C^*-algebras, groupoids, and symmetry

Tutte Institute for Mathematics and Computing
Communications Security Establishment

Mitch Haslehurst

Department of Mathematics & Statistics
University of Victoria

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Outline

1. C*-algebras
   - Definition(s) and examples
   - Main theorems and structure
   - Motivation for study
2. Groupoids
   - Definition and examples
   - Interlude: K-theory
   - Recent results in the Elliott classification program
3. Applications: dynamics and fractals
   - Iterated function systems
   - Data interpolation with fractals
   - Closing remarks

C*-algebras

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Abstract definition: A complete normed complex algebra with a conjugate linear involution * such that, if a and b are in A,

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Concrete definition: Any subalgebra of $\mathcal{B}(\mathcal{H})$, the bounded linear operators on a Hilbert space $\mathcal{H}$, that is closed in the uniform norm, and closed under the Hilbert space adjoint operation.

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$$C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous and vanishes at infinity}\}$$

with $f^*(x) = \overline{f(x)}$ and norm $\|f\| = \sup_{x \in X} |f(x)|$ is a C*-algebra.

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Example. $\mathcal{B}(\mathcal{H})$ with the Hilbert space operator adjoint and uniform norm. When $\mathcal{H}$ is finite-dimensional, we may identify $\mathcal{B}(\mathcal{H})$ with $M_n(\mathbb{C})$ for some n.

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If A is a commutative C^*-algebra, then there is a locally compact Hausdorff space X such that $A \cong C_0(X)$.

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All C*-algebras (commutative or not) look like a subalgebra of bounded linear operators.

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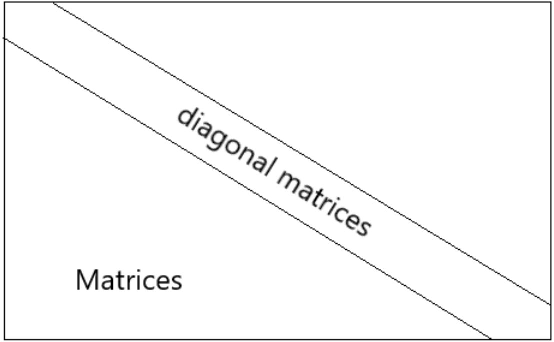
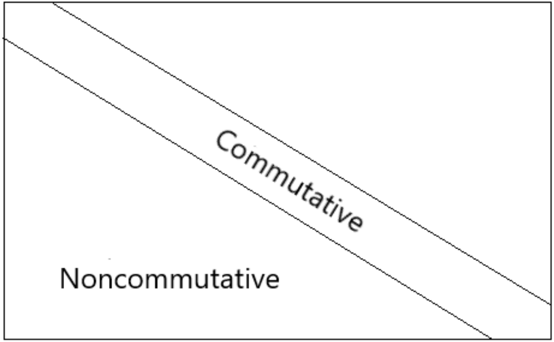
Classical mechanics $\longleftrightarrow$ Calculus and topology

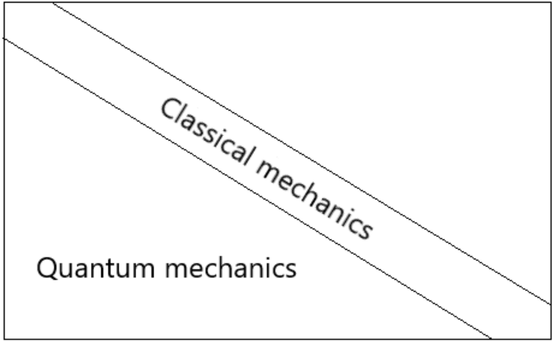
Quantum mechanics $\longleftrightarrow$ Operator algebras

"...Heisenberg postulated that the mathematics describing quantum physics should be the mathematics, **not of functions on a space**, but of **linear operators on a Hilbert space**, which, taken as an algebra, behaves, algebraically, much like the algebra of continuous functions on a space, but is not commutative..."

-Heath Emerson, *An introduction to C^*-algebras and Noncommutative Geometry*.


diagonal matrices
Matrices


Commutative
Noncommutative


Classical mechanics
Quantum mechanics

C^*-algebras have found interactions with:

1. Group theory
2. Harmonic analysis
3. Dynamical systems
4. Probability
5. Logic
6. Number theory
7. Graph theory
8. Geometry
9. Knot theory
10. Quantum information theory

Groupoids

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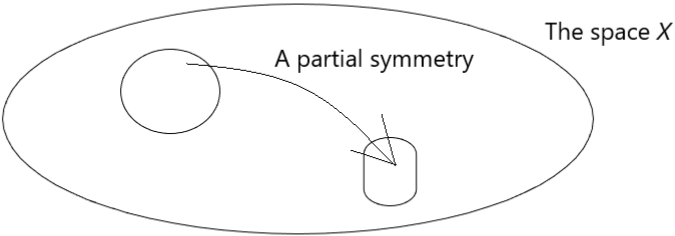
They are useful for dynamics, fractal geometry, quasicrystals, tilings.

Groupoids are collections of "partial symmetries".



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Example. Let X be a nonempty set and $R \subseteq X \times X$ an equivalence relation. Then R is a groupoid:

$$(x, y)(y', z) = (x, z) \qquad (x, y)^{-1} = (y, x)$$

the product being defined only when $y = y'$.

Given a groupoid with some nice topological properties, we can make a C^*-algebra out of it.

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$C_c(G)$ = all continuous, compactly supported functions $f : G \to \mathbb{C}$.

$$(f \star g)(x) = \sum_{yy^{-1}=xx^{-1}} f(y)g(y^{-1}x) \qquad f^*(x) = \overline{f(x^{-1})}$$

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To get a complete norm, represent $C_c(G)$ on a Hilbert space and take the closure to get the *reduced* C*-algebra of G, called $C_r^*(G)$.

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and $C_c(R) \cong M_n(\mathbb{C})$.

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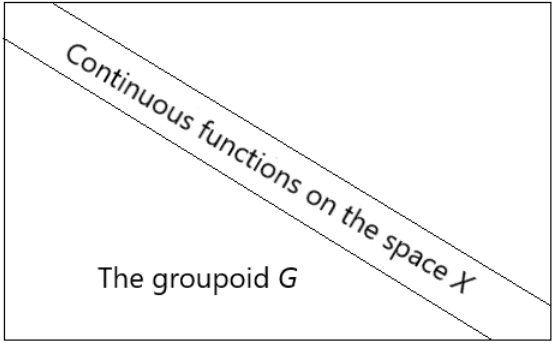
Example. If Γ is a group acting on a space X (a dynamical system), then $G = X \times \Gamma$ is a groupoid and $C_r^*(G)$ is the *crossed product* $C_0(X) \rtimes \Gamma$.



linear
operators

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C*-algebras have a useful homology theory: to each C*-algebra A there are two abelian groups called $K_0(A)$ and $K_1(A)$.

$K_0(A)$ consists of equivalence classes of projections p in $\bigcup_n M_n(A)$ that "have the same rank". $K_1(A)$ consists of equivalence classes of unitaries u in $\bigcup_n M_n(A)$ that are "stably homotopic".

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is exact, there are group homomorphisms δ_0 and δ_1 such that the sequence

$$\begin{tikzcd}
K_0(I) \arrow[r, "\iota_*"] & K_0(A) \arrow[r, "\pi_*"] & K_0(A/I) \arrow[d, "\delta_0"] \\
K_1(A/I) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "\pi_*"] & K_1(I) \arrow[l, "\iota_*"]
\end{tikzcd}$$

is exact.

Examples. $K_0(\mathbb{C}) \cong \mathbb{Z}$ (associate a projection in $\bigcup_n M_n(\mathbb{C})$ to its rank). $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$ for the same reason.

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$K_1(C(S^1)) \cong \mathbb{Z}$ where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ (associate a unitary to its winding number).

There is an immense amount of research being done into what K-theory can tell us about C^*-algebras.

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The Elliott classification program. Suppose A and B are two unital, simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebras that satisfy the Universal Coefficient Theorem. If A and B have the same K-theory, then they are isomorphic.

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Given some K-theory data D, there is a groupoid G such that $K_*(C_r^*(G)) \cong D$.

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Theorem (Putnam, 2017)

Given some torsion-free K-theory data D, there is an groupoid G on a Cantor set such that $K_*(C_r^*(G)) \cong D$.

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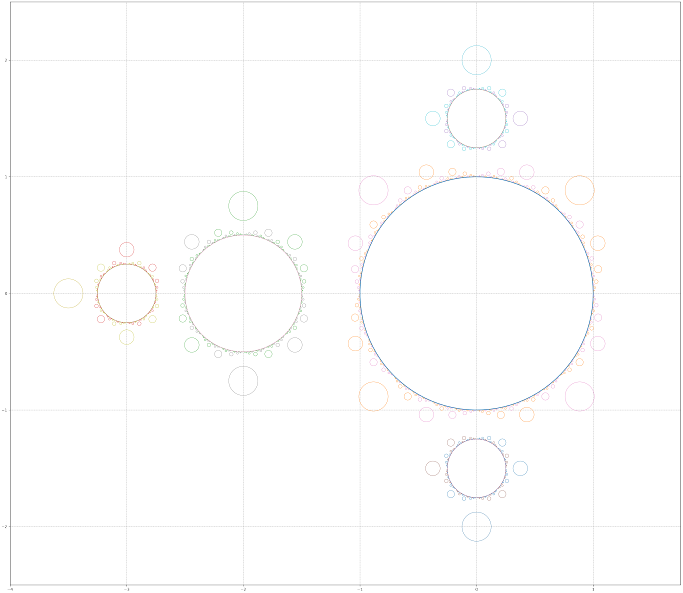
They also often have some self-similar fractal structure.


2
1
0
−1
−2
−4
−3
−2
−1
0
1

Applications: dynamics and fractals

An *iterated function system* $(X, \{f_j\}_{j=1}^n)$ (abbreviated IFS) is a complete metric space X with a finite set of functions $f_j : X \to X$ for $j = 1, 2, \ldots, n$.

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If the IFS is hyberbolic (every f_j is a contraction) then there is a unique compact subset $K \subseteq X$ such that

$$K = \bigcup_{j=1}^{n} f_j(K)$$

K is called the *attractor* of the IFS.

Example. Take $X = \mathbb{R}$ and $f_1(x) = \frac{1}{2}x$ and $f_2(x) = \frac{1}{2}x + \frac{1}{2}$. The attractor is $K = [0, 1]$.

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Example. Take $X = \mathbb{R}^2$ and

$$f_1(x) = \frac{1}{2}x \qquad f_2(x) = \frac{1}{2}x + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \qquad f_3(x) = \frac{1}{2}x + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}$$

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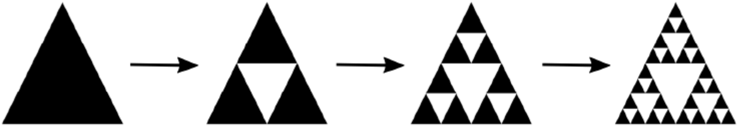
The attractor K is the Sierpiński triangle.



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Theorem (Korfanty, 2020)

Suppose $(K, \{f_j\}_{j=1}^n)$ and $(K', \{f_j'\}_{j=1}^n)$ are two compact hyberbolic IFS's with respective groupoids G and G'. If $(K, \{f_j\}_{j=1}^n)$ and $(K', \{f_j'\}_{j=1}^n)$ are topologically conjugate, then $C_r^*(G)$ and $C_r^*(G')$ are isomorphic.

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Conjecture. Most likely not.

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Theorem (Giordano, Putnam, Skau, 1995)

If K is a Cantor set and $\varphi, \psi : K \to K$ are two minimal homeomorphisms (every orbit is dense) and $C(K) \rtimes_{\varphi} \mathbb{Z}$ and $C(K) \rtimes_{\psi} \mathbb{Z}$ have the same K-theory, then (K, φ) and (K, ψ) are orbit-equivalent as dynamical systems.

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If C*-algebras are to give an invariant for IFS's:

1. K-theory will likely be crucial,
2. "topological conjugacy" may need to be weakened.

Theorem (Barnsley, 1986)

Let $x_1 < x_2 < \cdots < x_n$ be real numbers and $\{(x_j, y_j) \mid j = 1, 2, \ldots, n\} \subseteq \mathbb{R}^2$ be a data set. Then there is an IFS $(\mathbb{R}^2, \{f_j\}_{j=1}^n)$ such that the attractor K is the graph of a continuous function $F : [x_1, x_n] \to \mathbb{R}$ with $F(x_j) = y_j$ for all $j = 1, 2, \ldots, n$.

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Data interpolation using fractal interpolation functions:

1. Like smooth functions such as polynomials and trigonometric functions, can be approximated recurrently using formulae.
2. The fractal structure of the curve captures "irregularities" well as opposed to smooth functions.

Closing remarks

1. Groupoids provide the algebraic groundwork for studying local symmetry, while C^*-algebras provide an immense amount of structure and powerful tools.
2. Interplay between the dynamics of iterated function systems and C^*-algebras: only scratched the surface.

$$\begin{tikzcd} C^*\text{-algebras} \arrow[r, dashed, "??"] \arrow[dr] & \text{Fractal interpolation} \\ & \text{Iterated function systems} \arrow[u] \end{tikzcd}$$

Thank you!