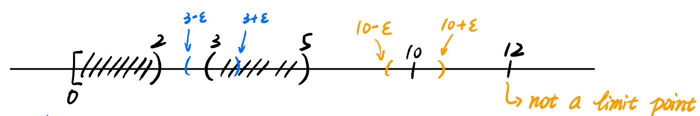
(Limits)
Continuity
Derivatives $\Big\{$ Proofs in this context
Integration

Def Let $X \in \mathbb{R}$, $c \in \mathbb{R}$ is a limit point of X if for any $\varepsilon > 0$, the nbhd $(c-\varepsilon, c+\varepsilon)$ contains some point of $X - \{c\}$

Note: We could have $c \in X$ or $c \notin X$

Ex: Identify the limit points of $X = [0,2) \cup (3,5) \cup \{10\} \cup \{12\}$



limit points = $[0,2] \cup [3,5]$

Def 2 $c \in X$ is an isolated point of X if it's not a limit point

Goal: Understand $\lim_{x \to c} f(x) = L$

"if we can make $f(x)$ arbitrary close to L by taking x arbitrary close to c"

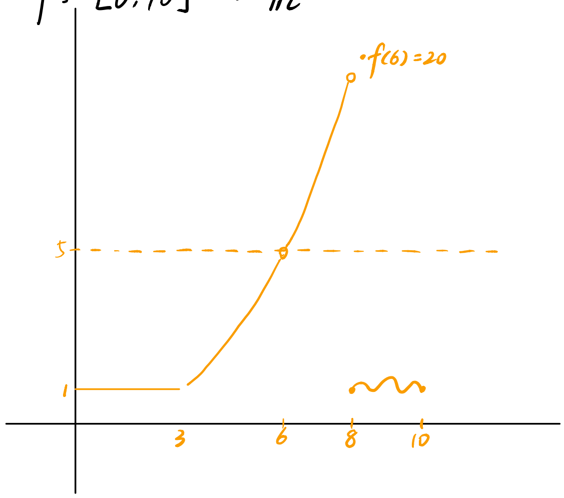
Def 1 Let $f: X \to \mathbb{R}$ and let c be a limit point of X

We say $\lim_{x \to c} f(x) = L$ if, for any $\varepsilon > 0$, there is a $\delta > 0$ s.t.

we never take $x = c$ if $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

if $x \in (c-\delta, c+\delta) - \{c\} \Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon)$

picture: $f: [0,10] \to \mathbb{R}$



$\lim_{x \to 6} f(x) = 5$

This value might not be defined

Note: ① The value $f(c)$ doesn't affect $\lim_{x \to c} f(x)$

② In general, taking a small $\varepsilon > 0$ implies taking a smaller $\delta > 0$.

③ $\lim_{x \to c} f(x)$ only depends on the behavior of $f(x)$ near the point c (flamework)

Exercise: Consider $f: \mathbb{R} \to \mathbb{R}$

Suppose $\lim_{x \to c} f(x) = L$ and $L > M$

Hint: Choose an $\varepsilon > 0$ s.t. $M < L - \varepsilon$

Proof: Take $\varepsilon = L - M > 0$

$$(L-\varepsilon, L+\varepsilon) = (M, 2L-M)$$

Since $\lim_{x \to c} f(x) = L$, we can find a $\delta > 0$ s.t.

$f(x) \in (M, 2L-M)$ whenever $x \in (c-\delta, c+\delta) - \{c\}$

In particular, $f(x) > M$ for all $x \in (c-\delta, c+\delta) - \{c\}$ $\square$

Thm 1: Let $f: X \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of X

$\lim_{x \to c} f(x) = L \iff$ For any sequence (x_n) in X with $(x_n) \to c$ and $x_n \neq c$ for all $n \in \mathbb{N}$, $\underbrace{\text{we have } f(x_n) \to L}_{(*)}$

Proof: $(\Rightarrow)$ Fix a sequence (x_n) in X s.t. $(x_n) \to c$ and $x_n \neq c$ for all $n \in \mathbb{N}$,

Now pick $\varepsilon > 0$. Show that $f(x_n)$ is eventually in $(L-\varepsilon, L+\varepsilon)$

① Since $\lim_{x \to c} f(x)$, we can find a $\delta > 0$ s.t. $x \in (c-\delta, c+\delta) - \{c\}$