Missy Shi

MATH 317: Final Exam

Instructions.

- **There are *eight* exercises in this exam.**
- **You must submit your solutions by 5:00 pm PST on Friday, March 20th. Please send me your solutions via email.**
- **My email address is `megomezl@uoregon.edu`. Please write 'MATH 317' and your name in the subject box.**

Exercise 1. Let $f : [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x < \frac{1}{2}, \\ 2 & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Can there exist a sequence of polynomials $p_n : [0,1] \to \mathbb{R}$ which converges uniformly to f? *You must justify your answer.*

Exercise 2. Let $f : [-10,10] \to \mathbb{R}$ be a differentiable function such that $f(\frac{1}{n}) = f(\frac{1}{m})$ for any $n, m \in \mathbb{N}$. Moreover, suppose that f' is continuous. Show that $f'(0) = 0$.

Exercise 3. Let $f : [a,b] \to \mathbb{R}$ be an integrable function.

(a) Show that there exists a point $d \in [a,b]$ such that $\int_a^c f\,dx \leq \int_a^d f\,dx$ for all $c \in [a,b]$.

(b) Now, suppose that f is continuous. Also, let d be the point we found in part (a), and suppose that we have

$$0 < \int_a^b f\,dx < \int_a^d f\,dx.$$

Show that we must have $f(d) = 0$.

Exercise 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $X \subseteq \mathbb{R}$ a closed set. Suppose that $f(x) \in X$ for all $x \in \mathbb{Q}$. Show that $f(\mathbb{R}) \subseteq X$.

Hint: For this exercise, the following result might be useful: a non-empty set $X \subseteq \mathbb{R}$ is closed if and only if, for any convergent sequence $x_n \in X$, we have that $\lim x_n \in X$.

Exercise 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$f(-20) > -20 \qquad f(0) < 0 \qquad f(20) > 20.$$

Show that there exists a point $c \in \mathbb{R}$ where $f'(c) = 1$.

Exercise 6. Consider the functions $f_1 : [-1,1] \to \mathbb{R}$ and $f_2 : [-1,1] \to \mathbb{R}$ defined respectively by $f_1(x) = |x|$ and

$$f_2(x) = \begin{cases} -x+2 & \text{if } x \leq 0, \\ x-1 & \text{if } x > 0. \end{cases}$$

(a) Does there exist a differentiable function $g : [-1,1] \to \mathbb{R}$ such that $g'(x) = f_1(x)$?

(b) Does there exist a differentiable function $h : [-1,1] \to \mathbb{R}$ such that $h'(x) = f_2(x)$?

Note: In both part (a) and part (b), you must provide a small argument which justifies your answer.

Exercise 7. Let $f_n : X \to \mathbb{R}$ be a sequence of functions that converges uniformly to a function $f : X \to \mathbb{R}$. Assume that each function f_n is uniformly continuous. Prove that f is also uniformly continuous.

Exercise 8. Define $f : [0,1] \to \mathbb{R}$ as follows:

- $f(0) = 0$.
- $f(x) = \frac{1}{2^{n-1}}$ if $\frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}$, where $n \in \mathbb{N}$.

Prove that f is integrable. Also, show that

$$\int_0^1 f\,dx = \sum_{n=1}^{\infty} \frac{1}{2^{2n-1}}.$$

Exercise 1. Let $f : [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x < \frac{1}{2}, \\ 2 & \text{if } x \geq \frac{1}{2}. \end{cases}$$

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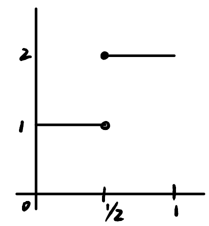
Can there exist a sequence of polynomials $p_n : [0,1] \to \mathbb{R}$ which converges uniformly to f? *You must justify your answer.*

No. Sequences of polynomials are continuous.

The uniform limit continuous function is also continuous.

But $f(x)$ is a discontinuous function,



thus it can not be an uniform limit of polynomials.

Ex 2

Suppose $f'(0) \neq 0$. Let $\varepsilon = |f'(0)| > 0$

Since f' is continuous, there is a $\delta > 0$ s.t. $|f'(x) - f'(0)| < \varepsilon$ whenever $|x| < \delta$

Let $n = \lfloor 1/\delta \rfloor + 1$, by the Mean Value Thm, there exists $\exists\, c \in (\frac{1}{n+1}, \frac{1}{n})$ s.t.

$$f'(c) = \frac{f(\frac{1}{n}) - f(\frac{1}{n+1})}{1/n - 1/(n+1)}$$

Then $f'(c) = 0$. Since $c \in (\frac{1}{n+1}, \frac{1}{n})$, $|c| < \frac{1}{n} < \delta$.

Thus, $\varepsilon = |f'(0)| = |0 - f'(0)| = |f'(c) - f'(0)| < \varepsilon$, which is a contradiction

Hence, $f'(0) = 0$ $\square$

Ex3

(a) Define $F : [a,b] \to \mathbb{R}$ as $F(c) = \int_a^c f\,dx$

F is differentiable $(F' = f) \Rightarrow F$ is continuous

Then F is continuous over a compact set

By the Extreme Value Thm, there exists $\exists\, d \in [a,b]$ s.t.

$F(c) \le F(d)$ for every $c \in [a,b]$

That is, for $\forall c \in [a,b]$, $\int_a^c f\,dx \le \int_a^d f\,dx$.

(b) Suppose $f(d) \neq 0$. Let $\varepsilon = \left|\frac{f(d)}{2}\right| > 0$.

Since f is continuous, there exists a $\delta > 0$ s.t.

$|f(x) - f(d)| < \varepsilon$ whenever $|x - d| < \delta$

Case 1 $f(d) > 0$

For every $x \in (d, d+\delta)$, $|f(x) - 2\varepsilon| < \varepsilon$

which is equivalent to $-\varepsilon < f(x) - 2\varepsilon < \varepsilon \Rightarrow \underline{\varepsilon < f(x)}$

Then $\int_d^{d+\delta} f\,dx \ge \int_d^{d+\delta} \varepsilon\,dx = \varepsilon \cdot \delta > 0$,

This means that $\int_a^{d+\delta} f\,dx = \int_a^d f\,dx + \int_d^{d+\delta} f\,dx > \int_a^d f\,dx$

which contradicts the definition of d.

Case 2 $f(d) < 0$

For every $x \in (d-\delta, d)$, $|f(x) + 2\varepsilon| < \varepsilon$

which is equivalent to $-\varepsilon < f(x) + 2\varepsilon < \varepsilon \Rightarrow \underline{f(x) < -\varepsilon}$

Then $\int_{d-\delta}^d f\,dx \le \int_{d-\delta}^d -\varepsilon\,dx = -\varepsilon \cdot \delta < 0$

This means that $\int_a^d f\,dx = \int_a^{d-\delta} f\,dx + \int_{d-\delta}^d f\,dx < \int_a^{d-\delta} f\,dx$

which contradicts the definition of d.

Ex 4.

Let $x \in \mathbb{R}$, let (x_n) be a sequence of rational numbers, and $(x_n) \to x$

Since f is continuous, $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$

Thus, $f(x)$ is the limit of a sequence in X.

X is closed $\Rightarrow f(x) \in X$.

Ex 5

Define $g: \mathbb{R} \to \mathbb{R}$ as $g(x) = f(x) - x$

Then, $g(-20) > 0$ $\quad g(0) < 0$ $\quad g(20) > 0$

Since f and x are continuous $\Rightarrow$ g is also continuous

By the Int. Value Thm, there exists $a \in (-20, 0)$, $b \in (0, 20)$ s.t.

$$g(a) = g(b) = 0$$

Thus, $f(a) = a$ and $f(b) = b$

By the Mean Value Thm, there exists $\exists\, c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b-a}{b-a} = 1$$

Exercise 6. Consider the functions $f_1 : [-1,1] \to \mathbb{R}$ and $f_2 : [-1,1] \to \mathbb{R}$ defined respectively by $f_1(x) = |x|$ and

$$f_2(x) = \begin{cases} -x+2 & \text{if } x \leq 0, \\ x-1 & \text{if } x > 0. \end{cases}$$

(a) Yes. Define $g : [-1,1] \to \mathbb{R}$ as $g(x) = \begin{cases} \frac{1}{2}x^2 & x \geq 0 \\ -\frac{1}{2}x^2 & x < 0 \end{cases}$

Then $g(x)$ is differentiable at $x \neq 0$ since polynomials are differentiable

$g'(0) = \lim_{h \to 0} \frac{g(h)-g(0)}{h} = 0$ because for any $\varepsilon > 0$,

there exists a $\delta = 2\varepsilon$ s.t. if $0 < |h| < \delta$, then

$$\left| \frac{g(h)-g(0)}{h} - 0 \right| = \frac{\pm\frac{1}{2}h^2 - 0}{h} = \left|\frac{h^2}{2h}\right| = \frac{|h|}{2} < \frac{\delta}{2} < \varepsilon$$

(b) No. Suppose there exists a differentiable function $h' = f_2$

Let $\varepsilon = 2$ and $\delta > 0$. Let $x = \min\{\delta/2, 1\}$.

Since f_2 is negative for $x \in (0,1)$, h is decreasing over $[0,1]$

Thus, $h(x) \leq h(0)$.

Then $\left|\frac{h(x)-h(0)}{x} - f_2(0)\right| = \left|\frac{h(x)-h(0)}{x} - 2\right| = \left|\frac{h(0)-h(x)}{x} + 2\right| = \frac{h(0)-h(x)}{x} + 2 \geq 2$

This contradicts the fact that $h'(0) = \lim_{h \to 0} \frac{h(x)-h(0)}{x} = f_2(0)$

Hence, the function h does not exist.

Ex 7

Let $\varepsilon > 0$. Since f_n converges uniformly to f, there exists an $N \in \mathbb{N}$ s.t.

if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon/3$ for $\forall x \in X$

Since f_N is uniformly continuous, there exists a $\delta > 0$ s.t.

for $x, y \in X$, if $|y - x| < \delta$, then $|f_N(x) - f_N(y)| < \varepsilon/3$

For any $x, y \in X$, if $|y-x| < \delta$, then by the triangle inequality,

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

Hence, $f(x)$ is uniformly continuous.

Ex 8

For $n \in \mathbb{N}$ and for each $i \in \{0, \cdots, 2^n\}$, define $x_i = x_{n,i} = \frac{i}{2^n}$

For $n \in \mathbb{N}$, let P_n be a partition of $[0,1]$ which uses the $x_{n,i}$

M_i is the max value of f over interval $[x_i, x_{i+1}]$, and

m_i is the min value of f over interval $[x_i, x_{i+1}]$

Then $M_0 = \frac{1}{2^n}$ and $m_0 = 0$

If $i > 0$ and i is not one more than a power of 2, then

M_i and m_i are equal.

If $i = 2^k+1$, then $M_i - m_i = \frac{2^{k+1}}{2^n} - \frac{2^k}{2^n} = \frac{2^k}{2^n}$

Thus $$U(f, P_n) - L(f, P_n) = \sum_{i=0}^{2^n-1} (M_i - m_i) \cdot \frac{1}{2^n} = \frac{1}{2^n}\left(\sum_{i=0}^{2^n-1}(M_i - m_i)\right)$$
$$= \frac{1}{2^n}\left(\frac{1}{2^n} + \sum_{i=1}^{2^n-1}(M_i - m_i)\right) = \frac{1}{2^n}\left(\frac{1}{2^n-1} + \sum_{k=0}^{n-1}\frac{2^k}{2^n}\right)$$
$$= \frac{1}{2^{2n}}\left(1 + \sum_{k=0}^{n-1} 2^k\right) = \frac{1}{2^{2n}}\left(1 + (2^n - 1)\right)$$
$$= \frac{1}{2^{2n}}(2^n)$$
$$= \frac{1}{2^n}$$

Thus $\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = \lim_{n\to\infty} \frac{1}{2^n} = 0$, which means f is integrable.

The integral is :

$$\int_0^1 f\,dx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \sum_{i=0}^{2^n-1} M_i \cdot \frac{1}{2^n} = \lim_{n\to\infty}\left[\frac{M_0}{2^n} + \sum_{k=1}^{n}\sum_{i=2^{k-1}}^{2^k-1} M_i \cdot \frac{1}{2^n}\right]$$
$$= \lim_{n\to\infty} M_0 \cdot \frac{1}{2^n} + \lim_{n\to\infty}\sum_{k=1}^{n}\sum_{i=2^{k-1}}^{2^k-1} M_i \cdot \frac{1}{2^n} = \lim_{n\to\infty}\frac{1}{2^n}\cdot\frac{1}{2^n} + \lim_{n\to\infty}\sum_{k=1}^{n}\sum_{i=2^{k-1}}^{2^k-1}\frac{2^{k+1}}{2^n}\cdot\frac{1}{2^n}$$
$$= 0 + \lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{2^k}{2^{2n}}\cdot\sum_{i=2^{k-1}}^{2^k-1} 1\right] = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2^k}{2^{2n}}\cdot 2^{k-1} = \lim_{n\to\infty}\frac{1}{2^{2n+1}}\cdot\sum_{k=1}^{n} 2^{2k}$$
$$= \lim_{n\to\infty}\frac{1}{2^{2n+1}}\cdot\frac{2^{2n+2}-2^2}{2^2-1} = \lim_{n\to\infty}\frac{1}{2^{2n+1}}\cdot\frac{2^{2n+2}-4}{3} = \lim_{n\to\infty}\frac{2}{3} + \frac{4}{3\cdot 2^{2n+1}}$$
$$= \frac{2}{3}$$

Also, $\sum_{n=1}^{\infty}\frac{1}{2^{2n-1}} = \frac{1/2}{1-1/4} = \frac{2}{3}$