Exercise 6.9

$f : \mathbb{R} \to \mathbb{R}$ $\quad c \in \mathbb{R}$

- f is cont. at $c \in \mathbb{R}$
- $f(c) > 0$

$\Rightarrow$ Show $\exists\, \delta > 0$ s.t. $f(x) > 0$ for each $x \in (c-\delta, c+\delta)$

<u>Proof:</u>

For any $\varepsilon > 0$ s.t. $f(x) \in (f(c)-\varepsilon, f(c)+\varepsilon)$ if $x \in (c-\delta, c+\delta)$

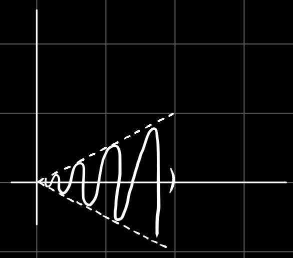
Take $\varepsilon = f(c) > 0$. Then, for this $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$x \in (c-\delta, c+\delta) \Rightarrow f(x) \in (f(c)-\varepsilon, f(c)+\varepsilon) = (0, 2f(c)) \subseteq (0, \infty)$$

In particular, if $x \in (c-\delta, c+\delta) \Rightarrow f(x) = 0$

Exercise 2.2

$f: (0,1) \to \mathbb{R}$ defined by $f(x) = x\sin(\frac{1}{x})$ is U.C. ?



Recall:
Thm If X is compact, and $f: X \to \mathbb{R}$ is cont. then f is U.C.

Strategy: Extend this f to a continuous function $g: [0,1] \to \mathbb{R}$
($\forall x \in (0,1)$, $f(x) = g(x)$)

$\sin(\frac{1}{x})$ ← NOT U.C.

Thm: $f: (a,b) \to \mathbb{R}$ is U.C. $\Leftrightarrow$ f can be extended continuously to $[a,b]$

How to define $g: [0,1] \to \mathbb{R}$?

$$g(x) \begin{cases} f(x) & x \in (0,1) \\ 0 & x = 0 \\ \sin(1) & x = 1 \end{cases}$$

This $g(x)$ is continuous on $[0,1]$ (Exercise prove this)

YES. $[0,1]$ is compact

Then, $g(x)$ is U.C. on $[0,1]$

Thus, $f(x)$ is also U.C. on on $(0,1)$

Exercise 6.13

$f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if if} < 0 \end{cases}$

Show that $f(x)$ is not cont. at 0.

Choose $x_n = \frac{1}{n} \longrightarrow 0$ $\quad f(x_n) = 1 \to 1$

$y_n = \frac{1}{n} \longrightarrow 0$ $\quad f(y_n) = 0 \to 0$

This $f(x)$ is not cont. at $x = 0$

<u>Thm</u>

$f: X \to \mathbb{R}$ is cont. at $c \in X$

$\Leftrightarrow$ For any seq. (x_n) in X with <u>$x_n \to c$</u>, we have <u>$f(x_n) \to f(c)$</u>

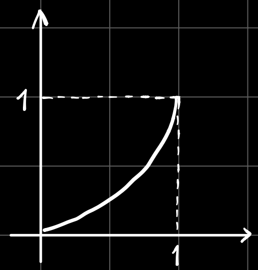
Exercise 6.48

$\to f: [0,1] \to \mathbb{R}$ and $f([0,1]) \subseteq [0,1]$

$f: [0,1] \to [0,1]$ cont. Show that $\exists\, c \in [0,1]$ s.t. $\underbrace{f(c) = c}$

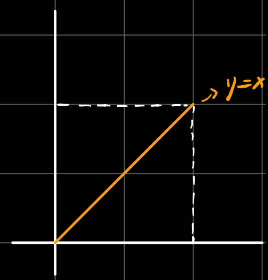
$\Leftrightarrow$ At some point the graph of $f(x)$ intersects the line $y = x$

Proof:



Case 1: Either $f(0) = 0$ or $f(1) = 1$

In this case, $c = 0$ or $c = 1$



Case 2: Suppose $\underbrace{f(0) \neq 0}$ and $\underbrace{f(1) \neq 1}$

$\cancel{f(0) < 0}$ or $f(0) > 0$ $\quad$ $\cancel{f(1) > 1}$ or $f(1) < 1$

We get $f(0) > 0$ and $f(1) = 1$

Apply the Int. Value Thm

$f(x) = x \iff \underbrace{f(x) - x} = 0$

Take $h: [0,1] \to \mathbb{R}$ defined by $h(x) = f(x) - x$ $\left(\text{find } c \in [0,1] \text{ s.t. } h(c) = 0\right)$

Take $h(0) = f(0) - 0 = f(0) > 0$

$h(1) = f(1) - 1 < 0$

By the Int. Value Thm, $\exists\, c \in (0,1)$ s.t. $h(c) = 0 \iff f(c) = c$