1.

Exercise 1. Let c be a limit point of a set $X \subseteq \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function with the property that

$$\lim_{x \to c} f(x) = L.$$

Now suppose that $g : X \to \mathbb{R}$ is some other function defined on X for which there exists a $\delta' > 0$ such that

$$g(x) = f(x) \quad \text{for all } x \in X \cap (c - \delta', c + \delta') - \{c\}.$$

Prove that we also have

$$\lim_{x \to c} g(x) = L.$$

Note: Essentially, this exercise is saying that the value of the limit $\lim_{x\to c} f(x)$ only depends on the behavior of $f(x)$ *near* c.

for every $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $|x - c| < \delta$

$\Rightarrow f(x) \in (L - \varepsilon, L + \varepsilon)$ whenever $x \in (c - \delta, c + \delta) - \{c\}$

$g(x) = f(x)$ for $\forall x \in X \cap (c - \delta', c + \delta') - \{c\}$

$\Rightarrow g(x) = f(x)$ $\forall x \in (c - \delta', c + \delta') - \{c\}$

Since for every $\varepsilon > 0$, there exists $\delta > 0$ s.t.

$$f(x) \in (L - \varepsilon, L + \varepsilon) \text{ whenever } x \in (c - \delta, c + \delta) - \{c\}$$

Choose $\delta = \delta'$, then $g(x) = f(x) \in (L - \varepsilon, L + \varepsilon)$ whenever $x \in (c - \delta', c + \delta') - \{c\}$

Thus, for every $\varepsilon > 0$, there exists $\delta' > 0$ s.t.

$$g(x) \in (L - \varepsilon, L + \varepsilon) \text{ whenever } x \in (c - \delta', c + \delta') - \{c\}$$

$\Rightarrow \lim_{x \to c} g(x) = L$ $\blacksquare$

2.

Exercise 2. Let c be a limit point of $X \subseteq \mathbb{R}$ and consider a function $f : X \to \mathbb{R}$. Prove that

$$\lim_{x \to c} f(x) = L$$

if and only if there exists a sequence of positive values $\delta_n > 0$ such that $\delta_n \to 0$ and

$$f\big(X \cap (c - \delta_n, c + \delta_n) - \{c\}\big) \subseteq (L - \frac{1}{n}, L + \frac{1}{n})$$

for all $n \in \mathbb{N}$.

Note: In my opinion, the 'right-hand' statement of this bi-implication captures the essence of the definition of limit: as the values x concentrate more and more around c, the images $f(x)$ will tend to get closer to L.

Proof: Let $\lim_{x \to c} f(x) = L$, choose $\varepsilon = \frac{1}{n}$

$|f(x) - L| < \frac{1}{n} \Rightarrow -\frac{1}{n} < f(x) - L < \frac{1}{n} \Rightarrow L - \frac{1}{n} < f(x) < \frac{1}{n} + L$

$\Rightarrow f(x) \subseteq (L - \frac{1}{n}, L + \frac{1}{n})$ (*)

But c is a limit point of $X \subseteq \mathbb{R}$

i.e. A point $c \in \mathbb{R}$ is a limit point of X if every neighborhood of c contains x different than c

i.e. $X \cap (c - \delta_n, c + \delta_n) \neq 0$

$x \in \{X \cap (c - \delta_n, c + \delta_n) - \{c\}\}$

thus, from (*), $f(X \cap (c - \delta_n, c + \delta_n) - \{c\}) \subseteq (L - \frac{1}{n}, L + \frac{1}{n})$

Conversely, Let $f(X \cap (c - \delta_n, c + \delta_n) - \{c\}) \subseteq (L - \frac{1}{n}, L + \frac{1}{n})$

$\Rightarrow f(x) \subseteq (L - \frac{1}{n}, L + \frac{1}{n})$

By definition of Limit Point,

$L - \frac{1}{n} < f(x) < L + \frac{1}{n}$

$\Rightarrow |f(x) - L| < \frac{1}{n}$ for $\forall n \in \mathbb{N}$

$\Rightarrow \lim_{x \to c} f(x) = L$ ▨

3.

Exercise 3. Suppose that, for each $n \in \mathbb{N}$, A_n is a finite set of numbers from $[0, 1]$. Further, assume that $A_n \cap A_m = \varnothing$ whenever $n \neq m$. Define $f : [0, 1] \to \mathbb{R}$ to be

$$f(x) = \begin{cases} 1/n & \text{if } x \in A_n \\ 0 & \text{if } x \notin A_n \text{ for any } n. \end{cases}$$

Prove that

$$\lim_{x \to c} f(x) = 0$$

for all $c \in [0, 1]$.

Proof: Let $\varepsilon > 0$, show that $\exists\, \delta > 0$ s.t. $\forall y \in (c-\delta, c+\delta)$, we have $|f(y)| < \varepsilon$

Choose $\frac{1}{p} < \varepsilon$ for $\forall p \in \mathbb{N}$

Then, $\delta_1 > b$ s.t. $(c-\delta_1, c+\delta_1) < [0,1]$

Now, A_1 is finite, so choose $\delta_2 > 0$ s.t. $(c-\delta_2, c+\delta_2) \subset (c-\delta_1, c+\delta_1)$

and $N(c, \delta_2) \cap A_1 = \varnothing$

Choose δ_p s.t. $N(c, \delta_p) \subset N(c, \delta_{p-1}) \subset \cdots \subset N(c, \delta_1) < [0,1]$

s.t. $N(c, \delta_p) \cap A_i = \varnothing$ for $\forall i \in \{1, \cdots, p\}$

Then, $\forall y \in N(c, \delta_p)$, $|f(y)| = 0$ or $\frac{1}{p+1}$

i.e. $|f(y)| < \varepsilon$, $\forall y \in N(c, \delta_p)$

Thus, $\lim_{x \to c} f(x) = 0$ for $\forall c \in [0,1]$ ▨

4. **Exercise 4.** Let $f, g : X \to \mathbb{R}$ be two functions that are continuous at a point $c \in X$. Show that the functions $\varphi, \psi : X \to \mathbb{R}$ defined by

$$\varphi(x) = \mathbf{max}\{f(x), g(x)\}, \quad \psi(x) = \mathbf{min}\{f(x), g(x)\} \text{ for all } x \in X$$

are also continuous at $c \in X$.

Prof: S'pose $f(c) \geq g(c)$

$\varphi(x) = \max\{f(x), g(x)\} = f(x)$, $\psi(x) = \min\{f(x), g(x)\} = g(x)$

Since f continuous at $c \in \mathbb{R}$, for $\forall \varepsilon > 0$, $\exists \delta_1 > 0$ s.t.

$|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon \quad \forall x \in X$

Since g continuous at $c \in \mathbb{R}$ for $\forall \varepsilon > 0$, $\exists \delta_2 > 0$ s.t.

$|x-c| < \delta \Rightarrow |g(x) - g(c)| < \varepsilon \quad \forall x \in X$

Let $\delta = \min\{\delta_1, \delta_2\}$, then $\delta \leq \delta_1$ or $\delta \leq \delta_2$

Suppose $|x-c| < \delta_2$ for $\forall x \in X$

Then $|f(x) - f(c)| < \varepsilon$ and $|g(x) - g(c)| < \varepsilon$

$\Rightarrow$ we have $\max\{f(x), g(x)\} = f(x) \, (\delta) \, g(x)$

$\Rightarrow \varphi(x) = f(x) \, (\delta) \, g(x)$

If $\varphi(x) = f(x) \ \forall x \in X$, then $|\varphi(x) - \varphi(c)| = |f(x) - f(c)| < \varepsilon$

Thus, for $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-c| < \delta \Rightarrow |\varphi(x) - \varphi(c)| < \varepsilon$, $\forall x \in X$

φ is continuous at $c \in \mathbb{R}$

If $\varphi(x) = g(x) \ \forall x \in X$, then $|\varphi(x) - \psi(c)| = |g(x) - f(c)|$

$g(x) - f(c) \leq g(x) - g(c)$

Thus $|g(x) - f(c)| \leq |g(x) - g(c)| < \varepsilon \Rightarrow |g(x) - f(c)| < \varepsilon \Rightarrow |\varphi(x) - \varphi(c)| < \varepsilon$

Hence, $\varphi(x) = \max\{f(x), g(x)\}$ is continuous at $c \in \mathbb{R}$

Prove $\psi(x) = \min\{f(x), g(x)\}$ continuous at $c \in \mathbb{R}$ is similar.

5.

Exercise 5. Suppose that $f, g : X \to \mathbb{R}$ are continuous on X.

(1) If X is open, show that the set $U = \{x \in X \mid f(x) \neq g(x)\}$ is also open.

(2) If X is closed, show that the set $F = \{x \in X \mid f(x) = g(x)\}$ is also closed.

For Exercise 6, we need to recall the following definition.

Definition. Let $A \subseteq \mathbb{R}$. The *closure of* A (denoted by $\overline{A}$) is the set defined by

$$\overline{A} = A \cup \{x \mid x \text{ is a limit point of } A\}.$$

Proof: $U = \{x \in X \mid f(x) \neq g(x)\} = \{x \in X \mid f(x) < g(x)\} \cup \{x \in X \mid f(x) > g(x)\}$

Now we know $A = \{x \in X : f(x) < g(x)\}$ is open

Let $h(x) = f(x) - g(x)$, then $h(x)$ is continuous

$A = \{x \in X \mid h(x) < 0\} = h^{-1}\{(-\infty, \infty)\}$, and $(-\infty, \infty)$ is open in $\mathbb{R}$

and h is continuous, thus, $h^{-1}\{(-\infty, \infty)\}$ is open

Hence, A is open

$B = \{x \in X \mid f(x) > g(x)\}$

Let $h(x) = f(x) - g(x)$, then $B = \{x \in X \mid h(x) > 0\}$

Similarly, $B = h^{-1}\{(0, \infty)\}$ is open in X

Thus, $U = A \cup B$ is open in X

(2) $F = \{x \in X \mid f(x) = g(x)\} = \{x \in X \mid (f-g)(x) = 0\}$

Then $F = (f-g)^{-1}\{0\}$

Since $\{0\}$ is closed in $\mathbb{R}$, and $f-g$ is continuous

Thus, we can say $F = (f-g)^{-1}\{0\}$ is closed in X

Hence, F is closed in X. $\boxtimes$

(Thm: Let $f : X \to \mathbb{R}$ be continuous in X. Then for every subset G of $\mathbb{R}$, $f^{-1}(G)$ is open in X

6.

Exercise 6. Prove that $f : \mathbb{R} \to \mathbb{R}$ if and only if $f(\overline{X}) \subset \overline{f(X)}$ for any $X \subseteq \mathbb{R}$.

Thm $A \subseteq \mathbb{R}$, $x \in \overline{A}$ $\iff$ there is a seq. (x_n) in A with $x_n \to x$

$(\Rightarrow)$ Take any $f(c) \in f(\overline{X})$ where $\underline{c \in \overline{X}}$

By Thm, $\exists (x_n) \subseteq X$ with $x_n \to c$

$\underbrace{f(x_n)}_{\in f(X)} \to f(c)$ $\Rightarrow$ By Thm, we have $f(c) \in \overline{f(X)}$

By contradiction

$(\Leftarrow)$ Assume that f is not cont. at $c \in X$

$\exists \varepsilon > 0$, $\forall \delta > 0$ ($\delta = 1/n$), there is an $x_\delta \in (c-\delta, c+\delta)$

but $f(x_\delta) \notin (f(c) - \varepsilon, f(c) + \varepsilon)$

$\Rightarrow$ For this $\varepsilon > 0$, $\forall n \in \mathbb{N}$ contradiction in this part

$\exists x_n \in (c - \frac{1}{n}, c + \frac{1}{n})$ but $\boxed{f(x_n) \notin (f(c) - \varepsilon, f(c) + \varepsilon)}$

$x_2 \in (c - \frac{1}{2}, c + \frac{1}{2})$

In particular, $\underset{\downarrow\, c}{c - \frac{1}{n}} < x_n < \underset{\downarrow\, c}{c + \frac{1}{n}}$

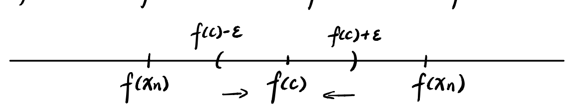
$x_n \to c \Rightarrow c \in \overline{X} \Rightarrow f(c) \in \overline{f(X)}$ $\qquad X = \{x_n \mid n \in \mathbb{N}\}$

$f(c) \in f(\overline{X}) \subseteq \overline{f(X)}$

$f(X) = \{f(x_n) \mid n \in \mathbb{N}\}$

$f(c) \in \overline{f(X)}$

$\forall f(x_n) \in f(X)$, $f(x_n) \notin$ $f(x_n) \notin (f(c) - \varepsilon, f(c) + \varepsilon)$



Hence, $f: \mathbb{R} \to \mathbb{R}$ iff $f(\overline{X}) \subset \overline{f(X)}$ for any $X \subseteq \mathbb{R}$

7.

Exercise 7. Suppose that both $f, g : X \to \mathbb{R}$ are continuous at $c \in X$. Suppose that, in any δ-neighborhood $(c - \delta, c + \delta)$ of c, we can find points x, y such that $f(x) < g(x)$ and $g(y) < f(y)$. Show that $f(c) = g(c)$.

Proof: Define $h = f - g$

Now, f & g are continuous at $c \Rightarrow h$ is continuous at c

For $x, y \in (c-\delta, c+\delta)$ for any $\delta > 0$

$h(x) = f(x) - g(x) < 0$, $h(y) = f(y) - g(y) > 0$

$\exists$ point $x' \Rightarrow f(x') - g(x') = 0 \Rightarrow f(x) = g(x')$

This is true for any $\delta > 0$

Thus, $h(c) = 0 \Rightarrow f(c) - g(c) = 0 \Rightarrow f(c) = g(c)$

8.

Exercise 8. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *periodic* if there exists a positive value $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Prove that, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous and periodic function, then f attains a maximal and minimal value. That is, show that there exists $a, b \in \mathbb{R}$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in \mathbb{R}$.

Given $f: \mathbb{R} \to \mathbb{R}$ is periodic

i.e. $f(x+p) = f(x)$ for $\forall x \in \mathbb{R}$, $\exists p > 0$

For any $x \in \mathbb{R}$, take the compact interval $[x, x+p]$

Since f is continuous

By Extreme Value Theorem, if f is continuous on a compact set,

then f will attain its maximum and minimum

We can write $\mathbb{R}$ as $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [x + np, x + (n+1)p]$

As f attains maximum and minimum on each compact set

$\Rightarrow f$ will attain maximal and minimal value

Then there exists $a, b \in \mathbb{R}$ for that f will attain minimal & maximal respectively

$\Rightarrow f(a) \leq f(x) \leq f(b)$ for $\forall x \in \mathbb{R}$

MATH 317: Homework 1

Deadline: Friday 01/17/2020

Exercise 1. Let c be a limit point of a set $X \subseteq \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function with the property that
$$\lim_{x \to c} f(x) = L.$$
Now suppose that $g : X \to \mathbb{R}$ is some other function defined on X for which there exists a $\delta' > 0$ such that
$$g(x) = f(x) \quad \text{for all } x \in X \cap (c - \delta', c + \delta') - \{c\}.$$
Prove that we also have
$$\lim_{x \to c} g(x) = L.$$

Note: Essentially, this exercise is saying that the value of the limit $\lim_{x \to c} f(x)$ only depends on the behavior of $f(x)$ *near* c.

Exercise 2. Let c be a limit point of $X \subseteq \mathbb{R}$ and consider a function $f : X \to \mathbb{R}$. Prove that
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if and only if there exists a sequence of positive values $\delta_n > 0$ such that $\delta_n \to 0$ and
$$f\big(X \cap (c - \delta_n, c + \delta_n) - \{c\}\big) \subseteq (L - \frac{1}{n}, L + \frac{1}{n})$$
for all $n \in \mathbb{N}$.

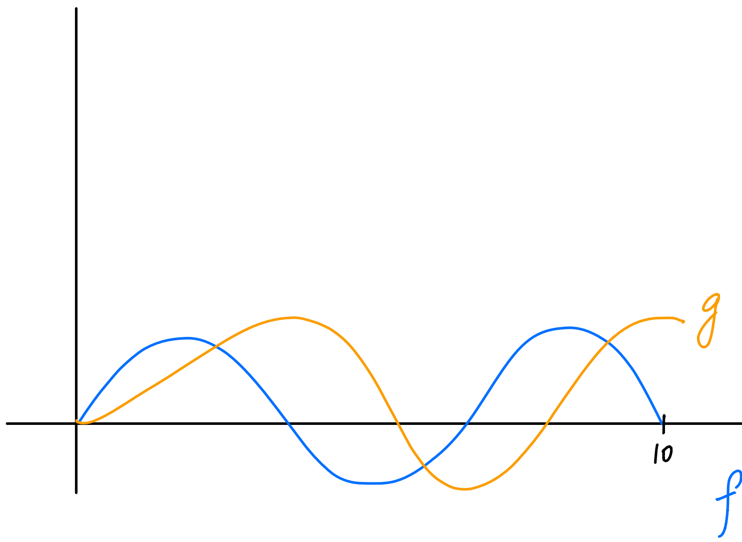
Note: In my opinion, the 'right-hand' statement of this bi-implication captures the essence of the definition of limit: as the values x concentrate more and more around c, the images $f(x)$ will tend to get closer to L.

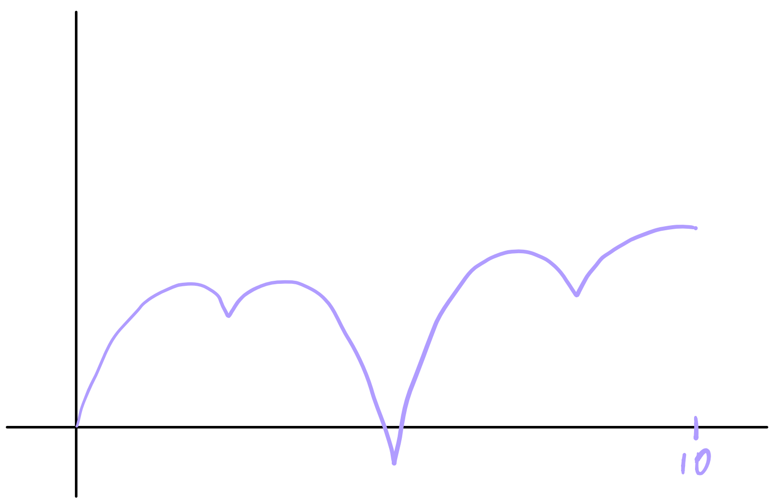
Exercise 3. Suppose that, for each $n \in \mathbb{N}$, A_n is a finite set of numbers from $[0, 1]$. Further, assume that $A_n \cap A_m = \varnothing$ whenever $n \neq m$. Define $f : [0, 1] \to \mathbb{R}$ to be
$$f(x) = \begin{cases} 1/n & \text{if } x \in A_n \\ 0 & \text{if } x \notin A_n \text{ for any } n. \end{cases}$$
Prove that
$$\lim_{x \to c} f(x) = 0$$
for all $c \in [0, 1]$.

Exercise 4. Let $f, g : X \to \mathbb{R}$ be two functions that are continuous at a point $c \in X$. Show that the functions $\varphi, \psi : X \to \mathbb{R}$ defined by

$$① \varphi(x) = \mathbf{max}\{f(x), g(x)\}, ② \psi(x) = \mathbf{min}\{f(x), g(x)\} \text{ for all } x \in X$$

are also continuous at $c \in X$.





Ex6: $f: X \to \mathbb{R}$ is cont. $\Leftrightarrow$ For any $X \subseteq \mathbb{R}$ $f(\overline{X}) \subseteq \overline{f(X)}$

$\left\{ \begin{array}{l} f(c) \in \overline{f(X)} \\ \Leftrightarrow \exists \text{ seq } (y_n) \text{ in } f(X) \\ y_n \to f(c) \end{array} \right.$

$f((c-\delta, c+\delta) \cap X) \subseteq$

<u>Thm</u> $A \subseteq \mathbb{R}$

$x \in \overline{A} \iff$ there is a seq. (x_n) in A with $x_n \to x$

$(\Rightarrow)$ Take any $f(c) \in f(\overline{X})$ where $\underline{c \in \overline{X}}$

By Thm, $\exists (x_n) \subseteq X$ with $x_n \to c$

$\underset{\in f(X)}{f(x_n)} \longrightarrow f(c) \quad \Rightarrow$ By Thm, we have $f(c) \in \overline{f(X)}$

(c is in the closure, $\cdots$)

By contradiction

$(\Leftarrow)$ Assume that f is not cont. at $c \in X$

$\exists \varepsilon > 0,\ \forall \delta > 0$, there is an $x_\delta \in (c-\delta, c+\delta)$ $\quad \delta = 1/n$

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$x_2 \in (c - \frac{1}{2}, c + \frac{1}{2})$ $\qquad$ closure ?

In particular, $\underset{\downarrow \atop c}{c - \frac{1}{n}} < x_n < \underset{\downarrow \atop c}{c + \frac{1}{n}}$ $\quad$ should think of Squeeze Thm.

$x_n \to c \Rightarrow c \in \overline{X} \Rightarrow f(c) \in \overline{f(X)}$ $\qquad X = \{x_n \mid n \in \mathbb{N}\}$

$f(c) \in f(\overline{X}) \subseteq \overline{f(X)}$

$f(X) = \{f(x_n) \mid n \in \mathbb{N}\}$

$f(c) \in \overline{f(X)}$

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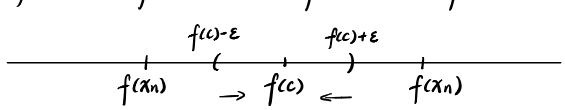

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are also continuous at $c \in X$.

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