Def of Closed

① X is closed if $\mathbb{R}-X$ is open

② X is closed if it contains all of its limit point

Def of limit point

$y \in \mathbb{R}$ is a limit point of X if for any $(y-\delta, y+\delta)$,
we have $(y-\delta, y+\delta)$ must contain points of X other than y

Def of Function Limit (6.8)

$f: A \to \mathbb{R}$, c a limit point of A

$\lim_{x\to c} f(x) = L$ if for $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

for every $x \in A$, $0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\lim_{x\to c} f(x)$ converges to L. $x \neq c$

Prop 6.11 Functional Limits are unique

$\lim_{x\to c} f(x)$ can converge to at most one value

$\lim_{x\to c} f(x) = L_1$, $\lim_{x\to c} f(x) = L_2 \Rightarrow L_1 = L_2$

Thm 6.12

$\lim_{x\to c} f(x) = L \iff$ every such $f(a_n) = L$ $a_n \neq c$, $(a_n) \to c$

Proposition: If $a_n \to c$, $b_n \to c$

$\Rightarrow$ the seq. $a_1\ b_1\ a_2\ b_2\ a_3\ b_3 \cdots$ also converges to c.

Cor 6.14 Functional Limit Laws

$f, g : X \to \mathbb{R}$, c a limit point of X

Assume $\lim_{x \to c} f(x) = L$ and $\lim g(x) = M$

① $\lim_{x \to c} f(x) + g(x) = L + M$

② $\lim_{x \to c} f(x) - g(x) = L - M$

③ $\lim_{x \to c} f(x) \cdot g(x) = L \cdot M$

④ $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ (as long as $M \neq 0$, $g(x) \neq 0$)

⑤ For any $\lambda \in \mathbb{R}$, $\lim_{x \to c} \lambda f(x) = \lambda L$

Cor 6.15 Functional Squeeze Thm

$f, g, h : X \to \mathbb{R}$, c a limit point of X

Suppose that there is a $\delta > 0$ s.t. for any $x \in (c-\delta, c+\delta) - \{c\}$

we have $f(x) \leq g(x) \leq h(x)$

If $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$

Def 6.16 Continuity

$f : X \to \mathbb{R}$ and $c \in X$, $f(x)$ is <u>continuous</u> at $c \in X$ if.

for any $\varepsilon > 0$ there exists a $\delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

$x \in (c-\delta, c+\delta) \Rightarrow f(x) \in (f(c) - \varepsilon, f(c) + \varepsilon)$

Thm 6.17 Continuity topologically & sequentially

$f: A \to \mathbb{R}$, $c \in A$

① f is cont. at c

② $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for $\forall x \in A$ where $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

③ $V_\varepsilon(f(c))$, $\exists V_\delta(c)$, for any $x \in A$, $x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(c))$

④ For all seq $(a_n) \subseteq A$. $(a_n) \to c \Rightarrow f(a_n) \to f(c)$

⑤ $\lim_{x \to c} f(x) = f(c)$ if c is a limit point of A.

④: If $a_n \to c$ while $f(a_n) \not\to f(c) \Rightarrow f$ is NOT cont. at c

Prop 6.20 Continuity Limit Laws

$f, g : A \to \mathbb{R}$ continuous at $c \in A$, Then

① $k \cdot f(x)$ cont. at c for $\forall k \in \mathbb{R}$

② $f(x) + g(x)$ cont. at c

③ $f(x) \cdot g(x)$ cont at c

④ $f(x) / g(x)$ cont at c for $\forall x \in A$ $(g(x) \neq 0)$

Prop 6.24 f, g continuous $\Rightarrow f \circ g$ continuous

$A, B \subseteq \mathbb{R}$, $f: A \to B$, $g: B \to \mathbb{R}$,

$$\left.\begin{array}{l} g \text{ cont. at } c \in A \\ f \text{ cont. at } g(c) \in B \end{array}\right\} \Rightarrow f \circ g \text{ continuous at } c.$$

Def 6.26 Topological Continuity

$X, Y \subseteq \mathbb{R}$, $f: X \to Y$,

for $B \subseteq \mathbb{R}$, $f^{-1}(B) = \{x \in X : f(x) \in B\}$

Thm 6.29 Continuity & Open

Fix $X \subseteq \mathbb{R}$, $U \subseteq X$ is open in X if

there exists an open set $V \subseteq \mathbb{R}$ s.t. $U = X \cap V$

Recall: $V \subseteq \mathbb{R}$ is open if, for any $x \in V$,

we can find an $\varepsilon > 0$ s.t. $(x - \varepsilon, x + \varepsilon) \subseteq V$

Thm 6.30 Continuity & Compact

$f: X \to \mathbb{R}$ cont., $A \subseteq X$ compact $\Rightarrow f(A)$ compact

Def of Compact: $X \subseteq \mathbb{R}$ is compact if in any open cover $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of X

we can find finitely many $U_{\lambda_1} \cup U_{\lambda_2} \cup \cdots \cup U_{\lambda_p} \in \mathcal{U}$

which still cover X

Def of Open Cover: A collection of sets $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of $X \subseteq \mathbb{R}$ if

(i) Each $U_\lambda \in \mathcal{U}$ is open

(ii) $X \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$

Thm 6.32 The Extreme Value Thm

Let $f: X \to \mathbb{R}$ be continuous

If X is compact, we can find $x_0, x_1 \in X$ s.t.

for any $x \in X$, $f(x_0) \leq f(x) \leq f(x_1)$

Lemma 6.36

f cont. & $f(c) > 0$

then $\exists\, \delta > 0$ s.t. $f(x) > 0$ for $\forall x \in (c-\delta, c+\delta)$

Prop 6.37 f cont., $f(a) > 0 > f(b) \Rightarrow \exists c \in (a,b), f(c) = 0$

Thm 6.38 Intermediate Value Thm

f cont. on $[a,b]$ & α any number between $f(a)$ and $f(b)$
then $\exists c \in (a,b)$ for which $f(c) = \alpha$

MATH 317: Practice Exercises for the First Midterm

(1) Review all the concepts and results listed in the indices of Weeks 1, 2, and 3.

(2) Exercise 6.5

(3) Exercise 6.9

(4) Exercise 6.13

(5) Exercise 6.15

(6) Exercise 6.16

(7) Exercise 6.18

(8) Exercise 6.23

(9) Exercise 6.28

(10) Exercise 6.33

(11) Exercise 6.40 (a)

(12) Exercise 6.41

(13) Exercise 6.44

Take any open set $U \subseteq \mathbb{R}$, $C = \mathbb{R} - U$

✓(14) Exercise 6.45 (b)

✓(15) Exercise 6.46

(16) Exercise 6.48

(17) Exercise 6.52

✓(18) Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(U)$ is open for any open set $U \subseteq \mathbb{R}$.

(19) Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions such that $f(x) < g(x)$ for all $x \in [a, b]$. Show that there is a $\lambda > 0$ so that $f(x) + \lambda < g(x)$ for all $x \in [a, b]$.

✓(20) Let $K \subseteq \mathbb{R}$ be compact and let $p \notin K$. Show that there is a point $c \in K$ whose distance to p is minimal.

✓✓(21) Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(C)$ is closed for any closed subset $C \subseteq \mathbb{R}$.

✓(22) Is the function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x\sin(\frac{1}{x})$ uniformly continuous?

✓ (23) If $f : [a, b] \to \mathbb{R}$ is continuous and $|f(x)| = 1$ for all $x \in [a, b]$, prove that f is a constant function.

(~~24~~) Show that $X \subseteq \mathbb{R}$ is disconnected if and only if there exists a continuous function $f : X \to \mathbb{R}$ such that $f(X) = \{0, 1\}$.

(~~25~~) Let $X_1, \ldots\ldots, X_n$ be pairwise disjoint closed and bounded subsets of $\mathbb{R}$. If

not on midterm

$$X = \bigcup_{j=1}^{n} X_j$$

and $f : X \to \mathbb{R}$ is continuous on X_j for $1 \leq j \leq n$, show that f is uniformly continuous on X.

(26) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos(x)$ is uniformly continuous.

(27) Write outlines of the proofs of the following theorems:

- The Extreme Value Theorem.
- The Intermediate Value Theorem.
- *Theorem.* $f : X \to \mathbb{R}$ is continuous $\iff$ for any open set $U \subseteq \mathbb{R}$, the pre-image $f^{-1}(U)$ is open in the domain of X (i.e., $f^{-1}(U) = X \cap V$ for some open set $V \subseteq \mathbb{R}$).

X not on midterm : f cont. bounded $\to$ bounded

Continuity & Compact

$f: X \to \mathbb{R}$ cont., $A \subseteq X$ compact $\Rightarrow f(A)$ compact

<u>Proof</u>: Assume that X is compact

Fix any open cover $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ for $f(X)$

<u>Step 1</u>: By <u>Thm 1</u>, each $f^{-1}(U_\lambda)$ is open in X

Meaning, for each $\lambda \in \Lambda$, $f^{-1}(U) = X \cap V_\lambda$ for some open set $V_\lambda \subseteq \mathbb{R}$

<u>Step 2</u>: $f(X) \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda \Rightarrow X \subseteq f^{-1}(\bigcup_{\lambda \in \Lambda} U_\lambda)$

$\Leftrightarrow X \subseteq \bigcup_{\lambda \in \Lambda} f^{-1}(U_\lambda) = \bigcup_{\lambda \in \Lambda} X \cap V_\lambda \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda$

In particular, $X \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda$

Then, $\{V_\lambda\}_{\lambda \in \Lambda}$ is an open cover for X

By the compactness of X, we can find $V_{\lambda_1}, V_{\lambda_2}, \cdots, V_{\lambda_p}$ in $\{V_\lambda\}_{\lambda \in \Lambda}$

s.t. $X \subseteq V_{\lambda_1}, V_{\lambda_2}, \cdots, V_{\lambda_p}$

<u>Fact</u> For each $\lambda \in \Lambda$, $f(V_\lambda) = U_\lambda$

$\Rightarrow f(X) \subseteq f(V_{\lambda_1} \cup V_{\lambda_2} \cup \cdots \cup V_{\lambda_p})$

$f(X) \subseteq f(V_{\lambda_1}) \cup f(V_{\lambda_2}) \cup \cdots \cup f(V_{\lambda_p})$

$\Leftrightarrow f(X) \subseteq U_{\lambda_1} \cup U_{\lambda_2} \cup \cdots \cup U_{\lambda_p}$

Thus, we have found finitely many sets in $\mathcal{U}$ which still cover $f(X)$

In other words, $f(X)$ is compact.

Extreme Value Thm

Let $f: X \to \mathbb{R}$ be continuous

If X is <u>compact</u>, we can find $x_0, x_1 \in X$ s.t.

for any $x \in X$, $f(x_0) \le f(x) \le f(x_1)$

Proof: Let $f: X \to \mathbb{R}$ be continuous function and let X be compact

By Thm 2. $f(X)$ is compact

Since $f(X)$ is bounded, $\sup f(X)$ and $\inf f(X)$ exists

Since $f(X)$ is closed, $\sup f(X), \inf f(X) \in f(X)$

In particular, there are $x_0, x_1 \in X$ s.t. $f(x_0) = \inf f(X)$, $f(x_1) = \sup f(X)$

Clearly: $\inf f(X) \le f(x) \le \sup f(X)$, $\forall x \in X$

This implies: for any $x \in X$, $f(x_0) \le f(x) \le f(x_1)$

6.44 If $f: [0,1] \to \mathbb{R}$

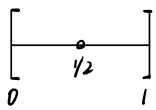
① Is it possible that $f([0,1]) = (0,1)$?

No Since $[0,1]$ is compact, $f([0,1])$ must also be compact

But $(0,1)$ is not compact.

② Is it possible that $f([0,1]) = (0,1) - \{\frac{1}{2}\}$?

No



$\leftarrow$ not compact because it's <u>not closed</u>

why? $\frac{1}{2}$ is a limit point of $[0,1] - \{1/2\}$

but $\frac{1}{2} \notin [0,1] - \{1/2\}$

6.45 (b) Two def's : Cauchy & U.C.

$f: X \to \mathbb{R}$ U.C. , (x_n) is a seq in X, which is Cauchy,

Prove that $(f(x_n))$ is also a Cauchy Seq.

~~Proof~~: Fix any $\varepsilon > 0$, we must find $N \in \mathbb{N}$ s.t. if $n, m \geq N$

$$\Rightarrow |f(x_n) - f(x_m)| < \varepsilon$$

① Since $f: X \to \mathbb{R}$ is U.C., for our $\varepsilon > 0$, there is a $\delta > 0$ s.t.

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

② Now since (x_n) is Cauchy, there is an $N \in \mathbb{N}$ s.t.

$$\text{if } n, m \geq N \Rightarrow |x_n - x_m| < \delta$$

The $N \in \mathbb{N}$ obtained in ② works!

$$\text{if } n, m \geq N \overset{\text{by ②}}{\Longrightarrow} |x_n - x_m| < \delta \overset{\text{by ①}}{\Longrightarrow} |f(x_n) - f(x_m)| < \varepsilon$$

6.46

$f: [a,b] \to \mathbb{R}$ cont., $f(a) < g(a)$ and $g(b) < f(b)$

Prove that $f(c) = g(c)$ for $\exists c \in [a,b]$

Proof: Apply the Int Value Thm

Observe that $f(x) = g(x) \Leftrightarrow f(x) - g(x) = 0$

Set $h(x) = f(x) - g(x)$

Find $c \in (a,b)$ so that $h(c) = 0$

$f(a) < g(a) \Leftrightarrow f(a) - g(a) < 0 \qquad h(a) < 0$

$f(b) > g(b) \Leftrightarrow f(b) - g(b) > 0 \qquad h(b) > 0$

We have $h(a) < 0 < h(b)$

Then by the Intermediate Value Thm, we can find $c \in (a,b)$

so that $h(c) = 0 \Leftrightarrow f(c) = g(c)$ $\square$

* (18) Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(U)$ is open for any open set $U \subseteq \mathbb{R}$.

Extreme Value Thm : $f: X \to \mathbb{R}$ cont., X compact, $\exists\, x_0, x_1 \in X$ s.t.

$$f(x_0) \le f(x) \le f(x_1) \quad \text{for } \forall x \in X$$

Def: Let $X \subseteq \mathbb{R}$. $U \subseteq X$ is <u>open in X</u> if $\exists$ open set $V \subseteq \mathbb{R}$ s.t. $U = X \cap V$

Prop: $X \subseteq \mathbb{R}$, $U \subseteq X$ is open in X

$\iff$ for any $x \in U$, $\exists\, \delta > 0$ s.t. $(x-\delta, x+\delta) \cap X \subseteq U$

<u>Proof</u>:

($\Rightarrow$) Assume $f: X \to \mathbb{R}$ cont. <u>and</u> $U \subseteq \mathbb{R}$ open

(20) Let $K \subseteq \mathbb{R}$ be compact and let $p \notin K$. Show that there is a point $c \in K$ whose distance to p is minimal.

$\exists x_0 \in K$ s.t. $|x_0 - p|$ is minimal $\Leftrightarrow |x_0 - p| \leq |x - p|$, $\forall x \in K$

cont. function

(21) Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(C)$ is closed for any closed subset $C \subseteq \mathbb{R}$.

Thm: $f: \mathbb{R} \to \mathbb{R}$ cont. $\Leftrightarrow$ $f^{-1}(U)$ open for any $U \subseteq \mathbb{R}$ open

($\Rightarrow$) f is cont., $C \subseteq \mathbb{R}$ closed $\Leftrightarrow$ $\exists U$ open, $C = \underbrace{\mathbb{R} - U}_{\text{closed}}$

*Fact: $g: X \to Y$ & $A \subseteq Y$ $\Rightarrow$ $g^{-1}(Y-A) = X - g^{-1}(A)$

then $f^{-1}(C) = f^{-1}(\mathbb{R}-U) = \mathbb{R} - f^{-1}(U)$, which is closed

($\Leftarrow$) Assume $f^{-1}(C)$ is closed if $C \subseteq \mathbb{R}$ is closed

Take any open set $U \subseteq \mathbb{R}$, $C = \mathbb{R} - U$

Show that $f^{-1}(U)$ is open for any U open

Observation: $U \subseteq \mathbb{R}$ open $\Rightarrow$ $\mathbb{R}-U$ closed, $U = \mathbb{R} - (\mathbb{R}-U)$

$f^{-1}(U) = f^{-1}(\mathbb{R} - (\mathbb{R}-U)) = \mathbb{R} - f^{-1}(\mathbb{R}-U)$

we know $f^{-1}(\mathbb{R}-U)$ is closed, then $\mathbb{R} - f^{-1}(\mathbb{R}-U)$ is open

$\Rightarrow$ $f^{-1}(U)$ is open

(22) Is the function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x\sin(\frac{1}{x})$ uniformly continuous?

$\lim_{x\to a} f(x) = L$

For any $\varepsilon > 0$,

$\exists\ \delta > 0$, $0 < |x-a| < \delta$

$|f(x) - L| < \varepsilon$

Yes. $f : (0,1) \to \mathbb{R}$, $f(x) = x\sin(\frac{1}{x})$

Obs: $\lim_{x\to 0 = a} \underbrace{x\sin(\frac{1}{x})}_{f(x)} = 0 = L$

Fix $\varepsilon > 0$. Find $\delta > 0$ s.t. if $0 < |x| < \delta$

$\Rightarrow |x\sin(\frac{1}{x}) - 0| = |x\sin(\frac{1}{x})| < \varepsilon$

$|\sin(\frac{1}{x})| = 1$ for any $x \neq 0$

$|x\sin(\frac{1}{x})| = |x| \cdot |\sin(\frac{1}{x})| < |x| \cdot 1 = |x|$

$|x\sin(\frac{1}{x}) \leq |x| < \varepsilon$

Take $\delta = \varepsilon$

$0 < |x| < \delta \Rightarrow |x| < \varepsilon$

But we have $|x\sin(\frac{1}{x})| \leq |x| < \varepsilon$

Thus $|x\sin(\frac{1}{x})| < \varepsilon$

$\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$ is cont. on open interval $(0, -1)$

$f : (0,1) \to \mathbb{R}$ $f(x) = x\sin(\frac{1}{x})$

Define $g : [0,1] \to \mathbb{R}$ as $g(x) \begin{cases} 0 & \text{if } x = 0 \\ x\sin(\frac{1}{x}) & \text{if } 0 < x \leq 1 \end{cases}$

This $g(x)$ is cont. on $[0,1]$

why? it is cont. on $(0,1]$

And at $x = 0$, we have $\lim_{x\to 0} g(x) = 0 = g(0)$ (Then $g(x)$ is cont. at $x=0$)

Since is cont. and $[a,b]$ is compact

$\Rightarrow g$ is U.C. on $[0,1]$

Show that $f(x) = x\sin(\frac{1}{x})$ is U.C. on $(0,1)$

Fix $\varepsilon > 0$, we want a $\delta > 0$, so that for any $x, y \in [0,1]$

with $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Since $g(x)$ is U.C. there is a $\delta>0$ s.t.

$x, y \in [0,1]$, $|x-y|<\delta \Rightarrow |g(x)-g(y)|<\varepsilon$

in particular, Then if $x, y \in (0,1)$ and $|x-y|<\delta$

$\Rightarrow |g(x)-g(y)| = |f(x)-f(y)|<\varepsilon$

$\sin(\frac{1}{x})$ is not U.C.

Thm: If X is compact and $f: X \to \mathbb{R}$ is cont., then f is U.C.

(23) If $f : [a, b] \to \mathbb{R}$ is continuous and $|f(x)| = 1$ for all $x \in [a, b]$, prove that f is a constant function.

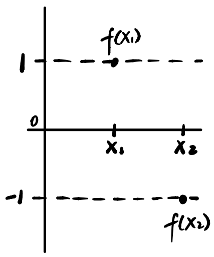
By Int. Value Thm

Proof: $|f(x)| = 1, \ \forall x \in [a,b] \Rightarrow f([a,b]) \subseteq \{-1, 1\}$ only 2 possible output

Show that $f(x)$ is constant.

Suppose not (prove by contradiction)

There exists x_1, x_2 s.t. $f(x_1) = 1$, $f(x_2) = -1$, with $x_1 < x_2$



By Int. Value Thm, $\exists\, c \in (x_1, x_2)$ s.t. $f(c) = 0$

which is a contradiction.

(24) Show that $X \subseteq \mathbb{R}$ is disconnected if and only if there exists a continuous function $f : X \to \mathbb{R}$ such that $f(X) = \{0, 1\}$.

Recall def of <u>disconnected</u>:

Exercise 19

Let $f, g : [a,b] \to \mathbb{R}$ cont. s.t. $f(x) < g(x)$, $\forall x \in [a,b]$

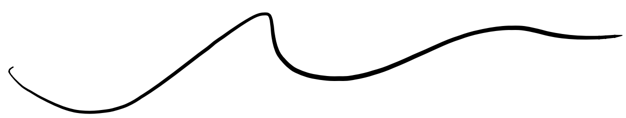
Proof: Consider $h(x) := g(x) - f(x)$

cont. on $[a,b]$ and $h(x) > 0$, $\forall x \in [a,b]$

Find $\lambda > 0$ s.t. $f(x) + \lambda < g(x)$ $\Leftrightarrow$ Find $\lambda > 0$ s.t. $h(x) > \lambda$, $\forall x \in [a,b]$

Apply the Extreme Value Thm,

we can find $x_0 \in [a,b]$ s.t.



Ex 645

Recall <u>Cauchy Sequence</u>

<u>Thm</u>: A seq. (x_n) is Cauchy $\Leftrightarrow$ (x_n) in $\mathbb{R}$ is convergent

<u>Ex</u>: $x_n = \frac{1}{n}$

<u>Def</u>: ① A seq (x_n) is Cauchy if, for any $\varepsilon > 0$, we can find $N \in \mathbb{N}$ s.t.

if $n, m \geq N \Rightarrow |x_n - x_m| < \varepsilon$

<u>Def</u>: ② $f: X \to \mathbb{R}$ is U.C.