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Recall: Thm: $f:[a,b]\to\mathbb{R}$ is integrable

$\iff$ For $\forall\,\varepsilon>0$, $U(f,P)-L(f,P)<\varepsilon$ for $\exists$ partition P

Def: $f_n: X\to\mathbb{R}$ converges uniformly to $f: X\to\mathbb{R}$ if, $(f_n\to_u f)$
for any $\varepsilon>0$, there is an $N\in\mathbb{N}$ s.t.
$|f_n(x)-f(x)|<\varepsilon$ for $\forall\, n\geq N$ & $\forall\, x\in X$

Lemma: If $f,g:[a,b]\to\mathbb{R}$ satisfy $|f(x)-g(x)|<\varepsilon'$ for $\forall\, x\in[a,b]$
and if P is any partition of $[a,b]$. Then

$$|U(f,P)-U(g,P)|\leq\varepsilon'(b-a)$$
$$|L(f,P)-L(g,P)|\leq\varepsilon'(b-a)$$

Proof of Lemma: (Mar 6)

<u>Thm A:</u> $f_n : [a,b] \to \mathbb{R}$ converges uniformly to $f : [a,b] \to \mathbb{R}$

Then : 1) If each f_n is integrable, then f is integrable

2) $\int_a^b f_n \, dx \longrightarrow \int_a^b f \, dx$

$\Updownarrow$

$\int_a^b \lim f_n \, dx = \lim \int_a^b f_n \, dx$

<u>Proof of Thm A:</u> $f_n \to_u f$

$\hookrightarrow$ integrable

Fix $\varepsilon > 0$

<u>Goal:</u> Find a partition P s.t. $U(f,P) - L(f,P) < \varepsilon$

<u>Step 1:</u> Since $f_n \to f$ we can find a $N \in \mathbb{N}$ s.t.

$$|f_N(x) - f| < \frac{\varepsilon}{3(b-a)}$$

Since $f_N : [a,b] \to \mathbb{R}$ is integrable, there is a partition P

$$U(f_N, P) - L(f_N, P) < \varepsilon/3$$

$$U(f,P) - L(f,P) = |U(f,P) - L(f,P)| \leftarrow (*)$$

$$(*) = |U(f,P) - U(f_N,P) + U(f_N,P)| - |L(f_N,P) - L(f_N,P) + L(f,P)|$$

$$\leq |U(f_N,P) - U(f,P)| + |U(f_N,P) - L(f_N,P)| + |L(f_N,P) - L(f,P)|$$

$$< \frac{\varepsilon\cancel{(b-a)}}{3\cancel{(b-a)}} + \frac{\varepsilon}{3} + \frac{\varepsilon\cancel{(b-a)}}{3\cancel{(b-a)}} = \varepsilon$$

2) Show that $\int_a^b f_n \, dx \longrightarrow \int_a^b f \, dx$

Fix $\varepsilon > 0$ Take $N \in \mathbb{N}$ s.t.

$|f_n(x) - f(x)| < \frac{\varepsilon}{2(b-a)}$ for $\forall x \in [a,b]$ & $\forall n \geq N$

$$\left|\int_a^b f_n \, dx - \int_a^b f \, dx\right| = \left|\int_a^b f_n - f \, dx\right|$$

$$\leq \int_a^b |f_n - f| \, dx$$

$$\leq \frac{\varepsilon}{2\cancel{(b-a)}} \cancel{(b-a)}$$

$$= \varepsilon/2 < \varepsilon$$

Fact: If $f_n \to_u f$ and each f_n is differentiable, then it might be the case that f is not differentiable

Thm B: Let $f_n : [a,b] \to \mathbb{R}$ be differentiable and $f_n \longrightarrow_u f$

Assume : 1) Each f_n' is continuous

2) $f_n' \to_u g$ for some function $g : [a,b] \to \mathbb{R}$

Then, f is differentiable and $f' = g$

Proof: Fundamental Thm of Calculus

For $\forall\, x \in [a,b]$ $\int_a^x f_n'\, dx = f_n(x) - f_n(a)$

$$\iff f_n(x) = \int_a^x f_n'\, dx + f_n(a)$$

$\to$ Thm A

$f_n \to_u f \qquad f_n' \longrightarrow g \longrightarrow g$ is continuous

$n \to \infty$

$$f(x) = \boxed{\int_a^x g\, dx} + f(a)$$

Since g is continuous, $G(x) = \int_a^x g\, dx$

$\hookrightarrow$ is differentiable, and $G'(x) = g(x)$

We have $f(x) = G(x) + f(a)$

Then, $f(x)$ is differentiable and $f'(x) = G'(x) = g(x)$