Index

- Recall Mean Value Thm
- Application of ↑
  - Proposition 1
  - Corollary
  - Proposition 2
  - Proposition 3
- Darboux's Thm
- HW3 Ex 2

Mean Value Thm:

Let $f: [a,b] \to \mathbb{R}$ be cont. on $[a,b]$ and differentiable at each $x \in (a,b)$

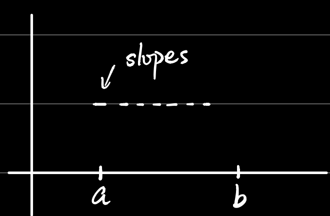
Then, there exists a $c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$

Application of this Thm:

interval

Proposition: Suppose $f: I \to \mathbb{R}$ is differentiable at each $x \in I$

f is constant function $\iff f'(c) = 0$ for any $c \in I$



Proof: $(\Rightarrow)$ in Feb 3 notes

$(\Leftarrow)$ Assume that $f'(c) = 0 \quad \forall c \in I$

Strategy: Take any $x \neq y$ in I. Show that $f(x) = f(y)$

Assume $x < y$

By Mean Value Thm, there is a $c \in (x,y)$ s.t.:

$$\frac{f(y)-f(x)}{y-x} = f'(c) = 0$$

$$\Rightarrow f(y) - f(x) = 0 \iff f(y) = f(x) \quad \blacksquare$$

interval

Corollary: If $f, g: I \to \mathbb{R}$ are differentiable at each $x \in I$ and if $f'(c) = g'(c)$ at each $c \in I$, then

$$f(x) = g(x) + C, \quad \text{where } C \text{ is a fixed constant}$$

Proof: Take $f-g$. This is differentiable at each $c \in I$, and $f'(c) - g'(c) = 0$. By Prop 1, $f(x) - f(c) = C$

Then, $f(x) = g(x) + C$

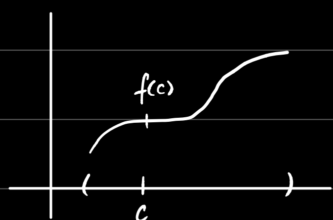
Def: $f: I \to \mathbb{R}$ is $\underset{\text{increasing}}{\text{decreasing}}$ if $x \le y \Rightarrow \underset{f(x) \le f(y)}{f(y) \le f(x)}$

Proposition 2: Let $f: I \to \mathbb{R}$ be differentiable at each $x \in I$

Then f is increasing $\Leftrightarrow f'(c) \ge 0$ for $\forall c \in I$

decreasing $\Leftrightarrow f'(c) \le 0$

Proof: $(\Rightarrow)$ Suppose f is increasing



Take a sequence (x_n) in I s.t.:

- $x_n < c$ for $\forall n \in \mathbb{N}$
- $(x_n) \to c$

$x_n \le c \Rightarrow f(x_n) \le f(c)$

Now, define $a_n = \dfrac{f(x_n) - f(c)}{\underbrace{x_n - c}_{< 0}} \le 0$ $\qquad \lim_{n \to \infty} a_n = f'(c)$

Since $a_n \ge 0$ for $\forall n \in \mathbb{N} \Rightarrow f'(c) \ge 0$

$(\Leftarrow)$ Assume $f'(c) \ge 0$ for $\forall c \in I$

Strategy: Take $x, y \in I$ with $x < y$. Show that

$f(x) \le f(y) \iff 0 \le f(y) - f(x)$

By the Mean Value Thm, there is a $c \in (x,y)$ s.t.

$$f'(c) = \frac{f(y)-f(x)}{\underbrace{y-x}_{>0}}$$

$$\Leftrightarrow f(y) - f(x) = \underbrace{f'(c)}_{\geq 0}\underbrace{(y-x)}_{>0}$$

Since $f'(c) \geq 0$ & $(y-x) > 0$,

it follows that $f(y) - f(x) \geq 0 \Leftrightarrow f(y) \geq f(x)$ ▨

Proposition 3: Let $f: I \to \mathbb{R}$ be differentiable at each $x \in I$

If $f'(c) > 0$ at each $c \in I \Rightarrow f$ is strictly increasing ($x<y \Rightarrow f(x)<f(y)$)

$f'(c) < 0$ strictly decreasing

Proof: Take $x<y$ in I

By the Mean Value Thm, $\exists\, c \in (x,y)$ s.t.

$$f'(c) = \frac{f(y)-f(x)}{y-x} \Leftrightarrow f(y) - f(x) = \overset{>0}{f'(c)}\,\overset{>0}{(y-x)} > 0$$

Then $f(y) - f(x) > 0 \Leftrightarrow f(y) > f(x)$ ▨

Note: In Prop 3 $(\Leftarrow)$ does not hold

Counterexample: $f(x) = x^3$ defined on $\mathbb{R}$

Def: $f: I \to \mathbb{R}$ (interval), let $D = \{x \in I \mid f \text{ is differentiable at } x\}$

Define $\left.\begin{array}{l} f': D \to \mathbb{R} \\ \quad x \mapsto f'(x) \end{array}\right\}$ Derivative function

If $D = I$, f is differentiable

Darboux's Thm:

Let $f : [a,b] \to \mathbb{R}$ be differentiable

Let λ be a value between $f'(a)$ & $f'(b)$ | either $f'(a) < \lambda < f'(b)$ or $f'(b) < \lambda < f'(a)$

Then there exists a $c \in (a,b)$ s.t. $f'(c) = \lambda$

(In other words, $f' \; [a,b] \to \mathbb{R}$ satisfies the <u>Intermediate Value Property</u>)

<u>HW3 Exercise 2</u> Assume $f'(a) < \lambda < f'(b) \to (*)$

<u>1^{st} Step:</u> Define $g : [a,b] \to \mathbb{R}$ as $g(x) = f(x) - \lambda x$

$g'(x) = f'(x) - \lambda$

focus

$(*) \Leftrightarrow \boxed{g'(a) < 0 < g'(b)}$

Rolle's Thm

<u>Goal</u>: Find a $c \in (a,b)$ s.t. $\boxed{g'(c) = 0}$

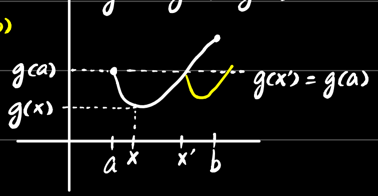
<u>Part (a)</u>: Show that there is an $x \in (a,b)$ s.t. $g(x) < g(a)$

and a $y \in (a,b)$ s.t. $g(x) < g(b)$

<u>Part (b)</u>: Find $c \in (a,b)$ s.t. $g'(c) = 0$

$g(a) < g(b)$

$g(x) < g(a) < g(b)$



while $x \to b$, we need to cross the height of $g(a)$ again by int. Value Thm

?

By Rolle's Thm

$\Rightarrow g'(c) = 0$

proof of $g(b) < g(a)$ & $g(a) = g(b)$ are similar to $g(a) < g(b)$