Convex Sets

1 Affine and convex sets

1.1 Lines and line segments

Suppose $x_1 \neq x_2$ are the points in $\mathbb{R}^n$. Points of the form

$$y = \theta x_1 + (1-\theta)x_2, \tag{1}$$

where $\theta \in \mathbb{R}$, are the *line* passing through x_1 and x_2.

1.2 Affine sets

A set $C \subseteq \mathbb{R}^n$ is an *affine* if the line through any two distinct points in C lies in C, i.e., if $x_1,\, x_2 \in C$ and $\theta \in \mathbb{R}$, we have that $\theta x_1 + (1-\theta)x_2 \in C$.

Generalizing this idea to more than two points, we refer to a point of the form $\theta_1 x_1 + \ldots + \theta_k x_k$ where $\sum_{i=1}^k \theta_k = 1$ as an *affine combination* of the points $x_1, \ldots, x_k$.

Definition 1. *An affine hull of C, denoted as* **aff** *C, is the set of all affine combinations of points in some set $C \subseteq \mathbb{R}^n$:*

$$\mathbf{aff}\ C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C, \theta_1 + \cdots + \theta_k = 1\}. \tag{2}$$

1.3 Affine dimension and relative interior

Problem Sets

Problem 1 (Distance Between Hyperplanes). What is the distance between the two parallel hyperplanes $\{x \in \mathbb{R}^n \mid a^\top x = b_1\}$ and $\{x \in \mathbb{R}^n \mid a^\top x = b_2\}$?

Solution 1. Let x_1 and x_2 denote points where a vector a, which is orthogonal to the given two planes, intersects the first and the second hyperplanes. That is, we have

$$x_1 = \frac{b_1}{\|a\|^2} a \quad \text{and} \quad x_2 = \frac{b_2}{\|a\|^2} a.$$

Hence, we can simply see that the distance between two points is given by

$$\|x_1 - x_2\| = \frac{|b_1 a - b_2 a|}{\|a\|^2} = \frac{|b_1 - b_2|}{\|a\|}.$$

Problem 2 (Voronoi Description of a Halfspace). Let a and b be distinct points in $\mathbb{R}^n$ and consider the set of points that are closer (in Euclidean norm) to a than b, i.e., $\mathcal{C} = \{x \mid \|x-a\|_2 \leq \|x-b\|_2\}$. The set $\mathcal{C}$ is a halfspace. Describe it explicitly as an inequality of the form $c^\top x \leq d$. Draw a picture.

Solution 2. In Euclidean norm, we can rewrite the inequality $\|x-a\|_2 \leq \|x-b\|_2$ as:

$$\begin{aligned}
(x-a)^\top (x-a) &\leq (x-b)^\top (x-b) \\
x^\top x - 2a^\top x + a^\top a &\leq x^\top x - 2b^\top x + b^\top b \\
\left(2b^\top - 2a^\top\right) x &\leq \left(b^\top b - a^\top a\right)
\end{aligned}$$

By letting $c = 2b - 2a$ and $d = b^\top b - a^\top a$, we can simply see that we reach at the following inequality:

$$c^\top x \leq d$$

which explains that $\mathcal{C}$ is a halfspace. In geometric point of view, consider the points that have the same distance to points a and b. We notice that the set of these points has to be normal to the directional vector $b - a$.

Problem 3 (Common Convex Sets). Which of the following sets is convex?

1. A slab, i.e., a set of the form $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$
2. A rectangle, i.e., a set of the form $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n\}$. A rectangle is sometimes called a hyperrectangle when $n > 2$.
3. A wedge, i.e., $\{x \in \mathbb{R}^n | a_1^\top x \leq b_1, a_2^\top x \leq b_2\}$
4. The set of points closer to a given point than a given set, i.e., $\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$ where $S \subseteq \mathbb{R}^n$
5. The set of points closer to one set than another, i.e., $\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\}$, where $S, T \subseteq \mathbb{R}^n$, and $\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$

Solution 3. We have that

1. Consider the points $x_1, x_2 \in \{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$. We have that
$$\alpha \leq a^\top x_1 \leq \beta \quad \text{and} \quad \alpha \leq a^\top x_2 \leq \beta$$
Given $\theta \in \mathbb{R}$ such that $0 \leq \theta \leq 1$, considering the point $x = \theta x_1 + (1 - \theta) x_2$, we have
$$\begin{aligned} \theta\alpha + (1 - \theta)\alpha \leq \theta\left(\alpha^\top x_1\right) + (1 - \theta)\left(\alpha^\top x_2\right) &\leq \theta\beta + (1 - \theta)\beta \\ \alpha \leq \alpha^\top \left(\theta x_1 + (1 - \theta) x_2\right) &\leq \beta \\ \alpha \leq \alpha^\top x &\leq \beta \end{aligned}$$
which indicates that $x \in \{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$. In other words, a slab is convex.
2. Consider the points $x_1, x_2 \in \{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n\}$. We have that
$$\alpha_i \leq x_{1,i} \leq \beta_i \quad \text{and} \quad \alpha_i \leq x_{2,i} \leq \beta_i$$
for all $i \in \{1, 2, \ldots, n\}$. Considering a convex combination for $0 \leq \theta \leq 1$ between x_1 and x_2, i.e. $y = \theta x_1 + (1 - \theta) x_2$, we have
$$\begin{aligned} \theta\alpha_i + (1 - \theta)\alpha_i \leq \theta x_{1,i} + (1 - \theta) x_{2,i} &\leq \theta\beta_i + (1 - \theta)\beta_i \\ \alpha_i \leq y_i &\leq \beta_i \end{aligned}$$
Hence, y is in the set of a rectangle. Therefore, a rectangle is convex.
3. N/A
4. N/A
5. N/A

Problem 4 (Some Sets of Probability Distributions). Let x be a real-valued random variable with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$, where $a_1 < a_2 < \cdots < a_n$. Of course $p \in \mathbb{R}^n$ lies in the standard probability simplex $P = \{p \mid \mathbf{1}^\top p = 1, p \succeq 0\}$. Which of the following conditions are convex in p? (That is, for which of the following conditions is the set of $p \in P$ that satisfy the condition convex?)

Solution 4. N/A

References