ISFAHAN UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF MATHEMATICAL SCIENCES

Applied Linear Algebra
Assignment #3

Due Date: 11 Khordad 1400

1. Reduce the matrix $A = \begin{bmatrix} 2 & 1 & -5 & -1 & 0 \\ -1 & -1 & 4 & 1 & 2 \\ 0 & -1 & 3 & 2 & 9 \\ -1 & 0 & 1 & 1 & 3 \end{bmatrix}$ to the reduced-echelon-form and find bases for $N(A)$ and $R(A)$ (the null-space and the image of A).

2. Let $A = \begin{bmatrix} 1 & -1 & -4 & 0 & 4 \\ 1 & 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & 0 & -2 \end{bmatrix}$.
   a) $N(A)$ is a subspace of $\mathbb{R}^k$. What is k?
   b) $R(A)$ is a subspace of $\mathbb{R}^p$. What is p?
   c) Find a nonzero vector $x \in N(A)$ if one exists. otherwise explain why it doesn't.
   d) Find a nonzero vector $x \in R(A)$ if one exists. otherwise explain why it doesn't.

3. Suppose A is a 5 by 3 matrix and Ax is never zero(except when x is the zero vector).
   a) What can you say about the columns and rows of A?
   b) Show that $A^T Ax$ is also never zero (except when $x = 0$).

4. a) Give an example of two symmetric matrices whose product is not symmetric.
   b) Now suppose that A and B are symmetric $n \times n$ matrices. Prove that AB is symmetric if and only if A commutes with B, i.e. $AB = BA$.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose action on the standard basis vectors of $\mathbb{R}^3$ is

$$T(1,0,0) = (1, -\tfrac{3}{2}, 2)$$
$$T(0,1,0) = (-3, \tfrac{9}{2}, -6)$$
$$T(0,0,1) = (2, -3, 4).$$

   Then what is $T(5,1,-1)$? The kernel of T is all the vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + ay + bz = 0$. Determine scalars a and b. The range of T is all the vectors $(x, y, z) \in \mathbb{R}^3$ such that $\frac{x}{2} = \frac{y}{c} = \frac{z}{d}$. Determine c and d.

6. Prove that one-to-one linear transformations preserve linear independence. That is: Let $T : V \longrightarrow W$ be one-to-one linear transformation between vector spaces and $\{x_1, x_2, \cdots, x_n\}$ be linearly independent, prove that $Tx_1, Tx_2, \cdots, Tx_n$ are linearly independent. Give an example that this does not hold for all linear transformation.
7. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(x, y, z) = (x + 2y - z, -y, x + 7z)$.
    a) Write the matrix representation of T with respect to the standard basis.
    b) Consider the basis $\mathscr{B} = \{(1,0,0), (1,1,0), (1,1,1)\}$ for $\mathbb{R}^3$ and write the matrix representation of T with respect to the basis $\mathscr{B}$, i.e. $[T]_{\mathscr{B},\mathscr{B}}$.
8. Suppose that A is an $n \times n$ matrix such that $A^2 = A$. Let $\mathscr{B}_1 = \{u_1, \ldots u_r\}$ be a basis for $R(A)$ and $\mathscr{B}_2 = \{v_1, \ldots, v_{n-r}\}$ be a basis for $N(A)$.
    a) Prove that for each $i \in \{1, \ldots, r\}$, $Au_i = u_i$.
    b) Prove that $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ is a basis for $\mathbb{R}^n$.
    c) Prove that $[A]_{\mathscr{B},\mathscr{B}} = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$.