Modal logic

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Notions covered: language and semantics of modal propositional logic, modal models .

Propositional logic and predicate logic give us formulas to talk about the *here and now* (the *hic et nunc*): they are to be interpreted as descriptions of how the actual world is. For example, the sentence:

$$\underbrace{\text{The earth is round.}}_{p}$$

is to be understood as a claim about how the world is like. If we wanted to check whether it is true or false, we would have to go search *this* world, the *actual world* that we live in.

Modal logics are different. They introduce a formal language with which we can also talk about other *possible worlds*. Modal logics let us express that something *should* be the case, or *must* be the case or *cannot possibly* be the case. For example, in order to evaluate whether this sentence is true:

$$\underbrace{\text{It is necessary that}}_{\Box}\ \underbrace{\text{the earth is round.}}_{p}$$

it does not suffice to check the actual shape of the earth; we need to consider different possibilities of how the world could have been.

Modal logics usually have (at least) these two *modal operators*:

operator	neutral paraphrase	natural language cues
$\Diamond$	It is possible that ...	*might*, *may*, *conceivably*, ...
$\Box$	It is necessary that ...	*must*, *have to*, *necessarily*, ...

There are many different kinds of *modalities*, or, as some linguists like to say: *modal flavors*. The necessity operator $\Box$ can be interpreted to be about:

(i) what one ought to do given one's parents rules and regulations;

(ii) what one ought to do given a countries law;

(iii) what is logically necessary;

(iv) what is believed or known to be true;

(v) ...

Here, we will have a look at the last mentioned modality to study a modal propositional logic for talking about beliefs and knowledge, a so-called *epistemic modal logic*.

1 The language of modal logic

Modal propositional logic (ModLog) extends propositional logic by including the new operators $\Diamond$ and $\Box$. In simple terms, whenever φ is a formula of PropLog, then φ, $\Box\varphi$ and $\Diamond\varphi$ are formulas of ModLog. To make things more interesting, we will have one pair of operators $\Diamond_i$ and $\Box_i$ where i is a variable for one of several *agents*.

1.1 Formulas

Let $\mathcal{P}$ be a set of proposition letters and let $\mathcal{A}$ be a set of agents. The language $\mathfrak{L}_{\mathcal{P},\mathcal{A}}$ of ModLog is the set of all *formulas* which are recursively defined as follows:

(i) Every proposition letter is a formula.

(ii) If φ is a formula, so is $\neg\varphi$, $\Box_i\varphi$ and $\Diamond_i\varphi$ for each $i \in \mathcal{A}$.

(iii) If φ and ψ are formulas, so are:
a. $(\varphi \wedge \psi)$ b. $(\varphi \vee \psi)$ c. $(\varphi \rightarrow \psi)$ d. $(\varphi \leftrightarrow \psi)$

(iv) Anything that cannot be constructed by (i)–(iii) is not a formula.

Here are examples of well-formed formulas of our multi-agent epistemic logic with paraphrases (where agent a is Alex, and b is Bo):

$\Box_a(p \rightarrow q)$	Alex believes that if p, then q.
$p \rightarrow \Diamond_a\Box_b q$	If p is true, then Alex considers it possible that Bo believes that q.
$\Box_a p \wedge \Box_b p \wedge \neg\Box_a\Box_b p$	Both Alex and Bo believe that p, but Alex doesn't believe that Bo believes it.

1.2 Semantics of modal logic

The semantics of ModLog is defined in terms of so-called *modal models*. Similar to PropLog a model will tell us whether any given formula is true or false. But while models in PropLog were simple valuation functions (where we interpreted a single valuation function as something like a possible world), modal models comprise (possibly: infinitely) many possible worlds.

Modal models. A *modal model* for the language $\mathfrak{L}_{\mathcal{P},\mathcal{A}}$ is a triple $\mathcal{M} = \langle W, V, (R_i)_{i\in\mathcal{A}}\rangle$ such that:[1]

W is a set of *(possible) worlds*,

$V : W \times \Phi \rightarrow \{0, 1\}$ is a *valuation function*, and

for each agent $i \in \mathcal{A}$, $R_i \subseteq W \times W$ is an *accessibility relation*.

[1] Modal models are also often called *Kripke structures*, named after the logician and philosopher of language Saul Kripke.

For each possible world w, a modal model defines w's own valuation function.[2] For each agent, there is moreover an accessibility relation R_i. We interpret this as follows: if the actual world is w, then agent i considers as live possibilities all the words that can be "accessed" or "seen" via relation R_i from w. We write wR_iv for $\langle w, v\rangle \in R_i$ and introduce the notation:

$$R_i(w) = \{v \in W \mid wR_iv\}$$

to refer to the set of *accessible worlds* for agent i from world w.

[2] There can be several worlds sharing the same valuation function.

Figure 1 below shows an example of a modal model for two agents ($\mathcal{A} = \{a, b\}$) and just two propositions letters ($\mathcal{P} = \{p, q\}$). This model only has three possible worlds. The diagram shows, for each possible world, the set of all proposition letters that are true in this world. Labelled arrows are used to represent the accessibility relation for each agent. Concretely, agent a "sees" worlds v and u in w (formally: $R_a(w) = \{u, v\}$), which is shown in the diagram in exactly two arrows with label a "leaving" world w.

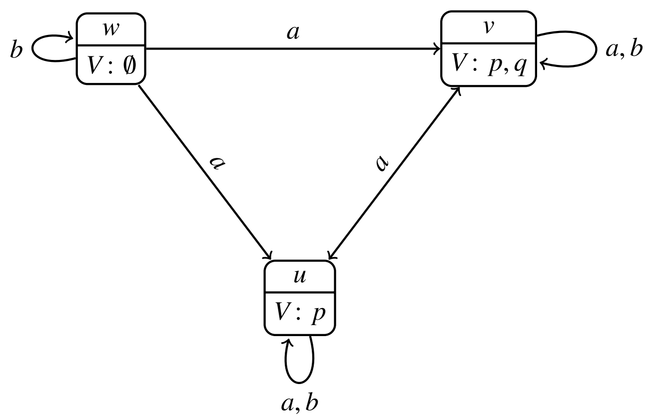

Figure 1: Example of a modal model with three worlds (w, v and u) for two agents (a and b) and two proposition letters (p and q).

Notation and terminology. In everything that follows, we implicitly restrict attention to what are called *serial models*, where $R_i(w) \neq \emptyset$ for all w and i. We say that a model is reflexive (r), transitive (t) or Euclidean (e) if all of its relations R_i are reflexive, transitive or Euclidean. For $C \subseteq \{r, t, e\}$ a set of properties, let $\mathfrak{M}^C_{\mathcal{P},\mathcal{A}}$ refer to the collection of all models $\mathcal{M}_{\mathcal{P},\mathcal{A}}$ that have the properties selected in C. Specifically:

$\mathfrak{M}^{\{t,e\}}_{\mathcal{P},\mathcal{A}}$ are *belief models*, and

$\mathfrak{M}^{\{r,t,e\}}_{\mathcal{P},\mathcal{A}}$ are *knowledge models*.

Truth conditions for ModLog. Truth conditions for formulas of ModLog are defined relative to *pointed models*. If $\mathcal{M} = \langle W, V, (R_a)_{a\in\mathcal{A}}\rangle$ and $w \in W$, the pair $\mathcal{M}, w$ is a pointed model. Truth of a formula in a pointed model $\mathcal{M}, w$ is defines as follows. For proposition letters $p \in \mathcal{P}$, the valuation function for world w decides on truth or falsity:

$$\mathcal{M}, w \models p \text{ iff } V(w, p) = 1$$

Formulas with main operators from propositional logic are treated as before:

$$\begin{array}{ll} \mathcal{M}, w \models \neg\varphi & \text{iff } \mathcal{M}, w \not\models \varphi \\ \mathcal{M}, w \models \varphi \wedge \psi & \text{iff } \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi \\ \vdots & \vdots \end{array}$$

What is new is the treatment of modal operators. For these, the accessibility relations are important.[3]

[3] Essentially, modal operators are like quantifiers from predicate logic, but quantifying over "accessible worlds."

$$\begin{array}{ll} \mathcal{M}, w \models \Box_i\varphi & \text{iff } \mathcal{M}, v \models \varphi \text{ for all } v \in R_i(w) \\ \mathcal{M}, w \models \Diamond_i\varphi & \text{iff } \mathcal{M}, v \models \varphi \text{ for some } v \in R_i(w) \end{array}$$

For example, the model shown in Figure 1 makes the following formulas true:

$\Box_j(\neg p \wedge \neg q)$
"j believes that p and q are false"

$\neg p \wedge \Box_i p$
"i falsely believes that p is true"

$\neg\Box_i q \wedge \neg\Box_i\neg q$
"i is uncertain about q"

$\neg\Box_j p \wedge \Box_i\Box_j p$
"i falsely believes that j believes p"

$\neg\Box_i\Box_j q \wedge \neg\Box_i\neg\Box_j q$
"i is uncertain whether j believes q"

$\Box_i(q \to \Box_j q \wedge \neg q \to \Box_j\neg q)$
"i believes that j knows whether q"