Factor groupoid constructions in C^*-algebras and iterated function systems

Nipissing University Topology Seminar

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- If $ab = ba$ for any a and b in A, then A is called *commutative*.

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- The analytic structure is completely determined by the algebraic structure.
- Every *-homomorphism $\varphi : A \to B$ is automatically continuous.
- If φ is injective, it is automatically isometric.
- There is at most one norm on a Banach *-algebra which makes it into a C*-algebra.

Example. The complex numbers $\mathbb{C}$, with its usual addition, multiplication, $z^* = \overline{z}$, and norm $|z|$ (absolute value). It is unital and commutative.

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$$C(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}$$

with $f^*(x) = \overline{f(x)}$ and norm $\|f\| = \sup_{x\in X} |f(x)|$ is a unital and commutative C*-algebra.

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Exercise: *Why is f necessarily bounded?*

Answer: X is compact, and continuous images of compact sets are compact, so $f(X)$ is bounded in $\mathbb{C}$.

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Think of:

$$C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is continuous and } \lim_{x\to\pm\infty} |f(x)| = 0\}$$

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$M_n(\mathbb{C})$ is unital, but never commutative when $n > 1$.

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When $\mathcal{H}$ is finite-dimensional, we may identify $\mathcal{B}(\mathcal{H})$ with $M_n(\mathbb{C})$ for some n.

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Exercise: $C_0(X, M_n(\mathbb{C}))$ is canonically isomorphic to $M_n(C_0(X))$. Can you see how?

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So all commutative C^*-algebras look like continuous functions and all C^*-algebras (commutative or not) look like a *-subalgebra of bounded linear operators.

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C*-algebras are often referred to as "noncommutative spaces"

"...Heisenberg postulated that the mathematics describing quantum physics should be the mathematics, **not of functions on a space**, but of **linear operators on a Hilbert space**, which, taken as an algebra, behaves, algebraically, much like the algebra of continuous functions on a space, but is not commutative..."

-Heath Emerson, *An introduction to C*-algebras and Noncommutative Geometry*.

K-theory

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v is called a *partial isometry* if v^*v is a projection.

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$K_0(A)$ consists of equivalence classes of projections p in $\bigcup_n M_n(A)$, where $p \sim q$ if $v^*v = p$ and $vv^* = q$ for some v in $\bigcup_n M_n(A)$. Add them by

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$K_1(A)$ consists of equivalence classes of unitaries u in $\bigcup_n M_n(A)$, where $u \sim v$ if there is a continuous path of unitaries connecting them. Add them by $[u] + [v] = [uv]$.

Examples. $K_0(\mathbb{C}) \cong \mathbb{Z}$ (associate a projection in $\bigcup_n M_n(\mathbb{C})$ to its rank). $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$ for the same reason.

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$K_1(C(S^1)) \cong \mathbb{Z}$ where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ (associate a unitary to its winding number).

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is exact, there are group homomorphisms δ_0 and δ_1 such that the sequence

$$\begin{tikzcd} K_0(I) \arrow[r, "\iota_*"] & K_0(A) \arrow[r, "\pi_*"] & K_0(A/I) \arrow[d, "\delta_0"] \\ K_1(A/I) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "\pi_*"] & K_1(I) \arrow[l, "\iota_*"] \end{tikzcd}$$

is exact.

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This is where groupoids come in.

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Every element x of G has an inverse x^{-1}, but there are many "units". The set of units is denoted $G^{(0)}$ and they are of the form $x^{-1}x$. Define the maps $r(x) = xx^{-1}$ and $s(x) = x^{-1}x$.

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Example. Let X be a nonempty set and $R \subseteq X \times X$ an equivalence relation. Then R is a groupoid:

$$(x, y)(y', z) = (x, z) \qquad (x, y)^{-1} = (y, x)$$

only when $y = y'$.

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$C_c(G) =$ all continuous, compactly supported functions $f : G \to \mathbb{C}$.

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To get a complete norm, represent $C_c(G)$ on a Hilbert space and take the closure to get the *reduced* C*-algebra of G, called $C_r^*(G)$.

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Exercise. *Do these formulae look familiar?*

Answer. It's matrix multiplication and the conjugate transpose, so $C_c(R) \cong M_n(\mathbb{C})$.

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Groupoids can give some fancy C*-algebras. If G is a dynamical system or *group action* $\Gamma \curvearrowright X$, then $C_r^*(G)$ is the *crossed product* $C_0(X) \rtimes \Gamma$.

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The idea is to arrange that one of $C_r^*(G')$ and $C_r^*(G)$ is familiar, while the other one is new and has some interesting K-theory.

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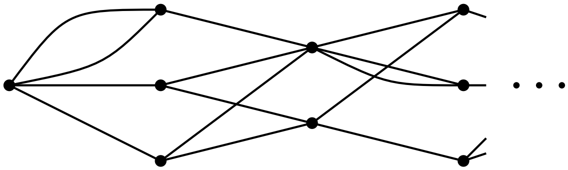

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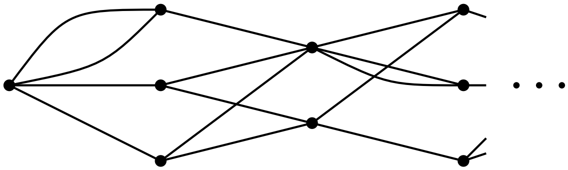

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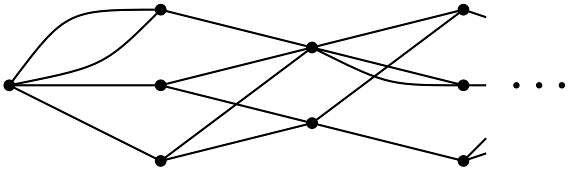

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Tail-equivalence $R_E \subseteq X_E \times X_E$ yields that $C^*_r(R_E)$ is an AF-algebra (approximately finite dimensional).

Bratteli diagrams

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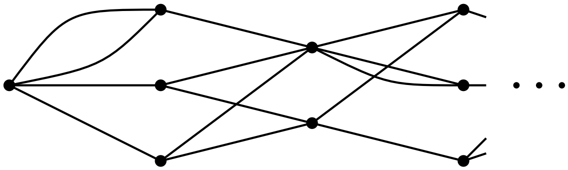

The *infinite path space* X_E of (V, E) is a totally disconnected compact metric space.

Tail-equivalence $R_E \subseteq X_E \times X_E$ yields that $C_r^*(R_E)$ is an AF-algebra (approximately finite dimensional).

Goal: make a factor groupoid of R_E.

The space X_ξ

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Equivalence relation $\sim_\xi$ on X_E:

$$
\begin{aligned}
& (x_1, x_2, \ldots, x_{n_0-1}, x_{n_0}, \xi^0(z_{n_0+1}), \xi^0(z_{n_0+2}), \ldots) && (1)\\
\sim_\xi\ & (x_1, x_2, \ldots, x_{n_0-1}, x'_{n_0}, \xi^1(z_{n_0+1}), \xi^1(z_{n_0+2}), \ldots) && (2)
\end{aligned}
$$

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Denote $X_\xi := X_E / \sim_\xi$ and $\rho : X_E \to X_\xi$ the quotient map.

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Facts:

1. X_ξ is a second-countable compact Hausdorff space,
2. the covering dimension of X_ξ is 1,
3. each connected component is either a single point or homeomorphic to S^1.

Many of the spaces X_ξ are fractal-like, sort of like attractors of iterated function systems.

Examples

Example 1. We let (V, E) be the Bratteli diagram with one vertex at each level and two edges at each level. Identify X_E with $\{0,1\}^\omega$.

(W, F) is a single path, and for f in F, $\xi^j(f) = j$ for $j = 0, 1$.

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$$(x_1, x_2, \ldots, x_n, 1, 0, 0, 0, 0, \ldots) \tag{3}$$
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The fibres are precisely the $\sim_\xi$ equivalence classes, so X_ξ is homeomorphic to S^1.

Examples

Example 2. Let (V, E) have one vertex and three edges at each level. Identify X_E with $\{0, 1, 2\}^\omega$.

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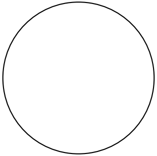
There is a nested sequence $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_E$ such that

$$X_E = \overline{\bigcup_{n=1}^{\infty} X_n}$$

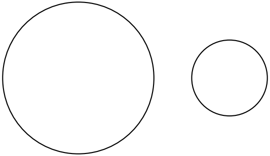
and each $\rho(X_n)$ is a disjoint union of finitely many circles.

Examples



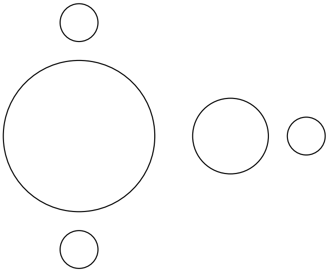
$\rho(X_1)$

Examples



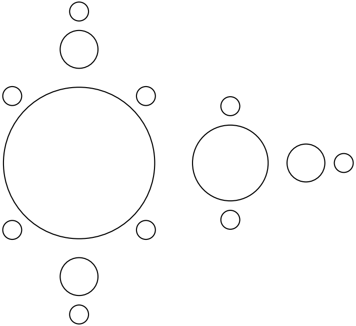
$\rho(X_2)$

Examples



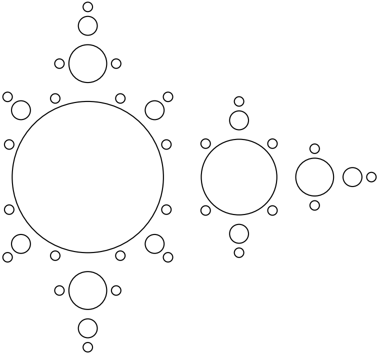
$$\rho(X_3)$$

Examples



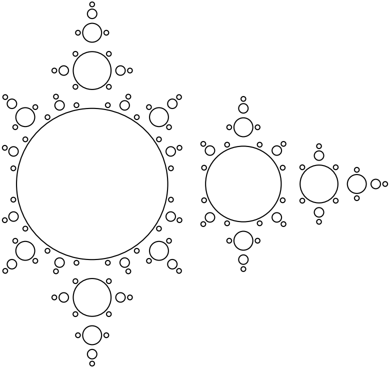
$\rho(X_4)$

Examples



$\rho(X_5)$

Examples



$\rho(X_6)$

The groupoid R_ξ

Let $R_\xi = \rho \times \rho(R_E)$.

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Through the set-up $\xi^0, \xi^1 : (W, F) \to (V, E)$, we can prescribe $K_*(C_r^*(R_\xi))$.

Further work

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Can K-theory tell us anything about fractals and iterated function systems? Vice versa?

Thank you!