MATH 317: Homework 4

Deadline: Monday 03/04/2020

Exercise 1. Assume $f : [a, b] \to \mathbb{R}$ is integrable.

(a) Show that if g satisfies $g(x) = f(x)$ for all but a finite number of points in $[a, b]$, then g is integrable as well. What is the integral of g?

(b) Find an example to show that g may fail to be integrable if it differs from f at a countable number of points.

Exercise 2. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable over $[0, 1]$ and compute $\int_0^1 f dx$.

Exercise 3. Let g be bounded on $[a, b]$ and assume there exists a partition P with $L(g, P) = U(g, P)$. Describe g. Is it integrable? If so, what is the value of $\int_a^b g$?

Exercise 4. Exercise 7.4.3. from the book *'Understanding Analysis'* by Stephen Abbott.

Exercise 5. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

(a) Show that if f is integrable on the interval $[a, b]$, then $|f|$ is also integrable on this interval.

(b) Also, prove that if f is integrable over $[a, b]$, then

$$\left| \int_a^b f dx \right| \leq \int_a^b |f| dx.$$

Exercise 6. Show that if $f(x) > 0$ for all $x \in [a, b]$ and f is integrable, then $\int_a^b f dx > 0$.

Exercise 7. Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\int_a^x f = 0$ for all $x \in [a, b]$, then $f(x) = 0$ everywhere on $[a, b]$. Provide an example to show that this conclusion does not follow if f is not continuous.

Exercise 8. Recall that a set $X \subseteq \mathbb{R}$ is said to have *measure zero* if, for any $\epsilon > 0$, there exists a countable collection of open intervals $\{(a_n, b_n)\}_n$ satisfying:

$$X \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n - a_n) < \epsilon.$$

Prove that any countable set $X \subseteq \mathbb{R}$ has *measure zero.*

Exercise 1. Assume $f : [a, b] \to \mathbb{R}$ is integrable.

(a) Show that if g satisfies $g(x) = f(x)$ for all but a finite number of points in $[a, b]$, then g is integrable as well. What is the integral of g?

(b) Find an example to show that g may fail to be integrable if it differs from f at a countable number of points.

(a) S'pose that $f : [a,b] \longrightarrow \mathbb{R}$ is bounded. f is cont. at every point in $[a,b]$ except a single point $c \in (a,b)$. Prove that f is integrable on $[a,b]$

Recall: Integrals Analytically

Proof: Fix $\varepsilon > 0$, let $m = \inf f(x)$, $M = \sup f(x)$

Pick an $n \in \mathbb{N}$ large enough so that $[x_0 - \frac{1}{n}, x_0 + \frac{1}{n}] \subseteq [a,b]$ and $\frac{2}{n} < \frac{\varepsilon}{3(M-m)}$

Consider a partition P s.t. $U(f,P) - L(U,P) < \varepsilon$

Step 1: Pick a partition P_1 of $[a, x_0 - \frac{1}{n}]$ s.t. $U(f,P_1) - L(U,P_1) < \varepsilon/3$

Step 2: Pick a partition P_2 of $[x_0 + \frac{1}{n}, b]$ s.t. $U(f,P_2) - L(U,P_2) < \varepsilon/3$

Step 3: Define $P = P_1 \cup P_2$, we have

$$U(f,P) - L(f,P)$$

$$= U(f,P_1) + U(f, \{x_0 - \tfrac{1}{n}, x_0 + \tfrac{1}{n}\}) + U(f,P_2) - [L(f,P_1) + L(f, \{x_0 - \tfrac{1}{n}, x_0 + \tfrac{1}{n}\}) + L(f,P_2)]$$

$$= (U(f,P_1) - L(f,P_1)) + (U(f,P_2) - L(f,P_2)) + (U(f, \{x_0 - \tfrac{1}{n}, x_0 + \tfrac{1}{n}\}) - L(f, \{x_0 - \tfrac{1}{n}, x_0 + \tfrac{1}{n}\}))$$

$$< \varepsilon/3 + \varepsilon/3 + (M-m) \cdot \frac{\varepsilon}{3(M-m)} = \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

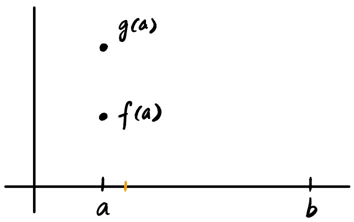
Then through induction,

1.(a) f is integrable $\Rightarrow$ f is bounded

$g = f \Rightarrow g$ is bounded $\Rightarrow$ $\inf g = m$ & $\sup g = M$ exist

<u>Base case</u>:

$x_1 \in [a, b]$, $x_1 = a$ (or $x_1 = b$)



> <u>Recall</u>:
>
> <u>Thm</u>:
>
> $g: [a,b] \longrightarrow \mathbb{R}$ is integrable
>
> $\Longleftrightarrow$ for any $\varepsilon > 0$, there is a partition $P = \{x_0, x_1, \cdots, x_n\}$
>
> s.t. $U(g,P) - L(g,P) < \varepsilon$

Fix $\varepsilon > 0$.

Pick an $x_1 \in (a,b)$ s.t.

$\boxed{x_1 - a < \frac{\varepsilon}{2(m-n)}}$

$[x_1, b] \frown f = g$

There is a partition $P' = \{x_1, \cdots, x_{n-1}, x_n\}$ of $[x_1, b]$ s.t.

$$\boxed{U(g, P') - L(g, P') < \frac{\varepsilon}{2}}$$

Take $P = \{a\} \cup P' = \{\overset{x_0 \,=}{a}, x_1, x_2, \cdots, x_n\}$

$$U(g,P) - L(g,P)$$

$$= \sup g([a,x_1])(x_1 - a) + U(g,P') - [\inf g([a,x_1])(x_1-a) + L(g,P')]$$

$$= \sup g([a,x_1])\underline{(x_1-a)} - \inf g([a,x_1])\underline{(x_1-a)} + [U(g,P') - L(g,P')]$$

$$= (\sup g([a,x_1]) - \inf g([a,x_1]))(x_1-a) + (U(g,P') - L(g,P'))$$

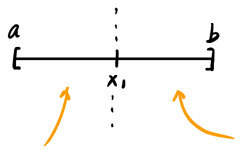
$$< (M-m)(x_1-a) + \frac{\varepsilon}{2}$$

$$< (\cancel{M-m})\frac{\varepsilon}{2\cancel{(M-m)}} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

__General Case__: $f \neq g$ at $x_1, x_2, \cdots, x_p \in [a,b]$

Then we can split $[a,b]$ into finitely many subintervals

g_1 (x_1) $\quad g_2$ (x_1, x_2) $\quad g_3$ (x_1, x_2, x_3) $\quad \cdots \quad g_p = g$ ($x_1, x_2, \cdots, x_p$)



g int. over $[a, x_1]$ $\quad$ g int. over $[x_1, b]$

$\Rightarrow g$ is integrable

Through base case, we know g is integrable over each small subinterval

Thus, we can conclude that g is integrable over $[a,b]$

$$\int_a^b g\,dx = \int_a^b f\,dx$$

(b) Let $g(x) = \begin{cases} 1 & \text{if } x \in Q \\ -1 & \text{if } x \in I \end{cases}$ & $f(x) = -1$

Then $g(x) = f(x)$ except when $x \in Q$, Q is countable

$f(x)$ is integrable

but $g(x)$ is not integrable

HW4 Ex1 (a) $f : [a,b] \to \mathbb{R}$ integrable ← bounded

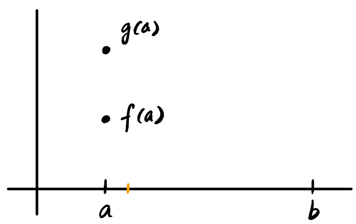
← bounded

$g : [a,b] \to \mathbb{R}$: $g(x) = f(x)$ everywhere on $[a,b]$ except $x_1, x_2, \cdots, x_p \in [a,b]$

$\Rightarrow$ g is integrable   bounded

(Prove by induction) $\Rightarrow$ $M = \sup g$ $m = \inf g$

Case 1: $x_1 \in [a,b]$, $x_1 = a$ or $x_1 = b$



Recall:

Thm:

$g : [a,b] \to \mathbb{R}$ is integrable

$\Longleftrightarrow$ for any $\varepsilon > 0$, there is a partition $P = \{x_0, x_1, \cdots, x_n\}$ s.t. $U(g,P) - L(g,P) < \varepsilon$

Fix $\varepsilon > 0$.

Pick an $x_1 \in (a,b)$ s.t.

$\boxed{x_1 - a < \frac{\varepsilon}{2(m-n)}}$

$[x_1, b]$ $f = g$

There is a partition $P' = \{x_1, \cdots, x_{n-1}, x_n\}$ of $[x_1, b]$ s.t.

$$\boxed{U(g,P') - L(g,P') < \frac{\varepsilon}{2}}$$

Take $P = \{a\} \cup P' = \{\overset{x_0 \,=}{a}, x_1, x_2, \cdots, x_n\}$

$$U(g,P) - L(g,P)$$

$$= \sup g([a,x_1])(x_1 - a) + U(g,P') - [\inf g([a,x_1])(x_1-a) + L(g,P')]$$

$$= \sup g([a,x_1])(x_1 - a) - \inf g([a,x_1])(x_1-a) + [U(g,P') - L(g,P')]$$

$$= \underbrace{(\sup g([a,x_1]) - \underbrace{\inf g([a,x_1])}_{\text{only over } [a,x_1] \Rightarrow < m})}_{\le M-m}(x_1 - a) + \underbrace{(U(g,P') - L(g,P'))}_{< \frac{\varepsilon}{2}}$$

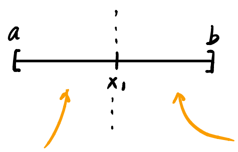
$$< (M-m)(x_1 - a) + \frac{\varepsilon}{2}$$

$$< \cancel{(M-m)}\frac{\varepsilon}{2\cancel{(M-m)}} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

<u>General Case</u>: $f \neq g$ at $x_1, x_2, \cdots, x_p \in [a,b]$

g_1 g_2 g_3 $\cdots$ $g_p = g$

x_1 x_1, x_2 x_1, x_2, x_3 $x_1, x_2, \cdots, x_p$



g int. over $[a, x_1]$ g int. over $[x_1, b]$

$\Rightarrow g$ is integrable

g is integrable

$$\int_a^b f\,dx = \int_a^b g\,dx$$

Q **Exercise 2.** Let $f : [0, 1] \to \mathbb{R}$ be defined by

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$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable over $[0, 1]$ and compute $\int_0^1 f\,dx$.

Thm: $f : [a, b] \to \mathbb{R}$

f is integrable $\Leftrightarrow$ There is a sequence of partition P_n of $[a,b]$ with $\boxed{U(f, P_n) - L(f, P_n)} \to 0$

By induction, if a function is 0 everywhere except finitely many points, then the integral will also be zero.

Exercise 2. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable over $[0,1]$ and compute $\int_0^1 f\,dx$.

Proof: Let $x_n = \frac{1}{n}$, $\{x_1, x_2, \cdots, x_n, \cdots\} = \{1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\}$

Claim: f is integrable over $[0,1]$

Recall: For any $\varepsilon > 0$, there is a sequence of partition P s.t.

$$U(f, P_n) - L(f, P_n) < \varepsilon$$

Given any $\varepsilon > 0$, since $(\frac{1}{n}) \to 0$, $\exists N \in \mathbb{N} \Rightarrow |\frac{1}{n}| < \varepsilon \quad \forall n \geq N$

Now take partition $P = \{x_1, x_2, \cdots, x_N, 0\}$

$$U(f,P) = \sum_{i=}^{N-1} M_i(f)(x_{i+1} - x_i) + M(f)(\tfrac{1}{N} - 0)$$

$$< (-\tfrac{1}{2} + 1) + (-\tfrac{1}{3} + \tfrac{1}{2}) + \cdots + (-\tfrac{1}{N} - \tfrac{1}{N-1}) + \varepsilon$$

$$< \varepsilon - \tfrac{1}{N} < \varepsilon$$

$\Rightarrow U(f,P) < \varepsilon$ implies that f is integrable over $[0,1]$

Recall f is integrable if $L(f) = U(f)$

$$\int_0^1 f\,dx = L(f) = U(f) = 0$$

Exercise 3. Let g be bounded on $[a, b]$ and assume there exists a partition P with $L(g, P) = U(g, P)$. Describe g. Is it integrable? If so, what is the value of $\int_a^b g$?

Since g is bounded on $[a,b]$, there exists $M>0$ s.t. $|g(x)| \le M$, $\forall x \in [a,b]$

Let $\varepsilon > 0$ and $\{O_1, O_2, \cdots, O_N\}$ be open intervals covers D_f s.t.

$$\sum_{n=1}^{N} |O_n| \le \frac{\varepsilon}{4M}$$

$(\bigcup_{n=1}^{N} O_n)^c$ is the finite collection of closed intervals on which g is continuous

then g is integrable

Thus, there exists a partition P_1 of $(\bigcup_{n=1}^{N} O_n)^c$ s.t.

$$U(g, P_1) - L(g, P_1) < \varepsilon/2$$

Consider a partition P from P_1 by adding finitely many end points of O_n

Then
$$\begin{aligned} U(g,P) - L(g,P) &= (U(g,P_1) - L(g,P_1)) + \sum_{n=1}^{N} (M_n - m_n)|O_n| \\ &\le U(g,P_1) - L(g,P_1) + \sum_{n=1}^{N} 2M|O_n| \\ &< \varepsilon/2 + 2M \cdot \sum_{n=1}^{N} |O_n| \\ &\le \varepsilon/2 + \cancel{2}\cancel{M} \cdot \frac{\varepsilon}{\cancel{4}_2 \cancel{M}} \\ &= \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

$\Rightarrow U(g,P) - L(g,P) < \varepsilon \iff g$ is integral (by Integral Analitically Thm)

Since g is integrable $\Rightarrow \int_a^b g = L(g) = U(g)$

$\Rightarrow \int_a^b g = m(b-a)$

Exercise 3. Let g be bounded on $[a, b]$ and assume there exists a partition P with $L(g, P) = U(g, P)$. Describe g. Is it integrable? If so, what is the value of $\int_a^b g$?

A bounded function $g : [a,b] \to \mathbb{R}$ is integrable if $L(g) = U(g)$

$U(g) = \inf \{ U(g,P) : P \in \mathcal{P} \}$

$L(g) = \sup \{ L(g,P) : P \in \mathcal{P} \}$

since $U(g,P) = L(g,P)$, $U(g) = L(g)$

and $\int_a^b g(x) = L(g) = U(g)$

Exercise 4. Exercise 7.4.3. from the book *'Understanding Analysis'* by Stephen Abbott.

Exercise 7.4.3. Decide which of the following conjectures is true and supply a short proof. For those that are not true, give a counterexample.

(a) If $|f|$ is integrable on $[a, b]$, then f is also integrable on this set.

(a) False. Counterexample: $f(x) = \begin{cases} 1 & \text{if } x \in Q \\ -1 & \text{if } x \in I \end{cases}$ $\Rightarrow$ not integrable

$|f(x)| = 1 \Rightarrow$ continuous $\Rightarrow$ integrable

(b) Assume g is integrable and $g(x) \geq 0$ on $[a, b]$. If $g(x) > 0$ for an infinite number of points $x \in [a, b]$, then $\int_a^b g > 0$.

False. Let $[a,b]$ be defined as $[0,1]$

Define $g(x) = \begin{cases} \frac{1}{n} & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$ then $\int_a^b g = 0$

(c) If g is continuous on $[a, b]$ and $g(x) \geq 0$ with $g(y_0) > 0$ for at least one point $y_0 \in [a, b]$, then $\int_a^b g > 0$.

Proof: Fix $\varepsilon = \frac{g(y_0)}{2} > 0$

By continuity at y_0, there exists a $\delta > 0$ s.t. $|x - y_0| < \delta$ s.t.

$|g(x) - g(y_0)| < \frac{g(y_0)}{2} \iff -\frac{g(y_0)}{2} < g(x) - g(y_0) < \frac{g(y_0)}{2}$

$\Rightarrow \frac{g(y_0)}{2} < g(x)$ for $|x - y_0| < \delta \Leftrightarrow \underset{m}{\frac{g(y_0)}{2}} < g(x)$ for $x \in \underset{[c,d]}{[y_0 - \delta, y_0 + \delta]}$

Since g is integrable, then $\int_{y_0-\delta}^{y_0+\delta} g \geq \frac{g(y_0)}{2}[(y_0+\delta) - (y_0 - \delta)]$

$\Rightarrow \int_{y_0-\delta}^{y_0+\delta} g \geq g(y_0) \cdot \delta > 0$

$\int_a^b g = \int_a^{y_0-\delta} g + \int_{y_0-\delta}^{y_0+\delta} g + \int_{y_0+\delta}^b g$, Then $g(x) \geq 0$ on $[a,b]$

$\Rightarrow \int_a^{y_0-\delta} g \geq 0$ & $\int_{y_0+\delta}^b g \geq 0$

$\int_a^{y_0-\delta} g + \int_{y_0-\delta}^{y_0+\delta} g + \int_{y_0+\delta}^b g \geq \int_{y_0-\delta}^{y_0+\delta} g \geq g(y_0)\delta > 0$

$\Rightarrow \int_a^b g > 0$

Exercise 5. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

(a) Show that if f is integrable on the interval $[a, b]$, then $|f|$ is also integrable on this interval.

(b) Also, prove that if f is integrable over $[a, b]$, then

$$\left|\int_a^b f\,dx\right| \leq \int_a^b |f|\,dx.$$

Proof: Let $\varepsilon > 0$. since f is integrable, f is bounded $\Rightarrow |f|$ is bounded

f is integrable $\Rightarrow$ there exists a partition $P_\varepsilon = \{x_0, x_1, \cdots, x_n\}$ s.t.

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

Define $\overline{m_i} = \inf\{|f(x)| : x \in [x_{i-1}, x_i]$

$\overline{M_i} = \sup\{|f(x)| : x \in [x_{i-1}, x_i]$

$f(x) \leq |f(x)|$ for $\forall x$, then $m_i \leq \overline{m_i}$ & $M_i \leq \overline{M_i}$ for $\forall i$

$\Rightarrow M_i - m_i \geq \overline{M_i} - \overline{m_i}$

if M_i & m_i are both positive / negative $\Rightarrow$ these two are equal

else: taking absolute value makes them getting closer

Then, $U(|f|, P_\varepsilon) - L(|f|, P_\varepsilon) = \sum_{i=1}^{n} (\overline{M_i} - \overline{m_i})(x_i - x_{i-1})$

$\leq \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$

$= U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$

Then by integral analytically thm, $|f|$ is integrable

(b) Consider $-|f(x)| \leq f(x) \leq |f(x)|$

Recall: $f \leq g \Rightarrow \int f \leq \int g$

Then $-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx \leftarrow (*)$

Recall: $-t \leq s \leq t \iff |s| \leq t$ for $\forall s, t \in \mathbb{R}$

Then $(*) \iff \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

Exercise 6. Show that if $f(x) > 0$ for all $x \in [a,b]$ and f is integrable, then $\int_a^b f\,dx > 0$.

Proof: Suppose $f(x) > 0$ for $\forall x \in [a,b]$ and f is integrable,

then f is bounded, and by following proposition:

Recall: $f: [a,b] \to \mathbb{R}$ is a bounded function, $m \le f(x) \le M$ for $\forall x \in [a,b]$

Then $m(b-a) \le L(f) \le U(f) \le M(b-a)$

Now we have $m \cdot (b-a) \le L(f) \Rightarrow 0 \cdot (b-a) < L(f) \Rightarrow 0 < L(f)$

By def of integrable $\Rightarrow \int_a^b f\,dx = L(f)$

$\Rightarrow \int_a^b f\,dx > 0$

Measure zero Any interval => sth. that has length cannot be measure 0

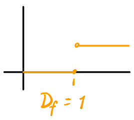
Being a measure 0 $\Rightarrow$ That set doesn't contain any length. (segment)

(No interval can be measure 0, unless the interval is a single point)

Exercise 6. Show that if $f(x) > 0$ for all $x \in [a,b]$ and f is integrable, then $\int_a^b f\,dx > 0$.

<u>Proof</u>: <u>Recall Thm</u>:

$f: [a,b] \to \mathbb{R}$ ($a<b$) is integrable $\Leftrightarrow$ The set of discontinuity points (D_f) has <u>measure zero</u>



<u>Step 1</u>: There is a $c \in [a,b]$ s.t. f is continuous at c

Since $a<b$, there exists a positive length between a, b

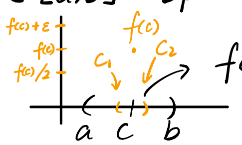
which means $[a,b]$ cannot be measure 0

(No interval can be measure 0, unless the interval is a single point)

Then $D_f \subset [a,b]$ ($D_f \neq [a,b]$),

There exists $\exists\, c \in [a,b] - D_f \Rightarrow f$ is continuous at c

WLOG, $c \in (a,b)$



$f(c) > 0$, f is positive at c

Find $[c_1, c_2]$ sufficiently small so that $\int_{c_1}^{c_2} f\,dx > 0$

$f(x) > 0$ for $\forall x \Rightarrow 0 \leq m(b-a) \leq \int_a^b f\,dx$

By continuity of f at c, there exists $\exists\, \delta$ s.t.

$$f(x) \in (f(c) - \varepsilon,\ f(c) + \varepsilon) \quad \text{if} \quad x \in (\overset{c_1}{c-\delta},\ \overset{c_2}{c+\delta})$$

$\Rightarrow f(c) - \varepsilon < f(x) \Rightarrow f(c)/2 < f(x)$

Show that $\int_a^b f\,dx > 0$:

$$\int_a^b f\,dx = \int_a^{c-\delta} f\,dx + \int_{c-\delta}^{c+\delta} f\,dx + \int_{c+\delta}^b f\,dx$$

$$\int_a^{c-\delta} f\,dx \geq 0, \quad \int_{c+\delta}^b f\,dx \geq 0, \quad \int_{c-\delta}^{c+\delta} f\,dx \geq 2\delta \frac{f(c)}{2} > 0$$

$$\Rightarrow \int_a^{c-\delta} f\,dx + \int_{c-\delta}^{c+\delta} f\,dx + \int_{c+\delta}^b f\,dx > 0$$

$$\Rightarrow \int_a^b f\,dx > 0$$

Exercise 7. Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\int_a^x f = 0$ for all $x \in [a, b]$, then $f(x) = 0$ everywhere on $[a, b]$. Provide an example to show that this conclusion does not follow if f is not continuous.

$f : [a,b] \to \mathbb{R}$ is cont.

Define $F : [a,b] \to \mathbb{R}$, $F(x) = \int_a^x f\,dt$

integrable $\Rightarrow$ continuous $\Rightarrow$ F is diff over $[a,b]$

For $\forall x \in [a,b]$, $\int_a^x f\,dt = 0$ $\Rightarrow$ $F(x) = 0$, F is constant

Recall Fundamental Thm of Calculus, part (ii)

$g : [a,b] \to \mathbb{R}$ is integrable, define $G : [a,b] \to \mathbb{R}$ by $G(x) = \int_a^x g(t)\,dt$

Then G is continuous, $\Rightarrow$ G is differentiable and $G'(x) = g(x)$

Thus, we can imply $f(x) = F'(x) = 0$

Counterexample:

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n},\ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Exercise 8. Recall that a set $X \subseteq \mathbb{R}$ is said to have *measure zero* if, for any $\epsilon > 0$, there exists a countable collection of open intervals $\{(a_n, b_n)\}_n$ satisfying:

$$X \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n - a_n) < \epsilon.$$

Prove that any countable set $X \subseteq \mathbb{R}$ has *measure zero.*

Let $X = \{x_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ be a countable set

Cosider $I_n = \{(a_n, b_n)\}_n$ covers each $\{x_n\}$

$x_n \in I_n \iff I_n = [x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}] \Rightarrow X \subseteq \bigcup_{n=1}^{\infty} I_n \Rightarrow X \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$

Now we can imply that $m(I_n) = |I_n| = (x_n + \frac{\varepsilon}{2^{n-1}}) - (x_n - \frac{\varepsilon}{2^{n-1}}) = \frac{2\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2^n}$

Thus, $\sum_{n=1}^{\infty} (a_n, b_n) = \sum_{n=1}^{\infty} |I_n| = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} < \varepsilon$

Hence $X \subseteq \mathbb{R}$ has measure zero.

Exercise 8. Recall that a set $X \subseteq \mathbb{R}$ is said to have *measure zero* if, for any $\epsilon > 0$, there exists a countable collection of open intervals $\{(a_n, b_n)\}_n$ satisfying:

$$X \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \quad \text{and} \quad \left(\sum_{n=1}^{\infty} (b_n - a_n)\right) < \epsilon.$$

Prove that any countable set $X \subseteq \mathbb{R}$ has *measure zero*.

$\{(x_n - \frac{1}{2^{N+n}}, x_n + \frac{1}{2^{N+n}})\}$

Q: is $x_n \in \{(x_n - \frac{1}{2^{N+n}}, x_n + \frac{1}{2^{N+n}})\}$? $\quad x_n \in X$

A: Yes

$$\sum_{n=1}^{\infty} (x_n + \frac{1}{2^{n+N}} - x_n + \frac{1}{2^{n+N}}) = \sum_{n=1}^{\infty} \frac{2}{2^{n+N}}$$

Consider $X = \{x_n \mid n \in \mathbb{N}\}$ is a countable set

For any $\varepsilon > 0$, take $(a_n, b_n) = (x_n - \varepsilon_n, x_n + \varepsilon_n)$ where $\varepsilon_n = \frac{\varepsilon}{2^{n+1}}$

This means the countable collection of open intervals $\{(a_n, b_n)\}_n$ covers X

$\Rightarrow X \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$

$$\sum_{n=1}^{\infty} |(a_n, b_n)| = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon\left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots\right)$$

$$= \frac{\varepsilon}{2}\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots\right)$$

$$= \frac{\varepsilon}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = \varepsilon$$

For each $n \in \mathbb{N}$, there exists (c_n, d_n) s.t.

$$x_n \in (c_n, d_n), \quad d_n - c_n = \frac{1}{2^{N+n-1}}$$

$$X \subseteq \bigcup_{n=1} (c_n, d_n)$$

$$\sum_{n=1}^{\infty} (d_n - c_n) = \sum_{n=1}^{\infty} \frac{1}{2^{N+n-1}} = \sum_{n=N}^{\infty} \frac{1}{2^n}$$

<u>one point</u> : ?