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Exercise 2.

Darboux's Thm: let $f: [a,b] \to \mathbb{R}$ be differentiable

If $f'(a) < \alpha < f'(b)$ (or $f'(b) < \alpha < f'(a)$),

then there exists a point $c \in (a,b)$ where $f'(c) = \alpha$

Step 1: Assume $f'(a) < \alpha < f'(b)$

WLOG, we can take $\alpha = 0$ $\quad \underbrace{f'(a) < 0 < f'(b)}_{(*)}$

$\boxed{\text{Goal: Find a } c \in (a,b) \text{ where } f'(c) = 0}$

Part (a): Assume $(*)$. Show that there is an $x \in (a,b)$ with $f(x) < f(a)$,

and a $y \in (a,b)$ with $f(y) < f(b)$

(Let's prove instead): Find a $\delta > 0$ s.t. for any $x \in (a, a+\delta)$ we have $f(x) < f(a)$

(best way: prove by contradiction) $\quad y \in (b-\delta, b)$ we have $f(y) < f(b)$

Suppose this isn't true: Then, for any $\frac{1}{n} > 0$, we can find an $x \in (a, a+\frac{1}{n})$

with $f(a) \le f(x_n)$

Obs: For $\forall n \in \mathbb{N}$, $x_n \in (a, a+\frac{1}{n})$.

Then $(x_n) \to a$

$\dfrac{f(x_n) - f(a)}{x_n - a} \begin{matrix} \ge 0 \\ > 0 \end{matrix} \longrightarrow f'(a)$

Since $\dfrac{f(x_n) - f(a)}{x_n - a} \ge 0$ for $\forall n \in \mathbb{N}$

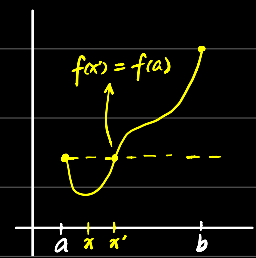
Then $f'(a) \ge 0$ (contradiction)

Part (b): We have $f'(a) < 0 < f'(b)$. Find $c \in (a,b)$ s.t. $f'(c) = 0$

We consider 3 cases:

1. $f(a) = f(b)$ $\quad$ 2. $f(a) < f(b)$ $\quad$ 3. $f(a) > f(b)$

Rolle's Thm: $f'(c) = 0$ for $\exists\, c \in (a,b)$



By Part (a), there's an $x \in (a,b)$ so that $f(x) < f(a) < f(b)$

Apply Int. Value Thm on $[x, b]$, $\exists\, x' \in [x, b]$ with $f(x') = f(a)$

Restruct f on $[a, x']$, we have $f(a) = f(x')$

Apply Rolle's Thm: $f'(c) = 0$ where $c \in [a, x']$

Generalized Mean Value Thm

Let $f, g : [a,b] \to \mathbb{R}$ be continuous and differentiable on $[a,b]$

Then there is a point $c \in (a,b)$ where $[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c)$

$f'(c) = \frac{f(b) - f(a)}{b - a} \iff 1.\ [f(b) - f(a)] = (b - a) f'(c)$

The Mean Value Thm Take $g(x) = x$ $(g'(x) = 1)$

Proof (Idea): Apply the Mean Value Thm to $h(x) = [f(b) - f(a)]\, g(x) = [g(b) - g(a)]\, f(x)$

L'Hospital's Rule (0/0 case): Let

- $f, g : I \to \mathbb{R}$ continuous (I = interval)
- $c \in I$

Suppose f, g are differentiable at each $x \in I - \{c\}$

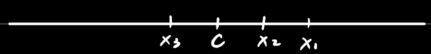
If $f(c) = g(c) = 0$ and $g'(x) \neq 0$ for $\forall x \in I - \{c\}$

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to c} \frac{f(x)}{g(x)} = L$$

Proof: Show that $\lim_{x \to c} \frac{f(x)}{g(x)} = L$

Fix any seq. $(x_n) \to c$ with $x_n \neq c$ for $\forall n \in \mathbb{N}$

Show that $\frac{f(x)}{g(x)} \to L$



Apply the Gen. Mean Value Thm

For each $n \in \mathbb{N}$, for $\exists\, x_n < c_n < c$ or $c < c_n < x_n$

we have : $(f(x_n) - \cancel{f(c)}^{0})\, g'(c_n) = (g(x_n) - \cancel{g(c)}^{0})\, f'(c_n)$

$\Leftrightarrow f(x_n)\, g'(c_n) = g(x_n)\, f'(c_n)$

$\Leftrightarrow \dfrac{f(x_n)}{g(x_n)} = \dfrac{f'(c_n)}{g'(c_n)} \xrightarrow[n \to \infty]{} L$ (Since $c_n \to c$)

Then, $\dfrac{f(x_n)}{g(x_n)} \longrightarrow L$

L'Hospital's Rule (∞/∞ case) $\longrightarrow$ Consequence of the Gen. Mean Value Thm

read in book