MATH 317: Homework 3

Deadline: Monday 02/17/2020

Exercise 1. Prove that $f : I \to \mathbb{R}$ is differentiable at $c \in I$ if and only if there exists a function $\eta : I \to \mathbb{R}$ which is continuous at c and such that $f(x) = f(a) + \eta(x)(x - a)$ for all $x \in I$.

Exercise 2. The goal of this exercise is to prove *Darboux's Theorem:*

Darboux's Theorem. *If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(b) < \alpha < f'(a)$), then there exists a point $c \in (a, b)$ with $f'(c) = \alpha$.*

First Step. From now on, let's suppose that $f'(a) < \alpha < f'(b)$. Now consider the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - \alpha x$. Note that $g'(x) = f'(x) - \alpha$, and it is straightforward to check that:

$$f'(a) < \alpha < f'(b) \iff g'(a) < 0 < g'(b).$$

Therefore, it is enough to prove that there is a $c \in (a, b)$ with $g'(c) = 0$.

(a) Show that there exist a point $x \in (a, b)$ where $g(a) > g(x)$, and a point $y \in (a, b)$ where $g(y) < g(b)$.

(b) Now prove that $g'(c) = 0$ for some $c \in (a, b)$.

Exercise 3. Let J be an open interval. If $f : J \to \mathbb{R}$ is differentiable at $c \in J$ and $f'(c) > 0$, then prove that there exists a $\delta > 0$ for which the following holds:

(a) $(c - \delta, c + \delta) \subseteq J$.
(b) If $x, y \in (c - \delta, c + \delta)$ and $x < c < y$, then $f(x) < f(c) < f(y)$.

Exercise 4. If $f : [a, b] \to \mathbb{R}$ is one-to-one, then there exists an *inverse function* f^{-1} defined on the image of f given by $f^{-1}(y) = x$ where $y = f(x)$. Suppose that f is differentiable and that $f'(x) \neq 0$ for all $x \in [a, b]$. Prove that f^{-1} is differentiable with

$$(f^{-1})'(y) = \frac{1}{f'(x)} \quad \text{where } y = f(x).$$

Exercise 5. Consider three functions $f, g, h : I \to \mathbb{R}$ defined on an interval I and suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in I$. If f and h are differentiable at $c \in I$, with $f(c) = h(c)$ and $f'(c) = h'(c)$, then prove that g is also differentiable at c and that $g'(c) = f'(c)$.

Exercise 6. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable at each $x \in (a, b)$. Suppose that $f'(x) \geq 0$ for all $x \in (a, b)$. If $f'(x) = 0$ at only finitely many points in (a, b), then prove that f is a strictly increasing function.

Exercise 7. Let $f : I \to \mathbb{R}$ be a differentiable function. Suppose that there is a constant $c > 0$ such that $|f(x) - f(y)| \leq c|x - y|^2$ for any $x, y \in I$. Prove that f must be constant.

We will need the following definition for Exercise 8:

Definition. A function $f : X \to \mathbb{R}$ is *bounded* if there is an interval $[m, p]$ so that $f(X) \subseteq [m, p]$.

Exercise 8. Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable and bounded. Moreover, suppose that the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to b} g(x)$$

do not exist. Prove that, for any $\alpha \in \mathbb{R}$, there is a point $c \in (a, b)$ such that $f'(c) = \alpha$.

5. **Exercise 5.** Consider three functions $f, g, h : I \to \mathbb{R}$ defined on an interval I and suppose that $f(x) \le g(x) \le h(x)$ for all $x \in I$. If f and h are differentiable at $c \in I$, with $f(c) = h(c)$ and $f'(c) = h'(c)$, then prove that g is also differentiable at c and that $g'(c) = f'(c)$.

$f, g, h : I \to \mathbb{R}$ $\begin{cases} \forall x \in I,\ f(x) \le g(x) \le h(x) \\ f \text{ and } h \text{ are differentiable at } c \in I \text{ with } f'(c) = h'(c) \end{cases}$

Show that g is also differentiable at $c \in I$ with $g'(c) = f'(c)$

<u>Step 1</u>: g is cont. at $c \in I$

Since $f(x) \le g(x) \le h(x)$, if $x_n \to c \Rightarrow g(x) \to f(c)$

$f(c) = g(c) = h(c)$

$$\frac{f(x)-f(c)}{x-c} \le \frac{g(x)-g(c)}{x-c} \le \frac{h(x)-h(c)}{x-c} \quad \to \quad \begin{matrix} x > c \\ x < c \end{matrix}$$

Exercise 1. Prove that $f : I \to \mathbb{R}$ is differentiable at $c \in I$ $\overset{\Leftrightarrow}{\underline{\text{if and only if}}}$ there exists a function $\eta : I \to \mathbb{R}$ which is continuous at c and such that $f(x) = f(a) + \eta(x)(x - a)$ for all $x \in I$.

<u>Proof</u>:

f is differentiable at $x = a \iff \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists finitely, and becomes $f'(a)$

Define $\eta(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & \text{if } x \neq a \\ f'(a) & \text{if } x = a \end{cases}$

Then $h(x)$ is a function s.t. $\lim_{x \to a} \eta(x) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a) = \eta(a)$

Which implies that η is continuous at a.

Now s'pose $x \neq a$, then $\frac{f(x)-f(a)}{x-a} = \eta(x) \Rightarrow f(x) - f(a) = \eta(x) \cdot (x-a)$

$$\Rightarrow f(x) = f(a) + \eta(x) \cdot (x-a)$$

$\Rightarrow f$ is differentiable at $a \iff \lim_{x \to a} \frac{f(x)-g(x)}{x-a}$ exists finitely

$\iff \lim_{x \to a} \eta(x)$ exists finitely & equal to $f'(a)$

$\iff$ at $x = a$, $\eta : I \to \mathbb{R}$ is continuous

and $f(x) = f(a) + \eta(x) \cdot (x-a)$ for $\forall x \in I$ $(*)$

Since if $x = a$, $f(a) = f(a) + \cancel{f'(a) \cdot (a-a)} \Rightarrow f(a) = f(a)$

Therefore, $f(x) = f(a) + \eta(x) \cdot (x-a)$ for $\forall x \in I$.

Exercise 2 The goal of this exercise is to prove *Darboux's Theorem:*

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Darboux's Theorem. *If $f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(b) < \alpha < f'(a)$), then there exists a point $c \in (a,b)$ with $f'(c) = \alpha$.*

First Step. From now on, let's suppose that $f'(a) < \alpha < f'(b)$. Now consider the function $g : [a,b] \to \mathbb{R}$ defined by $g(x) = f(x) - \alpha x$. Note that $g'(x) = f'(x) - \alpha$, and it is straightforward to check that:

$$f'(a) < \alpha < f'(b) \iff g'(a) < 0 < g'(b).$$

Therefore, it is enough to prove that there is a $c \in (a,b)$ with $g'(c) = 0$.

(a) Show that there exist a point $x \in (a,b)$ where $g(a) > g(x)$, and a point $y \in (a,b)$ where $g(y) < g(b)$.

(b) Now prove that $g'(c) = 0$ for some $c \in (a,b)$.

Proof: Suppose that $g'(b) > 0$, then choose $\varepsilon > 0$ s.t. $g'(b) - \varepsilon > 0$

Since $\lim_{x \to b} \frac{g(x)-g(b)}{x-b} = g'(b)$, then for any $\varepsilon > 0$, there exists $\exists\ \delta > 0$ s.t.

$$\left| \frac{g(x)-g(b)}{x-b} - g'(b) \right| < \varepsilon \text{ whenever } |x-b| < \delta$$

$$\Rightarrow g'(b) - \varepsilon < \frac{g(x)-g(b)}{x-b} < g'(b) + \varepsilon \text{ whenever } x \in (b-\delta, b+\delta)$$

$$\Rightarrow 0 < \frac{g(x)-g(b)}{x-b} < g'(b) + \varepsilon \text{ whenever } x \in (b-\delta, b+\delta)$$

$$\Rightarrow \frac{g(x)-g(b)}{x-b} > 0 \text{ whenever } x \in (b-\delta, b+\delta) \ \& \ x \neq b$$

$$\Rightarrow g(x) - g(b) > 0 \text{ if } x \in (b, b+\delta) \Rightarrow g(x) > g(b) \text{ for } \forall x \in (b, b+\delta)$$

and $g(x) - g(b) < 0$ if $x \in (b-\delta, b)$

Thus, $g(y) < g(b)$ if $y \in (b-\delta, b)$

i.e., there exists $y \in (a,b)$ where $g(y) < g(b)$

Now consider $g'(a) < 0$, choose any $\varepsilon > 0$ s.t. $g'(a) + \varepsilon < 0$

Since $\lim_{x \to a} \frac{g(x)-g(a)}{x-a} = g'(a)$, then there exists $\delta > 0$ s.t.

$$\left| \frac{g(x)-g(a)}{x-a} - g'(a) \right| < \varepsilon \text{ whenever } |x-a| < \delta$$

$$\Rightarrow g'(a) - \varepsilon < \frac{g(x)-g(a)}{x-a} < g'(a) + \varepsilon \text{ whenever } x \in (a-\delta, a+\delta)$$

$$\Rightarrow g'(a) - \varepsilon < \frac{g(x)-g(a)}{x-a} < 0 \text{ whenever } x \in (a-\delta, a+\delta)$$

Thus, $\frac{g(x)-g(a)}{x-a} < 0$ if $x \in (a-\delta, a+\delta)$

$$\Rightarrow \begin{cases} g(x) - g(a) < 0 & \text{if } x \in (a, a+\delta) \\ g(x) - g(a) > 0 & \text{if } x \in (a-\delta, a) \end{cases} \Rightarrow \begin{cases} g(x) < g(a) & \text{if } x \in (a, a+\delta) \\ g(x) > g(a) & \text{if } x \in (a-\delta, a) \end{cases}$$

(b) $g(x)$ is differentiable on $[a,b]$ $\Rightarrow$ $g(x)$ is continuous on $[a,b]$

This implies that g is bounded on $[a,b]$ and attains its bounds

Thus, there exists $c, d \in [a,b]$ s.t.

$$g(c) = \sup g \text{ on } [a,b] = U([a,b])$$

$$g(d) = \inf g \text{ on } [a,b] = L([a,b])$$

Now u is $\sup g$ on $[a,b]$, thus $g(x) \le u$ for $\forall x \in [a,b]$

Assume $c \ne a$ (similar as $c \ne b$)

- If $c = a$, then $g(c) = g(a) = u$

  From $g(x) > g(a)$ if $x \in (a-\delta, a)$ $\Rightarrow$ $f(x) > u$ for $\forall x \in (a-\delta, t)$

  which contradicts $g(x) \le u$ for $\forall x \in [a,b]$

- If $c = b$, then $g(c) = g(b) = u$

  then from $g(x) > g(b)$ for $\forall x \in (b, b+\delta)$ $\Rightarrow$ $g(x) > u$ for $\forall x \in (b, b+\delta)$

  which contradicts $g(x) \le u$ for $\forall x \in [a,b]$

Therefore, the assumption is correct.

Show that $g'(c) = 0$

Suppose $g'(c) > 0$, then there exists $\delta > 0$ s.t.

$g(x) > g(c)$ $\forall x \in (c, c+\delta)$ $\Rightarrow$ $g(x) > u$ for $\forall x \in (c, c+\delta)$

which is a contradiction.

Similarly for $g'(c) < 0$.

Thus, there exists $c \in (a,b)$ s.t. $g'(c) = 0$.

Exercise 3. Let J be an open interval. If $f : J \to \mathbb{R}$ is differentiable at $c \in J$ and $f'(c) > 0$, then prove that there exists a $\delta > 0$ for which the following holds:

(a) $(c - \delta, c + \delta) \subseteq J$.
(b) If $x, y \in (c - \delta, c + \delta)$ and $x < c < y$, then $f(x) < f(c) < f(y)$.

(a) Given that J is an open interval, which means J is an open set, since an open interval is also an open set
Then each point of J is an interior point of J
Given that $c \in J$, which means c is an interior point of J
Then there exists a $\delta > 0$ s.t. $c \in (c - \delta, c + \delta) \subseteq J$

(b) $f: J \to \mathbb{R}$ is differentiable at $c \in J$ and $f'(c) > 0$ implies $\lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c}\right) > 0$.
Then there exists a $\delta > 0$ s.t. $\frac{f(x) - f(c)}{x - c} > 0$ for $\forall x \in (c - \delta, c + \delta) \cap J$
Then, $\frac{f(x) - f(c)}{x - c} > 0$ for $\forall x \in (c - \delta, c) \cap J$ and $\forall x \in (c, c + \delta) \cap J$
Thus, assume $x < c < y$ and $x, y \in (c - \delta, c + \delta)$,
We can find $f(x) < f(c)$ for $\forall x \in (c - \delta, c) \cap J$ and $f(y) < f(c)$ for $\forall y \in (c, c + \delta) \cap J$
$\to f(x) < f(c) < f(y)$

Exercise 4. If $f : [a, b] \to \mathbb{R}$ is one-to-one, then there exists an *inverse function* f^{-1} defined on the image of f given by $f^{-1}(y) = x$ where $y = f(x)$. Suppose that f is differentiable and that $f'(x) \neq 0$ for all $x \in [a, b]$. Prove that f^{-1} is differentiable with

$$(f^{-1})'(y) = \frac{1}{f'(x)} \qquad \text{where } y = f(x).$$

Proof:

Suppose $f: [a, b] \to \mathbb{R}$ is a one-to-one function implies $f: [a, b] \to$ range $\mathbb{R}$ is a, b section
f^{-1} as also a bijection from range $\mathbb{R}$ to $[a, b]$
Suppose f is differentiable on $[a, b]$
Thus, f is continuous on $[a, b]$
Let $f^{-1} = g$ and suppose $y = f(x)$ for $x \in [a, b]$ and $d = f(c)$ for $c \in [a, b]$
Since $f^{-1} = g$, we have $x = f^{-1}(y) = g(y),\ c = f^{-1}(d) = g(d)$
Now $c \in [a, b]$, f is continuous on $[a, b]$ means f is continuous at $f(c)$
$f^{-1}(c)$ is continuous at $f(c)$ and g is continuous at d

$\lim\limits_{y \to c} \frac{g(y)-g(d)}{y-d} = \lim\limits_{y \to c} \frac{x-c}{f(x)-f(c)} = \frac{1}{\lim\limits_{y \to c} \frac{f(x)-f(c)}{x-c}} = \frac{1}{f'(c)}$

Thus, g is differentiable at d, that is f^{-1} is differentiable at $f(c)$
Also $g'(d) = \frac{1}{f'(c)} \Rightarrow f^{-1}(d) = \frac{1}{f'(c)}$
Since $f(c)$ is arbitrary, we have f^{-1} is differentiable at every point in range $\mathbb{R}$
Also $f^{-1}(y) = \frac{1}{f'(x)}$ where $f(x) = y$

Exercise 5. Consider three functions $f, g, h : I \to \mathbb{R}$ defined on an interval I and suppose that $f(x) \le g(x) \le h(x)$ for all $x \in I$. If f and h are differentiable at $c \in I$, with $f(c) = h(c)$ and $f'(c) = h'(c)$, then prove that g is also differentiable at c and that $g'(c) = f'(c)$.

Proof:
Suppose $x = c$, then $f(x) \le g(x) \le h(x)$ implies $f(c) \le g(c) \le h(c)$
Thus, $f(x) - f(c) \le g(x) - g(c) \le h(x) - h(c)$
Step 1:
Suppose $x \to c^+$, then $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \le \lim_{x\to c^+} \frac{g(x)-g(c)}{x-c} \le \lim_{x\to c^+} \frac{h(x)-h(c)}{x-c}$
$\Rightarrow f'(c) \le \lim_{x\to c^+} \frac{g(x)-g(c)}{x-c} \le h'(c)$
Step 2:
Consider $x \to c^-$, then $x - c < 0$
$\lim_{x\to c^-} \frac{h(x)-h(c)}{x-c} \le \lim_{x\to c^-} \frac{g(x)-g(c)}{x-c} \le \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$
$\Rightarrow h'(c) \le \lim_{x\to c^-} \frac{g(x)-g(c)}{x-c} \le f'(c)$
Since $f'(c) = h'(c)$
Therefore, $\lim_{x\to c^-} \frac{g(x)-g(c)}{x-c} = \lim_{x\to c^+} \frac{g(x)-g(c)}{x-c} = f'(c)$
Thus, g is differentiable at c and $g'(c) = f'(c)$

Exercise 6. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable at each $x \in (a, b)$. Suppose that $f'(x) \geq 0$ for all $x \in (a, b)$. If $f'(x) = 0$ at only finitely many points in (a, b), then prove that f is a strictly increasing function.

Consider $\alpha, \beta \in [a, b]$ s.t. $a \leq \alpha < \beta \leq b$, then f is continuous on $[\alpha, \beta]$ and differentiable on (α, β)
Thus, by Langrange's Mean Value Thm: $\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(c)$ for $\exists c \in (\alpha, \beta)$
But $f'(x) \geq 0$ for $\forall x \in (\alpha, \beta)$
Thus, $f'(c) \geq 0 \implies f(\beta) - f(\alpha) \geq 0 \implies f(\beta) \geq f(\alpha)$
Claim: $f(\beta) > f(\alpha)$
Assume $f(\beta) = f(\alpha) \implies f(\beta) - f(\alpha) = 0$
But α, β are any arbitrary points of $[a, b]$
Thus f is constant function in $[a, b] \implies f'(x) = 0$ for $\forall x \in [a, b]$
Which is a contradiction as given that $f'(x) = 0$ only for finitely many points in (a, b)
This implies that the assumption is not correct
i.e., $f(\alpha) \neq f(\beta)$, so $f(\beta) > f(\alpha)$
Then for $\forall\ \alpha, \beta \in [a, b]$ s.t. $\alpha < \beta \implies f(\alpha) < f(\beta)$
This implies that f is a strictly increasing function.

Exercise 7. Let $f : I \to \mathbb{R}$ be a differentiable function. Suppose that there is a constant $c > 0$ such that $|f(x) - f(y)| \leq c|x - y|^2$ for any $x, y \in I$. Prove that f must be constant.

Since f is differentiable at all points on I,

$$f'(x) = \lim_{y \to x} \frac{f(x) - f(y)}{x - y} \leq \lim_{y \to x} \frac{c \cdot |x - y|^2}{x - y} = \lim_{y \to x} c \cdot |x - y| = 0$$

Since x is arbitrary and $f'(x) = 0$ for any x

Then $f'(x) = 0$ for $\forall x \in I$

This implies that f is constant function.

We will need the following definition for Exercise 8:

Definition. A function $f : X \to \mathbb{R}$ is *bounded* if there is an interval $[m, p]$ so that $f(X) \subseteq [m, p]$.

Exercise 8. Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable and bounded. Moreover, suppose that the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to b} g(x)$$

do not exist. Prove that, for any $\alpha \in \mathbb{R}$, there is a point $c \in (a, b)$ such that $f'(c) = \alpha$.

By Darboux's Thm, we have that the image $f'((a,b))$ is $\exists$ interval I

Show that $I = \mathbb{R}$.

To show that $I = \mathbb{R}$, we need to show that I cannot be bounded below nor bounded above.

Recall Thm A: $f : (a,b) \to \mathbb{R}$ is continuous, bounded, and monotone function, then the limits $\lim_{x\to a} f(x)$ & $\lim_{x\to b} f(x)$ exist

Assume I is of the form (c, ∞) with $c \geq 0$

i.e. all the derivatives of f are non-negative

then $\lim_{x\to a} f'(a,b)$ & $\lim_{x\to b} f'(a,b)$ exist

which implies that I cannot be bounded

Similar contradiction can be reached by assuming all the derivatives of f are non-positive

Now consider f has both both positive and negative derivatives

Suppose I is of the form (c, ∞) with $c < 0$

Define $h : (a,b) \to \mathbb{R}$ as $h(x) = f(x) - cx$

Then all the derivatives of $h(x)$ are negative (contradiction)