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Conventions: ① If $f : [a,b] \to \mathbb{R}$ is integrable

$$\int_b^a f\,dx = -\int_a^b f\,dx$$

② $\int_a^a g\,dx = 0$

Thm A: Suppose $f : [a,b] \to \mathbb{R}$ is integrable and let $c \in (a,b)$

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$$

Exercise

Thm B: Suppose $f : [a,b] \to \mathbb{R}$ is integrable

Then, for any points $c, e, d \in [a,b]$ we have

$$\int_c^d f\,dx = \int_c^e f\,dx + \int_e^d f\,dx$$

The Fundamental Theorem of Calculus

Fund. Thm of Calculus Part A

Suppose $f : [a,b] \to \mathbb{R}$ is integrable and let $F : [a,b] \to \mathbb{R}$ s.t. for any $x \in [a,b]$, $F'(x) = f(x)$. Then

$$\int_a^b f\,dx = F(b) - F(a)$$

Proof: Take any partition $P = \{t_0, t_1, \cdots, t_n\}$ of $[a,b]$

Apply the Mean Value Thm to F and each interval of the form $[t_{i-1}, t_i]$

$$f(c_i) = F'(c_i) = \frac{F(t_i) - F(t_{i-1})}{t_i - t_{i-1}} \quad \text{for } \exists\, c_i \in (t_{i-1}, t_i)$$

$$\iff F(t_i) - F(t_{i-1}) = f(c_i)(t_{i-1}, t_i)$$

$$L(f,P) = \sum_{i=1}^n m_i(t_i - t_{i-1}) \le \sum_{i=1}^n f(c_i)(t_i - t_{i-1}) \le \sum_{i=1}^n M_i(t_i - t_{i-1}) = U(f,P)$$

$m_i \le f(c_i) \le M_i$

$$L(f,P) \le \sum_{i=1}^{n} \left(F(t_i) - F(t_{i-1}) \right) \le U(f,P)$$

$$\hookrightarrow = (\cancel{F(t_1)} - F(a)) + (\cancel{F(t_2)} - \cancel{F(t_1)}) + \cdots + (\cancel{F(t_{n-1})} - \cancel{F(t_{n-2})}) + (F(b) - \cancel{F(t_{n-1})}) = F(b) - F(a)$$

$\Leftrightarrow L(f,P) \le F(b) - F(a) \le U(f,P)$ $\longrightarrow$ This holds for any partition P of $[a,b]$

$\Rightarrow L(f) \le F(b) - F(a) \le U(f)$

Since $L(f) = U(f) = \int_a^b f\,dx$, then $\int_a^b f\,dx = F(b) - F(a)$ ▨

Fund. Thm of Calculus Part B

Let $g : [a,b]$ be integrable and

let $G : [a,b] \to \mathbb{R}$ be defined by: $G(x) = \int_a^x g(t)\,dt \quad x \in [a,b]$

Then (i) G is continuous on $[a,b]$

(ii) If g is continuous at $\exists\, c \in [a,b]$,

then G is differentiable at c and $G'(c) = g(c)$

Corollary: If g is continuous on $[a,b]$, then G is differentiable on $[a,b]$ and G is an antiderivative for g

Proof of (i): Pick any $x, y \in [a,b]$ with $x < y$

$$|G(y) - G(x)| = \left| \int_a^y g\,dt - \underbrace{\int_a^x g\,dt}_{\text{Apply Thm B to } a,x,y} \right| = \left| \cancel{\int_a^y g\,dt} - \left(\cancel{\int_a^y g\,dt} + \int_y^x g\,dt \right) \right| = \int_y^x g\,dt$$

? $$\left| \int_x^y g\,dt \right| \le \int_x^y |g|\,dt \le \underset{0 \le M = \sup|g|}{(y-x)\,M} = M\,\overset{|x-y|}{\overset{\parallel}{|y-x|}}$$

Then, for any $x, y \in [a,b]$, $|G(y) - G(x)| \le M|x-y|$

In particular, G is uniformly continuous.

Proof of (ii): Show that $\lim_{x \to c} \frac{G(x)-G(c)}{x-c} = g(c)$

For any $x \neq c$,

$$\frac{G(x)-G(c)}{x-c} = \frac{1}{x-c}\left(G(x)-G(c)\right) = \frac{1}{x-c}\left(\overbrace{\int_a^x}^{\text{Apply Thm B to } a,x,c} g\,dt - \int_a^c g\,dt\right)$$

$$= \frac{1}{x-c}\left(\left(\cancel{\int_a^c g\,dt} + \int_c^x g\,dt\right) - \cancel{\int_a^c g\,dt}\right) = \frac{1}{x-c}\int_c^x g\,dt$$

Show that $\lim_{x \to c} \frac{1}{x-c}\int_c^x g\,dt = g(c)$.

Fix $\varepsilon > 0$. Since g is continuous at $c \in [a,b]$, there is a $\delta > 0$ s.t.

$$|x-c| < \delta \Rightarrow |g(x)-g(c)| < \frac{\varepsilon}{2}$$

$$\left|\frac{1}{x-c}\int_c^x g\,dt - g(c)\right| = \left|\frac{1}{x-c}\int_c^x g\,dt - \frac{1}{x-c}\underbrace{(x-c)\,g(c)}_{\int_c^x g(c)\,dt}\right|$$

$$= \left|\frac{1}{x-c}\int_c^x g\,dt - \frac{1}{x-c}\int^x g(c)\,dt\right| = \left|\frac{1}{x-c}\int_c^x g(t)-g(c)\,dt\right| = \frac{1}{|x-c|}\cdot\left|\int_c^x g(t)-g(c)\,dt\right|$$

$$\leq \frac{1}{|x-c|}\int_c^x \underbrace{|g(t)-g(c)|\,dt}$$

t can only take values in the interval with end points c & x. Then $|t-c| \leq |x-c| < \delta$

Then, $|g(t)-g(c)| < \varepsilon/2$

$$\leq \cancel{\frac{1}{|x-c|}}\,\cancel{|x-c|}\cdot\frac{\varepsilon}{2} < \varepsilon$$