Relative K-theory for C^*-algebras and factor groupoids

Groups, Operators, and Banach Algebras Seminar

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The abelian group $K_0(A)$ consists of stable Murray-von Neumann equivalence classes of projections p in $M_\infty(\tilde{A}) = \bigcup_n M_n(\tilde{A})$, with group operation diagonal sum: $[p] + [q]$ is the class of $p \oplus q = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$.

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K_0 and K_1 are functors: if $\phi : A \to B$ is a *-homomorphism, then there are induced group homomorphisms $\phi_* : K_j(A) \to K_j(B)$ that satisfy

$$\phi_*([a]) = [\phi(a)]$$

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$$\begin{tikzcd} K_0(I) \arrow[r, "\iota_*"] & K_0(A) \arrow[r, "\pi_*"] & K_0(A/I) \arrow[d, "\delta_0"] \\ K_1(A/I) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "\pi_*"] & K_1(I) \arrow[l, "\iota_*"] \end{tikzcd}$$

is exact.

Relative K-Theory

Let $\phi : A \to B$ be a *-homomorphism. Let $\Gamma_0(\phi)$ be all triples (p, q, v) where $p, q \in M_\infty(\tilde{A})$ are projections and $v \in M_\infty(\tilde{B})$ is a partial isometry with $v^*v = \phi(p)$ and $vv^* = \phi(q)$.

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We can add them:

$$(p, q, v) \oplus (p', q', v') = \left(\begin{bmatrix} p & 0 \\ 0 & p' \end{bmatrix}, \begin{bmatrix} q & 0 \\ 0 & q' \end{bmatrix}, \begin{bmatrix} v & 0 \\ 0 & v' \end{bmatrix} \right)$$

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Say $(p, q, v) \cong (p', q', v')$ if there are partial isometries $c, d \in M_\infty(\tilde{A})$ with $c^*c = p$, $cc^* = p'$, $d^*d = q$, $dd^* = q'$, and $\phi(d)v = v'\phi(c)$.

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Say $(p, q, v) \sim (p', q', v')$ if there are elementary triples (r, r, c) and (s, s, d) such that

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Definition

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7. $[p,q,v] + [q,r,v'] = [p,r,v'v]$.

Let $\Gamma_1(\phi)$ be all triples (p, u, g) where $p \in M_\infty(\tilde{A})$ is a projection, u is a unitary in $pM_\infty(\tilde{A})p$, and g is a unitary in $C[0,1] \otimes \phi(p)M_\infty(\tilde{B})\phi(p)$ with $g(0) = \phi(p)$ and $g(1) = \phi(u)$.

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Relative K-theory

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Define the maps

$$\nu_0 : K_0(\phi) \to K_0(A) : [p, q, v] \mapsto [p] - [q]$$

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where $f_p(t) = e^{2\pi i t p}$. Then we have the six-term exact sequence

$$\begin{tikzcd}
K_1(B) \arrow[r, "\mu_0"] & K_0(\phi) \arrow[r, "\nu_0"] & K_0(A) \arrow[d, "\phi_*"] \\
K_1(A) \arrow[u, "\phi_*"] & K_1(\phi) \arrow[l, "\nu_1"] & K_0(B) \arrow[l, "\mu_1"]
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2. If $K_*(B) = 0$, then $K_*(\phi) = K_*(A)$. If $K_*(A) = 0$, then $K_j(\phi) = K_{1-j}(B)$.
3. If ϕ is a *-isomorphism, then $K_*(\phi) = 0$.

Example 2. Let D be a C^*-algebra, $B = M_2(D)$, and $A \subseteq B$ the subalgebra of diagonal matrices. With $\phi : A \to B$ being the inclusion map, denote $K_j(A; B) := K_j(\phi)$.

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$$\begin{tikzcd}
K_1(D) \arrow[r, "\mu_0"] & K_0(A;B) \arrow[r, "\nu_0"] & K_0(D) \oplus K_0(D) \arrow[d, "\phi_*"] \\
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The vertical maps are surjective, so $K_j(A; B) \cong \ker \phi_* \cong K_j(D)$.

Relative K-theory

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{M}$ a proper nonzero subspace. Regard $A = \mathcal{K}(\mathcal{M}) \oplus \mathcal{K}(\mathcal{M}^\perp)$ as a subalgebra of $B = \mathcal{K}(\mathcal{H})$ via $(a, b) \mapsto \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

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Let $\xi \in \mathcal{M}$ and $\eta \in \mathcal{M}^\perp$ be nonzero vectors, and p_ξ, p_η be the projections onto $\operatorname{span}\{\xi\}$ and $\operatorname{span}\{\eta\}$.

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Then $K_0(A; B) \cong \mathbb{Z}$ is generated by the triple (p_ξ, p_η, v), where v is a partial isometry in $\mathcal{K}(\mathcal{H})$ with $v^*v = p_\xi$ and $vv^* = p_\eta$.

Example 3. Let D be a C^*-algebra, $B = D \oplus D$, and $A = \{(a, a) \mid a \in D\} \subseteq B$, which we identify with D. This time the vertical maps are injective, and we have $K_j(A; B) \cong K_{1-j}(D)$.

Example 3. Let D be a C^*-algebra, $B = D \oplus D$, and $A = \{(a,a) \mid a \in D\} \subseteq B$, which we identify with D. This time the vertical maps are injective, and we have $K_j(A;B) \cong K_{1-j}(D)$.

In the case that $D = \mathcal{K}$, $K_1(A;B) \cong \mathbb{Z}$ is generated by the triple (p,p,g), where p is any rank one projection in $\mathcal{K}$ and $g(t) = (e^{2\pi it}p, p)$.

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Then there is a group homomorphism $\alpha_* : K_0(\phi) \to K_0(\psi)$ such that

$$\alpha_*([p,q,v]) = [\alpha(p), \alpha(q), \beta(v)]$$

and similarly for $K_1(\phi) \to K_1(\psi)$.

If

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\iota_A"] \arrow[d, "\psi"] & A \arrow[r, "\pi_A"] \arrow[d, "\phi"] & A/I \arrow[r] \arrow[d, "\gamma"] & 0 \\
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is commutative with exact rows, we have a six-term exact sequence

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\begin{tikzcd}
K_0(\psi) \arrow[r, "\iota_*"] & K_0(\phi) \arrow[r, "\pi_*"] & K_0(\gamma) \arrow[d, "\partial_0"] \\
K_1(\gamma) \arrow[u, "\partial_1"] & K_1(\phi) \arrow[l, "\pi_*"] & K_1(\psi) \arrow[l, "\iota_*"]
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Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_0(\mathbb{R}) \arrow[r, hook] \arrow[d, equal] & C(S^1) \arrow[r, "ev_1"] \arrow[d, "\phi"] & \mathbb{C} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & C_0(\mathbb{R}) \arrow[r, hook] & C[0,1] \arrow[r, "ev_{0,1}"] & \mathbb{C} \oplus \mathbb{C} \arrow[r] & 0
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$$\begin{tikzcd} 0 \arrow[r] & K_0(C(S^1); C[0,1]) \arrow[r, "ev_*"] & K_0(\mathbb{C}; \mathbb{C} \oplus \mathbb{C}) \arrow[d] \\ K_1(\mathbb{C}; \mathbb{C} \oplus \mathbb{C}) \arrow[u] & K_1(C(S^1); C[0,1]) \arrow[l, "ev_*"'] & 0 \arrow[l] \end{tikzcd}$$

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\end{tikzcd}$$

where $\phi(f)(t) = f(e^{2\pi i t})$ and $\gamma(z) = (z, z)$.

$$\begin{tikzcd}
0 \arrow[r] & K_0(C(S^1); C[0,1]) \arrow[r, "ev_*"] & K_0(\mathbb{C}; \mathbb{C} \oplus \mathbb{C}) \arrow[d] \\
K_1(\mathbb{C}; \mathbb{C} \oplus \mathbb{C}) \arrow[u] & K_1(C(S^1); C[0,1]) \arrow[l, "ev_*"'] & 0 \arrow[l]
\end{tikzcd}$$

So $K_*(C(S^1); C[0,1]) \cong K_*(\mathbb{C}; \mathbb{C} \oplus \mathbb{C})$.

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_0(\mathbb{R}^2) \arrow[r, hook] \arrow[d] & C(\mathbb{D}) \arrow[r] \arrow[d, "\phi"] & C(S^1) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & 0 \arrow[r] & C[0,1] \arrow[r, equal] & C[0,1] \arrow[r] & 0
\end{tikzcd}$$

where ϕ is the composition of the restriction to S^1 and the inclusion to $[0, 1]$.

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where ϕ is the composition of the restriction to S^1 and the inclusion to $[0,1]$.

$$\begin{tikzcd} K_0(C_0(\mathbb{R}^2)) \arrow[r] & 0 \arrow[r] & K_0(C(S^1); C[0,1]) \arrow[d] \\ K_1(C(S^1); C[0,1]) \arrow[u] & 0 \arrow[l] & K_1(C_0(\mathbb{R}^2)) \arrow[l] \end{tikzcd}$$

Relative K-theory

The map $\partial_1 : K_1(C(S^1); C[0,1]) \to K_0(C_0(\mathbb{R}^2))$ is an isomorphism.

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The map ∂_1 takes this element to $[p] - [1 \oplus 0]$ in $K_0(C_0(\mathbb{R}^2))$, where p is the Bott projection:

$$p(z) = \begin{bmatrix} |z|^2 & z(1-|z|^2)^{1/2} \\ \overline{z}(1-|z|^2)^{1/2} & 1-|z|^2 \end{bmatrix}$$

for z in the unit disk $\mathbb{D} \subseteq \mathbb{C}$

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Relative K-theory

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$$\begin{array}{ccc} K_j(\phi) & \xrightarrow{\alpha_*} & K_j(\psi) \\ \downarrow{\scriptstyle \Delta_j} & & \downarrow{\scriptstyle \Delta_j} \\ K_j(C_\phi) & \xrightarrow{\alpha_*} & K_j(C_\psi) \end{array}$$

Factor groupoids

Problem. Given some K-theory data, find a groupoid G such that the data is $K_*(C_r^*(G))$.

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Notable references:

Li, X. "Every classifiable simple C^*-algebra has a Cartan subalgebra". Invent. math. 219, 653–699 (2020).

Putnam, I.F. "Some classifiable groupoid C^*-algebras with prescribed K-theory". Math. Ann. 370, 1361–1387 (2018).

Factor groupoids

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Assume:

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3. $\pi|_{G^u} : G^u \to (G')^{\pi(u)}$ is bijective for all u in $G^{(0)}$.

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Obtain an inclusion $C^*_r(G') \subseteq C^*_r(G)$ via $b \mapsto b \circ \pi$ (b in $C_c(G')$)

Factor groupoids

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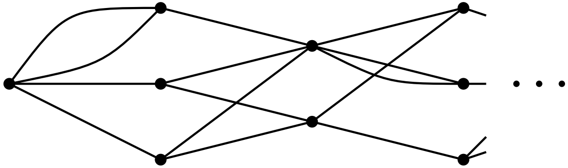

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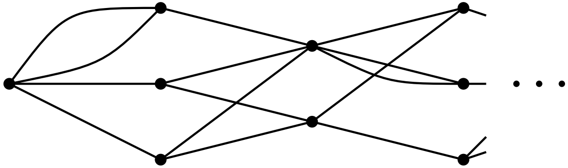

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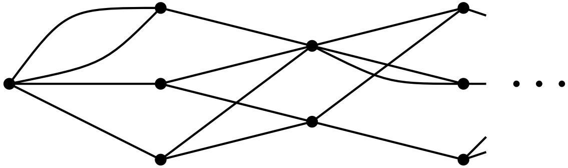

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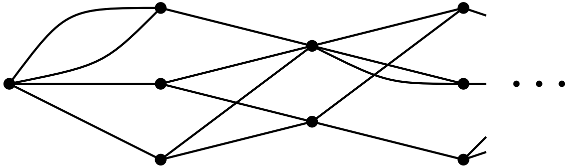

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Goal: make a factor groupoid of R_E.

Let (V, E) and (W, F) be two Bratteli diagrams.

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Two *graph embeddings* $\xi^0, \xi^1 : (W, F) \to (V, E)$ with $\xi^0|_W = \xi^1|_W$ and $\xi^0(F) \cap \xi^1(F) = \varnothing$.

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Equivalence relation $\sim_\xi$ on X_E:

$$\begin{aligned} & (x_1, x_2, \ldots, x_{n_0-1}, x_{n_0}, \xi^0(z_{n_0+1}), \xi^0(z_{n_0+2}), \ldots) && (1) \\ \sim_\xi\ & (x_1, x_2, \ldots, x_{n_0-1}, x'_{n_0}, \xi^1(z_{n_0+1}), \xi^1(z_{n_0+2}), \ldots) && (2) \end{aligned}$$

Factor groupoids

Denote $X_\xi := X_E / \sim_\xi$ and $\rho : X_E \to X_\xi$ the quotient map.

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Facts:

1. X_ξ is a second-countable compact Hausdorff space,
2. the covering dimension of X_ξ is 1,
3. each connected component is either a single point or homeomorphic to S^1.

Example 4. We let (V, E) be the Bratteli diagram with one vertex at each level and two edges at each level. Identify X_E with $\{0, 1\}^\omega$.

(W, F) is a single path, and for f in F, $\xi^j(f) = j$ for $j = 0, 1$.

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$$(x_1, x_2, \ldots, x_n, 1, 0, 0, 0, 0, \ldots) \tag{3}$$
$$\sim_\xi (x_1, x_2, \ldots, x_n, 0, 1, 1, 1, 1, \ldots) \tag{4}$$

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$$
\begin{aligned}
& (x_1, x_2, \ldots, x_n, 1, 0, 0, 0, 0, \ldots) && (3)\\
\sim_\xi \; & (x_1, x_2, \ldots, x_n, 0, 1, 1, 1, 1, \ldots) && (4)
\end{aligned}
$$

$$\{0,1\}^\omega \to S^1 : (x_n) \mapsto \exp\left(2\pi i \sum_{n=1}^{\infty} x_n 2^{-n}\right)$$

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$$\sim_\xi (x_1, x_2, \ldots, x_n, 0, 1, 1, 1, 1, \ldots) \tag{4}$$

$$\{0,1\}^\omega \to S^1 : (x_n) \mapsto \exp\left(2\pi i \sum_{n=1}^{\infty} x_n 2^{-n}\right)$$

The fibres are precisely the $\sim_\xi$ equivalence classes, so X_ξ is homeomorphic to S^1.

Example 5. Let (V, E) have one vertex and three edges at each level. Identify X_E with $\{0, 1, 2\}^\omega$.

(W, F) is again a single path, and for f in F, $\xi^0(f) = 0$ and $\xi^1(f) = 2$.

Example 5. Let (V, E) have one vertex and three edges at each level. Identify X_E with $\{0, 1, 2\}^\omega$.

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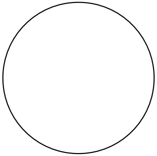
There is a nested sequence $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_E$ such that

$$X_E = \overline{\bigcup_{n=1}^{\infty} X_n}$$

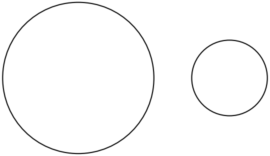
and each $\rho(X_n)$ is a disjoint union of finitely many circles.

Factor groupoids



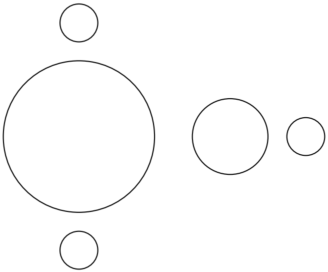
$\rho(X_1)$

Factor groupoids



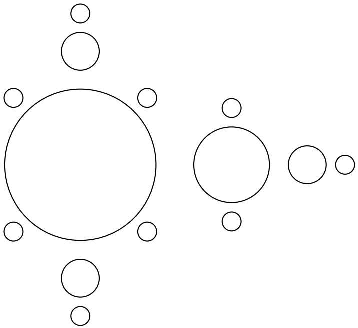
$\rho(X_2)$

Factor groupoids



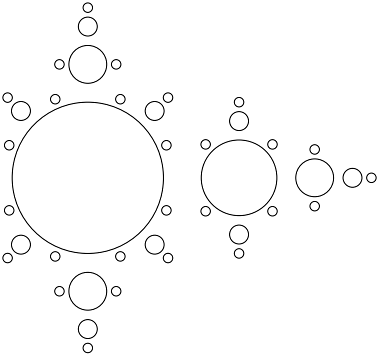
$$\rho(X_3)$$

Factor groupoids



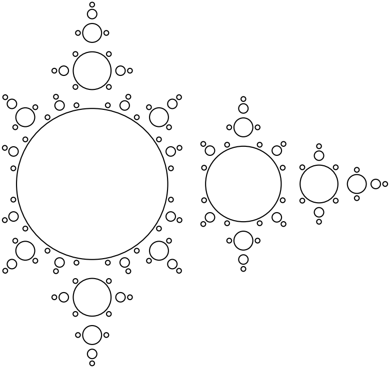
$\rho(X_4)$

Factor groupoids



$\rho(X_5)$

Factor groupoids



$\rho(X_6)$

Let $R_\xi = \rho \times \rho(R_E)$.

With the quotient topology, R_ξ is a second-countable locally compact Hausdorff étale groupoid, and a factor of R_E via $\rho \times \rho : R_E \to R_\xi$.

Factor groupoids

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With the quotient topology, R_ξ is a second-countable locally compact Hausdorff étale groupoid, and a factor of R_E via $\rho \times \rho : R_E \to R_\xi$.

We want to analyze the K-theory of $C_r^*(R_\xi) \subseteq C_r^*(R_E)$.

Factor groupoids

$$\begin{CD}
K_1(C_r^*(R_E)) @>>> K_0(C_r^*(R_\xi); C_r^*(R_E)) @>>> K_0(C_r^*(R_\xi)) \\
@A{\iota_*}AA @. @VV{\iota_*}V \\
K_1(C_r^*(R_\xi)) @<<< K_1(C_r^*(R_\xi); C_r^*(R_E)) @<<< K_0(C_r^*(R_E))
\end{CD}$$

Factor groupoids

$$\begin{tikzcd}
K_1(C_r^*(R_E)) \arrow[r] & K_0(C_r^*(R_\xi); C_r^*(R_E)) \arrow[r] & K_0(C_r^*(R_\xi)) \arrow[d, "\iota_*"] \\
K_1(C_r^*(R_\xi)) \arrow[u, "\iota_*"] & K_1(C_r^*(R_\xi); C_r^*(R_E)) \arrow[l] & K_0(C_r^*(R_E)) \arrow[l]
\end{tikzcd}$$

Let

$$H' = \{(x, y) \in R_\xi \mid \#(\rho \times \rho)^{-1}(x, y) > 1\}$$

and $H = (\rho \times \rho)^{-1}(H')$. Then we have *excision*: (Putnam, 2020)

$$K_*(C_r^*(R_\xi)); C_r^*(R_E)) \cong K_*(C_r^*(H'); C_r^*(H))$$

It turns out that $C_r^*(H')$ is stably isomorphic to $C_r^*(R_F)$, and $C_r^*(H)$ may be identified with $C_r^*(H') \oplus C_r^*(H')$. Moreover,

$$\begin{tikzcd} C_r^*(H') \arrow[r, hook] \arrow[d, equal] & C_r^*(H) \arrow[d, "\cong"] \\ C_r^*(H') \arrow[r, "a \oplus a"] & C_r^*(H') \oplus C_r^*(H') \end{tikzcd}$$

commutes.

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commutes.

By Example 3, we have

$$K_j(C_r^*(H'); C_r^*(H)) \cong K_{1-j}(C_r^*(H')) \cong K_{1-j}(C_r^*(R_F))$$

Factor groupoids

$$\begin{tikzcd}
K_1(C_r^*(R_E)) \arrow[r] & K_0(C_r^*(R_\xi); C_r^*(R_E)) \arrow[r] & K_0(C_r^*(R_\xi)) \arrow[d, "\iota_*"] \\
K_1(C_r^*(R_\xi)) \arrow[u, "\iota_*"] & K_1(C_r^*(R_\xi); C_r^*(R_E)) \arrow[l] & K_0(C_r^*(R_E)) \arrow[l]
\end{tikzcd}$$

Factor groupoids

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & K_0(C_r^*(R_\xi)) \arrow[d] \\
K_1(C_r^*(R_\xi)) \arrow[u] & K_0(C_r^*(R_F)) \arrow[l] & K_0(C_r^*(R_E)) \arrow[l]
\end{tikzcd}$$

$$\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & K_0(C_r^*(R_\xi)) \\
\uparrow & & & & \downarrow \\
K_1(C_r^*(R_\xi)) & \longleftarrow & K_0(C_r^*(R_F)) & \longleftarrow & K_0(C_r^*(R_E))
\end{array}$$

If one can show that the lower right map is zero...

Factor groupoids

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & K_0(C_r^*(R_\xi)) \arrow[d, "\cong"] \\
K_1(C_r^*(R_\xi)) \arrow[u] & K_0(C_r^*(R_F)) \arrow[l, "\cong"'] & K_0(C_r^*(R_E)) \arrow[l, "0"']
\end{tikzcd}$$

If one can show that the lower right map is zero...

Thus we obtain

$$K_0(C_r^*(R_\xi)) \cong K_0(C_r^*(R_E)) \qquad K_1(C_r^*(R_\xi)) \cong K_0(C_r^*(R_F))$$

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Thus we obtain

$$K_0(C_r^*(R_\xi)) \cong K_0(C_r^*(R_E)) \qquad K_1(C_r^*(R_\xi)) \cong K_0(C_r^*(R_F))$$

Through the set-up $\xi^0, \xi^1 : (W, F) \to (V, E)$, we can prescribe $K_*(C_r^*(R_\xi))$.

Thank you!

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