04 December 2018

Review: We ♥ Sequences

When we cannot express the solution to a problem in closed form, we seek an iterative means to characterize the existence of a solution **AND** to approximate it with arbitrarily fine precision: (x_k) a sequence, seek conditions for $\exists\, x^*$ s.t. $x_k \to x^*$.

- $x_k \to x^*$ in $(\mathcal{X}, \|\cdot\|)$ if $\forall \varepsilon > 0$, $\exists N(\varepsilon) < \infty$ s.t. $n \geq N \Rightarrow \|x_n - x^*\| < \varepsilon$.
- (x_k) is Cauchy if $\forall \varepsilon > 0$ $\exists N(\varepsilon) < \infty$ s.t. $n, m \geq N \Rightarrow \|x_n - x_m\| < \varepsilon$.
- $S \subset \mathcal{X}$ is a complete subset if ((x_k) Cauchy, and $\forall k \geq 1$, $x_k \in S$) $\Rightarrow \exists\, x^* \in S$ s.t. $x_k \to x^*$.

One says that every Cauchy sequence in S has a limit in S.

- $x_0 \in X$ is a limit point of $S \subset X$ if $\exists$(a sequence (x_k), $\forall k \geq 1$, $x_k \in S$) s.t. $x_k \to x_0$.

- S is closed $\Leftrightarrow$ S contains its limit points $[x_0 \in \overline{S} \Leftrightarrow \exists (x_k), x_k \in S, x_k \to x_0]$.

- $P: S \to S$ is a contraction mapping if $\exists\, 0 \leq c < 1$ s.t. $\forall\, x, y \in S$, $\|P(x) - P(y)\| \leq c\|x - y\|$.

- $x_0 \in S$, define $k \geq 1$, $x_k = P(x_{k-1})$. (x_k) is Cauchy.

- Today we will add to the above the notion of a subsequence and a compact set.

Def. (a) Let $(X, \|\cdot\|)$ and $(Y, |||\cdot|||)$ be two normed spaces. A function $f: X \to Y$ is continuous at $x_0 \in X$ if, $\forall \varepsilon > 0$, $\exists$ $\delta(x_0, \varepsilon) > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow |||f(x) - f(x_0)||| < \varepsilon.$$

$[\forall x \in B_\delta(x_0),\ f(x) \in B_\varepsilon(f(x_0))]$

(b) f is continuous on $S \subset X$ if it is continuous at each $x_0 \in S$.

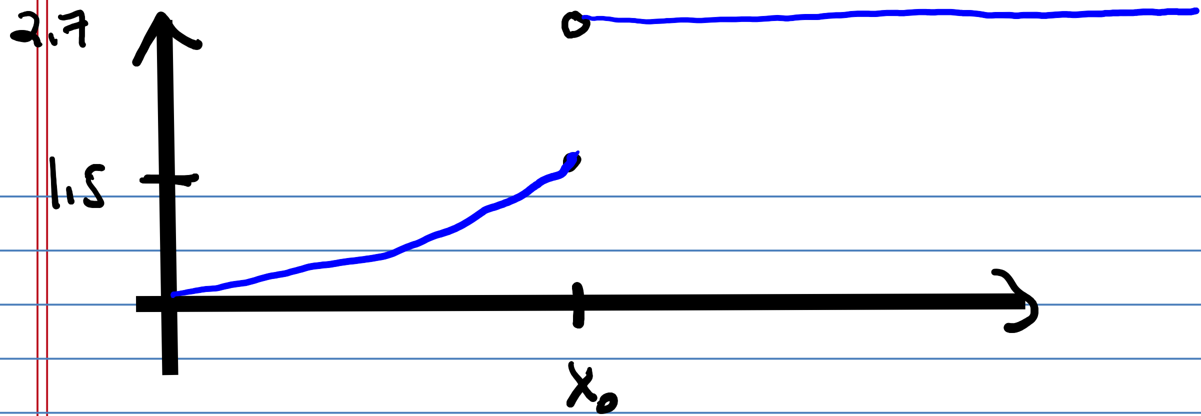
Negate the definition of f is continuous at x_0

Today

f is discontinuous at $x_0 \in X$ if $\exists \varepsilon > 0$, such that, $\forall \delta > 0$, $\exists$ $x \in X$ s.t.

$$\|x - x_0\| < \delta \text{ and } |||f(x) - f(x_0)||| \geq \varepsilon.$$

$[\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists$ $x \in B_\delta(x_0)$ such that $f(x) \notin B_\varepsilon(f(x_0))$. $]$



Let $\varepsilon = 1.0$. Then, $\forall \delta > 0$, $\exists\, x \in B_\delta(x_0)$ such that $|f(x) - f(x_0)| \geq \varepsilon = 1.0$. Indeed, $x = x_0 + \delta/2$ works.

<u>Thm</u> (HW#10, Prob 3). Let $(X, \|\cdot\|)$ and $(Y, |||\cdot|||)$ be normed spaces and let $f: X \to Y$ be a function.

a) If f is continuous at x_0, and (x_n) is a sequence such that $x_n \to x_0$, then the sequence $f(x_n)$ in Y converges to $f(x_0)$ [$y_n = f(x_n)$, $y_0 = f(x_0)$, $y_n \to y_0$]. [$\lim_{n\to\infty} f(x_n) = f(x_0)$]

b) If f is discontinuous at x_0, then $\exists$ (x_n) in X such that $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$.

Corollary f is conti at x_0 $\iff$ every convergent seq. in X is mapped by f into a convergent seq. in Y.

$\iff$ $\left(x_n \to x_0 \implies f(x_n) \to f(x_0) \right]$

Bottom line: Sequences characterize continuity at a point.

Def. Let (x_n) be a sequence. Let $1 \le n_1 < n_2 < n_3 < \cdots$ be an infinit set of increasing integers. Then (x_{n_i}) is a subsequence of (x_n).

Note: $n_i \geq i$.

Lemma (Exercise) Suppose $x_n \to x_0$ and (x_{n_i}) is a subsequence of (x_n). Then $x_{n_i} \longrightarrow x_0$.

Handy Inequality: $\forall x, y \in X$,

$$\|x-y\| \geq \big| \|x\| - \|y\| \big|$$

Why: $\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$

$\therefore \|x-y\| \geq \|x\| - \|y\|$

$\therefore$ Same argument gives

$$\|x-y\| \geq \|y\| - \|x\|$$

$$\boxed{\therefore \|x-y\| \geq \big| \|x\| - \|y\| \big|}$$

Def. Let $(X, \|\cdot\|)$ be a normed space. Then $C \subset X$ is COMPACT if every sequence in C has a convergent subsequence with limit in C.

Remark: Often called sequential compactness

Def. A $S \subset X$ is bounded if $\exists$ $r < \infty$ such that $S \subset B_r(0)$.

Exercise S is bounded $\iff$ $\sup_{x \in S} \|x\| < \infty$.

Fun Exercise: S is unbounded $\iff$ $\exists$ (x_k) such that, $\forall k \geq 1$, $x_k \in S$, and $\|x_{k+1}\| \geq \|x_k\| + 1$.

Fun Exercise: $\forall p \geq 1$, $\|x_{n+p} - x_n\| \geq p$

Note: $\|x_{n+p}\| \geq \|x_n\| + p$ + Handy Inequality.

Conclusion (x_n) has no convergent subsequence.

Bolzano-Weierstrass Theorem

In a finite-dimensional normed space, TFAE for a set $C \subset X$:

(a) C is closed and bounded.

(b) For every sequence (x_n) in C ($\forall n \geq 1$, $x_n \in C$) $\exists$ $x_0 \in C$ and a subsequence (x_{n_i}) of (x_n) such that $x_{n_i} \xrightarrow[i \to \infty]{} x_0$.

Proof

(a) $\Rightarrow$ (b) using a cute image from Wikipedia.

Case 1 Suppose (x_n) has only a finite number of distinct elements. Hence, at least one value is repeated an infinite number of times. Say that is x_s. Then $\exists$ $n_1 < n_2 < \cdots < n_k < \cdots$

such that $x_{n_i} = x_5$ $i = 1, 2, \ldots$

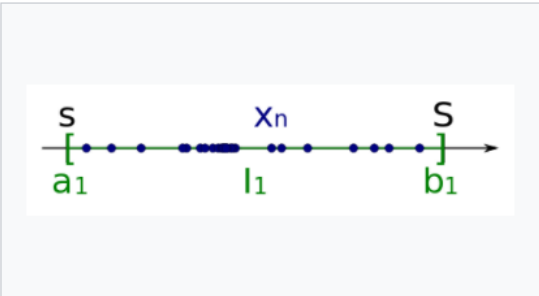
Hence $x_{n_i} \xrightarrow[i \to \infty]{} x_5$.

Case 2 (x_n) has an ∞ # of distinct elements

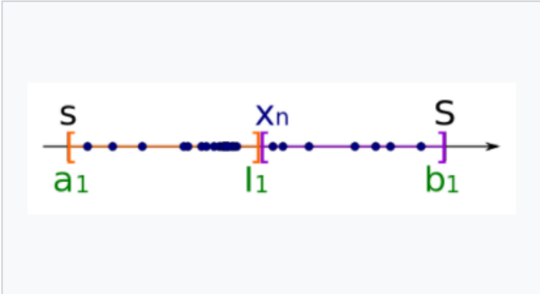
Alternative proof [edit]

There is also an alternative proof of the Bolzano–Weierstrass theorem using nested intervals. We start with a bounded sequence (x_n):

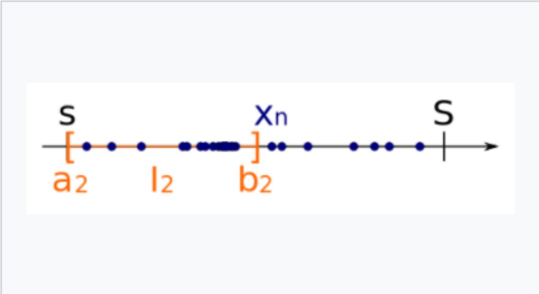
Because $(x_n)_{n\in\mathbb{N}}$ is bounded, this sequence has a lower bound s and an upper bound S.



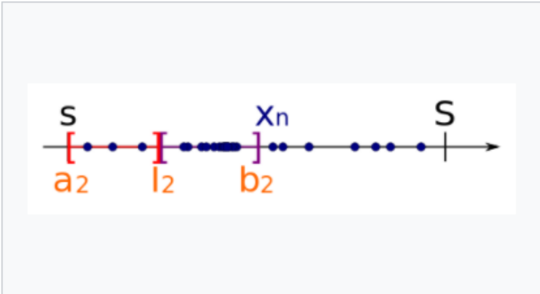
We take $I_1 = [s, S]$ as the first interval for the sequence of nested intervals.



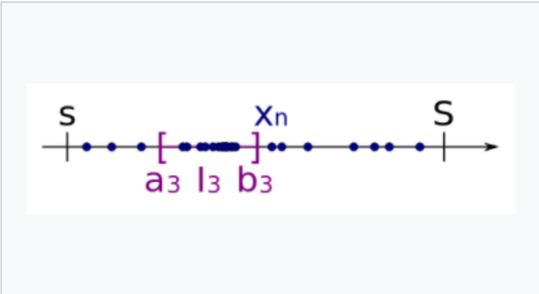
Then we split I_1 at the mid into two equally sized subintervals.



We take this subinterval as the second interval I_2 of the sequence of nested intervals which contains infinitely many members of $(x_n)_{n\in\mathbb{N}}$. Because each sequence has infinitely many members, there must be at least one subinterval which contains infinitely many members.



Then we split I_2 again at the mid into two equally sized subintervals.



Again we take this subinterval as the third subinterval I_3 of the sequence of nested intervals, which contains infinitely many members of $(x_n)_{n\in\mathbb{N}}$.

(b) $\Rightarrow$ (a) We do $\sim(a) \Rightarrow \sim(b)$

$\sim(a)$ C is either not closed **OR** not bounded.

$\sim(b)$ $\exists$ (x_n) with no convergent subsequence having a limit in C.

Case 1 C is unbounded.

Proof. Your exercise.

Case 2 C is not closed.

Proof. Hence, C does not contain all of its limit points. $\exists$ $x_0 \notin C$ and (x_n), with $x_n \in C$ and $x_n \to x_0$.

All subsequences (x_{n_i}) of (x_n) satisfy $x_{n_i} \to x_0$. Hence, we have built a sequence (x_n) for which none of its subsequences have a limit in C. $\square$