Gravitational Lensing by Point Masses

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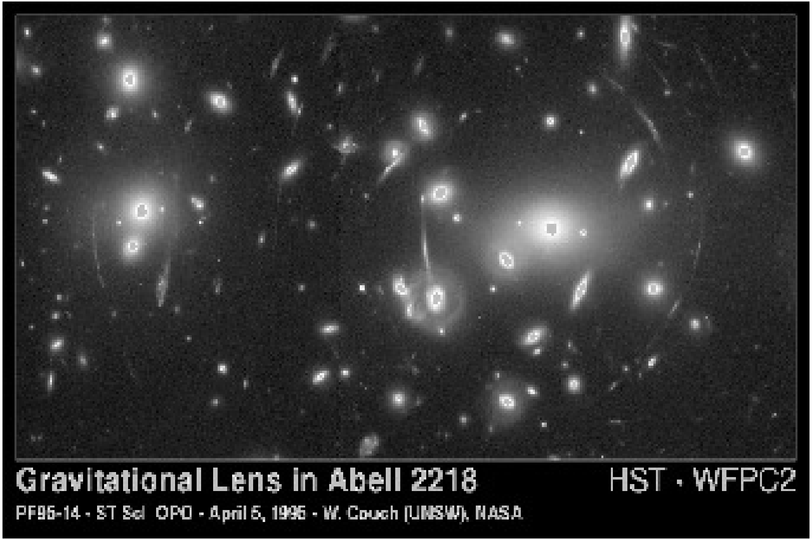
Gravitational Lens in Abell 2218



The Lens Equation

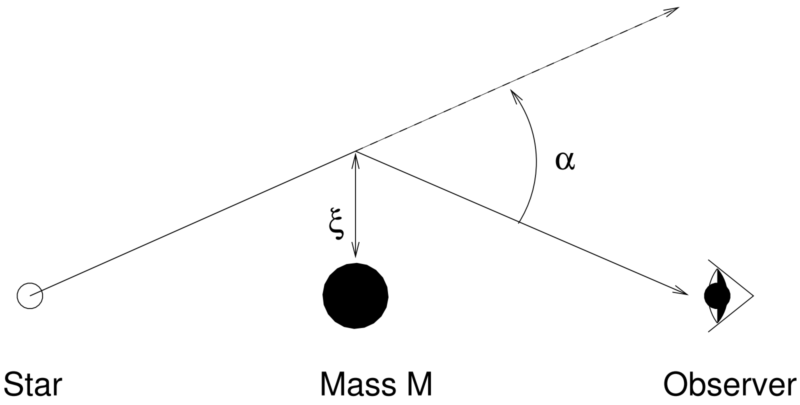
The angular deflection α of a gravitational lense is given by

$$\alpha = \frac{4GM}{c^2\xi} \tag{1}$$

where

- $M =$ mass of the deflecting object
- $G =$ gravitational constant
- $c =$ speed of light
- $\xi =$ impact radius of the incoming photon
- $\alpha =$ deflection angle

Angular Deflection of a Photon



Deflecting Mass

The deflecting mass may be viewed as an optical thin lens, made up of a two dimensional mass distribution.

$$M = \int_{R^2} \Sigma(\vec{\xi'}) \frac{\vec{\xi} - \vec{\xi'}}{|\vec{\xi} - \vec{\xi'}|^2} d^2\xi' \tag{2}$$

where

- $\Sigma =$ surface density
- $R^2 =$ surface area

If the deflecting mass, is perfectly symmetric, the deflection angle becomes

$$\vec{\alpha} = \frac{4GM(<\xi)\vec{\xi}}{c^2|\vec{\xi}|^2} \tag{3}$$

The Lens Equation

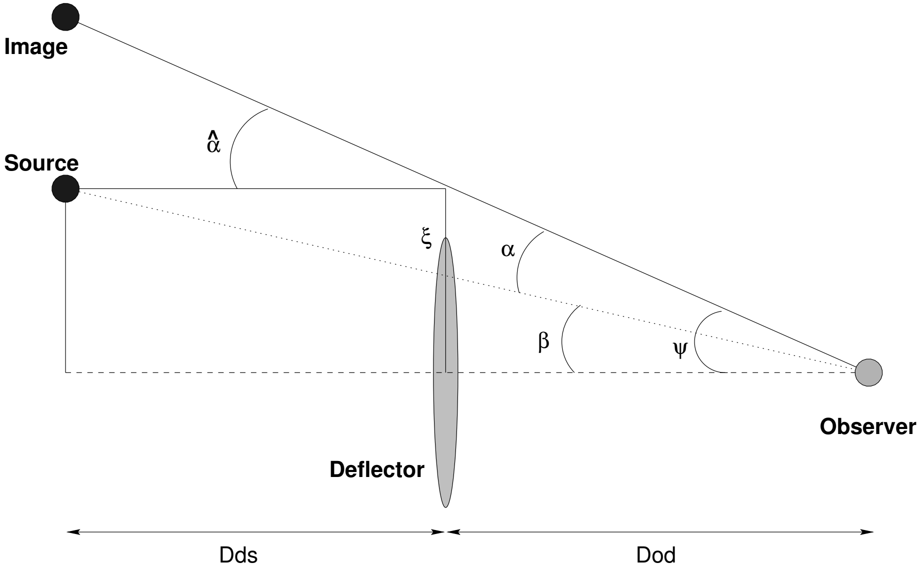
The lens equation is

$$\psi D_{os} + \alpha D_{ds} = \beta D_{os} \tag{4}$$

It is often the case in many gravitational lensing problems that the images form do not depend on the distances between source, observer and deflector directly. Rather a specific ratio of these distances is the quantity which needs to be considered. This is known as the effective distance D and is defined as:

$$D = \frac{D_{od} D_{ds}}{D_{os}} \tag{5}$$

Setup for the Gravitational Lens Equation



Einstein Ring

If a distant star is in perfect alignment with a point mass gravitational lens and an observer, the light from the distant star is lensed perfectly symmetrically forming a ring image of the star known as an Einstein ring. The radius of the Einstein ring is given by

$$\theta_E = \sqrt{\frac{4GM}{c^2} \frac{D_{ds}}{D_{od} D_{os}}} \tag{6}$$

If however there isn't a perfect alignment, for a point mass two images are produced for each point on the source plane. The angular positions at which these images is given by:

$$\theta_{\pm} = \frac{\xi}{D_{od}} = \frac{\beta}{2} \pm \sqrt{\beta^2 + 4\theta_E^2} \tag{7}$$

where β is the angular position of the source as shown in figure 3. The magnification of each of the images is given by

$$\mu_{\pm} = \frac{1}{4} \left[\frac{y}{\sqrt{y^2+4}} + \frac{\sqrt{y^2+4}}{y} \pm 2 \right] \tag{8}$$

where the source and image angles have been scaled such that $y = \psi/\alpha_0$ and $x = \alpha/\alpha_0$.

Reformulating the Lens Equation

Using this notation the lens equation can be rewritten as the scaled lens equation given by

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x}). \tag{9}$$

The vector notation used represents the coordinate in a cartesian plane. Therefore

- $\vec{y} = (y_1, y_2)$ is a coordinate in the source plane and
- $\vec{x} = (x_1, x_2)$ is a coordinate in the deflecting plane.

Complex Notation

This can also be written using complex number notation. A position in deflecting plane can be denoted as $z = x_1 + ix_2$, and a position in the source plane as $z_s = y_1 + iy_2$.
The magnification is given by:

$$\mu(\vec{x}) = \frac{1}{detA(\vec{x})} \tag{10}$$

where $detA(\vec{x})$ is the Jacobian determinant of the Hessian matrix given by

$$A(\vec{x}) = \frac{\partial \vec{y}}{\partial \vec{x}} \tag{11}$$

Critical Curves and Caustics

- Critical curves are the set of all points in the deflection plane where $A(\vec{x}) = 0$.
- The corresponding curves in the source plane (obtained from the lens equation) are known as caustics.
- Critical curves are where the gravitational lens infinitely magnifies the light passing through that point.
- Source plane caustics are the pre-image of the critical curves.
- Any light emitted near a caustic will be greatly magnified as it passes through the lens.

The Chang-Refsdal Lens

The Chang-Refsdal lens model describes gravitational lensing using a modification of the point mass model. This model says that when a source crosses a fold caustic the lensing is due to a point mass but with an additional external shear applied. The corresponding lens equation for the Chang-Refsdal model is:

$$\vec{y} = \begin{bmatrix} 1+\gamma & 0 \\ 0 & 1-\gamma \end{bmatrix} \vec{x} - \frac{\vec{x}}{|\vec{x}|^2} \tag{12}$$

This can also be written in the complex notation as $z_s = z + \gamma \bar{z} - \frac{\epsilon}{\bar{z}}$, where

- γ is a constant determining the amount shear.
- $\epsilon = (\frac{\kappa_s}{1-\kappa_c})$, where κ_s is the density of compact objects such as stars and κ_c is the surface density of continuously distributed matter.

The Effects of Distance on Gravitational Lensing

The Dyer-Roeder equation is given by:

$$(z+1)(\Omega z+1)\frac{d^2D}{dz^2}+\left(\frac{7}{2}\Omega z+\frac{1}{2}\Omega+3\right)\frac{dD}{dz}+\frac{3}{2}\tilde{\alpha}\Omega D=0 \tag{13}$$

This relates the angular diameter distance of a lensing system with the redshift z of the source object.

- Ω is the ratio of the mean mass density to the critical density of the universe and
- $\tilde{\alpha}$ is the clumpiness parameter, which determines the amount of matter between the source and the observer.

Initial Conditions for Dyer-Roeder Equation

The initial conditions of the Dyer-Roeder equation are:

$$D_{ii} = 0 \tag{14}$$

$$\left[\frac{dD_{ij}}{dz}\right]_{z_j=z_i} = \frac{\mathrm{sgn}(z_j - z_i)}{(z_i + 1)^2\sqrt{\Omega z_i + 1}} \tag{15}$$

The Dyer-Roeder equation needs to be solved for three different cases.

1. $\Omega = 0$ (D_I)
2. $\Omega = 1$, $\tilde{\alpha} = 1$ (D_{II})
3. $\Omega = 1$, $\tilde{\alpha} = 0$ (D_{III})

Solving the Dyer-Roeder Equation

This leads to three different equations which need to be solved.

$$(z+1)\frac{d^2D}{dz^2} + 3\frac{dD}{dz} = 0 \tag{16}$$

$$(z+1)^2\frac{d^2D}{dz^2} + \frac{7}{2}(z+1)\frac{dD}{dz} + \frac{3}{2}D = 0 \tag{17}$$

$$(z+1)^2\frac{d^2D}{dz^2} + \frac{7}{2}(z+1)\frac{dD}{dz} = 0 \tag{18}$$

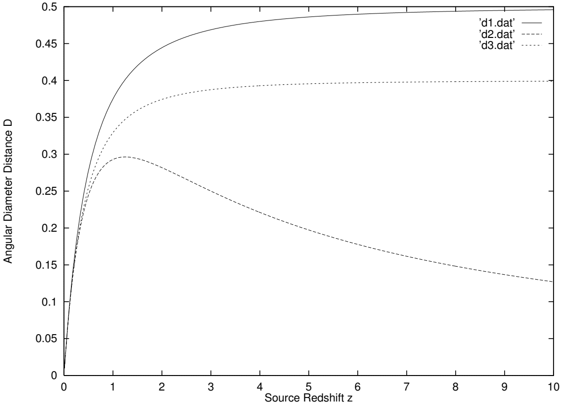
Computational Solution to the Dyer-Roeder Equation

- A FORTRAN program was written to numerically solve these equations.
- The algorithm used to solve these equation numerically was the *Runge-Kutta-Nyström* method.
- This method is a fourth order algorithm which is a general form of the standard *Runge-Kutta* method, used for solving second order ordinary differential equations.
- The equations were integrated from $z = 0$ to $z = 10$.
- The dimming factor $(D_{II}/D_{III})^2$ was determined and plotted as a function of redshift z.

Solutions to the Dyer-Roeder Equation

We solve the Dyer-Roeder equation for the three different cosmologies resulting in three solutions for the angular diameter distance D.

- D_I `d1.dat` (The top curve)
- D_{II} `d2.dat` (The bottom curve)
- D_{III} `d3.dat` (The centre curve)



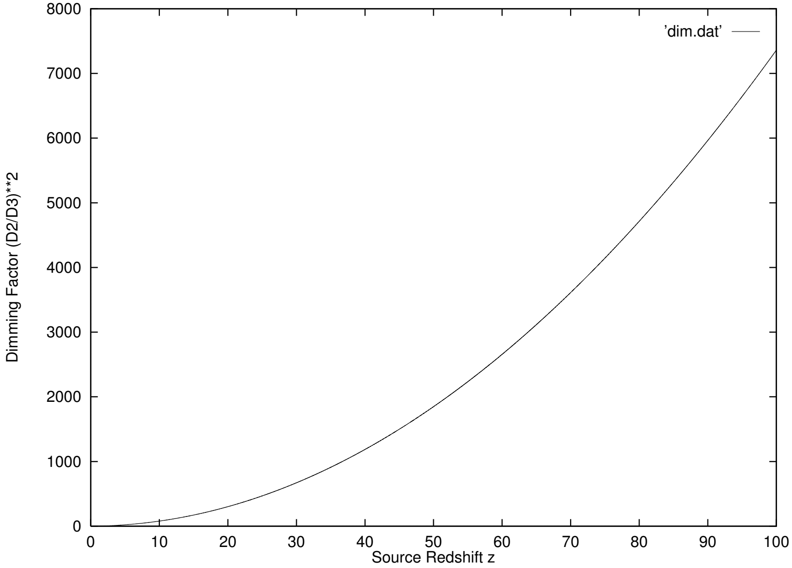
Increasing Redshift

- According to the big bang model of the universe, objects with large redshifts are further away.
- Hence the results of solving the Dyer-Roeder equation for the cases (1 and 3) when the "clumpiness" parameter $\tilde{\alpha}$ is ignored the angular diameter distance D increases as the redshift increases.
- In both cases we get asymptoting values:

$$\lim_{z \to \infty} D_I(z) = 0.5 \tag{19}$$

$$\lim_{z \to \infty} D_{II}(z) = 0.4 \tag{20}$$

Dimming factor against source redshift



Lensing an Extended Source by a Point Mass

Lensing Algorithm

procedure LENSE
 $img \leftarrow$ **load_image**()
 $x_c, y_c \leftarrow$ **find_centre**(img)
 $\theta_E \leftarrow$ **calc_einstein_radius**()
 for $c \in img.coordinates$ **do**
  $b_x, b_y \leftarrow$ **calc_location_lensing_body**()
  $\beta \leftarrow$ **impact_radius**()
  calc_angle_for_each_quadrant()
  $\theta_\pm \leftarrow$ **calc_new_angles**()
  $c \leftarrow$ **calc_new_deflected_coordinate**()
 end for
 save_image(img)
end procedure

Lensing Images

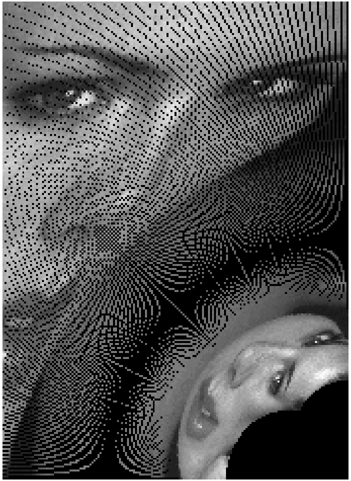
A number of different images were used in the simulation. Two were used for testing purposes and to observe the effect, and three were images of astronomical interest. The images used include:

- A human face
- The starship Enterprise
- The Andromeda Galaxy
- The Milky Way
- The Pleiades Star Cluster

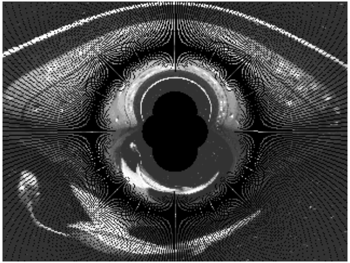
In all cases the distance ratio used was $D = 1$ and just the mass was varied. This is because when calculating the Einstein radius, the quantity DM appears as follows:

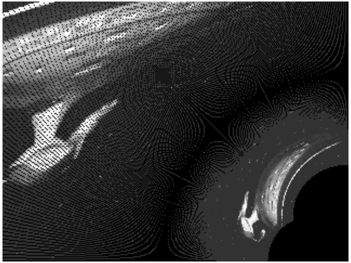
$$\theta_E = \sqrt{\frac{4GMD}{c^2}} \tag{21}$$

Gravitational lensing of image of a human face by a mass $M = 5 \times 10^{30}$ kg

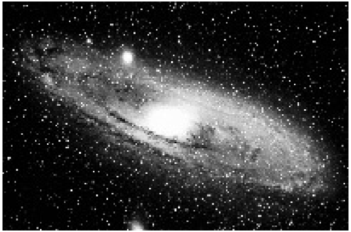


Gravitational lensing of an image of the starship Enterprise



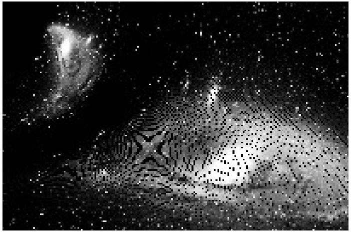


Gravitational Lensing of the Andromeda Galaxy



$M = 0kg$

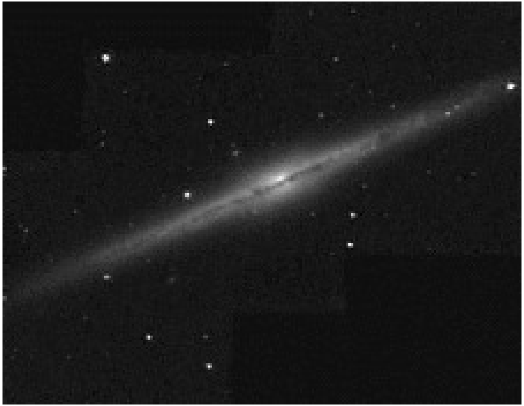
$M = 1 \times 10^{30}$ kg, $D = 1$

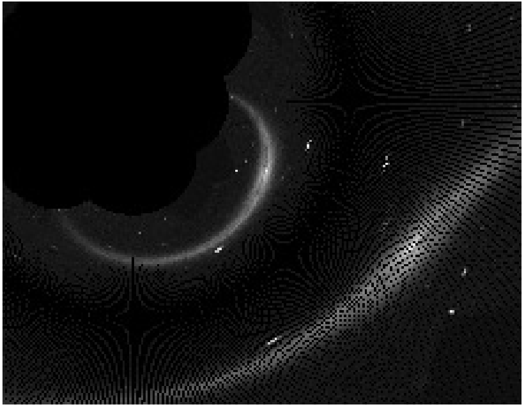


$M = 1 \times 10^{31}$ kg, $D = 1$

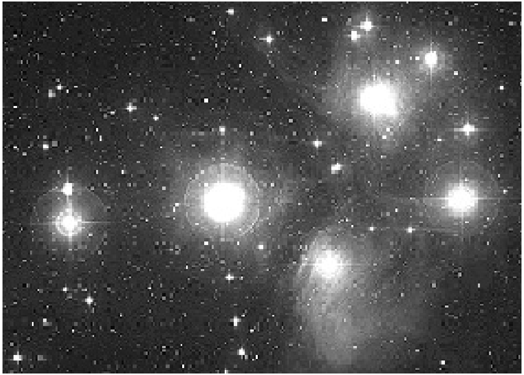
$M = 4 \times 10^{31}$ kg, $D = 1$

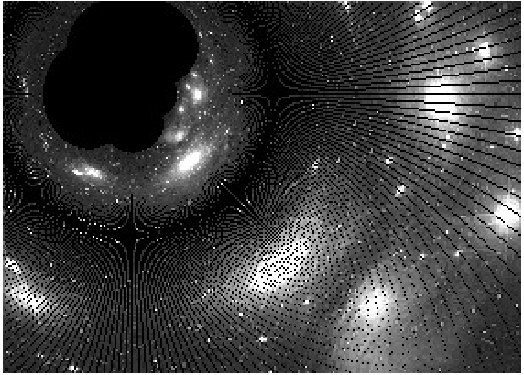
Gravitational Lensing of the Milky Way





Gravitational Lensing of the Pleiades Star Cluster





Caustics for the Chang-Refsdal Lens

$$\det A = 1 - \left(\gamma + \frac{\epsilon}{\bar{z}^2}\right)\left(\gamma + \frac{\epsilon}{\bar{z}^2}\right) = 0 \tag{22}$$

. Using complex polar coordinates letting $z = x\cos\phi + ix\sin\phi$ we get

$$x^4(1-\gamma^2) - 2\gamma\epsilon x^2(\cos^2\phi - \sin^2\phi) - 1 = 0 \tag{23}$$

The equation can be parameterised by letting:

$$\lambda = \cos^2\phi - \sin^2\phi$$

$$u = x^2$$

giving

$$u^2(1-\gamma^2) - 2\gamma\epsilon\lambda u - 1 = 0. \tag{24}$$

Caustics for the Change-Refsdal Lens (continued)

Solving this quadratic we obtain:

$$u = \frac{\gamma\epsilon\lambda \pm \sqrt{\gamma^2(\lambda^2 - 1) + 1}}{(1 - \gamma^2)} \tag{25}$$

For a given values of γ, ϵ and for $0 < \phi < 2\pi$ the program calculates u and then x. The (x, y) coordinates for the caustic curve is then given by $(x \cos\phi, x \sin\phi)$. The corresponding caustics are given by:

$$y_1 = \left[(1 + \gamma)x - \frac{\epsilon}{x}\right]\sqrt{\frac{1 + \lambda}{2}} \tag{26}$$

$$y_2 = \left[(1 - \gamma)x - \frac{\epsilon}{x}\right]\sqrt{\frac{1 - \lambda}{2}} \tag{27}$$

Selecting Values for ϵ and γ

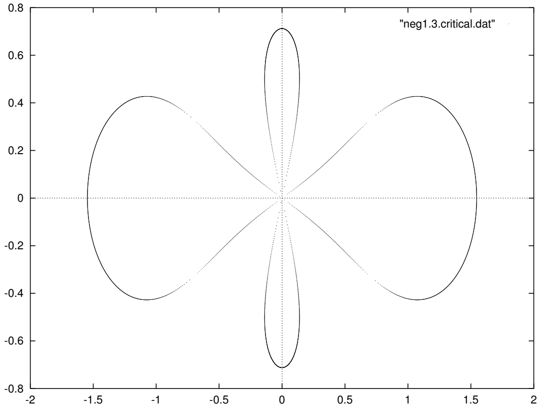
In the program a value of $\epsilon = 0.5$ was chosen. Values of gamma were chosen for the four representative regions in which caustics are found. The four regions are:

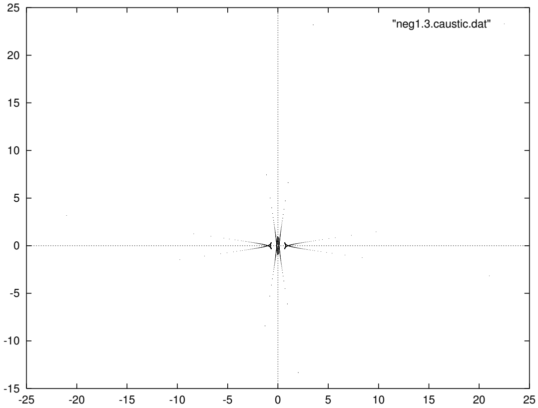
- $\gamma < -1$
- $-1 < \gamma < 0$
- $0 < \gamma < 1$
- $\gamma > 1$

The four values of γ selected are:

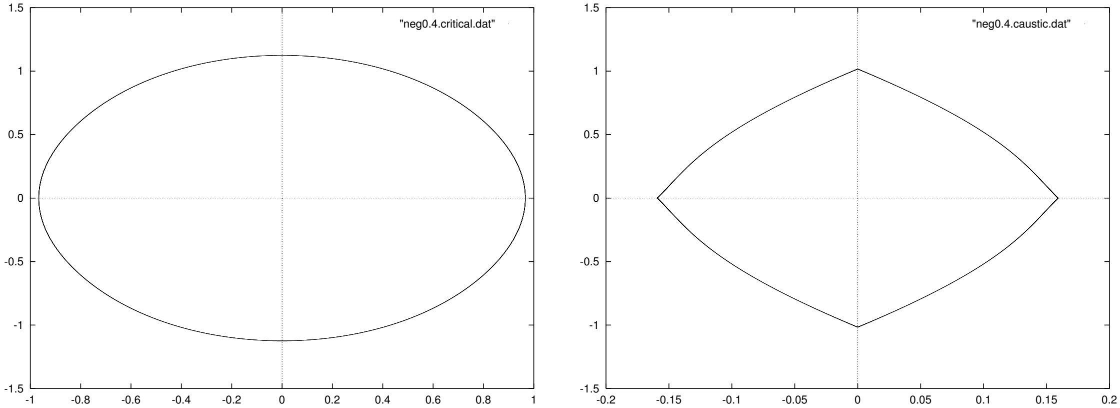
- $\gamma = -1.3$
- $\gamma = -0.4$
- $\gamma = +0.8$
- $\gamma = +1.6$

Critical curves and caustics for $\gamma = -1.3$

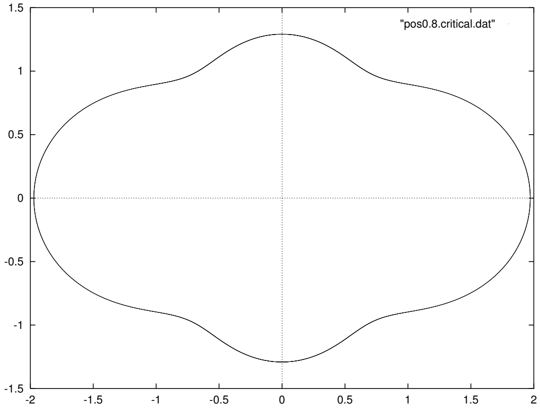


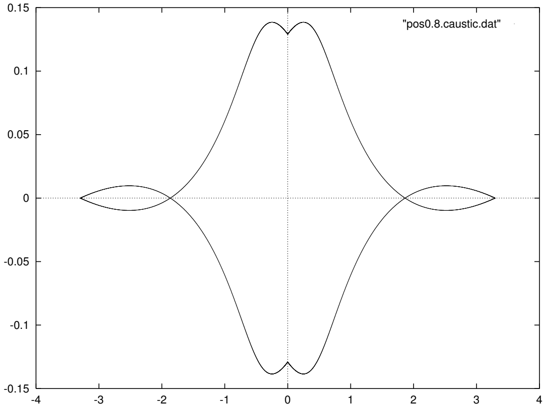


Critical curves and caustics for $\gamma = -0.4$

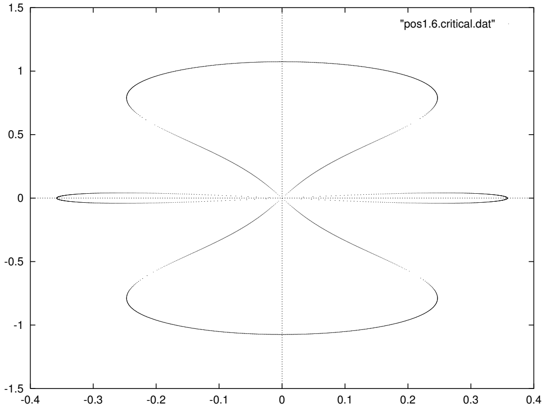


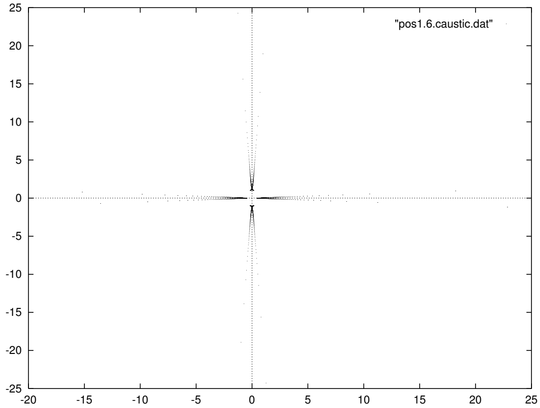
Critical curves and caustics for $\gamma = 0.8$





Critical curves and caustics for $\gamma = 1.6$





Gravitational Lensing by a Binary Star System