

Working notes on Automatic differentiation

October 17, 2020 — not for circulation

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1 The language

This paper is about automatic differentiation of functions, so we must be precise about the language in which those functions are written.

The syntax of our language is given in Figure 1. Note that

- Variables are divided into *functions*, f, g, h ; and *local variables*, x, y, z , which are either function arguments or let-bound.
- The language has a first order sub-language. Functions are defined at top level; functions always appear in a call, never (say) as an argument to a function; in a call $f(e)$, the function f is always a top-level-defined function, never a local variable.
- Functions have exactly one argument. If you want more than one, pass a pair.
- Pairs are built-in, with selectors $\pi_{1,2}$, $\pi_{2,2}$. In the real implementation, pairs are generalised to n -tuples, and we often do so informally here.
- Conditionals are a language construct.
- Let-bindings are non-recursive. For now, at least, top-level functions are also non-recursive.
- Lambda expressions and applications are present, so the language is higher order. AD will only accept a subset of the language, in which lambdas appear only as an argument to *build*. But the *output* of AD may include lambdas and application, as we shall see.

1.1 Built in functions

The language has built-in functions shown in Figure 2.

We allow ourselves to write functions infix where it is convenient. Thus $e_1 + e_2$ means the call $+(e_1, e_2)$, which applies the function $+$ to the pair (e_1, e_2) . (So, like all other

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Conference'17, July 2017, Washington, DC, USA

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ACM ISBN 978-x-xxxx-xxxx-x/YY/MM...\$15.00

<https://doi.org/10.1145/nnnnnnnn.nnnnnnn>

Atoms

$f, g, h ::=$ Function

$x, y, z ::=$ Local variable (lambda-bound or let-bound)

$k ::=$ Literal constants

Terms

$\text{pgm} ::= \text{def}_1 \dots \text{def}_n$

$\text{def} ::= f(x) = e$

$e ::= k$ Constant

$| x$ Local variable

$| f(e)$ Function call

$| (e_1, e_2)$ Pair

$| \lambda x. e$ Lambda

$| e_1 e_2$ Application

$| \text{let } x = e_1 \text{ in } e_2$

$| \text{if } b \text{ then } e_1 \text{ else } e_2$

Types

$\tau ::= \mathbb{N}$ Natural numbers

$| \mathbb{R}$ Real numbers

$| (\tau_1, \tau_2)$ Pairs

$| \text{Vec } \tau$ Vectors

$| \tau_1 \rightarrow \tau_2$ Functions

$| \tau_1 \multimap \tau_2$ Linear maps

Figure 1. Syntax of the language

functions, $(+)$ has one argument.) Similarly the linear map $m_1 \times m_2$ is short for $\times(e_1, e_2)$.

We allow ourselves to write vector indexing $ixR(i, a)$ using square brackets, thus $a[i]$.

Built-in functions		
$(+)$	$:: (\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$	
$(*)$	$:: (\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$	
$\pi_{1,2}$	$:: (t_1, t_2) \rightarrow t_1$	Selection
$\pi_{2,2}$	$:: (t_1, t_2) \rightarrow t_2$..ditto..
$build$	$:: (n :: \mathbb{N}, \mathbb{N} \rightarrow t) \rightarrow Vec\ t$	Vector build
ixR	$:: (\mathbb{N}, Vec\ t) \rightarrow t$	Indexing (NB arg order)
sum	$:: Vec\ t \rightarrow t$	Sum a vector
sz	$:: Vec\ t \rightarrow \mathbb{N}$	Size of a vector
Derivatives of built-in functions		
$\partial +$	$:: (\mathbb{R}, \mathbb{R}) \rightarrow ((\mathbb{R}, \mathbb{R}) \multimap \mathbb{R})$	
$\partial + (x, y)$	$= \mathbf{1} \bowtie \mathbf{1}$	
$\partial *$	$:: (\mathbb{R}, \mathbb{R}) \rightarrow ((\mathbb{R}, \mathbb{R}) \multimap \mathbb{R})$	
$\partial * (x, y)$	$= S(y) \bowtie S(x)$	
$\partial \pi_{1,2}$	$:: (t, t) \rightarrow ((t, t) \multimap t)$	
$\partial \pi_{1,2}(x)$	$= \mathbf{1} \bowtie \mathbf{0}$	
∂ixR	$:: (\mathbb{N}, Vec\ t) \rightarrow ((\mathbb{N}, Vec\ t) \multimap t)$	
$\partial ixR(i, v)$	$= \mathbf{0} \bowtie \mathcal{H}(build(sz(v), \lambda j. \text{if } i = j \text{ then } \mathbf{1} \text{ else } \mathbf{0}))$	
∂sum	$:: Vec\ \mathbb{R} \rightarrow (Vec\ \mathbb{R} \multimap \mathbb{R})$	
$\partial sum(v)$	$= lmhcatvbuild(sz(v), \lambda i. \mathbf{1})$	
\dots		

Figure 2. Built-in functions

Multiplication and addition are overloaded to work on any suitable type. On vectors they work element-wise; if you want dot-product you have to program it.

1.2 Vectors

The language supports one-dimensional vectors, of type $Vec\ T$, whose elements have type T (Figure 1). A matrix can be represented as a vector of vectors.

Vectors are supported by the following built-in functions (Figure 2):

- $build :: (\mathbb{N}, \mathbb{N} \rightarrow t) \rightarrow Vec\ t$ for vector construction.
- $ixR :: (\mathbb{N}, Vec\ t) \rightarrow t$ for indexing. Informally we allow ourselves to write $v[i]$ instead of $ixR(i, v)$.

- $sum :: Vec\ \mathbb{R} \rightarrow \mathbb{R}$ to add up the elements of a vector. We specifically do not have a general, higher order, fold operator; we say why in Section 4.1.
- $sz :: Vec\ t \rightarrow \mathbb{N}$ takes the size of a vector.
- Arithmetic functions $(*)$, $(+)$ etc are overloaded to work over vectors, always elementwise.

2 Linear maps and differentiation

If $f : S \rightarrow T$, then its derivative ∂f has type

$$\partial f : S \rightarrow (S \multimap T)$$

where $S \multimap T$ is the type of *linear maps* from S to T . That is, at some point $p : S$, $\partial f(p)$ is a linear map that is a good approximation of f at p .

By “a good approximation of f at p ” we mean this:

$$\forall p : S. f(p + \delta_p) \approx f(p) + \partial f(p) \odot \delta_p$$

Here the operation (\odot) is linear-map application: it takes a linear map $S \multimap T$ and applies it to an argument of type S , giving a result of type T (Figure 3).

The linear maps from S to T are a subset of the functions from S to T . We characterise linear maps more precisely in Section 2.1, but a good intuition can be had for functions $g : \mathbb{R}^2 \rightarrow \mathbb{R}$. This function defines a curvy surface $z = g(x, y)$. Then a linear map of type $\mathbb{R}^2 \multimap \mathbb{R}$ is a plane, and $\partial g(p_x, p_y)$ is the plane that best approximates g near (p_x, p_y) , that is a tangent plane passing through $z = g(p_x, p_y)$.

2.1 Linear maps

A *linear map*, $m : S \multimap T$, is a function from S to T , satisfying these two properties:

$$(LM1) \quad \forall x, y : S \quad m \odot (x + y) = m \odot x + m \odot y$$

$$(LM2) \quad \forall k : \mathbb{R}, x : S \quad k * (m \odot x) = m \odot (k * x)$$

Here $(\odot) : (s \multimap t) \rightarrow (s \rightarrow t)$ is an operator that applies a linear map $(s \multimap t)$ to an argument of type s . The type $s \multimap t$ is a type in the language (Figure 1).

Linear maps can be *built and consumed* using the operators in (see Figure 3). Indeed, you should think of linear maps as an *abstract type*; that is, you can *only* build or consume linear maps with the operators in Figure 3. We might *represent* a linear map in a variety of ways, one of which is as a matrix (Section 2.5).

2.1.1 Semantics of linear maps

The *semantics* of a linear map is completely specified by saying what ordinary function it corresponds to; or, equivalently, by how it behaves when applied to an argument by (\odot) . The semantics of each form of linear map are given in Figure 4

Operator	Type	Matrix interpretation
where $s = \mathbb{R}^m$, and $t = \mathbb{R}^n$		
Apply	$(\odot) : (s \multimap t) \rightarrow \delta s \rightarrow \delta t$	Matrix/vector multiplication
Reverse apply	$(\odot_R) : \delta t \rightarrow (s \multimap t) \rightarrow \delta s$	Vector/matrix multiplication
Compose	$(\circ) : (s \multimap t, r \multimap s) \rightarrow (r \multimap t)$	Matrix/matrix multiplication
Sum	$(\oplus) : (s \multimap t, s \multimap t) \rightarrow (s \multimap t)$	Matrix addition
Zero	$\mathbf{0} : s \multimap t$	Zero matrix
Unit	$\mathbf{1} : s \multimap s$	Identity matrix (square)
Scale	$\mathcal{S}(\cdot) : \mathbb{R} \rightarrow (s \multimap s)$	
VCat	$(\times) : (s \multimap t_1, s \multimap t_2) \rightarrow (s \multimap (t_1, t_2))$	Vertical juxtaposition
VCatV	$\mathcal{V}(\cdot) : \text{Vec } (s \multimap t) \rightarrow (s \multimap \text{Vec } t)$...vector version
HCat	$(\bowtie) : (t_1 \multimap s, t_2 \multimap s) \rightarrow ((t_1, t_2) \multimap s)$	Horizontal juxtaposition
HCatV	$\mathcal{H}(\cdot) : \text{Vec } (t \multimap s) \rightarrow (\text{Vec } t \multimap s)$...vector version
Transpose	$\cdot^\top : (s \multimap t) \rightarrow (t \multimap s)$	Matrix transpose
NB: We expect to have only \mathcal{L}/\mathcal{L}' but not both		
Lambda	$\mathcal{L} : (\mathbb{N} \rightarrow (s \multimap t)) \rightarrow (s \multimap (\mathbb{N} \rightarrow t))$	
TLambda	$\mathcal{L}' : (\mathbb{N} \rightarrow (t \multimap s)) \rightarrow ((\mathbb{N} \rightarrow t) \multimap s)$	Transpose of \mathcal{L}

Figure 3. Operations over linear maps

2.1.2 Properties of linear maps

Linear maps satisfy *properties* given in Figure 4. Note that (\circ) and \oplus behave like multiplication and addition respectively.

These properties can readily be proved from the semantics. To prove two linear maps are equal, we must simply prove that they give the same result when applied to any argument. So, to prove that $\mathbf{0} \circ m = m$, we choose an arbitrary x and reason thus:

$$\begin{aligned}
 & (\mathbf{0} \circ m) \odot x \\
 &= \mathbf{0} \odot (m \odot x) \quad \{\text{semantics of } (\circ)\} \\
 &= \mathbf{0} \quad \{\text{semantics of } \mathbf{0}\} \\
 &= \mathbf{0} \odot x \quad \{\text{semantics of } \mathbf{0} \text{ backwards}\}
 \end{aligned}$$

Note that the property

$$(m_1 \bowtie m_2) \circ (n_1 \times n_2) = (m_1 \circ n_1) \oplus (m_2 \circ n_2)$$

is the only reason we need the linear map (\oplus) .

Theorem: $\forall (m : S \multimap T). m \odot \mathbf{0} = \mathbf{0}$. That is, all linear maps pass through the origin. **Proof:** property (LM2) with $k = 0$. Note that the function $\lambda x.x + 4$ is not a linear map; its graph is a straight line, but it does not go through the origin.

2.2 Vector spaces

Given a linear map $m : S \multimap T$, we expect both S and T to be a *vector space with dot product* (aka inner product space¹). A vector space with dot product V has:

- *Vector addition* $(+_V) : V \rightarrow V \rightarrow V$.
- *Zero vector* $0_V : V$.
- *Scalar multiplication* $(*_V) : \mathbb{R} \rightarrow V \rightarrow V$
- *Dot-product* $(\bullet_V) : V \rightarrow V \rightarrow \mathbb{R}$.

We omit the V subscripts when it is clear which $(*)$, $(+)$, (\bullet) or 0 is intended.

¹https://en.wikipedia.org/wiki/Vector_space

Semantics of linear maps

$$(m_1 \circ m_2) \odot x = m_1 \odot (m_2 \odot x)$$

$$(m_1 \times m_2) \odot x = (m_1 \odot x, m_2 \odot x)$$

$$(m_1 \bowtie m_2) \odot (x_1, x_2) = (m_1 \odot x_1) + (m_2 \odot x_2)$$

$$(m_1 \oplus m_2) \odot x = (m_1 \odot x) + (m_2 \odot x)$$

$$0 \odot x = 0$$

$$1 \odot x = x$$

$$S(k) \odot x = k * x$$

$$\mathcal{V}(m) \odot x = \text{build}(\text{sz}(m), \lambda i. m[i] \odot x)$$

$$\mathcal{H}(m) \odot x = \Sigma_i (m[i] \odot x[i])$$

$$\mathcal{L}(f) \odot x = \lambda i. (f i) \odot x$$

$$\mathcal{L}'(f) \odot g = \Sigma_i (f i) \odot g(i)$$

Properties of linear maps

$$0 \circ m = 0$$

$$m \circ 0 = 0$$

$$1 \circ m = m$$

$$m \circ 1 = m$$

$$m \oplus 0 = m$$

$$0 \oplus m = m$$

$$m \circ (n_1 \bowtie n_2) = (m \circ n_1) \bowtie (m \circ n_2)$$

$$(m_1 \times m_2) \circ n = (m_1 \circ n) \times (m_2 \circ n)$$

$$(m_1 \bowtie m_2) \circ (n_1 \times n_2) = (m_1 \circ n_1) \oplus (m_2 \circ n_2)$$

$$S(k_1) \circ S(k_2) = S(k_1 * k_2)$$

$$S(k_1) \oplus S(k_2) = S(k_1 + k_2)$$

Figure 4. Linear maps: semantics and properties

These operations must obey the laws of vector spaces

$$v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$$

$$v_1 + v_2 = v_2 + v_1$$

$$v + 0 = 0$$

$$0 * v = 0$$

$$1 * v = v$$

$$r_1 * (r_2 * v) = (r_1 * r_2) * v$$

$$r * (v_1 + v_2) = (r * v_1) + (r * v_2)$$

$$(r_1 + r_2) * v = (r_1 * v) + (r_2 * v)$$

2.2.1 Building vector spaces

What types are vector spaces? Look the syntax of types in Figure 1.

- The real numbers \mathbb{R} is a vector space, using the standard $+$ and $*$ for reals; and $\bullet_{\mathbb{R}} = *$.
- If V is a vector space then $\text{Vec } V$ is a vector space, with
 - $v_1 + v_2$ is vector addition
 - $r * v$ multiplies each element of the vector v by the real r .
 - $v_1 \bullet v_2$ is a the usual vector dot-product.
 We often write $\text{Vec } \mathbb{R}$ as \mathbb{R}^N .
- If V_1 and V_2 are vector spaces, then the product space (V_1, V_2) is a vector space
 - $(v_1, v_2) + (w_1, w_2) = (v_1 + w_1, v_2 + w_2)$.
 - $r * (v_1, v_2) = (r * v_1, r * v_2)$
 - $(v_1, v_2) \bullet (w_1, w_2) = (v_1 \bullet w_1) + (v_2 \bullet w_2)$.

In all cases the necessary properties of the operations (associativity, distribution etc) are easy to prove.

2.3 Transposition

For any linear map $m : S \multimap T$ we can produce its transpose $m^\top : T \multimap S$. Despite its suggestive type, the transpose is *not* the inverse of m ! (In the world of matrices, the transpose of a matrix is not the same as its inverse.)

Definition 2.1. Given a linear map $m : S \multimap T$, its *transpose* $m^\top : T \multimap S$ is defined by the following property:

$$(TP) \quad \forall s : S, t : T. (m^\top \odot t) \bullet s = t \bullet (m \odot s)$$

This property *uniquely* defines the transpose, as the following theorem shows:

Theorem 2.2. If m_1 and m_2 are linear maps satisfying

$$\forall s t. (m_1 \odot s) \bullet t = (m_2 \odot s) \bullet t$$

then $m_1 = m_2$

Proof. It is a property of dot-product that if $v_1 \bullet x = v_2 \bullet x$ for every x , then $v_1 = v_2$. (Just use a succession of one-hot vectors for x , to pick out successive components of v_1 and v_2 .) So (for every t):

$$\forall s t. (m_1 \odot s) \bullet t = (m_2 \odot s) \bullet t$$

$$\Rightarrow \forall s. m_1 \odot s = m_2 \odot s$$

and that is the definition of extensional equality. So m_1 and m_2 are the same linear maps. \square

Figure 5 has a collection of laws about transposition. These identities are readily proved using the above definition. For example, to prove that $(m_1 \circ m_2)^\top = m_2^\top \circ m_1^\top$ we may

Laws for transposition of linear maps

$$(m_1 \circ m_2)^\top = m_2^\top \circ m_1^\top \quad \text{Note reversed order!}$$

$$(m_1 \times m_2)^\top = m_1^\top \bowtie m_2^\top$$

$$(m_1 \bowtie m_2)^\top = m_1^\top \times m_2^\top$$

$$(m_1 \oplus m_2)^\top = m_1^\top \oplus m_2^\top$$

$$\mathbf{0}^\top = \mathbf{0}$$

$$\mathbf{1}^\top = \mathbf{1}$$

$$\mathcal{S}(k)^\top = \mathcal{S}(k)$$

$$(m^\top)^\top = m$$

$$\mathcal{V}(v)^\top = \mathcal{H}(\text{map } (\cdot)^\top v)$$

$$\mathcal{H}(v)^\top = \mathcal{V}(\text{map } (\cdot)^\top v)$$

$$\mathcal{L}(\lambda i.m)^\top = \mathcal{L}'(\lambda i.m^\top)$$

$$\mathcal{L}'(\lambda i.m)^\top = \mathcal{L}(\lambda i.m^\top)$$

Laws for reverse-application

$$r \circ_R m = m^\top \circ r \quad \text{By definition}$$

$$r \circ_R (m_1 \circ m_2) = (r \circ_R m_1) \circ_R m_2$$

$$(r_1, r_2) \circ_R (m_1 \times m_2) = (r_1 \circ_R m_1) + (r_2 \circ_R m_2)$$

$$r \circ_R (m_1 \bowtie m_2) = (r \circ_R m_1, r \circ_R m_2)$$

$$r \circ_R (m_1 \oplus m_2) = (r \circ_R m_1) + (r \circ_R m_2)$$

$$r \circ_R \mathbf{0} = \mathbf{0}$$

$$r \circ_R \mathbf{1} = r$$

$$r \circ_R \mathcal{S}(k) = k * r$$

$$r \circ_R m^\top = m \circ r$$

$$r \circ_R \mathcal{V}(v) = \Sigma_i (r[i] \circ_R v[i])$$

$$r \circ_R \mathcal{H}(v) = \text{build}(\text{sz}(v), \lambda i. r \circ_R m[i])$$

Figure 5. Laws for transposition

reason as follows:

$$\begin{aligned} & ((m_2^\top \circ m_1^\top) \circ t) \bullet s \\ &= (m_2^\top \circ (m_1^\top \circ t)) \bullet s \quad \text{Semantics of } (\circ) \\ &= (m_1^\top \circ t) \bullet (m_2 \circ s) \quad \text{Use (TP)} \\ &= t \bullet (m_1 \circ (m_2 \circ s)) \quad \text{Use (TP) again} \\ &= t \bullet ((m_1 \circ m_2) \circ s) \quad \text{Semantics of } (\circ) \end{aligned}$$

And now the property follows by Theorem 2.2.

2.4 Reverse linear-map application

Rather than transpose the linear map (which is a rather boring operation), just replacing one operator with another, it's easier to define a reverse-application operator for linear maps:

$$(\circ_R) : \delta t \rightarrow (s \circ t) \rightarrow \delta s$$

It is defined by the following property:

$$(RP) \quad \forall s : \delta S, t : \delta T. (t \circ_R m) \bullet s = t \bullet (m \circ s)$$

2.5 Matrix interpretation of linear maps

A linear map $m : \mathbb{R}^M \rightarrow \mathbb{R}^N$ is isomorphic to a matrix $\mathbb{R}^{N \times M}$ with N rows and M columns.

Many of the operators over linear maps then have simple matrix interpretations; for example, composition of linear maps (\circ) is matrix multiplication, pairing (\times) is vertical juxtaposition, and so on. These matrix interpretations are all given in the final column of Figure 3.

You might like to check that matrix transposition satisfies property (TP).

When it comes to implementation, we do not want to *represent* a linear map by a matrix, because a linear map $\mathbb{R}^M \rightarrow \mathbb{R}^N$ is an $N \times M$ matrix, which is enormous if $N = M = 10^6$, say. The function might be very simple (perhaps even the identity function) and taking 10^{12} numbers to represent it is plain silly. So our goal will be to *avoid realising linear maps as matrices*.

2.6 Optimisation

In optimisation we are usually given a function $f : \mathbb{R}^N \rightarrow \mathbb{R}$, where N can be large, and asked to find values of the input that maximises the output. One way to do this is by *gradient descent*: start with a point p , make a small change to $p + \delta_p$, and so on. From p we want to move in the direction of maximum slope. (How *far* to move in that direction is another matter — indeed no one knows — but we will concentrate on the *direction* in which to move.)

Suppose $\delta(i, N)$ is the one-hot N -vector with 1 in the i 'th position and zeros elsewhere. Then $\delta_p[i] = \partial f(p) \circ \delta(i, N)$ describes how fast the output of f changes for a change in the i 'th input. The direction of maximum slope is just the vector

$$\delta_p = (\delta_p[1] \ \delta_p[2] \ \dots \ \delta_p[N])$$

How can we compute this vector? We can simply evaluate $\partial f(p) \circ \delta(i, N)$ for each i . But that amounts to running f N times, which is bad if N is large (say 10^6).

Suppose that we somehow had access to $\partial_R f$. Then we can use property (TP), setting $\delta_f = 1$ to get

$$\forall \delta_p. \partial f(p) \circ \delta_p = (\partial_R f(p) \circ 1) \bullet \delta_p$$

Then

$$\begin{aligned}\delta_p[i] &= \partial f(p) \odot \delta(i, N) \\ &= (\partial_R f(p) \odot 1) \bullet \delta(i, N) \\ &= (\partial_R f(p) \odot 1)[i]\end{aligned}$$

That is $\delta_p[i]$ is the i 'th component of $\partial_R f(p) \odot 1$, so $\delta_p = \partial_R f(p) \odot 1$.

That is, $\partial_R f(p) \odot 1$ is the N-vector of maximum slope, the direction in which to move if we want to do gradient descent starting at p . And *that* is why the transpose is important.

2.7 Lambdas and linear maps

Notice the similarity between the type of (\times) and the type of \mathcal{L} ; the latter is really just an infinite version of the latter. Their semantics in Figure 4 are equally closely related.

The transpositions of these two linear maps, (\bowtie) and \mathcal{L}' , are similarly related. *But*, there is a problem with the semantics of \mathcal{L}' :

$$\mathcal{L}'(f) \odot g = \Sigma_i (f \ i) \odot g(i)$$

This is an *infinite sum*, so there is something fishy about this as a semantics.

2.8 Questions about linear maps

- Do we need 1? After all $S(1)$ does the same job. But asking if $k = 1$ is dodgy when k is a float.
- Do these laws fully define linear maps?

Notes

- In practice we allow n-ary versions of $m \bowtie n$ and $m \times n$.

3 AD as a source-to-source transformation

To perform source-to-source AD of a function f , we follow the plan outlined in Figure 6. Specifically, starting with a function definition $f(x) = e$:

- Construct the full Jacobian ∂f , and transposed full Jacobian $\partial_R f$, using the transformations in Figure 6².
- Optimise these two definitions, using the laws of linear maps in Figure 4.
- Construct the forward derivative $\text{fwd}\$f$ and reverse derivative $\text{rev}\$f$, as shown in Figure 6³.
- Optimise these two definitions, to eliminate all linear maps. Specifically:
 - Rather than *calling* ∂f (in, say, $\text{fwd}\$f$), instead *inline* it.
 - Similarly, for each local let-binding for a linear map, of form $\text{let } \partial x = e \text{ in } b$, inline ∂x at each of

²We consider ∂f and $\partial_R f$ to be the names of two new functions. These names are derived from f , but distinct from f , rather like f' or f_1 in mathematics.

³Again $\text{fwd}\$f$ and $\text{rev}\$f$ are new names, derived from f

Original function	$f : S \rightarrow T$ $f(x) = e$
Full Jacobian	$\partial f : S \rightarrow (S \multimap T)$ $\partial f(x) = \text{let } \partial x = 1 \text{ in } \nabla_S[e]$
Forward derivative	$\text{fwd}\$f : (S, S) \rightarrow T$ $\text{fwd}\$f(x, dx) = \partial f(x) \odot dx$
Reverse derivative	$\text{rev}\$f : (S, T) \rightarrow S$ $\text{rev}\$f(x, dr) = dr \odot_R \partial f(x)$

Differentiation of an expression

If $e : T$ then $\nabla_S[e] : S \multimap T$	
$\nabla_S[k]$	$= 0$
$\nabla_S[x]$	$= \partial x$
$\nabla_S[f(e)]$	$= \partial f(e) \odot \nabla_S[e]$
$\nabla_S[(e_1, e_2)]$	$= \nabla_S[e_1] \times \nabla_S[e_2]$
$\nabla_S[\text{let } x=e_1 \text{ in } e_2]$	$= \text{let } x = e_1 \text{ in}$ $\text{let } \partial x = \nabla_S[e_1] \text{ in}$ $\nabla_S[e_2]$
$\nabla_S[\text{build}(e_n, \lambda i. e)]$	$= \mathcal{V}(\text{build}(e_n, \lambda i. \nabla_S[e]))$
$\nabla_S[\lambda i. e]$	$= \mathcal{L}(\lambda i. \nabla_S[e])$

Figure 6. Automatic differentiation

its occurrences in b . This may duplicate e ; but ∂x is a function that may be applied (via \odot) to many different arguments, and we want to specialise it for each such call. (I think.)

- Optimise using the rules of (\odot) in Figure 4.
- Use standard Common Subexpression Elimination (CSE) to recover any lost sharing.

Note that

- The transformation is fully compositional; each function can be AD'd independently. For example, if a user-defined function f calls another user-defined function g , we construct ∂g as described; and then construct ∂f . The latter simply calls ∂g .
- The AD transformation is *partial*; that is, it does not work for every program. In particular, it fails when applied to a lambda, or an application; and, as we will

see in Section 4, it requires that *build* appears applied to a lambda.

- We give the full Jacobian for some built-in functions in Figure 6, including for conditionals (*if*).

3.1 Forward and reverse AD

Consider

$$f(x) = p(q(r(x)))$$

Just running the algorithm above on *f* gives

$$\begin{aligned} f(x) &= p(q(r(x))) \\ \partial f(x) &= \partial p \circ (\partial q \circ \partial r) \\ fwd\$f(x, dx) &= (\partial p \circ (\partial q \circ \partial r)) \odot dx \\ &= \partial p \odot ((\partial q \circ \partial r) \odot dx) \\ &= \partial p \odot (\partial q \odot (\partial r \odot dx)) \\ \partial_R f(x) &= (\partial_R r \circ \partial_R q) \circ \partial_R p \\ rev\$f(x, dr) &= ((\partial_R r \circ \partial_R q) \circ \partial_R p) \odot dr \\ &= (\partial_R r \circ \partial_R q) \odot (\partial_R p \odot dr) \\ &= \partial_R r \odot (\partial_R q \odot (\partial_R p \odot dr)) \end{aligned}$$

In “*The essence of automatic differentiation*” Conal says (Section 12)

The AD algorithm derived in Section 4 and generalized in Figure 6 can be thought of as a family of algorithms. For fully right-associated compositions, it becomes forward mode AD; for fully left-associated compositions, reverse-mode AD; and for all other associations, various mixed modes.

But the forward/reverse difference shows up quite differently here: it has nothing to do with *right-vs-left association*, and everything to do with *transposition*.

This is mysterious. Conal is not usually wrong. I would like to understand this better.

4 AD for vectors

Like other built-in functions, each built-in function for vectors has its full Jacobian versions, defined in Figure 2. You may enjoy checking that ∂sum and ∂ixR are correct!

For *build* there are two possible paths, and it's not yet clear which is best

Direct path. Figure 6 includes a rule for $\nabla_S \llbracket build(e_n, \lambda i.e) \rrbracket$.

But *build* is an exception! It is handled specially by the AD transformation in Figure 6; there is no $\partial build$. Moreover the AD transformation only works if the second argument of the *build* is a lambda, thus *build*($e_n, \lambda i.e$). I tried dealing with *build* and *lambdas* separately, but failed (see Section ??).

I did think about having a specialised linear map for indexing, rather than using $\mathcal{H}()$, but then I needed its transposition, so just using $\mathcal{H}()$ seemed more economical. On the other hand, with the functions as I have them, I need the grotesquely delicate optimisation rule

$$\begin{aligned} sum(build(n, \lambda i. \text{if } i == e_i \text{ then } e \text{ else } 0)) \\ = \text{let } i = e_i \text{ in } b \\ \text{if } i \notin e_i \end{aligned}$$

I hate this!

4.1 General folds

We have $sum :: Vec \mathbb{R} \rightarrow \mathbb{R}$. What is ∂sum ? One way to define its semantics is by applying it:

$$\begin{aligned} \partial sum &:: Vec \mathbb{R} \rightarrow (Vec \mathbb{R} \multimap \mathbb{R}) \\ \partial sum(v) \odot dv &= sum(dv) \end{aligned}$$

That is OK. But what about product, which multiplies all the elements of a vector together? If the vector had three elements we might have

$$\begin{aligned} \partial product([x_1, x_2, x_3]) \odot [dx_1, dx_2, dx_3] \\ = (dx_1 * x_2 * x_3) + (dx_2 * x_1 * x_3) + (dx_3 * x_1 * x_2) \end{aligned}$$

This looks very unattractive as the number of elements grows. Do we need to use product?

This gives the clue that taking the derivative of *fold* is not going to be easy, maybe infeasible! Much depends on the particular lambda it appears. So I have left out product, and made no attempt to do general folds.

5 Avoiding duplication

5.1 ANF and CSE

We may want to ANF-ise before AD to avoid gratuitous duplication. E.g.

$$\begin{aligned} \nabla_S \llbracket sqrt(x + (y * z)) \rrbracket \\ = \partial sqrt(x + (y * z)) \circ \nabla_S \llbracket x + (y * z) \rrbracket \\ = \partial sqrt(x + (y * z)) \circ \partial + (x, y * z) \\ \quad \circ (\nabla_S \llbracket x \rrbracket \times \nabla_S \llbracket y * z \rrbracket) \\ = \partial sqrt(x + (y * z)) \circ \partial + (x, y * z) \\ \quad \circ (\partial x \times (\partial * (y, z) \circ (\partial y \times \partial z))) \end{aligned}$$

Note the duplication of $y * z$ in the result. Of course, CSE may recover it.

5.2 Tupling: basic version

A better (and well-established) path is to modify $\partial f : S \rightarrow (S \multimap T)$ so that it returns a pair:

$$\overline{\partial f} : \forall a. (a \multimap S, S) \rightarrow (a \multimap T, T)$$

Original function	$f : S \rightarrow T$
	$f(x) = e$
Full Jacobian	$\overline{\partial}f : S \rightarrow (T, S \multimap T)$
	$\overline{\partial}f(x) = \text{let } \overline{\partial}x = (x, \mathbf{1}) \text{ in } \overline{\nabla}_S \llbracket e \rrbracket$
Forward derivative	$\text{fwd}\$f : (S, \delta S) \rightarrow (T, \delta T)$
	$\text{fwd}\$f(x, dx) = \overline{\partial}f(x) \odot dx$
Reverse derivative	$\text{rev}\$f : (S, \delta T) \rightarrow (T, \delta S)$
	$\text{rev}\$f(x, dfr) = dr \odot_R \overline{\partial}f(x)$
Differentiation of an expression	
	If $e : T$ then $\overline{\nabla}_S \llbracket e \rrbracket : (S \multimap T, T)$
	$\overline{\nabla}_S \llbracket k \rrbracket = (k, \mathbf{0})$
	$\overline{\nabla}_S \llbracket x \rrbracket = \overline{\partial}x$
	$\overline{\nabla}_S \llbracket (e_1, e_2) \rrbracket = \overline{\nabla}_S \llbracket e_1 \rrbracket \overline{\times} \overline{\nabla}_S \llbracket e_2 \rrbracket$
	$\overline{\nabla}_S \llbracket f(e) \rrbracket = \text{let } a = \overline{\nabla}_S \llbracket e \rrbracket \text{ in}$
	let $r = \overline{\partial}f(\pi_1(a))$ in
	$(\pi_1(r), \pi_2(r) \circ \pi_2(a))$
	$\overline{\nabla}_S \llbracket \text{let } x = e_1 \text{ in } e_2 \rrbracket = \text{let } \overline{\partial}x = \overline{\nabla}_S \llbracket e_1 \rrbracket \text{ in } \overline{\nabla}_S \llbracket e_2 \rrbracket$
	$\overline{\nabla}_S \llbracket \text{build}(e_n, \lambda i. e) \rrbracket = \text{let } p = \Phi(\text{build}(e_n, \lambda i. \overline{\nabla}_S \llbracket e \rrbracket)) \text{ in}$
	$(\pi_1(p), \mathcal{V}(\pi_2(p)))$
Modified linear-map operations	
	$(\odot) : (r, s \multimap t) \rightarrow \delta s \rightarrow \delta t$
	$(v, m) \odot ds = m \odot ds$
	$(\odot_R) : \delta t \rightarrow (r, s \multimap t) \rightarrow \delta s$
	$dr \odot_R vm = dr \odot vm$
	$(\overline{\times}) : ((t_1, s \multimap t_1), (t_2, s \multimap t_2)) \rightarrow ((t_1, t_2), s \multimap (t_1, t_2))$
	$(t_1, m_1) \overline{\times} (t_2, m_2) = ((t_1, t_2), m_1 \times m_2)$
	$(\overline{\bowtie}) : ((t_1, t_1 \multimap s), (t_2, t_2 \multimap s)) \rightarrow ((t_1, t_2), (t_1, t_2) \multimap s)$
	$(t_1, m_1) \overline{\bowtie} (t_2, m_2) = ((t_1, t_2), m_1 \bowtie m_2)$
	$\Phi : \text{Vec } (a, b) \rightarrow (\text{Vec } a, \text{Vec } b)$
	$\cdot \overline{\top} : (r, s \multimap t) \rightarrow (r, t \multimap s)$
Derivatives of built-in functions	
	$\overline{\partial}+ :: (\mathbb{R}, \mathbb{R}) \rightarrow ((\mathbb{R}, \mathbb{R}) \multimap \mathbb{R}, \mathbb{R})$
	$\overline{\partial}+(x, y) = (\mathbf{1} \bowtie \mathbf{1}, x + y)$
	$\overline{\partial}* :: (\mathbb{R}, \mathbb{R}) \rightarrow ((\mathbb{R}, \mathbb{R}) \multimap \mathbb{R}, \mathbb{R})$
	$\overline{\partial}*(x, y) = (S(y) \bowtie S(x), x * y)$

Figure 7. Automatic differentiation: tupling

That is $\overline{\partial f}$ returns the “normal result” T as well as a linear map.

5.3 Polymorphic tupling: forward mode

Everything works much more compositionally if $\overline{\partial f}$ also takes a linear map as its input. The new transform is shown in Figure 8. Note that there is no longer any code duplications, even without ANF or CSE.

In exchange, though, all the types are a bit more complicated. So we regard Figure 6 as canonical, to be used when working things out, and Figure 8 as a (crucial) implementation strategy.

The crucial property are these:

$$(CP) \quad \overline{\partial f}(e) \odot dx = fwd\$f(e \odot dx)$$

Crucial because suppose we have

$$f(x) = g(h(x))$$

Then, we can transform as follows, using (CP) twice, on lines marked (\dagger):

$$\begin{aligned} \overline{\partial f}(\bar{x}) &= \overline{\partial g}(\overline{\partial h}(\bar{x})) \\ fwd\$f(x, dx) &= \overline{\partial g}(\overline{\partial h}(x, 1) \odot dx) \\ &= fwd\$g(\overline{\partial h}(x, 1) \odot dx) \quad (\dagger) \\ &= fwd\$g(fwd\$h(x, 1) \odot dx) \quad (\dagger) \\ &= fwd\$g(fwd\$h(x, 1 \odot dx)) \\ &= fwd\$g(fwd\$h(x, dx)) \end{aligned}$$

Why is (CP) true? It follows from a more general property of $\overline{\partial f}$:

$$\begin{aligned} \forall f:S \rightarrow T, x:S, m_1:A \multimap S, m_2:B \multimap A, db:\delta B. \\ \overline{\partial f}(x, m_1) \odot (m_2 \odot db) &= \overline{\partial f}(x, m_1 \circ m_2) \odot db \\ \forall f:S \rightarrow T, x:S, m_1:S \multimap A, m_2:A \multimap B, dr:\delta T. \\ m_2 \odot (\overline{\partial_R f}(x, m_1) \odot dr) &= \overline{\partial_R f}(x, m_2 \circ m_1) \odot dr \end{aligned}$$

Now we can prove our claim as follows

$$\begin{aligned} fwd\$f(e \odot dx) &= \{\text{by defn of } (\odot)\} \\ fwd\$f(\pi_1(e), \pi_2(e) \odot dx) &= \{\text{by defn of } fwd\$f\} \\ \overline{\partial f}(\pi_1(e), 1) \odot (\pi_2(e) \odot dx) &= \{\text{by crucial property}\} \\ \overline{\partial f}(\pi_1(e), \pi_2(e)) \odot dx &= \overline{\partial f}(e) \odot dx \end{aligned}$$

5.4 Polymorphic tupling: reverse mode

It turns out that things work quite differently for reverse mode. For a start the equivalent of (CP) for reverse-mode would look like this:

$$\overline{\partial_R f}(e) \odot dr = rev\$f(e \odot dr)$$

But this is not even well-typed!

How did we use (CP)? Suppose f is defined in terms of g and h :

$$f(x) = g(h(x))$$

Then we want $fwd\$f$ to be defined in terms of $fwd\$g$ and $fwd\$h$. That is, we want a *compositional* method, where we can create the code for $fwd\$f$ without looking at the code for g or h , simply by calling g and h 's derived functions. And that's just what we achieved:

$$fwd\$f(x, dx) = fwd\$g(fwd\$h(x, dx))$$

But for reverse mode, this plan is much less straightforward. Look at the types:

$$\begin{aligned} f &: R \rightarrow T \\ g &: S \rightarrow T \\ h &: R \rightarrow S \\ rev\$f &: (R, \delta T) \rightarrow (T, \delta R) \\ rev\$g &: (S, \delta T) \rightarrow (T, \delta S) \\ rev\$h &: (R, \delta S) \rightarrow (S, \delta R) \end{aligned}$$

How can we define $rev\$f$ by calling $rev\$g$ and $rev\$h$? It would have to look something like this

$$\begin{aligned} rev\$f(r, dt) &= \text{letrec } (t, ds) = rev\$g(s, dt) \\ &\quad (s, dr) = rev\$h(r, ds) \\ &\quad \text{in } (t, dr) \end{aligned}$$

We can't call $rev\$g$ before $rev\$h$, nor the other way around. That's why there is a *letrec*! Even leaving aside how we generate this code, We'd need lazy evaluation to execute it.

The obvious alternative is to change $fwd\$f$'s interface. Currently we have

$$rev\$f : (R, \delta T) \rightarrow (T, \delta R)$$

Instead, we can take that R value, but return a function $\delta T \rightarrow \delta R$, thus:

$$rev\$f : R \rightarrow (T, \delta T \rightarrow \delta R)$$

But that commits to returning a *function*, with its fixed, built-in representation. Instead, let's return linear map:

$$rev\$f : R \rightarrow (T, \delta T \multimap \delta R)$$

Now we can re-interpret the retuned linear map as some kind of record (trace) of all the things that f did. And if we insist on our compositional account we really must *manifest* that data structure, and later apply it to a value of type δT to

Original function	$f : S \rightarrow T$
	$f(x) = e$
Full Jacobian	$\overline{\partial f} : \forall a. (S, a \multimap S) \rightarrow (T, a \multimap T)$
	$\overline{\partial f}(\bar{x}) = \overline{\nabla}_a \llbracket e \rrbracket$
Transposed Jacobian	$\overline{\partial_R f} : \forall a. (S, S \multimap a) \rightarrow (T, T \multimap a)$
	$\overline{\partial_R f}(\bar{x}) = (\overline{\partial f}(\bar{x}))^\top$
Forward derivative	$fwd\$f : (S, \delta S) \rightarrow (T, \delta T)$
	$fwd\$f(x, dx) = \overline{\partial f}(x, 1) \odot dx$
Reverse derivative	$rev\$f : (S, \delta T) \rightarrow (T, \delta S)$
	$rev\$f(x, dr) = \overline{\partial_R f}(x, 1) \odot dr$
Differentiation of an expression	
	If $e : T$ then $\overline{\nabla}_a \llbracket e \rrbracket : (T, a \multimap T)$
	$\overline{\nabla}_a \llbracket k \rrbracket = (k, \mathbf{0})$
	$\overline{\nabla}_a \llbracket x \rrbracket = \bar{x}$
	$\overline{\nabla}_a \llbracket f(e) \rrbracket = \overline{\partial f}(\overline{\nabla}_a \llbracket e \rrbracket)$
	$\overline{\nabla}_a \llbracket (e_1, e_2) \rrbracket = \overline{\nabla}_a \llbracket e_1 \rrbracket \overline{\times} \overline{\nabla}_a \llbracket e_2 \rrbracket$
	$\overline{\nabla}_a \llbracket \text{let } x=e_1 \text{ in } e_2 \rrbracket = \text{let } \bar{x}=\overline{\nabla}_a \llbracket e_1 \rrbracket \text{ in } \overline{\nabla}_a \llbracket e_2 \rrbracket$
Modified linear-map operations	
	$(\odot) : (r, s \multimap t) \rightarrow \delta s \rightarrow (r, \delta t)$
	$(v, m) \odot ds = (v, m \odot ds)$
	$(\overline{\times}) : ((t_1, s \multimap t_1), (t_2, s \multimap t_2)) \rightarrow ((t_1, t_2), s \multimap (t_1, t_2))$
	$(t_1, m_1) \overline{\times} (t_2, m_2) = ((t_1, t_2), m_1 \times m_2)$
	$(\overline{\bowtie}) : ((t_1, t_1 \multimap s), (t_2, t_2 \multimap s)) \rightarrow ((t_1, t_2), (t_1, t_2) \multimap s)$
	$(t_1, m_1) \overline{\bowtie} (t_2, m_2) = (t_1 + t_2, m_1 \bowtie m_2)$
	$\cdot^\top : (t, s \multimap t) \rightarrow (t, t \multimap s)$
Derivatives of built-in functions	
	$\overline{\partial+} :: \forall a. ((\mathbb{R}, \mathbb{R}), a \multimap (\mathbb{R}, \mathbb{R})) \rightarrow (\mathbb{R}, a \multimap \mathbb{R})$
	$\overline{\partial+}((x, y), m) = (x + y, (1 \bowtie 1) \odot m)$
	$\overline{\partial*} :: \forall a. ((\mathbb{R}, \mathbb{R}), a \multimap (\mathbb{R}, \mathbb{R})) \rightarrow (\mathbb{R}, a \multimap \mathbb{R})$
	$\overline{\partial*}((x, y), m) = (x * y, (S(y) \bowtie S(x)) \odot m)$

Figure 8. Automatic differentiation: polymorphic tuples

Atoms $f, g, h ::= \text{Function}$ $k ::= \text{Literal constants}$ **Terms** $\text{pgm} ::= \text{def}_1 \dots \text{def}_n$ $\text{def} ::= f:S \Rightarrow T = c$ $c ::= I \quad \text{Identity}$ $| \mathcal{K}(k) \quad \text{Constant}$ $| \mathcal{P}[i_1, \dots, i_m/n] \quad \text{Pruning}(0 \leq m \leq n)$ $| \mathcal{F}(f) \quad \text{Function constant}$ $| c_1 ; c_2 \quad \text{Composition}$ $| (c_1, \dots, c_n) \quad \text{Tuple}$ $| \mathcal{IF}(c_1, c_2, c_3) \quad \text{Conditional}$ $| \mathcal{L}(x, c_r, c_b) \quad \text{Let}$ $| \mathcal{B}(c_s, i, c_e) \quad \text{Build}$ **Figure 9.** Syntax of CL

get a value of type δR . We could represent those linear maps as:

- A matrix
- A function closure that, when called, applies the linear map to an argument
- A syntax tree whose nodes are the constructors of the linear map type. When applying the linear map, we interpret that syntax tree.

Finally, notice that this final version of $\text{fwd}\$f$ is exactly $\overline{\partial_R f}$, just specialised with an input linear map of 1. So we may as well just use $\overline{\partial_R f}$, which *already* compositionally calls $\overline{\partial_R g}$ and $\overline{\partial_R h}$.

TL;DR: for reverse mode, we must simply compile $\overline{\partial_R f}$.

Notice that we can get quite a bit of optimisation by inlining $\overline{\partial_R g}$ into $\overline{\partial_R f}$, and so on. The more inlining the better. If we inline everything we'll eliminate all intermediate linear maps.

6 Compiling through categories

6.1 Splitting for reverse mode

Suppose f is defined in terms of g and h :

$$f(x) = g(h(x))$$

Semantics (aka conversion from CL): $e \diamond c = e$

$$t \diamond I = t$$

$$t \diamond \mathcal{P}[i_1, \dots, i_m/n] = (\pi_{i_1,n}(t), \dots, \pi_{i_m,n}(t))$$

$$t \diamond \mathcal{K}(k) = k$$

$$t \diamond \mathcal{F}(f) = f(t)$$

$$t \diamond (c_1 ; c_2) = (t \diamond c_1) \diamond c_2$$

$$t \diamond (c_1, \dots, c_n) = (t \diamond c_1, \dots, t \diamond c_n)$$

$$t \diamond \mathcal{IF}(c_1, c_2, c_3) = \text{if } (t \diamond c_1) \text{ then } (t \diamond c_2) \text{ else } (t \diamond c_3)$$

$$t \diamond \mathcal{L}(x, c_r, c_b) = \text{let } x = t \diamond c_r \text{ in } (t \diamond c_b)$$

$$t \diamond \mathcal{B}(c_s, i, c_e) = \text{build } (t \diamond c_s) \text{ } (\lambda i. (t \diamond i) \diamond c_e)$$

Conversion to CL

$$\Gamma ::= (x_1 : \tau_1, \dots, x_n : \tau_n)$$

$$\phi((x_1 : \tau_1, \dots, x_n : \tau_n), x_i) = i$$

$$T(x_1 : \tau_1, \dots, x_n : \tau_n) = (\tau_1, \dots, \tau_n)$$

$$C[f(x_1 : \tau_1, \dots, x_n : \tau_n) = e]$$

$$= \mathcal{F}(f) = C[e] (x_1 : \tau_1, \dots, x_n : \tau_n)$$

$$\boxed{\text{If } \Gamma \vdash e : \tau \text{ then } C[e] \Gamma : T(\Gamma) \Rightarrow \tau}$$

$$C[k] \Gamma = \mathcal{K}(k)$$

$$C[x] \Gamma = \mathcal{F}(\pi(\Gamma, x))$$

$$C[f(e)] \Gamma = C[e] \Gamma ; \mathcal{F}(f)$$

$$C[\text{if } e_1 \text{ then } e_2 \text{ else } e_3] \Gamma$$

$$= \mathcal{IF}(C[e_1] \Gamma, C[e_2] \Gamma, C[e_3] \Gamma)$$

$$C[(e_1, \dots, e_n)] \Gamma = (C[e_1] \Gamma, \dots, C[e_n] \Gamma)$$

$$C[\text{let } x:\tau = e_r \text{ in } e_b] \Gamma = \mathcal{L}(x, C[e_r] \Gamma, C[e_b] \Gamma, (\Gamma, x:\tau))$$

$$C[\text{build } e_s \text{ } (\lambda i. e_e)] \Gamma = \mathcal{B}(C[e_s] \Gamma, i, C[e_e] \Gamma, (\Gamma, i))$$

Pruning

$$C[e] \Gamma = \mathcal{P}[\phi(\Gamma, v_1), \dots, \phi(\Gamma, v_m)/sz(\Gamma)](C[e] \Gamma')$$

$$\text{where } \{v_1, \dots, v_m\} = fv(e)$$

$$\Gamma' = (v_1 : \Gamma(v_1), \dots, v_n : \Gamma(v_n))$$

Figure 10. Semantics of CL

Here are the types:

$$f : R \rightarrow T$$

$$g : S \rightarrow T$$

$$h : R \rightarrow S$$

$$\text{rev}\$f : (R, \delta T) \rightarrow (T, \delta R)$$

$$\text{rev}\$g : (S, \delta T) \rightarrow (T, \delta S)$$

$$\text{rev}\$h : (R, \delta S) \rightarrow (S, \delta R)$$

$\Gamma \vdash c : S \Rightarrow T$
$\Gamma \vdash I : S \Rightarrow S$
$\Gamma \vdash \mathcal{P}[i_1, \dots, i_m/n] : (s_1, \dots, s_n) \Rightarrow (s_{i_1}, \dots, s_{i_m})$
$\frac{f : S \rightarrow T \in \Gamma}{\Gamma \vdash \mathcal{F}(f) : S \Rightarrow T} \quad \frac{}{\Gamma \vdash \mathcal{K}(k) : () \Rightarrow \mathbb{R}}$
$\frac{\Gamma \vdash c_1 : S \Rightarrow R \quad \Gamma \vdash c_2 : R \Rightarrow T}{\Gamma \vdash c_1 ; c_2 : S \Rightarrow T}$
$\frac{\Gamma \vdash c_1 : S \Rightarrow T_1 \quad \dots \quad \Gamma \vdash c_n : S \Rightarrow T_n}{\Gamma \vdash (c_1, \dots, c_n) : S \Rightarrow (T_1, \dots, T_n)}$
$\frac{\Gamma \vdash c_1 : S \Rightarrow \mathbb{B} \quad \Gamma \vdash c_2 : S \Rightarrow T \quad \Gamma \vdash c_3 : S \Rightarrow T}{\Gamma \vdash \mathcal{IF}(c_1, c_2, c_3) : S \Rightarrow T}$
$\frac{\Gamma \vdash c_r : S \Rightarrow R \quad \Gamma \vdash c_b : (S \triangleright R) \Rightarrow T}{\Gamma \vdash \mathcal{L}(x, c_r, c_b) : S \Rightarrow T}$
$\frac{\Gamma \vdash c_s : S \Rightarrow \mathbb{N} \quad \Gamma \vdash c_e : (S \triangleright \mathbb{N}) \Rightarrow T}{\Gamma \vdash \mathcal{B}(c_s, i, c_e) : S \Rightarrow \text{Vec } T}$

Figure 11. Type system for CL

How can we define $\text{rev}\$f$ by calling $\text{rev}\$g$ and $\text{rev}\$h$? It would have to look something like this

$$\begin{aligned} \text{rev}\$f(r, dt) = & \text{letrec } (t, ds) = \text{rev}\$g(s, dt) \\ & (s, dr) = \text{rev}\$h(r, ds) \\ & \text{in } (t, dr) \end{aligned}$$

We can't call $\text{rev}\$g$ before $\text{rev}\$h$, nor the other way around. That's why there is a `letrec`! Even leaving aside how we generate this code, We'd need lazy evaluation to execute it.

The key idea for splitting is this. Given $f : S \rightarrow T$, produce two functions

$$\begin{aligned} \text{rev}\$f & : S \rightarrow (T, X) \\ \text{revr}\$f & : (X, \delta T) \rightarrow \delta S \end{aligned}$$

where the type X depends on the details of f 's definition. The idea is that X records all the stuff that f computed when running forward that is necessary for it to run backward.

Now we can write

$$\begin{aligned} \text{rev}\$f(s, dt) &= \text{letrec } (t, xf) = \text{rev}\$f(s) \\ & \quad ds = \text{rev}\$f(xf, dt) \\ & \quad \text{in } (t, ds) \\ \text{rev}\$f(r) &= \text{letrec } (s, xh) = \text{rev}\$h(r) \\ & \quad (t, xg) = \text{rev}\$g(r) \\ & \quad \text{in } (t, (xh, xg)) \\ \text{revr}\$f((xh, xg), dt) &= \text{revr}\$h(dh, \text{revr}\$g(dg, gt)) \end{aligned}$$

7 Implementation

The implementation differs from this document as follows:

- Rather than pairs, the implementation supports n -ary tuples. Similarly the linear maps (\times) and \bowtie are n -ary.
- Functions definitions can take n arguments, thus

$$f(x, y, z) = e$$

This is treated as equivalent to

$$\begin{aligned} f(t) &= \text{let } x = \pi_{1,3}(t) \\ & \quad y = \pi_{2,3}(t) \\ & \quad z = \pi_{3,3}(t) \\ & \quad \text{in } e \end{aligned}$$

8 Fold

9 Looping in split mode

10 Procedure language

The typing judgement for a procedure p is of the form $\Gamma \vdash p \dashv \Delta$.

Here the “context” Γ contains the types of the free variables, those which the procedure uses on the right hand side of an $=$ sign but that were not bound on the left hand side earlier in the procedure. The “co-context” Δ contains the types of the “cofree” variables, i.e. those which are bound on the left hand side of an $=$ sign but that are not used on the right hand side later in the procedure.

The requirement that there are no compound expressions in the procedure language leads to very verbose code, analogous to ANF. The reverse mode AD pass relies critically on this property, so although we could relax the condition and permit nested subexpressions we would have to apply an explicit ANF pass before applying the reverse mode transform. We will live with the verbosity for now.

11 BOG-style AD for ksc

See Figure 24 for BOG-style AD for ksc. This Figure assumes that the input language is a *single-use dialect* of ksc, in which every binder is used exactly once.

Typing rules for fold

$$\begin{aligned}
t &: (a, b) \\
e &: a \\
acc &: a \\
v &: \text{Vec } b \\
\text{fold } (\lambda t.e) \text{ acc } v &: a
\end{aligned}$$

Typing rules for lmFold

$$\begin{aligned}
t &: (a, b) \\
e &: a \\
e' &: (s, (a, b)) \multimap a \\
acc &: a \\
v &: \text{Vec } b \\
\text{lmFold } (\lambda t.e) (\lambda t.e') \text{ acc } v &: (s, (a, \text{Vec } b)) \multimap a
\end{aligned}$$

Typing rules for FFold and RFold

$$\begin{aligned}
t &: (a, b) \\
t_{dr} &: ((a, b), \delta a) \\
t_{dt} &: ((a, b), (\delta a, \delta b)) \\
e &: a \\
e_{dr} &: (\delta s, (\delta a, \delta b)) \\
e_{dt} &: \delta a \\
acc &: a \\
v &: \text{Vec } b \\
dr &: \delta a \\
d_{acc} &: \delta a \\
d_v &: \text{Vec } \delta b \\
\text{FFold } (\lambda t.e) \text{ acc } v (\lambda t_{dt}.e_{dt}) d_{acc} d_v &: \delta a \\
\text{RFold } (\lambda t.e) (\lambda t_{dr}.e_{dr}) \text{ acc } v dr &: (\delta s, (\delta a, \text{Vec } \delta b))
\end{aligned}$$

Figure 12. Rules for fold

Where is this assumption used? The reverse-pass transformation $\nabla_r \llbracket e \rrbracket$ creates a let-binding for every occurrence of a variable in e . If any variable occurs more than once, these bindings will conflict, or (equally bad) one will shadow the other so that the earlier one is ignored. Instead we rely on uses of `dup` to duplicate binders; in the reverse pass dup_{rbog} adds up the contribution of the two occurrences.

Note: In place of $\mathcal{B}[e]$ (a data structure) we could use a partial application of $e_{rbog}(\mathcal{B}[e]) : \delta T \rightarrow \delta S$.

Differentiation of fold

If $e : T$ then $\nabla_s \llbracket e \rrbracket : s \multimap T$
 $\nabla_s \llbracket \text{fold } (\lambda t.e) \text{ acc } v \rrbracket = \text{lmFold } (\lambda t.e) (\lambda t.e') \text{ acc } v \circ p$
 where $p : s \multimap (s, (a, \text{Vec } b))$
 $p = 1_s \times (\nabla_s \llbracket \text{acc} \rrbracket \times \nabla_s \llbracket v \rrbracket)$
 $e' = \text{let } \nabla x = \nabla x \circ (1_s \bowtie 0_s^{(a,b)})$
 \dots for each x occurring free in $\lambda t.e$
 $\text{let } \nabla t = 0_{(a,b)}^s \bowtie 1_{(a,b)}$
 $\text{in } \nabla_{(s, (a,b))} \llbracket e \rrbracket$

Applying an lmFold

$\text{lmFold } (\lambda t.e) (\lambda t.e') \text{ acc } v \odot dx = \text{FFold } (\lambda t.e) \text{ acc } v (\lambda t_{dt}.e_{dt}) d_{\text{acc}} d_v$
 where $e_{dt} = \text{let } t = \pi_1(t_{dt})$
 $\text{let } dt = \pi_2(t_{dt})$
 $\text{in } e' \odot (ds, dt)$
 $ds = \pi_1(dx)$
 $d_{\text{acc}} = \pi_1(\pi_2(dx))$
 $d_v = \pi_2(\pi_2(dx))$
 $dx \odot_R \text{lmFold } (\lambda t.e) (\lambda t.e') \text{ acc } v = \text{RFold } (\lambda t.e) (\lambda t_{dr}.e_{dr}) \text{ acc } v dx$
 where $e_{dr} = \text{let } t = \pi_1(t_{dr})$
 $\text{let } dr = \pi_2(t_{dr})$
 $\text{in } dr \odot_R e'$

Figure 13. Rules for fold

```

def FFold dA ((f : F) (acc : A) (v : Vec n B)
              (f_ : F_) (dacc : dA) (dv : Vec n dB))
  = FFold_recursive(0, f, acc, v f_, dacc, dv)

def FFold_recursive dA ((i : Integer) (f : F) (acc : A) (v : Vec n B)
                        (f_ : F_) (dacc : dA) (dv : Vec n dB))
  = if i == n
    then dacc
    else let fwd_f = f_((acc, v[i]), (dacc, dv[i]))
         in FFold_recursive(i + 1, f, f(acc, v[i]), v, f_, fwd_f, dv)

```

Figure 14. Forward mode derivative for fold


```

1541 def RFold (S, (dA, Vec n dB)) ((f : F) (f_ : F_) (acc : A) (v : Vec n B) (dr : dA))
1542   = let (ds, dv, da) = RFold_recursive(f, f_, 0, v, acc, dr)
1543     in (s, (da, dv))
1544
1545 def RFold_recursive (S, Vec n dB, dA) ((f : F) (f_ : F_) (i : Integer) (v : Vec n B)
1546   (acc : A) (dr : dA))
1547   = if i == n
1548     then (0, 0, dr)
1549     else let (r_ds, r_dv, r_dacc) = RFold_recursive(f, f_, i + 1, v, f(acc, v[i]), dr)
1550           (f_ds, (f_dacc, f_db)) = f_((acc, v[i]), r_dacc)
1551           in (r_ds + f_ds, r_dv + deltaVec(i, f_db), f_dacc)

```

Figure 15. Reverse mode derivative for fold

nTimes

$$\begin{array}{l}
 n : \text{Integer} \\
 sInitial : s \\
 f : s \rightarrow s \text{ (known function)} \\
 nTimes\ n\ sInitial\ f : s
 \end{array}$$
revf\$nTimes

$$\begin{array}{l}
 n : \text{Integer} \\
 sInitial : s \\
 f : s \rightarrow s \text{ (known function)} \\
 revf\$nTimes\ n\ sInitial\ f : (s, Vec\ BOG[f])
 \end{array}$$
revr\$nTimes

$$\begin{array}{l}
 dds' : \delta s \\
 bog' : Vec\ BOG[f] \\
 revr\$nTimes\ (bog', dds') : ((), \delta s, ())
 \end{array}$$

Figure 16. Typing rules for nTimes

Semantics of nTimes

$$\begin{aligned} \text{nTimes } 0 \text{ } sInitial \text{ } f &= sInitial \\ \text{nTimes } n \text{ } sInitial \text{ } f &= \text{nTimes } (n - 1) \text{ } f(sInitial) \text{ } f \quad \text{if } n > 0 \\ \text{nTimes } n \text{ } sInitial \text{ } f &= \text{undefined} \quad \text{if } n < 0 \end{aligned}$$
revf\$nTimes

```

revf$nTimes n s f =
  let (_, bog', s') = nTimes n (0, uninitializedVector n, s) (\(i, bog, s) ->
    let (s', bogf) = revf$f s
    bog'           = setAt i bog bogf
    i'             = i + 1
    in (i', bog', s'))
  in (bog', s')

```

revr\$nTimes

```

revr$nTimes (bog', dds') =
  let (_, _, dds) = nTimes (size bog') (size bog', bog', dds') n (\(i', bog', dds') ->
    let i      = i' - 1
    bogf = index i bog'
    bog  = bog'
    dds  = revr$f(bogf, dds')
    in (i, bog, dds)
  in ((), dds, ())

```

Figure 17. Behaviour of nTimes

ccl	
	$n : Integer$
	$sInitial : s$
	$f : (Integer, s) \rightarrow s$ (known function)
	$ccl\ n\ sInitial\ f : s$
revf\$ccl	
	$n : Integer$
	$sInitial : s$
	$f : (Integer, s) \rightarrow s$ (known function)
	$revf$ccl\ n\ sInitial\ f : (s, (n, s))$
revr\$ccl	
	$n : Integer$
	$s : s$
	$dds' : \delta s$
	$revr$ccl\ ((n, s), dds') : ((), \delta s, ())$

Figure 18. Typing rules for ccl**Semantics of ccl**

	$ccl\ 0\ sInitial\ f = sInitial$
	$ccl\ n\ sInitial\ f = ccl\ (n - 1)\ f(n - 1, sInitial)\ f \quad \text{if } n > 0$
	$ccl\ n\ sInitial\ f = \text{undefined} \quad \text{if } n < 0$
revf\$ccl	
	$revf$ccl\ n\ s\ f = (ccl\ n\ s\ f, (n, s))$
revr\$ccl	
	$revr$ccl\ ((n, s), dds') =$
	$\text{let } (_, dds) = ccl\ n\ (s, dds')\ (\backslash(i, (s, dds')) \rightarrow$
	$\text{let } (s', bogf) = revf$f\ s$
	$dds = revr$f(bogf, dds')$
	$\text{in } (s', dds)$
	$\text{in } ((), dds, ())$

This form of reverse pass is only correct if $revr\$f$ is commutative in the sense that

$$\forall bog1, bog2, dds' \quad revr\$f(bog1, revr\$f(bog2, dds')) = revr\$f(bog2, revr\$f(bog1, dds'))$$

Figure 19. Behaviour of ccl

Atoms		
f, g, h	$::=$ Function	
x, y, z	$::=$ Variable	
k	$::=$ Literal constant	
Terms		
$statement$	$::=$ $x = Call\ f\ y$	Function call
	$x = Const\ k$	Constant
	$Elim\ y$	Elimination
	$x = Dup\ y$	Duplication
	$x = (y, z)$	Tuple constructor
	$(x, y) = z$	Tuple pattern match
	$x = Inl\ y$	Sum constructor
	$x = Inr\ y$	Sum constructor
	$Case\ x\ (Inl\ x\ procedure)(Inr\ y\ procedure)$	Sum pattern match
$procedure$	$::=$ $statement$	
	$procedure\ ;\ procedure$	
Types		
τ	$::=$ \mathbb{R}	Real numbers
	(τ_1, τ_2)	Pairs
	$\tau_1 \oplus \tau_2$	Sums

Figure 20. Syntax of the procedure language

1981	$\boxed{\Gamma \vdash p \dashv \Delta}$	2036
1982		2037
1983		2038
1984	$\frac{f : S \rightarrow T}{a : S \vdash b = \text{Call } f \ a \dashv b : T}$	2039
1985		2040
1986	$\frac{}{a : S \vdash t = \text{Dup } s \dashv t : (S, S)}$	2041
1987		2042
1988	$\frac{k : T}{\vdash a = \text{Const } k \dashv a : T}$	2043
1989		2044
1990	$\frac{}{a : T \vdash \text{Elim } a \dashv}$	2045
1991		2046
1992	$\frac{}{a_1 : T_1, a_2 : T_2 \vdash b = (a_1, a_2) \dashv b : (T_1, T_2)}$	2047
1993		2048
1994	$\frac{}{b : (T_1, T_2) \vdash (a_1, a_2) = b \dashv a_1 : T_1, a_2 : T_2}$	2049
1995		2050
1996	$\frac{}{b : T_1 \vdash a = \text{Inl } b \dashv a : T_1 \oplus T_2}$	2051
1997		2052
1998	$\frac{}{b : T_2 \vdash a = \text{Inr } b \dashv a : T_1 \oplus T_2}$	2053
1999		2054
2000	$\frac{\Gamma_1 \vdash p1 \dashv \Xi, \Delta_1 \quad \Gamma_2, \Xi \vdash p2 \dashv \Delta_2}{\Gamma_1, \Gamma_2 \vdash p1; p2 \dashv \Delta_1, \Delta_2}$	2055
2001		2056
2002		2057

Figure 21. Type system for procedure language

" $y = 5 + 6$ "

$x_1 = \text{Const } 5$

$x_2 = \text{Const } 6$

$t = (x_1, x_2)$

$y = \text{Call add } t$

$\vdash \cdot \vdash y : \mathbb{R}$

" $(x, y) = (r \cos \theta, r \sin \theta)$ "

$(r_1, r_2) = \text{Dup } r$

$(\theta_1, \theta_2) = \text{Dup } \theta$

$ct = \text{Call cos } \theta_1$

$st = \text{Call sin } \theta_2$

$rct = (r_1, ct)$

$rst = (r_2, st)$

$x = \text{Call mul } rct$

$y = \text{Call mul } rst$

$r : \mathbb{R}, \theta : \mathbb{R} \vdash \cdot \vdash x : \mathbb{R}, y : \mathbb{R}$

" $r = x^2 + \sin xy + 1$ "

$(x_1, x_2) = \text{Dup } x$

$(x_3, x_4) = \text{Dup } x_1$

$xx = (x_2, x_3)$

$xsq = \text{Call mul } xx$

$xy = (x_4, y)$

$xmuly = \text{Call mul } xy$

$\sin xy = \text{Call sin } xmuly$

$o = \text{Const } 1$

$\sin xy1 = \text{Call add } \sin xy \ o$

$r = \text{Call add } xsq \ \sin xy1$

$x : \mathbb{R}, y : \mathbb{R} \vdash \cdot \vdash r : \mathbb{R}$

Figure 22. Example procedures

Γ	$\Gamma \vdash p \dashv \Delta$	Δ	$\Gamma \vdash F\llbracket p \rrbracket \dashv \Delta, \Upsilon_p$	Υ_p	$\bar{\Delta}, \Upsilon_p \vdash R\llbracket p \rrbracket \dashv \bar{\Gamma}$
a	$b = \text{Call } f \ a$	b	$bt = \text{Call } rf\$f \ a$ $(b, t_{b;a}) = bt$	$t_{b;a}$	$bt' = (\bar{b}, t_{b;a})$ $\bar{a} = \text{Call } rr\$f \ bt'$
s	$t = \text{Dup } s$	t	$t = \text{Dup } s$	\emptyset	$\bar{s} = \text{Call } add \ \bar{t}$
\emptyset	$a = \text{Const } k$	a	$a = \text{Const } k$	\emptyset	$\text{Elim } \bar{a}$
a	$\text{Elim } a$		$\text{Elim } a$	\emptyset	$\bar{a} = \text{Const } 0$
a_1, a_2	$b = (a_1, a_2)$	b	$b = (a_1, a_2)$	\emptyset	$(\bar{a}_1, \bar{a}_2) = \bar{b}$
b	$(a_1, a_2) = b$	a_1, a_2	$(a_1, a_2) = b$	\emptyset	$\bar{b} = (\bar{a}_1, \bar{a}_2)$
b	$a = \text{Inl } b$	a	$a = \text{Inl } b$	\emptyset	$\text{Case } \bar{a}$ $(\text{Inl } \bar{b})$ $(\text{Inr } \bar{z} (\text{Elim } \bar{z}; \bar{b} = \text{Const } 0))$
b	$a = \text{Inr } b$	a	$a = \text{Inr } b$	\emptyset	$\text{Case } \bar{a}$ $(\text{Inl } \bar{z} (\text{Elim } \bar{z}; \bar{b} = \text{Const } 0))$ $(\text{Inr } \bar{b})$
Γ_1, Γ_2	$p1; p2$	Δ_1, Δ_2	$F\llbracket p1 \rrbracket; F\llbracket p2 \rrbracket$	$\Upsilon_{p1}, \Upsilon_{p2}$	$R\llbracket p2 \rrbracket; R\llbracket p1 \rrbracket$

$t_{b;a}$ is a fresh variable name, $\bar{\cdot}$ bars all the variables in collection of variables it applies to

Figure 23. Procedure language reverse mode translation rules

This figure assumes the single-use dialect of ksc, in which every binder is used at most once.

Original function $f : S \rightarrow T$
 $f(s) = e$

Forward BOG

$$f_{fbog} : S \rightarrow (T, \mathcal{B}[e])$$

$$f_{fbog}(s) = \nabla_f[e]$$

If $e : T$ then $\nabla_f[e] : (T, \mathcal{B}[e])$

$$\nabla_f[k] = (k, ())$$

$$\nabla_f[x] = (x, ())$$

$$\begin{aligned} \nabla_f[f(e)] &= \text{let } (a, b_e) = \nabla_f[e] \text{ in} \\ &\quad \text{let } (r, b_f) = f_{fbog}(a) \text{ in} \\ &\quad (r, (b_e, b_f)) \end{aligned}$$

$$\begin{aligned} \nabla_f[(e_1, e_2)] &= \text{let } (a_1, b_1) = \nabla_f[e_1] \text{ in} \\ &\quad \text{let } (a_2, b_2) = \nabla_f[e_2] \text{ in} \\ &\quad ((a_1, a_2), (b_1, b_2)) \end{aligned}$$

$$\begin{aligned} \nabla_f[\text{let } x=e_1 \text{ in } e_2] &= \text{let } (x, b_1) = \nabla_f[e_1] \text{ in} \\ &\quad \text{let } (r, b_2) = \nabla_f[e_2] \text{ in} \\ &\quad (r, (b_1, b_2)) \end{aligned}$$

$$\begin{aligned} \nabla_f[\text{let } (x, y)=e_1 \text{ in } e_2] &= \text{let } (xy, b_1) = \nabla_f[e_1] \text{ in} \\ &\quad \text{let } (x, y) = xy \text{ in} \\ &\quad \text{let } (r, b_2) = \nabla_f[e_2] \text{ in} \\ &\quad (r, (b_1, b_2)) \end{aligned}$$

$$\nabla_f[\text{build } e_1 (\lambda i. e_2)] =$$

$$\begin{aligned} &\text{let } n = e_1 \text{ in} \\ &\text{let } (r, ba) = \text{unzip } (\text{build } n (\lambda i. \nabla_f[e_2])) \text{ in} \\ &(r, (n, ba)) \end{aligned}$$

$$\nabla_f[\text{case } e_1 \text{ of } \{ \text{Inl } x \rightarrow e_l; \text{Inr } y \rightarrow e_r \}] =$$

$$\begin{aligned} &\text{case } e_1 \text{ of} \\ &\quad \text{Inl } x \rightarrow \text{let } (r_l, b_l) = \nabla_f[e_l] \text{ in } (r_l, \text{Inl } b_l) \\ &\quad \text{Inr } y \rightarrow \text{let } (r_r, b_r) = \nabla_f[e_r] \text{ in } (r_r, \text{Inr } b_r) \end{aligned}$$

Reverse BOG

$$f_{rbog} : (\delta T, \mathcal{B}[e]) \rightarrow \delta S$$

$$f_{rbog}(\partial t, b) = \text{let } \nabla_r[e] \partial t \ b \text{ in } \delta s$$

If $\Gamma \vdash e : T$ and $\partial t : \delta T$ and $b : \mathcal{B}[e]$, then $\nabla_r[e] \partial t \ b$ is a set of bindings that, for every free variable $x : X$ of e , binds $\partial x : \delta X$

$$\nabla_r[k] \partial t \ b = \{ \}$$

$$\nabla_r[x] \partial t \ b = \{ \partial x = \partial t \}$$

$$\begin{aligned} \nabla_r[f(e)] \partial t \ b &= \{ \quad (b_e, b_f) = b \\ &\quad ; \quad \partial a = f_{rbog}(\partial t, b_f) \quad \} \\ &\quad ++ \nabla_r[e] \partial a \ b_e \end{aligned}$$

$$\begin{aligned} \nabla_r[(e_1, e_2)] \partial t \ b &= \{ \quad (\partial t_1, \partial t_2) = \partial t \\ &\quad ; \quad (b_1, b_2) = b \quad \} \\ &\quad ++ \nabla_r[e_1] \partial t_1 \ b_1 \\ &\quad ++ \nabla_r[e_2] \partial t_2 \ b_2 \end{aligned}$$

$$\begin{aligned} \nabla_r[\text{let } x=e_1 \text{ in } e_2] \partial t \ b &= \{ \quad (b_1, b_2) = b \quad \} \\ &\quad ++ \nabla_r[e_2] \partial t \ b_2 \\ &\quad ++ \nabla_r[e_1] \partial x \ b_1 \end{aligned}$$

$$\begin{aligned} \nabla_r[\text{let } (x, y)=e_1 \text{ in } e_2] \partial t \ b &= \{ \quad (b_1, b_2) = b \quad \} \\ &\quad ++ \nabla_r[e_2] \partial t \ b_2 \\ &\quad ++ \nabla_r[e_1] (\partial x, \partial y) \ b_1 \end{aligned}$$

$$\nabla_r[\text{build } e_1 (\lambda i. e_2)] \partial t \ b =$$

$$\begin{aligned} &\{ \quad (n, ba) = b \\ &\quad ; \quad \bar{v} = \text{sumbuild } n \\ &\quad \quad (\lambda i. \text{let } \nabla_r[e_2] (\partial t[i]) (ba[i]) \text{ in } \bar{v}) \quad \} \\ &\text{where } \bar{v} = fvs(e_2) \end{aligned}$$

$$\nabla_r[\text{case } e_1 \text{ of } \{ \text{Inl } x \rightarrow e_2; \text{Inr } y \rightarrow e_3 \}] \partial t \ b =$$

$$\begin{aligned} &\{ \quad (b_1, b_2) = b \\ &\quad ; \quad (\partial xy, \bar{v}) = \text{case } b_2 \text{ of} \\ &\quad \quad \text{Inl } b_l \rightarrow \text{let } \nabla_r[e_l] \partial t \ b_l \text{ in } (\text{Inl } \partial x, \bar{v}) \\ &\quad \quad \text{Inr } b_r \rightarrow \text{let } \nabla_r[e_r] \partial t \ b_r \text{ in } (\text{Inr } \partial y, \bar{v}) \quad \} \\ &\quad ++ \nabla_r[e_1] \partial xy \ b_1 \end{aligned}$$

Figure 24. BOG-style AD for ksc

Original function	Forward BOG	Reverse BOG
$f : S \rightarrow T$	$ffbog : S \rightarrow (T, \mathcal{B}[e])$	$frbog : (\delta T, \mathcal{B}[e]) \rightarrow \delta S$
$f(s) = e$	$ffbog(s) = \nabla_f[e]$ If $e : T$ then $\nabla_f[e] : (T, \mathcal{B}[e])$	$frbog(\partial t, b) = \nabla_r[e] \partial t \ b \ \partial s$ If $e : E$ and $\partial e : \delta E$ and $b : \mathcal{B}[e]$ and $\sigma : \delta S$ then $\nabla_r[e] \partial e \ b \ \sigma$ is ...
Expression e	Forward	Reverse
k	$\nabla_f[e] = (k, ())$	$\nabla_r[e] \partial e \ () \ \sigma = \sigma$
Note: Use of \underline{x} becomes binding of $\partial \underline{x}$, which will be used in σ . If x is unused, it won't be bound in the reverse pass. If it's used twice, there will be shadowing definitions, hence incorrect gradients, so we use dup.		
\underline{x}	$\nabla_f[e] = (x, ())$	$\nabla_r[e] \partial e \ b \ \sigma = \text{let } \partial \underline{x} = \partial e \text{ in } \sigma$
Note: Binding of \underline{x} becomes use of $\partial \underline{x}$ in the body of e_2 's reverse-pass.		
$\text{let } \underline{x} = e_1 \text{ in } e_2$	$\nabla_f[e] = \text{let } (x, b_1) = \nabla_f[e_1] \text{ in } \text{let } (r, b_2) = \nabla_f[e_2] \text{ in } (r, (b_1, b_2))$	$\nabla_r[e] \partial e_2 \ (b_1, b_2) \ \sigma = \nabla_r[e_2] \partial e_2 \ b_2 \ (\nabla_r[e_1] \partial \underline{x} \ b_1 \ \sigma)$
$\text{let } (\underline{x}, \underline{y}) = e_1 \text{ in } e_2$	$\nabla_f[e] = \text{let } (xy, b_1) = \nabla_f[e_1] \text{ in } \text{let } (x, y) = xy \text{ in } \text{let } (r, b_2) = \nabla_f[e_2] \text{ in } (r, (b_1, b_2))$	$\nabla_r[e] \partial e_2 \ (b_1, b_2) \ \sigma = \nabla_r[e_2] \partial e_2 \ b_2 \ (\nabla_r[e_1] (\partial \underline{x}, \partial \underline{y}) \ b_1 \ \sigma)$
$f(a)$	$\nabla_f[e] = \text{let } (v_a, b_a) = \nabla_f[a] \text{ in } \text{let } (r, b_f) = ffbog(v_a) \text{ in } (r, (b_a, b_f))$	$\nabla_r[e] \partial e \ (b_a, b_f) \ \sigma = \text{let } \partial f = frbog(\partial e, b_a) \text{ in } \nabla_r[a] \partial f \ b_f \ \sigma$
(e_1, e_2)	$\nabla_f[e] = \text{let } (a_1, b_1) = \nabla_f[e_1] \text{ in } \text{let } (a_2, b_2) = \nabla_f[e_2] \text{ in } ((a_1, a_2), (b_1, b_2))$	$\nabla_r[e] (\partial e_1, \partial e_2) \ (b_1, b_2) \ \sigma = \nabla_r[e_1] \partial e_1 \ b_1 \ (\nabla_r[e_2] \partial e_2 \ b_2 \ \sigma)$
$\text{if } p \ e_t \ e_f$	$\nabla_f[e] = \text{if } p \text{ let } (r, b_t) = \nabla_f[e_t] \text{ in } (r, (\text{true}, b_t)) \text{ let } (r, b_f) = \nabla_f[e_f] \text{ in } (r, (\text{false}, b_f))$	$\nabla_r[e] \partial e \ (b_p, b_{tf}) \ \sigma = \text{if } b_p \ \nabla_r[e_t] \partial e \ \pi_1(b_{tf}) \ \sigma \ \nabla_r[e_f] \partial e \ \pi_2(b_{tf}) \ \sigma$
Note: Build with "inlined" lambda has a special rule, build with top-level function doesn't need one ⁴		
build $e_1 \ (\lambda i. e_2)$	$\nabla_f[e] = \text{let } n = e_1 \text{ in } \text{let } \text{rb} = \text{build } n \ (\lambda i. \nabla_f[e_2]) \text{ in } \text{let } (r, b) = \text{unzip}(\text{rb}) \text{ in } (r, (n, b))$	$\nabla_r[e] \partial e \ (n, bs) \ \sigma = \text{sumbuild } n \ (\lambda i. \nabla_r[e_2] \partial e[i] \ bs[i] \ \sigma)$

Figure 25. BOG-style AD for ksc core language, continuation-passing style, for the single-use dialect of ksc, in which every binder is used exactly once.

Definition	Forward	Reverse
$\text{dup} : T \rightarrow (T, T)$ $\text{dup}(x) = (x, x)$	$\text{dup}_{fbog}(x) = ((x, x), ())$	$\text{dup}_{rbog}((\partial t_1, \partial t_2), ()) = \partial t_1 + \partial t_2$
$\text{build} : (E, \mathbb{N}, (E, \mathbb{N}) \rightarrow T) \rightarrow \text{Vec } T$ $\text{build}(env, n, f) = \dots$ $f : (E, \mathbb{N}) \rightarrow T$ $ffbog : (E, \mathbb{N}) \rightarrow (T, \mathcal{B}[e])$ $frbog : (\delta T, \mathcal{B}[e]) \rightarrow (\delta E, ())$	$\text{build}_{fbog}(env, n, f) = \text{let } bs = \text{build}(env, n, ffbog) \text{ in } \text{let } (r, ba) = \text{unzip}(bs) \text{ in } (r, (n, ba))$	$\text{build}_{rbog}(\partial t, (n, ba)) = \text{sumbuild } n \ (\lambda i. frbog(\partial t[i], ba[i]))$

Figure 26. BOG-style AD for some ksc primitive functions