¹ **Hidden Equivalence in the Operant Demand Framework: A Review and**

² **Evaluation of Multiple Methods for Evaluating Non-Consumption**

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⁶

Abstract

Operant translations of behavioral economic concepts and principles have enhanced the ability of researchers to characterize the effects of reinforcers on behavior. Operant behavioral economic models of choice (i.e., Operant Demand) have been particularly useful in evaluating how the consumption of reinforcers is affected by various ecological factors (e.g., price, limited resources). Prevailing perspectives in the Operant Demand Framework are derived from the framework presented in Hursh and Silberberg (2008). Few dispute the utility of this framework and model, though debate continues regarding how to address the challenges associated with logarithmic scaling. At present, there are competing views regarding the handling of non-consumption (i.e., 0 consumption values) and under which situations that alternative restatements of this framework are recommended. The purpose of this report was to review the shared mathematical bases for the Hursh and Silberberg (2008) and Koffarnus et al. (2015) models and how each can accommodate non-consumption values. Simulations derived from those featured in Koffarnus et al. (2015) were used to conduct tests of equivalence between modeling strategies while controlling for interpretations of residual error as well as the absolute lower asymptote. Simulations and proofs were provided to illustrate how neither the Hursh and Silberberg (2008) nor Koffarnus et al. (2015) models can characterize demand at 0 and how both ultimately arrive at the same upper and lower asymptotes. These findings are discussed and recommendations are provided to build consensus related to zero consumption values in the Operant Demand Framework.

Keywords: behavioral economics, operant demand, consumption, zero asymptotes

Word count: 5284

Hidden Equivalence in the Operant Demand Framework: A Review and Evaluation of Multiple Methods for Evaluating Non-Consumption

Introduction

Contemporary methods for evaluating the consumption of goods and services using the Operant Demand Framework are heavily influenced by the methodology proposed by Hursh and Silberberg (2008). This framework and methodology have evolved through several forms (e.g., Hursh et al., 1987) and the latest iteration takes a non-linear (i.e., "S"-type) shape and is driven by an exponential decay process (Hursh & Silberberg, 2008). This framework for evaluating the effects of unit price on the consumption of reinforcers has achieved widespread adoption and has also inspired derivatives that model consumption using varying scales, e.g., linear (Koffarnus et al., 2015), "log-like" (Gilroy, Kaplan, et al., 2021). Furthermore, this framework and manner of analysis has supported both basic and applied research, across a variety of real and hypothetical goods, within and across species [e.g., human and non-human animals; Hursh and Roma (2016)]. Although this framework has been effective in evaluating behavior across a range of applications, various modeling strategies are available and there is little consensus at present regarding which strategies are most appropriate for certain compositions of data.

The original implementation of the Hursh and Silberberg (2008) framework was modeled from the notion that the prototypical shape of the demand curve was an "S"-type form bounded by upper and lower limits. The original intent of Hursh and Silberberg (2008) was to have an upper asymptote defined at a price of zero (i.e., $\lim_{P \to -\infty}$) and a lower asymptote reached as prices approached infinity [i.e., $\lim_{P \to -\infty}$; Gilroy, Kaplan, et al. (2021)]. The upper asymptote is interpreted as the intensity of demand for a particular reinforcer (i.e., consumption [Q] at a price of $0 = Q_0$), and the rate by which demand progresses from the upper to lower asymptote refers to the overall sensitivity to price (i.e.,

62 rate of change in elasticity $= \alpha$).[1] This approach to characterizing the effects of reinforcers

63 has been effective for understanding patterns of choices related to abuse liability for drugs as

64 reinforcers (MacKillop et al., 2018) as well as substance use and abuse (Acuff et al., 2020;

65 Aston et al., 2016; González-Roz et al., 2019). Furthermore, this approach also provides a

66 means of evaluating reinforcer efficacy in behavioral interventions (Gilroy, Waits, et al., 2021;

67 Gilroy et al., 2018) as well as various other initiatives, e.g. environmental conservation

68 (Kaplan, Gelino, et al., 2018), consumption of evidence-based therapies (Gilroy et al., n.d.),

69 and COVID-19 vaccination (Hursh et al., 2020).

70   Models derived from this framework, such as Hursh and Silberberg (2008) and

71 Koffarnus et al. (2015), characterize the demand for reinforcers with non-zero upper and

72 lower asymptotes. Both models are bounded at an upper limit (i.e., Q_0) and progress

73 towards a non-zero lower limit in an "S"-type form. Non-zero upper and lower asymptotes in

74 these models make good sense because the original values of interest in the framework of

75 Hursh and Silberberg (2008) were positive real values (i.e., not including 0). The exclusion of

76 such quantities is expected because the logarithmic representation of consumption is

77 undefined at 0.

78   In response to the statistical and philosophical issues related to the omission of 0

79 consumption values, Koffarnus et al. (2015) introduced a restatement of the Hursh and

80 Silberberg (2008) model that accommodated these values during non-linear regression. This

81 procedure was made possible by exponentiatingterms such that the LHS (left-hand side) of

82 the original Hursh and Silberberg (2008) model reflected changes in consumption using the

83 linear scale. In this restatement, the LHS of the model (i.e., observed consumption) need not

84 be submitted to the log transformation that prevented the use of the original Hursh and

85 Silberberg (2008) model with non-consumption values.

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[1] Beyond the fitted estimates resulting from the framework of Hursh and Silberberg (2008), indicators of price elasticity of demand (e.g., P_{MAX}, O_{MAX}) are of primary interest in the Operant Demand Framework.

86 The approach presented in Koffarnus et al. (2015) drew considerable attention, as one

87 of the largest issues associated with the log scale could be avoided during nonlinear regression.

88 However, it warrants noting that most of the RHS (right-hand side) of the Koffarnus et al.

89 (2015) restatement remained on the log scale. Specifically, the span of the demand curve as

90 well as the rate of exponential decay remained in the log scale (Gilroy, Kaplan, et al., 2021).

91 It is for this reason that the span of the demand curve in this restated model cannot

92 characterize 0, despite including such quantities in non-linear regression. Additionally, it is

93 also relevant to note that the regressive process for logarithmic and linear implementations

94 of the model differs with respect to how residual error is interpreted and this introduces

95 behavior that varies between implementations (see Gilroy, Kaplan, et al., 2021).

Same Model But Different Error

97 The challenges associated with fitting models of operant demand (i.e., minimizing

98 residual error) are increasingly reviewed by researchers applying the Operant Demand

99 Framework (Gilroy, Kaplan, et al., 2021; Gilroy et al., 2020). Gilroy, Kaplan, et al. (2021)

100 noted, among other things, that residual error is reflected differently in log and linear scales

101 and that such differences affect model optimization, the resulting parameters, and even the

102 interpretations of such parameters. For example, changes in log scale represent relative

103 differences while changes in linear scale represent absolute differences. In most economic

104 applications, relative error is preferred because the usual quantities of interest and their

105 associated projections (i.e., $\hat{y}$) often span across multiple orders of magnitude (e.g., $\hat{y} = 1000$,

106 $\hat{y} = 10$, $\hat{y} = 0.1$). In these situations, the quantities observed at higher orders would be

107 weighted more heavily in absolute least squares regression (linear scale) than those at lower

108 orders unless some form of correction was applied (i.e., weighting). It is for this reason that

109 relative difference is often the default in these applications.

110 Whereas relative differences reference another quantity (e.g., predicted values,

111 weights), absolute differences are straightforward. That is, absolute error is simply the

112 difference from some observed quantity and $\hat{y}$ regardless of the order of magnitude. Although

113 more straightforward, the use of the linear scale in the Operant Demand Framework

114 introduces some variability in how parameters are optimized in these models. For example,

115 Gilroy, Kaplan, et al. (2021) noted how a departure from relative difference has led to

116 occasional inconsistencies wherein estimates across implementations of the Hursh and

117 Silberberg (2008) framework have led to different conclusions (e.g., shared vs. respective α

118 values across varying dose-response curves). In addition to fitted estimates, reflecting

119 residual error in terms of relative differences tends to yield more normalized patterns of error

120 variance. As such, differences in how residual error is handled represent one dimension along

121 which the two implementations of the Hursh and Silberberg (2008) framework differ.

Different Error but Same Asymptotes

123 There has been renewed attention regarding the asymptotes of models based upon

124 the framework of Hursh and Silberberg (2008). As currently designed, neither the Hursh and

125 Silberberg (2008) nor the Koffarnus et al. (2015) model can characterize demand at 0 and

126 this is because both reflect the span of the demand curve in log units (Gilroy, Kaplan, et al.,

127 2021). To address this fundamental issue, Gilroy, Kaplan, et al. (2021) presented a "log-like"

128 alternative to the log scale that preserves many of the desirable qualities of the log scale

129 while accommodating 0 consumption values in the Operant Demand Framework. For

130 example, this alternative (i.e., Inverse Hyperbolic Sine transformation) replicates the

131 behavior of logarithms across a range of values (e.g., 10, 100), normalizes residual error

132 variance, and supports a true lower bound at 0. This approach is not discussed at length in

133 this report, though interested readers are encouraged to consult Gilroy, Kaplan, et al. (2021)

134 for a discussion and demonstration of this "log-like" scale and its benefits (e.g., de-coupling

135 of α from the span of the curve, separate span parameter not necessary).

136 Revisiting the topic of asymptotes, two novel terms are introduced in this work:

137 A_{Upper} and A_{Lower}. The term A_{Upper} is used to refer to the absolute upper bound of the

138 demand curve. In models derived from the Hursh and Silberberg (2008) framework, this is

139 simply the fitted parameter Q_0. This is because parameter Q_0 is the absolute upper limit to

140 the demand curve when prices equals 0, i.e. $f(0) = A_{Upper}$. In contrast, the term A_{Lower}

141 refers the absolute lower bound of the demand curve and this is not reflected by any *single*

142 parameter. Mathematically, this absolute lower limit refers to the level of demand as price

143 approaches ∞, i.e. $\lim_{P \to \infty} f(x) = A_{Lower}$. These two upper and lower extremes are separated by

144 the span constant k, which specifies the distance in log units between these asymptotes

145 (Gilroy, Kaplan, et al., 2021). The notation of both A_{Upper} and A_{Lower} are noted below and

146 are proofed in greater detail across models in the Appendix of this work.

$$A_{Upper} = 10^{log_{10}Q_0}$$
$$A_{Lower} = 10^{log_{10}Q_0 - k}$$

(1)

147   Further inspection of A_{Lower} and its derivation evokes questions regarding how any

148 model based on the Hursh and Silberberg (2008) framework could characterize

149 non-consumption values. As noted in the bottom portion of *Equation 1*, neither the Hursh

150 and Silberberg (2008) and Koffarnus et al. (2015) models could represent a value of 0

151 because this value does not fall within the interval between these upper and lower extremes.

152 This introduces a complex situation wherein 0 consumption values could be included in

153 non-linear regression, but the predicted levels of demand could never characterize this value.

154 As such, the issue with non-zero lower asymptotes is one dimension along which derivatives

155 of the Hursh and Silberberg (2008) framework are the same.

Same Asymptotes and Same Spans

157   Understanding non-consumption in the Hursh and Silberberg (2008) framework

158 requires an appreciation of how the span parameter k influences the range of values that may

159 be predicted (i.e., $\hat{y}$). In the original implementation of the Hursh and Silberberg (2008)

160 framework, k represented the range of observed, non-zero consumption values. That is, k was

161 derived in log units from the upper and lower extremes of all positive, real numbers. Since 0

162 consumption values were not included in the original implementation, parameter k was

163 directly linked to the upper and lower limits of the observed data. Indeed, the specification

164 of this constant was straightforward and parameter k, A_{Upper}, and A_{Lower} were all directly

165 linked to positive real numbers. A visualization of parameter k is provided in Figure 1 with

166 respect to positive real consumption values.
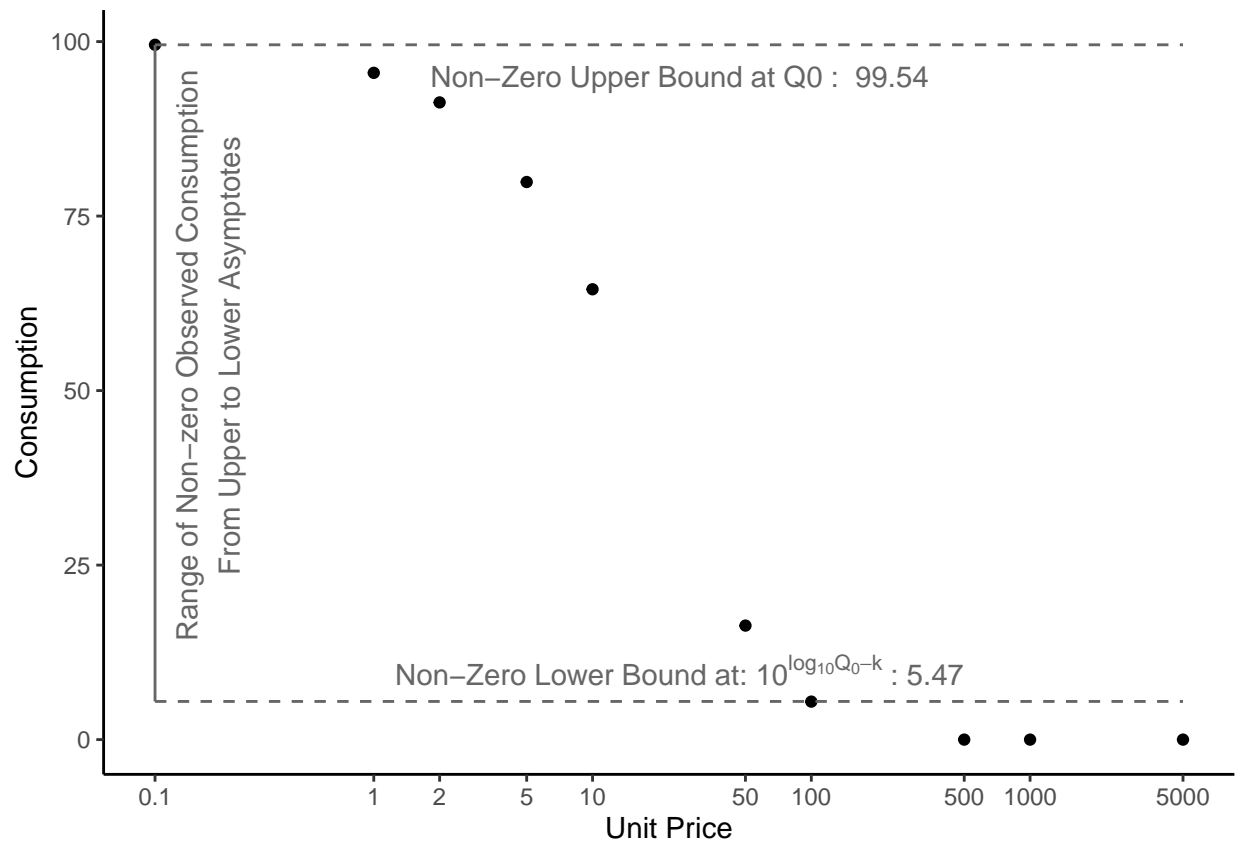
Figure 1

Span of Demand Curve with Non-zero Consumption

167   Whereas the determination of parameter k in the Hursh and Silberberg (2008)

168 implementation is linked to positive real numbers, the interpretation of parameter k became

169 more complicated in the implementation introduced by Koffarnus et al. (2015). This added

170 complexity emerged because parameter k was still based on positive real numbers but had to

171 be inflated to project A_{Lower} *beyond* the range of positive real values to a quantity nearer to

172 0. This represented novel behavior for parameter k and various teams have constructed

173 strategies to assist in driving A_{Lower} *beyond* the range of non-zero consumption. For example,

174 some have added a constant to parameter k (derived from positive real values) or allowed

175 this parameter to vary as a fitted parameter (Kaplan, Foster, et al., 2018). Regardless of the

176 method, the rationale was to inflate the span of the demand curve to and drive A_{Lower} to a

177 lower point. A visualization of this span-inflating behavior is illustrated in Figure 2.
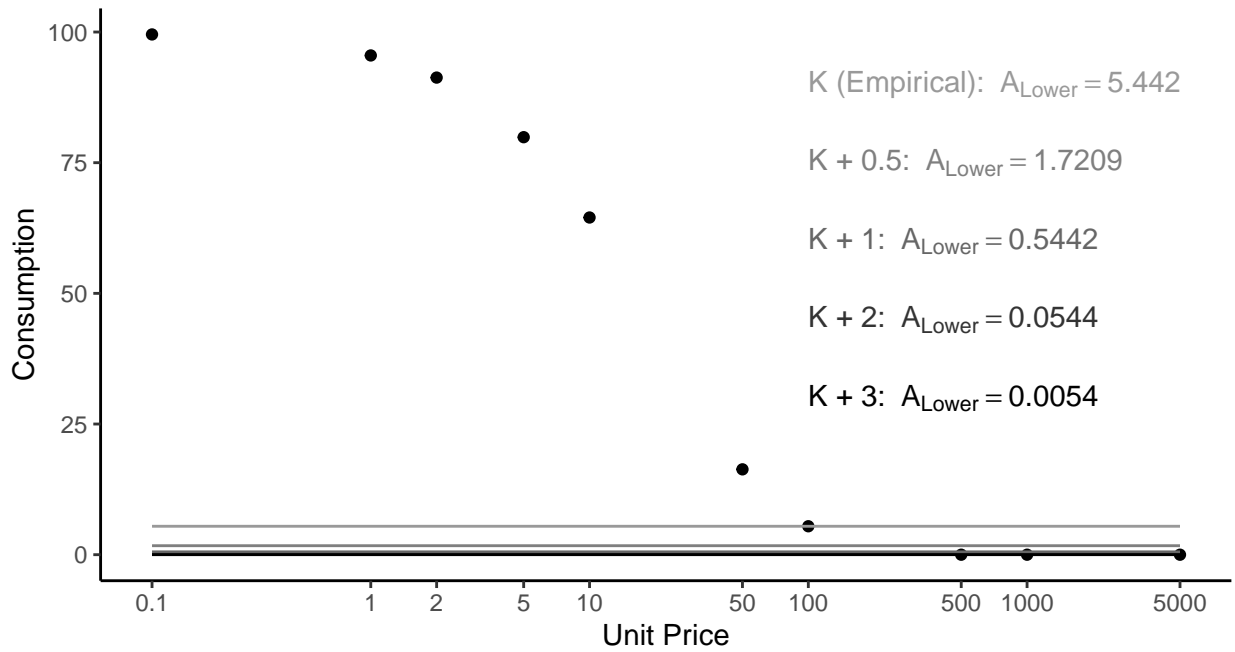
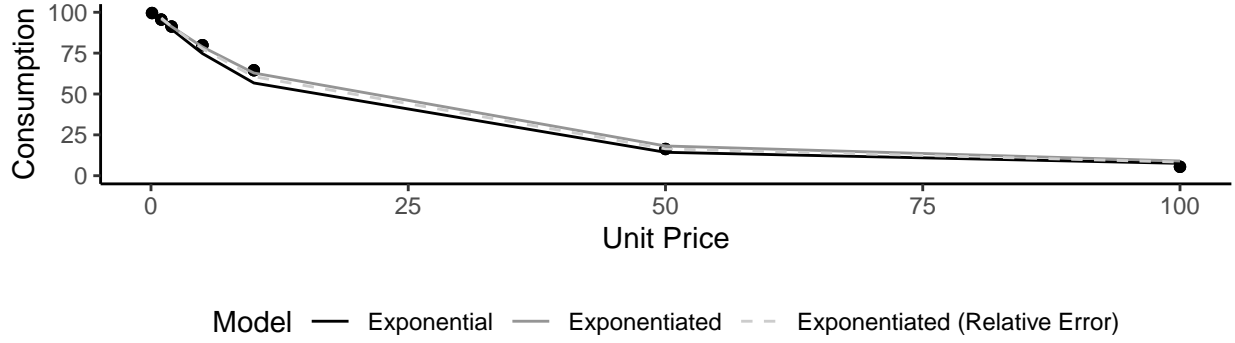**Figure 2**

Span of Empirical Demand Curve with Vary Spans

178   Figure 2 illustrates how inflating the span affects A_{Lower} and this has three

179 appreciable effects on the demand curve. First, the rate of change in elasticity is jointly

180 reflected by parameter α and the span of the demand curve (Gilroy et al., 2020). Given that

181 α is a unit-less quantity, it co-varies inversely with the size of the span constant. For example,

182 relatively greater α values reflect rapid changes in $\hat{y}$ across prices and relatively lesser values

183 reflecting gradual changes in $\hat{y}$ across prices. Second, k values (i.e., $k < \frac{e}{log(10)}$) influence both

184 the span of the demand curve as well as the range of elasticity and inelasticity observed in

185 models derived from the Hursh and Silberberg (2008) framework (Gilroy et al., 2019;

186 Newman & Ferrario, 2020). That is, k values below 1 log unit restrict the range of elasticity

187 values and render analytic solutions for unit elasticity impossible. Third, and most relevant

188 to the Koffarnus et al. (2015) model, inflated k values serve to lessen the absolute difference

189 between A_{Lower} and 0. That is, the distance between A_{Lower} and 0 is is lessened, but no k

190 value will drive the span to 0. Figure 2 provides an elegant display of how this gap decreases

191 proportionally with each unit increase in the span parameter k but will never reach 0.

192 **Hidden Model Equivalence**

193   The sections above outline the few ways in which two of the most popular derivatives

194 of the Hursh and Silberberg (2008) framework differ. These two modeling strategies differ in

195 terms of optimization (i.e., minimization of residual error) but share the limitations related

196 to asymptotes. Regarding the first point, residual error and optimization, the two models

197 can provide equivalent results when the handling of residual error is made *comparable*. That

198 is, re-weighting the errors (i.e., relative to $\hat{y}$) in the Koffarnus et al. (2015) model can yield

199 fits and estimates approximate to those resulting from the Hursh and Silberberg (2008)

200 model *in the absence of non-consumption*. Alternatively, the Hursh and Silberberg (2008)

201 model can be adjusted to yield estimates comparable to those from the Koffarnus et al.

202 (2015) model by adjusting residual error to be interpreted in terms of absolute difference,

203 i.e. $E_i = 10^{\hat{y}} - 10^{y}$. A visualization of inter-related model fits are illustrated in Figure 3.

204   Regarding the second point, A_{Lower} is seldom discussed in Operant Demand and this

205 has considerable influence on models derived from the Hursh and Silberberg (2008)

206 framework. This is an inherently complex topic, especially so in the Koffarnus et al. (2015)

207 restatement, because consumption values observed at 0 are a quantity that cannot be

208 predicted by models that reflect the range of consumption in log units. In attempts to

209 accommodate non-consumption, modeling based on the Hursh and Silberberg (2008)

210 framework must minimize *two* sources of error instead of one. That is, the non-linear
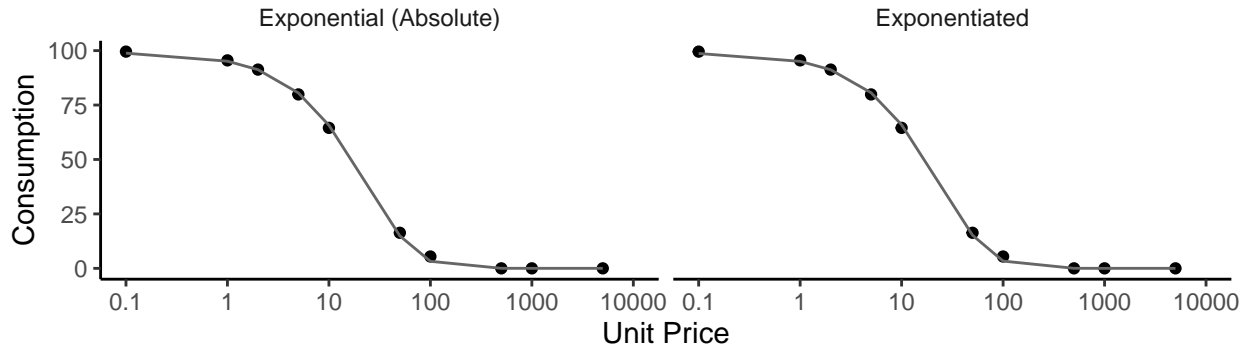
Figure 3

Comparable Model Fits with Comparable Error

211   regression must minimize residual error as well as the distance between A_{Lower} and 0 (i.e., k)

212   in an attempt to produce an A_{Lower} that *approximates* 0. For instance, an application of the

213   Koffarnus et al. (2015) model where k is included as a fitted parameter simultaneously

214   optimizes demand intensity, rates of change in elasticity, and a span constant (i.e., A_{Lower}).

215   As noted above, A_{Lower} is driven lower by inflating the span constant towards some non-zero

216   quantity that is *reasonably* close to 0. Pragmatically, proponents of the Koffarnus et al.

217   (2015) approach would likely argue that such a small amount of error calls for little concern

218   and that A_{Lower} could be considered *close enough* of an approximation of 0 to enable

219   analyses using the complete data set (non-consumption values included).

220   Revisiting the argument for a *close enough* approximation of non-consumption, let us

221   consider the following hypothetical. Let us say that the interpretation of a fitted Koffarnus

222   et al. (2015) model optimizes such that values at A_{Lower} are a close enough approximation of

223   0 to proceed with demand curve analyses using a complete data set. Following this logic (i.e.,

224   $A_{Lower} \cong 0$), it stands to reason that treating sufficiently low A_{Lower} values and 0

225   consumption values as the same *should* replicate the behavior of the Koffarnus et al. (2015)

226   model in the Hursh and Silberberg (2008) model. Assuming an inflated k parameter,

227   equivalent estimates should result because $\hat{y}$ can be predicted beyond the range of observed

228   non-zero levels and the resulting A_{Lower} should be *close enough* to 0 on the linear scale that

229 differences between A_{Lower} and 0 would be considered negligible. Controlling for differences

230 in terms of error representation, it stands to reason that the Hursh and Silberberg (2008)

231 model would provide equivalent estimates had non-consumption been replaced by respective

232 A_{Lower} values and error minimization been reflected in terms of absolute differences.

233   In a demonstration of this modified Hursh and Silberberg (2008) approach, the full

234 data set from Figure 1 was fitted with an inflated k parameter and non-consumption values

235 replaced with respective A_{Lower} values. Specifically, the most inflated span and corresponding

236 A_{Lower} from Figure 2 were used in this example demonstration, i.e. $A_{Lower} = 10^{log10Q_0 - (k+3)}$.

237 The results of this modified Hursh and Silberberg (2008) approach are illustrated along with

238 the Koffarnus et al. (2015) approach are illustrated in Figure 4.

Figure 4

Comparable Model Fits with Comparable Asymptotes

239   Controlling for differences in error handling (absolute difference) and A_{Lower} values

240 (non-consumption replaced by lower asymptotes in the Hursh and Silberberg (2008) model),

241 the model fits are functionally equivalent, see Table 1. This short example highlights several

242 details that often go unnoticed when using the Koffarnus et al. (2015) model. First, this

243 model does not characterize demand at 0. Rather, an inflated k parameter to drives A_{Lower}

244 to a quantity *close enough* to 0 that the absolute difference between 0 and $\hat{y}$ is negligible.

245 This is the best that this approach can achieve because 0 does not fall within the interval

246 between A_{Upper} and A_{Lower}. Second, this approach is functionally equivalent to the Hursh

Table 1

Comparison of Exponentiated and Absolute-Weighted Exponential

Models

P	Q	Exponentiated	Q.Mod	Exponential.Absolute
0.100	99.541	98.686	99.541	98.817
1.000	95.521	95.068	95.521	95.150
2.000	91.285	91.219	91.285	91.251
5.000	79.884	80.666	79.884	80.572
10.000	64.517	65.947	64.517	65.710
50.000	16.333	15.211	16.333	14.915
100.000	5.442	3.339	5.442	3.232
500.000	0.000	0.018	0.005	0.017
1,000.000	0.000	0.006	0.005	0.006
5,000.000	0.000	0.005	0.005	0.005

and Silberberg (2008) model when non-consumption values are replaced with by the respective A_{Lower} values and when residual errors are de-weighted (i.e., absolute). As such, both functional almost identically and differ in largely trivial aspects.

Research Questions

The Operant Demand Framework has achieved high regard as a robust approach for evaluating choices and behavior of societal significance (Hursh & Roma, 2013; Reed et al., 2013). Various labs and teams have been working towards expanding the scale and scope of this approach, moving from questions specific to individuals and groups to society-at-large (Hursh & Roma, 2013; Roma et al., 2017). This approach and its methods are increasingly represented in a range of scientific tools and packages as well (Gilroy et al., 2018; Kaplan et al., 2019). Despite increasing popularity and accessibility, few resources provide the

²⁵⁸ mathematical details necessary to support researchers in navigating between the available
²⁵⁹ options for performing demand curve analyses.

²⁶⁰   The purpose of this technical report was to review the mathematical underpinnings of
²⁶¹ two prevailing models derived from the framework of Hursh and Silberberg (2008) and
²⁶² present an argument as to why distinctions between such models create more confusion than
²⁶³ consensus.

²⁶⁴   Specifically, the shared mathematical bases between the two should allow for
²⁶⁵ modifications wherein both provide statistically equivalent estimates—even when
²⁶⁶ non-consumption values are present. The primary questions for the simulation study was
²⁶⁷ whether estimates resulting from the Hursh and Silberberg (2008) and Koffarnus et al.
²⁶⁸ (2015) models would be statistically equivalent when controlling for differences in handling
²⁶⁹ residual error (i.e., absolute, relative) and treating non-consumption values as respective
²⁷⁰ A_{Lower} values.

<div align="center">Methods</div>

²⁷¹

Data Generating Process

²⁷²

²⁷³   A total of 20000 hypothetical data series were simulated using using the R Statistical
²⁷⁴ Program (R Core Team, 2021). The specific syntax used to generate was featured in an R
²⁷⁵ package that was submitted to peer-review (Kaplan et al., 2019). Specifically, the
²⁷⁶ *SimulateDemand* method included in the *beezdemand* R package (Kaplan et al., 2019) was
²⁷⁷ used to simulate hypothetical purchase task data that included a large composition of
²⁷⁸ non-consumption values. The seed values and variance used to generate these data were
²⁷⁹ identical to those that were used in Koffarnus et al. (2015). This specific data generating
²⁸⁰ process was used as the basis for comparisons with the Hursh and Silberberg (2008) model
²⁸¹ given that the authors of the Koffarnus et al. (2015) study modeled their approach around
²⁸² "messy" data frequently observed in "real-world" purchase tasks that are often conducted on

283 "crowdsourced" platforms, e.g. Amazon's Mechanic Turk (mTurk).

Screening of Non-systematic Data Series

285   The three criteria for systematic data outlined in Stein et al. (2015) were applied to

286 all generated demand data. Specifically, individual series were screened for *bounce*, *trend*,

287 and *reversals from zero*. The first criterion, bounce, refers to local changes within an

288 expected downward trend as a function of increasing price. That is, it is unexpected to see

289 consumption increases immediately following a price increase. The second criterion, trend,

290 refers to the molar change in consumption from the lowest to the highest price. That is,

291 there is a certain amount of decrease in consumption expected across the full domain of price

292 increases. Lastly, reversals from zero refer to the return of consumption at a higher price

293 following the cessation of consumption at a lower price. Such trends are inconsistent with

294 expected patterns of consumption. Simulated data were carried forward into the final

295 analysis if each series met all indicators of systematic hypothetical purchase task data.

Modeling Strategies

297   A total of 4 modeling approaches were evaluated (2 models, 2 error interpretations).

298 Each approach was referenced as a specific strategy for conducting demand curve analysis

299 when non-consumption values were observed in the data. This facilitated two pairwise

300 comparisons when both models shared a comparable approach for handling residual error.

301 These comparisons were used to determine whether the various strategies provided

302 statistically equivalent estimates when asymptotes and error differences were comparable.

303 Consistent with efforts to maintain open and transparent science (Gilroy & Kaplan, 2019),

304 the source code necessary to reproduce these strategies and this report has been posted for

305 public review in a GitHub repository managed by the corresponding author, see Author

306 Note. Each of the strategies used in these comparisons are presented below in greater detail.

Strategy 1: Koffarnus et al. (2015) Model (Absolute Error)

The Koffarnus et al. (2015) model (absolute error difference) was fitted to simulated consumption data at the individual-level. The model was fit using the *optim* package included in the R Statistical Program (R Core Team, 2021) due to its considerable flexibility in performing ordinary least squares regression. Initial starts were derived based on the respective data for parameter Q_0 and both Q_0 and α were estimated on the log scale to 1) support more comparable step sizes in the optimization and 2) facilitate pairwise comparisons across strategies. The span constant k was derived from the empirical range of the full data set with an added constant (i.e., $k = k + 3$) to allow the span of the demand curve to extend below the lowest non-zero point of consumption, as is common practice (Kaplan, Foster, et al., 2018). The same span constant was used across all models to enable consistent comparisons between Q_0 and α. Non-consumption values remained at a value of 0 in this approach.

Strategy 2: Koffarnus et al. (2015) Model (Percentage Error)

The Koffarnus et al. (2015) model (percentage error difference) was evaluated consistent with Strategy 1 with the exception of how differences in residual error were reflected. In this approach, the absolute residuals simulated relative error by referencing $\hat{y}$, i.e. $e_i = (\hat{y} - y) * \frac{1}{\hat{y}} = \frac{\hat{y}-y}{\hat{y}}$. It warrants noting that this manner of weighting error is not identical to log difference. That is, the weighting of the absolute error difference against $\hat{y}$ is equivalent to reflecting residual error as percentage difference and this corresponds with log difference only certain circumstances, i.e. $ln\frac{Y_1}{Y_2} \approx \frac{Y_2-Y_1}{Y_1}$. Briefly, percentage difference is nearly identical to log difference with very small differences (e.g., 1% change) but the two diverge once the degree of difference between values grows larger (e.g., 50% change). As such, the varying approaches to reflecting relative differences are expected to vary and this source of error between the approaches is described more thoroughly in the Appendix. Regardless, the two approaches are expected to behave comparably but are not expected to be

333 equivalent. All other parameters were estimated consistent with Strategy 1.

Strategy 3: Hursh and Silberberg (2008) Model (Log Difference Error)

335 The Hursh and Silberberg (2008) model (log error difference) was fitted to simulated

336 consumption data at the individual-level. During the fitting, non-consumption values were

337 replaced by an A_{Lower} value that was generated dynamically based on parameters Q_0 and k

338 during parameter estimation. That is, a customized loss function was prepared for use with

339 the *optim* method. As noted in Strategy 2, both Strategy 2 and Strategy 3 reflected relative

340 difference in different ways. All other parameters were estimated consistent with the other

341 strategies.

Strategy 4: Hursh and Silberberg (2008) Model (Absolute Error)

343 The Hursh and Silberberg (2008) model (absolute error difference) was fitted to

344 simulated consumption data at the individual-level as well. This strategy was identical to

345 that of Strategy 3 with the exception of how residual error was interpreted during

346 optimization. Consistent with Strategy 3, non-consumption values were replaced by an

347 A_{Lower} value that was generated dynamically based on parameters Q_0 and k during

348 parameter estimation. A customized loss function was used to represent residual error in

349 terms of absolute differences, i.e. $e_i = 10^{\hat{y}} - 10^y$. All other parameters were estimated

350 consistent with that of the other strategies.

Analytical Strategy

352 Pairwise comparisons were conducted for parameters Q_0 and α resulting from each of

353 the four strategies while controlling for differences in how residual error was interpreted

354 during optimization. T-tests and tests of equivalence were used to compare estimates

355 resulting from the Koffarnus et al. (2015) model and the Hursh and Silberberg (2008) model

356 with and without modified error terms. T-tests were calculated using the base methods in R

357 and the *tost* method in the *equivalence* R package was used to perform two one-sided t-tests

358 (TOSTs, Robinson & Robinson, 2016). Specifically, t-tests were used first in each comparison

359 to test whether a significant difference was observed between estimates. Tests of equivalence

360 were performed if a non-significant difference was observed. The emphasis here was not on

361 determining a lack of *difference* between strategies but instead on determining whether these

362 were practically *equivalent.* The Smallest Effect Size of Interest (SESOI) was set to 0.01 (i.e.,

363 ~1% difference in log scale) and differences below this threshold were not considered

364 practically meaningful. Across all tests, corrections were applied due to presence of repeated

365 comparisons, i.e. $p = 0.05/2 = 0.025$.

Results

367   The data generating process was used to produce a total of 20000 distinct

368 consumption series that simulated hypothetical purchase task data. A range of series was

369 simulated but was restricted to those that met all indices of systematic purchase task data,

370 contained 50% or more non-zero consumption, and featured at least two unique positive real

371 consumption values (i.e., non-step data). Within these series, the R^2 metric was used as the

372 basis for selecting the 1,000 series that best represented the optimal performance across all

373 fitted models. The results of specific pairwise comparisons across these 1,000 cases are

374 presented below.

Strategy 1 vs. Strategy 4 (Absolute Error)

376   The primary comparison of interest in this report was between Strategy 1 and

377 Strategy 4 This comparison evaluated the correspondence between the Hursh and Silberberg

378 (2008) and Koffarnus et al. (2015) models when error differences were represented in terms of

379 absolute difference and when non-consumption was treated as A_{Lower} for the Hursh and

380 Silberberg (2008) model. Given the shared mathematical basis for each, the estimates

381 resulting from each were expected to be equivalent.

382   An evaluation of the relationship between Strategy 1 and 4 revealed perfect

Q$_0$ Estimates

α Estimates
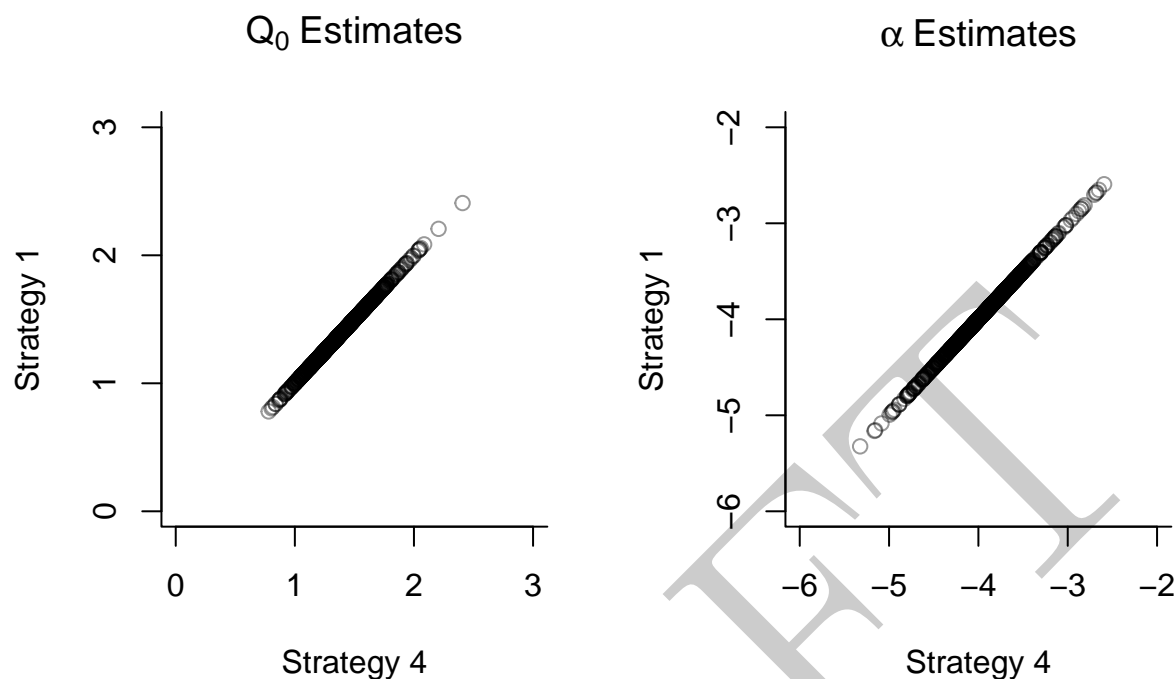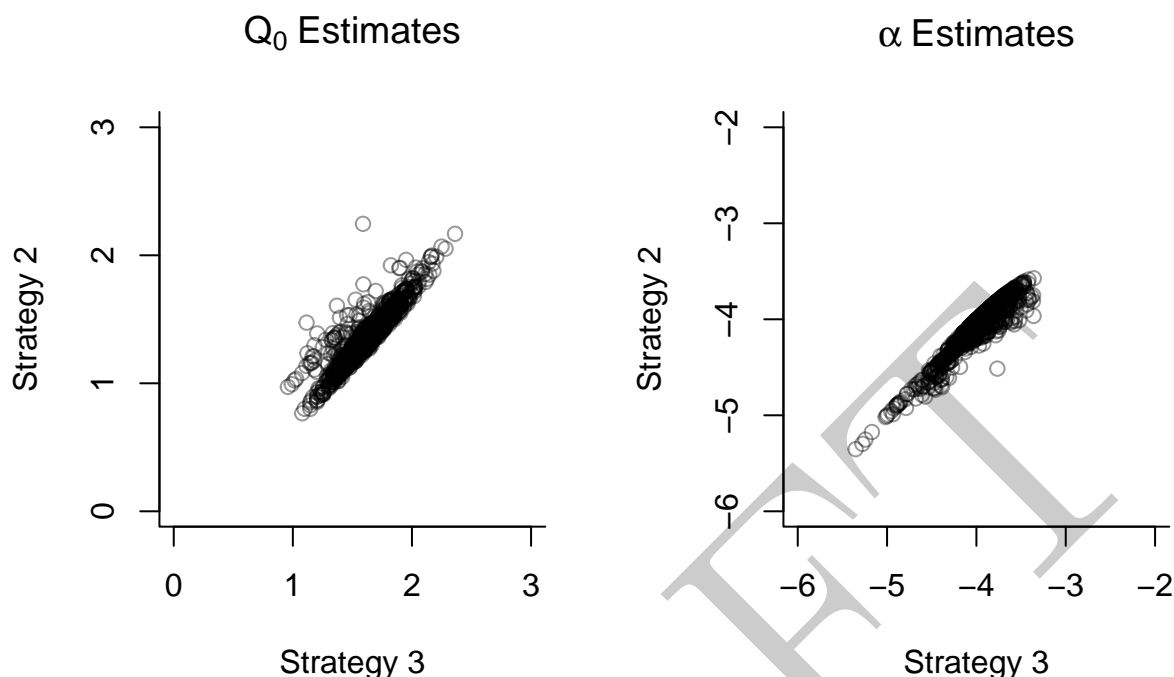

Figure 5

Comparisons of Strategy 1 and 4 (Absolute Error)

correlations for both Q_0 (r=1.00, t=612,004,386.43, df=998, p<0.025) and for α (r=1.00, t=80,650,327.30, df=998, p<0.025). That is, a perfect rank ordering was observed across strategies and across parameters. T-test comparisons were non-significant for Q_0 (t=0.00, df=1,998.00, p>0.975) and for α (t=0.00, df=1,998.00, p>0.975). Subsequent TOSTs were significant for Q_0 (p<0.025) and for α (p<0.025). Specifically, results of equivalence testing rejected the null hypothesis of statistical difference for both parameters and this indicated that estimates resulting from each strategy were statistically equivalent. A visualization of these corresponding estimates is illustrated in Figure 5.

Strategy 2 vs. Strategy 3 (Relative Error)

The secondary comparison of interest in this report was between Strategy 2 and Strategy 3. Comparisons between Strategy 2 and Strategy 3 evaluated the correspondence

Q_0 Estimates

α Estimates

Figure 6

Comparisons of Strategy 2 and 3 (Relative Error)

between the Hursh and Silberberg (2008) and Koffarnus et al. (2015) models when error

differences were interpreted in terms of relative difference and when non-consumption was

treated as A_{Lower} for the Hursh and Silberberg (2008) model. Specifically, the Hursh and

Silberberg (2008) model evaluated error using log difference and the Koffarnus et al. (2015)

model evaluated error using percentage difference. Given the varying methods of

representing residual error as relative, the estimates resulting from each were not expected to

be equivalent.

An evaluation of the relationship between Strategy 2 and Strategy 3 revealed strong,

but not perfect correlations for Q_0 (r=0.91, t=68.48, df=998, p<0.025) and for α (r=0.95,

t=101.43, df=998, p<0.025). T-test comparisons were significant for Q_0 (t=-27.42,

df=1,995.83, p<0.025) as well as for α (t=-7.46, df=1,989.75, p<0.025). No TOSTs were

performed given that t-tests indicated significant differences between estimates resulting

from each strategy. A visualization of these relationships are illustrated in Figure 6.

Discussion

This report provided an in-depth review of how non-consumption values (i.e., 0) have, thus far, been incorporated in models derived from the Hursh and Silberberg (2008) framework. As noted throughout this report, both the Hursh and Silberberg (2008) and the Koffarnus et al. (2015) approaches are unable to model demand at 0 and both are bounded by the non-zero lower asymptote, A_{Lower}. This is the case regardless of whether non-consumption values are included in the regression. As such, the approach put forward in Koffarnus et al. (2015) is not a complete solution for non-consumption values because the same limitations of the original approach remain in this regard. This is because the span of the demand curve in the Hursh and Silberberg (2008) framework remains in the log scale, despite LHS exponentiation, and the span in log scale cannot support 0. As an alternative to this issue with span, others have argued that a true solution to this issue would require deviating from the log scale altogether (Gilroy, Kaplan, et al., 2021).

The proofs and simulations featured in this study facilitated comparisons between the Hursh and Silberberg (2008) and Koffarnus et al. (2015) models when controlling for the common A_{Lower} and differences in how residual errors are interpreted during optimization. The goal of these comparisons was to advance the argument that the Exponential (Hursh & Silberberg, 2008) and Exponentiated (Koffarnus et al., 2015) models should not be so strongly distinguished. Indeed, it is quite trivial to arrive at statistically equivalent estimates in both approaches when the role of the span constant and A_{Lower} and the method of representing residual error are held perfectly constant. The results of planned comparisons confirmed that the two models provide statistically equivalent estimates when controlling for such differences perfectly—even when non-consumption values are included. However, this is not the case when different methods of addressing residual error are used. That is, similar methods for representing relative differences are closely correlated but not statistically

432 equivalent. This difference is mostly due to how percentage and log difference diverge as

433 differences grow larger (see Appendix).

434   Given that neither approach can characterize demand at 0, A_{Lower} is the *best*

435 approximation of 0 possible for models derived from the Hursh and Silberberg (2008)

436 framework. Following this logic, replacing non-consumption values with respective

437 A_{Lower} values often result in the Exponential model providing estimates that are at least

438 highly correlated with (potentially statistically equivalent to) the Exponentiated model. In

439 advancing this argument, it is necessary to state clearly that this claim is not presented with

440 the intent of favoring any specific approach as a de facto standard or a recommended default

441 when applying methods from the Operant Demand Framework. Rather, this work intended

442 to reveal how these supposedly opposing strategies are functionally interchangeable under

443 specific conditions. Indeed, they are so similar that distinguishing the two only serves to

444 obscure the many shared mathematical bases of each. That said, each approach has common

445 utility and future efforts should be directed towards improving the understanding of the

446 properties of the Hursh and Silberberg (2008) framework overall rather than reinforcing any

447 stance, position, or bias towards a specific implementation.

448   The final aim of this work was to reiterate the ways in which the proponents of each

449 approach have extended the Operant Demand Framework. That is, the proponents of each

450 approach were successful in advancing both the utility and scope of the Operant Demand

451 Framework. For example, the finding that the Hursh and Silberberg (2008) model can

452 replicate the behavior of the Koffarnus et al. (2015) without exponentiation terms in no way

453 detracts from the contributions of the Koffarnus et al. (2015) implementation of the

454 framework. Indeed, the Koffarnus et al. (2015) team led the charge towards addressing the

455 problematic issue of removing otherwise valid research data. For decades, substantial

456 portions of otherwise valid consumption data were never carried forward into analyses and it

457 is unclear how these prior analyses would compare had these data been included. Regardless

458 of whether analysts have an established preference for one approach or another, it is clear

459 that the methods included in Operant Demand Framework are better equipped now that

460 non-consumption values can now be considered in the analysis.

Future Directions in Operant Demand

462   This perspective and this framework currently reflect a range of consumption (and

463 non-consumption) and efforts are underway to leverage multilevel modeling as a

464 methodological extension (Kaplan et al., In Press). Indeed, various labs are working toward

465 increasing the applicability and generality of this approach. Towards this end, the intent and

466 mission of the original Koffarnus et al. (2015) study regarding non-consumption values is as

467 valid and valuable today as it was when this work was first published. However, debates and

468 conjecture regarding model superiority (or inferiority) in the absence of formal tests and

469 mathematical proofing do not enhance the Operant Demand Framework in any appreciable

470 manner. That said, the two approaches are functionally interchangeable (even in the presence

471 of non-consumption) and the reader is cautioned against thinking that any single model is

472 inherently "true," "better," or otherwise superior in the absence of careful and individualized

473 statistical evaluation. That said, it is unclear whether the prevailing approach in the Operant

474 Demand Framework will remain based on the framework presented in Hursh and Silberberg

475 (2008) well into the future. Indeed, it is possible that future research could explore to

476 deviations from the log scale (Gilroy, Kaplan, et al., 2021) or adopt a different framework

477 altogether (Newman & Ferrario, 2020). Regardless of the where the future takes the Operant

478 Demand Framework, future approaches and advances should be met with cautious optimism

479 and consideration rather than disregard in favor for what is preferred or familiar.

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566   **Appendix**

567   Several proofs are provided here to illustrate how the upper and lower asymptotes are

568   determined. Despite the shared mathematical basis, derivations of each are provided below.

569   **Modified Hursh & Silberburg (2008) Optimization (Relative Error)**

$$
e_i = \begin{cases} \hat{y}_i - log_{10}y_i & \text{if } y_i \neq 0 \\ \hat{y}_i - log_{10}A_{Lower} & \text{if } y_i = 0 \end{cases}
$$

570   **Modified Hursh & Silberburg (2008) Optimization (Absolute Error)**

$$
e_i = \begin{cases} 10^{\hat{y}_i} - 10^{log_{10}y_i} & \text{if } y_i \neq 0 \\ 10^{\hat{y}_i} - 10^{log_{10}A_{Lower}} & \text{if } y_i = 0 \end{cases}
$$

571   **Hursh & Silberburg (2008) Proofs**

572   A_{Upper} **at** $P = 0$

$$
\begin{aligned}
log_{10}A_{Upper} &= log_{10}Q_0 + k(e^{-\alpha*Q_0*0} - 1) \\
&= log_{10}Q_0 + k(e^0 - 1) \\
&= log_{10}Q_0 + k(1 - 1) \\
&= log_{10}Q_0 + k(0) \\
&= log_{10}Q_0 \\
A_{Upper} &= Q_0
\end{aligned}
$$

573   Note: Euler's constant raised to the power of 0 is equal to a value of 1. This

574   essentially zeroes out the k constant, leaving just the Q_0 parameter at 0 P.

575   A_{Lower} **at** $\lim\limits_{P \to \infty} f(x)$

$$log_{10}A_{Lower} = \lim\limits_{P \to \infty} f(x) = log_{10}Q_0 + k(e^{-\alpha*Q_0*\infty} - 1)$$

$$= log_{10}Q_0 + k(e^{-\infty} - 1)$$

$$= log_{10}Q_0 + k(0 - 1)$$

$$= log_{10}Q_0 + k(-1)$$

$$= log_{10}Q_0 - k$$

$$= log_{10}A_{Upper} - k$$

$$A_{Lower} = 10^{log_{10}A_{Upper} - k}$$

576   Note: Euler's constant raised to the power of $-\infty$ equates to a value of 0. That is,

577    $e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} \approx 0$. This is has effect of making the value in parentheses equal to -1, which

578   in turn results in the full subtraction of quantity k from $log_{10}Q_0$.

579   **Koffarnus et al. (2015) Proofs**

580   A_{Upper} **at** $P = 0$

$$A_{Upper} = Q_0 * 10^{k(e^{-\alpha*Q_0*0} - 1)}$$

$$= Q_0 * 10^{k(e^0 - 1)}$$

$$= Q_0 * 10^{k(1-1)}$$

$$= Q_0 * 10^{k(0)}$$

$$= Q_0 * 10^0$$

$$= Q_0 * 1$$

$$= Q_0$$

$$log_{10}A_{Upper} = log_{10}Q_0$$

581 A_{Lower} **at** $\lim_{P \to \infty} f(x)$

$$A_{Lower} = \lim_{P \to \infty} f(x) = Q_0 * 10^{k(e^{-\alpha * Q_0 * \infty} - 1)}$$

$$= Q_0 * 10^{k(e^{-\infty} - 1)}$$

$$= Q_0 * 10^{k(0 - 1)}$$

$$= Q_0 * 10^{k(-1)}$$

$$= Q_0 * 10^{-k}$$

$$log_{10} A_{Lower} = log_{10} Q_0 + (-k)$$

$$= log_{10} Q_0 - k$$

$$= log_{10} A_{Upper} - k$$

$$A_{Lower} = 10^{log_{10} A_{Upper} - k}$$

582 **Differences between Log and Percentage Difference**

583 *Logarithmic Difference*

$$V_1 = 100$$

$$V_2 = 90$$

$$ln(\frac{V_2}{V_1}) = -1 * ln(\frac{V_1}{V_2})$$

$$ln(\frac{90}{100}) = -1 * ln(\frac{100}{90})$$

$$ln(0.9) = -1 * ln(1.11)$$

$$-0.1053 = -1 * 0.1053$$

$$-0.1053 = -0.1053$$

$$V_1 = 100$$

$$V_2 = 50$$

$$ln(\frac{V_2}{V_1}) = -1 * ln(\frac{V_1}{V_2})$$

$$ln(\frac{50}{100}) = -1 * ln(\frac{100}{50})$$

$$ln(0.5) = -1 * ln(2)$$

$$-0.6931 = -1 * 0.6931$$

$$-0.6931 = -0.6931$$

584 *Percentage Difference*

$$V_1 = 100$$

$$V_2 = 90$$

$$\frac{V_2 - V_1}{V_1} \approx -1 * \frac{V_1 - V_2}{V_2}$$

$$\frac{90 - 100}{100} \approx -1 * \frac{100 - 90}{90}$$

$$\frac{-10}{100} \approx -1 * \frac{10}{90}$$

$$-0.1 \approx -1 * 0.11$$

$$-0.1 \approx -0.11$$

$$V_1 = 100$$

$$V_2 = 50$$

$$\frac{V_2 - V_1}{V_1} \approx -1 * \frac{V_1 - V_2}{V_2}$$

$$\frac{50 - 100}{100} \approx -1 * \frac{100 - 50}{50}$$

$$\frac{-50}{100} \approx -1 * \frac{50}{50}$$

$$-0.5 \approx -1 * 1$$

$$-0.5 \approx -1$$

585 Note: The examples provided above illustrate how log difference (-0.1053) and percentage

586 difference (-0.1) are quite close for small differences. However, the difference between log

587 difference (-0.6931) and percentage difference (-0.5) begins to differ considerably with

588 larger changes. As such, the two approaches to reflecting relative differences are unlikely to

589 be perfectly related outside of optimal conditions.