8-9 Q's

Exercises of midterm2/prac

MATH 317: Practice Exercises - Final Exam

not on final

(1) Recall that a sequence of functions $f_n : X \to \mathbb{R}$ is said to be *Cauchy on* X if, for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{for all } x \in X \text{ and for all } n, m \geq N.$$

Prove that $f_n : X \to \mathbb{R}$ is uniformly convergent if and only if f_n is Cauchy on X.

(2) Let $f_n : [-10, 10] \to \mathbb{R}$ be defined by $f_n(x) = \frac{x \sin(x)}{n}$. Prove that f_n converges uniformly.

(3) Let $f_n : X \to \mathbb{R}$ be a sequence of bounded functions and suppose that f_n converges uniformly to $f : X \to \mathbb{R}$. Show that $f : X \to \mathbb{R}$ is also bounded.

(4) Let $f_n(x) = x^n$ on $[0, 1]$, and let

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1), \\ 1 & \text{if } x = 1. \end{cases}$$

Prove that f_n converges pointwise, but not uniformly, to f.

(5) Assume that, for each $n \in \mathbb{R}$, the function $f_n : X \to \mathbb{R}$ is uniformly continuous. Also, assume that $f_n \to f$ uniformly. Prove that f is uniformly continuous.

(6) Exercise 9.24 in the book by Jay Cummings. 2 series of functions W. M-Test 01:08

(7) Let $g : [a, b] \to \mathbb{R}$ be integrable and fix a $c \in (a, b)$. Show that, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\left| \int_a^x g dx - \int_a^c g dx \right| < \epsilon$$

whenever $x \in (c - \delta, c + \delta)$.

(8) Suppose $f : [a, b] \to \mathbb{R}$ is integrable and let

$$A = \left\{ \int_a^c f dx \quad \middle| \quad c \in [a, b] \right\}.$$

Is it possible to have that $A = [0, 5] \cup [6, 10]$?

(9) Let $f : [a, b] \to \mathbb{R}$ be bounded and let $c \in (a, b)$. Suppose that f is integrable on $[a, c - \epsilon]$ and $[c + \epsilon, b]$ for any ϵ satisfying $0 < \epsilon < \min\{c - a, b - c\}$. Prove that f is integrable on $[a, b]$.

integrability fund thm of calculus

(10) Let $g : [a, b] \to \mathbb{R}$ be continuous and let $f : [a, b] \to \mathbb{R}$ be another function which differs from g at only finitely many points in $[a, b]$.

(a) Show that f is integrable.

(b) Show that the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f\, dt$$

is differentiable.

(11) Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $c < d$ are points in (a, b) such that

$$\int_a^c f\, dx = \int_a^d f\, dx = 0.$$

Show that there is an $e \in (c, d)$ such that $f(e) = 0$.

(12) Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that

$$\int_a^b f = f(x_0) \cdot (b - a)$$

for some $x_0 \in [a, b]$.

(13) Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable. Show that, for any $c \in (a, b)$, we can always find a sequence x_n in $(a, b) - \{c\}$ satisfying $x_n \to c$ and $f'(x_n) \to f'(c)$.

(14) Is

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in \mathbb{Q}, \\ x & \text{if } x \in \mathbb{I} \end{cases}$$

differentiable at $x = 0$?

(15) A function is called *Lipschitz* if there exists some $C \geq 0$ such that

$$|f(x) - f(y)| \leq C \cdot |x - y|$$

for all x and y.

Now suppose that $f : [a, b] \to \mathbb{R}$ is differentiable and that f' is continuous on $[a, b]$. Prove that f is Lipschitz.

(16) Suppose that f and g are differentiable functions with $f(a) = g(a)$ and $f'(x) < g'(x)$ for all $x > a$. Prove that $f(x) < g(x)$ for any $x > a$.

(17) Assume that $f : (a, b) \to \mathbb{R}$ is differentiable at some point $c \in (a, b)$. Prove that if $f'(c) \neq 0$, then there exists some $\delta > 0$ such that $f(x) \neq f(c)$ for all $x \in (c - \delta, c + \delta) - \{c\}$.

(18) Let $f : (a, b) \to \mathbb{R}$ be differentiable and suppose that f' is continuous at a point $c \in (a, b)$. If x_n and y_n is any pair of sequences with $x_n \neq y_n$ for all n and such that

$$\lim x_n = \lim y_n = c,$$

then show that

$$\frac{f(y_n) - f(x_n)}{y_n - x_n} \longrightarrow f'(c).$$

(19) Is it possible for a continuous function $f : \mathbb{R} \to \mathbb{R}$ to have the property that $f(\mathbb{R}) = \mathbb{Q}$?

(20) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with the property that $f(x) = 0$ for all $x \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

(21) *Review all the exercises from the first and second midterm.*

19. If $f: \mathbb{R} \to \mathbb{R}$ is cont. is it possible to have $f(\mathbb{R}) = \mathbb{Q}$?

A: $\mathbb{R} \to$ this is connected
since $\mathbb{Q}$ is not connected,
we can't have $f(\mathbb{R}) = \mathbb{Q}$

Seperation for X is a ...

(1) Recall that a sequence of functions $f_n : X \to \mathbb{R}$ is said to be *Cauchy on X* if, for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{for all } x \in X \text{ and for all } n, m \geq N.$$

Prove that $f_n : X \to \mathbb{R}$ is uniformly convergent ~~if and only if~~ f_n is Cauchy on X.

$\Longrightarrow$

pointwise convergent $\longrightarrow$ uniformly convergent

review (a_n) conv. $\Rightarrow$ a_n is Cauchy (from 316)

(2) Let $f_n : [-10, 10] \to \mathbb{R}$ be defined by $f_n(x) = \frac{x\sin(x)}{n}$. Prove that f_n converges uniformly.

$f_n : [-10, 10] \to \mathbb{R}$, $f_n(x) = \frac{x\sin(x)}{n}$ $\Rightarrow$ f_n converges uniformly

Define $f : [-10, 10] \to \mathbb{R}$

$|f_n - f| = \left|\frac{x\sin(x)}{n}\right| < \varepsilon$

$x\sin(x) < n\varepsilon$

domain compact $\to$ image compact

(3) Let $f_n : X \to \mathbb{R}$ be a sequence of bounded functions and suppose that f_n converges uniformly to $f : X \to \mathbb{R}$. Show that $f : X \to \mathbb{R}$ is also bounded.

$f_n : X \to \mathbb{R}$: seq of bounded function

$f_n \to_u f \Rightarrow f$ is bounded

def of bounded

(4) Let $f_n(x) = x^n$ on $[0, 1]$, and let

office + Mar 13 class

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1. \end{cases}$$

Prove that f_n converges pointwise, but not uniformly, to f.

it's NOT continuous

$f_n : [0,1] \longrightarrow \mathbb{R} \quad f_n(x) = x^n$

$f : [0,1] \longrightarrow \mathbb{R} \quad f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$

$f_n \longrightarrow f$ pointwise

$c = 1$, $f_n(1) = 1 = f(1)$

$0 \leq c < 1$

$c^n \longrightarrow 0$ as $n \to \infty$

$f_n(c) \to f(c)$

$f_n \to f$ uniformly? <u>No</u>

↳ Each f_n is continuous, but f isn't

(7) Let $g : [a, b] \to \mathbb{R}$ be integrable and fix a $c \in (a, b)$. Show that, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|\int_a^x g dx - \int_a^c g dx| < \epsilon$$

whenever $x \in (c - \delta, c + \delta)$.

$|x-c| < \delta$

(8) Suppose $f : [a, b] \to \mathbb{R}$ is integrable and let

$$A = \{ \int_a^c f\,dx \quad | \quad c \in [a, b]\}.$$

Is it possible to have that $A = [0, 5] \cup [6, 10]$?

(office)

similar to midterm 2 ex

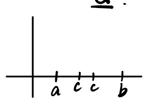
fund thm of Cal (b)

contradiction

$f : [a, b] \longrightarrow \mathbb{R}$ integrable

Let $A = \{ \int_a^c f\,dx \mid c \in [a,b] \} = F([a,b])$

$\underline{Q}$: Can we have that $A = [0,5] \cup [6,10]$?



A: $\underline{\text{No}}$. Because A must be an interval

Fund Thm of Calculus (Part B)

① $f : [a,b] \longrightarrow \mathbb{R}$ integrable

$F : [a,b] \longrightarrow \mathbb{R}$

$F(x) = \int_a^x f\,dx \quad \forall x \in [a,b]$

F is continuous

② If f is continuous, then

$F(x)$ is differentiable and

$F'(x) = f(x)$

* (9) Let $f : [a, b] \to \mathbb{R}$ be bounded and let $c \in (a, b)$. Suppose that f is integrable on $[a, c - \epsilon]$ and $[c + \epsilon, b]$ for any ϵ satisfying $0 < \epsilon < \min\{c - a, b - c\}$. Prove that f is integrable on $[a, b]$.

c is fixed in the middle ? No

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11. $f : [a,b] \to \mathbb{R}$ continuous

Take $c < d$ in (a,b) s.t. $\int_a^c f\,dx = \int_a^d f\,dx$

Show that $\exists\, c \in [a,b]$ s.t. $f(c) = 0$

Rolle's Thm + Fund Thm of Cal

<u>Proof:</u> $F(c) = F(d)$

By Rolle's Thm, $\exists\, c \in (c,d)$ s.t. $F'(c) = 0 \iff f(c) = 0$