HEP-PH Cheat Sheet

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Important notes

- Minkowski metric: $\eta = (+,-,-,-)$ unless otherwise noted.
- Levi-Civita symbol: $\epsilon^{12} = \epsilon_{12} = \epsilon^{123} = \epsilon_{123} = \epsilon^{1234} = \epsilon_{1234} = \cdots = 1$.
- Levi-Civita Lorentz tensor: $\varepsilon^{0123} = -\varepsilon_{0123} = 1$.
- Pauli matrices: $\sigma^i := \{\sigma_x, \sigma_y, \sigma_z\}$, hence $\sigma_i = -\sigma^i$ for $i = 1, 2, 3$, unless otherwise noted.
- Symbols with this color are under a locally-defined convention different from this CheatSheet.
- Elementary charge: $|e| \simeq 0.303$, always in absolute-value symbols. Note $\epsilon_0 = 1/(\mu_0 c^2) = 1$.

1 Notation and Convention

Convention

Pauli matrices: $\sigma_0 := \begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\sigma_x := \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_y := \begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}$, $\sigma_z := \begin{pmatrix}1&0\\0&-1\end{pmatrix}$; $\sigma^\mu := (1,\boldsymbol{\sigma})$, $\bar\sigma^\mu := (1,-\boldsymbol{\sigma})$;

$$\sigma_\pm := \frac{1}{2}(\sigma_x \pm \mathrm{i}\sigma_y) = \begin{pmatrix}0&1\\0&0\end{pmatrix} \Big/ \begin{pmatrix}0&0\\1&0\end{pmatrix}, \quad \sigma_x = \sigma_+ + \sigma_-, \quad \sigma_y = -\mathrm{i}(\sigma_+ - \sigma_-). \tag{1.1}$$

Fourier transf.: $$\tilde f(k) := \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}kx} f(x); \quad f(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}kx}\tilde f(k). \tag{1.2}$$

Minkowski metric: $$\eta_{\mu\nu} = \eta^{\mu\nu} := \mathrm{diag}(+,-,-,-), \quad \varepsilon^{0123} := 1, \quad \varepsilon_{0123} = -1. \tag{1.3}$$

coordinates: $$x^\mu := (t,x,y,z), \quad \partial_\mu = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right), \quad p^\mu = (E, p_x, p_y, p_z). \tag{1.4}$$

gamma matrices: $$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}, \quad \gamma_5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3; \quad \{\gamma^\mu,\gamma_5\} = 0, \quad \gamma^5\gamma^5 = 1. \tag{1.5}$$

chiral notation: $$\overline{\psi} := \psi^\dagger\gamma^0; \quad \gamma^\mu := \begin{pmatrix}0&\sigma^\mu\\\bar\sigma^\mu&0\end{pmatrix}, \quad \gamma_5 = \begin{pmatrix}-1&0\\0&1\end{pmatrix}; \quad P_{\mathrm{L}} = \frac{1-\gamma_5}{2}, \quad P_{\mathrm{R}} = \frac{1+\gamma_5}{2}. \tag{1.6}$$

Electromagnetism

$$A^\mu = (\phi, \boldsymbol{A}),^{\sharp 1} \quad \boldsymbol{E} = -\boldsymbol{\nabla}\phi - \dot{\boldsymbol{A}}, \quad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}. \tag{1.7}$$

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \{F_{01},F_{02},F_{03}\} = \boldsymbol{E}, \quad \{F_{23},F_{31},F_{12}\} = -\boldsymbol{B}; \quad F_{\mu\nu}F^{\mu\nu} = 2(\|\boldsymbol{B}\|^2 - \|\boldsymbol{E}\|^2). \tag{1.8}$$

Maxwell equations: $$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \iff \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{E} + \dot{\boldsymbol{B}} = 0,$$
$$\partial_\mu F^{\mu\nu} = j^\nu := (\rho, \boldsymbol{j}) \iff \boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho, \quad \boldsymbol{\nabla}\times\boldsymbol{B} - \dot{\boldsymbol{E}} = \boldsymbol{j}. \tag{1.9}$$

> **♯1:** The definition of A^μ is determined by that of x^μ (up to an overall sign). We cannot lower the index.

2 Kinematics

Notation here: $p_i^\mu = \begin{pmatrix} p_i^0 \\ \boldsymbol{p}_i \end{pmatrix}$, $\tilde{p}_i = \|\boldsymbol{p}_i\|$, $E_i = \sqrt{\tilde{p}_i^2 + m_i^2}$ (on-shell fixed), $Q_i = p_i^2 = p_i^\mu p_{i\mu}$.

$$\overline{2\pi\delta^{(4)}}\left(P^\mu - p^\mu\right) := (2\pi)^4\,\delta^{(4)}\left(P^\mu - p^\mu\right)$$

$$\overline{2\pi\delta}\left((p^0)^2 - E_p^2\right) := (2\pi)\,\delta\left((p^0)^2 - \|\boldsymbol{p}\|^2 - m^2\right)\Theta\left(p^0\right)$$

Källén function $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx = (x-y-z)^2 - 4yz$

LIPS $\mathrm{d}\Pi = \dfrac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\dfrac{1}{2E_p} = \dfrac{\mathrm{d}p^0\mathrm{d}^3\boldsymbol{p}}{(2\pi)^4}\,\overline{2\pi\delta}\left((p^0)^2 - E_p^2\right)$

$$\overline{\mathrm{d}\Pi^{(n)}} := \mathrm{d}\Pi_1\cdots\mathrm{d}\Pi_n\,\overline{2\pi\delta^{(4)}}\left(P^\mu - \sum p_n^\mu\right)$$

Decay rate and cross section ($\mathcal{M}$ has a mass dimension of $4 - N_\mathrm{i} - N_\mathrm{f}$.)

$$\text{decay rate (rest frame; } \sqrt{s} = M_0\text{):}\quad \mathrm{d}\Gamma = \frac{1}{2M_0}\overline{\mathrm{d}\Pi^{(N_\mathrm{f})}}\left|\mathcal{M}\left(M_0 \to \{p_1, p_2, \cdots, p_{N_\mathrm{f}}\}\right)\right|^2 \tag{2.1}$$

$$\text{cross section (Lorentz invariant):}\quad \mathrm{d}\sigma = \frac{1}{4E_A E_B\,v_\text{Møl}}\overline{\mathrm{d}\Pi^{(N_\mathrm{f})}}\left|\mathcal{M}\left(k_A, k_B \to \{p_1, p_2, \cdots, p_{N_\mathrm{f}}\}\right)\right|^2 \tag{2.2}$$

Møller parameter: $4E_A E_B v_\text{Møl} = 2s \cdot \lambda^{1/2}(1, m_A^2/s, m_B^2/s)$.

Mandelstam variables For $(k_A, k_B) \to (p_1, p_2)$ collision,

$$\begin{aligned}
&s = (k_A + k_B)^2 = (p_1 + p_2)^2 && (k_A - k_B)^2 = 2(m_A^2 + m_B^2) - s && k_A \cdot k_B = (s - m_A^2 - m_B^2)/2 \\
&t = (p_1 - k_A)^2 = (p_2 - k_B)^2 && (p_1 - p_2)^2 = 2(m_1^2 + m_2^2) - s && p_1 \cdot p_2 = (s - m_1^2 - m_2^2)/2 \\
&u = (p_1 - k_B)^2 = (p_2 - k_A)^2 && && k_A \cdot p_2 = (m_2^2 + m_A^2 - u)/2 \\
&s + t + u = m_A^2 + m_B^2 + m_1^2 + m_2^2 && && k_A \cdot p_1 = (m_1^2 + m_A^2 - t)/2
\end{aligned}$$

Two-body final state in the CM frame $p_1^\mu + p_2^\mu = \begin{pmatrix} E_1 \\ \boldsymbol{p} \end{pmatrix} + \begin{pmatrix} E_2 \\ -\boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \sqrt{s} \\ \boldsymbol{0} \end{pmatrix}$ with $\boldsymbol{p}$ directing to $\Omega = (\theta, \phi)$:

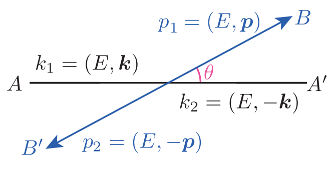
$$\overline{\mathrm{d}\Pi^{(2)}}\Big|_\text{CM} = \frac{\|\boldsymbol{p}\|}{4\pi\sqrt{s}}\frac{\mathrm{d}\Omega}{4\pi} = \frac{\|\boldsymbol{p}\|}{8\pi\sqrt{s}}\mathrm{d}\cos\theta = \frac{\lambda^{1/2}(1, m_1^2/s, m_2^2/s)}{16\pi}\mathrm{d}\cos\theta \qquad \left(\sqrt{s} = M_0 \text{ or } E_\text{CM}\right) \tag{2.3}$$

$$\|\boldsymbol{p}\| = \frac{\sqrt{s}}{2}\lambda^{1/2}\left(1, \frac{m_1^2}{s}, \frac{m_2^2}{s}\right), \quad E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, \quad E_2 = \frac{s - m_1^2 + m_2^2}{2\sqrt{s}}, \quad p_1 \cdot p_2 = \frac{s - (m_1^2 + m_2^2)}{2}.$$

Decay rates and 2-to-2 cross sections are

$$\mathrm{d}\Gamma \overset{\text{CM}}{=} \frac{\lambda^{1/2}(1, m_1^2/s, m_2^2/s)}{32\pi M_0}\mathrm{d}\cos\theta\,|\mathcal{M}|^2, \qquad \mathrm{d}\sigma \overset{\text{CM}}{=} \frac{1}{32\pi s}\frac{\lambda^{1/2}(1, m_1^2/s, m_2^2/s)}{\lambda^{1/2}(1, m_A^2/s, m_B^2/s)}\mathrm{d}\cos\theta\,|\mathcal{M}|^2 \tag{2.4}$$

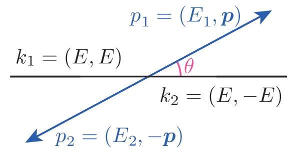
For collisions with the same mass $(m_A, m_A) \to (m_1, m_1)$:



$$t = m_A^2 + m_1^2 - s/2 + 2\tilde{k}\tilde{p}\cos\theta, \qquad \tilde{k} = \sqrt{s/4 - m_A^2},$$

$$u = m_A^2 + m_1^2 - s/2 - 2\tilde{k}\tilde{p}\cos\theta, \qquad \tilde{p} = \sqrt{s/4 - m_1^2}.$$

For collisions of massless initial particles $(0, 0) \to (m_1, m_2)$:



$$t = (m_1^2 + m_2^2 - s)/2 + \tilde{p}\sqrt{s}\cos\theta, \qquad \tilde{p} = (\sqrt{s}/2)\,\lambda^{1/2}(1, m_1^2/s, m_2^2/s),$$

$$u = (m_1^2 + m_2^2 - s)/2 - \tilde{p}\sqrt{s}\cos\theta.$$

Phase-space reduction With $Q' = p'^\mu p'_\mu$,

$$\overline{\mathrm{d}\Pi^{(n+m)}}(P;\, p_1, \ldots, p_{n+m}) = \overline{\mathrm{d}\Pi^{(n)}}(p';\, p_1, \ldots, p_n)\,\overline{\mathrm{d}\Pi^{(m+1)}}(P;\, p', p_{n+1}, \ldots, p_{n+m})\frac{1}{2\pi}\mathrm{d}Q' \tag{2.5}$$

Three-body final state Mandelstam-like variables can be defined, for $P \to (p_1, p_2, p_3)$, as

$$s_{ij} = (p_i + p_j)^2; \quad t_{0i} = (P - p_i)^2 = s_{jk}; \quad s_{12} + s_{23} + s_{31} = P^2 + p_1^2 + p_2^2 + p_3^2.$$

For spherically-symmetric processes, the phase-space integral is reduced to, at the center-of-mass frame,

$$\int \overline{\mathrm{d}\Pi^3}_{\text{sph. sym.}}(\sqrt{s}, \mathbf{0}) = \frac{1}{128\pi^3}\frac{1}{s}\int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} \mathrm{d}s_{23} \int \mathrm{d}s_{13}; \tag{2.6}$$

$$\begin{aligned}(s_{13})^{\max}_{\min} &= \frac{(s + m_3^2 - m_1^2 - m_2^2)^2}{4s_{23}} - \frac{1}{4s_{23}}\left[\lambda^{1/2}(s, m_1^2, s_{23}) \mp \lambda^{1/2}(s_{23}, m_2^2, m_3^2)\right]^2 \\ &= (E_1^* + E_3^*)^2 - \left(\sqrt{E_1^{*2} - m_1^2} \mp \sqrt{E_3^{*2} - m_3^2}\right)^2,\end{aligned} \tag{2.7}$$

where $E_1^* = \frac{s - s_{23} - m_1^2}{2\sqrt{s_{23}}}$, and $E_3^* = \frac{s_{23} - m_2^2 + m_3^2}{2\sqrt{s_{23}}}$.

2.1 Fundamentals

Lorentz-invariant phase space:

$$\mathrm{d}\Pi = \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_p} = \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\sqrt{m^2+\tilde{p}^2}} = \frac{\mathrm{d}p_0\mathrm{d}^3\boldsymbol{p}}{(2\pi)^4}2\pi\,\delta(p_0^2-\tilde{p}^2-m^2)\Theta(p^0) \equiv \frac{\mathrm{d}p_0\mathrm{d}^3\boldsymbol{p}}{(2\pi)^4}\overline{2\pi\delta}(p_0^2-E_p^2) \tag{2.8}$$

Källén function:

$$\lambda(x,y,z) = x^2+y^2+z^2-2xy-2yz-2zx = (x-y-z)^2-4yz;$$
$$\lambda(1;\alpha_1^2,\alpha_2^2) = (1-(\alpha_1+\alpha_2)^2)(1-(\alpha_1-\alpha_2)^2) = (1+\alpha_1+\alpha_2)(1-\alpha_1-\alpha_2)(1+\alpha_1-\alpha_2)(1-\alpha_1+\alpha_2).$$
$$\lambda^{1/2}\left(s;m_1^2,m_2^2\right) = s\,\lambda^{1/2}\left(1;\frac{m_1^2}{s},\frac{m_2^2}{s}\right); \qquad \lambda^{1/2}\left(1;\frac{m^2}{s},\frac{m^2}{s}\right) = \sqrt{1-\frac{4m^2}{s}},$$
$$\lambda^{1/2}\left(1;\frac{m_1^2}{s},\frac{m_2^2}{s}\right) = \sqrt{1-\frac{2(m_1^2+m_2^2)}{s}+\frac{(m_1^2-m_2^2)^2}{s^2}}, \qquad \lambda^{1/2}\left(1;\frac{m_1^2}{s},0\right) = \frac{s-m_1^2}{s}.$$

Two-body phase space If $f(p_1^\mu,p_2^\mu)$ is Lorentz invariant, $f\equiv f(p_1^2,p_2^2,p_1^\mu p_{2\mu})\equiv f(\tilde{p}_1,\tilde{p}_2,\cos\theta_{12})$. Meanwhile,

$$\mathrm{d}\Pi_1\mathrm{d}\Pi_2 = \frac{\mathrm{d}^3\boldsymbol{p}_1}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{p}_2}{(2\pi)^3}\frac{1}{2E_12E_2} = \frac{(4\pi)\mathrm{d}\tilde{p}_1\,\tilde{p}_1^2}{(2\pi)^3}\frac{(2\pi)\mathrm{d}\tilde{p}_2\,\tilde{p}_2^2\,\mathrm{d}\cos\theta_{12}}{(2\pi)^3}\frac{1}{2E_12E_2} = \frac{\mathrm{d}E_+\mathrm{d}E_-\mathrm{d}s}{128\pi^4} \tag{2.9}$$

with the replacement of the variables

$$E_\pm = E_1\pm E_2, \qquad s=(p_1+p_2)^2 = m_1^2+m_2^2+2E_1E_2-2\tilde{p}_1\tilde{p}_2\cos\theta_{12};$$
$$\left|\frac{\mathrm{d}(E_+,E_-,s)}{\mathrm{d}(\tilde{p}_1,\tilde{p}_2,\cos\theta_{12})}\right| = \frac{4\tilde{p}_1^2\tilde{p}_2^2}{E_1E_2}, \qquad \left|\frac{\mathrm{d}(E_1,E_2,s)}{\mathrm{d}(\tilde{p}_1,\tilde{p}_2,\cos\theta_{12})}\right| = \frac{2\tilde{p}_1^2\tilde{p}_2^2}{E_1E_2}.$$

Therefore,

$$\int\mathrm{d}\Pi_1\mathrm{d}\Pi_2 = \frac{1}{128\pi^4}\int_{(m_1+m_2)^2}^\infty\mathrm{d}s\int_{\sqrt{s}}^\infty\mathrm{d}E_+\int_{\min}^{\max}\mathrm{d}E_-, \tag{2.10}$$

where the boundary of E_- is given by

$$\cos\theta_{12} = \frac{E_+^2-E_-^2+2\left(m_1^2+m_2^2-s\right)}{\sqrt{(E_++E_-)^2-4m_1^2}\sqrt{(E_+-E_-)^2-4m_2^2}} \in [-1,1]$$
$$\therefore\quad \left|E_--\frac{m_1^2-m_2^2}{s}E_+\right| \le \sqrt{E_+^2-s}\cdot\lambda^{1/2}\left(1;\frac{m_1^2}{s},\frac{m_2^2}{s}\right).$$

Two-body phase space with momentum conservation We consider general two-body phase-space integrals,

$$\overline{\mathrm{d}\Pi^{(2)}}\times I = \mathrm{d}\Pi_1\mathrm{d}\Pi_2\,\overline{2\pi\delta^{(4)}}(P^\mu-p_1^\mu-p_2^\mu)\times I = \frac{1}{(2\pi)^2}\frac{\mathrm{d}^3\boldsymbol{p}_1}{2E_1}\frac{\mathrm{d}^3\boldsymbol{p}_2}{2E_2}\,\delta^{(4)}(P^\mu-p_1^\mu-p_2^\mu)\times I \tag{2.11}$$

with I being the integrand, $P^\mu=(E_0,\boldsymbol{P}_0)$, and $S_0=E_0^2-\tilde{P}_0^2$, assuming $S_0>0$. First, with carrying out $\boldsymbol{p}_2$ integral,

$$\overline{\mathrm{d}\Pi^{(2)}}\times I = \frac{1}{(2\pi)^2}\frac{\mathrm{d}^3\boldsymbol{p}_1}{2E_1}\mathrm{d}^4p_2^\mu\,\delta((p_2^0)^2-\|\boldsymbol{p}_2\|^2-m_2^2)\Theta(p_2^0)\,\delta^{(4)}(P^\mu-p_1^\mu-p_2^\mu)\times I \tag{2.12}$$
$$= \frac{1}{(2\pi)^2}\frac{\mathrm{d}^3\boldsymbol{p}_1}{2E_1}\,\delta((E_0-E_1)^2-\|\boldsymbol{P}_0-\boldsymbol{p}_1\|^2-m_2^2)\Theta(E_0-E_1)\times I \tag{2.13}$$
$$= \frac{1}{(2\pi)^2}\frac{\mathrm{d}^3\boldsymbol{p}_1}{2E_1}\,\delta(S_0-2E_0E_1+2\boldsymbol{P}_0\cdot\boldsymbol{p}_1+m_1^2-m_2^2)\Theta(E_0-E_1)\times I \tag{2.14}$$
$$= \frac{\tilde{p}_1\mathrm{d}E_1\mathrm{d}\cos\theta\mathrm{d}\phi}{8\pi^2}\,\delta(S_0-2E_0E_1+2\tilde{P}_0\tilde{p}_1\cos\theta+m_1^2-m_2^2)\Theta(E_0-E_1)\times I\Big|_{\tilde{p}_1=(E_1^2-m_1^2)^{1/2}}. \tag{2.15}$$

Now, we continue for a constant I. If $\tilde{P}_0=0$, then $\sqrt{S_0}=E_0$ and

$$\overline{\mathrm{d}\Pi^{(2)}} = \frac{(E_1^2-m_1^2)^{1/2}\mathrm{d}E_1}{2\pi}\,\delta(2\sqrt{S_0}\mathcal{E}-2E_0E_1)\Theta(E_0-E_1) = \frac{\phi}{4\pi\sqrt{S_0}} \tag{2.16}$$

to recover the CM result with the constants

$$\mathcal{E} = \frac{S_0+m_1^2-m_2^2}{2\sqrt{S_0}}, \qquad \phi = \frac{\lambda^{1/2}(S_0,m_1^2,m_2^2)}{2\sqrt{S_0}} = \sqrt{\mathcal{E}^2-m_1^2}. \tag{2.17}$$

Because the result is Lorentz scalar, the result is unchanged even if $\tilde{P}_0\neq 0$ as we explicitly check by integrating $\cos\theta$; the integration gives the Heaviside function and

$$\overline{\mathrm{d}\Pi^{(2)}} = \frac{1}{8\pi\tilde{P}_0}\int_{m_1}^{E_0}\mathrm{d}E_1\,\Theta\left(-1<c(E_1)<1\right); \qquad c(E_1) \coloneqq \frac{E_0E_1-\sqrt{S_0}\mathcal{E}}{\tilde{P}_0(E_1^2-m_1^2)^{1/2}}. \tag{2.18}$$

There,

$$E_{1;\,\max} = \frac{E_0\mathcal{E} + \tilde{P}_0\phi}{\sqrt{S_0}} \quad\Longleftrightarrow\quad \tilde{p}_1 = \frac{\tilde{P}_0\mathcal{E} + E_0\phi}{\sqrt{S_0}} \quad\Longrightarrow\quad c = +1, \tag{2.19}$$

$$E_{1;\,\min} = \frac{E_0\mathcal{E} - \tilde{P}_0\phi}{\sqrt{S_0}} \quad\Longleftrightarrow\quad \tilde{p}_1 = \left|\frac{\tilde{P}_0\mathcal{E} - E_0\phi}{\sqrt{S_0}}\right| \quad\Longrightarrow\quad c = \operatorname{sign}(\tilde{P}_0\mathcal{E} - E_0\phi). \tag{2.20}$$

If $\tilde{P}_0\mathcal{E} - E_0\phi < 0$, the first derivative $c'(E_1)$ cannot be zero between $E_{1;\,\min}$ and $E_{1;\,\max}$, so $c(E_1)$ is monotonically increasing and E_1 and $\cos\theta$ are in one-to-one correspondence. Meanwhile, if $\tilde{P}_0\mathcal{E} - E_0\phi > 0$, then $c(E_1)$ has a minimum

$$c_{\min} = c\left(\frac{E_0 m_1^2}{S_0\mathcal{E}}\right) = \frac{\sqrt{E_0^2 m_1^2 - S\mathcal{E}^2}}{\tilde{P}_0 m_1} = \sqrt{1 - \frac{S\phi^2}{m_1^2\tilde{P}_0^2}}. \tag{2.21}$$

In any cases, all E_1 between them are allowed, and we evaluate $\overline{\mathrm{d}\Pi^{(2)}} = (E_{1;\,\max} - E_{1;\,\min})/8\pi\tilde{P}_0 = \phi/4\pi\sqrt{S_0}$ as expected.

2.2 Decay rate and Cross section

As $\langle\text{out}|\text{in}\rangle = (2\pi)^4\,\delta^{(4)}(p_\mathrm{i} - p_\mathrm{f})\mathrm{i}\mathcal{M}$ (for in $\neq$ out) and $\langle\boldsymbol{p}|\boldsymbol{p}\rangle = 2E_{\boldsymbol{p}}(2\pi)^3\,\delta^{(3)}(\boldsymbol{0}) = 2E_{\boldsymbol{p}}V$ for one-particle state,

$$\frac{N_\text{ev}}{\prod_\text{in} N_\text{particle}} = \int \mathrm{d}\Pi^\text{out}\frac{|\langle\text{out}|\text{in}\rangle|^2}{\langle\text{in}|\text{in}\rangle} = \int \mathrm{d}\Pi^\text{out}\frac{(2\pi)^8|\mathcal{M}|^2}{\prod_\text{in}(2E)V}\frac{VT}{(2\pi)^4}\,\delta^{(4)}(p_\mathrm{i} - p_\mathrm{f}) = VT\int\overline{\mathrm{d}\Pi^{(N_\mathrm{f})}}\frac{|\mathcal{M}|^2}{\prod_\text{in}(2E)V}. \tag{2.22}$$

Therefore, decay rate (at the rest frame) is given by

$$\mathrm{d}\Gamma \coloneqq \frac{1}{T}\frac{\mathrm{d}N_\text{ev}}{N_\text{particle}} = \frac{1}{T}VT\overline{\mathrm{d}\Pi^{(N_\mathrm{f})}}\frac{|\mathcal{M}|^2}{(2E)V} = \frac{1}{2M_0}\overline{\mathrm{d}\Pi^{(N_\mathrm{f})}}|\mathcal{M}|^2. \tag{2.23}$$

We also define Lorentz-invariant cross section σ by $N_\text{ev} \eqqcolon (n_A v_\text{Møl}T\sigma)N_B = (n_A v_\text{Møl}T\sigma)(n_B V)$ with number density n, or

$$\mathrm{d}\sigma \coloneqq \frac{\mathrm{d}N_\text{ev}}{n_A v_\text{Møl} T N_B} = \frac{V}{v_\text{Møl}T}VT\overline{\mathrm{d}\Pi^{(N_\mathrm{f})}}\frac{|\mathcal{M}|^2}{2E_A 2E_B V^2} = \frac{1}{2E_A 2E_B\, v_\text{Møl}}\overline{\mathrm{d}\Pi^{(N_\mathrm{f})}}|\mathcal{M}|^2, \tag{2.24}$$

where the Møller parameter $v_\text{Møl}$ is equal to $v_\text{rel}^\text{NR} = \|\boldsymbol{v}_A - \boldsymbol{v}_B\|$ if $\boldsymbol{v}_A \parallel \boldsymbol{v}_B$ (cf. Ref. [1]). Generally,

$$v_\text{Møl} \coloneqq \frac{\sqrt{(p_A\cdot p_B)^2 - m_A^2 m_B^2}}{E_A E_B} = \frac{\sqrt{\lambda(s, m_A^2, m_B^2)}}{2E_A E_B} = \frac{p_A\cdot p_B}{E_A E_B}v_\text{rel} = (1 - \boldsymbol{v}_A\cdot\boldsymbol{v}_B)v_\text{rel}, \tag{2.25}$$

where v_rel is the actual relative velocity

$$v_\text{rel} = \sqrt{1 - \frac{(1 - v_A^2)(1 - v_B^2)}{1 - (\boldsymbol{v}_A\cdot\boldsymbol{v}_B)^2}} = \frac{\sqrt{\|\boldsymbol{v}_A - \boldsymbol{v}_B\|^2 - \|\boldsymbol{v}_A\times\boldsymbol{v}_B\|^2}}{1 - \boldsymbol{v}_A\cdot\boldsymbol{v}_B} = \frac{\lambda^{1/2}(s, m_A^2, m_B^2)}{s - (m_A^2 + m_B^2)} \neq v_\text{rel}^\text{NR}. \tag{2.26}$$

(Note that $p_A\cdot p_B/E_A E_B = 1$ if $\boldsymbol{p}_A = 0$ or $\boldsymbol{p}_B = 0$. Also, Each of v_rel, VT, and $E_A E_B v_\text{Møl}$ is Lorentz invariant.)

2.3 More than two bodies

The phase-space reduction formula (2.5) is proved by, with $p_{a,\dots,b} = \sum_{i=a}^b p_i$, (cf. Ref. [2])

$$\begin{aligned}
\overline{\mathrm{d}\Pi^{(n+m)}} &= \left[\frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3}\frac{\mathrm{d}p'^0}{2\pi}\,\overline{2\pi\delta^{(4)}}\left(p'^\mu - p^\mu_{1,\dots,n}\right)\right]\mathrm{d}\Pi_1\cdots\mathrm{d}\Pi_{n+m}\,\overline{2\pi\delta^{(4)}}\left(P^\mu - p^\mu_{1,\dots,n+m}\right)\\
&= \left[\overline{\mathrm{d}\Pi^{(n)}}(p';\,p_1,\dots,p_n)\,\overline{\mathrm{d}\Pi^{(m+1)}}(P;\,p',p_{n+1},\dots,p_{n+m})\right]\frac{\mathrm{d}p'^0}{2\pi}(2p'^0)\\
&= \left[\overline{\mathrm{d}\Pi^{(n)}}(p';\,p_1,\dots,p_n)\,\overline{\mathrm{d}\Pi^{(m+1)}}(P;\,p',p_{n+1},\dots,p_{n+m})\right]_{p'^\mu p'_\mu = Q'\,:\,\text{fixed}}\frac{\mathrm{d}Q'}{2\pi}.
\end{aligned}$$

The phase-space integral of massless n-body particles over a constant is obtained by, as we can set $P^\mu = \binom{\sqrt{s}}{\boldsymbol{0}}$,

$$\begin{aligned}
\int 1\,\overline{\mathrm{d}\Pi^{(n)}}(s) &= \frac{\mathrm{d}Q'}{2\pi}\overline{\mathrm{d}\Pi^{(2)}}(P;\,p',p_n)\,\overline{\mathrm{d}\Pi^{(n-1)}}(p';\,p_1,\dots,p_{n-1})\\
&= \frac{\mathrm{d}Q'}{2\pi}\frac{\tilde{p}'}{4\pi P^0}a_{n-1}Q'^{n-3} = \frac{\mathrm{d}Q'}{2\pi}\frac{s - Q'}{8\pi s}a_{n-1}Q'^{n-3} = \frac{a_{n-1}}{(4\pi)^2(n-1)(n-2)}s^{n-2}\\
\therefore\quad \int 1\,\overline{\mathrm{d}\Pi^{(n)}} &= \frac{1}{2(4\pi)^{2n-3}}\frac{s^{n-2}}{(n-1)!(n-2)!}
\end{aligned} \tag{2.27}$$

where the fact is used that $\overline{\mathrm{d}\Pi^{(n)}}(s) = a_n \times s^{n-2}$ with a_n being a constant.

Three-body: spherically-symmetric cases The three-body phase space integral is reduced as

$$\overline{\mathrm{d}\Pi^{(3)}} = \frac{\mathrm{d}Q_{23}}{2\pi}\overline{\mathrm{d}\Pi^{(2)}}(P;\ p_1, p_{23})\overline{\mathrm{d}\Pi^{(2)}}(p_{23};\ p_2, p_3), \tag{2.28}$$

$$\overline{\mathrm{d}\Pi^{(2)}}(p_{23};\ p_2, p_3) = \frac{\mathrm{d}\cos\theta_2}{8\pi}\frac{\tilde{p}_2^2}{E_{23}\tilde{p}_2 - \tilde{p}_{23}E_2\cos\theta_2}; \tag{2.29}$$

$$\tilde{p}_2 = \frac{(Q_{23}+m_2^2-m_3^2)\tilde{p}_{23}\cos\theta_2 + E_{23}\sqrt{\lambda(Q_{23}, m_2^2, m_3^2) - 4m_2^2\tilde{p}_{23}^2\sin^2\theta_2}}{2(Q_{23}+\tilde{p}_{23}^2\sin^2\theta_2)}, \tag{2.30}$$

where θ_2 is the angle between $\boldsymbol{p}_{23}$ and $\boldsymbol{p}_2$ (in the lab frame). If the matrix element to integrate is spherically symmetric, so as $\overline{\mathrm{d}\Pi^{(2)}}(p_{23}; p_2, p_3)|\mathcal{M}|^2$, i.e., it is independent of the angle of $\boldsymbol{p}_{23}$. Then one can simply evaluate $\int \mathrm{d}^3\boldsymbol{p}_{23}$, which leads to, in the center-of-mass frame,

$$\overline{\mathrm{d}\Pi^3}_{\text{sph. sym.}}(\sqrt{s}, \mathbf{0}) = \frac{\mathrm{d}s_{23}\mathrm{d}\cos\theta_2}{64\pi^3}\frac{p_1}{\sqrt{s}}\frac{p_2^2}{p_2\sqrt{s_{23}+p_1^2} - p_1\sqrt{p_2^2+m_2^2}\cos\theta_2}\bigg|_{p_1^2=\lambda(s,m_1^2,s_{23})/4s} = \frac{s}{128\pi^3}\mathrm{d}x_1\mathrm{d}x_2, \tag{2.31}$$

where we defined $x_i \coloneqq 2E_i/\sqrt{s}$. Noting that $s_{23} = s + m_1^2 - 2E_1\sqrt{s} = s(1-x_1) + m_1^2$ etc.,

$$\overline{\mathrm{d}\Pi^3}_{\text{sph. sym.}}(\sqrt{s}, \mathbf{0}) = \frac{1}{128\pi^3}\frac{1}{s}\int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2}\mathrm{d}s_{23}\int\mathrm{d}s_{13}; \tag{2.32}$$

$$(s_{13})^{\max}_{\min} = \frac{(s+m_3^2-m_1^2-m_2^2)^2}{4s_{23}} - \frac{1}{4s_{23}}\left[\lambda^{1/2}(s, m_1^2, s_{23}) \mp \lambda^{1/2}(s_{23}, m_2^2, m_3^2)\right]^2. \tag{2.33}$$

This is equal to the PDG-Eq. (47.23)[PDG2018].

2.4 Two-body decay of boosted particles

A particle with $(P, \Theta, \Phi; M)$ decaying to two particles; at the CM frame the momenta of the decay products are characterized by $\boldsymbol{q} = (q, \theta, \phi)$ with $q = (M_0/2)\,\lambda^{1/2}(1, m_1^2/M_0^2, m_2^2/M_0^2)$. Their lab-frame momenta are given by

$$P = \begin{pmatrix} E_0 \\ P_0 \mathrm{s}_\Theta \mathrm{c}_\Phi \\ P_0 \mathrm{s}_\Theta \mathrm{s}_\Phi \\ P_0 \mathrm{c}_\Theta \end{pmatrix}, \quad p_1 = \begin{pmatrix} (E_0\mathcal{E}_1 + P_0 q \mathrm{c}_\theta)/M_0 \\ q\mathrm{c}_\Theta\mathrm{c}_\Phi\mathrm{s}_\theta\mathrm{c}_\phi - q\mathrm{s}_\Phi\mathrm{s}_\theta\mathrm{s}_\phi + r_1\mathrm{s}_\Theta\mathrm{c}_\Phi \\ q\mathrm{c}_\Theta\mathrm{s}_\Phi\mathrm{s}_\theta\mathrm{c}_\phi + q\mathrm{c}_\Phi\mathrm{s}_\theta\mathrm{s}_\phi + r_1\mathrm{s}_\Theta\mathrm{s}_\Phi \\ -q\mathrm{s}_\Theta\mathrm{s}_\theta\mathrm{c}_\phi + r_1\mathrm{c}_\Theta \end{pmatrix}, \quad p_2 = \begin{pmatrix} (E_0\mathcal{E}_2 - P_0 q \mathrm{c}_\theta)/M_0 \\ -q\mathrm{c}_\Theta\mathrm{c}_\Phi\mathrm{s}_\theta\mathrm{c}_\phi + q\mathrm{s}_\Phi\mathrm{s}_\theta\mathrm{s}_\phi + r_2\mathrm{s}_\Theta\mathrm{c}_\Phi \\ -q\mathrm{c}_\Theta\mathrm{s}_\Phi\mathrm{s}_\theta\mathrm{c}_\phi - q\mathrm{c}_\Phi\mathrm{s}_\theta\mathrm{s}_\phi + r_2\mathrm{s}_\Theta\mathrm{s}_\Phi \\ q\mathrm{s}_\Theta\mathrm{s}_\theta\mathrm{c}_\phi + r_2\mathrm{c}_\Theta \end{pmatrix} \tag{2.34}$$

with $r_1 = (P_0\mathcal{E}_1 + E_0 q\mathrm{c}_\theta)/M_0$, $r_2 = (P_0\mathcal{E}_2 - E_0 q\mathrm{c}_\theta)$, and $\mathcal{E}_i = \sqrt{m_i^2 + q^2}$.

3 Spinors

Gamma matrices: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$; $\{\gamma^\mu, \gamma_5\} = 0$, $\gamma^5\gamma^5 = 1$.

Conjugates: $\overline{\psi} \coloneqq \psi^\dagger \beta$, $\psi^{\mathrm{c}} \coloneqq C(\overline{\psi})^{\mathrm{T}}$

Chiral notation

$$\overline{\psi} = \psi^\dagger \gamma^0;\ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},\ \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix};\ P_{\mathrm{L}} = \frac{1-\gamma_5}{2},\ P_{\mathrm{R}} = \frac{1+\gamma_5}{2}; \qquad C = -\mathrm{i}\gamma^2\gamma^0. \tag{3.1}$$

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0,\ (\gamma^\mu)^* = \gamma^2\gamma^\mu\gamma^2,\ (\gamma^\mu)^{\mathrm{T}} = \gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0; \tag{3.2}$$

$$C\gamma^\mu C^\dagger = -(\gamma^\mu)^\dagger,\ C = C^* = -C^{-1} = -C^\dagger = -C^{\mathrm{T}} \tag{3.3}$$

$$\psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix} = \begin{pmatrix} \xi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \bar{\chi}^{\dot{1}} \\ \bar{\chi}^{\dot{2}} \end{pmatrix} \qquad \xrightarrow{C} \qquad \psi^{\mathrm{c}} = C(\overline{\psi})^{\mathrm{T}} = -\mathrm{i}\gamma^2\psi^* = \begin{pmatrix} -(\bar{\chi}^{\dot{2}})^* \\ (\bar{\chi}^{\dot{1}})^* \\ (\xi_2)^* \\ -(\xi_1)^* \end{pmatrix} = \begin{pmatrix} -\chi^2 \\ \chi^1 \\ \bar{\xi}_{\dot{2}} \\ -\bar{\xi}_{\dot{1}} \end{pmatrix} = \begin{pmatrix} \chi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix} \tag{3.4}$$

$$\overline{\psi} = \begin{pmatrix} \chi^\alpha & \bar{\xi}_{\dot{\alpha}} \end{pmatrix} = (\,\chi^1\ \chi^2\ \bar{\xi}_{\dot{1}}\ \bar{\xi}_{\dot{2}}\,) \qquad \xrightarrow{C} \qquad \overline{\psi^{\mathrm{c}}} = \psi^{\mathrm{T}} C = \mathrm{i}\psi^{\mathrm{T}}\gamma^0\gamma^2 = (\,\xi_2\ -\xi_1\ -\bar{\chi}^{\dot{2}}\ \bar{\chi}^{\dot{1}}\,) = \begin{pmatrix} \xi^\alpha & \bar{\chi}_{\dot{\alpha}} \end{pmatrix} \tag{3.5}$$

It is instructive to write, e.g., for $e = e^-$ and $e^{\mathrm{c}} = e^+$,

$$\psi_e = \begin{pmatrix} e_{\mathrm{L}} \\ e_{\mathrm{R}} \end{pmatrix} = \begin{pmatrix} e_\alpha \\ (\bar{e}^{\mathrm{c}})^{\dot{\alpha}} \end{pmatrix}, \qquad \psi_{e^{\mathrm{c}}} = \begin{pmatrix} e^{\mathrm{c}}_\alpha \\ (\bar{e})^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} e^{\mathrm{c}}_\alpha \\ (\bar{e}^\alpha)^* \end{pmatrix}, \qquad \overline{\psi}_e = \begin{pmatrix} e^{\mathrm{c}\alpha} & \bar{e}_{\dot{\alpha}} \end{pmatrix}, \qquad \overline{\psi}_{e^{\mathrm{c}}} = \begin{pmatrix} e^\alpha & \bar{e}^{\mathrm{c}}_{\dot{\alpha}} \end{pmatrix},$$

where $\bar{e}^{\mathrm{c}}$ should be read as "bar of e^{c}", namely $(\bar{e}^{\mathrm{c}})^{\dot{\alpha}} = ((e^{\mathrm{c}})^\alpha)^*$ and $(\bar{e})^{\dot{\alpha}} = (e^\alpha)^*$. Then

$$A^\alpha B_\alpha = \overline{\psi}_{A^{\mathrm{c}}} P_{\mathrm{L}} \psi_B = \overline{\psi}_{B^{\mathrm{c}}} P_{\mathrm{L}} \psi_A, \qquad \bar{A}_{\dot{\alpha}} \bar{B}^{\dot{\alpha}} = \overline{\psi}_A P_{\mathrm{R}} \psi_{B^{\mathrm{c}}} = \overline{\psi}_B P_{\mathrm{R}} \psi_{A^{\mathrm{c}}}, \tag{3.6}$$

$$\bar{A}\bar{\sigma}B = \overline{\psi}_A \gamma^\mu P_{\mathrm{L}} \psi_B = -B\sigma^\mu \bar{A} = -\overline{\psi}_{B^{\mathrm{c}}} \gamma^\mu P_{\mathrm{R}} \psi_{A^{\mathrm{c}}} = (\bar{B}\bar{\sigma}A)^* = (\overline{\psi}_B \gamma^\mu P_{\mathrm{L}} \psi_A)^* = -(\overline{\psi}_{A^{\mathrm{c}}} \gamma^\mu P_{\mathrm{R}} \psi_{B^{\mathrm{c}}})^*, \tag{3.7}$$

$$(\overline{\psi_A}\psi_B)^* = (\psi_B)^\dagger(\overline{\psi}_A)^\dagger = \overline{\psi_B}\psi_A = A^\alpha B_\alpha + \bar{A}^{\mathrm{c}}_{\dot{\alpha}} \bar{B}^{\mathrm{c}\dot{\alpha}}, \tag{3.8}$$

$$\overline{\psi}_A \gamma^\mu \psi_B = \overline{\psi}_A \gamma^\mu P_{\mathrm{L}} \psi_B + \overline{\psi}_A \gamma^\mu P_{\mathrm{R}} \psi_B = \overline{\psi}_A \gamma^\mu P_{\mathrm{L}} \psi_B - \overline{\psi}_{B^{\mathrm{c}}} \gamma^\mu P_{\mathrm{L}} \psi_{A^{\mathrm{c}}}. \tag{3.9}$$

> **Remark:** Dirac notation is given by
>
> $$\hat{\gamma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \hat{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ \bar{\sigma}^i & 0 \end{pmatrix},\ \hat{\gamma}_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \hat{P}_{\mathrm{L}} = \frac{1-\hat{\gamma}_5}{2},\ \hat{P}_{\mathrm{R}} = \frac{1+\hat{\gamma}_5}{2}. \tag{3.10}$$
>
> $$(\hat{\gamma}^\mu)^\dagger = \hat{\gamma}^0\hat{\gamma}^\mu\hat{\gamma}^0,\ (\hat{\gamma}^\mu)^* = \hat{\gamma}^2\hat{\gamma}^\mu\hat{\gamma}^2,\ (\hat{\gamma}^\mu)^{\mathrm{T}} = \hat{\gamma}^0\hat{\gamma}^2\hat{\gamma}^\mu\hat{\gamma}^2\hat{\gamma}^0, \tag{3.11}$$

Defining

$$S^i = \{1\}, \qquad V^i = \{\gamma^\mu\}, \qquad T^i = \{\sigma^{\mu\nu}\}, \qquad A^i = \{\mathrm{i}\gamma^\mu\gamma_5\}, \qquad P^i = \{\gamma_5\}, \tag{3.12}$$

$$S_i = \{1\}, \qquad V_i = \{\gamma_\mu\}, \qquad T_i = \{\sigma_{\mu\nu}\}, \qquad A_i = \{\mathrm{i}\gamma_\mu\gamma_5\}, \qquad P_i = \{\gamma_5\}, \tag{3.13}$$

$$\tag{3.14}$$

Fierz Transformation

$$(AD)(BC) = -\frac{1}{2}(AB)(CD) - \frac{1}{8}(A\sigma^{\mu\nu}B)(C\sigma_{\mu\nu}D) \tag{3.15}$$

$$(\bar{A}\bar{D})(\bar{B}\bar{C}) = -\frac{1}{2}(\bar{A}\bar{B})(\bar{C}\bar{D}) - \frac{1}{8}(\bar{A}\bar{\sigma}^{\mu\nu}\bar{B})(\bar{C}\bar{\sigma}_{\mu\nu}\bar{D}) \tag{3.16}$$

$$(AD)(\bar{B}\bar{C}) = -\frac{1}{2}(A\sigma^\mu\bar{B})(\bar{C}\bar{\sigma}_\mu D) \tag{3.17}$$

3.1 Verbose derivation

We provide a discussion on the fermion convention, introducing various signs $h_i = \pm 1$ in order to keep generality as much as possible. We follow Ref. [3].

Lorentz group and Lorentz tensors The Lorentz transformation $\Lambda^\mu{}_\nu$ is defined as a linear transformation $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ conserving $x^2 = \eta_{\mu\nu}x^\mu x^\nu$, where x^μ is a spacetime point and η is the Minkowski metric:

$$\eta^{\mu\nu} = \eta_{\mu\nu} \stackrel{\text{def}}{=} h_\eta \times \operatorname{diag}(+1,-1,-1,-1), \qquad \eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu_\nu; \qquad \eta_{\rho\sigma} \stackrel{\text{def}}{=} \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma \text{ (defining equation)}. \tag{3.18}$$

Consequently, $\delta^\beta_\sigma = (\eta_{\alpha\mu}\eta^{\beta\nu}\Lambda^\mu{}_\nu)\Lambda^\alpha{}_\sigma =: (\Lambda^{-1})^\beta{}_\alpha\Lambda^\alpha{}_\sigma$ should define the inverse of Λ:

$$\Lambda_\beta{}^\alpha := (\Lambda^{-1})^\beta{}_\alpha = \eta_{\alpha\mu}\eta^{\beta\nu}\Lambda^\mu{}_\nu; \qquad x_\nu \mapsto x_\mu(\Lambda^{-1})^\mu{}_\nu = \Lambda_\mu{}^\nu x_\nu, \qquad \delta^\alpha_\beta = \Lambda_\mu{}^\alpha\Lambda^\mu{}_\beta = \Lambda^\alpha{}_\mu\Lambda_\beta{}^\mu, \tag{3.19}$$

They form a group $L \cong \mathrm{O}(1,3)$ (Lorentz group), which has four disconnected parts:

$$L_0 = \{\Lambda \mid \det\Lambda = +1, \Lambda^0{}_0 \ge 1\} \cong \mathrm{SO}^+(1,3), \qquad L_\mathrm{P} = \{\Lambda \mid \det\Lambda = -1, \Lambda^0{}_0 \ge 1\},$$
$$L_\mathrm{T} = \{\Lambda \mid \det\Lambda = +1, \Lambda^0{}_0 \le -1\}, \qquad L_\mathrm{PT} = \{\Lambda \mid \det\Lambda = -1, \Lambda^0{}_0 \le -1\}. \tag{3.20}$$

Tensors $T^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots}$ and pseudo-tensors $\tilde T^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots}$ are objects that satisfy

$$T^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots} \mapsto \Lambda^{\mu_1}{}_{\alpha_1}\cdots\Lambda_{\nu_1}{}^{\beta_1}\cdots T^{\alpha_1\alpha_2\cdots}_{\beta_1\beta_2\cdots}, \qquad \tilde T^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots} \mapsto (\det\Lambda)\Lambda^{\mu_1}{}_{\alpha_1}\cdots\Lambda_{\nu_1}{}^{\beta_1}\cdots \tilde T^{\alpha_1\alpha_2\cdots}_{\beta_1\beta_2\cdots}. \tag{3.21}$$

In particular, metrics and the antisymmetric tensors are constant (pseudo-)tensors.

$$\eta^{\mu\nu} \mapsto \Lambda^\mu{}_\alpha\Lambda^\mu{}_\beta\eta^{\alpha\beta} = \eta^{\mu\nu}, \qquad \varepsilon^{\mu\nu\rho\sigma} \mapsto \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Lambda^\sigma{}_\delta\varepsilon^{\alpha\beta\gamma\delta} = (\det\Lambda)\varepsilon^{\mu\nu\rho\sigma}; \tag{3.22}$$
$$\varepsilon^{0123} := 1, \qquad \varepsilon_{0123} = -1. \tag{3.23}$$

Infinitesimal transformation In order to consider infinitesimal transformation of a proper orthochronous Lorentz transformation $\Lambda \in L_0$, we identify Λ as a matrix:

$$\Lambda^\mu{}_\nu \mathrel{\hat=} \tilde\Lambda \stackrel{\text{def}}{\iff} \Lambda^\mu{}_\nu = \tilde\Lambda_{\mu+1,\nu+1}; \qquad \tilde\eta := h_\eta \operatorname{diag}(+1,-1,-1,-1).$$

Then, Eq. (3.18) is read by $\tilde\eta = \tilde\Lambda^\mathrm{T}\tilde\eta\tilde\Lambda$ and the infinitesimal transformation given by $\tilde\Lambda = 1 + \tilde\lambda + \mathcal{O}(\lambda^2)$ must satisfy

$$\lambda^\mu{}_\nu = -\eta_{\nu\alpha}\eta^{\mu\beta}\lambda^\alpha{}_\beta \iff \tilde\lambda^\mathrm{T} = -\tilde\eta\tilde\lambda\tilde\eta. \qquad \therefore\quad \tilde\lambda = \begin{pmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ -\omega_x & 0 & +\theta_z & -\theta_y \\ -\omega_y & -\theta_z & 0 & +\theta_x \\ -\omega_z & +\theta_y & -\theta_x & 0 \end{pmatrix} =: \mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{K} + \mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{J}, \tag{3.24}$$

where we can interpret θ_i as the **passive** rotation angle about i-axis and ω_i as the **passive** boost along i-axis with velocity $\beta = \tanh\omega$. The operators J_i and K_i are defined to be pure-imaginary; J_i is Hermitian, K_i is pseudo-Hermitian, and

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k, \qquad [J_i, K_j] = \mathrm{i}\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk}J_k. \tag{3.25}$$

Noticing that $\tilde\eta\tilde\lambda$ is anti-symmetric, we can express the infinitesimal parameters by anti-symmetric coefficients

$$d^{\mu\nu} := h_d h_\eta \lambda^\mu{}_\alpha\eta^{\alpha\nu}; \qquad d^{\mu\nu} = -d^{\nu\mu}, \qquad \{d^{01}, d^{02}, d^{03}\} = h_d\boldsymbol{\omega}, \qquad \{d^{32}, d^{13}, d^{21}\} = h_d\boldsymbol{\theta}.$$

The corresponding generators are then given by anti-symmetric matrices

$$\{\tilde M_{01}, \tilde M_{02}, \tilde M_{03}\} = -h_M\boldsymbol{K}, \qquad \{\tilde M_{32}, \tilde M_{13}, \tilde M_{21}\} = -h_M\boldsymbol{J}; \qquad \tilde M \mathrel{\hat=} (M_{\rho\sigma})^\mu{}_\nu := h_M h_\eta \cdot \mathrm{i}(\delta^\mu_\rho\eta_{\nu\sigma} - \delta^\mu_\sigma\eta_{\nu\rho}).$$

The infinitesimal transformation is given by

$$\tilde\lambda = -h_M h_d \frac{\mathrm{i}}{2} d^{\mu\nu}\tilde M_{\mu\nu}. \tag{3.26}$$

The Lorentz algebra is given by

$$h_M h_\eta [M_{\mu\nu}, M_{\rho\sigma}] = -\mathrm{i}\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\right). \tag{3.27}$$

In literature, the sign h_η is first determined; h_M is fixed by the Lorentz algebra (3.27) and then h_d is set by (3.26).

Isomorphism of Lorentz Algebra The commutation relations of $\boldsymbol{J}$ and $\boldsymbol{K}$ lead to

$$\boldsymbol{A} \stackrel{\text{def}}{=} \frac{\boldsymbol{J} + \mathrm{i}\boldsymbol{K}}{2}, \qquad \boldsymbol{B} \stackrel{\text{def}}{=} \frac{\boldsymbol{J} - \mathrm{i}\boldsymbol{K}}{2}; \qquad [A_i, A_j] = \mathrm{i}\epsilon_{ijk}A_k, \qquad [B_i, B_j] = \mathrm{i}\epsilon_{ijk}B_k, \qquad [A_i, B_j] = 0. \tag{3.28}$$

This means $\mathfrak{so}(1,3)$ is somewhat similar to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, or in fact, as discussed in Section A.4, $\mathfrak{so}(1,3)_\mathbb{C} \cong \mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C}$. Explicitly,

$$\tilde M^\pm_{\alpha\beta} = \frac{\mathrm{i}}{2}\left(\pm M_{\alpha\beta} + \frac{-1}{2}\eta_{\alpha\mu}\eta_{\beta\nu}\mathrm{i}\epsilon^{\mu\nu\rho\sigma}M_{\rho\sigma}\right), \qquad \{\tilde M^+_{01}, \tilde M^+_{02}, \tilde M^+_{03}\} = -h_M\boldsymbol{A}, \qquad \{\tilde M^-_{01}, \tilde M^-_{02}, \tilde M^-_{03}\} = -h_M\boldsymbol{B}.$$

♠TODO: reviewed upto here♠

Representation of Clifford algebra To construct an irreducible representation of $\mathfrak{C}_{1,3}$, we utilize the fact that we can form two sets of creation-annihilation operators

$$a^\pm = \sqrt{h_\eta}\frac{e^0 \pm e^3}{2}, \quad b^\pm = \sqrt{h_\eta}\frac{\pm e^2 - \mathrm{i}e^1}{2}; \qquad \{a^+, a^-\} = 1,\ \{b^+, b^-\} = 1,\ \{(\text{others})\} = 0. \tag{3.29}$$

These ladder operator allows us to construct four states starting from $|00\rangle$, which is a non-zero state with $a^-|00\rangle = b^-|00\rangle = 0$, and to construct an irreducible representation of $\mathfrak{C}_{1,3}$ (and, in fact, it is unique for even dimension):

$$|10\rangle = a^+|00\rangle, \quad |01\rangle = b^+|00\rangle, \quad |11\rangle = a^+b^+|00\rangle \quad \rightsquigarrow \quad a^+ = \begin{pmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad b^+ = \begin{pmatrix} 0&-1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \tag{3.30}$$

and $a^- = (a^+)^\dagger$, $b^- = (b^+)^\dagger$. We then obtain a representation γ, which is called "standard representation."[*1] They are not Hermitian, but as we will see, this non-Hermiticity is solved by amending the inner product by a matrix β: $(\psi, \gamma^\mu\psi) := \psi^\dagger\beta\gamma^\mu\psi$.

Although ψ forms an irreducible representation γ^μ of $\mathfrak{C}_{1,3}$, the resulting representation $S_{\mu\nu}$ (see the next paragraph) is a reducible representation of $\mathrm{Spin}(1,3)^+$. This is confirmed by

$$\gamma_5\gamma_5 = 1, \quad \{\gamma_5, \gamma^\mu\} = 0, \quad [\gamma_5, S_{\mu\nu}] = 0; \qquad \gamma_5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3, \tag{3.31}$$

and $P^{\mathrm{L}}_{\mathrm{R}} = (1 \mp \gamma_5)/2$ works as the projection operators. In addition, four state are eigenstates of $J_3 = h_\eta S_{12}$ because

$$[J_3, b^+b^-] = 0,\ J_3 = b^+b^- - \frac{1}{2}, \tag{3.32}$$

which also guarantees that spinors have spin 1/2. In summary,

$$|00\rangle = |-_{\mathrm{L}}\rangle, \quad |10\rangle =: |-_{\mathrm{R}}\rangle, \quad |01\rangle =: |+_{\mathrm{R}}\rangle, \quad |11\rangle =: |+_{\mathrm{L}}\rangle; \tag{3.33}$$

$$J_3|\pm_{\mathrm{H}}\rangle = \pm\tfrac{1}{2}|\pm_{\mathrm{H}}\rangle, \quad P_{\mathrm{L}}|\pm_{\mathrm{L}}\rangle = |\pm_{\mathrm{L}}\rangle,\ P_{\mathrm{R}}|\pm_{\mathrm{R}}\rangle = |\pm_{\mathrm{R}}\rangle; \quad P_{\mathrm{L}}|\pm_{\mathrm{R}}\rangle = P_{\mathrm{R}}|\pm_{\mathrm{L}}\rangle = 0. \tag{3.34}$$

For example, in chiral notation with $(+,-,-,-)$, the Lorentz generators $S_{\mu\nu}$ are block diagonal and $|\pm_{\mathrm{L}}\rangle$ ($|\pm_{\mathrm{R}}\rangle$) has non-zero component only in the upper (lower) two component:

$$|-_{\mathrm{L}}\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ |-_{\mathrm{R}}\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ |+_{\mathrm{R}}\rangle = \begin{pmatrix}0\\0\\\mathrm{i}\\0\end{pmatrix},\ |+_{\mathrm{L}}\rangle = \begin{pmatrix}\mathrm{i}\\0\\0\\0\end{pmatrix}. \tag{3.35}$$

Four-spinors and Lorentz transformation The above "theoretical" discussion can be seen more explicitly, starting from spinors and a matrix representation γ^μ given by

$$\overline{\psi} = \psi^\dagger\beta, \quad \beta\beta = 1, \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}; \quad \psi \mapsto T\psi, \quad \overline{\psi} \mapsto \overline{\psi}\beta T^\dagger\beta; \quad T \in \mathrm{Spin}(1,3)^+. \tag{3.36}$$

For $\overline{\psi}\psi$ and $\overline{\psi}\gamma^\mu\psi$ to be respectively scalar and vector, T should satisfy

$$T^{-1}\gamma^\mu T = \Lambda^\mu{}_\nu\gamma^\nu, \quad \beta T^\dagger\beta T = 1, \tag{3.37}$$

or in infinitesimal form $T = 1 + (-\mathrm{i}/2)d^{\mu\nu}S_{\mu\nu}$,

$$(S_{\mu\nu})^\dagger = \beta S_{\mu\nu}\beta, \quad [S_{\mu\nu}, \gamma^\alpha] = -(M_{\mu\nu})^\alpha{}_\beta\gamma^\beta; \qquad \therefore S_{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\nu]; \quad [\gamma_\mu^\dagger, \gamma_\nu^\dagger] = \beta[\gamma_\mu, \gamma_\nu]\beta; \tag{3.38}$$

the first condition leads to a representation of the Lorentz group

$$\Lambda \overset{\text{rep}}{=} \exp\left(\frac{-\mathrm{i}}{2}d^{\mu\nu}S_{\mu\nu}\right); \qquad [S_{\mu\nu}, S_{\rho\sigma}] = -\mathrm{i}\left(\eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\nu\sigma}S_{\mu\rho}\right) \tag{3.39}$$

as seen in Eq. (3.27), while the second condition determines what β should be, as given in, e.g., Eq. (3.1) and Eq. (3.10).

Charge conjugation and Majorana spinor The charge conjugation of ψ is something like ψ^* but should obey the same representation as ψ does, i.e., $C'\psi^*$ with C' being a unitary matrix such that $B\psi^* \to TB\psi^*$. Or it can be seen that the, the previous procedure with ψ^* may generate another irreducible representation and it should be related to γ^μ by an unitary matrix. Anyway, we define the charge conjugation by

$$\psi^{\mathrm{c}} = C(\overline{\psi})^{\mathrm{T}} = C\beta^{\mathrm{T}}\psi^*; \quad CC^\dagger = 1. \quad \therefore C^*\beta[\gamma_\mu^\dagger, \gamma_\nu^\dagger]\beta C^{\mathrm{T}} = [\gamma_\mu^{\mathrm{T}}, \gamma_\nu^{\mathrm{T}}]. \tag{3.40}$$

Combining with Eq. (3.38),

$$\beta\beta = 1, \quad CC^\dagger = 1, \quad \beta\gamma^\mu\beta = h_\beta(\gamma^\mu)^\dagger, \quad C\gamma_\mu^* C^\dagger = h_C\gamma_\mu^\dagger. \tag{3.41}$$

In even dimensions, the expressions have many choices as seen in the signs h_C and h_β; moreover, the sign h_{cc} defined in $(\psi^{\mathrm{c}})^{\mathrm{c}} = h_{\mathrm{cc}}\psi$ depends on the definition. It is thus useful to use a specific notation for further discussion.

For example, the construction of a Majorana spinor ψ_{M}, which satisfies $(\psi_{\mathrm{M}})^{\mathrm{c}} = \psi_{\mathrm{M}}$, is simply done as $\psi_{\mathrm{M}} \propto \psi + \psi^{\mathrm{c}}$ if $\eta_{\mathrm{cc}} = 1$, but needs some phases if $\eta_{\mathrm{cc}} = -1$.

fragments Lorentz tensor $M^{\mu_1\mu_2\cdots\mu_n} \propto \bar{\sigma}^{\mu_1\dot{\beta}_1\alpha_1}\cdots M_{\alpha_1\cdots\dot{\beta}_1\cdots}$

Especially $V^\mu =: \frac{1}{2}\bar{\sigma}^{\mu\dot{\beta}\alpha}V_{\alpha\dot{\beta}}$, $V_{\alpha\dot{\beta}} = V^\mu\sigma_{\mu\alpha\dot{\beta}}$; hermite $V_{\alpha\dot{\beta}} \Leftrightarrow realV^\mu$.

$$(V^{\mathrm{T}})_{\alpha\dot{\beta}} = V_{\dot{\beta}\alpha}, \text{♠TODO:(correct? possibly wrong dot-positions?)♠} \tag{3.42}$$

$$(V^*)_{\dot{\alpha}\beta} := (V_{\alpha\dot{\beta}})^*, \tag{3.43}$$

$$(V^\dagger)_{\alpha\dot{\beta}} := (V_{\beta\dot{\alpha}})^* = (V^*)_{\dot{\beta}\alpha} \tag{3.44}$$

♠TODO:anyway not very sure about the reasoning; though my old note says like this...♠

[*1] The chiral notation (3.1) and Dirac notation (3.10) are equivalent to this standard representation, i.e., related by unitary matrices.

3.2 Dirac spinors for massive fermions

Dirac equation $(\mathrm{i}\partial\!\!\!/ - m)\psi(x) = 0$ has plain-wave solutions $\tilde{u}(p)\mathrm{e}^{-\mathrm{i}px}$, i.e., $(p\!\!\!/ - m)\tilde{u}(p) = 0$ with $m > 0$. Non-zero solution is available if and only if $\det(p\!\!\!/ - m) = 0 \Leftrightarrow p^2 = m^2$. We hereafter fix $p^0 > 0$ and consider

$$u(\boldsymbol{p}) \text{ satisfying } (p\!\!\!/ - m)u(\boldsymbol{p}) = 0, \qquad v(\boldsymbol{p}) \text{ satisfying } (p\!\!\!/ + m)v(\boldsymbol{p}) = 0, \tag{3.45}$$

where the general solution of the Dirac equation is given by linear combination of $\{u(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}, v(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}\}_{p^0>0}$.

As $\operatorname{rank}(p\!\!\!/ \pm m) = 2$, each equation has $4 - 2 =$ two linear-independent solutions[*2], for which we introduce another label $s = 1, 2$. We use the following convention, noting the inner product of this vector space is $(\psi, \psi) = \overline{\psi}\psi$:

$$v^s(\boldsymbol{p}) := C[\overline{u}^s(\boldsymbol{p})]^{\mathrm{T}}, \qquad \overline{u}^s(\boldsymbol{p})u^t(\boldsymbol{p}) := 2m\delta^{st}. \tag{3.46}$$

Then $\gamma^\mu v = (h_\beta/h_C)C\beta^{\mathrm{T}}(\gamma^\mu u)^*$, for which $v(\boldsymbol{p})$ satisfies $(p\!\!\!/ + m)v(\boldsymbol{p}) = 0$ as requested. Orthogonality for $s \neq t$ is guaranteed by its definition given later. Also, as u and v have different eigenvalues of the matrix $p\!\!\!/$,

$$\overline{v}^s(\boldsymbol{p})v^t(\boldsymbol{p}) = -2m\delta^{st}, \qquad \overline{u}^s(\boldsymbol{p})v^t(\boldsymbol{p}) = \overline{v}^s(\boldsymbol{p})u^t(\boldsymbol{p}) = 0. \tag{3.47}$$

Several conventions for the label s are available:

Pauli--Lubański pseudovector operator gives the most generic definition for the spin of a moving particle:

$$w^\mu = \frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}. \tag{3.48}$$

With a reference vector e^μ such that $e^2 = -1$ and $e^\mu p_\mu = 0$,

$$e_\mu w^\mu = e_\mu \frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}p_\nu \frac{\mathrm{i}}{4}[\gamma_\rho, \gamma_\sigma] = \frac{\mathrm{i}}{4m}\epsilon^{\mu\nu\rho\sigma}e_\mu p_\nu \gamma_\rho\gamma_\sigma = \frac{1}{2m}\gamma_5 e\!\!\!/ p\!\!\!/ \tag{3.49}$$

commutes with $(p\!\!\!/ \pm m)$ and thus $s = \pm$ may denote the eigenvalue of this operator $\pm 1/2$.

Helicity operator A simpler option is the eigenvalues ± 1 of the helicity operator $h = \boldsymbol{\sigma}\cdot\boldsymbol{p}/\|\boldsymbol{p}\| =: \boldsymbol{\sigma}\cdot\boldsymbol{n}$. This is in fact a special case of Pauli–Lubański operator with $e^\mu = (\|\boldsymbol{p}\|/m, p_0\boldsymbol{n}/m)$, which is verified by

$$h = \epsilon^{0\nu\rho\sigma}(-n_\nu)S_{\rho\sigma} = \frac{1}{2\mathrm{i}}\epsilon^{0\nu\rho\sigma}n_\nu\gamma_\rho\gamma_\sigma = \frac{\eta^{\mu 0}}{2\mathrm{i}}\epsilon_{\mu\nu\rho\sigma}n^\nu\gamma^\rho\gamma^\sigma = \frac{\gamma_5}{2}\gamma_5[n\!\!\!/, \gamma^0] = \frac{\gamma_5}{\|\boldsymbol{p}\|}(\{p\!\!\!/, \gamma^0\}/2 - \gamma^0 p\!\!\!/); \tag{3.50}$$

$$w_\mu e^\mu = \frac{1}{2\|\boldsymbol{p}\|}\gamma_5(p^0 - \gamma^0 p\!\!\!/) \sim \frac{h}{2}. \tag{3.51}$$

♠TODO:Discuss Poincare group before introducing PL operator.: [4], [5]♠

♠TODO:We have not defined the relation between $S_{\mu\nu}$ and σ_i etc.; verify for more general case, including the proof of $\epsilon_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma = -\mathrm{i}\gamma_5[\gamma_\mu, \gamma_\nu]$.♠

Projection operators for the subspaces spanned by u^s (v^s) are given by

$$P_u = \frac{m + p\!\!\!/}{2m}, \quad P_v = \frac{m - p\!\!\!/}{2m}. \tag{3.52}$$

As $\{u^1, u^2, v^1, v^2\}$ is a basis of the vector space $\mathbb{C}^4$ with inner product $(\psi, \psi) = \overline{\psi}\psi$, we immediately have

$$\sum_{s=1,2} u^s(\boldsymbol{p})\overline{u}^s(\boldsymbol{p}) = P_u\overline{u}^s(\boldsymbol{p})u^s(\boldsymbol{p}) = p\!\!\!/ + m, \qquad \sum_{s=1,2} v^s(\boldsymbol{p})\overline{v}^s(\boldsymbol{p}) = p\!\!\!/ - m. \tag{3.53}$$

Similarly, using projection operator $P'_\pm = (1 \pm \gamma_5 e\!\!\!/ p\!\!\!/ /m)/2$,

$$u^s(\boldsymbol{p})\overline{u}^s(\boldsymbol{p}) = 2mP'_\pm P_u = \frac{1}{2}(p\!\!\!/ + m)(1 + s\gamma_5 e\!\!\!/), \tag{3.54}$$

$$v^s(\boldsymbol{p})\overline{v}^s(\boldsymbol{p}) = C(u^s\overline{u}^s)^{\mathrm{T}}\beta^* C^\dagger\beta = \frac{1}{2}(m + h_C p\!\!\!/)(1 - s h_C\gamma_5 e\!\!\!/)C\beta^* C^\dagger\beta \dashrightarrow \frac{1}{2}(p\!\!\!/ - m)(1 + s\gamma_5 e\!\!\!/) \tag{3.55}$$

for $s = \pm 1$.

3.3 Chiral notation

In the chiral notation, $\beta = \gamma^0$ and $C = -\mathrm{i}\gamma^2\gamma^0$, and $h_\beta = 1$ and $h_C = -1$, the plain-wave spinors are given by

$$u^s(\boldsymbol{p}) = \begin{pmatrix} \sqrt{\sigma\cdot p}\,\xi^s \\ \sqrt{\bar{\sigma}\cdot p}\,\xi^s \end{pmatrix}, \quad v^s(\boldsymbol{p}) = \begin{pmatrix} \sqrt{\sigma\cdot p}\,\eta^s \\ -\sqrt{\bar{\sigma}\cdot p}\,\eta^s \end{pmatrix}; \qquad \eta^s = \begin{pmatrix} \eta^s_1 \\ \eta^s_2 \end{pmatrix} := \begin{pmatrix} -(\xi^s_2)^* \\ (\xi^s_1)^* \end{pmatrix}, \quad (\xi^s)^\dagger(\xi^t) = (\eta^s)^\dagger(\eta^t) = \delta^{st}, \tag{3.56}$$

[*2] Let $X^\pm = p\!\!\!/ \pm m$. Denote $\det A$ by $|A|$, and $\operatorname{rank} A$ by $\langle A\rangle$. As Lorentz invariance guarantees $|X^\pm|$ is a function of p^2, $|A^+| = |A^-|$. Then $|A^+||A^-| = (p^2 - m^2)^4$ gives $|A^\pm| = (p^2 - m^2)^2$. One can show that $\langle A^\pm\rangle = 2$ if $p^2 = m^2$ as follows:
Frobenius inequality: $\langle AB\rangle + \langle BC\rangle \le \langle B\rangle + \langle ABC\rangle \qquad \therefore 2\langle A^\pm\rangle = \langle A^\pm\gamma_5\gamma_5\rangle + \langle\gamma_5 A^\mp\gamma_5\rangle \le \langle\gamma_5\rangle + \langle A^\pm\gamma_5\gamma_5 A^\mp\gamma_5\rangle = 4$,
rank subadditivity: $\langle A + B\rangle \le \langle A\rangle + \langle B\rangle \qquad \therefore 4 \le \langle A^+\rangle + \langle -A^-\rangle$.

Similar discussion applies for Pauli–Lubański operator: as $[\det(\gamma_5 e\!\!\!/ - x)]^2 = -e^2 - x^2$, it has eigenvalues $\pm 1/2$ for $u(\boldsymbol{p})$ and $v(\boldsymbol{p})$, and $B^\pm = (\gamma_5 e\!\!\!/ p\!\!\!/ \pm m/2)$ both have rank-2. Furthermore, ♠needs proofreeding♠as B^+ should have solutions in both subspaces spanned by $u(\boldsymbol{p})$ and $v(\boldsymbol{p})$, each subspace is spanned by eigenstates with different eigenvalues for the Pauli–Lubański operator and thus the label is used to distinguish two us (and vs).

where $\{\xi^1,\xi^2\}$ are the orthonormal basis, which fixes the definition of η, and[*3]

$$\sqrt{\sigma\cdot p} := \frac{m+p^\mu\sigma_\mu}{\sqrt{2(m+p^0)}}, \qquad \sqrt{\bar\sigma\cdot p} := \frac{m+p^\mu\bar\sigma_\mu}{\sqrt{2(m+p^0)}}. \tag{3.57}$$

[*4] So that the basis ξ has eigenvalue $\pm1/2$ for the Pauli–Lubański operator, it should satisfy

$$\frac{1}{2}\gamma_5\not{e}U\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix} = \left(\pm\frac{1}{2}\right)U\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix}; \qquad \text{where } U := \begin{pmatrix}\sqrt{\sigma\cdot p} & 0\\ 0 & \sqrt{\bar\sigma\cdot p}\end{pmatrix}, \quad U^{-1} = \frac{1}{m}\begin{pmatrix}\sqrt{\bar\sigma\cdot p} & 0\\ 0 & \sqrt{\sigma\cdot p}\end{pmatrix}. \tag{3.60}$$

Thus, by calculating the eigensystem of $U^{-1}\gamma_5\not{e}U$, one can get

$$\xi^+ = \frac{1}{\sqrt{2(1-\phi^3)}}\begin{pmatrix}\phi^1-\mathrm{i}\phi^2\\ 1-\phi^3\end{pmatrix}, \quad \xi^- = \frac{1}{\sqrt{2(1-\phi^3)}}\begin{pmatrix}1-\phi^3\\ -\phi^1-\mathrm{i}\phi^2\end{pmatrix}; \qquad \phi^\mu = e^\mu - \frac{e^0p^\mu}{m+p^0}. \tag{3.61}$$

If we use reference vector for the helicity,

$$\xi^+ = \frac{1}{\sqrt{2-2n^3}}\begin{pmatrix}n^1-\mathrm{i}n^2\\ 1-n^3\end{pmatrix} = \begin{pmatrix}\mathrm{e}^{-\mathrm{i}\phi}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix}, \quad \xi^- = \frac{1}{\sqrt{2-2n^3}}\begin{pmatrix}1-n^3\\ -n^1-\mathrm{i}n^2\end{pmatrix} = \begin{pmatrix}\sin(\theta/2)\\ -\mathrm{e}^{\mathrm{i}\phi}\cos(\theta/2)\end{pmatrix} \tag{3.62}$$

with $(n^1,n^2,n^3) = (\mathrm{s}_\theta\mathrm{c}_\phi, \mathrm{s}_\theta\mathrm{s}_\phi, \mathrm{c}_\theta)$ is the direction of the momentum.

[*3]The notation is justified by

$$\sqrt{\sigma\cdot p}\sqrt{\sigma\cdot p} = \sigma\cdot p, \qquad \sqrt{\bar\sigma\cdot p}\sqrt{\bar\sigma\cdot p} = \bar\sigma\cdot p, \qquad \sqrt{\sigma\cdot p}\sqrt{\bar\sigma\cdot p} = m.$$

[*4]It is straightforward to verify the above-given equations:

$$\bar u u = 2m\delta^{st}, \quad \bar v v = -2m\delta^{st}, \quad \bar u v = \bar v u = 0, \quad u^\dagger u = v^\dagger v = 2p^0\delta^{st}, \tag{3.58}$$

$$u^\dagger(\boldsymbol{p})u(-\boldsymbol{p}) = v^\dagger(\boldsymbol{p})v(-\boldsymbol{p}) = 2m\delta^{st}, \quad u^\dagger(\boldsymbol{p})v(-\boldsymbol{p}) = v^\dagger(\boldsymbol{p})u(-\boldsymbol{p}) = 0, \quad \bar u(\boldsymbol{p})u(-\boldsymbol{p}) = -\bar v(\boldsymbol{p})v(-\boldsymbol{p}) = 2p^0\delta^{st}. \tag{3.59}$$

4 Gauge theory

SU(N) Fundamental rep. $\boldsymbol{N} \sim (\tau^a)_{ij}$ (Hermitian), $\overline{\boldsymbol{N}} \sim (-\tau^{a*})_{ij}$, and adjoint rep. **adj.** $\sim (f^a)^{bc}$.[*5]

$$\mathrm{Tr}(\tau_a\tau_b) = \frac{1}{2}\delta_{ab}, \qquad [\tau_a, \tau_b] = \mathrm{i} f_{abc}\tau_c, \qquad [\tau_a, [\tau_b, \tau_c]] = [[\tau_a, \tau_b], \tau_c] + [\tau_b, [\tau_a, \tau_c]],$$
$$f^{abc} = -2\mathrm{i}\,\mathrm{Tr}([\tau^a, \tau^b]\tau^c) \,:\, \text{real, anti-symmetric}, \qquad f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0.$$

$$\boldsymbol{N}_i \mapsto [\exp(\mathrm{i}g\theta^a\tau^a)]_{ij}\boldsymbol{N}_j \simeq \boldsymbol{N}_i + \mathrm{i}g\theta^a\tau^a_{ij}\boldsymbol{N}_j \tag{4.1}$$

$$\begin{aligned}\overline{\boldsymbol{N}}_i &\mapsto \overline{\boldsymbol{N}}_j[\exp(-\mathrm{i}g\theta^a\tau^a)]_{ji} = [\exp(-\mathrm{i}g\theta^a\tau^{a*})]_{ij}\overline{\boldsymbol{N}}_j \qquad (\text{i.e., } \overline{\boldsymbol{N}}_i = \boldsymbol{N}_i^*)\\ &\simeq \overline{\boldsymbol{N}}_j - \mathrm{i}g\theta^a\overline{\boldsymbol{N}}_j\tau^a_{ji} \simeq \overline{\boldsymbol{N}}_j - \mathrm{i}g\theta^a\tau^{a*}_{ij}\overline{\boldsymbol{N}}_j\end{aligned} \tag{4.2}$$

SU(2) Fundamental representation $\mathbf{2} \sim T^a \equiv \sigma^a/2$ and adjoint representation $\mathbf{3} \sim \epsilon^{abc}$.

$$\mathrm{Tr}(T_aT_b) = \frac{1}{2}\delta_{ab}, \qquad [T_a, T_b] = \mathrm{i}\epsilon_{abc}T_c, \qquad \overline{\mathbf{2}} = \mathbf{2}^* = \epsilon\mathbf{2} \quad (\because T^* = \epsilon T\epsilon = -\epsilon T\epsilon^{-1}),$$

where the last identity comes as follows:

$$\epsilon_{ij}\mathbf{2}_j \mapsto \epsilon_{ij}\left([\exp(\mathrm{i}g\theta^aT^a)]_{jk}\mathbf{2}_k\right) = [\epsilon\exp(\mathrm{i}g\theta^aT^a)\epsilon^{-1}]_{ij}\epsilon\mathbf{2}_j = [\exp(-\mathrm{i}g\theta^aT^{a*})]_{ij}(\epsilon_{jk}\mathbf{2}_k). \tag{4.3}$$

SU(3) Fundamental rep. $\mathbf{3} \sim \tau^a \equiv \lambda^a/2$, $\overline{\mathbf{3}} \sim (-\tau^{a*})$, and adjoint rep. $\mathbf{8} \sim (f^a)^{bc}$.

$$\begin{aligned}\mathbf{3}:\ &\phi_a \to [\exp(\mathrm{i}g\theta^\alpha\tau^\alpha)]_{ab}\phi_b, & \overline{\mathbf{3}}:\ &\phi_a \to [\exp(-\mathrm{i}g\theta^\alpha\tau^{\alpha*})]_{ab}\phi_b,\\ &\phi_a^* \to [\exp(-\mathrm{i}g\theta^\alpha\tau^{\alpha*})]_{ab}\phi_b^*, & &\phi_a^* \to [\exp(\mathrm{i}g\theta^\alpha\tau^\alpha)]_{ab}\phi_b^*.\end{aligned} \tag{4.4}$$

[*5] Upper and lower gauge indices are equivalent, while Lorentz indices and Weyl-spinor indices are different for super- and subscripts because they are raised/lowered by, e.g., metric tensors.

4.1 Gell-Mann matrices

Gell-Mann matrices and a Mathematica code to generate them are:

$$\lambda_{1\text{–}8} = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&-\mathrm{i}&0\\ \mathrm{i}&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&-\mathrm{i}\\0&0&0\\ \mathrm{i}&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&-\mathrm{i}\\0&\mathrm{i}&0 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{pmatrix}. \tag{4.5}$$

```
GellMann[0] := DiagonalMatrix[{1,1,1}]/Sqrt[3/2]
GellMann[8] := DiagonalMatrix[{1,1,-2}]/Sqrt[3]
GellMann[a:1|2|3|4|5|6|7] := Module[
   {p=Switch[a,1|2|3,{1,2,0},4|5,{1,0,2},6|7,{0,1,2}]},
   Table[If[i*j==0, 0, PauliMatrix[{1,2,3,1,2,1,2}[[a]]][[i,j]]], {i,p}, {j,p}]]
```

5 Calculation techniques

Polarization sum

$$\text{massless:} \quad \sum_{\pm} \epsilon^*_\mu(k)\epsilon_\nu(k) = -\eta_{\mu\nu} - \frac{n^2 k_\mu k_\nu}{(n\cdot k)^2} + \frac{n_\mu k_\nu + n_\nu k_\mu}{n\cdot k} \quad \left(\dashrightarrow -\eta_{\mu\nu} \quad \text{with Ward id.}\right), \tag{5.1}$$

$$\text{massive:} \quad \sum_{\pm,0} \epsilon^*_\mu(k)\epsilon_\nu(k) = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}, \tag{5.2}$$

where $\epsilon\cdot k = 0$ is assumed and n^μ should satisfy $\epsilon\cdot n = 0$, and $k\cdot n \neq 0$ (usually $n^\mu = (1,0,0,0)^\mu$). ♠TODO: derivation♠

6 Loop calculation

Notation follows `LoopTools` [6]; capital Ms and Ps respectively denote squared masses and momenta.

Passarino--Veltman scalar integrals

$$A_0(M)/M = \Delta_\epsilon + \log\mu^2 + 1 - \log M, \tag{6.1}$$

$$B_0(P, M_0, M_1) = \Delta_\epsilon + \log\mu^2 - \int_0^1 \mathrm{d}x\ \log\left[-x(1-x)P + xM_1 + (1-x)M_0\right] \tag{6.2}$$

$$C_0(P_1, P_2, P_3, M_1, M_2, M_3) = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \frac{x}{Q_1} \tag{6.3}$$

$$= \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\, \frac{1}{Q_2}; \tag{6.4}$$

$$Q_1 = x(1-x)(1-y)P_2 + x^2y(1-y)P_3 + x(1-x)yP_1 - xyM_1 - (1-x)M_2 - x(1-y)M_3,$$
$$Q_2 = -P_2x^2 - P_1y^2 + (P_1 + P_2 - P_3)xy + (P_2 - M_2 + M_3)x + (M_2 - M_1 + P_3 - P_2)y - M_3.$$

Kinematical invariance:

$$B_0(P, M_0, M_1) = B_0(P, M_1, M_0), \quad \begin{aligned} C_0(P_1, P_2, P_3, M_1, M_2, M_3) &= C_0(P_2, P_3, P_1, M_2, M_3, M_1) \\ &= C_0(P_1, P_3, P_2, M_2, M_1, M_3) \end{aligned} \tag{6.5}$$

Special cases:

$$\begin{aligned} C_0(0, P, P, M, M, M') &= \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\, \frac{-1}{Px^2 - (P - M + M')x + M'}; \\ &= \int_0^1 \mathrm{d}x\, \frac{-x/P}{(x-\alpha)^2 - \lambda(P, M, M')/4P^2}; \qquad \alpha = (P - M + M')/2P. \end{aligned} \tag{6.6}$$

```
Q1 = {x(1-x)(1-y), x^2 y(1-y), x(1-x)y, -x y, -(1-x), -x(1-y)}.{P2, P3, P1, M1, M2, M3}
Q2 = -P2 x^2 - P1 y^2 + (P1+P2-P3) x y + (P2-M2+M3)x + (M2-M1+P3-P2)y - M3
```

6.1 Passarino--Veltman scalar integrals

See `calculator/loop/PaVeAnalytic.wl` for validation. We use the notation [7, 6]

$$\Delta_\epsilon = \frac{2}{4-d} - \gamma + \log 4\pi \equiv \texttt{GetDelta[]} \quad (= 0 \text{ in } \overline{\text{MS}}), \qquad \mu^2 \equiv \texttt{GetMudim[]}, \tag{6.7}$$

where μ is introduced due to the different mass dimension of vector and spinor fields in d-dimensional theory:

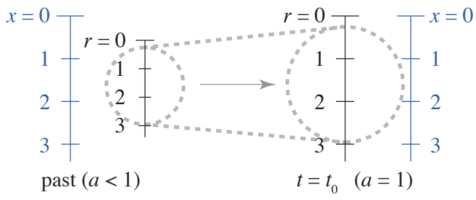
$$[A_\mu] = 1 - \frac{4-d}{2}, \quad [\psi] = \frac{3}{2} - \frac{4-d}{2}, \quad [\text{gauge couplings}] = \frac{4-d}{2} \quad \Rightarrow \quad e = (e)_{\text{4-dim.}}\mu^{(4-d)/2}. \tag{6.8}$$

The analytic form of scalar integrals are given in Refs. [8, 9, 7].

7 Cosmology

FLRW metric With a scale factor normalized by $a(t_0) = 1$,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right] \tag{7.1}$$



$$\begin{aligned}
&\text{comoving coordinate} && \boldsymbol{r} = (r\mathrm{s}_\theta\mathrm{c}_\phi, r\mathrm{s}_\theta\mathrm{c}_\phi, r\mathrm{c}_\theta),\\
&\text{proper coordinate} && \boldsymbol{x}(t) = a(t)\boldsymbol{r},\\
&\text{comoving distance} && \chi_{AB} = \int_{r_A}^{r_B}\frac{\mathrm{d}r}{\sqrt{1-Kr^2}},\\
&\text{proper distance} && d_{AB}(t) = a(t)\chi_{AB}.
\end{aligned}$$

Ricci tensor and scalar are given by

$$\begin{aligned}
&R_{00} = R^0{}_0 = \frac{3\ddot{a}}{a}, && R_{0i} = R_{i0} = R^0{}_i = R^i{}_0 = 0, \qquad R_{ij} \neq 0,\\
&R^i{}_j = \delta^i_j\left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2K}{a^2}\right); && R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right).
\end{aligned} \tag{7.2}$$

Particle density For a massless particle, with $L^\pm_n = \pm\,\texttt{PolyLog}(n, \pm\mathrm{e}^{\mu/T})$ and arrows denoting $\mu \to 0$,

$$n_{\mathrm{MB}} = \frac{\mathrm{e}^{\mu/T}}{\pi^2}gT^3 \quad \to \frac{1}{\pi^2}gT^3, \qquad \rho_{\mathrm{MB}} = 3Tn_{\mathrm{MB}} \quad \to \frac{3}{\pi^2}gT^4, \tag{7.3}$$

$$n_{\mathrm{BE}} = \frac{L^+_3}{\pi^2}gT^3 \quad \to \frac{\zeta_3}{\pi^2}gT^3, \qquad \rho_{\mathrm{BE}} = \frac{3L^+_4}{\pi^2}gT^4 \quad \to \frac{\pi^2}{30}gT^4, \tag{7.4}$$

$$n_{\mathrm{FD}} = \frac{L^-_3}{\pi^2}gT^3 \quad \to \frac{3}{4}\frac{\zeta_3}{\pi^2}gT^3, \qquad \rho_{\mathrm{FD}} = \frac{3L^-_4}{\pi^2}gT^4 \quad \to \frac{7}{8}\frac{\pi^2}{30}gT^4, \tag{7.5}$$

For massive particle, with $x = m/T$ and $K_n(x) = \texttt{BesselK}(n, x)$,

$$n_{\mathrm{MB}} = g\mathrm{e}^{\mu/T}\cdot\frac{T^3}{2\pi^2}x^2K_2(x) \quad \xrightarrow{x\gg1} g\mathrm{e}^{\mu/T}\frac{T^3}{(2\pi)^{3/2}}x^{3/2}\mathrm{e}^{-x}, \tag{7.6}$$

$$\rho_{\mathrm{MB}} = \left(3 + \frac{xK_1(x)}{K_2(x)}\right)Tn_{\mathrm{MB}} \quad \xrightarrow{x\gg1} \left(m + \frac{3}{2}T + \frac{15T^2}{8m}\right)n_{\mathrm{MB}}, \qquad p_{\mathrm{MB}} = Tn_{\mathrm{MB}}. \tag{7.7}$$

7.1 FLRW metric

Two conventions are known for FLRW (Фри́дман-Lemaître-Robertson-Walker) metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right] \qquad [r] = \text{(length)}, a \text{ is unitless with } a(t_0)=1 \tag{7.8}$$

$$= \mathrm{d}t^2 - R^2(t)\left[\frac{\mathrm{d}\tilde{r}^2}{1-\tilde{K}\tilde{r}^2} + \tilde{r}^2\mathrm{d}\theta^2 + \tilde{r}^2\sin^2\theta\mathrm{d}\phi^2\right] \qquad [R] = \text{(length)}, \tilde{r} \text{ is unitless}, \tilde{K}=\{0,\pm1\} \tag{7.9}$$

related by a rescaling, $R(t)/a(t) = R(t_0) \equiv R_0$, i.e., $r = \tilde{r}R_0$ and $K = \tilde{K}/R_0^2$. The curvature radius is given by $6K/a^2$ and a spherical, flat, and hyperspherical universe are respectively given by $K>0$, $K=0$, and $K<0$.

FLRW metric can have several forms. For $\{K>0, K=0, K<0\}$,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\Omega\right) \qquad \mathrm{d}\Omega = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2, \tag{7.10}$$

$$= \mathrm{d}t^2 - a^2(t)\left[\mathrm{d}\boldsymbol{r}^2 + \frac{K(\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r})^2}{1-K\|\boldsymbol{r}\|^2}\right] \qquad \boldsymbol{r} = (r\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi) \tag{7.11}$$

$$= \mathrm{d}t^2 - \left[\frac{a(t)}{1+(K/4)\rho^2}\right]^2(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega) \qquad \rho = R_0\tilde{\rho} := \frac{2r}{1+\sqrt{1-Kr^2}} = \frac{2\tilde{r}R_0}{1+\sqrt{1-\tilde{K}\tilde{r}^2}} \tag{7.12}$$

$$= \mathrm{d}t^2 - \left[\frac{R(t)}{1+(\tilde{K}/4)\tilde{\rho}^2}\right]^2(\mathrm{d}\tilde{\rho}^2 + \tilde{\rho}^2\mathrm{d}\Omega) \tag{7.13}$$

$$= \mathrm{d}t^2 - R^2(t)\Big(\mathrm{d}\tilde{\chi}^2 + \{\sin\tilde{\chi}, \tilde{\chi}, \sinh\tilde{\chi}\}^2\mathrm{d}\Omega\Big) \qquad \mathrm{d}\chi = R_0\mathrm{d}\tilde{\chi} = \frac{\mathrm{d}r'}{\sqrt{1-Kr'^2}} \quad \text{[comoving distance]} \tag{7.14}$$

$$= a^2(t)\Big(\mathrm{d}\eta^2 - \mathrm{d}\chi^2 - R_0^2\{\sin\tilde{\chi}, \tilde{\chi}, \sinh\tilde{\chi}\}^2\mathrm{d}\Omega\Big) \qquad \mathrm{d}\eta := \frac{\mathrm{d}t'}{a(t')} \quad \text{[conformal time]}. \tag{7.15}$$

Explicitly, χ is given by

$$\chi = \int_0^r \frac{\mathrm{d}r'}{\sqrt{1-Kr'^2}} = \int_0^{\tilde{r}} \frac{R_0\mathrm{d}\tilde{r}'}{\sqrt{1-\tilde{K}\tilde{r}'^2}} = R_0\{\sin^{-1}\tilde{r}, \tilde{r}, \sinh^{-1}\tilde{r}\} = 2R_0\left\{\tan^{-1}\frac{\tilde{\rho}}{2}, \frac{\tilde{\rho}}{2}, \tanh^{-1}\frac{\tilde{\rho}}{2}\right\}. \tag{7.16}$$

The Christofffel symbol, Riemann tensor, Ricci tensor, and Ricci scalar are given by

$$\Gamma^n{}_{ij} = \frac{g^{nk}}{2}(g_{jk,i} + g_{ik,j} - g_{ij,k}), \qquad R_{ijk}{}^l = \Gamma^l{}_{jk,i} - \Gamma^l{}_{ik,j} + \Gamma^a{}_{jk}\Gamma^l{}_{ai} - \Gamma^a{}_{ik}\Gamma^l{}_{aj}{}^{*6}, \qquad R_{ij} = R_{ikj}{}^k, \qquad R = g^{ij}R_{ij}.$$

7.2 Particle cosmology

The particle number density, pressure, and energy density are calculated from distribution functions:

$$f_{\mathrm{MB}}(\boldsymbol{k}) = \frac{g}{\mathrm{e}^{(E-\mu)/T}}, \qquad f_{\mathrm{BE}}(\boldsymbol{k}) = \frac{g}{\mathrm{e}^{(E-\mu)/T}-1}, \qquad f_{\mathrm{FD}}(\boldsymbol{k}) = \frac{g}{\mathrm{e}^{(E-\mu)/T}+1}; \tag{7.17}$$

$$n = \int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}f(\boldsymbol{k}), \qquad \rho = \int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}Ef(\boldsymbol{k}), \qquad p = \int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}k_zv_zf(\boldsymbol{k}) = \int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{k^2\cos^2\theta}{E}f(\boldsymbol{k}). \tag{7.18}$$

Note the pressure is (momentum)×(flux per time) on a "wall"; assuming MB, $p=\rho/3$ for $m \ll T$ and $p = T\rho/m$ for $m \gg T$.

A thermal average of a cross section $\sigma(s)$ is schematically given by

$$\langle\sigma v\rangle_{AB\to12\cdots n}(T) = \frac{1}{n_An_B}\int\frac{\mathrm{d}^3\boldsymbol{k}_A}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{k}_B}{(2\pi)^3}(f_Af_B)\Big\{\phi_1\cdots\phi_n\sigma(s)\Big\}v_{\mathrm{Møl}}; \qquad \phi_X = \mathrm{e}^{(E-\mu)/T}f_X/g, \tag{7.19}$$

Here, the final state statistical factor $\phi_1\cdots\phi_n$ are subject to the phase space integral of the calculation of $\sigma(s)$. They are specifically given by $\phi_{\mathrm{MB}}=1$, $\phi_{\mathrm{BE}} = 1+f_{\mathrm{BE}}/g$, and $\phi_{\mathrm{FD}} = 1-f_{\mathrm{FD}}/g$. Similarly, a thermal averaged decay rate is given by

$$\langle\Gamma\rangle_{A\to12\cdots n} = \frac{1}{n_A}\int\frac{\mathrm{d}^3\boldsymbol{k}_A}{(2\pi)^3}f_A\left\{\phi_1\cdots\phi_n\frac{m_A}{E_A}\Gamma\right\}. \tag{7.20}$$

With MB approximation,

$$\langle\sigma v\rangle = \frac{g_Ag_B}{n_An_B}\mathrm{e}^{(\mu_A+\mu_B)/T}\int\frac{\mathrm{d}^3\boldsymbol{k}_A}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{k}_B}{(2\pi)^3}\mathrm{e}^{-(E_A+E_B)/T}\sigma(s)v_{\mathrm{Møl}} \tag{7.21}$$

$$= \int\frac{\mathrm{d}s\mathrm{d}E_+\mathrm{d}E_-}{32m_A^2m_B^2T^2K_2(m_A/T)K_2(m_B/T)}\mathrm{e}^{-E_+/T}4E_AE_B\sigma(s)v_{\mathrm{Møl}} \quad (\times1/2 \text{ if } A=B) \tag{7.22}$$

$$= \frac{1}{16m_A^2m_B^2TK_2(m_A/T)K_2(m_B/T)}\int\frac{K_1(\sqrt{s}/T)\mathrm{d}s}{\sqrt{s}}\sqrt{\lambda(s,m_A^2,m_B^2)}\cdot 2E_A2E_Bv_{\mathrm{Møl}}\sigma(s) \quad (\times1/2), \tag{7.23}$$

$$\langle\Gamma\rangle = \frac{K_1(m_A/T)}{K_2(m_A/T)}\Gamma. \tag{7.24}$$

[*6] Overall sign is convention-dependent.

8 Standard Model

(summary page)

8.1 Particle content and convention

8.2 Lagrangian

8.3 Higgs mechanism

A general expression for composing a Dirac fermion from $\psi_{\mathrm{L}}(T_{3\mathrm{L}}, Y_{\mathrm{L}})$ and $\psi_{\mathrm{R}}(T_{3\mathrm{R}}, Y_{\mathrm{R}})$ is given by

$$
\begin{aligned}
&(g_2 \not{W}_3 T_{3\mathrm{L}} + g_Y \not{B} Y_{\mathrm{L}}) P_{\mathrm{L}} + (g_2 \not{W}_3 T_{3\mathrm{R}} + g_Y \not{B} Y_{\mathrm{R}}) P_{\mathrm{R}} && (8.1)\\
&= \Big[\big(|e|\not{A} + g_Z \mathrm{c}_{\mathrm{W}}^2 \not{Z}\big) T_{3\mathrm{L}} + \big(|e|\not{A} - g_Z \mathrm{s}_{\mathrm{W}}^2 \not{Z}\big) Y_{\mathrm{L}}\Big] P_{\mathrm{L}} + (\text{right}) && (8.2)\\
&= \frac{T_{3\mathrm{L}} + T_{3\mathrm{R}} + Y_{\mathrm{L}} + Y_{\mathrm{R}}}{2} |e|\not{A} + \frac{T_{3\mathrm{L}}\mathrm{c}_{\mathrm{W}}^2 - Y_{\mathrm{L}}\mathrm{s}_{\mathrm{W}}^2 + T_{3\mathrm{R}}\mathrm{c}_{\mathrm{W}}^2 - Y_{\mathrm{R}}\mathrm{s}_{\mathrm{W}}^2}{2} g_Z \not{Z} \\
&\quad + \frac{-T_{3\mathrm{L}} - Y_{\mathrm{L}} + T_{3\mathrm{R}} + Y_{\mathrm{R}}}{2} |e|\not{A}\gamma_5 + \frac{-\mathrm{c}_{\mathrm{W}}^2 T_{3\mathrm{L}} + \mathrm{s}_{\mathrm{W}}^2 Y_{\mathrm{L}} + \mathrm{c}_{\mathrm{W}}^2 T_{3\mathrm{R}} - \mathrm{s}_{\mathrm{W}}^2 Y_{\mathrm{R}}}{2} g_Z \not{Z}\gamma_5. && (8.3)
\end{aligned}
$$

In the SM, $T_{3\mathrm{L}} + Y_{\mathrm{L}} = Y_{\mathrm{R}} =: Q$ and $T_{3\mathrm{R}} = 0$ lead to

$$
Q|e|\not{A} + g_Z \not{Z}\big(T_{3\mathrm{L}} P_{\mathrm{L}} - Q\mathrm{s}_{\mathrm{W}}^2\big). \tag{8.4}
$$

8.4 Lagrangian in mass eigenstates

8.5 CKM matrix and Yukawa convention

We use the following convention for the Yukawa interaction terms:

$$\mathcal{L}_{\text{Yukawa}} = \overline{U}Y_u H P_{\mathrm{L}} Q - \overline{D}Y_d H^\dagger P_{\mathrm{L}} Q - \overline{E}Y_e H^\dagger P_{\mathrm{L}} L + \text{h.c.} \tag{8.5}$$

$$= \overline{U_i} Y_{uij} \epsilon^{ab} H^a P_{\mathrm{L}} Q^b_j - \overline{D_i} Y_{dij} H^{a*} P_{\mathrm{L}} Q^a_j - \overline{E_i} Y_{eij} H^{a*} P_{\mathrm{L}} L^a_j + \text{h.c.} \tag{8.6}$$

$$= -\overline{Q^a} Y_u^\dagger \epsilon^{ab} H^{b*} P_{\mathrm{R}} U - \overline{Q^a} Y_d^\dagger H^a P_{\mathrm{R}} D - \overline{L^a} Y_e^\dagger H^a P_{\mathrm{R}} E + \text{h.c.}, \tag{8.7}$$

where the last equality uses $(\overline{\psi_A} P_{\mathrm{L}} \psi_B)^* = \overline{\psi_B} P_{\mathrm{R}} \psi_A$.

These terms are diagonalized by the singular value decomposition $Y = UY^{\text{diag}}V^\dagger$ (see Section A.3):

$$\mathcal{L}_{\text{Yukawa}} = \epsilon^{ab}\overline{U}U_u Y_u^{\text{diag}} H^a P_{\mathrm{L}} V_u^\dagger Q^b - \overline{D}U_d Y_d^{\text{diag}} H^{a*} P_{\mathrm{R}} V_d^\dagger Q^a - \overline{E}U_e Y_e^{\text{diag}} H^{a*} P_{\mathrm{R}} V_e^\dagger L^a + \text{h.c.} \tag{8.8}$$

$$\dashrightarrow -\frac{v}{\sqrt{2}}\overline{E}U_u Y_u^{\text{diag}} V_u^\dagger P_{\mathrm{L}} Q^1 - \frac{v}{\sqrt{2}}\overline{D}U_d Y_d^{\text{diag}} V_d^\dagger P_{\mathrm{L}} Q^2 - \frac{v}{\sqrt{2}}\overline{E}U_e Y_e^{\text{diag}} V_e^\dagger P_{\mathrm{L}} L^2 + \text{h.c.} \tag{8.9}$$

under the EWSB with $v \simeq 246\,\text{GeV}$. Mass eigenstates are

$$\{Q^1, Q^2, L, \overline{U}, \overline{D}, \overline{E}\}^{\text{mass basis}} = \{V_u^\dagger Q^1, V_d^\dagger Q^2, V_e^\dagger L, \overline{U}U_u, \overline{D}U_d, \overline{E}U_e\} \tag{8.10}$$

and, since Q^1 and Q^2 are rotated by different matrices, the weak interaction receives flavor violation.amended as

$$\mathcal{L} \supset \overline{Q}\mathrm{i}\gamma^\mu(-\mathrm{i}g_2 W_\mu)P_{\mathrm{L}} Q \supset \frac{g_2}{\sqrt{2}}\left[\overline{Q^1}W^+ P_{\mathrm{L}} Q^2 + \overline{Q^2}W^- P_{\mathrm{L}} Q^1\right] \tag{8.11}$$

$$= \frac{g_2}{\sqrt{2}}\left[\overline{(Q^1)^{\text{mass}}}V_u^\dagger W^+ P_{\mathrm{L}} V_d (Q^2)^{\text{mass}} + \overline{(Q^2)^{\text{mass}}}V_d^\dagger W^- P_{\mathrm{L}} V_u (Q^1)^{\text{mass}}\right] \tag{8.12}$$

$$= \frac{g_2}{\sqrt{2}}\left[\overline{(Q^1)^{\text{mass}}}V_{\text{CKM}} W^+ P_{\mathrm{L}} (Q^2)^{\text{mass}} + \overline{(Q^2)^{\text{mass}}}V_{\text{CKM}}^\dagger W^- P_{\mathrm{L}} (Q^1)^{\text{mass}}\right], \tag{8.13}$$

where the CKM matrix are defined with positive angles:

$$V_{\text{CKM}} = V_u^\dagger V_d = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb}\end{pmatrix} = \begin{pmatrix} 1 & & \\ & \mathrm{c}_{23} & \mathrm{s}_{23} \\ & -\mathrm{s}_{23} & \mathrm{c}_{23}\end{pmatrix}\begin{pmatrix} \mathrm{c}_{13} & & \mathrm{s}_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ & 1 & \\ -\mathrm{s}_{13}\mathrm{e}^{\mathrm{i}\delta} & & \mathrm{c}_{13}\end{pmatrix}\begin{pmatrix} \mathrm{c}_{12} & \mathrm{s}_{12} & \\ -\mathrm{s}_{12} & \mathrm{c}_{12} & \\ & & 1\end{pmatrix} \tag{8.14}$$

$$= \begin{pmatrix} \mathrm{c}_{12}\mathrm{c}_{13} & \mathrm{s}_{12}\mathrm{c}_{13} & \mathrm{s}_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ -\mathrm{s}_{12}\mathrm{c}_{23} - \mathrm{c}_{12}\mathrm{s}_{23}\mathrm{s}_{13}\mathrm{e}^{\mathrm{i}\delta} & \mathrm{c}_{12}\mathrm{c}_{23} - \mathrm{s}_{12}\mathrm{s}_{23}\mathrm{s}_{13}\mathrm{e}^{\mathrm{i}\delta} & \mathrm{s}_{23}\mathrm{c}_{13} \\ \mathrm{s}_{12}\mathrm{s}_{23} - \mathrm{c}_{12}\mathrm{c}_{23}\mathrm{s}_{13}\mathrm{e}^{\mathrm{i}\delta} & -\mathrm{c}_{12}\mathrm{s}_{23} - \mathrm{s}_{12}\mathrm{c}_{23}\mathrm{s}_{13}\mathrm{e}^{\mathrm{i}\delta} & \mathrm{c}_{23}\mathrm{c}_{13}\end{pmatrix} \qquad [\mathrm{s}_{ij} > 0,\ \mathrm{c}_{ij} > 0]. \tag{8.15}$$

Here in the Standard Model, five phases in a unitary matrix (A.17) are removed by rotating fermion phases.

PDG convention [PDG2018, §12][PDG2020, §12]

$$\mathcal{L} \supset -Y_{ij}^d \overline{Q_{Li}^I}\phi d_{Rj}^I - Y_{ij}^u \overline{Q_{Li}^I}\epsilon\phi^* u_{Rj}^I, \qquad Y^{\text{diag}} = V_L Y V_R^\dagger, \qquad V_{\text{CKM}} = V_L^u V_L^{d\dagger}. \tag{8.16}$$

So, $Y^u = Y_u^\dagger$, $Y^d = Y_d^\dagger$; $Y^{\text{diag}} = V_R Y^\dagger V_L^\dagger = V_R Y V_L^\dagger$ leads $V_L = V^\dagger$, and the CKM matrix (and components) is in the same convention: $V_{\text{CKM}} = V_u^\dagger V_d = V_{\text{CKM}}$.

SLHA2 convention [10]

$$W \supset \epsilon_{ab}\left[(Y_E)_{ij}H_1^a L_i^b \bar{E}_j + (Y_D)_{ij}H_1^a Q_i^b \bar{D}_j + (Y_U)_{ij}H_2^b Q_i^a \bar{U}_j\right]; \tag{8.17}$$

$$\mathcal{L} \supset -\epsilon_{ab}\left[(Y_E)_{ij}H_1^a \psi_{Li}^b \psi_{\bar{E}j} + (Y_D)_{ij}H_1^a \psi_{Qi}^b \bar{\psi}_{\bar{D}j} + (Y_U)_{ij}H_2^b \psi_{Qi}^a \bar{\psi}_{\bar{U}j}\right] \tag{8.18}$$

$$\dashrightarrow -\left[\psi_{\bar{E}} v_{\mathrm{d}} Y_E^{\mathrm{T}} \psi_L^2 + \psi_{\bar{D}} v_{\mathrm{d}} Y_D^{\mathrm{T}} \psi_Q^2 + \psi_{\bar{U}} v_{\mathrm{u}} Y_U^{\mathrm{T}} \psi_Q^1\right]; \qquad Y^{\text{diag}} = U^\dagger Y^{\mathrm{T}} V, \qquad V_{\text{CKM}} = V_u^\dagger V_d. \tag{8.19}$$

Hence, $Y_E = Y_e^{\mathrm{T}}$, $Y_D = Y_d^{\mathrm{T}}$, $Y_U = Y_u^{\mathrm{T}}$; $Y^{\text{diag}} = U^\dagger Y V$, $V = V$ and $V_{\text{CKM}} = V_{\text{CKM}}$.

Wolfenstein parameterization The CKM matrix is precisely written in terms of λ, A, and $\bar\rho + \mathrm{i}\bar\eta$.

$$\lambda := \mathrm{s}_{12} = \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}, \qquad A := \frac{\mathrm{s}_{23}}{\lambda^2} = \lambda^{-1}\left|\frac{V_{cb}}{V_{us}}\right|, \qquad \bar\rho + \mathrm{i}\bar\eta := \frac{-V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}. \tag{8.20}$$

They are independent of the phase convention and used for SLHA2 input, i.e., `VCKMIN` should contain $(\lambda, A, \bar\rho, \bar\eta)$.

Also, $\bar\rho + \mathrm{i}\eta$ is approximately written by

$$R = \rho + \mathrm{i}\eta := \frac{s_{13}\mathrm{e}^{\mathrm{i}\delta}}{A\lambda^3} = \frac{V_{ub}^*}{A\lambda^3}\frac{V_{ud}}{|V_{ud}|} = \frac{(\bar\rho + \mathrm{i}\bar\eta)\sqrt{1 - A^2\lambda^4}}{\sqrt{1-\lambda^2}\,[1 - A^2\lambda^4(\bar\rho + \mathrm{i}\bar\eta)]} = (\bar\rho + \mathrm{i}\bar\eta)\left(1 + \frac{\lambda^2}{2} + \mathcal{O}(\lambda^4)\right), \tag{8.21}$$

with which

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3 R^* \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - R) & -A\lambda^2 & 1\end{pmatrix}\mathrm{e}^{\mathrm{i}\Theta} + \begin{pmatrix} \mathcal{O}(\lambda^4) & \mathcal{O}(\lambda^7) & 0 \\ \mathcal{O}(\lambda^5) & \mathcal{O}(\lambda^4) & \mathcal{O}(\lambda^8) \\ \mathcal{O}(\lambda^5) & \mathcal{O}(\lambda^4) & \mathcal{O}(\lambda^4)\end{pmatrix}. \tag{8.22}$$

8.6 General Higgs doublet and Nambu--Goldstone bosons

In linear parameterization,

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i}\sqrt{2}\phi^+ \\ v+h+\mathrm{i}\phi_3 \end{pmatrix}, \qquad D_\mu H = \begin{pmatrix} \mathrm{i}\partial_\mu\phi^+ - \frac{\mathrm{i}g_2}{2}(v+h+\mathrm{i}\phi_3)W^+_\mu + \left(|e|A_\mu + \frac{c_W^2-s_W^2}{2}g_Z Z_\mu\right)\phi^+ \\ \partial_\mu(h+\mathrm{i}\phi_3)/\sqrt{2} + \frac{\mathrm{i}g_Z}{2}Z_\mu(v+h+\mathrm{i}\phi_3)/\sqrt{2} + g_2 W^-_\mu\phi^+/\sqrt{2} \end{pmatrix}; \tag{8.23}$$

$$\begin{aligned} |D_\mu H|^2 = {} & \frac{(\partial_\mu h)^2 + (\partial_\mu\phi_3)^2}{2} + \partial_\mu\phi^+\partial^\mu\phi^- + \frac{(v+h)^2}{8}(2g_2^2 W^{+\mu}W^-_\mu + g_Z^2 Z^\mu Z_\mu) \\ & + \frac{\partial^\mu h}{2}\left[g_2 W^+_\mu\phi^- + g_2 W^-_\mu\phi^+ - g_Z Z_\mu\phi_3\right] + \frac{\partial^\mu\phi_3}{2}\left[g_Z(v+h)Z_\mu + \mathrm{i}g_2(W^+_\mu\phi^- - W^-_\mu\phi^+)\right] \\ & + \left\{\frac{\partial^\mu\phi^+}{2}\left[-g_2(v+h-\mathrm{i}\phi_3)W^-_\mu + (2|e|A_\mu + (c_W^2 - s_W^2)g_Z Z_\mu)\mathrm{i}\phi^-\right] + \text{H.c.}\right\} \\ & + \frac{\mathrm{i}g_2(v+h)}{2}(|e|A^\mu - g_Z s_W^2 Z^\mu)(W^-_\mu\phi^+ - W^+_\mu\phi^-) + \frac{g_2\phi_3}{2}(|e|A^\mu - g_Z s_W^2 Z^\mu)(W^-_\mu\phi^+ + W^+_\mu\phi^-) \\ & + \frac{\phi_3^2}{8}(2g_2^2 W^{+\mu}W^-_\mu + g_Z^2 Z^\mu Z_\mu) + \frac{\phi^+\phi^-}{4}\left[2g_2^2 W^{+\mu}W^-_\mu + (2|e|A_\mu + g_Z(c_W^2 - s_W^2)Z_\mu)^2\right]; \end{aligned} \tag{8.24}$$

$$V = \lambda|H|^4 - \mu^2|H|^2 = \frac{\lambda}{4}h^4 + \lambda v h^3 + \frac{2\lambda v^2}{2}h^2 - \frac{\lambda}{4}v^4 + \frac{\lambda}{4}(2\phi^+\phi^- + \phi_3^2)^2 + \frac{\lambda}{2}(h^2 + 2vh)(2\phi^+\phi^- + \phi_3^2), \tag{8.25}$$

where $v = \mu/\sqrt{\lambda} \sim 246\,\text{GeV}$, $\lambda \sim 0.13$, and $\mu \sim 89\,\text{GeV}$. In exponential parameterization,

$$H = \frac{1}{\sqrt{2}}\exp\left(\frac{\mathrm{i}}{v}\sigma_i\varphi_i\right)\begin{pmatrix} 0 \\ v+h \end{pmatrix}, \tag{8.26}$$

$$D_\mu H = \frac{1}{\sqrt{2}}e^{\mathrm{i}\sigma_i\varphi_i/v}\left[\mathrm{i}\sigma_i\partial_\mu\varphi_i\begin{pmatrix} 0 \\ 1+h/v \end{pmatrix} + \begin{pmatrix} 0 \\ \partial_\mu h \end{pmatrix}\right] + \frac{1}{\sqrt{2}}\frac{-\mathrm{i}}{2}(g_2\sigma_i W_{i\mu} + g_Y B_\mu)e^{\mathrm{i}\sigma_i\varphi_i/v}\begin{pmatrix} 0 \\ v+h \end{pmatrix} \tag{8.27}$$

$$= \frac{1}{\sqrt{2}}e^{\mathrm{i}\sigma_i\varphi_i/v}\left[\begin{pmatrix} 0 \\ \partial_\mu h \end{pmatrix} + \mathrm{i}\left(\sigma_i\partial_\mu\varphi_i - \frac{g_2 v}{2}e^{-\mathrm{i}\sigma_j\varphi_j/v}\sigma_i e^{\mathrm{i}\sigma_k\varphi_k/v}W_{i\mu} - \frac{g_Y v}{2}B_\mu\right)\begin{pmatrix} 0 \\ 1+h/v \end{pmatrix}\right], \tag{8.28}$$

$$V = \lambda|H|^4 - \mu^2|H|^2 = \frac{\lambda}{4}h^4 + \lambda v h^3 + \frac{2\lambda v^2}{2}h^2 - \frac{\lambda}{4}v^4. \tag{8.29}$$

These expressions have gauge degeneracy (i.e., without gauge-fixing terms and ghost terms) and thus not ready for calculations. If we choose the unitarity gauge, $\phi_i(x) = 0$,

$$\mathcal{L}_H = |D_\mu H|^2 - V(H) = \frac{1}{2}(\partial_\mu h)^2 - \frac{2\lambda v^2}{2}h^2 - \frac{\lambda}{4}h^4 - \lambda v h^3 + \frac{(v+h)^2}{8}(2g_2^2 W^{+\mu}W^-_\mu + g_Z^2 Z^\mu Z_\mu) + \frac{\lambda}{4}v^4. \tag{8.30}$$

8.7 CP-violating $F\tilde{F}$ terms

The Standard Model contains CP-violating terms

$$\mathcal{L}_{\text{gauge},\not{CP}} = \frac{g_3^2\Theta_g}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma} + \frac{g_2^2\Theta_W}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}W^a_{\mu\nu}W^a_{\rho\sigma} + \frac{g_Y^2\Theta_B}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}B_{\rho\sigma}. \tag{8.31}$$

We here discuss we can ignore Θ_W and Θ_B, while Θ_g causes the strong *CP* roblem.

One should first note that the value of Θ_i depends on the basis of the chiral fermions: in Section 8.5 fermions are redefined by rotations. These rotations generate these terms and the angles are modified. It is then found that Θ_W can be rotated away. Let us see this explicitly, starting from the mass basis, i.e., $Y_{u,d,e}$ are positive diagonal and SU(2) interactions are amended by V_{CKM}. As we do not introduce phases in, e.g., W–u–d interaction and fermion mass matrix, the possible rotation is limited to

$$(Q, U, D) \to e^{\mathrm{i}\theta}(Q, U, D), \qquad (L_i, E_i) \to e^{\mathrm{i}\theta_i}(L_i, E_i). \tag{8.32}$$

These rotations affect the CP-violating terms (Cf. Fujikawa method):[*7]

$$\Delta\Theta_W \propto 9\theta_Q + \sum\theta_{L_i} = 9\theta + \sum\theta_i \qquad \Delta\Theta_B \propto \frac{1}{2}\theta_Q + \frac{3}{2}\theta_L - (4\theta_U + \theta_D + 3\sum\theta_{E_i}) = -\frac{9}{2}\theta - \frac{1}{2}\sum\theta_i, \tag{8.33}$$

which means either Θ_W or Θ_B can be rotated away. As we discuss below, it is convenient to set $\Theta_W = 0$ and $\Theta_B \neq 0$.

Meanwhile, because $\Delta\Theta_g = 0$, we cannot remove Θ_g.[*8] We define $\Theta_{\text{QCD}} :=$ (Θ_g in the mass basis), which induces *CP*-violation in the strong sector. However, such *CP*-violation is not observed yet; this contradiction is called strong *CP* problem.

The form $\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\mu\nu}$ is a total derivative and the effect is pushed away to the surface.[*9] As discussed in [11, §23], the $\mathrm{U}(1)_Y$ surface term does not do anything (in the simple spacetime) but the SU(N) surface term corresponds to topologically

[*7] Fail-safe memo: chiral transformation $\psi \to \exp[\mathrm{i}\gamma_5\alpha(x)]\psi$ generates $\Delta\mathcal{L} = -(g^2/16\pi^2)\,\mathrm{Tr}[\alpha F\tilde{F}]$ (cf. Weinberg II Eq.(2.2.24) but the overall sign may differ). For a constant (and non-matrix) α, $\Delta\mathcal{L} = -(\alpha g^2/32\pi^2)F^a\tilde{F}^a$. Also, the absence of gauge anomaly means the corresponding gauge transformations do not induce additional Θ-terms.

[*8] If, e.g., u were massless, we can take $\theta_{u_R} \neq \theta_Q$ and rotate Θ_g away.

[*9] Sho thanks to Kyohei Mukaida and Teppei Kitahara for a very useful discussion.

non-trivial configuration of the gauge fields, labeled by a winding number ν. Such different configuration should be summed up in, e.g., the path integral formalism, and observed as the instanton effect ("sphaleron" for $\mathrm{SU}(2)_W$). If $\Theta_W \neq 0$, the processes $\nu \to \nu \pm 1$ would have different rate and CP would be violated in the processes. As Θ_B is not related to such process, we take $\Theta_W = 0$ and, though $\Theta_B \neq 0$, do not further consider Θ_B.

9 Standard Model Values[*10]

Mass and width

e : $0.510\,998\,950\,00(15)$ MeV

μ : $105.658\,3755(23)$ MeV, $2.196\,9811(22)$ μs = 659 m

τ : $1776.93(9)$ MeV, $2.903(5)\times10^{-13}$ s = 87.0 μm

t : $172.57(29)$ GeV[*11], $1.42^{(19)}_{(15)}$ GeV

h : $125.20(11)$ GeV, $3.7^{+1.9}_{-1.4}$ MeV

W : $80.3692(133)$ GeV, $2.085(42)$ GeV

Z : $91.1880(20)$ GeV, $2.4955(23)$ GeV

$(u,d,s)^{\overline{\rm MS}}_{2\,\rm GeV}$: $2.16(7)^\diamond, 4.70(7)^\diamond, 93.5(8)^\diamond$ MeV[*12]

$(\frac{u+d}{2}, \frac{u}{d}, \frac{2s}{u+d})^{\overline{\rm MS}}_{2\,\rm GeV}$: $3.49(7)^\diamond$ MeV, $0.462(20)^\diamond$, $27.33^{(18)\diamond}_{(14)}$

c : $1.2730(46)^\diamond$ GeV$^{\overline{\rm MS}}_{m_c}$ ($1.67(7)^\diamond$ GeV$^{\rm pole}$)

b : $4.183(7)^\diamond$ GeV$^{\overline{\rm MS}}_{m_b}$ ($4.78(6)^\diamond$ GeV$^{\rm pole}$)

$\pi^\pm$	139.570 39(18) MeV	$\rho^\pm_{770}$	775.11(34) MeV	$\eta_c(1S)$	2984.1(4) MeV
π^0	134.976 8(5) MeV	ρ^0_{770}	775.26(23) MeV	$J/\psi(1S)$	3096.900(6) MeV
η	547.862(17) MeV	ϕ_{1020}	1019.461(16) MeV	$\eta_b(1S)$	9398.7(2.0) MeV
η'	957.78(6) MeV	ω_{782}	782.66(13) MeV	$\Upsilon(1S)$	9460.40(10) MeV
$K^\pm$	493.677(15) MeV	$K^{*\pm}_{892}$	891.67(26) MeV	$\Upsilon(2S)$	10023.4(5) MeV
K^0	497.611(13) MeV	K^{*0}_{892}	895.55(20) MeV	$\Upsilon(3S)$	10355.1(5) MeV
D^0	1864.84(5) MeV	$B^\pm$	5279.41(7) MeV	$\Upsilon(4S)$	10579.4(1.2) MeV
$D^\pm$	1869.65(5) MeV	B^0	5279.72(8) MeV	p	938.272 088 16(29) MeV
$D^\pm_s$	1968.35(7) MeV	B_s	5366.93(10) MeV	n	939.565 420 5(5) MeV
		$B^\pm_c$	6274.47(32) MeV		

$\pi^\pm$: $26.033(5)$ ns = 7.81 m

π^0 : $8.43(13)\times10^{-17}$ s = 25.3 nm

$K^\pm$: $1.2380(20)\times10^{-8}$ s = 3.71 m

K^0_S : $8.954(4)\times10^{-11}$ s = 26.8 mm

K^0_L : $5.116(21)\times10^{-8}$ s = 15.3 m

$p > 9\times10^{29\diamond}$ yr

n : $878.4(5)$ s

$n\,^{2s+1}l_J J^{PC}$	$I=1$	$I=1/2$	$I=0$		$c\bar c$	$b\bar b$	charm		bottom		
$1\,^1S_0$ 0^{-+}	π	K	η	η'_{958}	$\eta_c(1S)$	$\eta_b(1S)$	D	D_s	B	B_s	B_c
$1\,^1S_1$ 1^{--}	ρ_{770}	K^*_{892}	ϕ_{1020}	ω_{782}	$J/\psi(1S)$	$\Upsilon(1S)$	D^*	D^*_s	B^*	B^*_s	

Electric and magnetic moment, important branching ratios, and neutrino property

$a_e = 0.001\,159\,652\,180\,62(12)$

$a_\mu = 0.001\,165\,920\,59(22)$

$a_\tau -0.057 - -0.024^{**}$

$\mu_p = 2.792\,847\,344\,6(8)\mu_N$

$\mu_n = -1.913\,042\,7(5)\mu_N$

$d_e < 4.1\times10^{-30\diamond}|e|$ cm

$d_\mu < 1.8\times10^{-19**}|e|$ cm

$d_p < 2.1\times10^{-25}|e|$ cm

$d_n < 1.8\times10^{-26\diamond}|e|$ cm

$\mathrm{Br}(\tau\to e,\mu) \simeq 35.2\%$

$\mathrm{Br}(\tau\to \mathrm{had}) \simeq 64.8\%$

$\mathrm{Br}(\tau; \text{1-prong}) \simeq 85.2\%$

$\mathrm{Br}(\tau; \text{3-prong}) \simeq 14.6\%$

$\mathrm{Br}(Z\to \mathrm{had}) = 69.911(56)\%$

$\mathrm{Br}(Z\to b\bar b) = 15.12(5)\%$

$\mathrm{Br}(Z\to \sum e,\mu,\tau) \simeq 10.10\%$

$\mathrm{Br}(Z\to \mathrm{inv}) = 20.000(55)\%$

$\mathrm{Br}(W\to \mathrm{had}) = 67.41(27)\%$

[*10] Data source: **PDG2020**, **PDG2022**, **PDG2024**. Confidence levels are shown by the marks $*$, $**$, $***$ (1–3σ), $\diamond$ (90%), and $\diamond\diamond$ (99%).

[*11] Cross section measurement gives $\overline{\rm MS}$ top mass $162.5^{+2.1}_{-1.5}$ GeV, equivalent to $172.4(7)$ GeV.

[*12] $m^{\overline{\rm MS}}_{1\,\rm GeV} = m^{\overline{\rm MS}}_{2\,\rm GeV}\times1.35$.

CKM matrix

$$|V_{\rm CKM}| = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 0.97435(16) & 0.22500(67) & 0.00369(11) \\ 0.22486(67) & 0.97349(16) & 0.04182^{(+85)}_{(-74)} \\ 0.00857^{(+20)}_{(-18)} & 0.04110^{(+83)}_{(-72)} & 0.999118^{(+31)}_{(-36)} \end{pmatrix}; \quad J = 3.08^{(+15)}_{(-13)} \times 10^{-5}$$

$$(\lambda, A, \bar\rho, \bar\eta) = (0.22500(67), 0.826^{(+18)}_{(-15)}, 0.159(10), 0.348(10))$$

$$(\sin\theta_{12}, \sin\theta_{13}, \sin\theta_{23}, \delta) = (0.22500(67), 0.00369(11), 0.04182^{(+85)}_{(-74)}, 1.144(27))$$

Astrophysical

$$\begin{aligned}
&T_0 = 2.7255(6)\,\mathrm{K} && H_0 = 100h\,\mathrm{km/s/Mpc},\ \ h = 0.674(5) && M_\odot = 1.98841(4)\times 10^{30}\,\mathrm{kg}\\
&n_\gamma = 410.73(27)\hat T_0^3\,\mathrm{cm}^{-3} && \rho_{\rm crit} = 1.053\,672(24)\times 10^{-5} h^2\,\mathrm{GeV/cm}^3 && M_\oplus = 5.97217(13)\times 10^{24}\,\mathrm{kg}\\
&\rho_\gamma = 0.2606(2)\hat T_0^4\,\mathrm{eV/cm}^3 && G_{\rm N} = 6.708\,83(15)\times 10^{-39}\,\mathrm{GeV}^{-2} && R_0 = 8.178(13)_{\rm stat}(22)_{\rm syst}\,\mathrm{kpc}\\
&s = 2891.2\hat T_0^3\,\mathrm{cm}^{-3} && M_{\rm Pl} = 1.220\,890(14)\times 10^{19}\,\mathrm{GeV} && v_0 = 240(8)\,\mathrm{km/s}\\
&\Omega_\gamma h^2 = 2.473\times 10^{-5}\hat T_0^4 && M_0 = 2.435\,323(28)\times 10^{18}\,\mathrm{GeV} && \rho_{\rm disk} = 3.7(5)\,\mathrm{GeV/cm}^3\\
&\quad [\hat T_0 = T_0/2.7255\,\mathrm{K}] && \eta = n_{\rm b}/n_\gamma = 6.14(19)\times 10^{-10}\ \text{(from BBN)} &&
\end{aligned}$$

Planck 2018 6-parameter fit to flat ΛCDM cosmology:

$$\begin{aligned}
&\{\Omega_{\rm b}h^2, \Omega_{\rm CDM}h^2\} = \{0.02237(15), 0.1200(12)\} && (z,t)_{\rm M=R} = 3402(26), 5.11(8)\times 10^4\,\mathrm{yr}\\
&\Omega_{\{\rm b,CDM,\Lambda\}} = \{0.0493(6), 0.265(7), 0.685(7)\} && (z,t)_* = 1089.92(25), 3.729(10)\times 10^5\,\mathrm{yr}\\
&\Lambda = 1.088(30)\times 10^{-56}\,\mathrm{cm}^{-2} && (z,t)_{\rm i} = 7.7(7), 6.90(90)\times 10^8\,\mathrm{yr}\\
&\Omega_K = 0.0007(19) && (z,t)_{\rm q} = 0.636(18), 7.70(10)\times 10^9\,\mathrm{yr}\\
&N_{\rm eff} = 2.99(17) && t_0 = 1.3797(23)\times 10^{10}\,\mathrm{yr}
\end{aligned}$$

Standard Model parameter fit

$$\begin{aligned}
&\alpha_{\rm EM}^{-1}(0) = 137.035\,999\,084(21) && \sin^2\theta^{\overline{\rm MS}}(M_Z) = 0.23122(4) && \alpha_{\rm s}(m_Z) = 0.1179(9)\\
&\hat\alpha^{(4)}(m_\tau)^{-1} = 133.471(7) && \sin^2\theta^{\overline{\rm MS}}(0) = 0.23863(5) && G_{\rm F} = 1.166\,3788(6)\times 10^{-5}\,\mathrm{GeV}^{-2}\\
&\hat\alpha^{(5)}(m_Z)^{-1} = 127.951(9) && \sin^2\theta^{\text{on-shell}} = 0.22339(10) && \overset{\rm tree}{=} g_2^2/(4\sqrt{2}m_W^2) = 1/(\sqrt{2}v^2)\\
&\Delta\alpha^{(5)}_{\rm had}(m_Z) = 0.02768(7) && \overset{\rm tree}{=} (g'/g_Z)^2 = 1-(m_W/m_Z)^2 &&
\end{aligned}$$

$\overline{\rm MS}$ parameters at $Q_0 = 173.1$ GeV based on Ref. [12] (cf. Ref. [13]):

$$\begin{aligned}
&g_s = 1.161\,8(4\,5) && v = 246.605(12)\,\mathrm{GeV} && \lambda = 0.126\,07(30)\\
&g = 0.647\,653(281) && -m^2 = 8612.0(22.8)\,\mathrm{GeV}^2 = (92.80(12)\,\mathrm{GeV})^2 &&\\
&g' = 0.358\,542(70) && y_{t,c,u} = \{0.931(4), 0.0341(10), 6.8(1.1)\times 10^{-6}\} &&\\
&|e| = 0.313\,68(18) && y_{b,s,d} = \{0.015\,53(14), 0.000\,293(25), 1.47(10)\times 10^{-5}\} &&\\
&g_Z = 0.740\,27(25) && y_{\tau,\mu,e} = \{0.009\,994\,4(8), 0.000\,588\,38(11), 2.793\,0(2\,6)\times 10^{-6}\} &&
\end{aligned}$$

10 Neutrino

(summary page)

10.1 Convention and Nomenclature

We define the neutrino mixing matrix U (or $U_{\alpha i}$) by, following the standard convention,

$$|\nu_\alpha^{\text{flavor}}\rangle = U_{\alpha i}^* |\nu_i^{\text{mass}}\rangle, \qquad \nu^{\text{flavor}} = U\nu^{\text{mass}}, \qquad [\alpha = e, \mu, \tau, \mathrm{r}_1, \mathrm{r}_2, \dots]. \tag{10.1}$$

The mass basis are labelled by $i = 1, 2, \dots$, while α labels the "flavor" basis. The flavor basis for $\alpha = e, \mu, \tau$ is defined by the charged leptons, i.e., so that the lepton doublets are aligned and $\nu_\alpha l_\beta^\pm W^\mp$ interactions are diagonal. Meanwhile, for other "extra" neutrinos, the flavor basis can be left arbitrary (undefined) because it is usually not of our interests.

We, in this note, use the term "PMNS (matrix)" only if the neutrino mixing matrix is a 3×3 unitary matrix.

10.2 Models

Dirac neutrino model The simplest model for neutrino masses are given by extending the SM Yukawa (8.5) with (n_{RHN} copies of) right-handed neutrino N:

$$\mathcal{L}_{\text{Dirac-}\nu} = \left(\overline{U}Y_u HP_{\mathrm{L}}Q - \overline{D}Y_d H^\dagger P_{\mathrm{L}}Q + \overline{N}Y_n HP_{\mathrm{L}}L - \overline{E}Y_e H^\dagger P_{\mathrm{L}}L\right) + \text{h.c.} \tag{10.2}$$

$$= \left(-\overline{Q^a}Y_u^\dagger \epsilon^{ab}H^{b*}P_{\mathrm{R}}U - \overline{Q^a}Y_d^\dagger H^a P_{\mathrm{R}}D - \overline{L^a}Y_n^\dagger \epsilon^{ab}H^{b*}P_{\mathrm{R}}N - \overline{L^a}Y_e^\dagger H^a P_{\mathrm{R}}E\right) + \text{h.c.} \tag{10.3}$$

Neutrinos become Dirac fermions and the discussion goes parallel to the CKM matrix:

$$Y_n = U_n Y_n^{\text{diag}} V_n^\dagger, \qquad \{\nu_{\mathrm{L}}, e_{\mathrm{L}}, \overline{N}, \overline{E}\}^{\text{mass basis}} = \{V_n^\dagger L^1, V_e^\dagger L^2, \overline{N}U_n, \overline{E}U_e\}, \tag{10.4}$$

where $Y_n \in \mathbb{C}^{n_{\mathrm{RHN}}\times 3}$, $V_n \in \mathbb{U}_{\mathbb{C}}^3$, and $U_n \in \mathbb{U}_{\mathbb{C}}^{n_{\mathrm{RHN}}}$. The Lagrangian is now given by

$$\mathcal{L} \supset \overline{L}\mathrm{i}\gamma^\mu(-\mathrm{i}g_2 W_\mu)P_{\mathrm{L}}L \supset \frac{g_2}{\sqrt{2}}\left[\overline{L^1}W\!\!\!/^+ P_{\mathrm{L}}L^2 + \overline{L^2}W\!\!\!/^- P_{\mathrm{L}}L^1\right] \tag{10.5}$$

$$= \frac{g_2}{\sqrt{2}}\left[\overline{(L^1)^{\text{mass}}}V_n^\dagger V_e W\!\!\!/^+ P_{\mathrm{L}}(L^2)^{\text{mass}} + \overline{(L^2)^{\text{mass}}}V_e^\dagger V_n W\!\!\!/^- P_{\mathrm{L}}(L^1)^{\text{mass}}\right] \tag{10.6}$$

$$= \frac{g_2}{\sqrt{2}}\left[\overline{\nu_{\mathrm{L}}^{\text{mass}}}U^\dagger W\!\!\!/^+ e_{\mathrm{L}}^{\text{mass}} + \overline{e_{\mathrm{L}}^{\text{mass}}}UW\!\!\!/^- \nu_{\mathrm{L}}^{\text{mass}}\right], \tag{10.7}$$

where the definition (10.1) provides the equality

$$U = V_e^\dagger V_n \equiv U_{\mathrm{PMNS}}. \tag{10.8}$$

According to our terminology, this 3×3 unitary matrix U is called "the PMNS matrix" for this model. Note that this is given in *the opposite manner* to the CKM matrix ($V_{\mathrm{CKM}} = V_u^\dagger V_d$).

The mass eigenstate Dirac fields are given by

$$\nu^{\text{mass}} = \begin{pmatrix}\nu_{\mathrm{L}}^{\text{mass}}\\ N^{\text{mass}}\end{pmatrix} = \nu_{\mathrm{L}}^{\text{mass}} + N^{\text{mass}} = V_n^\dagger L^1 + U_n^\dagger N \qquad \left(L^1 = P_{\mathrm{L}}V_n\nu^{\text{mass}},\ N = P_{\mathrm{R}}U_n\nu^{\text{mass}}\right). \tag{10.9}$$

Note that $n_{\mathrm{RHN}} = \{2, 3\}$ because models with $n_{\mathrm{RHN}} < 3$ yields $3 - n_{\mathrm{RHN}}$ massless left-handed neutrinos, while $n_{\mathrm{RHN}} > 3$ results in $n_{\mathrm{RHN}} - 3$ massless right-handed neutrinos.

Type-I see-saw The type-I see-saw models are given by

$$\mathcal{L}_{\text{type-I}} = \left(\overline{U}Y_u HP_{\mathrm{L}}Q - \overline{D}Y_d H^\dagger P_{\mathrm{L}}Q + \overline{N}Y_n HP_{\mathrm{L}}L - \overline{E}Y_e H^\dagger P_{\mathrm{L}}L - \frac{1}{2}\overline{N}M_N N^{\mathrm{c}}\right) + \text{h.c.} \tag{10.10}$$

$$= \left(-\overline{Q^a}Y_u^\dagger \epsilon^{ab}H^{b*}P_{\mathrm{R}}U - \overline{Q^a}Y_d^\dagger H^a P_{\mathrm{R}}D - \overline{L^a}Y_n^\dagger \epsilon^{ab}H^{b*}P_{\mathrm{R}}N - \overline{L^a}Y_e^\dagger H^a P_{\mathrm{R}}E - \frac{1}{2}\overline{N}M_N N^{\mathrm{c}}\right) + \text{h.c.}, \tag{10.11}$$

where M_N is a $n_{\mathrm{RHN}} \times n_{\mathrm{RHN}}$ complex symmetric matrix. Models with $n_{\mathrm{RHN}} \geq 2$ are viable, because $n_{\mathrm{RHN}} < 3$ gives $3 - n_{\mathrm{RHN}}$ massless (Weyl) neutrinos and $2n_{\mathrm{RHN}}$ massive (Majorana) neutrinos, while all neutrinos acquire Majorana mass if $n_{\mathrm{RHN}} \geq 3$.

We introduce left-handed Weyl spinors ξ and χ to avoid notational confusion and diagonalize the Dirac mass term M_{D}:

$$L^1 \equiv \begin{pmatrix}\xi\\ 0\end{pmatrix} \equiv \begin{pmatrix}\nu_{\mathrm{L}}\\ 0\end{pmatrix}, \quad N \equiv \begin{pmatrix}0\\ \bar\chi\end{pmatrix} \equiv \begin{pmatrix}0\\ n_{\mathrm{R}}\end{pmatrix}, \quad M_{\mathrm{D}} := \frac{v}{\sqrt{2}}Y_n, \quad M_{\mathrm{D}}^{\text{diag}} := U_n M_{\mathrm{D}} Y_n^\dagger. \tag{10.12}$$

Then the mass term is written in matrix form:

$$\mathcal{L}_{\text{type-I}} \supset -\frac{v}{\sqrt{2}}\overline{N}Y_n P_{\mathrm{L}}L^1 - \frac{1}{2}\overline{N}M_N N^{\mathrm{c}} + \text{h.c.} = -\frac{1}{2}\begin{pmatrix}\xi & \chi\end{pmatrix}\begin{pmatrix}0 & M_{\mathrm{D}}^{\mathrm{T}}\\ M_{\mathrm{D}} & M_N\end{pmatrix}\begin{pmatrix}\xi\\ \chi\end{pmatrix} + \text{h.c.} =: -\frac{1}{2}\tilde\nu^{\mathrm{T}}\tilde M\tilde\nu + \text{h.c.}, \tag{10.13}$$

where tildes denote "larger" objects. We can AT-diagonalize $\tilde M$, which is a symmetric $\mathbb{C}^{(3+n_{\mathrm{RHN}})\times(3+n_{\mathrm{RHN}})}$ matrix, as usual:

$$\tilde M = \tilde R\tilde M^{\text{diag}}\tilde R^{\mathrm{T}}; \qquad -\mathcal{L}_{\text{Dirac-}\nu} \supset \frac{1}{2}\tilde\nu^{\mathrm{T}}\tilde M\tilde\nu = \frac{1}{2}(\tilde\nu^{\text{mass}})^{\mathrm{T}}\tilde M^{\text{diag}}\tilde\nu^{\text{mass}}; \qquad \tilde\nu^{\text{mass}} = \tilde R^{\mathrm{T}}\tilde\nu. \tag{10.14}$$

Mass eigenstates ν^{mass} are Majorana fermions given by left-handed Weyl spinors. The neutrino mixing are given by

$$\begin{pmatrix}\xi\\ \chi\end{pmatrix} \equiv \begin{pmatrix}\nu_{\mathrm{L}}\\ (n_{\mathrm{R}})^\dagger\end{pmatrix} = \tilde R^*\begin{pmatrix}\nu_{1\text{–}3}\\ \nu_{4\text{–}}\end{pmatrix}, \qquad \therefore U = \begin{pmatrix}V_e^\dagger & 0\\ 0 & X\end{pmatrix}\tilde R^*, \tag{10.15}$$

where Y_e comes from the charged lepton mass diagonalization, while X is a unitary matrix (but unphysical if, as usual, the right-handed neutrinos are indistinguishable).

Connection of the above two models If we set $M_N=0$ in the Type-I see-saw ($n_{\mathrm{RHN}}=3$), the AT-diagonalization becomes

$$\tilde{M}=\frac{1}{2}\begin{pmatrix}0 & (U_nM_{\mathrm{D}}^{\mathrm{diag}}V_n^\dagger)^{\mathrm{T}}\\ U_nM_{\mathrm{D}}^{\mathrm{diag}}V_n^\dagger & 0\end{pmatrix},\quad \tilde{R}=\begin{pmatrix}V_n^* & \mathrm{i}V_n^*\\ U_n & -\mathrm{i}U_n\end{pmatrix},\quad \tilde{M}^{\mathrm{diag}}=\begin{pmatrix}M_{\mathrm{D}}^{\mathrm{diag}} & 0\\ 0 & M_{\mathrm{D}}^{\mathrm{diag}}\end{pmatrix}. \tag{10.16}$$

As three pairs of degenerate Weyl fermions form three Dirac neutrinos, the Dirac neutrino model is reproduced as expected.

10.3 Neutrino mixings

Experiments have revealed that the upper-left 3×3 submatrix is close to a unitary matrix, which we call "the PMNS matrix" U_{PMNS}. Thus[*13] we decompose the general neutrino mixing angle U to

$$U=U'\begin{pmatrix}U_{\mathrm{PMNS}} & 0\\ 0 & U_X\end{pmatrix}=\begin{pmatrix}U'_{11} & U'_{12}\\ U'_{21} & U'_{22}\end{pmatrix}\begin{pmatrix}U_{\mathrm{PMNS}} & 0\\ 0 & U_X\end{pmatrix} \tag{10.17}$$

with U_X (and hence U') being unitary. Here $U'_{11}\approx 1$ and $U'_{12},U'_{21}\approx 0$. We do not care much about U'_{22} but in type-I see-saw models $U'_{22}U_X$ inherits the redundancy of X and thus we can take, e.g., $U'_{22}\approx 1$ or $U_X=1$.

The matrix U' is theoretically given with some matrix Θ by [14][*14]

$$U'\equiv\exp\begin{pmatrix}0 & -\Theta^\dagger\\ \Theta & 0\end{pmatrix}=\sum_{n=0}^{\infty}\begin{pmatrix}(-\Theta^\dagger\Theta)^n/(2n)! & -(-\Theta^\dagger\Theta)^n\Theta^\dagger/(2n+1)!\\ +\Theta(-\Theta^\dagger\Theta)^n/(2n+1)! & (-\Theta\Theta^\dagger)^n/(2n)!\end{pmatrix} \tag{10.18}$$

but practically we derive it in perturbative expansion. For example, the type-I case is evaluated as[*15]

$$(\text{diag.})=U^{\mathrm{T}}\begin{pmatrix}0 & M_{\mathrm{D}}^{\mathrm{T}}\\ M_{\mathrm{D}} & M_N\end{pmatrix}U;\qquad U'\approx\begin{pmatrix}1-\frac{1}{2}\Theta_0^\dagger\Theta_0 & -\Theta_0^\dagger\\ \Theta_0 & 1-\frac{1}{2}\Theta_0\Theta_0^\dagger\end{pmatrix};\quad \Theta_0:=-M_N^{-1}M_{\mathrm{D}}, \tag{10.19}$$

$$(\text{block diag.})=U'^{\mathrm{T}}\begin{pmatrix}0 & M_{\mathrm{D}}^{\mathrm{T}}\\ M_{\mathrm{D}} & M_N\end{pmatrix}U'\approx\begin{pmatrix}-M_{\mathrm{D}}^{\mathrm{T}}M_N^{-1}M_{\mathrm{D}} & 0\\ 0 & M_N\end{pmatrix}, \tag{10.20}$$

where however note that Θ_0 is close but not equal to Θ.

If we focus only on the upper-left 3×3 part and are allowed to neglect Θ^4 (or Θ_0^4) terms, the above decompositions of a unitary matrix into two unitary matrices [15] can be expressed by

$$[U]_{\text{upperleft}}\simeq(1-\eta)U_{\mathrm{PMNS}};\qquad \eta:=\Theta^\dagger\Theta/2\text{ is a complex positive-semidefinite Hermitian matrix,} \tag{10.21}$$

where the positive-semidefiniteness yields

$$\eta_{ii}\ge 0,\quad |\eta_{ij}|\le\sqrt{\eta_{ii}\eta_{jj}}\le(\eta_{ii}+\eta_{jj})/2. \tag{10.22}$$

The QR decomposition is also useful [15]; together with the polar decomposition, it is summarized by

$$[U]_{\text{upperleft}}=(1-\alpha)U_{\mathrm{QR}};\qquad \text{QR decomposition; }\alpha\text{ is a complex upper-triangle matrix,} \tag{10.23}$$

$$[U]_{\text{upperleft}}=\eta' U_{\text{polar}};\qquad \text{polar decomposition; }\eta'\text{ is a complex positive-semidefinite Hermitian matrix,} \tag{10.24}$$

where $U_{\mathrm{QR}}\neq U_{\text{polar}}$. However, as experiments claim that $[U]_{\text{upperleft}}\approx U_{\mathrm{PMNS}}$ and η and α are still consistent with zero, they are consistent with U_{PMNS}. Note also that, as $[U]_{\text{upperleft}}$ seems to be invertible, these decompositions are given uniquely.

10.4 PMNS matrix

As experiments have not found any deviation from $[U]_{\text{upperleft}}$ being unitary, we express the mixing with the PMNS matrix (Pontecorvo–牧–中川–坂田) parameterized by

$$\begin{aligned}U_{\mathrm{PMNS}}&=\begin{pmatrix}1 & & \\ & \mathrm{c}_{23} & \mathrm{s}_{23}\\ & -\mathrm{s}_{23} & \mathrm{c}_{23}\end{pmatrix}\begin{pmatrix}\mathrm{c}_{13} & & \mathrm{s}_{13}\mathrm{e}^{-\mathrm{i}\delta_{\mathrm{CP}}}\\ & 1 & \\ -\mathrm{s}_{13}\mathrm{e}^{\mathrm{i}\delta_{\mathrm{CP}}} & & \mathrm{c}_{13}\end{pmatrix}\begin{pmatrix}\mathrm{c}_{12} & \mathrm{s}_{12} & \\ -\mathrm{s}_{12} & \mathrm{c}_{12} & \\ & & 1\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\eta_1} & & \\ & \mathrm{e}^{\mathrm{i}\eta_2} & \\ & & 1\end{pmatrix}\\ &=\begin{pmatrix}\mathrm{c}_{12}\mathrm{c}_{13} & \mathrm{s}_{12}\mathrm{c}_{13} & \mathrm{s}_{13}\mathrm{e}^{-\mathrm{i}\delta_{\mathrm{CP}}}\\ -\mathrm{s}_{12}\mathrm{c}_{23}-\mathrm{c}_{12}\mathrm{s}_{23}\mathrm{s}_{13}\mathrm{e}^{\mathrm{i}\delta_{\mathrm{CP}}} & \mathrm{c}_{12}\mathrm{c}_{23}-\mathrm{s}_{12}\mathrm{s}_{23}\mathrm{s}_{13}\mathrm{e}^{\mathrm{i}\delta_{\mathrm{CP}}} & \mathrm{s}_{23}\mathrm{c}_{13}\\ \mathrm{s}_{12}\mathrm{s}_{23}-\mathrm{c}_{12}\mathrm{c}_{23}\mathrm{s}_{13}\mathrm{e}^{\mathrm{i}\delta_{\mathrm{CP}}} & -\mathrm{c}_{12}\mathrm{s}_{23}-\mathrm{s}_{12}\mathrm{c}_{23}\mathrm{s}_{13}\mathrm{e}^{\mathrm{i}\delta_{\mathrm{CP}}} & \mathrm{c}_{23}\mathrm{c}_{13}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\eta_1} & & \\ & \mathrm{e}^{\mathrm{i}\eta_2} & \\ & & 1\end{pmatrix},\end{aligned} \tag{10.25}$$

where $\theta_{ij}\in[0,\pi/2]$ and $\delta_{\mathrm{CP}}\in[0,2\pi]$ as shown in Eq. (A.17). In the Dirac neutrino model, η_1 and η_2 ("Majorana phases") are unphysical because we can rotate n_{R} to eliminate them, while the rotation is not allowed in presence of the Majorana mass term.

It should be noted that the discussion in Section 8.7 holds. As far as the baryon number is conserved, we can remove the $\Theta_W W\tilde{W}$ term by quark rotation. Hence, the above-discussed models have CP violation only in the CKM matrix and the neutrino mixing matrix.

[*13] Sho thanks Josu Hernandez-Garcia for discussion useful in this whole section.

[*14] ♠TODO:♠proof? or obvious if considers $SU(6)/(SU(3)\times SU(3))$?

[*15] Cf. `calculator/neutrino/diagonalization_perturbative.wl`

PDG and NuFIT convention Our convention is the same with PDG [PDG2020, §14] [*16].

It also agrees with NuFIT [16, v5.0] (their convention is given in Ref. [17]) except for the Majorana phases, $\eta_i = \alpha_i$.

10.5 Casas--Ibarra parameterization

We start from the approximation given in Eq. (10.20):

$$m_{\rm L} \approx -U_{\rm PMNS}^{\rm T} M_{\rm D}^{\rm T} M_N^{-1} M_{\rm D} U_{\rm PMNS}, \qquad m_{\rm H} \approx U_X^{\rm T} M_N U_X, \tag{10.26}$$

where $m_{\rm L}$ and $m_{\rm H}$ are diagonal mass matrices for three light neutrinos and heavier neutrinos. We then obtain

$$m_{\rm L} \approx -U_{\rm PMNS}^{\rm T} M_{\rm D}^{\rm T} (U_X^* m_{\rm H} U_X^\dagger)^{-1} M_{\rm D} U_{\rm PMNS} = -U_{\rm PMNS}^{\rm T} M_{\rm D}^{\rm T} U_X m_{\rm H}^{-1} U_X^{\rm T} M_{\rm D} U_{\rm PMNS} \tag{10.27}$$

and decompose it as

$$[{\rm i}m_{\rm L}]^{1/2}[{\rm i}m_{\rm L}]^{1/2} \approx \left(\frac{v}{\sqrt{2}} m_{\rm H}^{-1/2} U_X^{\rm T} Y_n U_{\rm PMNS}\right)^{\rm T} \left(\frac{v}{\sqrt{2}} m_{\rm H}^{-1/2} U_X^{\rm T} Y_n U_{\rm PMNS}\right), \tag{10.28}$$

which is the master equation for Casas-Ibarra parameterization [18]. Note that U_X is unphysical, i.e., we can fix the basis of $n_{\rm R}$ so that $U_X = 1$. This basis is equivalent (under our approximation) to the basis in which M_N becomes diagonal.

Example: three right-handed neutrinos Let us assume all the neutrinos are massive thanks to three right-handed neutrinos. Then $M_{\rm L}^{\rm diag}$ is invertible and

$$R := -{\rm i} m_{\rm H}^{-1/2} U_X^{\rm T} M_{\rm D} U_{\rm PMNS} m_{\rm L}^{-1/2} \quad \Longrightarrow \quad R^{\rm T} R \approx 1. \tag{10.29}$$

Conversely, with a matrix R satisfying $R^{\rm T}R = 1$, the Yukawa matrix is given by

$$\frac{v}{\sqrt{2}} Y_n \approx {\rm i} U_X^* \sqrt{m_{\rm H}} R \sqrt{m_{\rm L}} U_{\rm PMNS}^\dagger \tag{10.30}$$

or

$$\frac{v}{\sqrt{2}} Y_n\Big|_{M_N \text{ being diagonal}} \approx {\rm i}\sqrt{m_{\rm H}} R \sqrt{m_{\rm L}} U_{\rm PMNS}^\dagger. \tag{10.31}$$

Now we successfully parameterized Y_n by a "complex orthogonal" matrix R, which can be parameterized by[*17]

$$R = \begin{pmatrix} {\rm c}_{12}{\rm c}_{13} & {\rm s}_{12}{\rm c}_{13} & {\rm s}_{13} \\ -\zeta {\rm s}_{12}{\rm c}_{23} - {\rm c}_{12}{\rm s}_{23}{\rm s}_{13} & \zeta {\rm c}_{12}{\rm c}_{23} - {\rm s}_{12}{\rm s}_{23}{\rm s}_{13} & {\rm s}_{23}{\rm c}_{13} \\ \zeta {\rm s}_{12}{\rm s}_{23} - {\rm c}_{12}{\rm c}_{23}{\rm s}_{13} & -\zeta {\rm c}_{12}{\rm s}_{23} - {\rm s}_{12}{\rm c}_{23}{\rm s}_{13} & {\rm c}_{23}{\rm c}_{13} \end{pmatrix}, \tag{10.32}$$

where ${\rm c}_{12} \equiv \cos\theta_{12}$ etc. and

$$\zeta = \pm 1; \qquad (\theta_{12}, \theta_{23}, \theta_{13}) \in \mathbb{C}, \quad |\,{\rm Re}\,\theta_{12}| \le \pi, \quad |\,{\rm Re}\,\theta_{23}| \le \pi, \quad |\,{\rm Re}\,\theta_{13}| \le \frac{\pi}{2}. \tag{10.33}$$

This R satisfies $RR^{\rm T} = 1$, which however is not general (as in the next example).

With this parameterization, the neutrino mixing matrix is given by, in the basis with M_N being diagonal,

$$U \approx \begin{pmatrix} U_{\rm PMNS} & M_{\rm D}^\dagger M_N^{*-1} U_X \\ -M_N^{-1} M_{\rm D} U_{\rm PMNS} & U_X \end{pmatrix} \approx \begin{pmatrix} U_{\rm PMNS} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -(-{\rm i} m_{\rm H}^{-1/2} R m_{\rm L}^{1/2})^\dagger \\ -{\rm i} m_{\rm H}^{-1/2} R m_{\rm L}^{1/2} & 1 \end{pmatrix}. \tag{10.34}$$

Example: two right-handed neutrinos For models with two right-handed neutrinos, one neutrino is massless and $M_{\rm L}^{\rm diag}$ is not invertible. However the parameterization

$$\frac{v}{\sqrt{2}} Y_n \approx {\rm i} U_X^* \sqrt{m_{\rm H}} R \sqrt{m_{\rm L}} U_{\rm PMNS}^\dagger \tag{10.35}$$

works with[*18]

$$R_{\text{normal hierarchy}} = \begin{pmatrix} 0 & \cos z & \zeta \sin z \\ 0 & -\sin z & \zeta \cos z \end{pmatrix}, \qquad R_{\text{inverse hierarchy}} = \begin{pmatrix} \cos z & \zeta \sin z & 0 \\ -\sin z & \zeta \cos z & 0 \end{pmatrix}, \tag{10.36}$$

where $z \in \mathbb{C}$ and $\zeta = \pm 1$.

[*16] Sho thinks Eq. (14.9) of PDG2020 lacks 1/2 in the right-most term.

[*17] For $w \in \mathbb{C}$, $\sin z_1 = w$ and $\cos z_2 = w$ always have solutions $z_{1,2} \in \mathbb{C}$. Meanwhile, $\tan z = w$ has no solution if and only if $w = \pm{\rm i}$. Then, using this fact, one first expresses R_{i3} components by $\zeta_{\rm A,B,C} = \pm 1$ and $\theta_{\rm A,B} \in \mathbb{C}$, restricting $0 \le {\rm Re}\,\theta_{\rm A,B} \le \pi/2$ ($\Leftrightarrow$ ${\rm Re}\sin\theta \ge 0 \wedge {\rm Re}\cos\theta \ge 0$), and then gets an expression of R with three angles and six signs. Five signs are absorbed by enlarging ${\rm Re}\,\theta$ and one sign remains, which is ζ.

[*18] See, e.g., Ref. [19]. Sho also thanks Kai Schmitz for his note.

11 Supersymmetry with $\eta = \mathrm{diag}(+,-,-,-)$

Convention Our convention follows DHM (except for D_μ):

$$\eta = \mathrm{diag}(1,-1,-1,-1);\quad \varepsilon^{0123} = -\varepsilon_{0123} = 1,\quad \varepsilon^{12}=\varepsilon_{21}=\varepsilon^{\dot1\dot2}=\varepsilon_{\dot2\dot1}=1\quad (\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma}=\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma}=\delta_\gamma^\alpha),$$

$$\psi^\alpha=\varepsilon^{\alpha\beta}\psi_\beta,\quad \psi_\alpha=\varepsilon_{\alpha\beta}\psi^\beta,\quad \bar\psi^{\dot\alpha}=\varepsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta},\quad \bar\psi_{\dot\alpha}=\varepsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta};$$

$$\sigma^\mu_{\alpha\dot\alpha} := (\mathbf{1},\boldsymbol\sigma)_{\alpha\dot\alpha},\quad \sigma^{\mu\nu}{}_\alpha{}^\beta := \frac{\mathrm{i}}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta,$$ [*19] $$(\sigma^\mu_{\alpha\dot\beta} = \varepsilon_{\alpha\delta}\varepsilon_{\dot\beta\dot\gamma}\bar\sigma^{\mu\dot\gamma\delta},\quad \bar\sigma^{\mu\dot\alpha\beta}=\varepsilon^{\dot\alpha\dot\delta}\varepsilon^{\beta\gamma}\sigma^\mu_{\gamma\dot\delta})$$

$$\bar\sigma^{\mu\dot\alpha\alpha} := (\mathbf{1}, -\boldsymbol\sigma)^{\dot\alpha\alpha},\quad \bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta} := \frac{\mathrm{i}}{4}(\bar\sigma^\mu\sigma^\nu-\bar\sigma^\nu\sigma^\mu)^{\dot\alpha}{}_{\dot\beta},$$ [*19]

$$(\psi\xi) := \psi^\alpha\xi_\alpha,\quad (\bar\psi\bar\chi) := \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha};\qquad \frac{\mathrm{d}}{\mathrm{d}\theta^\alpha}(\theta\theta) := \theta_\alpha \quad \text{[left derivative]}.$$

Especially, spinor-index contraction is done as ${}^\alpha{}_\alpha$ and ${}_{\dot\alpha}{}^{\dot\alpha}$ except for ε_{ab} (which always comes from left). Noting that complex conjugate reverses spinor order: $(\psi^\alpha\xi^\beta)^* := (\xi^\beta)^*(\psi^\alpha)^*$,

$$\bar\psi^{\dot\alpha} := (\psi^\alpha)^*,\quad \varepsilon^{\dot a\dot b} := (\varepsilon^{ab})^*,\quad (\psi\chi)^* = (\bar\psi\bar\chi),$$

$$(\sigma^\mu_{\alpha\dot\beta})^* = \bar\sigma^\mu{}_{\dot\alpha\beta} = \varepsilon_{\beta\delta}\varepsilon_{\dot\alpha\dot\gamma}\bar\sigma^{\mu\dot\gamma\delta},\quad (\sigma^{\mu\nu})^{\dagger\alpha}{}_\beta = \bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta},\quad (\sigma^{\mu\nu}{}_\alpha{}^\beta)^* = \bar\sigma^{\mu\nu\dot\beta}{}_{\dot\alpha} = \bar\sigma^{\mu\nu}{}_{\dot\alpha}{}^{\dot\beta} = \varepsilon_{\dot\alpha\dot\gamma}\varepsilon^{\dot\beta\dot\delta}\bar\sigma^{\mu\nu\dot\gamma}{}_{\dot\delta},$$

$$(\bar\sigma^{\mu\dot\alpha\beta})^* = \sigma^{\mu\alpha\dot\beta} = \varepsilon^{\dot\beta\dot\delta}\varepsilon^{\alpha\gamma}\sigma^\mu_{\gamma\dot\delta},\quad (\bar\sigma^{\mu\nu})^\dagger{}_{\dot\alpha}{}^{\dot\beta} = \sigma^{\mu\nu}{}_\alpha{}^\beta,\quad (\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta})^* = \sigma^{\mu\nu}{}_\beta{}^\alpha = \bar\sigma^{\mu\nu\alpha}{}_\beta = \varepsilon_{\beta\delta}\varepsilon^{\alpha\gamma}\sigma^{\mu\nu}{}_\gamma{}^\delta.$$

Contraction formulae

$$\begin{array}{llll}
\theta^\alpha\theta^\beta = -\frac12(\theta\theta)\varepsilon^{\alpha\beta} & \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}=\frac12(\bar\theta\bar\theta)\varepsilon^{\dot\alpha\dot\beta} & (\theta\xi)(\theta\chi) = -\frac12(\theta\theta)(\xi\chi) & (\theta\sigma^\nu\bar\theta)\theta^\alpha = \frac12(\theta\theta)(\bar\theta\bar\sigma^\nu)^\alpha \\
\theta_\alpha\theta_\beta=\frac12(\theta\theta)\varepsilon_{\alpha\beta} & \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac12(\bar\theta\bar\theta)\varepsilon_{\dot\alpha\dot\beta} & (\bar\theta\bar\xi)(\bar\theta\bar\chi) = -\frac12(\bar\theta\bar\theta)(\bar\xi\bar\chi) & (\theta\sigma^\nu\bar\theta)\bar\theta_{\dot\alpha} = -\frac12(\theta\sigma^\nu)_{\dot\alpha}(\bar\theta\bar\theta) \\
\theta^\alpha\theta_\beta = \frac12(\theta\theta)\delta^\alpha_\beta & \bar\theta_{\dot\alpha}\bar\theta^{\dot\beta}=\frac12(\bar\theta\bar\theta)\delta^{\dot\alpha}_{\dot\beta} & (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)=\frac12(\theta\theta)(\bar\theta\bar\theta)\eta^{\mu\nu} & \\
(\theta\sigma^\mu\bar\sigma^\nu\theta) = (\theta\theta)\eta^{\mu\nu} & (\bar\theta\bar\sigma^\mu\sigma^\nu\bar\theta) = (\bar\theta\bar\theta)\eta^{\mu\nu} & (\sigma^\mu\bar\theta)_\alpha(\theta\sigma^\nu\bar\theta) = \frac12(\bar\theta\bar\theta)(\sigma^\mu\bar\sigma^\nu\theta)_\alpha &
\end{array}$$

$$\begin{array}{ll}
\sigma^\mu\bar\sigma^\nu = \eta^{\mu\nu} - 2\mathrm{i}\sigma^{\mu\nu} & \sigma^\mu\bar\sigma^\nu\sigma^\rho + \sigma^\rho\bar\sigma^\nu\sigma^\mu = 2(\sigma^\mu\eta^{\rho\nu} - \sigma^\nu\eta^{\mu\rho} + \sigma^\rho\eta^{\mu\nu}) \\
\bar\sigma^\mu\sigma^\nu = \eta^{\mu\nu}-2\mathrm{i}\bar\sigma^{\mu\nu} & \sigma^\mu\bar\sigma^\nu\sigma^\rho - \sigma^\rho\bar\sigma^\nu\sigma^\mu = 2\mathrm{i}\sigma_\sigma\varepsilon^{\mu\nu\rho\sigma} \\
\mathrm{Tr}(\sigma^\mu\bar\sigma^\nu) = \mathrm{Tr}(\bar\sigma^\mu\sigma^\nu) = 2\eta^{\mu\nu} & \bar\sigma^\mu\sigma^\nu\bar\sigma^\rho + \bar\sigma^\rho\sigma^\nu\bar\sigma^\mu = 2(\bar\sigma^\mu\eta^{\rho\nu} - \bar\sigma^\nu\eta^{\mu\rho}+\bar\sigma^\rho\eta^{\mu\nu}) \\
\sigma^\mu_{\alpha\dot\alpha}\bar\sigma_\mu{}^{\dot\beta\beta} = 2\delta^{\dot\beta}_{\dot\alpha}\delta^\alpha_\beta & \bar\sigma^\mu\sigma^\nu\bar\sigma^\rho - \bar\sigma^\rho\sigma^\nu\bar\sigma^\mu = -2\mathrm{i}\bar\sigma_\sigma\varepsilon^{\mu\nu\rho\sigma} \\
\sigma_{\mu\alpha\dot\alpha}\sigma^\mu_{\beta\dot\beta} = 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} & \varepsilon_{\dot\beta\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha} = \varepsilon_{\dot\beta\dot\alpha}\varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\alpha\gamma}\sigma^\mu_{\gamma\dot\gamma} = \varepsilon^{\alpha\gamma}\sigma^\mu_{\gamma\dot\beta} \\
\bar\sigma_\mu{}^{\dot\alpha\alpha}\bar\sigma^{\mu\dot\beta\beta} = 2\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} & \varepsilon_{\beta\alpha}\bar\sigma^{\mu\dot\alpha\alpha} = \varepsilon_{\beta\alpha}\varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\alpha\gamma}\sigma^\mu_{\gamma\dot\gamma} = \varepsilon^{\dot\alpha\dot\gamma}\sigma^\mu_{\beta\dot\gamma} \\
\mathrm{Tr}(\sigma^{\mu\nu}) = \mathrm{Tr}(\bar\sigma^{\mu\nu}) = 0 & \mathrm{Tr}(\sigma^{\mu\nu}\sigma^{\rho\sigma}) = \frac12(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac12\mathrm{i}\varepsilon^{\mu\nu\rho\sigma} \\
\bar\sigma^{\mu\nu} = -\bar\sigma^{\nu\mu} \qquad \sigma^{\mu\nu} = -\sigma^{\nu\mu} & \mathrm{Tr}(\bar\sigma^{\mu\nu}\bar\sigma^{\rho\sigma}) = \frac12\mathrm{i}\varepsilon^{\mu\nu\rho\sigma} + \frac12(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\
\sigma^{\mu\nu}{}_\alpha{}^\beta\varepsilon_{\beta\gamma} = \sigma^{\mu\nu}{}_\gamma{}^\beta\varepsilon_{\beta\alpha} & \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} - \sigma^\nu_{\alpha\dot\alpha}\sigma^\mu_{\beta\dot\beta} = -2\mathrm{i}\varepsilon_{\dot\alpha\dot\gamma}\bar\sigma^{\mu\nu\dot\gamma}{}_{\dot\beta}\varepsilon_{\alpha\beta} - 2\mathrm{i}\sigma^{\mu\nu}{}_\alpha{}^\gamma\varepsilon_{\gamma\beta}\varepsilon_{\dot\alpha\dot\beta} \\
\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta}\varepsilon^{\dot\beta\dot\gamma} = \bar\sigma^{\mu\nu\dot\gamma}{}_{\dot\beta}\varepsilon^{\dot\beta\dot\alpha} & \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} + \sigma^\nu_{\alpha\dot\alpha}\sigma^\mu_{\beta\dot\beta} = 4\sigma^{\rho\mu}{}_\alpha{}^\gamma\varepsilon_{\gamma\beta}\varepsilon_{\dot\alpha\dot\gamma}\bar\sigma^{\sigma\nu\dot\gamma}{}_{\dot\beta}\eta_{\rho\sigma} + \varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\eta^{\mu\nu} \\
\bar\sigma_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} = -2\mathrm{i}\bar\sigma^{\mu\nu} & \bar\sigma^{\mu\dot\alpha\alpha}\bar\sigma^{\nu\dot\beta\beta} - \bar\sigma^{\nu\dot\alpha\alpha}\bar\sigma^{\mu\dot\beta\beta} = -2\mathrm{i}\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\gamma}\varepsilon^{\dot\gamma\dot\beta}\varepsilon^{\alpha\beta} - 2\mathrm{i}\varepsilon^{\alpha\gamma}\sigma^{\mu\nu}{}_\gamma{}^\beta\varepsilon^{\dot\alpha\dot\beta} \\
\sigma_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} = 2\mathrm{i}\sigma^{\mu\nu} & \bar\sigma^{\mu\dot\alpha\alpha}\bar\sigma^{\nu\dot\beta\beta} + \bar\sigma^{\nu\dot\alpha\alpha}\bar\sigma^{\mu\dot\beta\beta} = 4\varepsilon^{\alpha\gamma}\sigma^{\sigma\nu}{}_\gamma{}^\beta\bar\sigma^{\rho\mu\dot\alpha}{}_{\dot\gamma}\varepsilon^{\dot\gamma\dot\beta}\eta_{\rho\sigma} + \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\eta^{\mu\nu}
\end{array}$$

$$\bar\xi\bar\sigma^\mu\chi = -\chi\sigma^\mu\bar\xi \qquad \bar\xi\bar\sigma^\mu\sigma^\nu\bar\chi = \bar\chi\bar\sigma^\nu\sigma^\mu\bar\xi \qquad \xi\sigma^\mu\bar\sigma^\nu\chi = \chi\sigma^\nu\bar\sigma^\mu\xi \qquad \bar\xi\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho\chi = -\chi\sigma^\rho\bar\sigma^\nu\sigma^\mu\bar\xi$$

$$(\xi\sigma^\mu\bar\chi)^* = \chi\sigma^\mu\bar\xi \qquad (\bar\xi\bar\sigma^\mu\chi)^* = \bar\chi\bar\sigma^\mu\xi \qquad (\bar\chi\bar\sigma^\mu\sigma^\nu\bar\xi)^* = \chi\sigma^\mu\bar\sigma^\nu\xi = \xi\sigma^\nu\bar\sigma^\mu\chi$$

$$(\xi[\sigma\sigma\cdots]\chi)^* = \bar\chi[\sigma\sigma\cdots]_{\text{reversed}}\bar\xi \qquad (\xi\chi)\psi^\alpha = -(\psi\xi)\chi^\alpha - (\psi\chi)\xi^\alpha \qquad (\xi\chi)\bar\psi_{\dot\alpha} = \frac12(\xi\sigma^\mu\bar\psi)(\chi\sigma_\mu)_{\dot\alpha}$$

$$\mathrm{i}\psi_i\sigma^\mu\partial_\mu\bar\psi_j = -\mathrm{i}\partial_\mu\bar\psi_j\bar\sigma^\mu\psi_i \equiv \mathrm{i}\bar\psi_j\bar\sigma^\mu\partial_\mu\psi_i = -\mathrm{i}\partial_\mu\psi_i\sigma^\mu\bar\psi_j$$

[*19]As the definition of $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$ are not unified in literature, they are not used in this CheatSheet except for this page.

Superfields

$$\Phi = \phi(x) + \sqrt{2}\theta\psi(x) - \mathrm{i}\partial_\mu\phi(x)(\theta\sigma^\mu\bar\theta) + F(x)\theta^2 + \frac{\mathrm{i}}{\sqrt{2}}(\partial_\mu\psi(x)\sigma^\mu\bar\theta)\theta^2 - \frac{\theta^4}{4}\partial^2\phi(x), \tag{11.1}$$

$$\Phi^* = \phi^*(x) + \sqrt{2}\bar\psi(x)\bar\theta + F^*(x)\bar\theta^2 + \mathrm{i}\partial_\mu\phi^*(x)(\theta\sigma^\mu\bar\theta) - \frac{\mathrm{i}}{\sqrt{2}}[\theta\sigma^\mu\partial_\mu\bar\psi(x)]\bar\theta^2 - \frac{\theta^4}{4}\partial^2\phi^*(x), \tag{11.2}$$

$$V = (\bar\theta\bar\sigma^\mu\theta)A_\mu(x) + \bar\theta^2\theta\lambda(x) + \theta^2\bar\theta\bar\lambda(x) + \frac{\theta^4}{2}D(x) \qquad \text{(in Wess-Zumino supergauge).} \tag{11.3}$$

Without gauge symmetries

$$\mathcal{L} = \Phi_i^*\Phi_i\Big|_{\theta^4} + \left(W(\Phi_i)\Big|_{\theta^2} + \text{H.c.}\right); \tag{11.4}$$

$$\Phi_i^*\Phi_i\Big|_{\theta^4} = (\partial_\mu\phi_i^*)(\partial^\mu\phi_i) + \mathrm{i}\bar\psi_i\sigma^\mu\partial_\mu\psi_i + F_i^*F_i, \tag{11.5}$$

$$\begin{aligned} W(\Phi_i)\Big|_{\theta^2} &\dashrightarrow \Big[\kappa_i\Phi_i + m_{ij}\Phi_i\Phi_j + y_{ijk}\Phi_i\Phi_j\Phi_k\Big]_{\theta^2} \\ &= \kappa_iF_i + m_{ij}\left(-\psi_i\psi_j + F_i\phi_j + \phi_iF_j\right) \\ &\quad + y_{ijk}\Big[-(\psi_i\psi_j\phi_k + \psi_i\phi_j\psi_k + \phi_i\psi_j\psi_k) + \phi_i\phi_jF_k + \phi_iF_j\phi_k + F_i\phi_j\phi_k\Big]. \end{aligned} \tag{11.6}$$

With a U(1) gauge symmetry [*20]

$$\mathcal{L} = \Phi_i^*\mathrm{e}^{2gVQ_i}\Phi_i\Big|_{\theta^4} + \left[\left(\frac{1}{4} - \frac{\mathrm{i}g^2\Theta}{32\pi^2}\right)\mathcal{W}^\alpha\mathcal{W}_\alpha\Big|_{\theta^2} + W(\Phi_i)\Big|_{\theta^2} + \text{H.c.}\right] + \Lambda_{\text{FI}}D; \tag{11.7}$$

$$\Phi_i\mathrm{e}^{2gQ_iV}\Phi_i\Big|_{\theta^4} \equiv \mathrm{D}^\mu\phi_i^*\mathrm{D}_\mu\phi_i + \mathrm{i}\bar\psi_i\bar\sigma^\mu\mathrm{D}_\mu\psi_i + F_i^*F_i - \sqrt{2}gQ_i\phi_i^*\lambda\psi_i - \sqrt{2}gQ_i\bar\psi_i\bar\lambda\phi_i + gQ_i\phi_i^*\phi_iD, \tag{11.8}$$

$$\begin{aligned} \left(\frac{1}{4} - \frac{\mathrm{i}g^2\Theta}{32\pi^2}\right)\mathcal{W}^\alpha\mathcal{W}_\alpha\Big|_{\theta^2} + \text{H.c.} &= \frac{1}{2}\,\text{Re}\,\mathcal{W}\mathcal{W}|_{\theta^2} + \frac{g^2\Theta}{16\pi^2}\,\text{Im}\,\mathcal{W}\mathcal{W}|_{\theta^2} \\ &\equiv \mathrm{i}\bar\lambda\bar\sigma^\mu\mathrm{D}_\mu\lambda + \frac{1}{2}DD - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g^2\Theta}{64\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \end{aligned} \tag{11.9}$$

$$\mathrm{D}_\mu\phi_i = (\partial_\mu - \mathrm{i}gQ_iA_\mu)\phi_i, \qquad \mathrm{D}_\mu\psi_i = (\partial_\mu - \mathrm{i}gQ_iA_\mu)\psi_i,$$
$$\mathrm{D}^\mu\phi_i^* = (\partial^\mu + \mathrm{i}gQ_iA^\mu)\phi_i^*, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \mathrm{D}_\mu\lambda = \partial_\mu\lambda.$$

$$\{\phi,\psi,F\} \xrightarrow{\text{gauge}} \mathrm{e}^{\mathrm{i}gQ_i\theta}\{\phi,\psi,F\}, \qquad A_\mu \xrightarrow{\text{gauge}} A_\mu + \partial_\mu\theta, \qquad \lambda \xrightarrow{\text{gauge}} \lambda, \qquad D \xrightarrow{\text{gauge}} D. \tag{11.10}$$

[*20] We use the convention with $V \ni \lambda(x)\theta\bar\theta^2$, which corresponds to $\lambda = \mathrm{i}\lambda_{\text{SLHA}}$. In SLHA convention, the scalar-fermion-gaugino interaction is replaced to

$$-\sqrt{2}g\mathrm{i}\lambda^a_{\text{SLHA}}(\phi^*t^a\psi) - \sqrt{2}g(-\mathrm{i}\bar\lambda^a_{\text{SLHA}})(\bar\psi t^a\phi).$$

With an SU(*N*) gauge symmetry

$$\mathcal{L} = \Phi^* \mathrm{e}^{2gV}\Phi\Big|_{\theta^4} + \left[\left(\frac{1}{2} - \frac{\mathrm{i}g^2\Theta}{16\pi^2}\right)\mathrm{Tr}\,\mathcal{W}^\alpha\mathcal{W}_\alpha\Big|_{\theta^2} + W(\Phi)\Big|_{\theta^2} + \mathrm{H.c.}\right]; \tag{11.11}$$

$$\Phi^* \mathrm{e}^{2gV}\Phi\Big|_{\theta^4} := \Phi_i^* \left[\mathrm{e}^{2gV^a t^a_\Phi}\right]_{ij}\Phi_j\Big|_{\theta^4} \tag{11.12}$$

$$\begin{aligned} &= (\partial_\mu\phi_i^*)(\partial^\mu\phi_i) + \mathrm{i}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_i + F_i^*F_i - \sqrt{2}g\lambda^a(\phi^*t^a\psi) - \sqrt{2}g\bar\lambda^a(\bar\psi^*t^a\phi) \\ &\quad + gA^a_\mu\bar\psi\bar\sigma^\mu(t^a\psi) + 2\mathrm{i}gA^a_\mu\phi^*\partial_\mu(t^a\phi) + g^2A^{a\mu}A^b_\mu(\phi^*t^at^b\phi) + gD^a(\phi^*t^a\phi) \end{aligned} \tag{11.13}$$

$$= \mathrm{D}^\mu\phi^*\mathrm{D}_\mu\phi + \mathrm{i}\bar\psi_i\bar\sigma^\mu\mathrm{D}_\mu\psi_i + F^*F - \sqrt{2}g\lambda^a(\phi^*t^a\psi) - \sqrt{2}g\bar\lambda^a(\bar\psi t^a\phi) + gD^a(\phi^*t^a\phi) \tag{11.14}$$

$$\begin{aligned} \left(\frac{1}{2} - \frac{\mathrm{i}g^2\Theta}{16\pi^2}\right)\mathrm{Tr}\,\mathcal{W}^\alpha\mathcal{W}_\alpha\Big|_{\theta^2} + \mathrm{H.c.} &= \mathrm{Re}\,\mathrm{Tr}\,\mathcal{W}\mathcal{W}\big|_{\theta^2} + \frac{g^2\Theta}{8\pi^2}\,\mathrm{Im}\,\mathrm{Tr}\,\mathcal{W}\mathcal{W}\big|_{\theta^2} \\ &= \mathrm{i}\lambda^a\sigma^\mu\mathrm{D}_\mu\bar\lambda^a + \frac{1}{2}D^aD^a - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{g^2\Theta}{64\pi^2}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}; \end{aligned} \tag{11.15}$$

$$\mathrm{D}_\mu\phi_i = \partial_\mu\phi_i - \mathrm{i}gA^a_\mu t^a_{ij}\phi_j, \qquad \mathrm{D}_\mu\psi_i = \partial_\mu\psi_i - \mathrm{i}gA^a_\mu t^a_{ij}\psi_j, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gA^b_\mu A^c_\nu f^{abc},$$
$$\mathrm{D}^\mu\phi_i^* = \partial^\mu\phi_i^* + \mathrm{i}gA^{a\mu}\phi_j^*t^a_{ji}, \qquad \mathrm{D}_\mu\lambda^a_\alpha = \partial_\mu\lambda^a_\alpha + gf^{abc}A^b_\mu\lambda^c_\alpha.$$

$$\{\phi,\psi,F\} \xrightarrow{\text{gauge}} \mathrm{e}^{\mathrm{i}g\theta^at^a}\{\phi,\psi,F\},$$
$$A^a_\mu \xrightarrow{\text{gauge}} A^a_\mu + \partial_\mu\theta^a + gf^{abc}A^b_\mu\theta^c + \mathcal{O}(\theta^2), \qquad \lambda^a \xrightarrow{\text{gauge}} \lambda^a + gf^{abc}\lambda^b\theta^c + \mathcal{O}(\theta^2),$$
$$D^a \xrightarrow{\text{gauge}} D^a + gf^{abc}D^b\theta^c + \mathcal{O}(\theta^2), \qquad \bar\lambda^a \xrightarrow{\text{gauge}} \bar\lambda^a + gf^{abc}\bar\lambda^b\theta^c + \mathcal{O}(\theta^2).$$ [*21]

Auxiliary fields and Scalar potential In all of the above three theories,

$$\mathcal{L} \supset F_i^*F_i + F_i\frac{\partial W}{\partial\Phi_i}\Big|_{\text{scalar}} + F_i^*\frac{\partial W^*}{\partial\Phi_i^*}\Big|_{\text{scalar}} + \frac{1}{2}D^aD^a + gD^a(\phi^*t^a\phi); \tag{11.16}$$

$$\langle F_i^*\rangle = -\frac{\partial W}{\partial\Phi_i}\Big|_{\text{scalar}}, \qquad \langle D^a\rangle = -g\phi^*t^a\phi; \tag{11.17}$$

$$\mathcal{L} \supset -V_{\text{SUSY}} = -\left[\langle F_i^*\rangle\langle F_i\rangle + \frac{g^2}{2}(\phi^*t^a\phi)(\phi^*t^a\phi)\right]. \tag{11.18}$$

[*21] ♠TODO : give in non-infinitesimal form♠

11.1 Lorentz symmetry as SU(2)×SU(2)

11.2 Supersymmetry algebra

We define the generators as

$$P_\mu := \mathrm{i}\partial_\mu, \quad \{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot\alpha}\} = -2\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu = -2\sigma^\mu_{\alpha\dot\alpha}P_\mu, \quad \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = \{\bar{\mathcal{Q}}_{\dot\alpha}, \bar{\mathcal{Q}}_{\dot\beta}\} = 0, \tag{11.19}$$

which is realized by

$$\mathcal{Q}_\alpha = \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}(\sigma^\mu\bar\theta)_\alpha\partial_\mu, \quad \bar{\mathcal{Q}}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - \mathrm{i}(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu, \quad \mathcal{Q}^\alpha = -\frac{\partial}{\partial\theta_\alpha} - \mathrm{i}(\bar\theta\bar\sigma^\mu)^\alpha\partial_\mu, \quad \bar{\mathcal{Q}}^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} + \mathrm{i}(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu,$$

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} - \mathrm{i}(\sigma^\mu\bar\theta)_\alpha\partial_\mu, \quad \bar{\mathcal{D}}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \mathrm{i}(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu, \quad \mathcal{D}^\alpha = -\frac{\partial}{\partial\theta_\alpha} + \mathrm{i}(\bar\theta\bar\sigma^\mu)^\alpha\partial_\mu, \quad \bar{\mathcal{D}}^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - \mathrm{i}(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu;$$

$\mathcal{D}_\alpha$ etc. works as covariant derivatives because of the commutation relations

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = +2\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad \{\mathcal{Q}_\alpha, \mathcal{D}_\beta\} = \{\mathcal{Q}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\} = \{\bar{\mathcal{Q}}_{\dot\alpha}, \mathcal{D}_\beta\} = \{\bar{\mathcal{Q}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 0.$$

Derivative formulae

$$\varepsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta} = -\frac{\partial}{\partial\theta_\alpha} \qquad \frac{\partial}{\partial\theta^\alpha}\theta\theta = 2\theta_\alpha \qquad \frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta_\beta}\theta\theta = -2\delta_\alpha^\beta \qquad \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\frac{\partial}{\partial\bar\theta_{\dot\beta}}\bar\theta\bar\theta = 2\delta_{\dot\alpha}^{\dot\beta}$$

$$\varepsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} = -\frac{\partial}{\partial\theta^\alpha} \qquad \frac{\partial}{\partial\theta_\alpha}\theta\theta = -2\theta^\alpha \qquad \frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta_\beta}\theta\theta = 2\varepsilon^{\alpha\beta} \qquad \frac{\partial}{\partial\bar\theta_{\dot\alpha}}\frac{\partial}{\partial\bar\theta_{\dot\beta}}\bar\theta\bar\theta = -2\varepsilon^{\dot\alpha\dot\beta}$$

$$\varepsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\beta}} = -\frac{\partial}{\partial\bar\theta_{\dot\alpha}} \qquad \frac{\partial}{\partial\bar\theta_{\dot\alpha}}\bar\theta\bar\theta = 2\bar\theta^{\dot\alpha} \qquad \frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta^\beta}\theta\theta = 2\delta^\alpha_\beta \qquad \frac{\partial}{\partial\bar\theta_{\dot\alpha}}\frac{\partial}{\partial\bar\theta^{\dot\beta}}\bar\theta\bar\theta = -2\delta^{\dot\alpha}_{\dot\beta}$$

$$\varepsilon_{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar\theta_{\dot\beta}} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} \qquad \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta\bar\theta = -2\bar\theta_{\dot\alpha} \qquad \frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\theta\theta = -2\varepsilon_{\alpha\beta} \qquad \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\frac{\partial}{\partial\bar\theta^{\dot\beta}}\bar\theta\bar\theta = 2\varepsilon_{\dot\alpha\dot\beta}$$

In addition, we define

$$(y, \theta', \bar\theta') := (x - \mathrm{i}\theta\sigma^\mu\bar\theta, \theta, \bar\theta) : \tag{11.20}$$

$$\bar{\mathcal{D}}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta'^{\dot\alpha}}; \qquad \begin{pmatrix} \frac{\partial}{\partial x^\mu} \\ \frac{\partial}{\partial\theta^\alpha} \\ \frac{\partial}{\partial\bar\theta^{\dot\alpha}} \end{pmatrix} = \begin{pmatrix} \delta_\mu^\nu & 0 & 0 \\ -\mathrm{i}(\sigma^\nu\bar\theta)_\alpha & \delta_\alpha^\beta & 0 \\ \mathrm{i}(\theta\sigma^\nu)_{\dot\alpha} & 0 & \delta_{\dot\alpha}^{\dot\beta} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y^\nu} \\ \frac{\partial}{\partial\theta'^\beta} \\ \frac{\partial}{\partial\bar\theta'^{\dot\beta}} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial}{\partial y^\nu} \\ \frac{\partial}{\partial\theta'^\beta} \\ \frac{\partial}{\partial\bar\theta'^{\dot\beta}} \end{pmatrix} = \begin{pmatrix} \delta_\nu^\mu & 0 & 0 \\ \mathrm{i}(\sigma^\mu\bar\theta)_\beta & \delta_\beta^\alpha & 0 \\ -\mathrm{i}(\theta\sigma^\mu)_{\dot\beta} & 0 & \delta_{\dot\beta}^{\dot\alpha} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x^\mu} \\ \frac{\partial}{\partial\theta^\alpha} \\ \frac{\partial}{\partial\bar\theta^{\dot\alpha}} \end{pmatrix}, \tag{11.21}$$

and a function $f : \mathbb{C}^4 \to \mathbb{C}$ (independent of θ' and $\bar\theta'$) is expanded as

$$f(y) = f(x - \mathrm{i}\theta\sigma\bar\theta) = f(x) - \mathrm{i}(\theta\sigma^\mu\bar\theta)\partial_\mu f(x) - \frac{1}{4}\theta^4\partial^2 f(x). \tag{11.22}$$

Note that we differentiate $[f(y)]^*$ and $f^*(y)$:

$$[f(y)]^* = f(x) + \mathrm{i}(\theta\sigma^\mu\bar\theta)\partial_\mu f^*(x) - \frac{1}{4}\theta^4\partial^2 f^*(x) = f^*(y + \mathrm{i}\theta\sigma\bar\theta) = f^*(y^*). \tag{11.23}$$

11.3 Superfields

SUSY-invariant Lagrangian SUSY transformation is induced by $\xi\mathcal{Q} + \bar\xi\bar{\mathcal{Q}} = \xi^\alpha\partial_\alpha + \bar\xi_{\dot\alpha}\partial^{\dot\alpha} + \mathrm{i}(\xi\sigma^\mu\bar\theta + \bar\xi\bar\sigma^\mu\theta)\partial_\mu$. Therefore, for an object Ψ in the superspace,

$$\big[\Psi\big]_{\theta^4} \xrightarrow{\text{SUSY}} \big[\Psi + \xi^\alpha\partial_\alpha\Psi + \bar\xi_{\dot\alpha}\partial^{\dot\alpha}\Psi + \mathrm{i}(\xi\sigma^\mu\bar\theta + \bar\xi\bar\sigma^\mu\theta)\partial_\mu\Psi\big]_{\theta^4} = \big[\Psi + \mathrm{i}(\xi\sigma^\mu\bar\theta + \bar\xi\bar\sigma^\mu\theta)\partial_\mu\Psi\big]_{\theta^4}, \tag{11.24}$$

which means $\big[\Psi\big]_{\theta^4}$ is SUSY-invariant up to total derivative, i.e., $\int \mathrm{d}^4x\big[\Psi\big]_{\theta^4}$ is SUSY-invariant action. Also,

$$\big[\Psi\big]_{\theta^2} \xrightarrow{\text{SUSY}} \big[\Psi + \bar\xi_{\dot\alpha}\left(\partial^{\dot\alpha} + \mathrm{i}(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu\right)\Psi\big]_{\theta^2} = \big[\Psi + \bar\xi_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\Psi + 2\mathrm{i}(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu\Psi\big]_{\theta^2} \tag{11.25}$$

will be SUSY-invariant if $\bar{\mathcal{D}}_{\dot\alpha}\Psi = 0$, i.e., Ψ is a chiral superfield. Therefore, SUSY-invariant Lagrangian is given by

$$\mathcal{L} = \big[(\text{any real superfield})\big]_{\theta^4} + \big[(\text{any chiral superfield})\big]_{\theta^2} + \big[(\text{any chiral superfield})^*\big]_{\bar\theta^2}. \tag{11.26}$$

Chiral superfield A chiral superfield is a superfield that satisfies $\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0$, i.e., we find

$$\Phi = \phi(y) + \sqrt{2}\theta'\psi(y) + \theta'^2F(y) \tag{11.27}$$

$$= \phi(x) + \sqrt{2}\theta\psi(x) - \mathrm{i}\partial_\mu\phi(x)(\theta\sigma^\mu\bar\theta) + F(x)\theta^2 + \frac{\mathrm{i}}{\sqrt{2}}(\partial_\mu\psi(x)\sigma^\mu\bar\theta)\theta^2 - \frac{1}{4}\partial^2\phi(x)\theta^4 \tag{11.28}$$

$$\Phi^* = \phi^*(x) + \sqrt{2}\bar\psi(x)\bar\theta + F^*(x)\bar\theta^2 + \mathrm{i}\partial_\mu\phi^*(x)(\theta\sigma^\mu\bar\theta) - \frac{\mathrm{i}}{\sqrt{2}}[\theta\sigma^\mu\partial_\mu\bar\psi(x)]\bar\theta^2 - \frac{1}{4}\partial^2\phi^*(x)\theta^4; \tag{11.29}$$

their product is expanded as

$$\begin{aligned}\Phi_i^*\Phi_j &= \phi_i^*\phi_j + \sqrt{2}\phi_i^*(\theta\psi_j) + \sqrt{2}(\bar\psi_i\bar\theta)\phi_j + \phi_i^*F_j\theta^2 + 2(\bar\psi_i\bar\theta)(\theta\psi_j) - \mathrm{i}\left(\phi_i^*\partial_\mu\phi_j - \partial_\mu\phi_i^*\phi_j\right)(\theta\sigma^\mu\bar\theta) + F_i^*\phi_j\bar\theta^2\\
&\quad + \left[\sqrt{2}\bar\psi_i\bar\theta F_j - \frac{\mathrm{i}\left(\partial_\mu\phi_i^*\cdot\psi_j\sigma^\mu\bar\theta - \phi_i^*\partial_\mu\psi_j\sigma^\mu\bar\theta\right)}{\sqrt{2}}\right]\theta^2 + \left[\sqrt{2}F_i^*\theta\psi_j + \frac{\mathrm{i}\left(\theta\sigma^\mu\bar\psi_i\partial_\mu\phi_j - \theta\sigma^\mu\partial_\mu\bar\psi_i\phi_j\right)}{\sqrt{2}}\right]\bar\theta^2\\
&\quad + \frac14\left(4F_i^*F_j - \phi_i^*\partial^2\phi_j - (\partial^2\phi_i^*)\phi_j + 2(\partial_\mu\phi_i^*)(\partial^\mu\phi_j) + 2\mathrm{i}(\psi_j\sigma^\mu\partial_\mu\bar\psi_i) - 2\mathrm{i}(\partial_\mu\psi_j\sigma^\mu\bar\psi_i)\right)\theta^4
\end{aligned}\tag{11.30}$$

$$\begin{aligned}&\equiv \phi_i^*\phi_j + \sqrt{2}\phi_i^*(\theta\psi_j) + \sqrt{2}(\bar\psi_i\bar\theta)\phi_j + \phi_i^*F_j\theta^2 + 2(\bar\psi_i\bar\theta)(\theta\psi_j) - 2\mathrm{i}\left(\phi_i^*\partial_\mu\phi_j\right)(\theta\sigma^\mu\bar\theta) + F_i^*\phi_j\bar\theta^2\\
&\quad + \sqrt{2}\left(\bar\psi_i\bar\theta F_j + \mathrm{i}\phi_i^*\partial_\mu\psi_j\sigma^\mu\bar\theta\right)\theta^2 + \sqrt{2}\left(F_i^*\theta\psi_j - \mathrm{i}\theta\sigma^\mu\partial_\mu\bar\psi_i\phi_j\right)\bar\theta^2\\
&\quad + \left(F_i^*F_j + (\partial_\mu\phi_i^*)(\partial^\mu\phi_j) + \mathrm{i}\bar\psi_i\bar\sigma^\mu\partial_\mu\psi_j\right)\theta^4
\end{aligned}\tag{11.31}$$

$$\Phi_i\Phi_j\Big|_{\theta^2} = -\psi_i\psi_j + F_i\phi_j + \phi_iF_j \tag{11.32}$$

$$\Phi_i\Phi_j\Phi_k\Big|_{\theta^2} = -(\psi_i\psi_j)\phi_k - (\psi_k\psi_i)\phi_j - (\psi_j\psi_k)\phi_i + \phi_i\phi_jF_k + \phi_k\phi_iF_j + \phi_j\phi_kF_i \tag{11.33}$$

$$\mathrm{e}^{k\Phi} = \mathrm{e}^{k\phi}\left[1 + \sqrt{2}k\theta\psi + \left(kF - \frac{k^2}{2}\psi\psi\right)\theta^2 - \mathrm{i}k\partial_\mu\phi(\theta\sigma^\mu\bar\theta) + \frac{\mathrm{i}k\left(\partial_\mu\psi + k\psi\partial_\mu\phi\right)\sigma^\mu\bar\theta\theta^2}{\sqrt{2}} - \frac{k}{4}\left(\partial^2\phi + k\partial_\mu\phi\partial^\mu\phi\right)\theta^4\right]; \tag{11.34}$$

note that $\Phi_i\Phi_j$, $\Phi_i\Phi_j\Phi_k$, and $\mathrm{e}^{k\Phi}$ are all chiral superfields.

Vector superfield A vector superfield is a superfield V that satisfies $V = V^*$. It is given by real fields $\{C, M, N, D, A_\mu\}$ and Grassmann fields $\{\chi, \lambda\}$ as[*22]

$$\begin{aligned}V(x,\theta,\bar\theta) &= C(x) + \mathrm{i}\theta\chi(x) - \mathrm{i}\bar\theta\bar\chi(x) + \frac12\left(M(x)+\mathrm{i}N(x)\right)\theta^2 + \frac12\left(M(x)-\mathrm{i}N(x)\right)\bar\theta^2 + (\bar\theta\bar\sigma^\mu\theta)A_\mu(x)\\
&\quad \left(\lambda(x) + \frac12\partial_\mu\bar\chi(x)\bar\sigma^\mu\right)\theta\bar\theta^2 + \theta^2\bar\theta\left(\bar\lambda(x) + \frac12\bar\sigma^\mu\partial_\mu\chi(x)\right) + \frac12\left(D(x) - \frac12\partial^2C(x)\right)\theta^4.
\end{aligned}\tag{11.35}$$

With this convention,

$$V \to V - \mathrm{i}\Phi + \mathrm{i}\Phi^* \iff \begin{cases} C \to C - \mathrm{i}\phi + \mathrm{i}\phi^*, & \chi \to \chi - \sqrt{2}\psi, \qquad \lambda \to \lambda,\\ M + \mathrm{i}N \to M + \mathrm{i}N - 2\mathrm{i}F, & A_\mu \to A_\mu + \partial_\mu(\phi + \phi^*), \qquad D \to D.\end{cases} \tag{11.36}$$

The exponential of a vector superfield is also a vector superfield:

$$\begin{aligned}\mathrm{e}^{kV} = \mathrm{e}^{kC}\Big\{&1 + \mathrm{i}k(\theta\chi - \bar\theta\bar\chi) + \left(\frac{M+\mathrm{i}N}{2}k + \frac{\chi\chi}{4}k^2\right)\theta^2 + \left(\frac{M-\mathrm{i}N}{2}k + \frac{\bar\chi\bar\chi}{4}k^2\right)\bar\theta^2 + (k^2\theta\chi\bar\theta\bar\chi - k\theta\sigma^\mu\bar\theta A_\mu)\\
&+ \left[k\bar\theta\bar\lambda - \mathrm{i}k\bar\theta\bar\chi\left(\frac{M+\mathrm{i}N}{2}k + \frac{\chi\chi}{4}k^2\right) + \frac12 k\bar\theta\bar\sigma^\mu\left(\partial_\mu\chi - \mathrm{i}k\chi A_\mu\right)\right]\theta^2\\
&+ \left[k\theta\lambda + \mathrm{i}k\theta\chi\left(\frac{M-\mathrm{i}N}{2}k + \frac{\bar\chi\bar\chi}{4}k^2\right) - \frac12 k\theta\sigma^\mu\left(\partial_\mu\bar\chi + \mathrm{i}k\bar\chi A_\mu\right)\right]\bar\theta^2\\
&+ \left[\frac{k}{2}\left(D - \frac12\partial^2C\right) - \frac12\mathrm{i}k^2(\lambda\chi - \bar\lambda\bar\chi) + \left(\frac{M+\mathrm{i}N}{2}k + \frac{\chi\chi}{4}k^2\right)\left(\frac{M-\mathrm{i}N}{2}k + \frac{\bar\chi\bar\chi}{4}k^2\right)\right.\\
&\quad\left. + \frac{k^3}{4}\bar\chi\bar\sigma^\mu\chi A_\mu + \frac{k^2}{4}\left(\mathrm{i}\bar\chi\bar\sigma^\mu\partial_\mu\chi - \mathrm{i}\partial_\mu\bar\chi\bar\sigma^\mu\chi + A^\mu A_\mu\right)\right]\theta^4\Big\}.
\end{aligned}\tag{11.37}$$

Supergauge symmetry The gauge transformation $\phi(x) \to \mathrm{e}^{\mathrm{i}g\theta^a(x)t^a}\phi(x)$ is not closed in the chiral superfield. Namely, if the parameter $\theta(x)$ has x^μ-dependence, $\mathrm{e}^{\mathrm{i}g\theta^a(x)t^a}\Phi(x)$ is not a chiral superfield. Hence, in supersymmetric theories, it is extended to *supergauge symmetry* parameterized by a chiral superfield $\Omega(x)$, which is given by

$$\Phi \to \mathrm{e}^{2\mathrm{i}g\Omega^a(x)t^a}\Phi, \qquad \Phi^* \to \Phi^*\mathrm{e}^{-2\mathrm{i}g\Omega^{*a}(x)t^a} \tag{11.38}$$

for a chiral superfield Φ and an anti-chiral superfield Φ^*. The supergauge-invariant Lagrangian should be

$$\mathcal{L} \sim \Phi^*\cdot(\text{real superfield})\cdot\Phi; \tag{11.39}$$

we parameterize the "real superfield" as $\mathrm{e}^{2gV^a(x)t^a}$:

$$\mathcal{L} = \left[\Phi^*\mathrm{e}^{2gV^a(x)t^a}\Phi\right]_{\theta^4}; \qquad \mathrm{e}^{2gV^a(x)t^a} \to \mathrm{e}^{2\mathrm{i}g\Omega^{*a}(x)t^a}\mathrm{e}^{2gV^a(x)t^a}\mathrm{e}^{-2\mathrm{i}g\Omega^a(x)t^a}. \tag{11.40}$$

In Abelian case, t^a is replaced by the charge Q of Φ and

$$\mathcal{L} = \left[\Phi^*\mathrm{e}^{2gQV(x)}\Phi\right]_{\theta^4}; \qquad \Phi \to \mathrm{e}^{2\mathrm{i}gQ\Omega(x)}\Phi, \quad \Phi^* \to \Phi^*\mathrm{e}^{-2\mathrm{i}gQ\Omega^*(x)}, \tag{11.41}$$

$$\mathrm{e}^{2gQV(x)} \to \mathrm{e}^{2\mathrm{i}gQ\Omega^*(x)}\mathrm{e}^{2gQV(x)}\mathrm{e}^{-2\mathrm{i}gQ\Omega(x)} = \mathrm{e}^{2gQ(V-\mathrm{i}\Omega+\mathrm{i}\Omega^*)}. \tag{11.42}$$

[*22] Different coordination of "i"s are found in literature. Take care, especially, λ(ours) $= \mathrm{i}\lambda$(Wess-Bagger) $= \mathrm{i}\lambda$(SLHA).

The usual gauge transformation corresponds to the real part of the lowest component of Ω, i.e., $\theta \equiv 2\operatorname{Re}\phi = \phi + \phi^*$, and we use the other components to fix the supergauge so that C, M, N and χ are eliminated:

$$\text{supergauge fixing:}\quad V(x) \longrightarrow (\bar\theta\bar\sigma^\mu\theta)A_\mu(x) + \bar\theta^2\theta\lambda(x) + \theta^2\bar\theta\bar\lambda(x) + \frac{1}{2}D(x) \qquad \text{(Wess-Zumino gauge);} \tag{11.43}$$

$$e^{2gQV} \longrightarrow 1 + gQ\left(-2\theta\sigma^\mu\bar\theta A_\mu + 2\theta^2\bar\theta\bar\lambda + 2\bar\theta^2\theta\lambda + D\theta^4\right) + g^2Q^2A^\mu A_\mu\theta^4. \tag{11.44}$$

The gauge transformation is the remnant freedom: $\Theta = \phi(y) = \phi - i\partial_\mu\phi(\theta\sigma^\mu\bar\theta) - \partial^2\phi\theta^4/4$ with ϕ being real;

$$\Phi_i \to e^{2igQ\Theta}\Phi_i, \qquad e^{2gQV} \to e^{2gQ(V - i\Theta + i\Theta^*)}. \tag{11.45}$$

Rules for each component is obvious in $(y,\theta,\bar\theta)$-basis and given by

$$\{\phi,\psi,F\} \to e^{igQ\theta}\{\phi,\psi,F\}, \qquad A_\mu \to A_\mu + \partial_\mu\theta, \qquad \lambda\to\lambda, \qquad D\to D. \tag{11.46}$$

For non-Abelian gauges, the supergauge transformation for the real field is evaluated as

$$e^{2gV} \to e^{2ig\Omega^*}e^{2gV}e^{-2ig\Omega} \tag{11.47}$$

$$= \left(e^{2ig\Omega^*}e^{2gV}e^{-2ig\Omega^*}\right)\left(e^{2ig\Omega^*}e^{-2ig\Omega}\right) \tag{11.48}$$

$$= \exp\left(e^{[2ig\Omega^*,}2gV\right)e^{2ig(\Omega^*-\Omega)} + \mathcal{O}(\Omega^2) \tag{11.49}$$

$$= \exp\left(2gV + [2ig\Omega^*, 2gV]\right)e^{2ig(\Omega^*-\Omega)} + \mathcal{O}(\Omega^2); \tag{11.50}$$

$$= \exp\left[2gV + [2ig\Omega^*, 2gV] + \int_0^1 dt\, g(e^{[2gV,}\cdot)2ig(\Omega^*-\Omega)] + \mathcal{O}(\Omega^2)\right] \tag{11.51}$$

$$= \exp\left[2gV + [2ig\Omega^*, 2gV] + \sum_{n=0}^\infty \frac{B_n\left([2gV,)^n\right.}{n!}2ig(\Omega^*-\Omega)]\right] + \mathcal{O}(\Omega^2) \tag{11.52}$$

$$= \exp\left[2g\left(V + i(\Omega^*-\Omega) - [V, ig(\Omega^*+\Omega)] + \sum_{n=2}^\infty \frac{iB_n\left([2gV,)^n\right.}{n!}(\Omega^*-\Omega)]\right) + \mathcal{O}(\Omega^2)\right]. \tag{11.53}$$

Here, again we can use the "non-gauge" component of Ω to eliminate the C-term etc., i.e., we fix $i(\Omega^*-\Omega)$, the second term of the expansion, to remove those terms:

$$V - [V, ig(\Omega^*+\Omega)] + \left(i + \sum_{n=2}^\infty \frac{iB_n\left([2gV,)^n\right.}{n!}\right)(\Omega^*-\Omega)] + \mathcal{O}(\Omega^2) = (\bar\theta\bar\sigma^\mu\theta)A_\mu + \bar\theta^2\theta\lambda + \theta^2\bar\theta\bar\lambda + \frac{1}{2}D; \tag{11.54}$$

this defines the Wess-Zumino gauge:

$$\text{supergauge fixing:}\quad V^a(x) \longrightarrow (\bar\theta\bar\sigma^\mu\theta)A^a_\mu(x) + \bar\theta^2\theta\lambda^a(x) + \theta^2\bar\theta\bar\lambda^a(x) + \frac{1}{2}D^a(x), \tag{11.55}$$

$$e^{2gV^at^a} \longrightarrow 1 + g\left(-2\theta\sigma^\mu\bar\theta A^a_\mu + 2\theta^2\bar\theta\bar\lambda^a + 2\bar\theta^2\theta\lambda^a + D^a\theta^4\right)t^a + g^2A^{a\mu}A^b_\mu\theta^4t^at^b. \tag{11.56}$$

The gauge transformation is given by

$$\Phi \to e^{2ig\Theta^at^a}\Phi, \quad e^{2gV^at^a} \to e^{2ig\Theta^bt^b}e^{2gV^at^a}e^{-2ig\Theta^ct^c}. \tag{11.57}$$

For components in chiral superfields,

$$\{\phi,\psi,F\} \to e^{ig\theta^at^a}\{\phi,\psi,F\}, \tag{11.58}$$

while for vector superfield we can express as infinitesimal transformation:

$$V \to V' \simeq V + i(\Theta^*-\Theta) - [V, ig(\Theta^*+\Theta)] + \sum_{n=2}^\infty \frac{iB_n\left([2gV,)^n\right.}{n!}(\Theta^*-\Theta)] \tag{11.59}$$

$$= V + 2(\bar\theta\bar\sigma^\mu\theta)\partial_\mu\phi - \left[V, ig\left(2\phi - \frac{\theta^4}{2}\partial^2\phi\right)\right] + 2\sum_{n=2}^\infty \frac{B_n\left([2gV,)^n\right.}{n!}(\bar\theta\bar\sigma^\mu\theta)\partial_\mu\phi] \tag{11.60}$$

$$= V + 2(\bar\theta\bar\sigma^\mu\theta)\partial_\mu\phi + 2gf^{abc}V^b\phi^ct^a \qquad \text{(Wess-Zumino gauge)} \tag{11.61}$$

$$\begin{aligned} \therefore A^a_\mu &\to A^a_\mu + \partial_\mu\theta^a + gf^{abc}A^b_\mu\theta^c + \mathcal{O}(\theta^2), & \lambda^a &\to \lambda^a + gf^{abc}\lambda^b\theta^c + \mathcal{O}(\theta^2), \\ D^a &\to D^a + gf^{abc}D^b\theta^c + \mathcal{O}(\theta^2), & \bar\lambda^a &\to \bar\lambda^a + gf^{abc}\bar\lambda^b\theta^c + \mathcal{O}(\theta^2). \end{aligned} \tag{11.62}$$

Gauge-field strength The real superfield e^V is gauge-invariant in Abelian case and a candidate in Lagrangian term, but this is not case in non-Abelian case. We thus define a chiral superfield from e^V:

$$\mathcal{W}_\alpha = \frac{1}{4}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\left(e^{-2gV}\mathcal{D}_\alpha e^{2gV}\right); \tag{11.63}$$

$$\mathcal{W}_\alpha \xrightarrow{\text{gauge}} e^{2ig\Omega}\mathcal{W}_\alpha e^{-2ig\Omega} \quad \left(\mathcal{W}^a_\alpha \xrightarrow{\text{gauge}} [e^{+2g\tilde f^c\Omega^c}]^{ab}\mathcal{W}^b_\alpha \quad \text{with} \quad [\tilde f^c]_{ab} = f^{abc}\right);\text{[*23]} \tag{11.64}$$

it is not supergauge- or Lorentz-invariatn, but $\mathrm{Tr}(\mathcal{W}^\alpha\mathcal{W}_\alpha) = \mathrm{Tr}(\epsilon^{\alpha\beta}\mathcal{W}_\beta\mathcal{W}_\alpha)$ is supergauge- and Lorentz-invariant, and its θ^2-term is SUSY-invariant, which becomes a candidate in SUSY Lagrangian with its Hermitian conjugate.

[*23] ♠TODO:This equivalence should be checked/explained in gauge-theory section; especially, the sign is not verified and might be opposite.♠

In Wess-Zumino gauge, it is given by

$$\mathcal{W}_\alpha = \left\{\lambda^a_\alpha(y) + \theta_\alpha D^a(y) + \frac{[\mathrm{i}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)\theta]_\alpha}{4} F^a_{\mu\nu}(y) + \theta^2\left[\mathrm{i}\sigma^\mu \mathrm{D}_\mu\bar\lambda^a(y^*)\right]_\alpha\right\} t^a \tag{11.65}$$

$$= \left[\lambda^a_\alpha + \theta_\alpha D^a + \frac{\mathrm{i}}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha F^a_{\mu\nu} + \mathrm{i}\theta^2(\sigma^\mu\mathrm{D}_\mu\bar\lambda^a)_\alpha + \mathrm{i}(\bar\theta\bar\sigma^\mu\theta)\partial_\mu\lambda^a_\alpha - \frac{\theta^4}{4}\partial^2\lambda^a_\alpha \right.$$
$$\left. + \frac{\mathrm{i}\theta^2(\sigma^\mu\bar\theta)_\alpha}{2}\left(\partial_\mu D^a + \mathrm{i}\partial^\nu F^a_{\mu\nu} - g f^{abc}\varepsilon_{\mu\nu\rho\sigma}A^{\nu b}\partial^\rho A^{\sigma c}\right)\right] T^a, \tag{11.66}$$

where, as usual,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gA^b_\mu A^c_\nu f^{abc}, \qquad \mathrm{D}_\mu\lambda^a_\alpha = \partial_\mu\lambda^a_\alpha + gf^{abc}A^b_\mu\lambda^c_\alpha. \tag{11.67}$$

Also,

$$[\mathrm{Tr}(\mathcal{W}^\alpha\mathcal{W}_\alpha)]_{\theta^2} = \left[\mathrm{i}\lambda^a\sigma^\mu\mathrm{D}_\mu\bar\lambda^b + \mathrm{i}\lambda^b\sigma^\mu\mathrm{D}_\mu\bar\lambda^a + D^aD^b - \frac{1}{4}\left(\mathrm{i}\varepsilon^{\sigma\mu\nu\rho} + 2\eta^{\mu\rho}\eta^{\nu\sigma}\right)F^a_{\mu\nu}F^b_{\rho\sigma}\right]\mathrm{Tr}(t^at^b) \tag{11.68}$$

$$= \mathrm{i}\lambda^a\sigma^\mu\mathrm{D}_\mu\bar\lambda^a + \frac{1}{2}D^aD^a - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\mathrm{i}}{8}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}, \tag{11.69}$$

$$[\mathrm{Tr}(\mathcal{W}^\alpha\mathcal{W}_\alpha)]_{\theta^4} = \frac{\theta^4}{4}\left(2(\partial^\mu\lambda^a)(\partial_\mu\lambda^b) - \lambda^a\partial^2\lambda^b - (\partial^2\lambda^a)\lambda^b\right)\mathrm{Tr}(t^at^b) = \frac{\theta^4}{4}\left((\partial^\mu\lambda^a)(\partial_\mu\lambda^a) - \lambda^a\partial^2\lambda^a\right). \tag{11.70}$$

For Abelian theory,

$$\mathcal{W}_\alpha = \frac{1}{4}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\left(\mathrm{e}^{-2gV}\mathcal{D}_\alpha\mathrm{e}^{2gV}\right) = \frac{1}{4}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\mathcal{D}_\alpha(2gV), \tag{11.71}$$

$$\mathcal{W}^\alpha\mathcal{W}_\alpha\Big|_{\theta^2} = 2\left(\mathrm{i}\lambda\sigma^\mu\mathrm{D}_\mu\bar\lambda + \frac{1}{2}DD - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mathrm{i}}{8}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right). \tag{11.72}$$

11.4 Lagrangian blocks

Lagrangian construction The supergauge transformation is summarized as

$$\Phi_i \to [U_\Phi]_{ij}\Phi_j, \qquad \tilde\Phi_j \to \tilde\Phi_i[U_\Phi^{-1}]_{ij}, \qquad \mathcal{W}_\alpha \to U_\mathcal{W}\mathcal{W}_\alpha U_\mathcal{W}^{-1}, \tag{11.73}$$

where

$$\tilde\Phi^*_j := \Phi^*_i[\mathrm{e}^{2gVt^a_\Phi}]_{ij}, \qquad U_\Phi := \exp(2\mathrm{i}g\Omega^a t^a_\Phi), \qquad U_\mathcal{W} := \exp(2\mathrm{i}g\Omega^a t^a_\mathcal{W}), \tag{11.74}$$

t^a_Φ is the representation matrix or U(1) charge for the field Φ, and $t^a_\mathcal{W}$ is the representation matrix that is used to define $\mathcal{W}_\alpha$. To construct a Lagrangian, we should composite these ingredients in real and invariant under SUSY, supergauge, and Lorentz transformation. A sufficient condition for SUSY invariance is given by (11.26), so

$$\mathcal{L} = \left[K(\Phi_i, \tilde\Phi^*_j)\right]_{\theta^4} + \left\{\left[f_{ab}(\Phi_i)\mathcal{W}^a\mathcal{W}^b\right]_{\theta^2} + \text{H.c.}\right\} + \left\{\left[W(\Phi_i)\right]_{\theta^2} + \text{H.c.}\right\} + D \tag{11.75}$$

is one possible construction. The Kähler function K should be real and supergauge invariant, the gauge kinetic function f should be holomorphic and supergauge invariant with $\mathcal{W}^a\mathcal{W}^b$, and the superpotential W is holomorphic and supergauge invariant. The last term D (Fayet-Illiopoulos term) comes from V of an U(1) gauge boson; note that its supergauge invariance is due to the intentional definition of V.

One can construct more general Lagrangian; for example, one can introduce a vector superfield that is not associated to a gauge symmetry, but then the supergauge fixing is not available and one has to include C or M fields.

Renormalizable Lagrangian Since $[\Phi]_{\theta^4}$ is a total derivative, renormalizable Lagrangian is limited to

$$\mathcal{L} = \left[\Phi^*_i[\mathrm{e}^{2gVt^a_\Phi}]_{ij}\Phi_j\right]_{\theta^4} + \left\{[\mathcal{W}^a\mathcal{W}^a]_{\theta^2} + [W(\Phi_i)]_{\theta^2} + \text{H.c.}\right\} + D \tag{11.76}$$

up to numeric coefficients. With multiple gauge groups, the Kähler part is extended as $\Phi^*_i[\mathrm{e}^{2gVt^a_\Phi}\mathrm{e}^{2gV't'^a_\Phi}\cdots]_{ij}\Phi_j$, where the inner part is obviously commutable.

12 Minimal Supersymmetric Standard Model

Gauge symmetry: $\mathrm{SU}(3)_{\text{color}} \times \mathrm{SU}(2)_{\text{weak}} \times \mathrm{U}(1)_Y$

Particle content:

(a) Chiral superfields

	SU(3)	SU(2)	U(1)	B	L	scalar/spinor
Q_i	$\mathbf{3}$	$\mathbf{2}$	1/6	1/3		$\tilde{q}_\mathrm{L}, q_\mathrm{L}$ $[\to (u_\mathrm{L}, d_\mathrm{L})]$
L_i		$\mathbf{2}$	−1/2		1	$\tilde{l}_\mathrm{L}, l_\mathrm{L}$ $[\to (\nu_\mathrm{L}, l_\mathrm{L})]$
U_i^c	$\overline{\mathbf{3}}$		−2/3	−1/3		$\tilde{u}_\mathrm{R}^\mathrm{c}, u_\mathrm{R}^\mathrm{c}$
D_i^c	$\overline{\mathbf{3}}$		1/3	−1/3		$\tilde{d}_\mathrm{R}^\mathrm{c}, d_\mathrm{R}^\mathrm{c}$
E_i^c			1		−1	$\tilde{e}_\mathrm{R}^\mathrm{c}, e_\mathrm{R}^\mathrm{c}$
H_u		$\mathbf{2}$	1/2			$h_\mathrm{u}, \tilde{h}_\mathrm{u}$ $[\to (h_\mathrm{u}^+, h_\mathrm{u}^0)]$
H_d		$\mathbf{2}$	−1/2			$h_\mathrm{d}, \tilde{h}_\mathrm{d}$ $[\to (h_\mathrm{d}^0, h_\mathrm{d}^-)]$

(b) Vector superfields

	SU(3)	SU(2)	U(1)	ino/boson
g	adj.			$\tilde{g}, g_\mu$
W		adj.		$\tilde{w}, W_\mu$
B				$\tilde{b}, B_\mu$

Here, each of the column groups shows (from left to right) superfield name, charges for the gauge symmetries, other quantum numbers if relevant, and notation for corresponding fields (and SU(2) decomposition).

``c"-notation For scalars, $\tilde{\phi}_\mathrm{R}^\mathrm{c} := \phi_\mathrm{R}^* = \mathsf{C}\phi_\mathrm{R}\mathsf{C}$ (because the intrinsic phase for C is +1 for quarks and leptons.) For matter spinors, $\psi_\mathrm{R}^\mathrm{c} := \bar{\psi}_\mathrm{R}$ (and $\psi_\mathrm{R} = \bar{\psi}_\mathrm{R}^\mathrm{c}$); Dirac spinors are thus

$$\psi_\mathrm{L} = \begin{pmatrix}\psi_\mathrm{L}\\ 0\end{pmatrix}, \quad \overline{\psi_\mathrm{L}} = \begin{pmatrix}0 & \bar{\psi}_\mathrm{L}\end{pmatrix}, \quad \psi_\mathrm{R}^\mathrm{c} := \begin{pmatrix}\psi_\mathrm{R}^\mathrm{c}\\ 0\end{pmatrix} = C\begin{pmatrix}0\\ \psi_\mathrm{R}\end{pmatrix} = C\psi_\mathrm{R}, \quad \overline{\psi_\mathrm{R}^\mathrm{c}} = \begin{pmatrix}0 & \psi_\mathrm{R}\end{pmatrix} = \begin{pmatrix}\bar{\psi}_\mathrm{R} & 0\end{pmatrix} C = \overline{\psi_\mathrm{R}}C.$$

Superpotential and $\cancel{\mathrm{SUSY}}$-terms

$$W_\mathrm{RPC} = \mu H_\mathrm{u} H_\mathrm{d} - y_{uij} U_i^\mathrm{c} H_\mathrm{u} Q_j + y_{dij} D_i^\mathrm{c} H_\mathrm{d} Q_j + y_{eij} E_i^\mathrm{c} H_\mathrm{d} L_j, \tag{12.1}$$

$$W_\mathrm{RPV} = -\kappa_i L_i H_\mathrm{u} + \frac{1}{2}\lambda_{ijk} L_i L_j E_k^\mathrm{c} + \lambda'_{ijk} L_i Q_j D_k^\mathrm{c} + \frac{1}{2}\lambda''_{ijk} U_i^\mathrm{c} D_j^\mathrm{c} D_k^\mathrm{c}, \tag{12.2}$$

$$\mathcal{L}_{\cancel{\mathrm{SUSY}}} = -\frac{1}{2}\left(M_3 \tilde{g}_0 \tilde{g}_0 + M_2 \tilde{w}\tilde{w} + M_1 \tilde{b}\tilde{b} + \text{H.c.}\right) - V_{\cancel{\mathrm{SUSY}}}; \tag{12.3}$$

$$\begin{aligned} V_{\cancel{\mathrm{SUSY}}}^\mathrm{RPC} = {}& \left(\tilde{q}_\mathrm{L}^* m_Q^2 \tilde{q}_\mathrm{L} + \tilde{l}_\mathrm{L}^* m_L^2 \tilde{l}_\mathrm{L} + \tilde{u}_\mathrm{R}^* m_{U^\mathrm{c}}^2 \tilde{u}_\mathrm{R} + \tilde{d}_\mathrm{R}^* m_{D^\mathrm{c}}^2 \tilde{d}_\mathrm{R} + \tilde{e}_\mathrm{R}^* m_{E^\mathrm{c}}^2 \tilde{e}_\mathrm{R} + m_{H_\mathrm{u}}^2 |h_\mathrm{u}|^2 + m_{H_\mathrm{d}}^2 |h_\mathrm{d}|^2\right) \\ & + \left(-\tilde{u}_\mathrm{R}^* h_\mathrm{u} a_\mathrm{u} \tilde{q}_\mathrm{L} + \tilde{d}_\mathrm{R}^* h_\mathrm{d} a_\mathrm{d} \tilde{q}_\mathrm{L} + \tilde{e}_\mathrm{R}^* h_\mathrm{d} a_\mathrm{e} \tilde{l}_\mathrm{L} + b H_\mathrm{u} H_\mathrm{d} + \text{H.c.}\right) \\ & + \left(+\tilde{u}_\mathrm{R}^* h_\mathrm{d}^* c_\mathrm{u} \tilde{q}_\mathrm{L} + \tilde{d}_\mathrm{R}^* h_\mathrm{u}^* c_\mathrm{d} \tilde{q}_\mathrm{L} + \tilde{e}_\mathrm{R}^* h_\mathrm{u}^* c_\mathrm{e} \tilde{l}_\mathrm{L} + \text{H.c.}\right), \end{aligned} \tag{12.4}$$

$$\begin{aligned} V_{\cancel{\mathrm{SUSY}}}^\mathrm{RPV} = {}& \left(-b_i \tilde{l}_{\mathrm{L}i} H_\mathrm{u} + \frac{1}{2} T_{ijk} \tilde{l}_{\mathrm{L}i} \tilde{l}_{\mathrm{L}j} \tilde{e}_{\mathrm{R}k}^* + T'_{ijk} \tilde{l}_{\mathrm{L}i} \tilde{q}_{\mathrm{L}j} \tilde{d}_{\mathrm{R}k}^* + \frac{1}{2} T''_{ijk} \tilde{u}_{\mathrm{R}i}^* \tilde{d}_{\mathrm{R}j}^* \tilde{d}_{\mathrm{R}k}^* + \tilde{l}_{\mathrm{L}i}^* M_{Li}^2 H_\mathrm{d} + \text{H.c.}\right) \\ & + \left(C_{ijk}^1 \tilde{l}_{\mathrm{L}i}^* \tilde{q}_{\mathrm{L}j} \tilde{u}_{\mathrm{R}k}^* + C_i^2 h_\mathrm{u}^* h_\mathrm{d} \tilde{e}_{\mathrm{R}i}^* + C_{ijk}^3 \tilde{d}_{\mathrm{R}i} \tilde{u}_{\mathrm{R}j}^* \tilde{e}_{\mathrm{R}k}^* + \frac{1}{2} C_{ijk}^4 \tilde{d}_{\mathrm{R}i} \tilde{q}_{\mathrm{L}j} \tilde{q}_{\mathrm{L}k} + \text{H.c.}\right), \end{aligned} \tag{12.5}$$

($\lambda_{ijk} = -\lambda_{jik}$, $\lambda''_{ijk} = -\lambda''_{ikj}$, and $C_{ijk}^4 = C_{ikj}^4$.)

12.1 Notation

Our notation in this section (and the previous section) follows DHM [20, PhysRept] and Martin [21, v7] (but note that Martin uses $(-,+,+,+)$-metric) for RPC part and SLHA2 convention for RPV part. In particular, the sign of gauge bosons are fixed by $\mathrm{D}_\mu\phi = \partial_\mu\phi - \mathrm{i}gA^a_\mu t^a_{ij}\phi_j$, and the phase of gauginos are by $\mathcal{L} \ni \sqrt{2}g(\phi^* t^a\psi\lambda^a)$. Phases of ϕ and ψ in chiral superfields are not yet specified; they are later used to remove $F\tilde{F}$ terms and diagonalize Yukawa matrices.

12.2 Lagrangian construction

The most generic form of the Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{super}} + \mathcal{L}_{\text{FI}} + \mathcal{L}_{\cancel{\text{SUSY}}}; \tag{12.6}$$

$$\mathcal{L}_{\text{matter}} = \Phi_Q^* \exp\left(2g_Y(\tfrac{1}{6})V_B + 2g_2 V_W^a T^a + 2g_3 V_g^a \tau^a\right)\Phi_Q\Big|_{\theta^4} + \cdots; \tag{12.7}$$

$$\mathcal{L}_{\text{gauge}} = \left[\frac{1}{4}\left(1 - \frac{\mathrm{i}g_Y^2\Theta_B}{8\pi^2}\right)W_B W_B + \frac{1}{4}\left(1 - \frac{\mathrm{i}g_2^2\Theta_W}{8\pi^2}\right)W_W^a W_W^a + \frac{1}{4}\left(1 - \frac{\mathrm{i}g_3^2\Theta_g}{8\pi^2}\right)W_g^a W_g^a\right]_{\theta^2} + \text{H.c.}; \tag{12.8}$$

$$\mathcal{L}_{\text{super}} = W(\Phi)\Big|_{\theta^2} + \text{H.c.}, \tag{12.9}$$

$$W(\Phi) = W_{\text{RPC}} + W_{\text{RPV}}, \tag{12.10}$$

$$W_{\text{RPC}} = \mu H_{\text{u}} H_{\text{d}} - y_{uij} U_i^{\text{c}} H_{\text{u}} Q_j + y_{dij} D_i^{\text{c}} H_{\text{d}} Q_j + y_{eij} E_i^{\text{c}} H_{\text{d}} L_j, \tag{12.11}$$

$$W_{\text{RPV}} = -\kappa_i L_i H_{\text{u}} + \frac{1}{2}\lambda_{ijk} L_i L_j E_k^{\text{c}} + \lambda'_{ijk} L_i Q_j D_k^{\text{c}} + \frac{1}{2}\lambda''_{ijk} U_i^{\text{c}} D_j^{\text{c}} D_k^{\text{c}}; \tag{12.12}$$

$$\mathcal{L}_{\text{FI}} = \Lambda_{\text{FI}} D_B; \tag{12.13}$$

$$\mathcal{L}_{\cancel{\text{SUSY}}} = -\frac{1}{2}\left(M_3\tilde{g}_0\tilde{g}_0 + M_2\tilde{w}\tilde{w} + M_1\tilde{b}\tilde{b} + \text{H.c.}\right) - \left(V^{\text{RPC}}_{\cancel{\text{SUSY}}} + V^{\text{RPV}}_{\cancel{\text{SUSY}}}\right), \tag{12.14}$$

$$\begin{aligned} V^{\text{RPC}}_{\cancel{\text{SUSY}}} = {} & \left(\tilde{q}_{\text{L}}^* m_Q^2 \tilde{q}_{\text{L}} + \tilde{l}_{\text{L}}^* m_L^2 \tilde{l}_{\text{L}} + \tilde{u}_{\text{R}}^* m_{U^{\text{c}}}^2 \tilde{u}_{\text{R}} + \tilde{d}_{\text{R}}^* m_{D^{\text{c}}}^2 \tilde{d}_{\text{R}} + \tilde{e}_{\text{R}}^* m_{E^{\text{c}}}^2 \tilde{e}_{\text{R}} + m_{H_{\text{u}}}^2 |h_{\text{u}}|^2 + m_{H_{\text{d}}}^2 |h_{\text{d}}|^2\right) \\ & + \left(-\tilde{u}_{\text{R}}^* h_{\text{u}} a_{\text{u}} \tilde{q}_{\text{L}} + \tilde{d}_{\text{R}}^* h_{\text{d}} a_{\text{d}} \tilde{q}_{\text{L}} + \tilde{e}_{\text{R}}^* h_{\text{d}} a_{\text{e}} \tilde{l}_{\text{L}} + bH_{\text{u}} H_{\text{d}} + \text{H.c.}\right) \\ & + \left(\tilde{u}_{\text{R}}^* h_{\text{d}}^* c_{\text{u}} \tilde{q}_{\text{L}} + \tilde{d}_{\text{R}}^* h_{\text{u}}^* c_{\text{d}} \tilde{q}_{\text{L}} + \tilde{e}_{\text{R}}^* h_{\text{u}}^* c_{\text{e}} \tilde{l}_{\text{L}} + \text{H.c.}\right), \end{aligned} \tag{12.15}$$

$$\begin{aligned} V^{\text{RPV}}_{\cancel{\text{SUSY}}} = {} & \left(-b_i \tilde{l}_{\text{L}i} H_{\text{u}} + \frac{1}{2} T_{ijk} \tilde{l}_{\text{L}i} \tilde{l}_{\text{L}j} \tilde{e}_{\text{R}k}^* + T'_{ijk} \tilde{l}_{\text{L}i} \tilde{q}_{\text{L}j} \tilde{d}_{\text{R}k}^* + \frac{1}{2} T''_{ijk} \tilde{u}_{\text{R}i}^* \tilde{d}_{\text{R}j}^* \tilde{d}_{\text{R}k}^* + \tilde{l}_{\text{L}i}^* M_{Li}^2 H_{\text{d}} + \text{H.c.}\right) \\ & + \left(C^1_{ijk} \tilde{l}_{\text{L}i}^* \tilde{q}_{\text{L}j} \tilde{u}_{\text{R}k}^* + C^2_i h_{\text{u}}^* h_{\text{d}} \tilde{e}_{\text{R}i}^* + C^3_{ijk} \tilde{d}_{\text{R}i} \tilde{u}_{\text{R}j}^* \tilde{e}_{\text{R}k}^* + \frac{1}{2} C^4_{ijk} \tilde{d}_{\text{R}i} \tilde{q}_{\text{L}j} \tilde{q}_{\text{L}k} + \text{H.c.}\right). \end{aligned} \tag{12.16}$$

Hereafter we do not consider Θ_W and Θ_B as in the Standard Model (Section 8.7)[*24], while the SU(3) angle Θ_g forms QCD phase Θ_{QCD} together with the phases from Yukawa matrices. Also we assume the absence of Fayet-Illiopoulos term: $\Lambda_{\text{FI}} = 0$. Then,

$$\mathcal{L}_{\text{matter}} = \sum_{\text{matters}} \left[\mathrm{D}^\mu\phi^* \mathrm{D}_\mu\phi + \mathrm{i}\bar{\psi}\bar{\sigma}^\mu \mathrm{D}_\mu\psi - \sqrt{2}\sum_{\text{gauge}} g\left(\lambda^a(\phi^* t^a\psi) + \bar{\lambda}^a(\bar{\psi}t^a\phi)\right)\right] + (F\text{-terms}), \tag{12.17}$$

$$\mathcal{L}_{\text{gauge}} = \sum_{\text{gauges}} \left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \mathrm{i}\bar{\lambda}^a\bar{\sigma}^\mu \mathrm{D}_\mu\lambda^a\right) + \frac{g_3^2\Theta_g}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma} + (D\text{-terms}), \tag{12.18}$$

$$\begin{aligned} \mathcal{L}_{\text{super}} = {} & \epsilon^{ab}\left(-\mu\tilde{h}_{\text{u}}^a\tilde{h}_{\text{d}}^b - y_{dij} h_{\text{d}}^a d_{\text{R}i}^{\text{c}x} q_{\text{L}j}^{bx} - y_{dij}\tilde{d}_{\text{R}i}^{x*}\tilde{h}_{\text{d}}^a q_{\text{L}j}^{bx} + y_{dji}\tilde{q}_{\text{L}i}^{ax}\tilde{h}_{\text{d}}^b d_{\text{R}j}^{\text{c}x}\right. \\ & -y_{eij}\tilde{e}_{\text{R}i}^*\tilde{h}_{\text{d}}^a l_{\text{L}j}^b - y_{eij} h_{\text{d}}^a e_{\text{R}i}^{\text{c}} l_{\text{L}j}^b + y_{eji}\tilde{l}_{\text{L}i}^a\tilde{h}_{\text{d}}^b e_{\text{R}j}^{\text{c}} + y_{uij} h_{\text{u}}^a u_{\text{R}i}^{\text{c}x} q_{\text{L}j}^{bx} + y_{uij}\tilde{u}_{\text{R}i}^{x*}\tilde{h}_{\text{u}}^a q_{\text{L}j}^{bx} - y_{uji}\tilde{q}_{\text{L}i}^{ax}\tilde{h}_{\text{u}}^b u_{\text{R}j}^{\text{c}x} \\ & \left.-\kappa_i\tilde{h}_{\text{u}}^a l_{\text{L}i}^b - \lambda_{ikj}\tilde{l}_{\text{L}i}^a e_{\text{R}j}^{\text{c}} l_{\text{L}k}^b - \frac{1}{2}\lambda_{jki}\tilde{e}_{\text{R}i}^* l_{\text{L}j}^a l_{\text{L}k}^b - \lambda'_{ikj}\tilde{l}_{\text{L}i}^a d_{\text{R}j}^{\text{c}x} q_{\text{L}k}^{bx} + \lambda'_{kij}\tilde{q}_{\text{L}i}^{ax} d_{\text{R}j}^{\text{c}x} l_{\text{L}k}^b + \lambda'_{kji}\tilde{d}_{\text{R}i}^{x*} q_{\text{L}j}^{ax} l_{\text{L}k}^b\right) \\ & -\frac{1}{2}\epsilon^{xyz}\lambda''_{ijk}\tilde{u}_{\text{R}i}^{x*} d_{\text{R}j}^{\text{c}y} d_{\text{R}k}^{\text{c}z} + \epsilon^{xyz}\lambda''_{jik}\tilde{d}_{\text{R}i}^{x*} u_{\text{R}j}^{\text{c}y} d_{\text{R}k}^{\text{c}z} + \text{H.c.} + (F\text{-terms}), \end{aligned} \tag{12.19}$$

$$\mathcal{L}_{\cancel{\text{SUSY}}} = -\frac{1}{2}\left(M_3\tilde{g}_0\tilde{g}_0 + M_2\tilde{w}\tilde{w} + M_1\tilde{b}\tilde{b} + \text{H.c.}\right) - \left(V^{\text{RPC}}_{\cancel{\text{SUSY}}} + V^{\text{RPV}}_{\cancel{\text{SUSY}}}\right), \tag{12.20}$$

and the F- and D-terms form the supersymmetric scalar potential

$$V_{\text{SUSY}} = F_i^* F_i + \frac{1}{2}D^a D^a; \qquad F_i = -W_i^* = -\frac{\delta W^*}{\delta\phi_i^*}, \qquad D^a = -g(\phi^* t^a\phi), \tag{12.21}$$

$$V = V_{\text{SUSY}} + V^{\text{RPC}}_{\cancel{\text{SUSY}}} + V^{\text{RPV}}_{\cancel{\text{SUSY}}}, \tag{12.22}$$

where t_a corresponds to the gauge-symmetry generator relevant for each ϕ.

[*24] The rotations to remove Θ_W may generate phases in the RPV terms. In other words, we define the RPV terms in the $\Theta_W = 0$ basis.

Each auxiliary term is given by

$$-F^*_{h_\mathrm{u}^a} = \epsilon^{ab}\left(-\tilde{u}_\mathrm{R}^{x*}y_u\tilde{q}_\mathrm{L}^{bx} + \mu h_\mathrm{d}^b + \kappa_i \tilde{l}_{\mathrm{L}i}^b\right), \tag{12.23}$$

$$-F^*_{h_\mathrm{d}^a} = \epsilon^{ab}\left(\tilde{e}_\mathrm{R}^{*}y_e\tilde{l}_\mathrm{L}^{b} + \tilde{d}_\mathrm{R}^{x*}y_d\tilde{q}_\mathrm{L}^{bx} - \mu h_\mathrm{u}^b\right), \tag{12.24}$$

$$-F^*_{\tilde{q}_{\mathrm{L}i}^{ax}} = \epsilon^{ab}\left(-y_{dji}h_\mathrm{d}^b\tilde{d}_{\mathrm{R}j}^{x*} + y_{uji}h_\mathrm{u}^b\tilde{u}_{\mathrm{R}j}^{x*} - \lambda'_{kij}\tilde{d}_{\mathrm{R}j}^{x*}\tilde{l}_{\mathrm{L}k}^b\right), \tag{12.25}$$

$$-F^*_{\tilde{u}_{\mathrm{R}i}^{x*}} = -y_{uij}h_\mathrm{u}\tilde{q}_{\mathrm{L}j}^x + \frac{1}{2}\epsilon^{xyz}\lambda''_{ijk}\tilde{d}_{\mathrm{R}j}^{y*}\tilde{d}_{\mathrm{R}k}^{z*}, \tag{12.26}$$

$$-F^*_{\tilde{d}_{\mathrm{R}i}^{x*}} = y_{dij}h_\mathrm{d}\tilde{q}_{\mathrm{L}j}^x + \lambda'_{jki}\tilde{l}_{\mathrm{L}j}\tilde{q}_{\mathrm{L}k}^x - \lambda''_{jik}\epsilon^{xyz}\tilde{u}_{\mathrm{R}j}^{y*}\tilde{d}_{\mathrm{R}k}^{z*}, \tag{12.27}$$

$$-F^*_{\tilde{l}_{\mathrm{L}i}^a} = \epsilon^{ab}\left(-y_{eji}\tilde{e}_{\mathrm{R}j}^*h_\mathrm{d}^b - \kappa_i h_\mathrm{u}^b + \lambda_{ikj}\tilde{e}_{\mathrm{R}j}^*\tilde{l}_{\mathrm{L}k}^b + \lambda'_{ikj}\tilde{d}_{\mathrm{R}j}^{x*}\tilde{q}_{\mathrm{L}k}^{bx}\right), \tag{12.28}$$

$$-F^*_{\tilde{e}_{\mathrm{R}i}^*} = y_{eij}h_\mathrm{d}\tilde{l}_{\mathrm{L}j} + \frac{1}{2}\lambda_{jki}\tilde{l}_{\mathrm{L}j}\tilde{l}_{\mathrm{L}k}. \tag{12.29}$$

$$D^\alpha_{\mathrm{SU}(3)} = -g_3\sum_{i=1}^3\left(\sum_{a=1,2}\tilde{q}_{\mathrm{L}i}^{a*}\tau^\alpha\tilde{q}_{\mathrm{L}i}^a - \tilde{u}_{\mathrm{R}i}^*\tau^\alpha\tilde{u}_{\mathrm{R}i} - \tilde{d}_{\mathrm{R}i}^*\tau^\alpha\tilde{d}_{\mathrm{R}i}\right), \tag{12.30}$$

$$D^\alpha_{\mathrm{SU}(2)} = -g_2\left[\sum_{i=1}^3\left(\sum_{x=1}^3\tilde{q}_{\mathrm{L}i}^{x*}T^\alpha\tilde{q}_{\mathrm{L}i}^x + \tilde{l}_{\mathrm{L}i}^*T^\alpha\tilde{l}_{\mathrm{L}i}\right) + h_\mathrm{u}^*T^\alpha h_\mathrm{u} + h_\mathrm{d}^*T^\alpha h_\mathrm{d}\right], \tag{12.31}$$

$$D_{\mathrm{U}(1)} = -g_1\left(\frac{1}{6}|\tilde{q}_\mathrm{L}|^2 - \frac{1}{2}|\tilde{l}_\mathrm{L}|^2 - \frac{2}{3}|\tilde{u}_\mathrm{R}|^2 + \frac{1}{3}|\tilde{d}_\mathrm{R}|^2 + |\tilde{e}_\mathrm{R}|^2 + \frac{1}{2}|h_\mathrm{u}|^2 - \frac{1}{2}|h_\mathrm{d}|^2\right). \tag{12.32}$$

12.3 Full Lagrangian

Here the Lagrangian $\mathcal{L} = \mathcal{L}_\mathrm{vector} + \mathcal{L}_\mathrm{fermions} + \mathcal{L}_\mathrm{SFG} + \mathcal{L}_\mathrm{scalar}$ is explicitly given:

$$\mathcal{L}_\mathrm{vector} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{g_3^2\Theta_g}{64\pi^2}\varepsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}, \tag{12.33}$$

$$\mathcal{L}_\mathrm{fermions} = \mathrm{i}\bar{\psi}\bar{\sigma}^\mu\mathrm{D}_\mu\psi + \mathrm{i}\bar{\lambda}^a\bar{\sigma}^\mu\mathrm{D}_\mu\lambda^a - \frac{1}{2}\left(M_3\tilde{g}_0\tilde{g}_0 + M_2\tilde{w}\tilde{w} + M_1\tilde{b}\tilde{b} + \mathrm{H.c.}\right) + \mathcal{L}_\mathrm{super}\big|_{\text{no }F\text{-terms}}, \tag{12.34}$$

$$\mathcal{L}_\mathrm{SFG} = -\sqrt{2}g\lambda^a(\phi^*t^a\psi) - \sqrt{2}g\bar{\lambda}^a(\bar{\psi}t^a\phi), \tag{12.35}$$

$$\mathcal{L}_\mathrm{scalar} = \mathrm{D}^\mu\phi^*\mathrm{D}_\mu\phi - V. \tag{12.36}$$

12.3.1 Vector part

$$\begin{aligned}\mathcal{L}_\mathrm{vector} = &-\frac{1}{2}(\partial_\mu B_\nu - \partial_\nu B_\mu)\partial^\mu B^\nu - \frac{1}{2}(\partial_\mu g^a_\nu - \partial_\nu g^a_\mu)\partial^\mu g^{a\nu} - \frac{1}{2}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)\partial^\mu W^{a\nu}\\ &- g_2\epsilon^{abc}W^b_\mu W^c_\nu\partial^\mu W^{a\nu} - \frac{g_2^2}{4}\epsilon^{abe}\epsilon^{cde}W^a_\mu W^b_\nu W^{c\mu}W^{d\nu}\\ &- g_3 f^{abc}g^b_\mu g^c_\nu\partial^\mu g^{a\nu} - \frac{g_3^2}{4}f^{cde}f^{abe}g^a_\mu g^b_\nu g^{c\mu}g^{d\nu} + \frac{g_3^2\Theta_g}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma},\end{aligned} \tag{12.37}$$

$$\begin{aligned}= &\ (\text{gluons}) - \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial^\mu A^\nu - (\partial_\mu W^-_\nu - \partial_\nu W^-_\mu)\partial_\mu W^{+\nu} - \frac{1}{2}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)\partial^\mu Z^\nu\\ &+ \mathrm{i}g_2\mathrm{c_W}\left[(W^-_\mu Z_\nu - W^-_\nu Z_\mu)\partial^\mu W^{+\nu} - (W^+_\mu Z_\nu - W^+_\nu Z_\mu)\partial^\mu W^{-\nu} + (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu)\partial^\mu Z^\nu\right]\\ &+ \mathrm{i}|e|\left[(W^-_\mu A_\nu - W^-_\nu A_\mu)\partial^\mu W^{+\nu} - (W^+_\mu A_\nu - W^+_\nu A_\mu)\partial^\mu W^{-\nu} + (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu)\partial^\mu A^\nu\right]\\ &+ \frac{g_2^2}{2}W^{+\mu}W^+_\mu W^{-\nu}W^-_\nu - \frac{g_2^2}{2}W^{+\mu}W^{+\nu}W^-_\mu W^-_\nu - g_2^2W^{+\mu}W^-_\mu Z^\nu Z_\nu + g_2^2W^{+\mu}W^{-\nu}Z_\mu Z_\nu\\ &- e^2W^{+\mu}W^-_\mu A^\nu A_\nu + e^2W^{+\mu}W^-_\mu Z^\nu Z_\nu + e^2W^{+\mu}W^{-\nu}A_\mu A_\nu - e^2W^{+\mu}W^{-\nu}Z_\mu Z_\nu\\ &- 2g_2^2\mathrm{c_W s_W}W^{+\mu}W^-_\mu A^\nu Z_\nu + g_2^2\mathrm{c_W s_W}W^{+\mu}W^{-\nu}A_\mu Z_\nu + g_2^2\mathrm{c_W s_W}W^{+\mu}W^{-\nu}A_\nu Z_\mu,\end{aligned} \tag{12.38}$$

where

$$W^1_\mu = \frac{W^+_\mu + W^-_\mu}{\sqrt{2}},\quad W^2_\mu = \frac{\mathrm{i}(W^+_\mu - W^-_\mu)}{\sqrt{2}};\qquad W^\pm_\mu = \frac{W^1_\mu \mp \mathrm{i}W^2_\mu}{\sqrt{2}};$$

$$\begin{pmatrix}W^3_\mu\\ B_\mu\end{pmatrix} = \begin{pmatrix}\mathrm{c_W} & \mathrm{s_W}\\ -\mathrm{s_W} & \mathrm{c_W}\end{pmatrix}\begin{pmatrix}Z_\mu\\ A_\mu\end{pmatrix};\qquad \begin{pmatrix}Z_\mu\\ A_\mu\end{pmatrix} = \begin{pmatrix}\mathrm{c_W} & -\mathrm{s_W}\\ \mathrm{s_W} & \mathrm{c_W}\end{pmatrix}\begin{pmatrix}W^3_\mu\\ B_\mu\end{pmatrix};$$

$$|e| = g_2\mathrm{s_W} = g_Y\mathrm{c_W} = g_Z\mathrm{s_W c_W},\quad g_Z = g_2/\mathrm{c_W} = g_Y/\mathrm{s_W};\qquad g_Y = |e|/\mathrm{c_W} = g_Z\mathrm{s_W} = g_2\mathrm{t_W},\quad g_2 = |e|/\mathrm{s_W} = g_Z\mathrm{c_W}.$$

12.3.2 Fermion part

$$
\begin{aligned}
&\mathcal{L}_{\text{fermions}} \\
&= \mathrm{i}\bar{q}_{\mathrm{L}}\bar{\sigma}^{\mu}\left(\partial_{\mu} - \mathrm{i}g_3 g_{\mu}^{a}\tau^{a} - \mathrm{i}g_2 W_{\mu}^{a}T^{a} - \tfrac{1}{6}\mathrm{i}g_{Y}B_{\mu}\right)q_{\mathrm{L}} \\
&\quad + \mathrm{i}\bar{u}_{\mathrm{R}}^{\mathrm{c}}\bar{\sigma}^{\mu}\left(\partial_{\mu} + \mathrm{i}g_3 g_{\mu}^{a}\tau^{a*} + \tfrac{2}{3}\mathrm{i}g_{Y}B_{\mu}\right)u_{\mathrm{R}}^{\mathrm{c}} + \mathrm{i}\bar{d}_{\mathrm{R}}^{\mathrm{c}}\bar{\sigma}^{\mu}\left(\partial_{\mu} + \mathrm{i}g_3 g_{\mu}^{a}\tau^{a*} - \tfrac{1}{3}\mathrm{i}g_{Y}B_{\mu}\right)d_{\mathrm{R}}^{\mathrm{c}} \\
&\quad + \mathrm{i}\bar{l}_{\mathrm{L}}\bar{\sigma}^{\mu}\left(\partial_{\mu} - \mathrm{i}g_2 W_{\mu}^{a}T^{a} + \tfrac{1}{2}\mathrm{i}g_{Y}B_{\mu}\right)l_{\mathrm{L}} + \mathrm{i}\bar{e}_{\mathrm{R}}^{\mathrm{c}}\bar{\sigma}^{\mu}\left(\partial_{\mu} - \mathrm{i}g_{Y}B_{\mu}\right)e_{\mathrm{R}}^{\mathrm{c}} \\
&\quad + \mathrm{i}\bar{\tilde{h}}_{\mathrm{u}}\bar{\sigma}^{\mu}\left(\partial_{\mu} - \mathrm{i}g_2 W_{\mu}^{a}T^{a} - \tfrac{1}{2}\mathrm{i}g_{Y}B_{\mu}\right)\tilde{h}_{\mathrm{u}} + \mathrm{i}\bar{\tilde{h}}_{\mathrm{d}}\bar{\sigma}^{\mu}\left(\partial_{\mu} - \mathrm{i}g_2 W_{\mu}^{a}T^{a} + \tfrac{1}{2}\mathrm{i}g_{Y}B_{\mu}\right)\tilde{h}_{\mathrm{d}} \\
&\quad + \mathrm{i}\bar{\tilde{g}}_0^{a}\bar{\sigma}^{\mu}\left(\partial_{\mu}\tilde{g}_0^{a} + g_3 f^{abc}g_{\mu}^{b}\tilde{g}_0^{c}\right) + \mathrm{i}\bar{\tilde{w}}^{a}\bar{\sigma}^{\mu}\left(\partial_{\mu}\tilde{w}^{a} + g_2\epsilon^{abc}W_{\mu}^{b}\tilde{w}^{c}\right) + \mathrm{i}\bar{\tilde{b}}\bar{\sigma}^{\mu}\partial_{\mu}\tilde{b} \\
&\quad - \frac{1}{2}\left(M_3\tilde{g}_0^{a}\tilde{g}_0^{a} + M_2\tilde{w}^{a}\tilde{w}^{a} + M_1\tilde{b}\tilde{b} + \text{H.c.}\right) + \mathcal{L}_{\text{super}}\big|_{\text{no }F\text{-terms}}
\end{aligned}
\tag{12.39}
$$

$$
\begin{aligned}
&= \mathrm{i}\bar{\tilde{b}}\bar{\sigma}^{\mu}\partial_{\mu}\tilde{b} - \frac{1}{2}\left(M_1\tilde{b}\tilde{b} + M_1^{*}\bar{\tilde{b}}^{a}\bar{\tilde{b}}^{a}\right) + \mathrm{i}\bar{\tilde{g}}_0^{a}\bar{\sigma}^{\mu}\partial_{\mu}\tilde{g}_0^{a} - \frac{1}{2}\left(M_3\tilde{g}_0^{a}\tilde{g}_0^{a} + M_3^{*}\bar{\tilde{g}}_0^{a}\bar{\tilde{g}}_0^{a}\right) - \mathrm{i}g_3 f^{abc}(\bar{\tilde{g}}_0^{a}\bar{\sigma}^{\mu}\tilde{g}_0^{b})g_{\mu}^{c} \\
&\quad + \mathrm{i}\bar{\tilde{w}}^{+}\bar{\sigma}^{\mu}\partial_{\mu}\tilde{w}^{+} + \mathrm{i}\bar{\tilde{w}}^{-}\bar{\sigma}^{\mu}\partial_{\mu}\tilde{w}^{-} + \mathrm{i}\bar{\tilde{w}}^{3}\bar{\sigma}^{\mu}\partial_{\mu}\tilde{w}^{3} - \left(M_2\tilde{w}^{+}\tilde{w}^{-} + M_2^{*}\bar{\tilde{w}}^{+}\bar{\tilde{w}}^{-}\right) - \frac{1}{2}\left(M_2\tilde{w}^{3}\tilde{w}^{3} + M_2^{*}\bar{\tilde{w}}^{3}\bar{\tilde{w}}^{3}\right) \\
&\quad + g_2(\bar{\tilde{w}}^{3}\bar{\sigma}^{\mu}\tilde{w}^{-} - \bar{\tilde{w}}^{+}\bar{\sigma}^{\mu}\tilde{w}^{3})W_{\mu}^{+} - g_2(\bar{\tilde{w}}^{3}\bar{\sigma}^{\mu}\tilde{w}^{+} - \bar{\tilde{w}}^{-}\bar{\sigma}^{\mu}\tilde{w}^{3})W_{\mu}^{-} + g_2(\bar{\tilde{w}}^{+}\bar{\sigma}^{\mu}\tilde{w}^{+} - \bar{\tilde{w}}^{-}\bar{\sigma}^{\mu}\tilde{w}^{-})(\mathrm{c_W}Z_{\mu} + \mathrm{s_W}A_{\mu}) \\
&\quad + \bar{u}_{\mathrm{L}}\bar{\sigma}^{\mu}(\mathrm{i}\partial_{\mu} + g_3\tau^{a}g_{\mu}^{a})u_{\mathrm{L}} + \bar{u}_{\mathrm{R}}^{\mathrm{c}}\bar{\sigma}^{\mu}(\mathrm{i}\partial_{\mu} - g_3\tau^{a*}g_{\mu}^{a})u_{\mathrm{R}}^{\mathrm{c}} + \mathrm{i}\bar{\nu}_{\mathrm{L}}\bar{\sigma}^{\mu}\partial_{\mu}\nu_{\mathrm{L}} \\
&\quad + \bar{d}_{\mathrm{L}}\bar{\sigma}^{\mu}(\mathrm{i}\partial_{\mu} + g_3\tau^{a}g_{\mu}^{a})d_{\mathrm{L}} + \bar{d}_{\mathrm{R}}^{\mathrm{c}}\bar{\sigma}^{\mu}(\mathrm{i}\partial_{\mu} - g_3\tau^{a*}g_{\mu}^{a})d_{\mathrm{R}}^{\mathrm{c}} + \mathrm{i}\bar{e}_{\mathrm{L}}\bar{\sigma}^{\mu}\partial_{\mu}e_{\mathrm{L}} + \mathrm{i}\bar{e}_{\mathrm{R}}^{\mathrm{c}}\bar{\sigma}^{\mu}\partial_{\mu}e_{\mathrm{R}}^{\mathrm{c}} \\
&\quad + \mathrm{i}\bar{\tilde{h}}_{\mathrm{d}}^{-}\bar{\sigma}^{\mu}\partial_{\mu}\tilde{h}_{\mathrm{d}}^{-} + \mathrm{i}\bar{\tilde{h}}_{\mathrm{d}}^{0}\bar{\sigma}^{\mu}\partial_{\mu}\tilde{h}_{\mathrm{d}}^{0} + \mathrm{i}\bar{\tilde{h}}_{\mathrm{u}}^{+}\bar{\sigma}^{\mu}\partial_{\mu}\tilde{h}_{\mathrm{u}}^{+} + \mathrm{i}\bar{\tilde{h}}_{\mathrm{u}}^{0}\bar{\sigma}^{\mu}\partial_{\mu}\tilde{h}_{\mathrm{u}}^{0} \\
&\quad + \frac{g_2}{\sqrt{2}}\left(\bar{u}_{\mathrm{L}}\bar{\sigma}^{\mu}d_{\mathrm{L}} + \bar{\nu}_{\mathrm{L}}\bar{\sigma}^{\mu}e_{\mathrm{L}} + \bar{\tilde{h}}_{\mathrm{u}}^{+}\bar{\sigma}^{\mu}\tilde{h}_{\mathrm{u}}^{0} + \bar{\tilde{h}}_{\mathrm{d}}^{0}\bar{\sigma}^{\mu}\tilde{h}_{\mathrm{d}}^{-}\right)W_{\mu}^{+} + \frac{g_2}{\sqrt{2}}\left(\bar{d}_{\mathrm{L}}\bar{\sigma}^{\mu}u_{\mathrm{L}} + \bar{e}_{\mathrm{L}}\bar{\sigma}^{\mu}\nu_{\mathrm{L}} + \bar{\tilde{h}}_{\mathrm{u}}^{0}\bar{\sigma}^{\mu}\tilde{h}_{\mathrm{u}}^{+} + \bar{\tilde{h}}_{\mathrm{d}}^{-}\bar{\sigma}^{\mu}\tilde{h}_{\mathrm{d}}^{0}\right)W_{\mu}^{-} \\
&\quad + \frac{g_Z(3 - 4\mathrm{s_W^2})}{6}\bar{u}_{\mathrm{L}}\bar{\sigma}^{\mu}u_{\mathrm{L}}Z_{\mu} + \frac{g_Z\mathrm{s_W^2}}{3}\bar{u}_{\mathrm{R}}^{\mathrm{c}}\bar{\sigma}^{\mu}u_{\mathrm{R}}^{\mathrm{c}}Z_{\mu} + \frac{g_Z(2\mathrm{s_W^2} - 3)}{6}\bar{d}_{\mathrm{L}}\bar{\sigma}^{\mu}d_{\mathrm{L}}Z_{\mu} - \frac{g_Z\mathrm{s_W^2}}{3}\bar{d}_{\mathrm{R}}^{\mathrm{c}}\bar{\sigma}^{\mu}d_{\mathrm{R}}^{\mathrm{c}}Z_{\mu} \\
&\quad + \frac{g_Z}{2}\bar{\nu}_{\mathrm{L}}\bar{\sigma}^{\mu}\nu_{\mathrm{L}}Z_{\mu} + \frac{g_Z(2\mathrm{s_W^2} - 1)}{2}\bar{e}_{\mathrm{L}}\bar{\sigma}^{\mu}e_{\mathrm{L}}Z_{\mu} - g_Z\mathrm{s_W^2}\bar{e}_{\mathrm{R}}^{\mathrm{c}}\bar{\sigma}^{\mu}e_{\mathrm{R}}^{\mathrm{c}}Z_{\mu} \\
&\quad + \frac{g_Z(1 - 2\mathrm{s_W^2})}{2}\bar{\tilde{h}}_{\mathrm{u}}^{+}\bar{\sigma}^{\mu}\tilde{h}_{\mathrm{u}}^{+}Z_{\mu} + \frac{g_Z}{2}\bar{\tilde{h}}_{\mathrm{u}}^{0}\bar{\sigma}^{\mu}\tilde{h}_{\mathrm{u}}^{0}Z_{\mu} + \frac{g_Z(2\mathrm{s_W^2} - 1)}{2}\bar{\tilde{h}}_{\mathrm{d}}^{-}\bar{\sigma}^{\mu}\tilde{h}_{\mathrm{d}}^{-}Z_{\mu} + \frac{g_Z}{2}\bar{\tilde{h}}_{\mathrm{d}}^{0}\bar{\sigma}^{\mu}\tilde{h}_{\mathrm{d}}^{0}Z_{\mu} \\
&\quad + \frac{2|e|}{3}(\bar{u}_{\mathrm{L}}\bar{\sigma}^{\mu}u_{\mathrm{L}} - \bar{u}_{\mathrm{R}}^{\mathrm{c}}\bar{\sigma}^{\mu}u_{\mathrm{R}}^{\mathrm{c}})A_{\mu} - \frac{|e|}{3}(\bar{d}_{\mathrm{L}}\bar{\sigma}^{\mu}d_{\mathrm{L}} - \bar{d}_{\mathrm{R}}^{\mathrm{c}}\bar{\sigma}^{\mu}d_{\mathrm{R}}^{\mathrm{c}})A_{\mu} - |e|(\bar{e}_{\mathrm{L}}\bar{\sigma}^{\mu}e_{\mathrm{L}} - \bar{e}_{\mathrm{R}}^{\mathrm{c}}\bar{\sigma}^{\mu}e_{\mathrm{R}}^{\mathrm{c}})A_{\mu} \\
&\quad + |e|\bar{\tilde{h}}_{\mathrm{u}}^{+}\bar{\sigma}^{\mu}\tilde{h}_{\mathrm{u}}^{+}A_{\mu} - |e|\bar{\tilde{h}}_{\mathrm{d}}^{-}\bar{\sigma}^{\mu}\tilde{h}_{\mathrm{d}}^{-}A_{\mu} + \mathcal{L}_{\text{super}}\big|_{\text{no }F\text{-terms}};
\end{aligned}
\tag{12.40}
$$

here,

$$
\begin{aligned}
\mathcal{L}_{\text{super}}\big|_{\text{no F-terms}} &= -\mu\tilde{h}_{\mathrm{u}}^{+}\tilde{h}_{\mathrm{d}}^{-} + \mu\tilde{h}_{\mathrm{u}}^{0}\tilde{h}_{\mathrm{d}}^{0} + y_{uij}h_{\mathrm{u}}^{+}u_{\mathrm{R}i}^{\mathrm{c}}d_{\mathrm{L}j} - y_{uij}h_{\mathrm{u}}^{0}u_{\mathrm{R}i}^{\mathrm{c}}u_{\mathrm{L}j} + y_{uij}\tilde{d}_{\mathrm{L}j}\tilde{h}_{\mathrm{u}}^{+}u_{\mathrm{R}i}^{\mathrm{c}} - y_{uij}\tilde{u}_{\mathrm{L}j}\tilde{h}_{\mathrm{u}}^{0}u_{\mathrm{R}i}^{\mathrm{c}} \\
&\quad + y_{uji}\tilde{u}_{\mathrm{R}j}^{*}\tilde{h}_{\mathrm{u}}^{+}d_{\mathrm{L}i} - y_{uji}\tilde{u}_{\mathrm{R}j}^{*}\tilde{h}_{\mathrm{u}}^{0}u_{\mathrm{L}i} + y_{dij}h_{\mathrm{d}}^{-}d_{\mathrm{R}i}^{\mathrm{c}}u_{\mathrm{L}j} - y_{dij}h_{\mathrm{d}}^{0}d_{\mathrm{R}i}^{\mathrm{c}}d_{\mathrm{L}j} - y_{dij}\tilde{d}_{\mathrm{L}j}\tilde{h}_{\mathrm{d}}^{0}d_{\mathrm{R}i}^{\mathrm{c}} \\
&\quad + y_{dij}\tilde{u}_{\mathrm{L}j}\tilde{h}_{\mathrm{d}}^{-}d_{\mathrm{R}i}^{\mathrm{c}} + y_{dji}\tilde{d}_{\mathrm{R}j}^{*}\tilde{h}_{\mathrm{d}}^{-}u_{\mathrm{L}i} - y_{dji}\tilde{d}_{\mathrm{R}j}^{*}\tilde{h}_{\mathrm{d}}^{0}d_{\mathrm{L}i} + y_{eij}h_{\mathrm{d}}^{-}e_{\mathrm{R}i}^{\mathrm{c}}\nu_{\mathrm{L}j} - y_{eij}h_{\mathrm{d}}^{0}e_{\mathrm{R}i}^{\mathrm{c}}e_{\mathrm{L}j} \\
&\quad - y_{eij}\tilde{e}_{\mathrm{L}j}\tilde{h}_{\mathrm{d}}^{0}e_{\mathrm{R}i}^{\mathrm{c}} + y_{eij}\tilde{\nu}_{\mathrm{L}j}\tilde{h}_{\mathrm{d}}^{-}e_{\mathrm{R}i}^{\mathrm{c}} + y_{eji}\tilde{e}_{\mathrm{R}j}^{*}\tilde{h}_{\mathrm{d}}^{-}\nu_{\mathrm{L}i} - y_{eji}\tilde{e}_{\mathrm{R}j}^{*}\tilde{h}_{\mathrm{d}}^{0}e_{\mathrm{L}i} \\
&\quad - \kappa_i\tilde{h}_{\mathrm{u}}^{+}e_{\mathrm{L}i} + \kappa_i\tilde{h}_{\mathrm{u}}^{0}\nu_{\mathrm{L}i} - \lambda_{ijk}\tilde{e}_{\mathrm{R}k}^{*}\nu_{\mathrm{L}i}e_{\mathrm{L}j} - \lambda_{jki}\tilde{e}_{\mathrm{L}k}e_{\mathrm{R}i}^{\mathrm{c}}\nu_{\mathrm{L}j} + \lambda_{jki}\tilde{\nu}_{\mathrm{L}k}e_{\mathrm{R}i}^{\mathrm{c}}e_{\mathrm{L}j} \\
&\quad - \lambda'_{jik}\tilde{d}_{\mathrm{R}k}^{*}d_{\mathrm{L}i}\nu_{\mathrm{L}j} + \lambda'_{jik}\tilde{d}_{\mathrm{R}k}^{*}u_{\mathrm{L}i}e_{\mathrm{L}j} - \lambda'_{jki}\tilde{d}_{\mathrm{L}k}d_{\mathrm{R}i}^{\mathrm{c}}\nu_{\mathrm{L}j} + \lambda'_{jki}\tilde{u}_{\mathrm{L}k}d_{\mathrm{R}i}^{\mathrm{c}}e_{\mathrm{L}j} + \lambda'_{kji}\tilde{e}_{\mathrm{L}k}d_{\mathrm{R}i}^{\mathrm{c}}u_{\mathrm{L}j} \\
&\quad - \lambda'_{kji}\tilde{\nu}_{\mathrm{L}k}d_{\mathrm{R}i}^{\mathrm{c}}d_{\mathrm{L}j} - \epsilon^{xyz}\lambda''_{ijk}\tilde{d}_{\mathrm{R}k}^{x*}u_{\mathrm{R}i}^{\mathrm{c}y}d_{\mathrm{R}j}^{\mathrm{c}z} - \frac{1}{2}\epsilon^{xyz}\lambda''_{kij}\tilde{u}_{\mathrm{R}k}^{x*}d_{\mathrm{R}i}^{\mathrm{c}y}d_{\mathrm{R}j}^{\mathrm{c}z} + \text{H.c.}
\end{aligned}
\tag{12.41}
$$

12.3.3 Scalar-fermion-gaugino interaction

$$
\begin{aligned}
\mathcal{L}_{\mathrm{SFG}} = {} & -g_2\tilde{u}_{\mathrm{L}}^{*}d_{\mathrm{L}}\tilde{w}^{+} - g_2\tilde{u}_{\mathrm{L}}\bar{d}_{\mathrm{L}}\bar{\tilde{w}}^{+} - g_2\tilde{d}_{\mathrm{L}}^{*}u_{\mathrm{L}}\tilde{w}^{-} - g_2\tilde{d}_{\mathrm{L}}\bar{u}_{\mathrm{L}}\bar{\tilde{w}}^{-} \\
& - \sqrt{2}g_3\tilde{u}_{\mathrm{L}}^{*}\tau^{a}u_{\mathrm{L}}\tilde{g}_0^{a} + \sqrt{2}g_3\tilde{u}_{\mathrm{R}}^{*}\tau^{a}\bar{u}_{\mathrm{R}}^{\mathrm{c}}\bar{\tilde{g}}_0^{a} - \sqrt{2}g_3\tilde{u}_{\mathrm{L}}\tau^{a*}\bar{u}_{\mathrm{L}}\bar{\tilde{g}}_0^{a} + \sqrt{2}g_3\tilde{u}_{\mathrm{R}}\tau^{a*}u_{\mathrm{R}}^{\mathrm{c}}\tilde{g}_0^{a} \\
& - \frac{g_2}{\sqrt{2}}\tilde{u}_{\mathrm{L}}^{*}u_{\mathrm{L}}\tilde{w}^{3} - \frac{g_2}{\sqrt{2}}\tilde{u}_{\mathrm{L}}\bar{u}_{\mathrm{L}}\bar{\tilde{w}}^{3} - \frac{g_Y}{3\sqrt{2}}\tilde{u}_{\mathrm{L}}^{*}u_{\mathrm{L}}\tilde{b} + \frac{2\sqrt{2}g_Y}{3}\tilde{u}_{\mathrm{R}}^{*}\bar{u}_{\mathrm{R}}^{\mathrm{c}}\bar{\tilde{b}} - \frac{g_Y}{3\sqrt{2}}\tilde{u}_{\mathrm{L}}\bar{u}_{\mathrm{L}}\bar{\tilde{b}} + \frac{2\sqrt{2}g_Y}{3}\tilde{u}_{\mathrm{R}}u_{\mathrm{R}}^{\mathrm{c}}\tilde{b} \\
& - \sqrt{2}g_3\tilde{d}_{\mathrm{L}}^{*}\tau^{a}d_{\mathrm{L}}\tilde{g}_0^{a} + \sqrt{2}g_3\tilde{d}_{\mathrm{R}}^{*}\tau^{a}\bar{d}_{\mathrm{R}}^{\mathrm{c}}\bar{\tilde{g}}_0^{a} - \sqrt{2}g_3\tilde{d}_{\mathrm{L}}\tau^{a*}\bar{d}_{\mathrm{L}}\bar{\tilde{g}}_0^{a} + \sqrt{2}g_3\tilde{d}_{\mathrm{R}}\tau^{a*}d_{\mathrm{R}}^{\mathrm{c}}\tilde{g}_0^{a} \\
& + \frac{g_2}{\sqrt{2}}\tilde{d}_{\mathrm{L}}^{*}d_{\mathrm{L}}\tilde{w}^{3} + \frac{g_2}{\sqrt{2}}\tilde{d}_{\mathrm{L}}\bar{d}_{\mathrm{L}}\bar{\tilde{w}}^{3} - \frac{g_Y}{3\sqrt{2}}\tilde{d}_{\mathrm{L}}^{*}d_{\mathrm{L}}\tilde{b} - \frac{\sqrt{2}g_Y}{3}\tilde{d}_{\mathrm{R}}^{*}\bar{d}_{\mathrm{R}}^{\mathrm{c}}\bar{\tilde{b}} - \frac{g_Y}{3\sqrt{2}}\tilde{d}_{\mathrm{L}}\bar{d}_{\mathrm{L}}\bar{\tilde{b}} - \frac{\sqrt{2}g_Y}{3}\tilde{d}_{\mathrm{R}}d_{\mathrm{R}}^{\mathrm{c}}\tilde{b} \\
& - g_2\tilde{e}_{\mathrm{L}}\bar{\nu}_{\mathrm{L}}\bar{\tilde{w}}^{-} - g_2\tilde{\nu}_{\mathrm{L}}\bar{e}_{\mathrm{L}}\bar{\tilde{w}}^{+} - g_2\tilde{e}_{\mathrm{L}}^{*}\nu_{\mathrm{L}}\tilde{w}^{-} - g_2\tilde{\nu}_{\mathrm{L}}^{*}e_{\mathrm{L}}\tilde{w}^{+} \\
& - \frac{g_2}{\sqrt{2}}\tilde{\nu}_{\mathrm{L}}^{*}\nu_{\mathrm{L}}\tilde{w}^{3} + \frac{g_Y}{\sqrt{2}}\tilde{\nu}_{\mathrm{L}}^{*}\nu_{\mathrm{L}}\tilde{b} - \frac{g_2}{\sqrt{2}}\tilde{\nu}_{\mathrm{L}}\bar{\nu}_{\mathrm{L}}\bar{\tilde{w}}^{3} + \frac{g_Y}{\sqrt{2}}\tilde{\nu}_{\mathrm{L}}\bar{\nu}_{\mathrm{L}}\bar{\tilde{b}} \\
& + \frac{g_2}{\sqrt{2}}\tilde{e}_{\mathrm{L}}^{*}e_{\mathrm{L}}\tilde{w}^{3} + \frac{g_Y}{\sqrt{2}}\tilde{e}_{\mathrm{L}}^{*}e_{\mathrm{L}}\tilde{b} - \sqrt{2}g_Y\tilde{e}_{\mathrm{R}}^{*}\bar{e}_{\mathrm{R}}^{\mathrm{c}}\bar{\tilde{b}} + \frac{g_2}{\sqrt{2}}\tilde{e}_{\mathrm{L}}\bar{e}_{\mathrm{L}}\bar{\tilde{w}}^{3} + \frac{g_Y}{\sqrt{2}}\tilde{e}_{\mathrm{L}}\bar{e}_{\mathrm{L}}\bar{\tilde{b}} - \sqrt{2}g_Y\tilde{e}_{\mathrm{R}}e_{\mathrm{R}}^{\mathrm{c}}\tilde{b} \\
& - g_2h_{\mathrm{u}}^{+*}\tilde{h}_{\mathrm{u}}^{0}\tilde{w}^{+} - g_2h_{\mathrm{d}}^{0}\bar{\tilde{h}}_{\mathrm{d}}^{-}\bar{\tilde{w}}^{+} - g_2h_{\mathrm{d}}^{-*}\tilde{h}_{\mathrm{d}}^{0}\tilde{w}^{-} - g_2h_{\mathrm{u}}^{0}\bar{\tilde{h}}_{\mathrm{u}}^{+}\bar{\tilde{w}}^{-} \\
& - g_2h_{\mathrm{d}}^{0*}\tilde{h}_{\mathrm{d}}^{-}\tilde{w}^{+} - g_2h_{\mathrm{u}}^{0*}\tilde{h}_{\mathrm{u}}^{+}\tilde{w}^{-} - g_2h_{\mathrm{u}}^{+}\bar{\tilde{h}}_{\mathrm{u}}^{0}\bar{\tilde{w}}^{+} - g_2h_{\mathrm{d}}^{-}\bar{\tilde{h}}_{\mathrm{d}}^{0}\bar{\tilde{w}}^{-} \\
& - \frac{g_2}{\sqrt{2}}h_{\mathrm{u}}^{+}\bar{\tilde{h}}_{\mathrm{u}}^{+}\bar{\tilde{w}}^{3} - \frac{g_Y}{\sqrt{2}}h_{\mathrm{u}}^{+}\bar{\tilde{h}}_{\mathrm{u}}^{+}\bar{\tilde{b}} - \frac{g_2}{\sqrt{2}}h_{\mathrm{u}}^{+*}\tilde{h}_{\mathrm{u}}^{+}\tilde{w}^{3} - \frac{g_Y}{\sqrt{2}}h_{\mathrm{u}}^{+*}\tilde{h}_{\mathrm{u}}^{+}\tilde{b} + \frac{g_2}{\sqrt{2}}h_{\mathrm{d}}^{-*}\tilde{h}_{\mathrm{d}}^{-}\tilde{w}^{3} + \frac{g_Y}{\sqrt{2}}h_{\mathrm{d}}^{-*}\tilde{h}_{\mathrm{d}}^{-}\tilde{b} \\
& + \frac{g_2}{\sqrt{2}}h_{\mathrm{d}}^{-}\bar{\tilde{h}}_{\mathrm{d}}^{-}\bar{\tilde{w}}^{3} + \frac{g_Y}{\sqrt{2}}h_{\mathrm{d}}^{-}\bar{\tilde{h}}_{\mathrm{d}}^{-}\bar{\tilde{b}} + \frac{g_2}{\sqrt{2}}h_{\mathrm{u}}^{0*}\tilde{h}_{\mathrm{u}}^{0}\tilde{w}^{3} - \frac{g_Y}{\sqrt{2}}h_{\mathrm{u}}^{0*}\tilde{h}_{\mathrm{u}}^{0}\tilde{b} - \frac{g_2}{\sqrt{2}}h_{\mathrm{d}}^{0*}\tilde{h}_{\mathrm{d}}^{0}\tilde{w}^{3} + \frac{g_Y}{\sqrt{2}}h_{\mathrm{d}}^{0*}\tilde{h}_{\mathrm{d}}^{0}\tilde{b} \\
& + \frac{g_2}{\sqrt{2}}h_{\mathrm{u}}^{0}\bar{\tilde{h}}_{\mathrm{u}}^{0}\bar{\tilde{w}}^{3} - \frac{g_Y}{\sqrt{2}}h_{\mathrm{u}}^{0}\bar{\tilde{h}}_{\mathrm{u}}^{0}\bar{\tilde{b}} - \frac{g_2}{\sqrt{2}}h_{\mathrm{d}}^{0}\bar{\tilde{h}}_{\mathrm{d}}^{0}\bar{\tilde{w}}^{3} + \frac{g_Y}{\sqrt{2}}h_{\mathrm{d}}^{0}\bar{\tilde{h}}_{\mathrm{d}}^{0}\bar{\tilde{b}}
\end{aligned}
\tag{12.42}
$$

12.3.4 Scalar part

(see next page)

$$
\begin{aligned}
\mathcal{L}_{\text{scalar}} = {} & (\partial_\mu\tilde{u}_L^* + ig_3\tilde{u}_L^*\tau^a g_\mu^a)(\partial^\mu\tilde{u}_L - ig_3 g^{b\mu}\tau^b\tilde{u}_L) + (\partial_\mu\tilde{u}_R - ig_3\tilde{u}_R\tau^{a*}g_\mu^a)(\partial^\mu\tilde{u}_R^* + ig_3 g_\mu^b\tau^{b*}\tilde{u}_R^*) \\
& + (\partial_\mu\tilde{d}_L^* + ig_3\tilde{d}_L^*\tau^a g_\mu^a)(\partial^\mu\tilde{d}_L - ig_3 g^{b\mu}\tau^b\tilde{d}_L) + (\partial_\mu\tilde{d}_R - ig_3\tilde{d}_R^y\tau^{a*}g_\mu^a)(\partial^\mu\tilde{d}_R^* + ig_3 g_\mu^b\tau^{b*}\tilde{d}_R^*) \\
& + \sqrt{2}g_2g_3\tilde{u}_L^*\tau^a\tilde{d}_L W^{+\mu}g_\mu^a + \sqrt{2}g_2g_3\tilde{d}_L^*\tau^a\tilde{u}_L W^{-\mu}g_\mu^a + \frac{4}{3}g_3|e|(\tilde{u}_L^*\tau^a\tilde{u}_L + \tilde{u}_R^*\tau^a\tilde{u}_R)g^{a\mu}A_\mu \\
& - \frac{2}{3}g_3|e|(\tilde{d}_L^*\tau^a\tilde{d}_L + \tilde{d}_R^*\tau^a\tilde{d}_R)g^{a\mu}A_\mu + \frac{(3-4s_W^2)g_Z}{3}g_3\tilde{u}_L^*\tau^a\tilde{u}_L g^{a\mu}Z_\mu - \frac{4s_W^2 g_Z}{3}g_3\tilde{u}_R^*\tau^a\tilde{u}_R g^{a\mu}Z_\mu \\
& + \frac{(2s_W^2-3)g_Z}{3}g_3\tilde{d}_L^*\tau^a\tilde{d}_L g^{a\mu}Z_\mu + \frac{2s_W^2 g_Z}{3}g_3\tilde{d}_R^*\tau^a\tilde{d}_R g^{a\mu}Z_\mu \\
& + \frac{ig_2}{\sqrt{2}}W_\mu^+\left(\tilde{u}_L^*\partial^\mu\tilde{d}_L - \tilde{d}_L\partial^\mu\tilde{u}_L^*\right) - \frac{ig_2}{\sqrt{2}}W_\mu^-\left(\tilde{u}_L\partial^\mu\tilde{d}_L^* - \tilde{d}_L^*\partial^\mu\tilde{u}_L\right) \\
& + \frac{2i}{3}|e|A_\mu\left(\tilde{u}_L^*\partial^\mu\tilde{u}_L - \tilde{u}_L\partial^\mu\tilde{u}_L^* + \tilde{u}_R^*\partial^\mu\tilde{u}_R - \tilde{u}_R\partial^\mu\tilde{u}_R^*\right) - \frac{i}{3}|e|A_\mu\left(\tilde{d}_L^*\partial^\mu\tilde{d}_L - \tilde{d}_L\partial^\mu\tilde{d}_L^* + \tilde{d}_R^*\partial^\mu\tilde{d}_R - \tilde{d}_R\partial^\mu\tilde{d}_R^*\right) \\
& + \frac{i(4s_W^2-3)g_Z}{6}Z_\mu\left(\tilde{u}_L\partial^\mu\tilde{u}_L^* - \tilde{u}_L^*\partial^\mu\tilde{u}_L\right) + \frac{i(2s_W^2-3)g_Z}{6}Z_\mu\left(\tilde{d}_L^*\partial^\mu\tilde{d}_L - \tilde{d}_L\partial^\mu\tilde{d}_L^*\right) \\
& - \frac{2is_W^2 g_Z}{3}Z_\mu\left(\tilde{u}_R^*\partial^\mu\tilde{u}_R - \tilde{u}_R\partial^\mu\tilde{u}_R^*\right) + \frac{is_W^2 g_Z}{3}Z_\mu\left(\tilde{d}_R^*\partial^\mu\tilde{d}_R - \tilde{d}_R\partial^\mu\tilde{d}_R^*\right) \\
& + \frac{g_Z^2}{2}\left(|\tilde{u}_L|^2 + |\tilde{d}_L|^2\right)W^{+\mu}W_\mu^- - \frac{s_W^2 g_2 g_Z}{3\sqrt{2}}\tilde{u}_L^*\tilde{d}_L W^{+\mu}Z_\mu - \frac{s_W^2 g_2 g_Z}{3\sqrt{2}}\tilde{d}_L^*\tilde{u}_L W^{-\mu}Z_\mu \\
& + \frac{(3-4s_W^2)^2 g_Z^2}{36}|\tilde{u}_L|^2 Z^\mu Z_\mu + \frac{4s_W^4 g_Z^2}{9}|\tilde{u}_R|^2 Z^\mu Z_\mu + \frac{(3-2s_W^2)^2 g_Z^2}{36}|\tilde{d}_L|^2 Z^\mu Z_\mu + \frac{s_W^4 g_Z^2}{9}|\tilde{d}_R|^2 Z^\mu Z_\mu \\
& + \frac{4}{9}e^2\left(|\tilde{u}_L|^2 + |\tilde{u}_R|^2\right)A^\mu A_\mu + \frac{1}{9}e^2\left(|\tilde{d}_L|^2 + |\tilde{d}_R|^2\right)A^\mu A_\mu + \frac{|e|g_2}{3\sqrt{2}}\tilde{u}_L^*\tilde{d}_L W^{+\mu}A_\mu + \frac{|e|g_2}{3\sqrt{2}}\tilde{d}_L^*\tilde{u}_L W^{-\mu}A_\mu \\
& + \frac{2(3-4s_W^2)g_Z|e|}{9}|\tilde{u}_L|^2 A^\mu Z_\mu - \frac{8s_W^2 g_Z|e|}{9}|\tilde{u}_R|^2 A^\mu Z_\mu + \frac{(3-2s_W^2)g_Z|e|}{9}|\tilde{d}_L|^2 A^\mu Z_\mu - \frac{2s_W^2 g_Z|e|}{9}|\tilde{d}_R|^2 A^\mu Z_\mu \\[1ex]
& + \partial_\mu\tilde{e}_R\partial^\mu\tilde{e}_R^* + \partial_\mu\tilde{e}_L^*\partial^\mu\tilde{e}_L + \partial_\mu\tilde{\nu}_L^*\partial^\mu\tilde{\nu}_L + i\frac{g_2}{\sqrt{2}}W_\mu^+(\tilde{\nu}_L^*\partial^\mu\tilde{e}_L - \tilde{e}_L\partial^\mu\tilde{\nu}_L^*) + i\frac{g_2}{\sqrt{2}}W_\mu^-(\tilde{e}_L^*\partial^\mu\tilde{\nu}_L - \tilde{\nu}_L\partial^\mu\tilde{e}_L^*) \\
& - \frac{i(1-2s_W^2)g_Z}{2}Z_\mu(\tilde{e}_L^*\partial^\mu\tilde{e}_L - \tilde{e}_L\partial^\mu\tilde{e}_L^*) + \frac{ig_Z}{2}Z_\mu(\tilde{\nu}_L^*\partial^\mu\tilde{\nu}_L - \tilde{\nu}_L\partial^\mu\tilde{\nu}_L^*) + is_W^2 g_Z Z_\mu(\tilde{e}_R^*\partial^\mu\tilde{e}_R - \tilde{e}_R\partial^\mu\tilde{e}_R^*) \\
& + i|e|A_\mu(\tilde{e}_L\partial^\mu\tilde{e}_L^* - \tilde{e}_L^*\partial^\mu\tilde{e}_L + \tilde{e}_R\partial^\mu\tilde{e}_R^* - \tilde{e}_R^*\partial^\mu\tilde{e}_R) \\
& + \frac{g_Z^2}{2}(|\tilde{\nu}_L|^2 + |\tilde{e}_L|^2)W^{+\mu}W_\mu^- + \frac{g_2 g_Z s_W^2}{\sqrt{2}}\left(\tilde{e}_L^*\tilde{\nu}_L W_\mu^- Z^\mu + \tilde{\nu}_L^*\tilde{e}_L W_\mu^+ Z^\mu\right) - \frac{g_2|e|}{\sqrt{2}}\left(\tilde{\nu}_L^*\tilde{e}_L W_\mu^+ A^\mu + \tilde{e}_L^*\tilde{\nu}_L W_\mu^- A^\mu\right) \\
& + \frac{(1-2s_W^2)^2 g_Z^2}{4}|\tilde{e}_L|^2 Z^\mu Z_\mu + \frac{g_Z^2}{4}|\tilde{\nu}_L|^2 Z^\mu Z_\mu + g_Z^2 s_W^4|\tilde{e}_R|^2 Z^\mu Z_\mu \\
& + e^2(|\tilde{e}_L|^2 + |\tilde{e}_R|^2)A^\mu A_\mu + (1-2s_W^2)|e|g_Z|\tilde{e}_L|^2 A^\mu Z_\mu - 2s_W^2 g_Z|e||\tilde{e}_R|^2 A^\mu Z_\mu \\[1ex]
& + \partial_\mu h_d^{-*}\partial^\mu h_d^- + \partial_\mu h_d^{0*}\partial^\mu h_d^0 + \partial_\mu h_u^{+*}\partial^\mu h_u^+ + \partial_\mu h_u^{0*}\partial^\mu h_u^0 + i\frac{g_2}{\sqrt{2}}W_\mu^+\left(h_u^{+*}\partial^\mu h_u^0 - h_u^0\partial^\mu h_u^{+*}\right) \\
& + i\frac{g_2}{\sqrt{2}}W_\mu^-\left(h_u^{0*}\partial^\mu h_u^+ - h_u^+\partial^\mu h_u^{0*}\right) + i\frac{g_2}{\sqrt{2}}W_\mu^+\left(h_d^{0*}\partial^\mu h_d^- - h_d^-\partial^\mu h_d^{0*}\right) + i\frac{g_2}{\sqrt{2}}W_\mu^-\left(h_d^{-*}\partial^\mu h_d^0 - h_d^0\partial^\mu h_d^{-*}\right) \\
& + \frac{i(1-2s_W^2)g_Z}{2}Z_\mu\left(h_u^{+*}\partial^\mu h_u^+ - h_u^+\partial^\mu h_u^{+*}\right) + \frac{i(2s_W^2-1)g_Z}{2}Z_\mu\left(h_d^{-*}\partial^\mu h_d^- - h_d^-\partial^\mu h_d^{-*}\right) \\
& - \frac{ig_Z}{2}Z_\mu\left(h_u^{0*}\partial^\mu h_u^0 - h_u^0\partial^\mu h_u^{0*}\right) + \frac{ig_Z}{2}Z_\mu\left(h_d^{0*}\partial^\mu h_d^0 - h_d^0\partial^\mu h_d^{0*}\right) \\
& - i|e|A_\mu\left(h_d^{-*}\partial^\mu h_d^- - h_d^-\partial^\mu h_d^{-*}\right) + i|e|A_\mu\left(h_u^{+*}\partial^\mu h_u^+ - h_u^+\partial^\mu h_u^{+*}\right) \\
& + \frac{g_Z^2}{2}\left(|h_u^+|^2 + |h_u^0|^2 + |h_d^0|^2 + |h_d^-|^2\right)W^{+\mu}W_\mu^- + \frac{s_W^2 g_Z g_2}{\sqrt{2}}\left(h_d^{0*}h_d^- - h_u^{+*}h_u^0\right)W_\mu^+ Z^\mu \\
& + \frac{s_W^2 g_Z g_2}{\sqrt{2}}\left(h_d^{-*}h_d^0 - h_u^{0*}h_u^+\right)W_\mu^- Z^\mu + \frac{g_2|e|}{\sqrt{2}}\left(h_u^{+*}h_u^0 - h_d^{0*}h_d^-\right)W_\mu^+ A^\mu + \frac{g_2|e|}{\sqrt{2}}\left(h_u^{0*}h_u^+ - h_d^{-*}h_d^0\right)W_\mu^- A^\mu \\
& + \frac{(1-2s_W^2)^2 g_Z^2}{4}\left(|h_u^+|^2 + |h_d^-|^2\right)Z^\mu Z_\mu + \frac{g_Z^2}{4}\left(|h_u^0|^2 + |h_d^0|^2\right)Z^\mu Z_\mu + e^2\left(|h_u^+|^2 + |h_d^-|^2\right)A^\mu A_\mu \\
& + (1-2s_W^2)|e|g_Z\left(|h_u^+|^2 + |h_d^-|^2\right)A^\mu Z_\mu \\
& - (V_{\text{SUSY}} + V_{\cancel{\text{SUSY}}}),
\end{aligned}
$$

(12.43)

where the scalar potential is given by

$$
\begin{aligned}
V_{\text{SUSY}} = {} & |h_{\text{u}}|^2\left(|\mu|^2 + \sum_i |\kappa_i|^2\right) + |\mu|^2|h_{\text{d}}|^2 + \left(\kappa_i^* \mu \tilde{l}_{\text{L}i}^* h_{\text{d}} + \text{H.c.}\right) + \kappa_i^* \kappa_j \tilde{l}_{\text{L}i}^* \tilde{l}_{\text{L}j} \\
& + \Big[- y_{uij}\mu^* h_{\text{d}}^* \tilde{u}_{\text{R}i}^* \tilde{q}_{\text{L}j} - y_{uij}\kappa_k^* \tilde{u}_{\text{R}i}^* \tilde{q}_{\text{L}j} \tilde{l}_{\text{L}k}^* - \left(y_{dij}\mu^* + \lambda'_{kji}\kappa_k^*\right) h_{\text{u}}^* \tilde{d}_{\text{R}i}^* \tilde{q}_{\text{L}j} \\
& \qquad + y_{eij}\kappa_j^* \tilde{e}_{\text{R}i}^* h_{\text{u}}^* h_{\text{d}} + \left(\lambda_{jki}\kappa_k^* - y_{eij}\mu^*\right) h_{\text{u}}^* \tilde{e}_{\text{R}i}^* \tilde{l}_{\text{L}j} + \text{H.c.}\Big] \\
& + \frac{1}{8}\left(g_2^2 + g_Y^2\right)|h_{\text{d}}|^4 + \frac{1}{8}\left(g_2^2 + g_Y^2\right)|h_{\text{u}}|^4 + \left(-\frac{g_2^2}{4}|h_{\text{d}}|^2|h_{\text{u}}|^2 - \frac{g_Y^2}{4}|h_{\text{d}}|^2|h_{\text{u}}|^2 + \frac{g_2^2}{2}|h_{\text{d}}^* h_{\text{u}}|^2\right) \\
& + \left(-\frac{g_2^2}{4}|h_{\text{u}}|^2|\tilde{q}_{\text{L}}|^2 + \frac{g_Y^2}{12}|h_{\text{u}}|^2|\tilde{q}_{\text{L}}|^2 + \frac{g_2^2}{2}|h_{\text{u}}^b \tilde{q}_{\text{L}i}|^2 + \epsilon^{ac}\epsilon^{bd}(y_u^\dagger y_u)_{ji} h_{\text{u}}^a h_{\text{u}}^{b*} \tilde{q}_{\text{L}i}^c \tilde{q}_{\text{L}j}^{d*}\right) \\
& + \left(-\frac{g_2^2}{4}|h_{\text{d}}|^2|\tilde{q}_{\text{L}}|^2 - \frac{g_Y^2}{12}|h_{\text{d}}|^2|\tilde{q}_{\text{L}}|^2 + \frac{g_2^2}{2}|h_{\text{d}}^* \tilde{q}_{\text{L}i}|^2 + \epsilon^{ac}\epsilon^{bd}(y_d^\dagger y_d)_{ji} h_{\text{d}}^a h_{\text{d}}^{b*} \tilde{q}_{\text{L}i}^c \tilde{q}_{\text{L}j}^{d*}\right) \\
& + \left(-\frac{g_2^2}{4}|h_{\text{u}}|^2|\tilde{l}_{\text{L}}|^2 - \frac{g_Y^2}{4}|h_{\text{u}}|^2|\tilde{l}_{\text{L}}|^2 + \frac{g_2^2}{2}|h_{\text{u}}^* \tilde{l}_{\text{L}i}|^2\right) \\
& + \left(-\frac{g_2^2}{4}|h_{\text{d}}|^2|\tilde{l}_{\text{L}}|^2 + \frac{g_Y^2}{4}|h_{\text{d}}|^2|\tilde{l}_{\text{L}}|^2 + \frac{g_2^2}{2}|h_{\text{d}}^* \tilde{l}_{\text{L}i}|^2 + \epsilon^{ac}\epsilon^{bd}(y_e^\dagger y_e)_{ji} h_{\text{d}}^a h_{\text{d}}^{b*} \tilde{l}_{\text{L}i}^c \tilde{l}_{\text{L}j}^{d*}\right) \\
& + |h_{\text{u}}|^2\left(-\frac{g_Y^2}{3}|\tilde{u}_{\text{R}}|^2 + (y_u y_u^\dagger)_{ij} \tilde{u}_{\text{R}i}^* \tilde{u}_{\text{R}j}\right) + \frac{g_Y^2}{3}|h_{\text{d}}|^2|\tilde{u}_{\text{R}}|^2 \\
& + \frac{g_Y^2}{6}|h_{\text{u}}|^2|\tilde{d}_{\text{R}}|^2 + |h_{\text{d}}|^2\left(-\frac{g_Y^2}{6}|\tilde{d}_{\text{R}}|^2 + (y_d y_d^\dagger)_{ij} \tilde{d}_{\text{R}i}^* \tilde{d}_{\text{R}j}\right) - \left[(y_u y_d^\dagger)_{ij} \tilde{u}_{\text{R}i}^* \tilde{d}_{\text{R}j} (h_{\text{d}}^* h_{\text{u}}) + \text{H.c.}\right] \\
& + \frac{g_Y^2}{2}|h_{\text{u}}|^2|\tilde{e}_{\text{R}}|^2 + |h_{\text{d}}|^2\left(-\frac{g_Y^2}{2}|\tilde{e}_{\text{R}}|^2 + (y_e y_e^\dagger)_{ij} \tilde{e}_{\text{R}i}^* \tilde{e}_{\text{R}j}\right) \\
& + \Big[-\frac{1}{2}\epsilon^{ab}\epsilon^{xyz} y_{ulk} \lambda''^*_{lij} h_{\text{u}}^a \tilde{d}_{\text{R}i}^x \tilde{d}_{\text{R}j}^y \tilde{q}_{\text{L}k}^{bz} + \epsilon^{ab}\epsilon^{xyz} y_{dlk} \lambda''^*_{ijl} h_{\text{d}}^a \tilde{u}_{\text{R}i}^x \tilde{d}_{\text{R}j}^y \tilde{q}_{\text{L}k}^{bz} - y_{uil}\lambda'^*_{klj} h_{\text{u}}^a \tilde{u}_{\text{R}i}^* \tilde{d}_{\text{R}j} \tilde{l}_{\text{L}k}^{a*} \\
& \qquad + y_{dil}\lambda'^*_{klj} h_{\text{d}}^a \tilde{d}_{\text{R}i}^* \tilde{d}_{\text{R}j} \tilde{l}_{\text{L}k}^{a*} - \epsilon^{ab}\epsilon^{cd} y_{dli} \lambda'^*_{kjl} h_{\text{d}}^a \tilde{q}_{\text{L}i}^b \tilde{q}_{\text{L}j}^{c*} \tilde{l}_{\text{L}k}^{d*} + y_{eil}\lambda^*_{klj} h_{\text{d}}^a \tilde{e}_{\text{R}i}^* \tilde{e}_{\text{R}j} \tilde{l}_{\text{L}k}^{a*} \\
& \qquad - y_{ejl}\lambda'^*_{lki} h_{\text{d}}^a \tilde{e}_{\text{R}j}^* \tilde{d}_{\text{R}i} \tilde{q}_{\text{L}k}^{a*} + \frac{1}{2}\epsilon^{ab}\epsilon^{cd} y_{eli} \lambda^*_{jkl} h_{\text{d}}^a \tilde{l}_{\text{L}i}^b \tilde{l}_{\text{L}j}^{c*} \tilde{l}_{\text{L}k}^{d*} + \text{H.c.}\Big] \\
& + \left[\left(-\frac{g_3^2}{12} + \frac{g_Y^2}{72} - \frac{g_2^2}{8}\right)|\tilde{q}_{\text{L}}|^4 + \frac{g_2^2}{4}\tilde{q}_{\text{L}i}^{ax} \tilde{q}_{\text{L}j}^{by} \tilde{q}_{\text{L}i}^{bx*} \tilde{q}_{\text{L}j}^{ay*} + \frac{g_3^2}{4}\tilde{q}_{\text{L}i}^{ax} \tilde{q}_{\text{L}j}^{by} \tilde{q}_{\text{L}i}^{ay*} \tilde{q}_{\text{L}j}^{bx*}\right] \\
& + \left[\left(-\frac{g_3^2}{12} + \frac{2g_Y^2}{9}\right)|\tilde{u}_{\text{R}}|^4 + \frac{g_3^2}{4}\tilde{u}_{\text{R}i}^{x*} \tilde{u}_{\text{R}j}^{y*} \tilde{u}_{\text{R}i}^y \tilde{u}_{\text{R}j}^x\right] \\
& + \left[\left(-\frac{g_3^2}{12} + \frac{g_Y^2}{18}\right)|\tilde{d}_{\text{R}}|^4 + \frac{g_3^2}{4}\tilde{d}_{\text{R}i}^{x*} \tilde{d}_{\text{R}j}^{y*} \tilde{d}_{\text{R}i}^y \tilde{d}_{\text{R}j}^x + \frac{1}{2}\lambda''_{mij}\lambda''^*_{mkl} \tilde{d}_{\text{R}i}^{x*} \tilde{d}_{\text{R}j}^{y*} \tilde{d}_{\text{R}k}^x \tilde{d}_{\text{R}l}^y\right] \\
& + \left[\left(\frac{g_3^2}{6} - \frac{g_Y^2}{9}\right)|\tilde{u}_{\text{R}}|^2|\tilde{q}_{\text{L}}|^2 - \frac{g_3^2}{2}\tilde{u}_{\text{R}i}^{x*} \tilde{q}_{\text{L}j}^{ax} \tilde{u}_{\text{R}i}^y \tilde{q}_{\text{L}j}^{ay*} + y_{uik} y_{ujl}^* \tilde{u}_{\text{R}i}^{x*} \tilde{u}_{\text{R}j}^y \tilde{q}_{\text{L}k}^{ax} \tilde{q}_{\text{L}l}^{ay*}\right] \\
& + \left[\left(\frac{g_3^2}{6} + \frac{g_Y^2}{18}\right)|\tilde{d}_{\text{R}}|^2|\tilde{q}_{\text{L}}|^2 - \frac{g_3^2}{2}\tilde{d}_{\text{R}i}^{x*} \tilde{q}_{\text{L}j}^{ax} \tilde{d}_{\text{R}i}^y \tilde{q}_{\text{L}j}^{ay*} + \left(y_{dik} y_{djl}^* + \lambda'_{mki}\lambda'^*_{mlj}\right) \tilde{d}_{\text{R}i}^{x*} \tilde{q}_{\text{L}k}^{ax} \tilde{d}_{\text{R}j}^y \tilde{q}_{\text{L}l}^{ay*}\right] \\
& + \left[-\left(\frac{g_3^2}{6} + \frac{2g_Y^2}{9}\right)|\tilde{d}_{\text{R}}|^2|\tilde{u}_{\text{R}}|^2 + \left(\frac{g_3^2}{2} - \lambda''_{ikm}\lambda''^*_{ljm}\right) \tilde{u}_{\text{R}i}^{x*} \tilde{d}_{\text{R}j}^x \tilde{d}_{\text{R}j}^{y*} \tilde{u}_{\text{R}i}^y + \lambda''_{ikm}\lambda''^*_{jlm} \tilde{u}_{\text{R}i}^{x*} \tilde{u}_{\text{R}j}^x \tilde{d}_{\text{R}k}^{y*} \tilde{d}_{\text{R}l}^y\right] \\
& + \left[-\left(\frac{g_2^2}{4} + \frac{g_Y^2}{12}\right)|\tilde{l}_{\text{L}}|^2|\tilde{q}_{\text{L}}|^2 + \frac{g_2^2}{2}\tilde{q}_{\text{L}i}^{ax} \tilde{q}_{\text{L}i}^{bx*} \tilde{l}_{\text{L}j}^b \tilde{l}_{\text{L}j}^{a*} + \epsilon^{ac}\epsilon^{bd}\lambda'_{kim}\lambda'^*_{ljm} \tilde{q}_{\text{L}i}^{ax} \tilde{q}_{\text{L}j}^{bx*} \tilde{l}_{\text{L}k}^c \tilde{l}_{\text{L}l}^{d*}\right] + \frac{g_Y^2}{6}|\tilde{e}_{\text{R}}|^2|\tilde{q}_{\text{L}}|^2 \\
& + \left(-\frac{g_Y^2}{6}|\tilde{d}_{\text{R}}|^2|\tilde{l}_{\text{L}}|^2 + \lambda'_{lmi}\lambda'^*_{kmj} \tilde{d}_{\text{R}i}^* \tilde{d}_{\text{R}j} \tilde{l}_{\text{L}k}^* \tilde{l}_{\text{L}l}\right) + \frac{g_Y^2}{3}|\tilde{u}_{\text{R}}|^2|\tilde{l}_{\text{L}}|^2 - \frac{2g_Y^2}{3}|\tilde{u}_{\text{R}}|^2|\tilde{e}_{\text{R}}|^2 + \frac{g_Y^2}{3}|\tilde{d}_{\text{R}}|^2|\tilde{e}_{\text{R}}|^2 \\
& + \left[\left(-\frac{g_2^2}{8} + \frac{g_Y^2}{8}\right)|\tilde{l}_{\text{L}}|^4 + \frac{g_2^2}{4}\tilde{l}_{\text{L}i}^a \tilde{l}_{\text{L}j}^b \tilde{l}_{\text{L}i}^{b*} \tilde{l}_{\text{L}j}^{a*} + \frac{1}{4}\epsilon^{ab}\epsilon^{cd}\lambda_{ijm}\lambda^*_{klm} \tilde{l}_{\text{L}i}^a \tilde{l}_{\text{L}j}^b \tilde{l}_{\text{L}k}^{c*} \tilde{l}_{\text{L}l}^{d*}\right] \\
& + \frac{g_Y^2}{2}|\tilde{e}_{\text{R}}|^4 + \left[-\frac{g_Y^2}{2}|\tilde{e}_{\text{R}}|^2|\tilde{l}_{\text{L}}|^2 + \left(y_{eik} y_{ejl}^* + \lambda_{kmi}\lambda^*_{lmj}\right) \tilde{e}_{\text{R}i}^* \tilde{e}_{\text{R}j} \tilde{l}_{\text{L}k} \tilde{l}_{\text{L}l}^*\right] \\
& + \left[\left(y_{dik} y_{ejl}^* - \lambda'_{mki}\lambda^*_{lmj}\right) \tilde{d}_{\text{R}i}^{x*} \tilde{q}_{\text{L}k}^{ax} \tilde{e}_{\text{R}j} \tilde{l}_{\text{L}l}^{a*} - \epsilon^{ab}\epsilon^{xyz}\lambda'_{lkm}\lambda''^*_{ijm} \tilde{l}_{\text{L}l}^b \tilde{q}_{\text{L}k}^{az} \tilde{d}_{\text{R}j}^y \tilde{u}_{\text{R}i}^x + \text{H.c.}\right].
\end{aligned}
\tag{12.44}
$$

12.4 Higgs mechanism and fermion composition

The scalar potential includes

$$\begin{aligned}V_{\text{SUSY}} \supset{}& |h_{\text{u}}|^2\Bigl(|\mu|^2+\sum|\kappa_i|^2\Bigr)+|\mu|^2|h_{\text{d}}|^2+\frac{g_Z^2}{8}\bigl(|h_{\text{u}}|^2-|h_{\text{d}}|^2\bigr)^2+\frac{g_2^2}{2}|h_{\text{d}}^*h_{\text{u}}|^2\\&+\Bigl(\kappa_i^*\mu\tilde{l}_{\text{L}i}^*h_{\text{d}}+\text{H.c.}\Bigr)+\kappa_i^*\kappa_j\tilde{l}_{\text{L}i}^*\tilde{l}_{\text{L}j}\end{aligned}\tag{12.45}$$

$$V_{\cancel{\text{SUSY}}}\supset m_{H_{\text{u}}}^2|h_{\text{u}}|^2+m_{H_{\text{d}}}^2|h_{\text{d}}|^2+\epsilon^{ab}\left(bh_{\text{u}}^ah_{\text{d}}^b+b^*h_{\text{u}}^{a*}h_{\text{d}}^{b*}-b_i\tilde{l}_{\text{L}i}^ah_{\text{u}}^b-b_i^*\tilde{l}_{\text{L}i}^{a*}h_{\text{u}}^{b*}\right)+\tilde{l}_{\text{L}i}^*M_{Li}^2h_{\text{d}}+\tilde{l}_{\text{L}i}M_{Li}^{2*}h_{\text{d}}^*;\tag{12.46}$$

the Higgs mass term is given by

$$V\supset\begin{pmatrix}h_{\text{u}} & h_{\text{d}}^* & \tilde{l}_{\text{L}i}^*\end{pmatrix}\begin{pmatrix}|\mu|^2+m_{H_{\text{u}}}^2+\sum|\kappa_i|^2 & b & -b_j\\ b^* & |\mu|^2+m_{H_{\text{d}}}^2 & \kappa_j\mu^*+M_{Lj}^{2*}\\ -b_i^* & \kappa_i^*\mu+M_{Li}^2 & (m_L^2)_{ij}+\kappa_i^*\kappa_j\end{pmatrix}\begin{pmatrix}h_{\text{u}}^*\\ h_{\text{d}}\\ \tilde{l}_{\text{L}j}\end{pmatrix}\tag{12.47}$$

while corresponding fermion terms are

$$\mathcal{L}\supset\epsilon^{ab}\left(-\mu\tilde{h}_{\text{u}}^a\tilde{h}_{\text{d}}^b-\kappa_i\tilde{h}_{\text{u}}^al_{\text{L}i}^b\right).\tag{12.48}$$

If the R-parity is not conserved, we redefine (H_{d},L) superfields so that the mass matrix is block-diagonal, which corresponds to $\mathrm{U}(4)_{H_{\text{d}},L}\to\mathrm{U}(3)_L\times\mathrm{U}(1)_{H_{\text{d}}}$ (DOF counting: $16\to9+1$ to remove b_i'). Then lepton and $\tilde{h}_{\text{d}}$ are mixed.[*25] With R-parity conservation, we do not suffer from these mixings.

12.4.1 Higgs potential and induced mass in R-parity conserved case

We perform "SU(2)-notation fixing", i.e., use the freedom associated to T_1 and T_2 of SU(2), so that $\langle h_{\text{u}}^+\rangle=0$. Then $\langle h_{\text{d}}^-\rangle=0$ and effectively

$$V_{\text{pot}}=(|\mu|^2+m_{H_{\text{u}}}^2)|h_{\text{u}}^0|^2+(|\mu|^2+m_{H_{\text{d}}}^2)|h_{\text{d}}^0|^2+\frac{g_Z^2}{8}\left(|h_{\text{u}}^0|^2-|h_{\text{d}}^0|^2\right)^2-\left(bh_{\text{u}}^0h_{\text{d}}^0+\text{H.c.}\right).\tag{12.49}$$

We redefine H_{d} superfield so that $b>0$.[*26] Then $\arg\langle h_{\text{u}}^0\rangle=-\arg\langle h_{\text{d}}^0\rangle$ and, with T_3-rotation, $\langle h_{\text{u}}^0\rangle>0$ and $\langle h_{\text{d}}^0\rangle>0$:

$$\langle h_{\text{u}}^0\rangle=:v_{\text{u}}=:\frac{v_{\text{SM}}}{\sqrt{2}}\sin\beta,\qquad\langle h_{\text{d}}^0\rangle=:v_{\text{d}}=:\frac{v_{\text{SM}}}{\sqrt{2}}\cos\beta;\tag{12.50}$$

$$V_{\text{pot}}=\frac12(|\mu|^2+m_{H_{\text{u}}}^2)v_{\text{SM}}^2\sin^2\beta+\frac12(|\mu|^2+m_{H_{\text{d}}}^2)v_{\text{SM}}^2\cos^2\beta+\frac{g_Z^2}{32}v_{\text{SM}}^4\cos^22\beta-\frac12v_{\text{SM}}^2b\sin2\beta.\tag{12.51}$$

This potential can have two minima; one with $0<\beta\le\pi/4$ and the other with $\pi/4\le\beta<\pi/2$:

$$\tan\beta=\frac{B\mp\sqrt{B^2-4b^2}}{2b}\quad\left(\cos2\beta=\pm\frac{\sqrt{B^2-4b^2}}{B}\right),\qquad m_Z^2:=\frac{g_Z^2}{4}v_{\text{SM}}^2=\left(\pm\frac{m_{H_{\text{d}}}^2-m_{H_{\text{u}}}^2}{\sqrt{B^2-4b^2}}-1\right)B,\tag{12.52}$$

where $B:=2|\mu|^2+m_{H_{\text{u}}}^2+m_{H_{\text{d}}}^2>2b>0$ and m_Z is the Z-boson tree-level mass. Also,

$$\sin2\beta=\frac{2b}{2|\mu|^2+m_{H_{\text{u}}}^2+m_{H_{\text{d}}}^2},\qquad m_Z^2=\frac{-(m_{H_{\text{d}}}^2-m_{H_{\text{u}}}^2)}{\cos2\beta}-\left(2|\mu|^2+m_{H_{\text{u}}}^2+m_{H_{\text{d}}}^2\right)\tag{12.53}$$

are satisfied in both solutions.

Higgs sector The 南部-Goldstone–Higgs mixings and the mass terms for the charged Higgs bosons are given by

$$\begin{aligned}\mathcal{L}\supset{}&\partial_\mu h_{\text{d}}^{-*}\partial^\mu h_{\text{d}}^-+\partial_\mu h_{\text{u}}^{+*}\partial^\mu h_{\text{u}}^++\left(-b-\frac12g_2^2v_{\text{u}}v_{\text{d}}\right)(h_{\text{u}}^+h_{\text{d}}^-+h_{\text{u}}^{+*}h_{\text{d}}^{-*})\\&+\left[\frac{g_Y^2(v_{\text{u}}^2-v_{\text{d}}^2)-g_2^2(v_{\text{u}}^2+v_{\text{d}}^2)}{4}-|\mu|^2-m_{H_{\text{d}}}^2\right]|h_{\text{d}}^-|^2+\left[\frac{g_Y^2(v_{\text{d}}^2-v_{\text{u}}^2)-g_2^2(v_{\text{u}}^2+v_{\text{d}}^2)}{4}-|\mu|^2-m_{H_{\text{u}}}^2\right]|h_{\text{u}}^+|^2\\&+\frac{\mathrm{i}g_2}{\sqrt2}W_\mu^-\partial^\mu\left(v_{\text{u}}h_{\text{u}}^+-v_{\text{u}}h_{\text{u}}^{+*}-v_{\text{d}}h_{\text{d}}^{-*}+v_{\text{d}}h_{\text{d}}^-\right)\end{aligned}\tag{12.54}$$

and those for the neutral Higgs bosons are

$$\begin{aligned}\mathcal{L}\supset{}&\partial_\mu h_{\text{d}}^{0*}\partial^\mu h_{\text{d}}^0+\partial_\mu h_{\text{u}}^{0*}\partial^\mu h_{\text{u}}^0-\frac{g_Z^2v_{\text{d}}^2}{8}(h_{\text{d}}^0h_{\text{d}}^0+h_{\text{d}}^{0*}h_{\text{d}}^{0*})-\frac{g_Z^2v_{\text{u}}^2}{8}(h_{\text{u}}^0h_{\text{u}}^0+h_{\text{u}}^{0*}h_{\text{u}}^{0*})\\&+\left(b+\frac{g_Z^2v_{\text{u}}v_{\text{d}}}{4}\right)(h_{\text{u}}^0h_{\text{d}}^0+h_{\text{u}}^{0*}h_{\text{d}}^{0*})+\frac{g_Z^2v_{\text{u}}v_{\text{d}}}{4}(h_{\text{u}}^0h_{\text{d}}^{0*}+h_{\text{u}}^{0*}h_{\text{d}}^0)\\&+\left(\frac{g_Z^2(v_{\text{u}}^2-2v_{\text{d}}^2)}{4}-|\mu|^2-m_{H_{\text{d}}}^2\right)|h_{\text{d}}^0|^2+\left(\frac{g_Z^2(v_{\text{d}}^2-2v_{\text{u}}^2)}{4}-|\mu|^2-m_{H_{\text{u}}}^2\right)|h_{\text{d}}^0|^2\\&+\frac{\mathrm{i}g_Z}{2}Z_\mu\partial^\mu\left(v_{\text{d}}h_{\text{d}}^0-v_{\text{d}}h_{\text{d}}^{0*}-v_{\text{u}}h_{\text{u}}^0+v_{\text{u}}h_{\text{u}}^{0*}\right).\end{aligned}\tag{12.55}$$

[*25] If we separated leptons and $\tilde{h}_{\text{d}}$ first, sleptons would acquire VEVs and lepton-gaugino mixings would be induced.

[*26] Note that T_3-rotation induces $h_{\text{u}}^0\to\mathrm{e}^{\mathrm{i}\theta/2}h_{\text{u}}^0$ and $h_{\text{d}}^0\to\mathrm{e}^{-\mathrm{i}\theta/2}h_{\text{d}}^0$; it cannot remove the phase of b.

Therefore, with $m_W := c_W m_Z$ and

$$\begin{pmatrix} h_u^+ \\ h_d^{-*} \end{pmatrix} = \begin{pmatrix} s_\beta & c_\beta \\ -c_\beta & s_\beta \end{pmatrix}\begin{pmatrix} -iG^+ \\ H^+ \end{pmatrix}, \qquad \begin{pmatrix} h_u^0 \\ h_d^0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_u \\ \phi_d \end{pmatrix} + \frac{i}{\sqrt{2}}\begin{pmatrix} s_\beta & c_\beta \\ -c_\beta & s_\beta \end{pmatrix}\begin{pmatrix} G^0 \\ A^0 \end{pmatrix}, \tag{12.56}$$

we have

$$\begin{aligned}\mathcal{L} \supset{}& \partial_\mu G^{+*}\partial^\mu G^+ + \partial_\mu H^{+*}\partial^\mu H^+ + m_W(W_\mu^-\partial^\mu G^+ + W_\mu^+\partial^\mu G^{+*}) + \left(\frac{m_{H_d}^2 - m_{H_u}^2}{\cos 2\beta} + m_Z^2 s_W^2\right)|H^+|^2 \\ &+ \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + \frac{1}{2}(\partial_\mu A^0)^2 + \frac{1}{2}(\partial_\mu G^0)^2 + m_Z Z_\mu\partial^\mu G^0 - \frac{B}{2}A_0^2 \\ &- \frac{1}{4}\left(B + m_Z^2 + (B - m_Z^2)\cos 2\beta\right)\phi_u^2. - \frac{1}{4}\left(B + m_Z^2 - (B - m_Z^2)\cos 2\beta\right)\phi_d^2 + \frac{1}{2}(B + m_Z^2)(\sin 2\beta)\phi_u\phi_d.\end{aligned} \tag{12.57}$$

In particular, the tree-level masses are

$$m_{A_0}^2 = B = 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2, \tag{12.58}$$

$$m_{H^+}^2 = m_{A_0}^2 + m_W^2, \tag{12.59}$$

$$m_{h,H} = \frac{1}{2}\left(m_{A_0}^2 + m_Z^2 \mp \sqrt{\left(m_{A_0}^2 - m_Z^2\right)^2 + 4m_{A_0}^2 m_Z^2 \sin^2 2\beta}\right) \tag{12.60}$$

with

$$\begin{pmatrix} \phi_d \\ \phi_u \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} H \\ h \end{pmatrix}, \qquad \frac{\tan 2\alpha}{\tan 2\beta} = \frac{m_{A_0}^2 + m_Z^2}{m_{A_0}^2 - m_Z^2}. \tag{12.61}$$

The mixing α is stored in `ALPHA` block of SLHA, while `HMIX` stores

$$\mu = \mu, \quad \tan\beta = \tan\beta, \quad v = v_{SM}(\sim 246\,\mathrm{GeV}), \quad m_A^2 = \frac{2b}{\sin 2\beta}, \tag{12.62}$$

at the scale specified. The above discussion holds even with CP-violation, but quantum corrections mix the three Higgs bosons; such information should be stored in `(IM)VCHMIX`. ♠TODO:discuss when needed♠

Mass terms in the Lagrangian The other mass terms are given by

$$\begin{aligned}\mathcal{L} \supset{}& m_W^2 W^{+\mu}W_\mu^- + \frac{1}{2}m_Z^2 Z_\mu Z^\mu - \frac{1}{2}M_3\tilde{g}_0^a\tilde{g}_0^a - \frac{1}{2}M_3^*\bar{\tilde{g}}_0^a\bar{\tilde{g}}_0^a \\ &+ \left(-\frac{1}{2}M_1\tilde{b}\tilde{b} - \frac{1}{2}M_2\tilde{w}^3\tilde{w}^3 + \mu\tilde{h}_u^0\tilde{h}_d^0 + c_\beta m_Z s_W\tilde{h}_d^0\tilde{b} - c_W c_\beta m_Z\tilde{h}_d^0\tilde{w}^3 - m_Z s_W s_\beta\tilde{h}_u^0\tilde{b} + c_W m_Z s_\beta\tilde{h}_u^0\tilde{w}^3 + \text{h.c.}\right) \\ &- M_2\tilde{w}^+\tilde{w}^- - \mu\tilde{h}_u^+\tilde{h}_d^- - M_2^*\bar{\tilde{w}}^+\bar{\tilde{w}}^- - \mu^*\bar{\tilde{h}}_u^+\bar{\tilde{h}}_d^- - \sqrt{2}m_W\left(c_\beta\tilde{h}_d^-\tilde{w}^+ + s_\beta\tilde{h}_u^+\tilde{w}^- + c_\beta\bar{\tilde{h}}_d^-\bar{\tilde{w}}^+ + s_\beta\bar{\tilde{h}}_u^+\bar{\tilde{w}}^-\right) \\ &- v_u y_{uij}u_{Ri}^c u_{Lj} - v_d y_{dij}d_{Ri}^c d_{Lj} - v_d y_{eij}e_{Ri}^c e_{Lj} - v_u y_{uij}^*\bar{u}_{Ri}^c\bar{u}_{Lj} - v_d y_{dij}^*\bar{d}_{Ri}^c\bar{d}_{Lj} - v_d y_{eij}^*\bar{e}_{Ri}^c\bar{e}_{Lj} \\ &- \tilde{u}_L^*\left(m_Q^2 + v_u^2 y_u^\dagger y_u + \frac{3 - 4s_W^2}{6}c_{2\beta}m_Z^2\right)\tilde{u}_L - \tilde{u}_R^*\left(m_{U^c}^2 + v_u^2 y_u y_u^\dagger + \frac{4s_W^2}{6}c_{2\beta}m_Z^2\right)\tilde{u}_R \\ &- v_u a_{uij}\tilde{u}_{Ri}^*\tilde{u}_{Lj} + v_d\mu^* y_{uij}\tilde{u}_{Ri}^*\tilde{u}_{Lj} - v_u a_{uij}^*\tilde{u}_{Ri}\tilde{u}_{Lj}^* + v_d\mu y_{uij}^*\tilde{u}_{Ri}\tilde{u}_{Lj}^* \\ &- \tilde{d}_L^*\left(m_Q^2 + v_d^2 y_d^\dagger y_d + \frac{-3 + 2s_W^2}{6}c_{2\beta}m_Z^2\right)\tilde{d}_L - \tilde{d}_R^*\left(m_{D^c}^2 + v_d^2 y_d y_d^\dagger + \frac{-2s_W^2}{6}c_{2\beta}m_Z^2\right)\tilde{d}_R \\ &- v_d a_{dij}\tilde{d}_{Ri}^*\tilde{d}_{Lj} + v_u\mu^* y_{dij}\tilde{d}_{Ri}^*\tilde{d}_{Lj} - v_d a_{dij}^*\tilde{d}_{Ri}\tilde{d}_{Lj}^* + v_u\mu y_{dij}^*\tilde{d}_{Ri}\tilde{d}_{Lj}^* \\ &- \tilde{\nu}_L^*\left(m_L^2 + \frac{1}{2}c_{2\beta}m_Z^2\right)\tilde{\nu}_L \\ &- \tilde{e}_L^*\left(m_L^2 + v_d^2 y_e^\dagger y_e + \frac{-1 + 2s_W^2}{2}c_{2\beta}m_Z^2\right) - \tilde{e}_R^*\left(m_{E^c}^2 + v_d^2 y_e y_e^\dagger + (-s_W^2)c_{2\beta}m_Z^2\right)\tilde{e}_R \\ &- v_d a_{eij}\tilde{e}_{Ri}^*\tilde{e}_{Lj} + v_u\mu^* y_{eij}\tilde{e}_{Ri}^*\tilde{e}_{Lj} - v_d a_{eij}^*\tilde{e}_{Ri}\tilde{e}_{Lj}^* + v_u\mu y_{eij}^*\tilde{e}_{Ri}\tilde{e}_{Lj}^*,\end{aligned} \tag{12.63}$$

where, at the tree level, the gauge boson mass m_W and m_Z, the gluino mass M_3, and matter-fermion masses $v_u y_u$, $v_d y_d$, and $v_d y_e$ are given with the "correct" sign (as far as $M_3 > 0$, etc.).

Neutralinos and charginos The mass matrices for neutralinos and charginos are given by

$$\begin{aligned}-\mathcal{L} \supset{}& \frac{1}{2}\begin{pmatrix} \tilde{b} \\ \tilde{w}^3 \\ \tilde{h}_d^0 \\ \tilde{h}_u^0 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} M_1 & 0 & -c_\beta s_W m_Z & +s_\beta s_W m_Z \\ 0 & M_2 & +c_\beta c_W m_Z & -s_\beta c_W m_Z \\ -c_\beta s_W m_Z & +c_\beta c_W m_Z & 0 & -\mu \\ +s_\beta s_W m_Z & -s_\beta c_W m_Z & -\mu & 0 \end{pmatrix}\begin{pmatrix} \tilde{b} \\ \tilde{w}^3 \\ \tilde{h}_d^0 \\ \tilde{h}_u^0 \end{pmatrix} + \text{h.c.} \\ &+ \begin{pmatrix} \tilde{w}^- & \tilde{h}_d^- \end{pmatrix}\begin{pmatrix} M_2 & \sqrt{2}s_\beta m_W \\ \sqrt{2}c_\beta m_W & \mu \end{pmatrix}\begin{pmatrix} \tilde{w}^+ \\ \tilde{h}_u^+ \end{pmatrix} + \begin{pmatrix} \bar{\tilde{w}}^- & \bar{\tilde{h}}_d^- \end{pmatrix}\begin{pmatrix} M_2^* & \sqrt{2}s_\beta m_W \\ \sqrt{2}c_\beta m_W & \mu^* \end{pmatrix}\begin{pmatrix} \bar{\tilde{w}}^+ \\ \bar{\tilde{h}}_u^+ \end{pmatrix}.\end{aligned} \tag{12.64}$$

Note that the mass matrices themselves are the same as those in SLHA convention, $\mathcal{M}_{\tilde\psi^0}$ and $\mathcal{M}_{\tilde\psi^+}$, while the fields are in different convention. Therefore, we continue our discussion based only on the mass matrices so that the discussion is free from the choice of field convention.

As $\mathcal{M}_{\tilde\psi^0}$ is a complex symmetric matrix, there is a unitary matrix $\tilde N$ such that $M_{\tilde\psi^0} = \tilde N^*\mathcal{M}_{\tilde\psi^0}\tilde N^\dagger$, where $M_{\tilde\psi^0}$ is a *positive* diagonal matrix whose elements are (non-negative) singular values of $\mathcal{M}_{\tilde\psi^0}$ and in increasing order (Autonne-Takagi factorization). In SLHA2 convention with CP-violation, this matrix $\tilde N$ is stored as the `(IM)NMIX` blocks and the (positive) masses are stored in the `MASS` block. Meanwhile, if M_1, M_2 and μ are real, $\mathcal{M}_{\tilde\psi^0}$ is a real symmetric matrix and there is a real orthogonal matrix $\hat N$ such that $\hat M_{\tilde\psi^0} = \hat N^*\mathcal{M}_{\tilde\psi^0}\hat N^\dagger = \hat N\mathcal{M}_{\tilde\psi^0}\hat N^{\mathrm T}$, where $\hat M_{\tilde\psi^0}$ is a *real* diagonal matrix whose elements are the eigenvalues of $\mathcal{M}_{\tilde\psi^0}$ and in absolute-value-increasing order (spectral theorem). This matrix $\hat N$ is the `NMIX` block of SLHA convention and $\hat M_{ii}$ is stored in the `MASS` block, hence `MASS` block may have negative values for neutralinos.

The chargino mass matrix $\mathcal{M}_{\tilde\psi^+}$ is decomposed as $M_{\tilde\psi^+} = U^*\mathcal{M}_{\tilde\psi^+}V^\dagger$, where U and V are unitary matrices and the elements of the diagonal matrix $M_{\tilde\psi^+}$ are singular values of $\mathcal{M}_{\tilde\psi^+}$ (thus non-negative) and sorted in increasing order (singular value decomposition). These U and V are stored in `(IM)UMIX` and `(IM)VMIX`, and the singular values are stored in `MASS` block. Because the SVD theorem is closed in $\mathbb{R}$, if M_2 and μ are real, U and V can be real, and the `IM`-blocks are omitted.

In summary,

$$M_{\tilde\psi^0} = \tilde N^*\mathcal{M}_{\tilde\psi^0}\tilde N^\dagger, \qquad \tilde N = \texttt{(IM)NMIX}, \qquad (\texttt{MASS}) = [M_{\tilde\psi^0}]_{ii} \ge 0 \quad \text{(singular values)}; \tag{12.65}$$

$$\hat M_{\tilde\psi^0} = \hat N\mathcal{M}_{\tilde\psi^0}\hat N^{\mathrm T}, \qquad \hat N = \texttt{NMIX}, \qquad (\texttt{MASS}) = [\hat M_{\tilde\psi^0}]_{ii} \in \mathbb{R} \quad \text{(eigenvalues)}; \tag{12.66}$$

$$M_{\tilde\psi^+} = U^*\mathcal{M}_{\tilde\psi^+}V^\dagger, \qquad U = \texttt{(IM)UMIX}, \quad V = \texttt{(IM)VMIX}, \qquad (\texttt{MASS}) = [M_{\tilde\psi^+}]_{ii} \ge 0 \quad \text{(singular values)}. \tag{12.67}$$

Note that the singular values are equal to absolute values of the eigenvalues, which guarantees consistency of the two decomposition.

We then define matrix N by[*27]

$$N = \begin{cases}\tilde N \\ \mathrm{diag}(\varphi_i)\cdot\hat N\end{cases} = \mathrm{diag}(\varphi_i)\cdot\Big((\texttt{NMIX}) + \mathrm{i}(\texttt{IMNMIX})\Big); \qquad \varphi_i = \begin{cases}1 & \text{if } (\texttt{MASS})_i \ge 0,\\ \mathrm{i} & \text{if } (\texttt{MASS})_i < 0.\end{cases} \tag{12.68}$$

It gives the proper mass diagonalization in both of the NMIX convention:

$$N^*\mathcal{M}_{\tilde\psi^0}N^\dagger = \begin{cases}\tilde N^*\mathcal{M}_{\tilde\psi^0}\tilde N^\dagger = M_{\tilde\psi^0}, \\ \mathrm{diag}(\varphi_i^*)\hat N^*\mathcal{M}_{\tilde\psi^0}\hat N^\dagger\,\mathrm{diag}(\varphi_i^*) = \mathrm{diag}(\varphi_i^*)\hat M_{\tilde\psi^0}\,\mathrm{diag}(\varphi_i^*)\end{cases} = M_{\tilde\psi^0} \quad \text{(neutralino masses} \ge 0). \tag{12.69}$$

Noting that the discussion up here is irrelevant of the convention, we have the neutralino/chargino mass eigenstates,

$$\tilde\chi_i^0 = N_{ij}\begin{pmatrix}\tilde b\\ \tilde w^3\\ \tilde h_{\mathrm d}^0\\ \tilde h_{\mathrm u}^0\end{pmatrix}_j, \quad \tilde\chi_i^+ = V_{ij}\begin{pmatrix}\tilde w^+\\ \tilde h_{\mathrm u}^+\end{pmatrix}_j, \quad \tilde\chi_i^- = U_{ij}\begin{pmatrix}\tilde w^-\\ \tilde h_{\mathrm d}^-\end{pmatrix}_j, \tag{12.70}$$

in our convention and the mass terms are now

$$-\mathcal{L} \supset \frac12(\tilde\chi^0)^{\mathrm T}M_{\tilde\psi^0}\tilde\chi^0 + (\tilde\chi^-)^{\mathrm T}M_{\tilde\psi^+}\tilde\chi^+ + \text{h.c.} \tag{12.71}$$

Quarks, leptons, and super-CKM basis The Lagrangian in any basis (i.e., the ”original” basis) contains

$$-\mathcal{L} \supset u_{\mathrm R}^{\mathrm c}(v_{\mathrm u}y_u)u_{\mathrm L} + d_{\mathrm R}^{\mathrm c}(v_{\mathrm d}y_d)d_{\mathrm L} + e_{\mathrm R}^{\mathrm c}(v_{\mathrm d}y_e)e_{\mathrm L} + \text{h.c.} \tag{12.72}$$

$$= u_{\mathrm R}^{\mathrm c}(v_{\mathrm u}U_uy_u^{\mathrm{diag}}V_u^\dagger)u_{\mathrm L} + d_{\mathrm R}^{\mathrm c}(v_{\mathrm d}U_dy_d^{\mathrm{diag}}V_d^\dagger)d_{\mathrm L} + e_{\mathrm R}^{\mathrm c}(v_{\mathrm d}U_ey_e^{\mathrm{diag}}V_e^\dagger)e_{\mathrm L} + \text{h.c.}. \tag{12.73}$$

Thus, a conveneint basis (“super-CKM basis”) for the superfields is given by

$$\left[Q^1, Q^2, L, U^{\mathrm c}, D^{\mathrm c}, E^{\mathrm c}\right]_{\text{super-CKM}} = \left[V_u^\dagger Q^1, V_d^\dagger Q^2, V_e^\dagger L, U^{\mathrm c}U_u, D^{\mathrm c}U_d, E^{\mathrm c}U_e\right]_{\text{“original”}}. \tag{12.74}$$

Then the CKM mixings appear as, for example,

$$\left[\bar u_{\mathrm L}\bar\sigma^\mu d_{\mathrm L}W_\mu^+ + \bar d_{\mathrm L}\bar\sigma^\mu u_{\mathrm L}W_\mu^-\right]_{\text{”original”}} = \left[\bar u_{\mathrm L}V_u^\dagger V_d\bar\sigma^\mu d_{\mathrm L}W_\mu^+ + \bar d_{\mathrm L}V_d^\dagger V_u\bar\sigma^\mu u_{\mathrm L}W_\mu^-\right]_{\text{super-CKM}}; \tag{12.75}$$

where $V_{\mathrm{CKM}} = V_u^\dagger V_d$ as in Section 8.5. One may identify this operation as replacements $\bar u_{\mathrm L}d_{\mathrm L} \to \bar u_{\mathrm L}V_{\mathrm{CKM}}d_{\mathrm L}$, $\tilde d_{\mathrm L}^*\tilde u_{\mathrm L} \to \tilde d_{\mathrm L}^*V_{\mathrm{CKM}}^\dagger\tilde u_{\mathrm L}$, etc.

Squark masses in super-CKM basis Now we rewrite the squark mass terms, Eq. (12.63), with sfermions the super-CKM basis. Yukawa matrices are now replaced by diagonal quark masses, while the sfermion mass terms and a-terms also fit in the super-CKM basis:

$$[m_Q^2, m_{U^{\mathrm c}}^2, m_{D^{\mathrm c}}^2, m_L^2, m_{E^{\mathrm c}}^2]_{\text{super-CKM}} = [V_d^\dagger m_Q^2V_d, U_u^\dagger m_{U^{\mathrm c}}^2U_u, U_d^\dagger m_{D^{\mathrm c}}^2U_d, V_e^\dagger m_L^2V_e, U_e^\dagger m_{E^{\mathrm c}}^2U_e]_{\text{“original”}}, \tag{12.76}$$

$$[a_u, a_d, a_e]_{\text{super-CKM}} = [U_u^\dagger a_uV_u, U_d^\dagger a_dV_d, U_e^\dagger a_eV_e]_{\text{“original”}}. \tag{12.77}$$

[*27] The sign of φ_i is arbitrary and (should be) unphysical.

Using these, we obtain[*28]

$$
\begin{aligned}
-\mathcal{L} \supset{}& \tilde u_{\mathrm L}^* \left(V_u^\dagger (V_d m_Q^2 V_d^\dagger) V_u + m_u^2 + \frac{3-4s_{\mathrm W}^2}{6} c_{2\beta} m_Z^2 \right) \tilde u_{\mathrm L} + \tilde u_{\mathrm R}^* \left(m_{U^c}^2 + m_u^2 + \frac{4s_{\mathrm W}^2}{6} c_{2\beta} m_Z^2 \right) \tilde u_{\mathrm R} \\
&+ \tilde u_{\mathrm R}^* (v_{\mathrm u} a_{\mathrm u} - \mu^* m_u \cot\beta) \tilde u_{\mathrm L} + \tilde u_{\mathrm L}^* (v_{\mathrm u} a_{\mathrm u}^\dagger - \mu m_u \cot\beta) \tilde u_{\mathrm R} \\
&+ \tilde d_{\mathrm L}^* \left(m_Q^2 + m_d^2 + \frac{-3+2s_{\mathrm W}^2}{6} c_{2\beta} m_Z^2 \right) \tilde d_{\mathrm L} + \tilde d_{\mathrm R}^* \left(m_{D^c}^2 + m_d^2 + \frac{-2s_{\mathrm W}^2}{6} c_{2\beta} m_Z^2 \right) \tilde d_{\mathrm R} \\
&+ \tilde d_{\mathrm R}^* (v_{\mathrm d} a_d - \mu^* m_d \tan\beta) \tilde d_{\mathrm L} + \tilde d_{\mathrm L}^* (v_{\mathrm d} a_d^\dagger - \mu m_d \tan\beta) \tilde d_{\mathrm R} \\
&+ \tilde\nu_{\mathrm L}^* \left(m_L^2 + \frac{1}{2} c_{2\beta} m_Z^2 \right) \tilde\nu_{\mathrm L} \\
&+ \tilde e_{\mathrm L}^* \left(m_L^2 + m_e^2 + \frac{-1+2s_{\mathrm W}^2}{2} c_{2\beta} m_Z^2 \right) + \tilde e_{\mathrm R}^* \left(m_{E^c}^2 + m_e^2 + (-s_{\mathrm W}^2) c_{2\beta} m_Z^2 \right) \tilde e_{\mathrm R} \\
&+ \tilde e_{\mathrm R}^* (v_{\mathrm d} a_e - \mu^* m_e \tan\beta) \tilde e_{\mathrm L} + \tilde e_{\mathrm L}^* (v_{\mathrm d} a_e^\dagger - \mu m_e \tan\beta) \tilde e_{\mathrm R},
\end{aligned}
\tag{12.78}
$$

(note that m_Q^2 is diagonalied for down-type; not for up-type). In matrix form,

$$
\begin{aligned}
-\mathcal{L} \supset{}& \begin{pmatrix} \tilde u_{\mathrm L i}^* \\ \tilde u_{\mathrm R i}^* \end{pmatrix}^{\mathrm T} \begin{pmatrix} [V_{\mathrm{CKM}} m_Q^2 V_{\mathrm{CKM}}^\dagger]_{ij} + \left(m_u^2 + \frac{3-4s_{\mathrm W}^2}{6} c_{2\beta} m_Z^2 \right) \delta_{ij} & v_u [a_u^\dagger]_{ij} - (\mu m_u \cot\beta) \delta_{ij} \\ v_u [a_u]_{ij} - (\mu^* m_u \cot\beta) \delta_{ij} & [m_{U^c}^2]_{ij} + \left(m_u^2 + \frac{2s_{\mathrm W}^2}{3} c_{2\beta} m_Z^2 \right) \delta_{ij} \end{pmatrix} \begin{pmatrix} \tilde u_{\mathrm L j} \\ \tilde u_{\mathrm R j} \end{pmatrix} \\
&+ \begin{pmatrix} \tilde d_{\mathrm L i}^* \\ \tilde d_{\mathrm R i}^* \end{pmatrix}^{\mathrm T} \begin{pmatrix} [m_Q^2]_{ij} + \left(m_d^2 + \frac{-3+2s_{\mathrm W}^2}{6} c_{2\beta} m_Z^2 \right) \delta_{ij} & v_d [a_d^\dagger]_{ij} - (\mu m_d \tan\beta) \delta_{ij} \\ v_d [a_d]_{ij} - (\mu^* m_d \tan\beta) \delta_{ij} & [m_{D^c}^2]_{ij} + \left(m_d^2 - \frac{s_{\mathrm W}^2}{3} c_{2\beta} m_Z^2 \right) \delta_{ij} \end{pmatrix} \begin{pmatrix} \tilde d_{\mathrm L j} \\ \tilde d_{\mathrm R j} \end{pmatrix} \\
&+ \tilde\nu_{\mathrm L i}^* \left([m_L^2]_{ij} + \left(\tfrac{1}{2} c_{2\beta} m_Z^2 \right) \delta_{ij} \right) \tilde\nu_{\mathrm L j} \\
&+ \begin{pmatrix} \tilde e_{\mathrm L i}^* \\ \tilde e_{\mathrm R i}^* \end{pmatrix}^{\mathrm T} \begin{pmatrix} [m_L^2]_{ij} + \left(m_e^2 + \frac{-1+2s_{\mathrm W}^2}{2} c_{2\beta} m_Z^2 \right) \delta_{ij} & v_d [a_e^\dagger]_{ij} - (\mu m_e \tan\beta) \delta_{ij} \\ v_d [a_e]_{ij} - (\mu^* m_e \tan\beta) \delta_{ij} & [m_{E^c}^2]_{ij} + (m_e^2 - s_{\mathrm W}^2 c_{2\beta} m_Z^2) \delta_{ij} \end{pmatrix} \begin{pmatrix} \tilde e_{\mathrm L j} \\ \tilde e_{\mathrm R j} \end{pmatrix}
\end{aligned}
\tag{12.79}
$$

$$
= \begin{pmatrix} \tilde u_{\mathrm L i}^* \\ \tilde u_{\mathrm R i}^* \end{pmatrix}^{\mathrm T} \mathcal{M}_u \begin{pmatrix} \tilde u_{\mathrm L j} \\ \tilde u_{\mathrm R j} \end{pmatrix} + \begin{pmatrix} \tilde d_{\mathrm L i}^* & \tilde d_{\mathrm R i}^* \end{pmatrix}^{\mathrm T} \mathcal{M}_d \begin{pmatrix} \tilde d_{\mathrm L j} \\ \tilde d_{\mathrm R j} \end{pmatrix} + \tilde\nu_{\mathrm L}^* \mathcal{M}_\nu \tilde\nu_{\mathrm L} + (\tilde e_{\mathrm L i}^* \tilde e_{\mathrm R i}^*)^{\mathrm T} \mathcal{M}_e \begin{pmatrix} \tilde e_{\mathrm L j} \\ \tilde e_{\mathrm R j} \end{pmatrix} \tag{12.80}
$$

The sfermion mass matrices are diagonalized by unitary matrices as[*29]

$$
\mathcal{M}^{\mathrm{diag}} = R \mathcal{M} R^\dagger; \quad \tilde f_i = R_{ij} \begin{pmatrix} \tilde f_{\mathrm L} \\ \tilde f_{\mathrm R} \end{pmatrix}_j = \begin{pmatrix} R_{ij}^{\mathrm L} & R_{ij}^{\mathrm R} \end{pmatrix} \begin{pmatrix} \tilde f_{\mathrm L j} \\ \tilde f_{\mathrm R j} \end{pmatrix}; \qquad \tilde f_{\mathrm L i} = [R^{\mathrm L \dagger}]_{ij} \tilde f_j, \quad \tilde f_{\mathrm R i} = [R^{\mathrm R \dagger}]_{ij} \tilde f_j. \tag{12.81}
$$

where R_{ij} is 6×6 and $R_{ij}^{\mathrm{L,R}}$ are 3×6 matrices (except for sneutrinos).

These R-matrices are the same as `DSQMIX` etc. of SLHA2 format, but note that our notation for the other parameters is slightly different from SLHA's:

$$
m_{\tilde Q, \tilde L}^2 = m_{Q,L}^2|_{\text{“orig”}}, \qquad m_{\tilde u, \tilde d, \tilde e}^2 = (m_{U^c,D^c,E^c}^2|_{\text{“orig”}})^{\mathrm T}, \qquad T_{U,D,E} = (a_{u,d,e}|_{\text{“orig”}})^{\mathrm T}, \tag{12.82}
$$

$$
\hat m_{\tilde Q, \tilde L}^2 = m_{Q,L}^2|_{\mathrm{sCKM}}, \qquad \hat m_{\tilde u, \tilde d, \tilde e}^2 = U^\dagger T^{\mathrm T} V = m_{U^c,D^c,E^c}^2|_{\mathrm{sCKM}}, \qquad \hat T_{U,D,E} = U^\dagger T^{\mathrm T} V = a_{u,d,e}|_{\mathrm{sCKM}}, \tag{12.83}
$$

together with $Y_{u,d,e} = (y_{u,d,e})^{\mathrm T}$. Anyway, the SLHA2 blocks corresponds to the variable in our convention as

$$
\texttt{(IM)MSX2(IN)} = m_{Q,L,U^c,D^c,E^c}^2|_{\mathrm{sCKM};\,\overline{\mathrm{DR}}}, \quad \texttt{(IM)TX(IN)} = a_{u,d,e}|_{\mathrm{sCKM};\,\overline{\mathrm{DR}}}, \quad \texttt{YX} = y_{u,d,e}|_{\mathrm{sCKM};\,\overline{\mathrm{DR}}}, \tag{12.84}
$$

$$
\texttt{DSQMIX} = R_d, \quad \texttt{USQMIX} = R_u, \quad \texttt{SELMIX} = R_e. \tag{12.85}
$$

The sfermion mass matrices above are in super-CKM basis, so their off-diagonal entries immediately induce flavor violation or sfermion left-right mixing. In old SLHA format, we assume that flavor- and CP-violation is absent and left-right mixing is ignorable except for third generation, which leads

$$
\mathcal{M}_d = \begin{pmatrix} \tilde m_{dL11}^2 &&&&& \\ & \tilde m_{dL22}^2 &&&& \\ && \tilde m_{dL33}^2 &&& v_d a_{d33} - \mu m_b \tan\beta \\ &&& \tilde m_{dR11}^2 && \\ &&&& \tilde m_{dR22}^2 & \\ && v_d a_{d33} - \mu m_b \tan\beta &&& \tilde m_{dR33}^2 \end{pmatrix}; \quad \begin{pmatrix} \tilde d_{\mathrm L} \\ \tilde s_{\mathrm L} \\ \tilde b_1 \\ \tilde d_{\mathrm R} \\ \tilde s_{\mathrm R} \\ \tilde b_2 \end{pmatrix} = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && F_{11} &&& F_{12} \\ &&& 1 && \\ &&&& 1 & \\ && F_{21} &&& F_{22} \end{pmatrix} \begin{pmatrix} \tilde d_{\mathrm L j} \\ \tilde d_{\mathrm R j} \end{pmatrix} \tag{12.86}
$$

and these F_{ij} are stored in `SBOTMIX` etc.

Squark masses in super-CKM basis The gluino mass and phase is given by

$$
m_{\tilde g} = |M_3|; \quad M_3 = m_{\tilde g} e^{i\theta_3} = m_{\tilde g} \varphi_{\tilde g}^{-2}; \quad \tilde g_0 = \varphi_{\tilde g} \tilde g. \tag{12.87}
$$

[*28] 2021-Sep quick fix is given in this color.
[*29] 2021-Sep quick fix is given in this color.

12.4.2 Fermion composition

In super-CKM basis, the fermion-related interaction terms are

$$
\begin{aligned}
\mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{SFG}} &= \mathrm{i}\bar{\tilde{\chi}}_i^0\bar{\sigma}^\mu\partial_\mu\tilde{\chi}_i^0 + \mathrm{i}\bar{\tilde{\chi}}_i^-\bar{\sigma}^\mu\partial_\mu\tilde{\chi}_i^- + \mathrm{i}\bar{\tilde{\chi}}_i^+\bar{\sigma}^\mu\partial_\mu\tilde{\chi}_i^+ + \mathrm{i}\bar{\tilde{\mathrm{g}}}^a\bar{\sigma}^\mu\partial_\mu\tilde{\mathrm{g}}^a + \mathrm{i}\bar{u}_{\mathrm{L}i}\bar{\sigma}^\mu\partial_\mu u_{\mathrm{L}i} + \mathrm{i}\bar{u}^{\mathrm{c}}_{\mathrm{R}i}\bar{\sigma}^\mu\partial_\mu u^{\mathrm{c}}_{\mathrm{R}i} \\
&+ \mathrm{i}\bar{d}_{\mathrm{L}i}\bar{\sigma}^\mu\partial_\mu d_{\mathrm{L}i} + \mathrm{i}\bar{d}^{\mathrm{c}}_{\mathrm{R}i}\bar{\sigma}^\mu\partial_\mu d^{\mathrm{c}}_{\mathrm{R}i} + \mathrm{i}\bar{\nu}_{\mathrm{L}i}\bar{\sigma}^\mu\partial_\mu \nu_{\mathrm{L}i} + \mathrm{i}\bar{e}_{\mathrm{L}i}\bar{\sigma}^\mu\partial_\mu e_{\mathrm{L}i} + \mathrm{i}\bar{e}^{\mathrm{c}}_{\mathrm{R}i}\bar{\sigma}^\mu\partial_\mu e^{\mathrm{c}}_{\mathrm{R}i} - \frac{m_{\tilde{\mathrm{g}}}}{2}(\tilde{\mathrm{g}}^a\tilde{\mathrm{g}}^a + \text{h.c.}) \\
&- m_{\tilde{\chi}_i^\pm}(\tilde{\chi}_i^-\tilde{\chi}_i^+ + \bar{\tilde{\chi}}_i^-\bar{\tilde{\chi}}_i^+) - m_{di}\left(d^{\mathrm{c}}_{\mathrm{R}i}d_{\mathrm{L}i} + \bar{d}^{\mathrm{c}}_{\mathrm{R}i}\bar{d}^x_{\mathrm{L}i}\right) - m_{ui}\left(u^{\mathrm{c}}_{\mathrm{R}i}u_{\mathrm{L}j} + \bar{u}^{\mathrm{c}}_{\mathrm{R}i}\bar{u}_{\mathrm{L}i}\right) - m_{ei}\left(e^{\mathrm{c}}_{\mathrm{R}i}e_{\mathrm{L}i} + \bar{e}^{\mathrm{c}}_{\mathrm{R}i}\bar{e}_{\mathrm{L}i}\right) \\
&+ \left[\sqrt{2}g_3\varphi_{\tilde{\mathrm{g}}}\tilde{\mathrm{g}}^a\left(\tau^{yxa*}\tilde{d}^y_{\mathrm{R}}d^{\mathrm{c}x}_{\mathrm{R}} + \tau^{yxa*}\tilde{u}^y_{\mathrm{R}}u^{\mathrm{c}x}_{\mathrm{R}} - \tau^{yxa}\tilde{d}^{y*}_{\mathrm{L}}d^x_{\mathrm{L}} - \tau^{yxa}\tilde{u}^{y*}_{\mathrm{L}}u^x_{\mathrm{L}}\right) + \text{h.c.}\right] - \mathrm{i}g_3 f^{abc}g^a_\mu(\bar{\tilde{\mathrm{g}}}^b\bar{\sigma}^\mu\tilde{\mathrm{g}}^c) \\
&+ g_3 g^a_\mu\left(\bar{d}_{\mathrm{L}}\bar{\sigma}^\mu\tau^a d_{\mathrm{L}} + \bar{u}_{\mathrm{L}}\bar{\sigma}^\mu\tau^a u_{\mathrm{L}} - \bar{d}^{\mathrm{c}}_{\mathrm{R}}\bar{\sigma}^\mu\tau^{a*}d^{\mathrm{c}}_{\mathrm{R}} - \bar{u}^{\mathrm{c}}_{\mathrm{R}}\bar{\sigma}^\mu\tau^{a*}u^{\mathrm{c}}_{\mathrm{R}}\right) + A_\mu|e|\Big[\frac{2}{3}\bar{u}_{\mathrm{L}}\bar{\sigma}^\mu u_{\mathrm{L}} - \frac{2}{3}\bar{u}^{\mathrm{c}}_{\mathrm{R}}\bar{\sigma}^\mu u^{\mathrm{c}}_{\mathrm{R}} - \frac{1}{3}\bar{d}_{\mathrm{L}}\bar{\sigma}^\mu d_{\mathrm{L}} \\
&+ \frac{1}{3}\bar{d}^{\mathrm{c}}_{\mathrm{R}}\bar{\sigma}^\mu d^{\mathrm{c}}_{\mathrm{R}} - \bar{e}_{\mathrm{L}}\bar{\sigma}^\mu e_{\mathrm{L}} + \bar{e}^{\mathrm{c}}_{\mathrm{R}}\bar{\sigma}^\mu e^{\mathrm{c}}_{\mathrm{R}} + \bar{\tilde{\chi}}_i^+\bar{\sigma}^\mu\tilde{\chi}_i^+ - \bar{\tilde{\chi}}_i^-\bar{\sigma}^\mu\tilde{\chi}_i^-\Big] + g_Z Z_\mu\Big[\frac{3-4\mathrm{s}_{\mathrm{W}}^2}{6}\bar{u}_{\mathrm{L}}\bar{\sigma}^\mu u_{\mathrm{L}} + \frac{2\mathrm{s}_{\mathrm{W}}^2}{3}\bar{u}^{\mathrm{c}}_{\mathrm{R}}\bar{\sigma}^\mu u^{\mathrm{c}}_{\mathrm{R}} \\
&- \frac{3-2\mathrm{s}_{\mathrm{W}}^2}{6}\bar{d}_{\mathrm{L}}\bar{\sigma}^\mu d_{\mathrm{L}} - \frac{\mathrm{s}_{\mathrm{W}}^2}{3}\bar{d}^{\mathrm{c}}_{\mathrm{R}}\bar{\sigma}^\mu d^{\mathrm{c}}_{\mathrm{R}} + \frac{1}{2}\bar{\nu}_{\mathrm{L}}\bar{\sigma}^\mu\nu_{\mathrm{L}} + \frac{2\mathrm{s}_{\mathrm{W}}^2-1}{2}\bar{e}_{\mathrm{L}}\bar{\sigma}^\mu e_{\mathrm{L}} - \mathrm{s}_{\mathrm{W}}^2\bar{e}^{\mathrm{c}}_{\mathrm{R}}\bar{\sigma}^\mu e^{\mathrm{c}}_{\mathrm{R}} + \frac{N_{j3}N_{i3}^* - N_{j4}N_{i4}^*}{2}\bar{\tilde{\chi}}_j^0\bar{\sigma}^\mu\tilde{\chi}_i^0 \\
&+ \left[\mathrm{c}_{\mathrm{W}}^2 V_{j1}V_{i1}^* + V_{j2}V_{i2}^*(\mathrm{c}_{\mathrm{W}}^2 - \mathrm{s}_{\mathrm{W}}^2)/2\right]\bar{\tilde{\chi}}_j^+\bar{\sigma}^\mu\tilde{\chi}_i^+ + \left[U_{j2}U_{i2}^*(\mathrm{s}_{\mathrm{W}}^2 - \mathrm{c}_{\mathrm{W}}^2)/2 - \mathrm{c}_{\mathrm{W}}^2 U_{j1}U_{i1}^*\right]\bar{\tilde{\chi}}_j^-\bar{\sigma}^\mu\tilde{\chi}_i^-\Big] \\
&+ \frac{g_2}{\sqrt{2}}W_\mu^+\left[\bar{u}_{\mathrm{L}}\bar{\sigma}^\mu V_{\mathrm{CKM}}d_{\mathrm{L}} + \bar{\nu}_{\mathrm{L}}\bar{\sigma}^\mu e_{\mathrm{L}} + \left(V_{j2}N_{i4}^* - \sqrt{2}V_{j1}N_{i2}^*\right)\bar{\tilde{\chi}}_j^+\bar{\sigma}^\mu\tilde{\chi}_i^0 + \left(\sqrt{2}N_{j2}U_{i1}^* + N_{j3}U_{i2}^*\right)\bar{\tilde{\chi}}_j^0\bar{\sigma}^\mu\tilde{\chi}_i^-\right] \\
&+ \frac{g_2}{\sqrt{2}}W_\mu^-\left[\bar{d}_{\mathrm{L}}\bar{\sigma}^\mu V^\dagger_{\mathrm{CKM}}u_{\mathrm{L}} + \bar{e}_{\mathrm{L}}\bar{\sigma}^\mu \nu_{\mathrm{L}} + \left(N_{j4}V_{i2}^* - \sqrt{2}N_{j2}V_{i1}^*\right)\bar{\tilde{\chi}}_j^0\bar{\sigma}^\mu\tilde{\chi}_i^+ + \left(\sqrt{2}U_{j1}N_{i2}^* + U_{j2}N_{i3}^*\right)\bar{\tilde{\chi}}_j^-\bar{\sigma}^\mu\tilde{\chi}_i^0\right] \\
&+ \phi_{\mathrm{u}}\left(-\frac{y_{ui}}{\sqrt{2}}u^{\mathrm{c}}_{\mathrm{R}i}u_{\mathrm{L}i} + \frac{N_{i4}^*(g_2N_{j2}^* - g_Y N_{j1}^*)}{2}\tilde{\chi}_i^0\tilde{\chi}_j^0 - \frac{g_2U_{i1}^*V_{j2}^*}{\sqrt{2}}\tilde{\chi}_i^-\tilde{\chi}_j^+ + \text{h.c.}\right) \\
&+ \phi_{\mathrm{d}}\left(-\frac{y_{di}}{\sqrt{2}}d^{\mathrm{c}}_{\mathrm{R}i}d_{\mathrm{L}i} - \frac{y_{ei}}{\sqrt{2}}e^{\mathrm{c}}_{\mathrm{R}i}e_{\mathrm{L}i} + \frac{N_{i3}^*(g_Y N_{j1}^* - g_2N_{j2}^*)}{2}\tilde{\chi}_i^0\tilde{\chi}_j^0 - \frac{g_2U_{i2}^*V_{j1}^*}{\sqrt{2}}\tilde{\chi}_i^-\tilde{\chi}_j^+ + \text{h.c.}\right) \\
&+ \frac{A^0}{\sqrt{2}}\Big[-\mathrm{i}\mathrm{c}_{\beta0}y_{ui}u^{\mathrm{c}}_{\mathrm{R}i}u_{\mathrm{L}i} - \mathrm{i}\mathrm{s}_{\beta0}y_{di}d^{\mathrm{c}}_{\mathrm{R}i}d_{\mathrm{L}i} - \mathrm{i}\mathrm{s}_{\beta0}y_{ei}e^{\mathrm{c}}_{\mathrm{R}i}e_{\mathrm{L}i} - \frac{\mathrm{i}(\mathrm{s}_{\beta0}N_{i3}^* - \mathrm{c}_{\beta0}N_{i4}^*)(g_Y N_{j1}^* - g_2N_{j2}^*)}{\sqrt{2}}\tilde{\chi}_i^0\tilde{\chi}_j^0 \\
&+ \mathrm{i}g_2\left(\mathrm{c}_{\beta0}U_{i1}^*V_{j2}^* + \mathrm{s}_{\beta0}U_{i2}^*V_{j1}^*\right)\tilde{\chi}_i^-\tilde{\chi}_j^+ + \text{h.c.}\Big] + \frac{G^0}{\sqrt{2}}\Big[-\mathrm{i}\mathrm{s}_{\beta0}y_{ui}u^{\mathrm{c}}_{\mathrm{R}i}u_{\mathrm{L}i} + \mathrm{i}\mathrm{c}_{\beta0}y_{di}d^{\mathrm{c}}_{\mathrm{R}i}d_{\mathrm{L}i} + \mathrm{i}\mathrm{c}_{\beta0}y_{ei}e^{\mathrm{c}}_{\mathrm{R}i}e_{\mathrm{L}i} \\
&+ \frac{\mathrm{i}(g_Y N_{j1}^* - g_2N_{j2}^*)(\mathrm{c}_{\beta0}N_{i3}^* + \mathrm{s}_{\beta0}N_{i4}^*)}{\sqrt{2}}\tilde{\chi}_i^0\tilde{\chi}_j^0 - \mathrm{i}g_2(\mathrm{c}_{\beta0}U_{i2}^*V_{j1}^* - \mathrm{s}_{\beta0}U_{i1}^*V_{j2}^*)\tilde{\chi}_i^-\tilde{\chi}_j^+ + \text{h.c.}\Big] \\
&+ \Big[\left(\frac{U_{i2}^*(g_Y N_{j1}^* + g_2N_{j2}^*)}{\sqrt{2}} - g_2U_{i1}^*N_{j3}^*\right)\mathrm{s}_{\beta+}H^+\tilde{\chi}_i^-\tilde{\chi}_j^0 - \left(g_2V_{i1}^*N_{j4}^* + \frac{V_{i2}^*\left(g_Y N_{j1}^* + g_2N_{j2}^*\right)}{\sqrt{2}}\right)\mathrm{c}_{\beta+}H^-\tilde{\chi}_i^+\tilde{\chi}_j^0 \\
&+ \mathrm{c}_{\beta+}(u^{\mathrm{c}}_{\mathrm{R}}y_u V_{\mathrm{CKM}}d_{\mathrm{L}})H^+ + \mathrm{s}_{\beta+}(d^{\mathrm{c}}_{\mathrm{R}}y_d V^\dagger_{\mathrm{CKM}}u_{\mathrm{L}})H^- + \mathrm{s}_{\beta+}(e^{\mathrm{c}}_{\mathrm{R}}y_e\nu_{\mathrm{L}})H^- + \text{h.c.}\Big] \\
&+ \left[\mathrm{i}G^+\mathrm{c}_{\beta+}\left(\frac{U_{i2}^*(g_Y N_{j1}^* + g_2N_{j2}^*)}{\sqrt{2}} - g_2U_{i1}^*N_{j3}^*\right)\tilde{\chi}_i^-\tilde{\chi}_j^0 - \mathrm{i}G^-\mathrm{s}_{\beta+}\left(g_2V_{i1}^*N_{j4}^* + \frac{V_{i2}^*(g_Y N_{j1}^* + g_2N_{j2}^*)}{\sqrt{2}}\right)\tilde{\chi}_i^+\tilde{\chi}_j^0\right. \\
&\left.- \mathrm{i}\mathrm{s}_{\beta+}(u^{\mathrm{c}}_{\mathrm{R}}y_u V_{\mathrm{CKM}}d_{\mathrm{L}})G^+ - \mathrm{i}\mathrm{c}_{\beta+}(d^{\mathrm{c}}_{\mathrm{R}}y_d V^\dagger_{\mathrm{CKM}}u_{\mathrm{L}})G^- - \mathrm{i}\mathrm{c}_{\beta+}(e^{\mathrm{c}}_{\mathrm{R}}y_e\nu_{\mathrm{L}})G^- + \text{h.c.}\right] \\
&+ \Big[-\frac{\tilde{u}^*_{\mathrm{L}}u_{\mathrm{L}}\tilde{\chi}_j^0\left(g_Y N_{j1}^* + 3g_2N_{j2}^*\right)}{3\sqrt{2}} + \frac{2}{3}\sqrt{2}g_Y\tilde{u}_{\mathrm{R}}u^{\mathrm{c}}_{\mathrm{R}}\tilde{\chi}_j^0N_{j1}^* - y_{ui}u^{\mathrm{c}}_{\mathrm{R}i}\tilde{u}_{\mathrm{L}k}\tilde{\chi}_j^0N_{j4}^* - y_{ui}\tilde{u}^*_{\mathrm{R}i}u_{\mathrm{L}i}\tilde{\chi}_j^0N_{j4}^* \\
&- \frac{\tilde{d}^*_{\mathrm{L}i}d_{\mathrm{L}i}\tilde{\chi}_j^0\left(g_Y N_{j1}^* - 3g_2N_{j2}^*\right)}{3\sqrt{2}} - \frac{1}{3}\sqrt{2}g_Y\tilde{d}_{\mathrm{R}i}d^{\mathrm{c}}_{\mathrm{R}i}\tilde{\chi}_j^0N_{j1}^* - y_{di}\tilde{d}_{\mathrm{L}i}d^{\mathrm{c}}_{\mathrm{R}i}\tilde{\chi}_j^0N_{j3}^* - y_{di}\tilde{d}^*_{\mathrm{R}i}d_{\mathrm{L}i}\tilde{\chi}_j^0N_{j3}^* \\
&+ \frac{\tilde{\nu}^*_{\mathrm{L}i}\nu_{\mathrm{L}i}\tilde{\chi}_j^0\left(g_Y N_{j1}^* - g_2N_{j2}^*\right)}{\sqrt{2}} - \sqrt{2}g_Y\tilde{e}_{\mathrm{R}i}e^{\mathrm{c}}_{\mathrm{R}i}\tilde{\chi}_j^0N_{j1}^* + \frac{\tilde{e}^*_{\mathrm{L}i}e_{\mathrm{L}i}\tilde{\chi}_j^0\left(g_Y N_{j1}^* + g_2N_{j2}^*\right)}{\sqrt{2}} \\
&- y_{ei}\tilde{e}_{\mathrm{L}i}e^{\mathrm{c}}_{\mathrm{R}i}\tilde{\chi}_j^0N_{j3}^* - y_{ei}\tilde{e}^*_{\mathrm{R}i}e_{\mathrm{L}i}\tilde{\chi}_j^0N_{j3}^* + \text{h.c.}\Big] + \Big[V_{i2}^*(u^{\mathrm{c}}_{\mathrm{R}}y_u V_{\mathrm{CKM}}\tilde{d}_{\mathrm{L}})\tilde{\chi}_i^+ + V_{i2}^*(\tilde{u}^*_{\mathrm{R}}y_u V_{\mathrm{CKM}}d_{\mathrm{L}})\tilde{\chi}_i^+ \\
&+ U_{j2}^*(d^{\mathrm{c}}_{\mathrm{R}}y_d V^\dagger_{\mathrm{CKM}}\tilde{u}_{\mathrm{L}})\tilde{\chi}_j^- + U_{j2}^*(\tilde{d}^*_{\mathrm{R}}y_d V^\dagger_{\mathrm{CKM}}u_{\mathrm{L}})\tilde{\chi}_j^- - U_{j1}^*g_2(\tilde{d}^*_{\mathrm{L}}V^\dagger_{\mathrm{CKM}}u_{\mathrm{L}})\tilde{\chi}_j^- - V_{i1}^*g_2(\tilde{u}^*_{\mathrm{L}}V_{\mathrm{CKM}}d_{\mathrm{L}})\tilde{\chi}_i^+ \\
&- V_{i1}^*g_2\tilde{\nu}^*_{\mathrm{L}}e_{\mathrm{L}}\tilde{\chi}_i^+ - U_{j1}^*g_2\tilde{e}^*_{\mathrm{L}}\nu_{\mathrm{L}}\tilde{\chi}_j^- + U_{j2}^*y_{ei}\tilde{\nu}_{\mathrm{L}i}e^{\mathrm{c}}_{\mathrm{R}i}\tilde{\chi}_j^- + U_{j2}^*y_{ei}\tilde{e}^*_{\mathrm{R}i}\nu_{\mathrm{L}i}\tilde{\chi}_j^- + \text{h.c.}\Big]. \\
&- \epsilon^{xyz}\left(\lambda''_{ijk}\tilde{d}^{x*}_{\mathrm{R}k}u^{\mathrm{c}y}_{\mathrm{R}i}d^{\mathrm{c}z}_{\mathrm{R}j} + \frac{1}{2}\lambda''_{kij}\tilde{u}^{x*}_{\mathrm{R}k}d^{\mathrm{c}y}_{\mathrm{R}i}d^{\mathrm{c}z}_{\mathrm{R}j}\right) + \lambda_{ijk}\left(-\tilde{e}^*_{\mathrm{R}k}\nu_{\mathrm{L}i}e_{\mathrm{L}j} - \tilde{e}_{\mathrm{L}j}e^{\mathrm{c}}_{\mathrm{R}k}\nu_{\mathrm{L}i} + \tilde{\nu}_{\mathrm{L}j}e^{\mathrm{c}}_{\mathrm{R}k}e_{\mathrm{L}i}\right) \\
&+ \lambda'_{ijk}\left(-\tilde{d}^*_{\mathrm{R}k}d_{\mathrm{L}j}\nu_{\mathrm{L}i} + \tilde{d}^*_{\mathrm{R}k}u_{\mathrm{L}j}e_{\mathrm{L}i} - \tilde{d}_{\mathrm{L}j}d^{\mathrm{c}}_{\mathrm{R}k}\nu_{\mathrm{L}i} + \tilde{u}_{\mathrm{L}j}d^{\mathrm{c}}_{\mathrm{R}k}e_{\mathrm{L}i} + \tilde{e}_{\mathrm{L}i}d^{\mathrm{c}}_{\mathrm{R}k}u_{\mathrm{L}j} - \tilde{\nu}_{\mathrm{L}i}d^{\mathrm{c}}_{\mathrm{R}k}d_{\mathrm{L}j}\right) + \text{h.c.},
\end{aligned}
\tag{12.88}
$$

where R-parity violating terms are also shown as a reference; they are also redefined in super-CKM basis.

The full Lagrangian is given then by

$$\mathcal{L} = (\mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{SFG}}) + \mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{scalar}}, \tag{12.89}$$

where $\mathcal{L}_{\text{vector}}$ is given by Eq. (12.38), which should be amended with $m_W^2 W^{+\mu} W^-_\mu + (m_Z^2/2) Z^\mu Z_\mu$, mass and kinetic terms of $\mathcal{L}_{\text{scalar}}$ should be read from this section, and the interaction terms of $\mathcal{L}_{\text{scalar}}$ should be read from Eq. (12.43) and Eq. (12.44), replacing Higgs fields by VEVs, Higgs bosons, and Nambu-Goldstone bosons.

These Weyl fermions are combined to Dirac and Majorana fermions:

$$\tilde\chi_i^0 = \begin{pmatrix}\tilde\chi_i^0\\ \bar{\tilde\chi}_i^0\end{pmatrix}, \quad \tilde\chi_i^+ = \begin{pmatrix}\tilde\chi_i^+\\ \bar{\tilde\chi}_i^-\end{pmatrix}, \quad f = \begin{pmatrix}f_{\rm L}\\ \bar f_{\rm R}^{\rm c}\end{pmatrix}; \quad \overline{\tilde\chi_i^0} = \begin{pmatrix}\tilde\chi_i^0 & \bar{\tilde\chi}_i^0\end{pmatrix}, \quad \overline{\tilde\chi_i^+} = \begin{pmatrix}\tilde\chi_i^- & \bar{\tilde\chi}_i^+\end{pmatrix}, \quad \overline{f} = \begin{pmatrix}f_{\rm R}^{\rm c} & \bar f_{\rm L}\end{pmatrix}. \tag{12.90}$$

For example,

$$\overline{u}P_{\rm L}d = u_{\rm R}^{\rm c}d_{\rm L}, \quad \overline{u}P_{\rm R}d = \bar u_{\rm L}\bar d_{\rm R}^{\rm c}, \quad u_{\rm R}^{\rm c}d_{\rm L} + \text{h.c.} = \overline{u}P_{\rm L}d + \overline{d}P_{\rm R}u, \quad \overline{u}\gamma^\mu P_{\rm L}d = \bar u_{\rm L}\bar\sigma^\mu d_{\rm L}, \quad \overline{u}\gamma^\mu P_{\rm R}d = -\bar d_{\rm R}^{\rm c}\bar\sigma^\mu u_{\rm R}^{\rm c};$$
$$\tilde\chi^0 u_{\rm L} + \text{h.c.} = \overline{\tilde\chi^0}P_{\rm L}u + \overline{u}P_{\rm R}\tilde\chi^0, \quad \tilde\chi^0 u_{\rm R}^{\rm c} + \text{h.c.} = \overline{u}P_{\rm L}\tilde\chi^0 + \overline{\tilde\chi^0}P_{\rm R}u,$$
$$\overline{u}\gamma^\mu P_{\{\rm L,R\}}\tilde\chi^0 = \{\bar u_{\rm L}\bar\sigma^\mu\tilde\chi^0, -\bar{\tilde\chi}^0\bar\sigma^\mu u_{\rm R}^{\rm c}\}, \quad \overline{\tilde\chi^0}\gamma^\mu P_{\{\rm L,R\}}u = \{\bar{\tilde\chi}^0\bar\sigma u_{\rm L}, -\bar u_{\rm R}^{\rm c}\bar\sigma\tilde\chi^0\},$$
$$\bar{\tilde\chi}_i^0\bar\sigma\tilde\chi_j^0 = \overline{\tilde\chi_i^0}\gamma^\mu P_{\rm L}\tilde\chi_j^0 = -\overline{\tilde\chi_j^0}\gamma^\mu P_{\rm R}\tilde\chi_i^0.$$

With abbreviations

$$\begin{aligned}
(\mathcal{N}_{\phi_{\rm u}})_{ij} &:= N_{i4}^*(g_2N_{j2}^* - g_YN_{j1}^*)P_{\rm L} + N_{i4}(g_2N_{j2} - g_YN_{j1})P_{\rm R},\\
(\mathcal{N}_{\phi_{\rm d}})_{ij} &:= N_{i3}^*(g_YN_{j1}^* - g_2N_{j2}^*)P_{\rm L} + N_{i3}(g_YN_{j1} - g_2N_{j2})P_{\rm R},\\
(\mathcal{N}_{A^0})_{ij} &:= -(s_{\beta0}N_{i3}^* - c_{\beta0}N_{i4}^*)(g_YN_{j1}^* - g_2N_{j2}^*)P_{\rm L} + (s_{\beta0}N_{i3} - c_{\beta0}N_{i4})(g_YN_{j1} - g_2N_{j2})P_{\rm R},\\
(\mathcal{N}_{G^0})_{ij} &:= (g_YN_{j1}^* - g_2N_{j2}^*)(c_{\beta0}N_{i3}^* + s_{\beta0}N_{i4}^*)P_{\rm L} - (g_YN_{j1} - g_2N_{j2})(c_{\beta0}N_{i3} + s_{\beta0}N_{i4})P_{\rm R},\\
(\mathcal{C}_{\phi_{\rm u}})_{ij} &:= -U_{i1}^*V_{j2}^*P_{\rm L} - U_{i1}V_{j2}P_{\rm R},\\
(\mathcal{C}_{\phi_{\rm d}})_{ij} &:= -U_{i2}^*V_{j1}^*P_{\rm L} - U_{i2}V_{j1}P_{\rm R},\\
(\mathcal{C}_{A^0})_{ij} &:= (c_{\beta0}U_{i1}^*V_{j2}^* + s_{\beta0}U_{i2}^*V_{j1}^*)P_{\rm L} - (c_{\beta0}U_{i1}V_{j2} + s_{\beta0}U_{i2}V_{j1})P_{\rm R},\\
(\mathcal{C}_{G^0})_{ij} &:= -(c_{\beta0}U_{i2}^*V_{j1}^* - s_{\beta0}U_{i1}^*V_{j2}^*)P_{\rm L} + (c_{\beta0}U_{i2}V_{j1} - s_{\beta0}U_{i1}V_{j2})P_{\rm R},\\
(\mathcal{C}_{H^-})_{ij} &:= g_2V_{i1}^*N_{j4}^* + \frac{V_{i2}^*(g_YN_{j1}^* + g_2N_{j2}^*)}{\sqrt2}\\
,(\mathcal{C}_{H^+})_{ij} &:= \frac{U_{i2}^*(g_YN_{j1}^* + g_2N_{j2}^*)}{\sqrt2} - g_2U_{i1}^*N_{j3}^*
\end{aligned}$$

,

we have, noting that squarks and sleptons are in super-CKM basis and not in mass eigenstates: (next page)

$$
\begin{aligned}
&\mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{SFG}} \\
&= \frac{1}{2}\overline{\tilde{g}^a}\left[(\mathrm{i}\not{\partial} - m_{\tilde{g}})\delta^{ab} - \mathrm{i}g_3 f^{cab}\not{g}^c\right]\tilde{g}^b + \frac{1}{2}\overline{\tilde{\chi}^0_i}\left[\left(\mathrm{i}\not{\partial} - m_{\tilde{\chi}^0_i}\right)\delta_{ij} + g_Z\left(N_{i3}N^*_{j3} - N_{i4}N^*_{j4}\right)\not{Z}P_\mathrm{L}\right]\tilde{\chi}^0_j \\
&+ \overline{\tilde{\chi}^+_i}\left[\left(\mathrm{i}\not{\partial} - m_{\tilde{\chi}^\pm_i} + |e|\not{A}\right)\delta_{ij} + g_Z\not{Z}\left(\frac{2\mathrm{c}_\mathrm{W}^2 V_{i1}V^*_{j1} + \mathrm{c}_{2\mathrm{W}}V_{i2}V^*_{j2}}{2}P_\mathrm{L} + \frac{\mathrm{c}_{2\mathrm{W}}U_{i2}U^*_{j2} - 2\mathrm{c}_\mathrm{W}^2 U_{i1}U^*_{j1}}{2}P_\mathrm{R}\right)\right]\tilde{\chi}^+_j \\
&+ \overline{u_i}\left[\mathrm{i}\not{\partial} - m_{ui} + g_3\tau^a\not{g}^a + \frac{2|e|}{3}\not{A} + g_Z\not{Z}\left(\frac{3-4\mathrm{s}_\mathrm{W}^2}{6}P_\mathrm{L} - \frac{2\mathrm{s}_\mathrm{W}^2}{3}P_\mathrm{R}\right)\right]u_i \\
&+ \overline{d_i}\left[\mathrm{i}\not{\partial} - m_{di} + g_3\tau^a\not{g}^a - \frac{|e|}{3}\not{A} + g_Z\not{Z}\left(-\frac{3-2\mathrm{s}_\mathrm{W}^2}{6}P_\mathrm{L} + \frac{\mathrm{s}_\mathrm{W}^2}{3}P_\mathrm{R}\right)\right]d_i \\
&+ \overline{\nu_i}\left(\mathrm{i}\not{\partial} + \frac{g_Z}{2}\not{Z}P_\mathrm{L}\right)\nu_i + \overline{e_i}\left[\mathrm{i}\not{\partial} - m_{ei} - |e|\not{A} + g_Z\not{Z}\left(\frac{2\mathrm{s}_\mathrm{W}^2-1}{2}P_\mathrm{L} + \mathrm{s}_\mathrm{W}^2P_\mathrm{R}\right)\right]e_i \\
&+ \frac{g_2}{\sqrt{2}}W^+_\mu\left[\overline{u}\gamma^\mu V_\mathrm{CKM}P_\mathrm{L}d + \overline{\nu}\gamma^\mu P_\mathrm{L}e + \overline{\tilde{\chi}^+_j}\gamma^\mu\left(\left(-\sqrt{2}V_{j1}N^*_{i2} + V_{j2}N^*_{i4}\right)P_\mathrm{L} + \left(-\sqrt{2}N_{i2}U^*_{j1} - N_{i3}U^*_{j2}\right)P_\mathrm{R}\right)\tilde{\chi}^0_i\right] \\
&+ \frac{g_2}{\sqrt{2}}W^-_\mu\left[\overline{d}\gamma^\mu V^\dagger_\mathrm{CKM}P_\mathrm{L}u + \overline{e}\gamma^\mu P_\mathrm{L}\nu + \overline{\tilde{\chi}^0_i}\gamma^\mu\left(\left(-\sqrt{2}N_{i2}V^*_{j1} + N_{i4}V^*_{j2}\right)P_\mathrm{L} + \left(\sqrt{2}U_{j1}N^*_{i2} + U_{j2}N^*_{i3}\right)P_\mathrm{R}\right)\tilde{\chi}^+_j\right] \\
&+ \sqrt{2}g_3\Big[\overline{\tilde{g}^a}(\varphi^*_{\tilde{g}}\tilde{u}^*_\mathrm{R}P_\mathrm{R} - \varphi_{\tilde{g}}\tilde{u}^*_\mathrm{L}P_\mathrm{L})\tau^a u + \overline{u}\tau^a(\varphi_{\tilde{g}}\tilde{u}_\mathrm{R}P_\mathrm{L} - \varphi^*_{\tilde{g}}\tilde{u}_\mathrm{L}P_\mathrm{R})\tilde{g}^a \\
&\qquad + \overline{d}\tau^a(\varphi_{\tilde{g}}\tilde{d}_\mathrm{R}P_\mathrm{L} - \varphi^*_{\tilde{g}}\tilde{d}_\mathrm{L}P_\mathrm{R})\tilde{g}^a + \overline{\tilde{g}^a}(\varphi^*_{\tilde{g}}\tilde{d}^*_\mathrm{R}P_\mathrm{R} - \varphi_{\tilde{g}}\tilde{d}^*_\mathrm{L}P_\mathrm{L})\tau^a d\Big] + \frac{y_{ui}}{\sqrt{2}}\overline{u_i}\left(-\phi_\mathrm{u} + \mathrm{i}\gamma_5\mathrm{c}_{\beta0}A^0 + \mathrm{i}\gamma_5\mathrm{s}_{\beta0}G^0\right)u_i \\
&+ \frac{y_{di}}{\sqrt{2}}\overline{d_i}\left(-\phi_\mathrm{d} + \mathrm{i}\gamma_5\mathrm{s}_{\beta0}A^0 - \mathrm{i}\gamma_5\mathrm{c}_{\beta0}G^0\right)d_i + \frac{y_{ei}}{\sqrt{2}}\overline{e_i}\left(-\phi_\mathrm{d} + \mathrm{i}\gamma_5\mathrm{s}_{\beta0}A^0 - \mathrm{i}\gamma_5\mathrm{c}_{\beta0}G^0\right)e_i \\
&+ \frac{1}{2}\overline{\tilde{\chi}^0_i}\left(\phi_\mathrm{u}\mathcal{N}_{\phi_\mathrm{u}} + \phi_\mathrm{d}\mathcal{N}_{\phi_\mathrm{d}} + \mathrm{i}A^0\mathcal{N}_{A^0} + \mathrm{i}G^0\mathcal{N}_{G^0}\right)_{ij}\tilde{\chi}^0_j + \frac{g_2}{\sqrt{2}}\overline{\tilde{\chi}^+_i}\left(\phi_\mathrm{u}\mathcal{C}_{\phi_\mathrm{u}} + \phi_\mathrm{d}\mathcal{C}_{\phi_\mathrm{d}} + \mathrm{i}A_0\mathcal{C}_{A^0} + \mathrm{i}G_0\mathcal{C}_{G^0}\right)_{ij}\tilde{\chi}^+_j \\
&- (H^-\mathrm{c}_{\beta+} + \mathrm{i}G^-\mathrm{s}_{\beta+})(\mathcal{C}_{H^-})_{ij}\overline{\tilde{\chi}^0_j}P_\mathrm{L}\tilde{\chi}^+_i - (H^+\mathrm{c}_{\beta+} - \mathrm{i}G^+\mathrm{s}_{\beta+})(\mathcal{C}_{H^-})^*_{ij}\overline{\tilde{\chi}^+_i}P_\mathrm{R}\tilde{\chi}^0_j \\
&+ (H^+\mathrm{s}_{\beta+} + \mathrm{i}G^+\mathrm{c}_{\beta+})(\mathcal{C}_{H^+})_{ij}\overline{\tilde{\chi}^+_i}P_\mathrm{L}\tilde{\chi}^0_j + (H^-\mathrm{s}_{\beta+} - \mathrm{i}G^-\mathrm{c}_{\beta+})(\mathcal{C}_{H^+})^*_{ij}\overline{\tilde{\chi}^0_j}P_\mathrm{R}\tilde{\chi}^+_i \\
&+ (H^-\mathrm{s}_{\beta+} - \mathrm{i}G^-\mathrm{c}_{\beta+})\overline{d}y_dV^\dagger_\mathrm{CKM}P_\mathrm{L}u + (H^-\mathrm{c}_{\beta+} + \mathrm{i}G^-\mathrm{s}_{\beta+})\overline{d}V^\dagger_\mathrm{CKM}y_uP_\mathrm{R}u + (H^-\mathrm{s}_{\beta+} - \mathrm{i}G^-\mathrm{c}_{\beta+})\overline{e}y_eP_\mathrm{L}\nu \\
&+ (H^+\mathrm{c}_{\beta+} - \mathrm{i}G^+\mathrm{s}_{\beta+})\overline{u}y_uV_\mathrm{CKM}P_\mathrm{L}d + (H^+\mathrm{s}_{\beta+} + \mathrm{i}G^+\mathrm{c}_{\beta+})\overline{u}V_\mathrm{CKM}y_dP_\mathrm{R}d + (H^+\mathrm{s}_{\beta+} + \mathrm{i}G^+\mathrm{c}_{\beta+})\overline{\nu}y_eP_\mathrm{R}e \\
&+ \left(-\frac{g_YN^*_{j1} + 3g_2N^*_{j2}}{3\sqrt{2}}\tilde{u}^*_{\mathrm{L}i} - y_{ui}N^*_{j4}\tilde{u}^*_{\mathrm{R}i}\right)\overline{\tilde{\chi}^0_j}P_\mathrm{L}u_i + \left(\frac{2\sqrt{2}g_Y}{3}N_{j1}\tilde{u}^*_{\mathrm{R}i} - y_{ui}N_{j4}\tilde{u}^*_{\mathrm{L}i}\right)\overline{\tilde{\chi}^0_j}P_\mathrm{R}u_i \\
&+ \left(\frac{2\sqrt{2}g_Y}{3}N^*_{j1}\tilde{u}_{\mathrm{R}i} - y_{ui}N^*_{j4}\tilde{u}_{\mathrm{L}i}\right)\overline{u_i}P_\mathrm{L}\tilde{\chi}^0_j + \left(-\frac{g_YN_{j1} + 3g_2N_{j2}}{3\sqrt{2}}\tilde{u}_{\mathrm{L}i} - y_{ui}N_{j4}\tilde{u}_{\mathrm{R}i}\right)\overline{u_i}P_\mathrm{R}\tilde{\chi}^0_j \\
&+ \left(-\frac{g_YN^*_{j1} - 3g_2N^*_{j2}}{3\sqrt{2}}\tilde{d}^*_{\mathrm{L}i} - y_{di}N^*_{j3}\tilde{d}^*_{\mathrm{R}i}\right)\overline{\tilde{\chi}^0_j}P_\mathrm{L}d_i + \left(-\frac{\sqrt{2}g_Y}{3}N_{j1}\tilde{d}^*_{\mathrm{R}i} - y_{di}N_{j3}\tilde{d}^*_{\mathrm{L}i}\right)\overline{\tilde{\chi}^0_j}P_\mathrm{R}d_i \\
&+ \left(-\frac{\sqrt{2}g_Y}{3}N^*_{j1}\tilde{d}_{\mathrm{R}i} - y_{di}N^*_{j3}\tilde{d}_{\mathrm{L}i}\right)\overline{d_i}P_\mathrm{L}\tilde{\chi}^0_j + \left(-\frac{g_YN_{j1} - 3g_2N_{j2}}{3\sqrt{2}}\tilde{d}_{\mathrm{L}i} - y_{di}N_{j3}\tilde{d}_{\mathrm{R}i}\right)\overline{d_i}P_\mathrm{R}\tilde{\chi}^0_j \\
&+ \left(\frac{g_YN^*_{j1} + g_2N^*_{j2}}{\sqrt{2}}\tilde{e}^*_{\mathrm{L}i} - y_{ei}N^*_{j3}\tilde{e}^*_{\mathrm{R}i}\right)\overline{\tilde{\chi}^0_j}P_\mathrm{L}e_i + \left(-\sqrt{2}g_YN_{j1}\tilde{e}^*_{\mathrm{R}i} - y_{ei}N_{j3}\tilde{e}^*_{\mathrm{L}i}\right)\overline{\tilde{\chi}^0_j}P_\mathrm{R}e_i + \frac{g_YN_{j1} - g_2N_{j2}}{\sqrt{2}}\tilde{\nu}_{\mathrm{L}i}\overline{\nu_i}P_\mathrm{R}\tilde{\chi}^0_j \\
&+ \left(-\sqrt{2}g_YN^*_{j1}\tilde{e}_{\mathrm{R}i} - y_{ei}N^*_{j3}\tilde{e}_{\mathrm{L}i}\right)\overline{e_i}P_\mathrm{L}\tilde{\chi}^0_j + \left(+\frac{g_YN_{j1} + g_2N_{j2}}{\sqrt{2}}\tilde{e}_{\mathrm{L}i} - y_{ei}N_{j3}\tilde{e}_{\mathrm{R}i}\right)\overline{e_i}P_\mathrm{R}\tilde{\chi}^0_j + \frac{g_YN^*_{j1} - g_2N^*_{j2}}{\sqrt{2}}\tilde{\nu}^*_{\mathrm{L}i}\overline{\tilde{\chi}^0_j}P_\mathrm{L}\nu_i \\
&+ \left(U^*_{i2}(\tilde{d}^*_\mathrm{R}y_dV^\dagger_\mathrm{CKM})_j - U^*_{i1}g_2(\tilde{d}^*_\mathrm{L}V^\dagger_\mathrm{CKM})_j\right)\overline{\tilde{\chi}^+_i}P_\mathrm{L}u_j + V_{i2}(\tilde{d}^*_\mathrm{L}V^\dagger_\mathrm{CKM}y_u)_j\overline{\tilde{\chi}^+_i}P_\mathrm{R}u_j \\
&+ V^*_{i2}(y_uV_\mathrm{CKM}\tilde{d}_\mathrm{L})_j\overline{u_j}P_\mathrm{L}\tilde{\chi}^+_i + \left(U_{i2}(V_\mathrm{CKM}y_d\tilde{d}_\mathrm{R})_j - U_{i1}g_2(V_\mathrm{CKM}\tilde{d}_\mathrm{L})_j\right)\overline{u_j}P_\mathrm{R}\tilde{\chi}^+_i \\
&+ \left(-V^*_{i1}g_2(\tilde{u}^*_\mathrm{L}V_\mathrm{CKM})_j + V^*_{i2}(\tilde{u}^*_\mathrm{R}y_uV_\mathrm{CKM})_j\right)\overline{\tilde{\chi}^-_i}P_\mathrm{L}d_j + U_{i2}(\tilde{u}^*_\mathrm{L}V_\mathrm{CKM}y_d)_j\overline{\tilde{\chi}^-_i}P_\mathrm{R}d_j \\
&+ U^*_{i2}(y_dV^\dagger_\mathrm{CKM}\tilde{u}_\mathrm{L})_j\overline{d_j}P_\mathrm{L}\tilde{\chi}^-_i + \left(-V_{i1}g_2(V^\dagger_\mathrm{CKM}\tilde{u}_\mathrm{L})_j + V_{i2}(V^\dagger_\mathrm{CKM}y_u\tilde{u}_\mathrm{R})_j\right)\overline{d_j}P_\mathrm{R}\tilde{\chi}^-_i \\
&+ \left(-U^*_{i1}g_2\tilde{e}^*_\mathrm{L} + U^*_{i2}y_{ei}\tilde{e}^*_{\mathrm{R}i}\right)\overline{\tilde{\chi}^+_i}P_\mathrm{L}\nu + (-U_{i1}g_2\tilde{e}_\mathrm{L} + U_{i2}y_{ei}\tilde{e}_{\mathrm{R}i})\overline{\nu}P_\mathrm{R}\tilde{\chi}^+_i \\
&+ \overline{\tilde{\chi}^-_i}\left(-V^*_{i1}g_2\tilde{\nu}^*_\mathrm{L}P_\mathrm{L} + U_{i2}y_{ej}\tilde{\nu}^*_{\mathrm{L}j}P_\mathrm{R}\right)e_j + \overline{e_j}\left(U^*_{i2}y_{ej}\tilde{\nu}_{\mathrm{L}j}P_\mathrm{L} - V_{i1}g_2\tilde{\nu}_\mathrm{L}P_\mathrm{R}\right)\tilde{\chi}^-_i + (\text{RPV part}).
\end{aligned}
\tag{12.91}
$$

12.5 SLHA convention

The SLHA convention [22] is different from our notation. the reinterpretation rules for the parameters are given in the right table (**magenta color** for objects in other conventions), while

$$\mu, b, m^1_{Q,L,H_\mathrm{u},H_\mathrm{d}}, \text{RPV-trilinears } (\lambda\text{s and } T\text{s})$$

are in common.

SLHA		our notation		Martin/DHM
(H_1, H_2)	$=$	$(H_\mathrm{d}, H_\mathrm{u})$		
$Y_{\mathrm{u,d,e}}$	$=$	$(y_{\mathrm{u,d,e}})^\mathrm{T}$		
$T_{\mathrm{u,d,e}}$	$=$	$(a_{\mathrm{u,d,e}})^\mathrm{T}$		
$A_{\mathrm{u,d,e}}$	$=$	$(A_{\mathrm{u,d,e}})^\mathrm{T}$		
$m^2_{U^\mathrm{c},D^\mathrm{c},E^\mathrm{c}}$	$=$	$(m^2_{U^\mathrm{c},D^\mathrm{c},E^\mathrm{c}})^\dagger$		
$M_{1,2,3}$	$=$	$-M_{1,2,3}$		
m_3^2	$=$	b		
m_A^2	$=$	$m^2_{A_0}$ (tree)		
		κ_i	$=$	$-\mu'_i$ (rarely used)
D_i	$=$	b_i		
$m^2_{\tilde{L}_i H_1}$	$=$	M^2_{Li}		

In particular, the chargino/neutralino mass terms in RPC case are given by

$$\mathcal{L} \supset \left[\frac{1}{2}M_1\tilde{b}\tilde{b} + \frac{1}{2}M_2\tilde{w}\tilde{w} - \mu\tilde{h}_\mathrm{u}\tilde{h}_\mathrm{d} - \frac{g_Y}{2\sqrt{2}}\left(h_\mathrm{u}^*\tilde{h}_\mathrm{u} - h_\mathrm{d}^*\tilde{h}_\mathrm{d}\right)\tilde{b} - \sqrt{2}g_2\left(h_\mathrm{u}^* T^a\tilde{h}_\mathrm{u} + h_\mathrm{d}^* T_a\tilde{h}_\mathrm{d}\right)\tilde{w}\right] + \text{H.c.} \tag{12.92}$$

$$\to \frac{1}{2}\begin{pmatrix}\tilde{b}\\ \tilde{w}\\ h_\mathrm{u}^0\\ h_\mathrm{d}^0\end{pmatrix}^\mathrm{T}\begin{pmatrix} -M_1 & 0 & -m_Z\mathrm{c}_\beta\mathrm{s}_w & m_Z\mathrm{s}_\beta\mathrm{s}_w\\ 0 & -M_2 & m_Z\mathrm{c}_\beta\mathrm{c}_w & -m_Z\mathrm{s}_\beta\mathrm{c}_w\\ -m_Z\mathrm{c}_\beta\mathrm{s}_w & m_Z\mathrm{c}_\beta\mathrm{c}_w & 0 & -\mu\\ m_Z\mathrm{s}_\beta\mathrm{s}_w & -m_Z\mathrm{s}_\beta\mathrm{c}_w & -\mu & 0\end{pmatrix}\begin{pmatrix}\tilde{b}\\ \tilde{w}\\ h_\mathrm{u}^0\\ h_\mathrm{d}^0\end{pmatrix} \tag{12.93}$$

12.6 GMSB formulae

$$\Gamma^{-1}(\tilde{l} \to l\tilde{G}) = \frac{1}{48\pi M^2}\frac{m_{\tilde{l}}^5}{m_{\tilde{G}}^2} \times (\text{phase space}) \approx 5.9\times 10^{-7}\,\mathrm{s}\frac{(m_{3/2}/\mathrm{MeV})^2}{(m_{\tilde{l}}/\mathrm{TeV})^5} \tag{12.94}$$

$$= \frac{3m_{\tilde{l}}^5}{48\pi F_{\mathrm{tot}}^2} \times (\text{phase space}) \approx 3.3\times 10^{-9}\,\mathrm{s}\frac{(F_{\mathrm{tot}}/10^{12}\,\mathrm{GeV}^2)^2}{(m_{\tilde{l}}/100\,\mathrm{GeV})^5} = 1.0\,\mathrm{m}\frac{(F_{\mathrm{tot}}/10^{12}\,\mathrm{GeV}^2)^2}{(m_{\tilde{l}}/100\,\mathrm{GeV})^5} \tag{12.95}$$

$$\Gamma^{-1}(\tilde{B} \to \gamma\tilde{G}) = \frac{c_{\mathrm{W}}^2 m_{\tilde{B}}^5}{48\pi M^2 m_{\tilde{G}}^2}(1-r)^3(1+3r) \qquad \text{where} \quad r := \left(\frac{m_{\tilde{G}}}{m_{\tilde{B}}}\right)^2. \tag{12.96}$$

(cf. [23])

A Mathematics

A.1 Matrix exponential

Excerpted from §2 and §5 of Hall 2015 [24]:

$$e^X \coloneqq \sum_{m=0}^{\infty} \frac{X^m}{m!} \text{ (converges for any } X), \quad \log X \coloneqq \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-1)^m}{m} \text{ (conv. if } \|A-I\|<1). \tag{A.1}$$

$$e^{\log A} = A \text{ (if } \|A-I\|<1), \quad \log e^X = X \text{ and } \|e^X-1\|<1 \text{ (if } \|X\|<\log 2). \tag{A.2}$$

$$\text{Hilbert-Schmidt norm: } \|X\|^2 \coloneqq \sum_{i,j} |X_{ij}|^2 = \operatorname{Tr} X^\dagger X. \tag{A.3}$$

Properties:

$$e^{(X^{\mathrm{T}})} = (e^X)^{\mathrm{T}}, \qquad e^{(X^*)} = (e^X)^*, \qquad (e^X)^{-1} = e^{-X}, \qquad e^{YXY^{-1}} = Ye^XY^{-1},$$

$$\det\exp X = \exp\operatorname{Tr} X, \qquad e^{(\alpha+\beta)X} = e^{\alpha X}e^{\beta X} \text{ for } \alpha,\beta\in\mathbb{C};$$

Baker-Campbell-Hausdorff:

$$e^XYe^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots = e^{[X,]}Y; \tag{A.4}$$

$$e^Xe^Ye^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!}(e^XYe^{-X})^n = \exp\left(e^{[X,]}Y\right); \tag{A.5}$$

$$\log(e^Xe^Y) = X + \int_0^1 dt\, g(e^{[X,}e^{t[Y,})Y \qquad \left[g(z) = \frac{\log z}{1-z^{-1}} = 1 - \sum_{n=1}^{\infty}\frac{(1-z)^n}{n(n+1)}; \quad g(e^y) = \sum_{n=0}^{\infty}\frac{B_n y^n}{n!}\right] \tag{A.6}$$

$$= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots \quad \text{(Baker-Campbell-Hausdorff).} \tag{A.7}$$

$$\log(e^Xe^Y) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\sum_{m,n=0}^{\infty}\frac{X^mY^n}{m!n!} - 1\right)^k \quad = \sum_{k=1}^{\infty}\sum_{m_1+n_1>0}\cdots\sum_{m_k+n_k>0}\frac{(-1)^{k-1}}{k}\frac{X^{m_1}Y^{n_1}\cdots X^{m_k}Y^{n_k}}{m_1!n_1!\cdots m_k!n_k!} \tag{A.8}$$

$$\log(e^Xe^Y) = \sum_{k=1}^{\infty}\sum_{m_1+n_1>0}\cdots\sum_{m_k+n_k>0}\frac{(-1)^{k-1}}{k\sum_{i=1}^{k}(m_i+n_i)}\frac{\left([X,\right)^{m_1}\left([Y,\right)^{n_1}\cdots\left([X,\right)^{m_k}\left([Y,\right)^{n_k}]\cdots]}{m_1!n_1!\cdots m_k!n_k!} \tag{A.9}$$

with $[X]$ being X.

Derivative:

$$\frac{d}{dt}e^{tX} = Xe^{tX} = e^{tX}X \tag{A.10}$$

$$e^{-X(t)}\left(\frac{d}{dt}e^{X(t)}\right) = \frac{I-e^{-\mathrm{ad}_X}}{\mathrm{ad}_X}\left(\frac{dX}{dt}\right) = X' + \frac{[-X,X']}{2!} + \frac{[-X,[-X,X']]}{3!} + \cdots \tag{A.11}$$

$$\left(\frac{d}{dt}e^{X(t)}\right)e^{-X(t)} = X' + \frac{[X,X']}{2!} + \frac{[X,[X,X']]}{3!} + \cdots \tag{A.12}$$

where $X' = dX/dt$ and $\mathrm{ad}_X(Y) = [X,Y]$ is the adjoint action of a Lie algebra. Thus, explicitly,

$$\frac{d}{dt}e^{aX(t)} = e^{aX}\left\{\sum_{n=0}^{\infty}\frac{a^{n+1}}{(n+1)!}\left([-X,\right)^nX']\right\} = \left\{\sum_{n=0}^{\infty}\frac{a^{n+1}}{(n+1)!}\left([X,\right)^nX']\right\}e^{aX} \tag{A.13}$$

Component: If matrices t^a satisfies $[t^a,t^b] = if^{abc}t^c$ with totally-antisymmetric $f^{abc}\in\mathbb{R}$,

$$\left[e^{\theta^at^a}t_be^{-\theta^ct^c}\right]_{ij} = \left[e^{\theta^a[t^a,}t_b\right]_{ij} = \left[e^{i\theta^af^a}\right]^{bc}t^c_{ij} \tag{A.14}$$

holds for $\theta^a\in\mathbb{C}$, where $[f^a]_{bc} = f^{abc}$. ♠TODO:needs verification, generalization/restriction, and a nice proof or reference.♠

A.2 General unitary matrix

$$U_2 = \begin{pmatrix}1&0\\0&e^{i\alpha}\end{pmatrix}\begin{pmatrix}c_\theta&s_\theta\\-s_\theta&c_\theta\end{pmatrix}\begin{pmatrix}e^{i\beta}&0\\0&e^{i\gamma}\end{pmatrix} = \begin{pmatrix}c_\theta e^{i\beta}&s_\theta e^{i\gamma}\\-s_\theta e^{i(\alpha+\beta)}&c_\theta e^{i(\alpha+\gamma)}\end{pmatrix}, \qquad 0\le\theta\le\frac{\pi}{2}, \quad \alpha,\beta,\gamma\in\mathbb{R}; \tag{A.15}$$

$$U_3 = \begin{pmatrix}1&&\\&e^{ia}&\\&&e^{ib}\end{pmatrix}\begin{pmatrix}1&&\\&c_{23}&s_{23}\\&-s_{23}&c_{23}\end{pmatrix}\begin{pmatrix}c_{13}&&s_{13}e^{-i\delta}\\&1&\\-s_{13}e^{i\delta}&&c_{13}\end{pmatrix}\begin{pmatrix}c_{12}&s_{12}&\\-s_{12}&c_{12}&\\&&1\end{pmatrix}\begin{pmatrix}e^{ic}&&\\&e^{id}&\\&&e^{ie}\end{pmatrix} \tag{A.16}$$

$$= \begin{pmatrix}1&&\\&e^{ia}&\\&&e^{ib}\end{pmatrix}\begin{pmatrix}c_{12}c_{13}&s_{12}c_{13}&s_{13}e^{-i\delta}\\-s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta}&c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta}&s_{23}c_{13}\\s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta}&-c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta}&c_{23}c_{13}\end{pmatrix}\begin{pmatrix}e^{ic}&&\\&e^{id}&\\&&e^{ie}\end{pmatrix} \tag{A.17}$$

with $0\le\theta_{ij}\le\pi/2$ and $a,b,c,d,e,\delta\in\mathbb{R}$ (see, e.g., Ref. [25]).

```
U3 = Dot[
  DiagonalMatrix[Exp[I {0, a, b}]],
  RotationMatrix[θ23, {-1, 0, 0}],
  DiagonalMatrix[Exp[I {0,0,+δ}]],
  RotationMatrix[θ13, {0, 1, 0}],
  DiagonalMatrix[Exp[I {0,0,-δ}]],
  RotationMatrix[θ12, {0, 0, -1}],
  DiagonalMatrix[Exp[I {c, d, e}]]
]
```

A.3 Matrix diagonalization

In this section, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and $\mathbb{U}^n_{\mathbb{K}}\subset\mathbb{K}^{n\times n}$ is the set of the unitary matrices.

Diagonalization A matrix $M\in\mathbb{K}^{n\times n}$ is called diagonalizable if $\exists P$ and $\exists D$ s.t.

$$M=PDP^{-1};\qquad P\in\mathbb{K}^{n\times n},\quad D:\text{ diagonal matrix }(D_{ii}\in\mathbb{K}).\tag{A.18}$$

In particular,

$$M\text{ is normal}\overset{\text{def}}{\iff}M^\dagger M=MM^\dagger\iff\exists P\in\mathbb{U}^n_{\mathbb{K}}\text{ s.t. }M=PDP^{-1}.\tag{A.19}$$

Singular value decomposition Any $M\in\mathbb{K}^{m\times n}$ can be singular-value decomposed as

$$M=UDV^\dagger;\qquad U\in\mathbb{U}^m_{\mathbb{K}},\quad V\in\mathbb{U}^n_{\mathbb{K}},\quad D:\text{ non-negative real diagonal matrix }(D_{ii}\ge0).\tag{A.20}$$

Here, the matrix U (V) diagonalizes $MM^\dagger$ ($M^\dagger M$) and $(D_{ii})^2$ are the eigenvalues of $MM^\dagger$ (and $M^\dagger M$).

The calculation on Mathematica is straightforward for this convention:

```
{u, d, v} = SingularValueDecomposition[M]
```

Autonne-Takagi factorization If $M\in\mathbb{C}^{n\times n}$ is symmetric, it can be decomposed as

$$M=RDR^{\mathrm{T}};\qquad R\in\mathbb{U}^n_{\mathbb{C}},\quad D:\text{ non-negative real diagonal matrix }(D_{ii}\ge0).\tag{A.21}$$

Real symmetric matrices are normal and thus do not need this factorization; we can apply the above "diagonalization" method.

Sample Mathematica code to calculate $\{D,R\}$ (with ordering, if specified) is:

```
AutonneTakagi[M_, order_ : None] := Module[{v0, v, p, ord, R, D},
  ord = If[order === None, Range[Length[M]], order];
  v0 = Eigenvectors[Conjugate[M].M];
  v = Eigenvectors[v0.M.Transpose[v0]].v0; (*resolve degenerate eigenvalues*)
  p = DiagonalMatrix[If[Abs[#] > 0, (#/Abs[#])^(-1/2), 1] & /@ Diagonal[v.M.Transpose[v]]];
  R = ConjugateTranspose[Reverse[p.v][[ord]] // Orthogonalize];
  D = ConjugateTranspose[R].M.Conjugate[R];
  {D, R}];
```

A.4 Group theory

Lie Groups This section is mainly based on Ref. [24]. Also, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and $M^n_{\mathbb{K}}=\mathbb{K}^{n\times n}$.

With $\Omega=\begin{pmatrix}0&I_{n/2}\\-I_{n/2}&0\end{pmatrix}$ and $H=\begin{pmatrix}I_p&0\\0&-I_q\end{pmatrix}$,

$$\mathrm{GL}(n;\mathbb{K})=\{g\in M^n_{\mathbb{K}}\mid\det g\neq0\},\qquad\mathfrak{gl}(n;\mathbb{K})=\{a\in M^n_{\mathbb{K}}\},\tag{A.22}$$

$$\mathrm{SL}(n;\mathbb{K})=\{g\in M^n_{\mathbb{K}}\mid\det g=1\},\qquad\mathfrak{sl}(n;\mathbb{K})=\{a\in M^n_{\mathbb{K}}\mid\operatorname{Tr}a=0\},\tag{A.23}$$

$$\mathrm{SU}(n)=\{g\in M^n_{\mathbb{C}}\mid g^\dagger g=1\wedge\det g=1\},\qquad\mathfrak{su}(n)=\{a\in M^n_{\mathbb{C}}\mid a+a^\dagger=0\wedge\operatorname{Tr}a=0\},\tag{A.24}$$

$$\mathrm{SO}(n)=\{g\in M^n_{\mathbb{R}}\mid g^{\mathrm{T}}g=1\wedge\det g=1\},\qquad\mathfrak{so}(n)=\{a\in M^n_{\mathbb{R}}\mid a+a^{\mathrm{T}}=0\},\tag{A.25}$$

$$\mathrm{SO}(n;\mathbb{C})=\{g\in M^n_{\mathbb{C}}\mid g^{\mathrm{T}}g=1\wedge\det g=1\},\qquad\mathfrak{so}(n;\mathbb{C})=\{a\in M^n_{\mathbb{C}}\mid a+a^{\mathrm{T}}=0\},\tag{A.26}$$

$$\mathrm{SO}(p,q)=\{g\in M^{p+q}_{\mathbb{R}}\mid Hg^{\mathrm{T}}H=g^{-1}\wedge\det g=1\},\qquad\mathfrak{so}(p,q)=\{a\in M^{p+q}_{\mathbb{R}}\mid Ha^{\mathrm{T}}H=-a\},\tag{A.27}$$

$$\mathrm{Sp}(n;\mathbb{K})=\{g\in M^n_{\mathbb{K}}\mid\Omega g^{\mathrm{T}}\Omega=-g^{-1}\},\qquad\mathfrak{sp}(n;\mathbb{K})=\{g\in M^n_{\mathbb{K}}\mid\Omega a^{\mathrm{T}}\Omega=a\},\tag{A.28}$$

$$\mathrm{Sp}(n)=\{g\in M^n_{\mathbb{C}}\mid\epsilon g^{\mathrm{T}}\epsilon=-g^{-1}\wedge g^\dagger g=1\},\qquad\mathfrak{sp}(n)=\{g\in M^n_{\mathbb{C}}\mid\epsilon a^{\mathrm{T}}\epsilon=a\wedge a^\dagger+a=0\}.\tag{A.29}$$

Of course, $\mathrm{U}(n)$ and $\mathrm{O}(n;\mathbb{K})$ (and corresponding Lie algebra) are obtained by removing the condition $\det g=1$ (and $\operatorname{Tr}a=0$).

Important identities are:

$$\mathrm{Sp}(2;\mathbb{K})=\mathrm{SL}(2;\mathbb{K}),\quad\mathrm{Sp}(2)=\mathrm{SU}(2),\tag{A.30}$$

The complexification of a finite-dimensional *real* vector space V is defined by

$$V_{\mathbb{C}}=V\otimes\mathbb{C}=\{v_1+\mathrm{i}v_2\mid v_1,v_2\in V\}\cong V+\mathrm{i}V.\tag{A.31}$$

Then, complexification of a Lie algebra $\mathfrak{a}$ of a matrix Lie group is defined as follows. Since $\mathfrak{a}$ is a *real* Lie algebra, it can be complexified to $\mathfrak{a}_{\mathbb{C}}$, where its bracket operation has a unique extension so that $\mathfrak{a}_{\mathbb{C}}$ is a *complex* lie algebra. The complex Lie algebra $\mathfrak{a}_{\mathbb{C}}$ is called the complexification of $\mathfrak{a}$. In particular,

$$\mathfrak{gl}(n;\mathbb{R})_{\mathbb{C}} \cong \mathfrak{u}(n)_{\mathbb{C}} \cong \mathfrak{gl}(n;\mathbb{C}), \qquad \mathfrak{so}(n)_{\mathbb{C}} \cong \mathfrak{so}(n;\mathbb{C}),$$
$$\mathfrak{sl}(n;\mathbb{R})_{\mathbb{C}} \cong \mathfrak{su}(n)_{\mathbb{C}} \cong \mathfrak{sl}(n;\mathbb{C}), \qquad \mathfrak{sp}(n;\mathbb{R})_{\mathbb{C}} \cong \mathfrak{sp}(n)_{\mathbb{C}} \cong \mathfrak{sp}(n;\mathbb{C}). \tag{A.32}$$

On the other hand, a complex simple Lie algebra $\mathfrak{a}$ can be, on the other hand, viewed as a real Lie algebra, whose (real) dimension is twice as the original (complex) dimension. We describe such real Lie algebra by $\mathfrak{a}_{\mathbb{R}}$.

Important identities are:

$$\mathfrak{so}(3,1) \cong \mathfrak{sl}(2;\mathbb{C})_{\mathbb{R}}, \qquad \mathfrak{so}(3,1)_{\mathbb{C}} \cong \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C}). \tag{A.33}$$

A.5 Mathematical Foundations for Spinors

A.5.1 Clifford Algebra

Quadratic form Let [26, §6.3] V be an n-dimensional vector space over a field K. A quadratic form Q on V is defined by

$$\begin{aligned} &Q\colon\ V \to K, \qquad \forall k \in K, v \in V, Q(kv) = k^2 Q(v), \\ &\qquad\qquad B\colon\ (v,w) \mapsto Q(v+w) - Q(v) - Q(w) \text{ is linear in } v \text{ and } w. \end{aligned} \tag{A.34}$$

The bilinear form B satisfies $B(x,x) = 2Q(x)$. As long as the characteristic of F is not two, Q is determined by B.

Equivalently, Q is characterized by a symmetric "matrix" $\hat{Q}$ such that $Q(v) = v^{\mathrm{T}}\hat{Q}v$. Focusing on $K = \mathbb{R}$, $\hat{Q}$ is a real symmetric matrix and thus diagonalizable. Sylvester's law of inertia states that there exists a basis such that $\hat{Q} = \operatorname{diag}(1,1,\cdots,1,-1,-1,\cdots,-1,0,\cdots,0)$, where the number of $+1$ and -1 are respectively p and q. The pair (p,q) is unique and is called the signature of Q, where $\operatorname{rank}\hat{Q} = p+q \ (\le n)$. Meanwhile, if K is algebraically closed such as $\mathbb{C}$, we can take $\hat{Q} = \operatorname{diag}(1,1,\cdots,1,0,\cdots,0)$ and the algebra is characterized by $\operatorname{rank}\hat{Q}$.

Quadratic form induces orthogonality: a basis $\{e_i\}$ of V is called

$$\text{orthogonal under } Q \overset{\text{def}}{\iff} B(e_i,e_j) \propto \delta_{ij} \qquad \Big[\text{orthonormal under } Q \overset{\text{def}}{\iff} B(e_i,e_j) = c_i\delta_{ij} \text{ with } c_i \in \{2,0,-2\}\Big] \tag{A.35}$$

and (V,Q) has an orthogonal basis as long as the characteristic of K is not two. If K is a "spin field" such as $\mathbb{R}$ or $\mathbb{C}$ (or, precisely, $\forall\alpha \in K, \exists\beta \in K$ s.t. $\alpha = \beta^2$ or $-\beta^2$), (V,Q) has an orthonormal basis.

Geometric algebra Let [27] V be an n-dimensional vector space over a field K. Let Q be a quadratic form on V. Tensor algebra on V is defined by

$$T(V) = \bigoplus_k V^{\otimes k} = K \oplus V \oplus (V\otimes V) \oplus (V\otimes V\otimes V) \oplus \cdots. \tag{A.36}$$

It has an ideal K_Q generated by $\{v\otimes v - Q(v) \mid v \in V\}$:

$$K_Q = \left\{ \sum_i a_i \otimes (v_i \otimes v_i - Q(v_i)) \otimes b_i \,\middle|\, v_i \in V, a_i, b_i \in T(V) \right\}, \tag{A.37}$$

and the geometric algebra $\mathcal{G}(V,Q) \coloneqq T(V)/K_Q$. The product in $\mathcal{G}(V,Q)$ is written by $vw \coloneqq [v\otimes w] = v\otimes w + K_q$. Particularly, $v^2 = Q(v) \in K$ and $vw + wv = B(v,w) \in K$.

The pseudoscalar I is defined by $I \coloneqq e_1 e_2 \cdots e_n$ with the basis $\{e_i\}$. For $\mathcal{G}(\mathbb{R}^{s,t,u})$, $I^2 = (-1)^{n(n-1)/2+t}\delta_{0u}$ and I commutes (anti-commutes) with all e_i if n is even (odd).

Clifford Algebra For a finite set X and a commutative ring R with unit, the Clifford algebra $\mathrm{Cl}(X,R,r)$ is defined as the free R-module generated by the set 2^X of all subset of X. Here, $r\colon\ X \to R$ is an arbitrary function thought of as a signature. The Clifford algebra Cl has addition $\mathrm{Cl}\times\mathrm{Cl} \to \mathrm{Cl}$, scalar multiplication $R\times\mathrm{Cl} \to \mathrm{Cl}$, and product

$$\mathrm{Cl}\times\mathrm{Cl} \to \mathrm{Cl} \ni AB \coloneqq \tau(A,B) A \mathbin{\triangle} B \tag{A.38}$$

defined based on the following map $\tau\colon\ 2^X \times 2^X \to R$:

$$\begin{aligned} &\tau(\{x\},\{x\}) = r(x) \quad \forall x \in X, && \tau(A,B)\tau(A\mathbin{\triangle}B,C) = \tau(A,B\mathbin{\triangle}C)\tau(B,C) \quad \forall A,B,C \in 2^X, \\ &\tau(\{y\},\{x\}) = -\tau(\{x\},\{y\}) \quad \forall x,y \in X, x \neq y, && \tau(\varnothing,A) = \tau(A,\varnothing) = 1 \quad \forall A \in 2^X, \\ &\tau(A,B) \in \{-1,1\} \quad \text{if } A\cap B = \varnothing, \end{aligned} \tag{A.39}$$

where $A \mathbin{\triangle} B = (A\cup B)\setminus(B\cap A)$. Namely,

$$\{x\}\{x\} = r(x)\varnothing, \quad \{x\}\{y\} = -\{y\}\{x\}, \quad \varnothing A = A\varnothing = A, \quad (AB)C = A(BC), \quad A\cap B = \varnothing \Rightarrow AB = \pm 1(A\cup B). \tag{A.40}$$

Several standard operations are defined: for $A \in 2^X$ and $k = \binom{|A|}{2}$,

$$\text{Grade involution} \quad A^\star \coloneqq (-1)^{|A|}A, \qquad \text{Reversion} \quad A^\dagger \coloneqq (-1)^k A, \qquad \text{Clifford conjugate} \quad A^\square \coloneqq A^{\star\dagger}. \tag{A.41}$$

The sign of grade involution (reversion) is negative iff $|A| \bmod 4$ is 1 or 3 (2 or 3). Also, $(xy)^\star = x^\star y^\star$ and $(xy)^\dagger = y^\dagger x^\dagger$.

Let (V,Q) be an n-dimensional vector space V over a field K equipped with a quadratic form Q. Let $E = \{e_i\}$ be an orthogonal basis of (V,Q). Then

$$\mathrm{Cl}(E,K,Q|_E) \cong \mathcal{G}(V,Q). \tag{A.42}$$

The Clifford algebra $\mathrm{Cl}(X,R,r)$ has subspaces of k-vectors Cl^k and of even and odd vectors $\mathrm{Cl}^\pm$:

$$\mathrm{Cl}(X,R,r) = \mathrm{Cl}^+ \oplus \mathrm{Cl}^- = \mathrm{Cl}^0 \oplus \mathrm{Cl}^1 \oplus \mathrm{Cl}^2 \oplus \cdots \oplus \mathrm{Cl}^{|X|}.$$

In particular, Cl^+ is a subalgebra.

Isomorphisms There are several algebra-isomorphisms [27]:

$$\mathcal{G}(\mathbb{R}^{s,t}) \cong \mathcal{G}(\mathbb{R}^{t+1,s-1}), \quad \mathcal{G}^+(\mathbb{R}^{s,t}) \cong \mathcal{G}(\mathbb{R}^{s,t-1}) \cong \mathcal{G}(\mathbb{R}^{t,s-1}) \cong \mathcal{G}^+(\mathbb{R}^{t,s}),$$
$$\mathcal{G}(\mathbb{R}^{n+2,0}) \cong \mathcal{G}(\mathbb{R}^{0,n}) \otimes \mathcal{G}(\mathbb{R}^{2,0}), \quad \mathcal{G}(\mathbb{R}^{0,n+2}) \cong \mathcal{G}(\mathbb{R}^{n,0}) \otimes \mathcal{G}(\mathbb{R}^{0,2}), \quad \mathcal{G}(\mathbb{R}^{s+1,t+1}) \cong \mathcal{G}(\mathbb{R}^{s,t}) \otimes \mathcal{G}(\mathbb{R}^{1,1}),$$
$$\mathcal{G}(\mathbb{R}^{1,0}) \cong \mathbb{R} \oplus \mathbb{R}, \quad \mathcal{G}(\mathbb{R}^{0,1}) \cong \mathbb{C}, \quad \mathcal{G}(\mathbb{R}^{2,0}) \cong \mathbb{R}^{2\times 2}, \quad \mathcal{G}(\mathbb{R}^{0,2}) \cong \mathbb{H},$$
$$\mathcal{G}(\mathbb{C}^{s+t}) \cong \mathcal{G}(\mathbb{R}^{s,t}) \otimes_{\mathbb{R}} \mathbb{C}, \quad \mathcal{G}(\mathbb{C}^0) \cong \mathbb{C}, \quad \mathcal{G}(\mathbb{C}^1) \cong \mathbb{C} \oplus \mathbb{C}, \quad \mathcal{G}(\mathbb{C}^{n+2}) \cong \mathcal{G}(\mathbb{C}^2) \otimes_{\mathbb{C}} \mathcal{G}(\mathbb{C}^n) \cong \mathcal{G}(\mathbb{C}^n) \otimes_{\mathbb{C}} \mathbb{C}^{2\times 2},$$
$$s+t \text{ is odd and } I^2 = -1 \implies \mathcal{G}(\mathbb{R}^{s,t}) \cong \mathcal{G}^+(\mathbb{R}^{t,s}) \otimes \mathbb{C} \cong \mathcal{G}(\mathbb{C}^{s+t-1}).$$

Orthogonal Group The orthogonal (and related) group for (V,Q) is defined by

$$\mathrm{O}(V,Q) \coloneqq \{f: V \to V: \text{ linear bijection s.t. } Q \circ f = Q, \text{ i.e., } \forall v \in V, Q(f(v)) = f(v)\},$$
$$\mathrm{SO}(V,Q) \coloneqq \{f \in \mathrm{O}(V,Q) \mid \det f = 1\}, \qquad \mathrm{SO}(V,Q) \coloneqq \text{the part of } \mathrm{SO}(V,Q) \text{ connected to } 1,$$

where the determinant is defined with identifying $V \to V$ as a matrix.

Groups in Geometric Algebra Here we assume $K = \mathbb{R}$ and $\mathcal{G}$ is non-degenerate, having $\mathcal{G}(\mathbb{R}^{s,t})$ in mind. The geometric algebra $\mathcal{G}(V,Q)$ contains a group

$$\mathcal{G}^\times \coloneqq \{g \in \mathcal{G} \mid \exists g^{-1} \in \mathcal{G}, gg^{-1} = g^{-1}g = 1\} \tag{A.43}$$

with its addition and scalar multiplication being forgotten. Furthermore, with $V^\times = \{v \in V \mid Q(v) \neq 0\}$, it contains

$$\Gamma \coloneqq \{v_1 v_2 \cdots v_n \mid v_i \in V^\times\}, \qquad \tilde{\Gamma} \coloneqq \{g \in \mathcal{G}^\times \mid gVg^{-1} \in V\},$$
$$\mathrm{Pin} \coloneqq \{x \in \Gamma \mid xx^\dagger = \pm 1\}, \qquad \mathrm{Spin} \coloneqq \mathrm{Pin} \cap \mathcal{G}^+, \qquad \mathrm{Spin}^+ \coloneqq \{x \in \mathrm{Spin} \mid xx^\dagger = 1\}. \tag{A.44}$$

The Lipschitz group $\tilde{\Gamma}$ is actually equal to the versor group Γ and is the smallest group that contains $V^\times$. To summarize,

$$\mathcal{G} \supset \mathcal{G}^\times \supset \Gamma = \tilde{\Gamma} \supset V^\times; \qquad \Gamma \supset \mathrm{Pin} \supset \mathrm{Spin} \supset \mathrm{Spin}^+.$$

Consider a function $R: \mathbb{R} \to \mathrm{Spin}^+(V,Q)$. It is known that $R'(0) \in \mathcal{G}^2(V,Q)$, i.e., with considering the rotor group Spin^+ as a Lie group, we see the corresponding Lie algebra $\mathfrak{spin}$ is equal to $\mathcal{G}^2$ (bivector Lie algebra), where the product of $\mathfrak{spin}$ is the commutator in $\mathcal{G}$. Namely,

$$\mathfrak{so}(V,Q) \cong \mathfrak{spin}(V,Q) = \mathcal{G}^2(V,Q), \qquad \pm\exp\big(\mathcal{G}^2(V,Q)\big) \subset \mathrm{Spin}^+(V,Q). \tag{A.45}$$

There is an epimorphism with kernel $\mathbb{R}^\times$

$$\widetilde{\mathrm{Ad}}: \Gamma(V,Q) \to \mathrm{O}(V,Q), \quad g \to (v \mapsto g^\star v g^{-1}), \qquad (\therefore \quad \Gamma(V,Q)/\mathbb{R}^\times \cong \mathrm{O}(V,Q)). \tag{A.46}$$

Here, $v \in V$, $g \in \Gamma$, and it is easily shown that $g^\star v g^{-1} \in V$, which concludes $v \mapsto g^\star v g^{-1}$ is a linear bijection $V \to V$.

Focusing on $\mathcal{G}(\mathbb{R}^{s,t})$, we have epimorphisms with kernel ± 1:

$$\mathrm{Pin}(s,t) \to \mathrm{O}(s,t), \qquad \mathrm{Spin}(s,t) \to \mathrm{SO}(s,t), \qquad \mathrm{Spin}^+(s,t) \to \mathrm{SO}^+(s,t), \tag{A.47}$$

Furthermore, if $s \geq 1$ and $t \geq 1$, Spin^+ is of importance because, with any $e^\pm \in V$ s.t. $e^\pm e^\pm = \pm 1$ and $e^+e^- + e^-e^+ = 0$,

$$\mathrm{Pin}(s,t) = \mathrm{Spin}^+(s,t) \cdot \{1, e^+, e^-, e^+e^-\}, \qquad \mathrm{Spin}(s,t) = \mathrm{Spin}^+(s,t) \cdot \{1, e^+e^-\}. \tag{A.48}$$

It is also known that

$$\mathrm{Spin}^+(s,t) \text{ is simply connected for } (s,t) = (1 \text{ or } 0, n) \text{ and } (n, 1 \text{ or } 0) \text{ with } n \geq 3, \tag{A.49}$$
$$\text{if } s \leq 1 \text{ or } t \leq 1, \ \mathrm{Spin}^+(s,t) = \pm\exp\big(\mathcal{G}^2(\mathbb{R}^{s,t})\big) \tag{A.50}$$

with the minus part requires only for $(0,0)$, $(1, 0\text{–}3)$, and $(0\text{–}3, 1)$.

A.5.2 Representations

Primer As given in (3.20), a Lorentz transformation is an element of $\mathrm{O}(1,3)$ and the proper orthochronous part L_0 is isomorphic to $\mathrm{SO}^+(1,3)$, which is doubly covered by $\mathrm{Spin}^+(1,3) = \pm\exp(\mathcal{G}^2(\mathbb{R}^{1,3}))$. The corresponding geometric algebra $\mathcal{G}(\mathbb{R}^{1,3})$ can be embedded in $\mathcal{G}(\mathbb{C}^4) \cong \mathbb{C}^{4\times 4}$ and thus has a representation by complex 4×4 matrices:

$$L_0 \cong \mathrm{SO}^+(1,3) \cong \mathrm{Spin}^+(1,3)/\mathbb{Z}_2 \subset L \cong \mathrm{O}(1,3) \cong \mathrm{Pin}(1,3)/\mathbb{Z}_2 \subset \Gamma(\mathbb{R}^{1,3}) \subset \mathcal{G}^\times(\mathbb{R}^{1,3}) \qquad \text{(as groups)},$$
$$\mathcal{G}(\mathbb{R}^{1,3}) \cong \mathcal{G}(\mathbb{R}^4) \cong \mathcal{G}(\mathbb{R}^{0,2}) \otimes \mathcal{G}(\mathbb{R}^{2,0}) \cong \mathbb{R}^{2\times 2} \otimes \mathbb{H}, \qquad \mathcal{G}(\mathbb{R}^{1,3}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathcal{G}(\mathbb{C}^4) \cong \mathbb{C}^{4\times 4},$$
$$\mathcal{G}(\mathbb{C}^4) \cong \mathcal{G}(\mathbb{R}^{1,4}) \quad \text{if } (e_0e_1e_2e_3e_4)^2 = -1 \qquad \text{(as algebras)}.$$

The above equations are slightly modified for $(-{+}{+}{+})$-metric. In particular, $\mathcal{G}(\mathbb{R}^{3,1}) \cong \mathcal{G}(\mathbb{R}^{2,0}) \otimes \mathcal{G}(\mathbb{R}^{1,1}) \cong \mathbb{R}^{4\times 4}$ and $\mathcal{G}(\mathbb{R}^{3,1}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{4\times 4}$.

The isomorphism $\mathcal{G}(\mathbb{R}^{1,3}) \cong \mathbb{R}^{2\times 2} \otimes \mathbb{H}$ motivates the representation

$$g^\mu \mathrel{\hat{=}} \left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{j} \\ \mathrm{j} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{k} \\ \mathrm{k} & 0 \end{pmatrix}\right\}. \tag{A.51}$$

Here and hereafter, $\hat{=}$ denotes that the algebraic object in the left-hand side is represented by a matrix in the right-hand side. The basis of $\mathcal{G}(\mathbb{R}^{1,3})$ is represented by, with representing the elementary quaternions i, j, k by q,

$$1 \mathrel{\hat{=}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ g^\mu \mathrel{\hat{=}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}, \ \tfrac{1}{2}[g^0, g^i] \mathrel{\hat{=}} \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}, \ \tfrac{1}{2}[g^{i+1}, g^{i+2}] \mathrel{\hat{=}} \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, \ g^\mu I \mathrel{\hat{=}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \ I \mathrel{\hat{=}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which explicitly shows that they span $\mathbb{R}^{2\times 2} \otimes \mathbb{H}$. However, this representation has little connection to the popular ones. Although we get 4×4 matrices $g^\mu = \left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -\mathrm{i}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\right\}$ by replacing quaternions by Pauli matrices $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\} \mapsto -\mathrm{i}\{\sigma_1, \sigma_2, \sigma_3\}$, we need to perform a basis change, $\hat{\gamma}^\mu = \hat{R}^{-1} g^\mu \hat{R}$ with $\hat{R} = \mathrm{diag}(1, 1, \mathrm{i}, \mathrm{i})$, to reach the Dirac representation, but this is not allowed within $V = \mathbb{R}$.

Dirac and Chiral representations We instead use the isomorphism $\mathcal{G}(\mathbb{R}^{2,3}) \cong \mathcal{G}(\mathbb{C}^4)$. By defining $g^5 := \mathrm{i}g^0g^1g^2g^3$,

$$\{g^0, g^1, g^2, g^3, g^5\} \mathrel{\hat=} \left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, -\mathrm{i}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\right\}$$

forms a basis of $\mathcal{G}(\mathbb{R}^{2,3})$ and thus the 32 combinations among them form a basis of $\mathbb{C}^{4\times4}$. Then, we are allowed to use the above $\hat{R} = \mathrm{diag}(1, 1, \mathrm{i}, \mathrm{i})$ to get the Dirac representation and the Chiral representation:

$$\hat\gamma^0 = \hat{R}^{-1}g^0\hat{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat\gamma^i = \hat{R}^{-1}g^i\hat{R} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \hat\gamma^5 = \hat{R}^{-1}g^5\hat{R} = \mathrm{i}\hat\gamma^0\hat\gamma^1\hat\gamma^2\hat\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{A.52}$$

$$\gamma^0 = R^{-1}g^0R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = R^{-1}g^iR = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = R^{-1}g^5R = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad R := \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}. \tag{A.53}$$

The pseudoscalar is $I = -\mathrm{i}$ in both representations. (Notice that, for $(-{+}{+}{+})$-metric, we should use $\mathcal{G}(\mathbb{R}^{4,1}) \cong \mathcal{G}(\mathbb{C}^4)$ by defining $g^5 := g^0g^1g^2g^3$, where $I = -\mathrm{i}$; this choice is not included in the following discussion.)

In Chiral or Dirac representation, the conjugate operations are given by, with $A^{\mathrm{T}*}$ being the conjugate-transpose of A,

$$\text{Grade involution:} \quad -\gamma^2A^*\gamma^2, \qquad \text{Reversion} \quad \gamma^1\gamma^3A^{\mathrm{T}}\gamma^3\gamma^1, \qquad \text{Clifford conjugate} \quad \gamma^5\gamma^0A^{\mathrm{T}*}\gamma^0\gamma^5. \tag{A.54}$$

Note that γ^a and $\hat\gamma^a$ are all unitary (and traceless) and the 32 basis matrices are all unitary.

Let us give the explicit expressions for the spaces. The vector space V is given by

$$V = \{v_0\gamma^0 + \cdots + v_4\gamma^5 \mid v_i \in \mathbb{R}\}, \tag{A.55}$$

where all the elements are traceless. The inner product and norm are given by

$$B(v, w) = (vw + wv)/2 = (v_0w_0 - v_1w_1 - v_2w_2 - v_3w_3 + v_4w_4)E, \quad Q(v) = vv = (v_0^2 - v_1^2 - v_2^2 - v_3^2 + v_4^2)E, \tag{A.56}$$

where E is the 4×4 identity matrix. Since $v^\dagger = v$, $v^\star = -v$, and $vv^\dagger = Q(v)$, and $\det v = Q(v)^2$ for $v \in V$,

$$V^\times = \{v \in V \mid \det v > 0\}, \quad v^{-1} = v/Q(v). \tag{A.57}$$

The versor group Γ is characterized by the invertible matrices $v_1, \dots, v_n$, and thus $5n$ real numbers $v_{10}, \dots, v_{n4}$:

$$\Gamma(2,3) = \{v_1v_2\cdots v_n \mid v_i \in V, \det v_i > 0\}. \tag{A.58}$$

For $x = v_1\cdots v_n \in \Gamma$, $xx^\dagger = v_1\cdots v_nv_n^\dagger\cdots v_1^\dagger = \prod Q(v_i)$. Therefore,

$$\begin{aligned} &\mathrm{Pin}(2,3) = \{v_1v_2\cdots v_n \mid v_i \in V, \det v_i = 1\}, \\ &\mathrm{Spin}(2,3) = \{v_1v_2\cdots v_{2n} \mid v_i \in V, \det v_i = 1\}, \qquad \mathrm{Spin}^+(2,3) = \{v_1v_2\cdots v_{2n} \mid v_i \in V, Q(v_i) = 1\}. \end{aligned} \tag{A.59}$$

The epimorphism $\widetilde{\mathrm{Ad}}$, which induces the isomorphism $\Gamma(2,3)/\mathbb{R}^\times \cong \mathrm{Pin}(2,3)/\mathbb{Z}_2 \cong \mathrm{O}(2,3)$, is described as

$$x \in \Gamma, \quad x \mathrel{\hat=} x_1\cdots x_n, \quad x_i \in V,$$

$$v \in V, \quad v \mathrel{\hat=} v_0\gamma^0 + \cdots + v_4\gamma^5, \quad v_a \in \mathbb{R}, \qquad \widetilde{\mathrm{Ad}}_x(v) = x^\star vx^{-1} \in V$$

and it preserves the norm: $Q(\widetilde{\mathrm{Ad}}_x(v)) = \widetilde{\mathrm{Ad}}_x(v)\,\widetilde{\mathrm{Ad}}_x(v) = Q(v) = vv$. If $s \in \mathrm{Pin}(2,3)$,

$$\widetilde{\mathrm{Ad}}_s(v) = s^\star vs^{-1} = (-1)^ns_1\cdots s_n\,v\,s_n\cdots s_1.$$

Lorentz Group Consider $\widetilde{\mathrm{Ad}}_w(v)$ with $w = w_a\gamma^a$ limited to V. If $w_5 = 0$, $\widetilde{\mathrm{Ad}}_w$ does not modify the fifth component. So, we can restrict $\Gamma(2,3)$ to $\Gamma(1,3)$ and so on:

$$V_4 = \{v_0\gamma^0 + \cdots + v_3\gamma^3 \mid v_i \in \mathbb{R}\}; \tag{A.60}$$

$$\text{for } x \in \{x_1x_2\cdots x_n \mid x_i \in V_4, \det x_i = 1\}, \qquad \widetilde{\mathrm{Ad}}_x \text{ corresponds to an element in } \mathrm{O}(1,3) \quad \text{(double cover)}. \tag{A.61}$$

Note that, for $v \in V_4$,

$$v^{-1} = v/Q(v), \quad v^* = -\gamma^2v^\star\gamma^2 = \gamma^2A\gamma^2, \quad v^{\mathrm{T}} = \gamma^1\gamma^3v\gamma^3\gamma^1, \quad v^{\mathrm{T}*} = \gamma^0v\gamma^0.$$

The group $\mathrm{Spin}^+(1,3)$ is simply connected and isomorphic to $\pm\exp(\mathcal{G}^2(\mathbb{R}^{1,3}))$, where

$$\mathcal{G}^2(\mathbb{R}^{1,3}) = \mathrm{span}\left[\gamma^0\gamma^i \mathrel{\hat=} \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad \gamma^i\gamma^j \mathrel{\hat=} \epsilon^{ijk}\begin{pmatrix} -\mathrm{i}\sigma^k & 0 \\ 0 & -\mathrm{i}\sigma^k \end{pmatrix}.\right], \tag{A.62}$$

and, in this representation, the isomorphism ♠TODO:???♠ is explicit.

Spinor In Chiral representation, a matrix $v = v_0\gamma^0 + \cdots + v_3\gamma^3$ in V_4 can be expressed by

$$v = \psi\psi^{\mathrm{T}*}\gamma^0; \quad \psi := \frac{1}{\sqrt{2(v_0 - m)}}\left[(v_0 - m)\gamma^0 + v_1\gamma^1 + v_2\gamma^2 + v_3\gamma^3\right], \quad m := \sqrt{v_0^2 - v_1^2 - v_2^2 - v_3^2}.$$

Thus, the operation $\widetilde{\mathrm{Ad}}_s(v)$ is written by

$$\widetilde{\mathrm{Ad}}_s(v) = s^\star vs^{-1} = (-1)^ns_1\cdots s_n\psi\psi^{\mathrm{T}*}\gamma^0s_n\cdots s_1 = (-1)^n(s_1\cdots s_n\psi)(s_1\cdots s_n\psi)^{\mathrm{T}*}\gamma^0.$$

and each column of ψ is understood as a Dirac spinor. With identifying, as usual, v_μ as the momentum, we have

$$\psi = \frac{1}{\sqrt{2\tilde{E}}}\begin{pmatrix} 0 & 0 & \tilde{E} + p_z & \rho^* \\ 0 & 0 & \rho & \tilde{E} - p_z \\ \tilde{E} - p_z & -\rho^* & 0 & 0 \\ -\rho & \tilde{E} + p_z & 0 & 0 \end{pmatrix}; \qquad \rho := p_x + \mathrm{i}p_y \in \mathbb{C}, \quad \tilde{E} := E - m \in \mathbb{R}.$$

A.5.3 Fragments

For Minkowski spacetime Also, for our interest, there are isomorphisms [28, 29]

$$\mathcal{G}(\mathbb{R}^{4,1}) \cong \mathcal{G}(\mathbb{R}^{1,3})\otimes\mathbb{C} \cong \mathcal{G}(\mathbb{C}^4) \cong \mathbb{C}^{4\times4}, \qquad \mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Sp}(2,\mathbb{C}), \qquad L_0 \cong \mathrm{PSL}(2;\mathbb{C}) = \mathrm{SL}(2;\mathbb{C})/\mathbb{Z}_2. \tag{A.63}$$

Meanwhile, the Lorentz algebra $\mathfrak{so}(1,3)$ is isomorphic to $\mathfrak{sl}(2;\mathbb{C})$ viewed as a real Lie albegra [24, §7.8], and its complexification $\mathfrak{so}(1,3)_{\mathbb{C}}$ is isomorphic to $\mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$.

♠TODO:SO algebra and Grade-2 Clifford Algebra♠

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