SOME EXAMPLES OF FACTOR GROUPOIDS

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ABSTRACT. We examine two classes of examples of Hausdorff étale factor groupoids; one comes from taking a quotient space of the unit space of an AF-groupoid, and the other comes from certain nonhomogeneous extensions of Cantor minimal systems considered by Robin Deeley, Ian Putnam, and Karen Strung. The reduced C*-algebras of the factor groupoids are classifiable in the Elliott scheme, and we describe their K-theory and traces.

1. INTRODUCTION

In this paper, we consider the problem of constructing groupoid models for C*-algebras. This problem has attracted a great deal of attention in recent years, not least because groupoid models make tools from dynamics and symmetry available that can help to elucidate the structure of C*-algebras. If we restrict our attention to C*-algebras that are classifiable in the Elliott scheme, the problem becomes finding groupoid models while aiming for a specified Elliott invariant (in other words, "prescribed" K-theory). The most notable of recent results within this line of research is Xin Li's construction in [6], where a twisted groupoid is produced whose C*-algebra yields a given Elliott invariant. The idea is essentially to modify a similar construction due to George Elliott using inductive limits, while modifying the process so as to produce a Cartan subalgebra in the limit, hence a groupoid model. The results are quite extensive, and the complexity of the prescribed invariant is tied intimately with the dimension of the unit space of the constructed groupoid, as well as the necessity of a twist. In [10], Ian Putnam constructed groupoids whose C*-algebras have K-theory groups that can be realized as dimension groups using a procedure with subgroupoids. The idea is to enlarge an AF equivalence relation to regard it as an open subgroupoid of a larger groupoid, which produces two C*-algebras, one contained in the other as a subalgebra. Other subgroupoids constructed for similar purposes may be found in [2]. The methods in [2] improve upon those in [6] and [10] in that it is possible to prescribe K-theory with torsion without a twist.

In [2] and [10], the crucial tool in computing the K-theory is the excision theorem of [11]. While stated in a great deal of generality, the main result of [11] has two intended uses specific to groupoids. The first is the so-called "subgroupoid" situation mentioned above which is used in [2] and [10]. The second is the so-called "factor groupoid" situation, that is, two groupoids G and G' with a surjective groupoid homomorphism $\pi : G \to G'$ which results in an inclusion $C_r^*(G') \subseteq C_r^*(G)$. The term comes from the theory of dynamical systems via factor and extension systems, and therefore factor groupoids may be considered somewhat intuitively as the noncommutative version of the categorical equivalence between quotient maps $\pi : X \to Y$ of compact Hausdorff spaces and inclusions $C(Y) \subseteq C(X)$, see Proposition 2.1. The factor groupoid situation has seen little exploration to date, and here we determine some interesting examples of groupoids and C*-algebras that this situation supplies us with.

We will consider two distinct constructions of factor groupoids. The first construction begins with the same set-up as in [10]: two Bratteli diagrams (V, E) and (W, F) and two graph embeddings of (W, F) into (V, E) satisfying some specified conditions. The goal is to create a groupoid whose ordered K_0-group is the dimension group of (V,E), and whose K_1-group is the dimension group of (W,F). While the construction in [10] enlarges the usual tail equivalence relation R_E on the path space X_E of (V,E) to include pairs of paths in the two embeddings (which creates a new groupoid over the same unit space), we actually collapse these paths in X_E and pass R_E through the resulting quotient map, creating an equivalence relation on an entirely new space, and we denote the space and equivalence relation by X_ξ and R_ξ, respectively. It should also be noted that, unlike in [10], obtaining the étale topology and amenability on the new groupoids is straightforward (these properties essentially carry over right from R_E). It is the new unit spaces, rather than the new groupoids, that require more attention in order to describe them. We address this in section 3.

Theorem 1.1. *Let (V,E) and (W,F) be two Bratteli diagrams satisfying the embedding conditions described in section 3. There exists a metrizable quotient space X_ξ of X_E such that the resulting factor groupoid R_ξ of R_E satisfies the following.*

(i) R_ξ is second-countable, locally compact, Hausdorff, étale, and principal with dynamic asymptotic dimension zero,
(ii) if R_E is minimal, then so is R_ξ and hence $C^*_r(R_\xi)$ is classifiable,
(iii) $K_0(C^*_r(R_\xi))$ is order isomorphic to $K_0(C^*_r(R_E))$,
(iv) $K_1(C^*_r(R_\xi))$ is isomorphic to $K_0(C^*_r(R_F))$, and
(v) the map $\tau \mapsto \tau|_{C^*_r(R_\xi)}$, where τ is a tracial state on $C^*_r(R_E)$, is a homeomorphism between tracial state spaces.

While the order isomorphism between the K_0-groups in Theorem 1.1 is induced by the inclusion $C^*_r(R_\xi) \subseteq C^*_r(R_E)$, the K_1-group appears to receive its structure from the unit space of R_ξ, whose connected components are either points or circles. Indeed, the isomorphism between $K_1(C^*_r(R_\xi))$ and $K_0(C^*_r(R_F))$ is essentially given by a Bott map, and the elements of the group $K_0(C^*_r(R_F))$ seem to correspond to winding numbers on (at least a portion of) the space X_ξ. Curiously, these C*-algebras are isomorphic (by classification considerations) to those constructed in [10] using subgroupoid methods, but the two methods produce non-isomorphic Cartan subalgebras.

From Theorem 1.1 we obtain the following result of prescribed K-theory.

Corollary 1.2. *Let G_0 be a simple acyclic dimension group and G_1 a countable torsion-free abelian group. Then there exist Bratteli diagrams (V,E) and (W,F) satisfying the embedding conditions described in section 3 such that the resulting R_ξ from Theorem 1.1 has $K_0(C^*_r(R_\xi)) \cong G_0$ as ordered groups with order unit and $K_1(C^*_r(R_\xi)) \cong G_1$.*

The proof is essentially to find Bratteli diagrams (V,E) and (W,F) whose dimension groups are G_0 and G_1, respectively. By telescoping and symbol splitting the diagram (V,E), one can arrange that the embeddings described in section 3 exist without altering the dimension groups. Details may be found in [10].

Admittedly, the collection of Elliott invariants that are obtainable via Theorem 1.1 and Corollary 1.2 is not very extensive, as we are restricted to K-groups that are dimension groups and we have little control over the space of tracial states. However, it should be observed that the factor groupoids R_ξ have a relatively simple structure; indeed, the entire construction is essentially a generalized "noncommutative" version of the Cantor function (see the map ϕ in Proposition 3.1). This raises the question as to what K-theory data may be obtained with more complicated factor maps, or with additional structure(s) such as twists.

The second construction we consider arises from the nonhomogeneous extensions of Cantor minimal systems considered in [3], which we refer to as DPS extensions in acknowledgement of the authors. In these extensions, the fibres of the factor map are either a single point or homeomorphic to the attractor of a given iterated function system. Armed with the excision theorem from [11], we are in a position to provide a description of the K-theory of these extensions. A minor modification of the extension is necessary in order for the factor map to satisfy the regularity hypothesis needed to apply the excision theorem, and we describe the construction of the extension and its modification in section 4.

Theorem 1.3. *Let $(C, d_C, \mathcal{F})$ be a compact invertible iterated function system and (X_E, ϕ_E) a Bratteli-Vershik system on a properly ordered Bratteli diagram. There is a modified version of the associated DPS extension $(\tilde{X}, \tilde{\phi})$ of (X_E, ϕ_E), an AF subgroupoid R_E^C of R_E, a short exact sequence*

$$0 \longrightarrow K_0(C_r^*(R_E)) \longrightarrow K_0(C_r^*(\tilde{X} \times \mathbb{Z})) \longrightarrow K_0(C_r^*(R_E^C)) \otimes (K^0(C)/\mathbb{Z}) \longrightarrow 0$$

and a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_1(C_r^*(\tilde{X} \times \mathbb{Z})) \longrightarrow K_0(C_r^*(R_E^C)) \otimes K^{-1}(C) \longrightarrow 0.$$

By $K^0(C)/\mathbb{Z}$ we mean the quotient of $K^0(C)$ by the canonical copy of $\mathbb{Z}$ generated by the unit of $C(C)$. We refine the conclusion of Theorem 1.3 in some particular cases of interest, such as when C is a closed cube in Euclidean space (this case actually motivated the initial construction in [3]).

Corollary 1.4. *In the context of Theorem 1.3, we have the following.*

(i) If C is contractible, then the injective maps in both sequences are isomorphisms, and
(ii) if X_E consists of only one path $(x_1, x_2, \ldots)$ of edges such that $f_{x_n} = \mathrm{id}_C$ for all $n \geq 1$, then $K_0(C_r^*(R_E^C)) \cong \mathbb{Z}$ and hence

$$K_0(C_r^*(\tilde{X} \times \mathbb{Z}))/K_0(C_r^*(R_E)) \cong K^0(C)/\mathbb{Z} \qquad K_1(C_r^*(\tilde{X} \times \mathbb{Z}))/\mathbb{Z} \cong K^{-1}(C)$$

At this point it is not clear (at least to the author) whether or not the conclusions of Theorem 1.3 and Corollary 1.4 can be improved upon without knowing more about the attractor C of the iterated function system. For example, it is not clear precisely when the short exact sequences split, and hence when the quotients in Corollary 1.4 can be replaced with direct sums. Of course, this is trivially the case in part (i) of the corollary, and also when both $K^0(C)$ and $K^{-1}(C)$ are free abelian, such as when C is the Sierpiński triangle.

The paper is organized as follows. In section 2 we outline the necessary preliminary material. In section 3 we construct and analyze the spaces X_ξ and groupoids R_ξ. In section 4 we describe the DPS extensions and their modifications. In section 5 we analyze the K-theory and complete the proofs of Theorem 1.1 and Theorem 1.3.

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2. Preliminaries

We refer the reader to [12] for a detailed treatment of Hausdorff étale groupoids, but we outline our notation here. Let G be a locally compact Hausdorff groupoid with composable pairs $G^{(2)}$, unit space $G^{(0)}$, and range and source maps $r, s : G \to G^{(0)}$ defined by $r(x) = xx^{-1}$ and $s(x) = x^{-1}x$. We will only be concerned with étale groupoids, which means that r

and s are local homeomorphisms. An open subset U of G upon which r and s are local homeomorphisms is called an open bisection. For u in $G^{(0)}$, we denote $r^{-1}(u)$ and $s^{-1}(u)$ by G^u and G_u, respectively, which are discrete in the relative topology from G.

A groupoid homomorphism $\pi : G \to H$ is a map such that $\pi \times \pi(G^{(2)}) \subseteq H^{(2)}$ and $\pi(xy) = \pi(x)\pi(y)$ for every pair (x, y) in $G^{(2)}$. If G and H are topological groupoids, we will require that π be continuous. We have $\pi(x^{-1}) = \pi(x)^{-1}$ for all x in G, and $\pi(G^{(0)}) \subseteq H^{(0)}$, with equality if π is surjective. We also have $\pi \circ r = r \circ \pi$ and $\pi \circ s = s \circ \pi$. If π is bijective, we say it is a groupoid isomorphism.

We denote the *-algebra of all continuous compactly supported complex-valued functions on G by $C_c(G)$ with the operations

$$(f \star g)(x) = \sum_{y \in G^{r(x)}} f(y)g(y^{-1}x) \qquad f^*(x) = \overline{f(x^{-1})}$$

Since $G^{r(x)}$ is discrete and f and g are compactly supported, the above sum is finite for every x in G. For u in $G^{(0)}$ and x in G_u, let δ_x be the element of $l^2(G_u)$ that is 1 at x and 0 elsewhere. Define the representation $\phi_u : C_c(G) \to \mathcal{B}(l^2(G_u))$ by

$$\phi_u(f)\delta_x = \sum_{y \in G_{r(x)}} f(y)\delta_{yx}$$

for f in $C_c(G)$ and x in G_u. The representation $\bigoplus_{u \in G^{(0)}} \phi_u$ is called the left regular representation of $C_c(G)$, and it is faithful. The closure of $\bigoplus_{u \in G^{(0)}} \phi_u(C_c(G))$ in $\bigoplus_{u \in G^{(0)}} \mathcal{B}(l^2(G_u))$ is called the reduced C*-algebra of G, and we denote it by $C^*_r(G)$.

The two main examples of interest here are equivalence relations and integer actions. If R is an equivalence relation on a set X, then R is a groupoid with product $(x, y)(y', z) = (x, z)$ (only when $y = y'$) and inverse $(x, y)^{-1} = (y, x)$. If $\mathbb{Z}$ acts on the compact metric space X by a homeomorphism $\phi : X \to X$, then $X \times \mathbb{Z}$ is an étale groupoid with product $(x, n)(y, k) = (x, n + k)$ (only when $y = \phi^n(x)$) and inverse $(x, n)^{-1} = (\phi^n(x), -n)$. The algebra $C^*_r(X \times \mathbb{Z})$ is *-isomorphic to the (reduced) crossed product $C(X) \times_\phi \mathbb{Z}$, see Example 9.3.8 of [12].

It is a rather standard result that, given a continuous proper surjective map $\pi : X \to Y$ of locally compact Hausdorff spaces, the map $\alpha : C_0(Y) \to C_0(X)$ defined by $\alpha(f) = f \circ \pi$ is an injective *-homomorphism. We prove a generalization for groupoids, which is our "factor groupoid" situation.

Proposition 2.1. *Let G and G' be locally compact Hausdorff étale groupoids and $\pi : G \to G'$ a continuous proper surjective groupoid homomorphism. Suppose also that for all u in $G^{(0)}$, the map $\pi|_{G^u} : G^u \to (G')^{\pi(u)}$ is bijective. Then the map $\alpha : C_c(G') \to C_c(G)$ defined by $\alpha(f) = f \circ \pi$ is an injective *-homomorphism, and it extends to an injective *-homomorphism from $C^*_r(G')$ to $C^*_r(G)$ (which we also denote by α). If π is a groupoid isomorphism, then α is a *-isomorphism.*

Proof. That $\alpha(f)$ is continuous and compactly supported follows from the fact that π is continuous and proper. For any x in G, we have

$$\alpha(f \star g)(x) = (f \star g)(\pi(x)) = \sum_{z \in (G')^{\pi(r(x))}} f(z)g(z^{-1}\pi(x))$$

and

$$(\alpha(f) \star \alpha(g))(x) = \sum_{y \in G^{r(x)}} f(\pi(y))g(\pi(y^{-1}x))$$

Each z in the first sum corresponds to one and only one y in the second sum via the bijective map $\pi|_{G^{r(x)}}$. Thus both sums have precisely the same terms and they are equal. It is straightforward to check that $\alpha(f^*) = \alpha(f)^*$ and that it is injective. Now, for each u in $G^{(0)}$ the Hilbert spaces $l^2(G_u)$ and $l^2(G'_{\pi(u)})$ are isomorphic via the unitary $\delta_x \mapsto \delta_{\pi(x)}$ (notice that $\pi|_{G_u} : G_u \to (G')_{\pi(u)}$ is also bijective, for any u in $G^{(0)}$). It follows that the representation $\bigoplus_{u\in G^{(0)}}(\phi_u \circ \alpha)$ is unitarily equivalent (modulo extra direct factors over which the reduced norm is constant) to the left regular representation of $C_c(G')$, so we may identify $C_r^*(G')$ with the closure of $\bigoplus_{u\in G^{(0)}} \phi_u(\alpha(C_c(G')))$, which is contained in $C_r^*(G)$. If π is a groupoid isomorphism, then the map $\beta : C_c(G) \to C_c(G')$ defined by $\beta(f) = f \circ \pi^{-1}$ is obviously the inverse of α. Thus $C_c(G)$ is (algebraically) *-isomorphic to $C_c(G')$ via α, and the closures of $\bigoplus_{u\in G^{(0)}} \phi_u(C_c(G))$ and $\bigoplus_{u\in G^{(0)}} \phi_u(\alpha(C_c(G')))$ are equal. □

If π and α are as in Proposition 2.1, it is worth recording the easy but useful observation that

$$\alpha(C_c(G')) = \{f \in C_c(G) \mid f|_{\pi^{-1}(x')} \text{ is constant for all } x' \in G'\} \tag{1}$$

We now discuss relative K-theory. If A is a unital C*-algebra and A' is a C*-subalgebra of A that contains the unit of A, there is a homology theory $K_*(A'; A)$ in the sense that if $A' \subseteq A$ and $B' \subseteq B$, and $\phi : A \to B$ with $\phi(A') \subseteq B'$, then there is an induced homomorphism $\phi_* : K_*(A'; A) \to K_*(B'; B)$ which is an isomorphism if ϕ is a *-isomorphism and $\phi(A') = B'$. The theory satisfies Bott periodicity and there is an exact sequence

$$\begin{tikzcd}
K_1(A) \arrow[r, "\mu_0"] & K_0(A';A) \arrow[r, "\nu_0"] & K_0(A') \arrow[d] \\
K_1(A') \arrow[u] & K_1(A';A) \arrow[l, "\nu_1"'] & K_0(A) \arrow[l, "\mu_1"']
\end{tikzcd}$$

where the vertical maps are induced by the inclusion $A' \subseteq A$. The exact sequence above may be obtained by defining $K_*(A'; A)$ to be the K-theory of the mapping cone of the inclusion, but the portrait described in [5] via Karoubi's definitions seems to be more prudent if one wishes to use results from [11]. We now turn to these results.

Suppose that G has a metric d_G yielding its topology. Let

$$H' = \{x' \in G' \mid \#\pi^{-1}(x') \neq 1\}$$

and $H = \pi^{-1}(H')$. Give H' and H the metrics

$$d_{H'}(x', y') = d_G(\pi^{-1}(x'), \pi^{-1}(y'))$$

(via the Hausdorff distance) and

$$d_H(x, y) = d_G(x, y) + d_{H'}(\pi(x), \pi(y))$$

Definition 2.2 (7.7 of [11]). *$\pi : G \to G'$ is* regular *if, for all x' in H' and all $\varepsilon > 0$, there is an open set $U' \subseteq G'$ with x' in U' such that if y' is in U', at least one of the following holds*

$$d_G(\pi^{-1}(x'), \pi^{-1}(y')) < \varepsilon \qquad \textit{diam } \pi^{-1}(y') < \varepsilon$$

There is also a necessary notion of "measure regularity" for π. We will not record the definition because in our cases it will follow from regularity.

Definition 2.3 (5.5 of [11]). *Let $C \geq 1$. We say that a subset $X \subseteq H$ which is closed in G has the C-*extension property *if, for any f in $C_c(H)$ with support in X, there exists $\tilde{f}$ in $C_c(G)$ such that $\tilde{f}|_X = f|_X$ and $\|\tilde{f}\| \leq C\|f\|$, where $\|\cdot\|$ is the reduced norm in $C_r^*(G)$.*

The following will be our main tool for computing the K-theory of the factor groupoids. The assumptions are somewhat stronger than necessary, but sufficient for our purposes.

Theorem 2.4 (7.9, 7.18 and 7.19 of [11]). *Let $\pi : G \to G'$ be as in Proposition 2.1 and assume it is regular and measure regular. Then H' and H, with the metrics $d_{H'}$ and d_H, are locally compact Hausdorff groupoids with finer topologies than the relative topologies received from G' and G, respectively, and $\pi|_H : H \to H'$ also satisfies the hypotheses of Proposition 2.1. Moreoever, if there exists a $C \geq 1$ such that every subset $X \subseteq H$ which is closed in G has the C-extension property, then*

$$K_*(C_r^*(G'); C_r^*(G)) \cong K_*(C_r^*(H'); C_r^*(H))$$

We outline necessary definitions and notation for a particular class of groupoids known as AF-groupoids. Let (V, E) be a Bratteli diagram, that is, an infinite directed graph consisting of a set of vertices V and a set of edges E such that

(i) V and E are ordered and partitioned into countably many finite subsets V_n for $n \geq 0$ and E_n for $n \geq 1$,
(ii) V_0 consists of a single vertex v_0,
(iii) if e is in E_n, then $i(e)$ is in V_{n-1} and $t(e)$ is in V_n, and
(iv) $i^{-1}(v)$ is nonempty for each vertex v and $t^{-1}(v)$ is nonempty for each vertex v other than v_0.

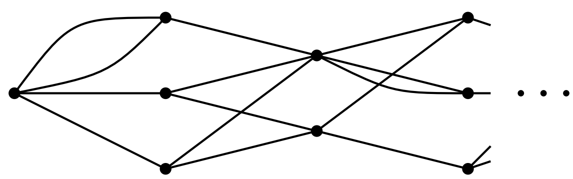


FIGURE 1. A Bratteli diagram

We say that (V, E) has full edge connections if, for every $n \geq 0$ and pair of vertices v in V_n and w in V_{n+1}, there is an edge e in E_{n+1} with $i(e) = v$ and $t(e) = w$. We denote by X_E the path space of (V, E): the set of all infinite paths, that is, all sequences $x = (x_1, x_2, x_3, \ldots)$, where x_n is in E_n and $t(x_n) = i(x_{n+1})$ for all $n \geq 1$. For a finite sequence of edges $p = (p_1, p_2, \ldots, p_n)$ where p_j is in E_j and $t(p_j) = i(p_{j+1})$, we denote

$$C(p) = C(p_1, p_2, \ldots, p_n) = \{x \in X_E \mid x_j = p_j \text{ for } 1 \leq j \leq n\}$$

called the cylinder set of p. For any p and q, either $C(p)$ and $C(q)$ are disjoint, or one is contained in the other. The path space X_E is a totally disconnected compact metric space in the metric

$$d_E(x, y) = \inf\{1, 2^{-n} \mid x_j = y_j \text{ for } 1 \leq j \leq n\}$$

and the cylinder sets $C(p)$ are closed and open (clopen) in this metric. If (V, E) has full edge connections, then X_E has no isolated points and is thus homeomorphic to the Cantor set. The equivalence relation $R_E \subseteq X_E \times X_E$ is defined by $(x, y) \in R_E$ if and only if there exists some $n \geq 1$ such that $x_k = y_k$ for all $k \geq n$, also known as tail-equivalence. For finite paths p and q with $t(p) = t(q) \in V_n$, let

$$\gamma(p, q) = \{(x, y) \in R_E \mid x \in C(p), y \in C(q), x_k = y_k \text{ for all } k \geq n+1\}$$

These sets form a base of compact open bisections which makes R_E into a second-countable locally compact Hausdorff étale groupoid. If (V, E) has full edge connections, then R_E is minimal, that is, every equivalence class is dense. This topology is concretely metrizable: write

$$R_n = \{(x, y) \in R_E \mid x_k = y_k \text{ for all } k \geq n\}$$

for $n \geq 1$ and $R_0 = \varnothing$, and note that $R_E = \bigcup_{n=0}^{\infty}(R_{n+1} - R_n)$, which is a disjoint union of compact open sets. Let $d_E^{(2)}((x, y), (x', y')) = \max\{d_E(x, x'), d_E(y, y')\}$ if (x, y) and (x', y') are both in $R_{n+1} - R_n$ for some n, and $d_E^{(2)}((x, y), (x', y')) = 1$ otherwise.

The C*-algebra $C_r^*(R_E)$ is a unital AF-algebra. Concretely, the linear span of the characteristic functions $\chi_{\gamma(p,q)}$ of the compact open sets $\gamma(p, q)$ act as "matrix units" and generate a dense union of finite dimensional algebras. If G is isomorphic to R_E for some Bratteli diagram (V, E), we say that G is an AF-groupoid.

We verify that we need not worry about the hypothesis involving the C-extension property in Theorem 2.4 when it comes to R_E.

Proposition 2.5. *Let R_E be the étale groupoid defined above, and suppose H is a subgroupoid of R_E equipped with a finer topology than that of the relative topology from R_E. Then every subset $X \subseteq H$ which is closed in R_E satisfies the C-extension property with $C = 1$.*

Proof. The technique outlined in [11] works, so we repeat it here. Suppose $X \subseteq H$ is closed in R_E and f is in $C_c(H)$ with support in X. As the support of f is compact in H, it is also compact in R_E, hence the support is contained in R_n for some n. Apply the Tietze extension theorem to $f|_{X \cap R_n}$ to obtain g in $C_c(R_n) = C(R_n)$ with $g|_{X \cap R_n} = f|_{X \cap R_n}$. Define the continuous function $h : [0, \infty) \to \mathbb{R}$ by

$$h(t) = \begin{cases} 1 & 0 \leq t \leq \|f\|^2 \\ \|f\| t^{-1/2} & t > \|f\|^2 \end{cases}$$

and let $\tilde{f} = g \star h(g^* \star g)$. As $C(R_n)$ is complete, this element is in $C(R_n) \subseteq C_c(R_E)$, satisfies $\tilde{f}|_X = f|_X$, and

$$\|\tilde{f}\|^2 = \|\tilde{f}^* \star \tilde{f}\| = \|h(g^* \star g) \star g^* \star g \star h(g^* \star g)\| = \|t h(t)^2\|_\infty \leq \|f\|^2$$

$\square$

3. Quotients of path spaces

In this section we describe an equivalence relation, denoted $\sim_\xi$, on the path space X_E of a Bratteli diagram (V, E) and examine the resulting quotient space of X_E.

Let (V, E) and (W, F) be two Bratteli diagrams, (V, E) with full edge connections, such that there exist two graph embeddings ξ^0 and ξ^1 of (W, F) into (V, E). We regard ξ^0 and ξ^1 as functions on both W and F. More precisely, for both $j = 0, 1$,

(i) $\xi^j(w)$ is in V_n for all w in W_n and all $n \geq 0$,
(ii) $\xi^j(f)$ is in E_n for all f in F_n and all $n \geq 1$,
(iii) $i(\xi^j(f)) = \xi^j(i(f))$ and $t(\xi^j(f)) = \xi^j(t(f))$ for all f in F,
(iv) $\xi^j(f) \neq \xi^j(f')$ for all $f \neq f'$ in F.

We also require that

(v) $\xi^0(w) = \xi^1(w)$ for all w in W,
(vi) $\xi^0(F) \cap \xi^1(F) = \varnothing$.

Due to (v), the functions $\xi^0, \xi^1 : W \to V$ are identical, so we may denote them both by ξ.

The equivalence relation $\sim_\xi$ on X_E is defined as follows. For $x = (x_1, x_2, x_3, \ldots)$ in X_E, suppose that there is a j in $\{0, 1\}$, an $n_0 \geq 0$, and edges z_n in F_n such that $x_n = \xi^j(z_n)$ for $n \geq n_0 + 1$. Moreover, suppose n_0 is the least integer with this property. If $n_0 = 0$, then

$$x \sim_\xi (\xi^{1-j}(z_1), \xi^{1-j}(z_2), \xi^{1-j}(z_3), \ldots).$$

If $n_0 \geq 1$ and x_{n_0} is not in $\xi^{1-j}(F)$, then

$$x \sim_\xi (x_1, x_2, \ldots, x_{n_0}, \xi^{1-j}(z_{n_0+1}), \xi^{1-j}(z_{n_0+2}), \ldots).$$

If $n_0 \geq 1$ and there is some f in F such that $x_{n_0} = \xi^{1-j}(f)$, then

$$x \sim_\xi (x_1, x_2, \ldots, x_{n_0-1}, \xi^j(f), \xi^{1-j}(z_{n_0+1}), \xi^{1-j}(z_{n_0+2}), \ldots).$$

If none of the above occurs for x, then $x \sim_\xi x$ only. Each equivalence class thus consists of either one or two points. If $x \sim_\xi x'$ and $x \neq x'$, we will refer to the edge level E_m, where m is the least integer such that $x_m \neq x'_m$, as the splitting level of the equivalence class $\{x, x'\}$. We denote the quotient space $X_E / \sim_\xi$ by X_ξ and let $\rho : X_E \to X_\xi$ denote the quotient map.

Proposition 3.1. *If $E = \xi^0(F) \cup \xi^1(F)$, then X_E is homeomorphic to $X_F \times \{0,1\}^{\mathbb{N}}$, and X_ξ is homeomorphic to $X_F \times S^1$, where S^1 is the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$.*

Proof. Define the map $\phi : \{0,1\}^{\mathbb{N}} \to S^1$ by $\phi(\{j_n\}) = \exp\left(2\pi i \sum_{n=1}^{\infty} j_n 2^{-n}\right)$, which is essentially the Cantor ternary function in a rather uncommon guise. Notice that two distinct sequences in $\{0,1\}^{\mathbb{N}}$ have the same image under ϕ if and only if they are $(0, 0, 0, \ldots)$ and $(1, 1, 1, \ldots)$, or of the form

$$(j_1, j_2, \ldots, j_m, 1, 0, 0, 0, 0, \ldots)$$
$$(j_1, j_2, \ldots, j_m, 0, 1, 1, 1, 1, \ldots)$$

By our assumption and properties (iv) and (vi) of the embeddings, each x in X_E is uniquely determined by a path z_x in X_F and a sequence $j_x = \{j_n\}$ in $\{0,1\}^{\mathbb{N}}$ with $x_n = \xi^{j_n}((z_x)_n)$ for all n. This gives an obvious continuous bijection from X_E to $X_F \times \{0,1\}^{\mathbb{N}}$, hence a homeomorphism. The composition of this homeomorphism with $\mathrm{id}_{X_F} \times \phi$ is a quotient map whose fibres are identical to the fibres of ρ. □

If the inclusion $E \supseteq \xi^0(F) \cup \xi^1(F)$ is proper, describing the space X_ξ is slightly more complicated. In the upcoming lemma we describe some open sets in X_E that remain open when passed to X_ξ.

Definition 3.2. *Let $p = (p_1, p_2, \ldots, p_n)$ be a finite path in (V, E). Denote*

$$D_j(p) = \{x \in C(p) \mid x_k \in \xi^j(F) \text{ for all } k \geq n+1\}$$

for $j = 0, 1$.

Recall that if $f : X \to Y$ is surjective, a subset V of X is called saturated if there is a subset U of Y with $V = f^{-1}(U)$.

Lemma 3.3. *Let $p = (p_1, p_2, \ldots, p_n)$ be a finite path in (V, E).*

(i) If p_n is not in $\xi^0(F) \cup \xi^1(F)$, then $C(p)$ is clopen and saturated.

(ii) If p_n is in $\xi^0(F) \cup \xi^1(F)$, then

$$U_p = C(p) - (D_0(p) \cup D_1(p))$$

is open and saturated.

(iii) *Suppose that $k \geq m+1$ and p and q are two finite paths either of the form*

$$p = (\xi^0(z_1), \xi^0(z_2), \ldots, \xi^0(z_k))$$
$$q = (\xi^1(z_1), \xi^1(z_2), \ldots, \xi^1(z_k))$$

or of the form

$$p = (x_1, x_2, \ldots, x_{m-1}, x_m, \xi^0(z_{m+1}), \xi^0(z_{n+2}), \ldots, \xi^0(z_k))$$
$$q = (x_1, x_2, \ldots, x_{m-1}, x'_m, \xi^1(z_{m+1}), \xi^1(z_{n+2}), \ldots, \xi^1(z_k))$$

where x_m is not in $\xi^0(F)$, x'_m is not in $\xi^1(F)$, $x_m = x'_m$ if x_m is not in $\xi^1(F)$, and $x'_m = \xi^0(f)$ if $x_m = \xi^1(f)$ for some f in F. Then

$$V_{p,q} = (C(p) - D_1(p)) \cup (C(q) - D_0(q))$$

is open and saturated.

Proof. (i) If x is in $C(p)$ and $x \sim_\xi x'$, then the splitting level of $\{x, x'\}$ must be past level E_n since p_n is not in $\xi^0(F) \cup \xi^1(F)$, so x' is in $C(p)$.

(ii) If x is in U_p and $x \sim_\xi x'$, the removal of the sets $D_j(p)$ for $j = 0, 1$ forces the splitting level of $\{x, x'\}$ to be past level E_n, so x and x' coincide on the first n edges and x' is thus in U_p. It is straightforward to check that the sets $D_j(p)$ are closed, so U_p is open.

(iii) The proof that $V_{p,q}$ is open is analogous to that for (ii). If $x \in C(p) - D_1(p)$, $x \sim_\xi x'$, and the splitting level of $\{x, x'\}$ is past level k, then x' is in $C(p) - D_1(p)$ similarly as in (ii). Otherwise, the splitting level must be either E_m or E_{m+1}, but then x' is in $C(q) - D_0(q)$. □

Definition 3.4. *(i) Let $\mathcal{P}$ denote all finite paths $p = (p_1, p_2, \ldots, p_n)$ in (V, E) such that p_n is not in $\xi^0(F) \cup \xi^1(F)$ but $t(p_n)$ is in $\xi(W)$.*

(ii) For p in $\mathcal{P}$, define

$$C_\xi(p) = \{x \in C(p) \mid x_k \in \xi^0(F) \cup \xi^1(F) \text{ for all } k \geq n+1\}$$

To simplify notation, we will assume that there is an "empty path" p_0 in $\mathcal{P}$ with the property that $C_\xi(p_0) = \{x \in X_E \mid x_k \in \xi^0(F) \cup \xi^1(F)$ for all $k \geq 1\}$.

(iii) For $w \in W_n$, define

$$X_F^{(w)} = \{z = (z_{n+1}, z_{n+2}, \ldots) \mid z_k \in F_k, t(z_k) = i(z_{k+1}) \text{ for all } k \geq n+1, \text{ and } i(z_{n+1}) = w\}$$

that is, all infinite paths in (W, F) that start at w.

Note that $C_\xi(p)$ is not equal to $D_0(p) \cup D_1(p)$, since edges in paths in $C_\xi(p)$ may alternate between the two embeddings. Regarding (iii) above, by fixing any finite path $p = (p_1, p_2, \ldots, p_n)$ in (W, F) with $t(p_n) = w$, we may identify $X_F^{(w)}$ with $C(p)$ by associating z with pz (the concatenation of p and z) and endow $X_F^{(w)}$ with the metric it receives from this identification. This makes $X_F^{(w)}$ a totally disconnected compact metric space.

Lemma 3.5. *Let $p = (p_1, p_2, \ldots, p_n)$ be a finite path in $\mathcal{P}$ with $t(p_n) = \xi(w)$. Then*

(i) $C(p)$ is clopen and saturated and $C_\xi(p)$ is closed and saturated,

(ii) $C_\xi(p)$ is homeomorphic to $X_F^{(w)} \times \{0,1\}^{\mathbb{N}}$ and $\rho(C_\xi(p))$ is homeomorphic to $X_F^{(w)} \times S^1$,

(iii) if $p \neq q$ in $\mathcal{P}$, there are disjoint clopen saturated sets U and V in X_E such that $C_\xi(p) \subseteq U$ and $C_\xi(q) \subseteq V$; in particular, $\rho(C_\xi(p))$ and $\rho(C_\xi(q))$ are disjoint,

(iv) for every x in X_E and $\delta > 0$, there is an open saturated subset U of X_E such that $\rho^{-1}(\rho(x)) \subseteq U \subseteq B(\rho^{-1}(\rho(x)), \delta)$, and

(v) *if x is in $X_E - \bigcup_{p\in\mathcal{P}} C_\xi(p)$, then U from part (iv) can be chosen to be clopen; it follows that $\{\rho(x)\}$ is a connected component of X_ξ in this case.*

Proof. (i) For $m \geq n+1$, let A_m be the finite union of all cylinder sets of the form

$$C(p_1, p_2, \ldots, p_n, q_{n+1}, q_{n+2}, \ldots, q_m)$$

where q_k is in $\xi^0(F) \cup \xi^1(F)$ for $n+1 \leq k \leq m$. Each A_m is closed and $C_\xi(p) = \bigcap_{m=n+1}^{\infty} A_m$. The set $C_\xi(p)$ is saturated since p_n is not in $\xi^0(F) \cup \xi^1(F)$.

(ii) Each x in $C_\xi(p)$ is uniquely determined by a path in $X_F^{(w)}$ and a sequence in $\{0,1\}^{\mathbb{N}}$, and the proof then proceeds analogously to that of Proposition 3.1.

(iii) If $C(p) \cap C(q) = \varnothing$, then $U = C(p)$ and $V = C(q)$ suffice. If $C(p) \subseteq C(q)$, take $U = C(p)$ and $V = X_E - C(p)$.

(iv) We consider two cases.
   (a) Suppose that $\rho^{-1}(\rho(x)) = \{x\}$. Choose $k \geq 1$ such that $2^{-k} < \delta$, let $p = (x_1, \ldots, x_k)$, and let $U = U_p$ as in part (ii) of Lemma 3.3. Observe that x is not in $D_0(p) \cup D_1(p)$ because otherwise $\rho^{-1}(\rho(x))$ would consist of two points.
   (b) If $\rho^{-1}(\rho(x)) = \{x, x'\}$ and E_m is the splitting level, choose $k \geq m$ with $2^{-k} < \delta$, let $p = (x_1, \ldots, x_k)$ and $q = (x'_1, \ldots, x'_k)$, and let $U = V_{p,q}$ from part (iii) of Lemma 3.3.

(v) There are infintely many edges $x_{n_1}, x_{n_2}, \ldots$ of x that are not in $\xi^0(F) \cup \xi^1(F)$. Choose $k \geq 1$ such that $2^{-n_k} < \delta$, let $U = C(x_1, \ldots, x_{n_k})$, and use part (i) of Lemma 3.3.

□

Proposition 3.6. *We have the following.*

(i) *The quotient map $\rho : X_E \to X_\xi$ is closed,*
(ii) *the space X_ξ is second-countable compact Hausdorff, hence metrizable,*
(iii) *every connected component of X_ξ is either a single point or homeomorphic to S^1, and*
(iv) *the covering dimension of X_ξ is 1.*

Proof. (i) Let F be closed in X_E and y a point not in $\rho(F)$. Then $F \cap \rho^{-1}(y) = \varnothing$, so $F \cap B(\rho^{-1}(y), \delta) = \varnothing$ for some $\delta > 0$. By part (iv) of Lemma 3.5, we can find an open saturated set U of X_E with $\rho^{-1}(y) \subseteq U$ and $F \cap U = \varnothing$. Then $\rho(U)$ is a neighbourhood of y disjoint from $\rho(F)$.

(ii) A closed continuous surjective map with compact fibres preserves the Hausdorff property and the second-countable property [7]. Metrizability follows from the Urysohn metrization theorem.

(iii) Let Y be a connected component of X_ξ. If Y consists of more than one point, it must be contained in $\bigcup_{p\in\mathcal{P}} \rho(C_\xi(p))$ by part (v) of Lemma 3.5. By part (iii) of Lemma 3.5, it must be contained in a single $\rho(C_\xi(p))$. The conclusion then follows from part (ii) of Lemma 3.5 since $X_F^{(w)}$ is totally disconnected.

(iv) By part (ii) there is a metric d which gives the topology of X_ξ. The space X_ξ contains at least one homeomorphic copy of S^1, so it suffices to prove that $\dim X_\xi \leq 1$. We can show this by showing that, for any $\varepsilon > 0$, we can find two collections $\mathcal{U}_0$ and $\mathcal{U}_1$ of pairwise disjoint open saturated subsets of X_E which, when taken together, cover X_E and diam $\rho(U) < \varepsilon$ for all U in $\mathcal{U}_0 \cup \mathcal{U}_1$. Choose $n \geq 1$ such that $d_E(x,y) < 2^{-n}$ implies that $d(\rho(x), \rho(y)) < \frac{\varepsilon}{2}$, and consider a finite path $p = (p_1, \ldots, p_n)$. If p_n is not in $\xi^0(F) \cup \xi^1(F)$, put $C(p)$ in $\mathcal{U}_0$. Otherwise, put the set U_p from part (ii) of Lemma 3.3 in $\mathcal{U}_0$. By the definition of the sets U_p, the remaining paths possibly not covered by the elements of $\mathcal{U}_0$ are the two-point equivalence classes whose splitting levels are among the levels $E_1, E_2, \ldots, E_n$. To cover these, let $\mathcal{U}_1$ consist of all sets of the form $V_{p,q}$ from part (iii) of Lemma 3.3, where $k = n$ and $1 \leq m \leq n-1$.

$\square$

Example 3.7. Let (V, E) be the Bratteli diagram with $\#V_n = 1$ for all n and $\#E_n = 2$ for all n (the 2^∞ diagram), while (W, F) is the diagram with $\#W_n = \#F_n = 1$ for all n. Embed one copy of (W, F) on the left and the other copy on the right. Then X_ξ is homeomorphic to S^1 by Proposition 3.1.

Example 3.8. Let (V, E) be the Bratteli diagram with $\#V_n = 1$ for all n and $\#E_n = 4$ for all n (the 4^∞ diagram), while (W, F) is the diagram with $\#W_n = 1$ and $\#F_n = 2$ for all n. Embed one copy of (W, F) on the left and the other copy on the right. Then X_ξ is homeomorphic to $X_F \times S^1$ by Proposition 3.1. By identifying X_F with the middle thirds Cantor set $X \subseteq [1, 2]$ (shifted to the right from its usual position), we can identify X_ξ with the planar set $\bigcup_{x \in X} xS^1$, where $xS^1 = \{xz \mid z \in S^1\}$.

Example 3.9. Consider the Bratteli diagram in Figure 2. Let (W, F) be the diagram with $\#W_n = \#F_n = 1$ for all n. Embed (W, F) down the bottom edges of (V, E). Then X_ξ is homeomorphic to $(\bigcup_{n=0}^\infty 2^{-n} S^1) \cup \{0\}$. Indeed, any finite path ending in a diagonal edge is in $\mathcal{P}$, and $\rho(C_\xi(p))$ is clopen and homeomorphic to S^1 for each such path. The only path not in $\bigcup_{p \in \mathcal{P}} C_\xi(p)$ is the path along the top of the diagram, which corresponds to the point 0.

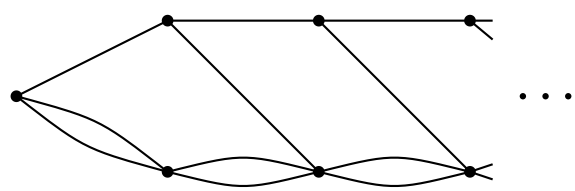


FIGURE 2. The Bratteli diagram (V, E) in Example 3.9

Example 3.10. Let (V, E) be the Bratteli diagram with $\#V_n = 1$ for all n and $\#E_n = 3$ for all n (the 3^∞ diagram), while (W, F) is the diagram with $\#W_n = \#F_n = 1$ for all n. For every finite path $p = (p_1, p_2, \ldots, p_n)$ in $\mathcal{P}$, $\rho(C_\xi(p))$ is homeomorphic to S^1 by part (ii) of Lemma 3.5. To help illustrate X_ξ, we note that $\bigcup_{p \in \mathcal{P}} C_\xi(p)$ is dense in X_E and construct a continuous injective map from $\bigcup_{p \in \mathcal{P}} C_\xi(p)$ to the plane $\mathbb{C}$ which is constant on the fibres of ρ. We will identify X_E with $\{0, 1, 2\}^\mathbb{N}$ and regard the edges labelled with j the edges in $\xi^j(F)$ for $j = 0, 1$.

First define $\theta_2 = 0$ and for a finite path $p = (p_1, p_2, \ldots, p_n)$ in $\{0, 1\}^{n-1} \times \{2\}$ for $n \geq 2$, set $\theta_p = 2^{-n} + \sum_{k=1}^{n-1} p_k 2^{-k}$. Define the function $f_p : \mathbb{C} \to \mathbb{C}$ by

$$f_p(z) = e^{2\pi i \theta_p}(2^{-n} z + (1 + 2^{-(n-1)}))$$

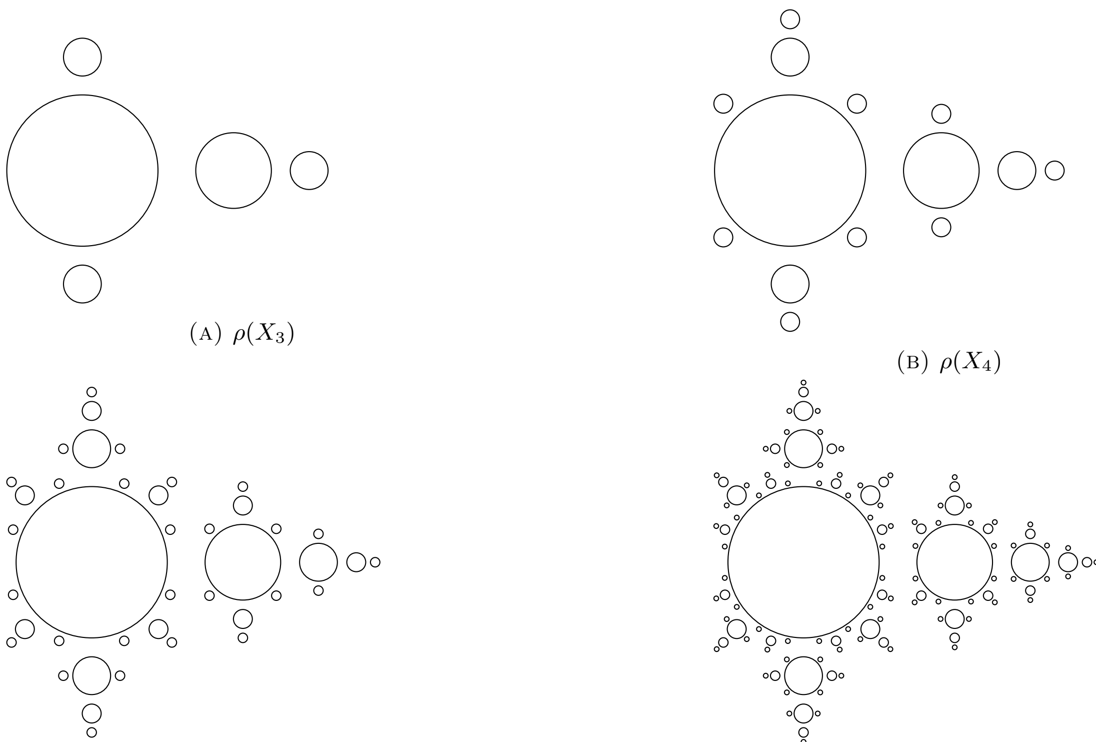
Now suppose that p in $\mathcal{P}$ is arbitrary, and partition $p = p^{(1)} p^{(2)} \cdots p^{(m)}$ where each $p^{(k)}$ is in $\{0, 1\}^{l_k} \times \{2\}$ for some integer $l_k \geq 0$ (note that such a partition exists and is unique). Define

$$f_p = f_{p^{(1)}} \circ f_{p^{(2)}} \circ \cdots \circ f_{p^{(m)}}$$

If p_0 is the empty path, let $f_{p_0}(z) = z$. If $x = (x_1, x_2, \ldots)$ is in $C_\xi(p)$, define

$$\rho(x) = f_p \left(\exp \left(2\pi i \sum_{k=n+1}^\infty x_k 2^{-k} \right) \right)$$

Letting $X_n = \bigcup_{\text{length}(p)\leq n-1} C_\xi(p)$ (where length(p) is the number of edges in p with the convention that length$(p_0) = 0$), we have $\rho(X_1)$ the unit circle and $\rho(X_2)$ is the disjoint union of the unit circle and the circle of radius $\frac{1}{2}$ centred at 2. Figure 3 illustrates $\rho(X_n)$ for $n = 3, 4, 5, 6$.



(A) $\rho(X_3)$

(B) $\rho(X_4)$

(C) $\rho(X_5)$

(D) $\rho(X_6)$

FIGURE 3. Geometric approximations of X_ξ from Example 3.10

We now turn to the factor groupoids R_ξ.

Definition 3.11. *Let $R_\xi = \rho \times \rho(R_E)$ and endow it with the quotient topology induced by the étale topology on R_E. Denote $\pi : R_E \to R_\xi$ to be the restriction of $\rho \times \rho$ to R_E.*

Clearly π is a continuous surjective groupoid homomorphism, and since tail-equivalent paths are never in the same fibre of ρ, π is bijective on the fibres of the range map $r : R_E \to R_E^{(0)}$. We observe that the sets R_n and $R_{n+1} - R_n$ are saturated with respect to π, for all $n \geq 1$.

Proposition 3.12. *We have the following.*

(i) R_ξ is a second-countable locally compact Hausdorff étale groupoid, and
(ii) the map $\pi : R_E \to R_\xi$ satisfies the hypotheses in Proposition 2.1, and is regular and measure regular.

Proof. We begin by proving that R_ξ is Hausdorff. Notice that π is continuous with respect to the relative product topologies from $X_E \times X_E$ and $X_\xi \times X_\xi$. It follows that the quotient topology R_ξ receives from R_E is finer than the relative topology from $X_\xi \times X_\xi$. The space X_ξ is Hausdorff by Proposition 3.6, thus R_ξ is too.

To see that R_ξ is locally compact, notice that $R_E = \bigcup_{n=1}^{\infty} R_n$ and each R_n is compact, open, and saturated. Therefore each element of R_ξ is contained in the compact open set $\pi(R_n)$ for some n.

Let $\mu_E : R_E^{(2)} \to R_E$ and $\mu_\xi : R_\xi^{(2)} \to R_\xi$ be the multiplication maps. We will show that μ_ξ is continuous. It is readily verified that $\pi \circ \mu_E = \mu_\xi \circ (\pi \times \pi)$. If U is open in R_ξ, then $\mu_E^{-1}(\pi^{-1}(U)) = (\pi \times \pi)^{-1}(\mu_\xi^{-1}(U))$ is open in $R_E^{(2)}$ since μ_E and π are continuous. Since R_E and R_ξ are locally compact Hausdorff and $R_E^{(2)}$ is a closed subset of $R_E \times R_E$ which is saturated (because $(\pi \times \pi)^{-1}(R_\xi^{(2)}) = R_E^{(2)}$), the map $\pi \times \pi|_{R_E^{(2)}}$ is a quotient map, so $\mu_\xi^{-1}(U) = \pi \times \pi((\pi \times \pi)^{-1}(\mu_\xi^{-1}(U)))$ is open in $R_\xi^{(2)}$. Continuity of inversion may be proven similarly.

To see that R_ξ is étale, first observe that, since $R_E^{(0)} = \pi^{-1}(R_\xi^{(0)})$, the groupoid is r-discrete (that is, $R_\xi^{(0)}$ is open), so it suffices to show that r is an open map. Let $U' \subseteq R_\xi$ be open. Then $U = \pi^{-1}(U')$ is open by continuity of π, and $r(U)$ is open since R_E is étale. Now $\pi(r(U)) = r(\pi(U)) = r(U')$, so we need only show that $r(U)$ is saturated with respect to π. If (x, x) is in $r(U)$ and (x', x') is in $R_E^{(0)}$ with $\rho(x) = \rho(x')$, pick y in X_E with (x, y) in U. Then there is some y' in X_E with $\rho(y) = \rho(y')$. Then (x', y') is in R_E and U is saturated, so (x', y') is in U. Then $(x', x') = r(x', y')$ is in $r(U)$.

Of the standing hypotheses, the only nontrivial one to check is that π is proper. Let $K \subseteq R_\xi$ be compact. As noted earlier, R_E is the union of the compact open saturated sets R_n. It follows that $K \subseteq \pi(R_n)$ for some n, so $\pi^{-1}(K) \subseteq R_n$. Being a closed subset of a compact set, $\pi^{-1}(K)$ is compact. At this point, we may conclude that π is a closed map, and hence that R_ξ is second-countable, using Lemma 7.5 of [11] and the proof of parts (i) and (ii) of Proposition 3.6.

Finally, we show that π is regular. Let (x, y) and (x', y') be two distinct pairs in R_E with $\pi(x, y) = \pi(x', y')$ and $\varepsilon > 0$. Let $n \geq 0$ be the integer such that (x, y) and (x', y') are in $R_{n+1} - R_n$, let E_m be the splitting level of $\{x, x'\}$, and let E_l be the splitting level of $\{y, y'\}$. As before, we assume that the edges of x and y are eventually in $\xi^0(F)$ and the edges of x' and y' are eventually in $\xi^1(F)$. Choose $k \geq \max\{l, m, n+1\}$ so that $2^{-k} < \varepsilon$. Let $p = (x_1, \ldots, x_k)$, $p' = (x'_1, \ldots, x'_k)$, $q = (y_1, \ldots, y_k)$, and $q' = (y'_1, \ldots, y'_k)$, and let

$$U = (\gamma(p, q) \cup \gamma(p', q')) \cap (V_{p,p'} \times V_{q,q'})$$

We must show that

(i) U is open,
(ii) U is saturated, and
(iii) $\pi(U)$ satisfies the definition of regularity.

For (i), since the étale topology on R_E is finer than the relative product topology, $V_{p,p'} \times V_{q,q'}$ is open in R_E, and therefore so is U.

(ii) and (iii) can be proved simultaneously, in two cases. Let (x'', y'') be in U (suppose, without loss of generality, that it is in $\gamma(p, q)$) and (x''', y''') be another pair with $\pi(x'', y'') = \pi(x''', y''')$. We must show that (x''', y''') is also in U and that either

$$d_E^{(2)}(\{(x, y), (x', y')\}, \{(x'', y''), (x''', y''')\}) < \varepsilon \tag{2}$$

(in the Hausdorff metric) or

$$d_E^{(2)}((x'', y''), (x''', y''')) < \varepsilon \tag{3}$$

The first case considers when $x''_j = y''_j$ is in $\xi^0(F)$ for all $j \geq k+1$. Here, the splitting level of $\{x'', x'''\}$ must be E_m and the splitting level of $\{y'', y'''\}$ must be E_l. This forces (x''', y''') into $\gamma(p', q') \cap (V_{p,p'} \times V_{q,q'}) \subseteq U$. Also, we have

$$\begin{aligned} d_E^{(2)}(\{(x,y),(x',y')\},\{(x'',y''),(x''',y''')\}) &= \max\{d_E^{(2)}((x,y),(x'',y'')), d_E^{(2)}((x',y'),(x''',y'''))\} \\ &= \max\{d_E(x,x''), d_E(y,y''), d_E(x',x'''), d_E(y',y''')\} \\ &\leq 2^{-k} \\ &< \varepsilon \end{aligned}$$

and hence (2) holds.

The second case considers when $x''_j = y''_j$ is not in $\xi^0(F)$ for some $j \geq k+1$. This assumption, together with (x'', y'') being in $V_{p,p'} \times V_{q,q'}$, forces the splitting levels of $\{x'', x'''\}$ and $\{y'', y'''\}$ to be level E_{k+1} or further. In this case, (x''', y''') is also in $\gamma(p, q) \cap (V_{p,p'} \times V_{q,q'}) \subseteq U$. Moreover,

$$\begin{aligned} d_E^{(2)}((x'',y''),(x''',y''')) &= \max\{d_E(x'',x'''), d_E(y'',y''')\} \\ &\leq 2^{-k} \\ &< \varepsilon \end{aligned}$$

and hence (3) holds. Finally, π is measure regular by Proposition 7.16 of [11]. □

Proposition 3.13. *The dynamic asymptotic dimension of R_ξ is zero. It follows that R_ξ is amenable, and that the nuclear dimension of $C_r^*(R_\xi)$ is at most one.*

Proof. Since R_ξ is equal to the union of the compact open subgroupoids $\pi(R_n)$, its dynamic asymptotic dimension is zero by Example 5.3 of [4]. The conclusion now follows from part (iv) of Proposition 3.6, Theorem 8.6 of [4], and Corollary 8.25 of [4]. □

All in all, the algebras $C_r^*(R_\xi)$ are unital, separable, have finite nuclear dimension, and satisfy the UCT (Theorem 10.1.7 of [12]). If, in addition, R_E is minimal, then so is R_ξ because π is continuous and surjective, hence dense equivalence classes map to dense equivalence classes. In this case, $C_r^*(R_\xi)$ is simple, and are therefore classifiable (see Theorem 14.2.1 in [12]).

Corollary 3.14. *If R_E is minimal, then the C^*-algebra $C_r^*(R_\xi)$ is classified by its Elliott invariant.*

We now prove a noncommutative version of Proposition 3.1.

Proposition 3.15. *If $E = \xi^0(F) \cup \xi^1(F)$, then $C_r^*(R_E)$ is isomorphic to $C_r^*(R_F) \otimes M_{2^\infty}$, and $C_r^*(R_\xi)$ is isomorphic to $C_r^*(R_F) \otimes B$, where B is the Bunce–Deddens algebra of type 2^∞.*

Proof. Let S denote the tail-equivalence relation on $\{0,1\}^{\mathbb{N}}$ with its usual étale topology, so that $C_r^*(S) \cong M_{2^\infty}$ (Example 9.2.6 of [12]). Let

$$T = \{(w,z) \in S^1 \times S^1 \mid w = e^{2\pi i\theta} z \text{ for some } \theta \in \mathbb{Z}[\tfrac{1}{2}]\}$$

Give T the topology with basic open sets $U_{W,\theta} = \{(z, e^{2\pi i\theta} z) \mid z \in W\}$, where $W \subseteq S^1$ is open and $\theta \in \mathbb{Z}[\frac{1}{2}]$. This makes T a second-countable locally compact Hausdorff étale principal groupoid, and $C_r^*(T) \cong B$, see 10.11.4 of [1]. It is not difficult to see that this topology is the same as the quotient topology received from $\tau : S \to T$, where $\tau = (\phi \times \phi)|_S$ and ϕ is as in Proposition 3.1.

Define the map $\Psi : R_E \to R_F \times S$ by $\Psi(x, y) = ((z_x, z_y), (j_x, j_y))$, using the notation in Proposition 3.1. It is straightforward to check that this is an isomorphism of groupoids (which proves the first statement of the proposition). The composition of this isomorphism with $\mathrm{id}_{R_F} \times \tau$ is a quotient map whose fibres are identical to the fibres of π, which therefore factors to an isomorphism from R_ξ to $R_F \times T$. □

4. Extensions of Cantor minimal systems

We now turn to the second construction. We begin with the definition of an iterated function system. A compact invertible iterated function system $(C, d_C, \mathcal{F})$ is a compact metric space (C, d_C) together with a finite set $\mathcal{F}$ of functions $f : C \to C$ such that

(i) there exists a real number λ with $0 < \lambda < 1$ and $d_C(f(x), f(y)) \leq \lambda d_C(x, y)$ for all points x and y in C and all f in $\mathcal{F}$,
(ii) each function in $\mathcal{F}$ is injective,
(iii) $\bigcup_{f \in \mathcal{F}} f(C) = C$.

In [3] it is shown that, given a Cantor minimal system (X, ϕ) and a compact invertible iterated function system $(C, d_C, \mathcal{F})$, there exists a minimal extension $(\tilde{X}, \tilde{\phi})$ with factor map $\tilde{\pi} : (\tilde{X}, \tilde{\phi}) \to (X, \phi)$ such that, for every x in X, $\tilde{\pi}^{-1}(x)$ is either a single point or homeomorphic to C. The construction is, briefly, as follows. We assume that the Cantor minimal system is given by a Bratteli-Vershik system (X_E, ϕ_E) on a properly ordered Bratteli diagram (see [8] for more information). To each edge e in the diagram one assigns a function f_e in $\mathcal{F} \cup \{\mathrm{id}_C\}$ such that

(i) if e is either maximal or minimal, then $f_e \neq \mathrm{id}_C$,
(ii) for every v in V, we have $\bigcup_{t(e)=v, f_e \neq \mathrm{id}_C} f_e(C) = C$, and
(iii) the set $\{e \mid f_e = \mathrm{id}_C\}$ contains an infinite path.

One then defines $\tilde{X} = \bigcap_{n=1}^{\infty} X_n$, where

$$X_n = \{(x, c) \in X_E \times C \mid c \in f_{x_1} \circ \cdots \circ f_{x_n}(C)\}$$

The factor map $\tilde{\pi} : \tilde{X} \to X_E$ (we reserve the letter π for our groupoid homomorphism) is defined by $\tilde{\pi}(x, c) = x$. The self map $\tilde{\phi}$ of $\tilde{X}$ is defined via two cases:

(i) if x has infinitely many n with $f_{x_n} \neq \mathrm{id}_C$, there is a unique point c_x in C with $\tilde{\pi}(x, c_x) = x$. Set $\tilde{\phi}(x, c) = (\phi_E(x), c_{\phi_E(x)})$, and
(ii) if there is an m with $f_{x_n} = \mathrm{id}_C$ for all $n \geq m$, then $\phi_E(x) = (y_1, y_2, \cdots, y_m, x_{m+1}, x_{m+2}, \ldots)$ for some edges $y_1, \ldots, y_m$ with $t(x_m) = t(y_m)$. Set $\tilde{\phi}(x, c) = (\phi_E(x), f_{y_1}^{-1} \circ \cdots \circ f_{y_m}^{-1} \circ f_{x_m} \circ \cdots \circ f_{x_1}(c))$.

Clearly $\tilde{\pi} \circ \tilde{\phi} = \phi_E \circ \tilde{\pi}$. Setting $G = \tilde{X} \times \mathbb{Z}$ and $G' = X_E \times \mathbb{Z}$ as group actions, we thus get a continuous surjective groupoid homomorphism $\pi = \tilde{\pi} \times \mathrm{id}_{\mathbb{Z}}$.

The only potential problem (if one hopes to compute the K-theory later) is that π may not be regular. To fix this, we adjust the edge assignment $e \mapsto f_e$ as follows. For $n \geq 1$ we set

$$\mathcal{F}^{(n)} = \{f_1 \circ f_2 \circ \cdots \circ f_n \mid f_j \in \mathcal{F} \text{ for } 1 \leq j \leq n\}$$

and we assign e in E_n to a function f_e in $\mathcal{F}^{(n)} \cup \{\mathrm{id}_C\}$, and the action $\tilde{X} \times \mathbb{Z}$ is constructed in the same way. It is not difficult to see that

Proposition 4.1. *We have the following.*

(i) G and G' are second-countable locally compact Hausdorff étale groupoids, and
(ii) the map $\pi : G \to G'$ is regular and satisfies the hypotheses of Proposition 2.1.

Proof. Both G' and G have the product topology and $\mathbb{Z}$ is discrete, so (i) is immediate. The only nontrivial thing to prove for (ii) is that π is regular. Since π does nothing to $\mathbb{Z}$, it suffices restrict our attention to $\tilde{\pi}$. By scaling the metric d_C, we may assume that diam $C = 1$. Fix $\varepsilon > 0$ and x in X_E; we seek an open set U of X_E such that for all y in U, either $d(\tilde{\pi}^{-1}(x), \tilde{\pi}^{-1}(y)) < \varepsilon$ or diam $\tilde{\pi}^{-1}(y) < \varepsilon$.

First suppose x has infinitely many n with $f_{x_n} \neq \mathrm{id}_C$. Choose N such that $\lambda^N < \varepsilon$ and $f_{x_N} \neq \mathrm{id}_C$, and set $U = C(x_1, x_2, \ldots, x_N)$. If y is in U and (y, c) and (y, d) are in $\tilde{\pi}^{-1}(y)$, then c and d are both in $f_{y_1} \circ \cdots \circ f_{y_N}(C) = f_{x_1} \circ \cdots \circ f_{x_N}(C)$ and since f_{x_N} has a contractive factor of at most λ^N by the modification above, we have $d_C(c, d) \leq \lambda^N < \varepsilon$ and hence diam $\tilde{\pi}^{-1}(y) < \varepsilon$.

Second, suppose that $f_{x_n} = \mathrm{id}_C$ eventually. Choose N such that $2^{-N} < \varepsilon$, $\lambda^N < \varepsilon$, and $f_{x_n} = \mathrm{id}_C$ for all $n \geq N$, and set $U = C(x_1, x_2, \ldots, x_N)$. Now let y be in U; if $f_{y_n} \neq \mathrm{id}_C$ for some $n \geq N$, then diam $f_{y_1} \circ \cdots \circ f_{y_n}(C) \leq \lambda^n \leq \lambda^N < \varepsilon$, so diam $\tilde{\pi}^{-1}(y) < \varepsilon$ as in the first case. Otherwise, $f_{x_n}(C) = f_{y_n}(C)$ for all $n \geq 1$, and $d_E(x, y) \leq 2^{-N} < \varepsilon$, which implies that $d(\tilde{\pi}^{-1}(x), \tilde{\pi}^{-1}(y)) < \varepsilon$. □

π is also measure regular in this construction, but it will be easier to prove it later (Proposition 5.11).

Proposition 4.2. *The dynamic asymptotic dimension of G is one, and if the covering dimension of C is finite, then $C^*_r(G)$ has finite nuclear dimension.*

Proof. This is immediate from Theorem 1.3(i) of [4] and Theorem 8.6 of [4]. □

Similarly as with Corollary 3.14, we obtain

Corollary 4.3. *If the covering dimension of C is finite, then the C^*-algebra $C^*_r(G)$ is classified by its Elliott invariant.*

5. K-theory

Up until now we have proven parts (i) and (ii) of Theorem 1.1. In this section we turn to the remaining parts, as well as Theorem 1.3.

5.1. **Quotients of path spaces.** We begin by analyzing the subgroupoids $H \subseteq R_E$ and $H' \subseteq R_\xi$ introduced in section 2. In these examples we have

$$H' = \{(x', y') \in R_\xi \mid \#\pi^{-1}(x', y') = 2\}$$

and $H = \pi^{-1}(H')$. Clearly (x, y) is in H if and only if both x and y are eventually in $\xi^0(F)$ or eventually in $\xi^1(F)$. We proceed to describe the topologies on H' and H. From [11], H and H' have the metrics

$$d_{H'}((x', y'), (w', z')) = d_E^{(2)}(\pi^{-1}(x', y'), \pi^{-1}(w', z'))$$

and

$$d_H((x, y), (w, z)) = d_E^{(2)}((x, y), (w, z)) + d_E^{(2)}(\pi^{-1}(\pi(x, y)), \pi^{-1}(\pi(w, z)))$$

The next result follows from the definition of d_H above.

Proposition 5.1. *A sequence $(x^{(n)}, y^{(n)}) \to (x, y)$ in d_H if and only if $x^{(n)} \to x$ and $y^{(n)} \to y$ in d_E and, if k is the least integer such that x_m and y_m are in $\xi^j(F)$ for $m \geq k$, then there is an l such that k is the least integer such that $x^{(n)}_m$ and $y^{(n)}_m$ are in $\xi^j(F)$ for all $n \geq l$ and all $m \geq k$.*

Lemma 5.2. *There is a *-isomorphism $\beta : C^*_r(H') \oplus C^*_r(H') \to C^*_r(H)$ such that the diagram*

$$\begin{array}{ccc} C_r^*(H') & \xrightarrow{\gamma} & C_r^*(H') \oplus C_r^*(H') \\ \| & & \downarrow \beta \\ C_r^*(H') & \xrightarrow{\alpha} & C_r^*(H) \end{array}$$

is commutative, where α is as in Proposition 2.1, and $\gamma(a) = (a, a)$.

Proof. Set

$$H_j = \{(x, y) \in H \mid x, y \in \xi^j(F) \text{ eventually}\}$$

for $j = 0, 1$. Then $H_0 \cup H_1$ is a partition of H into two nonempty disjoint sets, and by Proposition 5.1, they are clopen. H_0 and H_1 are therefore locally compact Hausdorff étale subgroupoids of H. Moreover, if f_j is in $C_c(H)$ with support contained in H_j for $j = 0, 1$, then for any (x, y) in H,

$$(f_0 \star f_1)(x, y) = \sum_{(x,z) \in R_E} f_0(x, z) f_1(z, y) = 0$$

since all (x, z) and (z, y) lie in one and only one H_j. It follows that $C_c(H) = C_c(H_0) \oplus C_c(H_1)$. By representing $C_c(H_0)$ and $C_c(H_1)$ on $\bigoplus_{x \in H^{(0)}} l^2(H_x)$ by restricting the functions, it is apparent that $C_r^*(H) = C_r^*(H_0) \oplus C_r^*(H_1)$ as well. Define

$$\beta' : C_c(H') \oplus C_c(H') \to C_c(H_0) \oplus C_c(H_1)$$

by $\beta'(f, g) = (f \circ (\pi|_{H_0}), g \circ (\pi|_{H_1}))$. Similarly as in Proposition 2.1, this is a *-isomorphism which extends to the reduced algebras. It is straightforward to check that the diagram commutes on continuous compactly supported functions, and therefore on the completions. □

Lemma 5.3. *$C_r^*(H')$ is Morita equivalent to $C_r^*(R_F)$, where R_F is tail-equivalence on the diagram (W, F). It follows that $K_*(C_r^*(H'))$ is isomorphic to $K_*(C_r^*(R_F))$.*

Proof. Let H_0 be as in the previous lemma, and let $J = \{(x, y) \in H_0 \mid x_n, y_n \in \xi^0(F) \text{ for all } n \geq 1\}$, endowed with the relative topology from H. The map $R_F \to J$ which sends (x, y) to $(\xi^0(x), \xi^0(y))$ (applying ξ^0 to each edge of x and y in the obvious way) is clearly an isomorphism of groupoids. $J^{(0)}$ is thus compact and open (in H), hence $\chi_{J^{(0)}}$ is a projection in $C_c(H_0)$. Then $\chi_{J^{(0)}} C_r^*(H_0)$ is a Morita equivalence bimodule between $C_r^*(H_0) \cong C_r^*(H')$ and $\chi_{J^{(0)}} C_r^*(H_0) \chi_{J^{(0)}} \cong C_r^*(J) \cong C_r^*(R_F)$. □

Lemma 5.4. *The map $\alpha_* : K_0(C_r^*(R_\xi)) \to K_0(C_r^*(R_E))$ is surjective.*

Proof. Since $C_r^*(R_E)$ is a unital AF-algebra, it suffices to show that, for a finite path $p = (p_1, p_2, \ldots, p_n)$ in the diagram (V, E), the characteristic function $\chi_{\gamma(p,p)}$ is equivalent to a projection in $\alpha(C_c(R_\xi))$. If p_n is not in $\xi^0(F) \cup \xi^1(F)$, then $\chi_{\gamma(p,p)}$ is in $\alpha(C_c(R_\xi))$, so there is nothing to do. Otherwise, let k be the least integer in $\{1, 2, \ldots, n\}$ such that p_j is in $\xi^0(F) \cup \xi^1(F)$ for all $k \leq j \leq n$. For a string $\omega = (j_1, j_2, \ldots, j_{n-k+1})$ in $\{0, 1\}^{n-k+1}$, let

$$q_\omega = (p_1, p_2, \ldots, p_{k-1}, \xi^{j_1}(z_k), \xi^{j_2}(z_{k+1}), \ldots, \xi^{j_{n-k+1}}(z_n))$$

and let $L = \bigcup_{\omega, \eta \in \{0,1\}^{n-k+1}} \gamma(q_\omega, q_\eta)$. It is easy to see that L is saturated. Let

$$v = \frac{1}{\sqrt{2^{n-k+1}}} \sum_{\omega \in \{0,1\}^{n-k+1}} \chi_{\gamma(p, q_\omega)}$$

which is a partial isometry such that $vv^* = \chi_{\gamma(p,p)}$ and $v^*v = \frac{1}{2^{n-k+1}} \chi_L$, the latter of which is in $\alpha(C_c(R_\xi))$. □

Readers with a background in quantum information theory may notice a similarity between the formula for v in the above proof and the one for the n-fold tensor product of the Hadamard operator $\tilde{H}$ acting on a computational basis state (written in ket notation),

$$\tilde{H}^{\otimes n} |\omega\rangle_n = \frac{1}{\sqrt{2^n}} \sum_{\eta \in \{0,1\}^n} (-1)^{\omega \cdot \eta} |\eta\rangle_n$$

where $\omega \cdot \eta$ is the componentwise dot product computed mod 2. Indeed, if one identifies the functions $\chi_{\gamma(p,q)}$ with matrix units as described in the preliminaries sections, the matrices vv^* and v^*v are respectively identified with the projections

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \frac{1}{2^n} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

in $M_{2^n}(\mathbb{C})$. A unitary equivalence between them is implemented by $\tilde{H}^{\otimes n}$.

We have the six-term exact sequence

$$\begin{tikzcd} K_1(C_r^*(R_E)) \arrow[r] & K_0(C_r^*(R_\xi); C_r^*(R_E)) \arrow[r] & K_0(C_r^*(R_\xi)) \arrow[d] \\ K_1(C_r^*(R_\xi)) \arrow[u] & K_1(C_r^*(R_\xi); C_r^*(R_E)) \arrow[l] & K_0(C_r^*(R_E)) \arrow[l] \end{tikzcd}$$

Using (in order) Theorem 2.4, Lemma 5.2, Example 2.6 from [5], and Lemma 5.3, we obtain

$$\begin{aligned} K_j(C_r^*(R_\xi); C_r^*(R_E)) &\cong K_j(C_r^*(H'); C_r^*(H)) \\ &\cong K_j(C_r^*(H'); C_r^*(H') \oplus C_r^*(H')) \\ &\cong K_{1-j}(C_r^*(H')) \\ &\cong K_{1-j}(C_r^*(R_F)) \end{aligned}$$

The exact sequence may therefore be simplified to

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & K_0(C_r^*(R_\xi)) \arrow[d] \\ K_1(C_r^*(R_\xi)) \arrow[u] & K_0(C_r^*(R_F)) \arrow[l] & K_0(C_r^*(R_E)) \arrow[l] \end{tikzcd}$$

By Lemma 5.4, the vertical map on the right is an order isomorphism. Exactness implies that the following map is zero, which in turn implies that the map after that is an isomorphism.

We have proven parts (iii) and (iv) of Theorem 1.1; it remains to prove part (v). As these groupoids are principal, the tracial states τ on their C*-algebras correspond exactly to invariant (that is, $\mu(r(U)) = \mu(s(U))$ for any open bisection U) Borel measures μ on their unit spaces via

$$\tau(f) = \int_{G^{(0)}} f \, d\mu$$

for f in $C_c(G)$. It therefore suffices to show that any invariant measure cannot see the non-one-to-one parts of π.

Proposition 5.5. *We have the following.*

(i) If μ is an invariant measure on X_E then $\mu(H^{(0)}) = 0$, and
(ii) if μ is an invariant measure on X_ξ then $\mu(H'^{(0)}) = 0$.

Proof. (i) Every such measure μ is uniquely determined by a function $\nu : V \to [0,1]$ such that $\nu(v_0) = 1$ and $\nu(v) = \sum_{e \in i^{-1}(v)} \nu(t(e))$, the correspondence being $\nu(v) = \mu(C(p))$, where p is any finite path ending at v (the invariance of μ makes the choice of p immaterial). It suffices to show that the set Y of all paths whose edges are eventually in $\xi^0(F)$ has measure zero; the proof is analogous for those paths eventually in $\xi^1(F)$, and $H^{(0)}$ is the union of these. Let

$$Y_n = \{x \in X_E \mid x_m \in \xi^0(F) \text{ for all } m \geq n\}$$

We have $Y = \bigcup_{n=1}^{\infty} Y_n$, so it further suffices to show that $\mu(Y_n) = 0$ for all n. To this end, fix n and, for $m \geq n+1$, let A_m be the union of all cylinder sets of the form $C(p_1, p_2, \ldots, p_n, q_{n+1}, q_{n+2}, \ldots, q_m)$ where q_k is in $\xi^0(F)$ for $n+1 \leq k \leq m$. For any cylinder set $C(p)$ in the union A_m, there are at least $2^{m-(n+1)}$ cylinder sets in total passing through the same vertices as the path p, due to the edges in $\xi^1(F)$ (and possibly more). Thus, by the sum given by ν, we have $\mu(A_m) \leq 2^{-m}$. Since $Y_n \subseteq A_m$ for every $m \geq n+1$, this completes the proof of (i).

(ii) The set $H'^{(0)}$ is contained in the countable union $\bigcup_{p \in \mathcal{P}} \rho(C_\xi(p))$, so it suffices to show that $\mu(H'^{(0)} \cap \rho(C_\xi(p))) = 0$ for every p in $\mathcal{P}$. Using part (ii) of Lemma 3.5, we may identify $\rho(C_\xi(p))$ with $X_F^{(w)} \times S^1$ for some w in W. Let $n \geq 1$ and let γ be any open arc on S^1 of normalized Lebesgue length 2^{-n}. By invariance of μ, the disjoint sets $X_F^{(w)} \times e^{2\pi i k/2^n}\gamma$ for $k = 0, 1, \ldots, 2^n - 1$ are each contained in an open set in X_ξ of equal measure, so we have

$$\mu(X_F^{(w)} \times \gamma) \leq 2^{-n} \mu(X_F^{(w)} \times S^1)$$

Thus we obtain that $\mu(X_F^{(w)} \times \{z\}) = 0$ for any z in S^1. As $H'^{(0)} \cap \rho(C_\xi(p))$ is identified with $X_F^{(w)} \times \exp(2\pi i \mathbb{Z}[\frac{1}{2}])$ and $\exp(2\pi i \mathbb{Z}[\frac{1}{2}])$ is countable, this completes the proof. □

Example 5.6. Consider the situation in Example 3.7. By Theorem 1.1, we have $K_0(C_r^*(R_\xi)) \cong \mathbb{Z}[\frac{1}{2}]$ with its usual order, and $K_1(C_r^*(R_\xi)) \cong \mathbb{Z}$ with generator induced by the identity function z in $C(S^1) = C(R_\xi^{(0)}) \subseteq C_r^*(R_\xi)$.

Example 5.7. Consider the situation in Example 3.8. By Theorem 1.1, we have $K_0(C_r^*(R_\xi)) \cong K_1(C_r^*(R_\xi)) \cong \mathbb{Z}[\frac{1}{2}]$ with the usual order in the K_0 case. The K_1-group is generated by the partial unitaries $\chi_U \otimes z$ in $C(X_F) \otimes C(S^1) = C(R_\xi^{(0)})$, where U is a clopen subset of X_F (see Proposition 3.4 of [5] and the discussion preceding it).

5.2. **Extensions of Cantor minimal systems.** We now prove Theorem 1.3. In this case we have

$$H' = \{(x,k) \mid f_{x_n} = \mathrm{id}_C \text{ eventually}\} \subseteq X_E \times \mathbb{Z}$$

and

$$H = \{(x,c,k) \mid f_{x_n} = \mathrm{id}_C \text{ eventually}\} \subseteq \tilde{X} \times \mathbb{Z} \subseteq X_E \times C \times \mathbb{Z}$$

We give a sequential description of the topologies on H' and H.

Proposition 5.8. *We have the following.*

(i) $(x^{(n)}, k_n) \to (x, k)$ *in* H' *if and only if* $k_n = k$ *eventually,* $x^{(n)} \to x$ *in* X_E*, and if* i_0 *is the least integer such that* $f_{x_i} = id_C$ *for all* $i \geq i_0$*, then* i_0 *is the least integer such that there exists an* n_0 *with* $f_{x_i^{(n)}} = id_C$ *for all* $i \geq i_0$ *and* $n \geq n_0$*,*

(ii) $(x^{(n)}, c_n, k_n) \to (x, c, k)$ *in* H *if and only if* $(x^{(n)}, k_n) \to (x, k)$ *in* H' *and* $c_n \to c$ *in* C*.*

Definition 5.9. *Let* R_E^C *be all pairs* (x, y) *in* $X_E \times X_E$ *such that* x *and* y *are tail-equivalent and* $f_{x_n} = f_{y_n} = id_C$ *eventually. We endow* R_E^C *with the relative topology from* R_E*.*

Proposition 5.10. *We have the following*

(i) R_E^C *is an AF-groupoid,*

(ii) the map $\Phi : H' \to R_E^C$ *defined by* $\Phi(x, k) = (x, \phi_E^k(x))$ *is an isomorphism of groupoids,*

(iii) the map $\Psi : H \to R_E^C \times C$ *defined by* $\Psi(x, c, k) = (x, \phi_E^k(x), (f_{x_n}^{-1} \circ \cdots \circ f_{x_1}^{-1}(c))$ *is well-defined as long as* $f_{x_i} = id_C$ *for* $i \geq n$*, and is an isomorphism of groupoids (*C *is regarded as the cotrivial groupoid* $\{(c, c) \mid c \in C\}$*), and*

(iv) if $\pi^C : R_E^C \times C \to R_E^C$ *is defined by* $\pi^C(x, y, c) = (x, y)$*, then* $\pi^C \circ \Psi = \Phi \circ \pi$*.*

Proof. To see (i), note that the diagram is properly ordered, so the orbit equivalence relation is identical to tail equivalence with the exception of the pairs $(x^{\min}, x^{\max})$ and $(x^{\max}, x^{\min})$. These pairs are not in R_E^C. Parts (ii), (iii), and (iv) are straightforward verifications. □

Proposition 5.11. $\pi : G \to G'$ *is measure regular.*

Proof. By Proposition 5.10 we may show instead that π^C is measure regular. Fix a point c_0 in C and define the map $\mu : R_E^C \to R_E^C \times C$ by $\mu(x, y) = (x, y, c_0)$. Clearly μ is a continuous groupoid homomorphism and $\pi^C \circ \mu = \mathrm{id}_{R_E^C}$, so Proposition 7.15 from [11] applies. □

We also need to verify that the C-extension property in Theorem 2.4 is not an obstacle.

Proposition 5.12. *Every subset* $X \subseteq H$ *which is closed in* G *satisfies the* C*-extension property with* $C = 1$*.*

Proof. Breaking the orbit $\{\phi_E^n(x^{\min})\}_{n\in\mathbb{Z}}$ at $x^{\min}$ and letting the two sets

$$\{\phi_E^n(x^{\min})\}_{n\geq 0} \qquad \{\phi_E^n(x^{\min})\}_{n<0}$$

be two separate equivalence classes results in a subgroupoid which is open in G' and isomorphic to R_E. Since H' does not intersect the orbit of $x^{\min}$, H' is contained in $R_E = \bigcup_{n=1}^\infty R_n$. Thus $X \subseteq \bigcup_{n=1}^\infty \pi^{-1}(R_n)$ and since π is continuous and proper, any f in $C_c(H)$ with support contained in X has support contained in some compact open subgroupoid $\pi^{-1}(R_n)$. The proof then proceeds analogously to that of Proposition 2.5. □

Lemma 5.13. *We have*

$$K_0(C_r^*(G'); C_r^*(G)) \cong K_0(C_r^*(R_E^C)) \otimes K^{-1}(C)$$

and

$$K_1(C_r^*(G'); C_r^*(G)) \cong K_0(C_r^*(R_E^C)) \otimes (K^0(C)/\mathbb{Z})$$

Proof. By Theorem 2.4 and Proposition 5.10, we have

$$K_*(C_r^*(G'); C_r^*(G)) \cong K_*(C_r^*(H'); C_r^*(H)) \cong K_*(C_r^*(R_E^C); C_r^*(R_E^C \times C))$$

By combining the obvious isomorphism $C_r^*(R_E^C \times C) \cong C_r^*(R_E^C) \otimes C(C)$ with the Künneth Theorem for tensor products [1], we have a natural isomorphism

$$K_*(C_r^*(R_E^C \times C)) \cong K_0(C_r^*(R_E^C)) \otimes K^*(C)$$

under which the K_0 map induced by the inclusion $C_r^*(R_E^C) \subseteq C_r^*(R_E^C \times C)$ becomes $K_0(C_r^*(R_E^C)) \to K_0(C_R^*(R_E^C)) \otimes K^0(C) : g \mapsto g \otimes 1$. We have the six-term exact sequence

$$\begin{tikzcd}
K_0(C_r^*(R_E^C)) \otimes K^{-1}(C) \arrow[r] & K_0(C_r^*(R_E^C); C_r^*(R_E^C \times C)) \arrow[r] & K_0(C_r^*(R_E^C)) \arrow[d] \\
K_1(C_r^*(R_E^C)) \arrow[u] & K_1(C_r^*(R_E^C); C_r^*(R_E^C \times C)) \arrow[l] & K_0(C_r^*(R_E^C)) \otimes K^0(C) \arrow[l]
\end{tikzcd}$$

As C is compact, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K^0(C) \longrightarrow K^0(C)/\mathbb{Z} \longrightarrow 0$$

and since $C_r^*(R_E^C)$ is an AF-algebra, $K_0(C_r^*(R_E^C))$ is a dimension group, hence torsion free. Thus we may tensor the sequence above with $K_0(C_r^*(R_E^C))$ and obtain that

$$0 \longrightarrow K_0(C_r^*(R_E^C)) \longrightarrow K_0(C_r^*(R_E^C)) \otimes K^0(C) \longrightarrow K_0(C_r^*(R_E^C)) \otimes (K^0(C)/\mathbb{Z}) \longrightarrow 0$$

is exact. In particular, the map $K_0(C_r^*(R_E^C)) \to K_0(C_r^*(R_E^C)) \otimes K^0(C)$ is injective, so the right vertical arrow in the six-term sequence is injective. Combining the fact that $K_1(C_r^*(R_E^C)) = 0$ with exactness gives the conclusion. $\square$

Lemma 5.14. *The map* $\alpha_* : K_j(C_r^*(G')) \to K_j(C_r^*(G))$ *is injective for* $j = 0, 1$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & C(X_E, \mathbb{Z}) \arrow[r, "\mathrm{id}-\phi_E"] & C(X_E, \mathbb{Z}) \arrow[ddd, bend left=60, "\alpha"] & & \\
K_1(C_r^*(G')) \arrow[r, "\delta_1"] \arrow[d, "\alpha_*"] & K^0(X_E) \arrow[u, "\dim"] \arrow[r, "\mathrm{id}-\phi_{E*}"] \arrow[d, "\tilde{\pi}^*"] & K^0(X_E) \arrow[u, "\dim"] \arrow[r, "\iota_*"] \arrow[d, "\tilde{\pi}^*"] & K_0(C_r^*(G')) \arrow[r] \arrow[d, "\alpha_*"] & 0 \\
K_1(C_r^*(G)) \arrow[r, "\delta_1"] & K^0(\tilde{X}) \arrow[r, "\mathrm{id}-\tilde{\phi}_*"] \arrow[d, "\dim"] & K^0(\tilde{X}) \arrow[r, "\iota_*"] \arrow[d, "\dim"] & K_0(C_r^*(G)) & \\
 & C(\tilde{X}, \mathbb{Z}) \arrow[r, "\mathrm{id}-\tilde{\phi}"] & C(\tilde{X}, \mathbb{Z}) & &
\end{tikzcd}$$

The second and third rows are extracted from the Pimsner-Voiculescu exact sequence (Theorem 10.2.1 of [1]), therefore they are exact. By the map $\alpha : C(X_E, \mathbb{Z}) \to C(\tilde{X}, \mathbb{Z})$, we mean $\alpha(f) = f \circ \tilde{\pi}$, which is clearly an injective group homomorphism.

We consider the $j = 0$ case first. Let x be in $K_0(C_r^*(G'))$ such that $\alpha_*(x) = 0$. Since the top ι_* is surjective (by exactness), there is a y in $K^0(X_E)$ with $\iota_*(y) = x$. We have $\iota_*(\tilde{\pi}^*(y)) = \alpha_*(\iota_*(y)) = \alpha_*(x) = 0$, so by exactness again, there is some z in $K^0(\tilde{X})$ such that $(\mathrm{id} - \tilde{\phi}_*)(z) = \tilde{\pi}^*(y)$. Denote $f = \dim(y)$ and $g = \dim(z)$. Commutativity of the diagram implies that $f \circ \tilde{\pi} = g - g \circ \tilde{\phi}^{-1}$ in $C(\tilde{X}, \mathbb{Z})$, which means that $g - g \circ \tilde{\phi}^{-1}$ is constant on the fibres of $\tilde{\pi}$. Thus, if w and w' are points in $\tilde{X}$ with $\tilde{\pi}(w) = \tilde{\pi}(w')$, we have

$$g(w) - g(w') = g(\tilde{\phi}^{-1}(w)) - g(\tilde{\phi}^{-1}(w'))$$

The above equation implies that the difference $g(w)-g(w')$ remains constant as w and w' run through their orbits under $\tilde{\phi}$ in tandem. Since $\tilde{\phi}$ is minimal and there are fibres that consist of a single point, we must have $g(w)-g(w')=0$, that is, g must also be constant on the fibres of $\tilde{\pi}$. Therefore there is a g_0 in $C(X_E,\mathbb{Z})$ with $g=g_0\circ\tilde{\pi}$. Then

$$f\circ\tilde{\pi}=g-g\circ\tilde{\phi}^{-1}=g_0\circ\tilde{\pi}-g_0\circ\tilde{\pi}\circ\tilde{\phi}^{-1}=(g_0-g_0\circ\phi_E^{-1})\circ\tilde{\pi}$$

Since $\alpha : C(X_E,\mathbb{Z})\to C(\tilde{X},\mathbb{Z})$ is injective, we have $f=g_0-g_0\circ\phi_E^{-1}=(\mathrm{id}-\phi_E)(g_0)$, and since $\dim : K^0(X_E)\to C(X_E,\mathbb{Z})$ is an isomorphism (X_E is totally disconnected), there is one and only one y' in $K^0(X_E)$ with $\dim(y')=g_0$. By commutativity we have

$$(\mathrm{id}-\phi_E*)(y')=\dim^{-1}\circ(\mathrm{id}-\phi_E)\circ\dim(y')=\dim^{-1}\circ(\mathrm{id}-\phi_E)(g_0)=\dim^{-1}(f)=y$$

Finally,

$$x=\iota_*(y)=\iota_*((\mathrm{id}-\phi_E*)(y'))=0$$

by exactness.

The $j=1$ case is simpler, since $K_1(C_r^*(G'))\cong\mathbb{Z}$ is generated by the class of the characteristic function $u=\chi_{\tilde{X}\times\{1\}}$ in $C_c(G')$ (Theorem 1.1 of [9]), so we need only show that $\alpha_*([u])$ is nonzero. We have

$$\tilde{\pi}^*(\delta_1([u]))=\tilde{\pi}^*(-[1_{X_E}])=-[1_{\tilde{X}}]$$

and both $[1_{X_E}]$ and $[1_{\tilde{X}}]$ are nonzero since both spaces are compact. Thus $\delta_1(\alpha_*([u]))=\tilde{\pi}^*(\delta_1([u]))$ is nonzero and hence $\alpha_*([u])$ cannot be zero. □

Returning to the six-term exact sequence with the relative groups $K_*(C_r^*(G');C_r^*(G))$,

$$\begin{tikzcd}
K_1(C_r^*(G)) \arrow[r] & K_0(C_r^*(G');C_r^*(G)) \arrow[r] & K_0(C_r^*(G')) \arrow[d] \\
K_1(C_r^*(G')) \arrow[u] & K_1(C_r^*(G');C_r^*(G)) \arrow[l] & K_0(C_r^*(G)) \arrow[l]
\end{tikzcd}$$

Lemma 5.13, Lemma 5.14, and the fact that $K_0(C_r^*(G'))\cong K_0(C_r^*(R_E))$ and $K_1(C_r^*(G'))\cong\mathbb{Z}$ simplify this to

$$\begin{tikzcd}
K_1(C_r^*(G)) \arrow[r] & K_0(C_r^*(R_E^C))\otimes K^{-1}(C) \arrow[r, "0"] & K_0(C_r^*(R_E)) \arrow[d] \\
\mathbb{Z} \arrow[u] & K_0(C_r^*(R_E^C))\otimes(K^0(C)/\mathbb{Z}) \arrow[l, "0"'] & K_0(C_r^*(G)) \arrow[l]
\end{tikzcd}$$

giving the conclusion of Theorem 1.3.

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