1 Theory of Interest

1.1 Interest

Definition 1.1 (Accumulation Function)**.**
When a principal of 1 dollar is deposited in an interest-paying account at time $t = 0$, it earns some interest over the time interval $[0, t]$.
The accumulated value of 1 dollar at time $t \geq 0$, denoted by $a(t)$, is known as the **accumulation function**. Clearly, $a(0) = 1$.

Definition 1.2 (Simple and Compound Interest)**.**
Let r be the annual rate of interest.
Based on the **simple-interest** method of calculating interest,

$$a(t) = 1 + rt \quad \text{for } t \geq 0$$

If the **compound interest** method is used,

$$a(t) = (1 + r)^t \quad \text{for } t \geq 0$$

Suppose the interest rate is r_i for the period $[\sum_{k=0}^{i-1} t_i, \sum_{k=1}^{i} t_i]$, where $t_0 = 0$,

$$a(t_j) = 1 + \sum_{i=1}^{j} r_i t_i \quad \text{when simple interest is used;}$$

$$a(t_j) = \prod_{i=1}^{j} (1 + r_i)^{t_i} \quad \text{when compound interest is used;}$$

Definition 1.3 (Frequency of Compounding)**.**
When an interest of $r = r^{(p)}$ is paid p times a year (or equivalently, $r^{(p)}$ is **convertible** p**thly** or $r^{(p)}$ is compounded p times a year), we call p the **frequency of compounding** and $r^{(p)}$ the **nominal** rate of interest.

The interest to be paid over the period, is $\frac{r^{(p)}}{p}$. Effectively, \$1 invested at time $t = 0$ will grow to $\left(1 + \frac{r^{(p)}}{p}\right)$ over a period of length $\frac{1}{p}$, so that the accumulated amount after one year is$\left(1 + \frac{r^{(p)}}{p}\right)^p$. **Remarks**

1. We write the superscript (p) for $r^{(p)}$ to indicate the frequency of compounding p.

2. We can drop the superscript (p) when $p = 1$.

3. $p = 2, 4, 12$ correspond to semi-annual, quarterly and monthly compounding respectively,

Definition 1.4 (Equivalent Interest Rates)**.**
Two nominal interest rates are said to be **equivalent** if and only if they yield same accumulation amount over a year. Hence, the nominal rates $r^{(p)}$ and $r^{(q)}$ are equivalent if and only if

$$\left(1 + \frac{r^{(p)}}{p}\right)^p = \left(1 + \frac{r^{(q)}}{q}\right)^q$$

In particular, the **effective** annual interest rate (when $p = 1$), denoted by r_e, is given by

$$1 + r_e = \left(1 + \frac{r^{(p)}}{p}\right)^p$$

The corresponding accumulation function is

$$a(t) = (1 + r_e)^t = \left(1 + \frac{r^{(p)}}{p}\right)^{pt}$$

It can be shown that $r_e \geq r^{(p)}$ for $p > 1$.

Definition 1.5 (Continuous Compounding)**.**
The interest is **compounded continuously** when the frequency of compounding tends to infinity.
Let $r^{(\infty)}$ denote the nominal rate of interest under continuous compounding. Then,

$$a(1) = \lim_{p \to \infty} \left(1 + \frac{r^{(\infty)}}{p}\right)^p = e^{r^{(\infty)}}$$

The number $r^{(\infty)}$ is known as the **continuously compounded** rate of interest. The corresponding accumulatio function is

$$a(t) = e^{r^{(\infty)} t}, \quad t \geq 0$$

Note that $e^{r^{(\infty)}} = 1 + r_e$.
It can be shown that

$$e^r > \left(1 + \frac{r}{p}\right)^p$$

for any $r > 0$ and for any $p \in \mathbb{Z}^+$.

1.2 Present Value

Definition 1.6 (Present Value, Time Value)**.**
Let $a(t)$ be the accumulation function. Let X be the amount that must be invested at time $t = 0$ to accumulate to 1 dollar at $t = T$. Then

$$X \cdot a(T) = 1$$

or equivalently, $X = \frac{1}{a(T)}$.
The amount $X = \frac{1}{a(T)}$ is the **present value** of 1 paid at time T.

It follows that the present value of a single payment of C at time $t + T$ is $\frac{C}{a(T)}$.
More generally, for a cash flow $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$ consisting of a series of payments, with c_i received at time t_i, for $i = 1, 2, 3, \ldots, n$, where $t_1 \geq 0$ and $t_i < t_j$ for $i < j$, the present value of this cash flow, denoted by $\mathrm{PV}(\mathbf{C})$, is defined by

$$\mathrm{PV}(\mathbf{C}) = \sum_{i=1}^{n} \frac{c_i}{a(t_i)}$$

Definition 1.7 (Time Value)**.**
The **time value** of the cash flow $\mathbf{C}$ at time $t \geq 0$, denoted by $\mathrm{TV}(\mathbf{C}, t)$, is given by

$$\mathrm{TV}(\mathbf{C}, t) = \mathrm{PV}(\mathbf{C}) \times a(t)$$

A consequence of the above definition is that for $0 < s < t$,

$$\mathrm{TV}(\mathbf{C}, t) = \frac{a(t)}{a(s)} \times \mathrm{TV}(\mathbf{C}, s)$$

Definition 1.8 (Principle of Equivalence).
In an environment where both the *interest rate* and its *method of accumulation* remain the same over any time period, two cash flows streams are **equivalent** if and only if they have the same present value.
(Alternatively, if and only if they have the same time value at $t = T$ for any $T \geq 0$).

It follows that the cash flow $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$ is equivalent to a single payment of $\mathrm{PV}(\mathbf{C}) = \sum_{i=1}^{n} \frac{c_i}{a(t_i)}$ at time $t = 0$.

Definition 1.9 (Deferred Cash Flow).
Let $k > 0$ and define the cash flow $\mathbf{C}_{(k)} = \{(c_1, t_1 + k), (c_2, t_2 + k), \ldots, (c_n, t_n + k)\}$ which is essentially the cash flow $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$ deferred by k years.
If the accumulation function is $a(t)$, then

$$\frac{\mathrm{PV}(\mathbf{C})}{\mathrm{PV}(\mathbf{C}_{(k)})} = a(k)$$

Notations:
For the special case when $t_i = i - 1$,

$$\mathbf{C} = \{(c_1, 0), (c_2, 1), \ldots, (c_n, n-1)\}$$

can be written as $(c_1, c_2, \ldots, c_n)$.

Definition 1.10 (Equation of Value).
Consider the cash flow stream $\mathbf{C} = \{(c_1, t_1),)c_2, t_2), \ldots, (c_n, t_n)\}$.
The equation

$$\mathrm{PV}(\mathbf{C}) = \sum_{i=1}^{n} \frac{c_i}{(1+r)^{t_i}} = 0$$

is known as the **equation of value**.

Definition 1.11 (Internal Rate of Return(IRR)).
Any non-negative root, r of the equation of value is called the **yield** or **internal rate of return (IRR)**, of the cash flow stream.

1.3 Annuities

Definition 1.12 (Annuities Immediate and Annuities Due).
An annuity is a series of payment made at regular intervals.
An **annuity-due** is one for which payments are made at the *beginning* of each period.
An **annuity-immediate** is one for which payments are made at the *beginning* of each period.

Definition 1.13 (Perpetuity).
A **perpetuity** is an annuity with an infinite number of payments.

Definition 1.14 (Loans).
Loans are normally repaid by a series of installment payments made at *periodic* intervals. The size of each installment can be determined using present-value analysis.
Specifically, if we let L be the amount of loan taken at time $t = 0$ and let $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$ be the series of repayments, then

$$L := \mathrm{PV}(\mathbf{C})$$

We can also compute the balance of the loan at any point in time.

Definition 1.15 (Loan Balance).
The **loan balance** L_m^{Balance} immediately after the mth installment has been paid is the **time value** at $t = m$ of the remaining $(n - m)$ installment payments.
Suppose installment is paid annually with effectively annual rate r and each repayment of value c_i for year $m + i$, the loan balance

$$L_m^{\text{Balance}} = \sum_{i=1}^{n-m} \frac{c_i}{(1+r)^i}$$

Suppose each annual repayment is of value A. In reality, the loan is usually fully paid with n repayment of A plus a final payment B made at time $t \geq n$, where B is determined from the equation

$$L = \mathrm{PV}(0, \underbrace{A, A, \ldots, A}_{n\text{payments}}) + \mathrm{PV}(\{(B, t)\})$$

2 Bonds and Term Structure

2.1 Bond Terminology

Definition 2.1 (Bond).
A **bond** is a written contract between the issuers(borrowers) and the investers(lenders) which specifies the following:

- **Face value**, F, of the bond: the amount based on which periodic interest payments are computed
- **Redemption/maturity value**, R, of the bond: the amount to be repaid at the end of the loan
- **Maturity date** of the bond: the date on which the loan will be fully repaid
- **Coupon rate**, c, (for coupon-paying bonds): the bond's interest payments, as a percentage of the par value, to be made to investors at regular intervals during the term of the loan

2.2 Bond Valuations

We use the following notations in connection with the bond pricing formula that follows.

- P = the current price of a bond
- F = face value of the bond

- R = redemption/maturity value of bond
- c = nominal coupon rate
- m = number of coupon payments per year
- n = total number of coupon payments (number of years $\times m$)
- λ = nomial yield

Theorem 2.1 (Price of a Bond).
The price of a bond equals to the present value of the cash flow consisting of all coupon payments and the redemption value at maturity, calculated at yield λ.
For the case when the cash flow is made up of:

- coupon payments of $\frac{cF}{m}$ at time $t = \frac{1}{m}, \frac{2}{m}, \ldots, \frac{n}{m}$ (a total of n payments)
- redemption value R at $t = \frac{n}{m}$

We have

$$P = \frac{R}{\left(1+\frac{\lambda}{m}\right)^n} + \sum_{i=1}^{n} \frac{\frac{cF}{m}}{\left(1+\frac{\lambda}{m}\right)^i}$$

When $F = R$,

$$P = F + F\left(\frac{c-\lambda}{\lambda}\right)\left[1 - \frac{1}{\left(1+\frac{\lambda}{m}\right)^n}\right]$$

A bond is said to be priced

- at a **premium** if $P > F$
- at **par** if $P = F$
- at a **discount** if $P < F$

From the proceding bond pricing formula, it is clear

- $P > F$ if and only if $c > \lambda$
- $P = F$ if and only if $c = \lambda$
- $P < F$ if and only if $c < \lambda$

Theorem 2.2 (Makeham Formula).
Let $K = \frac{F}{(1+\frac{\lambda}{m})^n}$, we have

$$P = K + \frac{c}{\lambda}(F-K)$$

Theorem 2.3.
Let P_k be the price immediately after the k the coupon payment. Then

$$P_{k+1} = P_k\left(1+\frac{\lambda}{m}\right) - \frac{cF}{m}$$

Definition 2.2 (Zero Coupon Bonds).
Zero coupon bonds are bonds that pay no coupons. The cash flow for a N-year zero-coupon bond is the maturity value, R at $t = N$. Hence, at an annual yield of λ,

$$P = \frac{R}{(1+\lambda)^N}$$

Definition 2.3 (Perpetual Bonds).
A bond that never matures (i.e., $n \to \infty$) is called a **perpetual bond**. Clearly,

$$P = \frac{cF}{\lambda}$$

Definition 2.4 (Bond Price Between Coupon Payments).
The price of a bond traded in $t = \frac{k+\varepsilon}{m}, (0 \le \varepsilon < 1$, which is between kth and $k+1$th coupon payment dates is

$$P_{k+\varepsilon} = (1+\mu)^{\varepsilon} P_k$$

where μ is the effective annual yield of the bond over the period $[k, k+1)$.

2.3 Macaulay Duration and Modified Duration

Definition 2.5 (Macaulay Duration).
The **Macaulay duration** is one of the commonly used measures of bond's price sensitivity to changes in interest rate. For cash flow stream $\mathbf{C} = \{(c_i, t_i) \mid i = 1, 2, \ldots, n\}$, the Macaulay duration, D. is defined by

$$D = \frac{\sum_{i=1}^{n} t_i \cdot \mathrm{PV}(c_i)}{\sum_{i=1}^{n} \mathrm{PV}(c_i)}$$

Equivalently, the Macaulay duration can be defined by the weighted average time to maturity of the cash flow stream:

$$D = \sum_{i=1}^{n} w_i t_i$$

where weight $w_i = \frac{\mathrm{PV}(c_i)}{\sum_{j=1}^{n} \mathrm{PV}(c_j)}$.

Theorem 2.4 (Properties of Macaulay Duration).

- If $c_i \ge 0$ for all i, then $t_0 \le D \le t_n$.
- For a zero-coupon bond, $D = t_n$.

We can extend definition of Macaulay duration D to any infinite cash flow stream $\mathbf{C} = \{(c_i, t_i) \mid i = 1, 2, \ldots\}$

$$D = \frac{\sum_{i=1}^{\infty} t_i \cdot \mathrm{PV}(c_i)}{\sum_{i=1}^{\infty} \mathrm{PV}(c_i)}$$

Theorem 2.5 (Macaulay Duration of bonds).
For a bond that pays a total of n coupons at a frequency of m payments a year. Let the nominal bond yield be λ and nominal coupon rate be c respectively. The cash flow stream in this case is

$$\mathbf{C} = \{(\frac{cF}{m}, t_1), \ldots, (\frac{cF}{m}, t_{n-1}), (\frac{cF}{m} + F, t_n)\}$$

as $t_i = \frac{i}{m}$, so that

$$D = \frac{1}{P}\left[\sum_{i=1}^{n} \frac{i}{m} \frac{\frac{cF}{m}}{\left(1+\frac{\lambda}{m}\right)^i} + \frac{n}{m}\frac{F}{\left(1+\frac{\lambda}{m}\right)^n}\right]$$

where

$$P = \sum_{i=1}^{n} \frac{\frac{cF}{m}}{\left(1+\frac{\lambda}{m}\right)^i} + \frac{F}{\left(1+\frac{\lambda}{m}\right)^n}$$

Let $\mu = \frac{\lambda}{m}$ and $\gamma = \frac{c}{m}$, then

$$D = \frac{\sum_{i=1}^{n} \frac{i}{m} \frac{\gamma}{(1+\mu)^i} + \frac{n}{m} \frac{1}{(1+\mu)^n}}{\sum_{i=1}^{n} \frac{\gamma}{(1+\mu)^i} + \frac{1}{(1+\mu)^n}}$$

It can be shown that

$$D = \frac{1+\mu}{m\mu} - \frac{1+\mu+n(\gamma-\mu)}{m\mu + m\gamma\left[(1+\mu)^n - 1\right]}$$

As the time to maturity tends to infinity, i.e. $n \to \infty$, for a perpetual bond,

$$D = \frac{1+\mu}{m\mu}$$

Macalay duration measures the sensitivity of bond prices to interest rates.
To see this, differentiate the pricing formula, we will have

$$\frac{\mathrm{d}\, P}{\mathrm{d}\, \lambda} = \left(-\frac{1}{1+\frac{\lambda}{m}} D\right) P$$

Definition 2.6 (Modified duration)**.**
The term $\frac{1}{1+\frac{\lambda}{m}} D$ is defined as the **modified duration** and is denoted by D_{M}.

In general, for a cash flow $\mathbf{C} = \{(c_i, t_i) \mid i = 1, 2, \ldots, n\}$ at an effective annual rate of r, the relation

$$\frac{\mathrm{d}\, P}{\mathrm{d}\, r} = -D_{\mathrm{M}}$$

still holds.

Theorem 2.6 (Linear Approximation of Price Change)**.**
If $\Delta\lambda$ is a small change in λ, then

$$\Delta P = -D_{\mathrm{M}} P \Delta\lambda$$

Definition 2.7 (Duration of Bond Portfolio)**.**
Consider a bond portfolio consisting of α_i units of bond i, $i = 1, 2, \ldots, n$, assuming that the bonds have a *common* effective annual yield to maturity.
Let P_i and D_i be respectively the price and duration of bond i. Then, the **duration D_p of a portfolio** of n bonds of equal yield to maturity, λ is given by

$$D_p = \sum_{i=1}^{n} w_i D_i$$

where the **portfolio weight** $w_i = \frac{\alpha_i P_i}{\sum_{i=1}^{n} \alpha_i P_i}$

Definition 2.8 (Convexity C)**.**
Convexity of the bond C, is defined as the second derivative of the bond price with respect to bond yield, divided by the price of the bond.

$$C := \frac{\frac{\mathrm{d}^2\, P}{\mathrm{d}\, \lambda^2}}{P}$$

By Taylor series, it can be show that

$$\Delta P \approx -D_M P \Delta\lambda + \frac{1}{2} \frac{\mathrm{d}^2\, P}{\mathrm{d}\, \lambda^2} (\Delta\lambda)^2$$

Therefore,

$$\Delta P \approx P\left[-D_M \Delta\lambda + \frac{1}{2} C (\Delta\lambda)^2\right]$$

This obtains a better approximation of the change in price.
Also, from the bond pricing formula $P = \sum_{i=1}^{n} \frac{c_i}{\left(1+\frac{\lambda}{m}\right)^i}$, we have

$$\begin{aligned}
C &= \frac{\frac{\mathrm{d}^2\, P}{\mathrm{d}\, \lambda^2}}{P} \\
&= \frac{1}{Pm^2\left(1+\frac{\lambda}{m}\right)^2} \sum_{i=1}^{n} i(i+1) \frac{c_i}{\left(1+\frac{\lambda}{m}\right)^i} \\
&= \frac{F}{P}\{\frac{2c}{\lambda^3}\left(1 - \frac{1}{\left(1+\frac{\lambda}{m}\right)^n}\right) \\
&\quad - \frac{2nc}{m\lambda^2\left(1+\frac{\lambda}{m}\right)^{n+1}} - \frac{n(n+1)(c-\lambda)}{\lambda m^2\left(1+\frac{\lambda}{m}\right)^{n+2}}\}
\end{aligned}$$

2.4 Yield curves and Term Structure of Interest Rates

Definition 2.9 (Spot Rates)**.**
A **spot rate** is the *annual* interest rate that begins today ($t = 0$) and lasts until some future time t. We denote this rate by s_t.
In effect the spot rate s_t is the yield to maturity of a zero-coupon bond that matures at t.

Definition 2.10 (Forward Rate)**.**
The interest rate observed at some future time $t_1 > 0$ and lasts until a time $t_2 > t_1$ is called a **forward rate**, denoted by f_{t_1,t_2}.
Note that $f_{0,t} = s_t$

Theorem 2.7.
In general,

$$(1+s_k)^k = (1+s_j)^j (1+f_{j,k})^{k-j}$$

and

$$(1+s_n)^n = (1+s_1)(1+f_{1,2})(1+f_{2,3})\cdots(1+f_{n-1,n})$$