Revision notes - MA1104

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Contents

1 Vectors, Lines and Planes

1.1 Distance between two points

The distance, d, between two points (x_1, y_1) and (x_2, y_2) on the same plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Similarly, the distance, d, between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in xyz-space is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

1.2 Introduction to Vectors

Definition 1.1 (Vector)**.**
A vector is completely defined by two things:

- Length

- Direction

Two vectors are **equal** if they have the same **length** and the same **direction**.

Definition 1.2 (Vector Addition)**.**
Geometrically, the sum $\mathbf{u} + \mathbf{v}$ is the resulting vector that starts at the initial point of $\mathbf{u}$ and ends at the terminal point of $\mathbf{v}$ when we place the initial point of $\mathbf{v}$ at the terminal point of $\mathbf{u}$.
Equivalently, vector addition can be defined *algebraically*:
If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle, \mathbf{b} = \langle b_1, b_2, b_3 \rangle$

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

The zero vector denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

Definition 1.3 (Scalar multiple)**.**
Let $c \in \mathbb{R}$ and $\mathbf{u}$ be a vector.
The **scalar multiple** $c\mathbf{u}$ is the vector

- whose length is $|c|$ times the length of $\mathbf{u}$ and

- whose direction is the same as $\mathbf{u}$ if $c > 0$ and is opposite to $\mathbf{u}$ if $c < 0$.

If $c = 0$ or $\mathbf{u} = \mathbf{0}$, then $c\mathbf{u} = 0$.
Clearly, If $c \in \mathbb{R}$ and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, then

$$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$$

1.3 Length of Vector

Definition 1.4 (Standard Basis Vector).
The **standard basis vectors** are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$

Any 3D vector can be written as a linear combination of standard basis vectors:

$$\langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Definition 1.5 (Length of Vector).
The **length** of the vector $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is

$$\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

A **unit vector** is a vector whose length is 1.

Theorem 1.1. Let $c \in \mathbb{R}$ and $\mathbf{u}$ be a vector. Then

$$\|c\mathbf{u}\| = |c| \, \|\mathbf{u}\|$$

Theorem 1.2. If $\mathbf{u} \neq \mathbf{0}$, then a unit vector in the same direction as $\mathbf{a}$ is given by

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

1.4 Dot product and Angle

Definition 1.6 (Dot Product).
The dot product of two vectos $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is defined to be

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Theorem 1.3 (Properties of Dot Product).
For vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ and any scalar d,

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

2. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

3. $(d\mathbf{a}) \cdot \mathbf{b} = d(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (d\mathbf{b})$

4. $\mathbf{0} \cdot \mathbf{a} = 0$

5. $\mathbf{a} \cdot \mathbf{a} = \|a\|^2$

Notice $\mathbf{a} \cdot \mathbf{b} = 0$ does not imply $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

Definition 1.7 (Angle between two vectors).
For two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, we define the **angle** θ between them to be the **smaller** angle between $\mathbf{a}$ and $\mathbf{b}$ when placing their initial points together.

Clearly, $0 \leq \theta \leq \pi$.
Some special cases:

- **a** and **b** have the same direction iff $\theta = 0/$

- **a** and **b** have opposite direction iff $\theta = \pi$.

- **a** and **b** are orthogonal iff $\theta = \frac{\pi}{2}$.

Theorem 1.4 (Dot Product Angle Formula).
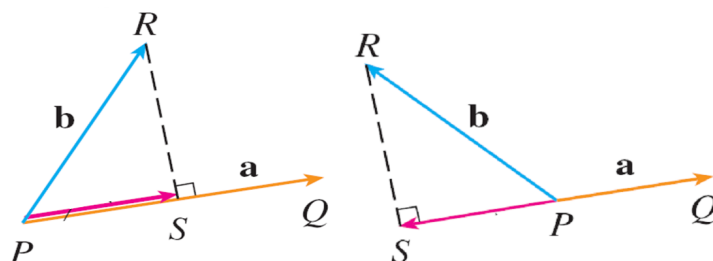Let θ be the angle between nonzero vectors **a** and **b**. Then

$$\mathbf{a} \cdot \mathbf{b} = \|a\| \, \|b\| \cos \theta$$

Theorem 1.5. Two nonzero vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

1.5   Projections

Definition 1.8 (Projection).
Let S be the foot of perpendicular line from R to the line containing $\overrightarrow{PQ}$. The vector $\overrightarrow{PS}$ is

called the **vector projection** of **b** onto **a**, denoted by

$$\text{proj}_\mathbf{a} \, \mathbf{b}$$

The **scalar projection** of **b** onto **a** is defined to be the *signed magnitude* of the vector projection:

$$\text{comp}_\mathbf{a} \, \mathbf{b} = \|\mathbf{b}\| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$$

Therefore,

$$\text{proj}_\mathbf{a} \, \mathbf{b} = \text{comp}_\mathbf{a} \, \mathbf{b} \times \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a}$$

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1.6 Cross Product

Definition 1.9 (Cross Product).
For two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, define the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ to be

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Theorem 1.6. The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

The vector $\mathbf{a} \times \mathbf{b}$ points in a direction perpendicular to $\mathbf{a}$ and $\mathbf{b}$. The direction can be given by the right-hand rule.

Theorem 1.7 (Cross product angle formula).
If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin \theta$$

Theorem 1.8 (Properties of cross product).
If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are vectors and d a scalar, then

- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

- $(d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (d\mathbf{b})$

- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

Theorem 1.9.
Suppose two adjacent sides of a parallelogram is $\mathbf{a}$ and $\mathbf{b}$, then the height is $\|\mathbf{a} \times \mathbf{b}\|$.
Suppose Q is a point and PR a line. The distance from Q to PR is

$$\left\| \overrightarrow{PQ} \right\| \sin \theta = \frac{\left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\|}{\left\| \overrightarrow{PR} \right\|}$$

1.7 Equation of a line

Definition 1.10 (Vector Equation of Line).

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R}$$

is called a **vector equation** of line, where $\mathbf{r}_0$ is coordinate vector of a point of the line and $\mathbf{v}$ a direction vector of the line.

Theorem 1.10 (Parametric Equation of Line).

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

1.8 Equation of a Plane

Theorem 1.11 (Vector Equation of Plane).

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

is a vector equation of plane, where $\mathbf{n}$ is the normal vector orthogonal to the plane and $\mathbf{r}_0$ a point on the plane.

Theorem 1.12 (Linear Equation of Plane).

$$ax + by + cz = d$$

is the linear equation of plane, where $\langle a, b, c \rangle$ is the normal vector.

Definition 1.11 (Angle between two planes).
An angle between two planes is the angle θ between their normal vectors. Notice $\pi - \theta$ is also an angle between the planes.

1.9 Vector Functions of One Variable

Definition 1.12 (Vector-valued Function).
A **vector-valued function** is

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{i}$$

The scalar function f, g, h are called the **component functions** of $\mathbf{r}$.

The vector function $\mathbf{r}(t)$ traces out the curve C. Therefore, $\mathbf{r}(t)$ is a **parametrization** of C.

1.10 Tangent Vectors

Definition 1.13 (Derivative of Vector-valued Functions).
The **derivative** of $\mathbf{r}(t)$ at $t = a$ is defined by

$$\mathbf{r}'(a) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(a + \Delta t) - \mathbf{r}(a)}{\Delta t}$$

It can be regarded as the rate of change of $\mathbf{r}(t)$ at $t = a$.

We also call $\mathbf{r}'(a)$ a **tangent vector** to the curve at $t = a$.

Theorem 1.13 (Derivative of Vector-valued Function).
Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and suppose that the components f, g, h are all differentiable at $t = a$.
Then $\mathbf{r}$ is differentiable at $t = a$ and its **derivative** is given by

$$\mathbf{r}'(a) = \langle f'(a), g'(a), h'(a) \rangle$$

Theorem 1.14 (Derivative Rules).

Suppose $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are differentiable vector-valued functions, $f(t)$ a differentiable scalar function and c is a scalar constant. Then

- $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}(t) + \mathbf{s}(t)) = \mathbf{r}'(t) + \mathbf{s}'(t)$

- $\frac{\mathrm{d}}{\mathrm{d}t}(c\mathbf{r}(t)) = c\mathbf{r}'(t)$

- $\frac{\mathrm{d}}{\mathrm{d}t}f(t)\mathbf{r}(t) = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$

- $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) \cdot \mathbf{s}(t) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r} \cdot \mathbf{s}'(t)$

- $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) \times \mathbf{s}(t) = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r} \times \mathbf{s}'(t)$

2 Functions of Two Variables, Quadric Surfaces, Limit and Continuity

2.1 Two-variable function $f(x, y)$

Definition 2.1 (Two-variable function).
A function f of two variables is a rule that assigns, to each *ordered pair* of real numbers (x, y) in a set $D \subseteq \mathbb{R}^2$, a *unique* real number denoted by $f(x, y)$.

If a function f is given by a formula and no domain is specified, then the **domain** of f is understood to be

> the set of all pairs (x, y) for which the given expression is a well-defined real number.

To visualise $f(x, y)$, we note that the graph of f is the **surface** S with equation $z = f(x, y)$. We can visualise the graph S of f lying directly above or below its domain D in the xy-plane. Visualisation can also be done through *traces*.

Definition 2.2 (Horizontal traces(level curves)).
Horizontal traces are resulting curves when we intersect the surface $z = f(x, y)$ with **horizontal** planes $z = k$.

Definition 2.3 (Vertical traces).
Vertical traces are resulting curves when we intersect the surface $z = f(x, y)$ with vertical planes $x = k$ or $y = k$.

Definition 2.4 (Level Curve).
A **level curve** of $f(x, y)$ is the **two-dimensional graph** of the equation $f(x, y) = k$ for some constant k.

Definition 2.5 (Contour Plot).
A **contour plot** of $f(x, y)$ is a graph of **numerous level curves** $f(x, y) = k$, for representative values of k.

2.2 Cylinder and Quadric Surfaces

Definition 2.6 (Cylinders).
A surface is a **cylinder** if there is a plane P such that *all* the planes parallel to P intersect the surface in the *same* curve (when viewed in 2-dimension).

In fact, any equation in x, y and z where one of the variable is missing is a cylinder.

Definition 2.7 (Quadric surface).
A **quadric surface** is the graph of a *second*-degree equation in three variables x, y and z:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, \ldots, J$ are constants.

By translation and rotation, a quadric surface can be brought into one of the two standard forms:

$$\boxed{Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0}$$

Excluding cylinders where one of the variable is missing, there are 6 basic quadric surfaces:

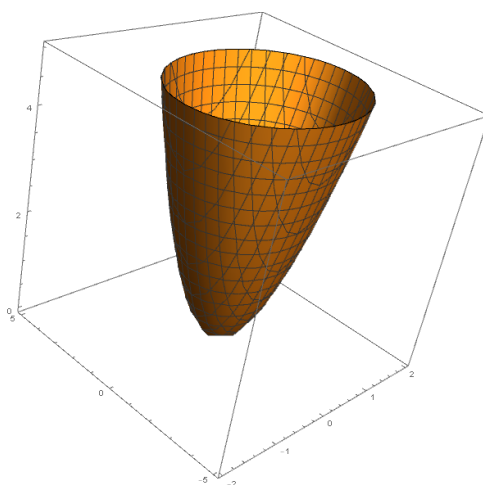| Equation	Standard form (symmetric about z-axis)
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$	Elliptic paraboloid
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$	Hyperbolic paraboloid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Ellipsoid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	Elliptic cone
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Hyperboloid of one sheet
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	Hyperboloid of two sheets

2.3 Elliptic Paraboliod

Definition 2.8 (Elliptic Paraboloid – symmetric about the z-axis).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

Horizontal traces: Ellipses
Vertical traces: Parabolas

The point $(0,0,0)$ is called the **vertex** of the elliptic paraboloid above.
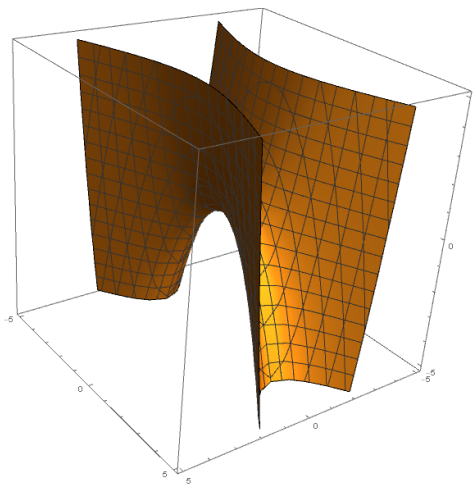The vertex will e shifted to (x_0, y_0, z_0) if the elliptic paraboloid is given by

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = \frac{(z - z_0)}{c}$$

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Definition 2.9 (Hyperbolic paraboloid – symmetric about the z-axis)**.**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

Horizontal traces: Hyperbolas
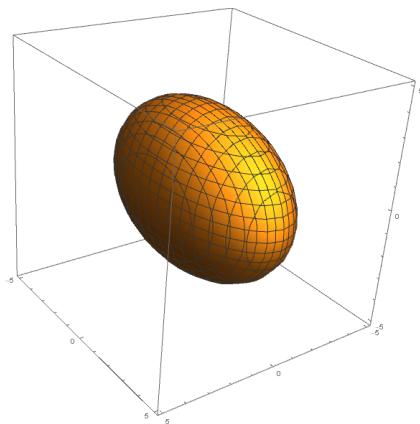**Vertical** traces: Parabolas

2.4   Ellipsoid, Cones and Hypeboloid

Definition 2.10 (Ellipsoid)**.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Horizontal traces: Ellipses
Vertical traces: Ellipses

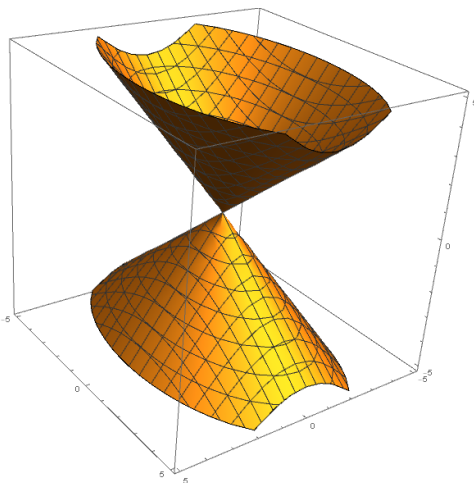**Definition 2.11** (Elliptic cone – symmetric about the z-axis)**.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Horizontal traces: Ellipses
Vertical traces: Hyperbolas

Definition 2.12 (Hyperboloid of one sheet – symmetric about the z-axis)**.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Horizontal traces: Ellipses
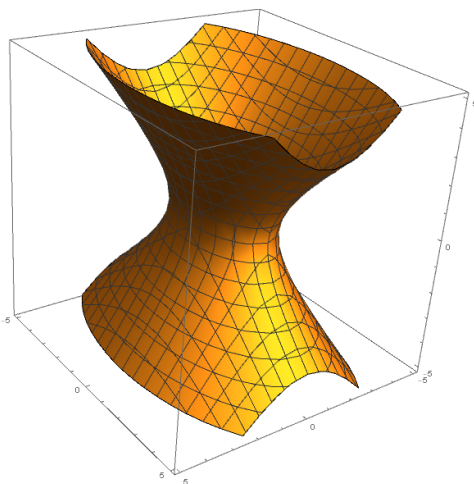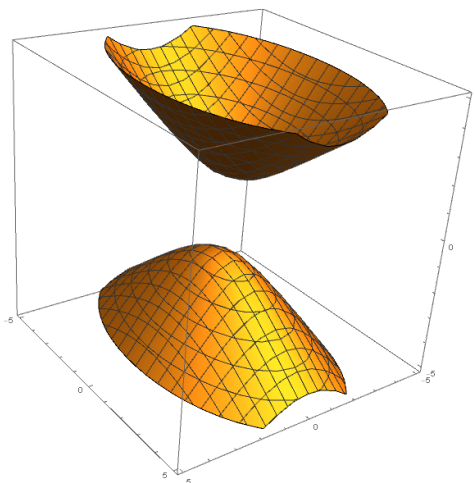**Vertical traces**: Hyperbolas

Definition 2.13 (Hyperboloid of two sheets – symmetric about the z-axis).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

Horizontal traces: Ellipses
Vertical traces: Hyperbolas

2.5 Function of three Variables

Definition 2.14.
A function f of three variables is a rule that assigns, to each **ordered triple** of real numbers (x, y, z) in a set $D \subseteq \mathbb{R}^3$, a *unique* real number denoted by $f(x, y, z)$.

Definition 2.15 (Level Surface).
A **level surface** of $f(x, y, z)$ is the three dimensional graph of the equation $f(x, y, z) = k$ for some constant k.

2.6 Limit of $f(x, y)$

Definition 2.16 (Limit).
Let f be a function of two variables whose domain D contains points arbitrarily close to (a, b). We say that the **limit** of $f(x, y)$ as (x, y) approaches (a, b) is $L \in \mathbb{R}$, denoted by

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if for any number $\varepsilon > 0$ there exists a number $\delta > 0$ such that $|f(x, y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

Remark: f is not required to be defined at (a, b).
It can be proven from the definition that if $\lim_{(x,y) \to (a,b)} f(x, y) = L$, then

- its value L is *unique*, and

- L is *independent* of the choice of path approaching (a, b).

2.7 How to show limit does not exist

Theorem 2.1.
If $f(x, y)$ approaches L_1 as (x, y) approaches (a, b) along a path P_1 and approaches L_2 as (x, y) approaches (a, b) along a path P_2, and $L_1 \neq L_2$, then

$$\lim_{(x,y) \to (a,b)} f(x, y)$$

does **not** exist.

In general, some of the paths that passes through a given point (a, b) to try include:

- $x = a, y \to b$ (vertical lines)

- $y = b, x \to a$ (horizontal lines)

- $y = g(x), x \to a$, where $g(x)$ is some simple function (usually linear and quadratic) such that $g(a) = b$.

- $x = g(y), y \to b$, where $g(x)$ is some simple function (usually linear and quadratic) such that $g(b) = a$.

2.8 How to show limit exists

To show limit exists:

- we can deduce it from known/simple functions using **properties of limit or continuity**; or

- we can use **squeeze theorem**

Theorem 2.2 (Limit Theorems)**.**
Suppose $f(x, y)$ and $g(x, y)$ both have limits as (x, y) approaches (a, b). Then

$$\lim_{(x,y) \to (a,b)} (f(x, y) \pm g(x, y)) = \lim_{(x,y) \to (a,b)} f(x, y) \pm \lim_{(x,y) \to (a,b)} g(x, y)$$

$$\lim_{(x,y) \to (a,b)} f(x, y)g(x, y) = \left(\lim_{(x,y) \to (a,b)} f(x, y) \right) \left(\lim_{(x,y) \to (a,b)} g(x, y) \right)$$

$$\lim_{(x,y) \to (a,b)} \frac{f(x, y)}{g(x, y)} = \frac{\lim_{(x,y) \to (a,b)} f(x, y)}{\lim_{(x,y) \to (a,b)} g(x, y)}$$

provided

$$\lim_{(x,y) \to (a,b)} g(x, y) \neq 0$$

Theorem 2.3 (Squeeze)**.**
Suppose

- $|f(x, y) - L| \leq g(x, y) \quad \forall (x, y)$ close to (a, b)

- $\lim_{(x,y) \to (a,b)} g(x, y) = 0$

Then,

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

2.9 Continuity of $f(x, y)$

Definition 2.17 (Continuity)**.**
We say f is **continuous at** (a, b) if

$$\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$$

Theorem 2.4 (Continuity Theorems)**.**
If $f(x, y)$ and $g(x, y)$ are continuous at (a, b), then

- $f \pm g$ is continuous at (a, b).

- $f \cdot g$ is continuous at (a, b).

- $\frac{f}{g}$ is continuous at (a, b), provided $g(a, b) \neq 0$.

Theorem 2.5 (Continuity of Composite Function)**.**
Suppose $f(x, y)$ is continuous at (a, b) and $g(x)$ is continuous at $f(a, b)$. Then

$$h(x, y) = (g \circ f)(x, y) = g(f(x, y))$$

is continuous at (a, b).

Subsequently, the following classes of functions are continuous **in its domain**.

- Polynomial in x and y.

- Trigonometric and exponential functions in x and y.

- Rational function in x and y.

3   Partial Derivatives, Chain Rule, Directional Derivatives

3.1   Partial Derivative

Definition 3.1 (Partial Derivative)**.**
If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by:

$$f_x(x, y) = \lim_{h=0} \frac{f(x + h, y) - f(x, y)}{h}$$
$$f_y(x, y) = \lim_{h=0} \frac{f(x, y + h) - f(x, y)}{h}$$

Other notations for partial derivatives:

$$f_x = \frac{\partial f}{\partial x} \quad f_y = \frac{\partial f}{\partial y}$$

3.2   Higher Order Partial Derivatives

Definition 3.2 (Second partial derivatives)**.**
Second partial derivatives of f is the partial derivatives of partial derivatives of f, i.e.

$$(f_x)_x, (f_x)_y, (f_y)_x, (f_y)_y$$

We use the following notation:

$$(f_x)_x = f_{xx} \ \ = \frac{\partial^2 f}{\partial x^2}$$
$$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$
$$(f_y)_x = f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$
$$(f_y)_y = f_{yy} \ \ = \frac{\partial^2 f}{\partial y^2}$$

Thus, the notation f_{xy} means that we *first* differentiate with respect to x and *then* with respect to y.

Theorem 3.1 (Clairaut's Theorem)**.**
Suppose f is defined on a disk D that contains (a, b). If the functions f_{xy} and f_{yx} are both *continuous* on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

In fact, so long as the number of the same variable occurring in the subscript are the same, the coresponding partial derivatives are the same.

3.3 Tangent Plane Equation

Definition 3.3 (Tangent Plane).
The **tangent plane** to the surface S at the point $P(a, b, c)$ is defined to be the plane that contains both tangent lines T_1 and T_2, where T_1 and T_2 are the tangent lines to the curves of intersections of the surface S and the vertical planes $y = b$ and $x = a$ respectively.
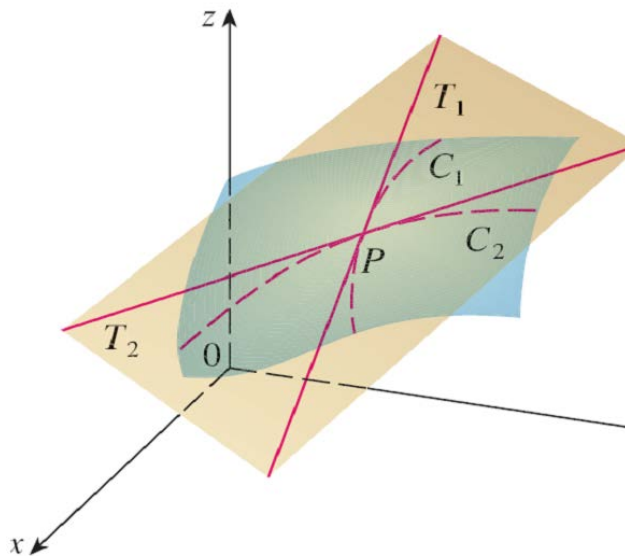
From the definition, we note that two vectors on the tangent plane are $\langle 1, 0, f_x(a, b) \rangle$ and $\langle 0, 1, f_y(a, b) \rangle$. Thus, a normal vector to the plane is

$$\mathbf{n} = \langle f_x(a, b), f_y(a, b), -1 \rangle$$

Theorem 3.2 (Equation of Tangent Plane).
Consider the surface S given by $z = f(x, y)$. A normal vector to the tangent plane to S at (a, b) is

$$\langle f_x(a, b), f_y(a, b), -1 \rangle$$

The tangent plane is given by

$$\langle x - a, y - b, z - f(a, b) \rangle \cdot \langle f_x(a, b), f_y(a, b), -1 \rangle = 0$$

Equivalently,

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

3.4 Differentiability of $f(x, y)$

In general, for $f(x, y)$ we have

$$\boxed{f \text{ differentiable} \Rightarrow f_x \text{ and } f_y \text{ exist}}$$

To define differentiability, we first define **increment**.

Definition 3.4 (Increment).
Let $z = f(x, y)$. Suppose Δx and Δy are increments in the *independent* variable x and y respectively from a fixed point (a, b). Then the **increment** in z at (a, b), Δz, is defined by

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Definition 3.5 (Differentiability - Two Variable).
Let $z = f(x, y)$. We say that f is **differentiable** at (a, b) if we can write

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and ϵ_2 are functions of Δx and Δy which vanish (i.e. $\epsilon_1, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$).[1]

We say that f is **differentiable on a region** $R \in \mathbb{R}^2$ if f is differentiable at every point in R.

3.5 Linear Approximation

Theorem 3.3 (Linear Approximation - Two Variable).
Suppose $z = f(x, y)$ is *differentiable* at (a, b). Let Δx and Δy be small increments in x and y respectively from (a, b). Then

$$\Delta z \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

This result can be extended to functions of more variables.

3.6 Chain Rule

Theorem 3.4 (Chain Rule - Case 1).
Suppose that $z = f(x, y)$ is a *differentiable* function of x and y, where $x = g(t)$ and $y = h(t)$ are both *differentiable* functions of t. Then, z is a **differentiable** function of t and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Theorem 3.5 (Chain Rule - Case 2).
Suppose that $z = f(x, y)$ is a *differentiable* function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both *differentiable* functions of s and t. Then, z is a **differentiable** function of s and t and

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Here there are three types of variables:

---

[1]Rearranging $\Delta z = (f_x(a, b) + \epsilon_1)\Delta x + (f_y(a, b) + \epsilon_2)\Delta y$, we will see the function is differentiable if the directional derivative at $(a, b, f(a, b))$ is will estimated in all direction when $\Delta x, \Delta y) \to 0$, which suggests that the tangent vector in all direction at $(a, b, f(a, b))$ will contain in the tangent plane.

- s and t are **independent** variables.

- x and y are called **intermediate** variables.

- z is the **dependent** variable.

Theorem 3.6 (Chain Rule - General Version)**.**
Suppose that u is a differentiable function of n variables $x_1, \ldots, x_n$, and each x_j is a diffe-rentiable function of m variables $t_1, \ldots, t_m$. Then u is a function of $t_1, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

3.7 Implicit Differentiation

Definition 3.6 (Implicit Function)**.**
z is an **implicit function** of x and y defined by $F(x, y, z) = 0$ if

> for every choice of x and y, the value of z is determined by $F(x, y, z) = 0$

Theorem 3.7 (Implicit Differentiation: Two Independent Variables)**.**
Suppose the equation $F(x, y, z) = 0$, where F is *differentiable*, defines z **implicitly** as a differentiable function of x and y. Then,

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}, \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$$

provided $F_z(x, y, z) \neq 0$.

3.8 Directional Derivatives

Definition 3.7 (Directional Derivative)**.**
The **directional derivative** of $f(x, y)$ at (x_0, y_0) in the direction of **unit** vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

The idea of directional derivative can be extended to functions of more variables.

Theorem 3.8 (Computing Directional Derivatives)**.**
If $f(x, y)$ is a *differentiable* function, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x_0, y_0) = \langle a, b \rangle \cdot \mathbf{u}$$

Definition 3.8 (Gradient)**.**
The **gradient** of $f(x, y)$ is the vector-valued function

$$\nabla f(x, y) = \langle f_x, f_y \rangle$$

provided that both partial derivatives exist.

4 Gradient Vector, Extrema, Langrange Multiplier

4.1 Gradient Vector and Level Curve

Theorem 4.1 (Level Curve vs ∇f)**.**
Suppose $f(x,y)$ is differentiable function of x and y at (x_0, y_0). Suppose $\nabla f(x_0, y_0) \neq \mathbf{0}$
Then $\nabla f(x_0, y_0)$ is **normal** to the level curve $f(x,y) = k$ that contains the point (x_0, y_0).

4.2 Gradient Vector and Level Surface

Theorem 4.2 (Level Surface vs ∇F)**.**
Suppose $F(x,y,z)$ is diffeerentiable function of x, y and z at (x_0, y_0, z_0). Suppose $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.
Then $\nabla F(x_0, y_0, z_0)$ is **normal** to the level surface $F(x,y,z) = k$ that contains the point (x_0, y_0, z_0).

Theorem 4.3 (Tangent Plane to Level Surface)**.**
The tangent plane to the level surface $F(x,y,z) = k$ on which (x_0, y_0, z_0) resides is given by

$$\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

4.3 Maximum/Minimum Rate of Change

At a given point (x_0, y_0, z_0), the rate of change of $f(x,y,z)$ is given by

$$\begin{aligned} D_{\mathbf{u}}f &= \nabla f \cdot \mathbf{u} \\ &= \|\nabla f\| \, \|\mathbf{u}\| \cos\theta \\ &= \|\nabla\| \cos\theta \end{aligned}$$

where θ is the ange between ∇f and $\mathbf{u}$.

Theorem 4.4 (Maximising rate of Increase/Decrease of f)**.**
Suppose f is a differentiable function of two or three variables. Let P denote a given point.
Assume $\nabla f(P) \neq \mathbf{0}$.

- $\nabla f(P)$ points in the direction of maximum rate of change of f at P, $\|\nabla f(P)\|$.

- $-\nabla f(P)$ points in the direction of minimum rate of change of f at P, $-\|\nabla f(P)\|$.

4.4 Critical Points of $f(x,y)$

Definition 4.1 (Local Maximum)**.**
Let $f(x,y) : D \to \mathbb{R}$. Then f has a **local maximum** at (a,b) if

$$f(x,y) \leq f(a,b) \text{for all points close to } (a,b)$$

The number $f(a,b)$ is called a local maximum value.

Definition 4.2 (Local Minimum).
Let $f(x, y) : D \to \mathbb{R}$. Then f has a **local minimum** at (a, b) if

$$f(x, y) \geq f(a, b) \text{ for all points close to } (a, b)$$

The number $f(a, b)$ is called a local minimum value.

Theorem 4.5 (A necessary condition).
If f has a local maximum or minimum at (a, b) and the first-order derivatives of f exist there, then

$$f_x(a, b) = f_y(a, b) = 0$$

Definition 4.3 (Saddle Point).
Let $f(x, y) : D \to \mathbb{R}$. Then a point (a, b) is called a **saddle point** of f if

1. $f_x(a, b) = f_y(a, b) = 0$; and

2. every neighbourhood at (a, b) contains points $(x, y) \in D$ for which $f(x, y) < f(a, b)$ and points $(x, y) \in D$ for which $f(x, y) > f(a, b)$.

4.5 Finding Absolute Maximum/Minimum

Definition 4.4 (Absolute Maximum).
Let $f(x, y) : D \to \mathbb{R}$. Then f has an **absolute maximum** at (a, b) if

$$f(x, y) \leq f(a, b) \text{ for all points in the domain } D$$

The number $f(a, b)$ is called a **absolute maximum value**.

Definition 4.5 (Absolute Minimum).
Let $f(x, y) : D \to \mathbb{R}$. Then f has an **absolute minimum** at (a, b) if

$$f(x, y) \geq f(a, b) \text{ for all points in the domain } D$$

The number $f(a, b)$ is called a **absolute minimum value**.

Definition 4.6 (Closed Set in $\mathbb{R}^2$).
A set $R \subseteq \mathbb{R}^2$ is **closed** if it contains all its boundary points.
A **boundary point** of R is a point (a, b) such that every disk with center (a, b) contains point in R and also points in $\mathbb{R}^2 \setminus R$.

Definition 4.7 (Bounded Set in $\mathbb{R}^2$).
A set $R \subseteq \mathbb{R}^2$ is **bounded** if it is contained within some disk. In other words, it is finite in extent.

Theorem 4.6 (Extreme Value Theorem).
If $f(x, y)$ is continuous on a closed and bounded set $D \subseteq \mathbb{R}^2$, then f attains

- an absolute maximum value $f(x_1, y_1)$ and

- an absolute minimum value $f(x_2, y_2)$

at some points (x_1, y_1) and (x_2, y_2) in D.

Theorem 4.7 (Closed Interval Method).
The following is the **closed interval method** for finding absolute maximum and minimum

Step 1 Find the values of f at its **critical points** in D.

Step 2 Find the extreme values of f on the **boundary** of D.

Step 3 The **largest**(resp. **smallest**) of the values from **Step 1** and **Step 2** is the **absolute maximum**(resp. **absolute minimum**).

4.6 Lagrange Multiplier – 2-Variable Case

Theorem 4.8 (Lagrange Multipliers for Function of Two Variables).
Suppose $f(x, y)$ and $g(x, y)$ are differentiable functions such that $\nabla g(x, y) \neq \mathbf{0}$ on the constraint curve $g(x, y) = k$.
Suppose that the **minimum/maximum** value of $f(x, y)$ subject to the constraint $g(x, y) = k$ occurs at (x_0, y_0). Then

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

for some constant λ.
The following are the steps of the method of Lagrange Multiplier for two variable functions:

Step 1 Find all values of x, y and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

and

$$g(x, y) = k$$

Step 2 Evaluate f at all points obtained in **Step 1**.

- The largest of these values is the maximum value of f;
- The smallest is the minimum value of f.

This theorem can be extended to functions of three variables.

5 Double Integral over region on the xy–plane

Definition 5.1 (Double Integral over Rectangle).
The **double integral** of f over the **rectangle** R is

$$\iint_R f(x,y)\mathrm{d}A = \lim_{m,n\to\infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*)\Delta A$$

provided that the limit exists and is the same for any choice of the sample points (x_{ij}^*, y_{ij}^*) in $R_{ij} := [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ whose region is $\Delta A = \Delta x \times \Delta y$, for $1 \le i \le m, 1 \le j \le n$.

Definition 5.2 (Double Integral over General Region).
Double integral of f over general region D is defined by

$$\iint_D f(x,y)\mathrm{d}A = \iint_R F(x,y)\mathrm{d}A$$

where

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R \setminus D \end{cases}$$

Double integrals are computed by means of iterated integrals.

Definition 5.3 (Iterated Integral).
The **iterated double integral** of f on the rectangle $R = [a,b] \times [c,d]$ in the **order** $\mathrm{d}y\mathrm{d}x$ is defined to be

$$\int_a^b \left(\int_c^d f(x,y)\mathrm{d}y \right) \mathrm{d}x$$

The **iterated double integral** of f on the rectangle $R = [a,b] \times [c,d]$ in the **order** $\mathrm{d}x\mathrm{d}y$ is defined to be

$$\int_c^d \left(\int_a^b f(x,y)\mathrm{d}x \right) \mathrm{d}y$$

Theorem 5.1 (Fubini's Theorem).
If f is **continuous** on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\mathrm{d}A = \int_a^b \int_c^d f(x,y)\mathrm{d}y\mathrm{d}x = \int_c^d (\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y$$

Definition 5.4 (Type I Region).
A plane region D is said to be of **Type I** if it lies between the graphs of two continuous functions of x, that is

$$D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where $g_1(x)$ and $g_2(x)$ are continuous on $[a,b]$.

Definition 5.5 (Type II Region)**.**
A plane region D is said to be of **Type II** if it lies between the graphs of two continuous functions of y, that is

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

where $h_1(y)$ and $h_2(y)$ are continuous on $[c, d]$.

Theorem 5.2 (Double Integral over Type I Domain)**.**
If f is continuous on a **Type I** domain D such that

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

then

$$\iint_D f(x, y)\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\mathrm{d}y\mathrm{d}x$$

Theorem 5.3 (Double Integral over Type II Domain)**.**
If f is continuous on a **Type II** domain D such that

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

then

$$\iint_D f(x, y)\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\mathrm{d}x\mathrm{d}y$$

Theorem 5.4 (Additivity with respect to Domain)**.**

$$\iint_D f(x, y)\mathrm{d}A = \sum_{i=1}^n \iint_{D_i} f(x, y)\mathrm{d}A$$

The theorem above allows us to decompose the domain into finitely many domains of Type I or II.

6 Double Integral over Polar Regions and Triple Integrals

Theorem 6.1 (Relationship between polar coordinates (r, θ) and cartesian coordinates (x, y)).

$$x = r \cos \theta \quad y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1} \frac{y}{x}, \ \textit{provided } x \neq 0$$

Definition 6.1 (Polar Rectangle).
A polar rectangle is a region

$$R = \{(r, \theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

Theorem 6.2 (Change to Polar Coordinates in Double Integrals).
If f is continuous on a polar rectangle R given by

$$R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x, y) \mathrm{d}A = \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta) r \mathrm{d}r \mathrm{d}\theta$$

Definition 6.2 (General Polar Regions).
General polar regions come in two different forms:

$$D_1 = \{(r, \theta) : a \leq r \leq b, g_1(r) \leq \theta \leq g_2(r)\}$$

or

$$D_2 = \{(r, \theta) : \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

Theorem 6.3.
If f is continuous on a polar regions D_1, then

$$\iint_D f(x, y) \mathrm{d}A = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r \cos \theta, r \sin \theta) r \mathrm{d}\theta \mathrm{d}r$$

If f is continuous on a polar region D_2, then

$$\iint_D f(x, y) \mathrm{d}A = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r \mathrm{d}r \mathrm{d}\theta$$

Definition 6.3 (Rectangular box).
A rectangular box is defined by

$$B = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$

Theorem 6.4 (Fubini's Theorem for Triple Integral).
If f is continuous on the rectangular box B, then

$$\iiint_B f(x, y, z) \mathrm{d}V = \int_r^s \int_c^d \int_a^b f(x, y, z) \mathrm{d}x \mathrm{d}y \mathrm{d}z$$

Furthermore, the iterated integral may be evaluated in any order.

7 Line Integrals

7.1 Line Integral of Scalar Field

Definition 7.1 (Line Integrals of Scalar Field).
If f is defined on a smooth curve C given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle, a \leq t \leq b$. Then the **line integral** of f along C is

$$\int_C f(x, y) \mathrm{d}s = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$

provided this limit exists and is the same for every choice of (x_i^*, y_i^*).

Theorem 7.1 (Formula for evaluation of Line Integral of Scalar Field).
Suppose C is given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle, a \leq t \leq b$, then

$$\int_C f(x, y) \mathrm{d}s = \int_a^b f(x(t), y(t)) \left\| \mathbf{r}'(t) \right\| \mathrm{d}t$$

$$= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t} \right)^2} \mathrm{d}t$$

Similarly, suppose that C is smooth space curve given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, a \leq t \leq b$. Then

$$\int_C f(x, y, z) \mathrm{d}s = \int_a^b f(x(t), y(t), z(t)) \left\| \mathbf{r}'(t) \right\| \mathrm{d}t$$

$$= \int_C f(x(t), y(t), z(t)) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t} \right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t} \right)^2} \mathrm{d}t$$

Remark: If $f(x, y) = 1$ for all x, y, the line integral formula evaluates the **arc length** of C. More generally, one can interpret the line integral $\int_C f(x, y) \mathrm{d}s$ as the **area** of the fence, whose base is C and height above the point (x, y) is $f(x, y)$.

Theorem 7.2 (Piecewise Parametrisation).
Suppose C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where the initial point of C_{i+1} is the terminal point of C_i. Then

$$\int_C f(x, y) \mathrm{d}s = \sum_{i=1}^{n} \int_{C_i} f(x, y) \mathrm{d}s$$

Theorem 7.3 (Parametrisation of Line Segment).
A vector parametrisation of line segment that starts from $\mathbf{r}_0$ and ends at $\mathbf{r}_i$ is given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0), \quad 0 \leq t \leq 1$$

7.2 Vector Field

Definition 7.2 (Vector field). *Let $D \subseteq \mathbb{R}^2$. A **vector field** on D is a function $\mathbf{F}$ that assigns to each point $(x, y) \in D$ a 2D vector $\mathbf{F}(x, y)$.*
*Let $D \subseteq \mathbb{R}^3$. A **vector field** on D is a function $\mathbf{F}$ that assigns to each point $(x, y, z) \in D$ a 3D vector $\mathbf{F}(x, y, z)$.*

Theorem 7.4 (Vector Field in Component Form).
For vector field $\mathbf{F}$ on $\mathbb{R}^2$,

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y) \rangle$$

or for short

$$\mathbf{F} = \langle P, Q \rangle$$

Similarly, for vector field on $\mathbf{F}$ on $\mathbb{R}^3$,

$$\mathbf{F} = \langle P, Q, R \rangle$$

Definition 7.3 (Line Integral of Vector Field).
Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, a \le t \le b$.
Then, the **line integral** of $\mathbf{F}(x, y, z)$ along C is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} := \int_C \mathbf{F} \cdot \mathbf{T}\mathrm{d}s = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\mathrm{d}t$$

Theorem 7.5.
We have

$$\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Theorem 7.6 (Component Form).
If $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ then we have

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$$

Sometimes a curve C is a union of finitely many smooth curves $C_1, \ldots, C_n$. In such cases,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \sum_{i=1}^n \int_{C_i} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

7.3 Conservative Vector Field

Definition 7.4 (Conservative Vector Field).
A vector field $\mathbf{F}$ is **conservative vector field** on D if we can write

$$\mathbf{F} = \nabla f$$

for some scalar function f on D.

The function f is called the **potential function** of $\mathbf{F}$.

Theorem 7.7 (Recovering f from $\mathbf{F}$)**.**

- Suppose $\mathbf{F} = \langle P, Q \rangle$, we have $f_x = P, f_y = Q$.

- Integrating f_x with respect to x, we have

$$f(x, y) = j(x, y) + h(y)$$

where $\frac{\partial}{\partial x} j(x, y) = f_x$.

- Differentiate $f(x, y)$ with respect to y, we have

$$f_y = \frac{\partial}{\partial y} j(x, y) + h'(y)$$

and by comparing the above expression with Q, we have

$$g'(y) = Q - \frac{\partial}{\partial y} j(x, y)$$

- Integrating $g'(y)$ with respect to y, we will have $g(y)$, which we substitute back to obtain $f(x, y)$.

Theorem 7.8 (Test for Conservative Field)**.**
Suppose $\mathbf{F}(x, y) = \langle P, Q \rangle$ is a vector field in an *open* and *simply-connected* region D and both P and Q have continuous first-order partial derivatives on D. Then

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

at *each* point (x, y) in D if and only if $\mathbf{F}$ is **conservative** on D.
Similarly, suppose $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ is a vector field in an open and simply-connected region D in space and both P, Q, R have continuous first-order partial derivatives on D. Then

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$$
$$\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$
$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

for each point (x, y, z) in D if and only if $\mathbf{F}$ is **conservative** on D.

7.4 Fundamental Theorem for Line Integral

The **fundamental theorem for line integrals** says that we can evaluate the line integral of a **conservative** vector field by only knowing the **potential function** at the **endpoint**.

Theorem 7.9 (Fundamental Theorem for Line Integrals)**.**
Suppose $\mathbf{F}$ is a conservative vector field with **potential function** f, and C a smooth curve with *initial* point A and *terminal* point B. Then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(B) - f(A)$$

The fundamental theorem of line integral says that the line integral of conservative field is **independent** of the paths with the same initial point and terminal point.

Theorem 7.10 (Another test for non-conservativeness)**.**
Vector field is **not** conservative if there are **two** paths with the same initial and terminal points but their line integrals are *different*.

8 Green's Theorem and Surface Integral of Scalar Field

8.1 Green's Theorem

There are three methods of computing surface integral, each of different applicability.
Method 1: Parametrize C by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, a \leq t \leq b$, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\mathrm{d}t$$

Method 2: If $\mathbf{F}$ is conservative, then apply Fundamental Theorem for Line Integrals:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(B) - f(A)$$

where $\mathbf{F} = \nabla f$, for some scalar function f, B is the terminal point of C and A the initial point. **Method 3**: **Green's theorem**.

Theorem 8.1 (Green's Theorem).
Let C be a positively oriented[2], piecewise-smooth, simply closed curve in the plane and let D be the region bounded by C.
Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$.
If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region that contains D, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A$$

Green's theorem can be considered as the Fundamental Theorem of Calculus for double integrals.

8.2 Application of Green's Theorem on Area of Plane Region

To evaluate the area of a plane region $\iint_D 1\mathrm{d}A$, we construct P and Q such that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

Therefore, we have the following possibilities

$$\iint_D 1\mathrm{d}A = \int_C x\mathrm{d}y = \int_C -y\mathrm{d}x = \frac{1}{2}\int_C x\mathrm{d}y - y\mathrm{d}x$$

8.3 Parametric Surfaces

Definition 8.1 (Parametric Surface).
The 2-variable vector function

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle, (u, v) \in D$$

parametrizes a surface S in the xyz-plane.

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[2]A curve has **positive orientation** if it is traversed in counterclockwise direction.

There are usually more than one ways to parametrize the surface. One special case is the parametrization of the sphere of radius *rho* centered at origin:

$$x = \rho \sin \phi \cos \theta$$
$$y = \rho \sin \phi \sin \theta$$
$$z \rho \cos \phi$$

8.4 Surface Integral of Scalar Field

Definition 8.2 (Surface Integral of Scalar Field).

$$\iint_S f(x, y, z)\mathrm{d}S = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

provided the limit exists and is the same for all choices of the evaluation points P_{ij}^*.

We also require the parametrix surface to be smooth.

Definition 8.3 (Smooth surface).
A surface S is **smooth** if it has a parametrization $\mathbf{r}(u, v), u, v \in D$, such that $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ for all $(u, v) \in D$.

Theorem 8.2 (Two important properties of $\mathbf{r}_u$ and $\mathbf{r}_v$).

1. $\mathbf{r}_u \times \mathbf{r}_v$ is normal to the surface D.

2. We can approximate the patch area ΔS_{ij} as follows:

$$\Delta S_{ij} \approx \|\mathbf{r}_u \times \mathbf{r}_v\| \Delta u \Delta v$$

From the two properties, we arrive at two useful theorems.

Theorem 8.3 (Tangent Plane of Smooth Surface).
Suppose a surface S is smooth and has a parametrization of $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle, (u, v) \in D$.
Then $\mathbf{r}_u(a, b) \times \mathbf{r}_v(a, b)$ is normal to the tangent plane of S at the point $(x(a, b), y(a, b), z(a, b))$.

Theorem 8.4 (Surface Integral Formula).

$$\iint_S f(x, y, z)\mathrm{d}S = \iint_D f(x(u, v), y(u, v), z(u, v)) \|\mathbf{r}_u \times \mathbf{r}_v\| \mathrm{d}A$$

If S is instead a **union** of finitely many smooth surfaces $S_1, \ldots, S_n$ that intersect only along their boundaries, then the surface integral of f over S is the sum of the surface integrals over each S_i.

Theorem 8.5.

$$\iint_S f(x, y, z)\mathrm{d}S = \sum_{i=1}^{n} \iint_{S_i} f(x, y, z)\mathrm{d}S$$

8.5 Surface Area

A special case of parametrization is that the surface is given by $z = g(x, y), (x, y) \in D$. Then we can take x, y as parameters

$$\mathbf{r}(x, y) = \langle x, y, g(x, y) \rangle$$

In this case we have a special formula for surface integral.

Theorem 8.6.

$$\iint_S f(x, y, z)\mathrm{d}S = \iint_D f(x, y, g(x, y))\sqrt{\left(\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1\right)}\mathrm{d}A$$

On a irrelevant note, surface area of any parametricx surface D traced by $\mathbf{r}(u, v)$ can be evaluated as

Theorem 8.7 (Surface Area).

$$A(S) = \iint_S 1\mathrm{d}S = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\,\mathrm{d}A$$

9 Surface Integral of Vector Field

9.1 Surface with Orientation

Definition 9.1 (Oriented Surface).
A surface S is **orientable** (or **two sided**) if it is possible to define a unit normal vector $\mathbf{n}$ at each point (x, y, z) (excluding boundary) of the surface such that $\mathbf{n}$ is a continuous function of (x, y, z).

There are two possible orientations for any orientable surfaces, as S will have two identifiable sides:

- **Open surface**: a top and a bottom

- **Closed surface**: an inside and an outside

Theorem 9.1 (Unit normal vector $\mathbf{n}$ of $\mathbf{r}(u, v)$).
One unit normal vector, which specifies a particular orientation of the parametrized surface $\mathbf{r}(u, v)$ is given by

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

and the opposite orientation is given by

$$-\mathbf{n} = \frac{\mathbf{r}_v \times \mathbf{r}_u}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

Theorem 9.2 (Unit normal vector $\mathbf{n}$ of $z = g(x, y)$).
When S is given by $z = g(x, y)$, the **upward** orientation of the surface is given by

$$\mathbf{n} = \frac{\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \rangle}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}$$

and the **downward** orientation is given by

$$-\mathbf{n} = \frac{\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1 \rangle}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}$$

Definition 9.2 (Positive Orientation for Closed Surfaces).
For a **closed** surface that is the boundary of a solid region E, the convention is that:
The **positive orientation** is the one for which the normal vectors point **outward** from E.
Inward pointing normals give the **negative orientation**.

9.2 Surface Integral of Vector Field

Definition 9.3 (Surface Integral of Vector Field).
If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with a unit normal vector $\mathbf{n}$, then the **surface integral** of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} := \iint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S$$

The integral is also called the **flux** of $\mathbf{F}$ across S.

Theorem 9.3 (Computation of Surface Integral over Vector Field).
To compute the integral, we first parametrize surface S by $\mathbf{r}(u,v)$. Then either $\frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$ or $\frac{\mathbf{r}_v \times \mathbf{r}_u}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$ will be chosen, depending on the given $\mathbf{n}$.
Suppose $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$, then

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}\,\mathrm{d}S$$
$$= \iint_D \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}\,\|\mathbf{r}_u \times \mathbf{r}_v\|\,\mathrm{d}A$$
$$= \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,\mathrm{d}A$$

where $\mathbf{r}_u \times \mathbf{r}_v$ has the same direction of $\mathbf{n}$.

Theorem 9.4.
Let $\mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$. Suppose S is given by

$$\mathbf{r}(x,y) = \langle x, y, g(x,y) \rangle \quad (x,y) \in D$$

Then the flux across S in the **upward** orientation is given by

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D \langle P, Q, R \rangle \cdot \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \rangle \mathrm{d}A$$

The flux across S in the **negative** orientation is given by

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D \langle P, Q, R \rangle \cdot \langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1 \rangle \mathrm{d}A$$