25 September 2018

Review a) Matrix Representation: $\{u\} = \{u^1, \dots, u^m\}$ a basis for $(\mathcal{X}, \mathcal{F})$, $\{v\} = \{v^1, \dots, v^n\}$ a basis for $(\mathcal{Y}, \mathcal{F})$ and $L: \mathcal{X} \to \mathcal{Y}$ a linear operator. A is a matrix rep. of L if $\forall x \in \mathcal{X}$, $[L(x)]_{\{v\}} = A[x]_{\{u\}}$.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "L"] \arrow[d, "{[x]_{\{u\}}}"'] & \mathcal{Y} \arrow[d, "{[y]_{\{v\}}}"] \\ \mathcal{F}^m \arrow[r, "A"] & \mathcal{F}^n \end{tikzcd}$$

Thm Always exist a matrix representation and $A = [A_1 | \cdots | A_m]$ with $A_i = [L(u^i)]_{\{v\}}$ ∎

(b) Remark: $u = \{u^1, \dots, u^m\}$ basis for $(\mathcal{X}, \mathcal{F})$ and $\bar{u} = \{\bar{u}^1, \dots, \bar{u}^m\}$ another basis for $(\mathcal{X}, \mathcal{F})$. Define $Id: \mathcal{X} \to \mathcal{X}$ by, $\forall x \in \mathcal{X}$, $Id(x) = x$. The change of basis matrix is the matrix is the matrix representation of Id.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "Id"] \arrow[d, "{\alpha = [x]_{\{u\}}}"'] & \mathcal{X} \arrow[d, "{[x]_{\{\bar{u}\}} = \bar{\alpha}}"] \\ \mathcal{F}^m \arrow[r, "P"] & \mathcal{F}^m \end{tikzcd}$$

$P = [P_1 | \cdots | P_m]$, $P_i = [u^i]_{\{\bar{u}\}} = [Id(u^i)]_{\{\bar{u}\}}$

(c) A = $n \times n$ matrix with entries in $\mathbb{C}$. $\lambda_i \in \mathbb{C}$, $v^i \in \mathbb{C}^n$, $v^i \neq 0$, $Av^i = \lambda_i v^i \iff \lambda_i$ = e-value and v^i = e-vector.

<u>Thm</u> $\{\lambda_1, \dots, \lambda_n\}$ distinct $\Rightarrow \{v^1, \dots, v^n\}$ linearly independent in $(\mathbb{C}^n, \mathbb{C})$.

Important (a) Converse is false (take $A = I$ for example) (b) You need to understand the proof.

Today

<u>Def.</u> Two square matrices A and B are <u>similar</u> if $\exists$ an invertible matrix P such that $B = PAP^{-1}$. $\{\iff BP = PA \iff P^{-1}BP = A\}$

<u>Remark:</u> If A and B are similar, then they represent the same linear operator, expressed in different bases. $\square$

Consider $A = n \times n$, with coeff. in $\mathbb{C}$, and suppose $\{\lambda_1, \ldots, \lambda_n\}$ not necessarily distinct BUT the corresponding e-vectors $\{v^1, \ldots, v^n\}$ are linearly independent.

$M = [v^1 | v^2 | \cdots | v^n]$ $n \times n$

$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ $n \times n$ $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$
\begin{aligned}
AM &= [Av^1 | Av^2 | \cdots | Av^n] \\
&= [\lambda_1 v^1 | \lambda_2 v^2 | \cdots | \lambda_n v^n] \\
&= M\Lambda \quad \text{(exercise 1)}
\end{aligned}
$$

Exercise 2: $\{v^1, \ldots, v^n\} \Longrightarrow M$ is invertible.

$\therefore$ $A = M \Lambda M^{-1}$ and $\Lambda = M^{-1} A M$

$\therefore$ $\{v^1, \ldots, v^n\}$ lin. indep $\Rightarrow$ A is similar to a diagonal matrix.

Exercise Find the matrix representation of $A: \mathbb{C}^n \to \mathbb{C}^n$ when you use the basis of e-vectors on both copies of $(\mathbb{C}^n, \mathbb{C})$.

Matrix Facts that we need (look up on the web if you care)

$A = n \times m$ with coeff. in $\mathbb{F}$ (for us, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, but is true in general)

Def. rank = # of linearly independent columns of A.

Theorem $\operatorname{rank}(A) = \operatorname{rank}(A^T) =$
$= \operatorname{rank}(A \cdot A^T) = \operatorname{rank}(A^T A)$

Note: # indep columns = # indep rows

$\text{rank}(A) \leq \min(n, m)$

M is symmetric if $M = M^T$.

$A \cdot A^T$ and $A^T A$ are symmetric.

Least Squares Handout

Normed Spaces

$\mathcal{F} = \mathbb{R}$ or $\mathcal{F} = \mathbb{C}$ $(\mathcal{X}, \mathcal{F})$ is a vector space.

Def. $\|\cdot\| : \mathcal{X} \longrightarrow \mathbb{R}$ is a **norm** if

a) $\forall x \in \mathcal{X}$, $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.

b) [Triangle Inequality] $\forall x, y \in \mathcal{X}$,

$$\|x+y\| \leq \|x\| + \|y\|$$

c) $\forall \alpha \in \mathcal{F}$, $\forall x \in \mathcal{X}$, $\|\alpha x\| = |\alpha| \cdot \|x\|$ where

$$|\alpha| = \begin{cases} \text{absolute value} & \alpha \in \mathbb{R} \\ \text{magnitude} & \alpha \in \mathbb{C} \end{cases}.$$

Examples

1. $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $\mathcal{X} = \mathbb{F}^n$

i) $\|x\|_2 := \sqrt{\sum_{i=1}^{n} |x_i|^2}$ Euclidean norm

ii) $\|x\|_p := \sqrt[p]{\sum_{i=1}^{n} |x_i|^p} =$

$$= \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \qquad 1 \le p < \infty$$

iii) $\|x\|_\infty := \max_{1 \le i \le n} |x_i|$

$\left.\right\}$ max-norm and also sup-norm and also ∞-norm

Def. $(\mathcal{X}, \mathbb{F}, \|\cdot\|)$ called a normed space.

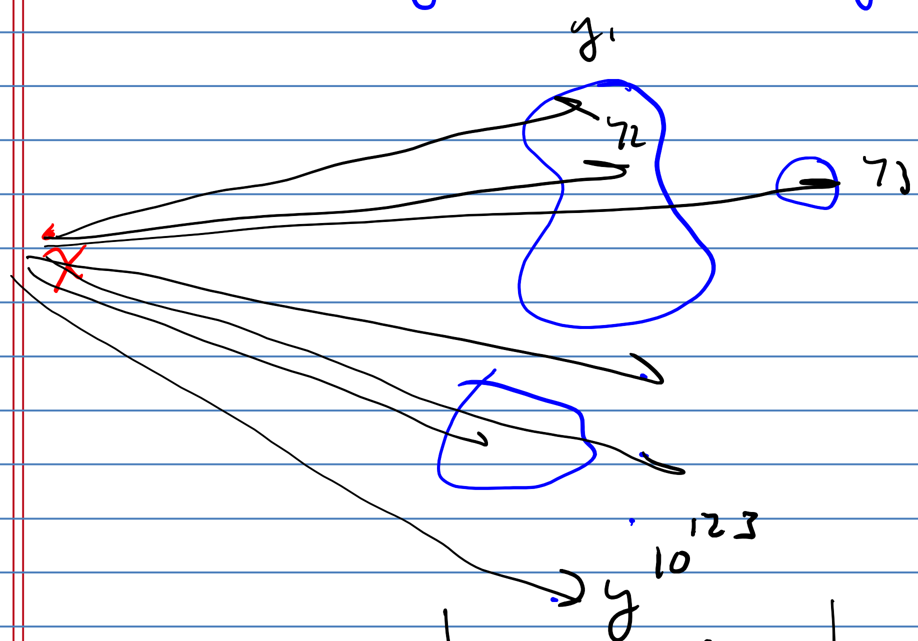
Def. For $x, y \in \mathcal{X}$, the distance from x to y is

$$d(x,y) := \|x - y\|$$

__Note:__ $d(x,y) = d(y,x)$

__Def.__ __Distance to a set__ Let $S \subset X$ be a subset.

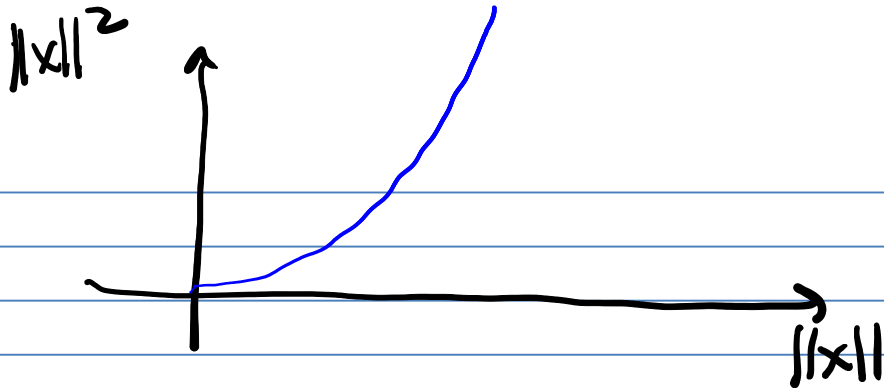
$$d(x,S) := \inf_{y \in S} d(x,y) := \inf_{y\in S} \|x-y\|$$



Working toward best approximation problems __or__ minimum distance problems.

$$x^* = \arg\min_{y \in S} \|x-y\|$$

means minimum exists, is unique and $\|x-x^*\| = \inf_{y\in S} \|x-y\|$, $x^* \in S$.



$$x^* = \arg\min_{y \in S} \|x-y\| \iff x^* = \arg\min_{y \in S} \|x-y\|^2$$

More Examples of Norms

$a, b \in \mathbb{R}$, $a < b$ $D = [a,b] \subset \mathbb{R}$

$\mathcal{X} = \{ f : [a,b] \mid \text{"f is continuous"} \}$

$\mathcal{F} = \mathbb{R}$

$$\|f\|_2 := \sqrt{\int_a^b |f(\tau)|^2 d\tau}$$

$$\|f\|_p := \left(\int_a^b |f(\tau)|^p d\tau \right)^{1/p} \qquad 1 \le p < \infty$$

$$\|f\|_\infty := \sup_{a \le t \le b} |f(t)| \overset{(\text{fact})}{=} \max_{a \le t \le b} |f(t)|$$

Let A be an $n \times m$ matrix.

$$\|A\| = \sqrt{tr(A^T A)}$$

Goal Make sense of this

picture