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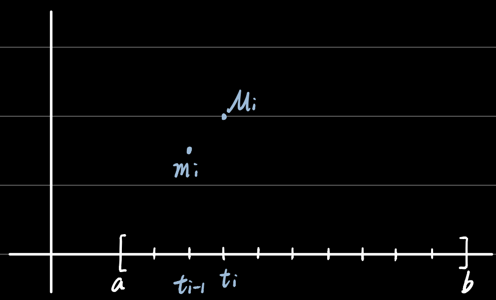
Fact: For any $\delta > 0$, we can find a partition $P = \{t_0, t_1, \cdots t_n\}$ of $[a, b]$ s.t. $t_i - t_{i-1} < \delta \quad \forall i$

Thm 1: f is integrable $\Leftrightarrow$ For any $\varepsilon > 0$, there is a partition P of $[a, b]$ s.t. $U(f, P) - L(f, P) < \varepsilon$



If $x, y \in [t_{i-1}, t_i]$
$\Rightarrow |f(x) - f(y)| < \varepsilon$

Thm 2: If $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable

Proof: Apply Thm 1: For any $\varepsilon > 0$

1) Since f is U.C., then there is a $\delta > 0$ s.t.

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$$

2) By Fact, there is partition $P = \{t_0, t_1, \cdots, t_n\}$ s.t.

$$0 < t_i - t_{i-1} < \delta \quad \text{for } \forall i = 1, \cdots, n$$

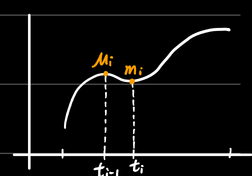
$x_i, y_i \in [t_{i-1}, t_i]$

$M_i = \sup f([t_{i-1}, t_i]) = f(x_i)$

$m_i = \inf f([t_{i-1}, t_i]) = f(y_i)$ $\rightsquigarrow$ Extreme Value Thm

Compute

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} M_i (t_i - t_{i-1}) - \sum_{i=1}^{n} m_i (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} (f(x_i) - f(y_i))(t_i - t_{i-1}) \\ &= \sum^{n} |f(x_i) - f(y_i)| (t_i - t_{i-1}) \qquad (*) \end{aligned}$$



since $x_i, y_i \in [t_{i-1}, t_i] \Rightarrow |x_i - y_i| < \delta$

$$(*) < \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(t_i - t_{i-1}) = \frac{\varepsilon}{b-a}\sum_{i=1}^{n}(t_i - t_{i-1}) = \frac{\varepsilon}{b-a}\cdot(b-a) = \varepsilon$$

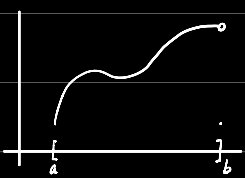
$$\Rightarrow U(f,P) - L(f,P) < \varepsilon \quad \blacksquare$$

$\longrightarrow$ Q: What about function with discontinuous ?

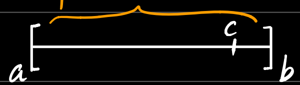
Case 1: Suppose $f : [a,b] \longrightarrow \mathbb{R}$ is continuous everywhere except at $x = b$

(same argument holds for $x = a$)

(This function is integrable)



Proof: Apply Thm 1: For any $\varepsilon > 0$



Step 1: Take a point $c \in (a,b)$ s.t. $b - c = \frac{\varepsilon}{2(M-m)}$

Step 2: Since $f : [a,c] \longrightarrow \mathbb{R}$ cont., then f is integrable over $[a,c]$

Step 3: We can find a partition $P' = \{t_0, t_1, \cdots, t_{n-1}\}$ of $[a,c]$ s.t.

$$U(f,P') - L(f,P') < \frac{\varepsilon}{2}$$

Step 4: $P = P' \cup \{b\}$ is a partition for $[a,b]$ $\longrightarrow$ $\begin{cases} M = \sup f([a,b]) \\ m = \inf f([a,b]) \\ M_n = \sup f([c,b]) \\ m_n = \inf f([c,b]) \end{cases}$

$$\begin{aligned} & U(f,P) - L(f,P) \\ &= \left(U(f,P') + M_n(t_n - t_{n-1})\right) - \left(L(f,P') + m_n(t_n - t_{n-1})\right) \\ &= \left(U(f,P') + M_n(b-c)\right) - \left(L(f,P') + m_n(b-c)\right) \\ &= \left(U(f,P') - L(f,P')\right) + \left(M_n(b-c) - m_n(b-c)\right) \\ &< \frac{\varepsilon}{2} + (M_n - m_n)(b-c) \le \frac{\varepsilon}{2} + (M-m)(b-c) \\ &< \frac{\varepsilon}{2} + \cancel{(M-m)}\cdot\frac{\varepsilon}{2\cancel{(M-m)}} = \varepsilon \end{aligned}$$

We get $U(f,P) - L(f,P) < \varepsilon$

Case 2: What if $f:[a,b] \to \mathbb{R}$ has m discontinuous, with $m > 1$?

Apply induction and use the following Thm

Thm 3: Let $f:[a,b] \longrightarrow \mathbb{R}$ be bounded and let $c \in (a,b)$

Then, f is integrable over $[a,b] \iff f$ is integrable over $[a,c]$ & $[c,b]$

In this case, $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$

But, what if $f:[a,b] \to \mathbb{R}$ is discontinuous at infinite countably many points?

A: Yes, it is integrable

Thm 4: $f:[a,b] \longrightarrow \mathbb{R}$ is integrable $\iff$ The set of discontinuity points has measure 0

Def of measure 0:

$X \subseteq \mathbb{R}$ has measure 0 if, for any $\varepsilon > 0$, we can find a countable collection of open intervals $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ s.t.

1) $X \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$

2) $\sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon$

Exercise in Homework 4:

Show that any countable set the measure 0

Hint: Use the series $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$