- Recall Fundamental Thm of Calculus
- Corollary
- Thm (Change of Variable - Substitution Rule) & Proof
- Thm (Integration by parts) & Proof

Midterm Review Exercises

- 8.18 ⟶ Midterm Prac

Fundamental Thm of Calculus: Fix an integral function $f: [a,b] \to \mathbb{R}$

- **Part (a)**: If $F: [a,b] \to \mathbb{R}$ is antiderivative of $f(x)$ ($F'(x) = f(x)$)

  then $\int_a^b f\,dx = F(b) - F(a)$

- **Part (b)**: Define $G: [a,b] \to \mathbb{R}$ by $G(x) = \int_a^x f(t)\,dt$ for $\forall x \in [a,b]$

  (1) G is continuous

  (2) If f is cont. at $c \in [a,b]$, then G is diff at c and $G'(c) = f(c)$

Corollary: If $f: [a,b] \to \mathbb{R}$ is continuous, then f has an antiderivative

Thm (Change of Variable - Substitution Rule)

Let: • $f: [a,b] \to \mathbb{R}$ be continuous

• $g: [c,d] \to \mathbb{R}$ be differentiable and $g([c,d]) \subseteq [a,b]$

• $g': [c,d] \to \mathbb{R}$ is integrable

Then, $\underbrace{\int_{g(c)}^{g(d)} f\,dx}_{①} = \underbrace{\int_c^d f(g(y))\,g'(y)\,dy}_{②}$

substitution: $x = g(y)$

Proof: Use the Fund Thm of Calculus

Let $F: [a,b] \to \mathbb{R}$ be an antiderivative of $f(x)$

① $\int_{g(c)}^{g(d)} f(x)\,dx = F(g(d)) - F(g(c))$

② Take $\frac{d}{dy}(F(g(y))) \overset{\text{chain rule}}{=} F'(g(y)) \cdot g'(y) = f(g(y)) \cdot g'(y)$

↳ $\int_c^d \boxed{f(g(y))\,g'(y)}\,dy = F(g(d)) - F(g(c)) = ①$ ▨

↳ integrable ?

① $f, g: [a,b] \to \mathbb{R}$ are integrable $\Rightarrow$ $f \cdot g: [a,b] \to \mathbb{R}$ is integrable

② $f: [a,b] \to \mathbb{R}$ is integrable & $h: [c,d] \to \mathbb{R}$ is cont.

$\Rightarrow$ $h \circ f$ is integrable

Thm (Integration by parts)

Let $f, g\ [a,b] \to \mathbb{R}$ be differentiable and f', g' are integrable

Then: $\int_a^b f g' \, dx = \underbrace{f \cdot g \big|_a^b}_{f \cdot g(b) - f \cdot g(a)} - \int_a^b f' \cdot g \, dx$

Proof: Note that $f \cdot g$ is an antiderivative of $f \cdot g' + f' \cdot g$

Then, by Part A of The Fund Thm of Calculus

$$\int_a^b f \cdot g' + f' \cdot g \, dx = f \cdot g(b) - f \cdot g(a) = f \cdot g \big|_a^b$$

$$\int_a^b f g' dx + \int_a^b f' g \, dx = f g \big|_a^b$$

$$\Rightarrow \int_a^b f g' dx = f g \big|_a^b - \int_a^b f' g \, dx \quad \blacksquare$$

office hour 补充:

$$\int_{g(c)}^{g(d)} f \, dx = \int_c^d \boxed{f(g(y))\, g'(y)} \, dy$$

(*)

Antiderivative:

$F'(x) = f(x)$

F = antiderivative of $f(x)$

$$\int_{g(c)}^{g(d)} f \, dx = F(g(d)) - F(g(c))$$

chain rule

Take $\frac{d}{dy}(F(g(y))) = F'(g(y)) \cdot g'(y)$

$= \boxed{f(g(y)) \cdot g'(y)}$

$F(g(y))$ is an antiderivative for (*)

$$\int_c^d f(g(y)) \cdot g'(y) \, dy = F(g(d)) - F(g(c))$$