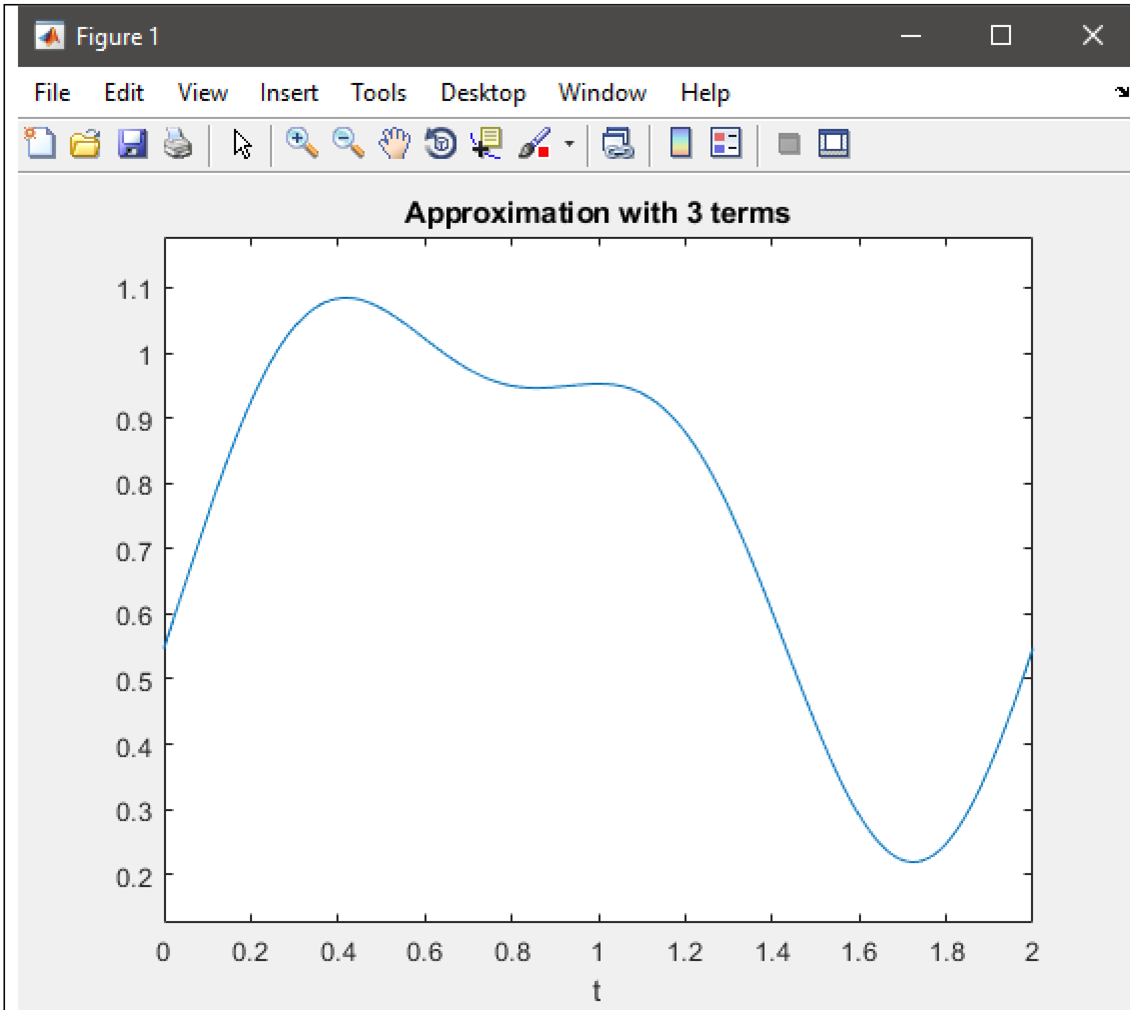
$$x(t)=\begin{cases}1, & 0\le t\le 1\\ 2-t, & 1\le t\le 2\end{cases}$$

Calculate the approximation percentage when the signal $x(t)$ is approximated by 3, 5, 7, and 17 terms of the trigonometric Fourier series. Furthermore, plot the signal in each case.

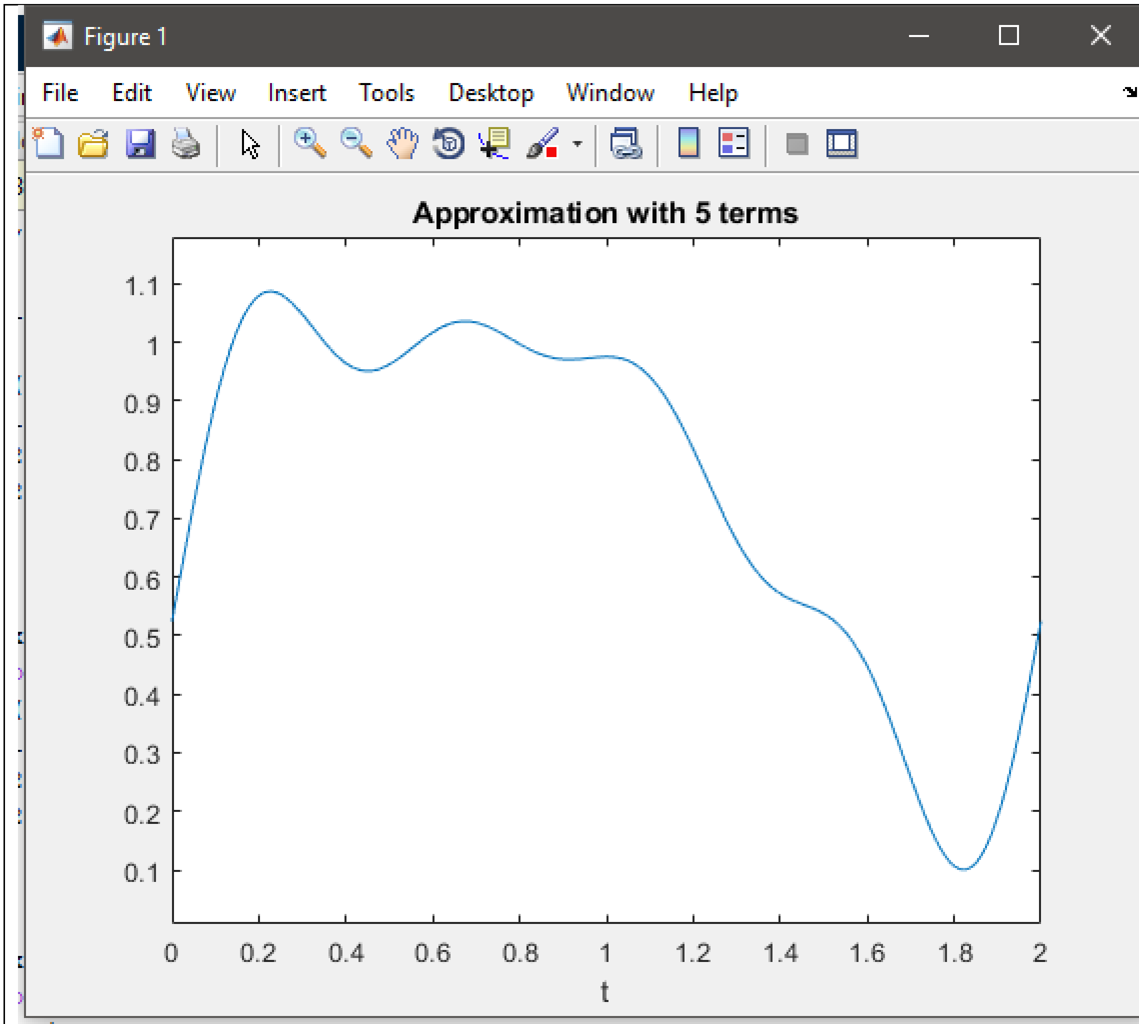
```
T = 2;
t0 = 0;
w = 2*pi/T;
syms t
x = heaviside(t)+((heaviside(t-1)).*(1-t));
ezplot(x,[t0 t0+T]),grid on
```




```
a0 = (1/T)*int(x,t,t0,t0+T);
for n = 1:2
b(n) = (2/T)*int(x*cos(n*w*t),t,t0,t0+T);
c(n) = (2/T)*int(x*sin(n*w*t),t,t0,t0+T);
end
k = 1:2;
xx1 = a0 + sum(b.*cos(k*w*t)) + sum(c.*sin(k*w*t));
ezplot(xx1, [t0 t0+T]);
title('Approximation with 3 terms')
```

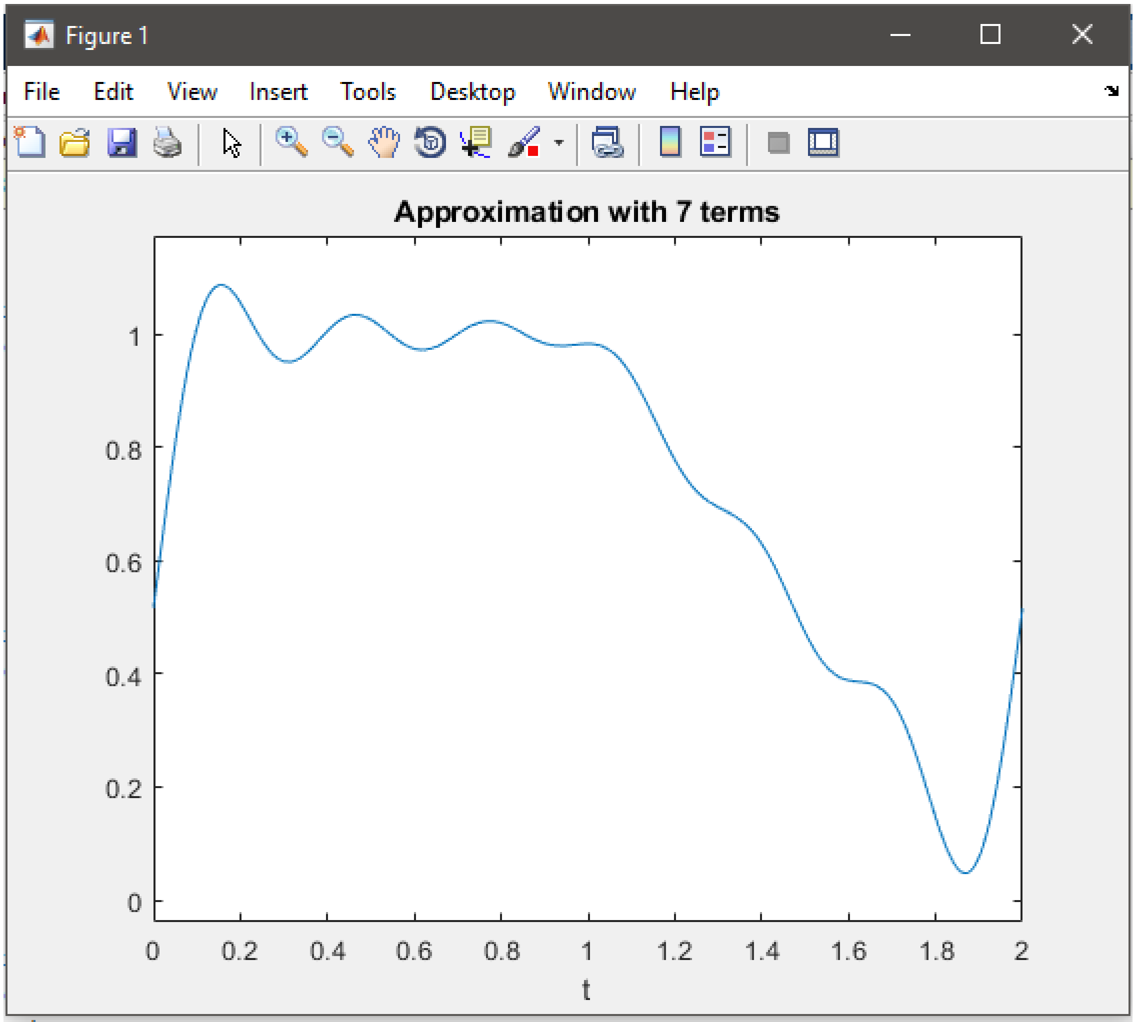



```
a0 = (1/T)*int(x,t,t0,t0+T);
for n = 1:4
b(n) = (2/T)*int(x*cos(n*w*t),t,t0,t0+T);
c(n) = (2/T)*int(x*sin(n*w*t),t,t0,t0+T);
end
k = 1:4;
xx1 = a0 + sum(b.*cos(k*w*t)) + sum(c.*sin(k*w*t));
ezplot(xx1, [t0 t0+T]);
title('Approximation with 5 terms')
```



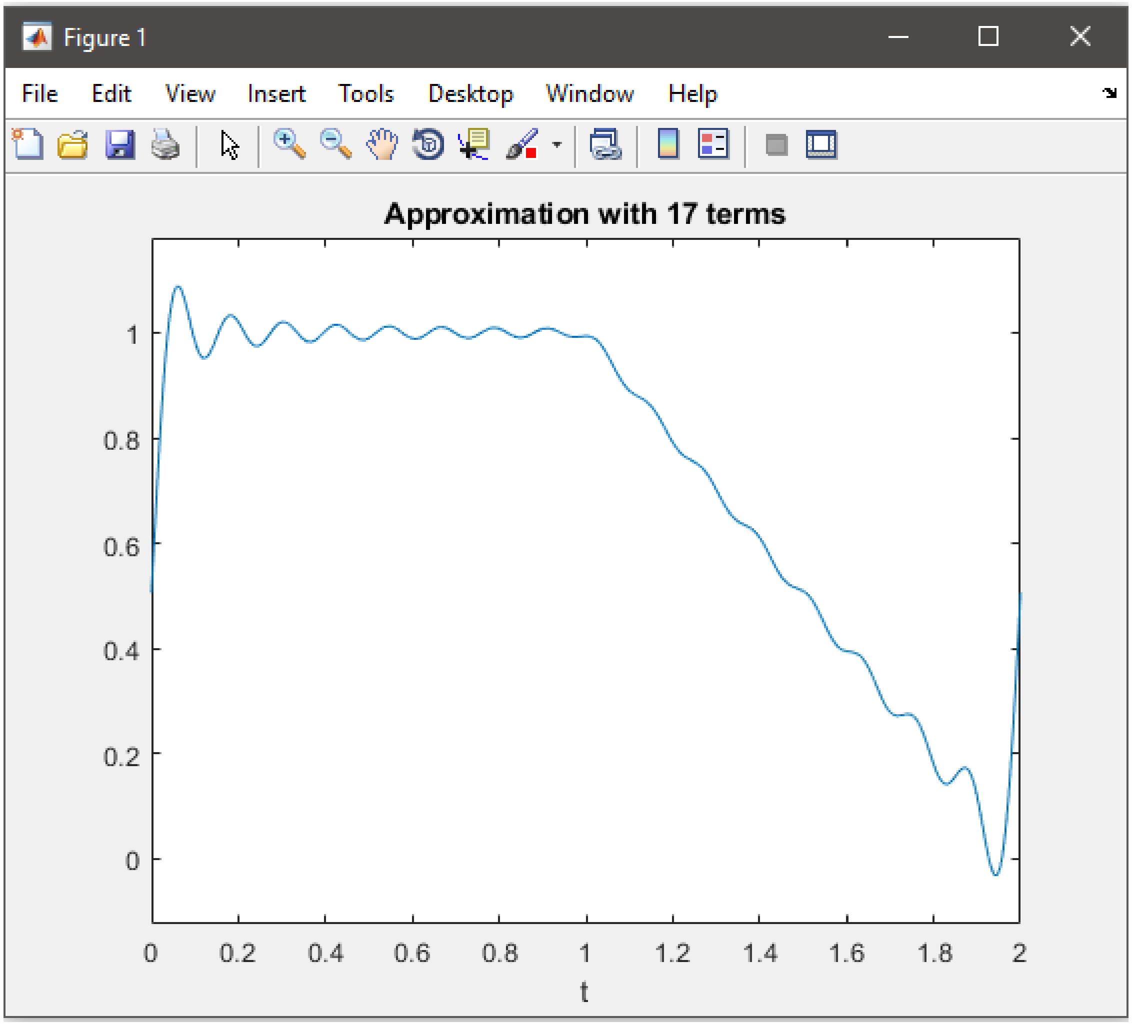

```
a0 = (1/T)*int(x,t,t0,t0+T);
for n = 1:6
b(n) = (2/T)*int(x*cos(n*w*t),t,t0,t0+T);
c(n) = (2/T)*int(x*sin(n*w*t),t,t0,t0+T);
end
k = 1:6;
xx1 = a0 + sum(b.*cos(k*w*t)) + sum(c.*sin(k*w*t));
ezplot(xx1, [t0 t0+T]);
```

```
title('Approximation with 7 terms')
```




```
a0 = (1/T)*int(x,t,t0,t0+T);
for n = 1:16
b(n) = (2/T)*int(x*cos(n*w*t),t,t0,t0+T);
c(n) = (2/T)*int(x*sin(n*w*t),t,t0,t0+T);
end
k = 1:16;
xx1 = a0 + sum(b.*cos(k*w*t)) + sum(c.*sin(k*w*t));
ezplot(xx1, [t0 t0+T]);
```

```
title('Approximation with 17 terms')
```



Post-Lab Task

Critical Analysis / Conclusion

In this lab, we learnt how to plot signals with approximations and observed the effect with increased number of terms and how to find coefficients “a_k” trigonometric Fourier series in MATLAB. Moreover, we plotted and observed the coefficients and angle of trigonometric exponential Fourier series.

Lab Assessment		
Pre-Lab	**/1**	**/10**
In-Lab	**/5**	
Critical Analysis	**/4**	
Instructor Signature and Comments		