Isfahan University of Technology
Department of Mathematical Sciences

Applied Linear Algebra
Assignment #4

Due Date: 14 Tir 1400

1. Assume that $A, B, C \in \mathbb{R}^{n\times n}$ with B invertible, and $A = BCB^{-1}$. Prove that $\det(A) = \det(C)$.

2. Show that if A is a 3×5 matrix, then $det(A^T A) = 0$. Note: you can't distribute the det, because A and A^T are not square!

3. Consider the system of linear equations

$$\begin{cases} kx + y + z = 1 \\ x + ky + z = 1 \\ x + y + kz = 1 \end{cases}$$

   For what value(s) of k does this system have
   (i) a unique solution?
   (ii) no solution?
   (iii) infinitely many solutions?

4. a) Find square matrices A, B such that $det(A+B) \neq det(A) + det(B)$.
   b) What possible determinant values does an orthogonal matrix have?

5. Let A and B be n×n matrices. If $A+B$ is invertible, show that $A(A+B)^{-1}B = B(A+B)^{-1}A$.

6. Let A be an $n \times n$ matrix. If $AB = BA$ for all invertible matrices B, show that $A = cI$ for some scalar c.

7. a) Let $trace(A)$ be the sum of all diagonal entries of the matrix A. For matrices $A_{m\times n}$ and $B_{n\times m}$, prove that $trace(AB) = trace(BA)$.
   b) Prove that for every $n \times n$ matrix A, the equation $AX - XA = I_{n\times n}$ has no solution.

8. Let $A : \mathbb{R}^l \to \mathbb{R}^k$ and $B : \mathbb{R}^k \to \mathbb{R}^l$. Prove that

$$rank(A) + rank(B) - l \leq rank(AB) \leq \min\{rank(A), rank(B)\}.$$