ISFAHAN UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF MATHEMATICAL SCIENCES

Applied Linear Algebra
Assignment #3

Due Date: 3 Bahman 1401

1. If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$ Find A^{-1}. Then, solve the system of linear equations

$$\begin{aligned} x - 2y &= 10, \\ 2x - y - z &= 8, \\ -2y + z &= 7, \end{aligned}$$

   using A^{-1}.

2. Consider the system of linear equations:

$$\begin{cases} kx + y + z = 1 \\ x + ky + z = 1 \\ x + y + kz = 1 \end{cases}$$

   For what value(s) of k does this system have:
   (i) a unique solution?
   (ii) no solution?
   (iii) infinitely many solutions?

3. If $a_1, a_2, a_3, \ldots$ are in G.P. (Geometric Progression, i.e. $a_{i+1} = a_i.q$), then prove that the determinant of $\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$ is independent of r.

4. In a triangle ABC, if

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 1+SinA & 1+SinB & 1+SinC \\ SinA+Sin^2A & SinB+Sin^2B & SinC+Sin^2C \end{pmatrix} = 0,$$

prove that $\triangle ABC$ is an isosceles triangle.

5. a) Prove that $\operatorname{rank}(A) = \operatorname{rank}(A^T A)$.
   b) Show that if A is a 3×5 matrix, then $\det(A^T A) = 0$. Note: you cannot distribute the det, because A and A^T are not square matrices!

6. Suppose $Q^{-1} = Q^T$ (transpose equals inverse, thus $Q^T Q = I$).

   a) Show that the columns $q_1, ..., q_n$ are unit vectors: $\|q_i\|^2 = 1$.
   b) Show that every two columns of Q are perpendicular: $q_i^T q_j = 0$.
   c) Without computing the determinant, prove that

$$\det\begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} = 0 \iff a = b = c = d = 0.$$

7. Let M be an n by n matrix. Split M into $S + A$ where S is symmetric, i.e. $S = S^T$ and A is anti-symmetric, i.e. $A = -A^T$. Find formulas for S and A involving M and M^T. We want $M = S + A$.
8. Let A be an $n \times n$ nonsingular matrix with integer entries. Prove that the inverse matrix A^{-1} contains only integer entries if and only if $det(A) = \pm 1$.

9. a) By an example show that $\det(A+B) = \det(A) + \det(B)$ is not necessarily correct.
   b) Let A, B and be two square matrices with real entries such that $(A+B)^{-1} = A^{-1} + B^{-1}$. Prove that $\det(A) = \det(B)$. Does the conclusion holds for complex entries?