1 Decision Theory Review

1.1 Loss functions

- We assume $L(\theta, a)$ is defined for all $(\theta, a) \in \Theta \times \mathcal{A}$, and $L(\theta, a) \geq B > -\infty$
- **Squared-error loss**
  - SEL: $L(\theta, a) = (\theta - a)^2$
  - Weighted SEL: $L(\theta, a) = w(\theta)(\theta - a)^2$
  - Quadratic loss: $L(\boldsymbol{\theta}, \mathbf{a}) = (\boldsymbol{\theta} - \mathbf{a})'\mathbf{Q}(\boldsymbol{\theta} - \mathbf{a})$ where $\mathbf{Q}$ is positive definite
- **Linear loss**
  - $\mathcal{A} = \Theta \subset \mathbf{R}$

$$L(\theta, a) = \begin{cases} K_0(\theta - a) & \theta - a \geq 0 \\ K_1(a - \theta) & \theta - a < 0 \end{cases} \tag{1}$$

- **Linex loss**
  - $\mathcal{A} = \Theta \subset \mathbf{R}$

$$L(\theta, a) = e^{c(\theta - a)} - c(\theta - a) - 1, \quad c \neq 0 \tag{2}$$

- **Entropy loss**
-
  - $\mathcal{A} = \Theta$

$$L(\theta, a) = -\operatorname{E}_\theta \left[\log \frac{p_a(X)}{p_\theta(X)}\right] \tag{3}$$

1.2 Unbiasedness

- Decision rule $\delta : \mathcal{X} \to \mathcal{A}$ is unbiased under $L(\theta, a)$ if

$$\operatorname{E}_\theta L(\theta, \delta) \leq \operatorname{E}_\theta L(\theta', \delta) \quad \forall \theta, \theta' \in \Theta \tag{4}$$

1.3 Bayes, Minimax

- Frequentist risk of rule δ:

$$R(\theta, \delta) = \operatorname{E}_\theta L(\theta, \delta) = \int_{\mathcal{X}} L(\theta, \delta(x)) dP_\theta(x) \tag{5}$$

- Bayes risk

$$r(\pi, \delta) = E^\pi R(\theta, \delta) = \int_\Theta \int_{\mathcal{X}} L(\theta, \delta(x)) dP_\theta(x) d\pi(\theta) \tag{6}$$

- Rule δ_1 is R-better than δ_2 if

$$R(\theta, \delta_1) \leq R(\theta, \delta_2) \quad \forall \theta \tag{7}$$

$$\exists \theta \text{ s.t.} \quad R(\theta, \delta_1) < R(\theta, \delta_2) \tag{8}$$

- δ_1, δ_2 R-equivalent if

$$R(\theta, \delta_1) = R(\theta, \delta_2) \quad \forall \theta \tag{9}$$

- Rule δ is a Bayes rule wrt proper prior π if

$$r(\pi, \delta) = \inf_{\delta^* \in \mathcal{D}} r(\pi, \delta^*) \tag{10}$$

  and we write it δ^π, and $r(\pi) = r(\pi, \delta^\pi)$
  - If $r(\pi) = \infty$, any rule is Bayes
- A Bayes rule can be found by choosing an action to minimize the posterior expected loss for all x in the support of the marginal

$$m(A) = \int P_\theta(A) d\pi(\theta) \tag{11}$$

  that is

$$\delta^\pi(x) = \arg \inf_{a \in \mathcal{A}} \int_\Theta L(\theta, a) d\pi(\theta | x) \quad \forall x \tag{12}$$

- If π is improper, a rule satisfying this condition is a generalized Bayes rule
- δ is admissible if there is no R-better rule, inadmissable if there is
- δ is minimax if it minimizes $\sup_\theta R(\theta, \delta^*)$ among all rules $\delta^* \in \mathcal{D}$, ie,

$$\sup_{\theta \in \Theta} R(\theta, \delta) = \inf_{\delta^* \in \mathcal{D}} \sup_{\theta \in \Theta} R(\theta, \delta^*) \tag{13}$$

- If Θ discrete, prior π s.t. $\pi(\theta) > 0 \ \forall \theta \in \Theta$, and $r(\pi) < \infty$, then δ^π admissible
- If Bayes rule unique, it is admissible
- **Blyth Theorem**: If Θ discrete, δ a rule, and $\exists$ sequence of generalized priors π_n s.t. $\liminf \pi_n(\theta) > 0 \ \forall \theta$, $r(\pi_n) < \infty$, $r(\pi_n, \delta) - r(\pi_n) \to 0$, then δ admissible
- If π proper such that $r(\pi) = \sup_{\theta \in \Theta} R(\theta, \delta^\pi)$, then δ^π minimax; if unique Bayes wrt π, it is also unique minimax
- Generalized Bayes rule with constant (finite) risk is minimax, called an equalizer
- Admissible rule with constant risk is minimax
- Unique minimax is admissible
- Minimax need not be admissible
- Admissible need not be Bayes
- Minimax need not be Bayes

1.4 Complete classes

- Class of rules $\mathcal{C}$ is essentially complete if $\forall \delta \notin \mathcal{C}$, $\exists \delta' \in \mathcal{C}$ R-better or R-equivalent to δ
- $\mathcal{C}$ complete if $\forall \delta \notin \mathcal{C}$, $\exists \delta' \in \mathcal{C}$ R-better than δ
- $\mathcal{C}$ is minimal complete if complete and no proper subset complete

2 Probability Background I

2.1 Maximum Likelihood Principle

- X a sample, $\theta \in \Theta$ unknown, $\mathcal{P} = \{P_\theta, \theta \in \Theta\}$ is the family of X
- Maximum likelihood principle says we should choose estimate

$$\hat{\theta} = \arg \max_{\theta \in \Theta} a(\theta) L_x(\theta) \tag{14}$$

- For point estimation, $a(\theta) = \text{constant} \neq 0$, so MLE is

$$\hat{\theta} = \arg \max_{\theta \in \Theta} L_x(\theta) \tag{15}$$

- For testing problem $a(\theta) = 0$ for incorrect decision

2.2 Sufficient statistics

- $T = T(X)$ is sufficient for X or $\mathcal{P} = \{P_\theta, \theta \in \Theta\}$ or θ if conditional distribution of X given $T(X)$ is independent of θ
- $T(X) \equiv X$ is always sufficient
- **Factorization Criterion**: If family $\mathcal{P}$ of dists of X is dominated by σ-finite measure μ, p_θ the pdf of P_θ wrt μ, then $T = T(X)$ is sufficient for $\mathcal{P}$ iff $\exists$ non-negative functions $q_\theta(t)$, $\theta \in \Theta$, and $h(x)$, such that

$$p_\theta(x) = q_\theta(T(x)) h(x) \quad (a.e. \mu) \tag{16}$$

- Under same conditions, if T sufficient, then $\forall \theta^*, \theta$, $p_\theta(x)/p_{\theta^*}(x)$ is a function only of $T(x)$. Conversely, if $\exists$ fixed θ^* with $p_{\theta^*}(x) > 0$ s.t. $\forall \theta$, $p_\theta(x)/p_{\theta^*}(x)$ is a function only of $T(x)$, then T sufficient
- **Koopman-Darmois family**: $\mathbf{X} = (X_1, \ldots, X_n)$, X_i iid $f_\theta(x)$, $f_\theta(x) = \exp\{P(\theta) + xQ(\theta) + R(x)\}$, Q a 1-1 function θ; $\sum_i X_i$ sufficient
- **Bayes sufficiency**: If under a prior π on Θ, $\pi(\theta) > 0 \ \forall \theta$, posterior $\pi(\theta|x)$ exists, depends only on $T(x)$ nd θ, then T is sufficient for θ. Conversely, if T is sufficient for θ, then under any prior π, the posterior $\pi(\theta|x)$ (if it exists) depends only on $T(x)$ and θ

2.3 Minimal sufficient statistics

- Sufficient statistic S is minimal sufficient for $\mathcal{P}$ if $\forall$ sufficient T, there is measurable function f of T such that $S = f(T)$ (a.e. $\mathcal{P}$)
- Any 1-1 measurable function of minimal suff. S is also minimal sufficient
- **Criterion for minimal sufficiency**: Let $S(X)$ be a statistic, π a prior on Θ s.t.

$$0 < m_\pi(x) = \int p_\theta(x) d\pi(\theta) < \infty \quad (a.e. \mu) \tag{17}$$

  If $\forall x, x'$,

$$\frac{p_\theta(x)}{m_\pi(x)} = \frac{p_\theta(x')}{m_\pi(x')} \iff S(x) = S(x') \tag{18}$$

  then S is minimal sufficient for θ
  - If $p_\theta(x) > 0 \ \forall \theta, x$, then we can pick arbitrary θ^*, π singular measure with mass 1 at θ^*
  - If $\pi(\theta) > 0 \ \forall \theta$, this is equivalent to checking if

$$\pi(\theta | x) = \pi(\theta | x') \iff S(x) = S(x') \tag{19}$$

- **Corollary**: Let S be a statistic.
  1. If $L_x(\theta) > 0 \ \forall (x, \theta)$, and if for any pair (x, x')

$$S(x) = S(x') \iff \frac{L_x(\theta)}{L_{x'}(\theta)} \text{ is constant} \tag{20}$$

     then S is minimal sufficient
  2. If S sufficient, $S(x) = S(x') \ \forall x, x'$ satisfying $L_x(\theta) = C L_{x'}(\theta)$, $C = C(x, x')$ a constant, then S minimal sufficient

3 Probability Background II

3.1 Expoenential family

- Exponential family has density

$$p_\theta(x) = C(\theta) \exp\left\{\sum_{j=1}^k Q_j(\theta) T_j(x)\right\} h(x) \tag{21}$$

  with respect to σ-finite measure μ over $\mathcal{X}$ such that $p_\theta > 0$ for all $x \in \mathcal{X}$. Also, the unnormalized density (ie, without $C(\theta)$) must have a finite integral over $\mathcal{X}$ for all $\theta \in \Theta$
- Then by factorization, a sufficient statistic for θ is $(\sum_{i=1}^n T_1(X_i), \ldots, \sum_{i=1}^n T_k(X_i))$
- We can get *canonical form* by absorbing $h(x)$ into $d\mu$, treat $(Q_1(\theta), \ldots, Q_k(\theta))$ as a parameter in $\mathbf{R}^k$ so that

$$p_\theta(x) = C(\theta) \exp\left\{\sum_{j=1}^k \theta_j T_j(x)\right\} \tag{22}$$

- Then, for the canonical exponential family, the natural parameter space is

$$\Omega = \left\{\theta \in \mathbf{R}^k : \int \exp[\theta \cdot T(x)] d\mu(x) < \infty\right\} \tag{23}$$

  and this is convex
- $T(\mathbf{X})$ is minimal sufficient for θ if there is $\theta^* \in \Theta$ such that $Q_1(\theta) - Q_1(\theta^*), \ldots, Q_k(\theta) - Q_k(\theta^*)$ are linearly independent functions of $\theta \in \Theta$.

3.2 Complete statistics

- Statistic $V(X)$ is ancillary for θ if its distribution does not depend on θ and first-order ancillary if $E_\theta V(X)$ is constant, independent of θ
- A statistic T is complete for θ if any real valued measurable function g of T is such that

$$E_\theta g(T) = 0 \quad \forall \theta \implies g(T) = 0 \quad (a.e. \mathcal{P}) \tag{24}$$

- If the implication holds for all bounded real valued measurable g, then T is boundedly complete for θ
- For the exponential family, $T(\mathbf{X})$ is complete if the interior of Ω is nonempty

3.3 Bounded completeness and sufficiency

- **Bahadur's Theorem**: If a sufficient statistic is boundedly complete, then it is minimal sufficient

3.4 Basu's Theorem

- Suppose $X \sim P_\theta$, and T is a complete and sufficient statistic for θ. Then T is independent of any ancillary statistic of X for θ

4 Uniformly Most Powerful Tests I

- The rejection region of a test ϕ is $\{x : \phi(x) = 1\}$, boundary region $\{x : 0 < \phi(x) < 1\}$, acceptance region $\{x : \phi(x) = 0\}$
- The power function of ϕ is

$$\beta(\theta) = \beta_\phi(\theta) = E_\theta \phi(X) \tag{25}$$

- The significance level / size α of ϕ is

$$\alpha = \alpha_\phi = \sup_{\theta \in \Theta_0} \beta_\phi(\theta) \tag{26}$$

4.1 Neyman-Pearson Lemma

- Consider hypotheses

$$H_0 : X \sim P_0 \text{ vs. } H_1 : X \sim P_1 \tag{27}$$

  Suppose P_0, P_1 have densities p_0, p_1 wrt dominating measure μ (eg $\mu = P_0 + P_1$).
  1. There exists test ϕ, constant k such that

$$E_0 \phi(X) = \alpha \tag{28}$$

$$\phi(x) = \begin{cases} 1 & p_1(x) > k p_0(x) \\ 0 & p_1(x) < k p_0(x) \end{cases} \tag{29}$$

  2. Any test satisfying these two for some k is most powerful at level α
- MP test uniquely determined except on boundary set $\{x : p_1(x) = k p_0(x)\}$ where it can be any function $0 \leq \phi(x) \leq 1$ as long as it has size α
- The test $\phi(x) = \mathbf{1}\{p_1(x)/p_0(x) > k\}$ is MP at level $P_0(p_1(X)/p_0(X) \leq k)$
- Critical value k is any $(1 - \alpha)$ quantile of $p_1(X)/p_0(X)$ under P_0, can be constant on the boundary where the ratio equals k
- For any $t \in [0, 1]$ the tth quantile of df F is any number x such that

$$F(x-) \leq t \leq F(x) \tag{30}$$

- x is a tth quantile $\iff x \in [F^*(t), F^\#(t)]$ where

$$F^*(t) = \inf\{x : F(x) \geq t\}, \quad F^\#(t) = \sup\{x : F(x) \leq t\} \tag{31}$$

- If F continuous and strictly increasing, then F^* and $F^\#$ are identical to F^{-1}. We refer to F^* as the inverse of F
- Also, if $U \sim U(0, 1)$, then $F^*(U) \sim F$
- Corollary: If $0 < \alpha < 1$, β is the power of the MP level α test, then $\alpha < \beta$ unless $P_0 = P_1$

5 Uniformly Most Powerful Tests II

5.1 MLR and UMP tests of composite hypotheses

- Let $C_\alpha = \{\phi : \phi \text{ is of size } \alpha\}$. ϕ_0 is uniformly most powerful of size α if it has size α and $\beta_{\phi_0}(\theta) \geq \beta_\phi(\theta)\ \forall\theta \in \Theta_1$ and $\phi \in C_\alpha$
- If there is real-valued function $T(X)$ such that for any $\theta < \theta'$, distributions $P_\theta, P_{\theta'}$ are distinct, the likelihood ratio $p_{\theta'}(x)/p_\theta(x) = g(T(X))$ where g is a nondecreasing function, then $\mathcal{P}$ has the MLR in T.
- It is trivial to see that T is sufficient
- If $\Theta = \{\theta_0, \theta_1\}$, then MLR is satisfied by choosing $T(x) = p_{\theta_1}(x)/p_{\theta_0}(x)$
- **Theorem:** Suppose X has density $p_\theta(x)$ with MLR in $T(x)$, $0 \leq \alpha \leq 1$. Then
  1. There is a UMP α-level test of $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$ of the form
  $$\phi(x) = \begin{cases} 1 & T(x) > C \\ \gamma & T(x) = C \\ 0 & T(x) < C \end{cases} \tag{32}$$
  where C, γ chosen so that $E_{\theta_0}\phi(X) = \alpha$.
  2. One such choice is $C = (1-\alpha)$th quantile of $T(X)$ under P_{θ_0}, and $\gamma = (\alpha - P_{\theta_0}\{T(X) > C\})/P_{\theta_0}\{T(X) = C\}$ if $P_{\theta_0}\{T(X) = C\} > 0$, else γ can be any number in $[0, 1]$.
  3. The power function of ϕ is strictly increasing for all θ for which $0 < \beta(\theta) < 1$
  4. For all θ' the test is UMP for $H_0' : \theta \leq \theta'$ vs $H_1' : \theta > \theta'$ where $\alpha' = \beta(\theta')$
  5. For any $\theta < \theta_0$, the test minimizes $\beta(\theta)$ among level α tests
- **Corollary:** If real-parameter family p_θ with df F_θ has MLR in $T(x) = x$, then for all x, $F_\theta(x)$ is strictly decreasing in θ for which $0 < F_\theta(x) < 1$
- **Theorem:** UMP test for $H_0 : \theta \geq \theta_0$ vs $H_1 : \theta < \theta_0$ is
$$\phi(x) = \begin{cases} 1 & T(x) < C \\ \gamma & T(x) = C \\ 0 & T(x) > C \end{cases} \tag{33}$$
and the power function $\beta(\theta)$ is strictly increasing, and for any $\theta < \theta_0$, it minimizes $\beta(\theta)$ among level α tests
- Note this is just $1 - \phi$ for the original UMP with the hypotheses switched

5.2 Generalized NP lemma

- For $H_0 : \theta \notin (\theta_1, \theta_2)$ vs $H_1 : \theta \in (\theta_1, \theta_2)$ where $\theta_1 < \theta_2$, if $p_\theta(x) = c(\theta)\exp[\theta T(x)]h(x)$, a UMP level α test is
$$\phi(x) = \begin{cases} 1 & C_1 < T(x) < C_2 \\ \gamma_i & T(x) = C_i, \quad i = 1, 2 \\ 0 & T(x) < C_1, \quad T(x) > C_2 \end{cases} \tag{34}$$
where $C_1, C_2, \gamma_1, \gamma_2$ chosen so that $\beta_\phi(\theta_1) = \beta_\phi(\theta_2) = \alpha$. Then ϕ minimizes $\beta_\phi(\theta)$ among level α tests. For $0 < \alpha < 1$, $\beta_\phi(\theta)$ has a maximum at a point $\theta_0 \in (\theta_1, \theta_2)$, and decreases strictly as θ tends away from θ_0 in either direction, unless there are two values t_1, t_2 such that $P_0(T(X) = t_1 \text{ or } t_2) = 1$ for all θ

5.3 Hellinger Distance and Consistency of NP-type tests

- The Hellinger distance $\mathcal{H}(P, Q)$ betwen P, Q is
$$\mathcal{H}^2(P, Q) = \frac{1}{2}\int (\sqrt{p} - \sqrt{q})^2 d\mu = 1 - \int \sqrt{pq}\, d\mu = 1 - \rho(P, Q) \tag{35}$$
where $\rho(P, Q) = \int \sqrt{pq}\, d\mu$ is the affinity between P, Q.
- The total variation distance between P, Q is
$$\|P - Q\|_1 = \int |p - q| d\mu \tag{36}$$
- $0 \leq \rho(P, Q) \leq 1$, $\mathcal{H}^2(P, Q) = 0 \iff p = q$ a.e. $\mu \iff \rho(P, Q) = 1$
- The following relationship holds
$$\mathcal{H}^2(P, Q) \leq \frac{1}{2}\|P - Q\|_1 \leq \mathcal{H}(P, Q)[2 - \mathcal{H}^2(P, Q)]^{1/2} = [1 - \rho^2(P, Q)]^{1/2} \tag{37}$$
- For NP test of $H_0 : X \sim P$ vs $H_1 : X \sim Q$ of the form
$$\phi(x) = \begin{cases} 1 & q(x) > Cp(x) \\ 0 & q(x) < Cp(x) \end{cases} \tag{38}$$
for $C > 0$, then
$$\alpha_\phi \leq C^{-1/2}\rho(P, Q), \quad 1 - \beta_\phi \leq C^{1/2}\rho(P, Q) \tag{39}$$

6 Unbiasedness for Hypothesis Testing I

6.1 Unifromly Most Powerful Unbiased Tests

- Suppose $X \sim P_\theta$ for some $\theta \in \Theta$, and ϕ is level α for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$ where $\{\Theta_0, \Theta_1\}$ is a partition of Θ.
- ϕ is unbiased if $\beta_\phi(\theta) \leq \alpha\ \forall\theta \in \Theta_0$ and $\beta_\phi(\theta) \geq \alpha\ \forall\theta \in \Theta_1$
- If ϕ is UMP, it is unbiased because it has size α and its power cannot be less than that of randomized $\phi^*(x) = \alpha$
- ϕ is uniformly most powerful unbiased at level α if it is unbiased and $\beta_\phi(\theta) \geq \beta_{\phi'}(\theta)$ $\forall\theta \in \Theta_1$ and all unbiased level-α tests ϕ'
- Any UMP test is UMPU
- Test ϕ is similar on the boundary at level α if $\beta_\phi(\theta) = \alpha\ \forall\theta \in \Theta_B := \overline{\Theta}_0 \cap \overline{\Theta}_1$
- If a test unbiased and has continuous power function, it is similar on the boundary
- If Θ is finite, or has no cluster points (eg Θ is set of integers), then $\beta_\phi(\theta)$ continuous
- Suppose P_θ such that power $\beta_\phi(\theta)$ is continuous for all tests and ϕ_0 is a level α test of $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$. If ϕ_0 is uniformly most powerful among all tests similar on the boundary at level α, then ϕ_0 UMPU

6.2 Application to one-parameter exponential families

- Consider $p_\theta(x) = c(\theta)\exp[\theta T(x)]h(x)$
  1. $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$
  2. $H_0 : \theta \notin (\theta_1, \theta_2)$ vs $H_1 : \theta \in (\theta_1, \theta_2)$
  3. $H_0 : \theta \in [\theta_1, \theta_2]$ vs $H_1 : \theta \notin [\theta_1, \theta_2]$
  4. $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$
- By NP lemma, there is a UMP test for 1, by generalized NP, we have UMP for 2.
- Note that for this exponential family, all power functions $\beta_\phi(\theta)$ are continuous and smooth in θ
- **Theorem:** The test
$$\phi(x) = \begin{cases} 1 & T(x) < C_1 \text{ or } T(x) > C_2 \\ \gamma_i & T(x) = C_i, \quad i = 1, 2 \\ 0 & C_1 < T(x) < C_2 \end{cases} \tag{40}$$
where C, γ satisfy $\beta_\phi(\theta_1) = \beta_\phi(\theta_2) = \alpha$ is UMPU for 3 at level α. If $\alpha \in (0, 1)$, no UMP test exists
- Further, if we also have $\mathrm{E}_{\theta_0}\phi(X) = \alpha$, $\mathrm{E}_{\theta_0}[T(X)\phi(X)] = \alpha\,\mathrm{E}_{\theta_0} T(X)$, then ϕ is UMPU for 4. If T has a symmetric distribution, then $\mathrm{E}_{\theta_0}\phi(X) = \alpha$, $C_2 = 2\alpha - C_1$ and $\gamma_1 = \gamma_2$ determine the constants. If $\alpha \in (0, 1)$, no UMP test exists

7 Unbiasedness for Hypothesis Testing II

7.1 Neyman Structure

- In the presence of nuisance parameter ϑ, we can condition on a sufficient statistic for ϑ—which frees the distribution from ϑ—we can write $\psi_t(x)$ if $T(x) = t$.
- We want to consider $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$ which is equivalent to $H_0 : (\theta, \vartheta) \in \Omega_0$ vs $H_1 : (\theta, \vartheta) \in \Omega_1$ where $\Omega_i = \{(\theta, \vartheta) : \theta \in \Theta_i\}$
- A test satisfying $\mathrm{E}_\theta(\phi|T) = \alpha\ \forall\theta \in \Theta_B$ has Neyman structure wrt T.
- Critical function ϕ is similar wrt $\mathcal{P}$ if $\mathrm{E}_P\,\phi = \alpha$ is independent of $P \in \mathcal{P}$
- Let T be sufficient for $\mathcal{P}$, and $\mathcal{P}^T = \{P^T : P \in \mathcal{P}\}$ where P^T is the dist of T when $X \sim P$. Critical function ϕ has Neyman structure wrt T if $\mathrm{E}(\phi|T) = \alpha$ a.e. $\mathcal{P}^T$
- If ϕ has Neyman structure wrt T, it is similar wrt $\mathcal{P}$
- **Theorem:** Let T be sufficient for $\mathcal{P}$. Then all critical functions that are similar wrt $\mathcal{P}$ have Neyman structure wrt $T \iff \mathcal{P}^T$ is bounded complete
- **Theorem:** The UMPU conditional test is: Suppose $X \sim P_{\theta,\vartheta}$, T a statistic of X, and we want to test $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$. Further, suppose that
  - The distributions $P_{\theta,\vartheta}$ have continuous power functions for every test
  - For each fixed θ, T is sufficient for ϑ
  - For each fixed $\theta \in \Theta_B$, T is boundedly complete for ϑ
  - For each t, θ being the only parameter for the conditional distribution of X given $T = t$, ψ_t is a level-α test of H_0 vs H_1, and is UMP among all tests that are similar on the boundary at level α

  Then $\phi(x) = \psi_{T(x)}(x)$ is UMPU.

7.2 UMPU tests for multiparameter exponential families

- For test 1, $\phi_1(x) = \phi(U(x), \mathbf{T}(x))$ with
$$\phi(u, \mathbf{t}) = \begin{cases} 1 & u > C(\mathbf{t}) \\ \gamma(\mathbf{t}) & u = C(\mathbf{t}) \\ 0 & u < C(\mathbf{t}) \end{cases} \tag{41}$$
is UMPU where $C(\mathbf{t}), \gamma(\mathbf{t})$ are such that $\mathrm{E}_{\theta_0}[\phi(U, \mathbf{t})|\mathbf{T} = \mathbf{t}] = \alpha\ \forall\mathbf{t}$
- For test 2, $\phi_2(x) = \phi(U(x), \mathbf{T}(x))$, with
$$\phi(u, \mathbf{t}) = \begin{cases} 1 & C_1(\mathbf{t}) < u < C_2(\mathbf{t}) \\ \gamma_i(\mathbf{t}) & u = C_i(\mathbf{t}), \quad i = 1, 2 \\ 0 & u < C_1(\mathbf{t}) \text{ or } u > C_2(\mathbf{t}) \end{cases} \tag{42}$$
where C, γ chosen so that $\mathrm{E}_{\theta_i}[\phi(U, \mathbf{t})|\mathbf{T} = \mathbf{t}] = \alpha\ \forall\mathbf{t}$, $i = 1, 2$
- For test 3, $\phi_3(x) = \phi(U(x), \mathbf{T}(x))$, with
$$\phi(u, \mathbf{t}) = \begin{cases} 1 & u < C_1(\mathbf{t}) \text{ or } u > C_2(\mathbf{t}) \\ \gamma_i(\mathbf{t}) & u = C_i(\mathbf{t}), \quad i = 1, 2 \\ 0 & C_1(\mathbf{t}) < u < C_2(\mathbf{t}) \end{cases} \tag{43}$$
where C, γ chosen so that $\mathrm{E}_{\theta_i}[\phi(U, \mathbf{t})|\mathbf{T} = \mathbf{t}] = \alpha\ \forall\mathbf{t}$, $i = 1, 2$
- For test 4, use the same as for test 3 with C, γ such that $\mathrm{E}_{\theta_0}[\phi(U, \mathbf{t})|\mathbf{T} = \mathbf{t}] = \alpha\ \forall\mathbf{t}$ and $\mathrm{E}_{\theta_0}[U\phi(U, \mathbf{t})|\mathbf{T} = \mathbf{t}] = \alpha\,\mathrm{E}_{\theta_0}(U|\mathbf{T} = \mathbf{t})\ \forall\mathbf{t}$.

7.3 UMPU unconditional tests

- Under the same assumptions as for the conditional tests, suppose we can find statistic $V = h(U, \mathbf{T})$ which is independent of $\mathbf{T}$ when $\theta \in \Theta_B$, such that $h(u, \mathbf{t})$ is strictly increasing in u for fixed $\mathbf{t}$:
  - Test 1: $\phi_1(x) = \psi(V(x))$,
  $$\phi(v) = \begin{cases} 1 & v > C \\ \gamma & v = C \\ 0 & v < C \end{cases} \tag{44}$$
  is UMPU where C, γ independent of $\mathbf{t}$ such that $E_{\theta_0}\phi(V) = \alpha$
  - Test 3: $\phi_3(x) = \phi(V(x))$ with
  $$\phi(v) = \begin{cases} 1 & v < C_1 \text{ or } v > C_2 \\ \gamma_i & v = C_i, \quad i = 1, 2 \\ 0 & C_1 < v < C_2 \end{cases} \tag{45}$$
  is UMPU where C, γ independent of $\mathbf{t}$ and $\mathrm{E}_{\theta_1}\phi(V) = \mathrm{E}_{\theta_2}\phi(V) = \alpha$
  - Test 2: $\phi_2(x; \alpha) = 1 - \phi_3(x; 1 - \alpha)$ is UMPU
  - Test 4: If $h(u, \mathbf{t}) = a(\mathbf{t})u + b(\mathbf{t})$, $a(\mathbf{t}) > 0$, then $\phi_4(x) = \phi(V(x))$ is UMPU, where ϕ is the same as for test 3 with C, γ independent of $\mathbf{t}$ such that $\mathrm{E}_{\theta_0}\phi(V) = \alpha$ and $\mathrm{E}_{\theta_0}[V\phi(V)] = \alpha\,\mathrm{E}_{\theta_0} V$