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- Thm: Suppose $f_n : X \to \mathbb{R}$ converges uniformly to $f : X \to \mathbb{R}$

  If each f_n is continuous at $c \in X$, then

  f is also continuous at $c \in X$.

Q: diff between def A & def B read Convergence

def A: This N might depend on $c \in X$ def B: This N works for $\forall x \in X$

less N than def B?

Q: Proof of ex of uniform convergence

c & N are independent? (33:00)

Sequences of functions

$X \subseteq \mathbb{R}$,

$f_1 : X \to \mathbb{R} \quad f_2 : X \to \mathbb{R} \quad \cdots \quad f_n : X \to \mathbb{R} \quad \cdots$

$f_1 \quad f_2 \quad \cdots \quad f_n \quad \cdots$

Ex 1: $X = [0,1] \quad f_n(x) = x^n$

$x \quad x^2 \quad x^3 \quad \cdots \quad x^n \quad \cdots$

Ex 2: $X = \mathbb{R}$

$$g_n : X \to \mathbb{R} \begin{cases} g_n(x) = \cos x & \text{if } x \text{ is odd} \\ g_n(x) = \sin x & \text{if } x \text{ is even} \end{cases}$$

g_1	g_2	g_3	$\cdots$	g_{2n}	g_{2n+1}
$\cos x$	$\sin x$	$\cos x$		$\sin x$	$\cos x$

Pointwise Convergence

Def A:

$f_n : X \to \mathbb{R}$ converges pointwise to $f : X \to \mathbb{R}$ if, for any $c \in X$, $\lim_{n\to\infty} f_n(c) = f(c)$

(For any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ s.t. if $n \geq N$ $|f_n(c) - f(c)| < \varepsilon$)

This N might depend on $c \in X$

Uniform Convergence

Def B:

$f_n : X \to \mathbb{R}$ converges uniformly to $f : X \to \mathbb{R}$ if, for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ for $\forall\, n \geq N$ and $\forall\, x \in X$

This N works for $\forall\, x \in X$

$c \in \mathbb{R}$
$g_n(c) = \frac{c}{n}$

Ex 1: $X = \mathbb{R}$, $g_n : X \to \mathbb{R}$, $g_n(x) = \frac{x}{n}$

For any $c \in \mathbb{R}$, $g_n(c) = \frac{c}{n} \to 0$

If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = 0$ for $\forall x \in \mathbb{R}$

$\boxed{g_n \to g \quad \text{pointwise}}$

Ex 2: $X = [0,1]$ $\quad f_n : X \to \mathbb{R}$ $\quad f_n(x) = x^n$

Let $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$

Claim: $f_n \to f$ pointwise

Recall: If $0 \le c < 1$, then $c^n \to 0$

Proof: $\cdot\ 0 \le c < 1$, $c^n = f_n(c) \to f(c)$

$\cdot\ c = 1$, $f_n(c) = c^n = 1$ & $f(c) = 1$

$f_n(c) \to f(c)$

obs: Uniform Convergence implies Pointwise Convergence

Ex: $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{1}{n(1+x^2)}$

For $c \in \mathbb{R}$, $f_n(c) = \frac{1}{n(1+c^2)} = \frac{1}{n} \cdot \frac{1}{(1+c^2)} \longrightarrow 0$ $\quad$ [$f_n \to 0$ pointwise

Claim: $f_n \to 0$ uniformly

Proof: Fix $\varepsilon > 0$ $\quad$ $n \le n(1+x^2) \iff \frac{1}{n(1+x^2)} \le \frac{1}{n}$

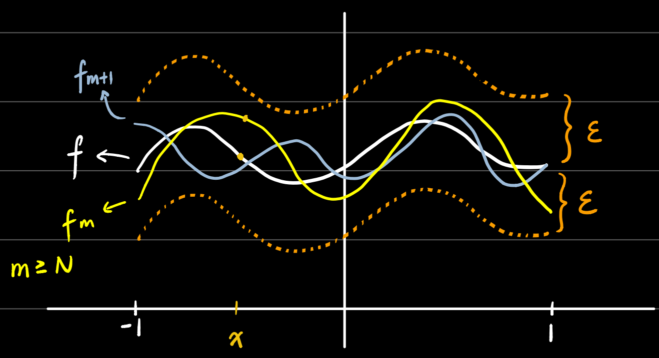
$|f_n(x) - 0| = \left|\frac{1}{n(1+x^2)}\right| \le \frac{1}{n} < \varepsilon$

For $\forall n \ge N$ (N sufficiently large)

$\boxed{|f_n(x) - 0| < \varepsilon \text{ if } n \ge N}$

Then, $f_n \to 0$ uniformly

Picture: $f : [-1,1] \longrightarrow \mathbb{R}$



Suppose $f_n \to f$ uniformly

Fix $\varepsilon > 0$, there is an $N \in \mathbb{N}$

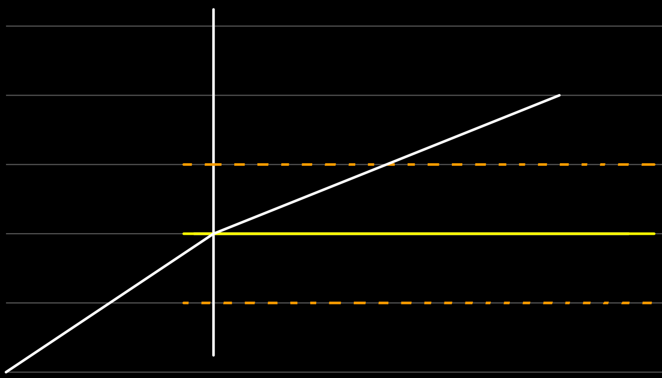
so that $|f_n(x) - f(x)| < \varepsilon$

for $\forall x \in X$ and $n \geq N$

Ex: $g_n : \mathbb{R} \to \mathbb{R}$ $\quad g_n(x) = \frac{x}{n}$

$g_n \to 0$ pointwise

Take any $n \in \mathbb{N}$ $\quad g_n(x) = \frac{x}{n}$



$g_n \not\to 0$ uniformly

Thm: Suppose $f_n : X \to \mathbb{R}$ converges uniformly to $f : X \to \mathbb{R}$

If each f_n is continuous at $c \in X$, then

f is also continuous at $c \in X$.

Proof:

Fix $\varepsilon > 0$, $|f(x) - f(c)|$

Step 1: Since $f_n \longrightarrow f$ uniformly,

there is an $N \in \mathbb{N}$ s.t. $|f_N(x) - f(x)| < \varepsilon/3$ for $\forall x \in X$

Step 2: By continuity of f_N, there is a $\delta > 0$ s.t.

$|x - c| < \delta \Rightarrow |f_N(x) - f_N(c)| < \varepsilon/3$

Step 3: Take $|x - c| < \delta$

$$
\begin{aligned}
|f(x) - f(c)| &= |(f(x) - f_N(x)) + (f_N(x) - f_N(c)) + (f_N(c) - f(c))| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\end{aligned}
$$