MATH 317: Practice Exercises - Integration

1) Let $f:[a,b] \to \mathbb{R}$ be continuous and suppose that $f(x) > 0$ for all $x \in [a,b]$. Prove that

   $\int_a^b f(x)dx > 0.$

2) Exercise 8.5.
3) Exercise 8.8
4) Exercise 8.11
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6) Exercise 8.12
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12) Exercise 8.21
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21) Bonus Exercise! Exercise 8.43

1) Let $f: [a, b] \to \mathbb{R}$ be continuous and suppose that $f(x) > 0$ for all $x \in [a, b]$. Prove that

$\int_a^b f(x)dx > 0$.

$\nearrow [a,b]$

Recall: $X \subseteq \mathbb{R}$ is compact $\Rightarrow$ $f: X \to \mathbb{R}$ is cont.

Assume $f(x) > 0$ for $\forall x \in X$

$\Rightarrow$ $m = \inf f(x) > 0$ and $b-a \geq 0$

Then, $\int_a^b f\,dx \geq m(b-a) > 0$

8.5: $a \le b \le c \le d$ <u>and</u> f is integrable on $[a,b]$

Prove if f is integrable on $[b,c]$

<u>Recall:</u> $f:[a,b]$ is bounded and $c \in (a,b)$

<u>Thm A:</u> f integrable over $[a,b] \Longleftrightarrow f$ integrable over both $[a,c]$ & $[c,b]$

In this case, $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$

<u>Proof:</u> Suppose $a \le b \le c \le d$ & f is integrable over $[a,d]$

By Thm above, $\Rightarrow f$ is integrable over $[a,c]$ & $[c,d]$

Apply the thm again $\Rightarrow f$ is integrable over $[a,b]$ & $[b,c]$

Ex 8.8: Prove if $f:[a,b]\to\mathbb{R}$ is integrable, then there exists a sequence P_n of partition of $[a,b]$ for which

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0$$

Recall: Thm B: $f:[a,b]\to\mathbb{R}$ is integrable $\Leftrightarrow$ for any $\varepsilon>0$, there is a partition P s.t. $U(f,P)-L(f,P)<\varepsilon$

Proof: Suppose $f:[a,b]\to\mathbb{R}$ integrable, fix $\varepsilon=\frac{1}{n}$, so that by Thm B, we can find a partition P_n s.t. $0\le U(f,P_n)-L(f,P_n)<\frac{1}{n}$

Apply the Squeeze Thm $\Rightarrow$ $\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0$

If there exists a sequence of partitions P_n of $[a,b]$ s.t.

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0$$

then f is integrable. (A direct consequence of Thm B)

③

(4) Ex 8.11 Prove that (a)(b)(c) by considering partitions into n equal subintervals

(a) $\int_0^b x\,dx = \frac{b^2}{2}$ (b) $\int_0^b x^2\,dx = \frac{b^3}{3}$ (c) $\int_0^b x^3\,dx = \frac{b^4}{4}$

$f(x) = x$ is continuous over $[0,b] \Rightarrow$ integrable over $[a,b]$

Prove that $L(f) = \frac{b^2}{2}$

For each $n \in \mathbb{N}$, consider the partition P of $[0,b]$ cosists

$$\{0, \frac{b}{n}, \frac{2b}{n}, \cdots, \frac{(n-1)b}{n}, \frac{nb}{n} = b\}$$

$$\Rightarrow L(f, P_n) = \sum_{i=1}^{n} \frac{ib}{n} \cdot \frac{b}{n} = \frac{b^2}{n^2} \cdot \sum_{i=1}^{n} i = \frac{b^2}{\cancel{n^2}} \cdot \frac{\cancel{n}(n+1)}{2} = \frac{b^2}{n} \cdot \frac{(n+1)}{2} = \frac{b^2}{2} \cdot \frac{n+1}{n}$$

As $n \to \infty$, $L(f, P_n) \to \frac{b^2}{2}$

Thus, we must have that $L(f, P_n) \to L(f)$

$$\Rightarrow L(f) = \frac{b^2}{2}$$

(b) Similarly but using $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(c) Similarly but using $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

(5) Ex 8.7 : modified Dirichlet function : $g(x) = \begin{cases} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$

(a) P : partition of $[0,4]$. compute $L(g,P)$

($\approx$ show that g is not integrable over the interval $[0,4]$)

Let $P = \{t_0, t_1, \cdots, t_n\}$ be any partition of $[0,4]$

Since $g(x) = 0$ for $\forall x \notin Q$, since Q is dense ,

Any $\inf g([t_{i-1}, t_i])$ of P will be 0

$\Rightarrow L(g,P) = 0$ for any partition P of $[0,4]$

$\Rightarrow L(g) = 0$

(b) Find $\inf\{U(g,P) : P \text{ a partition of } [0,4]\}$

Fact: $f(x) = x$ is integrable over $[0,4]$

For any $b > 0$, we have $\int_0^b x\,dx = \frac{b^2}{2}$ (from Ex 8.11)

Now $b = 4$, $\int_0^4 x\,dx = \frac{4^2}{2} = 8$

Take any arbitrary partition $P = \{t_0, t_1, \cdots, t_n\}$ of $[0,4]$

Since Q is dense $\Rightarrow$ $\sup g([t_{i-1}, t_i])$ of partition P is equal to t_i

Thus, $U(g,P) = \sum_{i=1}^{n} t_i \cdot (t_i - t_{i-1})$

Notice that $U(g,P) = U(f,P)$ that $f(x) = x$

Therefore, we have $U(g,P) = U(f,P)$ for any partition P of $[0,4]$

$\Rightarrow U(g) = U(f) = \int_0^4 x\,dx = 8$

Since $U(g) \neq L(g)$ $\Rightarrow$ g is not integrable on $[0,4]$

g is not integrable over any compact interval

(6) Ex 8.12 Consider $f : [0,3] \to \mathbb{R}$ given by $f(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ 2 & \text{if } x \in [1,2) \\ 3 & \text{if } x \in [2,3] \end{cases}$

Using the integrals analytically thm

Prove that f is integrable.

Recall: Thm B: $f : [a,b] \to \mathbb{R}$ is integrable $\Leftrightarrow$ for any $\varepsilon > 0$, there is a partition P s.t. $U(f,P) - L(f,P) < \varepsilon$

For each $n > 2$, define P_n of $[0,3]$: $P_n = \{0, 1-\frac{1}{n}, 1+\frac{1}{n}, 2-\frac{1}{n}, 2+\frac{1}{n}, 3\}$

For each $n > 2$, compute $U(f,P_n) - L(f,P_n)$

$$\begin{aligned} U(f,P_n) - L(f,P_n) &= (1-1)(1-\tfrac{1}{n}) + (2-1)\tfrac{2}{n} + (2-2)(1-\tfrac{2}{n})) + (3-2)\tfrac{2}{n} + (3-3)(1-\tfrac{1}{n}) \\ &= 0 + \tfrac{2}{n} + 0 + \tfrac{2}{n} + 0 = \tfrac{4}{n} \end{aligned}$$

$(U(f,P_n) - L(f,P_n)) \longrightarrow 0$

Then, given any $\varepsilon > 0$, we can find an $n > 2$ sufficiently large s.t.

$$U(f,P_n) - L(f,P_n) < \varepsilon$$

Therefore, by Thm B, f is integrable.

(7) Ex 8.14

(8) <u>Ex 8.15</u>: f & g are integrable on $[a,b]$

(a) Prove that if there exists $\exists\, c \in [a,b]$ s.t. $f(x) = g(x)$ for $\forall x \neq c$

then $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$

<u>Proof</u>: Let $c \in [a,b]$ and

suppose that $f(x) = g(x)$ for $\forall x \in [a,b] - \{c\}$ and $f(c) \neq g(c)$

Note: since f, g are integrable,

$\int_a^b f\,dx - \int_a^b g\,dx$ can be rewritten as $\int_a^b (f-g)\,dx \Rightarrow \int_a^b h\,dx$ $(h = f-g)$

$h(x) = 0$ for $\forall x \in [a,b]$ except at c

To prove $\int_a^b h\,dx = 0$, need to show that the upper sum $U(h) = 0$

WLOG, suppose that $c \in (a,b)$ and $h(c) > 0$

Assume $N \in \mathbb{N}$ large enough so that $1/N < c-a$ & $1/N < b-c$

For each $n \geq N$, define $P_n = \{a, c-\frac{1}{n}, c+\frac{1}{n}, b\}$

$U(h, P_n) = 0 \cdot (c - \frac{1}{n} - a) + h(c) \cdot \frac{2}{n} + 0 \cdot (b - c - \frac{1}{n}) = \frac{2h(c)}{n}$

$\frac{2h(c)}{n} \to 0$ as $n \to \infty$

$\Rightarrow$ for any $\varepsilon > 0$, we can find a partition P_n s.t. $0 < U(h, P_n) < \varepsilon$

$\Rightarrow$ $U(h) = 0$

$\Rightarrow$ $\int_a^b h\,dx = U(h) = 0$

$\Rightarrow$ $\int_a^b f\,dx = \int_a^b g\,dx$

(9) <u>Ex 8.16</u>: A function f is <u>strictly increasing</u> on $[a,b]$ if for any x_1 & x_2 from $[a,b]$ where $x_1 < x_2$, we have $f(x_1) < f(x_2)$

Prove that if f is strictly increasing on $[a,b]$, then f is integrable.

<u>Proof</u>:

Ex 8.18

(a) $f:[a,b]\to\mathbb{R}$ continuous

Assume that, for $\forall x\in[a,b]$, $\int_a^x f\,dx = 0$

Then, $f(x)=0$ for any $x\in[a,b]$

Proof: Since f is cont. it has an antiderivative $F(x)$

Part A, for each $x\in[a,b]$, $\int_a^x f(t)dt = F(x)-F(a) = 0 \quad \forall x\in[a,b]$

$\Rightarrow F(x)=F(a) \quad \forall x\in[a,b]$

In other words, F is constant on $[a,b]$.

Then, $F'(x) = f(x) = 0$ for $\forall x\in[a,b]$

(b) Now assume that, for $\forall x\in[a,b]$, we have $\int_a^x f\,dt = \int_x^b f\,dt$ $\to$ (*)

Then $f(x)=0$ for $\forall x\in[a,b]$

Proof: let $F(x)$ be an antiderivative of $f(x)$

From (*) & the Fund Thm of Calculus

$F(x)-F(a) = F(b)-F(x) \quad (\forall x\in[a,b])$

$\Leftrightarrow \quad 2F(x) = F(a)+F(b)$

$\Leftrightarrow \quad$ For $\forall x\in[a,b]$, $F(x) = \frac{F(a)+F(b)}{2}$

Then $F(x)$ is constantly equal to $\frac{F(a)+F(b)}{2}$

Then, $F'(x) = f(x) = 0$ for $\forall x$

Exercise 8.39

(a) $f : [a,b] \to \mathbb{R}$ continuous. (Mean Value Thm for Integral)

Show that we can find $c \in [a,b]$ s.t. $\int_a^b f\,dx = (b-a)\cdot f(c)$

$\Updownarrow$

$f(c) = \frac{1}{b-a}\int_a^b f\,dx$

Proof: Extreme Value Thm:

There exists $x_0, y_0 \in [a,b]$ s.t.

$$f(x_0) = M = \max f(x), \quad f(y_0) = m = \min f(x)$$

In particular, $m \le f(x) \le M$, $\forall x \in [a,b]$

$$m(b-a) \le \int_a^b f\,dx \le M(b-a)$$

Divide by $b-a$,

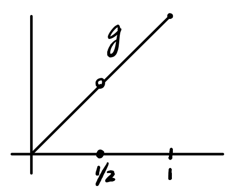
$$f(y_0) = m \le \frac{1}{b-a}\int_a^b f\,dx \le M = f(x_0)$$

$$\Leftrightarrow f(y_0) \le \frac{1}{b-a}\int_a^b f\,dx \le f(x_0)$$

Apply Intermediate Value Thm: $f(c) = \frac{1}{b-a}\int_a^b f\,dx$

(b) $g : [0,1] \to \mathbb{R}$

$$g(x) = \begin{cases} x & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2} \end{cases}$$



Proof:

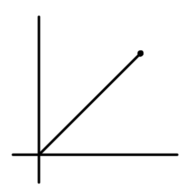
obs $g(x)$ is integrable

for this ex, we will use a result from an ex from prac 2

$\Rightarrow \int_0^1 g(x)\,dx$ exists

$f : [0,1] \to \mathbb{R}$ $\quad f(x) = x \quad \forall x \in [0,1]$

antiderivative $\to \frac{x^2}{2}$

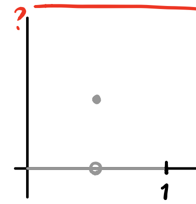


Claim: $\int_0^1 f\,dx = \int_0^1 g\,dx$ (functions are diff $\Rightarrow$ integrals are the same)

(b/c, f & g are same except one point) (exists b/c $\int_0^1 g\,dx$ ✓ $\Rightarrow \int_0^1 (-1)\cdot g\,dx$ ✓ $\Rightarrow \int_0^1 f+(-g)\,dx$ ✓)

☆ $\int_0^1 f-g\,dx = \int_0^1 f\,dx - \int_0^1 g\,dx = 0$ ← why 0 ? → ①

$$f-g(x) \begin{cases} 0 & x \neq \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \end{cases}$$



why 0 ? The actual def of integral, Lower Sums & Upper Sums

Find refinements of this partition of this interval

Antiderivative for $f(x) = x \Rightarrow \frac{x^2}{x}$

$\frac{d}{dx}\left(\frac{x^2}{x}\right) = \frac{1}{2}\frac{d}{dx}(x^2) = \frac{2x}{2} = x$ $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$

$$\int_0^1 g\,dx = \int_0^1 f\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$$

(c) $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ → might fail if we don't assume func is cont.

Counterexample

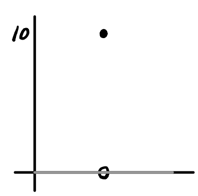
$\underbrace{\frac{1}{b-a}}_{\frac{1}{1-0}=1}\int_0^1 g(x)\,dx = \int_0^1 g(x)\,dx = \frac{1}{2} \neq g(c)$ for any $c \in [0,1]$

$\exists c \in [0,1]$ at $g(c) = \frac{1}{2}$? No.

① $h = f\cdot g$ $[a,b] = [0,1]$

$$h(x) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ c = 10 & \text{if } x = \frac{1}{2} \end{cases}$$

↳ $c \neq 0$



Thm: $h : [a,b] \to \mathbb{R}$

h is integrable $\Leftrightarrow$ There is a sequence of partition P_n of $[a,b]$ with $\boxed{U(h,P_n) - L(h,P_n)} \to 0$

In this case,

☆ review proofs for this part

$U(f,P_n) \longrightarrow \int_a^b h\,dx$

$L(f,P_n) \longrightarrow \int_a^b h\,dx$

How to choose P_n ?

Recall: $U(h,P) = \sum^{n} \overset{\text{sup of } h \text{ in } [t_{i-1},t_i]}{M_i}(t_i - t_{i-1})$

$P_1 = \{0, 1\}$ $(P_1 = P_2)$

$L(h,P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1})$

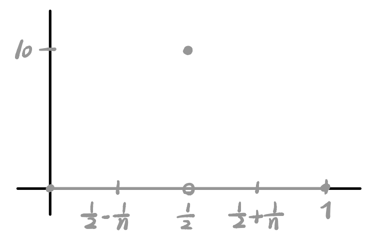
m_i: inf of h in $[t_{i-1}, t_i]$

$n \geq 2$ $P_n = \{0, \frac{1}{2} - \frac{1}{n}, \frac{1}{n} + \frac{1}{2}, 1\}$ $[0, \frac{1}{2} - \frac{1}{n}], [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}], [\frac{1}{2} + \frac{1}{n}, 1]$

$U(h, P_1) - L(h, P_1) = 10 - 0 = 10$

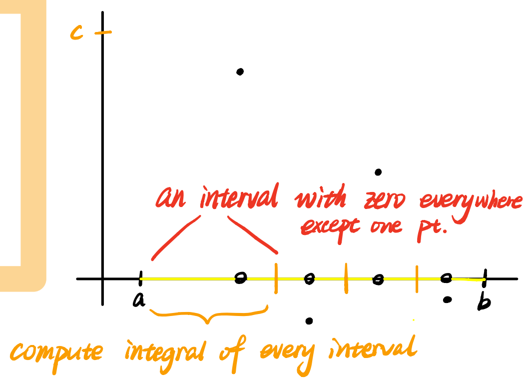
$n \geq 2$ $U(h, P_n) - L(h, P_n) =$

$(0 + 10 \cdot \underbrace{\frac{2}{n}}_{\frac{1}{2} + \frac{1}{n} - (\frac{1}{2} - \frac{1}{n})} + 0) - 0 = (\frac{20}{n}) \to 0$



$U(h, P_n) = \frac{20}{n} \to \int_0^1 h\,dx = 0$

By induction, if a function is 0 everywhere except finitely many points, then the integral will also be zero.



Ex 8.21: Suppose that f is continuous on $[0,1]$

Prove that $\lim_{n\to\infty} \int_0^1 f(x^n)\,dx = f(0)$

Lemma: For any $c \in (0,1)$, we have that $c^n \to 0$

Proof: Fix $\varepsilon > 0$,

Step 1: Fix an $M > \max\{\sup f(x), 0\}$ and pick a $c \in (0,1)$ s.t. $1-c < \frac{\varepsilon}{2M}$

Step 2: By Uniform Continuity, we can pick a $\delta > 0$ so that

$$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon/2$$

Step 3: Now pick an $N \in \mathbb{N}$ s.t. $0 < c^n < \delta$ for $\forall n \geq N$

In fact, we have that $0 < x^n < \delta$ for any $0 \leq x \leq c$

Step 4: Now, for $\forall n \geq N$, we have

$$\begin{aligned}
\left|\int_0^1 f(x^n)\,dx - f(0)\right| &= \left|\int_0^c f(x^n)\,dx - f(0) + \int_c^1 f(x^n)\,dx\right| \\
&\leq \left|\int_0^c f(x^n)\,dx - f(0)\right| + \left|\int_c^1 f(x^n)\,dx\right| \\
&= \left|\int_0^c (f(x^n) - f(0))\,dx\right| + \left|\int_c^1 f(x^n)\,dx\right| \\
&\leq \int_0^c |f(x^n) - f(0)|\,dx + (1-c)M \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2M}\cdot M < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Facts:

- $0 \leq x^n < \delta$ if $x \leq c$
- $|f(x^n) - f(0)| < \varepsilon/2$

Therefore, $\int_0^1 f(x^n)\,dx \to f(0)$

Ex 8.22:

Ex 8.24

Ex 8.26:

(a) f is integrable on $[a,b]$, and $f(x) \geq 0$ for $\forall x$

Moreover, assume that $\int_a^b f(x)dx > 0$.

Prove that there are infinitely many points x for which $f(x) > 0$.

Ex 8.28:

(a) S'pose f is integrable on $[a,b]$ and $f(x) \geq 0$ for $\forall x$

Prove that $\int_a^b f(x)\,dx \geq 0$

$\nearrow [a,b]$

Recall: $X \subseteq \mathbb{R}$ is compact $\Rightarrow f: X \to \mathbb{R}$ is cont.

Assume $f(x) > 0$ for $\forall x \in X$

$\Rightarrow m = \inf f(x) > 0$ and $b-a \geq 0$

Then, $\int_a^b f\,dx \geq m(b-a) > 0$

(b) f, g integrable on $[a,b]$ and $f(x) \geq g(x)$ for $\forall x$

Prove that $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Ex 8.34: S'pose that $f:[a,b]\longrightarrow\mathbb{R}$ is bounded. f is cont. at every point in $[a,b]$ except a single point $c\in(a,b)$. Prove that f is integrable on $[a,b]$

Recall: Integrals Analytically

Proof: Fix $\varepsilon>0$, let $m=\inf f(x)$, $M=\sup f(x)$

Pick an $n\in\mathbb{N}$ large enough so that $[x_0-\frac{1}{n}, x_0+\frac{1}{n}]\subseteq[a,b]$ and $\frac{2}{n}<\frac{\varepsilon}{3(M-m)}$

Consider a partition P s.t. $U(f,P)-L(U,P)<\varepsilon$

Step 1: Pick a partition P_1 of $[a, x_0-\frac{1}{n}]$ s.t. $U(f,P_1)-L(U,P_1)<\varepsilon/3$

Step 2: Pick a partition P_2 of $[x_0+\frac{1}{n}, b]$ s.t. $U(f,P_2)-L(U,P_2)<\varepsilon/3$

Step 3: Define $P=P_1\cup P_2$, we have

$$
\begin{aligned}
& U(f,P)-L(f,P) \\
&= U(f,P_1)+U(f,\{x_0-\tfrac{1}{n}, x_0+\tfrac{1}{n}\})+U(f,P_2) \\
&\quad -[L(f,P_1)+L(f,\{x_0-\tfrac{1}{n}, x_0+\tfrac{1}{n}\})+L(f,P_2)] \\
&= (U(f,P_1)-L(f,P_1))+(U(f,P_2)-L(f,P_2)) \\
&\quad +(U(f,\{x_0-\tfrac{1}{n}, x_0+\tfrac{1}{n}\})-L(f,\{x_0-\tfrac{1}{n}, x_0+\tfrac{1}{n}\})) \\
&< \varepsilon/3+\varepsilon/3+(M-m)\cdot\frac{\varepsilon}{3(M-m)} = \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon
\end{aligned}
$$

Extra Ex: Let $f : [0,1] \to \mathbb{R}$ be integrable.

Prove that $\lim_{n\to\infty} \int_{1/n}^{1} f\,dx = \int_0^1 f\,dx$

Proof: Let $M = \sup\{|f(x)| : x \in [0,1]\}$

For any $n \in \mathbb{N}$, we have

$$\left|\int_0^1 f\,dx - \int_{1/n}^1 f\,dx\right|$$

$$= \left|\int_0^{1/n} f\,dx + \cancel{\int_{1/n}^1 f}\,dx - \cancel{\int_{1/n}^1 f}\,dx\right|$$

$$= \left|\int_0^{1/n} f\,dx\right| = \int_0^{1/n} |f|\,dx$$

$$\leq M\cdot\left(\frac{1}{n} - 0\right) = \frac{M}{n}$$

$$\Rightarrow \left|\int_0^1 f\,dx - \int_{1/n}^1 f\,dx\right| \leq \frac{M}{n}$$

$\left(\frac{M}{n}\right) \to 0$ when $n \to \infty$

$$\Rightarrow \left|\int_0^1 f\,dx - \int_{1/n}^1 f\,dx\right| \to 0 \iff \int_{1/n}^1 f\,dx \to \int_0^1 f\,dx$$