ISFAHAN UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF MATHEMATICAL SCIENCES

Applied Linear Algebra
Assignment #5

Due Date: 14 Tir 1400

1. Let A be the 2×2 matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.
   a) Find the eigenvalues of A.
   b) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.
   c) Diagonalize the matrix A.
   d) Calculate A^7.
   e) Diagonalize the inverse matrix A^{-1} of A.
2. Suppose that A is a square matrix with nonnegative integers such that each of whose rows has the same sum s.
   a) Show that s is an eigenvalue of A.
   b) Prove that s is the maximum eigenvalue of A.
3. Suppose that v is a nonzero vector in $\mathbb{R}^3$, and suppose A is a 3×3 matrix with distinct real eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$. Suppose that $\|A^n v\|$ (the length of the vector $A^n v$) converges to 0 as $n \to \infty$. Find all possible values of λ_1. Hint: the answer will be an open interval in $\mathbb{R}$.
4. a) Suppose that A is a 2×2 matrix with $det(A) = 0$. Use Cayley-Hamilton theorem to show that $A^2 = tr(A)A$ and determine A^n explicitly for each integer $n \geq 2$.
   b) Let A and B be two 2×2 matrices with $A = AB - BA$. Prove that $A^2 = O$, where O is the all-zeros matrix. (Hint: Compute $trace(A)$ and then apply the Cayley-Hamilton theorem to A.)
5. A square matrix A is called nilpotent if there is a positive integer k such that $A^k = O$.
   a) Prove that the only diagonalizable nilpotent matrix is the all-zeros matrix (the matrix with all entries equal to zero).
   b) Prove that A is nilpotent if and only if all eigenvalues of A are zero.
   c) Let A be a nilpotent $n \times n$ matrix. Prove that $A^n = O$. Hint: Use Cayley-Hamilton Theorem.
6. Let $A = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$. Pick your favorite number x, and find the dimension of the null space of the matrix $A - xI$. Your score of this problem is equal to that dimension!