Thm A: $f: [a,b] \to \mathbb{R}$ is integrable $\Leftrightarrow$ For any $\varepsilon > 0$, there is a partition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$

Notation: $\mathcal{P}([a,b])$ = Set of partitions of $[a,b]$

Thm B

Thm C

(EXERCISE)

Obs: for any partition $P \in \wp([a,b])$, we have:

(1) $U(f+g, P) \le U(f,P) + U(g,P)$

Proof of (1): $P = \{t_0, t_1, \cdots, t_n\}$

Let $h = f+g$

$\sup(h([t_{i-1}, t_i])) < \sup(f([t_{i-1}, t_i])) + \sup(g([t_{i-1}, t_i]))$

$$U(h,P) = \sum_{i=1}^{n} (t_i - t_{i-1}) \sup(h([t_{i-1}, t_i]))$$

$$\le \sum_{i=1}^{n} (t_i - t_{i-1}) \sup(f([t_{i-1}, t_i])) + \sum_{i=1}^{n} (t_i - t_{i-1}) \sup(g([t_{i-1}, t_i]))$$

$$U(h,P) \le U(f,P) + U(g,P)$$

Recall: If R refines P, then $U(f,R) \le U(f,P)$

$L(f,R) \ge L(f,P) \iff -L(f,R) \le -L(f,P)$

In particular, $U(f,R) - L(f,R) \le U(f,P) - L(f,P)$

$$n \to \omega \quad \int_a^b \overset{\downarrow}{f}\, dx + \int_a^b \overset{\downarrow}{g}\, dx \leq \int_a^b f \overset{\downarrow}{+} g\, dx \leq \int_a^b f\, dx + \int_a^b g\, dx$$

Thm D

Thm E