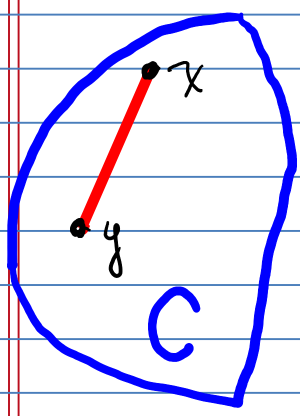
Lecture 06 Dec. 2018

Only pages 1 to 3 are on the ROB 501 Final Exam. This material is very much a part of modern engineering.

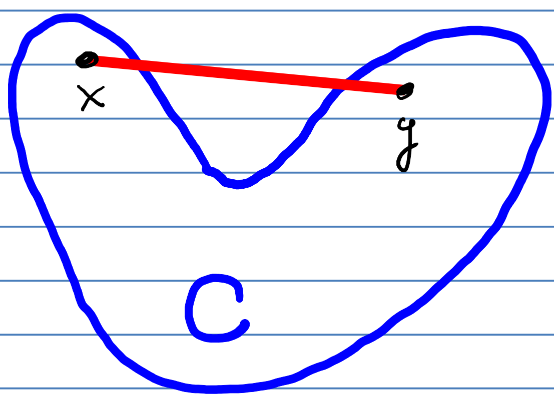
Convex Sets, Convex Functions,

Let $(\mathcal{V}, \mathbb{R})$ be a vector space.

Def. $C \subset \mathcal{V}$ is convex if $\forall\, x, y \in C$, and $\forall\, 0 \le \lambda \le 1$, the convex combination $\lambda x + (1-\lambda) y \in C$.



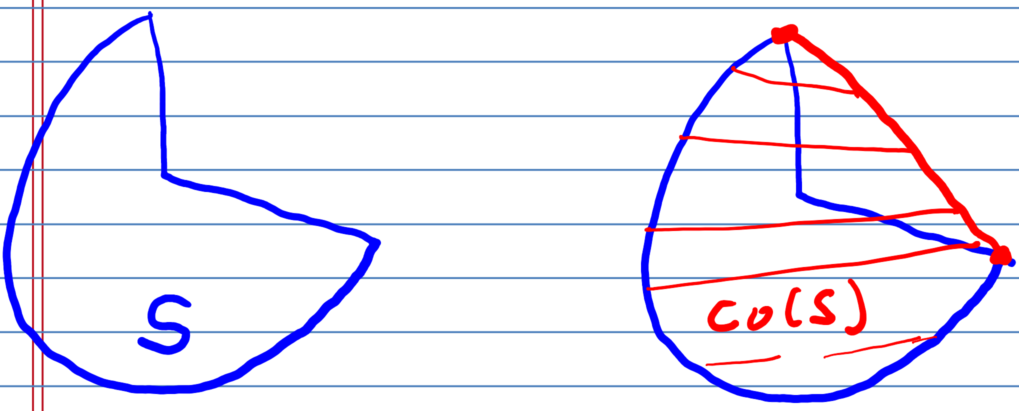
YES



NO

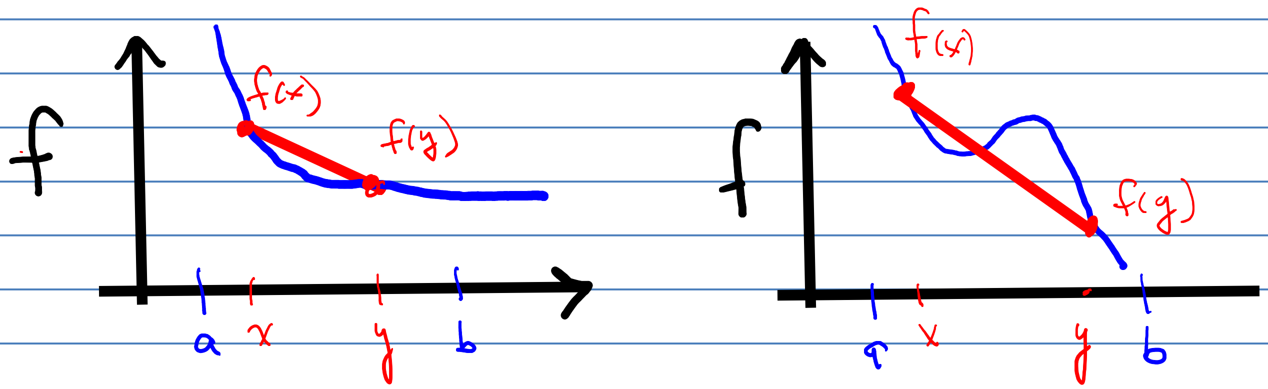
Def. The convex hull of a set $S \subset \mathcal{X}$ is

$$co(S) = \{ \lambda x + (1-\lambda) y \mid 0 \le \lambda \le 1,\ x, y \in S \}$$



Def. Suppose $C \subset \mathcal{X}$ is convex. A function $f : C \to \mathbb{R}$ is convex if $\forall\, x, y \in C$, $\forall\ 0 \le \lambda \le 1$,

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y).$$



Yes No

Let $(V, \mathbb{R}, \|\cdot\|)$ be a normed space, $D \subset V$ a subset.

Def. (a) $x^* \in D$ is a local minimum of $f: D \to \mathbb{R}$ if $\exists\, \delta > 0$ such that,

$$\forall\, x \in B_\delta(x^*) \cap D, \quad f(x^*) \le f(x).$$

(b) $x^* \in D$ is a global minimum of $f: D \to \mathbb{R}$ if, $\forall\, x \in D$, $f(x^*) \le f(x)$.

Theorem If D and $f: D \to \mathbb{R}$ are convex, then any local minimum is also a global minimum.

Proof. Contrapositive: (a) $\Rightarrow$ (b) $\Leftrightarrow$ $\sim$(b) $\Rightarrow$ $\sim$(a)

(a) $x \in D$ is a local minimum

(b) $x \in D$ is a global minimum

$\sim$(b) $\exists\, y \in D$ s.t. $f(y) < f(x)$

$\sim$(a) $\forall \delta > 0$, $\exists\, z \in B_\delta(x) \cap D$ such that $f(z) < f(x)$

Claim 1: If $f(y) < f(x)$, then $\forall\, 0 < \lambda \leq 1$, $z := (1-\lambda)x + \lambda y$ satisfies $f(z) < f(x)$.

Pf.
$$
\begin{aligned}
f(z) &= f((1-\lambda)x + \lambda y) \\
&\leq (1-\lambda) f(x) + \lambda f(y) \quad [\text{convexity}] \\
&< (1-\lambda) f(x) + \lambda f(x) \quad [f(x) > f(y)] \\
&= f(x)
\end{aligned}
$$

$\therefore\ f(z) < f(x)$ $\square$

Claim 2 $\forall \delta > 0$, $\exists\, 0 < \lambda < 1$ such that $z := (1-\lambda)x + \lambda y \in B_\delta(x) \cap D$.

Pf. $\|z - x\| = \|(1-\lambda)x + \lambda y - x\| = \|\lambda(y-x)\| = \lambda\|y-x\|$.

$\therefore$ If $0 < \lambda < \dfrac{\delta}{\|y-x\|}$, then $\|z-x\| < \delta$, and hence, $z \in B_\delta(x)$. Because D is convex, $z \in D$. $\therefore\ z \in B_\delta(x) \cap D$. $\square$

Claim 1 + Claim 2 $\Rightarrow [\sim(b) \Rightarrow \sim(a)]$

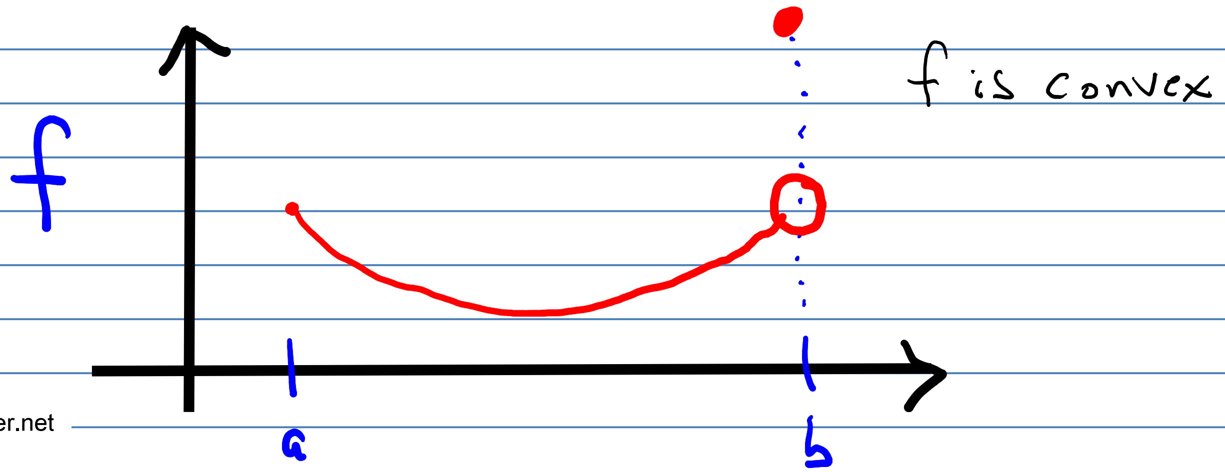
(End of material for Final Exam) $\square$

Theorem (Much harder to prove)

Suppose $(X, \mathbb{R}, \|\cdot\|)$ is a finite-dimensional normed space, C is convex, and $f: C \to \mathbb{R}$ is convex. Then f is continuous on $\mathring{C}$.

[Proof is posted on Canvas]

f can have jumps on the boundary of C $\left[\partial C = \overline{C} \cap \overline{(\sim C)} = \overline{C} \setminus \mathring{C} := \{x \in \overline{C} \mid x \notin \mathring{C}\}\right]$



Useful Facts

① All norms $\|\cdot\|: \mathcal{X} \to \mathbb{R}$ are convex

② $\forall 1 \leq \beta < \infty$, $\|\cdot\|^{\beta}$ is convex. Hence, on $\mathbb{R}^n$, $\forall 1 \leq p < \infty$, $\sum_{i=1}^{n} |x_i|^p$ is convex.

③ If K_1 and K_2 are convex, then so is $K_1 \cap K_2$ (by convention, $\emptyset$ is convex)

④ Consider $(\mathbb{R}^n, \mathbb{R})$ and let A be a real $m \times n$ matrix and $b \in \mathbb{R}^m$. Then

$$K_1 = \{x \in \mathbb{R}^n \mid Ax \leq b\} \text{ is convex}$$

(row wise)

$$K_2 = \{x \in \mathbb{R}^n \mid Ax = b\} \text{ is convex}$$

$$K_3 = \{x \in \mathbb{R}^n \mid A_1 x \leq b_1,\ A_2 x = b_2\} \text{ is convex}$$

Quadratic Programs (QPs)

$x \in \mathbb{R}^n$, $Q \geq 0$, f = $1 \times n$ row vector

$$\text{minimize } \frac{1}{2} x^T Q x + f x$$

subject to: $A_{in} x \leq b_{in}$
$A_{eq} x = b_{eq}$

You see these everywhere nowadays.

In ROB 501, we covered

$$\text{minimize } \frac{1}{2} x^T Q x \qquad Q > 0$$

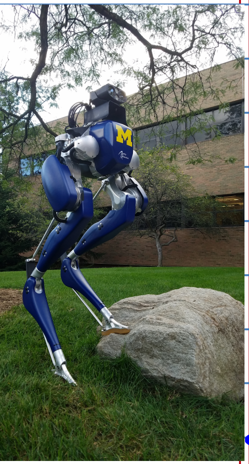
subject to: $A_{eq} x = b_{eq}$

Example use in robotics: distributing torque in a robot while assuring that important bounds on

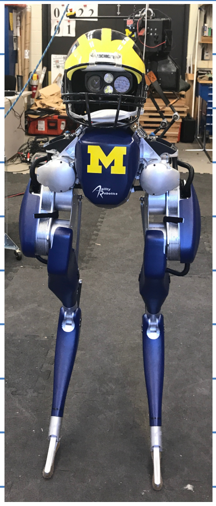
- gripping force
- friction cone
- motor torque limits
- stability (see MABEL & CLFs)

are respected.

Remark: $(x-x_0)^T Q (x-x_0) = x^T Q x \underbrace{-2x_0^T}_{f} Qx + x_0^T Q x_0$



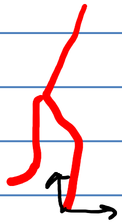
Robot Equations



$q \in \mathbb{R}^n$, $u \in \mathbb{R}^m$

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q)\, u$$

Ground Reaction Forces



$$F = \begin{bmatrix} F^h \\ F^v \end{bmatrix}$$

$$F = \Lambda_0(q,\dot{q}) + \Lambda_1(q)\, u$$

Suppose desired state variable feedback is $u = \gamma(q,\dot{q})$

but we need to respect GRF

$$F^v \geq 0.2 M_{total}\, g$$

$$|F^h| \leq 0.6 F^v$$

How to write as inequalities for a QP?

$$F^v \geq 0.2\, m_{total}\, g$$

$$\Updownarrow$$

$$\left. -F^v \leq -0.2\, m_{total}\, g \quad \right\}$$ Affine in u

$$|F^h| \leq 0.6\, F^v$$

$$\Updownarrow$$

$$F^h \leq 0.6\, F^v$$

$$-F^h \leq 0.6\, F^v$$

$$\Updownarrow$$

$$\left.\begin{aligned} F^h - 0.6\, F^v &\leq 0 \\ -F^h - 0.6\, F^v &\leq 0 \end{aligned}\right\}$$ Affine in u

Collect terms and rewrite to arrive at

$$A_{in}(q)\, u \leq b_{in}(q, \dot{q})$$

QP$_1$: $u^* = \arg\min \; (u - \gamma(q,\dot{q}))^T (u - \gamma(q,\dot{q}))$

subject to: $A_{in}(q)\, u \leq b_{in}(q, \dot{q})$

$d \in \mathbb{R}^m$ (relaxation parameter)

QP$_2$: $u^* = \arg\min \; u^T u + p\, d^T d$ ← Very big

$A_{in}(q)\, u \leq b_{in}(q, \dot{q})$

$u = \gamma(q, \dot{q}) + d$

Experts on Convex Optimization

Stephen Boyd
(Stanford)

Necmiye Ozay
(UM: ECE)

Ram Vasudevan
(UM: ME)

See also faculty in IOE.