In[1]:=

```
(* Time-Stamp: <2020-8-8 15:27:19> *)
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   You may not copy, distribute, or modify it, or create a derived work withou
```

In[2]:=

```
$Assumptions = {T>0, e≥0, m≥0, μ∈Reals};
kUp[exp_, max_] := (exp //. { BesselK[n_Integer /; n < max-1, z_] :→ BesselK[n
kDown[exp_]     := (exp //. { BesselK[n_Integer /; n ≥ 2, z_] :→ BesselK[n - 2,

fBE[e_, μ_, T_] := 1 / (Exp[(e-μ)/T] - 1)
fFD[e_, μ_, T_] := 1 / (Exp[(e-μ)/T] + 1)
fMB[e_, μ_, T_] := 1 / (Exp[(e-μ)/T] + 0)
```

In[8]:=

```
n[f_, m_, μ_, T_] := Integrate[(4Pi)Sqrt[e^2-m^2]e    f[e, μ, T] / (2Pi)^3, {e,
ρ[f_, m_, μ_, T_] := Integrate[(4Pi)Sqrt[e^2-m^2]e^2  f[e, μ, T] / (2Pi)^3, {e,
p[f_, m_, μ_, T_] := Integrate[(4Pi/3)Sqrt[e^2-m^2]^3 f[e, μ, T] / (2Pi)^3, {e,
Δint[f_, m_, μ_, T_, e_] :=       (4Pi)Sqrt[e^2-m^2]e    (f[e, μ, T]-f[e, -μ, T
Δ[f_, m_, μ_, T_] := Integrate[Δint[f,m,μ,T, e], {e,m,∞}]
```

In[13]:=

```
nMB = n[fMB, m, μ, T] /. {m→x T, μ→ ν T} // FullSimplify
ρMB = ρ[fMB, m, μ, T] /. {m→x T, μ→ ν T} // kUp[#, 2]& // Simplify
pMB = p[fMB, m, μ, T] /. {m→x T, μ→ ν T} // FullSimplify
```

Out[13]=

$$\frac{e^{\nu}\, T^3\, x^2\, \mathrm{BesselK}[2,\, x]}{2\, \pi^2} \quad \text{if} \quad x > 0$$

Out[14]=

$$\frac{e^{\nu}\, T^4\, x^2\, (x\, \mathrm{BesselK}[1,\, x] + 3\, \mathrm{BesselK}[2,\, x])}{2\, \pi^2} \quad \text{if} \quad x > 0$$

Out[15]=

$$\frac{e^{\nu}\, T^4\, x^2\, \mathrm{BesselK}[2,\, x]}{2\, \pi^2} \quad \text{if} \quad x > 0$$

In[16]:=

```
n[fFD, m, 0, 1]
ρ[fFD, m, 0, 1]
p[fFD, m, 0, 1]
n[fBE, m, 0, 1]
ρ[fBE, m, 0, 1]
p[fBE, m, 0, 1]
```

Out[16]=

$$\int_m^\infty \frac{e\sqrt{e^2-m^2}}{2\,(1+\mathbb{e}^e)\,\pi^2}\,\mathbb{d}e$$

Out[17]=

$$\int_m^\infty \frac{e^2\sqrt{e^2-m^2}}{2\,(1+\mathbb{e}^e)\,\pi^2}\,\mathbb{d}e$$

Out[18]=

$$\int_m^\infty \frac{\left(e^2-m^2\right)^{3/2}}{6\,(1+\mathbb{e}^e)\,\pi^2}\,\mathbb{d}e$$

Out[19]=

$$\int_m^\infty \frac{e\sqrt{e^2-m^2}}{2\,(-1+\mathbb{e}^e)\,\pi^2}\,\mathbb{d}e$$

Out[20]=

$$\int_m^\infty \frac{e^2\sqrt{e^2-m^2}}{2\,(-1+\mathbb{e}^e)\,\pi^2}\,\mathbb{d}e$$

Out[21]=

$$\int_m^\infty \frac{\left(e^2-m^2\right)^{3/2}}{6\,(-1+\mathbb{e}^e)\,\pi^2}\,\mathbb{d}e$$

In[22]:=

```
nFD = n[fFD, m, μ, T]
ρFD = ρ[fFD, m, μ, T]
pFD = p[fFD, m, μ, T]
nBE = n[fBE, m, μ, T]
ρBE = ρ[fBE, m, μ, T]
pBE = p[fBE, m, μ, T]
```

Out[22]=

$$\int_m^\infty \frac{e\sqrt{e^2-m^2}}{2\left(1+e^{\frac{e-\mu}{T}}\right)\pi^2}\,de$$

Out[23]=

$$\int_m^\infty \frac{e^2\sqrt{e^2-m^2}}{2\left(1+e^{\frac{e-\mu}{T}}\right)\pi^2}\,de$$

Out[24]=

$$\int_m^\infty \frac{\left(e^2-m^2\right)^{3/2}}{6\left(1+e^{\frac{e-\mu}{T}}\right)\pi^2}\,de$$

Out[25]=

$$\int_m^\infty \frac{e\sqrt{e^2-m^2}}{2\left(-1+e^{\frac{e-\mu}{T}}\right)\pi^2}\,de$$

Out[26]=

$$\int_m^\infty \frac{e^2\sqrt{e^2-m^2}}{2\left(-1+e^{\frac{e-\mu}{T}}\right)\pi^2}\,de$$

Out[27]=

$$\int_m^\infty \frac{\left(e^2-m^2\right)^{3/2}}{6\left(-1+e^{\frac{e-\mu}{T}}\right)\pi^2}\,de$$

In[28]:=

```
{nFD, ρFD, pFD} /. Integrate→Hold[Integrate] //. {e → e0+m, {e0+m, m, ∞} → {e0
{nFD0, ρFD0, pFD0} = ReleaseHold[%] // Collect[#, m, FullSimplify]&
Δ[fFD, m, μ, T] /. Integrate→Hold[Integrate] //. {e → e0+m, {e0+m, m, ∞} → {e0
ReleaseHold[%]
```

Out[28]=

$$\left\{\text{Hold}[\text{Integrate}]\left[\frac{1}{4\left(e^{e0}+e^{\mu}\right)^3\pi^2}\left(e^{3\mu}\left(2\,e0^2+4\,e0\,m+m^2\right)+e^{2\,e0+\mu}\left(-4\,e0\,(-1+m)\,m+m^2+e0^2\,(2+(-2+m)\,m)\right)+e^{e0+2\mu}\left(-4\,e0\,(-2+m)\,m+2\,m^2-e0^2\,(-4+m\,(2+m))\right)\right),\{e0,0,\infty\}\right],\right.$$

$$\text{Hold}[\text{Integrate}]\left[\frac{1}{4\left(e^{e0}+e^{\mu}\right)^3\pi^2}e^{\mu}\,e0\left(e^{2\mu}\left(2\,e0^2+6\,e0\,m+5\,m^2\right)+e^{2\,e0}\left(-6\,e0\,(-1+m)\,m+5\,m^2+e0^2\,(2+(-2+m)\,m)\right)+e^{e0+\mu}\left(-6\,e0\,(-2+m)\,m+10\,m^2-e0^2\,(-4+m\,(2+m))\right)\right),\{e0,0,\infty\}\right],$$

$$\left.\text{Hold}[\text{Integrate}]\left[\frac{1}{12\left(e^{e0}+e^{\mu}\right)^3\pi^2}e^{\mu}\,e0\left(e^{2\mu}\left(2\,e0^2+6\,e0\,m+3\,m^2\right)+e^{2\,e0}\left(-6\,e0\,(-1+m)\,m+3\,m^2+e0^2\,(2+(-2+m)\,m)\right)+e^{e0+\mu}\left(-6\,e0\,(-2+m)\,m+6\,m^2-e0^2\,(-4+m\,(2+m))\right)\right),\{e0,0,\infty\}\right]\right\}$$

Out[29]=

$$\left\{-\frac{m^2\,\text{Log}[1+e^{\mu}]}{4\pi^2}+\frac{m\left(\pi^2+6\,\mu\,\text{Log}[1+e^{\mu}]-3\,\text{Log}[1+e^{\mu}]^2+6\,\text{PolyLog}[2,-e^{\mu}]-6\,\text{PolyLog}\left[2,\frac{1}{1+e^{\mu}}\right]\right)}{6\pi^2}-\frac{\text{PolyLog}[3,-e^{\mu}]}{\pi^2},\right.$$

$$\frac{m^2\left(36\,\text{PolyLog}[2,-e^{\mu}]+5\left(\pi^2+6\,\mu\,\text{Log}[1+e^{\mu}]-3\,\text{Log}[1+e^{\mu}]^2-6\,\text{PolyLog}\left[2,\frac{1}{1+e^{\mu}}\right]\right)\right)}{24\pi^2}-\frac{3\,\text{PolyLog}[4,-e^{\mu}]}{\pi^2},$$

$$\left.\frac{m^2\left(\pi^2+6\,\mu\,\text{Log}[1+e^{\mu}]-3\,\text{Log}[1+e^{\mu}]^2+12\,\text{PolyLog}[2,-e^{\mu}]-6\,\text{PolyLog}\left[2,\frac{1}{1+e^{\mu}}\right]\right)}{24\pi^2}-\frac{\text{PolyLog}[4,-e^{\mu}]}{\pi^2}\right\}$$

Out[30]=

$$\text{Hold}[\text{Integrate}]\left[\frac{e0^2\,\text{Sinh}[\mu]}{2\pi^2\,(\text{Cosh}[e0]+\text{Cosh}[\mu])}+\frac{e0\,m\,(2\,(\text{Cosh}[e0]+\text{Cosh}[\mu])-e0\,\text{Sinh}[e0])\,\text{Sinh}[\mu]}{2\pi^2\,(\text{Cosh}[e0]+\text{Cosh}[\mu])^2}+\right.$$

$$\left(e^{3\,(e0+\mu)}\,m^2\left(2-3\,e0^2+\left(1+e0^2\right)\text{Cosh}[2\,e0]-2\left(-2+e0^2\right)\text{Cosh}[e0]\,\text{Cosh}[\mu]+\text{Cosh}[2\,\mu]-8\,e0\,(\text{Cosh}[e0]+\text{Cosh}[\mu])\,\text{Sinh}[e0]\right)\text{Sinh}[\mu]\right)\Big/$$

$$\left(\left(e^{e0}+e^{\mu}\right)^3\left(1+e^{e0+\mu}\right)^3\pi^2\right)-\left(e^{4\,(e0+\mu)}\,m^3\left(-6\,e0\,\text{Cosh}[3\,e0]+6\,e0\,\text{Cosh}[e0]\,(7+2\,\text{Cosh}[2\,\mu])+3\left(3-7\,e0^2\right)\text{Sinh}[e0]+2\left(3+e0^2\right)\text{Cosh}[2\,\mu]\,\text{Sinh}[e0]+\right.\right.$$

$$\left.\left.4\,\text{Cosh}[\mu]\left(12\,e0+\left(3-2\,e0^2\right)\text{Sinh}[2\,e0]\right)+\left(3+e0^2\right)\text{Sinh}[3\,e0]\right)\text{Sinh}[\mu]\right)\Big/$$

$$\left.\left(3\left(e^{e0}+e^{\mu}\right)^4\left(1+e^{e0+\mu}\right)^4\pi^2\right),\{e0,0,\infty\}\right]$$

Out[31]=

$$\frac{-3\,m^2\,\mu+2\,\mu\left(\pi^2+\mu^2\right)+m^3\,\text{Tanh}\left[\frac{\mu}{2}\right]}{12\pi^2}$$

In[32]:=
```
{nBE, ρBE, pBE} /. Integrate→Hold[Integrate] //. {e → e0+m, {e0+m, m, ∞} → {e0
ReleaseHold[%]
%/.m→0
```

Out[32]=

$$\left\{\text{Hold}[\text{Integrate}]\left[-\frac{e^{\mu}\, e0\left(e^{e0}\,(e0\,(-1+m)-2\,m)+e^{\mu}\,(e0+2\,m)\right)}{2\left(e^{e0}-e^{\mu}\right)^{2}\pi^{2}},\ \{e0,0,\infty\}\right],\right.$$
$$\text{Hold}[\text{Integrate}]\left[-\frac{e^{\mu}\, e0^{2}\left(e^{e0}\,(e0\,(-1+m)-3\,m)+e^{\mu}\,(e0+3\,m)\right)}{2\left(e^{e0}-e^{\mu}\right)^{2}\pi^{2}},\ \{e0,0,\infty\}\right],$$
$$\left.\text{Hold}[\text{Integrate}]\left[-\frac{e^{\mu}\, e0^{2}\left(e^{e0}\,(e0\,(-1+m)-3\,m)+e^{\mu}\,(e0+3\,m)\right)}{6\left(e^{e0}-e^{\mu}\right)^{2}\pi^{2}},\ \{e0,0,\infty\}\right]\right\}$$

Out[33]=

$$\left\{\boxed{-\frac{1}{6\,\pi^{2}}\left(m\left(\pi^{2}+3\,\text{Log}[1-e^{\mu}]\left(\text{Log}[1-e^{\mu}]+2\left(\mu+\text{Log}\left[\frac{1}{-1+e^{\mu}}\right]\right)\right)+6\,\text{PolyLog}[2,e^{\mu}]-6\,\text{PolyLog}\left[2,\frac{1}{1-e^{\mu}}\right]\right)-6\,\text{PolyLog}[3,e^{\mu}]\right)\ \text{if}\ \mu<0},\right.$$
$$\left.\boxed{\frac{3\,\text{PolyLog}[4,e^{\mu}]}{\pi^{2}}\ \text{if}\ \mu<0},\ \boxed{\frac{\text{PolyLog}[4,e^{\mu}]}{\pi^{2}}\ \text{if}\ \mu<0}\right\}$$

Out[34]=

$$\left\{\boxed{\frac{\text{PolyLog}[3,e^{\mu}]}{\pi^{2}}\ \text{if}\ \mu<0},\ \boxed{\frac{3\,\text{PolyLog}[4,e^{\mu}]}{\pi^{2}}\ \text{if}\ \mu<0},\ \boxed{\frac{\text{PolyLog}[4,e^{\mu}]}{\pi^{2}}\ \text{if}\ \mu<0}\right\}$$

In[35]:=
```
Series[Δint[fFD,m,μ,T,e], {μ,0, 1}] // Normal
Integrate[%, {e, m, ∞}]
Series[Δint[fBE,m,μ,T,e], {μ,0, 1}] // Normal
Integrate[%, {e, m, ∞}]
```

Out[35]=

$$\frac{e\, e^{e/T}\sqrt{e^{2}-m^{2}}\ \mu}{\left(1+e^{e/T}\right)^{2}\pi^{2}\,T}$$

Out[36]=

$$\int_{m}^{\infty}\frac{e\, e^{e/T}\sqrt{e^{2}-m^{2}}\ \mu}{\left(1+e^{e/T}\right)^{2}\pi^{2}\,T}\,de$$

Out[37]=

$$\frac{e\, e^{e/T}\sqrt{e^{2}-m^{2}}\ \mu}{\left(-1+e^{e/T}\right)^{2}\pi^{2}\,T}$$

Out[38]=

$$\int_{m}^{\infty}\frac{e\, e^{e/T}\sqrt{e^{2}-m^{2}}\ \mu}{\left(-1+e^{e/T}\right)^{2}\pi^{2}\,T}\,de$$

In[39]:=
```
Series[Δint[fFD,m,μ,T,e] /. {e→∈0+m}, {μ,0, 1}] // Normal // FullSimplify
List@@(Series[%, {m, 0, 2}] // Normal)
Integrate[%, {∈0, 0, ∞}]
```

Out[39]=

$$\frac{(m+\epsilon 0)\sqrt{\epsilon 0\,(2\,m+\epsilon 0)}\;\mu\,\mathrm{Sech}\left[\frac{m+\epsilon 0}{2\,T}\right]^2}{4\,\pi^2\,T}$$

Out[40]=

$$\left\{\frac{\epsilon 0\sqrt{\epsilon 0^2}\;\mu\,\mathrm{Sech}\left[\frac{\epsilon 0}{2\,T}\right]^2}{4\,\pi^2\,T},\ \frac{m\,\mu\left(2\,T\sqrt{\epsilon 0^2}\;\mathrm{Sech}\left[\frac{\epsilon 0}{2\,T}\right]^2-\epsilon 0\sqrt{\epsilon 0^2}\;\mathrm{Sech}\left[\frac{\epsilon 0}{2\,T}\right]^2\mathrm{Tanh}\left[\frac{\epsilon 0}{2\,T}\right]\right)}{4\,\pi^2\,T^2},\right.$$
$$\frac{1}{16\,\pi^2\,T^3\sqrt{\epsilon 0^2}}m^2\,\epsilon 0\,\mu\left(2\,T^2\,\mathrm{Sech}\left[\frac{\epsilon 0}{2\,T}\right]^2-\epsilon 0^2\,\mathrm{Sech}\left[\frac{\epsilon 0}{2\,T}\right]^2-\right.$$
$$\left.\left.8\,T\,\epsilon 0\,\mathrm{Sech}\left[\frac{\epsilon 0}{2\,T}\right]^2\mathrm{Tanh}\left[\frac{\epsilon 0}{2\,T}\right]+3\,\epsilon 0^2\,\mathrm{Sech}\left[\frac{\epsilon 0}{2\,T}\right]^2\mathrm{Tanh}\left[\frac{\epsilon 0}{2\,T}\right]^2\right)\right\}$$

Out[41]=

$$\left\{\frac{T^2\,\mu}{6},\ 0,\ -\frac{m^2\,\mu}{4\,\pi^2}\right\}$$

In[42]:=
```
Series[Δint[fBE,m,μ,T,e] /. {e→∈0+m}, {μ,0, 1}] // Normal // FullSimplify
List@@(Series[%, {m, 0, 2}] // Normal)
Integrate[%, {∈0, 0, ∞}]
```

Out[42]=

$$\frac{(m+\epsilon 0)\sqrt{\epsilon 0\,(2\,m+\epsilon 0)}\;\mu\,\mathrm{Csch}\left[\frac{m+\epsilon 0}{2\,T}\right]^2}{4\,\pi^2\,T}$$

Out[43]=

$$\left\{\frac{\epsilon 0\sqrt{\epsilon 0^2}\;\mu\,\mathrm{Csch}\left[\frac{\epsilon 0}{2\,T}\right]^2}{4\,\pi^2\,T},\ \frac{m\,\mu\left(2\,T\sqrt{\epsilon 0^2}\;\mathrm{Csch}\left[\frac{\epsilon 0}{2\,T}\right]^2-\epsilon 0\sqrt{\epsilon 0^2}\;\mathrm{Coth}\left[\frac{\epsilon 0}{2\,T}\right]\mathrm{Csch}\left[\frac{\epsilon 0}{2\,T}\right]^2\right)}{4\,\pi^2\,T^2},\right.$$
$$\frac{1}{16\,\pi^2\,T^3\sqrt{\epsilon 0^2}}m^2\,\epsilon 0\,\mu\left(2\,T^2\,\mathrm{Csch}\left[\frac{\epsilon 0}{2\,T}\right]^2-\epsilon 0^2\,\mathrm{Csch}\left[\frac{\epsilon 0}{2\,T}\right]^2-\right.$$
$$\left.\left.8\,T\,\epsilon 0\,\mathrm{Coth}\left[\frac{\epsilon 0}{2\,T}\right]\mathrm{Csch}\left[\frac{\epsilon 0}{2\,T}\right]^2+3\,\epsilon 0^2\,\mathrm{Coth}\left[\frac{\epsilon 0}{2\,T}\right]^2\mathrm{Csch}\left[\frac{\epsilon 0}{2\,T}\right]^2\right)\right\}$$

Integrate::idiv :

$$\frac{m^2\,\mu\,\mathrm{Csch}\left[\frac{\epsilon 0}{2\,T}\right]^4\left(-T^2+2\,\epsilon 0^2+\left(T^2+\epsilon 0^2\right)\mathrm{Cosh}\left[\frac{\epsilon 0}{T}\right]-4\,T\,\epsilon 0\,\mathrm{Sinh}\left[\frac{\epsilon 0}{T}\right]\right)}{16\,\pi^2\,T^3}$$ の積分は{0, ∞}で収束しません。

Out[44]=

$$\left\{\frac{T^2\,\mu}{3},\ -\frac{m\,T\,\mu}{\pi^2},\right.$$
$$\int_0^\infty\frac{1}{16\,\pi^2\,T^3\sqrt{\epsilon 0^2}}m^2\,\epsilon 0\,\mu\left(2\,T^2\,\mathrm{Csch}\left[\frac{\epsilon 0}{2\,T}\right]^2-\epsilon 0^2\,\mathrm{Csch}\left[\frac{\epsilon 0}{2\,T}\right]^2-8\,T\,\epsilon 0\,\mathrm{Coth}\left[\frac{\epsilon 0}{2\,T}\right]\right.$$
$$\left.\left.\mathrm{Csch}\left[\frac{\epsilon 0}{2\,T}\right]^2+3\,\epsilon 0^2\,\mathrm{Coth}\left[\frac{\epsilon 0}{2\,T}\right]^2\mathrm{Csch}\left[\frac{\epsilon 0}{2\,T}\right]^2\right)d\epsilon 0\right\}$$

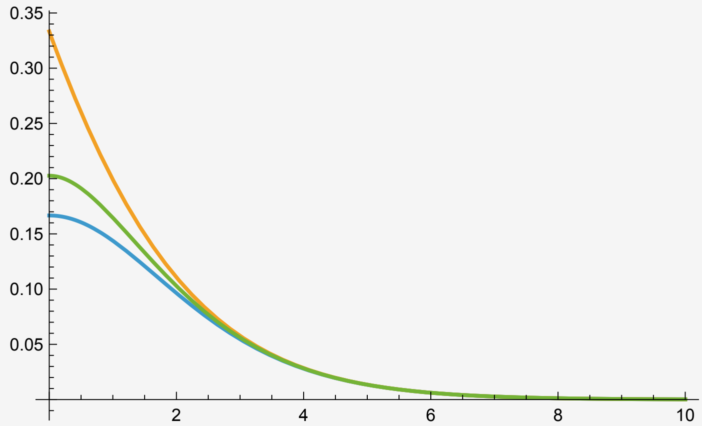
In[45]:=

```
ξ[sign:1|-1, x_] := NIntegrate[(k e^k Sqrt[k^2-x^2])/((e^k+sign)^2 π^2), {k,x,∞}]
ξMB = Integrate[(k e^k Sqrt[k^2-x^2])/((e^k+0)^2 π^2), {k,x,∞}]
Plot[{ξ[1,x], ξ[-1,x], ξMB}, {x, 0, 10}]
Plot[{ξ[1,x], (1/6 - x^2/(4Pi^2))},{x,0,4}]
Plot[{ξ[-1,x], (1/3 - x/(Pi^2))},{x,0,4}]
```

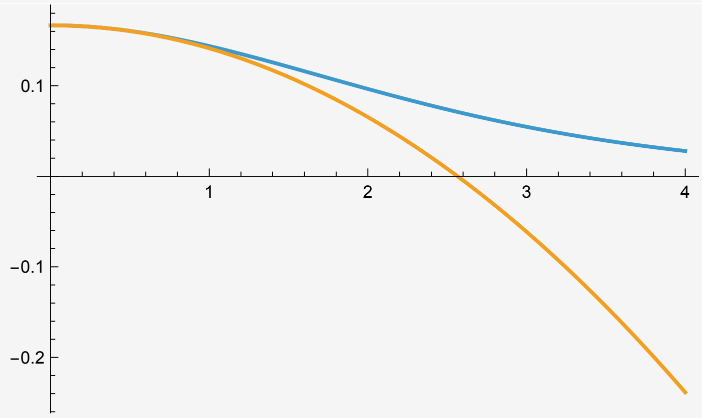
Out[46]=

$$\frac{x^2\,\mathrm{BesselK}[2, x]}{\pi^2} \quad \text{if } \mathrm{Re}[x] > 0 \;\&\&\; \mathrm{Im}[x] == 0$$

Out[47]=



Out[48]=



Out[49]=