Problems in structural dynamics

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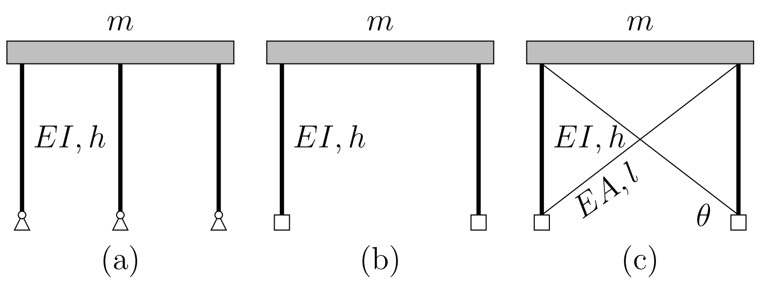
Take $E = 200\,\text{GPa}$ for steel and $E = 14\,\text{GPa}$ for concrete.

Free vibration of single degree of freedom (SDOF) structures

Exercise 1 For the structures shown, determine the natural frequency of vibration using simple structural concepts. Consider the frames below. Compute the stiffness and natural frequencies.



Answer : (a) $\omega = \sqrt{\frac{9EI}{mh^3}}$, (b) $\omega = \sqrt{\frac{24EI}{mh^3}}$ and (c) $\omega = \sqrt{\frac{24EI}{mh^3} + \frac{EA}{ml}\cos^2\theta}$.

Exercise 2 The portal frame structure shown has a weight of $100\,\text{kN}$. If the natural period of vibration is $0.9\,\text{s}$:

(a) determine the lateral stiffness of the structure;

(b) determine the diameter of the steel cross-braces required to strengthen the structure by reducing the period to $0.3\,\text{s}$;

(c) determine the period if a further load of $50\,\text{kN}$ is added to the strengthened structure.

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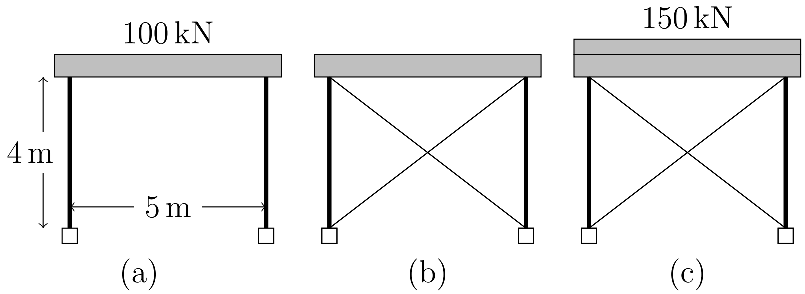
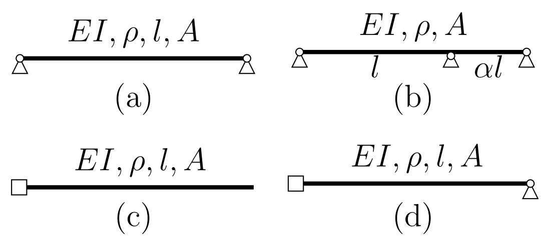


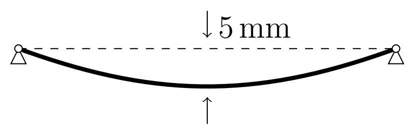
Answer : (a) $k = 487\,\mathrm{kN/m}$, (b) $D = 1.6\,\mathrm{cm}$, (c) $T = 0.37\,\mathrm{s}$.

Exercise 3 For the structures shown, determine the natural frequency of vibration using the Rayleigh method.



Answer : (a) $\omega \approx 10.95 \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$ rad/s, (c) $\omega \approx 4.47 \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$ rad/s.

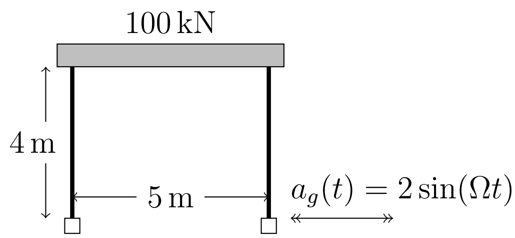
Exercise 4 To determine the dynamic properties of a simply supported bridge with a mass of 10^6 kg, the midpoint is displaced 5 mm by a jack and then suddenly released. At the end of 20 complete cycles, the time is 3 s and the peak displacement measured is 1 mm. Determine the natural period and damping ratio of the bridge.



Answer : $T = 0.15\,\mathrm{s}$, $\xi = 1.28\,\%$.v
References: Chopra, *Dynamics of structures, SI Edition*, p. 49; Blanco Díaz, *Análisis experimental de estructuras*, p. 287

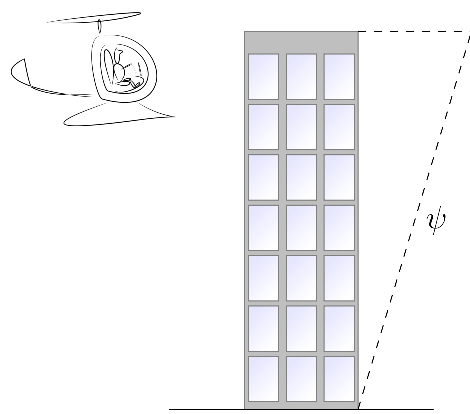
Forced vibration of SDOF structures

Exercise 5 The portal frame in Exercise 2 (a) is subject to a sinusoidal ground vibration with a horizontal acceleration amplitude of $2\,\mathrm{m/s^2}$. Assuming a damping ratio of 5 %, determine the maximum displacement and maximum total acceleration of the frame when the period of floor vibration is: (a) 0.1 s; (b) 0.9 s and (c) 5 s.



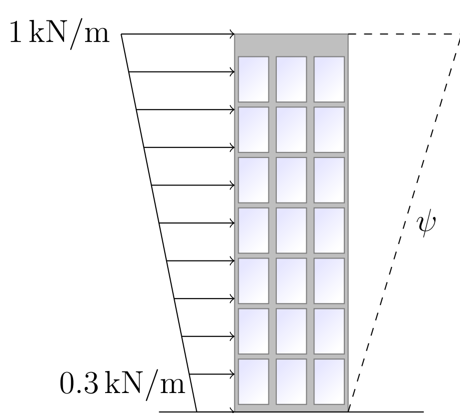
Answer : (a) $u_0 = 0.05\,\mathrm{cm}$, $\ddot{u}_0 = 0.025\,\mathrm{m/s^2}$, (b) $u_0 = 41\,\mathrm{cm}$, $\ddot{u}_0 = 20\,\mathrm{m/s^2}$ and (c) $u_0 = 4.2\,\mathrm{cm}$, $\ddot{u}_0 = 0.066\,\mathrm{m/s^2}$.

Exercise 6 A building has a height of 100 m, a square base measuring $20 \times 20\,\mathrm{m^2}$, an average specific weight of $1500\,\mathrm{N/m^3}$ and a natural period of vibration of 5 s. The top floor is hit by a helicopter with a mass of 10 000 kg and travelling at 30 m/s. Determine the maximum deflection at the top assuming conservation of linear momentum and a vibration shape function that increases linearly with the height.



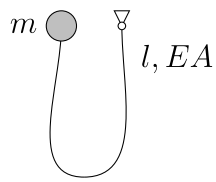
Answer : $u_0 = 8\,\mathrm{cm}$.

Exercise 7 The building in Exercise 6 is hit by a sharp wind gust that results in the sudden application of horizontal forces distributed along the height of the building, as shown in the picture. Assuming a vibration shape function that increases linearly with the height and neglecting damping, determine the maximum displacement at the top of the building.



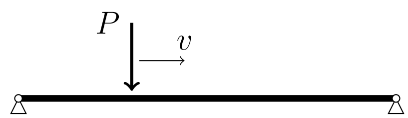
Answer : $u_0 = 2.5\,\text{cm}$.

Exercise 8 A mass m is released from a given height attached to a massless cable of length l, area A and Young's modulus E. If the cable is fixed at the point from which the mass is released, describe the motion/vibration of the mass. Determine the maximum stress in the cable and the lowest point reached by the mass.



Answer : $u_{max} = \left(\frac{mg}{EA} + \sqrt{\frac{2mg}{EA}}\right) l$, $\sigma_{max} = \frac{mg}{A} + \sqrt{\frac{2mgE}{A}}$.

Exercise 9 A point load $P = 1\,\text{kN}$ moves along with constant speed $v = 10\,\text{m/s}$ on a simply supported beam of length $l = 10\pi\,\text{m}$, as shown in the figure. The beam is made of concrete, has a rectangular section of 1 m width and 0.5 m height and an average density of $2800\,\text{kg/m}^3$. Determine the deflection of the beam as a function of time, the dynamic magnification factor and the maximum bending moment at the centre section.

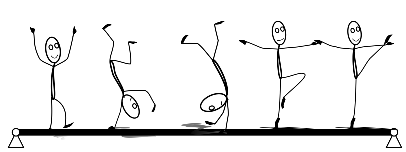


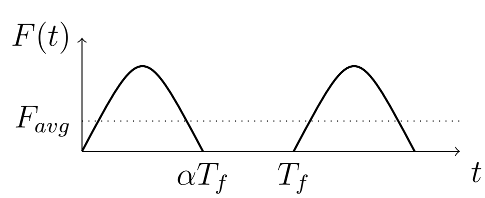
Answer : $H = 1.11$, $u = 0.5\sin(t)\,\text{cm}$, $M_{max} = 8.7\,\text{kN m}$.
References: Chopra, *Dynamics of structures, SI Edition*, p. 305

Exercise 10 A concrete ribbed slab floor spans 9 m and has an average mass of $500\,\text{kg/m}^2$. The floor is simply supported on either side and has a natural frequency of vibration of 6.3 Hz. The floor is to be used for aerobics and other similar rhythmic activities at frequencies ranging from 1.5 to 2.5 Hz and with contact ratios α between 0.5 and 1. During these activities the average imposed load will remain below $0.75\,\text{kN/m}^2$ (before dynamic magnification) and the damping ratio can be taken to be 3 %.

(a) Determine the maximum possible resonant displacement and the resulting peak acceleration and bending moment per unit width.

(b) If the floor has been designed for a service load of $5\,\text{kN/m}^2$, determine its suitability for the proposed use.



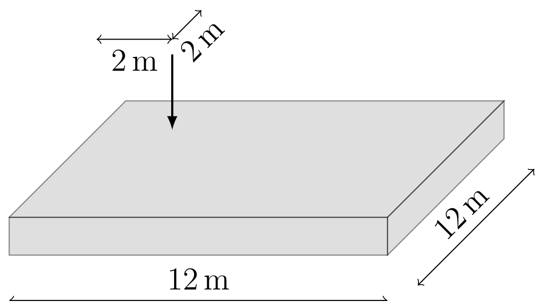


Answer : (a) $u_{\max} = 7\,\mathrm{mm}$, $M_{\max} = 250\,\mathrm{kN\,m/m}$, $a_{\max} = 9\,\mathrm{m/s^2}$. (b) The floor is not suitable for aerobics.
References: Chopra, *Dynamics of structures, SI Edition*

Exercise 11 A pneumatic hammer is to be used on a simply supported square concrete plate measuring $12 \times 12\,\mathrm{m}^2$ at a point $2\,\mathrm{m}$ away from each side. The hammer operates at $2\,\mathrm{Hz}$ and each blow corresponds to an instantaneous impulse of $100\,\mathrm{N\,s}$. The plate has a natural frequency of vibration of $8\,\mathrm{Hz}$, a damping ratio of $2\,\%$ and a mass of $500\,\mathrm{kg/m^2}$.

(a) Determine the equivalent static load.

(b) Obtain the deflection at the mid-span as a function of time, its peak value and the peack acceleration.



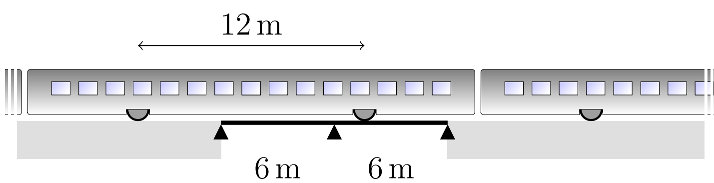
Answer : (a) ... (b) ...

Exercise 12 A $12\,\mathrm{m}$ long rail bridge has the continuous beam configuration shown in the figure, an average flexural stiffness $EI = 5\,\mathrm{GN\,m^2}$, a mass per unit length of $10\,000\,\mathrm{kg/m}$ and a damping ratio of $2\,\%$.

a. Using Ratleigh's method with an appropriate sinusoidal function, obtain its fundamental frequency of vibration.

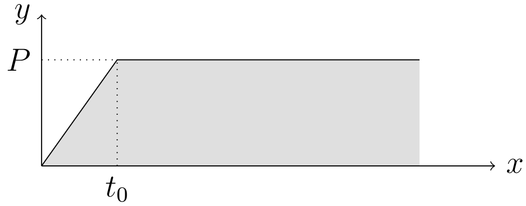
b. Determine the equivalent modal force that results from the motion of a train consisting of an 'infinite' number of $100\,\mathrm{kN}$ point loads separated by equal distances of $12\,\mathrm{m}$ and travelling at a constant speed of $225\,\mathrm{km/h}$.

c. Obtain the dynamic magnification factor for the above train of loads.



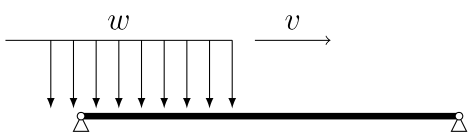
Answer : (a) ... (b) ... (c) ...

Exercise 13 A load is applied on a structure in such a way that its magnitude increases linearly with time until a maximum value P is reached at time t_0. Determine the dynamic magnification factor as a function of t_0 and the natural period of vibration T of the structure.



Answer : ...

Exercise 14 The infinitely long uniformly distributed load w shown in the figure moves along a simply supported bridge of length l at constant speed v. The total mass of the bridge is $2m$ and the natural period of vibration is T. Determine the resulting vibration and the dynamic magnification factor.



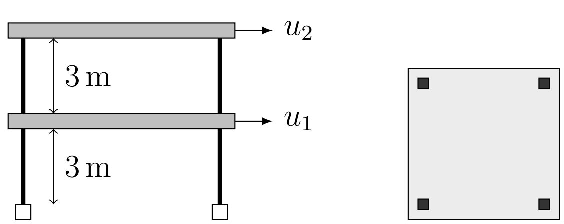
Answer : ...

Vibration of multiple degree of freedom (MDOF) structures

Exercise 15 The two-storey building shown is supported by four square concrete columns of dimensions $0.35 \times 0.35\,\mathrm{m}^2$. The total masses of the bottom and top floors are 150 and 100 t respectively.

(a) Determine the natural modes and frequencies of vibration in the horizontal direction shown.

(b) Determine the frequency of vibration that would be obtained, assuming that the fundamental mode of vibration increases linearly with height.



Answer : (a) $\omega_1 = 10\,\mathrm{rad/s}$, $\omega_2 = 24.5\,\mathrm{rad/s}$, $\mathbf{v}_1 = [2\ 3]^T$, $\mathbf{v}_2 = [1\ {-1}]^T$ (b) $\omega_1 = 10.4\,\mathrm{rad/s}$, $\mathbf{v}_1 = [1\ 2]^T$.

Exercise 16 The two-storey building from the previous Exercise is hit by a helicopter with a mass of 10 t traveling at 20 m/s.

(a) Determine the resulting vibration and the maximum displacement at the top of the building using both modes of vibration.

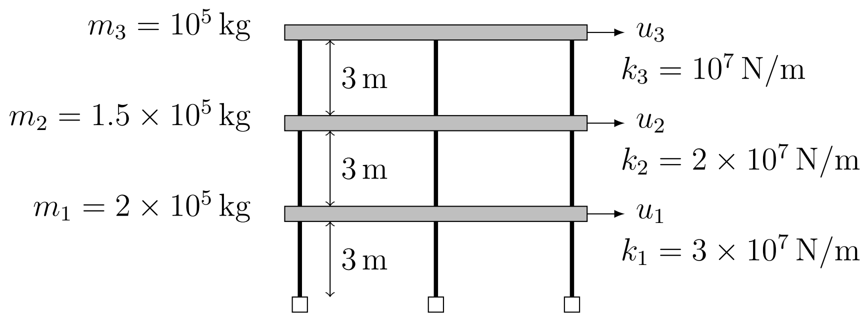
(b) Determine the resulting vibration and the maximum displacement at the top assuming that the linearly increasing mode absorbs the total momentum.

Answer : (a) $u_{max} = 15\,\text{cm}$, (b) $u_{max} = 14.5\,\text{cm}$.

Exercise 17 A three-storey building has the mass and stiffness distribution shown.

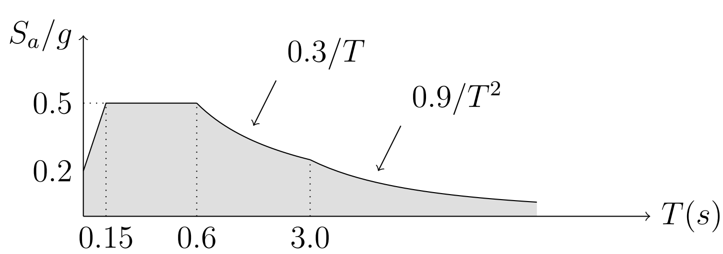
(a) Approximate the first period of vibration using a linearly increasing mode.

(b) Using a linearly increasing mode together with a second Ritz vector increasing quadratically with height, approximate the first two modes and frequencies of vibration.



Answer : (a) $\omega_1 = 5.9\,\text{rad/s}$, (b) $\omega_1 = 5.9\,\text{rad/s}$, $\omega_2 = 12.8\,\text{rad/s}$.

Exercise 18 The building of exercise 3 is subject to the elastic design earthquake with the response spectrum shown. Obtain the total base shear, overturning moment, maximum floor displacements and storey forces using (a) a single mode linear with height and (b) a complete combination of the three modes of vibration.



Answer : ...

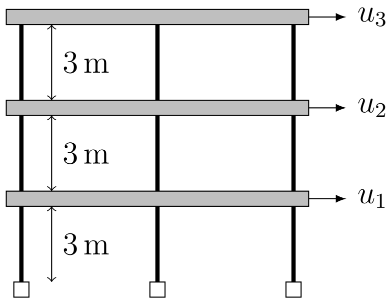
Exercise 19 A perfectly regular three storey building has a total weight of 800 kN per floor. Each floor has an equal height of 3 m and an inter-storey stiffness of 20 000 kN/m. Using a simplified shear building model with a single mode of vibration linearly increasing with height determine:

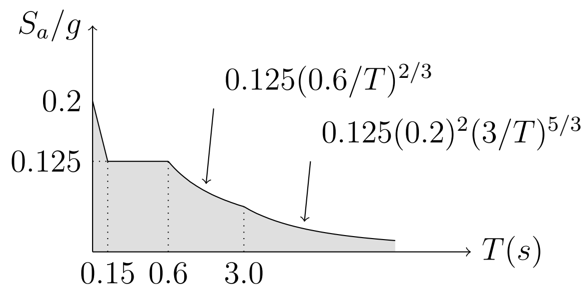
a. The main period of vibration.

b. The total base shear according to the inelsatic Eurocode 8 earthquake design spectrum.

c. The equivalent static loads at each floor for the same spectrum.

d. The peak displecment at the top if the above spectrum corresponds to a behavioural factor $q = 4$.





Answer : (a) $T = 0.86\,\mathrm{s}$; (b) $V_b = 240\,\mathrm{kN}$; (c) $T_1 = 40\,\mathrm{kN}$, $T_2 = 80\,\mathrm{kN}$, $T_3 = 120\,\mathrm{kN}$; (d) $u_{\mathrm{top}} = 0.11\,\mathrm{m}$

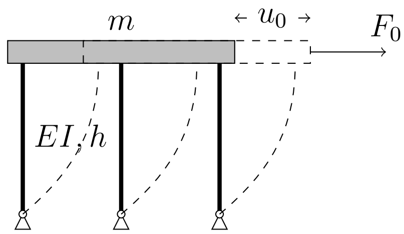
Solutions

Answer to Exercise 1 The natural frequency of a structure is obtained from the solution of the differential equation governing the displacement of a *mass spring* system without excitation:

$$m\ddot{u} + ku = 0$$

where m is the mass of the idealized system and k is the stiffness. The natural frequency depends on both constants, $\omega^2 = k/m$.

Every single structure can be broken down into its elements and each element can be analysed by any of the standard methods. Here, to obtain the stiffness of each element, we impose a unit displacement u_0 generated by the corresponding force F_0. The stiffness of the structure is the sum of the stiffness of its components.

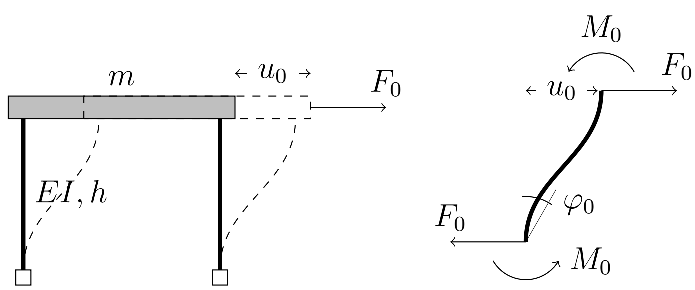


The displacement of the columns can be analysed as a cantilever using static analysis concepts:

$$u_0 = \frac{F_0 h^3}{3EI} \quad \rightarrow \quad k_{column} = \frac{3EI}{h^3}.$$

Finally, the stiffness and the frequency of the structure are

$$k = 3k_{column} = \frac{9EI}{h^3} \quad , \quad \omega = \sqrt{\frac{9EI}{mh^3}}.$$



Analogously, the second structure can be analysed by combining the stiffness of the columns. In this case, the rotation $\varphi_0 = u_0/h$ generated by the moment reaction M_0 has been imposed on the equivalent beams. The moment reaction must satisfy global equilibrium:

$$\sum M = 2M_0 - F_0 h = 0.$$

From static analysis, the rotation generated by the moment is

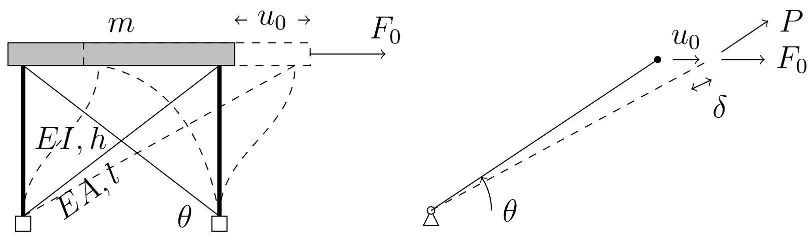
$$\varphi_0 = \frac{M_0 h}{6EI}$$

Substituting the moment and the rotation into the above expression gives

$$\frac{u_0}{h} = \frac{F_0 h}{12EI} \quad \rightarrow \quad k_{column} = \frac{12EI}{h^3}.$$

The lateral stiffness and frequency of the structure are

$$k = 2k_{column} = \frac{24EI}{h^3} \quad , \quad \omega = \sqrt{\frac{24EI}{mh^3}}.$$



The last structure adds two braces, and its stiffness will be added, but only one of them is contributing, because the bracing buckles under compression. The stiffness of a brace is

$$\delta = \frac{Pl}{EA} \quad \rightarrow \quad k_{brace} = \frac{EA}{l}\cos^2\theta$$

and the lateral stiffness and frequency of the structure are

$$k = 2k_{column} + k_{brace} = \frac{24EI}{h^3} + \frac{EA}{l}\cos^2\theta \quad , \quad \omega = \sqrt{\frac{24EI}{mh^3} + \frac{EA}{ml}\cos^2\theta}.$$

Answer to Exercise 2 Given that the natural period of vibration of the frame is 0.9 s, the frequency is computed as

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.9} = 6.98\,\mathrm{rad/s}$$

Then, the lateral stiffness is computed from the frequency and the mass of the structure,

$$\omega^2 = \frac{k}{m} \quad \rightarrow \quad k = \omega^2 m = \omega^2 \frac{P}{g} = 6.98^2 \frac{100}{10} = 487\,\mathrm{kN/m}$$

The goal of the second step is to determine the diameter of the steel cross-braces required to strengthen the structure by reducing the period to 0.3 s. First, the new stiffness is obtained following the same procedure,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.3} = 20.94\,\mathrm{rad/s}$$
$$k = \omega^2 \frac{P}{g} = 20.94^2 \frac{100}{10} = 4384\,\mathrm{kN/m}$$

The bracing system should provide the additional stiffness,

$$k = k_{frame} + k_{bracing} \quad \rightarrow \quad k_{bracing} = 4384 - 487 = 3897\,\text{kN/m}$$

We will consider the stiffness of one brace because the one under compression can buckle. Following the solution of Exercise 1 (c), the area can be obtained from

$$\begin{aligned}
k_{brace} &= \frac{EA}{l}\cos^2\theta \quad \rightarrow \quad A = \frac{k_{brace} l}{E\cos^2\theta} \\
l &= \sqrt{4^2+5^2} = 6.4\,\text{m} \\
\cos\theta &= \frac{5}{6.4} = 0.78 \\
A &\approx 2\cdot 10^{-4} m^2 = 2\,\text{cm}^2
\end{aligned}$$

and the diameter is obtained from the area,

$$D = 2\sqrt{\frac{A}{\pi}} = 2\sqrt{\frac{2}{\pi}} \approx 1.6\,\text{cm}$$

The last part of the example consists in computing the new period of vibration if a further load of 50 kN is added to the strengthened structure. Using the new lateral stiffness, the frequency is

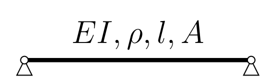
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4384}{15}} \approx 17.1\,\text{rad/s}$$

and the new period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{17.1} \approx 0.37\,\text{s}$$

Answer to Exercise 3 The Rayleigh method is based on assuming the vibration to be given in terms of a pre-determined shape function ψ as $u(x,t) = u_0(t)\psi(x)$. This assumption allows you to find an equivalent mass and stiffness associated with that mode of vibration and thus, the frequency.

For the simple beam, a possible shape function would be a parabola satisfying the kinematic boundary conditions, $\psi(0) = \psi(l) = 0$.



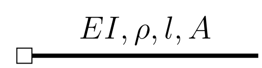
$$\begin{aligned}
\psi(x) &= \frac{4}{l^2}x(l-x) \\
\psi''(x) &= \frac{8}{l^2}
\end{aligned}$$

where the term $4/l^2$ is a scaling parameter and does not modify the solution. The equivalent mass and stiffness and frequency are

$$
\begin{aligned}
m &= \int_0^l \rho A \psi^2 dx = \int_0^l \rho A \left(\frac{4}{l^2} x(l-x) \right)^2 dx = \frac{8\rho A l}{15} \\
k &= \int_0^l EI \psi''^2 dx = \int_0^l EI \left(\frac{8}{l^2} \right)^2 dx = \frac{64EI}{l^3} \\
\omega &= \sqrt{\frac{k}{m}} = \sqrt{\frac{64EI}{l^3} \frac{15}{8\rho A l}} = \frac{1}{l^2} \sqrt{120 \frac{EI}{\rho A}} \approx 10.95 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}} \text{ rad/s.}
\end{aligned}
$$

The result obtained can be compared against the exact frequency, obtained by the same procedure applied to a fourth order polynomial satisfying both kinematic and dynamic boundary conditions. The analytical result is $\omega = \frac{1}{l^2}\sqrt{\frac{3024EI}{31\rho A}} \approx 9.86 \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$ rad/s.

The cantilever can be analysed using a second order polynomial. In that case, the kinematic boundary conditions are $\psi(0) = \psi'(0) = 0$.



$$
\begin{aligned}
\psi(x) &= \frac{1}{l^2} x^2 \\
\psi''(x) &= \frac{2}{l^2}.
\end{aligned}
$$

Again, the term $1/l^2$ is a scaling term providing consistency and does not modify the result of the calculations.

$$
\begin{aligned}
m &= \int_0^l \rho A \psi^2 dx = \int_0^l \rho A \left(\frac{1}{l^2} x^2 \right)^2 dx = \frac{\rho A l}{5} \\
k &= \int_0^l EI \psi''^2 dx = \int_0^l EI \left(\frac{2}{l^2} \right)^2 dx = \frac{4EI}{l^3} \\
\omega &= \sqrt{\frac{k}{m}} = \sqrt{\frac{4EI}{l^3} \frac{5}{\rho A l}} = \frac{1}{l^2} \sqrt{20 \frac{EI}{\rho A}} \approx 4.47 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}} \text{ rad/s.}
\end{aligned}
$$

The exact frequency of the first mode of vibration of a cantilever is $\omega \approx 3.52 \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$ rad/s.

The last two structures can be analysed using a cubic polynomial, because the shape function must fulfill three kinematic boundary conditions.

For the application of the Rayleigh method with periodic shape functions, see the solution to Exercise 9.

Answer to Exercise 4 The solution to the differential equation governing free vibration of a *mass spring damper* system is governed by the natural and damped frequencies and the damping ratio:

$$
u(t) = e^{-\xi \omega t} \sin(\omega_D t) \quad ; \quad \omega_D = \omega \sqrt{1 - \xi^2} \, .
$$

For simplicity, we will assume a small damping and $\sqrt{1-\xi^2} \to 1$. Thus, the natural period can be obtained directly from the measured values

$$T \approx T_D = \frac{3}{20} = 0.15\,\mathrm{s}$$

and the damping ratio ξ is directly related to the decay coefficient determined by the relation between two consecutive oscillations from a damped period

$$\frac{u(t)}{u(t+T_D)} = e^{\xi\omega T_D} = e^{\frac{2\pi\xi}{\sqrt{1-\xi^2}}} \quad \to \quad \frac{u_0}{u_1}\frac{u_1}{u_2}\dots\frac{u_{n-1}}{u_n} = \frac{u_0}{u_n} = e^{\frac{2n\pi\xi}{\sqrt{1-\xi^2}}}.$$

Because ξ is small, the damping ratio can be found as

$$2n\pi\xi = \log\left(\frac{u_0}{u_n}\right) \quad \to \quad \xi = \frac{1}{2n\pi}\log\left(\frac{u_0}{u_n}\right) = \frac{1}{2\cdot 20\pi}\log\left(\frac{5}{1}\right) = 0.0128$$
$$\xi = 1.28\,\%.$$

Answer to Exercise 5 The maximum displacements and accelerations of a vibrating structure under sinusoidal excitation are obtained from the stationary or homogeneous solution to the differential equation,

$$m\ddot{u} + ku = F_0 \sin(\Omega t + \phi)$$
$$u(t) = \frac{F_0}{k} H \sin(\Omega t + \phi - \Delta\phi)$$

where H is known as the magnification factor and is equal to

$$H = \frac{1}{\sqrt{(1-\gamma^2)^2 + 4\xi^2\gamma^2}} \quad ; \quad \gamma = \frac{\Omega}{\omega} = \frac{T_{struct}}{T_{force}} .$$

From the solution to Exercise 2 (a) we know that the period is $T = 0.9\,\mathrm{s}$, the frequency is $\omega = 2\pi/T \approx 7\,\mathrm{rad/s}$ and the lateral stiffness is $k = 487\,\mathrm{kN/m}$. The damping ratio is $\xi = 5\,\%$.

The amplitude F_0 of the external force is computed from the amplitude of the ground acceleration and the moving mass of the structure

$$F_0 = ma_0 = \frac{Pa_0}{g} = \frac{100\cdot 2}{10} = 20\,\mathrm{kN} .$$

Finally, the maximum displacements will depend on the period Ω of the external force,

$$u_0 = \frac{F_0}{k}H = \frac{20}{487}H = 0.041H\,\mathrm{m} = 4.1H\,\mathrm{cm}$$
$$\ddot{u}_0 = \frac{F_0}{m}\gamma^2 H = \Omega^2 u_0 .$$

When the period of the ground acceleration is $T_g = 0.1\,\mathrm{s}$, the maximum displacements and accelerations are

$$\gamma = \frac{0.9}{0.1} = 9 \quad ; \quad H = \frac{1}{\sqrt{(1-9^2)^2 + 4\cdot 0.05^2 \cdot 9^2}} = \frac{1}{\sqrt{80^2 + \cdots}} = 0.0125$$
$$u_0 = 4.1 \cdot 0.0125 = 0.05\,\mathrm{cm}$$
$$\ddot{u}_0 = \left(\frac{2\pi}{0.1}\right)^2 \cdot 0.00064 = 2.5\,\mathrm{cm/s^2} = 0.025\,\mathrm{m/s^2}\ .$$

This situation is *mass-dominated*, and is also known as *vibration isolation*.

When the period of the ground acceleration is $T_g = 0.9\,\mathrm{s}$, the maximum values are

$$\gamma = \frac{0.9}{0.9} = 1 \quad ; \quad H = \frac{1}{\sqrt{(1-1^2)^2 + 4\cdot 0.05^2 \cdot 1^2}} = \frac{1}{\sqrt{0 + 4\cdot 0.05^2}} = 10$$
$$u_0 = 4.1 \cdot 10 = 41\,\mathrm{cm}$$
$$\ddot{u}_0 = \left(\frac{2\pi}{0.9}\right)^2 \cdot 41 = 2000\,\mathrm{cm/s^2} = 20\,\mathrm{m/s^2}\ .$$

The *resonance* situation is *damping-dominated*.

For a period $T_g = 5\,\mathrm{s}$, the maximum values are

$$\gamma = \frac{0.9}{5} = 0.18 \quad ; \quad H = \frac{1}{\sqrt{(1-0.18^2)^2 + 4\cdot 0.05^2 \cdot 0.18^2}} = 1.033$$
$$u_0 = 4.1 \cdot 1.033 = 4.2\,\mathrm{cm}$$
$$\ddot{u}_0 = \left(\frac{2\pi}{5}\right)^2 \cdot 4.2 = 6.6\,\mathrm{cm/s^2} = 0.066\,\mathrm{m/s^2}\ ,$$

which is a practically *static* or *stiffness dominated* situation.

Answer to Exercise 6 When the building is hit by the helicopter, the impulse or linear momentum is preserved. To compute the linear momentum absorbed by the building, we can make use of the Raileygh method, it will be the integral value of the mass and velocity product. As suggested, we will use a linear shape function

$$\dot{u} = \dot{u}_0 \psi \quad ; \quad \psi = \frac{z}{h}$$
$$m\dot{u}_0 = \int_0^h \rho A \dot{u} dz = \int_0^h \rho A \dot{u}_0 \frac{z}{h} dz = \frac{1}{2}\rho A \dot{u}_0 h =$$
$$= \frac{1}{2} 150 \times 400 \times 100\ \dot{u}_0 = 3 \times 10^6\ \dot{u}_0\ \mathrm{kg\,m/s}\ .$$

The impulse of the helicopter is

$$I = (mv)_{helicopter} = 10^4 \times 30 = 3 \times 10^5\,\mathrm{kg\,m/s}\ ,$$

which is transferred to the building. Then, the maximum deflection at the top of the building is inferred according to the following expressions:

$$
\begin{aligned}
I &= m\dot{u}_0 \quad \rightarrow \quad u_0 = \frac{\dot{u}_0}{\omega} \\
\omega &= \frac{2\pi}{T} = \frac{2\pi}{5} = 1.25\,\mathrm{rad/s} \\
u_0 &= \frac{3 \times 10^5}{3 \times 10^6 \times 1.25} = 0.08\,\mathrm{m} = 8\,\mathrm{cm}\ .
\end{aligned}
$$

Answer to Exercise 7 Now, the building in Exercise 6 is exposed to a sharp wind gust. In this case, the dynamics of the structure is defined by a *mass-spring-damper* system under a constant force applied suddenly. The maximum amplitude is governed by the stationary or homogeneous solution to the differential equation

$$
\begin{aligned}
m\ddot{u} + c + ku &= F_0 \\
u(t) &= \frac{F_0}{k}(1 - e^{\xi\omega t}\cos(\omega t))\ .
\end{aligned}
$$

First, we need to determine the equivalent lateral force using the Rayleigh method from the wind pressure p_W, using the same shape function ψ as in Exercise 6:

$$
\begin{aligned}
F_0 = \int_0^h p_W \psi dz = \int_0^h \left(0.3 + (1 - 0.3)\frac{z}{h}\right)\frac{z}{h}dz = \left(\frac{0.3}{2} + \frac{0.7}{3}\right)h = \\
= \left(\frac{0.3}{2} + \frac{0.7}{3}\right)100 = 38.3\,\mathrm{kN}\ .
\end{aligned}
$$

Following the Rayleigh method and the linear shape function ψ, the modal mass is found to be 2×10^6 kg. The stiffness of the building can be estimated from the equivalent mass and the given period of the building

$$
k = \omega^2 m = \left(\frac{2\pi}{T}\right)^2 m = \left(\frac{2\pi}{5}\right)^2 2 \times 10^6 = 3.125 \times 10^6\,\mathrm{N/m} = 3125\,\mathrm{kN/m}
$$

Finally, the maximum displacement is obtained from the equation of motion

$$
u_{max} = \frac{F_0}{k}(1+1) = \frac{38.3}{3125}2 = 0.025\,\mathrm{m} = 2.5\,\mathrm{cm}\ .
$$

Answer to Exercise 8 The motion of the mass released from a given height and attached to a cable has two parts: first, a free fall; second, a vibration. The velocity at the end of the free fall is

$$
\frac{1}{2}m\dot{u}_0^2 = mgl \quad \rightarrow \quad \dot{u}_0 = \sqrt{gl}
$$

and may be taken as the initial condition of the vibration. Lastly, the elastic response of the cable is the superposition of the static elongation and the vibration. From the cable properties,

$$k = \frac{EA}{l}$$
$$m = \sqrt{\frac{k}{m}} = \sqrt{\frac{EA}{ml}}$$
$$u_{stat} = \frac{F}{M} = \frac{mgl}{EA}\ .$$

Then, neglecting damping, we can consider the motion as the transient solution to the differential equation satisfying the appropriate initial conditions,

$$u(t) = C\sin(\omega t + \phi) + u_{stat}$$
$$u(0) = u_{stat} \rightarrow C\sin(\phi) = 0 \rightarrow \phi = 0$$
$$\dot{u}(0) = \sqrt{2gl} \rightarrow C\omega\cos(\phi) = \sqrt{2gl} \rightarrow C = \sqrt{\frac{2mg}{EA}}l$$

and the maximum displacement u_{max} and stress σ are

$$u_{max} = \left(\frac{mg}{EA} + \sqrt{\frac{2mg}{EA}}\right) l$$
$$\sigma = E\epsilon = E\frac{u_{max}}{l} = \frac{mg}{A} + \sqrt{\frac{2mgE}{A}}\ .$$

N.B. the maximum displacement can also be obtained by applying energy conservation.

Answer to Exercise 9 In this case, the magnitude of the external force $P = 1\,\mathrm{kN}$ does not vary with time but moves with constant speed $v = 10\,\mathrm{m/s}$. The Rayleigh method allows us to compute the generalized force applied as a function of time. For this purpose, we choose a periodic shape function ψ satisfying the kinematic boundary conditions

$$\psi = \sin\left(\frac{\pi x}{l}\right)$$
$$F = P\psi(x) = P\psi(vt) = P\sin\left(\frac{\pi vt}{l}\right).$$

Thus, the period of the external excitation is $\Omega = \pi v/l = 1\,\mathrm{rad/s}$.

From the structure dimensions, we know that the mechanical properties of the section are:

$$A = 0.5\,\mathrm{m}^2$$
$$I = \frac{1}{12}0.5^3 = \frac{0.125}{12} \approx 0.01\,\mathrm{m}^4$$
$$E = 14\,\mathrm{GPa}$$

and the generalized mass and stiffness can be computed with the Rayleigh method,

$$m = \int_0^l \rho A \psi^2 dx = \int_0^l \rho A \sin^2\left(\frac{\pi x}{l}\right) dx = \frac{\rho A l}{2} =$$
$$= \frac{2800 \cdot 0.5 \cdot 10\pi}{2} = 7000\pi Kg = 7\pi \, \mathrm{t}$$
$$k = \int_0^l EI \psi''^2 dx = \int_0^l EI \left(-\left(\frac{\pi}{l}\right)^2 \sin\left(\frac{\pi x}{l}\right)\right)^2 dx = \frac{EI\pi^4}{2l^3} =$$
$$= \frac{14 \cdot 10^6 \cdot 0.01 \cdot \pi^4}{2 \cdot (10\pi)^3} = 70\pi \, \mathrm{kN/m}$$

with frequency equal to

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{70\pi}{7\pi}} = \sqrt{10} = 3.16 \, \mathrm{rad/s} \ .$$

Finally, the structural response is obtained following the same steps as in Exercise 5, a structure under periodic loading. The deflection depends on the dynamic magnification factor H and assuming a small damping ratio:

$$\gamma = \frac{\Omega}{\omega} = \frac{\pi v}{l\omega}$$
$$H = \frac{1}{\sqrt{\left(1 - \frac{\pi^2 v^2}{l^2\omega^2}\right)^2 + 4\xi^2 \frac{\pi^2 v^2}{l^2\omega^2}}} \approx \frac{1}{1 - \frac{\pi^2 v^2}{l^2\omega^2}} = \frac{l^2\omega^2}{l^2\omega^2 - \pi^2 v^2} =$$
$$= \frac{10^2 \cdot \pi^2 \cdot 3.16^2}{10^2 \cdot \pi^2 \cdot 3.16^2 - 10^2 \cdot \pi^2} = \frac{10}{10-1} = 1.11 \ .$$

Thus, the deflection of the beam as a function of time is

$$u = \frac{P}{k} H \sin(\Omega t) = \frac{1}{70\pi} 1.11 \sin(t) = 0.005 \sin(t) \, f\mathrm{m} = 0.5 \sin(t) \, \mathrm{cm}$$

And the maximum bending moment at the centre section is

$$M_{max} = \frac{Pl}{4} H = \frac{1 \cdot 10\pi}{4} 1.11 = 8.7 \, \mathrm{kN\,m}$$

Solution to the differential equation Given that the load is a piecewise function in time, the dynamic magnification factor H may not be accurate. The full solution to the differential equation is also a piecewise function of the form

$$u = \begin{cases} C_1 \sin(\omega t + \varphi_1) + \frac{P}{k} H \sin(\Omega t + \phi - \Delta\phi) & \text{if } t < \frac{l}{v} \\ C_2 \sin(\omega t + \varphi_2) & \text{if } t \geq \frac{l}{v} \end{cases}$$

The first part of the solution must satisfy homogeneous boundary conditions

$$\begin{matrix} u_1(0) = 0 \\ \dot{u}_1(0) = 0 \end{matrix} \rightarrow \begin{matrix} \varphi_1 = 0 \\ C_1 = -\frac{P}{k} H \gamma \end{matrix}$$

with $\gamma = \frac{\Omega}{\omega} = \frac{\pi v}{\omega l}$. For the second part of the motion, the expression must satisfy continuity at $t = l/v$,

$$u_2(l/v) = C_2 \sin\left(\frac{\omega l}{v} + \varphi_2\right) = u_1(l/v) = -\frac{P}{k} H\gamma \sin\frac{\omega l}{v}$$

$$\dot{u}_2(l/v) = C_2\omega \cos\left(\frac{\omega l}{v} + \varphi_2\right) = \dot{u}_1(l/v) = -\frac{P}{k} H\gamma\omega \cos\frac{\omega l}{v} - \frac{P}{k} H\Omega$$

Note that $\Omega l/v = \pi$.

The values of the integration constants are obtained after applying some trigonometric identities:

$$\omega \cot\left(\frac{\omega l}{v} + \varphi_2\right) = \omega \cot\frac{\omega l}{v} + \omega \csc(\omega l/v)$$

$$\cot\left(\frac{\omega l}{v} + \varphi_2\right) = \cot\frac{\omega l}{2v}$$

$$\varphi_2 = -\frac{\omega l}{2v}$$

and

$$C_2 = -\frac{P}{k} H\gamma \frac{\sin(\omega l/v)}{\sin(\omega l/v + \varphi_2)} = -\frac{P}{k} H\gamma \frac{\sin(\omega l/v)}{\sin(\omega l/v - \omega l/(2v))}$$

$$C_2 = -\frac{P}{k} H\gamma \frac{2\sin(\omega l/(2v))\cos(\omega l/(2v))}{\sin(\omega l/(2v))}$$

$$C_2 = -2\frac{P}{k} H\gamma \cos\frac{\omega l}{2v}\ .$$

The result is

$$u = \frac{P}{k}\frac{\omega^2 l^2}{\omega^2 l^2 - \pi^2 v^2} \begin{cases} \sin\frac{\pi v t}{l} - \frac{\pi v}{\omega l}\sin(\omega t) & \text{if } t < \frac{l}{v} \\ -\frac{2\pi v}{\omega l}\cos\frac{\omega l}{2v}\sin(\omega t - \frac{\omega l}{2v}) & \text{if } t \geq \frac{l}{v}\ . \end{cases}$$

Answer to Exercise 10 The structure properties per unit width using a sine shape function are

$$m = \int_0^l w\psi^2 dx = \int_0^l w \sin^2\left(\frac{\pi x}{l}\right) dx = \frac{wl}{2} = \frac{500 \times 9}{2} = 2250\,\mathrm{kg/m},$$

$$k = \omega^2 m = (2\pi 6.3)^2 \times 2250 = 3500\,\mathrm{kN/m^2},$$

and the equivalent force for the shape function ψ is

$$\hat{F} = \int_0^l F\psi dx = \frac{2Fl}{\pi} = 4.3\,\mathrm{kN/m}.$$

The transient walking force considered is a periodic loading with a piecewise definition. Because the action is periodic, a different approach from that of Example 9 can be followed. The action will be approximated using a Fourier series, which is suitable for any periodic action.

Pre-computed Fourier terms Each component of a Fourier series is also known as a harmonic. For a half sine activity, the amplitudes of the first four harmonics are the following:

Activity	α	F_1/F_{avg}	F_2/F_{avg}	F_3/F_{avg}	F_4/F_{avg}
Walking	2/3	1.29	0.16	0.13	0.04
Exercise	1/2	1.57	0.67	0.00	0.13
Jumping	1/3	1.80	1.29	0.67	0.16
High jumping	1/4	1.89	1.57	1.13	0.67

Then, the dynamic force is the superposition of the static force and the harmonics multiplied by the corresponding dynamic magnification factor:

$$F_{\text{dyn}} = F_{\text{avg}} + H_1 F_1 + H_2 F_2 + H_3 F_3 + \dots$$

At this point is is crucial that finding the maximum excitation is not a direct problem, since it depends on two free parameters, the walking frequency and the contact ratio depending on the type of activity. Instead of performing a trial error evaluation of each possible combination of frequencies and contact ratios, we will select them carefully by inspecting the table of predefined Fourier terms. It is reasonable to assume that the maximum response will happen when some of the harmonics are in ressonance. Thus, we choose a base frequency of 2.1 for the contact ratio 2/3, and a base frequency of 1.58 for the contact ratio equal to 0.5.

When $\alpha = 2/3$ and the frequency $f_1 = 2.1\,\text{Hz}$, the third mode will be in ressonance. Then, we have the following magnification factors,

$$\begin{aligned}
\gamma_1 &= \frac{2.1}{6.3} = 0.33 \;\rightarrow\; H_1 = \frac{1}{\sqrt{(1-0.33^2)^2 + 4 \times 0.03^2 \times 0.33^2}} = 1.12 \\
\gamma_2 &= \frac{4.2}{6.3} = 0.66 \;\rightarrow\; H_1 = \frac{1}{\sqrt{(1-0.66^2)^2 + 4 \times 0.03^2 \times 0.66^2}} = 1.8 \\
\gamma_3 &= \frac{6.3}{6.3} = 1.0 \;\rightarrow\; H_1 = \frac{1}{\sqrt{(1-1.0^2)^2 + 4 \times 0.03^2 \times 1.0^2}} = 16.7 \\
\gamma_4 &= \frac{8.4}{6.3} = 1.33 \;\rightarrow\; H_1 = \frac{1}{\sqrt{(1-1.33^2)^2 + 4 \times 0.03^2 \times 1.33^2}} = 1.28\;.
\end{aligned}$$

When intersted in computing the maximum displacement corresponding to the moving load, the average deflection shall be taken into account in addition to the harmonic components,

$$u_{\max} = \frac{\hat{F}_{\text{avg}}}{k}(1 + 1.12 \cdot 1.29 + 1.8 \cdot 0.16 + 16.7 \cdot 0.13 + 1.28 \cdot 0.04) = 5.0\frac{\hat{F}_{\text{avg}}}{k}.$$

Similarly, when the type of activity is classified as exercise, the contact ratio α is 0.5 and we will consider ressonance at the fourth harmonic, i.e., $f_1 = 1.58\,\text{Hz}$. The maximum displacement according to it will be

$$u_{\max} = \frac{\hat{F}_{\text{avg}}}{k}(1 + 1.07 \cdot 1.57 + 1.34 \cdot 0.67 + 0.0 + 16.7 \cdot 0.13) = 5.7\frac{\hat{F}_{\text{avg}}}{k}.$$

After this analysis, it is possible to conclude that the maximum response will happen when the contact ratio $\alpha = 0.5$ and the frequency $f_1 = 1.58\,\mathrm{Hz}$. The corresponding displacement and bending moment per unit width will be found as

$$u_{\max} = 5.7\frac{\hat{F}_{\text{avg}}}{k} = 5.7\frac{4.3}{3500} = 7\times10^{-3}\,\mathrm{m} = 7\,\mathrm{mm},$$
$$M_{\max} = 5.7\frac{\hat{F}_{\text{avg}}\,l^2}{8} = 5.7\frac{0.75\times9^2}{8} \approx 250\,\mathrm{kN\,m/m}.$$

On top of the previously calculated values, accelerations are a very common measure for human confort. When interested in maximum accelerations, only the contribution from the harmonics will be considered, because the time derivative of the average value is zero by definition. The acceleration

$$a_{\max} = 4.7\frac{\hat{F}_{\text{avg}}}{m} = 4.7\frac{4.3}{2.25} = 9\,\mathrm{m/s^2},$$

is extremely high due to the lisghtweight floor. While the ribbed floor is suitable for this activity from the point of view of displacements and bending moments, the acceleration is -by far- over the threshold of 0.5 % of the gravity. Thus, the floor slab is not suited for this activity.

Calculation of the Fourier terms Most commonly, Fourier series are expressed as a linear combination of sin and cos,

$$F(t) \approx \frac{a_0}{2} + \sum_{n=0}^{\infty}\left(a_n\cos(n\Omega t) + b_n\sin(n\Omega t)\right)\,,$$

with coefficients

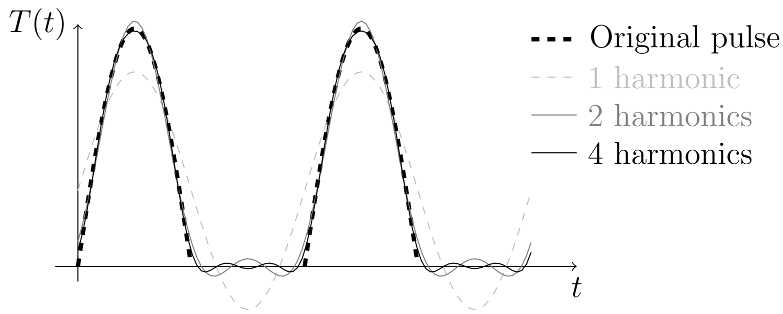
$$a_0 = \frac{2}{T}\int_0^T F(t)dt\,,$$
$$a_n = \frac{2}{T}\int_0^T F(t)\cos(n\Omega t)dt\,,$$
$$b_n = \frac{2}{T}\int_0^T F(t)\sin(n\Omega t)dt\,.$$

An alternative representation consists in combining the trigonometric functions as

$$F(t) \approx F_0 + \sum_{n=0}^{\infty} F_n\sin(n\Omega t + \phi_n)\,,$$

with the new coefficients

$$F_0 = \frac{a_0}{2}\,,$$
$$F_n = \sqrt{a_n^2 + b_n^2}\,,$$
$$\tan\phi_n = \frac{b_n}{a_n}\,.$$



Answer to Exercise 15 The inter-storey stiffness is obtained from the concrete column properties:

$$\begin{aligned}
E &= 14\,\mathrm{GPa} = 14\times 10^6\,\mathrm{KPa}\\
I &= \frac{1}{12}bc^3 = \frac{1}{12}0.35^4 = \frac{15}{12}\times 10^{-3}\mathrm{m}^4\\
k_{column} &= \frac{12EI}{h^3} = \frac{12\times 14\times 15\times 10^3}{12\times 27} = 7.77\times 10^3\,\mathrm{kN/m}\\
k &= 4k_{column} = 31.1\,\mathrm{kN/m}\ .
\end{aligned}$$

Using these values, the mass and stiffness matrices are

$$\mathbf{M} = \begin{bmatrix} m_1 & 0\\ 0 & m_2\end{bmatrix} = \begin{bmatrix} 150 & 0\\ 0 & 100\end{bmatrix}\mathrm{t}\ ,$$

$$\mathbf{K} = \begin{bmatrix} k_1+k_2 & -k_2\\ -k_2 & k_2\end{bmatrix} \approx \begin{bmatrix} 60 & -30\\ -30 & 30\end{bmatrix}\times 10^3\mathrm{kN/m}\ .$$

(a) Natural modes of vibration The natural modes of vibration can be obtained from the solution to the generalized eigenvalue problem

$$\begin{aligned}
|\mathbf{K}-\omega^2\mathbf{M}| &= 0\\
\begin{vmatrix} 60000-\omega^2 150 & -30000\\ -30000 & 30000-\omega^2 100\end{vmatrix} &= 0\\
(6000-15\omega^2)(3000-1\omega^2) - 3000\times 3000 &= 0\\
15\omega^2 - 10500\omega^2 + 900000 &= 0
\end{aligned}$$

$$\omega^2 = \begin{cases} 100 \rightarrow \omega_1 = 10\,\text{rad/s} \\ 600 \rightarrow \omega_2 = 24.5\,\text{rad/s} \ . \end{cases}$$

Because the determinant of the system is zero for the eigenvalues, the eigenvectors are found by assigning an arbitrary value to a component,

$$(\mathbf{K} - \omega_1^2 \mathbf{M})\mathbf{v}_1 = \mathbf{0}$$
$$\begin{bmatrix} 4500 & -3000 \\ -3000 & 2000 \end{bmatrix} \mathbf{v}_1 = \mathbf{0}$$
$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$(\mathbf{K} - \omega_2^2 \mathbf{M})\mathbf{v}_2 = \mathbf{0}$$
$$\begin{bmatrix} -3000 & -3000 \\ -3000 & -3000 \end{bmatrix} \mathbf{v}_2 = \mathbf{0}$$
$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

(b) The Ritz-Rayleigh method To show the possibilities of the Ritz-Rayleigh method, a linearly increasing mode of vibration $\mathbf{r}_1$ is chosen. If the structure were more complex, more trial modes of vibration $\mathbf{r}_i$ could be provided.

$$\mathbf{R} = [\mathbf{r}_1] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$\hat{\mathbf{M}} = \mathbf{R}^T \mathbf{M} \mathbf{R} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 550\,\text{t}$$
$$\hat{\mathbf{K}} = \mathbf{R}^T \mathbf{K} \mathbf{R} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} 60000 & -30000 \\ -30000 & 30000 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 6 \times 10^4\,\text{kN/m} \ .$$

The eigenvalue problem is reduced to a scalar equation,

$$|\hat{\mathbf{K}} - \omega^2 \hat{\mathbf{M}}| = 0 \quad \rightarrow \quad \omega_1^2 = \frac{6 \times 10^4}{550} \quad \rightarrow \quad \omega_1 = 10.4\,\text{rad/s} \ .$$

Answer to Exercise 16 The impulse of the helicopter from Exercise 6 is fully transmitted to the building with dynamic properties calculated in Exercise 15. The impulse of the helicopter is applied at the top level of the building,

$$I = (mv)_{helicopter} = 10 \times 20 = 200\,\text{t m/s}$$
$$\mathbf{I} = \begin{bmatrix} 0 \\ 200 \end{bmatrix} \text{t m/s} .$$

Both modes of vibration When several modes are present, the resulting vibration is calculated using modal decomposition. Finally, the full vibration os obtained by superposition of each mode of vibration. For the first mode of vibration,

$$\begin{aligned}
I_1 &= \mathbf{v}_1^T \mathbf{I} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} 0 \\ 200 \end{bmatrix} = 600\,\mathrm{t\,m/s} \\
m_1 &= \mathbf{v}_1^T \mathbf{M} \mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 1500\,\mathrm{t} \\
\omega_1 &= 10\,\mathrm{rad/s} \\
x_1 &= \frac{I_1}{m_1 \omega_1} = \frac{600}{1500 \times 10}\,\mathrm{m}\,.
\end{aligned}$$

For the second mode of vibration

$$\begin{aligned}
I_2 &= \mathbf{v}_1^T \mathbf{I} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 200 \end{bmatrix} = -200\,\mathrm{t\,m/s} \\
m_2 &= \mathbf{v}_1^T \mathbf{M} \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 150 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 250\,\mathrm{t} \\
\omega_2 &= 24.5\,\mathrm{rad/s} \\
x_2 &= \frac{I_2}{m_2 \omega_2} = \frac{-200}{250 \times 24.5}\,\mathrm{m}\,.
\end{aligned}$$

Thus, the resulting vibration is

$$\mathbf{u}(t) = \sum x_i(t) \mathbf{v}_i = \begin{bmatrix} 0.08 \\ 0.12 \end{bmatrix} \sin(10t) + \begin{bmatrix} -0.032 \\ 0.032 \end{bmatrix} \sin(24.5t)$$

where $u_{max} = 0.12 + 0.032 \approx 15\,\mathrm{cm}$.

Linearly increasing mode When the Ritz-Rayleigh method is applied, the equivalent action will be computed. The values are taken from Exercise 15.

$$\begin{aligned}
\hat{\mathbf{I}} &= \mathbf{R}^T \mathbf{I} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} 0 \\ 200 \end{bmatrix} = 400\,\mathrm{t\,m/s} \\
\hat{\mathbf{M}} &= 550\,\mathrm{t} \\
\omega_1 &= 10.4\,\mathrm{rad/s} \\
x_1 &= \frac{\hat{I}_1}{\hat{m}_1 \omega_1} = \frac{400}{550 \times 10.4}\,\mathrm{m}\,.
\end{aligned}$$

The resulting vibration is

$$\mathbf{u}(t) = x_1(t) \mathbf{r}_1 = \begin{bmatrix} 0.072 \\ 0.145 \end{bmatrix} \sin(10.4t)$$

where $u_{max} = 14.5\,\mathrm{cm}$.

Answer to Exercise 17 Approximate periods of vibrations can be found using the Ritz-Rayleigh method. Firstly, we need to construct the mass and stiffness matrices of the system that describes the three storey building:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times 10^5\,\text{kg}$$

$$\mathbf{K} = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \times 10^7\,\text{N/m}$$

One mode We start to apply the Ritz-Rayleigh method with a mode increasing linearly with height. The matrix of modes of deformation will contain only one vector,

$$\mathbf{R} = [\mathbf{r}_1] = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$$

and the reduced system is obtained by the following arithmetics:

$$\hat{\mathbf{M}} = \mathbf{R}^T\mathbf{M}\mathbf{R} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 17 \times 10^5\,\text{kg}$$

$$\hat{\mathbf{K}} = \mathbf{R}^T\mathbf{K}\mathbf{R} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 6 \times 10^7\,\text{N/m}$$

Thus, the eigenvalue problem is reduced to a scalar equation:

$$|\hat{\mathbf{K}} - \omega^2\hat{\mathbf{M}}| = 0 \quad \rightarrow \quad \omega_1^2 = \frac{6 \times 10^7}{17 \times 10^5} = 35.3 \quad \rightarrow \quad \omega_1 = 5.9\,\text{rad/s}$$

Two modes The Ritz-Rayleigh method can be enriched with more modes of vibration, in this case, with a mode varying quadratically with height,

$$\mathbf{R} = \begin{bmatrix} r_1 & r_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}^T .$$

The reduced system is then obtained by algebraic multiplications:

$$\hat{\mathbf{M}} = \mathbf{R}^T\mathbf{M}\mathbf{R} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{bmatrix} =$$

$$= \begin{bmatrix} 17 & 41 \\ 41 & 107 \end{bmatrix} \times 10^5\,\text{kg}$$

$$\hat{\mathbf{K}} = \mathbf{R}^T\mathbf{K}\mathbf{R} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{bmatrix} =$$

$$= \begin{bmatrix} 6 & 14 \\ 14 & 46 \end{bmatrix} \times 10^7\,\mathrm{N/m}$$

and the eigenvalue problem involves the calculation of the determinant of a 2×2 matrix:

$$|\hat{\mathbf{K}} - \omega^2\hat{\mathbf{M}}| = \begin{vmatrix} 600 - \omega^2 17 & 1400 - \omega^2 41 \\ 1400 - \omega^2 41 & 4600 - \omega^2 107 \end{vmatrix} \times 10^5 = 0\,.$$

The result is the second order polynomial $138\omega^4 - 27600\omega^2 + 800000 = 0$ with the following roots:

$$\begin{aligned} \omega_1^2 &= 35.1 \quad \rightarrow \quad \omega_1 = 5.9\,\mathrm{rad/s} \\ \omega_2^2 &= 164.8 \quad \rightarrow \quad \omega_2 = 12.8\,\mathrm{rad/s}\,. \end{aligned}$$

Answer to Exercise 19

Before approximating the main period of vibration assuming a linearly increasing shape function, the mass matrix and stiffness matrix should be built. Loads shall be converted into mass and the units must be consistent, e.g. tones and kilo Newton.

$$\mathbf{M} = \begin{bmatrix} 80 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 80 \end{bmatrix}\mathrm{t} \quad ; \quad \mathbf{K} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \times 10^4\,\mathrm{kN/m}$$

When assuming only a linearly increasing shape function, the Ritz-Rayleigh vectors will reduce to a single vector:

$$\Psi = \frac{z}{h_1} \quad ; \quad \mathbf{R} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

$$\hat{\mathbf{M}} = \mathbf{R}^T\mathbf{M}\mathbf{R} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 80 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 80 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1120\,\mathrm{t}$$

$$\hat{\mathbf{K}} = \mathbf{R}^T\mathbf{K}\mathbf{R} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} 10^4\,\mathrm{kN} = 6 \times 10^4\,\mathrm{kN}.$$

Finally, the main period of vibration is computed as if the building were a single-degree of freedom system,

$$\omega = \sqrt{\frac{\hat{\mathbf{K}}}{\hat{\mathbf{M}}}} = 7.3\,\mathrm{rad/s} \quad , \quad T = 0.86\,\mathrm{s}.$$

According to the inelastic spectrum proposed for the current building, the peak acceleration to be considered is

$$S_a = 0.125 \left(\frac{0.6}{0.86}\right)^{2/3} g = 1\,\mathrm{m/s^2}$$

and the base shear is then a simple multiplication of the peak acceleration and the total mass of the building,

$$V_b = m \cdot S_a = (3 \cdot 80) \cdot 1 = 240\,\mathrm{kN}.$$

With the base shear and the main period provided by the modal analysis (reduced by the Ritz-Rayleigh method), the simplified base shear model provides an expression that allows to compute the equivalent static forces at the center of each story:

$$T_i = \frac{m_i h_i}{\sum m_i h_i} V_b,$$

$$\begin{aligned}
T_1 &= \frac{80 \cdot 3}{80 \cdot 3 + 80 \cdot 6 + 80 \cdot 9} 240 = \frac{3}{18} 240 = 40\,\mathrm{kN}, \\
T_1 &= \frac{80 \cdot 6}{80 \cdot 3 + 80 \cdot 6 + 80 \cdot 9} 240 = \frac{6}{18} 240 = 80\,\mathrm{kN}, \\
T_1 &= \frac{80 \cdot 9}{80 \cdot 3 + 80 \cdot 6 + 80 \cdot 9} 240 = \frac{9}{18} 240 = 120\,\mathrm{kN}.
\end{aligned}$$

It should be noted that the cummulative equivalent static forces equals to the base shear.

Plastic dissipation allows to consider the inelastic spectrum with reduced forces. At the same time, plastic dissipation is equivalent of large strain rates. For that reason, the reduction of forces related to the inelastic spectrum is accompanied with a behavioural factor amplifying displacements. Then, the total displacement at the top is the cummulative inter-story drifts amplified by the behavioural factor,

$$u_{\text{top}} = q \left(\frac{T_1 + T_2 + T_3}{k_1} + \frac{T_2 + T_3}{k_2} + \frac{T_3}{k_3}\right) = 4 \left(\frac{240 + 200 + 120}{20000}\right) = 0.11\,\mathrm{m}.$$

Appendices

Modes of vibration of beams

Analyitical expressions for the frequencies of vibration for beams can be found by means of the Rayleigh's method. Generally, sinusoidal functions are used, since the appropriate combination of them allow to consider multiple boundary conditions and the main modes of vibration. The expressions shown in the table below coincide with the expressions proposed in most of the codes and application guidelines, see for instance A. L. Smith, *Design of floors for vibration: a new approach. Publication number: SCI P354*. All of them are the result of applying the Raileigh's method with appropriate shape functions.

Model	f (Hz)	M_Ψ
simply supp	$\frac{\pi}{2}\sqrt{\frac{EI}{L^3m}}$	$\frac{m}{2}$
cantilever	$\frac{\pi}{8\sqrt{3-8/\pi}}\sqrt{\frac{EI}{L^3m}}$	$0.64m$

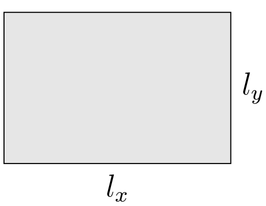
Modes of vibration of slabs

The natural stiffness of a floor slab can be approximated making use of the Rayleigh's method using an appropriate shape function satisfying the boundary conditions. From the static equilibrium equation of a slab, the modal mass and modal stiffness are rewritten according to the follwong expressions:

$$m_\psi = \int_0^{l_y}\int_0^{l_x} \rho t \psi^2 dxdy$$
$$k_\psi = \int_0^{l_y}\int_0^{l_x} D(\partial_{xx}\psi + \partial_{yy}\psi)^2 dxdy$$

where ρ is the density, t is the thickness and D is the flexural stiffness, calculated from the Young's modulus E and Poissins's ratio ν as

$$D = \frac{Et^3}{12(1-\nu^2)}.$$



For a rectangular floor *simply supported at the four sides*, the natural frequencies can be approximated using the following shape function, where n and m stand for the i^{th} mode of vibration at the x and y directions,

$$\psi = \sin\left(\frac{n\pi x}{l_x}\right)\sin\left(\frac{m\pi y}{l_y}\right).$$

The modal mass is

$$m_\psi = \int_0^{l_y}\int_0^{l_x} \rho t \sin^2\left(\frac{n\pi n x}{l_x}\right)\sin^2\left(\frac{m\pi y}{l_y}\right) dxdy =$$
$$= \rho t \int_0^{l_x} \sin^2\left(\frac{n\pi x}{l_x}\right) dx \int_0^{l_y} \sin^2\left(\frac{m\pi y}{l_y}\right) dy = \frac{\rho t l_x l_y}{4mn}.$$

The modal stiffness is

$$k_\psi = \int_0^{l_y}\int_0^{l_x} D\left[-\frac{n^2\pi^2}{l_x^2}\sin\left(\frac{n\pi x}{l_x}\right)\sin\left(\frac{m\pi y}{l_y}\right)\right.$$
$$\left. -\frac{m^2\pi^2}{l_y^2}\sin\left(\frac{n\pi x}{l_x}\right)\sin\left(\frac{m\pi y}{l_y}\right)\right]^2 dxdy =$$
$$= \pi^4 D\left[\frac{n^4}{l_x^4}\frac{l_x}{2n}\frac{l_y}{2m} + \frac{m^4}{l_y^4}\frac{l_x}{2n}\frac{l_y}{2m} + 2\frac{n^2m^2}{l_x^2l_y^2}\frac{l_x}{2n}\frac{l_y}{2m}\right] =$$
$$= \frac{\pi^4 D}{4}\left(\frac{l_y n^3}{l_x^3 m} + \frac{l_x m^3}{l_y^3 m} + \frac{2mn}{l_x l_y}\right).$$

Finally, the frequencies of the slab are obtained after division of the above results,

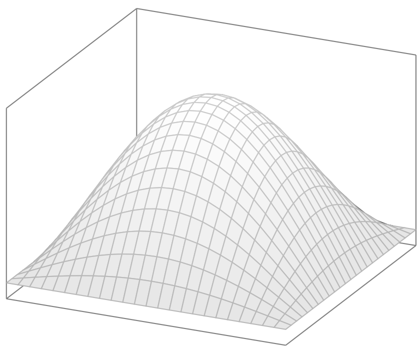
$$\omega^2 = \frac{k_\psi}{m_\psi} = \frac{\pi^4 D}{4}\frac{4}{\rho t}\left(\frac{n^4}{l_x^4} + \frac{m^4}{l_y^4} + 2\frac{n^2 m^2}{l_x^2 l_y^2}\right) = \frac{\pi^4 D}{\rho t}\left(\frac{n^2}{l_x^2} + \frac{m^2}{l_m^2}\right)^2,$$

$$\omega = \pi^2\sqrt{\frac{D}{\rho t}}\left(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}\right),$$

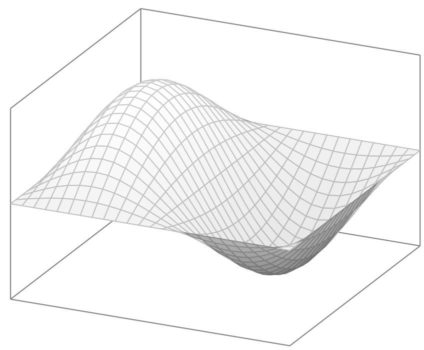
and the first natural frequency is

$$f_1 = \frac{\omega_1}{2\pi} = \frac{\pi}{2}\sqrt{\frac{D}{\rho t}}\left(\frac{1}{l_x^2} + \frac{1}{l_y^2}\right).$$

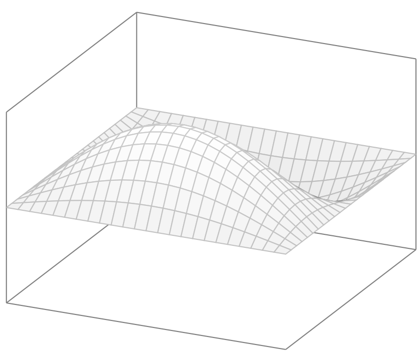
1^{st} mode ($n = 1$, $m = 1$)



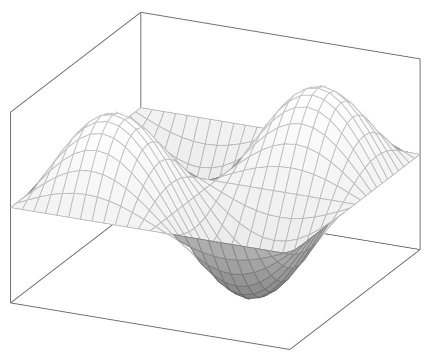
2^{nd} mode ($n = 2$, $m = 1$)



3^{rd} mode ($n = 1$, $m = 2$)

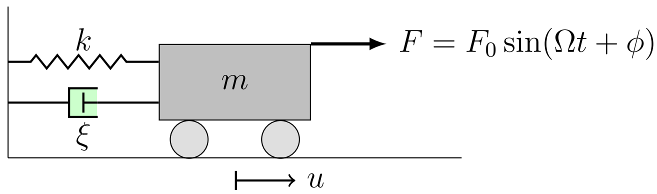


4^{th} mode ($n = 2$, $m = 2$)



Tuned mass dampers

A tuned mass damper (TMD) is primarily a single degree-of-freedom system, where a smaller and secondary single degree-of-freedom system is added. For that reason, the detailed analysis of the response begins by understanding by a simple *spring-mass-damper* system. Furthermore, we are interested in the steady state response, which depends on the external force applied to it, regadless of the initial conditions. Making use of the steady state solution, the full solution of the vibration can be obtained by adding the transient solution, which satisfies the initial conditions.



The governing equation of motion for the steady state solution is given by

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = \frac{F}{m}$$

Taking a trial function $u = \frac{F_0}{k} H \sin(\Omega t + \phi - \Delta\phi)$, the response is given by

$$\frac{F_0}{k} H \left((\omega^2 - \Omega^2) \sin(\Omega t + \phi - \Delta\phi) + 2\xi\omega\Omega \cos(\Omega t + \phi - \Delta\phi) \right) = \frac{F_0}{m} \sin(\Omega t + \phi).$$

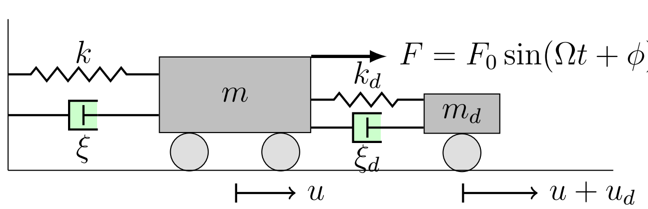
Using the notation $\gamma = \Omega/\omega$ and making use of the composition rule for sine and cosine functions, the previous expression leads to

$$H\sqrt{(1-\gamma^2)^2 + 4\xi^2\gamma^2} \sin\left(\Omega t + \phi - \Delta\phi + \tan^{-1} \frac{2\xi\gamma}{1-\gamma^2} \right) = \sin(\Omega t + \phi)$$

Finally, the dynamic magnification factor and phase lag are

$$H = \frac{1}{\sqrt{(1-\gamma^2)^2 + 4\xi^2\gamma^2}}$$
$$\tan\Delta\phi = \frac{2\xi\gamma}{1-\gamma^2}$$

Then, a secondary and smaller *spring-mass-damper* system can be attached to the main system in order to build a tuned mass damper. The system can be interpreted as the picture below, where the main components that are involved in the equations are shown.



Here, the subscript d refers to the tuned mass damper and the following notation is introduced,

$$\omega^2 = \frac{k}{m}$$
$$\omega_d^2 = \frac{k_d}{m_d}$$
$$\gamma = \frac{\Omega}{\omega}$$
$$\rho = \frac{m_d}{m} \ll 1$$
$$\eta = \frac{\omega_d}{\omega} \approx 1$$

The governing equations of motion for the steady state solution are given by

$$(1+\rho)\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = \frac{F}{m} - \rho\ddot{u}_d \qquad \text{(primary mass)}$$
$$\ddot{u}_d + 2\xi_d\omega_d\dot{u} + \omega^2 u_d = -\ddot{u} \qquad \text{(tuned mass)}$$

$\xi = 0$ Undamped structure, damped TMD

Using the trial functions

$$u = \frac{F_0}{k}\bar{H}e^{i\Omega t}$$
$$u_d = \frac{F_0}{k}\bar{H}_d e^{i\Omega t}$$

the response is given by

$$-(1+\rho)\Omega^2\frac{F_0}{k}\bar{H}e^{i\Omega t} + \omega^2\frac{F_0}{k}\bar{H}e^{i\Omega t} = \frac{F_0}{m}e^{i\Omega t} + \rho\Omega^2\frac{F_0}{k}\bar{H}_d e^{i\Omega t}$$
$$-\Omega^2\frac{F_0}{k}\bar{H}_d e^{i\Omega t} + i2\xi_d\omega_d\Omega\frac{F_0}{k}\bar{H}_d e^{i\Omega t} + \omega_d^2\frac{F_0}{k}\bar{H}_d e^{i\Omega t} = \Omega^2\frac{F_0}{k}\bar{H}e^{i\Omega t}$$

which yields the following expression, after simplification and cancelling the $e^{i\Omega t}$ terms

$$\bar{H} - (1+\rho)\gamma^2\bar{H} = 1 + \rho\gamma^2\bar{H}_d$$
$$\eta^2\bar{H}_d - \gamma^2\bar{H}_d + i2\xi_d\gamma\eta\bar{H}_d = \gamma\bar{H}$$

The solution is

$$\bar{H} = \frac{\eta^2 - \gamma^2 + i2\xi_d\gamma\eta}{\bar{H}^*}$$
$$\bar{H}_d = \frac{\gamma^2}{\bar{H}^*}$$

with

$$\bar{H}^* = (\eta^2 - \gamma^2)(1-\gamma^2) - \rho\gamma^2\eta^2 + i2\xi_d\gamma\eta(1-\gamma^2(1+\rho))$$

The complex numbers $\bar{H}$ and $\bar{H}_d$ are definig a modulus H and a phase angle $\Delta\phi$, both for the primary and the tuned mass. The real and imaginary parts correspond to a cosine and sinusoidal functions, so the polar form can be converted into a trigonometric function,

$$u = \frac{F_0}{k} H \sin(\Omega t + \Delta\phi - \Delta\phi_d)$$
$$u_d = \frac{F_0}{k} H_d \sin(\Omega t - \Delta\phi_d)$$

For the primary mass we have

$$H = \sqrt{\frac{(\eta^2-\gamma^2)^2 + 4\xi_d^2\gamma^2\eta^2}{((\eta^2-\gamma^2)(1-\gamma^2) - \rho\gamma^2\eta^2)^2 + (2\xi_d\gamma\eta(1-\gamma^2(1+\rho)))^2}}$$

$$\tan\Delta\phi = \frac{2\xi_d\gamma\eta}{\eta^2-\gamma^2}$$
$$\tan\Delta\phi_d = \frac{2\xi_d\gamma\eta(1-\gamma^2(1+\rho))}{(\eta^2-\gamma^2)(1-\gamma^2) - \rho\gamma^2\eta^2}$$

Figure 1 shows several response function at different tuned frequencies. It can be seen that the optimal TMD is achieved when the two maximums of H are symmetrics. It happens at

$$\eta_{\text{opt}} = \frac{1}{1+\rho}$$

Once the optimal tuned frequency is set, the tuned damping can modify the response. An excessive damping ratio may generate a monolitic response while a low damping ratio exibits two peaks of ressonance. Figure 2 shows the response, the optimal damping ratio is

$$\xi^2_{\text{opt}} = \frac{\rho(3-\sqrt{0.5\rho})}{8(1+\rho)(1-0.5\rho)}$$

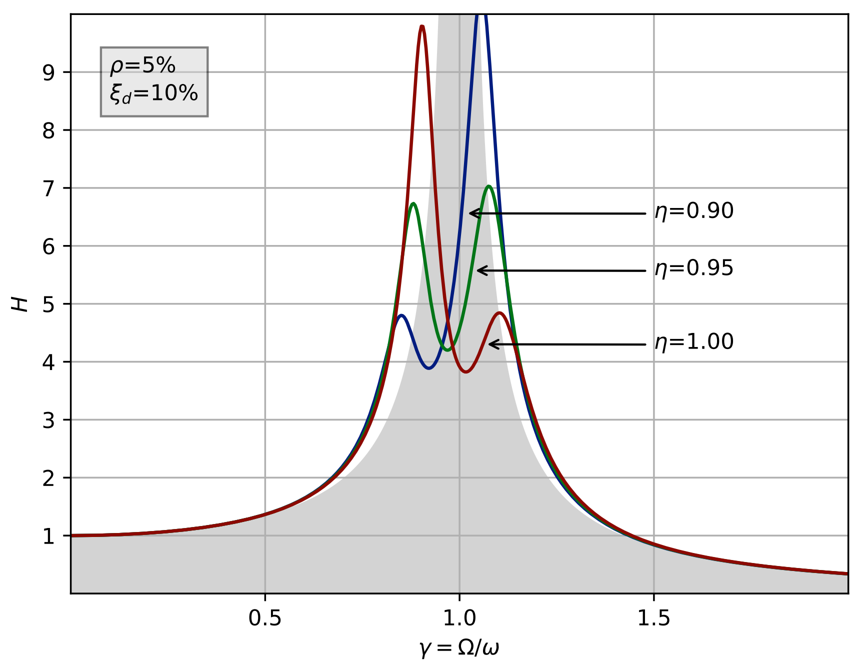


Figure 1: Plot of dynamic magnification factor H versus forcing frquency γ at different tuning frequencies η. The shadowed region shows dynamic magnification without TMD.

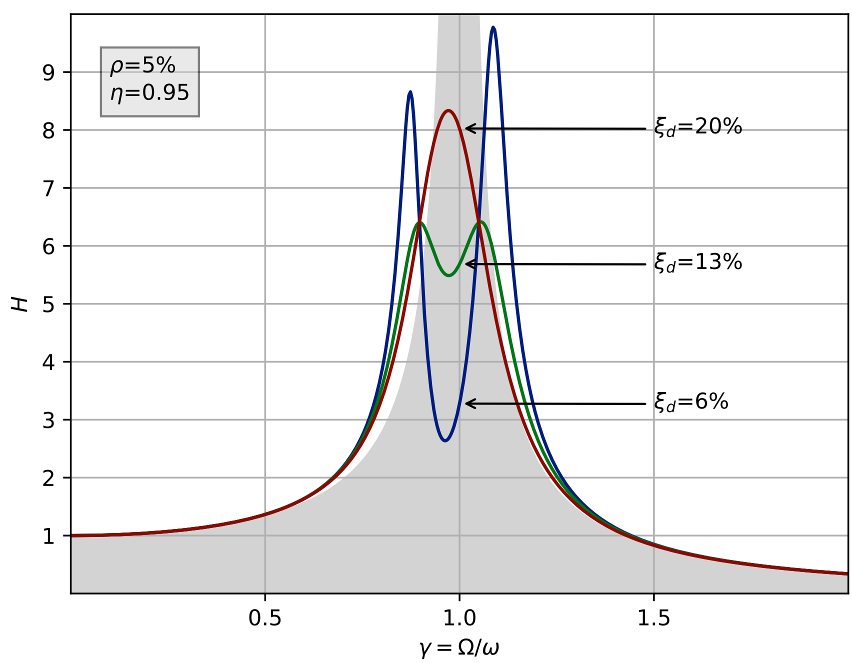


Figure 2: Plot of dynamic magnification factor H versus forcing frquency γ with different tuned dampings ξ_d. The shadowed region shows dynamic magnification without TMD.

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