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<u>Thm A</u>: $f: [a,b] \to \mathbb{R}$ is integrable $\Leftrightarrow$ For any $\varepsilon > 0$, there is a partition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$

<u>Properties of the integral</u> <u>Notation</u>: $\wp([a,b])$ = Set of partitions of $[a,b]$

<u>Thm B</u> (a) $f: [a,b] \to \mathbb{R}$ is integrable $\Leftrightarrow$ there is a sequence of partitions P_n of $[a,b]$ s.t. $U(f,P) - L(f,P) \to 0$

(b) In this case, $U(f,P) \to \int_a^b f\,dx$, $L(f,P_n) \to \int_a^b f\,dx$

<u>Proof</u>:

(a) ($\Rightarrow$) Assume f is integrable. Apply <u>Thm A</u> to values of the form $\varepsilon = \frac{1}{n}$:

For each $n \in \mathbb{N}$, there is a $P_n \in \wp([a,b])$ with $0 \le U(f,P) - L(f,P) < \frac{1}{n}$

By the <u>Squeeze Thm</u>: $U(f,P) - L(f,P) \to 0$

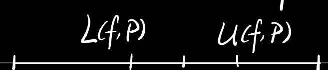
($\Leftarrow$) Suppose that we have a sequence $P_n \in \wp([a,b])$ with $U(f,P) - L(f,P) \to 0$

Apply <u>Thm A</u> Fix $\varepsilon > 0$, we can find a large enough $m \in \mathbb{N}$ s.t. $0 \le U(f,P) - L(f,P) < \varepsilon$.

Thus, by <u>Thm A</u>: f is integrable.

(b) Show that $U(f,P) \to \int_a^b f\,dx$

Fix $\varepsilon > 0$, $|U(f,P) - \int_a^b f\,dx| < \varepsilon$ if $n \ge N$ for $\exists N \in \mathbb{N}$



$\int_a^b dx = U(f) = L(f)$

Then, $U(f,P) \to \int_a^b f\,dx$. $\blacksquare$

Thm C: Suppose $f, g : [a,b] \to \mathbb{R}$ are integrable

(a) $f+g$ is also integrable. Also $\int_a^b f+g\,dx = \int_a^b f\,dx + \int_a^b g\,dx$

(b) $\int_a^b c f\,dx = c \cdot \int_a^b f\,dx$ (EXERCISE)

Proof:

(a) obs: for any partition $P \in \mathcal{P}([a,b])$, we have:

(1) $U(f+g, P) \leq U(f,P) + U(g,P)$

Proof of (1): $P = \{t_0, t_1, \cdots, t_n\}$

Let $h = f+g$ $\quad$ $\sup(h([t_{i-1}, t_i]) < \sup(f([t_{i-1}, t_i]) + \sup(g([t_{i-1}, t_i])$

$U(h,P) = \sum_{i=1}^{n} (t_i - t_{i-1}) \sup(h([t_{i-1}, t_i])$

$\leq \sum_{i=1}^{n} (t_i - t_{i-1}) \sup(f([t_{i-1}, t_i]) + \sum_{i=1}^{n} (t_i - t_{i-1}) \sup(g([t_{i-1}, t_i])$

$U(h,P) \leq U(f,P) + U(g,P)$

Since f & g are integrable over $[a,b]$, there are sequences (P_n) & (Q_n) in $\mathcal{P}([a,b])$ s.t.

$U(f,P_n) - L(f,P_n) \to 0 \qquad U(g,Q_n) - L(g,Q_n) \to 0$

Take $R_n = P_n \cup Q_n$.

Note that, for $\forall n \in \mathbb{N}$, R_n refines P_n & Q_n

Recall: If R refines P, then $U(f,R) \leq U(f,P)$

$L(f,R) \geq L(f,P) \iff -L(f,R) \leq -L(f,P)$

In particular, $U(f,R) - L(f,R) \leq U(f,P) - L(f,P)$

Then, for $\forall n \in \mathbb{N}$, we have

$0 \leq U(f,R_n) - L(f,R_n) \leq U(f,P_n) - L(f,P_n)$

$0 \leq U(g,R_n) - L(g,R_n) \leq U(g,Q_n) - L(g,Q_n)$

Proof of (a): we want to show that $U(f+g, P_n) - L(f+g, P_n) \to 0$

We have: $0 \le U(f+g, P_n) - L(f+g, P_n)$

$\le [U(f, P_n) + U(g, P_n)] - [L(f, P_n) + L(g, P_n)]$

$= [U(f, P_n) - L(f, P_n)] + [U(g, P_n) - L(g, P_n)]$

$\le [U(f, P_n) - L(f, P_n)] + [U(g, Q_n) - L(g, Q_n)]$

By the Squeeze Thm, $U(f+g, P_n) - L(f+g, P_n) \longrightarrow 0$

Note that for each $n \in \mathbb{N}$, we have

$L(f, P_n) + L(g, P_n) \le L(f+g, P_n) \le U(f+g, P_n) \le U(f, P_n) + U(g, P_n)$

$n \to \infty$ $\int_a^b f\,dx + \int_a^b g\,dx \le \int_a^b f+g\,dx \le \int_a^b f\,dx + \int_a^b g\,dx$

Then, $\int_a^b f+g\,dx = \int_a^b f\,dx + \int_a^b g\,dx$

Thm D: Suppose $f: [a,b] \to \mathbb{R}$ is integrable and Fix $c \in (a,b)$

Then, f is integrable $\Leftrightarrow$ f is integrable over $[a,c]$ & $[c,b]$

In this case, $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$

Thm E: $f, g : [a,b] \to \mathbb{R}$ are integrable

(a) If $m = \inf f$ and $M = \sup f$, then $m(b-a) \le \int_a^b f\,dx \le M(b-a)$

(b) If $f(x) \ge 0$ for $\forall x \in [a,b]$, then $\int_a^b f\,dx \ge 0$

(c) If $f(x) \ge g(x)$ for $\forall x \in [a,b]$, then $\int_a^b g\,dx + \int_a^b f\,dx$

(d) $|f|$ is also integrable and $\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$

Proof: (a) let $Q = \{a,b\}$ - partition of $[a,b]$

$$\underset{m(b-a)}{\underset{\parallel}{L(f,Q)}} \le \int_a^b f\,dx \le \underset{M(b-a)}{\underset{\parallel}{U(f,Q)}}$$

(b) Since $f(x) \geq 0$ for $\forall\, x \in [a,b]$,

we have $0 \leq m(b-a) \leq \int_a^b f \, dx$

(c) Apply part (b) to this function $f-g$:

$f(x) - g(x) \geq 0 \quad \forall\, x \in [a,b]$