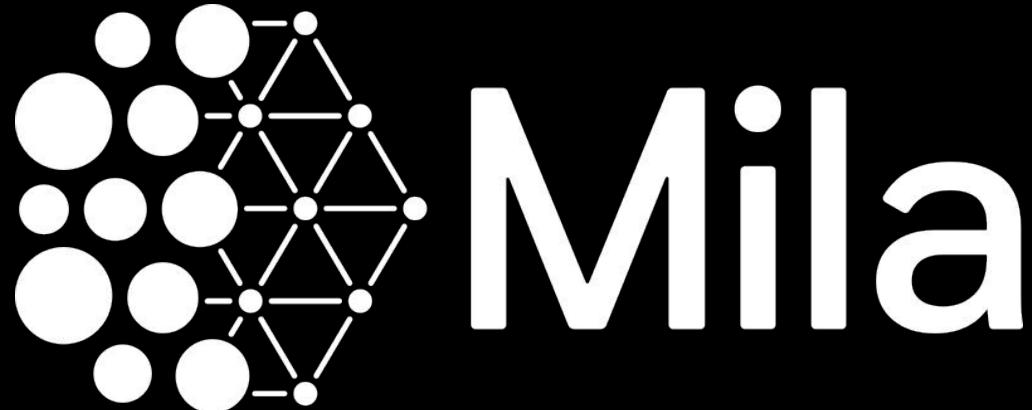


Quebec
Artificial
Intelligence
Institute



Optimization in deep learning

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Ingredients for supervised deep learning

- A task definition
- An evaluation metric
- A large amount of high-quality labeled data (>100,000)
- A learning algorithm:
 - End-to-end differentiable computational graph
 - A gradient calculation algorithm: backpropagation
 - **A parameter optimizer**

Why optimization is important in ML?

- We want to maximize the model performance evaluated by the metric.
- Usually, the evaluation metric is not differentiable (e.g., accuracy).
- We need a differentiable surrogate function for training the model.

Objectives of the presentation

Give the intuition to the basic concepts of optimization for deep learning.

Three concepts to remember:

- Preconditioning
- Momentum
- Stochastic optimization

Global minimum

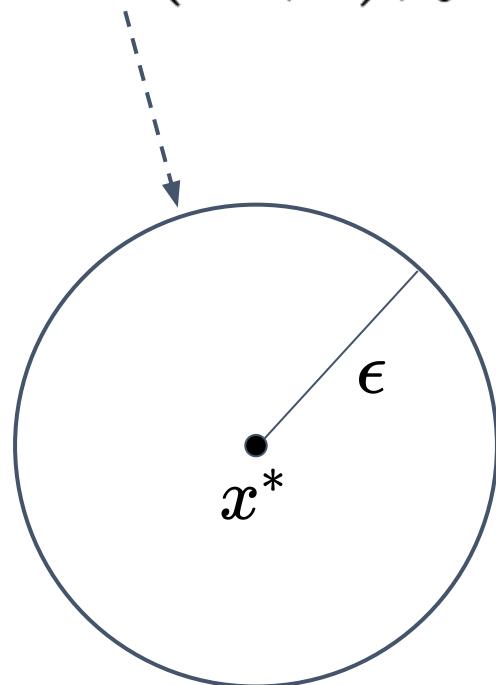
x^* is a *global minimum* of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ iif

$$\forall x \in \mathbb{R}^n, f(x^*) \leq f(x)$$

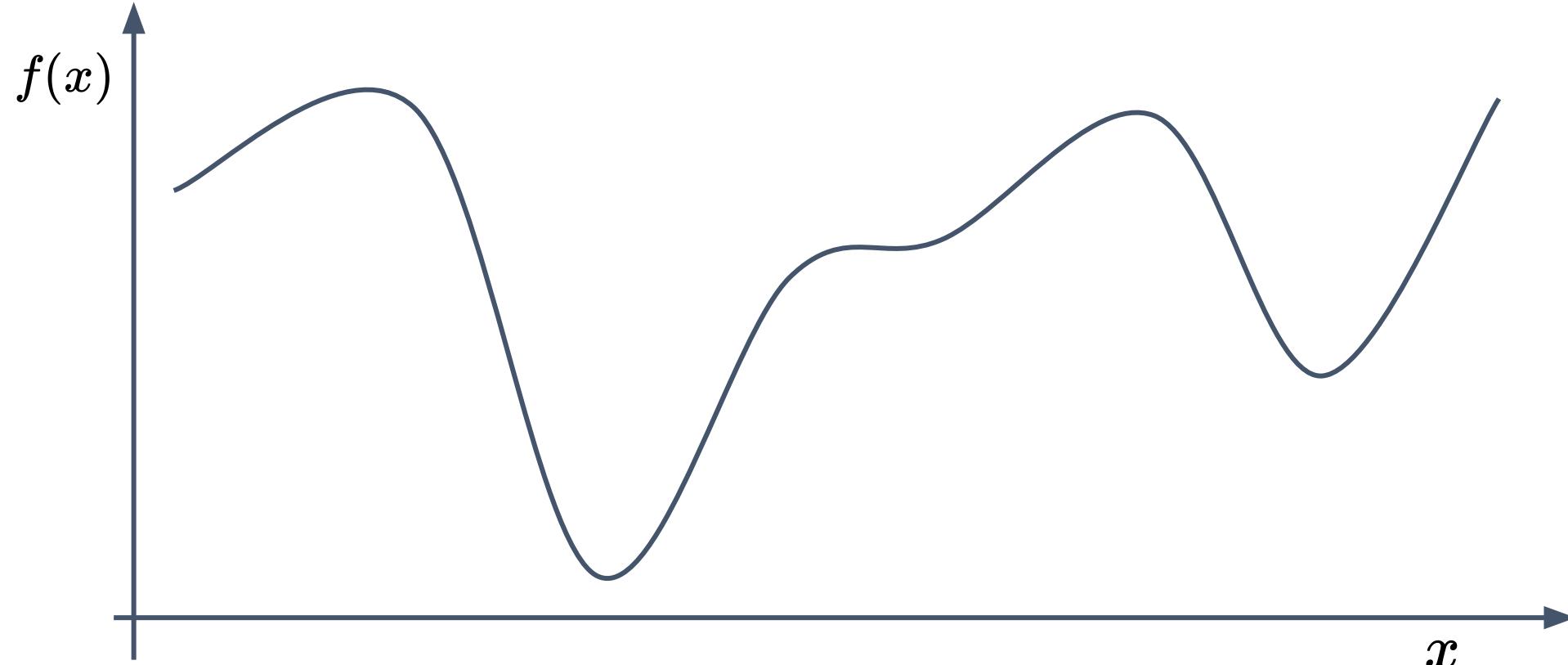
Local minimum

x^* is a *local minimum* of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ iif

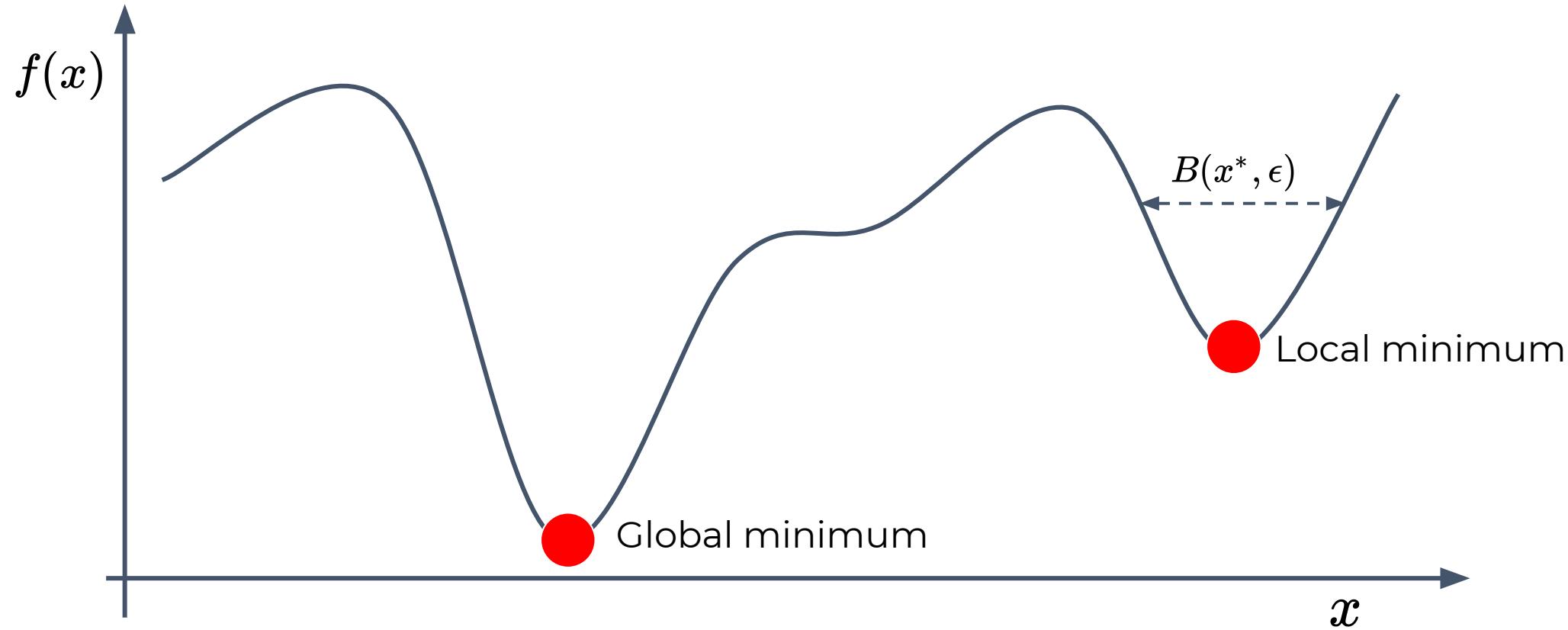
$$\exists \epsilon > 0, \forall x \in B(x^*, \epsilon), f(x^*) \leq f(x)$$



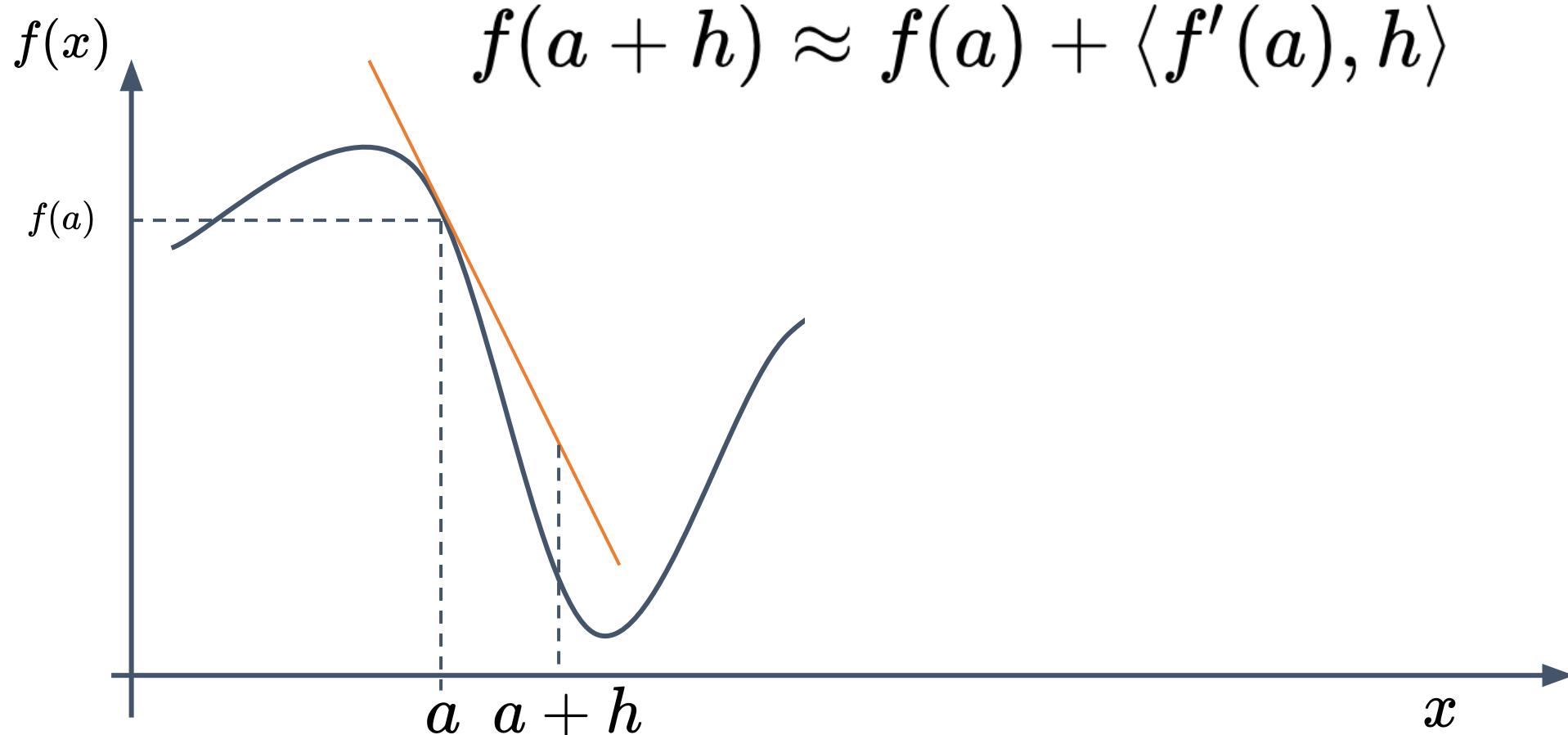
Example in 1D



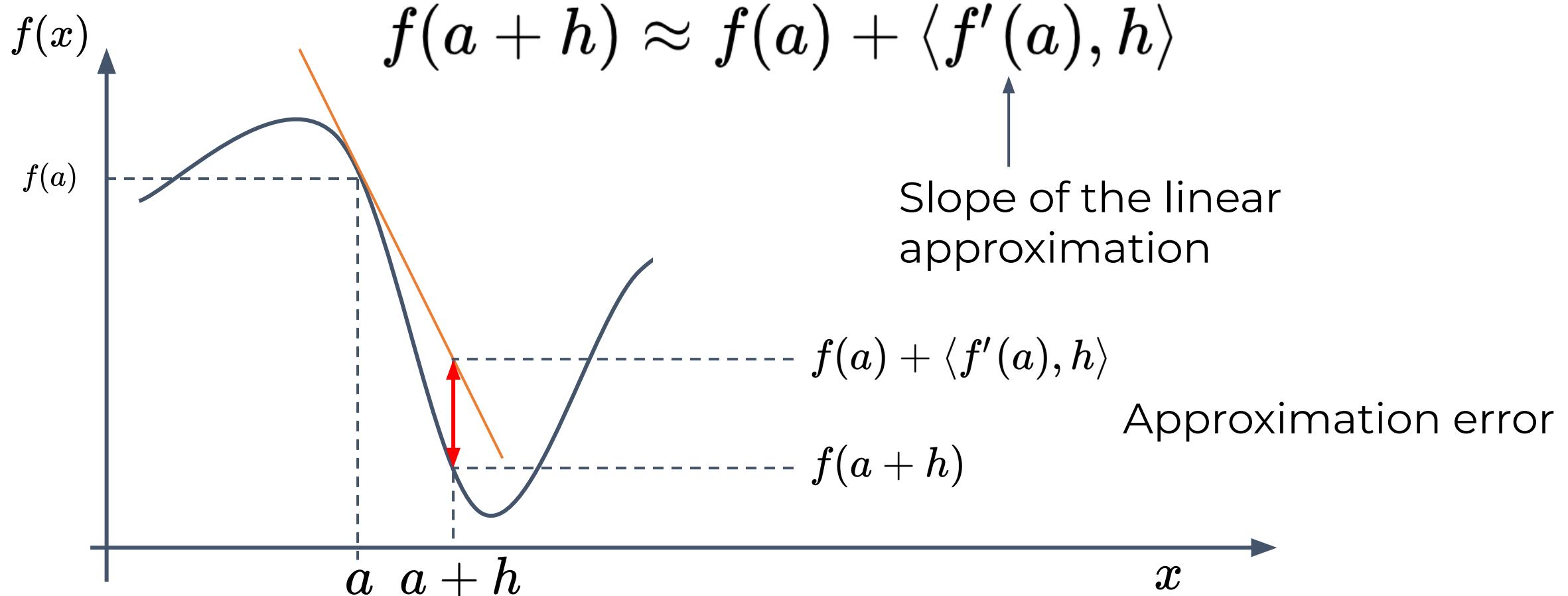
Example



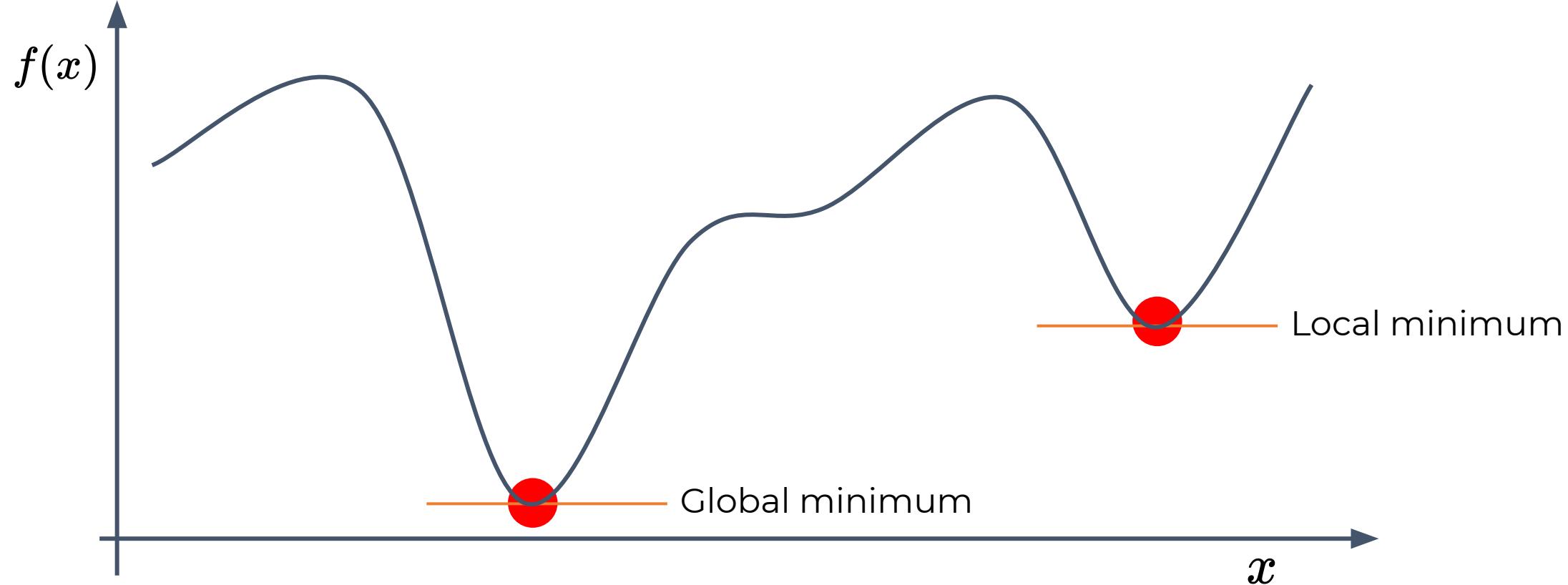
Linear approximation



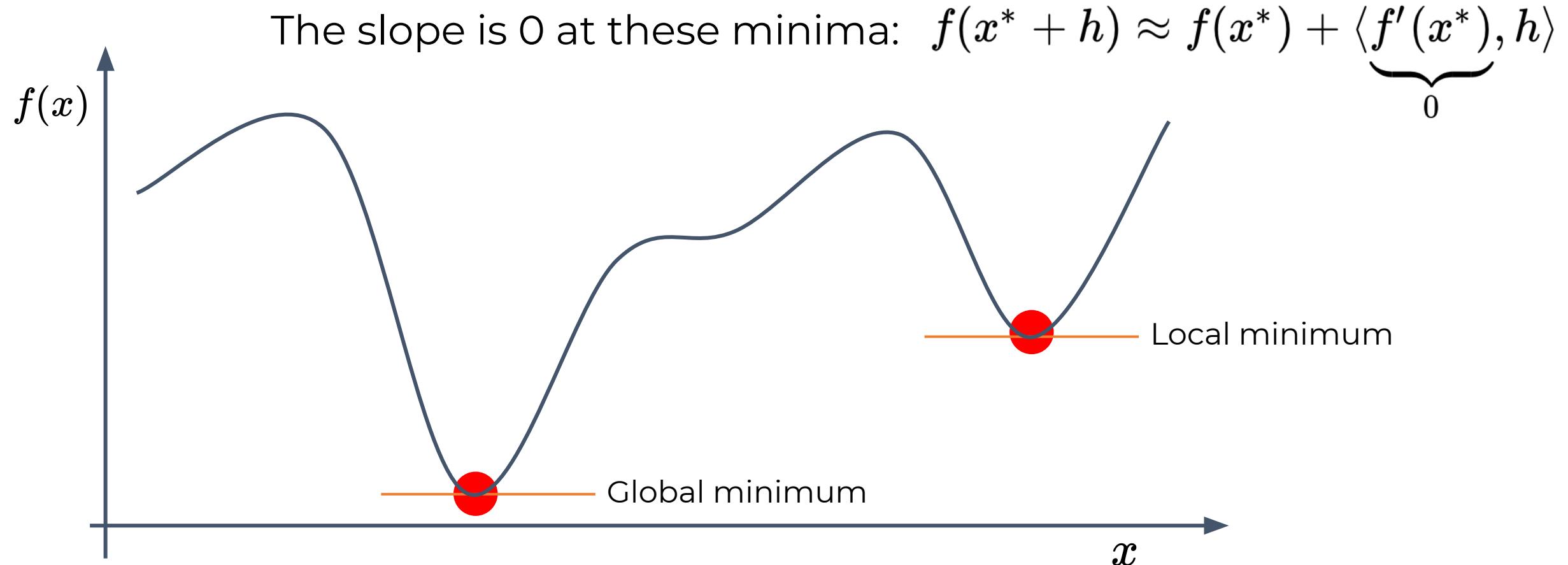
Linear approximation



How to find extrema?

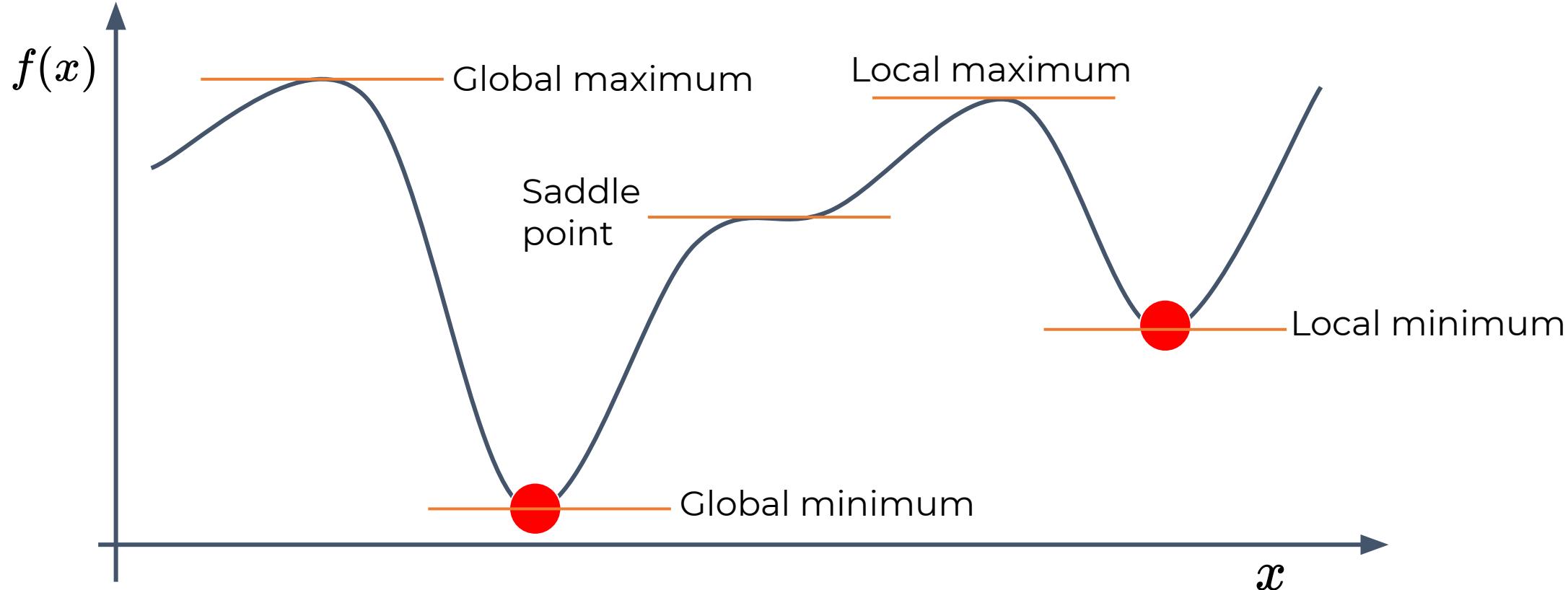


How to find extrema?



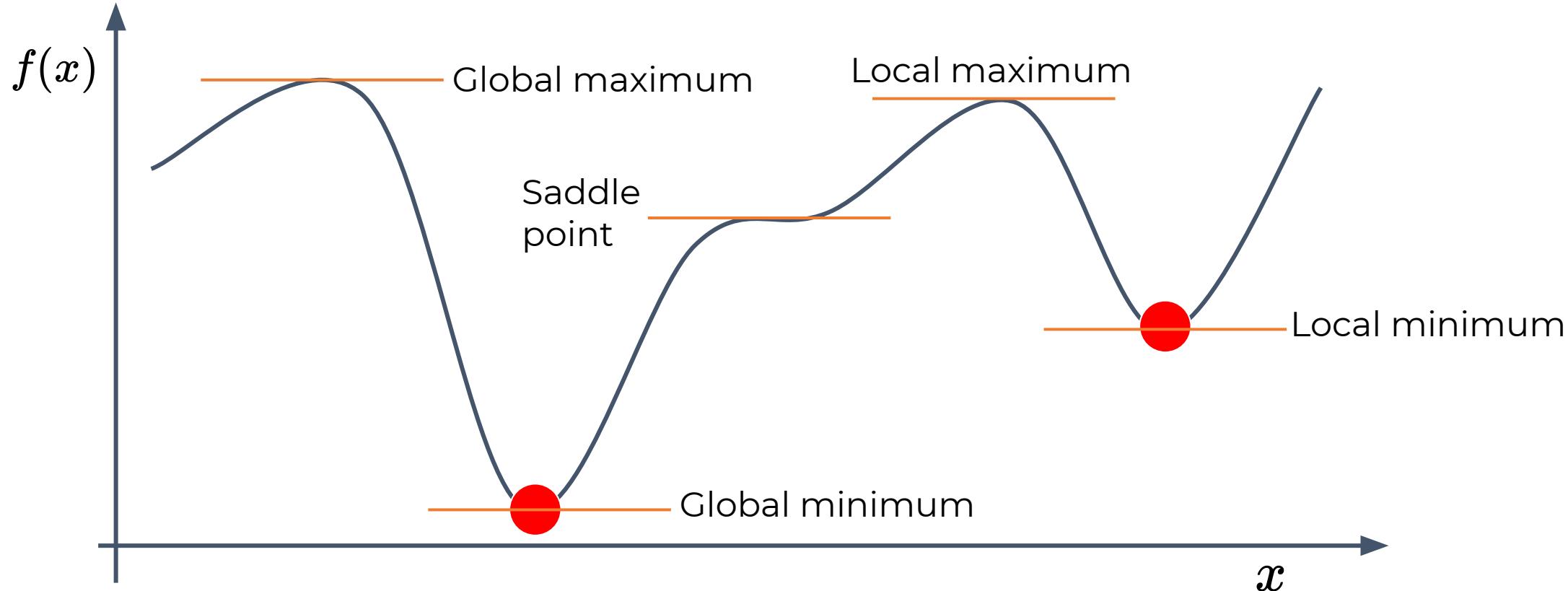
How to find extrema?

However, there are other points with a slope that equals 0.



Critical point classification

x^* is a critical point iif $f'(x^*) = 0$



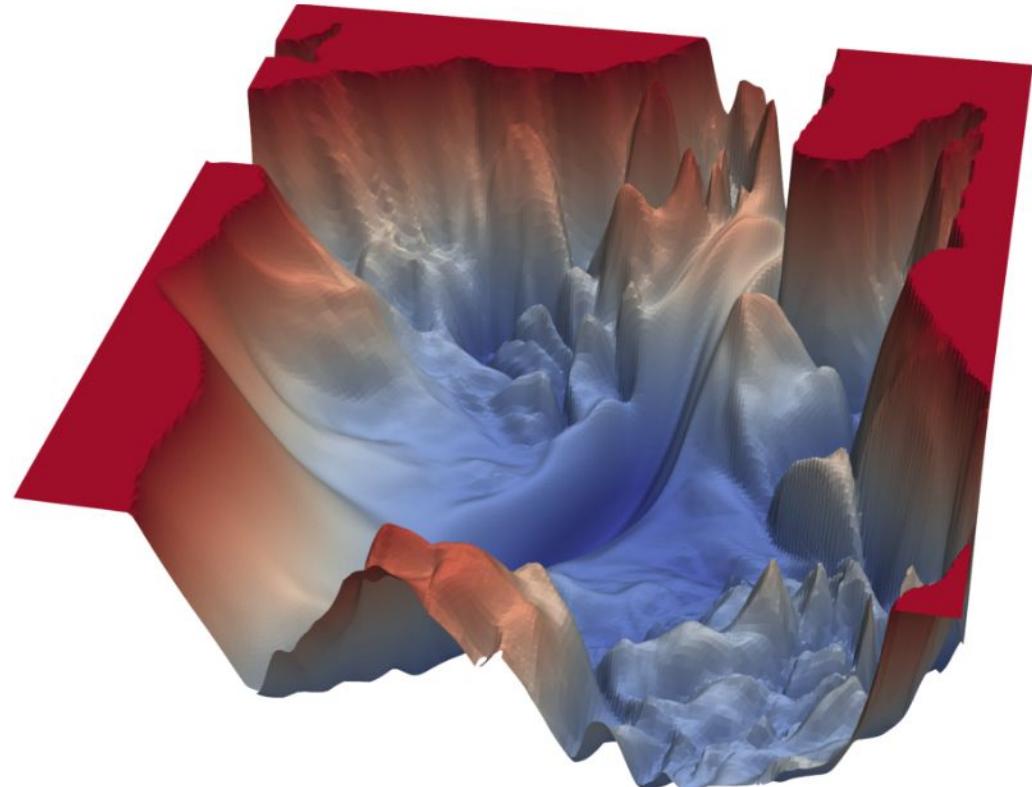
Fermat's theorem

If x^* is an extremum (local or global), then it is a critical point.

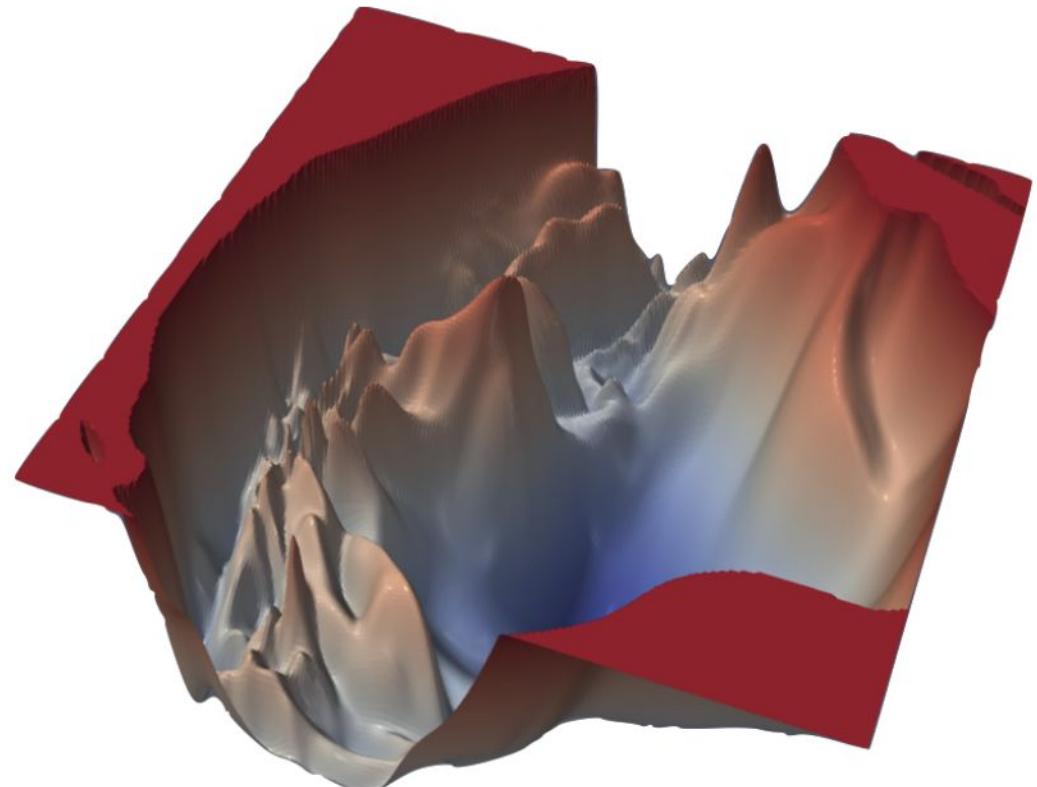
However, a critical point does not imply an extremum (e.g., a saddle point is not an extremum).

Parameter landscape for DL models

VGG-56

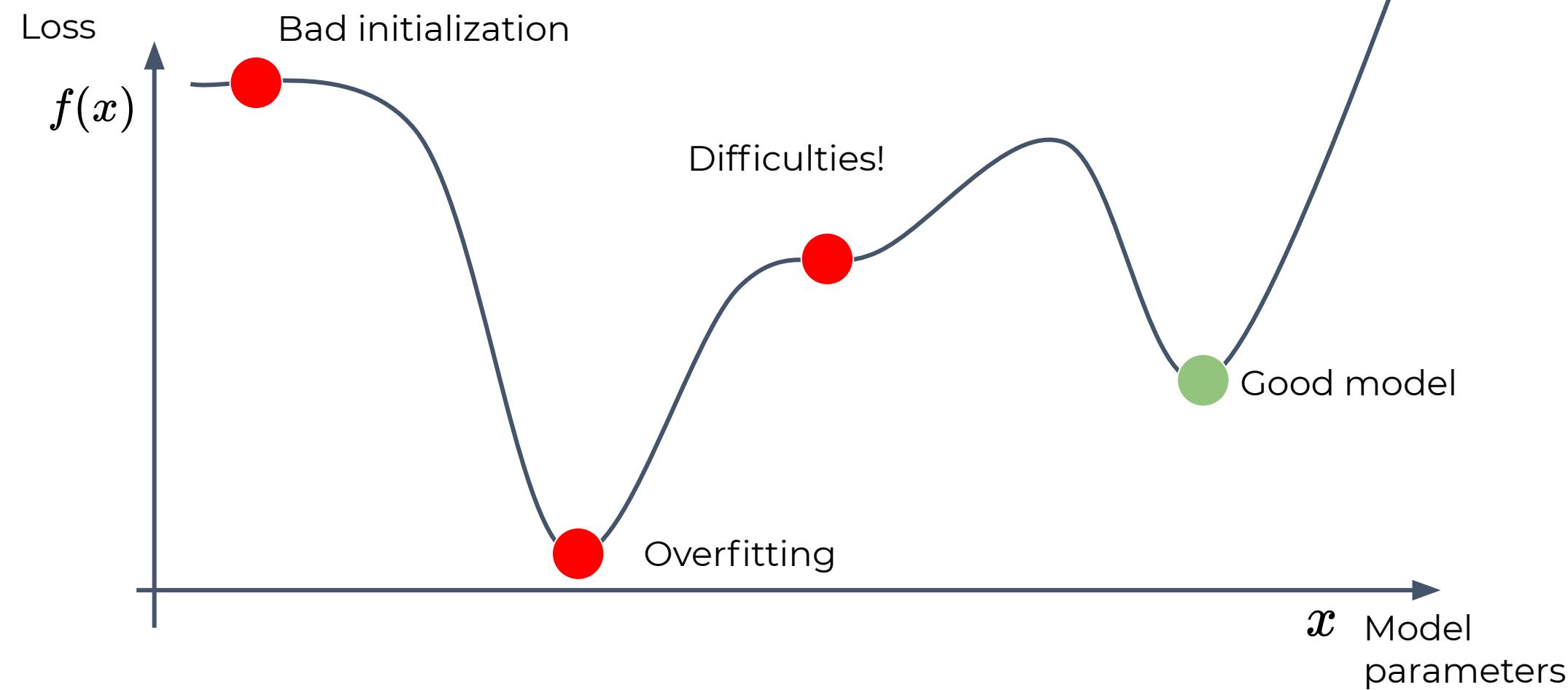


VGG-110



Li, Hao, et al. "Visualizing the loss landscape of neural nets." arXiv:1712.09913 (2017).

Learning is not only optimizing!



Critical point classification

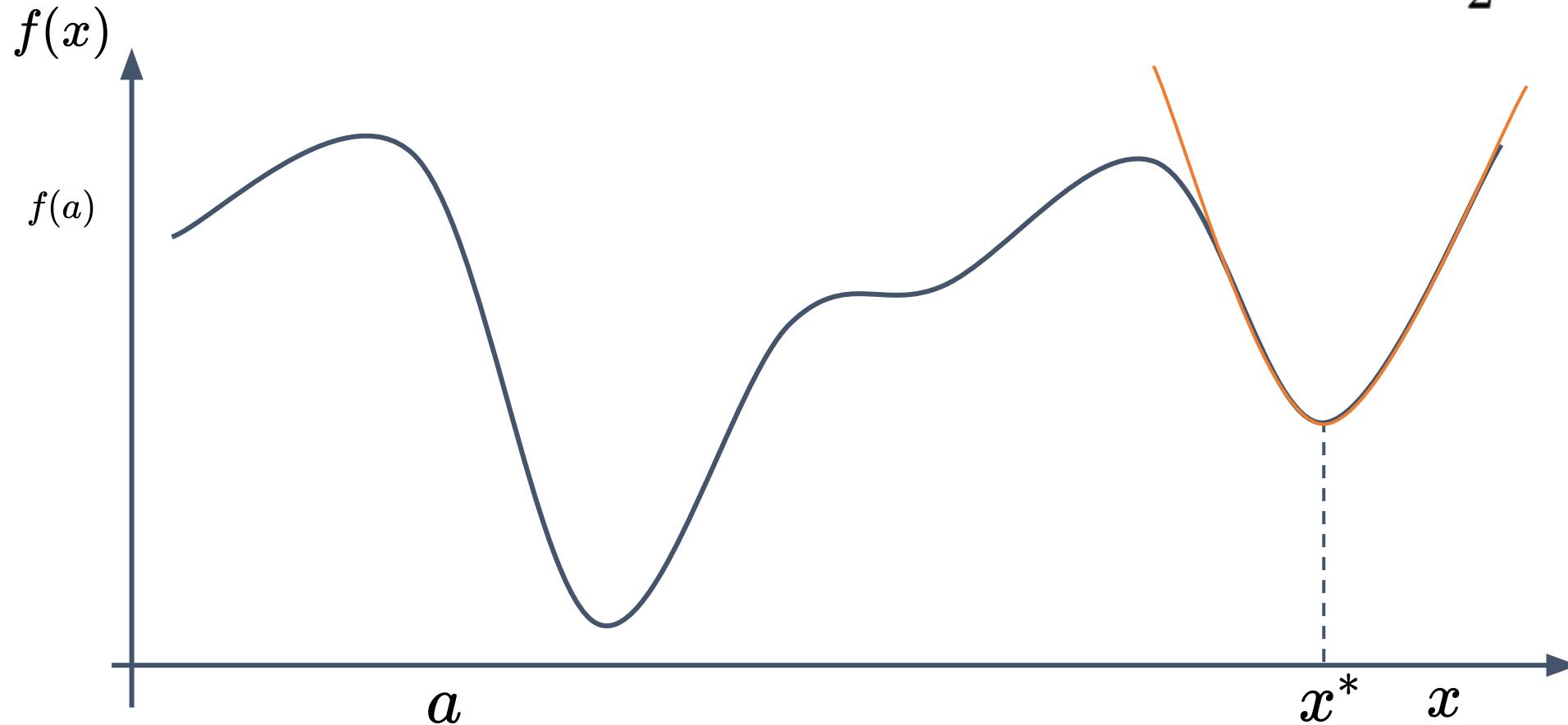
$$f(a + h) \approx f(a) + \langle f'(a), h \rangle + \frac{1}{2} \langle f''(a)h, h \rangle$$

Suppose that x^* is a minimum, then

$$f(x^* + h) \approx f(x^*) + \underbrace{\langle f'(x^*), h \rangle}_0 + \underbrace{\frac{1}{2} \langle f''(x^*)h, h \rangle}_{q(h)}$$

Example

$$f(x^* + h) \approx f(x^*) + \frac{1}{2} \langle f''(x^*)h, h \rangle$$



Quadratic function in 2D

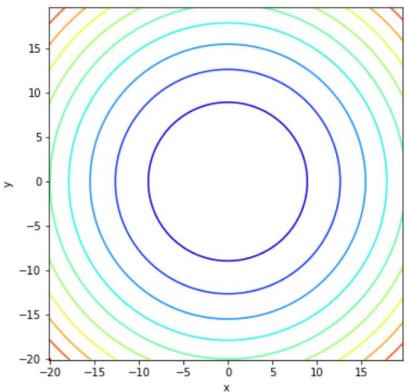
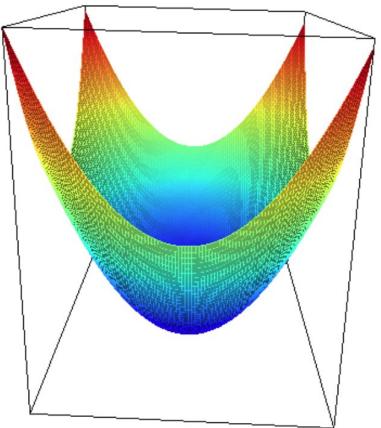
$$q(h) = \frac{1}{2} \langle f''(a)h, h \rangle$$

$$= \frac{1}{2} h^\top H h = \frac{1}{2} \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} \partial_{11} f(a) & \partial_{12} f(a) \\ \partial_{21} f(a) & \partial_{22} f(a) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

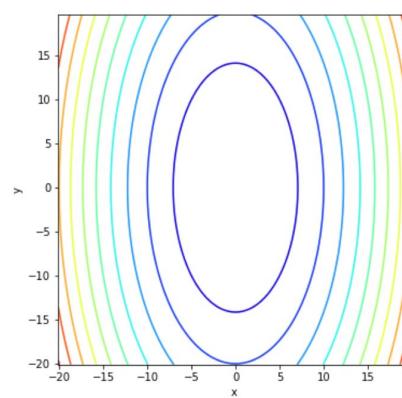
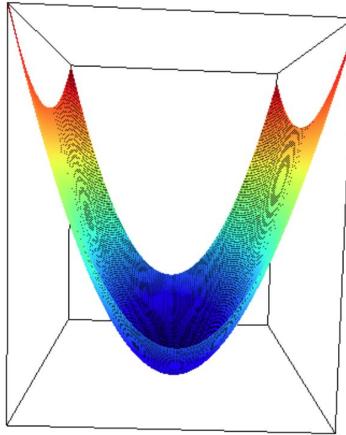
$$q'(h) = f''(a)h$$

$$q''(h) = f''(a)$$

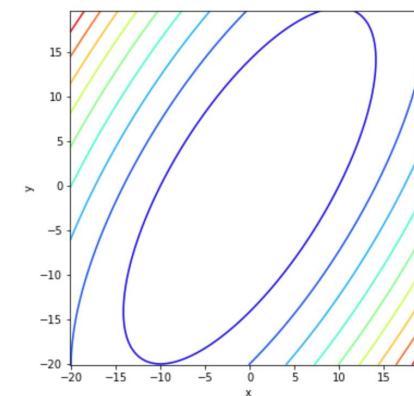
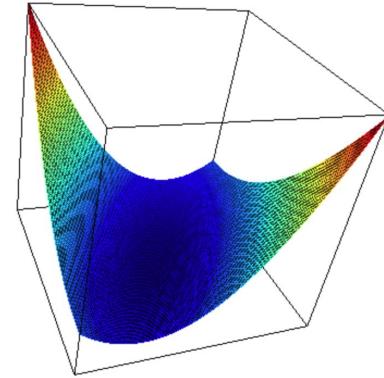
Positive definite in 2D



$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

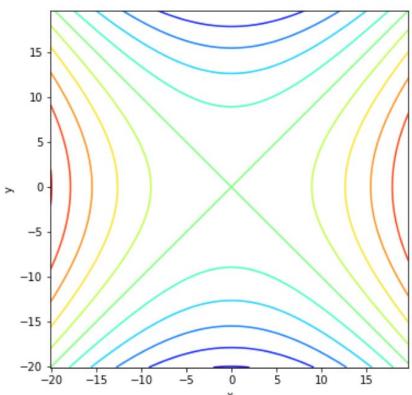
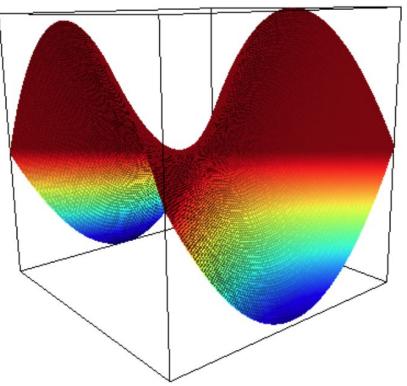


$$H = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

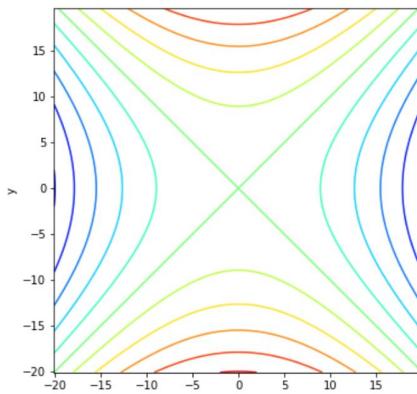
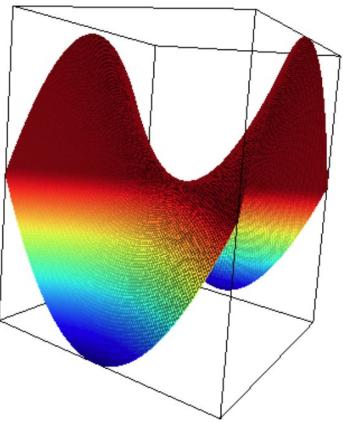


$$H = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

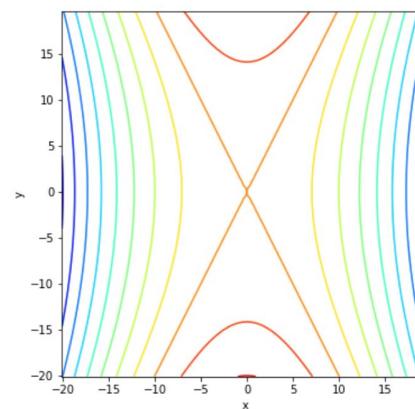
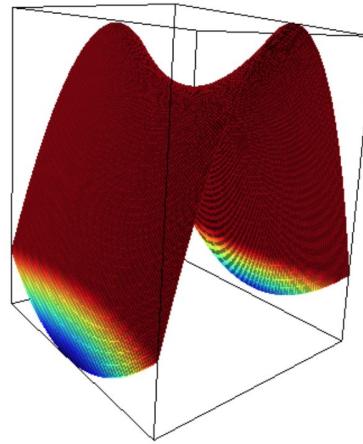
Saddle point in 2D



$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



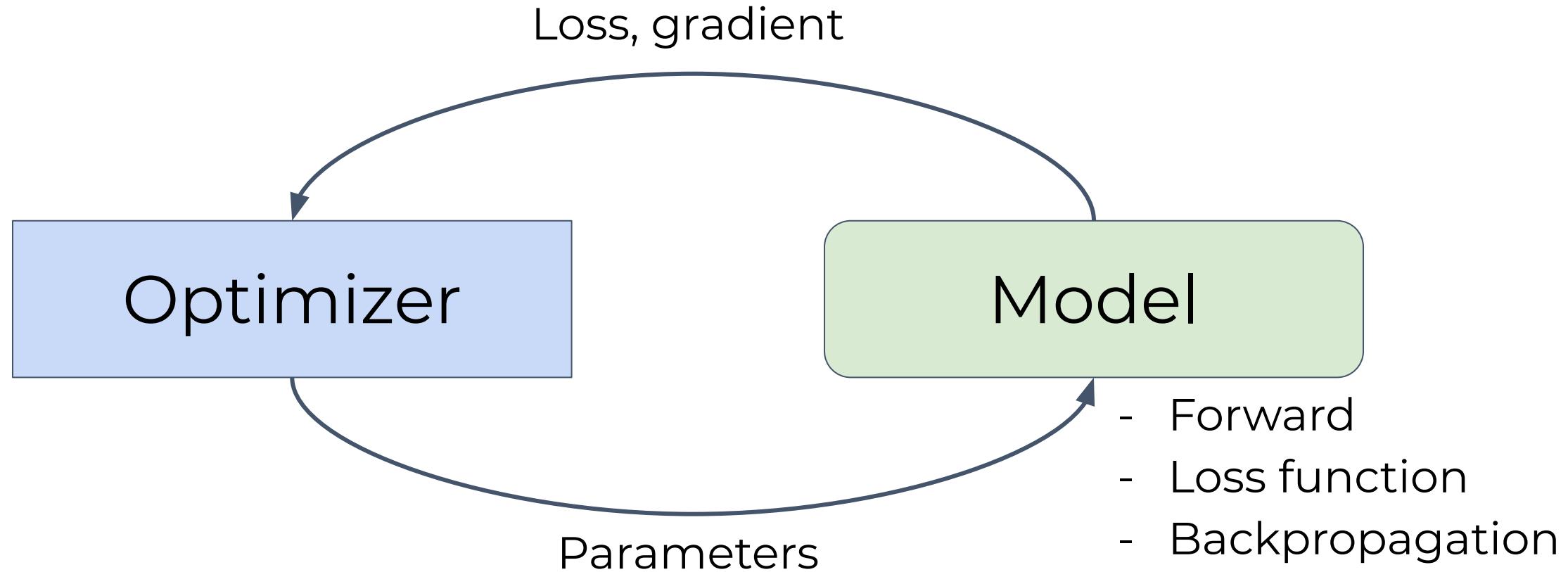
$$H = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$H = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$$

Optimization algorithm

Iterative procedure



Why gradient is important for optimization?

$$f(a + h) - f(a) \approx \langle f'(a), h \rangle$$

h is a vector that gives the direction to go.

$$\begin{aligned} \min_{h, \|h\|=1} \langle f'(a), h \rangle &= \|f'(a)\| \|h\| \cos(\angle(f'(a), h)) \\ &= C \cos(\angle(f'(a), h)) \end{aligned}$$

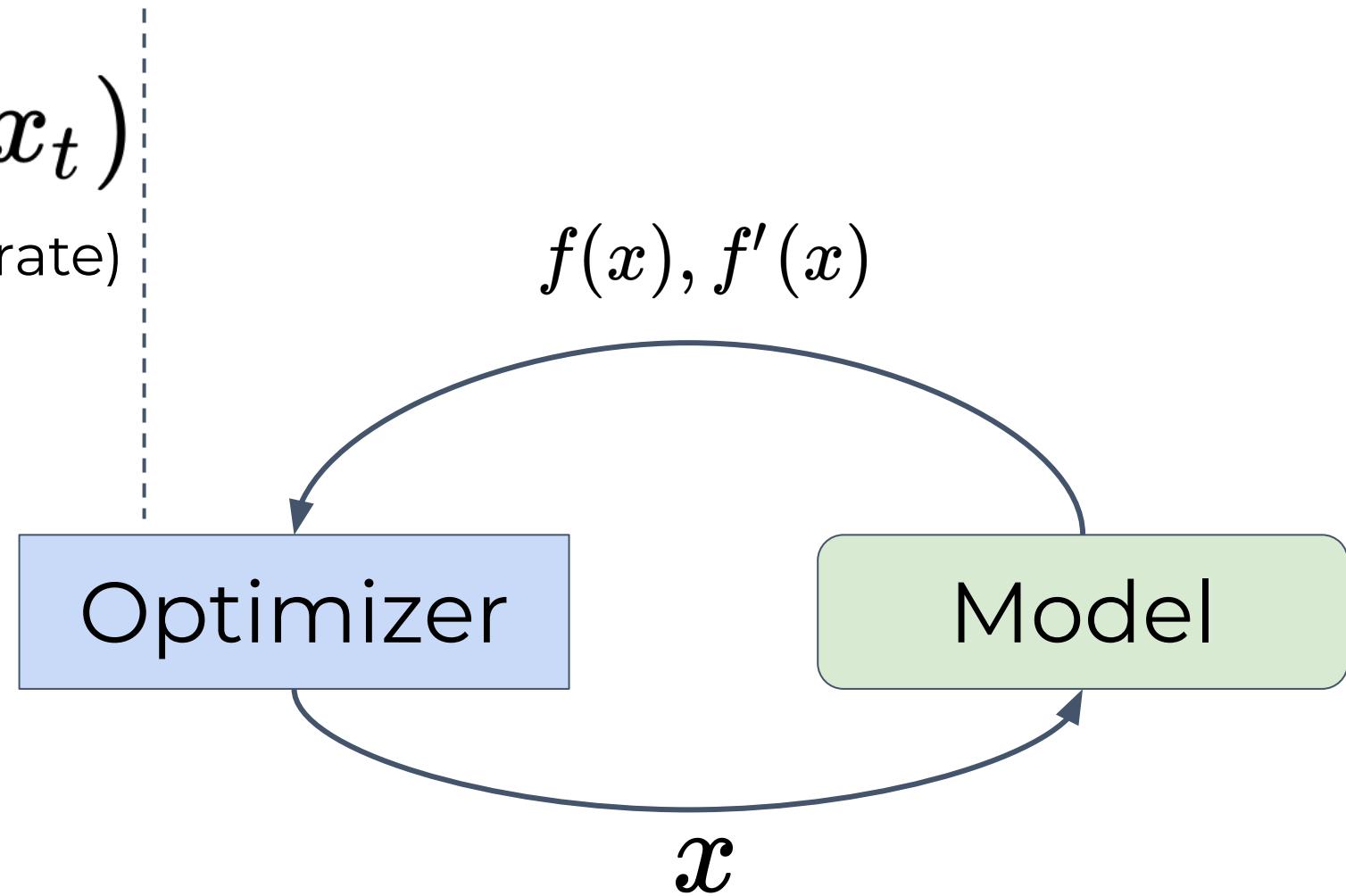
The gradient indicates
the direction of the **steepest slope** at point a

Steepest descent method (Gradient descent)

$$x_{t+1} \leftarrow x_t - \eta f'(x_t)$$

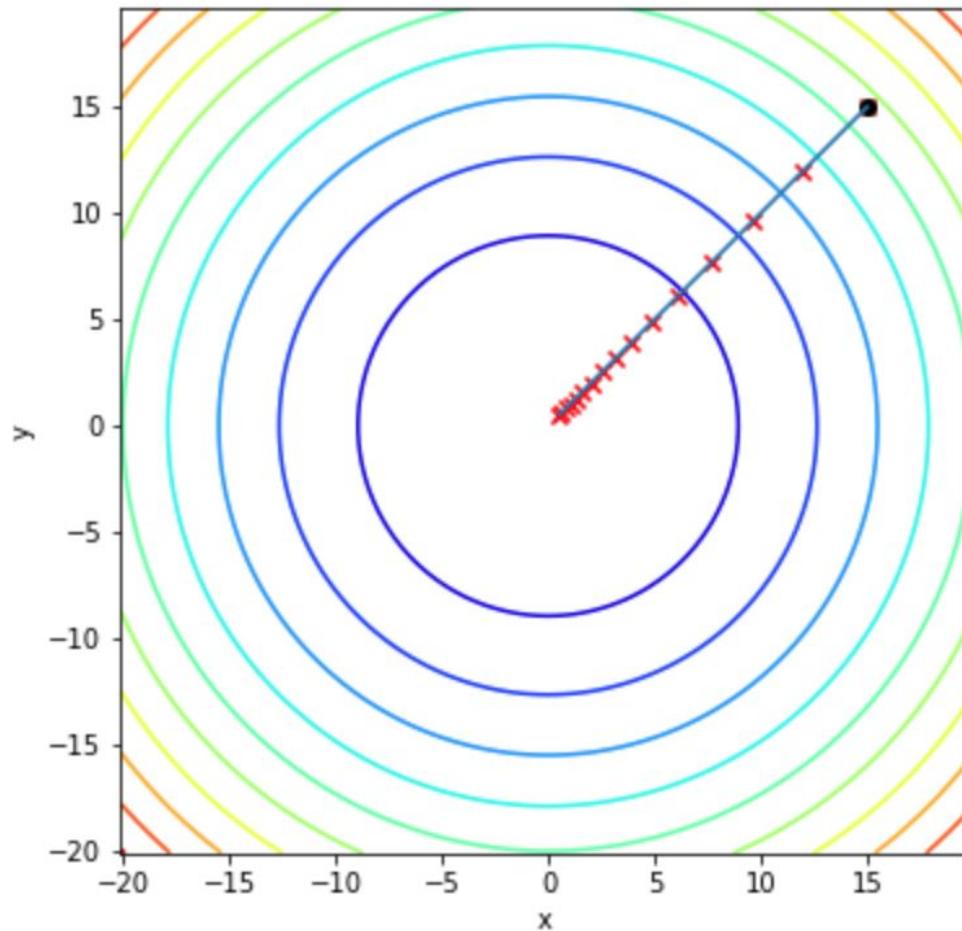
$\eta > 0$ (step-size, learning rate)

$$f(x), f'(x)$$

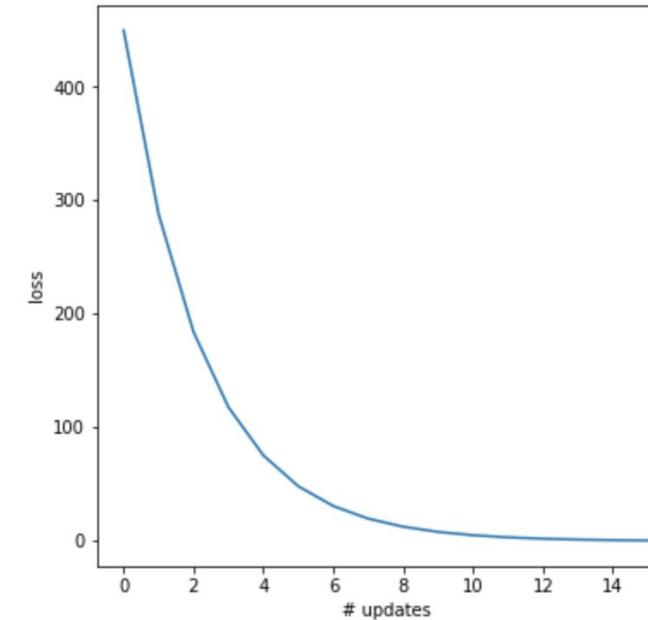


Gradient descent

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

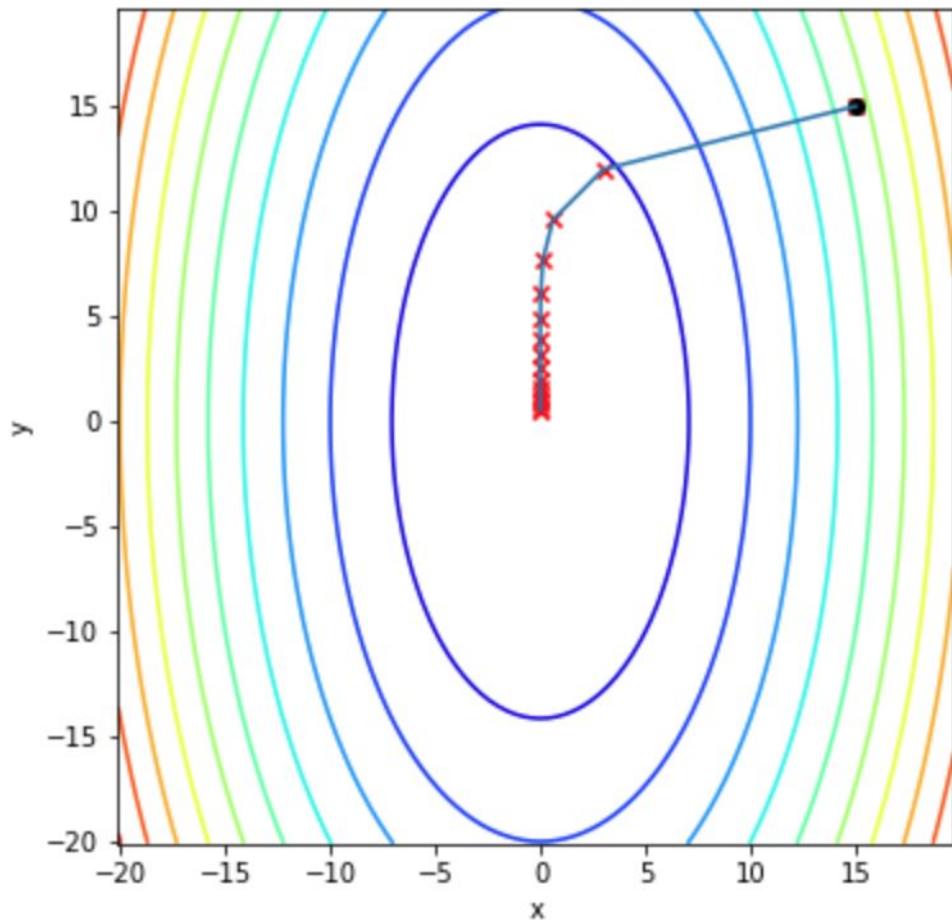


$$\eta = 0.1$$

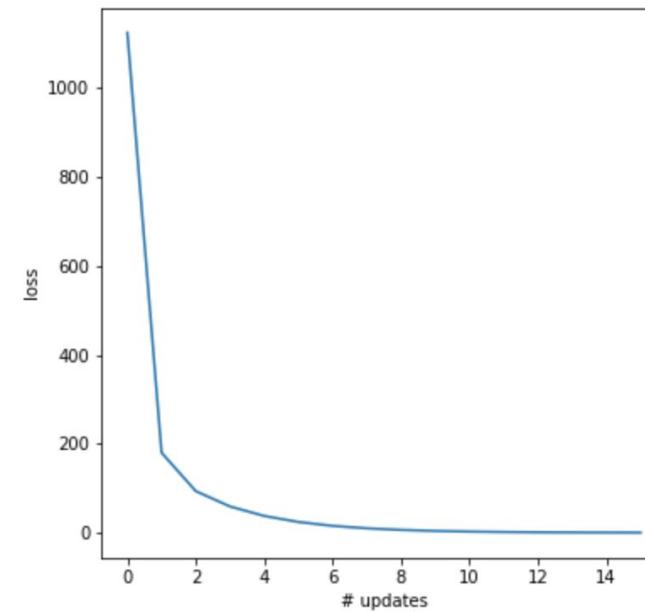


Gradient descent

$$H = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

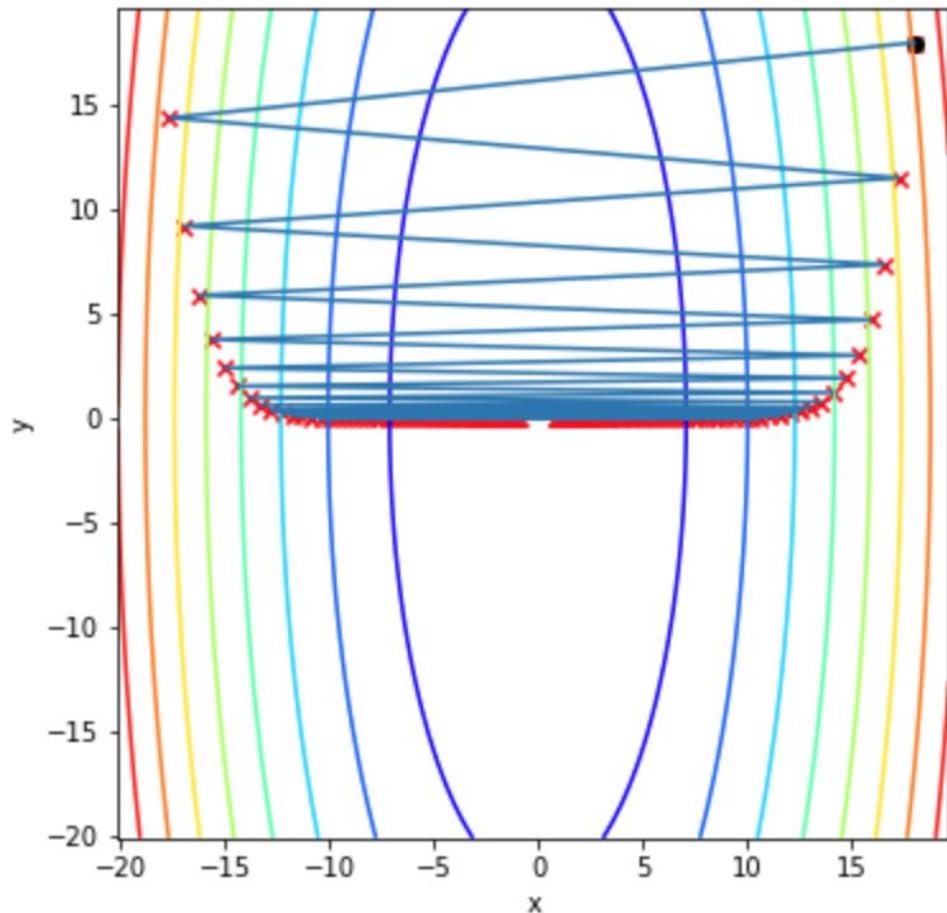


$$\eta = 0.1$$

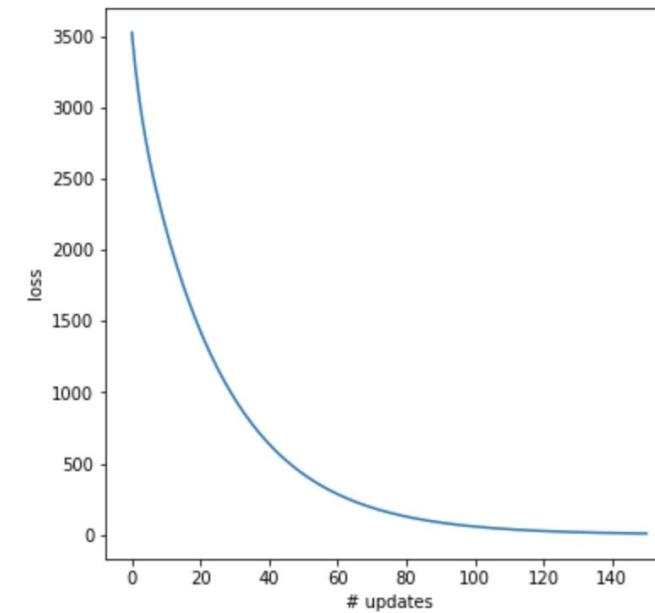


Gradient descent

$$H = \begin{bmatrix} 9.9 & 0 \\ 0 & 1 \end{bmatrix}$$



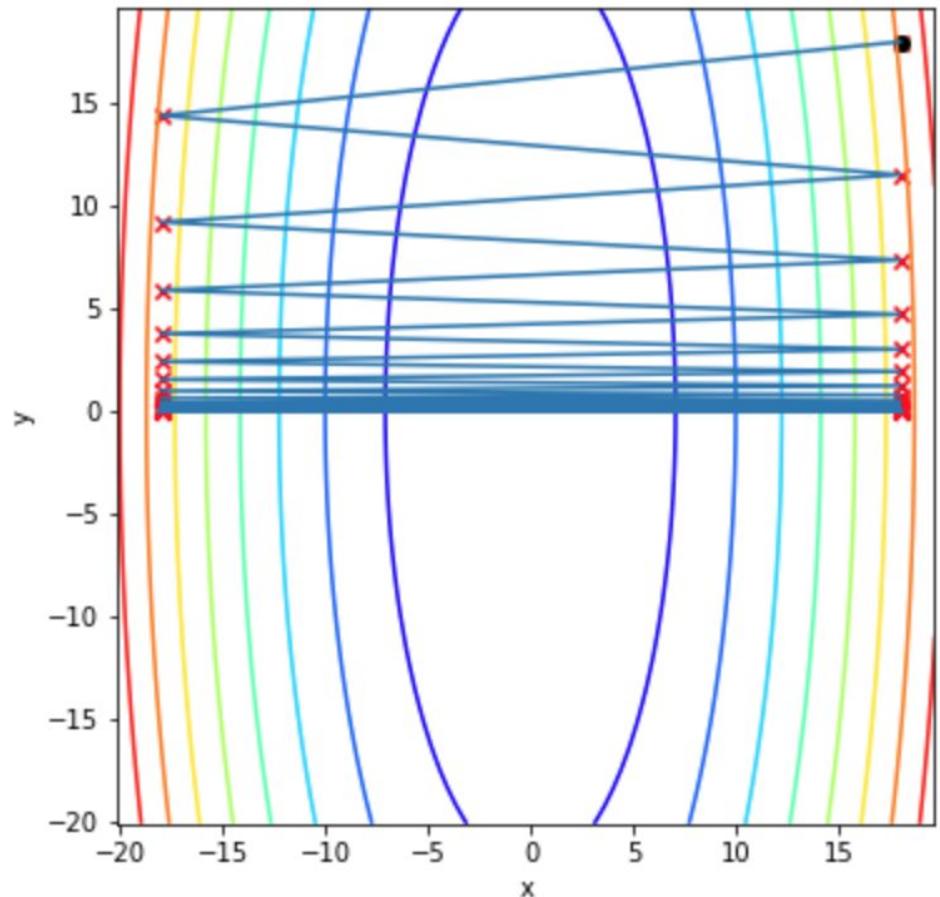
$$\eta = 0.1$$



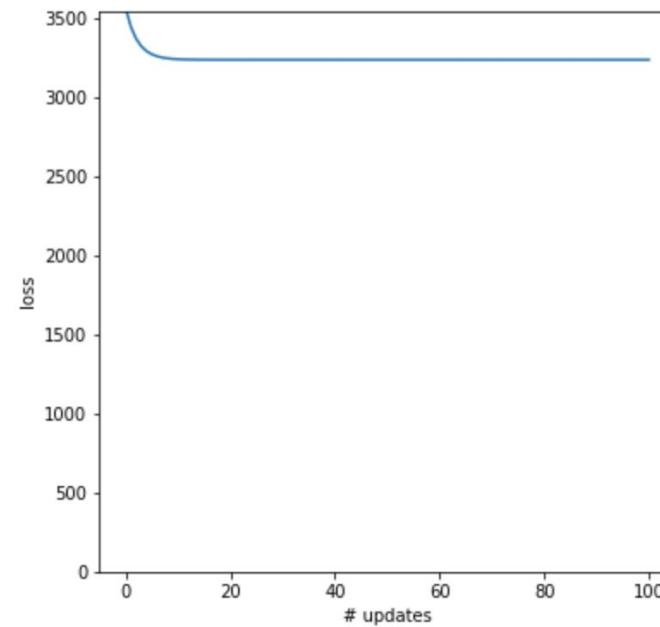
Notice the
rescaling of
the axis.

Gradient descent

$$H = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

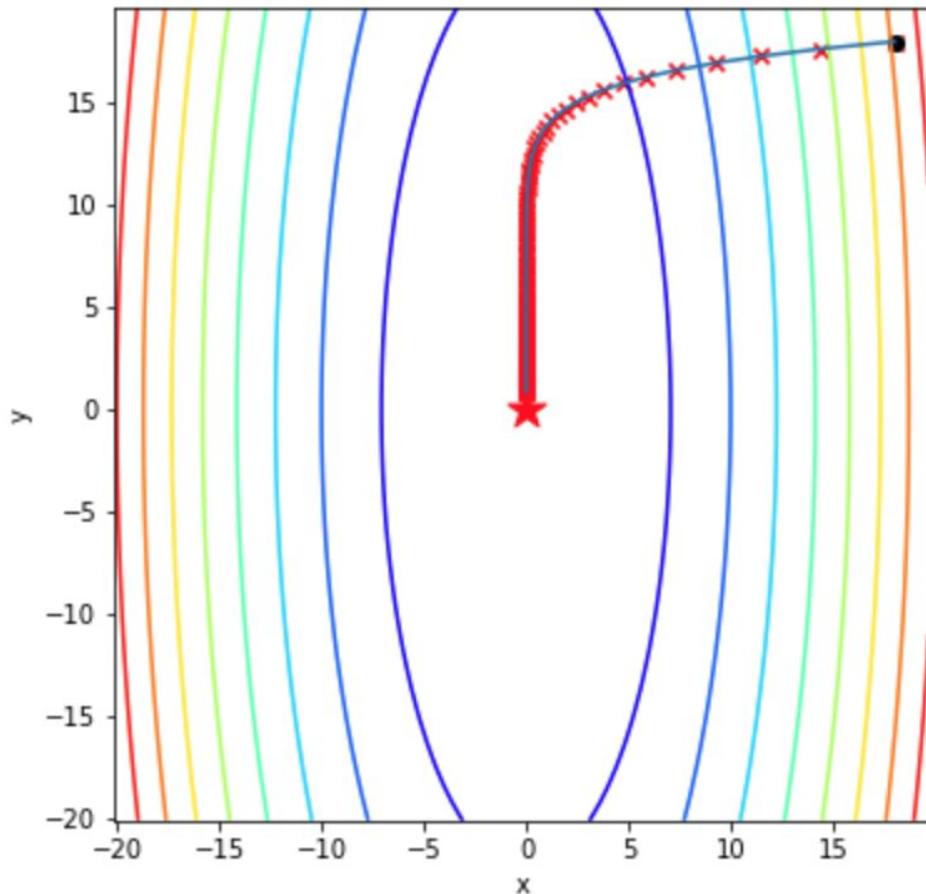


$$\eta = 0.1$$

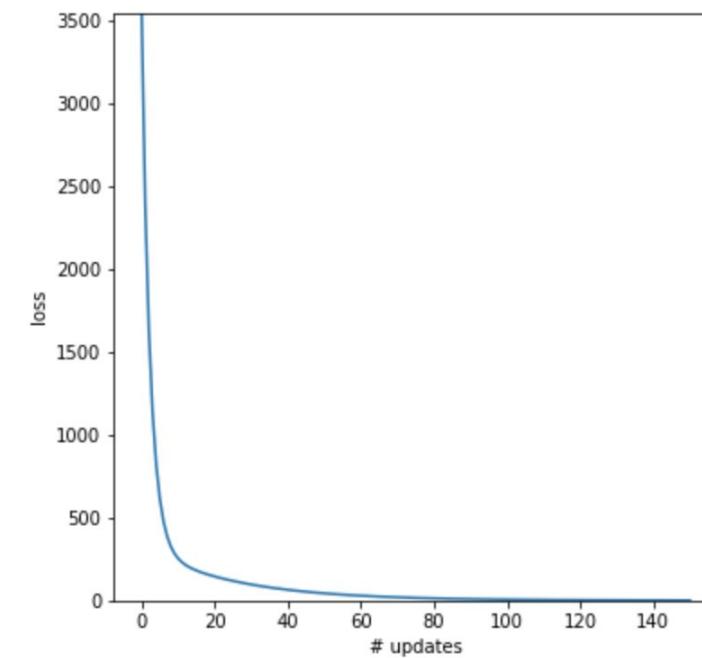


Gradient descent

$$H = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\eta = 0.01$$



Problem of physical units

What are the physical units of this equation?

$$x_{t+1} \leftarrow x_t - \eta f'(x_t)$$

2D example:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \eta \begin{bmatrix} \partial_1 f(x) \\ \partial_2 f(x) \end{bmatrix}$$

$$x_1 : [\text{km}] \quad x_2 : [\text{mm}] \quad \eta : [?]$$

$$\partial_1 f(x) : [\text{J}/\text{km}] \quad \partial_2 f(x) : [\text{J}/\text{mm}]$$

Problem of physical units

What are the physical units of this equation?

$$x_{t+1} \leftarrow x_t - \eta M_t f'(x_t)$$

2D example:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \eta \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix} \begin{bmatrix} \partial_1 f(x) \\ \partial_2 f(x) \end{bmatrix}$$

$x_1 : [\text{km}]$

$x_2 : [\text{mm}]$

$M_{11} : [\text{km}^2/\text{J}]$

$\partial_1 f(x) : [\text{J}/\text{km}] \quad \partial_2 f(x) : [\text{J}/\text{mm}]$

$M_{22} : [\text{mm}^2/\text{J}]$

Preconditioning

General update equation for gradient descent.

$$x_{t+1} \leftarrow x_t - \eta M_t f'(x_t)$$

M : preconditioning matrix

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \eta \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \partial_1 f(x) \\ \partial_2 f(x) \end{bmatrix}$$

Preconditioning

General update equation for gradient descent.

$$x_{t+1} \leftarrow x_t - n M_t f'(x_t)$$

M : preconditioner

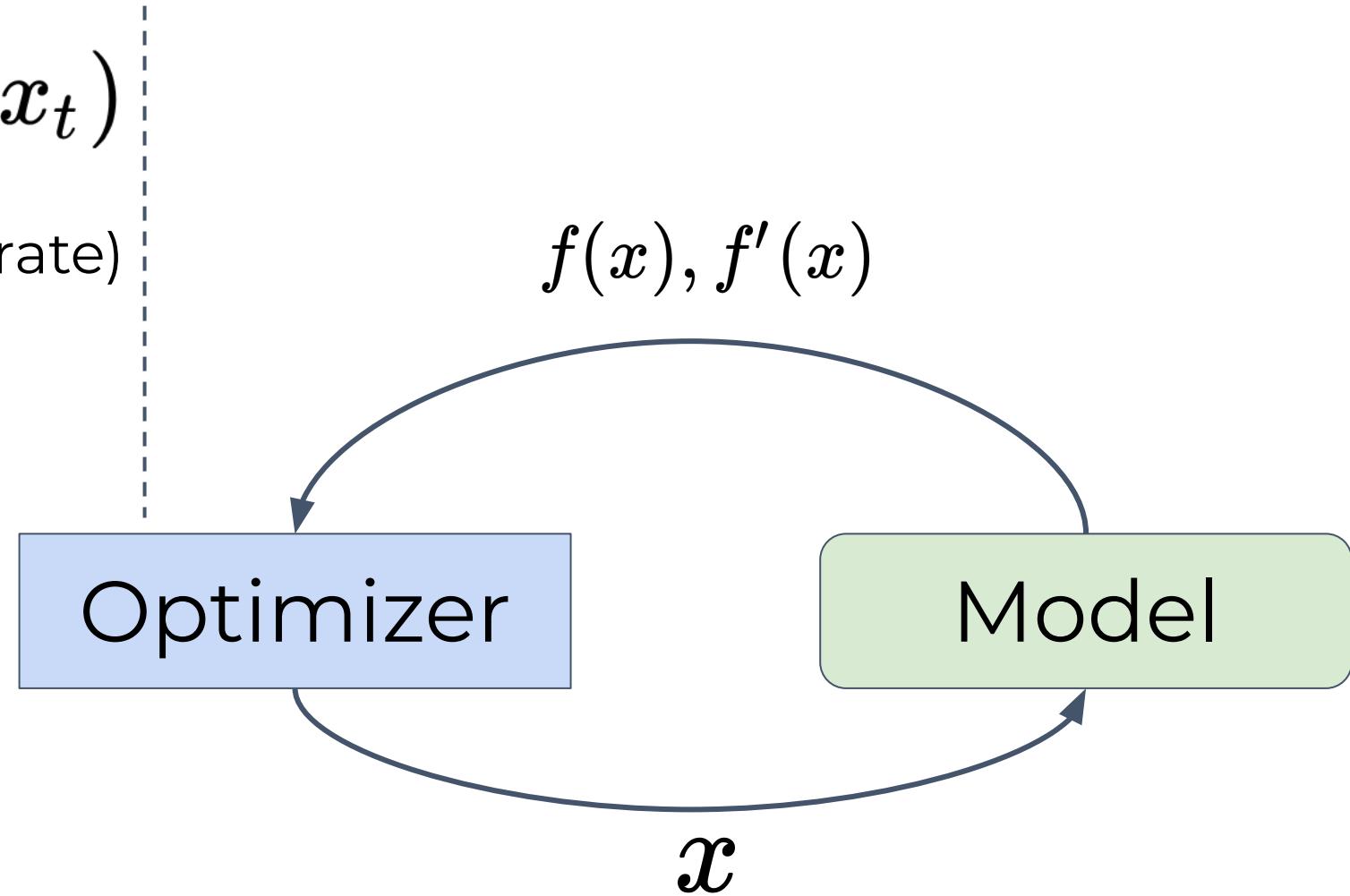
The size of M_t is (number of parameters)²!

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \eta \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \partial_1 f(x) \\ \partial_2 f(x) \end{bmatrix}$$

Preconditioned gradient descent

$$x_{t+1} \leftarrow x_t - \eta M_t f'(x_t)$$

$\eta > 0$ (step-size, learning rate)



RMSProp

Component-wise preconditioning

$$v_t = \beta v_{t-1} + (1 - \beta) f'(x_t)^2 \quad \text{Preconditioning term}$$

$$x_{t+1} = x_t + \frac{1}{\sqrt{v_t} + \epsilon} f'(x_t) \quad \text{Update rule}$$

Momentum

- Potential energy $U(x) = mgh(x)$

$$F = ma = -U'(x) = -mgh'(x)$$

Initial conditions: x_0, v_0

Physics

$$a_t \leftarrow -gh'(x_t)$$

$$v_{t+1} \leftarrow \mu v_t + a_t \Delta t$$

$$x_{t+1} \leftarrow x_t + v_{t+1} \Delta t$$

Optimization

$$a_t \leftarrow -\eta f'(x_t)$$

$$v_{t+1} \leftarrow \mu v_t + a_t$$

μ : friction



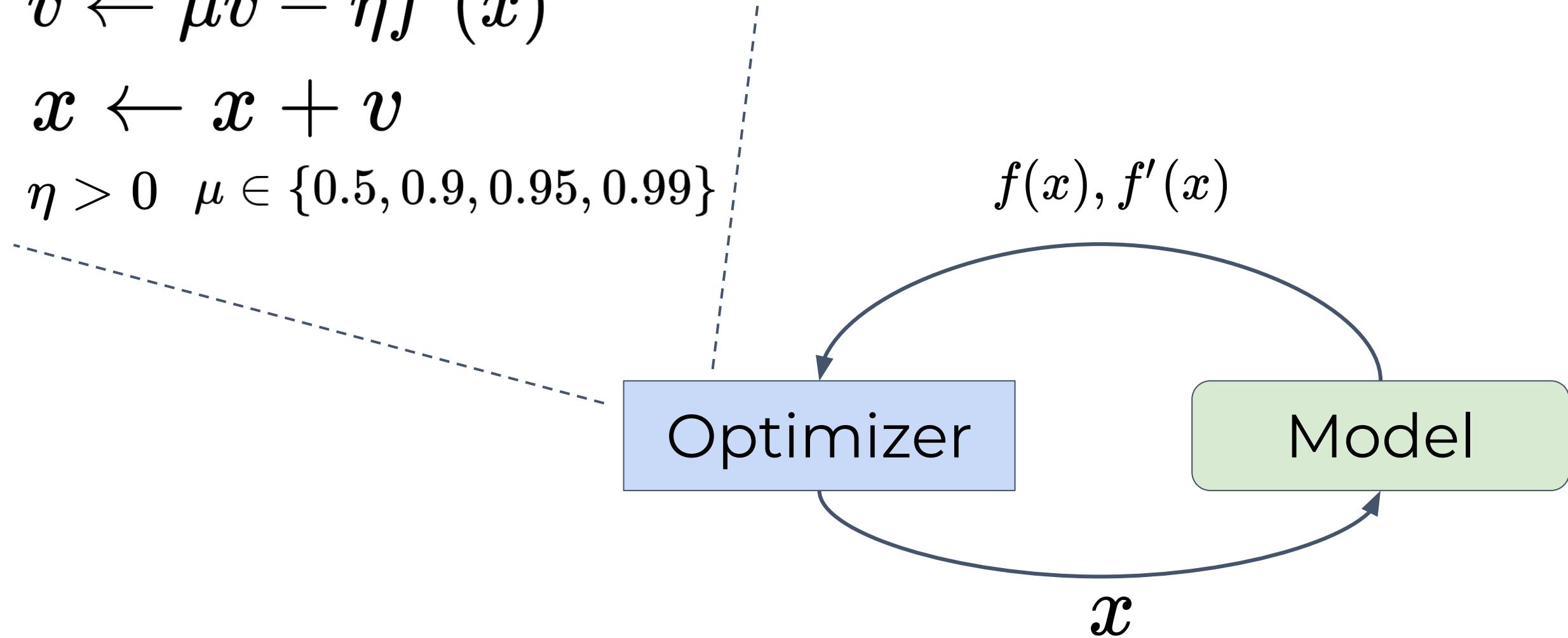
Source: Sam Poullain, Unsplash

Gradient descent with momentum

$$v \leftarrow \mu v - \eta f'(x)$$

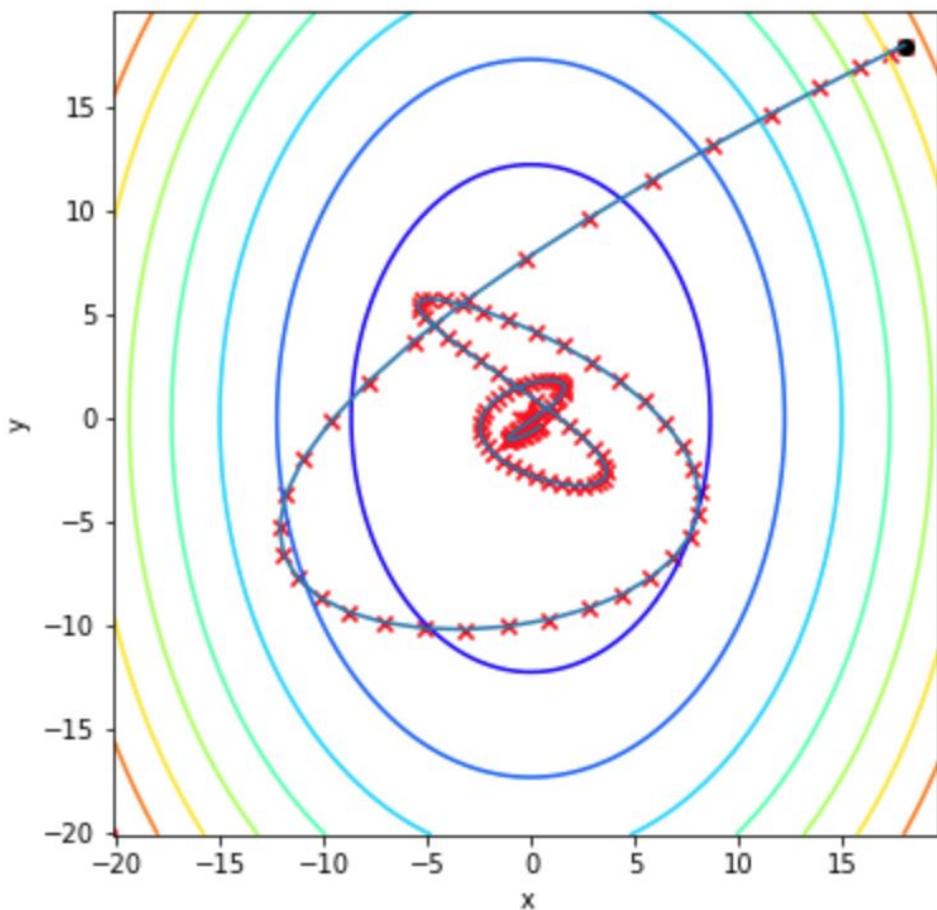
$$x \leftarrow x + v$$

$$\eta > 0 \quad \mu \in \{0.5, 0.9, 0.95, 0.99\}$$

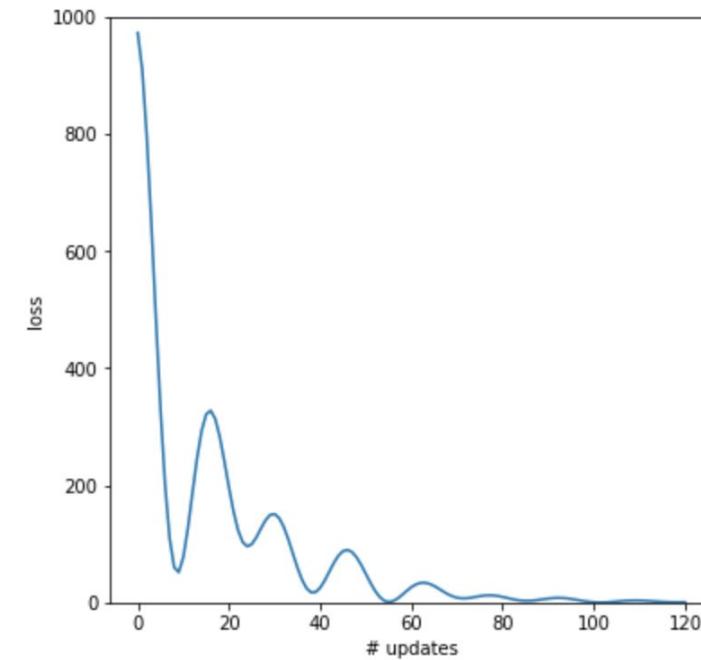


Momentum

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

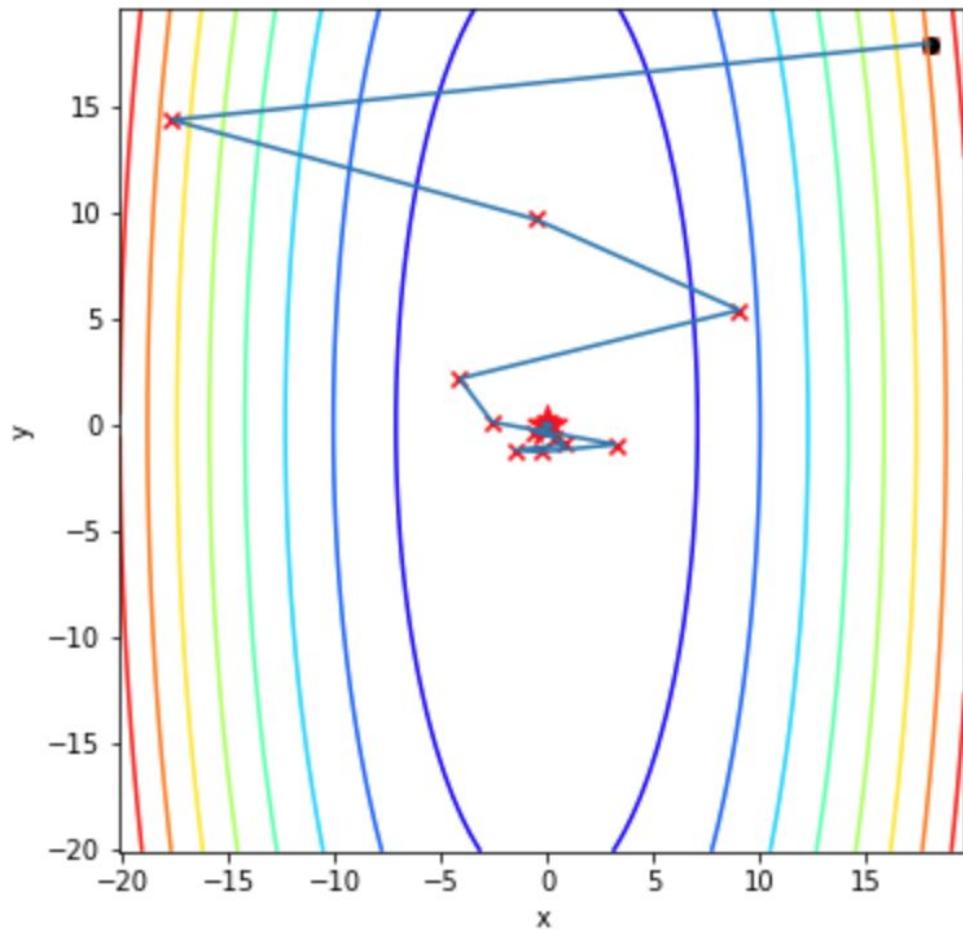


$$\mu = 0.95$$
$$\eta = 0.01$$

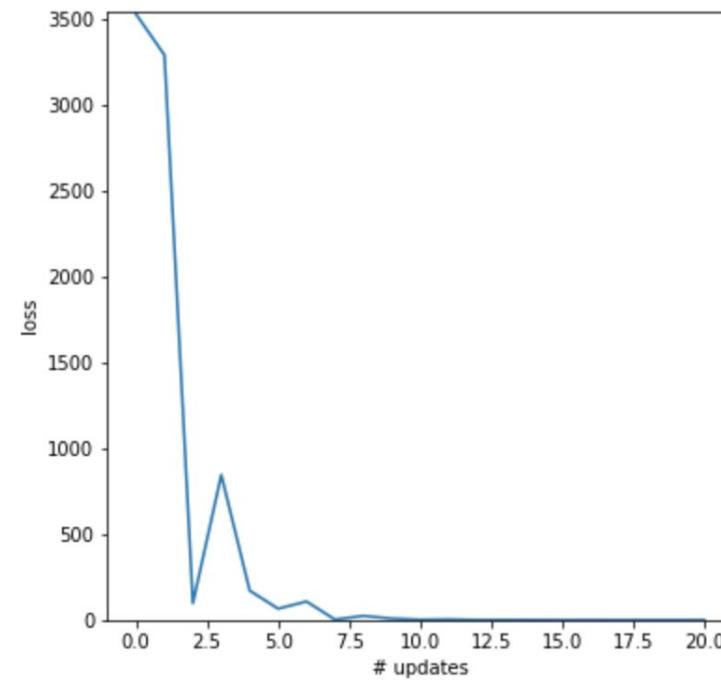


Momentum

$$H = \begin{bmatrix} 9.9 & 0 \\ 0 & 1 \end{bmatrix}$$

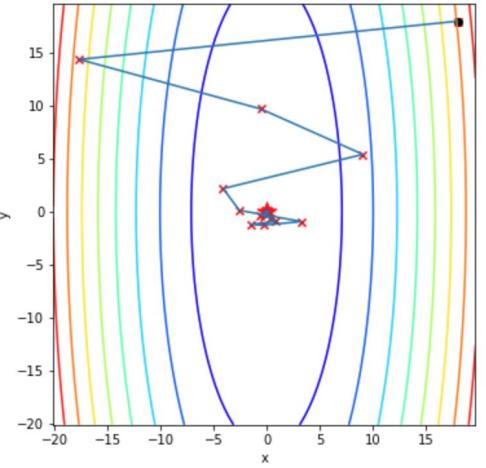


$$\mu = 0.5$$
$$\eta = 0.1$$

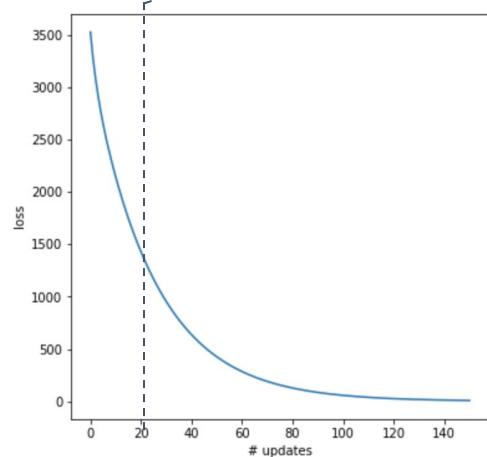
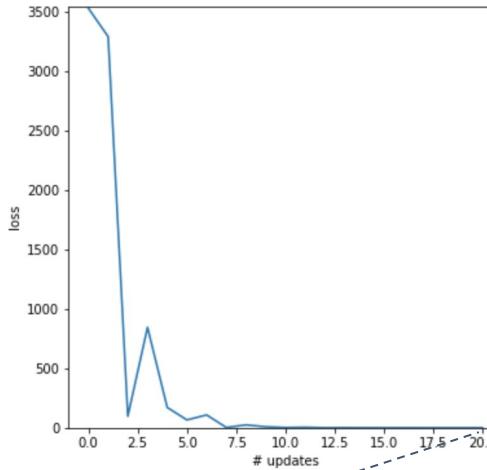
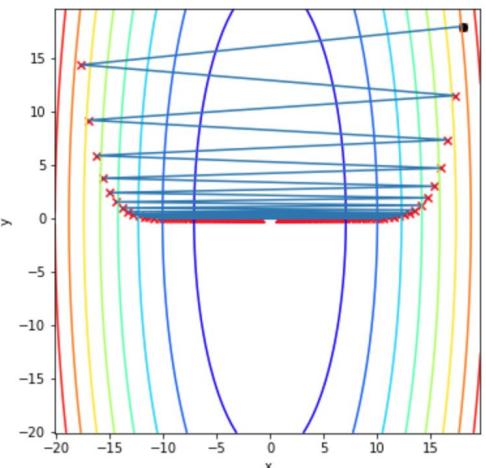


Different dynamics

With
Momentum



Without
Momentum



$$H = \begin{bmatrix} 9.9 & 0 \\ 0 & 1 \end{bmatrix}$$

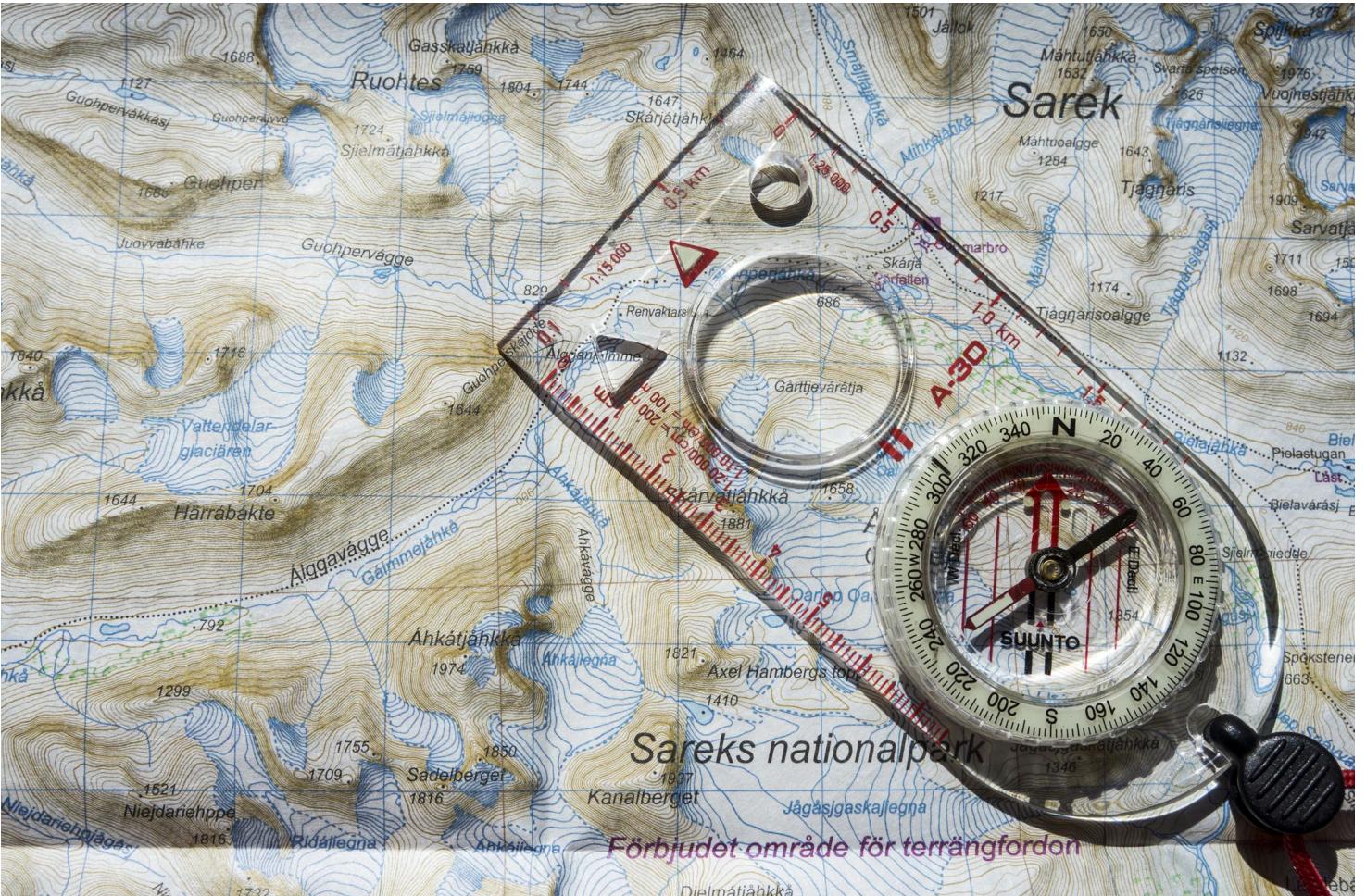
$$\eta = 0.1$$

$$\mu = 0.5$$

$$\eta = 0.1$$

A family of algorithms

- Adagrad
- Adadelta
- AdaMax
- RMSProp
- Adam**
- AMSGrad
- ...



Source: Hendrik Morkel, Unsplash

Adaptive moment estimation (Adam)

RMSProp with momentum

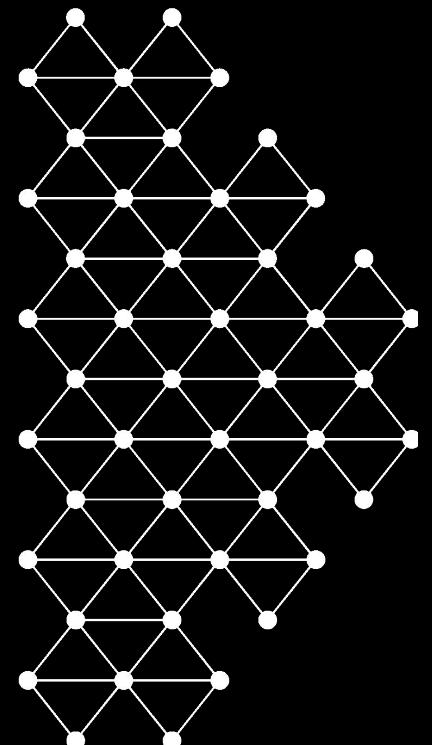
Component-wise preconditioning

$$m_t = \beta_1 m_{t-1} + (1 - \beta_1) f'(x_t) \quad \text{Momentum term}$$

$$v_t = \beta_2 v_{t-1} + (1 - \beta_2) f'(x_t)^2 \quad \text{Preconditioning term}$$

$$x_{t+1} = x_t + \frac{1}{\sqrt{v_t} + \epsilon} m_t \quad \text{Update rule}$$

This is a simplified version
without the bias correction
terms!



Stochastic optimization

Stochastic gradient descent

Risk

$$L(\theta) = \mathbb{E}_{z \sim \mathcal{D}} [l(\theta; z)]$$

Stochastic gradient descent

$$\begin{array}{ccc}
 \text{Risk} & & \text{Empirical risk} \\
 L(\theta) = \mathbb{E}_{z \sim \mathcal{D}} [l(\theta; z)] & \xrightarrow{\text{-----}} & \hat{L}(\theta) = \frac{1}{N} \sum_{i=1}^N l(\theta; z_i) \\
 & & \text{Approximation}
 \end{array}$$

Stochastic gradient descent

Risk

$$L(\theta) = \mathbb{E}_{z \sim \mathcal{D}} [l(\theta; z)]$$

Empirical risk

$$\hat{L}(\theta) = \frac{1}{N} \sum_{i=1}^N l(\theta; z_i)$$

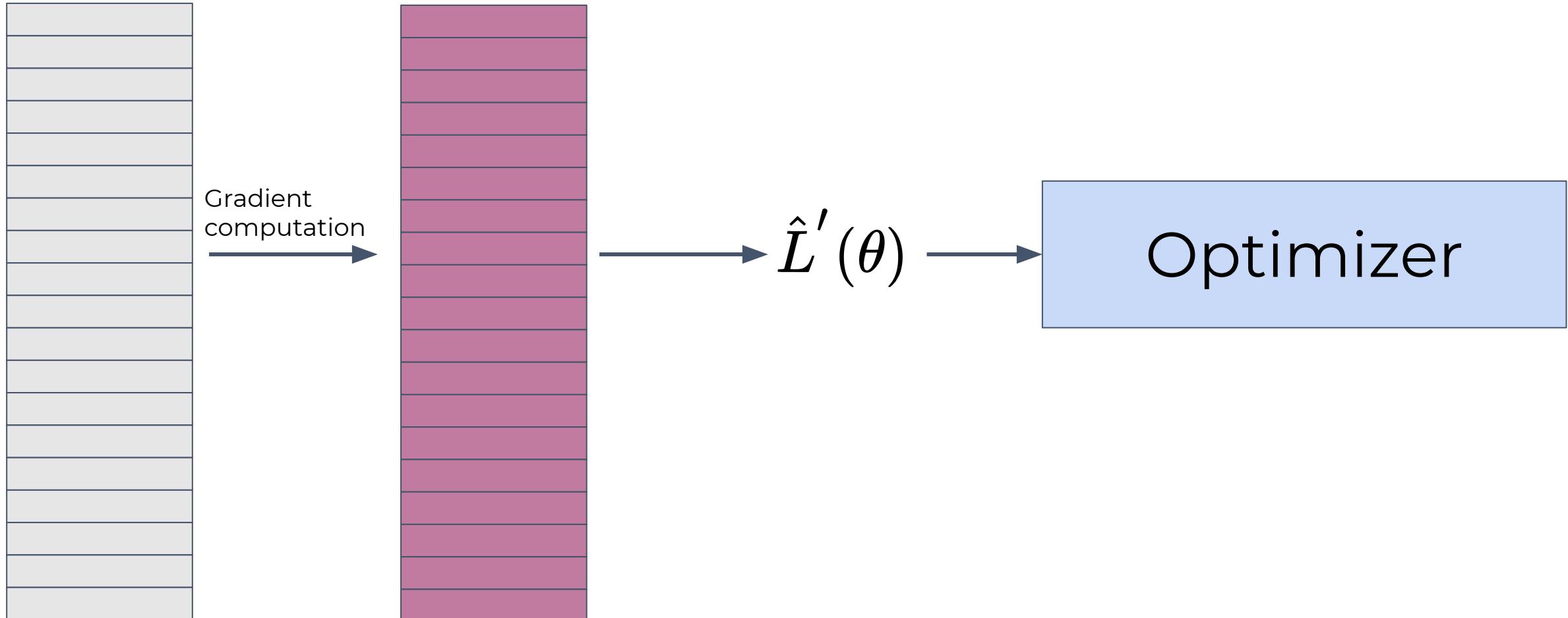
Approximation

Approximation

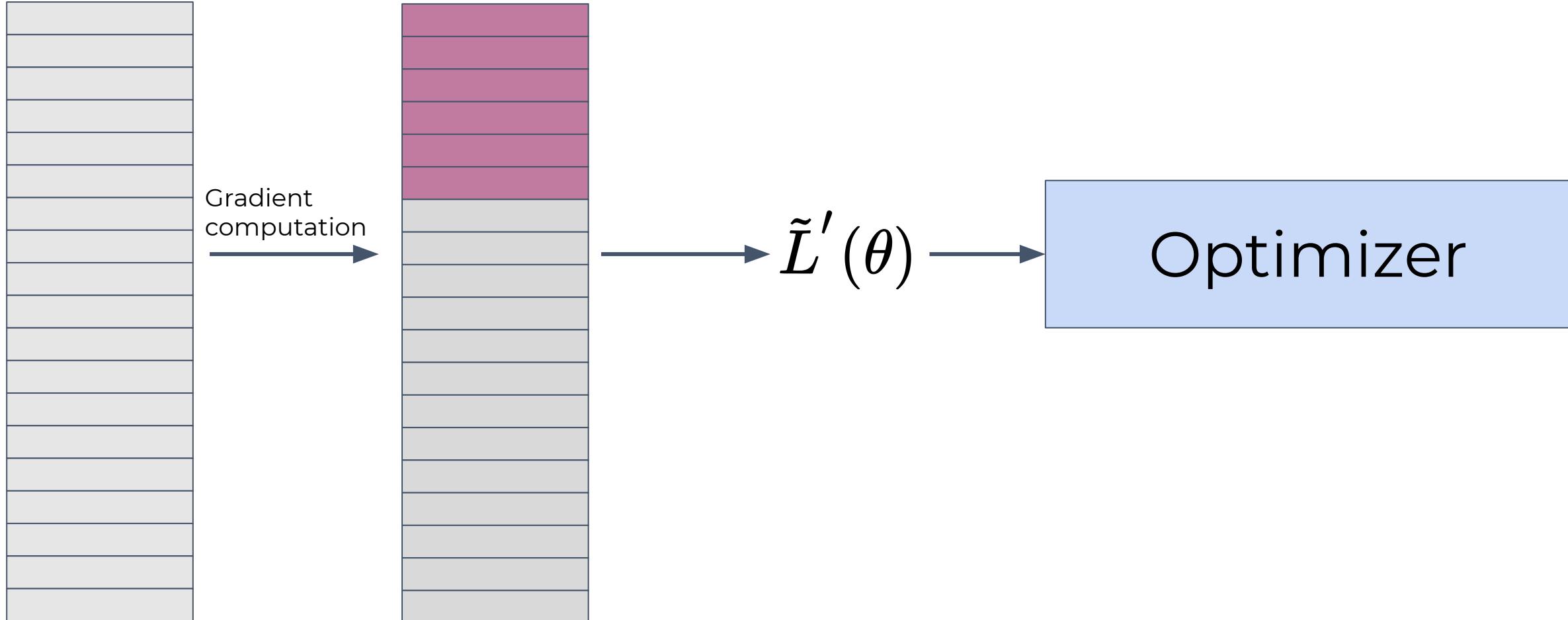
$$\tilde{L}(\theta) = \frac{1}{K} \sum_{i=1}^K l(\theta; z_{\sigma(i)})$$

K<<N : Number of examples (Mini-batch)

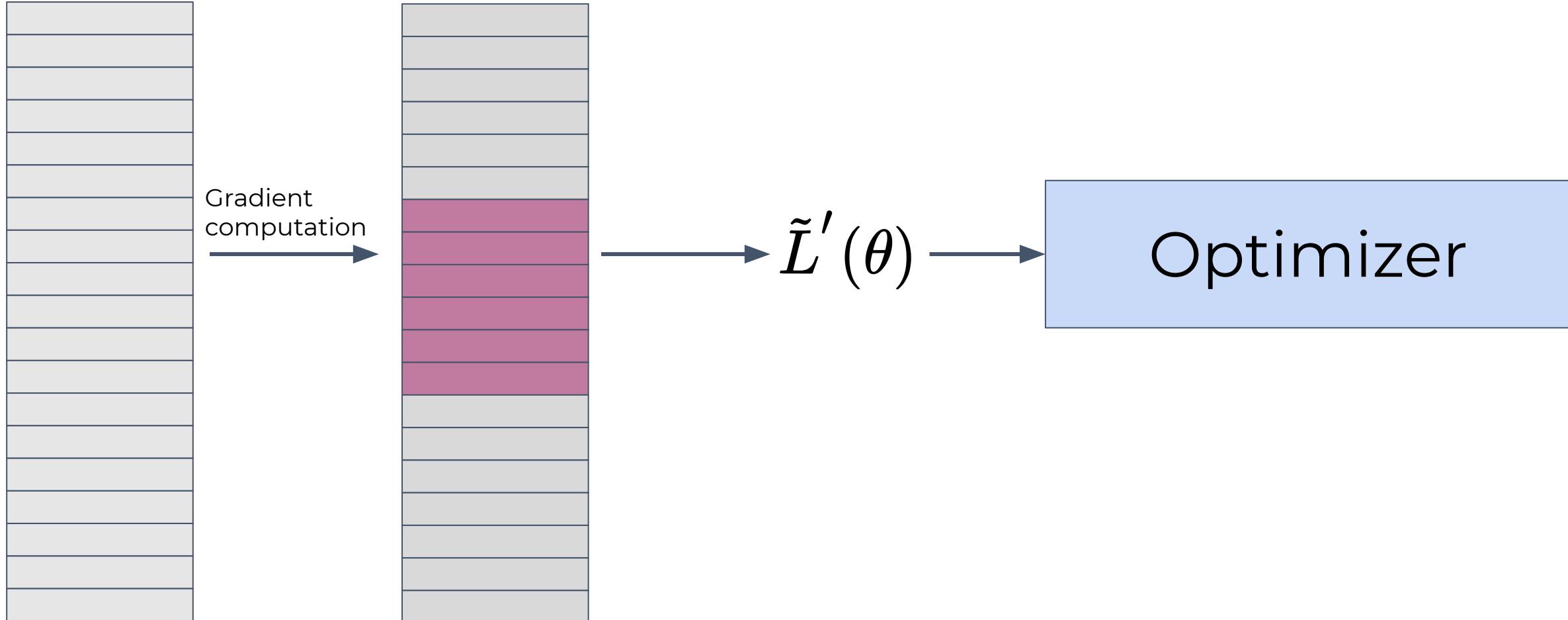
Gradient descent



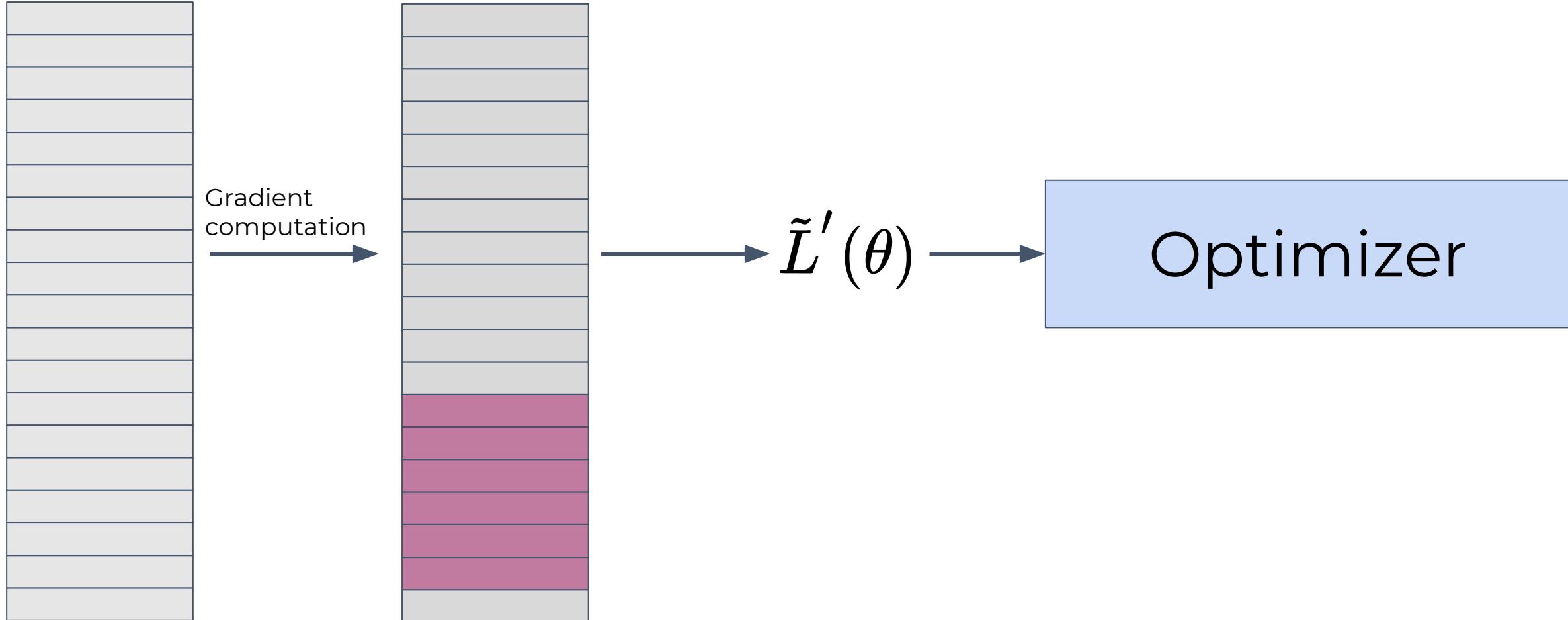
Stochastic gradient descent



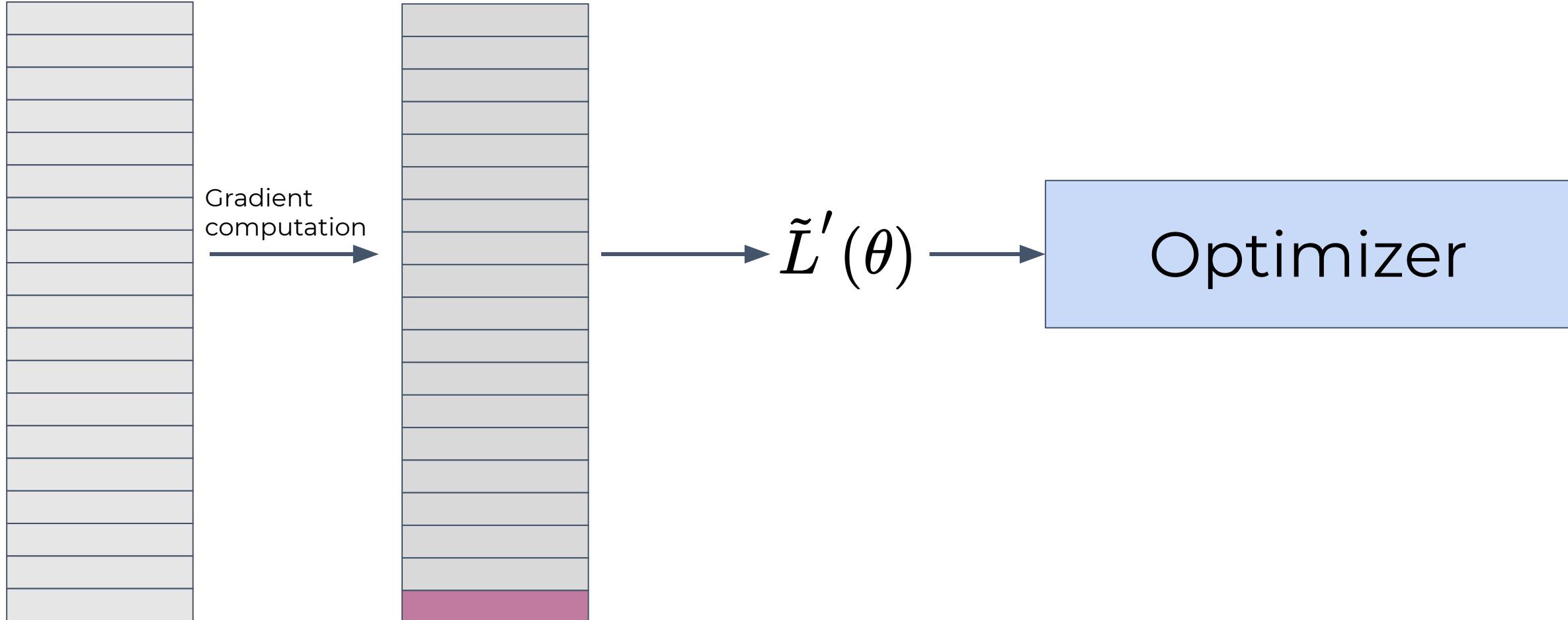
Stochastic gradient descent



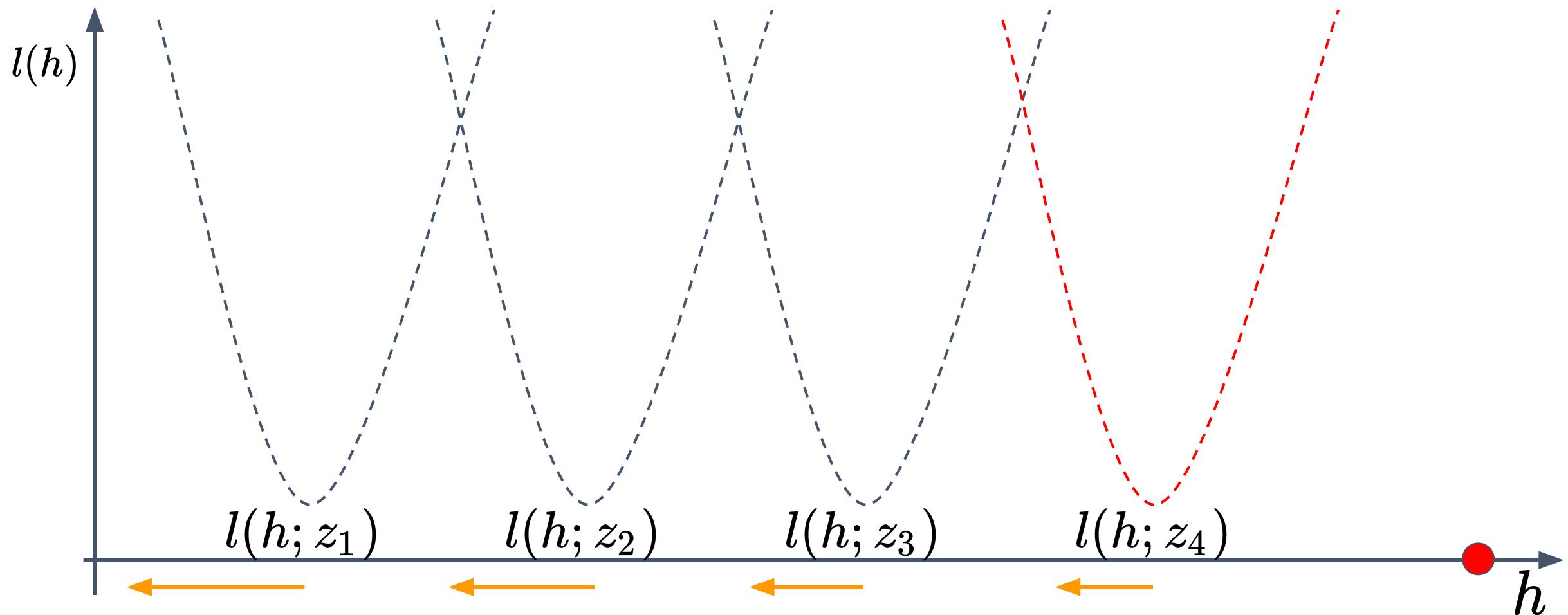
Stochastic gradient descent



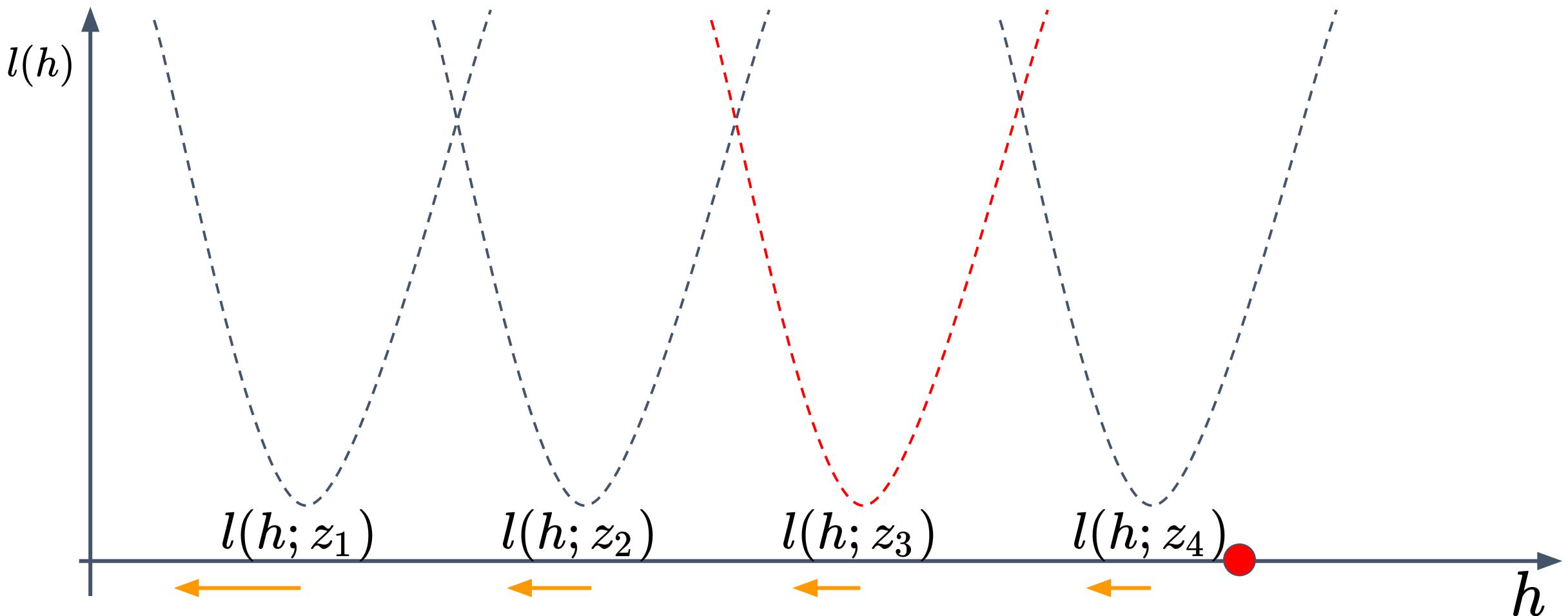
Stochastic gradient descent



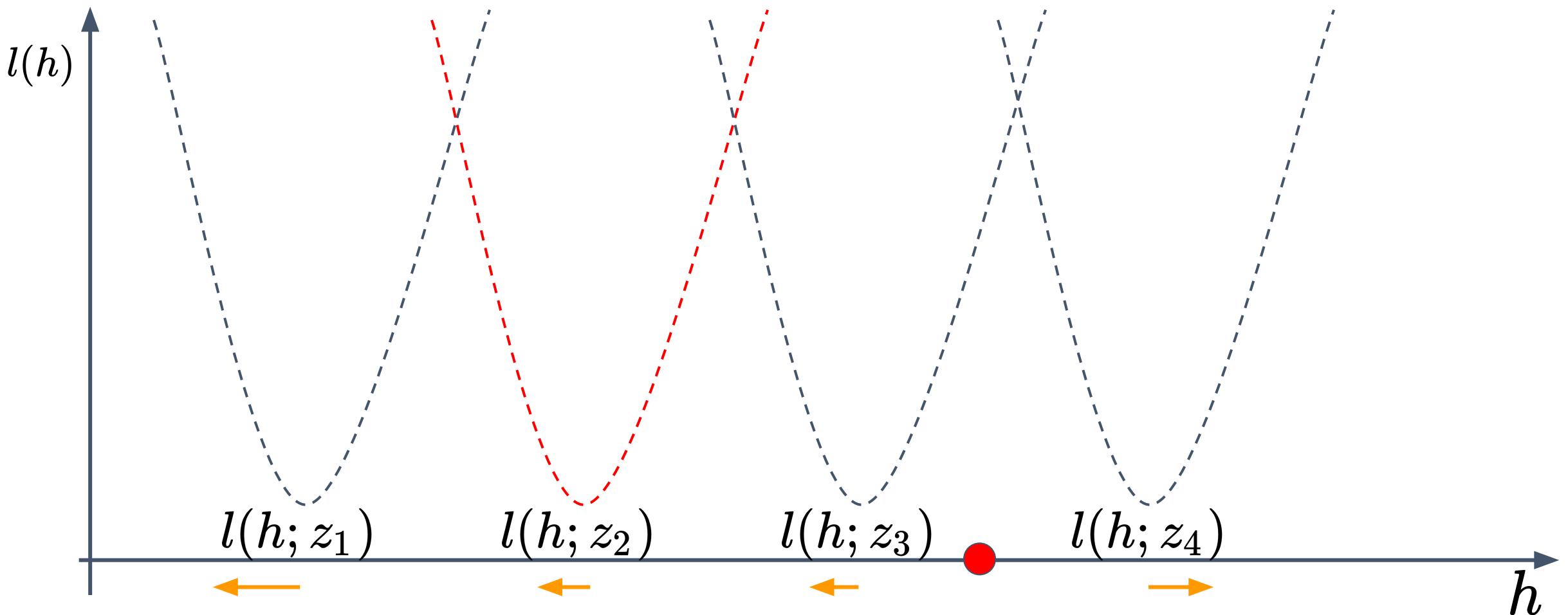
Convergence problem



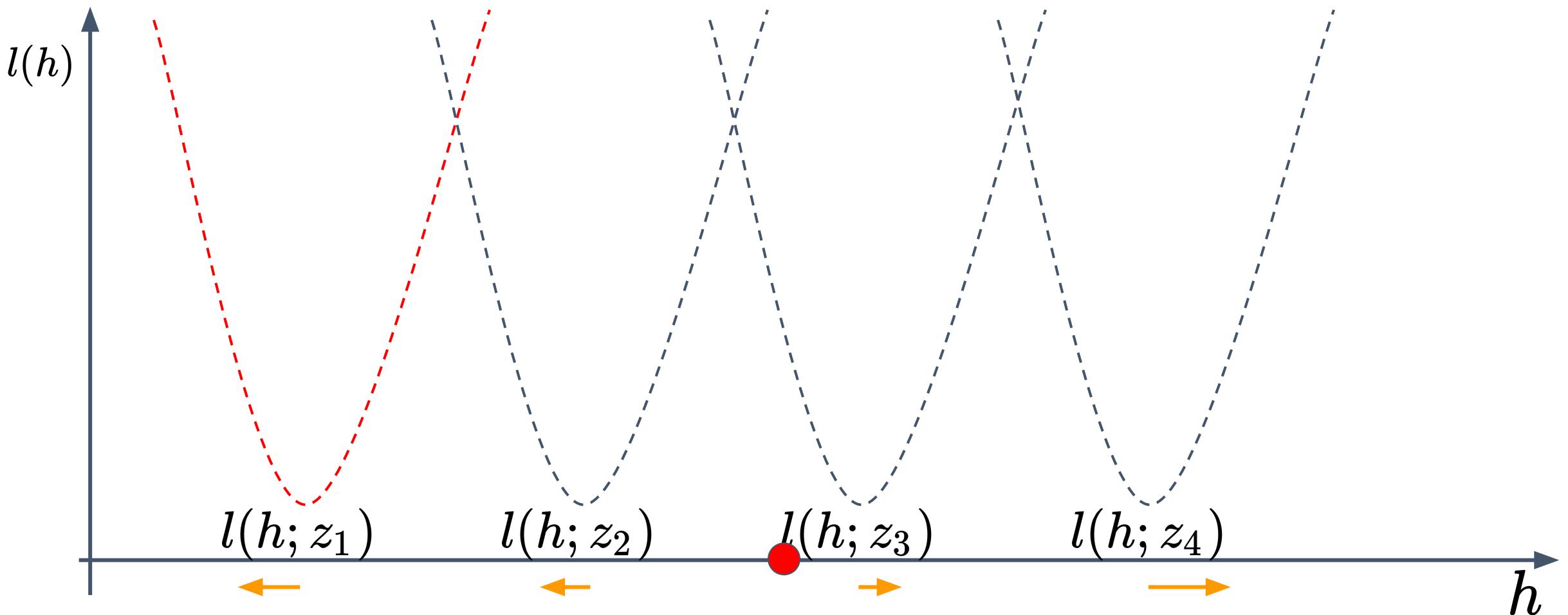
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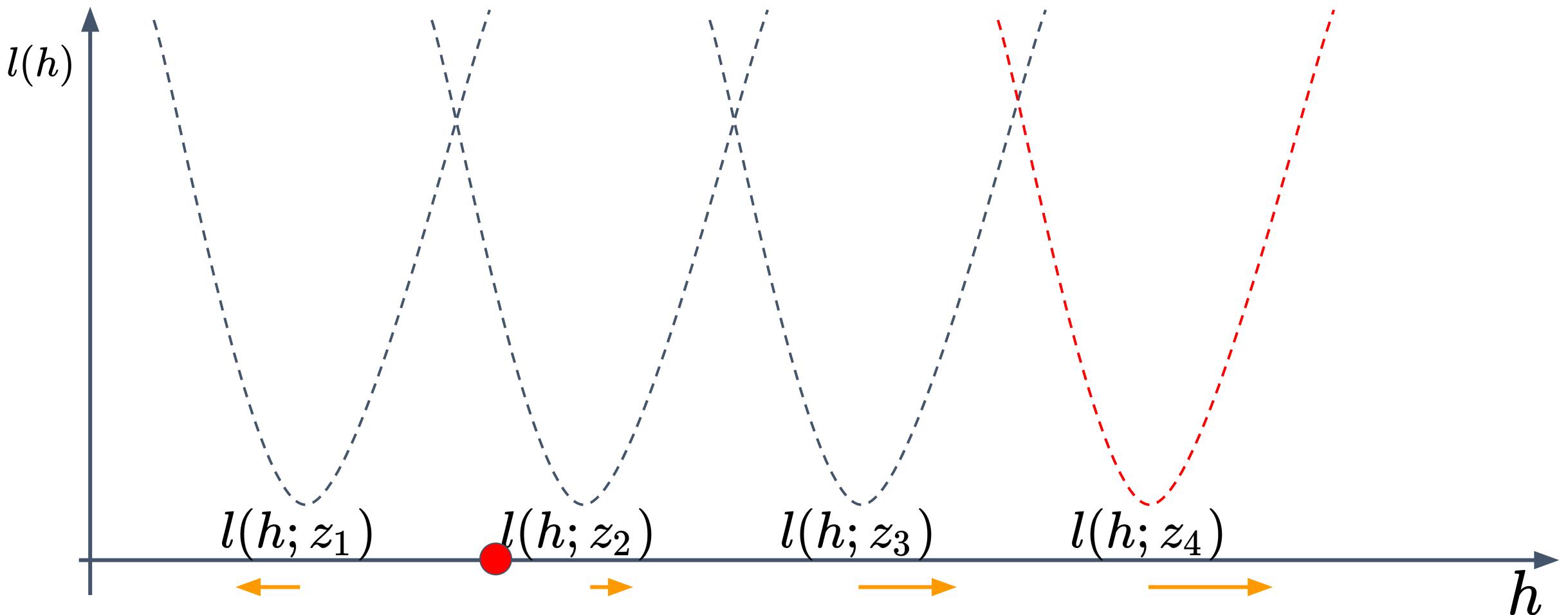
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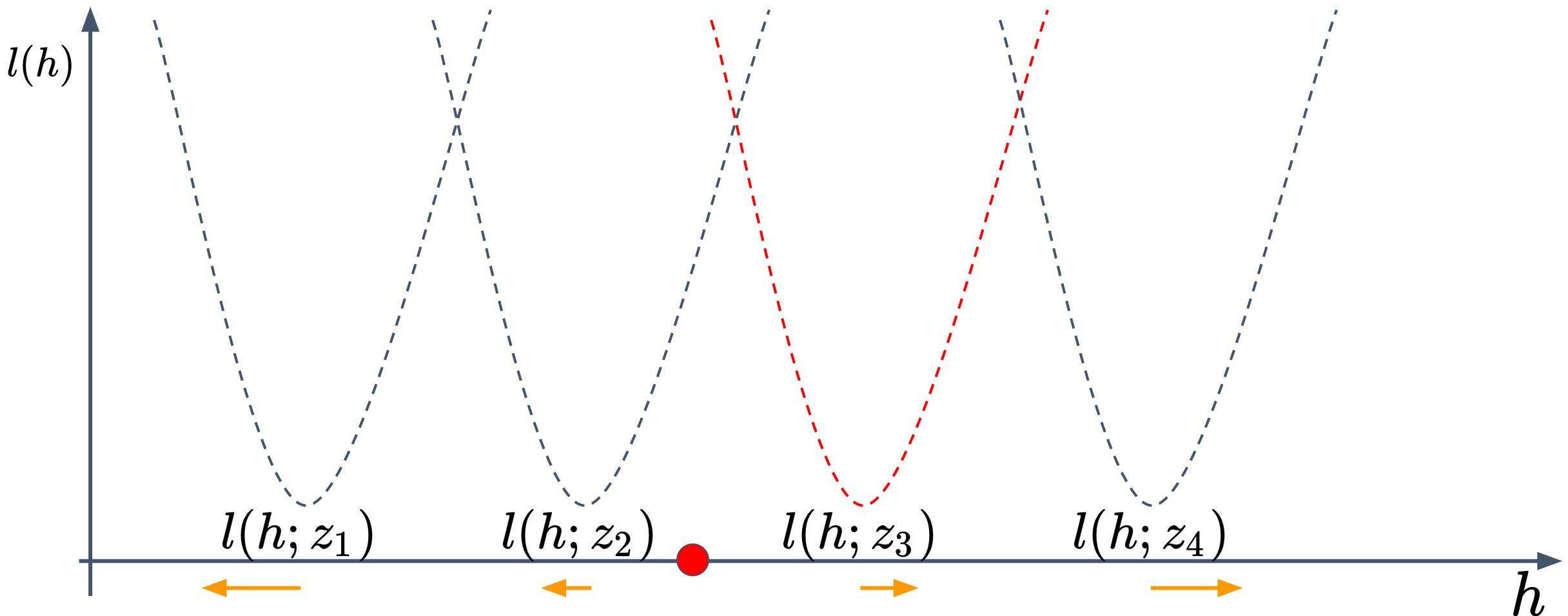
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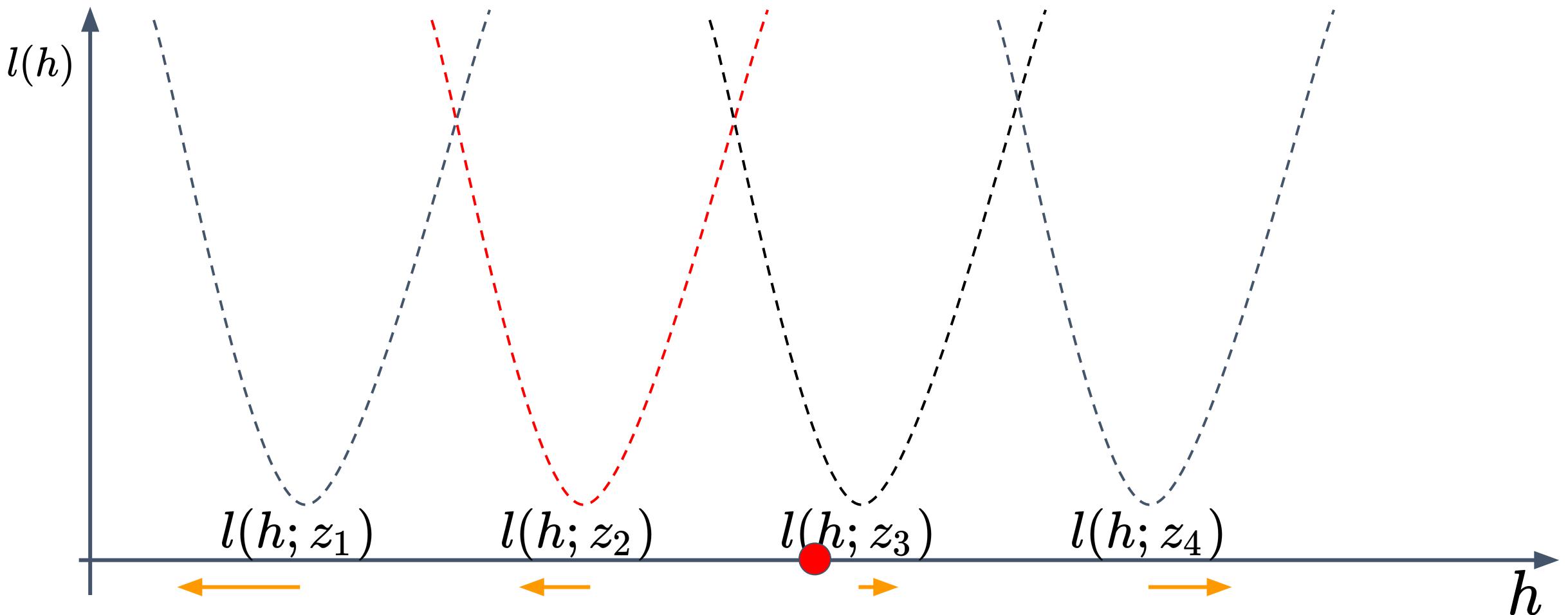
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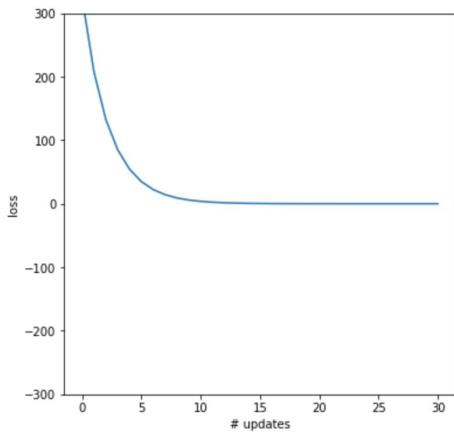
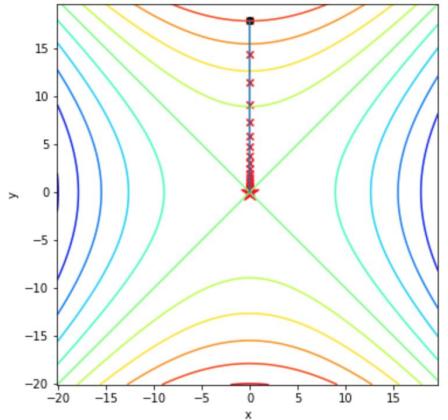
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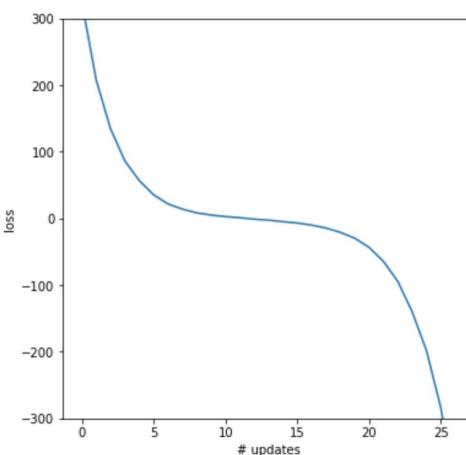
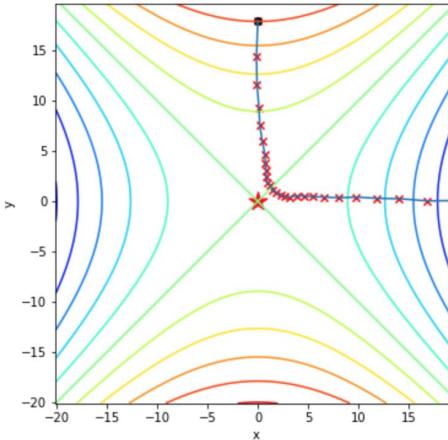
Convergence problem



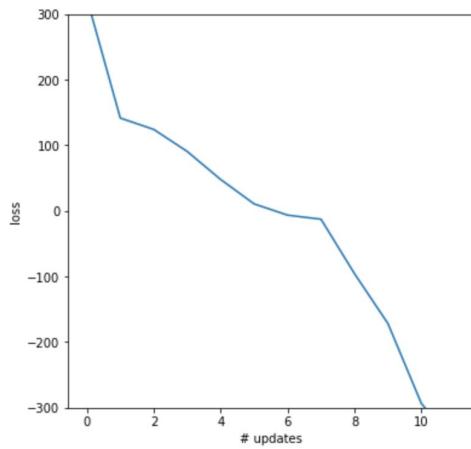
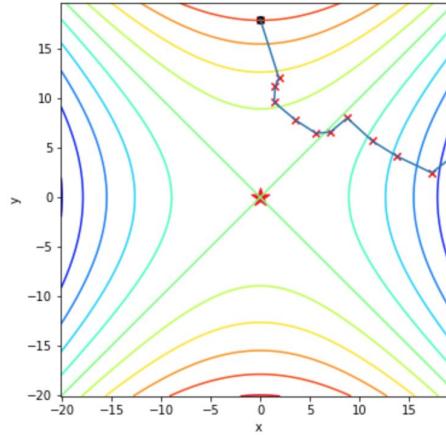
Escape the saddle points



$$\sigma^2 = 0$$



$$\sigma^2 = 0.1$$



$$\sigma^2 = 1$$

Take-home message

- Machine learning is not only about optimization.
- Preconditioning facilitates optimization.
- Momentum can leads to better results.

References

- [Explanation of the effect of noise on saddle points](#)
- [Explanation of the effect of momentum](#)
- [The chapter on numerical calculations in the book Deep Learning by Goodfellow, Bengio and Courville](#)
- [Explanation of Adam by Andrew Ng](#)
- [How to reduce learning rate by Andrew Ng](#)
- [Description of the different optimizers in DL](#)

