MATH 317: Homework 2

Deadline: Friday 02/03/2020

Exercise 1. A function $f : X \to \mathbb{R}$ is said to be *locally constant* if, for any $x \in X$, there exists a $\delta > 0$ such that f is constant on $X \cap (c - \delta, c + \delta)$. Prove that any locally constant function $f : I \to \mathbb{R}$ defined on an interval I is constant.

For Exercise 2, we will need the following definition.

Definition A. A function $f : I \to \mathbb{R}$ defined on an interval I is said to have *the intermediate value property* if, for any interval $J \subseteq I$, the image $f(J)$ is also an interval.

Exercise 2. Let I be an interval and let $f : I \to \mathbb{R}$ be a function which satisfies *the intermediate value property.* Also, suppose that for any $c \in \mathbb{R}$ there exists at most finitely many points $x \in I$ such that $f(x) = c$. Prove that f must be continuous.

Exercise 3. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$.

(a) Show that there exists an $x \in [0, 1]$ with the property that $f(x + \frac{1}{2}) = f(x)$.

(b) Prove the previous result using $\frac{1}{3}$ instead of $\frac{1}{2}$.

(c) Find a statement that generalizes the results from parts (a) and (b) and prove it.

Exercise 4. Let $f : K \to \mathbb{R}$ be a continuous defined on a compact set $K \subset \mathbb{R}$. If $f(x) > 0$ for all $x \in K$, prove that there is a value $c > 0$ such that $f(x) \geq c$ for all $x \in K$.

For Exercises 5, 6, and 7, we need to understand what it means to take *limits at infinity.*

Definition B. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on all of $\mathbb{R}$ and let $L \in \mathbb{R}$. We say that

$$\lim_{x \to +\infty} f(x) = L \qquad \text{(resp. } \lim_{x \to -\infty} f(x) = L\text{)}$$

if, for any $\epsilon > 0$, there is an $M > 0$ (resp. $M < 0$) such that $|f(x) - L| < \epsilon$ if $x > M$ (resp. $x < M$).

Definition C. For a function $f : \mathbb{R} \to \mathbb{R}$, we write

$$\lim_{x \to +\infty} f(x) = +\infty \qquad \text{(resp. } \lim_{x \to -\infty} f(x) = +\infty\text{)}$$

if, for any value $R > 0$, we can find an $M > 0$ (resp. $M < 0$) such that $f(x) > R$ whenever $x > M$ (resp. $x < M$). We can also formulate a similar definition in the case when the limit is equal to $-\infty$.

Exercise 5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = +\infty.$$

Prove that there is an $x_0 \in \mathbb{R}$ such that $f(x_0) \leq f(x)$ for all $x \in \mathbb{R}$.

Exercise 6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\lim_{x \to +\infty} f(x) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty.$$

Prove that, for any $c \in \mathbb{R}$, the set of values $\{|x| : x \in f^{-1}(c)\}$ has a *minimum* (in other words, you have to show that this set is bounded below and that the infimum is in the set).

Exercise 7. Consider a continuous function $f : \mathbb{R} \to \mathbb{R}$. Suppose that the limits

$$\lim_{x \to +\infty} f(x) \quad \text{and} \quad \lim_{x \to -\infty} f(x)$$

exist. Prove that f is uniformly continuous.

Exercise 8. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x^2)$ is not uniformly continuous.

Exercise 9. Suppose that $f, g : X \to \mathbb{R}$ are uniformly continuous.

(a) Prove that $f + g$ is uniformly continuous.

(b) Prove that the functions $\varphi(x) = \mathbf{max}\{f(x), g(x)\}$ and $\psi(x) = \mathbf{min}\{f(x), g(x)\}$ are also uniformly continuous.

1. **Exercise 1.** A function $f : X \to \mathbb{R}$ is said to be *locally constant* if, for any $x \in X$, there exists a $\delta > 0$ such that f is constant on $X \cap (c - \delta, c + \delta)$. Prove that any locally constant function $f : I \to \mathbb{R}$ defined on an interval I is constant.

Proof: let $n_0 \in X$, then $f(n_0) \in \mathbb{R}$

S'pose some open set $U \subseteq X$ s.t. $f(y) = f(n_0)$ for $\forall y \in U$

$U = \{ n : f(n) = f(n_0) \}$

let $V = \{ n : f(n) \neq f(n_0) \}$

Then V is union of open sets, $U \cup V = X$ & $U \cap V = \emptyset$

$\Rightarrow$ X is disconnected, which is a contradiction.

Thus $V = \emptyset$ and $U = X$, in particular, f is constant

The function $f : I \to \mathbb{R}$ is constant since I is connected.

2. **Exercise 2.** Let I be an interval and let $f : I \to \mathbb{R}$ be a function which satisfies *the intermediate value property*. Also, suppose that for any $c \in \mathbb{R}$ there exists at most finitely many points $x \in I$ such that $f(x) = c$. Prove that f must be continuous.

Definition A. A function $f : I \to \mathbb{R}$ defined on an interval I is said to have *the intermediate value property* if, for any interval $J \subseteq I$, the image $f(J)$ is also an interval.

Recall f is continuous on I $\iff$ f is continuous at all points on I

Let $c \in I$, then f is continuous at x if

for $\forall \varepsilon > 0$. $\exists \delta > 0$ s.t. $0 < |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Let $x_0 \in I$, show that f is continuous at x_0

Let $\varepsilon > 0$, find $\delta > 0$ s.t. $0 < |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Since pre-image of a point under f only have finitely many points,

let $f^{-1}(f(x_0)+\varepsilon) = \{x_1, \cdots, x_n\}$ and $f^{-1}(f(x_0)-\varepsilon) = \{y_1, \cdots, y_m\}$

Let $$\delta = \min \left\{ \begin{array}{c|c} |x_0 - x_i| & i = 1, \cdots, n \\ |y_0 - y_j| & j = 1, \cdots, m \end{array} \right\}$$

Note: $f(J) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, where $J = (x_0 - \delta, x_0 + \delta)$

Thus, for $\forall x$ satisfying $0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

Hence, f is continuous at x_0.

$f: \mathbb{R} \to \mathbb{R} \Rightarrow \forall c \in \mathbb{R},\ f^{-1}(c)$ is finite

(For each $c \in \mathbb{R}$, there are only finitely many points x s.t. $f(x) = c$)

$\Rightarrow f$ is continuous

Proof: Fix $x_0 \in \mathbb{R}$. Show that f is cont. at x_0

Fix $\varepsilon > 0$. Find a $\delta > 0$ s.t. $f(x_0-\delta, x_0+\delta) \subseteq (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$

By assumption, $f^{-1}(f(x_0)+\varepsilon) = \{x_1, \cdots, x_p\}$

$f^{-1}(f(x_0)-\varepsilon) = \{y_1, \cdots, y_p\}$

Note: $x_0 \notin \{x_1, \cdots, x_p\}$ $\Rightarrow$ Obs: $\exists\, \delta > 0$ s.t.

$x_0 \notin \{y_1, \cdots, y_p\}$ no x_i & y_j is in $(x_0-\delta, x_0+\delta)$

$$\delta = \min \left\{ \begin{array}{c|c} |x_0 - x_i| & i = 1, \cdots, p \\ |y_0 - y_j| & j = 1, \cdots, q \end{array} \right\}$$

For this $\delta > 0$, we have $f((x_0-\delta, x_0+\delta)) \subseteq (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$

Suppose not, $f((x_0-\delta, x_0+\delta))$

this is an interval

In particular, $f(x_0)+\varepsilon \in f((x_0-\delta, x_0+\delta))$

3.

Exercise 3. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$.

(a) Show that there exists an $x \in [0,1]$ with the property that $f(x + \frac{1}{2}) = f(x)$.

(b) Prove the previous result using $\frac{1}{3}$ instead of $\frac{1}{2}$.

(c) Find a statement that generalizes the results from parts (a) and (b) and prove it.

Proof: Define $g(x) = f(x) - f(x + \frac{1}{2})$

Let $S = \{f(0), f(\frac{1}{2}), f(1)\}$

Case 1: $f(\frac{1}{2}) > f(0) = f(1)$

Consider $g(0) = f(0) - f(\frac{1}{2}) < 0$, $g(\frac{1}{2}) = f(\frac{1}{2}) - f(0) > 0$

Then by Intermediate Value Thm, $\exists\, c \in [0, \frac{1}{2}]$ s.t. $g(c) = 0$

Thus, $f(c) = f(c + \frac{1}{2})$ and hence the result

Case 2: If all values are equal $(= S)$,

then $f(0) = f(0 + \frac{1}{2}) = f(\frac{1}{2})$

4. **Exercise 4.** Let $f : K \to \mathbb{R}$ be a continuous defined on a compact set $K \subset \mathbb{R}$. If $f(x) > 0$ for all $x \in K$, prove that there is a value $c > 0$ such that $f(x) \geq c$ for all $x \in K$.

Proof: Since f is continuous on the compact set K

then there exists a point $x_1 \in K$ s.t.

$$f(x_1) = \inf_{x \in K} f(x)$$

we have $f(x) \geq f(x_1)$ for $\forall x \in K$ $(*)$ ($f(x_1)$ is the inf of $f(x)$ for $\forall x \in K$)

Let $f(x_1) = c$,

Since $x_1 \in K$, $f(x_1) > 0$

Thus, $c > 0$, $f(x_1) = c$

from $(*)$, $\Rightarrow f(x) \geq c$ for $\forall x \in K$

Hence, there is a value $c > 0$ s.t. $f(x) \geq c$ for $\forall x \in K$.

5. **Exercise 5.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty.$$

Prove that there is an $x_0 \in \mathbb{R}$ such that $f(x_0) \le f(x)$ for all $x \in \mathbb{R}$.

Proof: $f: \mathbb{R} \to \mathbb{R}$ is a continuous function

Let $f(0) = M$

As $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = +\infty$, there exists M_1, M_2 s.t.

$$f(x) > M \qquad M_2 < \forall x < M_1$$

$\Rightarrow f(x) > f(0) \quad \forall x > M_1, x < M_2$

$\Rightarrow \min\limits_{x\in\mathbb{R}} f(x) = \min\limits_{x\in[M_2,M_1]} f(x)$ as $f(0)$ is the minimum of $\{f(x): x > M_1, x < M\}$

As $[M_2, M_1]$ is a compact set so there exists $x_0 \in [M_2, M_1]$ s.t.

$$f(x_0) \le f(x) \quad \forall x \in \mathbb{R}$$

As $f(x) > f(0) \quad \forall x > M_1, x < M_2$

$\Rightarrow f(x_0) \le f(x) \quad \forall x \in \mathbb{R}$

6. **Exercise 6.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\lim_{x\to+\infty} f(x) = +\infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

Prove that, for any $c \in \mathbb{R}$, the set of values $\{|x| : x \in f^{-1}(c)\}$ has a *minimum* (in other words, you have to show that this set is bounded below and that the infimum is in the set).

Proof: $f : \mathbb{R} \to \mathbb{R}$ continuous satisfying $\lim_{x\to\infty} f(x) = +\infty$ & $\lim_{x\to-\infty} f(x) = -\infty$

Then f is not bounded. i.e. f maps every real numbers

Then Image of $f = \mathbb{R}$ ①

Now, we have $|x| \geq 0$ for $\forall x \in \mathbb{R}$

$\Rightarrow$ for $\forall c \in \mathbb{R}$, $|f^{-1}(c)| \geq 0$

From ①, we know the set $\{|x| : x \in f^{-1}(c)\}$ is non-empty and bounded below by 0.

Thus, $\{|x| : x \in f^{-1}(c)\}$ has an inf ②

Since f is cont. $\Rightarrow$ pre-image of closed set is closed,

$\{c\}$ is closed in $\mathbb{R}$ $\Rightarrow$ $\{|x| : x \in f^{-1}(c)\}$ is closed in $\mathbb{R}$

Thus, $\{|x| : x \in f^{-1}(c)\}$ contains all limit points ③

If $\{|x| : x \in f^{-1}(c)\}$ is a set of isolated points and since it's bounded below, it always has a minimum.

Through ②, let $m = \inf\{|x| : x \in f^{-1}(c)\}$

Show that $m \in \{|x| : x \in f^{-1}(c)\}$, Suppose <u>not</u>

Assume $m \notin \{|x| : x \in f^{-1}(c)\}$ $\Rightarrow$ m is a limit point of $\{|x| : x \in f^{-1}(c)\}$

By ③, $m \in \{|x| : x \in f^{-1}(c)\}$, which is a contradiction

Hence, $m \in \{|x| : x \in f^{-1}(c)\}$

7. **Exercise 7.** Consider a continuous function $f : \mathbb{R} \to \mathbb{R}$. Suppose that the limits

$$\lim_{x\to+\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x)$$

exist. Prove that f is uniformly continuous.

Proof: let $\lim\limits_{x\to\infty} f(x) = M$ & $\lim\limits_{x\to-\infty} f(x) = L$

For $\forall \varepsilon > 0$, $\exists\, a_1, a_2$ s.t. $|f(x) - M| < \varepsilon/2$ $\forall x \le a_2$

f is cont at a_1 & a_2, $\exists\, \delta_1, \delta_2 > 0$ s.t.

$$|f(x) - f(a_1)| < \tfrac{\varepsilon}{2} \quad \forall x \text{ s.t. } |x - a_1| < \delta_1$$

$$|f(x) - f(a_2)| < \tfrac{\varepsilon}{2} \quad \forall x \text{ s.t. } |x - a_2| < \delta_2$$

Since f cont. on $[a_2, a_1]$, we have f is U.C. on $[a_2, a_1]$

Thus, $\exists\, \delta_3$ s.t. $|f(x) - f(y)| < \varepsilon/2$ $\forall x, y \in [a_2, a_1]$ s.t. $|x - y| < \delta_3$ ①

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and $|a - b| < \delta$

Case 1: $a, b \in [a_1, \infty) \Rightarrow |f(a) - f(b)| \le |f(a) - M| + |f(b) - M| < \varepsilon$

Case 2: $a, b \in (-\infty, a_2] \Rightarrow |f(a) - f(b)| \le |f(a) - L| + |f(b) - L| < \varepsilon$

Case 3: $a, b \in [a_2, a_1]$, then $|a - b| < \delta \Rightarrow |f(a) - f(b)| < \varepsilon$

$|a - b| < \delta \Rightarrow |a - b| < \delta_3$ and ① implies this

Case 4: $a \in [a_2, a_1]$, $b \in [a_1, \infty)$. Then $|a - b| < \delta \Rightarrow |a - a_1| < \delta$ & $|b - b_1| < \delta$

Now we have $|a - a_1| < \delta \Rightarrow |a - a_1| < \delta_1 \Rightarrow |f(a) - f(a_1)| < \varepsilon/2$

and $|b - a_1| < \delta \Rightarrow |b - a_1| < \delta_2 \Rightarrow |f(b) - f(a_1)| < \varepsilon/2$

so $|f(a) - f(b)| \le |f(a) - f(a_1)| + |f(a_1) - f(b)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

Case 5: $a \in (-\infty, a_2]$ and $b \in [a_2, a_1]$

Then $|a - b| < \delta \Rightarrow |a - a_2| < \delta_2 \Rightarrow |f(a) - f(a_2)| < \varepsilon/2$

and $|a - b| < \delta \Rightarrow |b - a_2| < \delta_2 \Rightarrow |f(b) - f(a_2)| < \varepsilon/2$

so $|f(a) - f(b)| \le |f(a) - f(a_2)| + |f(a_2) - f(b)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

Through 5 cases we have for $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|a - b| < \delta \Rightarrow |f(a) - f(b)| < \varepsilon$

Hense, f is U.C. on $\mathbb{R}$

8. **Exercise 8.** Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x^2)$ is not uniformly continuous.

Proof: Suppose $x = \sqrt{n\pi}$, $y = \sqrt{n\pi + \frac{\pi}{2}}$

Then $|x-y| = \sqrt{n\pi+\frac{\pi}{2}} - \sqrt{n\pi} = \frac{n\pi+\frac{\pi}{2} - n\pi}{\sqrt{n\pi+\frac{\pi}{2}} + \sqrt{n\pi}} = \frac{\pi/2}{\sqrt{n\pi+\pi/2} + \sqrt{n\pi}} < \frac{2}{2\sqrt{n\pi}} < \frac{1}{\sqrt{n}}$

If $n > \frac{1}{\delta^2}$ then $|x-y| < \delta$, but $|f(x) - f(y)| \geq 1$

The values of x, y are very close but $f(x)$ and $f(y)$ are far apart.

Choose $\varepsilon = 1$, there does not exist a δ that allow you to know $f(x)$ within precision ε if you know x within precision δ.

The oscillations in $\sin(x^2)$ get faster, on arbitrary small intervals, the function changes its value from $0 \to 1$

Therefore, $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x^2)$ is not U.C.

9. **Exercise 9.** Suppose that $f, g : X \to \mathbb{R}$ are uniformly continuous.

(a) Prove that $f + g$ is uniformly continuous.

(b) Prove that the functions $\varphi(x) = \mathbf{max}\{f(x), g(x)\}$ and $\psi(x) = \mathbf{min}\{f(x), g(x)\}$ are also uniformly continuous.

(a) Proof: Given $\varepsilon > 0$, there exists $\delta_1, \delta_2 > 0$ s.t. for any $x, y \in E$,

if $|x-y| < \delta_1$, then $|f(x)-f(y)| < \frac{\varepsilon}{2}$ and

if $|x-y| < \delta_2$, then $|g(x)-g(y)| < \frac{\varepsilon}{2}$

Let $\delta = \min\{\delta_1, \delta_2\}$,

if $|x-y| < \delta$, then $|(f+g)(x) - (f-g)(x)| \le |f(x)-f(y)| + |g(x)-g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Hence, $f+g$ is U.C.

(b) Proof: Fix $\varepsilon > 0$. find a $\delta > 0$ s.t. $|x-y| < \delta \Rightarrow |\varphi(x) - \varphi(y)| < \varepsilon$

Since f & g are U.C. $\exists\, \delta_1, \delta_2 > 0$ s.t.

$|x-y| < \delta_1 \Rightarrow |f(x) - f(y)| < \varepsilon$

$|x-y| < \delta_2 \Rightarrow |g(x) - g(y)| < \varepsilon$

Take $\delta = \min\{\delta_1, \delta_2\}$

Claim: This $\delta > 0$ works

Suppose $|x-y| < \delta$

① $\varphi(x) = f(x)$, $\varphi(y) = f(y)$ Then $|\varphi(x) - \varphi(y)| = |f(x) - f(y)| < \varepsilon$

② $\varphi(x) = g(x)$ and $\varphi(y) = g(y)$ Then $|\varphi(x) - \varphi(y)| = |f(x) - f(y)| < \varepsilon$

③ S'pose $\varphi(x) = f(x)$ & $\varphi(y) = g(y)$

Case 3.1: $\varphi(x) = \varphi(y)$, $|\varphi(x) - \varphi(y)| < \varepsilon$

Case 3.2: $\varphi(x) < \varphi(y) \Rightarrow g(x) \le f(x) < g(y) \Rightarrow |\varphi(x) - \varphi(y)| \le |g(x) - g(y)| < \varepsilon$

Case 3.3: $\varphi(y) < \varphi(x)$

(b) <u>Proof</u>: if $\varphi(x) = \max\{f(x), g(x)\}$

Case 1: $\max\{f(x), g(x)\} = f(x) \Rightarrow \varphi(x) = f(x)$

Case 2: $f(x)$ is U.C. $\Rightarrow \varphi(x)$ is U.C.

Similarly, when $\max\{f(x), g(x)\} = g(x)$ and $\psi(x) = \min\{f(x), g(x)\} = f(x)$

Case 1: when $\min\{f(x), g(x)\} = f(x) \Rightarrow \psi(x) = f(x)$ for $\forall x \in X$

Since $f(x)$ is U.C. $\Rightarrow \psi(x)$ is U.C.

Case 2: when $\min\{f(x), g(x)\} = g(x) \Rightarrow \psi(x) = g(x)$

$\Rightarrow \psi(x)$ is also U.C. for $\forall x \in X$