ISFAHAN UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF MATHEMATICAL SCIENCES

Applied Linear Algebra
Assignment #2

Due Date: 19 Azar 1401

1. For the space $\mathbb{R}^4$, let $w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix}$, $y = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ and let $W = sp\{w_1, w_2\}$.
   a) Find a basis for W consisting of two orthogonal vectors.
   b) express y as the sum of a vector in W and a vector in $W^\perp$.
2. Let u and v be orthogonal vectors. If $u+v$ and $u-v$ are orthogonal, show that $|u| = |v|$. (A rectangle with orthogonal diagonals is an square)
3. Let $V = \mathbb{R}_2[x]$ be the vector space of all polynomials of $degree \leq 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace consisting of those polynomials $f(x)$ such that $f(0) = 0$. Find the orthogonal projection of the polynomial $x+2$ onto the subspace W, with respect to the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

4. Let V be the vector space $\mathbb{R}^3$, equipped with the inner product

$$\langle (a,b,c),(d,e,f) \rangle = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

   a) Prove that this is an inner product.
   b) Using this inner product rather than the natural dot product, find the orthogonal projection of the vector $(1,0,0)$ onto the plane spanned by $(0,1,0)$ and $(0,0,1)$.

5. Using least squares techniques, fit the following data

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	2	7	9	12	13	14	14	13	10	8	4

with a line $y = \alpha_0 + \alpha_1 x$ and then fit the data with a quadratic $y = \alpha_0 + \alpha_1 x + \alpha_2 x^2$. Determine which of these two curves best fits with the data by computing the sum of the squares of the errors in each case.

6. Let S^3 denote the vector space of 3×3 symmetric matrices.

   a) Prove that $\langle S, T \rangle = tr(ST)$ is an inner product on S^3.

   b) Let $S_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $S_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Apply the Gram-Schmidt process in S^3 to find a matrix orthogonal to both S_1 and S_2.

   c) Find an orthonormal basis for S^3 using the above three matrices.

   d) Let $T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. Find the projection of T on the span of $\{S_1, S_2\}$.

7. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(x, y, z) = (x + 2y - z, -y, x + 7z)$.

   a) Write the matrix representation of T with respect to the standard basis.

   b) Consider the basis $\mathscr{B} = \{(1,0,0), (1,1,0), (1,1,1)\}$ for $\mathbb{R}^3$ and write the matrix representation of T with respect to the basis $\mathscr{B}$, i.e. $[T]_{\mathscr{B},\mathscr{B}}$.

8. Suppose that A is an $n \times n$ matrix such that $A^2 = A$. Let $\mathscr{B}_1 = \{u_1, \dots u_r\}$ be a basis for $R(A)$ and $\mathscr{B}_2 = \{v_1, \dots, v_{n-r}\}$ be a basis for $N(A)$.

   a) Prove that for each $i \in \{1, \dots, r\}$, $Au_i = u_i$.

   b) Prove that $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ is a basis for $\mathbb{R}^n$.

   c) Prove that $[A]_{\mathscr{B},\mathscr{B}} = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$.