Revision notes - MA1104

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1 Vectors, Lines and Planes

1.1 Distance between two points

The distance, d, between two points (x_1, y_1) and (x_2, y_2) on the same plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Similarly, the distance, d, between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in xyz-space is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

1.2 Introduction to Vectors

Definition 1.1 (Vector)**.**
A vector is completely defined by two things:

- Length
- Direction

Two vectors are **equal** if they have the same **length** and the same **direction**.

Definition 1.2 (Vector Addition)**.**
Geometrically, the sum $\mathbf{u} + \mathbf{v}$ is the resulting vector that starts at the initial point of $\mathbf{u}$ and ends at the terminal point of $\mathbf{v}$ when we place the initial point of $\mathbf{v}$ at the terminal point of $\mathbf{u}$.
Equivalently, vector addition can be defined *algebraically*:
If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$,$\mathbf{b} = \langle b_1, b_2, b_3 \rangle$

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

The zero vector denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

Definition 1.3 (Scalar multiple)**.**
Let $c \in \mathbb{R}$ and $\mathbf{u}$ be a vector.
The **scalar multiple** $c\mathbf{u}$ is the vector

- whose length is $|c|$ times the length of $\mathbf{u}$ and
- whose direction is the same as $\mathbf{u}$ if $c > 0$ and is opposite to $\mathbf{u}$ if $c < 0$.

If $c = 0$ or $\mathbf{u} = \mathbf{0}$, then $c\mathbf{u} = 0$.
Clearly, If $c \in \mathbb{R}$ and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, then

$$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$$

1.3 Length of Vector

Definition 1.4 (Standard Basis Vector)**.**
The **standard basis vectors** are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$

Any 3D vector can be written as a linear combination of standard basis vectors:

$$\langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Definition 1.5 (Length of Vector)**.**
The **length** of the vector $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is

$$\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

A **unit vector** is a vector whose length is 1.

Theorem 1.1. Let $c \in \mathbb{R}$ and $\mathbf{u}$ be a vector. Then

$$\|c\mathbf{u}\| = |c| \, \|\mathbf{u}\|$$

Theorem 1.2. If $\mathbf{u} \neq \mathbf{0}$, then a unit vector in the same direction as $\mathbf{a}$ is given by

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

1.4 Dot product and Angle

Definition 1.6 (Dot Product)**.**
The dot product of two vectos $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is defined to be

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Theorem 1.3 (Properties of Dot Product)**.**
For vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ and any scalar d,

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
3. $(d\mathbf{a}) \cdot \mathbf{b} = d(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (d\mathbf{b})$
4. $\mathbf{0} \cdot \mathbf{a} = 0$
5. $\mathbf{a} \cdot \mathbf{a} = \|a\|^2$

Notice $\mathbf{a} \cdot \mathbf{b} = 0$ does not imply $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

Definition 1.7 (Angle between two vectors)**.**
For two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, we define the **angle** θ between them to be the **smaller** angle between $\mathbf{a}$ and $\mathbf{b}$ when placing their initial points together.

Clearly, $0 \leq \theta \leq \pi$.
Some special cases:

- **a** and **b** have the same direction iff $\theta = 0/$
- **a** and **b** have opposite direction iff $\theta = \pi$.
- **a** and **b** are orthogonal iff $\theta = \frac{\pi}{2}$.

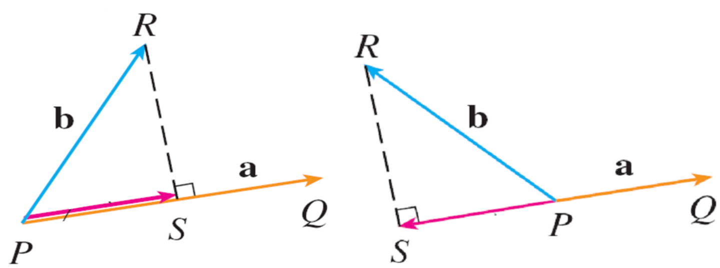
Theorem 1.4 (Dot Product Angle Formula)**.**
Let θ be the angle between nonzero vectors **a** and **b**. Then

$$\mathbf{a} \cdot \mathbf{b} = \|a\| \, \|b\| \cos\theta$$

Theorem 1.5. Two nonzero vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

1.5 Projections

Definition 1.8 (Projection)**.**
Let S be the foot of perpendicular line from R to the line containing $\overrightarrow{PQ}$. The vector $\overrightarrow{PS}$ is



called the **vector projection** of **b** onto **a**, denoted by

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b}$$

The **scalar projection** of **b** onto **a** is defined to be the *signed magnitude* of the vector projection:

$$\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \|\mathbf{b}\| \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$$

Therefore,

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \operatorname{comp}_{\mathbf{a}} \mathbf{b} \times \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a}$$

1.6 Cross Product

Definition 1.9 (Cross Product)**.**
For two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, define the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ to be

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Theorem 1.6. The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

The vector $\mathbf{a} \times \mathbf{b}$ points in a direction perpendicular to $\mathbf{a}$ and $\mathbf{b}$. The direction can be given by the right-hand rule.

Theorem 1.7 (Cross product angle formula)**.**
If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta$$

Theorem 1.8 (Properties of cross product)**.**
If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are vectors and d a scalar, then

- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $(d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (d\mathbf{b})$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

Theorem 1.9.
Suppose two adjacent sides of a parallelogram is $\mathbf{a}$ and $\mathbf{b}$, then the height is $\|\mathbf{a} \times \mathbf{b}\|$.
Suppose Q is a point and PR a line. The distance from Q to PR is

$$\left\|\overrightarrow{PQ}\right\| \sin\theta = \frac{\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\|}{\left\|\overrightarrow{PR}\right\|}$$

1.7 Equation of a line

Definition 1.10 (Vector Equation of Line)**.**

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R}$$

is called a **vector equation** of line, where $\mathbf{r}_0$ is coordinate vector of a point of the line and $\mathbf{v}$ a direction vector of the line.

Theorem 1.10 (Parametric Equation of Line)**.**

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

1.8 Equation of a Plane

Theorem 1.11 (Vector Equation of Plane).

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

is a vector equation of plane, where $\mathbf{n}$ is the normal vector orthogonal to the plane and $\mathbf{r}_0$ a point on the plane.

Theorem 1.12 (Linear Equation of Plane).

$$ax + by + cz = d$$

is the linear equation of plane, where $\langle a, b, c\rangle$ is the normal vector.

Definition 1.11 (Angle between two planes).
An angle between two planes is the angle θ between their normal vectors. Notice $\pi - \theta$ is also an angle between the planes.

1.9 Vector Functions of One Variable

Definition 1.12 (Vector-valued Function).
A **vector-valued function** is

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{i}$$

The scalar function f, g, h are called the **component functions** of $\mathbf{r}$.

The vector function $\mathbf{r}(t)$ traces out the curve C. Therefore, $\mathbf{r}(t)$ is a **parametrization** of C.

1.10 Tangent Vectors

Definition 1.13 (Derivative of Vector-valued Functions).
The **derivative** of $\mathbf{r}(t)$ at $t = a$ is defined by

$$\mathbf{r}'(a) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(a + \Delta t) - \mathbf{r}(a)}{\Delta t}$$

It can be regarded as the rate of change of $\mathbf{r}(t)$ at $t = a$.

We also call $\mathbf{r}'(a)$ a **tangent vector** to the curve at $t = a$.

Theorem 1.13 (Derivative of Vector-valued Function).
Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ and suppose that the components f, g, h are all differentiable at $t = a$.
Then $\mathbf{r}$ is differentiable at $t = a$ and its **derivative** is given by

$$\mathbf{r}'(a) = \langle f'(a), g'(a), h'(a)\rangle$$

Theorem 1.14 (Derivative Rules).
Suppose $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are differentiable vector-valued functions, $f(t)$ a differentiable scalar function and c is a scalar constant. Then

- $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}(t) + \mathbf{s}(t)) = \mathbf{r}'(t) + \mathbf{s}'(t)$
- $\frac{\mathrm{d}}{\mathrm{d}t}(c\mathbf{r}(t)) = c\mathbf{r}'(t)$
- $\frac{\mathrm{d}}{\mathrm{d}t}f(t)\mathbf{r}(t) = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$
- $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) \cdot \mathbf{s}(t) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r} \cdot \mathbf{s}'(t)$
- $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) \times \mathbf{s}(t) = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r} \times \mathbf{s}'(t)$

2 Functions of Two Variables, Quadric Surfaces, Limit and Continuity

2.1 Two-variable function $f(x,y)$

Definition 2.1 (Two-variable function)**.**
A function f of two variables is a rule that assigns, to each *ordered pair* of real numbers (x,y) in a set $D \subseteq \mathbb{R}^2$, a *unique* real number denoted by $f(x,y)$.

If a function f is given by a formula and no domain is specified, then the **domain** of f is understood to be

the set of all pairs (x,y) for which the given expression is a well-defined real number.

To visualise $f(x,y)$, we note that the graph of f is the **surface** S with equation $z = f(x,y)$. We can visualise the graph S of f lying directly above or below its domain D in the xy-plane. Visualisation can also be done through *traces*.

Definition 2.2 (Horizontal traces(level curves))**.**
Horizontal traces are resulting curves when we intersect the surface $z = f(x,y)$ with **horizontal** planes $z = k$.

Definition 2.3 (Vertical traces)**.**
Vertical traces are resulting curves when we intersect the surface $z = f(x,y)$ with vertical planes $x = k$ or $y = k$.

Definition 2.4 (Level Curve)**.**
A **level curve** of $f(x,y)$ is the **two-dimensional graph** of the equation $f(x,y) = k$ for some constant k.

Definition 2.5 (Contour Plot)**.**
A **contour plot** of $f(x,y)$ is a graph of **numerous level curves** $f(x,y) = k$, for representative values of k.

2.2 Cylinder and Quadric Surfaces

Definition 2.6 (Cylinders)**.**
A surface is a **cylinder** if there is a plane P such that *all* the planes parallel to P intersect the surface in the *same* curve (when viewed in 2-dimension).

In fact, any equation in x, y and z where one of the variable is missing is a cylinder.

Definition 2.7 (Quadric surface)**.**
A **quadric surface** is the graph of a *second*-degree equation in three variables x, y and z:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, \dots, J$ are constants.

By translation and rotation, a quadric surface can be brought into one of the two standard forms:

$$\boxed{Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0}$$

Excluding cylinders where one of the variable is missing, there are 6 basic quadric surfaces:

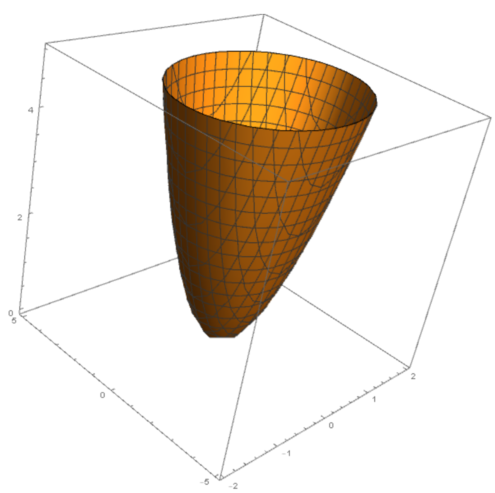
Equation	Standard form (symmetric about z-axis)
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$	Elliptic paraboloid
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$	Hyperbolic paraboloid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Ellipsoid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	Elliptic cone
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Hyperboloid of one sheet
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	Hyperboloid of two sheets

2.3 Elliptic Paraboliod

Definition 2.8 (Elliptic Paraboloid – symmetric about the z-axis)**.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

Horizontal traces: Ellipses
Vertical traces: Parabolas



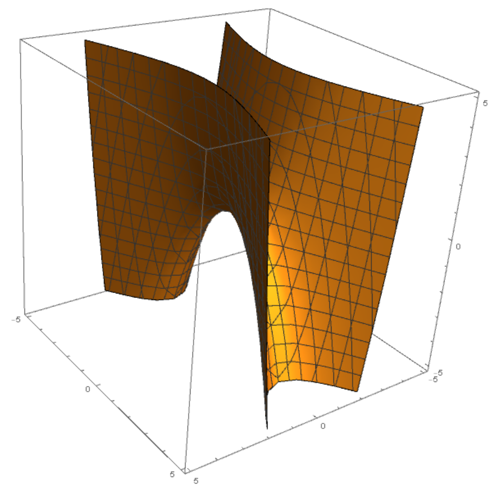
The point $(0, 0, 0)$ is called the **vertex** of the elliptic paraboloid above.
The vertex will e shifted to (x_0, y_0, z_0) if the elliptic paraboloid is given by

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = \frac{(z - z_0)}{c}$$

Definition 2.9 (Hyperbolic paraboloid – symmetric about the z-axis)**.**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

Horizontal traces: Hyperbolas
Vertical traces: Parabolas

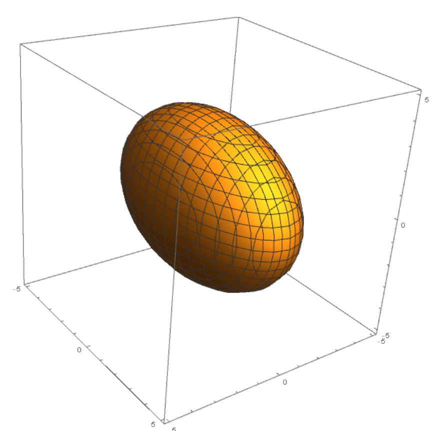


2.4 Ellipsoid, Cones and Hypeboloid

Definition 2.10 (Ellipsoid)**.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

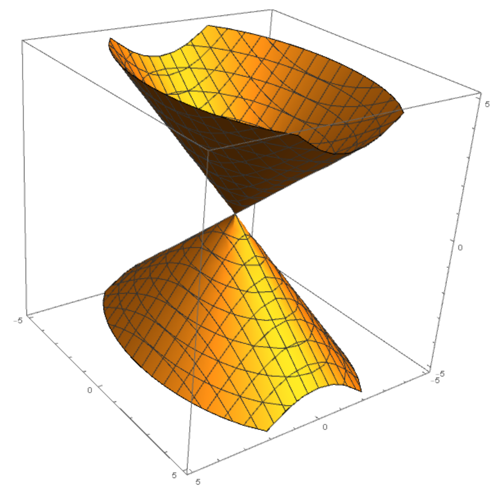
Horizontal traces: Ellipses
Vertical traces: Ellipses



Definition 2.11 (Elliptic cone – symmetric about the z-axis)**.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

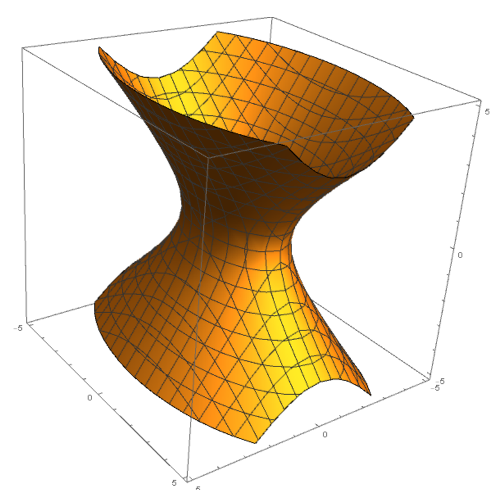
Horizontal traces: Ellipses
Vertical traces: Hyperbolas



Definition 2.12 (Hyperboloid of one sheet – symmetric about the z-axis)**.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

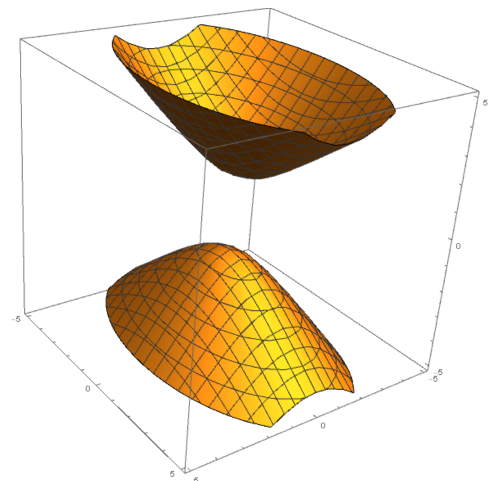
Horizontal traces: Ellipses
Vertical traces: Hyperbolas



Definition 2.13 (Hyperboloid of two sheets – symmetric about the z-axis)**.**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

Horizontal traces: Ellipses
Vertical traces: Hyperbolas



2.5 Function of three Variables

Definition 2.14.
A function f of three variables is a rule that assigns, to each **ordered triple** of real numbers (x, y, z) in a set $D \subseteq \mathbb{R}^3$, a *unique* real number denoted by $f(x, y, z)$.

Definition 2.15 (Level Surface)**.**
A **level surface** of $f(x, y, z)$ is the three dimensional graph of the equation $f(x, y, z) = k$ for some constant k.

2.6 Limit of $f(x, y)$

Definition 2.16 (Limit)**.**
Let f be a function of two variables whose domain D contains points arbitrarily close to (a, b). We say that the **limit** of $f(x, y)$ as (x, y) approaches (a, b) is $L \in \mathbb{R}$, denoted by

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for any number $\varepsilon > 0$ there exists a number $\delta > 0$ such that $|f(x, y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

Remark: f is not required to be defined at (a, b).
It can be proven from the definition that if $\lim_{(x,y)\to(a,b)} f(x, y) = L$, then

- its value L is *unique*, and
- L is *independent* of the choice of path approaching (a, b).

2.7 How to show limit does not exist

Theorem 2.1.
If $f(x,y)$ approaches L_1 as (x,y) approaches (a,b) along a path P_1 and approaches L_2 as (x,y) approaches (a,b) along a path P_2, and $L_1 \neq L_2$, then

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

does **not** exist.

In general, some of the paths that passes through a given point (a,b) to try include:

- $x = a, y \to b$ (vertical lines)
- $y = b, x \to a$ (horizontal lines)
- $y = g(x), x \to a$, where $g(x)$ is some simple function (usually linear and quadratic) such that $g(a) = b$.
- $x = g(y), y \to b$, where $g(x)$ is some simple function (usually linear and quadratic) such that $g(b) = a$.

2.8 How to show limit exists

To show limit exists:

- we can deduce it from known/simple functions using **properties of limit or continuity**; or
- we can use **squeeze theorem**

Theorem 2.2 (Limit Theorems)**.**
Suppose $f(x,y)$ and $g(x,y)$ both have limits as (x,y) approaches (a,b). Then

$$\lim_{(x,y)\to(a,b)} (f(x,y) \pm g(x,y)) = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y)$$

$$\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = \left(\lim_{(x,y)\to(a,b)} f(x,y)\right)\left(\lim_{(x,y)\to(a,b)} g(x,y)\right)$$

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$$

provided

$$\lim_{(x,y)\to(a,b)} g(x,y) \neq 0$$

Theorem 2.3 (Squeeze)**.**
Suppose

- $|f(x,y) - L| \leq g(x,y) \quad \forall (x,y)$ close to (a,b)
- $\lim_{(x,y)\to(a,b)} g(x,y) = 0$

Then,

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

2.9 Continuity of $f(x, y)$

Definition 2.17 (Continuity)**.**
We say f is **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

Theorem 2.4 (Continuity Theorems)**.**
If $f(x, y)$ and $g(x, y)$ are continuous at (a, b), then

- $f \pm g$ is continuous at (a, b).
- $f \cdot g$ is continuous at (a, b).
- $\frac{f}{g}$ is continuous at (a, b), provided $g(a, b) \neq 0$.

Theorem 2.5 (Continuity of Composite Function)**.**
Suppose $f(x, y)$ is continuous at (a, b) and $g(x)$ is continuous at $f(a, b)$. Then

$$h(x, y) = (g \circ f)(x, y) = g(f(x, y))$$

is continuous at (a, b).

Subsequently, the following classes of functions are continuous **in its domain**.

- Polynomial in x and y.
- Trigonometric and exponential functions in x and y.
- Rational function in x and y.

3 Partial Derivatives, Chain Rule, Directional Derivatives

3.1 Partial Derivative

Definition 3.1 (Partial Derivative)**.**
If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by:

$$f_x(x, y) = \lim_{h=0} \frac{f(x+h, y) - f(x, y)}{h}$$
$$f_y(x, y) = \lim_{h=0} \frac{f(x, y+h) - f(x, y)}{h}$$

Other notations for partial derivatives:

$$f_x = \frac{\partial f}{\partial x} \quad f_y = \frac{\partial f}{\partial y}$$

3.2 Higher Order Partial Derivatives

Definition 3.2 (Second partial derivatives)**.**
Second partial derivatives of f is the partial derivatives of partial derivatives of f, i.e.

$$(f_x)_x, (f_x)_y, (f_y)_x, (f_y)_y$$

We use the following notation:

$$(f_x)_x = f_{xx} = \frac{\partial^2 f}{\partial x^2}$$
$$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$
$$(f_y)_x = f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$
$$(f_y)_y = f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

Thus, the notation f_{xy} means that we *first* differentiate with respect to x and *then* with respect to y.

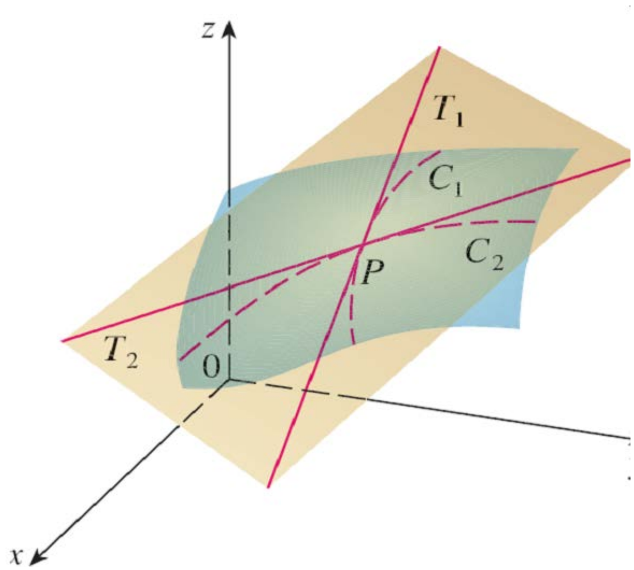
Theorem 3.1 (Clairaut's Theorem)**.**
Suppose f is defined on a disk D that contains (a, b). If the functions f_{xy} and f_{yx} are both *continuous* on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

In fact, so long as the number of the same variable occurring in the subscript are the same, the coresponding partial derivatives are the same.

3.3 Tangent Plane Equation

Definition 3.3 (Tangent Plane)**.**
The **tangent plane** to the surface S at the point $P(a, b, c)$ is defined to be the plane that contains both tangent lines T_1 and T_2, where T_1 and T_2 are the tangent lines to the curves of intersections of the surface S and the vertical planes $y = b$ and $x = a$ respectively.



From the definition, we note that two vectors on the tangent plane are $\langle 1, 0, f_x(a, b)\rangle$ and $\langle 0, 1, f_y(a, b)\rangle$. Thus, a normal vector to the plane is

$$\mathbf{n} = \langle f_x(a, b), f_y(a, b), -1\rangle$$

Theorem 3.2 (Equation of Tangent Plane)**.**
Consider the surface S given by $z = f(x, y)$. A normal vector to the tangent plane to S at (a, b) is

$$\langle f_x(a, b), f_y(a, b), -1\rangle$$

The tangent plane is given by

$$\langle x - a, y - b, z - f(a, b)\rangle \cdot \langle f_x(a, b), f_y(a, b), -1\rangle = 0$$

Equivalently,

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

3.4 Differentiability of $f(x, y)$

In general, for $f(x, y)$ we have

$$\boxed{f \text{ differentiable} \Rightarrow f_x \text{ and } f_y \text{ exist}}$$

To define differentiability, we first define **increment**.

Definition 3.4 (Increment).
Let $z = f(x, y)$. Suppose Δx and Δy are increments in the *independent* variable x and y respectively from a fixed point (a, b). Then the **increment** in z at (a, b), Δz, is defined by

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Definition 3.5 (Differentiability - Two Variable).
Let $z = f(x, y)$. We say that f is **differentiable** at (a, b) if we can write

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and ϵ_2 are functions of Δx and Δy which vanish (i.e. $\epsilon_1, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$).[1]

We say that f is **differentiable on a region** $R \in \mathbb{R}^2$ if f is differentiable at every point in R.

3.5 Linear Approximation

Theorem 3.3 (Linear Approximation - Two Variable).
Suppose $z = f(x, y)$ is *differentiable* at (a, b). Let Δx and Δy be small increments in x and y respectively from (a, b). Then

$$\Delta z \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

This result can be extended to functions of more variables.

3.6 Chain Rule

Theorem 3.4 (Chain Rule - Case 1).
Suppose that $z = f(x, y)$ is a *differentiable* function of x and y, where $x = g(t)$ and $y = h(t)$ are both *differentiable* functions of t. Then, z is a **differentiable** function of t and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Theorem 3.5 (Chain Rule - Case 2).
Suppose that $z = f(x, y)$ is a *differentiable* function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both *differentiable* functions of s and t. Then, z is a **differentiable** function of s and t and

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Here there are three types of variables:

[1] Rearranging $\Delta z = (f_x(a, b) + \epsilon_1)\Delta x + (f_y(a, b) + \epsilon_2)\Delta y$, we will see the function is differentiable if the directional derivative at $(a, b, f(a, b))$ is will estimated in all direction when $\Delta x, \Delta y) \to 0$, which suggests that the tangent vector in all direction at $(a, b, f(a, b))$ will contain in the tangent plane.

- s and t are **independent** variables.
- x and y are called **intermediate** variables.
- z is the **dependent** variable.

Theorem 3.6 (Chain Rule - General Version)**.**
Suppose that u is a differentiable function of n variables $x_1, \ldots, x_n$, and each x_j is a differentiable function of m variables $t_1, \ldots, t_m$. Then u is a function of $t_1, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

3.7 Implicit Differentiation

Definition 3.6 (Implicit Function)**.**
z is an **implicit function** of x and y defined by $F(x, y, z) = 0$ if

for every choice of x and y, the value of z is determined by $F(x, y, z) = 0$

Theorem 3.7 (Implicit Differentiation: Two Independent Variables)**.**
Suppose the equation $F(x, y, z) = 0$, where F is *differentiable*, defines z **implicitly** as a differentiable function of x and y. Then,

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}, \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$$

provided $F_z(x, y, z) \neq 0$.

3.8 Directional Derivatives

Definition 3.7 (Directional Derivative)**.**
The **directional derivative** of $f(x, y)$ at (x_0, y_0) in the direction of **unit** vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

The idea of directional derivative can be extended to functions of more variables.

Theorem 3.8 (Computing Directional Derivatives)**.**
If $f(x, y)$ is a *differentiable* function, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}} f(x_0, y_0) = \langle a, b\rangle \cdot \mathbf{u}$$

Definition 3.8 (Gradient)**.**
The **gradient** of $f(x, y)$ is the vector-valued function

$$\nabla f(x, y) = \langle f_x, f_y\rangle$$

provided that both partial derivatives exist.

4 Gradient Vector, Extrema, Langrange Multiplier

4.1 Gradient Vector and Level Curve

Theorem 4.1 (Level Curve vs ∇f)**.**
Suppose $f(x, y)$ is differentiable function of x and y at (x_0, y_0). Suppose $\nabla f(x_0, y_0) \neq \mathbf{0}$
Then $\nabla f(x_0, y_0)$ is **normal** to the level curve $f(x, y) = k$ that contains the point (x_0, y_0).

4.2 Gradient Vector and Level Surface

Theorem 4.2 (Level Surface vs ∇F)**.**
Suppose $F(x, y, z)$ is diffeerentiable function of x, y and z at (x_0, y_0, z_0). Suppose $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.
Then $\nabla F(x_0, y_0, z_0)$ is **normal** to the level surface $F(x, y, z) = k$ that contains the point (x_0, y_0, z_0).

Theorem 4.3 (Tangent Plane to Level Surface)**.**
The tangent plane to the level surface $F(x, y, z) = k$ on which (x_0, y_0, z_0) resides is given by

$$\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

4.3 Maximum/Minimum Rate of Change

At a given point (x_0, y_0, z_0), the rate of change of $f(x, y, z)$ is given by

$$\begin{aligned} D_{\mathbf{u}} f &= \nabla f \cdot \mathbf{u} \\ &= \|\nabla f\| \, \|\mathbf{u}\| \cos\theta \\ &= \|\nabla\| \cos\theta \end{aligned}$$

where θ is the ange between ∇f and $\mathbf{u}$.

Theorem 4.4 (Maximising rate of Increase/Decrease of f)**.**
Suppose f is a differentiable function of two or three variables. Let P denote a given point. Assume $\nabla f(P) \neq \mathbf{0}$.

- $\nabla f(P)$ points in the direction of maximum rate of change of f at P,$\|\nabla f(P)\|$.
- $-\nabla f(P)$ points in the direction of minimum rate of change of f at P,$-\|\nabla f(P)\|$.

4.4 Critical Points of $f(x, y)$

Definition 4.1 (Local Maximum)**.**
Let $f(x, y) : D \to \mathbb{R}$. Then f has a **local maximum** at (a, b) if

$$f(x, y) \leq f(a, b) \text{for all points close to } (a, b)$$

The number $f(a, b)$ is called a local maximum value.

Definition 4.2 (Local Minimum).
Let $f(x,y) : D \to \mathbb{R}$. Then f has a **local minimum** at (a,b) if

$$f(x,y) \geq f(a,b) \text{for all points close to } (a,b)$$

The number $f(a,b)$ is called a local minimum value.

Theorem 4.5 (A necessary condition).
If f has a local maximum or minimum at (a,b) and the first-order derivatives of f exist there, then

$$f_x(a,b) = f_y(a,b) = 0$$

Definition 4.3 (Saddle Point).
Let $f(x,y) : D \to \mathbb{R}$. Then a point (a,b) is called a **saddle point** of f if

1. $f_x(a,b) = f_y(a,b) = 0$; and
2. every neighbourhood at (a,b) contains points $(x,y) \in D$ for which $f(x,y) < f(a,b)$ and points $(x,y) \in D$ for which $f(x,y) > f(a,b)$.

4.5 Finding Absolute Maximum/Minimum

Definition 4.4 (Absolute Maximum).
Let $f(x,y) : D \to \mathbb{R}$. Then f has an **absolute maximum** at (a,b) if

$$f(x,y) \leq f(a,b) \text{ for all points in the domain } D$$

The number $f(a,b)$ is called a **absolute maximum value**.

Definition 4.5 (Absolute Minimum).
Let $f(x,y) : D \to \mathbb{R}$. Then f has an **absolute minimum** at (a,b) if

$$f(x,y) \geq f(a,b) \text{ for all points in the domain } D$$

The number $f(a,b)$ is called a **absolute minimum value**.

Definition 4.6 (Closed Set in $\mathbb{R}^2$).
A set $R \subseteq \mathbb{R}^2$ is **closed** if it contains all its boundary points.
A **boundary point** of R is a point (a,b) such that every disk with center (a,b) contains point in R and also points in $\mathbb{R}^2 \setminus R$.

Definition 4.7 (Bounded Set in $\mathbb{R}^2$).
A set $R \subseteq \mathbb{R}^2$ is **bounded** if it is contained within some disk. In other words, it is finite in extent.

Theorem 4.6 (Extreme Value Theorem).
If $f(x,y)$ is continuous on a closed and bounded set $D \subseteq \mathbb{R}^2$, then f attains

- an absolute maximum value $f(x_1, y_1)$ and

- an absolute minimum value $f(x_2, y_2)$

at some points (x_1, y_1) and (x_2, y_2) in D.

Theorem 4.7 (Closed Interval Method)**.**
The following is the **closed interval method** for finding absolute maximum and minimum

Step 1 Find the values of f at its **critical points** in D.

Step 2 Find the extreme values of f on the **boundary** of D.

Step 3 The **largest**(resp. **smallest**) of the values from **Step 1** and **Step 2** is the **absolute maximum**(resp. **absolute minimum**).

4.6 Lagrange Multiplier – 2-Variable Case

Theorem 4.8 (Lagrange Multipliers for Function of Two Variables)**.**
Suppose $f(x, y)$ and $g(x, y)$ are differentiable functions such that $\nabla g(x, y) \neq \mathbf{0}$ on the constraint curve $g(x, y) = k$.
Suppose that the **minimum/maximum** value of $f(x, y)$ subject to the constraint $g(x, y) = k$ occurs at (x_0, y_0). Then

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

for some constant λ.
The following are the steps of the method of Lagrange Multiplier for two variable functions:

Step 1 Find all values of x, y and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

and

$$g(x, y) = k$$

Step 2 Evaluate f at all points obtained in **Step 1**.

- The largest of these values is the maximum value of f;
- The smallest is the minimum value of f.

This theorem can be extended to functions of three variables.

5 Double Integral over region on the xy–plane

Definition 5.1 (Double Integral over Rectangle)**.**
The **double integral** of f over the **rectangle** R is

$$\iint_R f(x,y)\mathrm{d}A = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

provided that the limit exists and is the same for any choice of the sample points (x_{ij}^*, y_{ij}^*) in $R_{ij} := [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ whose region is $\Delta A = \Delta x \times \Delta y$, for $1 \leq i \leq m, 1 \leq j \leq n$.

Definition 5.2 (Double Integral over General Region)**.**
Double integral of f over general region D is defined by

$$\iint_D f(x,y)\mathrm{d}A = \iint_R F(x,y)\mathrm{d}A$$

where

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R \setminus D \end{cases}$$

Double integrals are computed by means of iterated integrals.

Definition 5.3 (Iterated Integral)**.**
The **iterated double integral** of f on the rectangle $R = [a,b] \times [c,d]$ in the **order** $\mathrm{d}y\mathrm{d}x$ is defined to be

$$\int_a^b \left(\int_c^d f(x,y)\mathrm{d}y\right)\mathrm{d}x$$

The **iterated double integral** of f on the rectangle $R = [a,b] \times [c,d]$ in the **order** $\mathrm{d}x\mathrm{d}y$ is defined to be

$$\int_c^d \left(\int_a^b f(x,y)\mathrm{d}x\right)\mathrm{d}y$$

Theorem 5.1 (Fubini's Theorem)**.**
If f is **continuous** on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\mathrm{d}A = \int_a^b \int_c^d f(x,y)\mathrm{d}y\mathrm{d}x = \int_c^d (\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y$$

Definition 5.4 (Type I Region)**.**
A plane region D is said to be of **Type I** if it lies between the graphs of two continuous functions of x, that is

$$D = \{(x,y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

where $g_1(x)$ and $g_2(x)$ are continuous on $[a,b]$.

Definition 5.5 (Type II Region)**.**
A plane region D is said to be of **Type II** if it lies between the graphs of two continuous functions of y, that is

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

where $h_1(y)$ and $h_2(y)$ are continuous on $[c, d]$.

Theorem 5.2 (Double Integral over Type I Domain)**.**
If f is continuous on a **Type I** domain D such that

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

then

$$\iint_D f(x, y)\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\mathrm{d}y\mathrm{d}x$$

Theorem 5.3 (Double Integral over Type II Domain)**.**
If f is continuous on a **Type II** domain D such that

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

then

$$\iint_D f(x, y)\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\mathrm{d}x\mathrm{d}y$$

Theorem 5.4 (Additivity with respect to Domain)**.**

$$\iint_D f(x, y)\mathrm{d}A = \sum_{i=1}^{n} \iint_{D_i} f(x, y)\mathrm{d}A$$

The theorem above allows us to decompose the domain into finitely many domains of Type I or II.

6 Double Integral over Polar Regions and Triple Integrals

Theorem 6.1 (Relationship between polar coordinates (r, θ) and cartesian coordinates (x, y))**.**

$$x = r\cos\theta \quad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}\frac{y}{x}, \ \textit{provided } x \neq 0$$

Definition 6.1 (Polar Rectangle)**.**
A polar rectangle is a region

$$R = \{(r, \theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

Theorem 6.2 (Change to Polar Coordinates in Double Integrals)**.**
If f is continuous on a polar rectangle R given by

$$R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x, y)\mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r \mathrm{d}r \mathrm{d}\theta$$

Definition 6.2 (General Polar Regions)**.**
General polar regions come in two different forms:

$$D_1 = \{(r, \theta) : a \leq r \leq b, g_1(r) \leq \theta \leq g_2(r)\}$$

or

$$D_2 = \{(r, \theta) : \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

Theorem 6.3.
If f is continuous on a polar regions D_1, then

$$\iint_D f(x, y)\mathrm{d}A = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta) r \mathrm{d}\theta \mathrm{d}r$$

If f is continuous on a polar region D_2, then

$$\iint_D f(x, y)\mathrm{d}A = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r \mathrm{d}r \mathrm{d}\theta$$

Definition 6.3 (Rectangular box)**.**
A rectangular box is defined by

$$B = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$

Theorem 6.4 (Fubini's Theorem for Triple Integral)**.**
If f is continuous on the rectangular box B, then

$$\iiint_B f(x, y, z)\mathrm{d}V = \int_r^s \int_c^d \int_a^b f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Furthermore, the iterated integral may be evaluated in any order.

7 Line Integrals

7.1 Line Integral of Scalar Field

Definition 7.1 (Line Integrals of Scalar Field)**.**
If f is defined on a smooth curve C given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle, a \leq t \leq b$. Then the **line integral** of f along C is

$$\int_C f(x,y)\mathrm{d}s = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

provided this limit exists and is the same for every choice of (x_i^*, y_i^*).

Theorem 7.1 (Formula for evaluation of Line Integral of Scalar Field)**.**
Suppose C is given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle, a \leq t \leq b$, then

$$\begin{aligned}\int_C f(x,y)\mathrm{d}s &= \int_a^b f(x(t), y(t))\,\|\mathbf{r}'(t)\|\,\mathrm{d}t \\ &= \int_a^b f(x(t), y(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\mathrm{d}t\end{aligned}$$

Similarly, suppose that C is smooth space curve given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, a \leq t \leq b$. Then

$$\begin{aligned}\int_C f(x,y,z)\mathrm{d}s &= \int_a^b f(x(t), y(t), z(t))\,\|\mathbf{r}'(t)\|\,\mathrm{d}t \\ &= \int_C f(x(t), y(t), z(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\mathrm{d}t\end{aligned}$$

Remark: If $f(x,y) = 1$ for all x, y, the line integral formula evaluates the **arc length** of C. More generally, one can interpret the line integral $\int_C f(x,y)\mathrm{d}s$ as the **area** of the fence, whose base is C and height above the point (x,y) is $f(x,y)$.

Theorem 7.2 (Piecewise Parametrisation)**.**
Suppose C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where the initial point of C_{i+1} is the terminal point of C_i. Then

$$\int_C f(x,y)\mathrm{d}s = \sum_{i=1}^{n}\int_{C_i} f(x,y)\mathrm{d}s$$

Theorem 7.3 (Parametrisation of Line Segment)**.**
A vector parametrisation of line segment that starts from $\mathbf{r}_0$ and ends at $\mathbf{r}_i$ is given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0), \qquad 0 \leq t \leq 1$$

7.2 Vector Field

Definition 7.2 (Vector field)**.** *Let $D \subseteq \mathbb{R}^2$. A* ***vector field*** *on D is a function $\mathbf{F}$ that assigns to each point $(x, y) \in D$ a 2D vector $\mathbf{F}(x, y)$.*
Let $D \subseteq \mathbb{R}^3$. A ***vector field*** *on D is a function $\mathbf{F}$ that assigns to each point $(x, y, z) \in D$ a 3D vector $\mathbf{F}(x, y, z)$.*

Theorem 7.4 (Vector Field in Component Form)**.**
For vector field $\mathbf{F}$ on $\mathbb{R}^2$,

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

or for short

$$\mathbf{F} = \langle P, Q\rangle$$

Similarly, for vector field on $\mathbf{F}$ on $\mathbb{R}^3$,

$$\mathbf{F} = \langle P, Q, R\rangle$$

Definition 7.3 (Line Integral of Vector Field)**.**
Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, a \leq t \leq b.$
Then, the **line integral** of $\mathbf{F}(x, y, z)$ along C is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} := \int_C \mathbf{F} \cdot \mathbf{T}\mathrm{d}s = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\mathrm{d}t$$

Theorem 7.5.
We have

$$\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Theorem 7.6 (Component Form)**.**
If $\mathbf{F}(x, y, z) = \langle P, Q, R\rangle$ then we have

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$$

Sometimes a curve C is a union of finitely many smooth curves $C_1, \ldots, C_n$. In such cases,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \sum_{i=1}^{n} \int_{C_i} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

7.3 Conservative Vector Field

Definition 7.4 (Conservative Vector Field)**.**
A vector field $\mathbf{F}$ is **conservative vector field** on D if we can write

$$\mathbf{F} = \nabla f$$

for some scalar function f on D.

The function f is called the **potential function** of $\mathbf{F}$.

Theorem 7.7 (Recovering f from $\mathbf{F}$)**.**

- Suppose $\mathbf{F} = \langle P, Q \rangle$, we have $f_x = P, f_y = Q$.
- Integrating f_x with respect to x, we have

$$f(x, y) = j(x, y) + h(y)$$

  where $\frac{\partial}{\partial x} j(x, y) = f_x$.
- Differentiate $f(x, y)$ with respect to y, we have

$$f_y = \frac{\partial}{\partial y} j(x, y) + h'(y)$$

  and by comparing the above expression with Q, we have

$$g'(y) = Q - \frac{\partial}{\partial y} j(x, y)$$

- Integrating $g'(y)$ with respect to y, we will have $g(y)$, which we substitute back to obtain $f(x, y)$.

Theorem 7.8 (Test for Conservative Field)**.**
Suppose $\mathbf{F}(x, y) = \langle P, Q \rangle$ is a vector field in an *open* and *simply-connected* region D and both P and Q have continuous first-order partial derivatives on D. Then

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

at *each* point (x, y) in D if and only if $\mathbf{F}$ is **conservative** on D.
Similarly, suppose $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ is a vector field in an open and simply-connected region D in space and both P, Q, R have continuous first-order partial derivatives on D. Then

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$$
$$\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$
$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

for each point (x, y, z) in D if and only if $\mathbf{F}$ is **conservative** on D.

7.4 Fundamental Theorem for Line Integral

The **fundamental theorem for line integrals** says that we can evaluate the line integral of a **conservative** vector field by only knowing the **potential function** at the **endpoint**.

Theorem 7.9 (Fundamental Theorem for Line Integrals)**.**
Suppose $\mathbf{F}$ is a conservative vector field with **potential function** f, and C a smooth curve with *initial* point A and *terminal* point B. Then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(B) - f(A)$$

The fundamental theorem of line integral says that the line integral of conservative field is **independent** of the paths with the same initial point and terminal point.

Theorem 7.10 (Another test for non-conservativeness)**.**
Vector field is **not** conservative if there are **two** paths with the same initial and terminal points but their line integrals are *different*.

8 Green's Theorem and Surface Integral of Scalar Field

8.1 Green's Theorem

There are three methods of computing surface integral, each of different applicability.
Method 1: Parametrize C by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, a \leq t \leq b$, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) \mathrm{d}t$$

Method 2: If $\mathbf{F}$ is conservative, then apply Fundamental Theorem for Line Integrals:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(B) - f(A)$$

where $\mathbf{F} = \nabla f$, for some scalar function f, B is the terminal point of C and A the initial point. **Method 3**: **Green's theorem**.

Theorem 8.1 (Green's Theorem)**.**
Let C be a positively oriented[2], piecewise-smooth, simply closed curve in the plane and let D be the region bounded by C.
Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$.
If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region that contains D, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A$$

Green's theorem can be considered as the Fundamental Theorem of Calculus for double integrals.

8.2 Application of Green's Theorem on Area of Plane Region

To evaluate the area of a plane region $\iint_D 1\mathrm{d}A$, we construct P and Q such that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

Therefore, we have the following possibilities

$$\iint_D 1\mathrm{d}A = \int_C x\mathrm{d}y = \int_C -y\mathrm{d}x = \frac{1}{2} \int_C x\mathrm{d}y - y\mathrm{d}x$$

8.3 Parametric Surfaces

Definition 8.1 (Parametric Surface)**.**
The 2-variable vector function

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle, (u, v) \in D$$

parametrizes a surface S in the xyz-plane.

[2]A curve has **positive orientation** if it is traversed in counterclockwise direction.

There are usually more than one ways to parametrize the surface. One special case is the parametrization of the sphere of radius rho centered at origin:

$$\begin{aligned}
x &= \rho \sin\phi \cos\theta \\
y &= \rho \sin\phi \sin\theta \\
z\rho &\cos\phi
\end{aligned}$$

8.4 Surface Integral of Scalar Field

Definition 8.2 (Surface Integral of Scalar Field)**.**

$$\iint_S f(x,y,z)\mathrm{d}S = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

provided the limit exists and is the same for all choices of the evaluation points P_{ij}^*.

We also require the parametrix surface to be smooth.

Definition 8.3 (Smooth surface)**.**
A surface S is **smooth** if it has a parametrization $\mathbf{r}(u,v), u,v \in D$, such that $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ for all $(u,v) \in D$.

Theorem 8.2 (Two important properties of $\mathbf{r}_u$ and $\mathbf{r}_v$)**.**

1. $\mathbf{r}_u \times \mathbf{r}_v$ is normal to the surface D.
2. We can approximate the patch area ΔS_{ij} as follows:

$$\Delta S_{ij} \approx \|\mathbf{r}_u \times \mathbf{r}_v\| \, \Delta u \Delta v$$

From the two properties, we arrive at two useful theorems.

Theorem 8.3 (Tangent Plane of Smooth Surface)**.**
Suppose a surface S is smooth and has a parametrization of $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, (u,v) \in D$.
Then $\mathbf{r}_u(a,b)\times\mathbf{r}_v(a,b)$ is normal to the tangent plane of S at the point $(x(a,b), y(a,b), z(a,b))$.

Theorem 8.4 (Surface Integral Formula)**.**

$$\iint_S f(x,y,z)\mathrm{d}S = \iint_D f(x(u,v), y(u,v), z(u,v)) \, \|\mathbf{r}_u \times \mathbf{r}_v\| \, \mathrm{d}A$$

If S is instead a **union** of finitely many smooth surfaces $S_1, \ldots, S_n$ that intersect only along their boundaries, then the surface integral of f over S is the sum of the surface integrals over each S_i.

Theorem 8.5.

$$\iint_S f(x,y,z)\mathrm{d}S = \sum_{i=1}^{n} \iint_{S_i} f(x,y,z)\mathrm{d}S$$

8.5 Surface Area

A special case of parametrization is that the surface is given by $z = g(x, y), (x, y) \in D$. Then we can take x, y as parameters

$$\mathbf{r}(x, y) = \langle x, y, g(x, y)\rangle$$

In this case we have a special formula for surface integral.

Theorem 8.6.

$$\iint_S f(x, y, z)\mathrm{d}S = \iint_D f(x, y, g(x, y))\sqrt{\left(\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1\right)}\mathrm{d}A$$

On a irrelevant note, surface area of any parametricx surface D traced by $\mathbf{r}(u, v)$ can be evaluated as

Theorem 8.7 (Surface Area)**.**

$$A(S) = \iint_S 1\mathrm{d}S = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| \,\mathrm{d}A$$

9 Surface Integral of Vector Field

9.1 Surface with Orientation

Definition 9.1 (Oriented Surface)**.**
A surface S is **orientable** (or **two sided**) if it is possible to define a unit normal vector $\mathbf{n}$ at each point (x, y, z) (excluding boundary) of the surface such that $\mathbf{n}$ is a continuous function of (x, y, z).

There are two possible orientations for any orientable surfaces, as S will have two identifiable sides:

- **Open surface**: a top and a bottom
- **Closed surface**: an inside and an outside

Theorem 9.1 (Unit normal vector $\mathbf{n}$ of $\mathbf{r}(u, v)$)**.**
One unit normal vector, which specifies a particular orientation of the parametrized surface $\mathbf{r}(u, v)$ is given by

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

and the opposite orientation is given by

$$-\mathbf{n} = \frac{\mathbf{r}_v \times \mathbf{r}_u}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

Theorem 9.2 (Unit normal vector $\mathbf{n}$ of $z = g(x, y)$)**.**
When S is given by $z = g(x, y)$, the **upward** orientation of the surface is given by

$$\mathbf{n} = \frac{\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\rangle}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}$$

and the **downward** orientation is given by

$$-\mathbf{n} = \frac{\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1\rangle}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}$$

Definition 9.2 (Positive Orientation for Closed Surfaces)**.**
For a **closed** surface that is the boundary of a solid region E, the convention is that:
The **positive orientation** is the one for which the normal vectors point **outward** from E.
Inward pointing normals give the **negative orientation**.

9.2 Surface Integral of Vector Field

Definition 9.3 (Surface Integral of Vector Field)**.**
If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with a unit normal vector $\mathbf{n}$, then the **surface integral** of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} := \iint_S \mathbf{F} \cdot \mathbf{n} \mathrm{d}S$$

The integral is also called the **flux** of $\mathbf{F}$ across S.

Theorem 9.3 (Computation of Surface Integral over Vector Field)**.**
To compute the integral, we first parametrize surface S by $\mathbf{r}(u, v)$. Then either $\frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$ or $\frac{\mathbf{r}_v \times \mathbf{r}_u}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$ will be chosen, depending on the given $\mathbf{n}$.
Suppose $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$, then

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \mathrm{d}S \\
&= \iint_D \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \|\mathbf{r}_u \times \mathbf{r}_v\| \, \mathrm{d}A \\
&= \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \mathrm{d}A
\end{aligned}$$

where $\mathbf{r}_u \times \mathbf{r}_v$ has the same direction of $\mathbf{n}$.

Theorem 9.4.
Let $\mathbf{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$. Suppose S is given by

$$\mathbf{r}(x, y) = \langle x, y, g(x, y) \rangle \quad (x, y) \in D$$

Then the flux across S in the **upward** orientation is given by

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D \langle P, Q, R \rangle \cdot \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \rangle \mathrm{d}A$$

The flux across S in the **negative** orientation is given by

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D \langle P, Q, R \rangle \cdot \langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1 \rangle \mathrm{d}A$$

10 Divergence and Curl

10.1 Divergence

Divergence can be regarded as a measurement of the net *outward* flux of vector field.

Definition 10.1 (Divergence)**.**
Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field in space, where P, Q, R have first order derivatives in some region D. The **divergence** of $\mathbf{F}$ is the scalar function defined by

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Definition 10.2 (Del Operator)**.**
The vector differential operator ∇ is defined by

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

Divergence can be expressed in terms of the del operator:

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$$

10.2 Gauss/Divergence Theorem

Theorem 10.1 (Divergence Theorem)**.**
Let E be a solid region where the boundary surface S of E is piecewise smooth with positive (outward) orientation.
Let $\mathbf{F}(x, y, z)$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then,

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_E \div \mathbf{F} \mathrm{d}V$$

10.3 Curl

Curl is the measure of microscopic circulation of the underlying vector field.

Definition 10.3 (Curl)**.**

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times \mathbf{F}$$

10.4 Stokes' Theorem

Definition 10.4 (Positive Orientation of Boundary Curve)**.**
Let C be the boundary curve of an open surface S in space with unit normal $\mathbf{n}$. The positive orientation of the bounary curve is given by the **right hand rule**.

Theorem 10.2 (Stoke's Theorem)**.**
Let C be the boundary curve(simple closed curve) of a surface S with unit normal $\mathbf{n}$. Suppose that C is positively oriented with respect to $\mathbf{n}$.
Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

Green's Theorem is a special case of Stoke's Theorem.