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# Operator Algebras and Dynamics: Groupoids, Crossed Products, and Rokhlin Dimension



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# Operator Algebras and Dynamics: Groupoids, Crossed Products, and Rokhlin Dimension

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ISSN 2297-0304                      ISSN 2297-0312 (electronic)  
Advanced Courses in Mathematics - CRM Barcelona  
ISBN 978-3-030-39712-8              ISBN 978-3-030-39713-5 (eBook)  
<https://doi.org/10.1007/978-3-030-39713-5>

Mathematics Subject Classification (2010): 46L05, 46L10, 46L35, 46L40, 46L55, 22A22, 47L65

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The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

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## Foreword

Recent developments in Elliott's classification program are the culmination of a burst of activity in the area of simple nuclear  $C^*$ -algebras. These breakthroughs have been driven by techniques directly analogous to those used in the classification of amenable von Neumann factors. At the same time, the scope of the theory has led to a demand for good models for the class of  $C^*$ -algebras encompassed by classification results. A very good source of such models is naturally found in  $C^*$ -dynamical systems and their crossed products.

Taking this into account, an intensive research program was set up at the Centre de Recerca Matemàtica CRM-Barcelona in the Spring of 2017, centered around Operator Algebras, Dynamical Systems, and the interactions between these. The first part of the semester featured an intensive Advanced Course on Dynamical Systems and Operator Algebras, and a workshop on  $C^*$ -algebras and Dynamical Systems. The overall aim was to bring together renowned international experts as well as prominent young researchers to share their insight, and provide a good opportunity for young mathematicians with research interests in this area to be exposed to the latest developments, to interact with researchers from other countries, and to initiate future collaborations.

The Advanced Course, entitled *Operator Algebras: Groupoids, Crossed Products, and Rokhlin Dimension*, took place from March 13 to March 17, 2017. It consisted of three series of lectures, delivered by Aidan Sims (University of Wollongong) on *Hausdorff étale Groupoids and their  $C^*$ -algebras*, Gabor Szabó (KU Leuven) on *Introduction to Rokhlin Dimension*, and Dana Williams (Dartmouth College) on *A Primer on Crossed Products*. This volume contains the set of notes the authors used in the lectures. Thus, this text provides a very solid introduction to the theory of crossed products, groupoids and the notion of Rokhlin dimension, along with the important role played by the Elliott Classification program.

The different aspects of the volume are presented in a logical sequence, hence the definition and main technical features of crossed product  $C^*$ -algebras are given and developed in the first part, with special emphasis on their representation theory and their primitive ideal structure.

Groupoids and  $C^*$ -algebras associated to them constitute the main theme of the second part. As general constructions turn out to be quite involved, the focus is put on the class of étale groupoids. This reduces the amount of technicalities and, at the same time, gives access to a wealth of examples. A discussion on basic structural properties of groupoid  $C^*$ -algebras, in the light of amenability, is given, as well as applications of groupoids to the classification of Fell algebras.

The final part of the volume is devoted to the notion of Rokhlin dimension as introduced by I. Hirshberg, W. Winter and J. Zacharias, and its applications to crossed products. Connections with the notion of finite nuclear dimension and the classification program are also established. Special attention is devoted to the case of Rokhlin dimension for single automorphisms and the case for finite group

actions. Various generalizations of Rokhlin dimension are outlined towards the end, some of which sit at the edge of current research.

The Scientific Committee for this program consisted of Nathanial P. Brown (The Pennsylvania State University), Francesc Perera (Universitat Autònoma de Barcelona), Aidan Sims (University of Wollongong), Stuart White (University of Glasgow), and Wilhelm Winter (Münster Universität).

The said Intensive Research Program was made possible not only by the support of the CRM, but also from the Simons Foundation, the National Science Foundation, the Clay Mathematics Institute, by the Ministerio de Ciencia e Innovación (MTM2014-5364 4-P), and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.

Bellaterra, August 2018

Francesc Perera

**Part I**

# **A Primer on Crossed Products**

**Dana Williams**



# Introduction

This monograph is an expanded version of a lecture series I gave at the Centre de Recerca Matemàtica at the Universitat Autònoma de Barcelona during the week of March 13 to 17, 2017 as an “Advanced Course on Crossed Products”. The lectures were based largely on my 2007 text *Crossed Products of  $C^*$ -Algebras* [Wil07]. Hence the lectures and this monograph are meant to be a short introduction to the tools needed to work with crossed products in serious way. While the theory of crossed products in analysis can trace its origins back to statistical mechanics and the Murray–von Neumann group measure space construction, here we will simply require as a prerequisite the motivation and the desire to study interesting examples of  $C^*$ -algebras constructed from a given  $C^*$ -algebra  $A$  and a locally compact group  $G$  of automorphisms of  $A$ .

Nowadays, the subject of crossed products is much too vast to be profitably discussed in a single volume – let alone in a week long lecture series. Hence I have used my own interests as a guide and been limited by my own ignorance. I have discussed other primary sources and other flavors of the subject in Section 6.

As a result, these notes, just as is [Wil07], are primarily motivated by the representation theory of crossed products. Since the representation theory of non-type I  $C^*$ -algebras is intractable, the focus shifts to what I call the fine ideal structure of crossed products as reflected in the primitive ideal space together with its Jacobson topology. In the basic case where  $B = C(X)$ , the primitive ideal space can be canonically identified with  $X$  as a topological space and this completely characterizes  $B$ . In general,  $\text{Prim } B$  is a modest invariant but nevertheless contains a great deal of structural information. For example, the lattice of ideals in  $B$  is reflected exactly by the lattice of open sets in  $\text{Prim } B$ . (Hence the term, “ideal structure”.)

Our primary tools for studying primitive ideals in the crossed product  $B = A \rtimes_{\alpha} G$  is the process of inducing from subsystems  $A \rtimes_{\alpha} G_P$  associated to stability groups  $G_P$  for the natural action of  $G$  on  $\text{Prim } A$ , and Morita equivalence in the form of what we will call “imprimitivity theorems”. Hence we will need to spend some time considering the theory of induced representations of crossed products as well as the associated Morita equivalence results.

I have tried to keep the required background to a minimum. For the details and proper references, I have mostly referred to the corresponding result in [Wil07] and sometimes [RW98].

Please keep in mind that it is a running assumption that every  $C^*$ -algebra that can be is separable. (We have to allow that  $B(\mathcal{H})$  and multiplier algebras will not be separable in the best of circumstances.) Similarly, topological spaces are assumed to be second countable unless they obviously are not. Homomorphisms between  $C^*$ -algebras are assumed to be  $*$ -preserving, and representations of  $C^*$ -algebras are presumed to be nondegenerate.

## Errata

Since [Wil07] is extensively referenced here, any reader is advised to print out a copy of its errata <https://math.dartmouth.edu/cpcsa> and tuck it inside the cover. The length of that errata page only makes it more obvious that there will be more than the occasional typo, and perhaps thinko, in this document. Once I am aware of them, I will update the list at <https://math.dartmouth.edu/2017crm/>.



# Chapter 1

## What is a Crossed Product?

### 1.1 Dynamical systems

As defined in [Wil07], a  $C^*$ -Dynamical System  $(A, G, \alpha)$  is a  $C^*$ -algebra  $A$  together with a locally compact group  $G$  and a homomorphism  $\alpha : G \rightarrow \text{Aut } A$ . If  $G$  is not discrete, then we require that  $s \mapsto \alpha_s(a)$  is continuous for each  $a \in A$ . Since we are all operator folks here, we will drop the “ $C^*$ –” in the sequel.

Dynamical systems have their origin in statistical mechanics. (Let  $G = \mathbf{R}$  and think of  $\alpha$  as the time evolution of the system.) Here we will simply agree that interest in dynamical systems and their associated  $C^*$ -algebras is a prerequisite for this course.

**Example 1.1.1** (Single Automorphisms). Let  $\phi \in \text{Aut } A$ . Then we can define  $\alpha : \mathbf{Z} \rightarrow \text{Aut } A$  by composition. That is,  $\alpha_n = \phi^n = \phi \circ \cdots \circ \phi$  and  $\alpha_{-n} = (\phi^{-1})^n$  if  $n \geq 1$ . The continuity condition is vacuous here as  $\mathbf{Z}$  is discrete.

A very significant class of examples arises from a group  $G$  acting on  $X$ . Recall that  $G$  acts continuously on  $X$  if the map  $(s, x) \mapsto s \cdot x$  is continuous from  $G \times X$  to  $X$ . The pair  $(G, X)$  is often called a *transformation group*.

**Lemma 1.1.2.** *If  $(G, X)$  is a transformation group, we can define  $\text{lt}_s \in \text{Aut } C_0(X)$  by  $\text{lt}_s(f)(x) := f(s^{-1} \cdot x)$ . It follows that  $(C_0(X), G, \text{lt})$  is a dynamical system.*

The proof boils down to checking that  $s \mapsto \alpha_s(f)$  is continuous from  $G$  into  $C_0(X)$ . The details are worked out in [Wil07, Lemma 2.5].

With a bit more work we can show that all dynamical systems with  $A$  commutative arise from transformation groups.

**Proposition 1.1.3.** *Suppose that  $(C_0(X), G, \alpha)$  is a dynamical system. Then there is an action of  $G$  on  $X$  so that  $\alpha = \text{lt}$ ; that is  $\alpha_s(f)(x) = f(s^{-1} \cdot x)$ .*

The key to this result is the observation that if  $\alpha \in \text{Aut } C_0(X)$ , then there is a homeomorphism  $h \in \text{Homeo}(X)$  such that  $\alpha(f)(x) = f(h(x))$ . A proof is provided

in [Wil07, Proposition 2.7]. For this reason, we almost always write  $\text{lt}$  in place of  $\alpha$  when working with dynamical systems with  $A = C_0(X)$ .

### 1.1.1 The induced action on $\text{Prim } A$

The existence of an underlying group action on the maximal ideal space of  $A$  in Proposition 1.1.3 has an important generalization. If  $A = C_0(X)$  is a commutative  $C^*$ -algebra, then the space  $X$  arises as the spectrum or primitive ideal space of  $A$ . Recall that if  $A$  is a possibly noncommutative  $C^*$ -algebra, then a (two-sided closed) ideal  $P \in \mathcal{I}(A)$  is called *primitive* if it is the kernel of an irreducible representation. In a separable  $C^*$ -algebra, these are exactly the prime ideals (see [RW98, Proposition A.19 and A.49]). The set  $\text{Prim } A$  of primitive ideals in  $A$  has a natural topology called the *hull-kernel* or *Jacobson* topology. If  $F \subset \text{Prim } A$ , then  $\overline{F} = \{P \in \text{Prim } A : P \supset \bigcap_{K \in F} K\}$ .

On the other hand, the *spectrum* of  $A$  is the collection  $\hat{A}$  of unitary equivalence classes  $[\pi]$  of irreducible representations  $\pi$  of  $A$ . The map  $[\pi] \mapsto \ker \pi$  is a well-defined surjection. We get a topology on  $\hat{A}$  by pulling back the Jacobson topology. I recommend [RW98, §A.2] for an elementary review of the spectrum and primitive ideal space.

To get back on point, if  $A = C_0(X)$ , then basic spectral theory shows that every irreducible representation of  $C_0(X)$  is given by a point evaluation  $\text{ev}_x$ :  $\text{ev}_x(f) = f(x)$  [RW98, Example A.19]. Hence we can identify both  $\hat{A}$  and  $\text{Prim } A$  with  $X$ :  $x \mapsto \{f \in C_0(X) : f(x) = 0\}$ . An application of the Stone–Weierstrass Theorem shows that this identification is a homeomorphism. Hence the next result is a generalization of Proposition 1.1.3.

**Proposition 1.1.4.** *Let  $(A, G, \alpha)$  be a dynamical system. Then  $G$  acts continuously on  $\text{Prim } A$  via  $s \cdot P := \alpha_s(P)$ . Alternatively,  $G$  acts continuously on  $\hat{A}$  via  $s \cdot [\pi] = [\pi \circ \alpha_s^{-1}]$ .*

This result is not immediate and requires that we realize  $\text{Prim } A$  as a quotient of the set of pure states on  $A$ . For this, I suggest looking at [RW98, Lemma 7.1].

## 1.2 Covariant representations

As operator algebraists, we like to put things on Hilbert space. Given a  $C^*$ -algebra, we find a representation  $\pi : A \rightarrow B(\mathcal{H}_\pi)$  which we can require to be faithful if we wish. If we have an automorphism  $\phi \in \text{Aut } A$ , then we could wish that it be implemented by a unitary  $U \in B(\mathcal{H}_\pi)$  so that  $U\pi(a)U^* = \pi(\phi(a))$ . This kind of wishing leads naturally to the following definition.

**Definition 1.2.1.** A *covariant representation* of  $(A, G, \alpha)$  on a Hilbert space  $\mathcal{H}$  is a pair  $(\pi, U)$  consisting of a representation  $\pi : A \rightarrow B(\mathcal{H})$  and a unitary representation  $U : G \rightarrow U(\mathcal{H})$  such that  $\pi(\alpha_s(a)) = U_s\pi(a)U_s^*$  for all  $a \in A$  and  $s \in G$ .



A reasonable person could be forgiven for guessing that it might not be possible to always find covariant representations, or for suspecting that they might be very rare. Nevertheless, the next example provides a large class of examples. First recall that the algebraic tensor product  $\mathcal{H} \odot \mathcal{V}$  of Hilbert spaces is naturally an inner-product space [RW98, Lemma 2.59]. However, unless one of the spaces is finite-dimensional,  $\mathcal{H} \odot \mathcal{V}$  is unlikely to be complete. The Hilbert space completion is what we denote by  $\mathcal{H} \otimes \mathcal{V}$ . It is usually enough to work with the dense subspace  $\mathcal{H} \odot \mathcal{V}$ .

Here we want to look at the case of  $L^2(G) \otimes \mathcal{H}$ . We can view  $L^2(G) \odot \mathcal{H}$  as almost everywhere equivalence classes of functions from  $G$  to  $\mathcal{H}$  of the form  $s \mapsto \sum_{i=1}^n f_i(s)h_i$  with each  $f_i$  a square integrable Borel function, and each  $h_i \in \mathcal{H}$ . It is not a great leap of faith to think of  $L^2(G) \otimes \mathcal{H}$  as almost everywhere equivalence classes of Borel square integrable  $\mathcal{H}$ -valued functions on  $G$ . The details are worked out in [RW98, Example 2.62]. For this reason, here, I will often write  $L^2(G, \mathcal{H})$  in place of  $L^2(G) \otimes \mathcal{H}$ . (In practice, we can usually work with  $C_c(G) \odot \mathcal{H}$  or  $C_c(G, \mathcal{H})$  as both are dense in  $L^2(G) \otimes \mathcal{H}$ .)

**Example 1.2.2** (Regular representations). Let  $\rho : A \rightarrow B(\mathcal{H}_\rho)$  be any representation of  $A$ . Let  $\mathcal{H} = L^2(G, \mathcal{H}_\rho)$ . Define  $\text{Ind } \rho := (\tilde{\rho}, \lambda)$  on  $\mathcal{H}$  by  $\tilde{\rho}(a)h(s) = \rho(\alpha_s^{-1}(a))h(s)$  and  $\lambda_r h(s) = h(r^{-1}s)$ .

It is straightforward to check that  $\text{Ind } \rho$  is covariant ([Wil07, Example 2.14]). We call  $\text{Ind } \rho$  the *regular representation* associated to  $\rho$ . The notation  $\text{Ind } \rho$  will be explained in due course.

**Example 1.2.3** (A Basic Regular Representation). Let  $\phi \in \text{Aut } A$  be of period three so that we get a dynamical system  $(A, \mathbf{Z}_3, \alpha)$ . We can assume  $A \subset B(\mathcal{H})$  and let  $\rho$  be the identity representation. To make the formulas prettier, let  $\phi = \alpha_1^{-1}$ . Then  $\text{Ind id}$  is equivalent to the covariant representation  $(\pi, U)$  on  $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$  given by

$$\pi(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & \phi(a) & 0 \\ 0 & 0 & \phi^2(a) \end{pmatrix} \quad \text{and} \quad U_n = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n.$$

## 1.3 The universal crossed product

To get a  $C^*$ -algebra from a dynamical system, the first step is to make  $C_c(G, A)$  into a  $*$ -algebra. For motivation, we start with  $A = \mathbf{C}$  and the convolution product on  $C_c(G)$ . If  $G = \mathbf{R}$ , we are all familiar with the convolution product on  $L^1(\mathbf{R})$ :

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y) dy. \quad (1.1)$$

In order that this be a binary operation on  $L^1(\mathbf{R})$  and that it be associative, we require the invariance of Lebesgue measure under translation.

If we want an analogue of (1.1) on  $C_c(G)$ , this suggests we need a Radon measure on  $G$  that is left-invariant so that  $\mu(rE) = \mu(E)$  for all Borel sets  $E$  and  $r \in G$ . (For more details, see [Wil07, §1.3].) For us, left invariance amounts to formulas such as

$$\int_G f(rs) d\mu(s) = \int_G f(s) d\mu(s) \quad \text{provided } f \in C_c(G) \text{ and } r \in G.$$

Every locally compact group  $G$  has such a measure – called a *Haar measure* – which is unique up to multiplication by a positive constant [Wil07, Theorem 1.57]. Although we will try to ignore it in these notes, we have to live with the fact that there are groups whose Haar measures are not also right-invariant. However, if  $\mu$  is a Haar measure, so is  $\mu \cdot r$  where  $\mu \cdot r(E) = \mu(Er)$ . Hence there is a constant  $\Delta(r) \in (0, \infty)$  such that  $\mu \cdot r = \Delta(r)\mu$ . We call  $\Delta$  the *modular function* associated to  $\mu$  and note that  $\Delta : G \rightarrow (0, \infty)$  is a continuous homomorphism (see [Wil07, Lemma 1.61]).

All we care about is that a Haar measure exists. Other than that, the subject is (almost) devoid of measure theory. If  $G$  is finite or even discrete, then a Haar measure is just a counting measure. Then, if  $f \in C_c(G)$ ,  $f$  is finitely supported and

$$\int_G f(s) d\mu(s) = \sum_{s \in G} f(s).$$

If  $G = \mathbf{R}^n \times \mathbf{T}^m \times \mathbf{Z}^k \times F$  – where  $F$  is a finite abelian group – then Haar measure is exactly what you think it is. (Such groups are called *elementary abelian Lie groups*.)

Note that Haar measures on discrete groups and abelian groups are bi-invariant. Note that a Haar measure is bi-invariant exactly when the modular function is always identically one. It is not hard to see that compact groups are unimodular, and it is known that large classes of Lie groups – such as connected reductive and nilpotent – are unimodular.

**Example 1.3.1** ([Wil07, Example 1.60]). A simple example where a Haar measure is not bi-invariant is given by the so-called “ax+b” group  $G = \{ (a, b) \in \mathbf{R}^2 : a > 0 \}$  thought of as linear maps from  $\mathbf{R}$  to itself:  $x \mapsto ax+b$ . Then  $(a, b)(c, d) = (ac, ad+b)$  and  $(a, b)^{-1} = (\frac{1}{a}, -\frac{b}{a})$ . You can work out for yourself that a Haar measure on  $G$  is given by

$$\mu(f) = \int_{-\infty}^{\infty} \int_0^{\infty} f(x, y) \frac{1}{x^2} dx dy,$$

and that  $\Delta(a, b) = \frac{1}{a}$ .

### 1.3.1 Vector-valued integrals

Another issue is to make sense of integrals

$$\int_G f(s) d\mu(s) \tag{1.2}$$

where  $f$  is  $A$ -valued and  $f \in C_c(G, A)$ . The details are worked out in [Wil07, §1.5]. (A more general theory for suitably measurable functions is discussed in [Wil07, Appendix B], but we almost never need that level of generality.) For example, if  $G$  is finite or discrete, then  $f \in C_c(G, A)$  is finite valued, and (1.2) is just the finite sum

$$\sum_{s \in G} f(s) \in A.$$

If  $G = \mathbf{R}$  or  $\mathbf{T}$  (or even  $\mathbf{R}^n \times \mathbf{T}^m$ ), then  $f$  is compactly supported we can form the analogue of a Riemann sum for (1.2). Letting the mesh go to zero, we obtain Cauchy sequences with a unique limit we can define to be the integral. Should you be silly enough to look at more general groups, then we take it on faith that (1.2) defines an element of  $a_f \in A$  such that if  $\pi : A \rightarrow B(\mathcal{H}_\pi)$  is any representation of  $A$  then

$$\left( \pi \left( \int_G f(s) d\mu(s) \right) h \mid k \right) = \left( \pi(a_f) h \mid k \right) = \int_G \left( \pi(f(s)) h \mid k \right) d\mu(s).$$

The philosophy here is to treat vector-valued integrals like scalar-valued ones. The additional properties follow naturally if one keeps the discrete case in mind. For example, if  $L : A \rightarrow B$  is a linear map between  $C^*$ -algebras, then

$$L \left( \int_G f(s) d\mu(s) \right) = \int_G L(f(s)) d\mu(s).$$

If  $a, b \in A$  or even  $M(A)$ , then

$$a \int_G f(s) d\mu(s) b = \int_G a f(s) b d\mu(s).$$

In the case where  $A = C_0(X)$ , recall that  $\text{ev}_x : C_0(X) \rightarrow \mathbf{C}$  – given by evaluation,  $\text{ev}_x(\phi) = \phi(x)$  – is a bounded linear map (even a homomorphism). Hence

$$\int_G f(s) d\mu(s)(x) = \int_G f(s)(x) d\mu(s) \quad \text{if } f \in C_c(G, C_0(X)).$$

More to the point, if  $F \in C_c(G \times X)$  viewed as a subset of  $C_c(G, C_0(X))$ , then

$$\int_G F(s, \cdot) d\mu(s)(x) = \int_G F(s, x) d\mu(s).$$

### 1.3.2 A $*$ -algebra

We use our Haar measure to define a convolution on  $C_c(G, A)$  that generalizes the group case. The key is to include the  $G$ -action on  $A$ :

$$f * g(s) = \int_G f(r) \alpha_r(g(r^{-1}s)) d\mu(r).$$

This makes sense as  $r \mapsto f(r) \alpha_r(g(r^{-1}s))$  is in  $C_c(G)$  for each  $s$ . Showing that  $f * g \in C_c(G, A)$  takes only a bit of work (see [Wil07, Corollary 1.104]). The

involution is straightforward: we replace complex conjugation with the adjoint so that  $f^*(s) = \Delta(s^{-1})f(s^{-1})^*$ .<sup>1</sup> For the moment, I will put the modular functions in the formulas for reference. But we really should be ignoring them on a first pass.

**Example 1.3.2.** If  $A = C_0(X)$ , then we can dispense with vector-valued integrals completely by recalling as above that evaluation at  $x$  is a bounded linear map on  $C_0(X)$ . To further simplify things, we note that while in general,

$$C_c(G \times X) \not\subseteq C_c(G, C_c(X)) \not\subseteq C_c(G, C_0(X)),$$

$C_c(G \times X)$  is suitably dense in  $C_c(G, C_0(X))$ . Then convolution is given on  $C_c(G \times X)$  by

$$f * g(s, x) = \int_G f(r, x) g(r^{-1} \cdot s, r^{-1} \cdot x) d\mu(r),$$

and the involution is  $f^*(s, x) = \Delta(s^{-1})f(s^{-1}, s^{-1} \cdot x)$ .

If you are the careful sort, whether working with transformation groups or the general case, you will want to check that convolution is associative and that our  $*$ -operation is an involution: that is,  $(f * g) * h = f * (g * h)$  and  $(f * g)^* = g^* * f^*$ . For example if  $G$  is discrete, then associativity follows from

$$(f * g) * h(s) = \sum_{r,t} f(t) \alpha_t(g(t^{-1}r)) \alpha_r(h(r^{-1}s))$$

which, replacing  $r$  by  $tr$ , is

$$\begin{aligned} &= \sum_t f(t) \alpha_t \left( \sum_r g(r) \alpha_r(h(r^{-1}t^{-1}s)) \right) \\ &= \sum_t f(t) \alpha_t(g * h(t^{-1}s)) \\ &= f * (g * h)(s). \end{aligned}$$

For more details see [Wil07, §2.3].

### 1.3.3 From covariant representation to representation

Suppose that  $(\pi, U)$  is a covariant representation of  $(A, G, \alpha)$  on  $\mathcal{H}$  and that  $f \in C_c(G, A)$ . If  $G$  is discrete, then  $s \mapsto \pi(f(s))U_s$  is in  $C_c(G, B(\mathcal{H}))$  and we can make sense out of

$$\pi \rtimes U(f) := \int_G \pi(f(s))U_s d\mu(s) = \sum_{s \in G} \pi(f(s))U_s \in B(\mathcal{H}). \quad (1.3)$$

---

<sup>1</sup>The modular function is there only so that the involution is  $\|\cdot\|_1$ -norm preserving. Groupoid aficionados might point out that the modular function is not necessary here. But if we leave it out, it reappears in (1.3). It is traditional to include it here since we are only concerned with group actions.

We call  $\pi \rtimes U$  the *integrated form* of  $(\pi, U)$ . It is not hard to verify that  $\pi \rtimes U$  defines a  $*$ -homomorphism of  $C_c(G, A)$  into  $B(\mathcal{H})$ .

If  $G$  is not discrete, then  $s \mapsto \pi(f(s))U_s$  is likely not in  $C_c(G, B(\mathcal{H}))$ . But it is continuous with compact support into  $B(\mathcal{H})$  equipped with the strict topology (viewed as the multiplier algebra of  $\mathcal{K}(\mathcal{H})$ ). This is close enough that we can still make sense out of

$$\pi \rtimes U(f) := \int_G \pi(f(s))U_s d\mu(s)$$

as an element of  $B(\mathcal{H})$ . The details are in and around [Wil07, Lemma 1.101]. The key thing is that if  $h, k \in \mathcal{H}$ , then

$$(\pi \rtimes U(f)h | k) = \int_G (\pi(f(s))U_s h | k) d\mu(s). \quad (1.4)$$

Using (1.4), it is not hard to check that, as an operator on  $B(\mathcal{H})$ ,

$$\|\pi \rtimes U(f)\| \leq \|f\|_1 := \int_G \|f(s)\| d\mu(s),$$

and that as in the discrete case,  $\pi \rtimes U$  defines a  $*$ -homomorphism of  $C_c(G, A)$  into  $B(\mathcal{H})$ . Just as above, we call  $\pi \rtimes U$  the integrated form of  $(\pi, U)$ .

### 1.3.4 The universal norm

Now that we have a  $*$ -algebra structure on  $C_c(G, A)$ , we want a norm. Just as in Harmonic analysis, the  $\|\cdot\|_1$ -norm is not satisfactory as it does not satisfy the  $C^*$ -norm identity. However, if  $(\pi, U)$  is a covariant representation, then  $\|f\|_{(\pi, U)} := \|\pi \rtimes U(f)\|$  is a seminorm on  $C_c(G, A)$  that does satisfy the  $C^*$ -norm identity. Motivated by this, if  $f \in C_c(G, A)$ , then we define its *universal norm* by

$$\|f\| := \sup \{ \|\pi \rtimes U(f)\| : (\pi, U) \text{ is covariant} \}.$$

Even though the supremum is taken over a class rather than a set, there is no real problem. (For details, see [Wil07, Remark 2.28] the upshot of which is that you should not worry about it. Basically, a subclass of a set is a set.) Using regular representations, we see that the universal norm is a norm. Moreover,  $\|f^* * f\| = \|f\|^2$ . Hence the completion, denoted by  $A \rtimes_\alpha G$  or  $C^*(G, A, \alpha)$  is a  $C^*$ -algebra called the *crossed product* of  $A$  by  $G$ . (Recall that if  $X$  is a normed vector space, we obtain its completion as the closure of its isometric image  $\iota(X)$  in  $X^{**}$ .)

### 1.3.5 A first look at $A \rtimes_\alpha G$

Naïvely,  $A \rtimes_\alpha G$  is generated by a “covariant copy” of  $A$  and  $G$ . If  $1 \in A$  and  $G$  is discrete, define  $\iota_G : G \rightarrow C_c(G, A)$  and  $\iota_A : A \rightarrow C_c(G, A)$  by  $\iota_G(s) = 1_A \cdot \delta_s$  and  $\iota_A(a) = a \cdot \delta_e$ . Clearly  $C_c(G, A) \subset A \rtimes_\alpha G$  is generated by the ranges of  $\iota_G$  and  $\iota_A$ . Moreover,  $(\iota_A, \iota_G)$  is covariant:  $\iota_G(s) * \iota_A(a) * \iota_G(s^{-1}) = \iota_A(\alpha_s(a))$ . In

general, we have similar maps  $\iota_G : G \rightarrow UM(A \rtimes_\alpha G)$  and  $\iota_A : A \rightarrow M(A \rtimes_\alpha G)$  such that  $\text{span}\{\iota_G(\phi)\iota_G(a) : \phi \in C_c(G) \text{ and } a \in A\}$  is dense in  $A \rtimes_\alpha G$ , and such that  $\iota_G(s)\iota_A(a)\iota_G(s^{-1}) = \iota_A(\alpha_s(a))$ . The details are worked out in [Wil07, Proposition 2.34 and Lemma 2.35]. If  $f \in C_c(G, A)$ , then  $\iota_G(s)(f)(r) = \alpha_s(f(s^{-1}r))$  and  $\iota_A(a)(f)(r) = af(r)$ .

The essential feature of the crossed product is that it is universal with respect to covariant representations. This statement is made considerably more formal in [Wil07, Theorem 2.61]. The version we are stressing here is as follows.

**Theorem 1.3.3** ([Wil07, Propositions 2.39 and 2.40]). *The map  $(\pi, U) \mapsto \pi \rtimes U$  is a bijection from the class of covariant representations of  $(A, G, \alpha)$  onto the class of representations of  $A \rtimes_\alpha G$ . The map preserves equivalence, direct sums and irreducibility.*

*Proof.* Let  $L : A \rtimes_\alpha G \rightarrow B(\mathcal{H})$  be a representation. Define  $\pi := \bar{L} \circ \iota_A$  and  $U = \bar{L} \circ \iota_G$  where  $\bar{L}$  is the canonical extension of  $L$  to  $M(A \rtimes_\alpha G)$ . Check that  $L = \pi \rtimes U$ .  $\square$

**Example 1.3.4.** Let  $\phi \in \text{Aut } A$ . Let  $\rho : A \rightarrow B(\mathcal{H}_\rho)$  be any representation of  $A$  such that there is a unitary  $U \in U(\mathcal{H}_\rho)$  such that  $U\rho(a)U^* = \rho(\phi(a))$ . Then there is a homomorphism  $j : A \rtimes_\phi \mathbf{Z} \rightarrow C^*(\{\rho(A), U\})$  such that  $\bar{j}(\iota_A(a)) = \rho(a)$  and  $\bar{j}(\iota_G(1)) = U$ .

**Example 1.3.5** ([Wil07, Lemma 2.73]). Suppose that  $\alpha : G \rightarrow \text{Aut } A$  is trivial:  $\alpha_s = \text{id}_A$  for all  $A$ . Then covariant representations of  $(A, G, \alpha)$  correspond exactly to commuting pairs of representations of  $A$  and  $C^*(G)$ . The universal properties of the crossed product and maximal tensor product imply that  $A \rtimes_\alpha G \cong A \otimes_{\max} C^*(G)$ .

**Remark 1.3.6.** As an aside we note that the same conclusion holds if  $u : G \rightarrow U(A)$  is a strictly continuous homomorphism such that  $\alpha_s(a) = u_s a u_s^*$ . We say that  $\alpha$  is *unitarily implemented* in this case. More generally, two  $G$  systems  $(A, G, \alpha)$  and  $(A, G, \beta)$  are said to be *exterior equivalent* if there is a strictly continuous  $u : G \rightarrow UM(A)$  such that

- (a) for all  $a \in A$  and all  $s \in G$ , we have  $\alpha_s(a) = u_s \beta_s(a) u_s^*$ , and
- (b) for all  $s, t \in G$ ,  $u_{st} = u_s \bar{\beta}_s(u_t)$ .

We often call  $u$  a *unitary 1-cocycle*. If  $(A, G, \alpha)$  and  $(A, G, \beta)$  are exterior equivalent, then the map sending  $f \in C_c(G, A)$  to  $(s \mapsto f(s)u_s)$  extends to an isomorphism of  $A \rtimes_\alpha G$  onto  $A \rtimes_\beta G$  by [Wil07, Lemma 2.68].

**Remark 1.3.7.** If  $(C_0(X), G, \text{lt})$  is a dynamical system, then we can form a locally compact groupoid  $\underline{G} = G \times X$  with unit space  $G^{(0)}$  equal to  $\{e\} \times X$  (which we identify with  $X$ ). The range map is given by  $r(g, x) = x$  and the source map by  $s(g, x) = g^{-1} \cdot x$ . Then multiplication is given by  $(g, x)(h, g^{-1} \cdot x) = (gh, x)$  and  $(g, x)^{-1} = (g^{-1}, g^{-1} \cdot x)$ . The groupoid  $\underline{G}$  is étale if and only if  $G$  is discrete. The corresponding groupoid  $C^*$ -algebra  $C^*(\underline{G})$ , as defined in A. Sims' lecture series [Sim17] in this volume (pp. 59–120), is naturally isomorphic to the crossed product  $C_0(X) \rtimes_{\text{lt}} G$ .

## 1.4 Examples

Unfortunately, it is not so easy to provide nontrivial examples – especially with no technology at hand. Part of the explanation for this is that even with  $A = \mathbf{C}$  – and  $\alpha$  is necessarily trivial – we get the group  $C^*$ -algebra  $\mathbf{C} \rtimes G := C^*(G)$ . Those of you familiar with noncommutative harmonic analysis will know that, with the exception of abelian and compact groups, these algebras are notoriously hard to compute.

### 1.4.1 Abelian groups

If  $f \in C_c(G)$ , then its universal norm in the group  $C^*$ -algebra is  $\|f\| = \sup\{\|U(f)\| : U \text{ is unitary}\}$ . (We have adopted the annoying but widespread convention of using  $U$  to denote both a unitary representation of  $G$  as well as its integrated form on  $C_c(G)$  or  $C^*(G)$ .) General  $C^*$ -nonsense says we can take attain the supremum by restricting to irreducible  $U$ . Hence if  $G$  is abelian, so that its irreducible representations  $\omega \in \widehat{G}$  are all characters, then

$$\|f\| = \sup_{\omega \in \widehat{G}} \left| \int_G f(s) \omega(s) d\mu(s) \right| = \sup_{\omega \in \widehat{G}} |\hat{f}(\omega)|,$$

where  $\hat{f}$  is the Fourier transform of  $f$ . Hence  $\|f\| = \|\hat{f}\|_\infty$ . It follows that we can identify  $C^*(G)$  with  $C_0(\widehat{G})$  (see [Wil07, §2.5]).

### 1.4.2 Finite group actions

Finite group actions are discussed at the beginning of [Wil07, §2.5]. The key idea is that if  $C(G, A)$  admits a complete  $C^*$ -norm, then it has to be the universal norm since all representations of  $C(G, A)$  would have to be norm reducing.

**Example 1.4.1.** Let  $G = \mathbf{Z}_2$  so that any  $\alpha : \mathbf{Z}_2 \rightarrow \text{Aut } A$  is given by a period two automorphism  $\phi$ . Let

$$D = \left\{ \begin{pmatrix} a & b \\ \phi(b) & \phi(a) \end{pmatrix} : a, b \in A \right\}.$$

Then  $D$  is a  $C^*$ -subalgebra of  $M_2(A)$ . The map  $\Phi : C(\mathbf{Z}_2, A) \rightarrow D$  given by

$$\Phi(f) = \begin{pmatrix} f(0) & f(1) \\ \phi(f(1)) & \phi(f(0)) \end{pmatrix}$$

is a  $*$ -isomorphism onto  $D$ . This equips  $C(\mathbf{Z}_2, A)$  with a complete  $C^*$ -norm which has to be the universal norm (since every representation is norm decreasing). Hence  $A \rtimes_\alpha \mathbf{Z}_2 \cong D$ .

Now plug in your favorite finite group and construct  $D$ . For more on finite group actions, see [Phi17, §12].

In the previous example suppose that  $A = C(X)$  for a compact Hausdorff space and that  $\phi$  is given by a period two homeomorphism  $\sigma$  without fixed points. (For example, we could have  $X = S^n = \{\mathbf{x} \in \mathbf{R}^{n+1} : \|\mathbf{x}\| = 1\}$  and  $\sigma(\mathbf{x}) = -\mathbf{x}$ .) As in the previous example,  $C(X) \rtimes_{\text{lt}} \mathbf{Z}_2 \cong C(\mathbf{Z}_2 \times X)$ , and

$$\Phi(f)(x) = \begin{pmatrix} f(0, x) & f(1, x) \\ f(1, \sigma(x)) & f(0, \sigma(x)) \end{pmatrix}$$

is a  $*$ -isomorphism of  $C(X) \rtimes_{\text{lt}} \mathbf{Z}_2$  onto the 2-homogeneous  $C^*$ -algebra<sup>2</sup> with spectrum  $\mathbf{Z}_2 \backslash X$  given by

$$A = \{f \in C(X, M_2) : f(\sigma(x)) = Wf(x)W^*\}$$

where  $W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

Note that  $A$  is the section algebra of a locally trivial  $M_2$ -bundle over  $\mathbf{Z}_2 \backslash X$  which need not be globally trivial. For example in the case of the antipodal map on  $S^n$ , we get a nontrivial  $M_2$ -bundle over  $\mathbf{R}P_n$ . For more details, see [Wil07, Example 2.52].

**Remark 1.4.2.** For the sake of motivation, we will look ahead a bit. It turns out that the lack of global triviality in the previous example is more a symptom of the finiteness of  $G$  than anything else. Recall that  $G$  acts properly on  $P$  if the map  $(s, p) \mapsto (s \cdot p, p)$  is proper from  $G \times P$  to  $P \times P$ . (This is automatic if  $G$  is finite or compact.) If in addition the action is free, then we will see later that  $C_0(P) \rtimes_{\text{lt}} G$  is isomorphic to the generalized compacts on a Hilbert  $C_0(G \backslash X)$ -module  $X$ . If the orbit map  $q : P \rightarrow G \backslash P$  has local sections or  $G \backslash P$  has finite covering dimension, it can be shown that  $X$  is the section algebra of a locally trivial Hilbert space bundle with fibres  $L^2(G)$  (see [Gre77, Corollary 15]). If  $G$  is infinite, such a bundle must be trivial [RW98, Corollary 4.79].<sup>3</sup> Hence  $C_0(P) \rtimes_{\text{lt}} G \cong C_0(G \backslash P, \mathcal{K}(L^2(G)))$  in such cases.

**Example 1.4.3.** Let  $G$  act on itself by left translation. Then we get a covariant representation  $(M, \lambda)$  on  $L^2(G)$  given by  $M(f)h(s) = f(s)h(s)$  and  $\lambda_r h(s) = h(r^{-1}s)$ . If  $F \in C_c(G \times G)$ , then

$$\begin{aligned} M \rtimes \lambda(F)h(s) &= \int_G F(r, s)h(r^{-1}s) d\mu(r) \\ &= \int_G f(sr^{-1}, s)\Delta(r^{-1})h(r) d\mu(r) \\ &= \int_G K(s, r)h(r) d\mu(r), \end{aligned}$$

<sup>2</sup>A  $C^*$ -algebra is called  $n$ -homogeneous if all its irreducible representations are  $n$ -dimensional. These turn out to be the section algebras of fibre bundles with fibre  $M_n$  and structure group  $PU(n)$ . See [RW98, §4.91] for an explanation of this.

<sup>3</sup>The key fact is that the unitary group in  $B(\mathcal{H})$  is contractible only if  $\mathcal{H}$  is infinite-dimensional.



where  $K(s, r) = f(sr^{-1}, s)\Delta(r^{-1})$ . Hence  $M \rtimes \lambda(F)$  is a Hilbert–Schmidt operator. In fact,  $M \rtimes \lambda(C_c(G \times G))$  is dense in  $\mathcal{K}(L^2(G))$ . This provides part of the proof of our next result.

**Theorem 1.4.4** (Stone–von Neumann Theorem). *Suppose that  $G$  is a locally compact group. Then  $C_0(G) \rtimes_{\text{lt}} G$  is isomorphic to  $\mathcal{K}(L^2(G))$ . In particular, the representation  $M \rtimes \lambda$  in the previous example is an isomorphism.*

*Proof.* If  $G$  is finite, then the result follows from the existence of a complete  $C^*$ -norm on  $C(G \times G)$ . In general we need to know that every covariant representation of  $(C_0(G), G, \text{lt})$  is a direct sum of copies of  $(M, \lambda)$  or use representation theory to show that  $C_0(G) \rtimes_{\text{lt}} G$  is simple. We will do the later in Corollary 2.3.4. (For the former, see [Rie72].) The full details are given in [Wil07, Theorem 4.24].  $\square$

Just for completeness, we will acknowledge the following result due to Green.

**Theorem 1.4.5.** *Let  $H$  be a closed subgroup of  $G$ . Then  $C_0(G/H) \rtimes_{\text{lt}} G$  is isomorphic to the tensor product  $C^*(H) \otimes \mathcal{K}(L^2(G/H))$ .*

Green’s original proof – as given in [Wil07, Theorem 4.30] for example – is not very constructive. A much more constructive proof using groupoid techniques has been given by Buneci [Bun01]. It is interesting that all these approaches require measurable cross sections and working outside of the continuous compactly supported functions we traditionally work with in this subject.

### 1.4.3 Dos Equis’ most interesting example in the world

Back in the day, the study of the rotation algebras was a very hot topic. Although they are fairly well understood today, that is only because of the hard work of a number of people. I believe they still serve as a great example to motivate that study of crossed products.

**Example 1.4.6.** Let  $\theta \in [0, 1]$  and define  $h \in \text{Homeo}(\mathbf{T})$  by  $h(z) = \rho z$  where  $\rho = e^{2\pi i \theta}$ . The associated crossed product  $A_\theta := C(\mathbf{T}) \rtimes_h \mathbf{Z}$  is called a *rotation algebra*. It is not hard to see that  $A_\theta$  is generated by two unitaries  $u, v \in C_c(\mathbf{Z} \times \mathbf{T}) \subset C(\mathbf{T}) \rtimes_h \mathbf{Z}$  given by  $u(n, z) = \delta_1(n)$  and  $v(n, z) = z\delta_0(n)$ . Note that  $u * v = \rho \cdot v * u$ .

In fact, the content of [Wil07, Proposition 2.56] is that  $A_\theta$  is the universal  $C^*$ -algebra generated by two unitaries such that

$$UV = \rho VU.$$

**Theorem 1.4.7.** *If  $U, V \in B(\mathcal{H})$  and  $UV = \rho VU$ , then there is a surjective homomorphism  $j : A_\theta \rightarrow C^*(\{U, V\})$  such that  $j(u) = U$  and  $j(v) = V$ .*

Since we trivially have  $C^*(\{U, V\}) = C^*(\{V, U\})$ , we clearly have  $A_\theta \cong A_{1-\theta}$ . In §5.5.1 we will say something about the proof that if  $\theta_i \in [0, \frac{1}{2})$ , then we have  $A_{\theta_1} \cong A_{\theta_2}$  if and only if  $\theta_1 = \theta_2$ . Furthermore, if  $\theta \notin \mathbf{Q}$ , then  $A_\theta$  is simple (see [Wil07, Proposition 2.56]).

## 1.5 The reduced crossed product

Since the crossed product is a universal object, it comes with nice functorial properties. But a careful analysis requires a knowledge of *all* covariant representations, and this is usually too big an ask. Just as in the group case, we look for a more concrete alternative. If  $\rho$  is a faithful representation of  $A$ , then  $\text{Ind } \rho$  is a faithful representation of  $C_c(G, A)$ . We define the reduced norm  $\|f\|_r$  to be the operator norm of  $\|\text{Ind } \rho(f)\|$  as an operator on  $L^2(G, \mathcal{H}_\rho)$ . Later, we will see that this turns out to be independent of our choice of  $\rho$ . We define the *reduced crossed product*,  $A \rtimes_{\alpha, r} G$  to be the completion of  $(C_c(G, A), \|\cdot\|_r)$  (see [Wil07, §7.2]). Then  $A \rtimes_{\alpha, r} G$  is isomorphic to the closure of  $\{\text{Ind } \phi(f) : f \in C_c(G, A)\}$  in  $B(L^2(G, \mathcal{H}_\rho))$  (for any faithful  $\rho$ ).



## Chapter 2

# Induced Representations and Imprimitivity Theorems

### 2.1 Hilbert modules and the Rieffel machine

Some knowledge of Hilbert modules and their adjointable operators is a prerequisite for this lecture series. Chapter two of [RW98] and/or a familiarity with Lance's book [Lan95] is plenty of background. Here we recall that a (right) Hilbert  $B$ -module is a  $B$ -module  $X$  equipped with a  $B$ -valued inner product which is linear in the second variable and satisfies  $\langle x, y \rangle_B^* = \langle y, x \rangle_B$ ,  $\langle x, y \cdot b \rangle_B = \langle x, y \rangle_B b$  and  $\langle x, x \rangle_B \geq 0$  in  $B$ . Of course we require  $X$  be complete with respect to the norm  $\|x\|_B = \|\langle x, x \rangle_B\|_B^{\frac{1}{2}}$ . An operator  $T : X \rightarrow X$  is called *adjointable* if there is an operator  $T^* : X \rightarrow X$  such that  $\langle T(x), y \rangle_B = \langle x, T^*(y) \rangle_B$  for all  $x, y \in X$ . The collection  $\mathcal{L}(X)$  of adjointable operators is a  $C^*$ -algebra with respect to the operator norm. Like  $B(\mathcal{H})$ ,  $\mathcal{L}(X)$  is often not separable even if  $X$  is. A more tractable object is the closed linear span of the “rank-one” operators  $\Theta_{x,y}$  – where  $\Theta_{x,y}(z) := x \cdot \langle y, z \rangle_B$ . This is a  $C^*$ -subalgebra  $\mathcal{K}(X)$  called the *generalized compact operators* on  $X$ . It is straightforward to see that  $\mathcal{K}(X)$  is an ideal in  $\mathcal{L}(X)$ .

Hilbert modules emerged in the literature via Hilbert  $C_0(T)$ -modules. For each  $t \in T$  we get a pre-inner product on  $X$  via  $(x | y) = \langle x, y \rangle_*(t)$ . The completion  $\mathcal{H}(t)$  is a Hilbert space and we can profitably view  $X$  as the  $C_0$ -sections of a continuous Hilbert bundle  $H = \coprod \mathcal{H}(t)$  with fibres  $\mathcal{H}(t)$ . (To actually topologize the bundle and make all this formal, see [KMRW98, §2].) Similarly, we view  $\mathcal{K}(X)$  as a field of  $C^*$ -algebras with fibres  $\mathcal{K}(\mathcal{H}(t))$ . If  $H$  is locally trivial, then  $\mathcal{K}(X)$  is locally of the form  $C_0(U, \mathcal{K}(\mathcal{H}))$ . Using the fact the unit ball of an infinite-dimensional separable Hilbert space is contractible, it turns out that if  $H$  is locally trivial with infinite-dimensional fibres, then  $H$  is trivial [RW98, Corollary 4.79].

## 2.2 Induced representations – the Rieffel machine

Rieffel’s approach to induced representations of  $C^*$ -algebras is treated in detail in [RW98, §2.4]. In particular, a Hilbert  $B$ -module  $X$  is called a *right Hilbert  $A$ – $B$ -bimodule* or a *correspondence from  $A$  to  $B$*  if there is a homomorphism  $\phi : A \rightarrow \mathcal{L}(X)$  which is *nondegenerate* in that  $\phi(A) \cdot X$  is dense in  $X$ . We write  $a \cdot x$  in place of  $\phi(a)(x)$  and view  $X$  as a bimodule. If  $\pi : B \rightarrow B(\mathcal{H}_\pi)$  is a representation of  $B$ , then it is shown in [RW98, Proposition 2.64] that we get a pre-inner product on the algebraic tensor product<sup>1</sup>  $X \otimes \mathcal{H}_\pi$  such that

$$(x \otimes h \mid y \otimes k) = (\pi(\langle y, x \rangle_B)h \mid k).$$

If we write  $X \otimes_B \mathcal{H}_\pi$  for the Hilbert space completion, then the representation of  $A$  induced from  $\pi - \text{Ind}_B^A \pi$  or  $X\text{-Ind } \pi$  acts on  $X \otimes_B \mathcal{H}_\pi$  via

$$\text{Ind}_B^A \pi(a)[x \otimes h] = [a \cdot x \otimes h].$$

The details are in [RW98, Proposition 2.66].

The great advantage of Rieffel’s bimodule approach is its algebraic simplicity. Nevertheless, it transpires that it is the “right thing” in all the classical situations.

**Example 2.2.1** ([RW98, Example 2.68]). Suppose that  $I \in \mathcal{I}(A)$  and view  $I$  as a Hilbert  $A$ – $I$ -bimodule. If  $\pi$  is a representation of  $I$ , then  $b \otimes h \mapsto \pi(b)h$  induces a unitary from  $I \otimes_I \mathcal{H}_\pi$  onto  $\mathcal{H}_\pi$  intertwining  $\text{Ind}_I^A \pi$  with the canonical extension  $\bar{\pi}$  of  $\pi$  to  $A$ :  $\bar{\pi}(a)(\pi(b)h) = \pi(ab)h$ .

### 2.2.1 The real example

The fundamental example comes from considering a closed subgroup  $H$  of a locally compact group  $G$ . The full details are worked out in [RW98, Appendices C.4 and C.5]. The idea is to equip  $C_c(G)$  with a  $C_c(H)$ -valued sesquilinear form which we view as a  $C^*(H)$ -valued pre-inner product. Then we complete as in [RW98, Lemma 2.16] to get a Hilbert  $C^*(H)$ -module  $X_H^G$ . So as not to obscure the exposition, we will assume both  $H$  and  $G$  are *unimodular*. (The formulas in [RW98, Appendix C] are appropriately decorated with modular functions for those interested.) Then our inner product is

$$\begin{aligned} \langle f, g \rangle_\star(h) &= f^* \star g(h) = \int_G \overline{f(s^{-1})} g(s^{-1}h) d\mu(s) \\ &= \int_G \overline{f(s)} g(sh) d\mu(s). \end{aligned}$$

<sup>1</sup>I generally reserve the symbol “ $\otimes$ ” for complete objects. When the mundane algebraic tensor product is used, then I use “ $\odot$ ”.

It is nontrivial to see that  $\langle f, f \rangle_\star \geq 0$  in  $C^*(H)$ .<sup>2</sup> But it is, and we can complete to form our  $C^*(H)$ -module  $\mathsf{X}_H^G$ . Then we get a homomorphism  $\phi : C^*(G) \rightarrow \mathcal{L}(\mathsf{X}_H^G)$ :  $\phi(f)([g]) = [f * g]$ . Now  $\mathsf{X}_H^G$  is a Hilbert  $C^*(G)$ – $C^*(H)$ -Hilbert bimodule and we have a Rieffel machine to induce representations from  $H$  to  $G$ .

## 2.2.2 The “right thing” for groups

What is impressive about Example 2.2.1 is that the Rieffel machine produces the “right thing” when  $H$  is a subgroup of  $G$  in the sense that we recover Mackey’s measure theoretic definition – without the measure theory. Here we are simplifying by taking  $H$  and  $G$  unimodular. For the full details, consult [RW98, Appendix C.5]. To begin with, there is a unique Radon measure  $\mu_{G/H}$  on  $G/H$  such that

$$\int_G f(s) d\mu(s) = \int_{G/H} \int_H f(sh) \mu_H(h) d\mu_{G/H}(sH).$$

If  $G$  is discrete, then counting measure will do. If  $H$  is normal in  $G$ , then a suitable multiple of Haar measure is the thing. Otherwise, we will take its existence on faith. Let  $u : H \rightarrow U(\mathcal{H}_u)$  be a unitary representation of  $H$ . Let  $\mathcal{V}_1$  be the bounded continuous functions from  $G$  to  $\mathcal{H}_u$  such that  $f(sh) = u_h^{-1}(f(s))$  for all  $s \in G$  and  $h \in H$ . Note that  $s \mapsto \|f(s)\|$  is constant on  $H$  cosets. Let  $\mathcal{V}_c$  be those  $f \in \mathcal{V}_1$  such that  $sH \mapsto \|f(s)\|$  is in  $C_c(G/H)$ . We can equip  $\mathcal{V}_c$  with an inner product given by

$$(f | g) := \int_{G/H} (f(s) | g(s)) d\mu_{G/H}(sH).$$

We denote the Hilbert space completion of  $\mathcal{V}_c$  by  $L_u^2(G, \mathcal{H}_u)$ . Although it suffices to work with the dense subspace  $\mathcal{V}_c$ , as the notation suggests, we can realize  $L_u^2(G, \mathcal{H}_u)$  as the collection of almost everywhere equivalence classes of Borel functions  $f : G \rightarrow \mathcal{H}_u$  such that  $f(sh) = u_h^{-1}(f(s))$  for all  $s \in G$  and  $h \in H$  that satisfy

$$\int_{G/H} \|f(s)\|^2 d\mu_{G/H}(sH) < \infty.$$

The details are similar to those in [Wil07, §I.4] and are formally contained in the discussion preceding [Wil07, Proposition 9.18].

In any event, we have the following.

**Theorem 2.2.2** ([RW98, Theorem C.33]). *If  $u$  is a representation of  $H$ , then Rieffel’s induced representation  $\text{Ind}_H^G u$  is equivalent to the usual Mackey induced representation  $U$  on  $L_u^2(G, \mathcal{H}_u)$  given by left-translation:  $U_r h(s) = h(r^{-1}s)$ .*

<sup>2</sup>In fact, in the general case if  $\Delta_G$  and  $\Delta_H$  are not equal on  $H$ , then this form is not even self-adjoint, let alone positive. In general, we have to define  $\langle f, g \rangle_\star(h) = \Delta_G(h)^{\frac{1}{2}} \Delta_H(s)^{-\frac{1}{2}} f^* * g(h)$  which seems like more than enough reason to work in the unimodular case here.

*Proof.* The unitary from  $\mathsf{X}_H^G \otimes_H \mathcal{H}_u$  to  $L_u^2(G, \mathcal{H}_u)$  is given by

$$W(f \otimes h)(s) = \int_H f(st) u_t(h) d\mu_H(t). \quad \square$$

### 2.2.3 Kernels of induced representations

Another advantage of the Rieffel approach is that the following observation is routine.

**Proposition 2.2.3** ([RW98, Proposition 3.34]). *Suppose that  $\mathsf{X}$  is an  $A$ – $B$ -Hilbert bimodule. Then  $a \in \ker(\mathrm{Ind}_B^A L)$  if and only if  $\langle a \cdot x, y \rangle_B \in \ker L$  for all  $x, y \in \mathsf{X}$ . In particular,  $\ker(\mathrm{Ind}_B^A L)$  depends only on  $\ker L$ , and we get a map  $\mathrm{Ind}_B^A : \mathcal{I}(B) \rightarrow \mathcal{I}(A)$  such that  $\ker(\mathrm{Ind}_B^A L) = \mathrm{Ind}_B^A(\ker L)$ .*

*Proof.* Observe that  $(\mathrm{Ind}_B^A(a)([x \otimes h]) \mid y \otimes k) = (L(\langle y, a \cdot x \rangle_B)h \mid k)$ .  $\square$

We give  $\mathcal{I}(A)$  the topology generated by the sets

$$\mathcal{O}_J := \{I \in \mathcal{I}(A) : I \not\supset J\}$$

for all  $J \in \mathcal{I}(A)$ . (More precisely, these sets form a subbasis for a topology on  $\mathcal{I}(A)$ .) It is not such an attractive topology, but it has two advantages. The relative topology on  $\mathrm{Prim} A \subset \mathcal{I}(A)$  is the Jacobson topology. (In  $\mathrm{Prim} A$ , the  $\mathcal{O}_J$  form a basis because primitive ideals are prime ideals.)

Furthermore, the map  $\mathrm{Ind}_B^A : \mathcal{I}(B) \rightarrow \mathcal{I}(A)$  in Proposition 2.2.3 given by

$$\mathrm{Ind}_B^A(\ker \pi) := \ker(\mathrm{Ind}_B^A \pi)$$

is not only well defined but *continuous from  $\mathcal{I}(B)$  to  $\mathcal{I}(A)$*  (see [RW98, Corollary 3.35]).

## 2.3 Morita equivalence

Some knowledge of Morita equivalence is also assumed in these lectures. Naturally, I will suggest [RW98, §3] as a reference. Here we should recall that, although Morita equivalence of  $C^*$ -algebras is a strictly weaker equivalence relation on  $C^*$ -algebras than is isomorphism, it nevertheless implies an intimate relationship between  $A$  and  $B$ . Two  $C^*$ -algebras  $A$  and  $B$  are (strongly) *Morita equivalent* or Rieffel Morita equivalent if there is a Hilbert  $B$ -module  $\mathsf{X}$  such that  $A$  is isomorphic to  $\mathcal{K}(\mathsf{X})$  (see [RW98, Proposition 3.8]).<sup>3</sup> If  $A$  is Morita equivalent to  $B$ ,

<sup>3</sup>Originally, “ $A$  Morita equivalent to  $B$ ” meant that their representation theories were naturally equivalent as in [RW98, Theorem 3.29]. What we are here calling “Morita equivalent”, Rieffel called “strongly Morita equivalent”. Since the subject has little use for the former property, in our book Raeburn and I chose to single the representation theory property out as “categorical”

then  $\text{Ind}_B^A$  establishes a natural equivalence between the representations of  $B$  and the representations of  $A$  (see [RW98, Theorem 3.29]). Furthermore, [RW98, Theorem 3.22] implies  $\text{Ind}_B^A$  is a lattice isomorphism of  $\mathcal{I}(B)$  onto  $\mathcal{I}(A)$  (and therefore a homeomorphism), and then  $\text{Ind}_B^A$  restricts to a homeomorphism of  $\text{Prim } B$  onto  $\text{Prim } A$  (see [RW98, Corollary 3.33]). Separable  $C^*$ -algebras  $A$  and  $B$  are Morita equivalent if and only if  $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$  (see [RW98, Theorem 5.55]).

Perhaps the first of the fundamental Morita equivalences to be studied arises when we have a closed subgroup  $H$  in  $G$ . The full details are in [RW98, Appendix C.4]. Here we will sketch the details assuming everything in sight is unimodular.

**Example 2.3.1.** Let  $H$  be a subgroup of  $G$  with both  $H$  and  $G$  unimodular, and consider Rieffel's Hilbert  $C^*(H)$ -module  $\mathsf{X}_H^G = \overline{C_c(G)}$ . Then  $C_c(G \times G/H) \subset C_0(G/h) \rtimes_{\text{lt}} G$  maps into  $\mathcal{L}(\mathsf{X}_H^G)$  via

$$\Phi(F)h(s) = \int_G F(r, s \cdot H)h(r^{-1}s) d\mu(r).$$

Check that  $\Theta_{f,g} = \Phi(F)$  where

$$F(r, s \cdot H) = \int_H f(sh)g(r^{-1}sh) d\mu_H(h).$$

We can prove the following.

**Theorem 2.3.2** ([RW98, Theorem C.23]). *The map  $\Phi$  above determines an isomorphism of  $C_0(G/H) \rtimes_{\text{lt}} G$  onto  $\mathcal{K}(\mathsf{X}_H^G)$ . Hence  $C_0(G/H) \rtimes_{\text{lt}} G$  is Morita equivalent to  $C^*(H)$ .*

This is a major part of the proof of the result  $C_0(G/H) \rtimes_{\text{lt}} G \cong C^*(H) \otimes \mathcal{K}(L^2(G/H))$  mentioned in the previous chapter.

### 2.3.1 Mackey's imprimitivity theorem

To see the significance of Theorem 2.3.2, we record a couple of corollaries. The first is Mackey's classical imprimitivity theorem.

**Corollary 2.3.3** ([RW98, Theorem C.27]). *Let  $U$  be a representation of  $G$ . Then  $U$  is equivalent to a representation induced from a closed subgroup  $H$  in  $G$  if and only if there is a representation  $M : C_0(G/H) \rightarrow B(\mathcal{H}_u)$  such that  $U_s M(\phi) U_s^* = M(\text{lt}_s(\phi))$ .*

---

or “weak” Morita equivalence and reserve the unqualified term “Morita equivalent” for the more useful concept. This is not universally accepted terminology. Hence one still sees strong Morita equivalence and Rieffel Morita equivalent in the literature to distinguish it from Categorical Morita equivalence.

*Proof.* If  $u$  is a representation of  $H$ , then Theorem 2.2.2 implies  $\text{Ind}_H^G u$  is equivalent to  $U$  on  $L_u^2(G, \mathcal{H}_u)$  where  $U_r h(s) = h(r^{-1}s)$ . Then

$$M : C_0(G/H) \rightarrow B(L_u^2(G, \mathcal{H}_u))$$

can be defined by  $M(\phi)h(s) = \phi(sH)h(s)$ .

On the other hand, given  $M$ ,  $(M, U)$  is a covariant representation of

$$(C_0(G/H), G, \text{lt}).$$

Hence  $M \rtimes U$  must be induced since every representation of  $C_0(G/H) \rtimes_{\text{lt}} G$  is.  $\square$

Incidentally, it is because of Corollary 2.3.3 (aka Mackey’s imprimitivity theorem) that Morita equivalent results like Theorem 2.3.2 are often called *imprimitivity theorems*.<sup>4</sup>

Note that we also obtain the following nontrivial result we referred to in our proof of the Stone–von Neumann Theorem 1.4.4.

**Corollary 2.3.4.** *The crossed product  $C_0(G) \rtimes_{\text{lt}} G$  is Morita equivalent to  $\mathcal{C}$  and hence simple.*

## 2.4 Induced representations for crossed products

The theory of induced representations for crossed products is worked out in full detail in [Wil07, §5.1]. The goal is to produce a theory of induced representations for crossed products that generalizes the traditional one for groups. Here, as in Section 2.2.1, we restrict to unimodular groups to avoid the distraction of decorating our formulas with modular functions that usually are not there anyway. If  $(A, G, \alpha)$  is a dynamical system and  $H$  is a subgroup of  $G$  (unimodular), then we can build a Hilbert  $A \rtimes_{\alpha|H} H$ -module  $X_H^G(A)$  out of  $C_c(G, A)$  via

$$\langle f, g \rangle_\star(h) = f^* * g(h) = \int_G \alpha_s^{-1}(f(s)^* g(sh)) d\mu(s).$$

Just as in the scalar case, we have a homomorphism of  $A \rtimes_\alpha G$  into  $\mathcal{L}(X_H^G(A))$  via  $\phi(f)[g] = [f * g]$  making  $X_H^G(A)$  into a  $A \rtimes_\alpha G - A \rtimes_\alpha G$ -Hilbert bimodule. Hence, we have a Rieffel machine for induction that generalizes the classical one for groups.

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<sup>4</sup>This begs the question as to where the word “imprimitivity” comes from in Mackey’s Theorem. It was suggested to me that a representation of  $G$  that is not induced would be “primitive”. Hence a result telling us which representations are not primitive could be called an imprimitivity result?



### 2.4.1 Regular representations revisited

Let  $(A, G, \alpha)$  be a dynamical system. Any representation  $\rho : A \rightarrow B(\mathcal{H}_\rho)$  can be viewed as a covariant representation of the trivial dynamical system  $(A, \{e\}, \alpha)$ . The induced representation acts on the Hilbert space completion  $\mathsf{X}_H^G(A) \otimes \mathcal{H}_\rho$  of  $C_c(G, A) \odot \mathcal{H}_\rho$ . The map  $f \otimes h \mapsto (s \mapsto \rho(f(s))h)$  extends to a unitary mapping  $\mathsf{X}_H^G(A) \otimes \mathcal{H}_\rho$  onto  $L^2(G, \mathcal{H}_\rho)$  intertwining the representation  $\text{Ind}_e^G \rho$  induced from  $\rho$  with our regular representation  $\text{Ind} \rho$  from Example 1.2.2 (see [Wil07, Remark 5.7]). Hence the notation.

Having realized regular representations as actual induced representations, we can take advantage of the basic properties of the Rieffel machine. In particular, we now know that the kernel of a regular representation  $\text{Ind} \rho$  depends only on the kernel of  $\rho$ . This finally justifies the assertions we made regarding the definition of the reduced norm.

**Corollary 2.4.1.** *If  $\rho$  and  $\pi$  are faithful representations of  $A$ , then  $\ker \text{Ind} \rho = \text{Ind}_e^G(\ker \rho) = \text{Ind}_e^G\{0\} = \ker \text{Ind} \pi$  (hence  $\|\cdot\|_r$  is well defined). In particular,  $\|\text{Ind} \rho(f)\| \leq \|f\|_r$  for all representations  $\rho$  of  $A$  faithful or not.*

The connection to induced representations of groups goes much deeper than just a similarity of notation.

**Example 2.4.2** ([Wil07, Proposition 5.4 and Corollary 5.6]). Suppose  $H$  is a closed subgroup of  $G$  (both unimodular) and that  $L = \pi \rtimes u$  is a representation of  $A \rtimes_{\alpha|_H} H$ . Then  $\text{Ind}_H^G(\pi \rtimes u)$  is equivalent to the representation  $\Pi \rtimes U$  on  $L_u^2(G, \mathcal{H}_u)$  given by

$$\Pi(a)h(s) = \pi(\alpha_s^{-1}(a))(h(s)) \quad \text{and} \quad U_r h(s) = h(r^{-1}s).$$

*Note that in the above example,  $U = \text{Ind}_H^G u$ .*

**Example 2.4.3** ([Wil07, Proposition 8.24]). Consider  $(C_0(X), G, \text{lt})$  and let  $\text{ev}_x$  be evaluation at  $x$ . Then  $\text{Ind} \text{ev}_x$  is (equivalent to) the representation on  $L^2(G)$  given by

$$\text{Ind} \text{ev}_x(f)h(s) = \int_G f(r, s \cdot x) h(r^{-1}s) d\mu(s).$$

Since induction preserves direct sums and  $\bigoplus_{x \in X} \text{ev}_x$  is certainly faithful on  $C_0(X)$ , we see that

$$\|f\|_r = \sup_{x \in X} \|\text{Ind} \text{ev}_x(f)\|.$$

Clearly, it would suffice to take the supremum over a dense subset  $D \subset X$ .

### 2.4.2 Representations induced from stability groups

As we shall emphasize, representations induced from stability groups are centrally important in the general structure theory for crossed products.

**Example 2.4.4.** Consider  $(C_0(X), G, \text{lt})$  and  $H$  is a closed subgroup of  $G$  (both unimodular). Suppose that  $H \subset G_{x_0} = \{s \in G : s \cdot x_0 = x_0\}$ . (The important case here is  $H = G_{x_0}$ .) Then  $(\text{ev}_{x_0}, u)$  is a covariant representation of  $(C_0(X), H, \text{lt})$ . It is not hard to work out that  $\text{Ind}_H^G \text{ev}_{x_0} \rtimes u$  is equivalent to  $L_H^G(x_0, u) = M \rtimes \text{Ind}_H^G u$  on  $L_u^2(G, \mathcal{H}_u)$  where  $M(\phi)h(s) = \phi(s \cdot x_0)h(s)$  (and as we have seen before  $\text{Ind}_H^G u(r)h(s) = h(r^{-1}s)$ ) (see [Wil07, Proposition 8.24]).

## 2.5 Induced algebras and imprimitivity theorems

The importance of general imprimitivity theorems and the structure of crossed products probably starts with [Gre77, Theorem 14]. Stated in modern language, Green's result amounts to the following.

**Example 2.5.1** (Green). Suppose that  $G$  acts freely and properly on  $P$ . Then the orbit space  $G \backslash P$  is locally compact Hausdorff and we can make  $C_c(P)$  into a pre-Hilbert  $C_0(G \backslash P)$ -module:

$$\langle f, g \rangle_\star (G \cdot p) := \int_G \overline{f(s^{-1} \cdot p)} g(s^{-1} \cdot p) d\mu(s).$$

Let  $X$  be the Hilbert  $C_0(G \backslash P)$ -module completion. We get a map  $\Phi : C_c(G \times P) \subset C_0(P) \rtimes_{\text{lt}} G \rightarrow \mathcal{L}(X)$  by

$$\Phi(F)(h)(p) = \int_G F(s, p) h(s^{-1} \cdot p) d\mu(s).$$

Furthermore, if  $F(s, p) = f(p) \overline{g(s^{-1} \cdot p)}$ , then  $\Phi(F) = \Theta_{f, g}$ . With a bit of work, we can conclude that  $C_0(P) \rtimes_{\text{lt}} G$  is Morita equivalent to  $C_0(G \backslash P)$  (see [Wil07, Corollary 4.11]).

As we outlined earlier, this means that  $X$  is the section algebra of a bundle of Hilbert  $H$  spaces with fibres  $L^2(G)$  (see [Gre77, Corollary 15]). Hence  $C_0(P) \rtimes G \cong \mathcal{K}(X)$  is a bundle of  $C^*$ -algebras with fibres  $\mathcal{K}(L^2(G))$ . If either  $G \backslash P$  has finite covering dimension or the orbit map has local sections, then the underlying Hilbert bundle is locally trivial. If  $G$  is infinite, then the bundle of Hilbert spaces has to be trivial and  $C_0(P) \rtimes_{\text{lt}} G \cong C_0(G \backslash P, \mathcal{K}(L^2(G)))$ .

### 2.5.1 Green's symmetric imprimitivity theorem

Green was later able to prove a symmetric version of Example 2.5.1 now known as Green's symmetric imprimitivity theorem. At the time, it subsumed many of the known Morita equivalence results (see [Rie82]), and is still the gold standard for transformation groups.

**Theorem 2.5.2** ([Wil07, Corollary 4.11]). *Suppose  $K$  and  $H$  act freely and properly on the left and right, respectively, of  $P$  and that the actions commute. Then  $C_0(P/H) \rtimes_{\text{lt}} K$  is Morita equivalent to  $C_0(K \backslash P) \rtimes_{\text{rt}} H$ .*

The following corollary is an immediate and important consequence.

**Corollary 2.5.3.** *Suppose that  $H$  and  $K$  are closed subgroups of  $G$ .*

*Then  $C_0(G/H) \rtimes_{\text{lt}} K$  is Morita equivalent to  $C_0(K \backslash G) \rtimes_{\text{rt}} H$ .*

**Example 2.5.4** ([Wil07, Example 4.14]). Let  $G = \mathbf{R}$ ,  $K = \mathbf{Z}$  and  $H = \theta \mathbf{Z}$ . The usual identification of  $\mathbf{Z} \backslash \mathbf{R}$  with  $\mathbf{T}$  shows that  $C_0(\mathbf{Z} \backslash \mathbf{R}) \rtimes_{\text{rt}} \theta \mathbf{Z} \cong A_\theta$ . On the other hand, the usual identification of  $\mathbf{R}/\theta \mathbf{Z}$  with  $\mathbf{T}$  gives an isomorphism of  $C_0(\mathbf{R}/\theta \mathbf{Z}) \rtimes_{\text{lt}} \mathbf{Z}$  with  $A_{\frac{1}{\theta}}$ . Hence we get the curious result that  $A_\theta$  is always Morita equivalent to  $A_{\frac{1}{\theta}}$ .

## 2.5.2 Induced algebras

Green's symmetric imprimitivity theorem has a number of interesting consequences, and we will run into it again rather soon. However, just now, we want look at generalizations of Green's symmetric imprimitivity theorem to noncommutative algebras. For this, we need to recall what an *induced algebra* is. For more details, see [Wil07, §3.6]. Suppose that  $H$  acts freely and properly on the right of  $P$  and that  $(A, H, \alpha)$  is a dynamical system. The corresponding *induced algebra*  $\text{Ind}_H^P(A, \alpha)$  is the collection of continuous  $f : P \rightarrow A$  such that  $f(p \cdot h) = \alpha_h^{-1}(f(p))$  and such that  $p \cdot H \mapsto \|f(p)\|$  is in  $C_0(P/H)$ . If  $K$  acts freely and properly on the left of  $P$  such that  $k \cdot (p \cdot h) = (k \cdot p) \cdot h$ , then we get a dynamical system  $(\text{Ind}_H^P(A, \alpha), K, \text{lt})$  via left-translation  $\text{lt}_k(f)(p) = f(k^{-1} \cdot p)$ .

An interesting special case arises when  $P = G$  and  $H$  is a closed subgroup. For example, we can let  $H$  be a closed subgroup of  $G$  and let  $P = K = G$ . Then if  $(A, H, \alpha)$  is a dynamical system we can form  $\text{Ind}_H^G(A, \alpha)$  as above. If  $\alpha$  extends to all of  $G$ , then  $f \mapsto \tilde{f}$  where  $\tilde{f}(sH) = \alpha_s(f(s))$  is an isomorphism of  $\text{Ind}_H^G(A, \alpha)$  with  $C_0(G/H, A)$ . In general,  $\text{Ind}_H^G(A, \alpha)$  will not so trivialize, but we can identify  $\text{Prim}(\text{Ind}_H^G(A, \alpha))$  with the orbit space  $(G \times \text{Prim } A)/H$  where  $(s, P) \cdot h = (sh, \alpha_h^{-1}(P))$ . The induced  $G$ -action on  $(G \times \text{Prim } A)/H$  is left-translation on the first factor. Hence we have a  $G$ -equivariant map of  $\text{Prim}(\text{Ind}_H^G(A, \alpha))$  onto  $G/H$ .

Induced algebras are not just pretty definitions. They arise quite naturally in the subject whenever  $\text{Prim } A$  admits a continuous equivariant map onto  $G/H$  for some closed subgroup  $H$ .

**Example 2.5.5** ([Wil07, Proposition 3.53]). Suppose  $(A, G, \alpha)$  admits a continuous equivariant map  $\sigma : \text{Prim } A \rightarrow G/H$  for  $H < G$ . Let

$$I := \bigcap \{ P \in \text{Prim } A : \sigma(P) = eH \} \in \mathcal{I}(A).$$

Then  $I$  is  $H$ -invariant and we get a dynamical system  $(I, H, \alpha^I)$  where  $\alpha_h^I(a + I) = \alpha_h(a) + I$ . Then  $\Phi(f)(s) := \alpha_s^{-1}(a) + I$  defines an equivariant isomorphism of  $(A, G, \alpha)$  onto  $(\text{Ind}_H^G(D, \alpha^I), G, \text{lt})$ .

### 2.5.3 Raeburn's symmetric imprimitivity theorem

Suppose that we have commuting free and proper left and right actions of  $K$  and  $H$ , respectively, on  $P$ . Suppose also that we have commuting dynamical systems  $(A, K, \alpha)$  and  $(A, H, \beta)$ .

Then we can form the dynamical system  $(\text{Ind}_H^P(A, \beta), K, \sigma)$  where  $\sigma_k(f)(p) = \alpha_k(f(k^{-1} \cdot p))$ . We can proceed similarly for  $(\text{Ind}_K^P(A, \alpha), H, \tau)$ . Here  $\tau_h(f)(p) = \beta_h(f(p \cdot h))$ .

Raeburn's symmetric imprimitivity theorem is the generalization of Green's symmetric imprimitivity theorem to noncommutative  $A$ .

**Theorem 2.5.6** ([Wil07, Theorem 4.1]). *The crossed products*

$$\text{Ind}_K^P(A, \alpha) \rtimes_\tau H \quad \text{and} \quad \text{Ind}_H^P(A, \beta) \rtimes_\sigma K$$

*are Morita equivalent.*

It is fair to admit that Raeburn's theorem is a bit much to absorb all at once. No one be surprised to learn that, historically, it was only formulated after a series of preliminary results. One of the first versions for noncommutative  $A$  is due to Green. It can be derived from Raeburn's theorem as follows. Suppose  $H$  is a closed subgroup of  $G$  and let  $K = P = G$  in the symmetric theorem so that  $(A, K, \alpha) = (A, G, \alpha)$  and  $(A, H, \beta) = (A, H, \alpha|_H)$ . Now  $(\text{Ind}_H^P(A, \alpha), K, \tau) = (C_0(G/H, A), G, \text{lt} \otimes \alpha)$  and  $(\text{Ind}_K^P(A, \beta), H, \sigma) = (A, H, \alpha)$ .

**Theorem 2.5.7** ([Wil07, Theorem 4.22]).  *$C_0(G/H, A) \rtimes_{\text{lt} \otimes \alpha} G$  is Morita equivalent to  $A \rtimes_\alpha H$  via  $X_H^G(A)$ .*

This is the analogue for crossed products of Mackey's imprimitivity theorem for groups and hence pairs naturally with our theory of induced representations.

However, Raeburn's theorem allows a number of other applications as well.

**Example 2.5.8.** Again, we suppose that  $H$  is a closed subgroup of  $G$  and let  $G = P = K$ . Suppose that  $(A, G, \alpha)$  is a dynamical system admitting a continuous equivariant map  $\sigma : \text{Prim } A \rightarrow G/H$ . Then, as we have seen,  $(A, G, \alpha) \cong (\text{Ind}_H^G(D, \alpha^I), G, \text{lt})$ . We can give  $G$  the trivial action on  $D$  so that  $\text{Ind}_G^G(D, \text{id}) \cong D$ . Now by the symmetric imprimitivity theorem,  $A \rtimes_\alpha G$  is Morita equivalent to  $D \rtimes_{\alpha^I} H$ .

The previous example is just a special case of the following corollary to the symmetric imprimitivity theorem.

**Corollary 2.5.9** ([Wil07, Corollary 4.17]). *If  $H$  is a closed subgroup of  $G$  and  $(D, H, \beta)$  is dynamical system, then  $\text{Ind}_H^G(D, \beta) \rtimes_{\text{lt}} G$  is Morita equivalent to  $D \rtimes_\beta H$ .*

Our next example hints pretty strongly that stability groups will be important in the sequel.

**Example 2.5.10** ([Wil07, Proposition 8.7]). Let  $(A, G, \alpha)$  be such that  $G$  acts transitively on  $\text{Prim } A$ . Let  $G_P$  be the *stability group*  $\{s \in G : s \cdot P = P\}$  for some  $P \in \text{Prim } A$ . Then the continuous bijection  $sG_P \mapsto s \cdot P$  is (usually) a homeomorphism of  $G/G_P$  onto  $\text{Prim } A$ . If  $A$  is GCR, then  $A/P \cong \mathcal{K}$ , and  $A \rtimes_\alpha G$  is Morita equivalent to  $A/P \rtimes_{\bar{\alpha}} G_P \cong \mathcal{K} \rtimes_\sigma G_P$ .

We will have more to say about this observation when we talk about “regular systems” in Section 3.2.4.

## Chapter 3

### Ideal Structure

#### 3.1 Amenability and the reduced norm

The theory of amenable groups is classical and vast. A very short summary can be found in [Wil07, Appendix A]. Here we recall just a few details. A group  $G$  is called *amenable* if there is a left invariant mean on  $L^\infty(G)$ . (A mean is a positive linear functional of norm one.) We know that  $G$  is amenable if and only if the left-regular representation is faithful on  $C^*(G)$ . Equivalently,  $G$  is amenable if and only if the reduced and universal norms agree on  $C_c(G)$ . Abelian and compact groups are amenable as well as closed subgroups, quotients and extensions of amenable groups. The key property we will need is that if  $G$  is amenable, then there are  $a_n \in C_c(G)$  such that  $\|a_n\|_2 \leq 1$  and  $e_n = a_n^* * a_n \rightarrow 1$  uniformly on compact subsets of  $G$ .

Here, our interest in amenable groups starts with the following.

**Theorem 3.1.1** ([Wil07, Theorem 7.13]). *If  $(A, G, \alpha)$  is a dynamical system with  $G$  amenable, then the reduced norm and the universal norm agree on  $C_c(G, A)$ . That is,  $A \rtimes_\alpha G = A \rtimes_{\alpha, r} G$ .*

*Proof.* It suffices to see that if  $L = \pi \rtimes U$  is a representation of  $A \rtimes_\alpha G$  and  $h \in \mathcal{H}_L$  is a unit vector, then there is a unit vector  $k \in L^2(G, \mathcal{H}_L)$  such that  $|(L(f)h | h)| \leq |(\text{Ind } \pi(f)k | k)|$ . In fact, we can take  $f \in C_c(G, A)$ . Then you can compute that

$$(\text{Ind } \pi(f)k | k) = \int_G \int_G (\pi(\alpha_r^{-1}(f(rs)))k(s^{-1}) | k(r)) dr ds.$$

On the other hand, if we let  $e \cdot f(s) = e(s)f(s)$  with  $e(s) = a^* * a(s)$ , then

$$\begin{aligned} (L(e \cdot f)h | h) &= \int_G \int_G \overline{a(r^{-1})} a(r^{-1}s) (\pi(f(rs))U_{rs}h | h) dr ds \\ &= \int_G \int_G \overline{a(r^{-1})} a(r^{-1}s) (\alpha_r^{-1}(f(rs))U_s h | U_r^* h) dr ds \\ &= \int_G \int_G (\pi(\alpha_r^{-1}(f(rs)))k(s^{-1}) | k(r)) dr ds \end{aligned}$$

where  $k(r) = a(r^{-1})U_r^*h$ . The amenability of  $G$  allows us to pick the  $e$  so that  $f$  is closely approximated by  $e \cdot f$ .  $\square$

If  $A \neq C$ , then the converse to Theorem 3.1.1 can fail. For example, for any group  $G$ ,  $C_0(G) \rtimes_{\text{lt}} G \cong \mathcal{L}(L^2(G))$  is always simple, and this forces  $\|\cdot\| = \|\cdot\|_r$ . Determining general conditions under which  $\|\cdot\| = \|\cdot\|_r$  on  $C_c(G, A)$  seems to be very difficult – even for  $A$  commutative and/or  $G$  discrete and/or  $G$  abelian. We will make a small step in this direction in §5.1.

### 3.1.1 Equivariant maps

Suppose that  $(A, G, \alpha)$  and  $(B, G, \beta)$  are dynamical systems. Then we say  $\phi: A \rightarrow B$  is *equivariant* if  $\phi(\alpha_s(a)) = \beta_s(\phi(a))$ . Alternatively, we could say that  $A$  and  $B$  are  $G$ - $C^*$ -algebras and that  $\phi$  is a  $G$ -map or a  $G$ -homomorphism. Note that if  $(\pi, U)$  is a covariant representation of  $(B, G, \beta)$ , then  $(\pi \circ \phi, U)$  is a (possibly degenerate) covariant representation of  $(A, G, \alpha)$ . It follows that the induced map  $f \mapsto \phi \circ f$  from  $C_c(G, A)$  to  $C_c(G, B)$  is norm reducing and determines a homomorphism  $\phi \rtimes \text{id}: A \rtimes_\alpha G \rightarrow B \rtimes_\beta G$ . Note that  $\text{Ind } \rho \circ (\phi \rtimes \text{id}) = \text{Ind}(\rho \circ \phi)$ . Hence we also get a homomorphism  $\phi \rtimes_r \text{id}: A \rtimes_{\alpha, r} G \rightarrow B \rtimes_{\beta, r} G$ . As a consequence, we can show that the crossed product and the reduced crossed product are functors.

### 3.1.2 Invariant ideals

As detailed in [Wil07, §3.4], an ideal  $I \in \mathcal{I}(A)$  is *invariant* –  $I \in \mathcal{I}_G(A)$  – if  $\alpha_s(I) \subset I$  for all  $s$ . Note that  $I \in \mathcal{I}_G(A)$  if and only if the corresponding open subset  $\mathcal{O}_I := \{P \in \text{Prim } A : P \not\supset I\}$  is  $G$ -invariant for the induced action of  $G$  on  $\text{Prim } A$ . If  $I \in \mathcal{I}_G(A)$ , then we get dynamical systems  $(I, G, \alpha|_I)$  and  $(A/I, G, \alpha^I)$  where  $\alpha_s^I(a + I) := \alpha_s(a) + I$ . It is a nice exercise to see that if  $q: A \rightarrow A/I$  is the quotient map, then  $q \rtimes \text{id}(C_c(G, A)) = C_c(G, A/I)$ . Hence  $q \rtimes \text{id}$  and  $q \rtimes_r \text{id}$  are surjective. If  $(\rho, U)$  is a covariant representation of  $(I, G, \alpha)$  and if  $\bar{\rho}$  is the canonical extension of  $\rho$  to  $A$ , then  $(\bar{\rho}, U)$  is a representation of  $(A, G, \alpha)$ . It follows that  $\iota \rtimes \text{id}$  is faithful from  $I \rtimes_\alpha G$  onto the ideal  $\text{Ex}(I) = \overline{C_c(G, I)}$  in  $A \rtimes_\alpha G$ .

**Theorem 3.1.2.** *If  $I \in \mathcal{I}_G(A)$ , then we have a short exact sequence*

$$0 \longrightarrow I \rtimes_\alpha G \xrightarrow{\iota \rtimes \text{id}} A \rtimes_\alpha G \xrightarrow{q \rtimes \text{id}} A/I \rtimes_{\alpha^I} G \longrightarrow 0.$$

*Proof.* The only issue is exactness at  $A \rtimes_\alpha G$ . Clearly  $\iota \rtimes \text{id}(I \rtimes_\alpha G) = \text{Ex}(I) \subset \ker q \rtimes \text{id}$ . We require the universal property of  $A \rtimes_\alpha G$ . If  $(\pi, U)$  is a covariant and  $I \subset \ker \pi$ , then  $(\pi \circ q, U)$  is a representation of  $(A/I, G, \alpha^I)$ . Hence  $\ker q \rtimes \text{id} \subset \ker \pi \rtimes U$ . But if  $I \not\subset \ker \pi$ , then  $\text{Ex}(I) \not\subset \ker \pi \rtimes U$ . This suffices. For more details, see [Wil07, Proposition 3.19].  $\square$

### 3.1.3 Exactness

The proof of Theorem 3.1.2 for exactness at  $A \rtimes_{\alpha,r} G$  of

$$0 \longrightarrow I \rtimes_{\alpha,r} G \xrightarrow{\iota \rtimes_r \text{id}} A \rtimes_{\alpha,r} G \xrightarrow{q \rtimes_r \text{id}} A/I \rtimes_{\alpha I,r} G \longrightarrow 0$$

does not work for the reduced norm. A group for which

$$0 \longrightarrow A \rtimes_{\alpha,r} G \xrightarrow{\iota \rtimes_r \text{id}} B \rtimes_{\beta,r} G \xrightarrow{q \rtimes_r \text{id}} C \rtimes_{\gamma,r} G \longrightarrow 0$$

is exact for any short exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{q} C \longrightarrow 0$$

of  $G$ - $C^*$ -algebras is called *exact*. Of course, any amenable group is exact, but there are exact groups which are not amenable. In fact, all free groups are exact. For a time, there was speculation that all groups might be exact, but there are now examples known of nonexact groups. However more detail here would take us too far afield (see [Ech11, §8] for additional information and references).

The special case of Theorem 3.1.2 when  $A = C_0(X)$  is very useful.

**Example 3.1.3.** Let  $U$  be a  $G$ -invariant open subset of  $X$ . Let  $F = X \setminus U$ . Then we have an exact sequence

$$0 \longrightarrow C_0(U) \rtimes_{\text{lt}} G \longrightarrow C_0(X) \rtimes_{\text{lt}} G \longrightarrow C_0(F) \rtimes_{\text{lt}} G \longrightarrow 0.$$

In the sequel, we will want to apply this in the following situation. First recall that a *composition series* in a  $C^*$ -algebra is a family of ideals  $\{I_\lambda\}$  indexed by ordinals  $0 \leq \lambda \leq \lambda_0$  such that  $I_0 = \{0\}$ ,  $I_{\lambda_0} = A$ ,  $I_\lambda \subsetneq I_{\lambda+1}$  and if  $\alpha$  is a limit ordinal, then  $I_\alpha = \bigcup_{\lambda < \alpha} I_\lambda$ . For example, by definition, a  $C^*$ -algebra  $A$  is GCR if and only if it has a composition series  $\{I_\lambda\}_{0 \leq \lambda \leq \lambda_0}$  so that each subquotient  $I_{\lambda+1}/I_\lambda$  is a CCR  $C^*$ -algebra.<sup>1</sup> Equivalently, we could require that each subquotient has Hausdorff spectrum or even that each subquotient be a continuous trace [Dix77, Theorem 4.5.5].

The point of the next example is that we easily build composition series in a transformation group  $C^*$ -algebra using nested  $G$ -invariant open sets in  $X$  together with Theorem 3.1.2.

**Example 3.1.4.** Let  $\{U_i\}_{i=0}^n$  be open  $G$ -invariant subsets of  $X$  with  $U_i \subset U_{i+1}$  for  $1 \leq i \leq n-1$ . Assume that  $U_0 = \emptyset$  and  $U_n = X$ . Let  $I_k$  be the ideal  $C_0(U_k) \rtimes_{\text{lt}} G$ . Then  $I_k \subset I_{k+1}$  and  $I_{k+1}/I_k \cong C_0(U_{k+1} \setminus U_k) \rtimes_{\text{lt}} G$ . More generally, we could assume the  $U_i$  are indexed by ordinals  $0 \leq i \leq \alpha_0$ . But if  $\alpha$  is a limit ordinal, assume  $U_\alpha = \bigcup_{j < \alpha} U_j$ . Then we get a composition series  $\{I_\lambda\}$  as above such that  $I_{k+1}/I_k \cong C_0(U_{k+1} \setminus U_k) \rtimes_{\text{lt}} G$ .

As we will see, this will allow us to decompose  $C_0(X) \rtimes_{\text{lt}} G$  over spaces where the  $G$ -action is especially well behaved.

<sup>1</sup>A  $C^*$ -algebra is CCR if every irreducible representation  $\pi$  satisfies  $\pi(A) = \mathcal{K}(\mathcal{H}_\pi)$ .



## 3.2 Orbit spaces

We need some technicalities on orbit spaces, and on the space  $G \backslash \text{Prim } A$  in particular. First let us dredge up a fact or two about  $\text{Prim } A$ .

Recall that if  $(A, G, \alpha)$  is a dynamical system, then  $G$  acts continuously on  $\text{Prim } A$ . The topology, such as it is, on  $\text{Prim } A$  has as a basis the open sets  $\mathcal{O}_J = \{P \in \text{Prim } A : P \not\supset J\}$  for all  $J \in \mathcal{I}(A)$ . As a topological space,  $\text{Prim } A$  is only guaranteed to satisfy the  $T_0$ -axiom of separability: if  $\overline{\{P\}} = \overline{\{Q\}}$ , then  $P = Q$ . However,  $\text{Prim } A$  is nevertheless *locally compact* in that every point has a neighborhood basis consisting of compact sets. (If  $a \notin P$ , then  $\{Q : \|a+Q\| \geq \frac{\|a+P\|}{2} > 0\}$  is a compact neighborhood of  $P$ .) Since there is no Hausdorffness in sight, some authors – especially those that follow Bourbaki – would call  $\text{Prim } A$  locally quasi-compact as these authors include Hausdorffness as part of the definition of compactness.

Orbit spaces are discussed in [Wil07, §3.5]. Here we recall that if  $G$  acts on a topological space  $X$ , then the *orbit space*  $G \backslash X = \{G \cdot x : x \in X\}$  carries the largest topology – called the *quotient topology* – such that the *quotient map*  $q : X \rightarrow G \backslash X$  is continuous. Since  $G$  is a group, the quotient map  $q$  is open as well as continuous. As a result, even if  $X$  is a not necessarily Hausdorff locally compact space, then  $G \backslash X$  is locally compact and second countable if  $X$  is. Provided  $X$  is a  $T_0$ -space, the stability groups  $G_x = \{s \in G : s \cdot x = x\}$  are closed in  $G$ . The orbit space  $G \backslash X$  need not be  $T_0$  even if  $X$  is. Consider the action of  $\mathbf{Z}$  on  $\mathbf{T}$  by an irrational rotation: every orbit is dense in  $\mathbf{T}$ . Hence the topology on  $\mathbf{Z} \backslash \mathbf{T}$  is the coarsest possible: the only open sets are the empty set and the either space itself.

### 3.2.1 The orbit space $G \backslash \text{Prim } A$

A good deal of the structure of  $A \rtimes_\alpha G$  is reflected in the orbit space  $G \backslash \text{Prim } A$ . As we will see, things are much more straightforward when  $G \backslash \text{Prim } A$  is  $T_0$ . When it is not, we will want to talk about the *quasi-orbit space*  $\mathcal{Q} = \mathcal{Q}_G(\text{Prim } A)$  which is the topological quotient of  $G \backslash \text{Prim } A$  where we identify  $G \cdot P$  and  $G \cdot Q$  if  $\overline{G \cdot P} = \overline{G \cdot Q}$ . This is equivalent to identifying  $P$  and  $Q$  in  $\text{Prim } A$  if  $\cap s \cdot P = \cap s \cdot Q$  in  $\mathcal{I}_G(A)$ .

**Lemma 3.2.1** ([Wil07, Lemma 6.12]). *The natural map  $q : \text{Prim } A \rightarrow \mathcal{Q}_G(\text{Prim } A)$  is continuous and open. In particular, when  $A$  is separable,  $\mathcal{Q}_G(\text{Prim } A)$  is a second countable locally compact (not necessarily Hausdorff) space.*

### 3.2.2 The Res map

If  $L = \pi \rtimes U$  is a representation of  $A \rtimes_\alpha G$ , then we write  $\pi = \text{Res } L$ . We get a continuous well-defined map  $\text{Res} : \mathcal{I}(A \rtimes_\alpha G) \rightarrow \mathcal{I}(A)$  satisfying  $\text{Res}(\ker L) = \ker(\text{Res } L)$  (see [Wil07, Corollary 6.15 and Lemma 6.16]). Since  $U_s \pi(a) U_s^* = \pi(\alpha_s(a))$ , it is not hard to see that  $\text{Res } K \in \mathcal{I}_G(A)$ . The  $G$ -invariant closed set  $F = \{P \in \text{Prim } A : P \supset \ker \text{Res } L\}$  is called the *support of  $L$* .

**Example 3.2.2.** Suppose that  $L$  is a representation of  $A \rtimes_\alpha G$ . Let  $I = \text{Res}(\ker L)$  and  $\text{supp } L = F = C_1 \cup C_2$  where each  $C_k$  is a nonempty, proper, closed  $G$ -invariant subset of  $F$ . Let  $I_k = \bigcap_{P \in C_k} P$ . Since each  $C_k$  is proper,  $\text{Ex}(I_k) \not\subset \ker L$ . Since  $I = I_1 \cap I_2 = I_1 I_2$ ,  $\text{Ex}(I_1) \text{Ex}(I_2) \subset \text{Ex}(I) \subset \ker L$ . This means  $\ker L$  is not a prime ideal.

Recall that we have a standing assumption that  $A$  is separable.

**Theorem 3.2.3** ([Wil07, Proposition 6.21]). *Suppose that  $L$  is an irreducible representation of  $A \rtimes_\alpha G$ . Then the support  $F = \{Q : Q \supset \text{Res}(\ker L)\}$  of  $L$  is an orbit closure  $\overline{G \cdot P}$ . Alternatively, there is a  $P \in \text{Prim } A$  such that  $\ker(\text{Res } L) = \bigcap s \cdot P$ .*

*Proof.* Since primitive ideals are prime, the support of an irreducible representation cannot be the union of nontrivial closed invariant subsets. Because  $\text{Prim } A$  is second countable and has the Baire property, this forces the support to be an orbit closure.  $\square$

In this case, we say that the representation *lives on the quasi-orbit* associated to  $P$ . When every irreducible representation lives on a quasi-orbit, we say that  $(A, G, \alpha)$  is *quasi-regular*. In the separable world, *every* dynamical system is quasi-regular (see [Wil07, Proposition 6.21]).

### 3.2.3 Almost Hausdorff spaces

**Definition 3.2.4.** A locally compact space  $X$  is called *almost Hausdorff* if every nonempty locally compact subset has a nonempty relatively open Hausdorff subset.

It is not hard to work out that  $X$  is almost Hausdorff if and only if locally compact subset has a relatively open *dense* Hausdorff subset.

**Example 3.2.5** ([Dix77, Theorem 4.4.5]). If  $A$  is a GCR  $C^*$ -algebra, then  $A$  has an essential ideal  $I$  with Hausdorff spectrum. Every locally closed subset of  $\text{Prim } A = \hat{A}$  is the primitive ideal space of an ideal in a quotient of  $A$ . Hence  $\text{Prim } A = \hat{A}$  is almost Hausdorff if  $A$  is GCR.

In  $C^*$ -algebra theory, there is clear dichotomy between good and evil. Without being precise, we call  $A$  “a good  $C^*$ -algebra” if there is some hope of understanding its irreducible representations, and  $A$  is “evil” if there is no such hope. A bit more formally, the good  $C^*$ -algebras are the GCR or Type I  $C^*$ -algebras – “evil” simply means not GCR.<sup>2</sup> The underlying principle has to do with orbit spaces: either the orbit space for a given  $G$ -action is nice or it is terrible. This often goes under the title of the “Mackey–Glimm” Dichotomy [Eff81, Eff65, Gli61, Mac57]. A useful version for us is the following taken from [Wil07, Theorem 6.2].

<sup>2</sup>A  $C^*$ -algebra is called “type I” if  $\pi(A)''$  is always a type I von Neumann algebra for every representation  $\pi$ . This is a remarkable bit of bad terminology, a type I von Neumann algebra is almost never a type I  $C^*$ -algebra. Fortunately, a  $C^*$ -algebra is type I if and only if it is GCR so I am using the less controversial term here.

**Theorem 3.2.6** (Mackey–Glimm Dichotomy). *Let  $X$  be a second countable almost Hausdorff locally compact  $G$ -space. Then the following are equivalent.*

- (a)  $G \backslash X$  is  $T_0$ .
- (b) Every orbit is open in its own closure (i.e., every orbit is locally closed).
- (c) Each orbit is a Baire space: the countable intersection of relatively open dense subsets is dense.
- (d) For all  $x \in X$ ,  $sG_x \mapsto s \cdot x$  is a homeomorphism of  $G/G_x$  onto  $G \cdot x$ .
- (e)  $G \backslash X$  is almost Hausdorff.

### 3.2.4 Regular dynamical systems

As we have seen, orbit spaces can be very complicated. This certainly remains true of  $G \backslash \text{Prim } A$ . We always know that the map  $sG_P \mapsto s \cdot P$  is a continuous bijection of  $G/G_P$  onto the orbit  $G \cdot P$ . It is often, but not always, a homeomorphism. There is also the question of what sort of subset of  $\text{Prim } A$  is  $G \cdot P$ . When  $\text{Prim } A$  is locally Hausdorff, then Theorem 3.2.6 shows these two questions are related. In general, we make the following definition.

**Definition 3.2.7.** We say that an orbit  $G \cdot P \subset \text{Prim } A$  is *regular* if it is locally closed and  $s \cdot G_P \mapsto s \cdot P$  is a homeomorphism of  $G/G_P$  onto  $G \cdot P$ . We say that  $(A, G, \alpha)$  is regular if (it is quasi-regular) and every orbit is regular.

**Remark 3.2.8.** If  $A$  is GCR and separable, then the Mackey–Glimm Dichotomy implies  $(A, G, \alpha)$  is regular if and only if  $G \backslash \text{Prim } A$  is  $T_0$ .

**Some Notation** If  $F \subset \text{Prim } A$  is closed, we will write  $A(F)$  for the quotient  $A/I$  with  $I = \bigcap_{P \in F} P$ . Note that  $A(\overline{F}) = A/I$  where  $I = \bigcap_{P \in F} P$ . More generally, if  $F$  is locally closed, we will also write  $A(F)$  for the ideal of  $A(\overline{F})$  given by  $\bigcap_{P \in \overline{F} \setminus F} P/I$  where  $I = \bigcap_{P \in F} P$ . Of course,  $P \mapsto P/I$  is a homeomorphism of  $F$  onto  $\text{Prim } A(F)$ .

**Example 3.2.9.** Suppose that  $G \cdot P$  is a regular orbit. Then we can form  $A(G \cdot P) \rtimes_{\bar{\alpha}} G$  which is Morita equivalent to  $A/P \rtimes_{\bar{\alpha}} G_P$ . (If  $I$  is as above,  $A(\overline{G \cdot P}) = A/I$  and we can identify  $(A/I)/(P/I)$  with  $A/P$ .)

## 3.3 $C_0(X)$ -algebras

A summary of  $C_0(X)$ -algebras and their corresponding upper semicontinuous  $C^*$ -bundles is given in [Wil07, Appendix C]. As usual, we summarize just some of the basic results here.

We call  $A$  a  $C_0(X)$ -algebra if there is a homomorphism  $\Phi_A$  from  $C_0(X)$  into the center of the multiplier algebra  $ZM(A)$  such that  $\Phi_A(C_0(X))A$  is dense in  $A$ . Normally, we simply think of  $A$  as a central  $C_0(X)$ -module such that  $C_0(X) \cdot A$  is dense. If  $J_x = \{f \in C_0(X) : f(x) = 0\}$  then we set  $A(x) := A/J_x \cdot A$ . If  $a \in A$ , let  $a(x) = a + J_x \cdot A$  be the image of  $a$  in  $A(x)$ . Then  $x \mapsto \|a(x)\|$  is upper semicontinuous. The idea is to view  $A$  as a field of  $C^*$ -algebras with fibres  $A(x)$ .

More formally,  $A$  is  $C_0(X)$ -isomorphic to the section algebra  $\Gamma_0(X, \mathcal{A})$  of an upper semicontinuous  $C^*$ -bundle  $\mathcal{A} = \coprod A(x)$  equipped with a suitable topology (see [Wil07, Theorem C.26]).

As it turns out,  $C_0(X)$ -algebras are a very easy and common way to view a  $C^*$ -algebra as being fibred over a space. To gain further insight into this claim, we recall that the Dauns–Hofmann Theorem [RW98, Theorem A.34] provides an isomorphism  $\Psi$  from the bounded continuous functions  $C^b(\text{Prim } A)$  onto  $ZM(A)$ . The map  $\Psi$  is characterized as follows: if  $\pi$  is an irreducible representation of  $A$ , then  $\pi(\Psi(f)a) = f(\ker \pi)\pi(a)$ . If  $\sigma : \text{Prim } A \rightarrow X$  is continuous, then we get a  $C_0(X)$ -algebra structure on  $A$  via  $f \cdot a = \Psi(f \circ \sigma)(a)$ . Since every irreducible representation of a  $C_0(X)$ -algebra must factor through a fibre  $A(x)$ ,  $C_0(X)$ -algebra structures on  $A$  correspond exactly to continuous maps  $\sigma : \text{Prim } A \rightarrow X$  as above. Furthermore,  $A$  is a continuous field over  $X$  – in that  $x \mapsto \|a(x)\|$  is continuous for all  $a \in A$  – exactly when  $\sigma$  is open. This is all contained in [Wil07, Theorem C.26]. In my opinion, this is a much tidier way of dealing with what Dixmier called continuous fields of operators in [Dix77]. The bundle approach for continuous fields is due to Fell. The fact that there is a bundle approach for upper semicontinuous fields was also known to Fell and others [Hof74], but only seems to appear in print in [Wil07].

### 3.4 Stability groups

To begin to see how  $C_0(X)$ -algebras arise in the theory, we have the following which can be extracted from [Wil07, Proposition 8.7 and Corollary 8.9] (together with Note J.6 in the errata for [Wil07]).

**Example 3.4.1.** Suppose that  $G \backslash \text{Prim } A$  is Hausdorff. Then we easily see that the action of  $C^b(\text{Prim } A)$  on  $A$  gives us a central action of  $C_0(G \backslash \text{Prim } A)$  on  $C_c(G, A)$  which gives us a  $C_0(G \backslash \text{Prim } A)$ -algebra structure on  $A \rtimes_\alpha G$ . The fibre over  $G \cdot P$  is  $A(G \cdot P) \rtimes_{\tilde{\alpha}} G$ . If  $(A, G, \alpha)$  is regular, (which is automatic if  $A$  is GCR), then the fibres are Morita equivalent to  $A/P \rtimes_{\tilde{\alpha}} G_P$ . In particular, every irreducible representation of  $A \rtimes_\alpha G$  is of the form  $\text{Ind}_{G_P}^G L$  where  $L$  is an irreducible representation of  $(A, G_P, \alpha)$  with  $\ker(\text{Res } L) = P$ . If  $G$  is amenable, then  $A \rtimes_\alpha G$  is a continuous field over  $G \backslash \text{Prim } A$ .

**Remark 3.4.2.** The above example is much simpler to grasp when  $A = C_0(X)$  and  $G \backslash X$  is Hausdorff. Then the fibre over  $G \cdot x$  is Morita equivalent to  $C^*(G_x)$  and the irreducible representations of  $C_0(X) \rtimes_{\text{lt}} G$  are all of the form  $L(x, \omega) \cong \text{ev}_x \rtimes \omega$  with  $\omega \in \hat{G}_x$ .

The previous examples motivate the following definitions.

**Definition 3.4.3.** We say that an irreducible representation  $L$  of  $A \rtimes_\alpha G$  is *induced from a stability group* if there is a  $P \in \text{Prim } A$  such that  $L \cong \text{Ind}_{G_P}^G R$  where  $R$  is an irreducible representation of  $(A, G_P, \alpha)$  with  $\ker \text{Res } R = P$ .

Similarly, we say that a primitive ideal  $K \in \text{Prim } A \rtimes_\alpha G$  is *induced from a stability group* if there is a primitive ideal  $J \in \text{Prim } A \rtimes_\alpha G_P$  such that  $K = \text{Ind}_{G_P}^G J$  and  $\text{Res } J = P$ .

### 3.5 Regular systems

Clearly, if every irreducible representation is induced from a stability group, then every primitive ideal is. Furthermore, “when things are regular”, we recover just about everything we want from stability groups. This is best illustrated in the transformation group case.

**Example 3.5.1.** Suppose that  $G \backslash X$  is  $T_0$  and that  $q : X \rightarrow G \backslash X$  is the quotient map. Then  $G \backslash X$  is almost Hausdorff. Hence there is dense relatively open Hausdorff set  $V_1 \subset G \backslash X$ . Let  $U_1 = q^{-1}(V_1)$ . Then  $I_1 := C_0(U_1) \rtimes_{\text{lt}} G$  is a  $C_0(V_1)$ -algebra with fibres Morita equivalent to  $C^*(G_x)$  for each  $G \cdot x \in V_1$ . If  $(G \backslash X) \setminus U_1$  is nonempty, let  $V_2$  be a relatively open Hausdorff subset of  $(G \backslash X) \setminus U_1$ . Let  $U_2 = q^{-1}(V_1 \cup V_2)$  and  $I_2 := C_0(U_2) \rtimes_{\text{lt}} G$ . Then  $I_2/I_1 \cong C_0(U_2 \setminus U_1) \rtimes_{\text{lt}} G$  is a  $C_0(V_2)$ -algebra with fibres Morita equivalent to  $C^*(G_x)$  for all  $G \cdot x \in V_2$ . If  $U_2 \neq X$ , continue as necessary.

**Remark 3.5.2.** In any sane example, the process in Example 3.5.1 terminates in a small (finite) number of steps. In general we get a composition series  $\{I_\lambda\}_{\lambda \leq \lambda_0}$  indexed by ordinals (with  $\lambda_0$  countable due to our separability hypotheses) such that  $I_{\lambda+1}/I_\lambda$  is a  $C_0(V_{\lambda+1})$ -algebra with fibres Morita equivalent to an appropriate stability group.

**Theorem 3.5.3.** *If  $G \backslash X$  is  $T_0$ , then every irreducible representation of  $C_0(X) \rtimes_{\text{lt}} G$  is of the form  $L(x, \omega) \cong \text{ev}_x \rtimes \omega$  for  $x \in X$  and  $\omega \in \hat{G}_x$ .*

Theorem 3.5.3 is the special case of the following observation proved in [Wil07, Theorem 8.16] (see also note J.6 in the errata).

**Theorem 3.5.4.** *If  $(A, G, \alpha)$  is a regular system with  $G \backslash \text{Prim } A$  almost Hausdorff – which is automatic if  $A$  is GCR and  $G \backslash \hat{A}$  is  $T_0$  – then every irreducible representation of  $A \rtimes_\alpha G$  is induced from a stability group.*



## Chapter 4

# Type Theory, Actions on the Compacts and the Effros–Hahn Theory

### 4.1 Type theory

There is a very long list of conditions on a  $C^*$ -algebra  $A$  that are equivalent to  $A$  being GCR – see [Ped79, Theorem 6.8.7] for one such. This is related to the Mackey–Glimm dichotomy we talked about earlier. For our current purposes, we should recall that a *separable*  $C^*$ -algebra  $A$  is GCR if and only if every irreducible representation  $\pi$  of  $A$  satisfies  $\mathcal{K}(\mathcal{H}_\pi) \subset \pi(A)$ . Equivalently,  $A$  is GCR if two irreducible representations with the same kernel are equivalent. Note that these two properties are preserved when we lift a representation from a subfactor – that is, an ideal of a quotient of  $A$ . This property is also preserved by Morita equivalence: if  $A$  is Morita equivalent to  $B$  and  $A$  is GCR, then so is  $B$  (see [Wil07, Proposition I.44]).

**Theorem 4.1.1** (Type Theory for Transformation Groups). *The transformation group  $C^*$ -algebra  $C_0(X) \rtimes_{\text{lt}} G$  is GCR if and only if  $G \backslash X$  is  $T_0$  and each  $G_x$  is a GCR group.*

*Proof.* The first part of the proof is given in [Wil07, Proposition 7.30]. If  $G \backslash X$  is  $T_0$ , then  $G \backslash X$  is almost Hausdorff. This gives us a filtration  $\{U_\lambda\}$  of  $X$  by  $G$ -invariant open sets such that  $G \backslash (U_{\lambda+1} \setminus U_\lambda)$  is Hausdorff. Then the ideals  $I_\lambda := C_0(U_\lambda) \rtimes_{\text{lt}} G$  are such that  $I_{\lambda+1}/I_\lambda$  is a nice field with fibres  $C_0(G \cdot x) \rtimes_{\text{lt}} G$ . Hence every irreducible representation of  $C_0(X) \rtimes_{\text{lt}} G$  factors through an ideal of a quotient Morita equivalent to  $C^*(G_x)$ . For the converse, if  $G$  is abelian, we show in [Wil07, Theorem 8.43] that if  $G \backslash X$  is not  $T_0$ , then there are inequivalent irreducible representations with the same kernel. The same proof works in general, or we can refer to [Goo73, Theorem 3.3].  $\square$

### 4.1.1 Type structure – the general case

For general crossed products, a tidy result characterizing when  $A \rtimes_\alpha G$  is GCR is out of reach. We can make some better progress in the case  $A$  is GCR. recall that a  $C^*$ -algebra is called elementary if it is (isomorphic to) the compact operators on some Hilbert space.

**Remark 4.1.2.** If  $A$  is GCR and  $G \backslash \hat{A}$  is  $T_0$ , then the irreducible representations are lifted from a fibre of a subfactor of the form  $A(G \cdot P) \rtimes_\alpha G$ . Since these fibres are Morita equivalent to  $\mathcal{K} \rtimes_{\tilde{\alpha}} G_P$  for an elementary  $C^*$ -algebra  $\mathcal{K}$ , the type structure is determined by such crossed products.

## 4.2 Actions on the compacts

These comments on actions on the compact operators are taken from [Wil07, §7.3]. Hence we consider a dynamical system  $(\mathcal{K}(\mathcal{H}), G, \alpha)$ . (The homomorphism  $\alpha : G \rightarrow \text{Aut } \mathcal{K}(\mathcal{H})$  is often called a *projective representation*.) Each  $\alpha_s \in \text{Aut } \mathcal{K}(\mathcal{H})$  must be of the form  $\alpha_s = \text{Ad } U$  for some unitary  $U \in U(\mathcal{H})$ . Form the group

$$G' = \{ (s, U) \in G \times U(\mathcal{H}) : \alpha_s = \text{Ad } U \}.$$

We get a short exact sequence

$$1 \longrightarrow \mathbf{T} \xrightarrow{j} G' \xrightarrow{j} G \longrightarrow e.$$

Hence  $G'$  is locally compact. Let  $\tau : \mathbf{T} \rightarrow \mathbf{T}$  be given by  $\tau(z) = \bar{z}$ . A representation  $\pi$  of  $G'$  preserves  $\tau$  if  $\pi(e, zI) = \tau(z)I_{\mathcal{H}_\pi} = \bar{z}I_{\mathcal{H}_\pi}$ . Let  $I$  be the ideal in  $M(C^*(G'))$  generated by the set  $\{ \iota_{G'}((e, z1_{\mathcal{H}})) - \bar{z}1_{G'} \}$  and  $I_\tau$  of  $C^*(G')$  given by  $I_\tau := I \cap C^*(G')$ . Then  $I_\tau$  is such that  $\pi$  preserves  $\tau$  if and only if  $I_\tau \subset \ker \pi$ . Hence the representations of  $C^*(G', \tau) := C^*(G')/I_\tau$  are in one-to-one correspondence with  $\tau$ -preserving representations of  $G'$ .

We can also realize  $C^*(G, \tau)$  more concretely as the completion of the  $*$ -subalgebra  $C_c(G', \tau) = \{ f \in C_c(G') : f(s, zU) = zf(s, U) \}$  with respect to the quotient norm  $\| \cdot \|_\tau$ . If  $f \in C_c(G', \tau)$  and  $T \in \mathcal{K}(\mathcal{H})$ , then the map  $(s, U) \mapsto f(s, U)TU^*$  depends only on  $s$ . Hence we get a  $*$ -homomorphism  $\psi : C_c(G', \tau) \odot \mathcal{K}(\mathcal{H}) \rightarrow C_c(G, \mathcal{K}(\mathcal{H}))$  by sending  $\psi(f \otimes T)(s) = f(s, U)TU$ .

**Theorem 4.2.1** ([Wil07, Theorem 7.21]). *The map  $\psi$  extends to an isomorphism of*

$$C^*(G', \tau) \otimes \mathcal{K}(\mathcal{H}) \quad \text{with} \quad \mathcal{K}(\mathcal{H}) \rtimes_\alpha G.$$

### 4.2.1 The Mackey obstruction

The introduction of the extension  $G'$  is disconcerting, but it is a fundamental issue in addressing the representation theory of dynamical systems with  $A$  non-commutative. One way to avoid mention of  $G'$  and “stick with  $G$ ” is to work with

multiplier representations of  $G$  as in [Wil07, §D.3]. The idea is to take a Borel section  $c : G \rightarrow G'$  for the quotient map  $j : G' \rightarrow G$  above. Then, after composing with the projection from  $G'$  to  $U(\mathcal{H})$ , we get a Borel map  $U : G \rightarrow U(\mathcal{H})$  and a Borel function  $\omega : G \times G \rightarrow \mathbf{T}$  such that  $U_s U_r = \omega(s, r) U_{sr}$  and  $\alpha_s = \text{Ad } U_s$ . We call  $\omega$  a 2-cocycle or *multiplier* and  $U$  an  $\omega$ -representation of  $G$ . The class of  $\omega$  in  $H^2(G, \mathbf{T})$  is called the *Mackey Obstruction* for  $\alpha$  (and depends only on  $\alpha$ ). It is the obstruction to  $U$  being a bonafide homomorphism (see [Wil07, Proposition D.27]).

Note that the  $\tau$ -preserving representations  $\pi$  of  $G'$  are in one-to-one correspondence with  $\bar{\omega}$ -representations of  $G$ : let  $U_s = \pi(c(s))$  where  $c$  is the section for  $j : G' \rightarrow G$  in the previous paragraph.

If  $\omega$  is a multiplier, then we can make a  $*$ -algebra out of  $L^1(G)$  via<sup>1</sup>

$$f * g(s) = \int_G f(r) g(r^{-1}s) \omega(r, r^{-1}s) d\mu(r), \quad (4.1)$$

(and, if you insist,  $f^*(s) = \overline{\omega(s, s^{-1}) \Delta(s^{-1}) f(s^{-1})}$ .) We can complete  $L^1(G)$  to a  $C^*$ -algebra  $C^*(G, \omega)$  whose representations correspond exactly to  $\omega$ -representations.

In the previous discussion,  $C^*(G, \bar{\omega}) \cong C^*(G', \tau)$  and

$$\mathcal{K}(\mathcal{H}) \rtimes_{\alpha} G \cong C^*(G, \bar{\omega}) \otimes \mathcal{K}(\mathcal{H})$$

where  $\omega$  is the Mackey obstruction corresponding to  $\alpha$ .

## 4.2.2 The type question again

Let  $(A, G, \alpha)$  be a regular dynamical system with  $A$  GCR. Then the representations are all induced and factor through a subquotient Morita equivalent to  $\mathcal{K} \rtimes_{\alpha} G_P \cong \mathcal{K} \otimes C^*(G_P, \bar{\omega}_P)$  where  $\omega_P$  represents the Mackey obstruction. Then  $A \rtimes_{\alpha} G$  is GCR if and only if every  $C^*(G_P, \bar{\omega}_P)$  is ([Wil07, Proposition 7.33]).

## 4.3 Non-regular systems

If  $A$  is any (separable)  $C^*$ -algebra and  $\mathcal{H}$  is a separable infinite-dimensional Hilbert space, the set  $\text{Irr}(A, \mathcal{H})$  of irreducible representations of  $A$  on  $\mathcal{H}$  is a Polish space (see [Arv76, §4.1]). Of course, every (infinite-dimensional) irreducible representation of  $A$  has multiple representatives in  $\text{Irr}(A, \mathcal{H})$  and the spectrum is the orbit space  $\hat{A} = U(\mathcal{H}) \backslash \text{Irr}(A, \mathcal{H})$ . The adult version of the Mackey–Glimm Dichotomy implies that if  $A$  fails to be GCR, then there is no Borel set in  $\text{Irr}(A, \mathcal{H})$  containing exactly one representative of each equivalence class (see [Arv76, §4]). Many people regard this as a “proof” that you are not going to be able to give a coherent

<sup>1</sup>It is necessary to switch to  $L^1$  here since multipliers  $\omega$  are only guaranteed to be Borel and not necessarily continuous. Hence (4.1) is no longer necessarily back in  $C_c(G)$  even if  $f$  and  $g$  are.



description of the set  $\hat{A}$  of unitary equivalence classes of irreducible representations of  $A$ .

If  $(A, G, \alpha)$  is not regular, then  $A \rtimes_\alpha G$  is almost certainly not GCR. Hence the above discussion implies that we are not likely to attain a coherent description of the irreducible representations of  $A \rtimes_\alpha G$  in the non-regular case.

### 4.3.1 The generalized Effros–Hahn conjecture

But mathematics does not ban all hope – just unreasonable hope. In 1967, Effros and Hahn conjectured that we could at least recover all the primitive ideals of  $C_0(X) \rtimes_{\text{lt}} G$  via induction from the stability groups provided  $G$  is *amenable*. Note that a description of  $\text{Prim } A \rtimes_\alpha G$  together with its Jacobson topology gives a complete description of the ideal structure of  $A \rtimes_\alpha G$ .

This motivates the following definition. We say that  $(A, G, \alpha)$  is *EH-regular* if every primitive ideal of  $A \rtimes_\alpha G$  is induced from a stability group. That is, given  $K \in \text{Prim } A \rtimes_\alpha G$ , there is a  $P \in \text{Prim } A$  and  $J \in \text{Prim } A \rtimes_\alpha G_P$  such that  $\text{Ind}_{G_P}^G J = K$  and  $\text{Res } J = P$ .

### 4.3.2 The regular case

**Example 4.3.1** ([Wil07, Theorem 8.16 plus note J.6 in the errata]). Suppose that  $A$  is GCR and  $G \backslash \text{Prim } A$  is  $T_0$ . (More generally, assume that  $(A, G, \alpha)$  is regular with  $G \backslash \text{Prim } A$  almost Hausdorff.) Then  $(A, G, \alpha)$  is EH-regular.

**Remark 4.3.2.** Of course, we have a lot more in these cases. Every irreducible representation is induced. The point is that perhaps EH-regularity is not such a big ask.

### 4.3.3 Minimal actions

We say that a  $G$ -space  $X$  is *minimal* if  $X$  admits no nontrivial closed  $G$ -invariant sets. Thus  $X$  is minimal if and only if every orbit is dense. If  $G$  does not act minimally on  $\text{Prim } A$ , then there is a nontrivial closed invariant set  $F$ . The corresponding ideal  $I$  is invariant and  $I \rtimes_\alpha G$  is a nontrivial ideal in  $A \rtimes_\alpha G$ . Hence if  $A \rtimes_\alpha G$  is simple, then  $G$  acts minimally on  $\text{Prim } A$ .

**Proposition 4.3.3** ([Wil07, Theorem 8.20]). *Suppose that  $(A, G, \alpha)$  is EH-regular and that  $G$  acts freely on  $\text{Prim } A$ . Then  $A \rtimes_\alpha G$  is simple if and only if  $G$  acts minimally on  $\text{Prim } A$ .*

*Proof.* We only need to see that  $G$  acting minimally implies  $A \rtimes_\alpha G$  is simple. Note that every primitive ideal of  $A \rtimes_\alpha G$  is of the form  $\text{Ind}_e^G P$  for  $P \in \text{Prim } A$ . Since each orbit is dense, given  $Q \in \text{Prim } A$ , we have  $Q \in \overline{G \cdot P}$  or  $Q \supset \bigcap s \cdot P$ . Since  $\text{Ind}_e^G s \cdot P = \text{Ind}_e^G P$  (because  $\text{Ind } \rho$  and  $\text{Ind } \rho \circ \alpha_s$  are equivalent [Wil07,

Lemma 5.8]), we have  $\text{Ind}_e^G Q \supset \text{Ind}_e^G P$ . By symmetry,  $\text{Ind}_e^G P = \text{Ind}_e^G Q$  for any  $P$  and  $Q$  in  $\text{Prim } A$ . This suffices.  $\square$

After considerable activity, Gootman and Rosenberg succeeded in establishing the validity of the Effros–Hahn conjecture in 1979. Their work depended on a considerable foundation built by Sauvageot [Sau78, Sau79].

**Theorem 4.3.4** (Gootman–Rosenberg–Sauvageot).

*Every separable system  $(A, G, \alpha)$  with  $G$ -amenable is EH-regular.*

Suffice it to say that the proof is somewhat involved. There is enough measure theory for everyone as well as a goodly dose of direct integrals in the form of an old decomposition result of Effros – a so-called ideal center decomposition – which expresses an arbitrary representation into a nice direct integral of homogeneous representations. The full details can be found in [Wil07, Theorem 8.21 and Chapter 9].

#### 4.3.4 The converse

**Remark 4.3.5.** Saying that  $(A, G, \alpha)$  is EH-regular simply says that every primitive ideal is of the form  $\text{Ind}_{G_P}^G J$  for  $J$  primitive with  $\text{Res } J = P$ . Neither the definition nor the proof of the GRS-Theorem say anything about the converse: *is an induced primitive ideal primitive?* For that matter, is an induced irreducible representation irreducible?

Proving the converse turns out to be subtle and we don’t have a result without additional hypotheses. Suppose  $A$  is a primitive GCR algebra – that is,  $\{0\}$  is a primitive ideal. Then  $A$  contains an essential elementary ideal  $K$ , and every (primitive) ideal in  $A$  contains  $K$ . Thus,  $\{0\} = \mathcal{O}_K = \{P \in \text{Prim } A : P \not\supset K\}$  is open and dense in  $\text{Prim } A$ . As a consequence, if  $A$  is GCR, then points are locally closed in  $\text{Prim } A$ :  $\overline{\{P\}} \cong \text{Prim } A/P$ . There do exist separable  $C^*$ -algebras  $A$  and  $P \in \text{Prim } A$  which are not locally closed [EW08].

**Theorem 4.3.6.** *Let  $(A, G, \alpha)$  be separable. Suppose either that points in  $\text{Prim } A$  are locally closed, or that each stability group  $G_P$  is normal in  $G$ . (It suffices for  $A$  to be GCR or for  $G$  to be abelian.) Then if  $L$  is an irreducible representation of  $A \rtimes_\alpha G_P$  with  $\ker \text{Res } L = P$ , then  $\text{Ind}_{G_P}^G L$  is irreducible.*

**Remark 4.3.7.** This certainly shows that  $\text{Ind}_{G_P}^G J$  is primitive if  $J$  primitive and  $\text{Res } J = P$ .

*Proof.* The proof at this level of generality uses the full power of Effros’ ideal center direct integral decomposition. The details can be found in [EW08, Propositions 1.8 and 1.9]. In the case  $A = C_0(X)$ , the proof is much more straightforward – see [Wil07, Proposition 8.27].  $\square$

I believe the result – at least at the level of ideals – should be true without any hypotheses on  $A$  (other than separability), but the question remains open.

## 4.4 Transformation groups with $G$ Abelian

We can be a lot more aggressive when we restrict to transformation groups with  $G$  abelian. If  $G$  is abelian, every irreducible representation is a character and the characters of  $G_x$  are all restrictions of characters of  $G$ . Recall that if  $(x, \omega) \in X \times \widehat{G}$ , then we get an induced representation  $L(x, \omega|_{G_x}) = \text{ev}_x \rtimes \omega|_{G_x}$  of  $C_0(X) \rtimes_{\text{lt}} G$ . Theorem 4.3.6 (or [Wil07, Proposition 8.27] in this case) implies that

$$\ker L(x, \omega|_{G_x}) \in \text{Prim } C_0(X) \rtimes_{\text{lt}} G.$$

Combined with the GRS-Theorem, we see that the map  $\Phi$  sending  $(x, \omega) \mapsto \ker L(x, \omega|_{G_x})$  is a surjection from  $X \times \widehat{G}$  onto  $\text{Prim } C_0(X) \rtimes_{\text{lt}} G$ .

**Theorem 4.4.1** ([Wil07, Theorem 8.39]). *Suppose  $(C_0(X), G, \text{lt})$  is separable with  $G$  abelian. Then  $\Phi$  induces a homeomorphism of the quotient space*

$$\mathcal{Q}_G(X) \rtimes \widehat{G}/\sim \quad \text{with} \quad \text{Prim } C_0(X) \rtimes_{\text{lt}} G$$

where  $(\overline{G \cdot x}, \omega) \sim (\overline{G \cdot y}, \rho)$  if  $\overline{G \cdot x} = \overline{G \cdot y}$  and  $\bar{\rho}\omega \in G_x^\perp$ .

*Proof.* For the proof we need to check that  $\Phi$  is continuous as a map from  $X \times \widehat{G}$  to  $\text{Prim } C_0(X) \rtimes_{\text{lt}} G$ . Then we check that  $\Phi(x, \omega) = \Phi(y, \rho)$  if and only if  $\overline{G \cdot x} = \overline{G \cdot y}$  and  $\bar{\rho}\omega \in G_x^\perp$ . Finally, one shows that  $\Phi$  is open.  $\square$

Note that the quotient map  $X \times \widehat{G}$  to  $\mathcal{Q}_G(X) \rtimes \widehat{G}/\sim$  is open. This means understanding the quotient topology on  $\mathcal{Q}_G(X) \rtimes \widehat{G}/\sim$  is pretty straightforward.

### 4.4.1 Pushing the envelope

Suppose now that  $G$  is arbitrary with  $(C_0(X), G, \text{lt})$  EH-regular. By the GRS-Theorem, it suffices for  $G$  to be amenable or the system to be regular. Suppose that there is an abelian normal subgroup  $H$  such that  $G_x \subset H$  for all  $x$ . (For example, suppose that  $G$  acts freely on  $\text{Prim } A$ .) Then  $G$  acts on  $X \times \widehat{H}$ :  $s \cdot (x, \omega) = (s \cdot x, s \cdot \omega)$  where  $s \cdot \omega(h) = \omega(shs^{-1})$ . The  $G$ -action factors through  $\mathcal{Q}_H(X) \times \widehat{H}/\sim$  and  $\text{Prim } C_0(X) \rtimes_{\text{lt}} G$  is homeomorphic to the quasi-orbit space

$$\mathcal{Q}_G(\mathcal{Q}_H(X) \times \widehat{H}/\sim).$$

For details, see [Wil81, Corollaries 5.10 and 5.11].

**Corollary 4.4.2.** *Suppose that  $(C_0(X), G, \text{lt})$  is EH-regular and that  $G$  acts freely on  $X$ . Then  $C_0(X) \rtimes_{\text{lt}} G$  is simple if and only if  $\mathcal{Q}_G(X) = \{pt\}$  if and only if  $G$  acts minimally.*

### 4.4.2 The right result

For questions of simplicity when  $G$  is not abelian, we need to relax the requirement of freeness. This is one of the central themes of [Phi17, Part 4]. We will just scratch the surface here.

If  $G$  is not abelian, then examples show that freeness is too much to ask for as a necessary condition for simplicity. We say that  $X$  is E-free as a  $G$ -space if  $\{x \in X : G_x = \{e\}\}$  is dense.<sup>2</sup> There are actions that are E-free but not free, but they take a bit of work to describe. In fact, Phillips gives an example of a minimal action which is E-free but not free in [Phi17, Example 2.35]. This is not possible for abelian groups. If  $G$  acts minimally on  $X$ , then the action is E-free if and only if there is a point with  $G_x$  trivial.

**Theorem 4.4.3** ([Ren91, Corollary 4.6]). *Suppose that  $G$  is discrete and acts minimally on  $\text{Prim } A$ . If there is a  $P \in \text{Prim } A$  with  $G_P = \{e\}$ , then  $A \rtimes_{\alpha,r} G$  is simple.*

**Remark 4.4.4.** Even E-freeness is not required. If  $G = \mathbf{F}_2$ , then  $C_r^*(\mathbf{F}_2) \cong C \rtimes_r \mathbf{F}_2$  is simple (see [Phi17, Theorem 6.6]).

**Theorem 4.4.5** ([Ren91, Corollary 4.9]). *Suppose that  $G$  is discrete, amenable and the collection of  $P \in F$  with  $G_P = \{e\}$  is dense in every  $G$ -invariant closed subset  $F \subset \text{Prim } A$ , then  $\text{Prim } A \rtimes_{\alpha} G$  is homeomorphic to  $\mathcal{Q}_G(\text{Prim } A)$ .*

**Remark 4.4.6.** Similar assertions for Theorems 4.4.3 and 4.4.5 for non-discrete  $G$  are not known. Furthermore, Theorem 4.4.5 probably fails if  $G$  is not amenable. Nevertheless, there are many positive results in [Phi17, Part 4].

## 4.5 Takai duality

While we are basking in the glory of abelian groups, let us recall one of the key tools in abelian harmonic analysis: *duality*. Recall that if  $G$  is a locally compact abelian group, the collection  $\widehat{G}$  of continuous homomorphisms  $\omega : G \rightarrow \mathbf{T}$  is a locally compact abelian group with respect to the compact-open topology [Wil07, §1.4]. If  $s \in G$ , then  $\iota_s$  gives us a character on  $\widehat{G}$  given by  $\iota_s(\omega) = \omega(s)$ . The Pontryagin Duality Theorem implies  $s \mapsto \iota_s$  is a topological group isomorphism of  $G$  onto the dual of  $\widehat{G}$ . The result that  $\widehat{\widehat{H}} \cong \widehat{G}/H^\perp$  – and hence that every character of  $H$  extends to one on  $G$  – is a consequence.

### 4.5.1 The dual action

To find the dynamical system analogue of Duality for abelian  $G$ , we need the concept of the *dual action*: if  $f \in C_c(G, A)$ , then define  $\widehat{\alpha}_\gamma(f)(s) = \gamma(s)f(s)$ . Since

<sup>2</sup>If  $G$  is a countable discrete group, then E-free is the same as what Phillips calls essentially free in [Phi17]. Unfortunately, the term “essentially free” is used to mean different things in various places in the literature. I have opted for E-free here to connect up with the groupoid literature and [Ren91] in particular.

$\pi \rtimes U(\widehat{\alpha}_\gamma(f)) = \pi \rtimes \bar{\gamma}U(f)$ ,  $\widehat{\alpha}_\gamma$  is isometric and extends to a strongly continuous action  $\widehat{\alpha} : \widehat{G} \rightarrow \text{Aut}(A \rtimes_\alpha G)$  (see [Wil07, §7.1]).

**Example 4.5.1.** If  $A = \mathbf{C}$ , then  $C^*(G) = \mathbf{C} \rtimes_{\text{id}} G \cong C_0(\widehat{G})$ . The dual action is then left-translation:  $\widehat{\text{id}}_\gamma(f)(\omega) = f(\bar{\gamma}\omega)$ . Hence the iterated crossed product  $(\mathbf{C} \rtimes G) \rtimes_{\widehat{\text{id}}} \widehat{G} \cong C_0(\widehat{G}) \rtimes_{\text{lt}} \widehat{G} \cong \mathcal{K}(L^2(\widehat{G})) \cong \mathcal{K}(L^2(G))$ .

**Theorem 4.5.2** (Takai Duality – [Wil07, Theorem 7.1]). *If  $(A, G, \alpha)$  is a dynamical system with  $G$  abelian, then there is an isomorphism  $\Psi$  of the iterated crossed product  $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$  onto  $A \otimes \mathcal{K}(L^2(G))$ . This isomorphism is equivariant for the double dual action of  $G$  on  $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$  and the  $G$ -action on  $A \otimes \mathcal{K}(L^2(G))$  given by  $\alpha \otimes \text{Ad } \rho$  where  $\rho$  is the right-regular representation of  $G$  on  $L^2(G)$ .*

*Proof.* We can view  $C_c(\widehat{G} \times G, A)$  as a dense subalgebra of  $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$ . Then  $\Phi(F)(s, \gamma) := \gamma(s)F(\gamma, s)$  defines an equivariant map into  $C_c(G \times \widehat{G}, A) \subset C^*(\widehat{G}) \otimes A \rtimes_{\widehat{\text{id}}^{-1} \otimes \alpha} G$  where  $\widehat{\text{id}}$  is the dual  $G$ -action on  $C^*(\widehat{G})$ . But  $C^*(\widehat{G}) \otimes A \cong C_0(G, A)$  and an argument that it absorbs  $\alpha$  gives us a map into

$$C_0(G, A) \rtimes_{\text{lt} \otimes \text{id}} G \cong C_0(G) \rtimes_{\text{lt}} G \otimes_{\max} A \cong \mathcal{K}(L^2(G)) \otimes A. \quad \square$$

## Chapter 5

# Tensor Products, $K$ -Theory and Rotation Algebras

### 5.1 Metric amenability

We have already commented (Theorem 3.1.1) that if  $G$  is amenable, then  $\|\cdot\| = \|\cdot\|_r$  on  $C_c(G)$  and  $A \rtimes_\alpha G = A \rtimes_{\alpha,r} G$ . In general, characterizing when  $\|\cdot\| = \|\cdot\|_r$  seems out of reach. However, using some of our ideal theory, we can make some additional comments for regular systems without assuming that  $G$  is amenable. Recall that  $(A, G, \alpha)$  is EH-regular when  $G \setminus \text{Prim } A$  is locally Hausdorff. This always happens when  $A$  is GCR and  $G \setminus \text{Prim } A$  is  $T_0$ . Hence our next result applies to many systems where  $G$  is *not* amenable. For example, it applies to any system where  $G$  acts freely on  $\text{Prim } A$  and  $G \setminus \text{Prim } A$  is almost Hausdorff.

**Theorem 5.1.1.** *Suppose that  $(A, G, \alpha)$  is EH-regular and that every stability group  $G_P$  is amenable. Then  $A \rtimes_\alpha G = A \rtimes_{\alpha,r} G$ .*

*Proof.* It suffices to show that  $\text{Ind}_e^G \{0\} = \{0\}$ . Let  $K \in \text{Prim } A \rtimes_\alpha G$ . By assumption, there is a  $J \in \text{Prim } A \rtimes_\alpha G_P$  such that  $\text{Ind}_{G_P}^G J = K$ . Since  $G_P$  is amenable,  $\text{Ind}_{e^{G_P}}^G \{0\} = \{0\}$ . By Induction in Stages [Wil07, Theorem 5.9],

$$K = \text{Ind}_{G_P}^G J \supset \text{Ind}_{G_P}^G \{0\} \supset \text{Ind}_{G_P}^G (\text{Ind}_e^{G_P} \{0\}) = \text{Ind}_e^G \{0\}.$$

Since  $K$  is arbitrary,  $\text{Ind}_e^G \{0\} = \{0\}$ . □

### 5.2 Tensor products

A primer on  $C^*$ -tensor products can be found in [RW98, Appendix B]. Recall that the maximal tensor product  $A \otimes_{\max} B$  is universal for commuting representations of  $A$  and  $B$ . Thus if  $\pi_A : A \rightarrow B(\mathcal{H})$  and  $\pi_B : B \rightarrow B(\mathcal{H})$  have commuting ranges, then we get a representation  $\pi_A \otimes_{\max} \pi_B : A \otimes_{\max} B \rightarrow B(\mathcal{H})$ . If  $\rho_A : A \rightarrow$

$B(\mathcal{H}_A)$  and  $\rho_B : G \rightarrow B(\mathcal{H}_B)$  are faithful representations, then we get a faithful representation  $\rho_A \otimes \rho_B : A \otimes_{\min} B \rightarrow B(\mathcal{H}_A \otimes \mathcal{H}_B)$ . (In [RW98, Appendix B] the minimal tensor product  $\otimes_{\min}$  is also called the spatial tensor product and denoted  $\otimes_{\sigma}$ .) If  $\kappa = \kappa_{A,B} : A \otimes_{\max} B \rightarrow A \otimes_{\min} B$  is the natural map and we let  $\pi_A = \rho_A \otimes 1$  and  $\pi_B = 1 \otimes \rho_B$ , then  $\pi_A \otimes_{\max} \pi_B = (\rho_A \otimes \rho_B) \circ \kappa_{A,B}$ .

### 5.2.1 Maximal tensor product and trivial actions

We have already mentioned the result that  $A \rtimes_{\text{id}} G \cong A \otimes_{\max} C^*(G)$ . The idea of the proof was that if  $\pi_A$  and  $V$  are commuting representations of  $A$  and  $C^*(G)$ , respectively, then the map sending  $f \otimes a \in C_c(G, A) \subset A \rtimes_{\alpha} G$  to  $f \otimes_{\max} a \in A \otimes_{\max} C^*(G)$  intertwines  $\pi_A \otimes_{\max} V$  and  $(\pi_A \rtimes V)$  where we have conveniently thought of  $V$  as both a unitary representation of  $G$  and as a representation of  $C^*(G)$ .

This is a special case of the following.

**Theorem 5.2.1** ([Wil07, Lemma 2.75]). *The algebraic isomorphism  $j$  of  $C_c(G) \odot (A \odot B)$  onto  $(C_c(G) \odot A) \odot B$  taking  $f \otimes (a \otimes_{\max} b)$  to  $(f \otimes a) \otimes_{\max} b$  induces an isomorphism of  $(A \otimes_{\max} B) \rtimes_{\alpha \otimes \text{id}} G$  with  $(A \rtimes_{\alpha} G) \otimes_{\max} B$ .*

*Proof.* The map intertwines representations  $(\pi_A \otimes_{\max} \pi_B) \rtimes V$  of  $(A \otimes_{\max} B) \rtimes_{\alpha \otimes \text{id}} G$  with representations  $(\pi_A \rtimes V) \otimes_{\max} \pi_B$  of  $(A \rtimes_{\alpha} G) \otimes_{\max} B$ .  $\square$

### 5.2.2 The reduced norm and trivial actions

The corresponding result for reduced crossed products is a bit more subtle. This is another example – similar to exactness – where the non-universality of the reduced norm makes waves. Nevertheless, we can prove the following. Note that the maximal tensor product has been replaced with the spatial or minimal tensor product.

**Theorem 5.2.2** ([Wil07, Lemma 7.16]).  *$(A \otimes_{\min} B) \rtimes_{\alpha \otimes \text{id}, r} G \cong (A \rtimes_{\alpha, r} G) \otimes_{\min} B$ . In particular,  $A \rtimes_{\text{id}, r} G \cong A \otimes_{\min} C^*(G)$ .*

*Proof.* Let  $\rho_A : A \rightarrow B(\mathcal{H}_A)$  and  $\phi_B : B \rightarrow B(\mathcal{H}_B)$  be faithful representations. Let  $\tilde{\pi}_Z \rtimes U = \text{Ind}_e^G(\rho_A \otimes 1_{\mathcal{H}_B}) = \text{Ind}_e^G \rho_A \otimes 1_{\mathcal{H}_B}$ . Let  $\tilde{\pi}_B = 1_{L^2(G, \mathcal{H}_A)} \otimes \rho_B$ . Check that  $(\tilde{\pi}_A \rtimes U) \otimes_{\max} \tilde{\pi}_B$  maps  $(A \rtimes_{\alpha} G) \otimes_{\max} B$  onto  $(A \rtimes_{\alpha, r} G) \otimes_{\min} B$ . Check that  $\text{Ind}_e^G(\rho_A \otimes \rho_B) \circ (\kappa_{A,B} \rtimes \text{id}) = (\tilde{\pi}_A \otimes_{\max} \tilde{\pi}_B) \rtimes U$  maps  $(A \otimes_{\max} B) \rtimes_{\alpha \otimes \text{id}}$  onto  $(A \otimes_{\min} B) \rtimes_{\alpha \otimes \text{id}} G$ .

Let  $j : C_c(G) \odot (A \odot B) \rightarrow (C_c(G) \odot A) \odot B$  be the usual algebraic isomorphism. The upshot of the above constructs is that the following diagram commutes on

elementary tensors.

$$\begin{array}{ccc}
 (A \otimes_{\max} B) \rtimes_{\alpha \otimes \text{id}} G & \xrightarrow[\cong]{j} & (A \rtimes_{\alpha} G) \otimes_{\max} B \\
 \downarrow \kappa' \rtimes \text{id} & & \downarrow \kappa \\
 (A \otimes_{\min} B) \rtimes_{\alpha \otimes \text{id}} G & & (A \rtimes_{\alpha} G) \otimes_{\min} B \\
 \downarrow \text{Ind}_e^G(\rho_A \otimes \rho_B) & & \downarrow \text{Ind}_e^G(\rho_A) \otimes \rho_B \\
 (A \otimes_{\min} B) \rtimes_{\alpha \otimes \text{id}, r} G & \cdots \cdots \cdots & (A \rtimes_{\alpha, r} G) \otimes_{\min} B.
 \end{array}$$

It follows that the algebraic map  $j : C_c(G) \odot (A \odot B) \rightarrow (C_c(G) \odot A) \odot B$  is also isometric for the minimal and reduced norms. Hence  $j$  extends to an isomorphism  $j_r$  as claimed.  $\square$

### 5.3 Nuclear crossed products

We say that  $A$  is *nuclear* if the natural map  $\kappa_{A,B} : A \otimes_{\max} B \rightarrow A \otimes_{\min} B$  is injective for all  $B$ . This just means that the maximal and spacial norms agree on the algebraic tensor product  $A \odot B$ . This is not quite the same thing as saying that  $A \otimes_{\max} B$  and  $A \otimes_{\min} B$  are isomorphic for all  $B$ ! All GCR algebras are nuclear [RW98, Corollary B.49], but the converse is false. If  $G$  is amenable, then  $C^*(G)$  is nuclear. If  $G$  is discrete, then  $G$  is amenable if and only if  $C_r^*(G)$  is nuclear.

**Corollary 5.3.1** ([Wil07, Corollary 7.18]). *If  $A$  is nuclear and  $G$  is amenable, then  $A \rtimes_{\alpha} G$  is nuclear.*

*Proof.* For any  $B$ , we have the diagram from the proof of Theorem 5.2.2:

$$\begin{array}{ccc}
 (A \otimes_{\max} B) \rtimes_{\alpha \otimes \text{id}} G & \xleftarrow[\cong]{} & (A \rtimes_{\alpha} G) \otimes_{\max} B \\
 \downarrow \kappa' \rtimes \text{id} & & \downarrow \kappa \\
 (A \otimes_{\min} B) \rtimes_{\alpha \otimes \text{id}} G & & (A \rtimes_{\alpha} G) \otimes_{\min} B \\
 \downarrow \text{Ind}_e^G(\rho_A \otimes \rho_B) & & \downarrow \text{Ind}_e^G(\rho_A) \otimes \rho_B \\
 (A \otimes_{\min} B) \rtimes_{\alpha \otimes \text{id}, r} G & \xleftarrow[\cong]{} & (A \rtimes_{\alpha, r} G) \otimes_{\min} B.
 \end{array}$$

If  $G$  is amenable, the two lower vertical arrows are isomorphisms. If  $A$  is nuclear, the upper left vertical arrow is an isomorphism. Hence  $\kappa$  is an isomorphism as required.  $\square$

### 5.4 $K$ -theory for crossed products

$K$ -theory and crossed products have an uneasy relationship. There isn't much to say in any generality. Recently, we were seeing a bit more progress, but the



classical results require we restrict to  $\mathbf{Z}$ - or  $\mathbf{R}$ -actions, and for these we rely on Blackadar's book [Bla98]. I will assume that we all have a working familiarity with the subject. In particular, recall that  $K_*$  is a half-exact functor from  $C^*$ -algebras to abelian groups. If  $A$  is a unital  $C^*$ -algebra,  $K_0(A)$  is the Grothendick group of the semigroup of projections in  $M_\infty(A) := \varinjlim M_n(A)$  with respect to Murray–von Neumann equivalence. We have  $K_*(A \otimes \mathcal{K}) \cong K_*(A)$  and either by a theorem or by definition, we have  $K_1(A) \cong K_0(SA)$  where  $SA := C_0(\mathbf{R}, A)$ . By the Bott periodicity,  $K_1(SA) := K_2(A) \cong K_0(A)$ .

### 5.4.1 The six term sequence in $K$ -theory

Every thing we do here depends critically on the ubiquitous six term exact sequence in  $K$ -theory.

**Theorem 5.4.1** ([Bla98, Theorem 9.3.1]). *If  $0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$  is a short exact sequence of  $C^*$ -algebras, then we get a six term exact sequence in  $K$ -theory:*

$$\begin{array}{ccccc} K_1(A) & \xrightarrow{i_*} & K_1(B) & \xrightarrow{j_*} & K_1(C) \\ \uparrow \epsilon & & & & \downarrow \partial \\ K_0(C) & \xleftarrow{j_*} & K_0(B) & \xleftarrow{i_*} & K_0(A). \end{array}$$

### 5.4.2 Connes' Thom isomorphism theorem

Our first intersection with crossed products here is the Connes Thom Isomorphism Theorem for crossed products by  $\mathbf{R}$ .

**Theorem 5.4.2** ([Bla98, Theorem 10.2.2]). *Suppose that  $(A, \mathbf{R}, \alpha)$  is a dynamical system. Then  $K_0(A \rtimes_\alpha \mathbf{R}) \cong K_1(A)$  and  $K_1(A \rtimes_\alpha \mathbf{R}) \cong K_0(A)$ .*

For the proof, let  $C = C_0(\mathbf{R} \cup \{\infty\})$  and let  $\mathbf{R}$  act on  $\mathbf{R} \cup \{\infty\}$  by translation:  $r \cdot c = c + r$  (with  $\infty + r = \infty$  for all  $r$ ). Let  $CA := C \otimes A$  so that we get a short exact sequence

$$0 \longrightarrow SA \rtimes_{\text{lt} \otimes \alpha} \mathbf{R} \longrightarrow CA \rtimes_{\text{lt} \otimes \alpha} \mathbf{R} \longrightarrow A \rtimes_\alpha \mathbf{R} \longrightarrow 0,$$

where  $SA = C_0(\mathbf{R}) \otimes A$  as usual.

**Lemma 5.4.3** ([Bla98, Lemma 10.9.1]).  $SA \rtimes_{\text{lt} \otimes \alpha} \mathbf{R} \cong SA \rtimes_{\text{lt} \otimes \text{id}} \mathbf{R} \cong A \otimes \mathcal{K}$ .

*Proof of Lemma 5.4.3.* To see that  $\alpha$  is absorbed by  $\text{lt}$ , consider the map  $\phi$  from  $C_c(\mathbf{R} \times \mathbf{R}, A)$  to itself given by  $\phi(F)(s, r) = \alpha_r^{-1}(F(s, r))$ . For the second isomorphism  $SA \rtimes_{\text{lt} \otimes \text{id}} \mathbf{R} = (C_0(\mathbf{R}) \otimes A) \rtimes_{\text{lt} \otimes \alpha} \mathbf{R} \cong A \otimes (C_0(\mathbf{R}) \rtimes_{\text{lt}} \mathbf{R})$ .  $\square$

*Proof of the Thom Isomorphism Theorem.* The short exact sequence for  $CA \rtimes_{\text{lt} \otimes \alpha} \mathbf{R}$  yields the six term sequence

$$\begin{array}{ccccc} K_1(SA \rtimes_{\text{lt} \otimes \alpha} \mathbf{R}) & \longrightarrow & K_1(CA \rtimes_{\text{lt} \otimes \alpha} \mathbf{R}) & \longrightarrow & K_1(A \rtimes_{\alpha} \mathbf{R}) \\ \epsilon \uparrow & & & & \downarrow \partial \\ K_0(A \rtimes_{\alpha} \mathbf{R}) & \longleftarrow & K_0(CA \rtimes_{\text{lt} \otimes \alpha} \mathbf{R}) & \longleftarrow & K_0(SA \rtimes_{\text{lt} \otimes \alpha} \mathbf{R}) \end{array}$$

Lemma 5.4.3 allows us to replace  $SA \rtimes_{\text{lt} \otimes \alpha} \mathbf{R}$  with  $A \otimes \mathcal{K}$ . Since  $K_*(A \otimes \mathcal{K}) \cong K_*(A)$  we have

$$\begin{array}{ccccc} K_1(A) & \longrightarrow & K_1(CA \rtimes_{\text{lt} \otimes \alpha} \mathbf{R}) & \longrightarrow & K_1(A \rtimes_{\alpha} \mathbf{R}) \\ \epsilon \uparrow & & & & \downarrow \partial \\ K_0(A \rtimes_{\alpha} \mathbf{R}) & \longleftarrow & K_0(CA \rtimes_{\text{lt} \otimes \alpha} \mathbf{R}) & \longleftarrow & K_0(A). \end{array}$$

With these observations, the theorem amounts to showing that  $\partial$  and  $\epsilon$  are always isomorphisms. If we can show that  $\partial$  is always surjective for any  $A$  and  $\alpha$ , then a diagram chase and replacing  $A$  with  $CA$  and  $\alpha$  with  $\text{lt} \otimes \alpha$  allows us to claim that  $K_*(CA \rtimes_{\text{lt} \otimes \alpha} \mathbf{R}) = \{0\}$ . Hence we have

$$\begin{array}{ccccc} K_1(A) & \longrightarrow & 0 & \longrightarrow & K_1(A \rtimes_{\alpha} \mathbf{R}) \\ \epsilon \uparrow & & & & \downarrow \partial \\ K_0(A \rtimes_{\alpha} \mathbf{R}) & \longleftarrow & 0 & \longleftarrow & K_0(A). \end{array}$$

The result follows from this. Showing the general surjectivity of  $\partial$  is hard and the details are in [Bla98].  $\square$

### 5.4.3 Pimsner–Voiculescu six term exact sequence

Perhaps the most famous result in the intersection of  $K$ -theory and crossed products is the Pimsner–Voiculescu Six Term Exact Sequence for actions by  $\mathbf{Z}$ .

**Theorem 5.4.4** ([Bla98, Theorem 10.2.1]). *Suppose that  $\phi \in \text{Aut } A$ . Then*

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-\phi_*} & K_0(A) & \xrightarrow{(\iota_A)_*} & K_0(A \rtimes_{\phi} \mathbf{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_{\phi} \mathbf{Z}) & \xleftarrow{(\iota_A)_*} & K_1(A) & \xleftarrow{1-\phi_*} & K_1(A). \end{array}$$

For the proof, we need the *mapping cone* for  $\phi \in \text{Aut } A$ :

$$M_{\phi} = \{ f \in C([0, 1], A) : f(1) = \phi(f(0)) \}.$$

Here we let  $SA = C_0((0, 1), A)$ . We get a short exact sequence

$$0 \longrightarrow SA \longrightarrow M_\phi \longrightarrow A \longrightarrow 0.$$

**Proposition 5.4.5** ([Bla98, Proposition 10.3.2]). *Let  $(B, \mathbf{T}, \hat{\alpha})$  be a dynamical system, and let  $(B, \mathbf{R}, \beta)$  be the lift to an  $\mathbf{R}$ -action. Let  $\phi = \widehat{\hat{\alpha}}_1$  be the dual action in  $\text{Aut } B \rtimes_{\hat{\alpha}} \mathbf{T}$ . Then  $B \rtimes_{\beta} \mathbf{R} \cong M_\phi$ .*

Using the proposition, the short exact sequence for the mapping cone  $M_\phi$  becomes

$$0 \longrightarrow S(B \rtimes_{\hat{\alpha}} \mathbf{T}) \longrightarrow B \rtimes_{\beta} \mathbf{R} \longrightarrow B \rtimes_{\hat{\alpha}} \mathbf{T} \longrightarrow 0.$$

If  $B = A \rtimes_{\alpha} \mathbf{Z}$  and  $\hat{\alpha}$  is the dual action, then  $B \rtimes_{\hat{\alpha}} \mathbf{T} \cong A \otimes \mathcal{K}$  by the Takai Duality, and we get a short exact sequence

$$0 \longrightarrow S(A \otimes \mathcal{K}) \longrightarrow (A \rtimes_{\alpha} \mathbf{Z}) \rtimes_{\beta} \mathbf{R} \longrightarrow A \otimes \mathcal{K} \longrightarrow 0.$$

*Proof of the Pimsner–Voculescu Theorem.* The corresponding six term exact sequence is

$$\begin{array}{ccccc} K_1(S(A \otimes \mathcal{K})) & \longrightarrow & K_1(A \rtimes_{\alpha} \mathbf{Z} \rtimes_{\beta} \mathbf{R}) & \longrightarrow & K_1(A \otimes \mathcal{K}) \\ \uparrow & & & & \downarrow \\ K_0(A \otimes \mathcal{K}) & \longleftarrow & K_0(A \rtimes_{\alpha} \mathbf{Z} \rtimes_{\beta} \mathbf{R}) & \longleftarrow & K_0(S(A \otimes \mathcal{K})), \end{array}$$

which after applying  $K_*(A \otimes \mathcal{K}) \cong K_*(A)$ ,  $K_*(SA) = K_{*+1}(A)$  and  $K_*(B \rtimes_{\beta} \mathbf{R}) \cong K_{*+1}(B)$ , as well as the nontrivial business of computing the connecting maps, is

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{(\iota_A)_*} & K_0(A \rtimes_{\alpha} \mathbf{Z}) & \longrightarrow & K_1(A) \\ \uparrow 1-\alpha_* & & & & \downarrow 1-\alpha_* \\ K_0(A) & \longleftarrow & K_1(A \rtimes_{\alpha} \mathbf{Z}) & \xleftarrow{(\iota_A)_*} & K_1(A). \end{array}$$

This is what we wanted to prove. □

#### 5.4.4 Crossed products by $\mathbf{T}$

The Pimsner–Voculescu result is limited to  $\mathbf{Z}$ -actions. Using the Takai Duality, we get a result for  $\mathbf{T} = \widehat{\mathbf{Z}}$ -actions but only at the expense of  $K_*(A \rtimes_{\alpha} \mathbf{T})$  appearing four times in the exact sequence which limits its usefulness.

**Theorem 5.4.6** ([Bla98, §10.6]). *If  $(A, \mathbf{T}, \alpha)$  is a dynamical system and  $\hat{\alpha}_* \in \text{Aut } A \rtimes_{\alpha} \mathbf{T}$  generates the dual action, then*

$$\begin{array}{ccccc} K_0(A \rtimes_{\alpha} \mathbf{T}) & \xrightarrow{1-\hat{\alpha}_*} & K_0(A \rtimes_{\alpha} \mathbf{T}) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(A \rtimes_{\alpha} \mathbf{T}) & \xleftarrow{1-\hat{\alpha}_*} & K_1(A \rtimes_{\alpha} \mathbf{T}). \end{array}$$

*Proof.* Let  $B = A \rtimes_{\alpha} \mathbf{T}$  and apply the Pimsner–Voiculescu Theorem to the dual action and  $B \rtimes_{\hat{\alpha}} \mathbf{Z}$ :

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{1-\hat{\alpha}_*} & K_0(B) & \longrightarrow & K_0(B \rtimes_{\hat{\alpha}} \mathbf{Z}) \\ \uparrow & & & & \downarrow \\ K_1(B \rtimes_{\hat{\alpha}} \mathbf{Z}) & \longleftarrow & K_1(B) & \xleftarrow{1-\hat{\alpha}_*} & K_1(B). \end{array}$$

Now employ the Takai Duality. □

## 5.5 Loose ends

### 5.5.1 Telling the rotation algebras apart

Once upon a time, considerable effort went into trying to decide whether the rotation algebras were all isomorphic. One of the many early applications of  $K$ -theory was to show that – except for the obvious relations – they are all different. The material here is taken from [Yin86].

The rotation algebra  $A_{\theta}$  always has a tracial state  $\tau$  given on  $C_c(\mathbf{Z} \times \mathbf{T})$  by

$$\tau(f) = \int_0^1 f(0, e^{2\pi i t}) dt.$$

The Pimsner–Voiculescu Theorem tells us that  $K_0(A_{\theta}) \cong \mathbf{Z}^2 \cong \mathbf{Z} + \mathbf{Z}[p]$  for an appropriate projection  $p$ . Every trace (the trace is unique if  $\theta$  is irrational) extends to a unique map  $\tau: K_0(A_{\theta}) \rightarrow \mathbf{R}$  such that  $\tau([1]) = 1$  and  $\tau([p]) = \theta$ . If  $\phi: A_{\theta_1} \rightarrow A_{\theta_2}$  is an isomorphism, then we get a commutative diagram

$$\begin{array}{ccc} \mathbf{Z} + \mathbf{Z}[p_1] & \xrightarrow{\phi_*} & \mathbf{Z} + \mathbf{Z}[p_2] \\ & \searrow \tau_1 & \swarrow \tau_2 \\ & \mathbf{R} & \end{array}$$

Since  $\phi_* \in \text{GL}_2(\mathbf{Z})$  and  $\phi_*([1]) = [1]$ ,  $\phi_* \sim \begin{pmatrix} 1 & n \\ 0 & \pm 1 \end{pmatrix}$ . But  $\theta_1 = \tau_1([p_1]) = \tau_2 \circ \phi_*([p_2]) = n \pm \theta_2$ . Hence  $\theta_1 = \pm \theta_2 \pmod{1}$ .

We have proved.

**Theorem 5.5.1.** *The rotation algebras  $A_{\theta_1}$  and  $A_{\theta_2}$  are isomorphic if and only if  $\theta_1 = \pm \theta_2 \pmod{1}$ .*

### 5.5.2 Morita equivalence of rotation algebras

We can also ask when two rotation algebras are Morita equivalent. We already know something more subtle is happening. Recall that if  $K$  and  $H$  are closed subgroups of  $G$ , then  $C_0(G/H) \rtimes_{\text{lt}} K$  and  $C_0(K \backslash G) \rtimes_{\text{rt}} H$  are Morita equivalent. We used  $G = \mathbf{R}$ ,  $K = \mathbf{Z}$  and  $H = \theta \mathbf{Z}$  to see that  $A_\theta$  and  $A_{\frac{1}{\theta}}$  were Morita equivalent. As it happens, this result is a special case of the following.

**Theorem 5.5.2** ([Rie81, Theorem 4]). *The rotation algebras  $A_{\theta_1}$  and  $A_{\theta_2}$  are Morita equivalent if and only if there is a matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbf{Z})$  such that  $\theta_2 = \frac{a\theta_1+b}{c\theta_1+d}$ .*

(It helps to note that  $\text{GL}_2(\mathbf{Z})$  is generated by  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  which correspond to  $\theta \mapsto \frac{1}{\theta}$  and  $\theta \mapsto \theta + 1$ .)

### 5.5.3 A special case

As a special case, we have the following.

**Corollary 5.5.3.** *If  $\theta$  is rational, then  $A_\theta$  is Morita equivalent to  $C(\mathbf{T}^2)$ .*

*Proof.* Just as in showing  $A_\theta$  Morita equivalent to  $A_{\frac{1}{\theta}}$ , we invoke Green's symmetric imprimitivity theorem in the special case  $K$  and  $H$  are closed subgroups of  $G$ , then we know that  $C_0(G/H) \rtimes_{\text{lt}} K$  and  $C_0(K \backslash G) \rtimes_{\text{rt}} H$  are Morita equivalent. Let  $\theta = \frac{p}{q}$  with  $(p, q) = 1$ . Let  $a, b \in \mathbf{Z}$  be such that  $ap + bq = 1$ . Let  $G = \mathbf{R} \times \mathbf{Z}_a$ ,  $K = \{(nq, [n]) \in G : n \in \mathbf{Z}\}$  and  $H = \{(n, [bn]) \in G : n \in \mathbf{Z}\}$ . Check that  $C_0(K \backslash G) \rtimes_{\text{rt}} H \cong A_{\frac{p}{q}}$  while  $C_0(G/H) \rtimes_{\text{lt}} K \cong C_0(\mathbf{T}^2)$ .  $\square$

### 5.5.4 More rotation games

In general, Green's symmetric imprimitivity theorem implies that if  $K$  and  $H$  have commuting free and proper actions on the left and right, respectively, of  $P$ , then  $C_0(P/H) \rtimes_{\text{lt}} K$  and  $C_0(G \backslash P) \rtimes_{\text{rt}} H$  are Morita equivalent.

Let  $Y$  be an  $H$ -space and  $P = G \times Y$ . Let  $s \cdot (g, y) = (sg, y)$  and  $(g, y) \cdot h = (gh, h^{-1} \cdot y)$ . Let  $G \times_H Y$  be the orbit space. Conclude that  $C_0(G \times_H Y) \rtimes_{\text{lt}} G$  is Morita equivalent to  $C_0(Y) \rtimes_{\text{lt}} H$ .

Now let  $H = \mathbf{Z}$  act on  $Y = \mathbf{T}$  by rotation through  $2\pi i\theta$ . Then  $(r, e^{2\pi i\alpha}) \mapsto (e^{2\pi ir}, e^{2\pi i(\alpha - r\theta)})$  is a homeomorphism of  $\mathbf{R} \times_{\mathbf{Z}} \mathbf{T}$  onto  $\mathbf{T}^2$  intertwining the left  $\mathbf{R}$ -action with the flow on the torus  $\mathbf{T}^2$  with slope  $2\pi i\theta$ . Hence  $C(\mathbf{T}^2) \rtimes_{\text{lt}} \mathbf{R}$  is Morita equivalent to  $A_\theta$ .

If  $\theta$  is irrational, then  $A_\theta$  is a unital simple NGCR-algebra<sup>1</sup> which is Morita equivalent to a nonunital simple NGCR-algebra.

<sup>1</sup>A  $C^*$ -algebra is called NGCR if all of its nonzero ideals fail to be GCR.

## Chapter 6

### The Roads Not Taken

Since these notes were meant only as a gentle introduction to a vast subject, I have not tried to push the boundary all the way to current research problems. I have not even mentioned the numerous generalizations actively studied in the literature. An important such example are (Green) twisted crossed products. Specifically, and twisted dynamical system  $(A, G, N, \alpha, \tau)$  is an ordinary dynamical system  $(A, G, \alpha)$  together with a strictly continuous<sup>1</sup> homomorphism  $\tau : N \rightarrow UM(A)$  (called *twisting map*) such that

$$\tau_n a \tau_n^{-1} = \alpha_n(a) \quad \text{and} \quad \bar{\alpha}_s(\tau_n) = \tau_{s n s^{-1}}$$

where  $\bar{\alpha}_s$  is the canonical extension of  $\alpha_s$  to  $M(A)$ . Then there is a quotient of  $A \rtimes_{\alpha} G$ , denoted by  $A \rtimes_{\alpha, \tau} (G, N)$ , such that a representation  $\pi \rtimes V$  factors through  $A \rtimes_{\alpha, \tau} (G, N)$  exactly when it “preserves  $\tau$ ” in that  $\bar{\pi}(\tau_n) = V_n$ . (For details and references, see [Ech11, §8].) This is exactly analogous to the quotient  $C^*(G', \tau)$  of  $C^*(G')$  where  $G'$  is the group extension corresponding to a projective representation  $\alpha : G \rightarrow \mathcal{K}(\mathcal{H})$  as we constructed in Section 4.2. In fact, when we consider  $C^*(G')$  as the crossed product  $\mathbf{C} \rtimes G'$ , then the algebra  $C^*(G', \tau)$  constructed in Section 4.2 is the twisted crossed product for the twisting map  $\tau : \mathbf{T} \rightarrow \mathbf{T} = U(M(\mathbf{C}))$  given by  $\tau(z) = \bar{z}$ .

Another motivating example for twisted crossed products relates to normal subgroups  $N$  of  $G$ . Then we get a new dynamical system  $(A \rtimes_{\alpha} N, G, \beta)$  where on  $C_c(N, A)$  we have  $\beta_s(f)(n) = \delta(s) \alpha_s(f(s^{-1}ns))$  where  $\delta$  is the continuous homomorphism determined by

$$\delta(s) \int_n f(s^{-1}ns) d\mu_N(n) = \int_N f(n) d\mu_N(n).$$

When there is a splitting *homomorphism* for the quotient map  $G \rightarrow G/H$ , then  $G$  is the semidirect product of  $N$  and another subgroup  $K$  isomorphic to

<sup>1</sup>In this context, strict continuity simply means that  $n \mapsto \tau_n a$  is continuous for all  $a \in A$ . See [RW98, Corollary C.8] for more detail and additional references.

$G/N$ . Then we can restrict  $\beta$  to  $K$  and view it as an action of  $G/N$ . The resulting iterated crossed product is isomorphic to  $A \rtimes_{\alpha} G$ . That is,

$$(A \rtimes_{\alpha} H) \rtimes_{\beta} G/N \cong A \rtimes_{\alpha} G. \quad (6.1)$$

Details of this assertion are worked out in [Wil07, §3.3].

Of course, in many cases, there will be no such splitting map. Nevertheless, our intuition suggests that something like (6.1) should be true.

Twisted crossed products provide a sort of generalization of (6.1). We observe that the natural map  $\iota_N : N \rightarrow UM(A \rtimes_{\alpha} N)$  (as defined in Section 1.3.5) is a twisting map for  $\beta$ . Then it turns out that

$$A \rtimes_{\alpha} G \cong (A \rtimes_{\alpha} N) \rtimes_{\beta, \iota_N} (G, N). \quad (6.2)$$

This result was originally due to Green [Gre78, Proposition 1]. A more complete summary can be found in [Ech11, §8].

As with the construction of  $C^*(G', \tau)$  above, it is annoying that  $G/N$  itself does not appear in (6.2). It is obscured by the twist. As we did in Section 4.2.1, we can recover  $G/N$  at the expense of using a Borel section for the quotient map  $G \rightarrow G/N$ . At this level of generality, this results in a suitably Borel  $U(M(A))$ -valued cocycle – called Busby–Smith twisted crossed products.<sup>2</sup> This is analogous to the algebra  $C^*(G, \omega)$  constructed in Section 4.2.1. For the definitions, results and further references pertaining to these sorts of twisted crossed products consult [Lep68, BS70, PR89].

A different sort of generalization comes from the observation that if  $G$  is a discrete group, then  $C_0(X) \rtimes_{\text{lt}} G$  is the same as the groupoid  $C^*$ -algebra associated to the étale groupoid  $X \times G$  as discussed in Aidan Sims lectures in this same series [Sim17], p. 59. In general, one can define a  $C^*$ -algebra  $C^*(G)$  associated to any locally compact groupoid with a Haar system [Ren80]. From there one can introduce the notion of a groupoid dynamical system (even a twisted groupoid system) and the associated crossed product [Ren87, Ren91, Goe10, Goe09, LaL15, MW08b]. All these varieties and flavors are subsumed by the  $C^*$ -algebras associated to (saturated) Fell bundles over groupoids [MW08a, IW12, IW11, IKS16].

Whether or not one wants to pursue any of these generalizations, anyone wanting to study crossed products should be aware that other than these notes and [Wil07], there are nice general treatments by Echterhoff [Ech11] and also by Phillips [Phi17]. The treatment by Phillips has a very little overlap with the treatment here and is focused on the active area of classification problems for  $C^*$ -algebras. Among its many attributes it is an extensive family of detailed examples (a facet too often missing in this subject.) Phillips also provides an extensive reading list of other sources for crossed products and related topics in his introduction.

<sup>2</sup>The “ $L^1$ -version of these sort of crossed products originated with Leptin. But the name “Busby-Smith” is more common in the literature.

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**Part II**

# **Hausdorff Étale Groupoids and Their $C^*$ -algebras**

**Aidan Sims**



## Foreword

These notes were written as supplementary material for a five-hour lecture series presented at the Centre de Recerca Matemàtica at the Universitat Autònoma de Barcelona from the 13th to the 17th of March 2017. This was part of the Intensive Research Program *Operator algebras: dynamics and interactions* which ran from March to July 2017. I thank the CRM for the excellent support, facility, and atmosphere it provided. The whole IRP provided an exceptional environment for research and research interactions, and it was a real pleasure to be a part of the enthusiastic and productive research activity that was going on. The place was really buzzing.

The intention of these notes is to give a brief overview of some key topics in the area of  $C^*$ -algebras associated to étale groupoids. The scope has been deliberately contained to the case of étale groupoids with the intention that much of the representation-theoretic technology and measure-theoretic analysis required to handle general groupoids can be suppressed in this simpler setting. Because these notes are based on a short course, they feature only a small selection of topics, chosen for their relevance and interest to participants in the masterclass. My choice to include or omit any particular topic is not a comment on the interest of that topic in general – it was a question of developing something consistent and coherent that could sensibly be presented in a week’s worth of lectures, and would hopefully seem sensible in the context of the masterclass and the other two lecture series being presented. So, for example, I have not included any discussion here of inverse semigroups and their connections with étale groupoids even though, arguably, inverse semigroups and étale groupoids are more or less inseparable. I apologize wholeheartedly to all those people whose very nice work on groupoids and groupoid  $C^*$ -algebras has every right to be discussed in a set of notes like this but has not been mentioned.

Finally, my thanks to those participants in the masterclass, and to those subsequent readers, who pointed out errors and possible improvements on the draft version of the notes that was circulated at the time of the lecture series. Special thanks to Kevin Brix from the University of Copenhagen for a number of helpful corrections and suggestions, to Valentin Deaconu for spotting a number of typos, and to Becky Armstrong for many helpful suggestions and corrections; and also to Mikael Rørdam for suggesting the neat proof of Lemma 9.2.2.

**Errata.** There are doubtless errors in these notes, despite my best efforts to weed them out and the generous help I have had from others. I will maintain an up-to-date list of errata for these notes as part of a common errata page for these notes and the other two sets of notes in this volume at [math.dartmouth.edu/2017crm/](http://math.dartmouth.edu/2017crm/). Please let me know by email to [asims@uow.edu.au](mailto:asims@uow.edu.au) if you find any typos or other errors – I will be very grateful.



# Chapter 7

## Introduction

Groupoids are algebraic objects that behave like a group except that the multiplication operation is only partially defined. Topological groupoids provide a useful unifying model for groups and group actions, and equivalence relations induced by continuous maps between topological spaces. They also provide a good algebraic model for the quotient of a topological space by a group or semigroup action in instances where the quotient space itself is, topologically, poorly behaved – for example, the quotient of a shift-space determined by the shift map, or the quotient of the circle by an irrational rotation.

The collection  $\mathcal{G}^{(0)}$  of idempotent elements in a groupoid  $\mathcal{G}$  is called its unit space, since these are precisely the elements  $x$  that satisfy  $x\gamma = \gamma$  and  $\eta x = \eta$  whenever these products are defined. This leads to a natural fibred structure of  $\mathcal{G}$  over  $\mathcal{G}^{(0)}$ : the fibre over a unit  $x$  is the collection  $\mathcal{G}_x$  of elements  $\gamma$  for which the product  $\gamma x$  is defined. If  $\mathcal{G}$  is a topological groupoid, then  $\mathcal{G}^{(0)}$ , as well as each  $\mathcal{G}_x$ , is a topological space in the subspace topology, and it is often helpful to think of the subspaces  $\mathcal{G}_x$  as transverse to  $\mathcal{G}^{(0)}$ . In the special case that  $\mathcal{G}$  is a group, its unit space has just one element  $e$ , and then  $\mathcal{G} = \mathcal{G}_e$ . So the analogues, in the setting of groupoids, of topological properties of groups, typically involve corresponding topological conditions on the sets  $\mathcal{G}_x$ . In particular, the analogue of a discrete group is a groupoid in which the sets  $\mathcal{G}_x$  are all discrete in a coherent way. More specifically, we ask that the map that sends each  $\gamma$  to the unique element  $s(\gamma)$  for which  $\gamma s(\gamma)$  makes sense should be a local homeomorphism. Renault [37] called such groupoids “étale” (which means something like “loose” or “spread out”), and the terminology has stuck.

The study of  $C^*$ -algebras associated to groupoids was initiated by Renault in his PhD thesis [37], and was motivated by earlier work, particularly that of Feldman and Moore [19, 20, 21] for von Neumann algebras. Groupoid  $C^*$ -algebras have been studied intensively ever since, and provide useful and concrete models of many classes of  $C^*$ -algebras. The construction and study of groupoid  $C^*$ -algebras in general is fairly involved and requires a significant amount of representation-theoretic background. But just as the study of  $C^*$ -algebras of discrete groups and their

crossed products requires less background than the study of group  $C^*$ -algebras in general, so the restriction to the situation of étale groupoids significantly reduces the overheads involved in studying groupoid  $C^*$ -algebras. And, as Renault realized from the outset, étale-groupoid  $C^*$ -algebras are sufficient to capture a lot of examples. For example, all crossed products of commutative  $C^*$ -algebras by discrete-group actions; all AF algebras; all Cuntz–Krieger algebras and graph algebras; all Kirchberg algebras in the UCT class; and many others.

My intention in preparing these notes was to develop a concise account of the elementary theory of étale groupoids and their  $C^*$ -algebras that minimises the representation-theoretic and analytic background needed. For this reason, I have chosen, for example, not to include any discussion of non-Hausdorff groupoids, even though there are many good reasons for studying these (for example, Exel and Pardo’s  $C^*$ -algebras associated to self-similar actions of groups on graphs admit étale-groupoid models, but these groupoids are frequently non-Hausdorff). I have also skimmed over some of the more technical aspects of groupoid theory (for example the question of amenability). I have tried to include enough examples along the way, together with two sections that outline a couple of important applications of the theory, to illustrate what is going on with key concepts.

We start in Chapter 8 with a discussion of groupoids themselves: the axioms and set up, a number of illustrative examples, the definition of a topological groupoid, and a discussion of the étale condition.

In Chapter 9, we describe the construction of the convolution algebra of an étale groupoid, then of its two  $C^*$ -algebras – the full  $C^*$ -algebra and the reduced  $C^*$ -algebra – and finally of the notion of equivalence of étale groupoids and Renault’s equivalence theorem. We already see advantages to sticking to étale groupoids here, since we are able to get through all of this material using fairly elementary techniques, and in particular without having to include a treatment of Renault’s Disintegration Theorem, which is one of the mainstays of groupoid  $C^*$ -algebra theory in general, but quite a complicated piece of work.

In Chapter 10, we discuss some of the elementary structure theory of groupoid  $C^*$ -algebras. As mentioned earlier, I have chosen to skim over the details of amenability for groupoids, though I have tried to give enough references to help the interested reader find out more. There is a whole book on the subject of amenability for groupoids [2], and unlike the situation for groups, it is far from a done deal. My focus in discussing amenability has been to describe its  $C^*$ -algebraic consequences, and some standard techniques for showing that a given groupoid is amenable. We then go on to discuss effective groupoids (these are like topologically free group actions). Again, things simplify significantly in the étale setting, and we are able to present a short and self-contained proof that every nonzero ideal of the reduced  $C^*$ -algebra of an effective étale (Hausdorff) groupoid must have nonzero intersection with the abelian subalgebra of  $C_0$ -functions on the unit space. We follow this with a discussion of invariant sets of units and the ideal structure of the  $C^*$ -algebra of an amenable étale groupoid. We use the results we have put to-

gether on ideal structure to characterize the amenable étale (Hausdorff) groupoids whose  $C^*$ -algebras are simple. To finish Chapter 10, we describe Anantharaman-Delaroche's notion of locally contracting groupoids, and prove that the reduced  $C^*$ -algebra of a locally contracting groupoid is purely infinite.

Interestingly, modulo treating amenability as a black box, all of the structure theory developed in these first few chapters is done without recourse to any heavy machinery like Renault's Disintegration theorem. Though probably known to, or at least expected by, experts, I am not aware of such an approach having appeared in print previously.

In Chapter 11 we discuss beautiful results of Renault, extending earlier work of Kumjian, that provide a  $C^*$ -algebraic version of Feldman–Moore theory, and then go on to discuss an application of this machinery to the classification of Fell algebras.

In Section 11.1, we discuss the notion of a twist  $\mathcal{E}$  over an étale groupoid  $\mathcal{G}$ , and of the associated full and reduced  $C^*$ -algebras. These can be thought of as the analogue, for groupoids, of the reduced and full twisted  $C^*$ -algebras of a discrete group with respect to a  $\mathbb{T}$ -valued 2-cocycle. The details here begin to get significantly more complicated than in the previous three sections, and so I have given an overview with sketches of proofs rather than a detailed treatment of all the results. We discuss Renault's definition of a Cartan pair of  $C^*$ -algebras, indicate how a twist over an effective étale groupoid gives rise to a Cartan pair of  $C^*$ -algebras, and outline Renault's proof that the twist can be recovered from its Cartan pair, so that Cartan pairs are in bijection with twists over effective étale groupoids. This implies, for example, due to the work of Barlak and Li, that any separable nuclear  $C^*$ -algebra that admits a Cartan subalgebra belongs to the UCT class.

We then wrap up in Section 11.2 by outlining an application of the groupoid technology we have developed to the classification of Fell algebras. Fell algebras are Type I  $C^*$ -algebras that generalize the continuous trace  $C^*$ -algebras, which in turn are the subject of the famous Dixmier–Douady classification theorem. The classical approach to the Dixmier–Douady theorem does not work well for Fell algebras, but another approach is available: we sketch how to construct, from each Fell algebra  $A$ , a Cartan pair  $(C, D)$  in which  $C$  is Morita equivalent to  $A$ . We then outline how to make the collection of isomorphism classes of twists over a given groupoid  $\mathcal{G}$  into a group  $\text{Tw}_{\mathcal{G}}$ . If  $\mathcal{G}$  is the equivalence relation determined by a local homeomorphism of a locally compact Hausdorff space  $Y$  onto a locally compact, locally Hausdorff space  $X$ , then its twist group is isomorphic to a second sheaf-cohomology group of  $X$ . In particular, the pair  $(C, D)$  discussed above determines an element of  $H^2(\widehat{A}, \mathcal{S})$ ; moreover this class is independent of any of the choices involved in our constructions, so we can regard it as an invariant  $\delta(A)$  of  $A$ . The main result discussed in the section says that  $\delta(A)$  is a complete invariant of  $A$ , and also that the range is exhausted in the sense that every element of  $H^2(X, \mathcal{S})$  can be obtained as  $\delta(A)$  for some Fell algebra  $A$  with spectrum  $X$ .



In this section, I have been very brief. I provide no detailed proofs, and very few proof sketches, and instead try to present the big picture. The details can be found in [25].

These notes are just a brief introduction to a small part of the theory of groupoid  $C^*$ -algebras. There are many useful references for the more general theory. The theory began with Renault [37], and this remains the definitive text. Exel [18] and Paterson [31] have both given excellent discussions of étale groupoids – particularly as they relate to inverse semigroups – but in the non-Hausdorff setting, where the details are a little trickier. Renault’s equivalence theorem for (full) groupoid  $C^*$ -algebras first appeared in print in the work of Muhly–Renault–Williams [29], but the approach used here, via linking groupoids, is based on [42] and also owes a lot to many valuable conversations I have had with Alex Kumjian. It is also closely related to [30, 45, 49]. A trove of information about amenability of groupoids is contained in Anantharaman-Delaroche and Renault’s book [2] on the topic. There are countless other very useful references that I have forgotten, or that are hard to come by (for example Paul Muhly’s excellent but lamentably unfinished book on the subject). I apologize to the surely long list of people whose work I have overlooked (despite its being eminently worthy of mention and attention) in this brief and far-from-comprehensive discussion.

What I believe *is* missing from the literature is an elementary and self-contained introduction to the  $C^*$ -algebras of étale Hausdorff groupoids (though Exel’s paper [18] does contain an excellent treatment of the construction of the groupoid  $C^*$ -algebra for non-étale Hausdorff groupoids); these notes should go a little way to filling this gap. I hope that they give a flavour of the subject and a useful reference for those who find themselves in the enviable position of having all their groupoids turn out to be Hausdorff and étale. I think that most of the arguments in the first three sections here (with the exceptions of anything about amenability, and of Anantharaman-Delaroche’s pure-infiniteness result) were developed from scratch; but of course the results are not new, and the treatment reflects the many ideas and techniques that I have accumulated both from the literature, and from discussion and collaboration with many people including Jon Brown, Lisa Clark, Valentin Deaconu, Ruy Exel, Astrid an Huef, Alex Kumjian, Paul Muhly and Dana Williams. Some parts are more identifiably attributable to ideas I learned from others: in particular, the elementary approach to the construction of the universal  $C^*$ -algebra  $C^*(\mathcal{G})$  was showed to me by Robin Deeley during a series of beautiful graduate-level lectures he gave at the research event *Refining  $C^*$ -algebraic invariants for dynamics using  $KK$ -theory* at the MATRIX facility of the University of Melbourne in July 2016. Robin tells me that the idea came to him in turn from lecture notes of Ian Putnam.

## Chapter 8

# Étale Groupoids

### 8.1 What is a groupoid?

The following definition of a groupoid comes from [24] (see [31, page 7]); Hahn himself attributes it to a conversation with G. Mackey. This is a fairly minimal set of axioms, so optimal for the purposes of checking whether a given object is a groupoid, but I refer the reader forward to Remark 8.1.5 for an equivalent, but less efficient, list of axioms that might provide a little more intuition.

**Definition 8.1.1.** A *groupoid* is a set  $\mathcal{G}$  together with a distinguished subset  $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ , a multiplication map  $(\alpha, \beta) \mapsto \alpha\beta$  from  $\mathcal{G}^{(2)}$  to  $\mathcal{G}$  and an inverse map  $\gamma \mapsto \gamma^{-1}$  from  $\mathcal{G}$  to  $\mathcal{G}$  such that

1.  $(\gamma^{-1})^{-1} = \gamma$  for all  $\gamma \in \mathcal{G}$ ;
2. if  $(\alpha, \beta)$  and  $(\beta, \gamma)$  belong to  $\mathcal{G}^{(2)}$ , then  $(\alpha\beta, \gamma)$  and  $(\alpha, \beta\gamma)$  belong to  $\mathcal{G}^{(2)}$ , and  $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ ; and
3.  $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$  for all  $\gamma \in \mathcal{G}$ , and for all  $(\gamma, \eta) \in \mathcal{G}^{(2)}$ , we have  $\gamma^{-1}(\gamma\eta) = \eta$  and  $(\gamma\eta)\eta^{-1} = \gamma$ .

Axiom (2) shows that for products of three groupoid elements, there is no ambiguity in dropping the parentheses (as we do in groups), and simply writing  $\alpha\beta\gamma := (\alpha\beta)\gamma$ .

To get a feeling for groupoids, we begin by exploring some of the consequences of the above axioms.

Given a groupoid  $\mathcal{G}$  we shall write  $\mathcal{G}^{(0)} := \{\gamma^{-1}\gamma \mid \gamma \in \mathcal{G}\}$  and refer to elements of  $\mathcal{G}^{(0)}$  as *units* and to  $\mathcal{G}^{(0)}$  itself as the *unit space*. Since  $(\gamma^{-1})^{-1} = \gamma$  for all  $\gamma$ , we also have  $\mathcal{G}^{(0)} = \{\gamma\gamma^{-1} \mid \gamma \in \mathcal{G}\}$ . We define  $r, s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  by

$$r(\gamma) := \gamma\gamma^{-1} \quad \text{and} \quad s(\gamma) := \gamma^{-1}\gamma$$

for all  $\gamma \in \mathcal{G}$ .

**Lemma 8.1.2.** *If  $\mathcal{G}$  is a groupoid and  $\gamma \in \mathcal{G}$ , then  $(r(\gamma), \gamma)$  and  $(\gamma, s(\gamma))$  belong to  $\mathcal{G}^{(2)}$ , and*

$$r(\gamma)\gamma = \gamma = \gamma s(\gamma).$$

*We have  $r(\gamma^{-1}) = s(\gamma)$  and  $s(\gamma^{-1}) = r(\gamma)$ . Moreover,  $\gamma^{-1}$  is the unique element such that  $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$  and  $\gamma\gamma^{-1} = r(\gamma)$ , and also the unique element such that  $(\gamma^{-1}, \gamma) \in \mathcal{G}^{(2)}$  and  $\gamma^{-1}\gamma = s(\gamma)$ .*

*Proof.* The first statement, is just axiom (3) with  $\eta = \gamma^{-1}$ .

We have  $r(\gamma^{-1}) = \gamma^{-1}(\gamma^{-1})^{-1} = \gamma^{-1}\gamma = s(\gamma)$  by axiom (1).

If  $(\gamma, \alpha) \in \mathcal{G}^{(2)}$  and  $\gamma\alpha = r(\gamma) = \gamma\gamma^{-1}$ , then axiom (2) shows that  $(\gamma^{-1}\gamma, \alpha) \in \mathcal{G}^{(2)}$  and  $\alpha = \gamma^{-1}\gamma\alpha = \gamma^{-1}r(\gamma) = \gamma^{-1}s(\gamma^{-1}) = \gamma^{-1}$ . A similar argument shows that  $\alpha\gamma = s(\gamma)$  only for  $\alpha = \gamma^{-1}$ .  $\square$

We also quickly see that groupoids have cancelation.

**Lemma 8.1.3.** *Let  $\mathcal{G}$  be a groupoid. Suppose that  $(\alpha, \gamma), (\beta, \gamma) \in \mathcal{G}^{(2)}$  and that  $\alpha\gamma = \beta\gamma$ . Then  $\alpha = \beta$ . Similarly if  $(\gamma, \alpha), (\gamma, \beta) \in \mathcal{G}^{(2)}$  and  $\gamma\alpha = \gamma\beta$  then  $\alpha = \beta$ .*

*Proof.* If  $\alpha\gamma = \beta\gamma$ , then axioms (2) and (3) show that  $\alpha = \alpha\gamma\gamma^{-1} = \beta\gamma\gamma^{-1} = \beta$ .  $\square$

**Lemma 8.1.4.** *Let  $\mathcal{G}$  be a groupoid. Then  $(\alpha, \beta) \in \mathcal{G}^{(2)}$  if and only if  $s(\alpha) = r(\beta)$ . We have*

1.  $r(\alpha\beta) = r(\alpha)$  and  $s(\alpha\beta) = s(\beta)$  for all  $(\alpha, \beta) \in \mathcal{G}^{(2)}$ ;
2.  $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$  for all  $(\alpha, \beta) \in \mathcal{G}^{(2)}$ ; and
3.  $r(x) = x = s(x)$  for all  $x \in \mathcal{G}^{(0)}$ .

*Proof.* First suppose that  $s(\alpha) = r(\beta)$ ; that is,  $\alpha^{-1}\alpha = \beta\beta^{-1}$ . In particular, we have  $(\alpha, \beta\beta^{-1}) = (\alpha, \alpha^{-1}\alpha) \in \mathcal{G}^{(2)}$ . Since  $(\beta\beta^{-1}, \beta) \in \mathcal{G}^{(2)}$  as well, we deduce from axiom (2) that  $(\alpha, \beta\beta^{-1}\beta)$ , which is just  $(\alpha, \beta)$ , belongs to  $\mathcal{G}^{(2)}$ . On the other hand, if  $(\alpha, \beta) \in \mathcal{G}^{(2)}$ , then  $(\alpha^{-1}, \alpha\beta) \in \mathcal{G}^{(2)}$  with  $\alpha^{-1}\alpha\beta = \beta = r(\beta)\beta$ , and so Lemma 8.1.3 shows that  $s(\alpha) = \alpha^{-1}\alpha = r(\beta)$ .

For (1) observe that axiom (2) shows that  $(r(\alpha), \alpha\beta)$  belongs to  $\mathcal{G}^{(2)}$  and  $r(\alpha)(\alpha\beta) = (r(\alpha)\alpha)\beta = \alpha\beta = r(\alpha\beta)(\alpha\beta)$ . So Lemma 8.1.3 shows that  $r(\alpha) = r(\alpha\beta)$ . A similar argument gives  $s(\beta) = s(\alpha\beta)$ .

For (2) we use (1) to see that  $(\beta^{-1}, \alpha^{-1})$  and  $(\alpha\beta, \beta^{-1}\alpha^{-1})$  belong to  $\mathcal{G}^{(2)}$ . Since  $r(\beta^{-1}) = s(\beta)$  and since (1) gives  $s(\alpha\beta\beta^{-1}) = s(\beta^{-1}) = r(\alpha^{-1})$ , we can use (1) twice more to see that the products  $(\alpha\beta\beta^{-1})\alpha^{-1}$  and  $(\alpha\beta)(\beta^{-1}\alpha^{-1})$  make sense and are equal. We have  $(\alpha\beta\beta^{-1})\alpha^{-1} = \alpha\alpha^{-1} = r(\alpha) = r(\alpha\beta)$ , and so uniqueness in the final statement of Lemma 8.1.2 implies that  $\beta^{-1}\alpha^{-1} = (\alpha\beta)^{-1}$ .

For (3), fix  $x \in \mathcal{G}^{(0)}$ , say  $x = \gamma^{-1}\gamma$ . Then (1) shows that  $r(x) = r(\gamma^{-1}) = s(\gamma) = \gamma^{-1}\gamma = x$ . Similarly,  $s(x) = s(\gamma^{-1}\gamma) = s(\gamma) = \gamma^{-1}\gamma = x$ .  $\square$

In many places you will find the definition of a groupoid summarized with the pithy “a groupoid is a small category with inverses.” The above results should convince you that this is equivalent to Definition 8.1.1. It is a slick definition, but if it means much to you, then you probably already knew what a groupoid was anyway. . . .

Lemma 8.1.4 shows that if  $\mathcal{G}$  is a groupoid, then  $\mathcal{G}^{(2)} = \{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} \mid s(\alpha) = r(\beta)\}$ . When describing a groupoid (in an example) it is often convenient, and more helpful, to make use of this: we typically specify the set  $\mathcal{G}$ , the distinguished subset  $\mathcal{G}^{(0)}$  and the maps  $r, s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  first; and then specify an associative multiplication map from  $\{(\alpha, \beta) \mid s(\alpha) = r(\beta) \in \mathcal{G}^{(0)}\}$  to  $\mathcal{G}$  satisfying  $r(\alpha\beta) = r(\alpha)$ ,  $s(\alpha\beta) = s(\beta)$  and  $r(\alpha)\alpha = \alpha = \alpha s(\alpha)$ , and specify an inverse map satisfying  $s(\alpha^{-1}) = r(\alpha)$ ,  $r(\alpha^{-1}) = s(\alpha)$  and  $\alpha^{-1}\alpha = s(\alpha)$  and  $\alpha\alpha^{-1} = r(\alpha)$ . Using the results above, you should be able to convince yourself that this is equivalent to Definition 8.1.1. We will specify groupoids this way throughout these notes.

**Remark 8.1.5.** An earlier version of these notes contained the following definition of a groupoid: A groupoid is a set  $\mathcal{G}$  with a distinguished subset  $\mathcal{G}^{(0)}$ , maps  $r, s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ , a map  $(\alpha, \beta) \mapsto \alpha\beta$  from  $\{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} \mid s(\alpha) = r(\beta)\}$  to  $\mathcal{G}$  and a map  $\gamma \mapsto \gamma^{-1}$  from  $\mathcal{G}$  to  $\mathcal{G}$  with the following properties:

- (G1)  $r(x) = x = s(x)$  for all  $x \in \mathcal{G}^{(0)}$ ;
- (G2)  $r(\gamma)\gamma = \gamma = \gamma s(\gamma)$  for all  $\gamma \in \mathcal{G}$ ;
- (G3)  $r(\gamma^{-1}) = s(\gamma)$  and  $s(\gamma^{-1}) = r(\gamma)$  for all  $\gamma \in \mathcal{G}$ ;
- (G4)  $\gamma^{-1}\gamma = s(\gamma)$  and  $\gamma\gamma^{-1} = r(\gamma)$  for all  $\gamma \in \mathcal{G}$ ;
- (G5)  $r(\alpha\beta) = r(\alpha)$  and  $s(\alpha\beta) = s(\beta)$  whenever  $s(\alpha) = r(\beta)$ ; and
- (G6)  $(\alpha\beta)\gamma = \alpha(\beta\gamma)$  whenever  $s(\alpha) = r(\beta)$  and  $s(\beta) = r(\gamma)$ .

Lemmas 8.1.2 and 8.1.4 show that every groupoid has these properties, and it is not hard to check that given the structure above, putting  $\mathcal{G}^{(2)} := \{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} \mid s(\alpha) = r(\beta)\}$  yields a groupoid according to Definition 8.1.1. (The only thing to check is that (G1)–(G6) force  $(\gamma^{-1})^{-1} = \gamma$ ; and since (G2), (G4) and (G6) ensure cancellativity just as in Lemma 8.1.3, this follows from the observation that (G3) and (G4) give  $(\gamma^{-1})^{-1}\gamma^{-1} = s(\gamma^{-1}) = r(\gamma) = \gamma\gamma^{-1}$ .) In the end, I decided to stick with Definition 8.1.1 because it is the definition that appears in the literature. But I still feel that (G1)–(G6), while not maximally efficient, are a good way to present groupoids to an audience unfamiliar with them and trying to come to grips with them: this presentation emphasizes the point of view of a groupoid as a collection of reversible arrows, carrying an associative composition operation, between points in  $\mathcal{G}^{(0)}$  (see Example 8.1.15 below).

**Corollary 8.1.6.** *If  $\mathcal{G}$  is a groupoid then  $\mathcal{G}^{(0)} = \{\gamma \in \mathcal{G} \mid (\gamma, \gamma) \in \mathcal{G}^{(2)} \text{ and } \gamma^2 = \gamma\}$ .*

*Proof.* Lemma 8.1.2 combined with Lemma 8.1.4(3) gives  $\subseteq$ . For  $\supseteq$ , suppose that  $(\gamma, \gamma) \in \mathcal{G}^{(2)}$  and  $\gamma^2 = \gamma$ . Then  $\gamma^2 = \gamma = \gamma s(\gamma)$  by Lemma 8.1.2, and then Lemma 8.1.3 shows that  $\gamma = s(\gamma) \in \mathcal{G}^{(0)}$ .  $\square$

**Example 8.1.7.** Every group  $\Gamma$  can be viewed as a groupoid, with  $\Gamma^{(0)} = \{e\}$ , multiplication given by the group operation, and inversion the usual group inverse. A groupoid is a group if and only if its unit space is a singleton.

**Example 8.1.8** (Group bundles). Let  $X$  be a set, and for each  $x \in X$ , let  $\Gamma_x$  be a group. Let  $\mathcal{G} := \bigcup_{x \in X} \{x\} \times \Gamma_x$ . This is a groupoid with  $\mathcal{G}^{(0)} = \{(x, e_{\Gamma_x}) \mid x \in X\}$  identified with  $X$ ,  $r(x, g) = x = s(x, g)$ ,  $(x, g)(x, h) = (x, gh)$  and  $(x, g)^{-1} = (x, g^{-1})$ .

**Example 8.1.9** (Matrix groupoids). Fix  $N \geq 1$ . Define

$$R_N := \{1, \dots, N\} \times \{1, \dots, N\}.$$

Put  $R_N^{(0)} = \{(i, i) \mid i \leq N\}$ ,  $r(i, j) = (i, i)$ ,  $s(i, j) = (j, j)$  and  $(i, j)(j, k) = (i, k)$ . Then  $R_N$  is a groupoid, and  $(i, j)^{-1} = (j, i)$  for all  $i, j$ . We usually identify  $R_N^{(0)}$  with  $\{1, \dots, N\}$  in the obvious way.

**Example 8.1.10.** There was nothing special about  $\{1, \dots, N\}$ . For any set  $X$ , the set  $R_X := X \times X$  is a groupoid with operations analogous to those above. Again, we identify  $R_X^{(0)}$  with  $X$ .

**Example 8.1.11** (Equivalence relations). More generally again, if  $R$  is an equivalence relation on a set  $X$ , then  $R^{(0)} := \{(x, x) \mid x \in X\}$  is contained in  $R$  by reflexivity; we identify  $R^{(0)}$  with  $X$  again. The maps  $r(x, y) = x$ ,  $s(x, y) = y$ ,  $(x, y)(y, z) = (x, z)$  and  $(x, y)^{-1} = (y, x)$  make  $R$  into a groupoid.

If  $R$  is an equivalence relation on  $X$ , then the map  $\gamma \mapsto (r(\gamma), s(\gamma))$  from  $R$  to  $X \times X$  is the identity map from  $R$  to  $R$ .

Given groupoids  $\mathcal{G}$  and  $\mathcal{H}$ , we call a map  $\phi: \mathcal{G} \rightarrow \mathcal{H}$  a *groupoid homomorphism* if  $(\phi \times \phi)(\mathcal{G}^{(2)}) \subseteq \mathcal{H}^{(2)}$  and  $\phi(\alpha)\phi(\beta) = \phi(\alpha\beta)$  for all  $(\alpha, \beta) \in \mathcal{G}^{(2)}$ .

**Lemma 8.1.12.** *If  $\mathcal{G}$  and  $\mathcal{H}$  are groupoids and  $\phi: \mathcal{G} \rightarrow \mathcal{H}$  is a groupoid homomorphism, then  $\phi(\mathcal{G}^{(0)}) \subseteq \mathcal{H}^{(0)}$ . We have  $\phi(r(\gamma)) = r(\phi(\gamma))$ ,  $\phi(s(\gamma)) = s(\phi(\gamma))$  and  $\phi(\gamma^{-1}) = \phi(\gamma)^{-1}$  for all  $\gamma \in \mathcal{G}$ .*

*Proof.* For  $u \in \mathcal{G}^{(0)}$  we have  $\phi(u)^2 = \phi(u^2) = \phi(u)$ , so  $\phi(u)$  is idempotent and therefore a unit by Corollary 8.1.6. For  $\gamma \in \mathcal{G}$  we have

$$\phi(\gamma)\phi(s(\gamma)) = \phi(\gamma s(\gamma)) = \phi(\gamma) = \phi(\gamma)s(\phi(\gamma))$$

and similarly  $\phi(r(\gamma))\phi(\gamma) = r(\phi(\gamma))\phi(\gamma)$ . So Lemma 8.1.3 shows that  $\phi(s(\gamma)) = s(\phi(\gamma))$  and  $\phi(r(\gamma)) = r(\phi(\gamma))$ . We then have  $\phi(\gamma)\phi(\gamma^{-1}) = \phi(\gamma\gamma^{-1}) = \phi(s(\gamma)) = s(\phi(\gamma))$ , and so the uniqueness assertion in Lemma 8.1.2 shows that  $\phi(\gamma^{-1}) = \phi(\gamma)^{-1}$ .  $\square$

**Lemma 8.1.13.** *If  $\mathcal{G}$  is a groupoid, then  $R = R(\mathcal{G}) \subseteq \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$  defined by  $R = \{(r(\gamma), s(\gamma)) \mid \gamma \in \mathcal{G}\}$  is an equivalence relation on  $\mathcal{G}^{(0)}$ , and  $\gamma \mapsto (r(\gamma), s(\gamma))$  is a surjective groupoid homomorphism from  $\mathcal{G}$  to  $R$ . We call  $R$  the equivalence relation of  $\mathcal{G}$ .*

*Proof.* We have  $(x, x) = (r(x), s(x)) \in R$  for all  $x \in \mathcal{G}^{(0)}$ , so  $R$  is reflexive. We have  $(s(\gamma), r(\gamma)) = (r(\gamma^{-1}), s(\gamma^{-1}))$  for each  $\gamma$ , so  $R$  is symmetric. And if  $(x, y), (y, z) \in R$ , say  $x = r(\alpha)$ ,  $y = s(\alpha)$ , and  $y = r(\beta)$ ,  $z = s(\beta)$ , then  $(x, z) = (r(\alpha\beta), s(\alpha\beta)) \in R$ , so  $R$  is transitive. That is,  $R$  is an equivalence relation. The map  $\gamma \mapsto (r(\gamma), s(\gamma))$  is surjective by definition of  $R$ , and is a homomorphism by Lemma 8.1.4(1).  $\square$

We say that a groupoid  $\mathcal{G}$  is *principal* if  $\gamma \mapsto (r(\gamma), s(\gamma))$  is injective.

**Lemma 8.1.14.** *A groupoid  $\mathcal{G}$  is algebraically isomorphic to an equivalence relation if and only if it is principal, in which case it is algebraically isomorphic to  $R(\mathcal{G})$ .*

*Proof.* We just saw that equivalence relations are always principal, so the “only if” implication is clear. For the “if” implication, suppose that  $\mathcal{G}$  is principal. Lemma 8.1.13 shows that  $\gamma \mapsto (r(\gamma), s(\gamma))$  is a surjective groupoid homomorphism onto  $R(\mathcal{G})$ , and it is injective because  $\mathcal{G}$  is principal.  $\square$

From the preceding lemma, it may seem a little strange, in the first instance, to make the distinction between an “equivalence relation” and a “principal groupoid.” Indeed, algebraically there is no difference. But when we start introducing topology into the mix the distinction makes sense. The term “equivalence relation” is reserved for principal groupoids that have the relative topology inherited from the product topology on  $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ , whereas “principal groupoids” may have finer topologies.

**Example 8.1.15** (Transformation groupoids). Let  $X$  be a set, and let  $\Gamma$  be a group acting on  $X$  by bijections. Let  $\mathcal{G} := \Gamma \times X$ , put  $\mathcal{G}^{(0)} = \{e\} \times X$  (identified with  $X$  in the obvious way), define  $r(g, x) := g \cdot x$  and  $s(g, x) := x$ , and define  $(g, h \cdot x)(h, x) := (gh, x)$  and  $(g, x)^{-1} := (g^{-1}, g \cdot x)$ . Then  $\mathcal{G}$  is a groupoid, called the *transformation groupoid*.

**Example 8.1.16** (Deaconu–Renault groupoids). Let  $X$  be a set,  $\Gamma$  an abelian group, and  $S \subseteq \Gamma$  a subsemigroup of  $\Gamma$  that contains 0. Suppose that  $S$  acts on  $X$  in the sense that we have maps  $x \mapsto s \cdot x$  from  $X \rightarrow X$  satisfying  $s \cdot (t \cdot x) = (s + t) \cdot x$ , and  $0 \cdot x = x$  for all  $x$ . Let

$$\mathcal{G} := \{(x, s - t, y) \mid s \cdot x = t \cdot y\}.$$

Put  $\mathcal{G}^{(0)} = \{(x, 0, x) \mid x \in X\}$  and identify it with  $X$ . Define  $r(x, g, y) = x$  and  $s(x, g, y) = y$ , and put  $(x, g, y)^{-1} = (y, -g, x)$ . Suppose that  $s_1 \cdot x = t_1 \cdot y$  and  $s_2 \cdot y = t_2 \cdot z$ . Then

$$(s_1 + s_2) \cdot x = s_2 \cdot (s_1 \cdot x) = s_2 \cdot (t_1 \cdot y) = (t_1 + s_2) \cdot y = t_1 \cdot (s_2 \cdot y) = t_1 \cdot (t_2 \cdot z) = (t_1 + t_2) \cdot z.$$

So we can define  $(x, g, y)(y, h, z) = (x, g + h, z)$ . Under these operations,  $\mathcal{G}$  is a groupoid.

## 8.2 Isotropy

Before discussing the isotropy of a groupoid, we introduce some standard notation.

**Notation.** For  $x \in \mathcal{G}^{(0)}$  we write  $\mathcal{G}_x := \{\gamma \in \mathcal{G} \mid s(\gamma) = x\}$ , and  $\mathcal{G}^x := \{\gamma \in \mathcal{G} \mid r(\gamma) = x\}$ . In some articles you might see these sets denoted  $\mathcal{G}x$  and  $x\mathcal{G}$ ; this is helpful and sensible notation, but the literature is well established and the superscript-subscript notation is standard, so we will stick with it here. We write  $\mathcal{G}_x^y := \mathcal{G}_x \cap \mathcal{G}^y$ .

We will, however, write  $UV$  for  $\{\alpha\beta \mid \alpha \in U, \beta \in V, s(\alpha) = r(\beta)\}$  for any pair of subsets  $U, V$  of a groupoid  $\mathcal{G}$ .

If  $\mathcal{G}$  is a groupoid, then the *isotropy subgroupoid* of  $\mathcal{G}$ , or just the isotropy of  $\mathcal{G}$ , is the subset  $\text{Iso}(\mathcal{G}) = \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x = \{\gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma)\}$ . It is straightforward to see that the isotropy subgroupoid really is a subgroupoid; indeed, it is a group bundle as in Example 8.1.8.

Clearly  $\mathcal{G}^{(0)} \subseteq \text{Iso}(\mathcal{G})$ .

**Lemma 8.2.1.** *A groupoid  $\mathcal{G}$  is principal if and only if  $\text{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$ .*

*Proof.* If  $\mathcal{G}$  is principal and  $\gamma \in \text{Iso}(\mathcal{G})$ , then  $x = r(\gamma)$  satisfies  $(r(\gamma), s(\gamma)) = (r(x), s(x))$ , and since  $\mathcal{G}$  is principal it follows that  $\gamma = x \in \mathcal{G}^{(0)}$ . Now suppose that  $\text{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$ , and that  $(r(\gamma), s(\gamma)) = (r(\alpha), s(\alpha))$ . Then  $\alpha\gamma^{-1} \in \text{Iso}(\mathcal{G})$  and therefore  $\alpha\gamma^{-1} = r(\alpha)$ . So Lemma 8.1.2 forces  $\alpha = \gamma$ .  $\square$

**Example 8.2.2.** If  $X$  is a set, and  $\Gamma$  is a group acting on  $X$ , then, as usual, the isotropy subgroup of  $\Gamma$  at  $x \in X$  is  $\Gamma_x := \{g \in \Gamma \mid g \cdot x = x\}$ . The isotropy subgroupoid of the transformation groupoid  $\mathcal{G}$  is then  $\bigcup_{x \in X} \{x\} \times \Gamma_x$ , the union of the isotropy subgroups associated to points  $x \in X$ .

**Lemma 8.2.3.** *If  $\mathcal{G}$  is a groupoid and  $\gamma \in \mathcal{G}$ , then the map  $\text{Ad}_\gamma : \alpha \mapsto \gamma\alpha\gamma^{-1}$  is a group isomorphism from  $\mathcal{G}_{s(\gamma)}^{s(\gamma)}$  to  $\mathcal{G}_{r(\gamma)}^{r(\gamma)}$ .*

*Proof.* The map  $\text{Ad}_{\gamma^{-1}}$  is plainly an inverse for  $\text{Ad}_\gamma$ ; and

$$\text{Ad}_\gamma(\alpha) \text{Ad}_\gamma(\beta) = \gamma\alpha\gamma^{-1}\gamma\beta\gamma^{-1} = \text{Ad}_\gamma(\alpha\beta). \quad \square$$

## 8.3 Topological groupoids

Since we are going to be interested here in  $C^*$ -algebras, we will want to topologise our groupoids.

**Definition 8.3.1.** A *topological groupoid* is a groupoid  $\mathcal{G}$  endowed with a locally compact topology under which  $\mathcal{G}^{(0)} \subseteq \mathcal{G}$  is Hausdorff in the relative topology, the maps  $r, s$  and  $\gamma \mapsto \gamma^{-1}$  are continuous, and the map  $(g, h) \rightarrow gh$  is continuous with respect to the relative topology on  $\mathcal{G}^{(2)}$  as a subset of  $\mathcal{G} \times \mathcal{G}$ .

We naturally expect that the unit space should be closed in a topological groupoid; but in fact, this is true only when  $\mathcal{G}$  is Hausdorff.

**Lemma 8.3.2.** *If  $\mathcal{G}$  is a topological groupoid, then  $\mathcal{G}^{(0)}$  is closed in  $\mathcal{G}$  if and only if  $\mathcal{G}$  is Hausdorff.*

*Proof.* First suppose that  $\mathcal{G}$  is Hausdorff, and suppose that  $(x_i)_{i \in I}$  is a net in  $\mathcal{G}^{(0)}$  such that  $x_i \rightarrow \gamma \in \mathcal{G}$ . Since  $r$  is continuous, we have  $x_i = r(x_i) \rightarrow r(\gamma) \in \mathcal{G}^{(0)}$ . Since  $\mathcal{G}$  is Hausdorff, this limit point is unique, and we deduce that  $\gamma = r(\gamma)$ .

Now suppose that  $\mathcal{G}^{(0)}$  is closed. To see that  $\mathcal{G}$  is Hausdorff, it suffices to show that convergent nets have unique limit points. For this, suppose that  $(\gamma_i)_{i \in I}$  is a net and that  $\gamma_i \rightarrow \alpha$  and  $\gamma_i \rightarrow \beta$ . By continuity, we then have  $\gamma_i^{-1}\gamma_i \rightarrow \alpha^{-1}\beta$ . Since each  $\gamma_i^{-1}\gamma_i = s(\gamma_i) \in \mathcal{G}^{(0)}$  and  $\mathcal{G}^{(0)}$  is closed, we deduce that  $\alpha^{-1}\beta \in \mathcal{G}^{(0)}$ . Hence  $\alpha = \beta$ .  $\square$

Although there are many interesting and important examples without these properties, in these notes, all the topological groupoids that I discuss will be second-countable and Hausdorff as topological spaces. So in these notes,  $\mathcal{G}^{(0)}$  is always a closed subset of  $\mathcal{G}$ .

**Example 8.3.3** (Discrete groupoids). Every groupoid is a topological groupoid in the discrete topology.

**Example 8.3.4** (Topological equivalence relations). If  $X$  is a second-countable Hausdorff space, and  $R$  is an equivalence relation on  $X$ , then  $R$  is a topological groupoid in the relative topology inherited from  $X \times X$ .

The previous example, combined with Lemma 8.1.14, shows that if  $\mathcal{G}$  is a principal groupoid, then any second-countable Hausdorff topology on  $\mathcal{G}^{(0)}$  induces a topological groupoid structure on  $\mathcal{G}$ . Moreover, if  $\mathcal{G}$  is a principal topological groupoid, then  $\gamma \mapsto (r(\gamma), s(\gamma))$  is a bijective continuous map from  $\mathcal{G}$  to the topological equivalence relation  $R(\mathcal{G})$ . So the topology on  $\mathcal{G}$  must be finer than the one inherited from the product topology on  $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ . It can be strictly finer:

**Example 8.3.5.** Let  $X := \prod_{i=1}^{\infty} \{0, 1\}$ , viewed as right-infinite strings of 0's and 1's, and given the product topology. Define an equivalence relation  $R$  on  $X$  by  $(x, y) \in R$  if and only if there exists  $n \in \mathbb{N}$  such that  $x_j = y_j$  for all  $j \geq n$ . Let  $\mathcal{R}_{2^\infty} := \{(x, y) \mid x \sim y\}$  be an algebraic copy of  $R$ . For finite words  $v, w \in \{0, 1\}^n$ , define  $Z(v, w) = \{(vx, wx) \mid x \in X\} \subseteq \mathcal{R}_{2^\infty}$ . Observe that

$$Z(v, w) \cap Z(v', w') = \begin{cases} Z(v, w) & \text{if } v = v'u \text{ and } w = w'u \text{ for some } u \\ Z(v', w') & \text{if } v' = vu \text{ and } w' = wu \text{ for some } u \\ \emptyset & \text{otherwise.} \end{cases}$$

So the  $Z(v, w)$  form a base for a topology.

We claim that  $\mathcal{R}_{2^\infty}$  is a topological groupoid in this topology. It is Hausdorff because if  $(x, y) \neq (x', y')$  then there is an  $n$  such that  $(x(0, n), y(0, n)) \neq (x'(0, n), y'(0, n))$ , and then  $Z(x(0, n), y(0, n))$  and  $Z(x'(0, n), y'(0, n))$  separate



these points. The sets  $Z(v, w)$  are also compact: the map  $x \mapsto (vx, wx)$  is a bijective continuous map from the compact space  $X$  to the Hausdorff space  $Z(v, w)$ , and therefore a homeomorphism. The maps  $r, s$  restrict to homeomorphisms of  $Z(v, w)$  onto  $Z(v)$  and  $Z(w)$ , so they are continuous. Inversion is clearly continuous. Multiplication is continuous because the pre-image of  $Z(v, w)$  is

$$\bigcup_{|y|=|v|} \bigcup_{u \in X} (Z(vu, yu) \times Z(yu, wu)) \cap \mathcal{R}_{2^\infty}.$$

This proves the claim.

We claim that this topology is finer than the one inherited from the product topology. To see this, consider the sequence  $\gamma_n = ((0^n 10^\infty, 0^\infty))_{n=1}^\infty$ . In the product topology, we have  $\gamma_n \rightarrow (0^\infty, 0^\infty)$ . But the basic neighbourhoods of  $(0^\infty, 0^\infty)$  are of the form  $Z(0^m, 0^m)$ , and we have  $\gamma_n \notin Z(0^m, 0^m)$  for  $n > m$ . So  $\gamma_n \not\rightarrow (0^\infty, 0^\infty)$  in  $\mathcal{R}_{2^\infty}$ .

As we will see later, this example is important: its  $C^*$ -algebra is  $M_{2^\infty}(\mathbb{C})$ .

**Example 8.3.6.** Suppose that  $X$  is a second-countable locally compact Hausdorff space and that  $\Gamma$  is a locally compact group acting on  $X$  by homeomorphisms as in [52, Section 1.1]. Then the transformation groupoid  $\mathcal{G}$  is a topological groupoid in the product topology.

In these notes, a *local homeomorphism* from  $X$  to  $Y$  is a continuous map  $h : X \rightarrow Y$  such that every  $x \in X$  has an open neighbourhood  $U$  such that  $h(U) \subseteq Y$  is open and  $h : U \rightarrow h(U)$  is a homeomorphism.

**Example 8.3.7.** If  $X$  is a second-countable locally compact Hausdorff space,  $\Gamma$  is a discrete abelian group,  $S$  is a subsemigroup of  $\Gamma$  containing 0, and  $S$  acts on  $X$  by local homeomorphisms, then the Deaconu–Renault groupoid becomes a locally compact Hausdorff groupoid in the topology with basic open sets

$$Z(U, p, q, V) = \{(x, p - q, y) \mid x \in U, y \in V, p \cdot x = q \cdot y\}$$

indexed by pairs  $U, V$  of open subsets of  $X$  and pairs  $p, q \in S$ .

## 8.4 Étale groupoids

In these notes, we will focus on étale groupoids. These are the analogue, in the groupoid world, of discrete groups.

**Definition 8.4.1.** A topological groupoid  $\mathcal{G}$  is *étale* if the range map  $r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  is a local homeomorphism.

The first important consequence of this condition is the following.

**Lemma 8.4.2.** *If  $\mathcal{G}$  is an étale groupoid, then  $\mathcal{G}^{(0)}$  is open in  $\mathcal{G}$ .*

*Proof.* We prove the contrapositive: that is, we suppose that  $\mathcal{G}^{(0)}$  is not open in  $\mathcal{G}$  and show that  $r$  is not a local homeomorphism. Since  $\mathcal{G}^{(0)}$  is not open, there

is a sequence  $(\gamma_n)_{n=1}^\infty$  in  $\mathcal{G} \setminus \mathcal{G}^{(0)}$  such that  $\gamma_n \rightarrow x \in \mathcal{G}^{(0)}$ . Since  $r$  is continuous,  $r(\gamma_n) \rightarrow x$  as well. Fix an open neighbourhood  $U$  of  $x$  in  $\mathcal{G}$ . Then there exists  $n$  such that  $\gamma_n$  and  $r(\gamma_n)$  both belong to  $U$ . Since  $\gamma_n$  is not a unit, we have  $\gamma_n \neq r(\gamma_n)$ , but  $r(\gamma_n) = r(r(\gamma_n))$ . Hence  $r|_U$  is not injective, and we deduce that  $r$  is not locally injective and therefore not a local homeomorphism.  $\square$

**Example 8.4.3.** Every discrete groupoid is étale.

**Example 8.4.4.** The principal groupoid of Example 8.3.5 is étale: we proved that  $r$  is a homeomorphism of  $Z(u, v)$  onto  $Z(u) \subseteq \mathcal{G}^{(0)}$  for each  $(u, v)$ .

**Example 8.4.5.** A transformation groupoid is étale if and only if the acting group  $\Gamma$  is discrete.

**Example 8.4.6.** Deaconu–Renault groupoids associated to actions by local homeomorphisms are always étale: For each  $(x, p, q, y) \in \mathcal{G}$  we can choose open neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $u \mapsto p \cdot u$  is a homeomorphism from  $U$  to  $p \cdot U$ , and  $v \mapsto q \cdot v$  is a homeomorphism of  $V$  onto  $p \cdot V$ . Let  $W = p \cdot U \cap q \cdot V$  and let  $U' := \{u \in U \mid p \cdot u \in W\}$  and  $V' := \{v \in V \mid q \cdot v \in W\}$ . Then  $Z(U', p, q, V')$  is an open neighbourhood of  $(x, p, q, y)$ , and the range map restricts to a homeomorphism of this neighbourhood onto  $U'$ .

**Example 8.4.7** (Graph groupoids). Let  $E$  be a directed graph with vertex set  $E^0$ , edge set  $E^1$  and direction of edges described by range and source maps  $r, s : E^1 \rightarrow E^0$ . Assume that  $E$  is row-finite with no sources (so  $r^{-1}(v)$  is finite and nonempty for every vertex  $v$ ). See [33] for background on graphs in the context of graph  $C^*$ -algebras; we will use Raeburn’s conventions and notation for graphs throughout. Let  $E^\infty$  denote the space of right-infinite paths in  $E$ , so  $E^\infty = \{x_1 x_2 x_3 \cdots \mid x_i \in E^1, r(x_{i+1}) = s(x_i)\}$ . Give  $E^\infty$  the topology inherited from the product space  $\prod_{i=1}^\infty E^1$ . Then  $E^\infty$  is a totally-disconnected locally compact Hausdorff space, and the sets  $Z(\mu) = \{\mu x \mid x \in E^\infty, r(x_1) = s(\mu)\}$  form a base of compact open sets for the topology. The map  $\sigma : E^\infty \rightarrow E^\infty$  given by  $\sigma(x)_i = x_{i+1}$  is a local homeomorphism (it restricts to a homeomorphism on  $Z(\mu)$  whenever  $|\mu| \geq 1$ ), so induces an action of  $\mathbb{N}$  by local homeomorphisms. The associated Deaconu–Renault groupoid  $\mathcal{G}_E = \{(x, m - n, y) \mid \sigma^m(x) = \sigma^n(y)\}$  is called the *graph groupoid* of  $E$ .

Since  $\gamma \mapsto \gamma^{-1}$  is continuous and self-inverse, if  $\mathcal{G}$  is étale, then  $s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  is also a local homeomorphism. So there are plenty of open sets on which  $r, s$  are both homeomorphisms.

**Definition 8.4.8.** A subset  $B$  of an étale groupoid  $\mathcal{G}$  is a *bisection* if there is an open set  $U$  containing  $B$  such that  $r : U \rightarrow r(U)$  and  $s : U \rightarrow s(U)$  are both homeomorphisms onto open subsets of  $\mathcal{G}^{(0)}$ .

**Lemma 8.4.9.** Let  $\mathcal{G}$  be a second-countable Hausdorff étale groupoid. Then  $\mathcal{G}$  has a countable base of open bisections.

*Proof.* Choose a countable dense subset  $\{\gamma_n\}$  of  $\mathcal{G}$ . For each  $\gamma_n$ , choose countable neighbourhood bases  $\{U_{n,i}\}_i$  and  $\{V_{n,i}\}_i$  at  $\gamma_n$  such that  $r$  is a homeomorphism of each  $U_{n,i}$  onto an open set, and  $s$  is a homeomorphism of each  $V_{n,i}$  onto an open set. Then  $\{U_{n,i} \cap V_{n,i} \mid n, i \in \mathbb{N}\}$  is a countable base of open bisections.  $\square$

**Corollary 8.4.10.** *If  $\mathcal{G}$  is an étale groupoid, then each  $\mathcal{G}_x$  and each  $\mathcal{G}^x$  is discrete in the relative topology.*

*Proof.* For each  $\gamma \in \mathcal{G}_x$ , choose an open bisection  $U_\gamma$  containing  $\gamma$ . Then  $U_\gamma \cap \mathcal{G}_x = \{\gamma\}$ , and so  $\{\gamma\}$  is open in  $\mathcal{G}_x$ .  $\square$

We finish this section with the important observation that multiplication is an open map in an étale groupoid. The following quick proof was shown to me by Dana Williams.

**Lemma 8.4.11.** *If  $\mathcal{G}$  is a topological groupoid and  $r$  is an open map, then the multiplication map on  $\mathcal{G}$  is open. In particular, if  $\mathcal{G}$  is étale, then multiplication is an open map.*

*Proof.* Fix open sets  $U, V \subseteq \mathcal{G}$ , and an element  $(\alpha, \beta) \in (U \times V) \cap \mathcal{G}^{(2)}$ . Fix a sequence  $\gamma_i$  converging to  $\alpha\beta$ ; it suffices to show that the  $\gamma_i$  eventually belong to  $UV$ . Fix a descending neighbourhood base  $\{U_j\}_{j \in \mathbb{N}}$  for  $\alpha$  contained in  $U$ . Since  $r$  is an open map, each  $r(U_j)$  is an open neighbourhood of  $r(\alpha)$ . Since  $\gamma_i \rightarrow \alpha\beta$ , we have  $r(\gamma_i) \rightarrow r(\alpha\beta) = r(\alpha)$ , so for each  $j$  we eventually have  $r(\gamma_i) \in r(U_j)$ . Choose  $\alpha_i$  in  $U$  with  $r(\alpha_i) = r(\gamma_i)$  and  $\alpha_i \in U_j$  whenever  $r(\gamma_i) \in r(U_j)$ . Then  $\alpha_i \rightarrow \alpha$ . Hence  $\alpha_i^{-1}\gamma \rightarrow \beta$ , and therefore  $\alpha_i^{-1}\gamma \in V$  for large  $i$ . But then  $\gamma_i = \alpha_i(\alpha_i^{-1}\gamma) \in UV$  for large  $i$ .  $\square$

## Chapter 9

### $C^*$ -algebras and Equivalence

In this section, we will associate two  $C^*$ -algebras to each étale groupoid. As with groups and dynamical systems, each groupoid has both a reduced  $C^*$ -algebra and a full  $C^*$ -algebra. We first discuss the convolution product on  $C_c(\mathcal{G})$  and then its two key  $C^*$ -completions,  $C^*(\mathcal{G})$  and  $C_r^*(\mathcal{G})$ . At the end of the chapter, we discuss equivalence of groupoids and Renault's equivalence theorem for their  $C^*$ -algebras.

#### 9.1 The convolution algebra

**Proposition 9.1.1.** *Let  $\mathcal{G}$  be a second-countable locally compact Hausdorff étale groupoid. For  $f, g \in C_c(\mathcal{G})$  and  $\gamma \in \mathcal{G}$ , the set*

$$\{(\alpha, \beta) \in \mathcal{G}^{(2)} \mid \alpha\beta = \gamma \text{ and } f(\alpha)g(\beta) \neq 0\}$$

*is finite. The complex vector space  $C_c(\mathcal{G})$  is a  $*$ -algebra with multiplication given by  $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$  and involution  $f^*(\gamma) = \overline{f(\gamma^{-1})}$ . For  $f, g \in C_c(\mathcal{G})$  we have  $\text{supp}(f * g) \subseteq \text{supp}(f)\text{supp}(g)$ .*

*Proof.* If  $\alpha\beta = \gamma$ , then  $\alpha \in G^{r(\gamma)}$  and  $\beta \in G_{s(\gamma)}$ . We saw in Corollary 8.4.10 that these are discrete sets, so their intersections with the compact sets  $\text{supp}(f)$  and  $\text{supp}(g)$  are finite. The rest is routine.  $\square$

**Remark 9.1.2.** The convolution formula can be (and often is) equivalently reformulated as

$$(f * g)(\gamma) = \sum_{\alpha \in G^{r(\gamma)}} f(\alpha)g(\alpha^{-1}\gamma).$$

We will see later that the multiplication operation, and also the  $C^*$ -norms, are simpler for elements of  $C_c(\mathcal{G})$  whose supports are contained in a bisection than for general elements. So it is helpful to know that when  $\mathcal{G}$  is étale, such functions span the convolution algebra.

**Lemma 9.1.3.** *Suppose that  $\mathcal{G}$  is a second-countable locally compact Hausdorff étale groupoid. Then*

$$C_c(\mathcal{G}) = \text{span}\{f \in C_c(\mathcal{G}) \mid \text{supp}(f) \text{ is a bisection}\}.$$

*Proof.* Fix  $f \in C_c(\mathcal{G})$ . By Lemma 8.4.9, we can cover  $\text{supp}(f)$  with open bisections, and then use compactness to pass to a finite subcover  $U_1, \dots, U_n$ . Choose a partition of unity  $\{h_i\}$  on  $\text{supp}(f)$  subordinate to the  $U_i$ . The pointwise products  $f_i := f \cdot h_i$  belong to  $C_c(\mathcal{G})$  with  $\text{supp}(f_i) \subseteq U_i$ , and we have  $f = \sum_i f_i$ .  $\square$

One reason why the preceding lemma is so useful is because convolution is very easy to compute for functions supported on bisections.

**Lemma 9.1.4.** *Suppose that  $\mathcal{G}$  is a second-countable locally compact Hausdorff étale groupoid. If  $U, V \subseteq \mathcal{G}$  are open bisections and  $f, g \in C_c(\mathcal{G})$  satisfy  $\text{supp}(f) \subseteq U$  and  $\text{supp}(g) \subseteq V$ , then  $\text{supp}(f * g) \subseteq UV$  and for  $\gamma = \alpha\beta \in UV$ , we have*

$$(f * g)(\gamma) = f(\alpha)g(\beta). \quad (9.1)$$

*We have  $C_c(\mathcal{G}^{(0)}) \subseteq C_c(\mathcal{G})$ . If  $f \in C_c(\mathcal{G})$  is supported on a bisection, then  $f^* * f \in C_c(\mathcal{G}^{(0)})$  is supported on  $s(\text{supp}(f))$  and  $(f^* * f)(s(\gamma)) = |f(\gamma)|^2$  for all  $\gamma \in \text{supp}(f)$ . Similarly  $f * f^*$  is supported on  $r(\text{supp}(f))$  and  $(f * f^*)(r(\gamma)) = |f(\gamma)|^2$  for  $\gamma \in \text{supp}(f)$ . For  $f \in C_c(\mathcal{G})$  and  $h \in C_c(\mathcal{G}^{(0)})$ , we have*

$$(h * f)(\gamma) = h(r(\gamma))f(\gamma) \quad \text{and} \quad (f * h)(\gamma) = f(\gamma)h(s(\gamma)).$$

*Proof.* We have  $f(\gamma) = \sum_{\eta\zeta=\gamma} f(\eta)g(\zeta)$ . For any  $\eta, \zeta$  appearing in the sum, we have  $r(\eta) = r(\gamma)$  and  $s(\zeta) = s(\gamma)$ . Since  $f$  and  $g$  are supported on bisections, and since  $\alpha \in \mathcal{G}^{r(\gamma)}$  and  $\beta \in \mathcal{G}^{s(\gamma)}$  are contained in  $\text{supp}(f)$  and  $\text{supp}(g)$  respectively, it follows that the only term in the sum that can be nonzero is  $f(\alpha)g(\beta)$ .

Since  $\mathcal{G}^{(0)}$  is open by Lemma 8.4.2, we can regard  $C_c(\mathcal{G}^{(0)})$  as a subalgebra of  $C_c(\mathcal{G})$  in the usual way: for  $f \in C_c(\mathcal{G}^{(0)})$  the corresponding element of  $C_c(\mathcal{G})$  agrees with  $f$  on  $\mathcal{G}^{(0)}$  and vanishes on its complement. The remaining statements follow from the convolution formula (9.1).  $\square$

**Example 9.1.5.** Consider the matrix groupoid  $R_N$ . Since this is a finite discrete groupoid, we have  $C_c(R_N) = \text{span}\{1_{(i,j)} \mid i, j \leq N\}$ . Lemma 9.1.4 shows that  $1_{(i,j)}1_{(k,l)} = \delta_{j,k}1_{(i,l)}$ . So the  $1_{(i,j)}$  are matrix units, and  $C_c(R_N) \cong M_N(\mathbb{C})$ .

**Example 9.1.6.** If  $\Gamma$  is a discrete group, regarded as a groupoid, then its convolution algebra as described above is identical to the usual group algebra  $C_c(\Gamma)$ .

**Example 9.1.7.** Let  $\Gamma$  be a discrete group acting on a compact space  $X$ , and let  $\mathcal{G}$  be the associated transformation groupoid. Let  $\alpha$  be the action of  $\Gamma$  on  $C(X)$  induced by the  $\Gamma$  action on  $X$ . Let  $C_c(\Gamma, C(X))$  be the convolution algebra of the  $C^*$ -dynamical system  $(C(X), \Gamma, \alpha)$  described in [52, Section 1.3.2]. Then there is an isomorphism  $\omega : C_c(\mathcal{G}) \rightarrow C_c(\Gamma, C(X))$  given by  $\omega(f)(g)(x) = f(g, x)$  for all  $f \in C_c(\mathcal{G})$ , all  $g \in \Gamma$  and all  $x \in X$ .

## 9.2 The full $C^*$ -algebra

There are two ways to describe the full  $C^*$ -algebra of a discrete group. The first is as the universal  $C^*$ -algebra generated by a unitary representation of  $\Gamma$ . The second is as the universal  $C^*$ -algebra generated by a  $*$ -representation of  $C_c(\Gamma)$ .

There is a version of the first description for groupoids, which we will discuss briefly later; but it is a little technical. The second description, on the other hand, generalizes nicely, and is the one we will make use of in these notes – a luxury that we can afford because we are sticking to étale groupoids throughout. The following elementary construction of the full  $C^*$ -norm on the convolution algebra of an étale groupoid was shown to me by Robin Deeley. It appears, for non-Hausdorff groupoids, in Ruy Exel's paper [18, Definition 3.17]; thanks to Eduardo Scarparo for reminding me.

**Proposition 9.2.1.** *Let  $\mathcal{G}$  be a second-countable locally compact Hausdorff étale groupoid. For each  $f \in C_c(\mathcal{G})$ , there is a constant  $K_f \geq 0$  such that  $\|\pi(f)\| \leq K_f$  for every  $*$ -representation  $\pi : C_c(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{H})$  of  $C_c(\mathcal{G})$  on Hilbert space. If  $f$  is supported on a bisection, we can take  $K_f = \|f\|_\infty$ .*

To prove Proposition 9.2.1, we need the following standard lemma. Thanks to Mikael Rørdam for suggesting this clever little proof.

**Lemma 9.2.2.** *Let  $X$  be a locally compact Hausdorff space, let  $A$  be a  $C^*$ -algebra, and let  $\pi : C_c(X) \rightarrow A$  be a  $*$ -homomorphism. Then  $\|\pi(f)\| \leq \|f\|_\infty$  for all  $f \in C_c(X)$ .*

*Proof.* Fix  $f \in C_c(X)$ . Let  $U := \{x \in X : f(x) \neq 0\}$ . Then  $\pi$  restricts to a homomorphism from the  $C^*$ -algebra  $C_0(U) \subseteq C_c(X)$  to  $A$ . Since  $C^*$ -homomorphisms are norm-decreasing,  $\pi|_{C_0(U)}$  is norm-decreasing, and since  $f \in C_0(U)$ , we deduce that  $\|\pi(f)\| \leq \|f\|_\infty$ .  $\square$

*Proof of Proposition 9.2.1.* Fix  $f \in C_c(\mathcal{G})$  and use Lemma 9.1.3 to write  $f = \sum_{i=1}^n f_i$  with each  $f_i$  supported on a bisection. Define  $K_f := \sum_{i=1}^n \|f_i\|_\infty$ .

Suppose that  $\pi$  is a  $*$ -representation. Then  $\pi|_{C_c(\mathcal{G}^{(0)})}$  is a  $*$ -representation, so Lemma 9.2.2 implies that  $\|\pi(h)\| \leq \|h\|_\infty$  for every  $h \in C_c(\mathcal{G}^{(0)})$ . Lemma 9.1.4 implies first that each  $f_i^* * f_i$  is supported on  $\mathcal{G}^{(0)}$  and second that  $\|f_i^* * f_i\|_\infty = \|f_i\|_\infty^2$ . So

$$\|\pi(f_i)\|^2 = \|\pi(f_i^* * f_i)\| \leq \|f_i^* * f_i\|_\infty = \|f_i\|_\infty^2,$$

and so each  $\|\pi(f_i)\| \leq \|f_i\|_\infty$ . Now the triangle inequality gives  $\|\pi(f)\| \leq K_f$ . If  $f$  is supported on a bisection, then there is just one term in the sum, so  $K_f = \|f\|_\infty$ .  $\square$

This allows us to define the universal  $C^*$ -algebra of an étale groupoid.

**Theorem 9.2.3.** *Suppose that  $\mathcal{G}$  is a second-countable locally compact Hausdorff étale groupoid. There exist a  $C^*$ -algebra  $C^*(\mathcal{G})$  and a  $*$ -homomorphism  $\pi_{\max} : C_c(\mathcal{G}) \rightarrow C^*(\mathcal{G})$  such that  $\pi_{\max}(C_c(\mathcal{G}))$  is dense in  $C^*(\mathcal{G})$ , and such that for*

every  $*$ -representation  $\pi : C_c(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{H})$  there is a representation  $\psi$  of  $C^*(\mathcal{G})$  such that  $\psi \circ \pi_{\max} = \pi$ . The norm on  $C^*(\mathcal{G})$  satisfies

$$\|\pi_{\max}(f)\| = \sup \{ \|\pi(f)\| \mid \pi \text{ is a } * \text{-representation of } C_c(\mathcal{G}) \}$$

for all  $f \in C_c(\mathcal{G})$ .

*Proof.* For each  $a \in C_c(\mathcal{G})$ , Proposition 9.2.1 shows that the set

$$\{ \pi(a) \mid \pi \text{ is a } * \text{-representation of } C_c(\mathcal{G}) \}$$

is bounded above, and it is nonempty because of the zero representation. So we can define  $\rho : C_c(\mathcal{G}) \rightarrow [0, \infty)$  by

$$\rho(f) := \sup \{ \|\pi(f)\| \mid \pi : C_c(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{H}) \text{ is a } * \text{-representation} \}.$$

It is routine to check that  $\rho$  is a  $*$ -seminorm satisfying the  $C^*$ -identity using that each  $f \mapsto \|\pi(f)\|$  has the same properties. So we can define  $C^*(\mathcal{G})$  to be the completion of the quotient of  $C_c(\mathcal{G})$  by  $N := \{f \in C_c(\mathcal{G}) \mid \|f\| = 0\}$  in the pre- $C^*$ -norm  $\|\cdot\|$  induced by  $\rho$ . We define  $\pi_{\max}(f) := f + N \in C^*(\mathcal{G})$ .

By construction of  $\rho$ , if  $\pi$  is a  $*$ -representation of  $C^*(\mathcal{G})$ , then  $\|\pi(f)\| \leq \rho(f) = \|\pi_{\max}(f)\|$  for all  $f \in C_c(\mathcal{G})$ . This implies that there is a well-defined norm-decreasing linear map  $\psi : C^*(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{H})$  satisfying  $\psi \circ \pi_{\max} = \pi$ . Continuity then ensures that this  $\psi$  is a  $C^*$ -homomorphism.  $\square$

Of course, we expect that  $\pi_{\max}$  is injective; we will prove this in the next section.

It is not immediately obvious that the norm defined in Theorem 9.2.3 agrees with Renault's definition. This is because, to deal with non-étale groupoids, Renault defines the universal norm, not as the supremum over all  $*$ -representations of  $C_c(\mathcal{G})$ , but as the supremum only over  $*$ -representations of  $C_c(\mathcal{G})$  that are bounded with respect to the “ $I$ -norm” on  $C_c(\mathcal{G})$ . When  $\mathcal{G}$  is étale, the  $I$ -norm is given by

$$\|f\|_I = \sup_{x \in \mathcal{G}^{(0)}} \max \left\{ \sum_{\gamma \in \mathcal{G}_x} |f(\gamma)|, \sum_{\gamma \in \mathcal{G}^x} |f(\gamma)| \right\}.$$

(Think of it like a fibrewise 1-norm.) Renault shows that boundedness in the  $I$ -norm is equivalent to continuity in the inductive-limit topology on  $C_c(\mathcal{G})$ : the topology obtained by regarding  $C_c(\mathcal{G})$  as the inductive limit of the subspaces  $X_K := (\{f \in C(\mathcal{G}) \mid \text{supp}(f) \subseteq K\}, \|\cdot\|_\infty)$  indexed by compact subsets  $K$  of  $\mathcal{G}$  (see, for example, [13, Definition 5.4 and Example 5.10]). In the general non-étale setting, this equivalence between  $I$ -norm boundedness and continuity in the inductive-limit topology is nontrivial, and requires an appeal to Renault's Disintegration Theorem, which we will discuss later.

So to make sure we are talking about the same universal  $C^*$ -algebra as Renault, we must verify that every  $*$ -representation of  $C_c(\mathcal{G})$  is continuous in the

inductive-limit topology when  $\mathcal{G}$  is étale; and for completeness we should also prove that continuity in this topology implies boundedness with respect to the  $I$ -norm. The nice proof of this latter fact given below was developed by Ben Maldon in his honours thesis; it does not seem to have appeared in the literature previously.

**Lemma 9.2.4.** *Suppose that  $\mathcal{G}$  is a second-countable locally compact Hausdorff étale groupoid. Then every  $*$ -representation  $\pi$  of  $C_c(\mathcal{G})$  is both continuous in the inductive-limit topology, and bounded in the  $I$ -norm.*

*Proof.* Fix a nondegenerate  $*$ -representation of  $C_c(\mathcal{G})$ . By, for example, [13, Proposition 5.7], to see that  $\pi$  is continuous in the inductive-limit topology, we just have to check that  $\pi|_{X_K}$  is continuous for each compact  $K \subseteq \mathcal{G}$ . To see this, fix a compact  $K \subseteq \mathcal{G}$ . We can cover  $K$  by open bisections, and then use compactness to obtain a finite subcover  $K \subseteq \bigcup_{i=1}^n U_i$ . Fix a partition of unity  $\{h_i\}$  for  $K$  subordinate to the  $U_i$ . Then for  $f \in X_K$ , Lemma 9.2.1 gives

$$\|\pi(f)\| = \left\| \sum_i \pi(h_i \cdot f) \right\| \leq \sum_{i=1}^n \|\pi(h_i \cdot f)\| \leq \sum_{i=1}^n \|h_i \cdot f\|_\infty \leq n \|f\|_\infty.$$

So  $\pi$  is Lipschitz on  $X_K$  with Lipschitz constant at most  $n$ . This shows that  $\pi$  is continuous in the inductive-limit topology.

To see that it is  $I$ -norm bounded, observe that if  $f \in C_c(\mathcal{G})$ , then  $\|f\|_\infty \leq \|f\|_I$ . So the inductive-limit topology is weaker than the  $I$ -norm topology, and we deduce that  $\pi$  is continuous for the  $I$ -norm. Since continuity is equivalent to boundedness for linear maps on normed spaces, we deduce that  $\pi$  is  $I$ -norm bounded. Routine calculations show that the  $I$ -norm is a  $*$ -algebra norm, so  $\pi$  extends to a  $*$ -homomorphism from the Banach  $*$ -algebra completion  $\overline{C_c(\mathcal{G})}^I$  of  $C_c(\mathcal{G})$  in the  $I$ -norm into  $\mathcal{B}(\mathcal{H})$ . Now we can apply spectral theory: Write  $\rho_A : A \rightarrow [0, \infty)$  for the spectral-radius function on a Banach algebra  $A$ . For each  $f \in C_c(\mathcal{G})$ , we have  $\|\pi(f)\|^2 = \|\pi(f^* f)\| = \rho_{\mathcal{B}(\mathcal{H})}(\pi(f^* f)) \leq \rho_{\overline{C_c(\mathcal{G})}^I}(f^* f) \leq \|f^* f\|_I \leq \|f\|_I^2$ .  $\square$

We will not really need the preceding result from here on in, but we can take comfort that we are discussing the same family of representations as Renault is; so we can appeal to his theorems at need.

**Example 9.2.5.** If  $\mathcal{G}$  is a group, then  $C^*(\mathcal{G})$  is the usual full group  $C^*$ -algebra.

**Example 9.2.6.** Let  $X$  be a compact Hausdorff space, and  $\Gamma$  a discrete group acting on  $X$ . Then the elements  $U_g := 1_{\{g\} \times X}$  indexed by  $g \in \Gamma$  belong to  $C_c(\mathcal{G})$ , and there is an inclusion  $\pi : C(\mathcal{G}^{(0)}) \rightarrow C_c(\mathcal{G})$  such that  $\pi(f)(g, x) = \delta_{e, g} f(x)$ . For  $f \in C(X)$  and  $g \in \Gamma$  we have  $U_g \pi(f) U_g^*(h, x) = \sum_{\alpha \beta \gamma = (h, x)} 1_{\{g\} \times X}(\alpha) \pi(f)(\beta) 1_{\{g\} \times X}^*(\gamma)$ . This can only be nonzero if  $h = 0$ , and then the only nonzero term occurs when  $\alpha = (g, g^{-1} \cdot x)$  and  $\gamma = (g^{-1}, x)$ . For this  $\alpha, \beta$  we have  $1_{\{g\} \times X}(\alpha) \pi(f)(\beta) 1_{\{g\} \times X}^*(\gamma) = \pi(f)(e, \gamma^{-1} \cdot x) = f(\gamma^{-1} \cdot x)$ . So  $U_g \pi(f) U_g^* \in \pi(C(X))$ , and agrees with  $\pi(x \mapsto f(g^{-1} \cdot x))$ . So the universal property of the crossed product  $C(X) \rtimes \Gamma$  (see [52,



Theorem 1.3.3] and [51, Theorem 2.6.1]) gives a homomorphism  $C(X) \rtimes \Gamma \rightarrow C^*(\mathcal{G})$  that takes each  $i_\Gamma(g)$  to  $U_g$  and each  $i_{C(X)}(f)$  to  $\pi(f)$ . Conversely, for  $f \in C_c(\mathcal{G})$  and  $g \in \mathcal{G}$ , define  $f_g \in C(X)$  by  $f_g(x) = f(g, x)$ . Easy calculations show that the map  $\psi(f) := \sum_{g \in \Gamma} i_{C(X)}(f_g) i_\Gamma(g)$  gives a  $*$ -homomorphism  $\psi : C_c(\mathcal{G}) \rightarrow C(X) \rtimes \Gamma$ , and that  $\psi$  is inverse to  $\pi \times U$ . So  $\psi$  is an isomorphism  $C^*(\mathcal{G}) \cong C(X) \rtimes \Gamma$ .

**Example 9.2.7.** Consider the groupoid  $\mathcal{R}_{2^\infty}$  of Example 8.3.5. For each  $n \geq 0$  and each pair  $u, v \in \{0, 1\}^n$ , let  $\theta_n(u, v) := 1_{Z(u, v)} \in C_c(\mathcal{R}_{2^\infty})$ . Easy calculations using Lemma 9.1.4 show that  $\theta_n(u, v)^* = \theta_n(v, u)$  and  $\theta_n(u, v)\theta_n(w, y) = \delta_{v, w}\theta_n(u, y)$ . So  $A_n := \text{span}\{\theta_n(u, v) \mid u, v \in \{0, 1\}^n\}$  is isomorphic to  $M_{\{0, 1\}^n}(\mathbb{C})$  via the map  $\sum_{u, v} a_{u, v} \theta_n(u, v) \mapsto (a_{u, v})_{u, v \in \{0, 1\}^n}$ . Since  $Z(u, v) = Z(u0, v0) \sqcup Z(u1, v1)$ , we have  $\theta_n(u, v) = \theta_{n+1}(u0, v0) + \theta_{n+1}(u1, v1)$ , and so  $A_n \subseteq A_{n+1}$ . If we identify  $M_{\{0, 1\}^{n+1}}(\mathbb{C})$  with  $M_2(M_{\{0, 1\}^n}(\mathbb{C}))$  via

$$(a_{u, v})_{u, v \in \{0, 1\}^{n+1}} \mapsto ((a_{wi, yj})_{w, y \in \{0, 1\}^n})_{i, j \in \{0, 1\}},$$

then the inclusion  $A_n \rightarrow A_{n+1}$  is compatible with the canonical block-diagonal inclusion  $M_{\{0, 1\}^n}(\mathbb{C}) \rightarrow M_{\{0, 1\}^{n+1}}(\mathbb{C})$  with multiplicity 2. So the uniqueness of  $M_{2^\infty}$  shows that  $\overline{\bigcup_n A_n} \cong M_{2^\infty}$ . A straightforward application of the Stone–Weierstrass theorem shows that  $C(Z(u, v)) \subseteq \overline{\bigcup_n A_n}$  for all  $|u| = |v|$ , and we deduce that  $C_c(\mathcal{R}_{2^\infty}) = \overline{\bigcup_n A_n}$ . Hence  $C^*(\mathcal{R}_{2^\infty}) \cong M_{2^\infty}(\mathbb{C})$ .

**Example 9.2.8.** If  $\mathcal{G}$  is the graph groupoid of a directed graph  $E$  as in Example 8.4.7, then the characteristic functions  $\{1_{Z(v, v)} \mid v \in E^0\}$  and  $\{1_{Z(e, s(e))} \mid e \in E^1\}$  constitute a Cuntz–Krieger family for  $E$  that generates  $C^*(\mathcal{G})$ . The homomorphism  $C^*(E) \rightarrow C^*(\mathcal{G})$  induced by this family is an isomorphism.

**Remark 9.2.9** (Unitary representations). To make sense of a unitary representation of  $\mathcal{G}$ , we proceed, very roughly, as follows. A *unitary representation* of  $\mathcal{G}$  is a triple  $(\mathcal{H}, \mu, U)$  where  $\mu$  is a Borel measure on  $\mathcal{G}^{(0)}$ ,  $\mathcal{H} = \bigsqcup_{x \in \mathcal{G}^{(0)}} \mathcal{H}_x$  is a  $\mu$ -measurable field of Hilbert spaces, and  $U = \{U_\gamma \mid \gamma \in \mathcal{G}\}$  is a family of unitary operators  $U_\gamma : \mathcal{H}_{s(\gamma)} \rightarrow \mathcal{H}_{r(\gamma)}$  satisfying  $U_\alpha U_\beta = U_{\alpha\beta}$ ,  $U_{\alpha^{-1}} = U_\alpha^*$ , and  $\gamma \mapsto (U_\gamma \xi(s(\gamma)) \mid \eta(r(\gamma)))$  is measurable for each pair  $\xi, \eta$  of measurable sections of  $\mathcal{H}$ . Every unitary representation  $(\mathcal{H}, \mu, U)$  of  $\mathcal{G}$  induces a  $*$ -representation  $\pi_{(\mathcal{H}, \mu, U)}$  of  $C_c(\mathcal{G})$  on the direct integral  $\int_{\mathcal{G}^{(0)}}^\oplus \mathcal{H}_x d\mu(x)$  characterized by

$$(\pi_{(\mathcal{H}, \mu, U)}(f)\xi \mid \eta) = \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}_x} (f(\gamma)U_\gamma \xi(x) \mid \eta(r(\gamma))) d\mu(x).$$

This representation is called the *integrated form* of  $(\mathcal{H}, \mu, U)$ . Renault’s *Disintegration Theorem* [37, Theorem II.1.21] says that every representation is unitarily equivalent to the integrated form of a unitary representation.

### 9.3 The reduced $C^*$ -algebra

To show that the map  $\pi_{\max} : C_c(\mathcal{G}) \rightarrow C^*(\mathcal{G})$  of Theorem 9.2.3 is injective, we need to construct a representation that is injective on  $C_c(\mathcal{G})$ .

If  $\mathcal{G}$  were a group, we would use the regular representation of  $\mathcal{G}$ . We aim to do the same thing for groupoids, but there are multiple regular representations to consider.

**Proposition 9.3.1.** *Let  $\mathcal{G}$  be a second-countable locally compact Hausdorff étale groupoid. For each  $x \in \mathcal{G}^{(0)}$ , there is a  $*$ -representation  $\pi_x : C_c(\mathcal{G}) \rightarrow \mathcal{B}(\ell^2(\mathcal{G}_x))$  such that*

$$\pi_x(f)\delta_\gamma = \sum_{\alpha \in \mathcal{G}_r(\gamma)} f(\alpha)\delta_{\alpha\gamma}.$$

This  $\pi_x$  is called the regular representation of  $C_c(\mathcal{G})$  associated to  $x$ .

For each  $\eta \in \mathcal{G}$  the map  $U_\eta : \ell^2(\mathcal{G}_{s(\eta)}) \rightarrow \ell^2(\mathcal{G}_{r(\eta)})$  given by  $U_\eta\delta_\gamma = \delta_{\gamma\eta^{-1}}$  is a unitary operator and we have  $\pi_{r(\eta)} = U_\eta\pi_{s(\eta)}U_\eta^*$ .

*Proof.* The first assertion is relatively straightforward to check using that the formula given for  $\pi_x(f)\delta_\gamma$  is really just the formula for  $f * \delta_\gamma$  if the convolution product is extended to not-necessarily-continuous functions.

The operators  $U_\eta$  are certainly unitary, and for  $f \in C_c(\mathcal{G})$ ,

$$\begin{aligned} U_\eta\pi_{s(\eta)}(f)U_\eta^*\delta_\gamma &= U_\eta\pi_{s(\eta)}(f)\delta_{\gamma\eta} \\ &= \sum_{\alpha \in \mathcal{G}_r(\gamma\eta)} f(\alpha)U_\eta\delta_{\alpha\gamma\eta} = \sum_{\alpha \in \mathcal{G}_r(\gamma\eta)} f(\alpha)\delta_{\alpha\gamma} = \pi_{r(\eta)}(f)\delta_\gamma. \quad \square \end{aligned}$$

**Definition 9.3.2.** Let  $\mathcal{G}$  be a second-countable locally compact Hausdorff étale groupoid. The *reduced  $C^*$ -algebra*, denoted  $C_r^*(\mathcal{G})$ , of  $\mathcal{G}$  is the completion of

$$\left( \bigoplus_{x \in \mathcal{G}^{(0)}} \pi_x \right) (C_c(\mathcal{G})) \subseteq \bigoplus_{x \in \mathcal{G}^{(0)}} \mathcal{B}(\ell^2(\mathcal{G}_x)).$$

We write  $\|\cdot\|_r$  for the  $C^*$ -norm on  $C_r^*(\mathcal{G})$ .

The universal property of  $C^*(\mathcal{G})$  yields a homomorphism, which we denote  $\pi_r$ , from  $C^*(\mathcal{G})$  to  $C_r^*(\mathcal{G})$  such that  $\pi_r \circ \pi_{\max} = \bigoplus_x \pi_x$ . In particular,  $\|\cdot\|_r \leq \|\cdot\|$  on  $C_c(\mathcal{G})$ .

**Proposition 9.3.3.** *Let  $\mathcal{G}$  be a second-countable locally compact Hausdorff étale groupoid. There is an injective, norm-decreasing map  $j : C_r^*(\mathcal{G}) \rightarrow C_0(\mathcal{G})$  such that*

$$j(a)(\gamma) = (\pi_{s(\gamma)}(a)\delta_{s(\gamma)} \mid \delta_\gamma)$$

for all  $a \in C_r^*(\mathcal{G})$  and  $\gamma \in \mathcal{G}$ . For  $f \in C_c(\mathcal{G})$ , we have  $j(f) = f$ .

*Proof.* For  $f \in C_c(\mathcal{G})$ , we have

$$j(f)(\gamma) = \left( \sum_{\alpha \in \mathcal{G}_{r(\gamma)}} f(\alpha) \delta_{\alpha s(\gamma)} \mid \delta_\gamma \right) = f(\gamma).$$

The Cauchy–Schwarz inequality shows that  $j$  is norm-decreasing from  $\|\cdot\|_r$  to  $\|\cdot\|_\infty$ . An  $\varepsilon/3$ -argument shows that each  $j(a)$  belongs to  $C_0(\mathcal{G})$ . If  $a \in C_r^*(\mathcal{G})$  is nonzero, then there exists  $x \in \mathcal{G}^{(0)}$  such that  $\pi_x(a) \neq 0$ , and then there exist  $\alpha, \beta \in \mathcal{G}_x$  such that  $(\pi_x(a) \delta_\alpha \mid \delta_\beta) \neq 0$ . Now Proposition 9.3.1 gives

$$j(a)(\beta \alpha^{-1}) = (\pi_{r(\alpha)}(a) \delta_{r(\alpha)} \mid \delta_{\beta \alpha^{-1}}) = (U_\alpha^* \pi_{r(\alpha)}(a) U_\alpha \delta_\alpha \mid \delta_\beta) = (\pi_{s(\alpha)} \delta_\alpha \mid \delta_\beta) \neq 0.$$

Hence  $j$  is injective.  $\square$

**Corollary 9.3.4.** *Suppose that  $\mathcal{G}$  is a second-countable locally compact Hausdorff étale groupoid. Then the homomorphisms  $\pi_{\max} : C_c(\mathcal{G}) \rightarrow C^*(\mathcal{G})$  and  $\bigoplus_x \pi_x : C_c(\mathcal{G}) \rightarrow C_r^*(\mathcal{G})$  are both injective. For  $f \in C_c(\mathcal{G})$ , we have  $\|f\|_\infty \leq \|f\|_r \leq \|f\|$ . If  $f$  is supported on a bisection, then we have equality throughout.*

*Proof.* That  $\|f\|_r \leq \|f\|$  is by definition of the universal norm. That  $\|f\|_\infty \leq \|f\|_r$  follows from Proposition 9.3.3. Since  $j(f) = f$  for all  $f \in C_c(\mathcal{G})$ , it follows that  $\pi_{\max}$  and  $\bigoplus_x \pi_x$  are injective on  $C_c(\mathcal{G})$ . If  $f$  is supported on a bisection, then we have  $\|f\|^2 = \|f^* * f\|$ . Since  $f^* * f \in C_c(\mathcal{G}^{(0)})$ , the uniqueness of the  $*$ -algebra norm (see Lemma 9.2.2) on  $C_c(\mathcal{G}^{(0)})$  gives  $\|f\|^2 = \|f^* * f\|_\infty$ , and this is precisely  $\|f\|_\infty^2$  by Lemma 9.1.4.  $\square$

Using the above corollary, we see that we can apply the Stone–Weierstrass theorem to establish surjectivity of a homomorphism into either  $C^*(\mathcal{G})$  or  $C_r^*(\mathcal{G})$ .

**Corollary 9.3.5.** *Suppose that  $\mathcal{G}$  is a second-countable locally compact Hausdorff étale groupoid. Let  $A$  be a  $C^*$ -algebra, and suppose that  $\pi : A \rightarrow C^*(\mathcal{G})$  (or  $\pi : A \rightarrow C_r^*(\mathcal{G})$ ) is a homomorphism. Suppose that for each open bisection  $U \subseteq \mathcal{G}$  and each pair of distinct points  $\beta, \gamma \in U$ , there exists  $a \in A$  such that  $\pi(a) \in C_0(U)$ ,  $\pi(a)(\beta) = 0$  and  $\pi(a)(\gamma) = 1$ . Then  $\pi$  is surjective.*

*Proof.* A straightforward application of the Stone–Weierstrass theorem shows that  $\pi(A)$  contains  $C_0(\mathcal{G}^{(0)})$ . Using this and the convolution formula in Lemma 9.1.4, we see that for  $U \subseteq \mathcal{G}$  an open bisection, the set  $\pi(A) \cap C_0(U)$  is closed under pointwise multiplication (identify  $C_0(U)$  with  $C_0(r(U))$ ), and then note that if  $f, g \in \pi(A) \cap C_0(U)$ , then  $f \circ r^{-1} \in C_0(r(U)) \subseteq C_0(\mathcal{G}) \subseteq \pi(A)$ , and  $f \cdot g = (f \circ r^{-1}) * g$ . So another application of the Stone–Weierstrass theorem combined with the fact that  $\|\cdot\|_{C^*(\mathcal{G})}$  agrees with  $\|\cdot\|_\infty$  on  $C_c(U)$  shows that  $C_c(U) \subseteq \pi(A)$ . Now Lemma 9.1.3 shows that  $C_c(\mathcal{G}) \subseteq \pi(A)$ . Since  $\pi$  is a  $C^*$ -homomorphism, it has closed range, and we deduce that  $C^*(\mathcal{G}) \subseteq \pi(A)$ .  $\square$

**Example 9.3.6.** If  $\mathcal{G}$  is a group, then there is just one unit  $e$ , and the regular representation  $\pi_e$  is the usual left-regular representation of the group algebra. So  $C_r^*(\mathcal{G})$  is the usual reduced  $C^*$ -algebra. In particular,  $C_r^*(\mathcal{G}) = C^*(\mathcal{G})$  if and only if  $\mathcal{G}$  is amenable.

**Example 9.3.7.** Let  $R$  be a discrete equivalence relation. For distinct equivalence classes  $S, T \subseteq R^{(0)}$  under  $R$ , we have  $C_c(S \times S) \perp C_c(T \times T)$  inside  $C_c(R)$ , and so  $C_c(R)$  is the direct sum of the  $*$ -subalgebras  $C_c(S)$ . It follows that the completions of these  $*$ -subalgebras are direct summands in each of  $C^*(R)$  and  $C_r^*(R)$ . For a fixed  $S$ , the elements  $\{1_{(s,s')} \mid s, s' \in S\}$  form a complete set of nonzero matrix units indexed by  $S$ , and generate both  $C^*(S \times S)$  and  $C_r^*(S \times S)$  as  $C^*$ -algebras. Since  $\mathcal{K}(\ell^2(S))$  is the unique  $C^*$ -algebra generated by a family of nonzero matrix units indexed by  $S$ , we deduce that  $C_r^*(S \times S) \cong C^*(S \times S) \cong \mathcal{K}(\ell^2(S))$ .

**Example 9.3.8.** If  $\mathcal{G}$  is the transformation groupoid for an action of a discrete group  $\Gamma$  on a compact Hausdorff space  $X$ , then the unit space  $\mathcal{G}^{(0)}$  is just  $X$ . Fix  $x \in X = \mathcal{G}^{(0)}$ . The isomorphism  $\omega : C_c(\mathcal{G}) \rightarrow C_c(\Gamma, C(X))$  of Example 9.1.7 intertwines the regular representation  $\pi_x$  of  $C_c(\mathcal{G})$  with the induced representation of  $C_c(\Gamma, C(X))$  associated the character of  $C(X)$  given by evaluation at  $x$ , denoted  $\text{Ind} \text{ev}_x$  in [52, Example 2.4.3]. So as discussed in that example, since  $\bigoplus_x \text{ev}_x$  is a faithful representation of  $C_0(X)$ , we see that  $C_r^*(\mathcal{G})$  is isomorphic to the reduced crossed product  $C(X) \rtimes_{\alpha, r} \Gamma$ .

**Remark 9.3.9.** There is an alternative, slightly slicker, approach to defining  $C_r^*(\mathcal{G})$ . Define  $\langle \cdot, \cdot \rangle_{C_0(\mathcal{G}^{(0)})} : C_c(\mathcal{G}) \times C_c(\mathcal{G}) \rightarrow C_0(\mathcal{G}^{(0)})$  by  $\langle f, g \rangle_{C_0(\mathcal{G}^{(0)})} := (f * g)|_{\mathcal{G}^{(0)}}$ . It is straightforward to check that this is a  $C_0(\mathcal{G}^{(0)})$ -valued inner-product (in particular, positive definite) on  $C_c(\mathcal{G})$ , so we can form the corresponding Hilbert-module completion  $X_{\mathcal{G}}$ . Now the action of  $C_c(\mathcal{G})$  on itself by left multiplication extends to a homomorphism  $L : C_c(\mathcal{G}) \rightarrow \mathcal{L}(X_{\mathcal{G}})$ , the  $C^*$ -algebra of adjointable operators on  $X_{\mathcal{G}}$ . It is a fairly straightforward exercise, if you are familiar with Hilbert modules, to verify that  $\|L(f)\| = \|f\|_r$  for all  $f \in C_c(\mathcal{G})$ . So the completion of  $L(C_c(\mathcal{G}))$  in  $\mathcal{L}(X_{\mathcal{G}})$  is isomorphic to  $C_r^*(\mathcal{G})$ ; indeed, we could have taken this as the definition of  $C_r^*(\mathcal{G})$ .

## 9.4 Equivalence of groupoids

Renault's notion of equivalence of groupoids closely reflects Morita equivalence for  $C^*$ -algebras.

**Definition 9.4.1.** Suppose that  $\mathcal{G}$  is an étale groupoid. A *left  $\mathcal{G}$ -space* is a locally compact Hausdorff space  $X$  endowed with a continuous map  $r : X \rightarrow \mathcal{G}^{(0)}$  and a continuous map  $\cdot : \mathcal{G} * X := \{(\gamma, \xi) \in \mathcal{G} \times X \mid s(\gamma) = r(\xi)\} \rightarrow X$  such that

1.  $r(\gamma \cdot \xi) = r(\gamma)$  for all  $(\gamma, \xi) \in \mathcal{G} * X$ ;
2.  $r(\xi) \cdot \xi = \xi$  for all  $\xi \in X$ ; and

3.  $\alpha \cdot (\beta \cdot \xi) = (\alpha\beta) \cdot \xi$  for all  $(\beta, \xi) \in \mathcal{G} * X$  and  $\alpha \in \mathcal{G}_{r(\beta)}$ .

A *right  $\mathcal{G}$ -space* is defined similarly, but with a map  $s : X \rightarrow \mathcal{G}^{(0)}$  and the roles of  $s$  and  $r$  reversed in (1)–(3).

We say that  $X$  is a *proper left  $\mathcal{G}$ -space* if the map  $(\gamma, \xi) \mapsto (\gamma \cdot \xi, \xi)$  from  $\mathcal{G} * X \rightarrow X \times X$  is a proper map. It is *free* if  $\gamma \cdot \xi = \xi$  implies  $\gamma = r(\xi)$ .

If  $\mathcal{G}$  and  $\mathcal{H}$  are two étale groupoids, then a  $\mathcal{G}$ – $\mathcal{H}$ -equivalence is a locally compact Hausdorff space  $X$  that is simultaneously a free and proper left  $\mathcal{G}$ -space and a free and proper right  $\mathcal{H}$ -space such that the left and right actions commute, and such that  $r : X \rightarrow \mathcal{G}^{(0)}$  and  $s : X \rightarrow \mathcal{H}^{(0)}$  are open maps and induce homeomorphisms  $\tilde{r} : X/\mathcal{H} \cong \mathcal{G}^{(0)}$ , and  $\tilde{s} : \mathcal{G} \backslash X \rightarrow \mathcal{H}^{(0)}$ .

**Lemma 9.4.2.** *Suppose that  $\mathcal{G}, \mathcal{H}$  are second-countable locally compact Hausdorff étale groupoids. Let  $X$  be a  $\mathcal{G}$ – $\mathcal{H}$ -equivalence. If  $\xi, \eta \in X$  satisfy  $r(\xi) = r(\eta)$ , then there is a unique element  $[\xi, \eta]_{\mathcal{H}} \in \mathcal{H}$  such that  $\xi \cdot [\xi, \eta]_{\mathcal{H}} = \eta$ . Likewise, if  $s(\xi) = s(\eta)$ , then there is a unique  ${}_{\mathcal{G}}[\xi, \eta] \in \mathcal{G}$  such that  ${}_{\mathcal{G}}[\xi, \eta] \cdot \eta = \xi$ .*

*Proof.* It suffices to prove the first statement; the second is symmetric. That  $r$  descends to a homeomorphism  $X/\mathcal{H} \rightarrow \mathcal{G}^{(0)}$  shows that there exists an element  $\lambda \in \mathcal{H}$  such that  $\xi \cdot \lambda = \eta$ . Freeness shows that this  $\lambda$  is unique.  $\square$

Given a  $\mathcal{G}$ – $\mathcal{H}$ -equivalence  $X$ , there is a corresponding  $\mathcal{H}$ – $\mathcal{G}$ -equivalence  $X^* := \{\xi^* \mid \xi \in X\}$  defined by  $r(\xi^*) = s(\xi)$ ,  $s(\xi^*) = r(\xi)$ ,  $\lambda \cdot \xi^* := (\xi \cdot \lambda^{-1})^*$  and  $\xi^* \cdot \gamma = (\gamma^{-1} \cdot x)^*$ . Clearly  $X^{**} \cong X$  via the map  $\xi^{**} \mapsto \xi$ .

**Proposition 9.4.3.** *Suppose that  $\mathcal{G}, \mathcal{H}$  are second-countable locally compact Hausdorff étale groupoids. Let  $X$  be a  $\mathcal{G}$ – $\mathcal{H}$ -equivalence. Let  $L := \mathcal{G} \sqcup X \sqcup X^* \sqcup \mathcal{H}$ , with the relative topology. Then  $L$  is an étale groupoid with*

- $L^{(0)} = \mathcal{G}^{(0)} \sqcup \mathcal{H}^{(0)}$ ,
- range and source maps inherited from those on  $\mathcal{G}, X, X^*, \mathcal{H}$ ,
- multiplication inherited from multiplication in  $\mathcal{G}$  and  $\mathcal{H}$ , the actions of  $\mathcal{G}$  and  $\mathcal{H}$  on  $X$  and  $X^*$  and with  $\xi^* \eta := [\xi, \eta]_{\mathcal{H}}$  for  $\xi, \eta \in X$  with  $r(\xi) = r(\eta)$ , and  $\xi \eta^* := {}_{\mathcal{G}}[\xi, \eta]$  for  $\xi, \eta \in X$  with  $s(\xi) = s(\eta)$ , and
- $\xi^{-1} = \xi^*$  and  $(\xi^*)^{-1} = \xi$  for  $\xi \in X$ .

We have  $\mathcal{G}^{(0)} L \mathcal{G}^{(0)} = \mathcal{G}$  and  $\mathcal{H}^{(0)} L \mathcal{H}^{(0)} = \mathcal{H}$ .

*Proof.* The proof that  $L$  is a topological groupoid is routine but tedious. To see that it is étale, we show that  $r : X \rightarrow \mathcal{G}^{(0)}$  is a local homeomorphism; that  $s$  is a local homeomorphism follows from a similar argument.

We already know that  $r$  is an open map, so we need only show that it is locally injective. So fix  $\xi \in X$  and sequences  $\xi_n, \xi'_n \rightarrow \xi$  such that  $r(\xi_n) = r(\xi'_n)$  for all  $n$ . We must show that  $\xi_n = \xi'_n$  for large  $n$ . We have  $[\xi_n, \xi'_n]_{\mathcal{H}} \rightarrow [\xi, \xi]_{\mathcal{H}} = s(\xi)$ . Since  $s(\xi) \in \mathcal{H}^{(0)}$  and since  $\mathcal{H}^{(0)}$  is open, we therefore have  $[\xi_n, \xi'_n]_{\mathcal{H}} \in \mathcal{H}^{(0)}$  for large  $n$ ; that is  $[\xi_n, \xi'_n]_{\mathcal{H}} = s(\xi_n)$  for large  $n$ , and therefore  $\xi'_n = \xi_n \cdot s(\xi_n) = \xi_n$  for large  $n$ .  $\square$

We call the groupoid of Proposition 9.4.3 the *linking groupoid* of  $X$ .

**Theorem 9.4.4.** *Suppose that  $\mathcal{G}, \mathcal{H}$  are second-countable locally compact Hausdorff étale groupoids. Let  $X$  be a  $\mathcal{G}$ - $\mathcal{H}$ -equivalence. Let  $L$  be the linking groupoid of  $X$ . Then  $P := 1_{\mathcal{G}^{(0)}}$  and  $Q := 1_{\mathcal{H}^{(0)}}$  belong to  $\mathcal{M}(C^*(L))$  and to  $\mathcal{M}(C_r^*(L))$ , and are complementary full projections. We have  $PC^*(L)P \cong C^*(\mathcal{G})$  and  $QC^*(L)Q \cong C^*(\mathcal{H})$ , and similarly  $PC_r^*(L)P \cong C_r^*(\mathcal{G})$  and  $QC_r^*(L)Q \cong C_r^*(\mathcal{H})$ . In particular,  $C^*(\mathcal{G})$  and  $C^*(\mathcal{H})$  are Morita equivalent via the imprimitivity bimodule  $PC^*(L)Q$  and likewise  $C_r^*(\mathcal{G})$  and  $C_r^*(\mathcal{H})$  are Morita equivalent via the imprimitivity bimodule  $PC_r^*(L)Q$ .*

*Proof.* Fix an increasing sequence  $K_n$  of compact subsets of  $\mathcal{G}^{(0)}$  with  $\bigcup_n K_n = \mathcal{G}^{(0)}$  and choose functions  $e_n \leq 1$  in  $C_0(\mathcal{G}^{(0)})$  with  $e_n \equiv 1$  on  $K_n$ . For  $f \in C_c(L)$ , we have  $e_n f = f|_{\mathcal{G}^{(0)}L}$  for large  $n$  (just take  $n$  large enough so that  $r(\text{supp}(f)) \subseteq K_n$ ). So the  $e_n$  converge strictly to a multiplier projection  $P$  with the property that  $Pf = 1_{\mathcal{G}^{(0)}} * f$  for  $f \in C_c(\mathcal{G})$ . A similar argument gives  $Q$ , and it is clear that  $P + Q = 1$ . The map  $\pi_r$  is clearly nondegenerate, and its extension to  $\mathcal{M}(C^*(\mathcal{G}))$  takes  $P$  and  $Q$  to projections in  $\mathcal{M}(C_r^*(\mathcal{G}))$  with the same properties, and which we continue to call  $P$  and  $Q$ .

To see that  $PC_r^*(L)P \cong C_r^*(\mathcal{G})$ , first note that since  $\mathcal{G}$  is an open subgroupoid of  $L$ , there is an inclusion of  $C_c(\mathcal{G})$  in  $C_c(L)$  that extends a function  $f \in C_c(\mathcal{G})$  to  $L$  by defining  $f(\eta) = 0$  for  $\eta \notin L$ . We just have to show that this inclusion is isometric for the reduced norms. For a fixed  $x \in \mathcal{G}^{(0)}$ , consider the regular representation  $\pi_x^L$  of  $C_c(L)$  on  $\ell^2(L_x)$ . Let  $R \in \mathcal{B}(\ell^2(L_x))$  be the orthogonal projection onto  $\overline{\text{span}}\{\delta_\eta \mid r(\eta) \in \mathcal{G}^{(0)}\}$ . We have  $R = \tilde{\pi}_x^L(P)$ , and since  $Pf = fP = f$  for  $f \in C_c(\mathcal{G})$ , we see that  $\pi_x^L(f) = R\pi_x^L(f)R$  for  $f \in C_c(\mathcal{G})$ . Since  $x \in \mathcal{G}^{(0)}$ , the set  $\{\eta \in L_x \mid r(\eta) \in \mathcal{G}^{(0)}\}$  is precisely  $\mathcal{G}_x$ . Using this, it is easy to see that  $R\pi_x^L|_{C_c(\mathcal{G})}R$  is a copy of the regular representation  $\pi_x^{\mathcal{G}}$  of  $C_c(\mathcal{G})$ . So  $\|\pi_x^{\mathcal{G}}(f)\| = \|R\pi_x^L(f)R\| \leq \|\pi_x^L(f)\|$  for all  $f \in C_c(\mathcal{G})$ . This immediately shows that for  $f \in C_c(\mathcal{G})$  we have  $\|f\|_{C_r^*(\mathcal{G})} \leq \|f\|_{C_r^*(L)}$ . For the reverse inequality, it suffices to show that if  $y \in L^{(0)} \setminus \mathcal{G}^{(0)}$  then there exists  $x \in \mathcal{G}^{(0)}$  such that  $\|\pi_y^L(f)\| = \|\pi_x^L(f)\|$  for all  $f \in C_c(\mathcal{G})$ . For this, first note that since  $s : X \rightarrow \mathcal{H}^{(0)}$  induces a homeomorphism of  $\mathcal{G} \setminus X$  onto  $\mathcal{H}^{(0)}$ , it is surjective, so we can fix  $\eta \in X$  with  $s(\eta) = y$ . By Proposition 9.3.1, the representation  $\pi_y^L$  is unitarily equivalent to  $\pi_{r(\eta)}^L$ , and so  $x := r(\eta) \in \mathcal{G}^{(0)}$  has the desired property. An identical argument shows that  $QC^*(L)Q \cong C^*(\mathcal{H})$ .

We now turn to universal  $C^*$ -algebras; again, it suffices to show that the inclusion  $C_c(\mathcal{G}) \hookrightarrow C_c(L)$  is isometric for the universal norm. Certainly the inclusion  $C_c(\mathcal{G}) \hookrightarrow C_c(L) \hookrightarrow C^*(L)$  determines a  $*$ -representation of  $C_c(\mathcal{G})$ , and so the universal property of  $C^*(\mathcal{G})$  shows that there is a homomorphism  $C^*(\mathcal{G}) \rightarrow C^*(L)$  that agrees with the canonical inclusion of  $C_c(\mathcal{G})$ . Thus the inclusion is norm-decreasing. For the reverse inequality, the rough idea is to use that the pair  $(C_c(L\mathcal{G}^{(0)}), C_c(\mathcal{G}^{(0)}L))$  is a  $*$ -Morita context in the sense of Ara [3] between  $C_c(L)$  and  $C_c(\mathcal{G})$ . Fix a nondegenerate  $*$ -representation  $\pi$  of  $C_c(\mathcal{G})$  in  $\mathcal{B}(\mathcal{H})$ . It suffices

to show that there is a  $*$ -representation  $\tilde{\pi}$  of  $C_c(L)$  such that

$$\|\tilde{\pi}(f)\| \geq \|\pi(f)\| \quad \text{for all } f \in C_c(\mathcal{G}). \quad (9.2)$$

For this, define a positive semidefinite sesquilinear form on  $C_c(L\mathcal{G}^{(0)}) \odot \mathcal{H}$  by  $(f \odot h \mid g \odot k) = (h \mid \pi(f^*g)k)$ . Let  $\tilde{\mathcal{H}}$  denote the Hilbert space obtained by quotienting out the space of vectors satisfying  $(\xi \mid \xi) = 0$  and then completing in the norm coming from the inner-product. For  $f \in C_c(L\mathcal{G}^{(0)})$  and  $h \in \mathcal{H}$ , we write  $f \otimes_{C_c(\mathcal{G})} h$  for the image of  $f \odot h$  in  $\tilde{\mathcal{H}}$ . It is not hard to check that  $\tilde{\pi}(f)(g \otimes_{C_c(\mathcal{G})} h) := (fg) \otimes_{C_c(\mathcal{G})} h$  defines a  $*$ -representation of  $C_c(L)$  on  $\tilde{\mathcal{H}}$ . To establish (9.2), fix  $f \in C_c(\mathcal{G})$  and  $\varepsilon > 0$ , and choose  $h \in \mathcal{H}$  such that  $\|h\| = 1$  and  $\|\pi(f)h\| > \|\pi(f)\| - \varepsilon$ . Fix an increasing approximate identity  $e_n$  for  $C_0(\mathcal{G}^{(0)})$  in  $C_c(\mathcal{G}^{(0)})$  as in the first paragraph of the proof. Then  $\pi(e_n)h \rightarrow h$  because  $\pi$  is nondegenerate. It then follows from the definition of the inner product on  $\tilde{\mathcal{H}}$  that  $e_n \otimes_{C_c(\mathcal{G})} h$  is Cauchy in  $\tilde{\mathcal{H}}$  and so converges to some  $\tilde{h}$ . Since

$$(e_n \otimes_{C_c(\mathcal{G})} h \mid e_n \otimes_{C_c(\mathcal{G})} h) = (h \mid \pi(e_n^*e_n)h) \rightarrow 1,$$

we have  $\|\tilde{h}\| = 1$ . Since  $fe_n = f$  for large  $n$ , we have

$$\begin{aligned} \|\tilde{\pi}(f)\tilde{h}\|^2 &= \lim_n \|\pi(fe_n) \otimes_{C_c(\mathcal{G})} h\|^2 \\ &= \lim_n (h \mid \pi((fe_n e_n^* f^*))h) = (h \mid \pi(ff^*)h) = \|\pi(f)h\|^2 \end{aligned}$$

So  $\|\tilde{\pi}(f)\tilde{h}\| = \|\pi(f)h\| > \|\pi(f)\| - \varepsilon$ , and since  $\|\tilde{h}\| = 1$ , we deduce that  $\|\tilde{\pi}(f)\| > \|\pi(f)\| - \varepsilon$ . Letting  $\varepsilon \rightarrow 0$  gives (9.2).  $\square$

**Remark 9.4.5.** Recall that, by Brown's theorem [9], if  $A$  is a  $\sigma$ -unital  $C^*$ -algebra, and  $P$  is a multiplier projection of  $A$ , then  $PAP \otimes \mathcal{K} \cong APA \otimes \mathcal{K}$ , and then the Brown–Green–Rieffel theorem [10] says that  $\sigma$ -unital  $C^*$ -algebras  $A$  and  $B$  are Morita equivalent if and only if they are stably isomorphic. We now have a version of equivalence for groupoids, and we know that the (discrete) full equivalence relation  $R_{\mathbb{N}} := \mathbb{N} \times \mathbb{N}$  has  $C^*$ -algebra isomorphic to  $\mathcal{K}$ . It is also not too difficult to see, using universal properties, that if  $\mathcal{G}$  and  $\mathcal{H}$  are étale groupoids, then  $\mathcal{G} \times \mathcal{H}$  is a groupoid, and that  $C^*(\mathcal{G} \times \mathcal{H}) \cong C^*(\mathcal{G}) \otimes C^*(\mathcal{H})$ . So it is not unreasonable to ask whether Brown's theorem and the Brown–Green–Rieffel theorem could carry over to groupoids. The answer is a qualified yes. Specifically, if  $\mathcal{G}$  and  $\mathcal{H}$  are étale groupoids such that  $\mathcal{G}^{(0)}$  and  $\mathcal{H}^{(0)}$  are totally disconnected as topological spaces, then for every compact open  $K \subseteq \mathcal{G}^{(0)}$ , we have  $K\mathcal{G}K \times R_{\mathbb{N}} \cong \mathcal{G}K\mathcal{G} \times R_{\mathbb{N}}$ ; and  $\mathcal{G}$  and  $\mathcal{H}$  are equivalent if and only if  $\mathcal{G} \times R_{\mathbb{N}} \cong \mathcal{H} \times R_{\mathbb{N}}$ ; the proof follows, almost exactly, the proofs of Brown's theorem and the Brown–Green–Rieffel theorem [12].



## Chapter 10

# Fundamental Structure Theory

In this section, we will discuss the structural properties of  $C^*(\mathcal{G})$ . When is it nuclear and when does it satisfy the UCT? When is it/isn't it simple, and more generally what is its ideal structure? When is it purely infinite?

### 10.1 Amenability, nuclearity and the UCT

The theory of amenability for groupoids is complicated; it could easily be a five-hour course all by itself. So we are going to skate over the top of it here. Most of what appears here is taken from [2].

Recall that a discrete group  $\Gamma$  is *amenable* if it admits a finitely additive probability measure  $\mu$  with the property that  $\mu(gA) = \mu(A)$  for all  $A \subseteq \Gamma$  and  $g \in \Gamma$ . A discrete group  $\Gamma$  is amenable if and only if  $C^*(\Gamma) = C_r^*(\Gamma)$ .

Amenability for groupoids is intended as an analogue of amenability for groups, but unfortunately, the analogies are not so well behaved as we might like. There are a number of notions of amenability for groupoids, two of the most prominent being measurewise amenability and topological amenability. These two coincide for étale groupoids, by results of Anantharaman-Delaroche–Renault [2], but as we shall see, they are not equivalent to coincidence of the full and reduced  $C^*$ -algebras, even for group bundles.

We need some set-up. If  $\mathcal{G}$  is an étale groupoid, then a *continuous system of probability measures for  $\mathcal{G}$*  is a system  $\{\lambda^x \mid x \in \mathcal{G}^{(0)}\}$  of Radon probability measures  $\lambda^x$  on  $\mathcal{G}$  with the support of  $\lambda^x$  contained in  $\mathcal{G}^x$  such that for  $f \in C_c(\mathcal{G})$ , the function  $x \mapsto \int_{\mathcal{G}} f d\lambda^x$  is continuous.

**Remark 10.1.1.** Since each  $\mathcal{G}^x$  is discrete, a Radon probability measure  $\lambda^x$  on  $\mathcal{G}^x$  amounts to a function  $\lambda^x : \mathcal{G}^x \rightarrow [0, \infty)$  with  $\sum_{\gamma} w^x(\gamma) = 1$ .

An *approximate invariant continuous mean* for  $\mathcal{G}$  is a net  $\lambda_i$  of continuous systems of probability measures for  $\mathcal{G}$  such that the net  $(M_i : \gamma \mapsto \|\lambda_i^{r(\gamma)}(\gamma \cdot) -$



$\lambda_i^{s(\gamma)}(\cdot)\|_1)_i$  of functions from  $\mathcal{G}$  to  $[0, \infty)$  has the property that  $M_i|_K \rightarrow 0$  uniformly for every compact  $K \subseteq \mathcal{G}$ .

**Definition 10.1.2.** Let  $\mathcal{G}$  be an étale groupoid. We say that  $\mathcal{G}$  is (topologically) *amenable* if  $\mathcal{G}$  admits an approximate invariant continuous mean.

There are a number of equivalent formulations of amenability, particularly in the setting of étale groupoids. Perhaps one of the most useful is the following:

**Lemma 10.1.3** ([2, Proposition 2.2.13]). *Let  $\mathcal{G}$  be an étale groupoid. Then  $\mathcal{G}$  is amenable if and only if there is a sequence  $(h_i)_{i=1}^\infty$  in  $C_c(\mathcal{G})$  such that*

1. *the maps  $x \mapsto \sum_{\gamma \in \mathcal{G}^x} |h_i(\gamma)|^2$  (indexed by  $i$ ) converge uniformly to 1 on every compact subset of  $\mathcal{G}^{(0)}$ ; and*
2. *the maps  $\alpha \mapsto \sum_{\gamma \in \mathcal{G}^{r(\alpha)}} |h_i(\alpha^{-1}\gamma) - h_i(\gamma)|$  (indexed by  $i$ ) converge uniformly to 0 on every compact subset of  $\mathcal{G}$ .*

The key point of amenability is the following:

**Theorem 10.1.4** ([2, Proposition 6.1.8]). *Let  $\mathcal{G}$  be an étale groupoid, and suppose that  $\mathcal{G}$  is amenable. Then  $\pi_r : C^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G})$  is injective.*

From [52, Theorem 3.1.1], we know that if  $\Gamma$  is a discrete amenable group acting on a locally compact Hausdorff space  $X$  and  $\mathcal{G}$  is the transformation groupoid, then  $C^*(\mathcal{G}) \cong C_0(X) \rtimes \Gamma$  and  $C_r^*(\mathcal{G}) \cong C_0(X) \rtimes_r \Gamma$  coincide. Indeed, if  $\Gamma$  is amenable, then so is  $\mathcal{G}$ : just pull back a mean on  $\Gamma$  to each  $\mathcal{G}^x \cong \{x\} \times \Gamma$  to obtain a (constant) approximately invariant continuous mean. It is possible for  $\mathcal{G}$  to be amenable even when  $\Gamma$  is not: for example, the transformation groupoid of the free group acting on its boundary.

Amenability also has some other very important consequences.

**Theorem 10.1.5** ([2, Corollary 6.2.14 and Theorem 3.3.7]). *If  $\mathcal{G}$  is an étale groupoid, then the following are equivalent:  $\mathcal{G}$  is amenable;  $C^*(\mathcal{G})$  is nuclear;  $C_r^*(\mathcal{G})$  is nuclear.*

It is also possible to study the nuclear dimension of a groupoid  $C^*$ -algebra in terms of dynamical properties of the groupoid. For example, if  $\mathcal{G}$  is a transformation groupoid, then finite Rokhlin dimension of the action as discussed in [46, Section 16.1] implies finite nuclear dimension for  $C^*(\mathcal{G})$ . For more general étale groupoids there is a generalization of Rokhlin dimension, called *dynamic asymptotic dimension* [23] (see [46, Section 17.2]) which, for principal groupoids, guarantees finite nuclear dimension for the associated  $C^*$ -algebra.

**Corollary 10.1.6.** *If  $\mathcal{G}$  and  $\mathcal{H}$  are equivalent étale groupoids, then  $\mathcal{G}$  is amenable if and only if  $\mathcal{H}$  is.*

*Proof.* A direct groupoid-theoretic proof of this can be found in [2], but we will take a shortcut: since  $\mathcal{G}$  and  $\mathcal{H}$  are equivalent,  $C^*(\mathcal{G})$  and  $C^*(\mathcal{H})$  are Morita

equivalent, and therefore  $C^*(\mathcal{G})$  is nuclear if and only if  $C^*(\mathcal{H})$  is nuclear, so the result follows from Theorem 10.1.5.  $\square$

A beautiful result of Tu also relates amenability to the UCT.

**Theorem 10.1.7.** *Let  $\mathcal{G}$  be an étale groupoid. If  $\mathcal{G}$  is amenable, then  $C_r^*(\mathcal{G})$  satisfies the UCT.*

**Remark 10.1.8.** It is a very important question whether every nuclear  $C^*$ -algebra satisfies the UCT. The previous two theorems say that every nuclear groupoid  $C^*$ -algebra (associated to an étale groupoid) satisfies the UCT. In fact, results of Barlak and Li [4] show that this result extends – significantly – to twisted groupoid  $C^*$ -algebras. Moreover, results of Renault [39] show that we can characterize twisted  $C^*$ -algebras associated to étale effective groupoids amongst arbitrary  $C^*$ -algebras in purely  $C^*$ -algebraic terms: they are the ones that admit a *Cartan subalgebra*. We will discuss this further in Chapter 11.

Generally speaking, checking amenability using the definition is hard work. Fortunately, there is a fairly extensive bag of tricks available, and usually the best approach is to see if any of them apply or can be adapted to the example at hand. Theorem 10.1.5 certainly belongs to this bag; we will list a few more that come up particularly frequently.

**Proposition 10.1.9.** *If  $\mathcal{G}$  is a principal étale groupoid and is an  $F_\sigma$  set in  $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ , then the orbit space  $\mathcal{G}^{(0)}/\mathcal{G}$  is a  $T_0$  space if and only if each orbit  $[x] := \{r(\gamma) \mid s(\gamma) = x\}$  is locally closed (that is, each  $[x]$  is relatively open in its closure), and these equivalent conditions imply that  $\mathcal{G}$  is amenable.*

*Proof.* Since  $\mathcal{G}$  is principal, it is algebraically isomorphic to  $\mathcal{R}(\mathcal{G})$ , and so  $\mathcal{R}(\mathcal{G})$  is an  $F_\sigma$  in  $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ . So all of the conditions (1)–(14) in the Ramsay–Mackey–Glimm dichotomy [36, Theorem 2.1] are equivalent, and in particular (4)  $\iff$  (5) of that theorem shows that  $\mathcal{G}^{(0)}/\mathcal{G}$  is  $T_0$  if and only if each  $[x]$  is locally closed. It then follows from [2, Examples 2.1.4(2)] that  $\mathcal{G}$  is a proper Borel groupoid, and therefore amenable by [2, Examples 3.2.2(2) and Theorem 3.3.7].  $\square$

It follows, in particular, that every discrete equivalence relation is amenable (though we could also deduce this from nuclearity of  $\mathcal{K}$ ).

**Proposition 10.1.10** ([2, Proposition 5.3.37]). *Suppose that  $\mathcal{G}$  is an étale groupoid, and that for each  $n \in \mathbb{N}$ ,  $\mathcal{G}_n$  is a closed subgroupoid of  $\mathcal{G}$  with  $\mathcal{G}^{(0)} \subseteq \mathcal{G}_n \subseteq \mathcal{G}_{n+1}$  for all  $n$ . Further suppose that each  $\mathcal{G}_{n+1}$  is a proper  $\mathcal{G}_n$ -space, and that  $\mathcal{G} = \bigcup_n \mathcal{G}_n$ . If each  $\mathcal{G}_n$  is amenable, then  $\mathcal{G}$  is amenable.*

As an example of this, consider the groupoid  $\mathcal{R}_{2^\infty}$  of Example 8.3.5. For each  $n$ , let  $\mathcal{G}_n := \{(vx, wx) \mid |v| = |w| = n, x \in X\}$ . Then each  $\mathcal{G}_n$  is closed (in fact clopen) in  $\mathcal{G}$  and contains  $\mathcal{G}^{(0)}$ , and the  $\mathcal{G}_n$  are nested. In a fixed  $\mathcal{G}_n$  each orbit is finite, of size  $2^n$ , and so  $\mathcal{G}^{(0)}/\mathcal{G}_n$  is a standard Borel space. So [2, Examples 2.1.4(2),

Examples 3.2.2(2) and Theorem 3.3.7] above show that each  $\mathcal{G}_n$  is amenable. Since  $\mathcal{G} = \bigcup_n \mathcal{G}_n$ , we deduce that  $\mathcal{G}$  is amenable.

**Proposition 10.1.11** ([40, Corollary 4.5]). *Suppose that  $\mathcal{G}$  is an étale groupoid and  $c : \mathcal{G} \rightarrow \Gamma$  is a continuous homomorphism into a discrete amenable group. If the clopen subgroupoid  $\ker(c) \subseteq \mathcal{G}$  is amenable, then  $\mathcal{G}$  is amenable.*

The above result for  $\Gamma$  a discrete *abelian* group was first proved by Spielberg in [44, Proposition 9.3]. Spielberg's proof passed through  $C^*$ -algebra theory, proving that  $C^*(\mathcal{G})$  is nuclear, and deducing that  $\mathcal{G}$  is amenable from Theorem 10.1.5. The Renault–Williams proof, by contrast, is entirely groupoid theoretic. Moreover, the Renault–Williams result is more general even than the one stated above: see [40, Theorem 4.2].

**Example 10.1.12.** Deaconu–Renault groupoids for actions of  $\mathbb{N}^k$  are amenable: if  $\mathcal{G}$  is a Deaconu–Renault groupoid over  $\mathbb{N}^k$ , then the map  $c(x, p - q, y) := p - q$  is a continuous cocycle into the abelian, and hence amenable, group  $\Gamma$ . An argument very similar to the one used above to see that the  $\mathcal{G}_n$  in Example 8.3.5 are amenable, shows that for each  $n \in \mathbb{N}^k$  the subgroupoid  $\mathcal{G}_n := \{(x, 0, y) \mid T^n x = T^n y\}$  is a proper Borel groupoid, and then that  $\ker(c) = \bigcup \mathcal{G}_n$  is amenable (see [43] for details).

**Example 10.1.13.** It follows that every graph groupoid is amenable, since it is the Deaconu–Renault groupoid of the shift map on  $E^\infty$ .

**Proposition 10.1.14.** *If  $\mathcal{G}$  is an amenable étale groupoid and  $\mathcal{H}$  is an open or a closed subgroupoid of  $\mathcal{G}$ , then  $\mathcal{H}$  is also amenable.*

The rough idea here is to verify that an approximate invariant continuous mean for  $\mathcal{G}$  restricts to one for  $\mathcal{H}$ .

We finish the section with an example due to Willett that shows that, unlike the situation for groups, in the setting of groupoids it is not the case that amenability is equivalent to coincidence of the full and reduced  $C^*$ -algebras.

**Example 10.1.15** ([50, Lemma 2.8]). Let  $\mathbb{F}_2$  denote the free group on two generators. For  $n \in \mathbb{N}$ , let  $K_n$  denote the intersection of all the normal subgroups of  $\mathbb{F}_2$  that have index at most  $n$  in  $\mathbb{F}_2$ . Willett shows that  $\mathbb{F}_2$  is the infinite union of the  $K_n$ . For each  $n$ , let  $\Gamma_n := \mathbb{F}_2/K_n$ , and let  $\Gamma_\infty = \mathbb{F}_2$ . Let  $\mathcal{G}^{(0)} := \mathbb{N} \cup \{\infty\}$ , the 1-point compactification of  $\mathbb{N}$ , and let  $\mathcal{G}$  be the group bundle  $\bigcup_{x \in \mathcal{G}^{(0)}} \Gamma_x \times \{x\}$ . For each  $\gamma \in \mathcal{G}_\infty = \mathbb{F}_2$ , and each  $n \in \mathbb{N}$ , let  $W(\gamma, n) := \{(\pi_m(\gamma), m) \mid m \geq n\}$ . Then

$$\{W(\gamma, n) \mid \gamma \in \mathbb{F}_2, n \in \mathbb{N}\} \cup \{\{(\gamma, n)\} \mid n \in \mathbb{N}, \gamma \in \Gamma_n\}$$

is a basis for a locally compact Hausdorff topology on  $\mathcal{G}$  under which it is étale. (Each fibre  $\mathcal{G}_x$  is discrete in the relative topology.) This groupoid is not amenable, because an approximate invariant mean on  $\mathcal{G}$  would restrict to an invariant mean on  $\mathbb{F}_2$ . However, Willett proves that  $C^*(\mathcal{G}) = C_r^*(\mathcal{G})$  by showing that the universal norm of  $f \in C_c(\mathcal{G})$  is given by  $\sup_{n \in \mathbb{N}} \|f|_{\Gamma_n}\|$ .

It is still not known, for example, whether a minimal, or even transitive, action for which the full and reduced  $C^*$ -algebras coincide must have an amenable transformation groupoid.

## 10.2 Effective groupoids and uniqueness

Recall that an action of a discrete group  $\Gamma$  on a locally compact Hausdorff space  $X$  is *effective* if, for each  $g \in \Gamma$ , the set  $\{x \in X \mid g \cdot x = x\}$  has empty interior.

In the corresponding transformation groupoid, the basic open sets are the bisections of the form  $\{g\} \times U$  where  $U$  ranges over a base for the topology on  $\mathcal{G}^{(0)}$ . So we can reinterpret effectiveness of an action in terms of the transformation groupoid as follows: the action of  $\Gamma$  on  $X$  is effective if and only if the interior of the isotropy in  $\mathcal{G}$  is equal to  $\mathcal{G}^{(0)}$ . This leads us to a definition.

**Definition 10.2.1.** Let  $\mathcal{G}$  be an étale groupoid. We say that  $\mathcal{G}$  is *effective* if  $\text{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$ .

A Baire category argument shows that an action of a countable discrete group is effective if and only if the points in  $X$  at which the isotropy is trivial are dense in  $X$ . We will need the equivalent for second-countable étale groupoids. We first need a technical lemma that will come up again later.

**Lemma 10.2.2.** *Let  $\mathcal{G}$  be an étale groupoid, and suppose that  $\gamma \in \mathcal{G}$  satisfies  $r(\gamma) \neq s(\gamma)$  and that  $U$  is a bisection containing  $\gamma$ . Then there is an open neighbourhood  $V$  of  $s(\gamma)$  such that  $r(UV) \cap V = \emptyset$ .*

*Proof.* We prove the contrapositive. That is, we suppose that  $U$  is an open bisection and that  $\gamma \in U$ , and that for every neighbourhood  $V$  of  $s(\gamma)$ , we have  $r(UV) \cap V \neq \emptyset$  and prove that  $r(\gamma) = s(\gamma)$ . Choose a descending neighbourhood base  $V_i$  at  $s(\gamma)$  with each  $V_i \subseteq s(U)$ . Since each  $r(UV_i) \cap V_i$  is nonempty, for each  $i$  we can choose  $\gamma_i \in UV_i$  with  $r(\gamma_i) \in V_i$ . Since  $s|_U$  is a homeomorphism, the  $UV_i$  form a neighbourhood base at  $\gamma$ , and so  $\gamma_i \rightarrow \gamma$ . In particular,  $r(\gamma) = \lim_i r(\gamma_i)$ . Since each  $r(\gamma_i) \in V_i$  and the  $V_i$  are a neighbourhood base at  $s(\gamma)$ , we deduce that  $r(\gamma_i) \rightarrow s(\gamma)$ ; so  $s(\gamma) = r(\gamma)$ .  $\square$

**Lemma 10.2.3** ([39, Proposition 3.6]). *Let  $\mathcal{G}$  be a second-countable étale groupoid. Then  $\mathcal{G}$  is effective if and only if  $\{x \mid \mathcal{G}_x^x = \{x\}\}$  is dense in  $\mathcal{G}^{(0)}$ .*

*Proof.* First suppose that  $\text{Iso}(\mathcal{G})^\circ \neq \mathcal{G}^{(0)}$ . Then there is an open  $U \subseteq \text{Iso}(\mathcal{G})$  that is not contained in  $\mathcal{G}^{(0)}$ . Since  $\mathcal{G}^{(0)}$  is closed,  $U \setminus \mathcal{G}^{(0)}$  is open and nonempty, so we can assume that  $U$  has trivial intersection with  $\mathcal{G}^{(0)}$ . Since  $s$  is an open map,  $s(U)$  is an open subset of  $\mathcal{G}^{(0)}$  such that  $\mathcal{G}_x^x \setminus \{x\} \supseteq Ux$  is nonempty for all  $x \in U$ . So  $\{x \mid \mathcal{G}_x^x = \{x\}\}$  is not dense in  $\mathcal{G}^{(0)}$ .

Now suppose that  $\text{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$ . Let  $U$  be an open bisection that does not intersect  $\mathcal{G}^{(0)}$ . We claim that  $U \setminus \text{Iso}(\mathcal{G})$  is open. To see this, suppose that  $\gamma \in U \setminus \text{Iso}(\mathcal{G})$ . We have  $r(\gamma) \neq s(\gamma)$ , and so Lemma 10.2.2 shows that there

is an open neighbourhood  $V$  of  $s(\gamma)$  contained in  $s(U)$  such that  $r(UV) \cap V$  is empty. Now  $UV$  is an open neighbourhood of  $\gamma$  in  $U \setminus \text{Iso}(\mathcal{G})$ . Since  $U \cap \text{Iso}(\mathcal{G})$  has empty interior, we see that  $U \setminus \text{Iso}(\mathcal{G})$  is dense in  $U$ . It follows that  $A_U := r(U \setminus \text{Iso}(\mathcal{G})) \cup (\mathcal{G}^{(0)} \setminus \overline{r(U)})$  is an open dense subset of  $\mathcal{G}^{(0)}$ . By definition, we have  $A_U \subseteq \mathcal{G}^{(0)} \setminus \{x \in \mathcal{G}^{(0)} \mid \mathcal{G}_x^x \cap U \neq \emptyset\}$ .

Now take a countable cover  $\{U_i\}$  of  $\mathcal{G} \setminus \mathcal{G}^{(0)}$  by open bisections. By the preceding paragraph, the sets  $A_{U_i} \subseteq \mathcal{G}^{(0)}$  are open dense sets. So the Baire category theorem implies that  $\bigcap_i A_{U_i}$  is dense in  $\mathcal{G}^{(0)}$ . By construction of the  $A_{U_i}$ , we have  $\mathcal{G}_x^x \cap U_j = \emptyset$  for every  $x \in \bigcap_i A_{U_i}$  and every  $j \in \mathbb{N}$ . Since the  $U_i$  cover  $\mathcal{G} \setminus \mathcal{G}^{(0)}$ , it follows that  $\bigcap_i A_i \subseteq \{x \in \mathcal{G}^{(0)} \mid \mathcal{G}_x^x = \{x\}\}$ , and so the latter is dense as claimed.  $\square$

**Remark 10.2.4.** In the literature, the condition that  $\{x \in \mathcal{G}^{(0)} \mid \mathcal{G}_x^x = \{x\}\}$  is dense has gone by many names, including “topologically principal,” and “topologically free;” but both of these terms have also been used elsewhere for different concepts. So one has to be careful with these terms in the literature: in any given article, check what definition is being used.

Since, in an étale groupoid, the unit space is a clopen subset of  $\mathcal{G}$ , the map  $f \mapsto f|_{\mathcal{G}^{(0)}}$  is a map from  $C_c(\mathcal{G})$  to  $C_c(\mathcal{G}^{(0)})$ . We regard  $C_c(\mathcal{G}^{(0)})$  as an abelian subalgebra of  $C_c(\mathcal{G})$ . We will see later that this restriction map extends to a faithful conditional expectation of  $C_r^*(\mathcal{G})$  onto  $C_0(\mathcal{G}^{(0)})$ . But to exploit this, we need some preliminary work.

**Lemma 10.2.5.** *Let  $\mathcal{G}$  be an effective étale groupoid, and suppose that  $\pi : C_c(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{H})$  is a  $*$ -representation that is injective on  $C_c(\mathcal{G}^{(0)})$ . For each  $f \in C_c(\mathcal{G})$  and each  $\varepsilon > 0$ , there exists  $h \in C_c(\mathcal{G}^{(0)})$  such that  $\|h\| = 1$ ,  $hfh = hf|_{\mathcal{G}^{(0)}}h \in C_c(\mathcal{G}^{(0)})$  and  $\|\pi(hfh)\| \geq \|f|_{\mathcal{G}^{(0)}}\| - \varepsilon$ . In particular,  $\|\pi(f|_{\mathcal{G}^{(0)}})\| \leq \|\pi(f)\|$ .*

*Proof.* Fix  $f \in C_c(\mathcal{G})$ , and  $\varepsilon > 0$ . It suffices to show that  $\|\pi(f)\| \geq \|f|_{\mathcal{G}^{(0)}}\| - \varepsilon$ .

Since  $\pi$  is injective on  $C_c(\mathcal{G}^{(0)})$ , we have  $\|\pi(f|_{\mathcal{G}^{(0)}})\| = \|f|_{\mathcal{G}^{(0)}}\|_\infty$ . Thus, since  $\mathcal{G}$  is effective, Lemma 10.2.3 shows that there exists  $x \in \mathcal{G}^{(0)}$  such that  $f(x) \geq \|f|_{\mathcal{G}^{(0)}}\| - \varepsilon$  and  $\mathcal{G}_x^x = \{x\}$ .

Let  $f_0 := f|_{\mathcal{G}^{(0)}}$ . Then  $f - f_0 \in C_c(\mathcal{G})$ , and so Lemma 9.1.3, shows that we can write  $f - f_0 = \sum_{i=1}^n f_i$  where each  $f_i$  is supported on a bisection  $U_i$  that does not intersect  $\mathcal{G}^{(0)}$ . For each  $i \neq 0$  such that  $x \notin s(\text{supp}(f_i))$ , choose an open neighbourhood  $V_i$  of  $x$  in  $\mathcal{G}^{(0)}$  such that  $V_i \cap s(\text{supp}(f_i)) = \emptyset$ . Then for any  $h \in C_c(V_i)$  we have  $hf_ih = 0$ . For each  $i \neq 0$  such that  $x \in s(\text{supp}(f_i))$ , the unique element  $\gamma$  of  $U_i x$  belongs to  $\mathcal{G}_x^x \setminus \{x\}$ . Since  $\mathcal{G}_x^x = \{x\}$ , we deduce that  $r(\gamma) \neq x$ . By Lemma 10.2.2, we can choose a neighbourhood  $V_i$  of  $x$  with  $V_i \subseteq s(U_i)$  and  $r(U_i V_i) \cap V_i = \emptyset$ . In particular,  $r(\text{supp}(f_i)V_i) \cap V_i = \emptyset$ . Hence Lemma 9.1.4 shows that for any  $h \in C_c(V_i)$  we have  $hf_ih = 0$ .

Now let  $V := \bigcap_{i=1}^n V_i$ . Then  $V$  is an open neighbourhood of  $x$  and so we can choose  $h \in C_c(V) \subseteq C_c(\mathcal{G}^{(0)})$  such that  $h(x) = 1$  and  $\|h\|_\infty = 1$ . Since  $\pi$  is injective on  $C_c(\mathcal{G}^{(0)})$ , it is isometric on  $C_c(\mathcal{G}^{(0)})$ . We have  $h \in C_c(V_i)$  for each  $i$ , and so

$hf_ih = 0$  for  $i \neq 0$  by choice of the  $V_i$ . Hence  $hfh = hf_0h$ , giving

$$\|\pi(f)\| \geq \|\pi(hfh)\| = \|\pi(hf_0h)\| = \|hf_0h\| \geq (hf_0h)(x) = f(x) \geq \|\pi(f_0)\| - \varepsilon. \quad \square$$

**Proposition 10.2.6.** *Let  $\mathcal{G}$  be an étale groupoid. There is a faithful conditional expectation  $\Phi : C_r^*(\mathcal{G}^{(0)}) \rightarrow C_0(\mathcal{G}^{(0)})$  such that  $\Phi(f) = f|_{\mathcal{G}^{(0)}}$  for all  $f \in C_c(\mathcal{G})$ . The map  $j$  of Proposition 9.3.3 satisfies  $j(\Phi(a)) = j(a)|_{\mathcal{G}^{(0)}}$  for all  $a \in C_r^*(\mathcal{G})$ .*

*Proof.* Proposition 9.3.3 shows that  $\|f|_{\mathcal{G}^{(0)}}\|_\infty \leq \|f\|_r$  for every  $f \in C_c(\mathcal{G})$ . By Corollary 9.3.4, we have  $\|f|_{\mathcal{G}^{(0)}}\|_\infty = \|f|_{\mathcal{G}^{(0)}}\|_r$ , and we deduce that  $f \mapsto f|_{\mathcal{G}^{(0)}}$  is a norm-decreasing idempotent linear map from  $C_c(\mathcal{G})$  to  $C_0(\mathcal{G}^{(0)})$ . It therefore extends to a idempotent linear map  $\Phi$  of norm 1 of  $C_r^*(\mathcal{G})$  onto  $C_0(\mathcal{G}^{(0)})$ . By [5, Theorem II.6.10.2], this  $\Phi$  is a conditional expectation. Since  $j(\Phi(f)) = j(f|_{\mathcal{G}^{(0)}})$  for  $f \in C_c(\mathcal{G})$ , continuity of  $j$  and  $\Phi$  give  $j(\Phi(a)) = j(a)|_{\mathcal{G}^{(0)}}$  for all  $a \in C_r^*(\mathcal{G})$ .

To see that  $\Phi$  is faithful, suppose that  $a \neq 0$ . Then there exist  $x \in \mathcal{G}^{(0)}$  and  $\gamma \in \mathcal{G}_x$  such that  $(\pi_x(a^*a)\delta_\gamma | \delta_\gamma) \neq 0$ . Applying the unitary equivalence between  $\pi_x$  and  $\pi_{r(\gamma)}$  obtained in Proposition 9.3.1, we see that  $(\pi_{r(\gamma)}(a^*a)\delta_{r(\gamma)} | \delta_{r(\gamma)}) \neq 0$ . That is,  $j(a^*a)(r(\gamma)) \neq 0$ . Hence  $j(\Phi(a^*a)) = \Phi(a^*a)|_{\mathcal{G}^{(0)}} \neq 0$ , and we conclude that  $\Phi(a^*a) \neq 0$ .  $\square$

We can now prove our main theorem for this section.

**Theorem 10.2.7.** *Let  $\mathcal{G}$  be an effective étale groupoid. If  $\phi : C_r^*(\mathcal{G}) \rightarrow A$  is a  $C^*$ -homomorphism that is injective on  $C_0(\mathcal{G}^{(0)})$ , then it is injective on all of  $C_r^*(\mathcal{G})$ .*

*Proof.* By Lemma 10.2.5, we have  $\|\pi(f)\| \geq \|\pi(f|_{\mathcal{G}^{(0)}})\|$  for all  $f \in C_c(\mathcal{G})$ , and so there is a well-defined linear map  $\Psi : \pi(C_r^*(\mathcal{G})) \rightarrow \pi(C_0(\mathcal{G}^{(0)}))$  such that  $\Psi(\pi(f)) = \pi(f|_{\mathcal{G}^{(0)}})$  for  $f \in C_c(\mathcal{G})$ . It follows from continuity that the faithful conditional expectation  $\Phi$  of Proposition 10.2.6 satisfies  $\Psi \circ \pi = \pi \circ \Phi$ . Now we follow the standard argument, using injectivity of  $\phi$  on  $C_0(\mathcal{G}^{(0)})$  at the third implication:

$$\phi(a) = 0 \Rightarrow \Psi(\phi(a^*a)) = 0 \Rightarrow \phi(\Phi(a^*a)) = 0 \Rightarrow \Phi(a^*a) = 0 \Rightarrow a = 0. \quad \square$$

**Remark 10.2.8.** An equivalent restatement of Theorem 10.2.7 is that if  $\mathcal{G}$  is an effective étale groupoid, then every nontrivial ideal of  $C_r^*(\mathcal{G})$  has nonzero intersection with  $C_0(\mathcal{G}^{(0)})$ .

## 10.3 Invariant sets, ideals, and simplicity

Our aim in this section is to shed some light on the ideal structure of  $C^*(\mathcal{G})$ , and to characterize simplicity of  $C^*(\mathcal{G})$  when  $\mathcal{G}$  is an amenable étale groupoid. When  $\mathcal{G}$  is not assumed amenable, things become more complicated: most of the statements in the section remain true for either the full or the reduced  $C^*$ -algebra, but typically not both, and care is required.

We will say that a subset  $U$  of  $\mathcal{G}^{(0)}$  is *invariant* if  $r(\mathcal{G}U) \subseteq U$ . Observe that if  $\mathcal{G}$  is a transformation groupoid, then an open  $U \subseteq \mathcal{G}^{(0)}$  is invariant precisely if  $C_0(U) \subseteq C_0(\mathcal{G}^{(0)})$  is an invariant ideal as in [52, Section 3.1.2].

**Lemma 10.3.1.** *Let  $\mathcal{G}$  be an étale groupoid, and let  $I$  be an ideal of  $C^*(\mathcal{G})$ . Then there is an open invariant subset  $\text{supp}(I) \subseteq \mathcal{G}^{(0)}$  such that  $I \cap C_0(\mathcal{G}^{(0)}) = \{f \in C_0(\mathcal{G}^{(0)}) \mid f(x) = 0 \text{ for all } x \in \mathcal{G}^{(0)} \setminus \text{supp}(I)\} \cong C_0(\text{supp}(I))$ .*

*Proof.* Since  $I \cap C_0(\mathcal{G}^{(0)})$  is an ideal of a commutative  $C^*$ -algebra, it has the form  $I \cap C_0(\mathcal{G}^{(0)}) = C_0(\text{supp}(I))$  for an open set  $\text{supp}(I) \subseteq \mathcal{G}^{(0)}$ . We just have to show that this set is invariant. For this, suppose that  $x \in \text{supp}(I)$ . Choose  $f \in I \cap C_0(\mathcal{G}^{(0)})$  such that  $f(x) \neq 0$ . Fix  $\gamma \in G_x$ ; we must show that  $r(\gamma) \in \text{supp}(I)$ . For this, take an open bisection  $U$  containing  $\gamma$  and fix  $h \in C_c(U)$  with  $h(\gamma) = 1$ . Lemma 9.1.4 shows that  $hfh^*$  is supported on  $U\mathcal{G}^{(0)}U^{-1} = r(U) \subseteq \mathcal{G}^{(0)}$ . So  $hfh^* \in I \cap C_0(\mathcal{G}^{(0)})$ . Since  $(hfh^*)(r(\gamma)) = h(\gamma)f(x)h^*(\gamma^{-1}) = f(x) \neq 0$ , we deduce that  $r(\gamma) \in \text{supp}(I)$ .  $\square$

If  $U$  is an open invariant subset of  $\mathcal{G}^{(0)}$ , then  $\mathcal{G}U$  is an open subgroupoid of  $\mathcal{G}$  and so a locally compact Hausdorff étale groupoid in the relative topology. Similarly,  $\mathcal{G} \setminus \mathcal{G}U$  is a closed subgroupoid of  $\mathcal{G}$ , and hence again a locally compact étale groupoid in the subspace topology.

**Proposition 10.3.2.** *Let  $\mathcal{G}$  be an étale groupoid, and let  $U$  be an open invariant subset of  $\mathcal{G}^{(0)}$ . Define  $W := \mathcal{G}^{(0)} \setminus U$ . The inclusion  $C_c(\mathcal{G}U) \hookrightarrow C_c(\mathcal{G})$  extends to an injective  $C^*$ -homomorphism  $i_U : C^*(\mathcal{G}U) \hookrightarrow C^*(\mathcal{G})$ . The image  $I_U := i_U(C^*(\mathcal{G}U))$  is an ideal of  $C^*(\mathcal{G})$ , and is generated as an ideal by  $C_c(U) \subseteq C_0(\mathcal{G}^{(0)})$ . There is a homomorphism  $\pi_U : C^*(\mathcal{G}) \rightarrow C^*(\mathcal{G}W)$  satisfying  $\pi_U(f) = f|_{\mathcal{G}W}$  for all  $f \in C_c(\mathcal{G})$ . Moreover the sequence*

$$0 \longrightarrow C^*(\mathcal{G}U) \xrightarrow{i_U} C^*(\mathcal{G}) \xrightarrow{\pi_U} C^*(\mathcal{G}W) \longrightarrow 0$$

*is exact.*

*Proof.* The inclusion  $C_c(\mathcal{G}U) \hookrightarrow C_c(\mathcal{G}) \hookrightarrow C^*(\mathcal{G})$  is a  $*$ -homomorphism, so the universal property of  $C^*(\mathcal{G})$  shows that there is a homomorphism  $i_U : C^*(\mathcal{G}U) \rightarrow C^*(\mathcal{G})$  as required.

To see that the image of  $i_U$  is an ideal, observe that if  $f \in C_c(\mathcal{G}U)$ , and if  $g \in C_c(\mathcal{G})$  is supported on a bisection, then  $\text{supp}(g * f) \subseteq \text{supp}(g)\text{supp}(f) \subseteq \mathcal{G}(\mathcal{G}U) = \mathcal{G}U$ ; similarly (or by taking adjoints) we have  $\text{supp}(f * g) \subseteq \mathcal{G}U$ . So  $i_U(C_c(\mathcal{G}U))$  is an algebraic 2-sided ideal of  $C^*(\mathcal{G})$ , and hence  $I_U$  is an ideal by continuity. Since  $C_c(U)$  contains an approximate identity for  $C^*(\mathcal{G}U)$ , the ideal  $I_U$  is generated as an ideal by  $C_c(U)$ .

Since every  $*$ -homomorphism  $\pi$  of  $C_c(\mathcal{G})$  into a  $C^*$ -algebra  $B$  can be composed with a faithful representation of  $B$  to obtain a  $*$ -representation of  $C_c(\mathcal{G})$  that achieves the same norm on every element, it suffices to show that there is a  $*$ -homomorphism  $\pi : C_c(\mathcal{G}) \rightarrow B$  for some  $C^*$ -algebra  $B$  such that  $\|\pi(f)\| \geq \|f\|_{C^*(\mathcal{G}U)}$  for every  $f \in C_c(\mathcal{G}_u)$ . To see this observe that  $C_c(\mathcal{G}U)$  is an algebraic



ideal of  $C_c(\mathcal{G})$ , and so for each  $f \in C_c(\mathcal{G})$ , there is a linear map  $\pi(f) : C_c(\mathcal{G}U) \rightarrow C_c(\mathcal{G}U)$  given by  $\pi(f)g = f * g$ . If we regard  $C^*(\mathcal{G}U)$  as a Hilbert bimodule over itself with inner product  $\langle a, b \rangle_{C^*(\mathcal{G}U)} = a^*b$ , then  $\langle \pi(f)a, b \rangle_{C^*(\mathcal{G}U)} = a^*\pi(f^*)b = \langle a, \pi(f^*)b \rangle_{C^*(\mathcal{G}U)}$ ; so  $\pi(f)$  is an adjointable operator on  $C^*(\mathcal{G}U)_{C^*(\mathcal{G}U)}$ . From this we see that  $\pi$  is a  $*$ -homomorphism into the algebra  $\mathcal{L}(C^*(\mathcal{G}U)_{C^*(\mathcal{G}U)})$  of adjointable operators on  $C^*(\mathcal{G}U)_{C^*(\mathcal{G}U)}$ . For  $f \in C_c(\mathcal{G}U)$ , we have

$$\|\pi(f)\| \geq \|\pi(f) \frac{f^*}{\|f\|}\| = \|f\|$$

as required.

The map  $f \mapsto f|_W$  is a  $*$ -homomorphism of  $C_c(\mathcal{G})$  onto  $C_c(\mathcal{G}W)$  and hence determines a  $*$ -homomorphism from  $C_c(\mathcal{G})$  to  $C^*(\mathcal{G}W)$ ; so once again the universal property gives a homomorphism  $\pi : C^*(\mathcal{G}) \rightarrow C^*(\mathcal{G}W)$  that extends restriction of functions. Clearly  $\ker \pi$  contains  $C_c(\mathcal{G}U)$  and hence the image of  $i_U$ . In particular,  $\pi$  induces a homomorphism  $\tilde{\pi} : C^*(\mathcal{G})/I_U \rightarrow C^*(\mathcal{G}W)$ . To see that this homomorphism is injective, observe that since  $C_c(\mathcal{G}U) \subseteq I_U$ , if  $f, g \in C_c(\mathcal{G})$  satisfy  $f|_{\mathcal{G}W} = g|_{\mathcal{G}W}$ , then  $f - g \in I_U$ . Hence there is a well-defined  $*$ -homomorphism  $\phi : C_c(\mathcal{G}W) \rightarrow C^*(\mathcal{G})/I_U$  such that  $\phi(f|_W) = f + I_U$  for all  $f \in C_c(\mathcal{G})$ . The universal property of  $C^*(\mathcal{G}W)$  shows that  $\phi$  extends to a homomorphism  $\phi : C^*(\mathcal{G}W) \rightarrow C^*(\mathcal{G})/I_U$ . The image of  $\phi$  contains the image of  $C_c(\mathcal{G}W)$  in the quotient and so  $\phi$  is surjective. Since  $\tilde{\pi}_U \circ \phi$  is the identity map on  $C_c(\mathcal{G}W)$  we see that  $\tilde{\pi}_U \circ \phi$  is the identity homomorphism; so surjectivity of  $\phi$  ensures that  $\tilde{\pi}_U$  is injective, and therefore that  $\ker(\pi_U) = I_U$ .  $\square$

The preceding proposition holds for general groupoids, but the proof requires the Disintegration Theorem. For reduced  $C^*$ -algebras, the corresponding maps  $i_U^r$  and  $\pi_U^r$  between reduced  $C^*$ -algebras exist,  $i_U^r$  is injective,  $\pi_U^r$  is surjective, and  $\ker \pi_U^r$  contains the image of  $i_U^r$  (these statements are all relatively easy to prove using the properties of regular representations). But the sequence is not necessarily exact. The first example of this was given by Skandalis [38, Appendix, p. 35], but Willett's example, Example 10.1.15, also gives an instance of the failure of exactness: let  $\mathcal{G}$  be Willett's groupoid. Since  $\mathcal{G}$  is a group bundle, the set  $\mathbb{N} \subseteq \mathbb{N} \cup \{\infty\} = \mathcal{G}^{(0)}$  is an open invariant subset. Since  $\mathcal{G}\mathbb{N}$  is a (discrete) bundle of finite groups, it is amenable (just take normalized counting measure on each fibre), so  $C^*(\mathcal{G}\mathbb{N}) = C_r^*(\mathcal{G}\mathbb{N})$ , and Willett's result says that  $C^*(\mathcal{G}) = C_r^*(\mathcal{G})$ . So Proposition 10.3.2 shows that the sequence

$$0 \rightarrow C_r^*(\mathcal{G}\mathbb{N}) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C^*(\mathbb{F}_2) \rightarrow 0$$

is exact. Since  $C_r^*(\mathbb{F}_2)$  is a proper quotient of  $C^*(\mathbb{F}_2)$ , we deduce that

$$0 \rightarrow C_r^*(\mathcal{G}\mathbb{N}) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C_r^*(\mathbb{F}_2) \rightarrow 0$$

is not exact.

We will say that  $\mathcal{G}$  is *strongly effective* if  $\mathcal{G}W$  is effective for every closed invariant subset of  $\mathcal{G}^{(0)}$ . This is a strictly stronger condition than effectiveness:



Consider the action of  $\mathbb{Z}$  on its own 1-point compactification  $\mathbb{Z} \cup \{\infty\}$  determined by continuous extension of the translation action of  $\mathbb{Z}$  on itself. The resulting transformation groupoid  $\mathcal{G}$  is effective because the only point with nontrivial isotropy is  $\infty$ ; but  $\{\infty\}$  is a closed invariant subset of  $\mathcal{G}^{(0)}$  and clearly  $\mathcal{G}\{\infty\} \cong \mathbb{Z}$  is not effective.

**Theorem 10.3.3.** *Let  $\mathcal{G}$  be an amenable étale groupoid. The map  $U \mapsto I_U$  is an injection from the set of open invariant subsets of  $\mathcal{G}^{(0)}$  to the set of ideals of  $C^*(\mathcal{G})$ . It is bijective if and only if  $\mathcal{G}$  is strongly effective.*

*Proof.* If  $U, V$  are distinct open invariant sets, then  $I_U$  and  $I_V$  are distinct because  $I_U \cap C_0(\mathcal{G}^{(0)}) = C_0(U) \neq C_0(V) = I_V \cap C_0(\mathcal{G}^{(0)})$ . So  $U \mapsto I_U$  is injective.

First suppose that  $\mathcal{G}$  is strongly effective. Fix an ideal  $I$  of  $C^*(\mathcal{G})$ . We must show that  $I = I_{\text{supp}(I)}$ . To see this, first observe that Proposition 10.3.2 shows that  $I_{\text{supp}(I)}$  is generated as an ideal by  $C_c(\text{supp}(I))$ , which is a subset of  $I \cap C_0(\mathcal{G}^{(0)})$  by definition. So  $I_{\text{supp}(I)} \subseteq I$ . Thus the quotient map induces a homomorphism  $\tilde{q} : C^*(\mathcal{G})/I_{\text{supp}(I)} \rightarrow C^*(\mathcal{G})/I$ . Let  $W := \mathcal{G}^{(0)} \setminus \text{supp}(I)$ . Proposition 10.3.2 gives an isomorphism  $\tilde{\pi} : C^*(\mathcal{G})/I_{\text{supp}(I)} \cong C^*(\mathcal{G}W)$  extending restriction of functions on  $C_c(\mathcal{G})$ . Since  $I \cap C_0(\mathcal{G}^{(0)}) = C_0(\text{supp}(I))$ , we see that  $\tilde{q} \circ \tilde{\pi}$  is injective on  $C_0(W)$ . Since  $\mathcal{G}$  is strongly effective,  $\mathcal{G}W$  is effective, and it then follows from Theorem 10.2.7 that  $\tilde{\pi}$  is injective. So  $I = I_{\text{supp}(I)}$ .

Now suppose that  $\mathcal{G}$  is not strongly effective. Fix a closed invariant set  $W \subseteq \mathcal{G}^{(0)}$  such that  $\mathcal{H} := \mathcal{G}W$  is not effective. We will construct a representation  $\psi$  of  $C^*(\mathcal{H})$  such that  $\psi$  is faithful on  $C_0(\mathcal{H}^{(0)})$  but is not faithful on  $C^*(\mathcal{H})$ .

Recall that the orbit  $[x]$  of  $x \in \mathcal{H}^{(0)}$  is the set  $\{r(\gamma) \mid \gamma \in \mathcal{H}_x\}$ . For each  $x \in \mathcal{H}^{(0)}$ , there is a linear map  $\epsilon_x : C_c(\mathcal{H}) \rightarrow \mathcal{B}(\ell^2([x]))$  given by  $\epsilon_x(f)\delta_y = \sum_{\gamma \in \mathcal{H}_y} f(\gamma)\delta_{r(\gamma)}$ . For  $f, g \in C_c(\mathcal{H})$  and  $y, z \in [x]$ , we have

$$(\epsilon_x(f)\epsilon_x(g)\delta_y \mid \delta_z) = \sum_{\beta \in \mathcal{H}_y} g(\beta)(\epsilon_x(f)\delta_{r(\alpha)} \mid \delta_z) = \sum_{\beta \in \mathcal{H}_y, \alpha \in \mathcal{H}_{r(\beta)}^z} f(\alpha)g(\beta).$$

On the other hand,

$$(\epsilon_x(f * g)\delta_y \mid \delta_z) = \sum_{\gamma \in \mathcal{H}_y^z} (f * g)(\gamma) = \sum_{\gamma \in \mathcal{H}_y^z} \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta) = \sum_{\gamma \in \mathcal{H}_y^z, \alpha \in \mathcal{H}^z} f(\alpha)g(\alpha^{-1}\gamma).$$

The map  $(\alpha, \beta) \mapsto (\alpha, \alpha\beta)$  is a bijection from  $\{(\alpha, \beta) \mid \beta \in \mathcal{H}_y, \alpha \in \mathcal{H}_{r(\beta)}^z\}$  to  $\{(\alpha, \gamma) \mid \gamma \in \mathcal{H}_y^z, \alpha \in \mathcal{H}^z\}$  (the inverse is  $(\alpha, \gamma) \mapsto (\alpha, \alpha^{-1}\gamma)$ ). So the two sums are equal, and therefore  $\epsilon_x$  is multiplicative. Likewise,

$$(\epsilon_x(f^*)\delta_y \mid \delta_z) = \sum_{\gamma \in \mathcal{H}_y^z} f^*(\gamma) = \sum_{\gamma \in \mathcal{H}_y^z} \overline{f(\gamma^{-1})} = \sum_{\eta \in \mathcal{H}_z^y} \overline{f(\eta)} = (\delta_y \mid \epsilon_x(g)\delta_z).$$

So  $\epsilon_x$  is a  $*$ -representation of  $C_c(\mathcal{H})$ . The universal property of  $C^*(\mathcal{H})$  therefore shows that  $\epsilon_x$  extends to a representation of  $C^*(\mathcal{H})$ . Let  $\psi := \bigoplus_{x \in \mathcal{H}^{(0)}} \epsilon_x$ . If  $f \in$

$C_0(\mathcal{H}^{(0)}) \setminus \{0\}$ , say  $f(x) \neq 0$ , then  $\epsilon_x(f)\delta_x = f(x)\delta_x \neq 0$ , so  $\psi$  is faithful on  $C_0(\mathcal{H}^{(0)})$ .

Since  $\mathcal{H}$  is not effective, there is an open bisection  $U$  contained in  $\text{Iso}(\mathcal{H}) \setminus \mathcal{H}^{(0)}$ . Fix  $f \in C_c(r(U))$ , and define  $\tilde{f} \in C_c(U)$  by

$$\tilde{f}(\gamma) = f(r(\gamma)) \quad \text{for all } \gamma \in U.$$

Since  $U$  and  $r(U)$  are open in  $\mathcal{H}$ , we can regard  $f$  and  $\tilde{f}$  as elements of  $C_c(\mathcal{H})$ . Since  $U$  is contained in  $\text{Iso}(\mathcal{H})$ , for  $x \in \mathcal{H}^{(0)}$  and  $y \in [x]$ , we have

$$\begin{aligned} \epsilon_x(\tilde{f})\delta_y &= \begin{cases} \tilde{f}(Uy)\delta_y & \text{if } y \in r(U) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} f(y)\delta_y & \text{if } y \in r(U) \\ 0 & \text{otherwise} \end{cases} \\ &= \epsilon_x(f)\delta_y. \end{aligned}$$

So  $f - \tilde{f} \in \ker(\psi)$ , and  $f - \tilde{f} \neq 0$  because  $\text{supp}(f) \subseteq \mathcal{G}^{(0)}$  and  $\text{supp}(\tilde{f}) \subseteq U \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$ .

Now consider the ideals  $I_{\mathcal{G}^{(0)} \setminus W}$  and  $\ker(\psi \circ \pi_{\mathcal{G}^{(0)} \setminus W})$ . We have just seen that they have identical intersection with  $C_0(\mathcal{G}^{(0)})$  (namely  $C_0(\mathcal{G}^{(0)} \setminus W)$ ), but are not equal. So  $U \mapsto I_U$  is not a bijection.  $\square$

**Remark 10.3.4.** • Whether or not  $\mathcal{G}$  is amenable or strongly effective, the map  $U \mapsto I_U$  is an injection from the collection of open invariant sets of  $\mathcal{G}^{(0)}$  to the space of ideals of  $C^*(\mathcal{G})$ .

- We could replace amenability of  $\mathcal{G}$  with the requirement that  $C^*(\mathcal{G}W) = C_r^*(\mathcal{G}W)$  for every closed invariant  $W$  in both Proposition 10.3.2 and Theorem 10.3.3. Conversely, if there exists a closed invariant set  $W$  such that  $C^*(\mathcal{G}W) \neq C_r^*(\mathcal{G}W)$ , then  $U \mapsto I_U$  is not surjective because the kernel of  $\pi_r^{\mathcal{G}W} \circ \pi_{\mathcal{G}^{(0)} \setminus W}$  is not in its range.
- In the non-amenable case, the map  $U \mapsto \ker(\pi_U^r)$  remains an injection from open invariant sets to ideals of the reduced  $C^*$ -algebra. It is possible for this map to be bijective even if  $\mathcal{G}$  is not effective: for example, the reduced  $C^*$ -algebra of the free group is simple.

We say that a groupoid  $\mathcal{G}$  is *minimal* if for every  $x \in \mathcal{G}^{(0)}$  the orbit  $[x]$  is dense in  $\mathcal{G}^{(0)}$ .

**Lemma 10.3.5.** *Let  $\mathcal{G}$  be a topological groupoid. Then  $\mathcal{G}$  is minimal if and only if the only open invariant subsets of  $\mathcal{G}^{(0)}$  are  $\emptyset$  and  $\mathcal{G}^{(0)}$ .*

*Proof.* If  $\mathcal{G}$  is minimal, then the only nonempty closed invariant subset of  $\mathcal{G}^{(0)}$  is  $\mathcal{G}^{(0)}$ . Hence the only open invariant subsets of  $\mathcal{G}^{(0)}$  are  $\emptyset$  and  $\mathcal{G}^{(0)}$ .

Now suppose that  $\mathcal{G}$  is not minimal, say  $x \in \mathcal{G}^{(0)}$  satisfies  $\overline{[x]} \neq \mathcal{G}^{(0)}$ . We claim that  $\overline{[x]}$  is invariant. To see this, suppose that  $y \in \overline{[x]}$  and  $\gamma \in \mathcal{G}_y$ ; we must

show that  $r(\gamma) \in \overline{[x]}$ . Choose  $\alpha_n \in \mathcal{G}_x$  with  $r(\alpha_n) \rightarrow y$ . Choose an open bisection  $U$  containing  $\gamma$ . Since  $s(U)$  is an open neighbourhood of  $s(\gamma) = y$ , we have  $r(\alpha_n) \in U$  for large  $n$ , so by passing to a subsequence we can assume that  $r(\alpha_n) \in U$  for all  $n$ . For each  $n$ , let  $\gamma_n$  be the unique element of  $U$  such that  $s(\gamma_n) = r(\alpha_n)$ . Since  $s(\gamma_n) \rightarrow s(\gamma)$  and since  $s|_U$  is a homeomorphism, we have  $\gamma_n \rightarrow \gamma$  and so  $r(\gamma_n) \rightarrow r(\gamma)$ . Now each  $\gamma_n \alpha_n \in \mathcal{G}_x$  and we have  $r(\gamma_n \alpha_n) = r(\gamma_n) \rightarrow r(\gamma)$ , and we deduce that  $r(\gamma) \in \overline{[x]}$ . That is,  $\overline{[x]}$  is  $\mathcal{G}$ -invariant as claimed. Hence  $\mathcal{G}^{(0)} \setminus \overline{[x]}$  is a nontrivial open invariant subset of  $\mathcal{G}^{(0)}$ .  $\square$

We therefore obtain the following characterization of simplicity [8].

**Theorem 10.3.6.** *Let  $\mathcal{G}$  be an amenable étale groupoid. Then  $C^*(\mathcal{G})$  is simple if and only if  $\mathcal{G}$  is effective and minimal.*

*Proof.* First suppose that  $\mathcal{G}$  is effective and minimal. Since it is minimal, the only nonempty closed invariant set is  $\mathcal{G}^{(0)}$  and so  $\mathcal{G}$  is (trivially) strongly effective. So Theorem 10.3.3 and Lemma 10.3.5 show that the only ideals of  $C^*(\mathcal{G})$  are  $I_\emptyset = \{0\}$  and  $I_{\mathcal{G}^{(0)}} = C^*(\mathcal{G})$ .

If  $\mathcal{G}$  is not minimal, then Lemma 10.3.5 gives a nontrivial open invariant subset of  $\mathcal{G}^{(0)}$  and so Theorem 10.3.3 gives a nontrivial ideal. Likewise, if  $\mathcal{G}$  is not effective, then  $U \mapsto I_U$  is injective but not bijective by Theorem 10.3.3; so it is not surjective. Since  $\{0\} = I_\emptyset$  and  $C^*(\mathcal{G}) = I_{\mathcal{G}^{(0)}}$  are in its range, it follows that  $C^*(\mathcal{G})$  has a nontrivial ideal.  $\square$

We also obtain a sufficient condition for reduced  $C^*$ -algebras.

**Proposition 10.3.7.** *If  $\mathcal{G}$  is an effective, minimal, étale groupoid, then  $C_r^*(\mathcal{G})$  is simple.*

*Proof.* Let  $I$  be a nonzero ideal of  $C_r^*(\mathcal{G})$ ; we must show that  $I = C_r^*(\mathcal{G})$ . By Theorem 10.2.7, we have  $I \cap C_0(\mathcal{G}^{(0)}) \neq \{0\}$ . Now if  $q_I : C_r^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G})/I$  is the quotient map, and  $\pi_r : C^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G})$  is the canonical surjection, then  $J := \ker(q_I \circ \pi_r) \cap C_0(\mathcal{G}^{(0)}) \supseteq I \cap C_0(\mathcal{G}^{(0)})$  is nonzero. So Lemma 10.3.1 shows that  $\text{supp}(J)$  is a nonempty open invariant set. Since  $\mathcal{G}$  is minimal, it follows that  $\text{supp}(J) = \mathcal{G}^{(0)}$  and so  $C_0(\mathcal{G}^{(0)}) \subseteq J$ . So  $\pi_r(C_0(\mathcal{G}^{(0)})) = C_0(\mathcal{G}^{(0)})$  is contained in  $I$ , and we deduce that  $I = C_r^*(\mathcal{G})$ .  $\square$

## 10.4 Pure infiniteness

In this section we briefly discuss a result of Anantharaman-Delaroche giving a sufficient condition for  $C^*(\mathcal{G})$  to be purely infinite. There is no improving on her argument, so the treatment here is more or less exactly the same as in [1].

**Definition 10.4.1** ([1, Definition 2.1]). Let  $\mathcal{G}$  be an étale groupoid. We say that  $\mathcal{G}$  is *locally contracting* at  $x \in \mathcal{G}^{(0)}$  if for every open neighbourhood  $V$  of  $x$ , there is an

open set  $W \subseteq V$  and an open bisection  $U$  such that  $\overline{W} \subseteq s(U)$  and  $r(U\overline{W}) \not\subseteq W$ . We say that  $\mathcal{G}$  is *locally contracting* if it is locally contracting at  $x$  for every  $x \in \mathcal{G}^{(0)}$ .

**Theorem 10.4.2** ([1, Proposition 2.4]). *Suppose that  $\mathcal{G}$  is an effective, locally contracting, étale groupoid. Then  $C_r^*(\mathcal{G})$  is purely infinite.*

*Proof.* It suffices to show that for every  $a \in A_+$ , the hereditary subalgebra generated by  $a$  contains an infinite projection. So fix  $a \in A_+$ . We may assume without loss of generality that the faithful conditional expectation  $\Phi : C_r^*(\mathcal{G}) \rightarrow C_0(\mathcal{G}^{(0)})$  of Proposition 10.2.6 satisfies  $\|\Phi(a)\| = 1$ . Since  $C_c(\mathcal{G})$  is dense in  $C^*(\mathcal{G})$ , we can choose  $b \in C_c(\mathcal{G}) \cap A_+$  such that  $b \leq a$  and  $\|a - b\| < \frac{1}{4}$ . Since  $\Phi$  is norm decreasing we deduce that  $\|\Phi(a) - \Phi(b)\| < \frac{1}{4}$ , and so  $\|\Phi(b)\| > \frac{3}{4}$ . Lemma 10.2.5 with  $\varepsilon = (\|\Phi(b)\| - 3/4)/2$  gives a function  $h \in C_c(\mathcal{G}^{(0)})$  such that  $\|h\| = 1$ ,  $hbh = h\Phi(b)h \in C_0(\mathcal{G}^{(0)})$ , and  $\|hbh\| > \frac{3}{4}$ . Since  $hbh \leq b \leq a$ , it suffices to find an infinite projection  $p$  and a partial isometry  $w$  in  $C^*(\mathcal{G})$  such that  $wpw^* \leq h\Phi(b)h =: b_0$ .

Using that  $\mathcal{G}$  is locally contracting, we choose an open  $V$  with  $\overline{V} \subseteq \{x \in \mathcal{G}^{(0)} \mid b_0(x) > 3/4\}$  and a bisection  $B$  with  $\overline{V} \subseteq s(B)$  and  $r(B\overline{V}) \not\subseteq V$ . Let  $T_B : s(B) \rightarrow r(B)$  be the homeomorphism  $T_B(s(\gamma)) = r(\gamma)$  for  $\gamma \in B$ . Then  $T_B(\overline{V}) = r(B\overline{V})$  is a compact subset of  $V$  and is not all of  $V$ . So we can choose  $k \in C_c(V)$  such that  $k$  is identically 1 on  $T_B(\overline{V})$ . Define  $x \in C_c(BV)$  by  $x(\gamma) = k(s(\gamma))$  for  $\gamma \in BV$ . We have  $x^*x = k^2$ , and in particular  $x^*x$  is identically 1 on  $r(B\overline{V}) \supseteq r(\text{supp } x)$ . Hence  $x^*xx = x$ .

So  $x$  is a scaling element. There is a standard trick for constructing a projection from such an element: Define  $v$  in the minimal unitization of  $C_r^*(\mathcal{G})$  by  $v = x + (1 - x^*x)^{1/2}$ . We have

$$v^*v = x^*x + x^*(1 - x^*x)^{1/2} + (1 - x^*x)^{1/2}x + (1 - x^*x).$$

Since  $(1 - x^*x)x = 0$ , every  $(1 - x^*x)^n x = 0$  and then by continuity  $f(1 - x^*x)x = 0$  for every  $f \in C(\sigma(1 - x^*x))$ . In particular,  $(1 - x^*x)^{1/2}x = 0 = x^*(1 - x^*x)^{1/2}$ , and so  $v^*v = 1$ . Consequently,  $vv^*$  is a projection. We compute

$$\begin{aligned} vv^* &= xx^* + x(1 - x^*x)^{1/2} + (1 - x^*x)^{1/2}x^* + (1 - x^*x) \\ &= 1 + xx^* - x^*x + x(1 - x^*x)^{1/2} + (1 - x^*x)^{1/2}x^* \\ &= 1 - (x^*x - xx^* - x(1 - x^*x)^{1/2} - (1 - x^*x)^{1/2}x^*). \end{aligned}$$

So  $p := x^*x - xx^* - x(1 - x^*x)^{1/2} - (1 - x^*x)^{1/2}x^* = 1 - vv^*$  is a projection in  $C_r^*(\mathcal{G})$ . We have  $\Phi(p) = x^*x - xx^*$  which is nonzero, and so  $p$  is nonzero. Also, since  $r(\text{supp}(x)) \subseteq s(\text{supp}(x)) \subseteq V \subseteq \{z \in \mathcal{G}^{(0)} \mid b_0(z) > 3/4\}$ , we see that  $p$  belongs to the hereditary subalgebra generated by  $b$ .

To see that  $p$  is an infinite projection, argue exactly as above, but with  $V$  replaced by a nonempty open subset of  $\text{supp}(x^*x) \setminus \text{supp}(xx^*)$  to obtain a scaling element  $y$  in  $pC_r^*(\mathcal{G})p$ . Then the calculations we performed above for  $v$  show that  $w := y + (p - y^*y)^{1/2}$  is a partial isometry with  $w^*w = p$  and  $ww^* < p$ . So  $p$  is infinite as required.  $\square$

When  $\mathcal{G}$  is also minimal, we can verify that  $\mathcal{G}$  is locally contracting by verifying it at any one unit  $x$ .

**Lemma 10.4.3.** *Let  $\mathcal{G}$  be an étale groupoid. If  $\mathcal{G}$  is minimal, then  $\mathcal{G}$  is locally contracting at some point  $x$  if and only if  $\mathcal{G}$  is locally contracting.*

*Proof.* The “if” implication is trivial.

For the “only if”, suppose that  $\mathcal{G}$  is locally contracting at  $x$  and fix  $y \in \mathcal{G}^{(0)}$ . Fix an open neighbourhood  $V$  of  $y$ . Since  $\mathcal{G}$  is minimal, there is an open bisection  $B$  such that  $r(B) \subseteq V$  and  $x \in s(B)$ . Since  $\mathcal{G}$  is locally contracting at  $x$ , there is an open  $W$  containing  $x$  with  $\overline{W} \subseteq s(B)$  and an open bisection  $U$  such that  $\overline{W} \subseteq s(U)$  and  $r(U\overline{W}) \not\subseteq W$ . Now  $r(BW) = BWB^{-1}$  is an open neighbourhood of  $y$  with  $r(BW) \subseteq V$ , and  $BUB^{-1}$  is a bisection satisfying

$$\begin{aligned} r(BUB^{-1}\overline{r(BW)}) &= r(BUB^{-1}B\overline{W}B^{-1}) \\ &= r(BU\overline{W}B^{-1}) \\ &= r(Br(U\overline{W})) \not\subseteq r(BW). \end{aligned} \quad \square$$

**Remark 10.4.4.** Brown, Clark and Sierakowski [7] have proved that if  $\mathcal{G}$  is an étale, effective, minimal groupoid, then  $C_r^*(\mathcal{G})$  is purely infinite if and only if every element of  $C_0(\mathcal{G}^{(0)})$  is infinite in  $C_r^*(\mathcal{G})$ .



## Chapter 11

# Cartan Pairs, and Dixmier–Douady Theory for Fell Algebras

In this chapter we first discuss the beautiful reconstruction theorem of Renault [39] that shows that an effective groupoid and twist can be recovered from the associated twisted groupoid algebra. This builds on previous work of Kumjian [26], and develops ideas that go back to Feldman and Moore in the context of von Neumann algebras [19, 20, 21]. We will then discuss an application of this theory to the classification of Fell algebras up to spectrum-preserving Morita equivalence [25].

### 11.1 Kumjian–Renault theory

The aim in this section is to outline Renault’s construction for recovering an étale groupoid from its reduced  $C^*$ -algebra together with the canonical abelian subalgebra  $C_0(\mathcal{G}^{(0)})$ . This is a  $C^*$ -algebraic analogue of Feldman–Moore theory for von Neumann algebras of Borel equivalence relations.

We will omit almost all of the proofs in this section. The details are due to Kumjian and then Renault and can be found in [26, 39].

To get the most out of this theory, we need to introduce twisted groupoid  $C^*$ -algebras. In Renault’s original work [37], twisted groupoid  $C^*$ -algebras were determined by continuous normalized 2-cocycles on  $\mathcal{G}$ ; that is, continuous maps  $\sigma : \mathcal{G}^{(2)} \rightarrow \mathbb{T}$  satisfying  $\sigma(r(\gamma), \gamma) = 1 = \sigma(\gamma, s(\gamma))$  for all  $\gamma$  and satisfying the cocycle identity  $\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\beta, \gamma)\sigma(\alpha, \beta\gamma)$  for every composable triple  $(\alpha, \beta, \gamma)$ .

The twisted convolution algebra is then defined as  $C_c(\mathcal{G}, \sigma) = C_c(\mathcal{G})$  as a vector space, but with multiplication and involution given by  $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} \sigma(\alpha, \beta)f(\alpha)g(\beta)$ , and  $f^*(\gamma) = \sigma(\gamma^{-1}, \gamma)\overline{f(\gamma^{-1})}$ . However, Kumjian subsequently observed that the notion of a twisted groupoid  $C^*$ -algebra that most naturally leads to an analogue of Feldman–Moore theory comes from a *twist*.

**Definition 11.1.1.** Let  $\mathcal{G}$  be an étale groupoid. A *twist* over  $\mathcal{G}$  is a sequence

$$\mathcal{G}^{(0)} \times \mathbb{T} \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} \mathcal{G},$$

where  $\mathcal{G}^{(0)} \times \mathbb{T}$  is regarded as a trivial group bundle with fibres  $\mathbb{T}$ ,  $\mathcal{E}$  is a locally compact Hausdorff groupoid, and  $i$  and  $\pi$  are continuous groupoid homomorphisms that restrict to homeomorphisms of unit spaces (we identify  $\mathcal{E}^{(0)}$  with  $\mathcal{G}^{(0)}$  via  $i$ ) such that

- $i$  is injective,
- $\mathcal{E}$  is a locally trivial  $\mathcal{G}$ -bundle in the sense that every point  $\alpha \in \mathcal{G}$  has a bisection neighbourhood  $U$  on which there exists a continuous section  $S : U \rightarrow \mathcal{E}$  satisfying  $\pi \circ S = \text{id}_U$ , and such that the map  $(\alpha, z) \mapsto i(r(\alpha), z)S(\alpha)$  is a homeomorphism of  $U \times \mathbb{T}$  onto  $\pi^{-1}(U)$ ;
- $i(\mathcal{G}^{(0)} \times \mathbb{T})$  is central in  $\mathcal{E}$  in the sense that  $i(r(\varepsilon), z)\varepsilon = \varepsilon i(s(\varepsilon), z)$  for all  $\varepsilon \in \mathcal{E}$  and  $z \in \mathbb{T}$ ; and
- $\pi^{-1}(\mathcal{G}^{(0)}) = i(\mathcal{G}^{(0)} \times \mathbb{T})$ .

If  $\mathcal{G} = \Gamma$  is a discrete group, then a twist over  $\mathcal{G}$  as defined above is precisely a central extension of  $\Gamma$ .

**Notation 11.1.2.** If  $\mathcal{E}$  is a twist over  $\mathcal{G}$ ,  $\varepsilon \in \mathcal{E}$  and  $z \in \mathbb{T}$ , we will write  $z \cdot \varepsilon := i(r(\varepsilon), z)\varepsilon$ , and  $\varepsilon \cdot z = \varepsilon i(s(\varepsilon), z)$ ; so  $z \cdot \varepsilon = \varepsilon \cdot z$  because  $i(\mathcal{G}^{(0)} \times \mathbb{T})$  is central in  $\mathcal{E}$ .

The local triviality of  $\mathcal{E}$  guarantees that  $\pi$  is a quotient map, so that  $\mathcal{G}$  is topologically identical to the quotient of  $\mathcal{E}$  by the action of  $\mathbb{T}$  just described.

**Lemma 11.1.3.** *If  $\mathcal{E} \rightarrow \mathcal{G}$  is a twist, and  $\varepsilon, \delta \in \mathcal{E}$  satisfy  $\pi(\varepsilon) = \pi(\delta)$ , then there is a unique  $z \in \mathbb{T}$  such that  $z \cdot \varepsilon = \delta$ .*

*Proof.* We have  $\pi(\varepsilon^{-1}\delta) = s(\delta) \in \mathcal{G}^{(0)}$ , so  $\varepsilon^{-1}\delta = i(\{s(\delta)\} \times z)$  for some  $z \in \mathbb{T}$ ; there is just one such  $z$  because  $i$  is injective. We then have  $(z \cdot \varepsilon)^{-1}\delta = (\varepsilon^{-1}\delta)i(s(\delta), \bar{z}) = s(\delta)$ . Multiplying on the right by  $z \cdot \varepsilon$  gives the result.  $\square$

**Example 11.1.4.** The Cartesian product groupoid  $\mathcal{G} \times \mathbb{T}$  is a twist over  $\mathcal{G}$  in the obvious way. This is called the trivial twist over  $\mathcal{G}$ .

**Example 11.1.5.** More generally, if  $\sigma$  is a continuous normalized 2-cocycle on  $\mathcal{G}$ , then  $\mathcal{G} \times \mathbb{T}$  can be made into a groupoid  $\mathcal{E}_\sigma$  with the usual unit space and range and source maps, but with multiplication and inversion given by  $(\alpha, w)(\beta, z) = (\alpha\beta, \sigma(\alpha\beta)wz)$  and  $(\alpha, w)^{-1} = (\alpha, \sigma(\alpha^{-1}, \alpha)w)$ . Since  $\sigma(r(\gamma), \gamma) = 1 = \sigma(\gamma, s(\gamma))$  for all  $\gamma$ , the set inclusion  $\mathcal{G}^{(0)} \times \mathbb{T} \hookrightarrow \mathcal{E}_\sigma$  is a groupoid homomorphism, as is the projection  $\pi : \mathcal{E}_\sigma \rightarrow \mathcal{G}$  given by  $\pi(\gamma, z) = \gamma$ . It is routine to check that  $\mathcal{E}_\sigma$  is then a twist over  $\Gamma$  with respect to  $i$  and  $\pi$ .

**Remark 11.1.6.** We can recover the cohomology class of  $\sigma$  from the twist  $\mathcal{E}_\sigma \rightarrow \mathcal{G}$  as follows: choose any continuous section  $S$  for  $\sigma$ . For  $(\alpha, \beta) \in \mathcal{G}^{(2)}$ , we have

$\sigma(S(\alpha)S(\beta)S((\alpha\beta)^{-1}) = r(\alpha) \in \mathcal{G}^{(0)}$ , and so Lemma 11.1.3 shows that there is a unique element  $\omega(\alpha, \beta) \in \mathbb{T}$  such that  $S(\alpha)S(\beta)S(\alpha\beta)^{-1} = (r(\alpha), \omega(\alpha, \beta))$ . The map  $\omega$  defined in this way is a continuous 2-cocycle. If  $S'$  is another continuous section for  $\sigma$ , then  $\omega^{-1}\omega'$  is equal to the 2-coboundary obtained from the 1-cochain  $b$  determined by  $S(\alpha)^{-1}S'(\alpha) = (r(\alpha), b(\alpha))$ . Thus the cocycles obtained from distinct choices of  $S$  are cohomologous. Taking  $S(\gamma) = (\gamma, 1)$  for all  $\gamma$ , yields  $\omega = \sigma$ , so the cohomology class  $[\sigma]$  of  $\sigma$  is equal to that of any cocycle obtained from a continuous section  $S: \mathcal{G} \rightarrow \mathcal{E}_\sigma$ .

More generally, if  $\mathcal{E}$  is a twist over  $\mathcal{G}$  that admits a continuous section  $S: \mathcal{G} \rightarrow \mathcal{E}$  for the quotient map, then there is a 2-cocycle  $\sigma$  on  $\mathcal{G}$  defined by  $S(\alpha)S(\beta)S(\alpha\beta)^{-1} = i(s(\alpha), \sigma(\alpha, \beta))$ . There is then an isomorphism  $\mathcal{E} \cong \mathcal{E}_\sigma$  that is equivariant for  $i$  and  $q$ . So  $\mathcal{E}$  is isomorphic to a twist coming from a cocycle. But it is not clear that every  $\mathcal{E}$  admits a continuous section, so the notion of a twist is formally more general than that of a continuous 2-cocycle.

**Definition 11.1.7.** If  $\mathcal{E}$  is a twist over the étale groupoid  $\mathcal{G}$ , then we write

$$\Sigma_c(\mathcal{G}; \mathcal{E}) := \{f \in C_c(\mathcal{E}) \mid f(z \cdot \varepsilon) = zf(\varepsilon) \text{ for all } \varepsilon \in \mathcal{E} \text{ and } z \in \mathbb{T}\}.$$

**Remark 11.1.8.** Each twist  $\mathcal{E}$  over  $\mathcal{G}$  determines a complex line bundle  $\tilde{\mathcal{E}}$  over  $\mathcal{G}$  as follows: Define an equivalence relation  $\sim$  on  $\mathcal{E} \times \mathbb{C}$  by  $(\delta, w) \sim (\varepsilon, z)$  if  $\pi(\delta) = \pi(\varepsilon)$ ,  $|w| = |z|$  and either  $w = z = 0$  or  $(w/|w|) \cdot \delta = (z/|z|) \cdot \varepsilon$ . Then  $\tilde{\mathcal{E}} := \mathcal{E}/\sim$  is a line-bundle over  $\mathcal{G}$  with respect to the fibre map  $p: \tilde{\mathcal{E}} \rightarrow \mathcal{G}$  given by  $p([\delta, w]) = \pi(\delta)$ .

For  $\gamma \in \mathcal{G}$ , any choice of  $\delta \in \pi^{-1}(\gamma)$  determines a homeomorphism  $\mathbb{T} \cong \pi^{-1}(\gamma) \subseteq \mathcal{E}$  given by  $z \mapsto z \cdot \delta$ . Since the Haar measure on  $\mathbb{T}$  is rotation invariant, the measure on  $\pi^{-1}(\gamma)$  obtained by pulling back the Haar measure on  $\mathbb{T}$  is independent of our choice of  $\delta \in \pi^{-1}(\gamma)$ . We endow each  $\mathcal{E}^x$  with the measure  $\lambda^x$  that agrees with this pulled back copy of the Haar measure on  $\pi^{-1}(\gamma)$  for each  $\gamma \in \mathcal{G}^x$  (so each  $\pi^{-1}(\gamma)$  has measure 1).

**Lemma 11.1.9.** The space  $\Sigma_c(\mathcal{G}; \mathcal{E})$  is a  $*$ -algebra under the operations

$$f * g(\varepsilon) = \int_{\mathcal{E}^{r(\varepsilon)}} f(\delta)g(\delta^{-1}\varepsilon) d\lambda^{r(\varepsilon)} \text{ and } f^*(\varepsilon) = \overline{f(\varepsilon^{-1})}.$$

For any  $\varepsilon \in \mathcal{G}$ ,  $f, g \in \Sigma_c(\mathcal{G}; \mathcal{E})$  and any choice of (not necessarily continuous) section  $\alpha \mapsto \tilde{\alpha}$  for  $\pi|_{\mathcal{G}^{r(\varepsilon)}}$ , we have

$$f * g(\varepsilon) = \sum_{\beta \in \mathcal{G}^{r(\varepsilon)}} f(\tilde{\beta})g(\tilde{\beta}^{-1}\varepsilon). \quad (11.1)$$

There is an isomorphism

$$C_c(\mathcal{G}^{(0)}) \cong D_0 := \{f \in \Sigma_c(\mathcal{G}; \mathcal{E}) \mid \text{supp}(f) \subseteq i(\mathcal{G}^{(0)} \times \mathbb{T})\}$$

that carries  $f \in C_c(\mathcal{G}^{(0)})$  to the function  $\tilde{f}: i(x, z) \mapsto zf(x)$ .



*Proof.* We verify (11.1): if  $\beta \in \mathcal{G}^{r(\varepsilon)}$  and  $\delta, \delta' \in \pi^{-1}(\beta)$ , then  $\delta' = z\delta$  for some  $z \in \mathbb{T}$ , and hence

$$f(\delta)g(\delta^{-1}\varepsilon) = f(\bar{z} \cdot \delta')g(z \cdot (\delta')^{-1}\varepsilon) = \bar{z}f(\delta')zg((\delta')^{-1}\varepsilon) = f(\delta')g((\delta')^{-1}\varepsilon).$$

So each  $\int_{\delta \in \pi^{-1}(\beta)} f(\delta)g(\delta^{-1}\varepsilon)d\lambda^{r(\beta)}(\delta)$  collapses to  $f(\tilde{\beta})g(\tilde{\beta}^{-1}\varepsilon)$ .

From here, that  $\Sigma_c(\mathcal{G}; \mathcal{E})$  is a  $*$ -algebra follows from calculations similar to the ones that show that  $C_c(\mathcal{G})$  is a  $*$ -algebra.

Since  $x \mapsto i(x, 1)$  is a section for  $\pi$  on  $\mathcal{G}^{(0)}$ , the final assertion follows from (11.1).  $\square$

**Remark 11.1.10.** Kumjian points out that there is an isomorphism of  $\Sigma_c(\mathcal{G}; \mathcal{E})$  with the space of compactly supported continuous sections of the complex line bundle  $\tilde{\mathcal{E}}$  over  $\mathcal{G}$  described in Remark 11.1.8. This isomorphism carries  $f \in \Sigma_c(\mathcal{G}; \mathcal{E})$  to the section  $\tilde{f}$  given by  $\tilde{f}(\alpha) = [\tilde{\alpha}, f(\tilde{\alpha})]$  for any choice of  $\tilde{\alpha}$  in  $\pi^{-1}(\alpha)$ .

We define the regular representations  $\pi_x$ ,  $x \in \mathcal{G}^{(0)}$  of  $C_c(\mathcal{G}; \mathcal{E})$  on the spaces  $L^2(\mathcal{G}_x; \mathcal{E}_x)$  of square-integrable  $\mathbb{T}$ -equivariant functions on  $\mathcal{E}_x$  by extension of the convolution formula. We define  $C_r^*(\mathcal{G}; \mathcal{E})$  to be the completion of the (injective) image of  $\Sigma_c(\mathcal{G}; \mathcal{E})$  in the direct sum of these representations, and  $\|\cdot\|_r$  the  $C^*$ -norm in this  $C^*$ -algebra. Arguments very similar to the ones for untwisted algebras give the following:

**Theorem 11.1.11.** *For any  $f \in \Sigma_c(\mathcal{G}; \mathcal{E})$ , the set*

$$\{\|\pi(f)\| \mid \pi \text{ is a } * \text{-representation of } \Sigma_c(\mathcal{G}; \mathcal{E})\}$$

*is bounded above. Taking the supremum gives a pre- $C^*$ -norm  $\|\cdot\|$  on  $\Sigma_c(\mathcal{G}; \mathcal{E})$ , and we define  $C^*(\mathcal{G}; \mathcal{E})$  to be the completion in this norm. We have  $\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_r$  on  $\Sigma_c(\mathcal{G}; \mathcal{E})$ , with equality on functions supported on  $\pi^{-1}(U)$  for any bisection  $U$ . If  $\mathcal{G}$  is amenable, then  $\|\cdot\|_r$  and  $\|\cdot\|$  agree. The map  $f \mapsto f|_{\mathcal{E}^{(0)}}$  from  $D_0$  to  $C_c(\mathcal{G}^{(0)})$  extends to an isomorphism of the completion of  $D_0$ , in either norm, with  $C_0(\mathcal{G}^{(0)})$ .*

We will write  $D$  for the completion of  $D_0$  in  $C^*(\mathcal{E}; \mathcal{G})$  and  $D_r$  for the completion in  $C_r^*(\mathcal{G})$ .

**Remark 11.1.12.** For the trivial twist  $\mathcal{G} \times \mathbb{T}$ , the map  $\gamma \mapsto (\gamma, 1)$  is a continuous section for  $\pi : \mathcal{G} \times \mathbb{T} \rightarrow \mathcal{G}$ . The cocycle obtained from this section as in Example 11.1.5 is the trivial one. So we can use the formula (11.1) to see that  $C_r^*(\mathcal{G}; \mathcal{G} \times \mathbb{T}) \cong C_r^*(\mathcal{G})$  in the canonical way.

In this section, we are interested in  $C_r^*(\mathcal{G}; \mathcal{E})$  and the subalgebra  $D_r$ .

**Proposition 11.1.13.** *Let  $\mathcal{G}$  be an effective étale groupoid and  $\mathcal{E}$  a twist over  $\mathcal{G}$ . There is a faithful conditional expectation  $\Phi : C_r^*(\mathcal{G}; \mathcal{E}) \rightarrow D_r$  that extends restriction of functions in  $\Sigma_c(\mathcal{G}; \mathcal{E})$  to  $i(\mathcal{G}^{(0)} \times \mathbb{T})$ . This is the only conditional expectation from  $C_r^*(\mathcal{G}; \mathcal{E})$  to  $D_r$ .*

*Proof sketch.* The proof of existence follows the outline of Proposition 10.2.6. To see that  $\Phi$  is the unique conditional expectation onto  $D_r$ , first observe that the expectation property says that if  $\Psi : C_r^*(\mathcal{G}; \mathcal{E}) \rightarrow D_r$  is a conditional expectation, then for any  $a \in C_r^*(\mathcal{G}; \mathcal{E})$  and any  $b \in D_r$ , we have

$$\Psi(ab) = \Psi(a)b = b\Psi(a) = \Psi(ba).$$

Arguing as in Lemma 10.2.5, we show that for each  $a \in \Sigma_c(\mathcal{G})$  whose support does not intersect  $i(\mathcal{G}^{(0)} \times \mathbb{T})$ , and each unit  $x \in \mathcal{G}^{(0)}$ , we can find an element  $h \in D_r$  such that  $hah = 0$  and  $h(x) > 0$ . Using that  $s(\text{supp}(a))$  is compact and a partition-of-unity argument, we find finitely many  $h_i$  such that  $a \sum_i h_i^2 = a$  and  $h_i ah_i = 0$  for all  $i$ . This gives  $\Psi(a) = \Psi(a \sum_i h_i^2) = \sum \Psi(h_i ah_i) = 0$ . So  $\Psi$  agrees with  $\Phi$  on the space  $D_0^\perp$  of elements of  $\Sigma_c(\mathcal{G}; \mathcal{E})$  whose support does not intersect  $i(\mathcal{G}^{(0)} \times \mathbb{T})$ , and it agrees with  $\Phi$  on  $D_0$  because every conditional expectation is the identity map on its range. Since  $\Sigma_c(\mathcal{G}; \mathcal{E}) = D_0 + D_0^\perp$ , we deduce that  $\Phi$  and  $\Psi$  agree on all of  $\Sigma_c(\mathcal{G}; \mathcal{E})$ , and so are equal.  $\square$

As in the untwisted case, each element of  $C_r^*(\mathcal{G}; \mathcal{E})$  determines a  $\mathbb{T}$ -equivariant function  $j \in C_0(\mathcal{E})$ . One way to see this is to fix a section (we do not require continuity)  $\gamma \mapsto \tilde{\gamma}$  for the map  $\pi : \mathcal{E} \rightarrow \mathcal{G}$ , so that  $\pi(\tilde{\gamma}) = \gamma$  for all  $\gamma$ . If  $\delta, \varepsilon \in \mathcal{E}$  satisfy  $\pi(\delta) = \pi(\varepsilon)$ , then there is a unique  $[\delta, \varepsilon] \in \mathbb{T}$  such that  $\delta = [\delta, \varepsilon] \cdot \varepsilon$ . In particular, if  $(\alpha, \beta) \in \mathcal{G}^{(2)}$ , then  $\pi(\tilde{\alpha}\tilde{\beta}) = \alpha\beta = \pi(\tilde{\alpha\beta})$ . It is not too hard to see that for each  $x \in \mathcal{G}^{(0)}$  there is a representation of  $\Sigma_c(\mathcal{G}; \mathcal{E})$  on  $\ell^2(\mathcal{G}_x)$  satisfying

$$\tilde{\pi}_x(f)\delta_\beta = \sum_{\alpha \in \mathcal{G}_{r(\beta)}} [\tilde{\alpha}\tilde{\beta}, \widetilde{\alpha\beta}] f(\tilde{\alpha}) \delta_{\alpha\beta},$$

and this representation is unitarily equivalent to the regular representation  $\pi_x$  of  $\Sigma_c(\mathcal{G}; \mathcal{E})$ . With this representation in hand, the argument of Proposition 9.3.3 carries across to the twisted setting.

If  $A$  is a  $C^*$ -algebra and  $B$  is a subalgebra of  $A$ , we shall say that  $n \in A$  is a *normalizer* of  $B$  if  $nBn^* \cup n^*Bn \subseteq B$ . We write  $N(B)$  for the collection of all normalizers of  $B$ . We say that  $B$  is *regular* in  $A$  if  $A$  is generated as a  $C^*$ -algebra by  $N(B)$ .

**Proposition 11.1.14.** *If  $\mathcal{G}$  is an étale, effective groupoid, and  $\mathcal{E}$  is a twist over  $\mathcal{G}$ , then  $D_r$  is a regular maximal abelian subalgebra of  $C_r^*(\mathcal{G}; \mathcal{E})$  that contains an approximate unit for  $C^*(\mathcal{G}; \mathcal{E})$ .*

*Proof sketch.* Clearly  $D_r$  is an abelian algebra. To see that it is maximal abelian, suppose that  $a$  belongs to its complement. Then  $j(a)$  must be nonzero at some  $\varepsilon \in \mathcal{E} \setminus \pi^{-1}(\text{Iso}(\mathcal{G})) = \text{Iso}(\mathcal{E})$ . So we can choose  $h \in D_0 \subseteq D_r$  such that  $h(r(\varepsilon)) = 1$  and  $h(s(\varepsilon)) = 0$ . Now  $j(ah)(\varepsilon) = 0$  whereas  $j(ha)(\varepsilon) = a(\varepsilon) \neq 0$ . To see that  $D_r$  is regular, we use the multiplication formula to see that if  $n \in C_c(\pi^{-1}(U))$  for some bisection  $U$  of  $\mathcal{G}$ , then for  $h \in C_c(\mathcal{G}^{(0)})$  (regarded as an element of  $D_0$  using the isomorphism of Theorem 11.1.11) we have  $\text{supp}(nhn^*) \subseteq \text{supp}(n)\text{supp}(h)\text{supp}(n)^{-1} \subseteq \pi^{-1}(UU^{-1}) \subseteq \pi^{-1}(\mathcal{G}^{(0)})$ , and similarly for  $n^*hn$ .  $\square$

Following Renault [39], we shall say that a pair  $(A, B)$  of separable  $C^*$ -algebras is a *Cartan pair* and say that  $B$  is a *Cartan subalgebra* of  $A$  if  $A$  is a  $C^*$ -algebra,  $B$  is a  $C^*$ -subalgebra of  $A$  containing an approximate unit for  $A$ , there is a faithful conditional expectation of  $A$  onto  $B$ ,  $B$  is a maximal abelian subalgebra of  $A$ , and  $B$  is regular in  $A$ . We can reinterpret the preceding result as saying that  $(C^*(\mathcal{G}; \mathcal{E}), D_r)$  is a Cartan pair. Our main objective here is to prove that every Cartan pair has this form.

Given a Cartan pair  $(A, B)$ , and given  $n \in N(B)$ , we write

$$\text{dom}(n) := \overline{\{\phi \in \widehat{B} \mid \phi(n^*n) > 0\}} \quad \text{and} \quad \text{ran}(n) := \overline{\{\phi \in \widehat{B} \mid \phi(nn^*) > 0\}}.$$

We have  $\text{ran}(n) = \text{dom}(n^*)$ .

**Proposition 11.1.15.** *Let  $(A, B)$  be a Cartan pair of  $C^*$ -algebras. For each  $n \in N(B)$ , there is a homeomorphism  $\alpha_n : \text{dom}(n) \rightarrow \text{ran}(n)$  satisfying  $\alpha_n(\phi)(nbn^*) = \phi(bn^*n)$ . There is an equivalence relation  $\sim$  on  $\{(n, \phi) \mid n \in N(B), \phi \in \text{dom}(n)\}$  such that  $(n, \phi) = (m, \psi)$  if and only if  $\phi = \psi$  and there is a neighbourhood  $U$  of  $\phi$  such that  $\alpha_n|_U = \alpha_m|_U$ . The set*

$$\mathcal{G}_{(A,B)} := \{[n, \phi] \mid n \in N(B), \phi \in \text{dom}(n)\}$$

*is a groupoid with unit space*

$$\mathcal{G}_{(A,B)}^{(0)} = \{[b, \phi] \mid b \in B, \phi \in \text{supp}(B)\},$$

*and groupoid structure given by*

$$\begin{aligned} r([n, \phi]) &= [nn^*, \alpha_n(\phi)], & s([n, \phi]) &= [n^*n, \phi], \\ [m, \alpha_n(\phi)][n, \phi] &= [mn, \phi] & \text{and} & \quad [n, \phi]^{-1} = [n^*, \alpha_n(\phi)]. \end{aligned}$$

*Proof sketch.* Take the polar decomposition  $n = v|n|$  of  $n$  in  $A^{**}$  and observe that if  $f$  belongs to the dense subalgebra  $C_c(\text{dom}(n)^\circ)$  of  $I_{n^*n} := C_0(\text{dom}(n)^\circ)$ , then  $|n|$  is invertible on  $\text{supp}(f)$ , and so we can write  $f = |n|g|n|^*$  for some  $g \in I_{n^*n}$ . Hence  $vf v^* = v|n|g|n|^*v^* = aga^* \in B$ . Applying the same reasoning to  $n^*$  we see that conjugation by  $v$  determines an isomorphism of commutative  $C^*$ -algebras  $I_{n^*n} \cong I_{nn^*}$ , and therefore induces a homeomorphism  $\alpha_n$  between their spectra. It is clear that  $\sim$  is an equivalence relation, and that  $r, s$  are well defined. To check that the multiplication is well defined, we use the observation that if  $v, w$  are the partial isometries appearing in the polar decompositions of  $m, n$ , and if  $\text{dom}(m) = \text{ran}(n)$ , then  $vw$  is the partial isometry appearing in the polar decomposition of  $mn$ ; and also that for  $h \in C_c(\text{supp}(n))$  and  $\phi$  satisfying  $\phi(h) \neq 0$ , we have  $[n, \phi] = [nh, \phi] = [(h \circ \alpha_n^{-1})n, \phi]$ . It is easy to check that inversion is well defined using that if  $n = v|n|$ , then  $n^* = v^*|n^*|$ . Associativity of multiplication comes from associativity of multiplication in  $A$ , and the inverse property follows directly from the definitions of  $r$  and  $s$ .  $\square$

**Theorem 11.1.16.** *The groupoid  $\mathcal{G}_{(A,B)}$  becomes an étale groupoid under the topology with basic open sets  $Z(n, U) := \{[n, \phi] \mid \phi \in U\}$  indexed by elements  $n \in N(B)$  and open sets  $U \subseteq \text{dom}(n)$ . If  $A = C^*(\mathcal{G}; \mathcal{E})$  and  $B = D_0$  for some twist  $\mathcal{G}^{(0)} \times \mathbb{T} \rightarrow \mathcal{E} \rightarrow \mathcal{G}$ , then there is an isomorphism  $\theta : \mathcal{G} \cong \mathcal{G}_{(A,B)}$  such that for any  $\gamma \in \mathcal{G}$ , any open bisection  $U$  containing  $\gamma$  whose closure is a compact bisection and any  $n \in \Sigma_c(\mathcal{G}; \mathcal{E})$  that is nonzero everywhere on  $\pi^{-1}(U)$ , we have  $\theta(\gamma) = [n, \widehat{s(\gamma)}]$ .*

*Proof sketch.* It is fairly straightforward to check that the sets  $Z(n, U)$  are a base for a locally compact Hausdorff topology, and that  $\mathcal{G}_{(A,B)}$  is étale in this topology. If  $A = C^*(\mathcal{G}; \mathcal{E})$  and  $B = D_r$ , then the defining property of  $\alpha_n$  (namely  $\alpha_n(\phi)(nhn^*) = \phi(hn^*n)$ ) shows that if  $n$  is nonzero on all of  $\pi^{-1}(U)$ , then  $\alpha_n(\widehat{s(\gamma)}) = \widehat{r(\gamma)}$  for  $\gamma \in U$ . It follows from direct computations that  $\theta$  is an algebraic isomorphism. The pre-image of a given  $Z(n, U)$  under  $\theta$  is the open bisection  $\text{supp}(n)U$ , so  $\theta$  is continuous. Moreover, for a given  $\gamma \in \mathcal{G}$  we can choose a compact bisection  $K$  containing  $\gamma$ , an open bisection  $U$  containing  $K$  and an element  $n \in C_c(U)$  that is identically 1 on  $K$ , and  $\theta$  is then a continuous bijection of the compact set  $K$  onto  $Z(n, K)$ , and hence a homeomorphism between these sets. Since the interiors of compact bisections form a base for the topology on  $\mathcal{G}$ , it follows that  $\theta$  is open.  $\square$

As in [39], we call  $\mathcal{G}_{(A,B)}$  the *Weyl groupoid* of the Cartan pair  $(A, B)$ .

**Proposition 11.1.17.** *Let  $(A, B)$  be a Cartan pair of  $C^*$ -algebras. There is an equivalence relation  $\approx$  on the set  $\{(n, \phi) \in N(B) \times \widehat{B} \mid \psi(n^*n) > 0\}$  such that  $(m, \phi) \approx (n, \phi)$  if  $\phi = \psi$ , and there exist  $b, b' \in B$  with  $\phi(b), \phi(b') > 0$  and  $mb = nb'$ . The set*

$$\mathcal{E}_{(A,B)} := \{(n, \phi) \in N(B) \times \widehat{B} \mid \psi(n^*n) > 0\} / \approx$$

*is a groupoid with unit space  $\{[h, \phi] \mid h \in C_0(\mathcal{G}^{(0)}), \phi \in \text{dom}(h)\}$ , range and source maps  $r([n, \phi]) = [nn^*, \phi]$ ,  $s([n, \phi]) = [n^*n, \phi]$ , multiplication  $[[m, \alpha_n(\phi)][n, \phi] = [[mn, \phi]$ , and inversion  $[[n, \phi]]^{-1} = [n^*, \alpha_n(\phi)]$ .*

*Proof sketch.* This is relatively straightforward; the only potential sticking point is well-definedness of multiplication, but this follows from the fact that if  $(n, \phi) = (n', \phi)$  and  $(m, \alpha_n(\phi)) = (m', \alpha_{n'}(\phi))$ , say  $nc = n'c'$  and  $mb = m'b'$ , then we can assume (by multiplying by some  $h > 0$  supported on  $\alpha_n(\text{supp}(c) \cap \text{supp}(c'))$ ) that  $\text{supp}(b) = \text{supp}(b') \subseteq \text{ran}(n)$ , and then note that  $bn = n(b \circ \alpha_n)$  and  $b'n' = n'(b' \circ \alpha_{n'})$ , so that  $mn(b \circ \alpha_n c) = m'n'(b' \circ \alpha_{n'} c')$ .  $\square$

**Proposition 11.1.18.** *Let  $(A, B)$  be a Cartan pair. There is a locally compact Hausdorff topology on  $\mathcal{E}_{(A,B)}$  with basic open sets  $\widetilde{Z}(n, U) = \{[n, \phi] \mid \phi \in U\}$  indexed by  $n \in N(B)$  and open sets  $U \subseteq \text{dom}(n)$ . The groupoid  $\mathcal{E}_{(A,B)}$  is a topological groupoid in this topology. There is an injective continuous groupoid homomorphism  $i_{(A,B)} : \widehat{B} \times \mathbb{T} \rightarrow \mathcal{E}_{(A,B)}$  given by  $i_{(A,B)}(\phi, z) = [b, \phi]$  for any*

$b \in B$  such that  $\phi(b) = z$ , and there is continuous surjective groupoid homomorphism  $\pi_{(A,B)} : \mathcal{E}_{(A,B)} \rightarrow \mathcal{G}_{(A,B)}$  such that  $\pi(\llbracket n, \phi \rrbracket) = [n, \phi]$ . The sequence  $\widehat{B} \times \mathbb{T} \rightarrow \mathcal{E}_{(A,B)} \rightarrow \mathcal{G}_{(A,B)}$  is a twist over  $\mathcal{G}_{(A,B)}$ .

*Proof sketch.* This is largely a matter of straightforward checking of details. The key point is that if  $(m, \phi) \approx (n, \phi)$ , say  $mb = nb'$  where  $b, b' \in B$ , then by multiplying  $b, b'$  by some positive  $h$  with  $\phi(h) = 1$ , we can assume that  $\text{supp}(b) = \text{supp}(b') = U$ , say, and then  $\alpha_m|_U = \alpha_{mb}|_U = \alpha_{nb'}|_U = \alpha_n|_U$ . This shows that the map  $\llbracket n, \phi \rrbracket \rightarrow [n, \phi]$  makes sense and is well defined. To see why  $i_{(A,B)}$  is injective, and its image is precisely the kernel of  $\pi_{(A,B)}$ , observe that for any  $b, c \in C_0(\mathcal{G}^{(0)})$  with  $\phi(b), \phi(b') \neq 0$ , we have  $[b, \phi] = [c, \phi]$ , but we have  $\llbracket b, \phi \rrbracket = \llbracket c, \phi \rrbracket$  if and only if  $\phi(b)/|\phi(b)| = \phi(c)/|\phi(c)|$ : for if so, then  $z = \phi(b)/|\phi(b)|$  satisfies  $\phi(\bar{z}b), \phi(\bar{z}c) > 0$  and  $b(\bar{z}c) = c(\bar{z}b)$ .  $\square$

The following theorem, which is the desired Feldman–Moore-type theorem, in our setting, is due to Renault. For  $\mathcal{G}$  principal, it was first proved by Kumjian [26].

**Theorem 11.1.19** (Renault, [39, Theorem 5.9]). *Let  $\mathcal{G}$  be a second-countable effective étale groupoid. Suppose that  $\mathcal{E}$  is a twist over  $\mathcal{G}$ . Then there is an isomorphism  $\zeta : \mathcal{E}_{C_r^*(\mathcal{G}; \mathcal{E}), D_r} \rightarrow \mathcal{E}$  such that the diagram*

$$\begin{array}{ccccc}
 \widehat{D_r} \times \mathbb{T} & \xrightarrow{i_{(C_r^*(\mathcal{G}; \mathcal{E}), D_r)}} & \mathcal{E}_{C_r^*(\mathcal{G}; \mathcal{E}), D_r} & \xrightarrow{\pi_{(C_r^*(\mathcal{G}; \mathcal{E}), D_r)}} & \mathcal{G}_{C_r^*(\mathcal{G}; \mathcal{E}), D_r} \\
 \downarrow \cong & & \downarrow \zeta & & \downarrow \theta \\
 \mathcal{G}^{(0)} \times \mathbb{T} & \xrightarrow{i} & \mathcal{E} & \xrightarrow{\pi} & \mathcal{G}
 \end{array}$$

commutes. In particular, the map

$$(\mathcal{G}^{(0)} \times \mathbb{T} \rightarrow \mathcal{E} \rightarrow \mathcal{G}) \mapsto (C_r^*(\mathcal{G}; \mathcal{E}), D_r)$$

induces a bijection between isomorphism classes of twists and isomorphism classes of Cartan pairs of  $C^*$ -algebras.

**Corollary 11.1.20.** *If  $(A, B)$  is a Cartan pair, then there is only one conditional expectation of  $A$  onto  $B$ .*

*Proof.* We proved in Proposition 11.1.13 that the expectation extending restriction of functions is the unique expectation from  $C_r^*(\mathcal{G}; \mathcal{E})$  to  $D_r$ .  $\square$

In [4], Barlak and Li showed how to extend Tu’s result (Theorem 10.1.7) that the  $C^*$ -algebras of amenable groupoids always belong to the UCT class to twisted groupoids. Building on this, work of Takeishi and van Erp–Williams on nuclearity of  $C^*$ -algebras of groupoid Fell bundles, and combining this with Theorem 11.1.19, Barlak and Li made substantial progress on “UCT question”: does every nuclear  $C^*$ -algebra belong to the UCT class?

**Theorem 11.1.21** (Barlak–Li [4, Theorem 1.1 and Corollary 1.2]). *If  $\mathcal{G}$  is an étale groupoid and  $\mathcal{E}$  is a twist over  $\mathcal{G}$ , and if  $C_r^*(\mathcal{G}; \mathcal{E})$  is nuclear, then  $C_r^*(\mathcal{G}; \mathcal{E})$  belongs to the UCT class. In particular, if  $A$  is a separable nuclear  $C^*$ -algebra that is Morita equivalent to a  $C^*$ -algebra with a Cartan subalgebra, then it is in the UCT class.*

Another interesting application of Theorem 11.1.19 is a beautiful theorem of Matsumoto and Matui [28]. Recall that the Cuntz–Krieger algebra of an irreducible  $\{0, 1\}$  matrix  $A$  (that is not a permutation matrix) agrees with the  $C^*$ -algebra of the graph whose adjacency matrix is  $A$  [17, 27], and therefore with the groupoid  $C^*$ -algebra associated to the graph groupoid described in Example 8.4.7. A fundamental result of Franks [22], building on his previous work with Bowen [6] and on work of Parry and Sullivan [32], says that the (two-sided) shift spaces of  $\{0, 1\}$ -matrices  $A$  and  $B$  are flow equivalent if and only if  $\text{coker}(1 - A^t) \cong \text{coker}(1 - B^t)$  and, if nonzero,  $\det(1 - A^t)$  and  $\det(1 - B^t)$  have the same sign (positive or negative). Building on previous work of Cuntz [14], Rørdam proved [41] that the Cuntz–Krieger algebras  $\mathcal{O}_A$  and  $\mathcal{O}_B$  are stably isomorphic if and only if  $\text{coker}(1 - A^t) = \text{coker}(1 - B^t)$ . So the Cuntz–Krieger algebra “forgets” some information about flow-equivalence. But, using Renault’s results and previous work of Matui, Matsumoto and Matui proved part (1) and the equivalence of (2a) and (2b) in the following theorem in 2013. The final two equivalences, proved later, require the groupoid equivalence theorem of [12], and the results of [11].

Recall that if  $E$  is a directed graph, then the Tomforde stabilisation graph  $SE$  is obtained by attaching an infinite head at each vertex of  $E$  [47]. The resulting “stabilized” symbolic dynamical system can be described as follows: if  $X_E$  is the shift space of  $E$  (namely  $E^\infty$  endowed with the usual shift map), then  $X_{SE}$  is given as a space by  $X_{SE} = X_E \times \{0, 1, 2, \dots\}$ , and the dynamics  $\bar{\sigma}$  on  $X_{SE}$  is given by

$$\bar{\sigma}(x, n) = \begin{cases} (x, n-1) & \text{if } n \geq 1 \\ (\sigma(x), 0) & \text{if } n = 1. \end{cases}$$

We will call  $(X_{SE}, \bar{\sigma})$  the *stabilisation* of  $(X_E, \sigma_E)$ , and say that  $(X_E, \sigma_E)$  and  $(X_F, \sigma_F)$  are *stably orbit equivalent* if their stabilisations are orbit equivalent.

**Theorem 11.1.22.** *Let  $A, B$  be irreducible  $\{0, 1\}$ -matrices that are not permutation matrices.*

1. *The following are equivalent:*

- (a) *The one-sided shift-spaces determined by  $A$  and  $B$  are continuously orbit equivalent;*
- (b) *The Deaconu–Renault groupoids of the one-sided shift maps associated to  $A$  and  $B$  are isomorphic;*
- (c) *There is an isomorphism  $\mathcal{O}_A \cong \mathcal{O}_B$  that carries  $C_0(\mathcal{G}_A^{(0)})$  to  $C_0(\mathcal{G}_B^{(0)})$ .*

2. The following are equivalent

- (a) The two-sided shift-spaces determined by  $A$  and  $B$  are flow equivalent;
- (b) The Cartan pairs  $(\mathcal{O}_A \otimes \mathcal{K}, C_0(\mathcal{G}_A^{(0)}) \otimes c_0)$  and  $(\mathcal{O}_B \otimes \mathcal{K}, C_0(\mathcal{G}_B^{(0)}) \otimes c_0)$  are isomorphic as Cartan pairs;
- (c) The groupoids  $\mathcal{G}_A \times \mathcal{R}_{\mathbb{N}}$  and  $\mathcal{G}_B \times \mathcal{R}_{\mathbb{N}}$  are isomorphic.
- (d) The dynamical systems  $(X_E, \sigma_E)$  and  $(X_F, \sigma_F)$  are stably orbit equivalent.

## 11.2 A Dixmier–Douady theorem for Fell algebras

In what follows, given a  $C^*$ -algebra  $A$ , we shall write  $\hat{A}$  for the space of unitary equivalence classes of irreducible representations of  $A$ . We give  $\hat{A}$  the initial topology obtained from the quotient map from  $\hat{A}$  to  $\text{Prim}(A)$ , where  $\text{Prim}(A)$  is given the Jacobson topology.

Recall that a  $C^*$ -algebra  $A$  is *liminary* or Type I if every irreducible representation  $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$  has image  $\mathcal{K}(\mathcal{H})$ . A positive element  $a$  of a liminary  $C^*$ -algebra is a *continuous trace element* if  $\pi(a)$  has finite trace for every  $\pi \in \hat{A}$  and the map  $\pi \mapsto \text{Tr}(\pi(a))$  is continuous. A *continuous-trace  $C^*$ -algebra* is a liminary  $C^*$ -algebra that is generated as an ideal by its continuous-trace elements. The spectrum  $\hat{A}$  of a continuous-trace  $C^*$ -algebra is always Hausdorff. The Dixmier–Douady theorem [16] says that for a given locally compact Hausdorff space  $X$ , the continuous-trace  $C^*$ -algebras with spectrum  $X$  are classified up to spectrum-preserving Morita equivalence by the Dixmier–Douady invariant, which is an element of the cohomology group  $H^3(X, \mathbb{Z})$ ; moreover, the invariant is exhausted in the sense that each class in  $H^3(X, \mathbb{Z})$  occurs as the Dixmier–Douady invariant of some continuous-trace algebra with spectrum  $X$ .

Raeburn and Taylor subsequently gave a very appealing description of the continuous-trace  $C^*$ -algebra with given Dixmier–Douady invariant  $\delta \in H^3(X, \mathbb{Z})$  using groupoids:

**Example 11.2.1** (Raeburn–Taylor [34]). Recall that a Čech 2-cocycle on a locally compact Hausdorff space  $X$  consists of a cover of  $X$  by open sets  $U_i$  and a collection of continuous  $\mathbb{T}$ -valued functions  $c_{ijk}$  defined on triple-overlaps  $U_{ijk} = U_i \cap U_j \cap U_k$  such that  $c_{iij}, c_{ijj} \equiv 1$  for each  $i, j$ , and such that on nonempty quadruple overlaps  $U_{ijkl}$  we have  $c_{ijk}c_{ikl} = c_{ijl}c_{jkl}$ . A coboundary is a cocycle of the form  $(\delta b)_{ijk}(x) := b_{ij}(x)b_{ik}(x)^{-1}b_{jk}(x)$  for some collection of continuous functions  $b_{ij} : U_{ij} \rightarrow \mathbb{T}$  defined on double-overlaps. The Čech cohomology group is the quotient  $\check{H}(X, \mathbb{T})$  of the group of 2-cocycles by the subgroup of 2-coboundaries. Given a Čech 2-cocycle on  $X$ , we can form an equivalence relation  $R$  with unit space  $\bigsqcup \{i\} \times U_i \times \{i\}$  and with elements  $\{(i, x, j) \mid x \in U_{ij}\}$ , where  $r(i, x, j) = (i, x, i)$  and  $s(i, x, j) = (j, x, j)$ . We then construct a twist over  $R$  by putting  $\mathcal{E} = R \times \mathbb{T}$  and defining multiplication on  $\mathcal{E}$  by  $((i, x, j), w)((j, x, k), z) = ((i, x, k), c_{ijk}(x)wz)$ . Note that this twist comes from

a continuous 2-cocycle on  $R$ . Raeburn and Taylor [34] proved that the  $C^*$ -algebra of this twist has Dixmier–Douady invariant equal to the cohomology class of  $c$ .

In this section we will give a brief overview of how, using groupoids and the construction of the preceding section, we can obtain a version of the Dixmier–Douady theorem for Fell algebras based on the Raeburn–Taylor construction of the preceding example. The details appear in [25], though of course the ideas there owe a great deal to the previous work of Dixmier–Douady [16], Raeburn–Taylor [34], and the excellent monograph on Dixmier–Douady theory by Raeburn–Williams [35]. The details of the material in this section involve significant extra background and set-up, so I will give almost no proofs, and just touch on the main points of the construction.

**Definition 11.2.2.** A  $C^*$ -algebra is called a *Fell algebra* if it is liminary, and for every  $[\pi] \in \hat{A}$ , there exists  $b \in A^+$  and a neighbourhood  $U$  of  $[\pi]$  such that  $\psi(b)$  is a rank-1 projection whenever  $[\psi] \in U$ .

Roughly speaking, this says that lots of elements of  $A$  have the same rank under nearby irreducible representations. So it should be related to the continuous-trace condition. Indeed, it turns out that a Fell algebra is a continuous-trace algebra if and only if it has Hausdorff spectrum. Theorem 3.3 of [25] says that  $A$  is a Fell algebra if and only if it is liminary and generated as an ideal by elements  $a \in A_+$  such that  $aAa$  is abelian, and that this in turn happens if and only if there is a set  $S$  of ideals of  $A$  each element of which is Morita equivalent to a commutative  $C^*$ -algebra and such that  $\bigcup S$  spans a dense subspace of  $A$ .

We now show how to obtain an equivalence relation from a Fell algebra.

**Proposition 11.2.3.** *Let  $A$  be a Fell algebra. Choose a sequence  $d_i$  of positive elements of  $A$  with  $\|d_i\| = 1$  such that each  $d_i A d_i$  is abelian, and such that the  $A d_i A$  generate  $A$ . For each  $i$ , let  $a_i := d_i \otimes \theta_{i,i} \in A \otimes \mathcal{K}$ . Then  $\sum_i a_i$  converges to an element  $a$  of  $\mathcal{M}(A \otimes \mathcal{K})$ . This element is full in the sense that  $\overline{\text{span}}(A \otimes \mathcal{K})a(A \otimes \mathcal{K}) = A \otimes \mathcal{K}$ . Moreover  $C := a(A \otimes \mathcal{K})a$  and  $D := \sum_i a_i(A \otimes \mathcal{K})a_i \cong \bigoplus_i d_i A d_i$  form a Cartan pair  $(C, D)$ .*

By Corollary 11.1.20, if  $(C, D)$  is a Cartan pair, then there is a unique expectation  $\Phi : A \rightarrow B$ . Given  $\phi \in \widehat{D}$ , the composition  $\phi \circ \Phi$  gives a pure state of  $C$ , and then the GNS construction yields an irreducible representation. So we obtain a well-defined map  $\sigma : \widehat{D} \rightarrow \widehat{C}$ , which we call the *spectral map*.

**Proposition 11.2.4.** *Let  $A$  be a Fell algebra, and choose a sequence  $d_i$  as in the preceding proposition. The Weyl groupoid  $\mathcal{G}_{C,D}$  of Theorem 11.1.16 is isomorphic to the equivalence relation  $R(\sigma)$  determined by the spectral map:  $R(\sigma) = \{(\phi, \psi) \in \widehat{D} \mid \sigma(\phi) = \sigma(\psi)\}$ .*

Using this, we are able to characterize diagonal-preserving Morita equivalence of Fell algebras in terms of groupoid equivalence.



We shall say that twists  $\mathcal{E}_1 \rightarrow \mathcal{G}_1$  and  $\mathcal{E}_2 \rightarrow \mathcal{G}_2$  are equivalent if there is a linking groupoid  $L$  for  $\mathcal{G}_1$  and  $\mathcal{G}_2$  and a twist  $\mathcal{L}$  over  $L$  such that reduction of  $L$  and  $\mathcal{L}$  to  $\mathcal{G}_i^{(0)} \subseteq L^{(0)}$  yields a twist  $\mathcal{G}_i^{(0)} \times \mathbb{T} \rightarrow \mathcal{G}_i^{(0)} \mathcal{L} \mathcal{G}_i^{(0)} \rightarrow \mathcal{G}_i^{(0)} L \mathcal{G}_i^{(0)}$  that is isomorphic to  $\mathcal{G}_i^{(0)} \times \mathbb{T} \rightarrow \mathcal{E}_i \rightarrow \mathcal{G}_i$ . This is the natural extension of the notion of groupoid equivalence to twists.

**Proposition 11.2.5.** *If  $(C_1, D_1)$  and  $(C_2, D_2)$  are Cartan pairs in which  $C_1$  and  $C_2$  are Fell algebras, then  $C_1$  and  $C_2$  are Morita equivalent if and only if the twists  $\mathbb{T} \times \hat{D}_i \rightarrow \mathcal{E}_{C_i, D_i} \rightarrow \mathcal{G}_{C_i, D_i}$  ( $i = 1, 2$ ) are equivalent twists.*

The Dixmier–Douady invariant of a continuous-trace  $C^*$ -algebra with spectrum  $X$  is an element of a cohomology group. For us, the collection of equivalence classes of twists will act as a proxy for this cohomology group. (Theorem 11.2.7 describes how these two groups are related in the continuous-trace setting.)

We say that twists  $\mathcal{G}^{(0)} \times \mathbb{T} \rightarrow \mathcal{E} \rightarrow \mathcal{G}$  and  $\mathcal{G}^{(0)} \times \mathbb{T} \rightarrow \mathcal{F} \rightarrow \mathcal{G}$  over the same étale groupoid  $\mathcal{G}$  are *isomorphic* if there is a groupoid isomorphism  $\zeta : \mathcal{E} \rightarrow \mathcal{F}$  such that the diagram

$$\begin{array}{ccccc} \mathcal{G}^{(0)} \times \mathbb{T} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{G} \\ \downarrow = & & \downarrow \zeta & & \downarrow = \\ \mathcal{G}^{(0)} \times \mathbb{T} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} \end{array}$$

commutes.

For the following result, we need to describe the pullback construction for twists over a given relation  $R$ . Let  $R$  be an equivalence relation, and suppose that  $\mathcal{E}$  and  $\mathcal{E}'$  are twists over  $R$ . Define an equivalence relation  $\sim$  on

$$\mathcal{E} \times_{\pi \times \pi'} \mathcal{E}' := \{(\varepsilon, \varepsilon') \in \mathcal{E} \times \mathcal{E}' \mid \pi(\varepsilon) = \pi'(\varepsilon')\}$$

by  $(\varepsilon, \varepsilon') \sim (\delta, \delta')$  if and only if there exists  $z \in \mathbb{T}$  such that  $z \cdot \varepsilon = \delta$  and  $\bar{z} \cdot \varepsilon' = \delta'$ .

The pullback  $\mathcal{E} * \mathcal{E}'$  is defined as

$$\mathcal{E} * \mathcal{E}' := (\mathcal{E} \times_{\pi \times \pi'} \mathcal{E}') / \sim.$$

This is a twist over  $R$  with respect to the map  $\pi * \pi' : \mathcal{E} * \mathcal{E}' \rightarrow R$  given by  $(\pi * \pi')([\varepsilon, \varepsilon']) = \pi(\varepsilon)$ , and the map  $i * i' : R^{(0)} \times \mathbb{T} \rightarrow \mathcal{E} * \mathcal{E}'$  given by  $(i * i')(x, z) := [i(x, z), i'(x, 1)]$ .

**Lemma 11.2.6.** *Let  $R$  be a topological equivalence relation. Then the collection of isomorphism classes of twists over  $R$  becomes an abelian group  $\text{Tw}_R$  with identity element equal to the class of the trivial twist, and with group operation given by  $[\mathcal{E}] + [\mathcal{E}'] := [\mathcal{E} * \mathcal{E}']$ .*

If  $A$  is a Fell algebra, we write  $\mathcal{S}$  for the sheaf of germs of continuous  $\mathbb{T}$ -valued functions on  $\hat{A}$ . One can then form the sheaf cohomology group  $H^2(\hat{A}, \mathcal{S})$ . If  $\hat{A}$  is Hausdorff, then  $H^2(\hat{A}, \mathcal{S})$  is isomorphic to  $H^3(\hat{A}, \mathbb{Z})$ .

**Theorem 11.2.7.** *Let  $R$  be a topological equivalence relation. Then there is a homomorphism  $\rho_R : \mathrm{Tw}_R \rightarrow H^2(\widehat{A}, \mathcal{S})$ . If  $A$  is a Fell algebra,  $(C_1, D_1)$  and  $(C_2, D_2)$  are two Cartan pairs constructed as in Proposition 11.2.3 and  $\mathcal{E} := \mathcal{E}_{C_1, D_1} \rightarrow \mathcal{G}_{C_1, D_1}$  and  $\mathcal{E}' := \mathcal{E}_{C_2, D_2} \rightarrow \mathcal{G}_{C_2, D_2}$  are the twists obtained from these two pairs, then  $\rho_R([\mathcal{E} \rightarrow R]) = \rho_{R'}([\mathcal{E}' \rightarrow R'])$ .*

We denote the element  $\rho_R([\mathcal{E} \rightarrow R]) \in H^2(\widehat{A}, \mathcal{S})$  obtained from any Cartan pair constructed as in Proposition 11.2.3 by  $\delta(A)$ . For our final result, we need to recall that if  $A$  is a  $C^*$ -algebra, then its spectrum  $\widehat{A}$  is a locally compact locally Hausdorff space whose every open subset is again locally compact. Dixmier calls such spaces *quasi locally compact* [15], but I am going to call them *locally locally compact*.

**Theorem 11.2.8.**

1. *Let  $A$  and  $A'$  be Fell algebras. Then  $A$  and  $A'$  are Morita equivalent if and only if there is a homeomorphism  $\widehat{A} \cong \widehat{A}'$  such that the induced isomorphism  $H^2(\widehat{A}, \mathcal{S}) \cong H^2(\widehat{A}', \mathcal{S})$  carries  $\delta(A)$  to  $\delta(A')$ .*
2. *If  $X$  is a locally locally compact, locally Hausdorff space, and  $\delta \in H^2(X, \mathcal{S})$ , then there exist a Fell algebra  $A$  and a homeomorphism  $\widehat{A} \rightarrow X$  such that the induced isomorphism  $H^2(\widehat{A}, \mathcal{S}) \cong H^2(X, \mathcal{S})$  carries  $\delta(A)$  to  $\delta$ .*

More or less by definition of the invariant  $\delta(A)$ , the proof of Theorem 11.2.8(2) is very closely related to the Raeburn–Taylor construction: Take  $\delta \in H^2(X, \mathcal{S})$ , represent it by a Čech cocycle  $c$  defined on an open cover  $X = \bigcup_{i \in I} U_i$  by Hausdorff neighbourhoods. Let  $Y := \bigsqcup_{i \in I} \{i\} \times U_i$ , and define  $\psi : Y \rightarrow X$  by  $\psi(i, x) = x$ . The cocycle  $c$  then determines a continuous cocycle  $\sigma$  on  $R(\psi)$  as in Example 11.2.1, and the  $C^*$ -algebra of the resulting twist is then a Fell algebra with invariant  $\delta$ .

**Remark 11.2.9.** I have refrained from calling  $\delta(A)$  the Dixmier–Douady invariant of  $A$  because, unfortunately, if  $A$  is a continuous-trace  $C^*$ -algebra, it is not clear that  $\delta(A)$  is equal to the classical Dixmier–Douady invariant.



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## **Part III**

# **Introduction to Rokhlin Dimension**

**Gábor Szabó**





## Foreword

These notes were written as supplementary material for a 5-hour lecture series presented during the Masterclass on *groupoids, crossed products, and Rokhlin dimension* at the Centre de Recerca Matemàtica, Universitat Autònoma de Barcelona from the 13th to the 17th of March 2017. This was part of the greater research program *Operator algebras: dynamics and interactions* that took place from March to July 2017. I would like to express my gratitude to the CRM for the hospitality and support, to the organizers of this event for doing an excellent job, and to all the participants of the Masterclass.

The intent of these notes is to give an introduction to Rokhlin-type properties, but with a particular emphasis on Rokhlin dimension and its applications to crossed products. Being based on a short course, the core of these notes centers around the heuristically important special cases of either finite group actions or single automorphisms treated in the seminal work of Hirshberg, Winter and Zacharias. The more advanced theory, including generalizations and connections to groupoids, is touched upon near the end, but has otherwise been cut in order to end up with a selection of topics that can coherently be presented during a week's worth of lectures.



# Chapter 12

## Introduction

The classical Rokhlin lemma, sometimes also called Kakutani–Rokhlin lemma, is a fundamental result in ergodic theory. It states that an aperiodic measure-preserving dynamical system can be decomposed into an arbitrarily high tower of measurable sets and a remainder of arbitrarily small measure. It was proven by Vladimir Abramovich Rokhlin and independently by Shizuo Kakutani. The lemma is used extensively in ergodic theory and has many generalizations.

Near the last quarter of the 20th century, Rokhlin-type properties emerged in operator algebras through the work of Connes and have since played an important role. Both in the theory of  $W^*$ -algebras and  $C^*$ -algebras, Rokhlin-type properties serve as useful technical gadgets leading to beautiful theorems.

On the one hand, there exist various notions of *the Rokhlin property* for group actions on operator algebras. The main purpose of such a notion is to study the fine structure of the underlying dynamical systems, ideally leading to hard classification results for group actions. This is most impressive for  $W^*$ -algebras: The very first *Rokhlin property* was due to Connes [7, 5] for cyclic group actions, and was used to classify such actions up to cocycle conjugacy, which in turn played a crucial part in his award-winning classification of injective factors [6]. This was later generalized by Jones [26] and Ocneanu [38], leading to the classification of amenable group actions on injective factors that we know today.

On the  $C^*$ -algebraic side, Rokhlin properties exist for actions of finite groups and poly- $\mathbb{Z}$  groups, and are similarly used for the purpose of classification [23, 31, 37, 34, 24, 25].<sup>1</sup> However, the results are not on par with the  $W^*$ -algebraic ones because  $C^*$ -algebras do not exhibit the same rigid behavior, but offer much richer dynamical structures. Still, this direction of research has a long history, and arguably there is still much to do. Although there is much more to say about this topic, this direction shall **not** be the focus of these notes.

Instead, we will focus on the notion of Rokhlin dimension. This was invented by Hirshberg–Winter–Zacharias [21] a few years ago, and has since then spawned

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<sup>1</sup>This list of references is far from complete, but these papers can serve as suitable starting points to get into the subject.

and inspired a host of other papers studying either generalizations or new applications. Conceptually speaking, the purpose of Rokhlin dimension is to gain a firm footing in our understanding of the fine structure of certain crossed products, rather than the underlying dynamical systems. Although this means that it is a priori less powerful than *the Rokhlin property*, it has the great advantage in being far less restrictive. As of writing these notes, it is one of the main tools in order to show that certain crossed products fall within the scope of the Elliott classification program.

It is worthwhile to mention that the various versions of *tracial* [40] or *weak* [35, 36] Rokhlin properties play similar roles, but they mostly apply to actions on simple  $C^*$ -algebras and will not be covered here. For the reader interested in this direction, it might be a good idea to have a look at another recent lecture series on Rokhlin-type properties given by Gardella.<sup>2</sup>

Rokhlin dimension is not only applicable for non-simple  $C^*$ -algebras, but its most impressive applications in fact concern crossed products that one associates to classical topological dynamical systems [42, 44, 22, 19]. Apart from the Rokhlin dimension theory itself, these applications shall be our secondary focus within these notes.

In order to minimize some of the technical background needed for further generalizations, we shall for the most part develop the theory in the same generality as in Hishberg–Winter–Zacharias’ original work, namely for either single automorphisms or actions of finite groups on unital  $C^*$ -algebras. Heuristically speaking, these cases already showcase the main ideas of the theory in a nice fashion.

We start in Chapter 13 with a math history lesson, in reviewing the classical Rokhlin lemma in ergodic theory as well as the main ideas leading to its proof. In order to motivate the key feature of Rokhlin dimension for  $C^*$ -algebras, we outline how the Rokhlin lemma can be used to show that aperiodic measure-preserving transformations have approximately finite-dimensional (hyperfinite)  $W^*$ -crossed products. We will also dip our toes into the topological setup by proving a Rokhlin lemma for aperiodic homeomorphisms on the Cantor set, which serves as a warm-up for what comes later.

In Chapter 14 we will introduce and discuss the nuclear dimension of  $C^*$ -algebras and its importance to the classification of simple  $C^*$ -algebras. We will also go over some technical preparations involving the sequence algebra construction that come in handy later, most notably some sufficient conditions for obtaining upper estimates on the nuclear dimension of a  $C^*$ -algebra.

In Chapter 15, we will discuss the Rokhlin property and Rokhlin dimension for actions of finite groups. The main result is that actions with finite Rokhlin dimension are well behaved with respect to nuclear dimension when passing to the crossed product.

In Chapter 16, we will discuss the Rokhlin dimension for single automorphisms (i.e.,  $\mathbb{Z}$ -actions) and outline its consequences for the nuclear dimension of

<sup>2</sup><http://math.ipm.ac.ir/conferences/2016/OpeAlg2016-2/Lecture%20Notes-Gardella.pdf>

crossed products, building on similar (but more sophisticated) ideas as in the finite group case. We will also discuss how certain topological dynamical systems serve as a huge class of examples for the main theorem in this context.

In Chapter 17, we will briefly review generalizations of Rokhlin dimension theory in various directions. This includes Rokhlin dimension for actions of  $\mathbb{Z}^d$  or residually finite groups, Rokhlin dimension for flows, and the connection to dynamic asymptotic dimension for groupoids.

Throughout these notes, our standing assumption is that our  $C^*$ -algebras of interest are separable and unital, unless they are obviously not. Although this is never really important for obtaining the desired results, this is convenient because it spares us some dry technicalities that have little to do with the main ideas communicated throughout.

# Chapter 13

## Classical Dynamical Systems

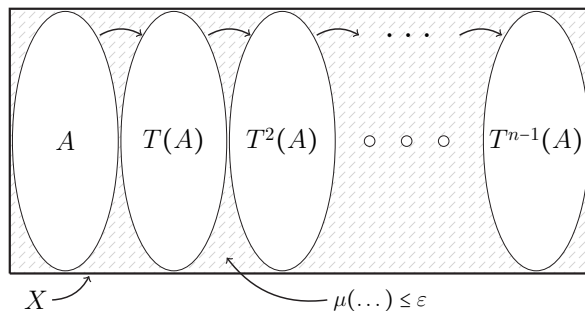
### 13.1 Measureable dynamics

In this part, we will treat the classical Rokhlin lemma. For its generalizations and importance within ergodic theory, the interested reader may consult the excellent recent book of Kerr–Li [27].

**Definition 13.1.1.** Let  $(X, \mu)$  be a Borel probability space and  $T : X \rightarrow X$  an invertible, measure-preserving transformation.  $T$  is called aperiodic, if  $x \neq T^j(x)$  for all  $j > 0$  and  $\mu$ -almost all  $x \in X$ .  $T$  is called ergodic, if any  $T$ -invariant Borel set  $Y \subseteq X$  either satisfies  $\mu(Y) = 0$  or  $\mu(Y) = 1$ . Note that if  $X$  is infinite, then every ergodic transformation is aperiodic.

**Remark 13.1.2** (see [27, Proposition 2.4]). A transformation  $T$  is aperiodic if and only if the following holds: For every  $k > 0$  and every Borel set  $W \subseteq X$  with  $\mu(W) > 0$ , there exists a Borel set  $E \subseteq W$  with  $\mu(E) > 0$  and  $E \cap T^j(E) = \emptyset$  for all  $j = 1, \dots, k-1$ .

**Theorem 13.1.3** (Rokhlin’s Lemma). *Let  $(X, \mu)$  be a Borel probability space and  $T : X \rightarrow X$  an invertible, measure-preserving transformation. Suppose that  $T$  is aperiodic. Let  $n \in \mathbb{N}$  and  $\varepsilon > 0$  be given. Then there exists a Borel set  $A \subset X$  with  $A \cap T^j(A) = \emptyset$  for all  $j = 1, \dots, n-1$  and such that  $\mu\left(\bigcup_{j=0}^{n-1} T^j(A)\right) \geq 1 - \varepsilon$ .*



**Definition 13.1.4.** Let  $T : X \rightarrow X$  be a measure-preserving transformation on a Borel probability space  $(X, \mu)$ . Let  $E \subseteq X$  be a Borel set. Consider the first return time function

$$r_E : E \rightarrow \mathbb{N} \cup \{\infty\}, \quad r_E(x) = \min\{l > 0 \mid T^l(x) \in E\}.$$

**Proposition 13.1.5.** Up to a  $\mu$ -nullset, one has  $E = \bigsqcup_{l \in \mathbb{N}} r_E^{-1}(l)$ .

*Proof.* Suppose this were false. Then one has  $\mu(A) > 0$  for  $A = r_E^{-1}(\infty) \subset E$ . By definition, all  $T$ -translates of  $A$  are pairwise disjoint, and thus

$$1 \geq \sum_{j=0}^{\infty} \mu(T^j(A)) = \sum_{j=0}^{\infty} \mu(A),$$

which leads to a contradiction.  $\square$

*Proof of Rokhlin's Lemma – Step 1.* Without loss of generality, assume  $\varepsilon < 1$ . Choose  $m > \varepsilon^{-1}$ . Suppose that  $B \subseteq X$  is some Borel set with  $\mu(B) > 0$  such that  $B \cap T^j(B) = \emptyset$  for all  $j = 1, \dots, nm-1$ . Consider the  $T$ -invariant set  $Y = \bigcup_{j=0}^{\infty} T^j(B)$ . We claim that we can then find a Borel set  $E \subset Y$  with

$$E \cap T^j(E) = \emptyset \text{ for } j = 1, \dots, n-1 \quad \text{and} \quad \mu\left(\bigcup_{j=0}^{n-1} T^j(E)\right) \geq (1-\varepsilon)\mu(Y).$$

Write  $B_k = r_B^{-1}(k)$  for  $k > 0$ , which forms a partition of  $B$  up to a nullset. Note that by assumption, we have  $B_k = \emptyset$  for  $k < nm$ . Also write  $B_k^j = T^j(B_k)$  for brevity. Set

$$E = \bigsqcup_{k=nm}^{\infty} \bigsqcup_{j=0}^{\lfloor k/n \rfloor - 1} B_k^j.$$

Then indeed  $E \cap T^j(E) = \emptyset$  for  $j = 1, \dots, n-1$  by our assumption on  $B$  and the definition of the first return time function. Note that we have the equation

$$Y = \bigcup_{j=0}^{\infty} T^j(B) = \bigsqcup_{k=nm}^{\infty} \bigsqcup_{j=0}^{k-1} B_k^j$$

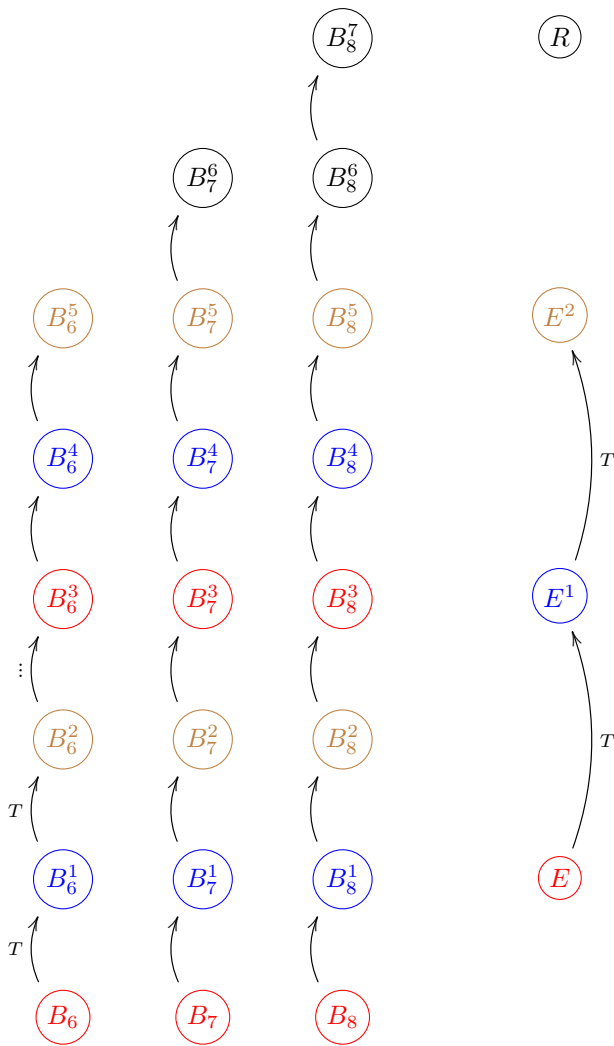
and thus

$$R := Y \setminus \left( \bigcup_{l=0}^{n-1} T^l(E) \right) = \bigsqcup_{k=nm}^{\infty} \bigsqcup_{j=\lfloor k/n \rfloor n-1}^{k-1} B_k^j.$$

We may estimate

$$\begin{aligned} \mu(R) &= \sum_{k=nm}^{\infty} (k - \lfloor k/n \rfloor n) \cdot \mu(B_k) = \sum_{k=nm}^{\infty} \frac{k - \lfloor k/n \rfloor n}{k} \cdot \mu\left(\bigsqcup_{j=0}^{k-1} B_k^j\right) \\ &\leq \frac{1}{m} \cdot \sum_{k=nm}^{\infty} \mu\left(\bigsqcup_{j=0}^{k-1} B_k^j\right) \leq \varepsilon \mu(Y), \end{aligned}$$

which shows our intermediate claim.  $\square$





**Remark 13.1.6.** In the case that  $T$  is ergodic, we would automatically have  $\mu(Y) = 1$  above, so we would now be done upon merely choosing some  $B \subset X$  as in the outset of Step 1. Let us finish the general case.

On the next page is a sketch for the case  $n = 3, m = 2$  and  $B = B_6 \sqcup B_7 \sqcup B_8$ .

*Proof of Rokhlin's Lemma – Step 2.* We combine Step 1 with an argument involving Zorn's Lemma.

Define  $\mathfrak{C}$  to be all pairs  $(A, Y)$  where  $Y \subseteq X$  is a  $T$ -invariant Borel set and  $A \subset Y$  is a Borel set satisfying  $A \cap T^j(A) = \emptyset$  for  $j = 1, \dots, n-1$  and  $\mu\left(\bigcup_{j=0}^{n-1} T^j(A)\right) \geq (1-\varepsilon)\mu(Y)$ . We define a preorder on  $\mathfrak{C}$  by writing  $(A_1, Y_1) \leq (A_2, Y_2)$  if  $A_1 \subseteq A_2$  and  $Y_1 \subseteq Y_2$  hold up to a nullset.

If  $\mathcal{P} \subseteq \mathfrak{C}$  is a totally ordered subset, then we may find a sequence  $(A_m, Y_m) \in \mathcal{P}$  satisfying

$$\sup_{(A,Y) \in \mathcal{P}} \mu(A) = \sup_{m \in \mathbb{N}} \mu(A_m) \quad \text{and} \quad \sup_{(A,Y) \in \mathcal{P}} \mu(Y) = \sup_{m \in \mathbb{N}} \mu(Y_m).$$

Then  $A^0 = \bigcup_{m=1}^{\infty} A_m$  and  $Y^0 = \bigcup_{m=1}^{\infty} Y_m$  define a pair  $(A^0, Y^0) \in \mathfrak{C}$  that is an upper bound for  $\mathcal{P}$ . It now follows from Zorn's Lemma that  $\mathfrak{C}$  must have some maximal element  $(A, Y)$ .

We claim that  $\mu(Y) = 1$ . Note that this would finish the proof of Rokhlin's lemma as then  $A$  is a set satisfying the properties we are looking for. Suppose that  $\mu(Y) < 1$ . Then  $Z = X \setminus Y$  is a  $T$ -invariant Borel set with  $\mu(Z) > 0$ . Since  $T$  is aperiodic on  $Z$ , it follows from Step 1 that we find a  $T$ -invariant Borel set  $Z_0 \subseteq Z$  with  $\mu(Z_0) > 0$  and  $E \subset Z_0$  satisfying  $E \cap T^j(E) = \emptyset$  for  $j = 1, \dots, n-1$  and  $\mu\left(\bigcup_{j=0}^{n-1} T^j(E)\right) \geq (1-\varepsilon)\mu(Z_0)$ . But this implies that  $(A \sqcup E, Y \sqcup Z_0) \in \mathfrak{C}$  is strictly bigger than the maximal element  $(A, Y)$ , which leads to a contradiction.  $\square$

## 13.2 Hyperfiniteness of crossed products

**Remark 13.2.1.** Let  $(X, \mu)$  be a Borel probability space with a measure-preserving transformation  $T$ . Recall that the  $W^*$ -crossed product  $L^\infty(X) \rtimes_T \mathbb{Z}$  is a von Neumann algebra generated by  $L^\infty(X)$  and a unitary  $u$  satisfying  $ufu^* = f \circ T^{-1}$  for all  $f \in L^\infty(X)$ .<sup>1</sup>  $L^\infty(X) \rtimes_T \mathbb{Z}$  carries a faithful normal tracial state  $\tau$ , which arises as the continuous extension of the assignment  $\sum_{j=-N}^N f_j \lambda_T^j \mapsto \int_X f_0 d\mu$  on the algebraic crossed product. The strong operator topology on bounded sets is then given by the distance in the tracial 2-norm, which is defined by  $\|x\|_{2,\tau} = \sqrt{\tau(x^*x)}$ .

As a consequence of the Rokhlin lemma, we can extract the following structure result for the  $W^*$ -crossed product:

**Theorem 13.2.2.** *Let  $(X, \mu)$  be a Borel probability space with an aperiodic, measure-preserving transformation  $T$ . Then  $L^\infty(X) \rtimes_T \mathbb{Z}$  is hyperfinite. In particular, if we assume that  $T$  is ergodic, then  $L^\infty(X) \rtimes_T \mathbb{Z}$  is isomorphic to the (unique) hyperfinite  $\text{II}_1$ -factor.*

**Remark 13.2.3.** We note that in the above theorem, the hyperfiniteness can also be deduced from injectivity and Connes' famous theorem that injectivity implies hyperfiniteness [6]. The point here is that hyperfiniteness can be seen *on the nose* with the help of Rokhlin's lemma.

*Sketch of proof.* Let  $\mathcal{F} \subset L^\infty(X) \rtimes_T \mathbb{Z}$  be a finite set. Without loss of generality, we may assume that  $\mathcal{F}$  consists of elements in some generating set, e.g., let us assume that there is some  $n$  such that the elements in  $\mathcal{F}$  have the form  $\chi_A \cdot u^r$  for some Borel set  $A \subseteq X$  and  $|r| \leq n$ . Let  $\mathcal{B}$  be the finite set of all such  $A$ . By subdividing and adding sets, if necessary, let us moreover assume that  $\mathcal{B}$  is a partition of  $X$ .

Let  $N \gg n$ . By the Rokhlin Lemma, there exists a Borel set  $E \subset X$  such that  $E, T(E), \dots, T^{N-1}(E)$  are pairwise disjoint and  $1 \approx \mu\left(\bigcup_{j=0}^{N-1} T^j(E)\right)$ .

For all maps  $B : \{0, \dots, N-1\} \rightarrow \mathcal{B}$ , consider

$$E_B = E \cap \bigcap_{j=0}^{N-1} T^{-j}(B_j).$$

These form a partition of  $E$ . By construction, it is clear that for any map  $B$  and  $j = 0, \dots, N-1$ , the translate  $T^j(E_B)$  is either empty or contained in precisely one member of the partition  $\mathcal{B}$ , namely  $B_j$ . By changing the index set, let us denote this partition of  $E$  simply as  $\{E_k\}_{k=1}^m$  for some  $m \geq 1$ .

Since all the translates  $T^j(E_k)$  for  $k \leq m$  and  $j \leq N-1$  are pairwise disjoint and make up the entire space up to a small measure defect, we have that for every  $A \in \mathcal{B}$  there are some  $k_i \leq m$  and  $\mathcal{J}_i \subseteq \{0, \dots, N-1\}$  such that

$$\mu\left(A \Delta \bigsqcup_i \bigsqcup_{j \in \mathcal{J}_i} T^j(E_{k_i})\right) \approx 0.$$

<sup>1</sup>It can be defined as the weak closure of the image of the reduced crossed product under the left-regular covariant representation on  $\ell^2(\mathbb{Z}) \otimes L^2(X)$ .

If we consider the projections

$$P_k^j = \chi_{T^j(E_k)} \in L^\infty(X), \quad k = 1, \dots, m, \quad j = 0, \dots, N-1,$$

then we see that

$$\sum_{k,j} P_k^j \approx_{\|\cdot\|_{2,\tau}} 1, \quad P_k^j \circ T^{-1} = P_k^{j+1} \quad (j < n-1)$$

and every characteristic function  $\chi_A$  for  $A \in \mathcal{B}$  is  $\|\cdot\|_{2,\tau}$ -approximately a sum of these projections.

By the properties of the above projections, the assignment

$$\bigoplus_{k=1}^m M_N \rightarrow L^\infty(X) \rtimes_T \mathbb{Z}, \quad e_{j_1, j_2}^{(k)} \mapsto P_k^{j_1} u^{j_2 - j_1} \quad (0 \leq j_1, j_2 \leq N-1)$$

defines a  $*$ -homomorphism. Moreover, one can see that by the choice of  $P_k^j$  and as  $N \gg n$ , the elements of  $\mathcal{F}$  are approximately contained in the image of this homomorphism: For example, if an element  $x \in \mathcal{F}$  has the form  $x = \chi_A \cdot u^r$  for some  $0 \leq r \leq n$ , then there are numbers  $o(k, j) \in \{0, 1\}$  with

$$\chi_A \approx_{\|\cdot\|_{2,\tau}} \sum_{k=1}^m \sum_{j=0}^{N-1} o(k, j) \cdot P_k^j,$$

and hence

$$x \approx_{\|\cdot\|_{2,\tau}} \sum_{k=1}^m \sum_{j=0}^{N-1} o(k, j) \cdot P_k^j u^r \stackrel{N \gg n \geq r}{\approx_{\|\cdot\|_{2,\tau}}} \sum_{k=1}^m \sum_{j=0}^{N-1-r} o(k, j) \cdot P_k^j \cdot u^{(r+j)-j},$$

which is in the image of the above  $*$ -homomorphism. This finishes the proof of hyperfiniteness.  $\square$

### 13.3 Clopen Rokhlin lemma on the Cantor set

Let us now turn to a topological variant of Rokhlin's lemma. Arguably, the most immediate analog can be found on the Cantor set. It is difficult to put a name to the following theorem, as it appears to be a folklore result well known in topological dynamics. To my knowledge this precise form has first implicitly appeared in a paper of Bezuglyi–Dooley–Medynets [3], although I would not be surprised if it was known long before that.

**Theorem 13.3.1.** *Let  $X$  be the Cantor set and let  $T : X \rightarrow X$  be an aperiodic homeomorphism. For every  $n \in \mathbb{N}$ , there exist clopen sets  $Z_1, Z_2 \subset X$  such that*

$$X = \bigsqcup_{j=0}^{n-1} T^j(Z_1) \sqcup \bigsqcup_{l=0}^n T^l(Z_2).$$

*Proof.* Let us first recall a number-theoretic fact: Given  $n$ , there exists some large number  $N \in \mathbb{N}$  such that every natural number  $L \geq N-1$  can be expressed uniquely as  $L = an + b(n+1)$  for  $a, b \geq 0$ .

Now let  $x \in X$  be an arbitrary point. By freeness and continuity of  $T$ , we can find a clopen neighbourhood  $U_x$  of  $x$  such that  $U_x \cap T^j(U_x) = \emptyset$  for  $j = 1, \dots, 2(N-1)$ . Having made such a choice over every point, one obviously obtains an open cover. As  $X$  is compact, we obtain a finite cover  $X = \bigcup_{k=0}^m U_k$  satisfying  $U_k \cap T^j(U_k) = \emptyset$  for  $j = 1, \dots, 2(N-1)$ . Let us inductively define

$$B_0 = U_0 \quad \text{and} \quad B_{k+1} = B_k \cup \left[ U_{k+1} \setminus \left( \bigcup_{j=1}^{N-1} T^j(B_k) \right) \right], \quad k < m.$$

Inductively, one sees that  $B_k \cap T^j(B_k) = \emptyset$  for  $j = 1, \dots, N-1$  and  $U_1 \cup \dots \cup U_k \subseteq \bigcup_{j=1}^{N-1} T^j(B_k)$ . As the  $U_k$  form an open cover, this means that  $B = B_m$  will be a clopen set with  $B \cap T^j(B) = \emptyset$  for  $j = 1, \dots, N-1$  and  $X = \bigcup_{j=0}^{2(N-1)} T^j(B)$ .

Now  $B$  is almost what we are looking for. We set  $B_k = r_B^{-1}(k)$  for  $k \in \mathbb{N}$  and  $B_k^j = T^j(B_k)$ , which are clopen sets because  $B$  was clopen. Note that  $B_k \neq \emptyset$  only for  $N \leq k \leq 2N-1$ . Thus we have a partition

$$X = \bigcup_{k=N}^{2N-1} \bigcup_{j=0}^{k-1} B_k^j.$$

Finally, we can obtain  $Z_1$  and  $Z_2$  via the following rearrangement of this partition: Writing  $k-1 = k_1n + k_2(n+1)$  uniquely for each  $k \geq N$ , we set

$$Z_1 = \bigcup_{k=N}^{2N-1} \bigcup_{l=0}^{k_1-1} B_k^{ln}, \quad Z_2 = \bigcup_{k=N}^{2N-1} \bigcup_{l=0}^{k_2-1} B_k^{k_1n+l(n+1)}.$$

The reason why this works is analogous to the proof of the measurable Rokhlin lemma. We leave the details to the reader as an exercise.  $\square$

# Chapter 14

## Classification Theory and Nuclear Dimension

Let us now switch to our main field of interest, namely  $C^*$ -algebra theory. Although it would have been easy to fill the rest of these notes with applications of the Rokhlin property to classification of automorphisms on  $C^*$ -algebras, we will instead focus on its applications to the structure of crossed products, and the key ideas giving rise to the notion of Rokhlin dimension.

### 14.1 Nuclear dimension

For this purpose we should of course familiarize ourselves with some concepts that are relevant for the fine structure of  $C^*$ -algebras, in particular with regard to the Elliott program. Our main focus shall be on the nuclear dimension of  $C^*$ -algebras. For basics about completely positive maps between  $C^*$ -algebras, see Brown–Ozawa’s book [4].

**Definition 14.1.1** (Winter–Zacharias [49]). A completely positive map  $\varphi : A \rightarrow B$  between  $C^*$ -algebras is said to be of order zero, if for every pair of positive elements  $a, b \in A$  with  $ab = 0$  it follows that  $\varphi(a)\varphi(b) = 0$ . In other words,  $\varphi$  is orthogonality-preserving in the sense that  $a \perp b$  implies  $\varphi(a) \perp \varphi(b)$ .

Instead of defining nuclear dimension of  $C^*$ -algebras directly, it can often be useful to consider more generally the nuclear dimension of a homomorphism, as first done by Tikuisis–Winter [46]. Although this did not appear in the original paper of Winter–Zacharias, it is not a stretch to say that this was implicit in their work and likely only left out because it was not needed at the time.

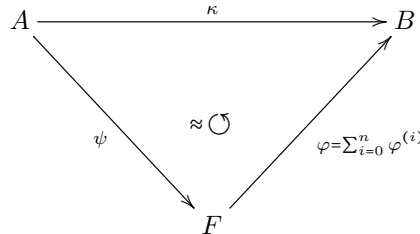
**Definition 14.1.2** (Winter–Zacharias). Let  $A$  and  $B$  be two  $C^*$ -algebras. Let  $\kappa : A \rightarrow B$  be a  $*$ -homomorphism and  $n \geq 0$ . The map  $\kappa$  is said to have nuclear dimension  $n$ , written  $\dim_{\text{nuc}}(\kappa) = n$ , if  $n$  is the smallest natural number satisfying the following property:

For every  $\mathcal{F} \subset A$  and  $\varepsilon > 0$ , there exist

- a finite-dimensional  $C^*$ -algebra  $F$ ;
- a completely positive and contractive map  $\psi : A \rightarrow F$ ;

- completely positive contractive maps of order zero  $\varphi^{(0)}, \dots, \varphi^{(n)} : F \rightarrow B$  (denote  $\varphi = \varphi^{(0)} + \dots + \varphi^{(n)}$ )

such that  $(F, \psi, \varphi)$  yields a finite-dimensional completely positive  $(\mathcal{F}, \varepsilon)$ -approximation of  $\kappa$ , meaning that  $\|\kappa(a) - \varphi(\psi(a))\| \leq \varepsilon$  for all  $a \in \mathcal{F}$ .



If no such  $n$  exists, then  $\kappa$  is said to have infinite nuclear dimension, written  $\dim_{\text{nuc}}(\kappa) = \infty$ .

**Definition 14.1.3** (Winter–Zacharias [50]). For a  $C^*$ -algebra  $A$ , one defines

$$\dim_{\text{nuc}}(A) = \dim_{\text{nuc}}(\text{id}_A : A \rightarrow A).$$

**Remark 14.1.4.** Note that, if one ignores the bound  $n$  in the above definition and merely requires the upward map to be uniformly bounded in norm, one recovers the notion of a nuclear map. The above is thus a natural strengthening of nuclearity. Its main conceptual role however is a noncommutative notion of covering dimension. Winter–Zacharias have proved that

$$\dim_{\text{nuc}}(\mathcal{C}_0(X)) = \dim(X)$$

for all metrizable, locally compact spaces  $X$ .

## 14.2 Classification

Soon after its invention, the notion of nuclear dimension became more prominent through the Toms–Winter regularity conjecture [11, 48] for simple  $C^*$ -algebras, linking it to a priori different properties like Jiang–Su absorption and good behavior of the Cuntz semigroup. On the conceptual level, it is now clear that finite nuclear dimension plays the role of hyperfiniteness for von Neumann algebras. The most impressive demonstration of its importance is now captured by the combination of some recent breakthrough results [15, 10, 45] that may be considered the finite continuation of the Kirchberg–Phillips classification [28, 39]:

**Theorem 14.2.1** ( $\dots$ , Winter, Lin, (Elliott–)Gong–Lin–Niu, Tikuisis–White–Winter). *If  $A$  and  $B$  are two separable, unital, simple  $C^*$ -algebras that have finite nuclear dimension and satisfy the UCT, then*

$$A \cong B \quad \text{if and only if} \quad \text{Ell}(A) \cong \text{Ell}(B).$$

Here,  $\text{Ell}(\cdot)$  denotes the so-called Elliott invariant, consisting of  $K$ -theory and traces. The following shall be the main motivation for studying Rokhlin-type properties of group actions:

**Question 14.2.2.** Let  $\alpha : G \curvearrowright A$  be an action of a group on a  $C^*$ -algebra. When can we say that  $\dim_{\text{nuc}}(A) < \infty$  implies  $\dim_{\text{nuc}}(A \rtimes_{\alpha} G) < \infty$ ?

### 14.3 Some technical tools

Before we delve into this, some preparation is necessary. The following is seen fairly easily from the definition of the nuclear dimension of maps, and we shall leave it as an exercise to the reader.

**Proposition 14.3.1.** Let  $\kappa : A \rightarrow B$  and  $\mu : C \rightarrow D$  be two  $*$ -homomorphisms between  $C^*$ -algebras. Suppose that  $\psi : A \rightarrow C$  is a contractive completely positive map, and  $\varphi^{(0)}, \dots, \varphi^{(d)} : D \rightarrow B$  are order zero maps such that  $\kappa = \sum_{l=0}^d \varphi^{(l)} \circ \mu \circ \psi$ . Then

$$\dim_{\text{nuc}}^{+1}(\kappa) \leq (d+1) \dim_{\text{nuc}}^{+1}(\mu).$$

**Definition 14.3.2.** Let  $A$  be a  $C^*$ -algebra. Its sequence algebra is defined as the quotient

$$A_{\infty} = \prod_{\mathbb{N}} A / \bigoplus_{\mathbb{N}} A = \ell^{\infty}(\mathbb{N}, A) / c_0(\mathbb{N}, A).$$

The norm is given by  $\|[(x_n)]\| = \limsup_{n \rightarrow \infty} \|x_n\|$ . We can identify  $\iota_A : A \rightarrow A_{\infty}$  by embedding as constant sequences. This identification  $A \subset A_{\infty}$  will be the standard one. The central sequence algebra is denoted

$$A_{\infty} \cap A' = \{x \in A_{\infty} \mid [x, A] = 0\}$$

and is of particular interest when  $A$  is separable.

When  $\alpha : G \curvearrowright A$  is a group action, then it gives rise to an action  $\alpha_{\infty} : G \curvearrowright A_{\infty}$  by componentwise application of  $\alpha$ , which in turn restricts to an action on  $A_{\infty} \cap A'$ .

**Lemma 14.3.3.** For any  $C^*$ -algebra, one has  $\dim_{\text{nuc}}(A) = \dim_{\text{nuc}}(\iota_A)$ .

*Proof.* The  $\geq$  part is clear, the interesting bit is the  $\leq$  part. Let  $n = \dim_{\text{nuc}}(\iota_A)$ .

Let  $\varepsilon > 0$  and  $\mathcal{F} \subset A$  be given. Consider a completely positive  $(\mathcal{F}, \varepsilon)$ -approximation

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & A_{\infty} \\ & \searrow \psi & \nearrow \varphi = \sum_{i=0}^n \varphi^{(i)} \\ & F & \end{array} \quad \approx \quad \bigcirc$$

as required by the definition. Now it is known (cf. [29, 2.4]) that order zero maps with finite-dimensional domains can be lifted under any quotient maps. As a consequence we may find, for every  $i = 0, \dots, n$ , order zero maps  $\varphi_k^{(i)} : F \rightarrow A$  fitting into the following commutative diagram:

$$\begin{array}{ccc} & & \ell^\infty(\mathbb{N}, A) \\ & \nearrow (\varphi_k^{(i)})_k & \downarrow \\ F & \xrightarrow{\varphi^{(i)}} & A_\infty \end{array}$$

By the definition of the sequence algebra, we can then consider for every  $a \in \mathcal{F}$  that

$$\varepsilon \geq \|a - \sum_{i=0}^n \varphi^{(i)} \circ \psi(a)\| = \limsup_{k \rightarrow \infty} \|a - \sum_{i=0}^n \varphi_k^{(i)} \circ \psi(a)\|.$$

For large  $k$ , this results in a completely positive  $(\mathcal{F}, 2\varepsilon)$ -approximation

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ & \searrow \psi & \nearrow \varphi_k = \sum_{i=0}^n \varphi_k^{(i)} \\ & F & \end{array} \quad \approx \quad \circlearrowright$$

which shows our claim. □



# Chapter 15

## Rokhlin Dimension for Finite Groups

In order to capture the main idea for the relevance of Rokhlin-type properties to nuclear dimension, it is useful to step back from the single automorphism case and focus first on finite groups. (Note that everything in this section can be generalized to compact groups [20, 14, 13].)

### 15.1 The Rokhlin property

The following definition is due to Izumi [23] in this generality.

**Definition 15.1.1** (Izumi). Let  $A$  be a unital  $C^*$ -algebra and  $G$  a finite group. Let  $\alpha : G \curvearrowright A$  be an action. We say that  $\alpha$  has the Rokhlin property, if for every  $\mathcal{F} \subset A$  and  $\varepsilon > 0$ , there exist projections  $\{e_g\}_{g \in G} \subset A$  satisfying

- $1 = \sum_{g \in G} e_g$ ;
- $\|\alpha_g(e_h) - e_{gh}\| \leq \varepsilon$  for all  $g, h \in G$ ;
- $\|[a, e_g]\| \leq \varepsilon$  for all  $g \in G$  and  $a \in \mathcal{F}$ .

We can consider the following characterization that easily follows from its definition and the use of sequence algebras.

**Proposition 15.1.2.** *A finite group action  $\alpha : G \curvearrowright A$  on a separable, unital  $C^*$ -algebra has the Rokhlin property if and only if there exists a unital and equivariant  $*$ -homomorphism from  $(\mathcal{C}(G), \sigma)$  to  $(A_\infty \cap A', \alpha_\infty)$ . (Here  $\sigma : G \curvearrowright \mathcal{C}(G)$  is the  $G$ -shift action given by  $\sigma_g(f) = f(g^{-1} \cdot -)$ .)*

*Proof.* “ $\Leftarrow$ ”: Let  $\varphi$  be such a  $*$ -homomorphism. Let  $\varepsilon > 0$  and  $\mathcal{F} \subset A$ . Given  $g \in G$ , the element  $e_g = \varphi(\chi_{\{g\}}) \in A_\infty \cap A'$  is a projection, and hence can be represented by a sequence of projections  $e_g^{(n)}$ . For sufficiently large  $n$ , it then follows from the definition of the central sequence algebra that the family  $\{e_g^{(n)}\}_{g \in G}$  satisfies the required properties for  $(\mathcal{F}, \varepsilon)$ , possibly after a slight manipulation using functional calculus to make them orthogonal.

“ $\Rightarrow$ ”: Assume  $\alpha$  has the Rokhlin property. As  $A$  is separable, let  $\mathcal{F}_n \subset A$  be an increasing sequence with dense union. Let  $\varepsilon_n > 0$  be some decreasing null sequence. For every  $n \in \mathbb{N}$ , we may find  $\{e_g^{(n)}\}_{g \in G}$  as required by the definition for  $(\mathcal{F}_n, \varepsilon_n)$ . Then one can check directly (small exercise) that  $\varphi(f) = \sum_{g \in G} f(g) \cdot [(e_g^{(n)})_n]$  defines a unital  $*$ -homomorphism from  $(\mathcal{C}(G), \sigma)$  to  $(A_\infty \cap A', \alpha_\infty)$ .  $\square$

**Remark 15.1.3.** For a separable, unital  $C^*$ -algebra  $A$ , one of the main features of its central sequence algebra is that there is a canonical  $*$ -homomorphism

$$(A_\infty \cap A') \otimes_{\max} A \rightarrow A_\infty, \quad x \otimes a \mapsto x \cdot a.$$

It is clear that this is equivariant if we equip  $A$  with any group action.

Suppose now that  $\alpha : G \curvearrowright A$  has the Rokhlin property. Using the previous characterization of the Rokhlin property, we may obtain a commutative diagram of  $*$ -homomorphisms

$$\begin{array}{ccc} (A, \alpha) & \xrightarrow{\text{id}_A} & (A_\infty, \alpha_\infty) \\ & \searrow 1 \otimes \text{id}_A & \nearrow \\ & (\mathcal{C}(G) \otimes A, \sigma \otimes \alpha) & \end{array} \quad \text{with } \text{Circled } \sigma \text{ in the center.}$$

We may now apply the crossed product functor and obtain a commutative diagram of the form

$$\begin{array}{ccc} A \rtimes_\alpha G & \xrightarrow{\quad} & A_\infty \rtimes_{\alpha_\infty} G \xrightarrow{\cong} (A \rtimes_\alpha G)_\infty \\ & \searrow (1 \otimes \text{id}_A) \rtimes G & \nearrow \\ & (\mathcal{C}(G) \otimes A) \rtimes_{\sigma \otimes \alpha} G & \end{array}$$

The relevance to nuclear dimension becomes clear upon examining the bottom half of this diagram further:

**Lemma 15.1.4** (cf. Theorem 1.4.4 on page 15). *Let  $\alpha : G \curvearrowright A$  be a finite group action on a  $C^*$ -algebra. Then*

$$\Phi : (\mathcal{C}(G) \otimes A, \sigma \otimes \text{id}_A) \rightarrow (\mathcal{C}(G) \otimes A, \sigma \otimes \alpha)$$

given by

$$\Phi(f)(g) = \alpha_g(f(g))$$

is an equivariant isomorphism. In particular

$$(\mathcal{C}(G) \otimes A) \rtimes_{\sigma \otimes \alpha} G \cong (\mathcal{C}(G) \rtimes_\sigma G) \otimes A \cong M_{|G|} \otimes A.$$

**Slogan:** Tensoring with the shift on  $\mathcal{C}(G)$  kills the dynamical structure!

*Proof.* It is clear that  $\Phi$  is an isomorphism. For equivariance, let us compute

$$\begin{aligned}
 \Phi((\sigma \otimes \text{id}_A)_g(f))(h) &= \alpha_h((\sigma \otimes \text{id}_A)_g(f)(h)) \\
 &= \alpha_h(f(g^{-1}h)) \\
 &= \alpha_g \circ \alpha_{g^{-1}h}(f(g^{-1}h)) \\
 &= \alpha_g(\Phi(f)(g^{-1}h)) \\
 &= ((\sigma \otimes \alpha)_g \circ \Phi)(f)(h).
 \end{aligned}$$

This finishes the proof. □

As a consequence of our observations so far, we see:

**Theorem 15.1.5.** *Let  $\alpha : G \curvearrowright A$  be a finite group action with the Rokhlin property. Then  $\dim_{\text{nuc}}(A \rtimes_{\alpha} G) \leq \dim_{\text{nuc}}(A)$ .*

## 15.2 Rokhlin dimension

**Remark 15.2.1.** The problem with the Rokhlin property is that it is too restrictive and rigid in general. Given the projections in the definition, one often faces  $K$ -theoretic obstructions. For example, there are no Rokhlin actions of finite groups on  $\mathcal{O}_{\infty}$ :

If there was, then the projections  $\{e_g\}_{g \in G}$  in the definition of the Rokhlin property would give rise to elements in the  $K_0$ -group with  $[1] = \sum_{g \in G} [e_g]$ . Since these projections are also approximately shifted with automorphisms, which are always approximately inner on  $\mathcal{O}_{\infty}$ , we see that the projections are equivalent, leading to  $[1] = |G| \cdot [e_0]$ . However, this is impossible, as  $[1] \in K_0(\mathcal{O}_{\infty}) \cong \mathbb{Z}$  represents a generator.

What we may now observe is that the upward map in the diagram above is a  $*$ -homomorphism, which is a bit overkill to just see that the nuclear dimension of  $A \rtimes_{\alpha} G$  is finite. It would be enough to arrange for the upward map to be a finite sum of order zero maps (see Proposition 14.3.1). This and other reasons give rise to the following definition.

**Definition 15.2.2** (Hirshberg–Winter–Zacharias [21]). A finite group action  $\alpha : G \curvearrowright A$  on a unital  $C^*$ -algebra has Rokhlin dimension  $d$ , written  $\dim_{\text{Rok}}(\alpha) = d$ , if  $d$  is the smallest natural number with the following properties:

Given  $\varepsilon > 0$  and  $\mathcal{F} \subset A$ , there are positive contractions  $\{f_g^{(l)}\}_{g \in G}^{l=0, \dots, d} \subset A$  satisfying

- $\|1 - \sum_{l=0}^d \sum_{g \in G} f_g^{(l)}\| \leq \varepsilon$ ;
- $\|f_g^{(l)} f_h^{(l)}\| \leq \varepsilon$  for  $g \neq h$  and  $0 \leq l \leq d$ ;

- $\|\alpha_g(f_h^{(l)}) - f_{gh}^{(l)}\| \leq \varepsilon$  for all  $g, h \in G$  and  $0 \leq l \leq d$ ;
- $\|[a, f_g^{(l)}]\| \leq \varepsilon$  for all  $g \in G$ ,  $0 \leq l \leq d$  and  $a \in \mathcal{F}$ .

Once again, there is a neat way to express this in terms of the central sequence algebra. As the proof is almost the same as before, we omit it.

**Proposition 15.2.3.** *Let  $\alpha : G \curvearrowright A$  be a finite group action on a separable, unital  $C^*$ -algebra and  $d \geq 0$ . One has  $\dim_{\text{Rok}}(\alpha) \leq d$  if and only if there exist equivariant order zero maps*

$$\varphi^{(0)}, \dots, \varphi^{(d)} : (\mathcal{C}(G), \sigma) \rightarrow (A_\infty \cap A', \alpha_\infty)$$

with  $\varphi^{(0)}(1) + \dots + \varphi^{(d)}(1) = 1$ .

**Example 15.2.4.** In contrast to the rigidity and  $K$ -theoretic obstructions of the Rokhlin property due to using projections, this definition only utilizes positive elements and thus avoids such obstructions. In fact, every outer action of a finite group on  $\mathcal{O}_\infty$  (or in fact any Kirchberg algebra) has Rokhlin dimension at most one – a stark contrast to the restrictive nature of the Rokhlin property. See also [1, 13, 43, 33, 32].

Let us now come to our first major theorem about Rokhlin dimension:

**Theorem 15.2.5** (Hirshberg–Winter–Zacharias). *For a finite group action  $\alpha : G \curvearrowright A$ , we have*

$$\dim_{\text{nuc}}(A \rtimes_\alpha G) \leq \dim_{\text{Rok}}^{+1}(\alpha) \dim_{\text{nuc}}^{+1}(A).$$

**Remark 15.2.6.** For convenience, we shall prove this result under the assumption that  $A$  is separable and unital. This makes the proof shorter, although it is no essential assumption.

**Remark 15.2.7.** There is one important fact that we will use without proof: For any locally compact group  $G$ , the full crossed product  $\left[ (A, \alpha, G) \mapsto A \rtimes_\alpha G \right]$  is functorial with respect to equivariant order zero maps. In the case that  $G$  is discrete, this means that if  $\varphi : (A, \alpha) \rightarrow (B, \beta)$  is an equivariant order zero map, then the assignment  $au_g \mapsto \varphi(a)u_g$  extends to an order zero map  $\varphi \rtimes G : A \rtimes_\alpha G \rightarrow B \rtimes_\beta G$ . For details, see [19, Lemma 1.4].

*Proof of the theorem.* Consider  $d = \dim_{\text{Rok}}(\alpha)$ , which we may assume to be finite. Let  $\varphi^{(0)}, \dots, \varphi^{(d)}$  be the maps from above. For every  $l = 0, \dots, d$ , we can consider the equivariant order zero map  $\tilde{\varphi}^{(l)}$  given by the composition

$$\mathcal{C}(G) \otimes A \xrightarrow{\varphi^{(l)} \otimes \text{id}_A} (A_\infty \cap A') \otimes_{\max} A \xrightarrow{x \otimes a \mapsto x \cdot a} A_\infty.$$

Then the relation  $\varphi^{(0)}(1) + \dots + \varphi^{(d)}(1) = 1$  will give us the commutativity of the following diagram of equivariant maps:

$$\begin{array}{ccc}
 (A, \alpha) & \xrightarrow{\iota_A} & (A_\infty, \alpha_\infty) \\
 \searrow 1 \otimes \text{id} & \circlearrowleft & \nearrow \sum_{l=0}^d \tilde{\varphi}^{(l)} \\
 & (\mathcal{C}(G) \otimes A, \sigma \otimes \alpha) &
 \end{array}$$

We may now apply the crossed product functor and obtain a commutative diagram of the form

$$\begin{array}{ccccc}
 A \rtimes_\alpha G & \xrightarrow{\quad} & A_\infty \rtimes_{\alpha_\infty} G & \xrightarrow{\cong} & (A \rtimes_\alpha G)_\infty \\
 \searrow (1 \otimes \text{id}_A) \rtimes G & & \nearrow \sum_{l=0}^d \tilde{\varphi}^{(l)} \rtimes G & & \\
 & (\mathcal{C}(G) \otimes A) \rtimes_{\sigma \otimes \alpha} G & & & \\
 \downarrow \cong & & & & \\
 & M_{|G|} \otimes A & & &
 \end{array}$$

Now the upward map is a sum of  $(d+1)$  order zero maps, and thus we obtain

$$\dim_{\text{nuc}}^{+1}(A \rtimes_\alpha G) = \dim_{\text{nuc}}^{+1}(\iota_{A \rtimes_\alpha G}) \leq (d+1) \dim_{\text{nuc}}^{+1}(A)$$

via Proposition 14.3.1. □

# Chapter 16

## Single Automorphisms

Let us now come back to single automorphisms, or  $\mathbb{Z}$ -actions. The basic idea behind the Rokhlin dimension for  $\mathbb{Z}$ -actions is to utilize its finite quotients, which are cyclic groups of the form  $\mathbb{Z}/n\mathbb{Z}$ . If  $n$  gets large, the quotient map  $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  sees a *large* part of  $\mathbb{Z}$ .

### 16.1 Rokhlin dimension

The following version of Rokhlin dimension for single automorphisms is the naive generalization of Rokhlin dimension for finite group actions we have just seen. Note that there is some freedom of choice how to set up this definition, and there are several versions treated in the original paper of Hirshberg–Winter–Zacharias [21]. However, they all have in common that each can be estimated from above by a multiple of another, and thus none of the choices involved matter for the finiteness of the resulting value.

**Definition 16.1.1** (Hirshberg–Winter–Zacharias). Let  $\psi$  be an automorphism on a unital  $C^*$ -algebra  $A$ . We say that  $\psi$  has Rokhlin dimension  $d$ , written  $\dim_{\text{Rok}}(\psi) = d$ , if  $d$  is the smallest natural number satisfying the following property:

For every  $\varepsilon > 0$ ,  $\mathcal{F} \subset A$  and  $n \in \mathbb{N}$ , there are positive contractions

$$\{f_j^{(l)}\}_{j=0, \dots, n-1}^{l=0, \dots, d} \subset A$$

satisfying

- $\|1 - \sum_{l=0}^d \sum_{j=0}^{n-1} f_j^{(l)}\| \leq \varepsilon$ ;
- $\|f_i^{(l)} f_j^{(l)}\| \leq \varepsilon$  for  $i \neq j$  and  $0 \leq l \leq d$ ;
- $\|\psi(f_j^{(l)}) - f_{j+1}^{(l)}\| \leq \varepsilon$  for all  $j = 0, \dots, n-1$  and  $0 \leq l \leq d$ ; (denoting  $f_n^{(l)} = f_0^{(l)}$ .)
- $\|[a, f_j^{(l)}]\| \leq \varepsilon$  for all  $g \in G$ ,  $j = 0, \dots, n-1$ ,  $0 \leq l \leq d$  and  $a \in \mathcal{F}$ .

Conceptually speaking, this property is interesting for large  $n$  because a large chunk of the acting group  $\mathbb{Z}$  is reflected in the index shift of the Rokhlin towers. For a fixed  $n$ , this property mimics the Rokhlin dimension of a  $\mathbb{Z}/n\mathbb{Z}$ -action.

In a similar way as before, this can be characterized in terms of the central sequence algebra:<sup>1</sup>

**Lemma 16.1.2.** *Let  $\psi$  be an automorphism on a separable, unital  $C^*$ -algebra  $A$ . Let  $d \geq 0$ . The following are equivalent:*

- (i)  $\dim_{\text{Rok}}(\psi) \leq d$ ;
- (ii) *For every  $n \in \mathbb{N}$ , there are equivariant order zero maps*

$$\varphi^{(0)}, \dots, \varphi^{(d)} : (\mathcal{C}(\mathbb{Z}/n\mathbb{Z}), \sigma^{(n)}) \rightarrow (A_\infty \cap A', \psi_\infty)$$

*with  $\varphi^{(0)}(1) + \dots + \varphi^{(d)}(1) = 1$ . Here  $\sigma^{(n)} : \mathbb{Z} \curvearrowright \mathcal{C}(\mathbb{Z}/n\mathbb{Z})$  is induced by the  $\mathbb{Z}$ -shift.*

- (iii) *There exist equivariant order zero maps*

$$\Phi^{(0)}, \dots, \Phi^{(d)} : (\mathcal{C}(X), \beta) \rightarrow (A_\infty \cap A', \psi_\infty)$$

*with  $\Phi^{(0)}(1) + \dots + \Phi^{(d)}(1) = 1$ , where one defines*

$$(X, \beta) = \varprojlim (\mathbb{Z}/n!\mathbb{Z}, \mathbb{Z}\text{-shift}).$$

*(This is the universal odometer on the Cantor set.)*

The following goes by a completely analogous argument as we discussed for finite group actions.

**Proposition 16.1.3.** *Let  $\psi$  be an automorphism on a separable, unital  $C^*$ -algebra  $A$ . Assume that  $d = \dim_{\text{Rok}}(\psi)$  is finite. Let  $(X, \beta)$  be as above. Then there exist order zero maps  $\tilde{\Phi}^{(0)}, \dots, \tilde{\Phi}^{(d)}$  so that the following diagram commutes:*

$$\begin{array}{ccc} A \rtimes_\psi \mathbb{Z} & \xrightarrow{\hspace{2cm}} & (A \rtimes_\psi \mathbb{Z})_\infty \\ & \searrow (1 \otimes \text{id}_A) \rtimes \mathbb{Z} & \nearrow \sum_{l=0}^d \tilde{\Phi}^{(l)} \\ & (C(X) \otimes A) \rtimes_{\beta \otimes \psi} \mathbb{Z} & \end{array}$$

*In particular, we have*

$$\dim_{\text{nuc}}^{+1}(A \rtimes_\psi \mathbb{Z}) \leq (d+1) \cdot \dim_{\text{nuc}}^{+1}((1 \otimes \text{id}_A) \rtimes \mathbb{Z}).$$

<sup>1</sup>We omit the details here once again, but carrying this proof out on one's own may serve as an excellent exercise for getting used to sequence algebras. Note that the equivalence between (ii) and (iii) has nothing to do with dynamical systems, but is rather a general property of sequence algebras that generalizes to arbitrary inductive limits.

## 16.2 Permanence of finite nuclear dimension

Now comes the main theorem relating Rokhlin dimension and nuclear dimension for single automorphisms:

**Theorem 16.2.1** (Hirshberg–Winter–Zacharias). *Let  $\psi$  be an automorphism on a  $C^*$ -algebra  $A$ . Then*

$$\dim_{\text{nuc}}^{+1}(A \rtimes_{\psi} \mathbb{Z}) \leq 2 \cdot \dim_{\text{Rok}}^{+1}(\psi) \cdot \dim_{\text{nuc}}^{+1}(A).$$

By what we already know, we can reduce this theorem to the following statement:

**Lemma 16.2.2.** *Let  $(X, \beta) = \varprojlim (\mathbb{Z}/n!\mathbb{Z}, \mathbb{Z}\text{-shift})$  be as before. For an automorphism  $\psi$  on a  $C^*$ -algebra  $A$ , we consider the embedding*

$$(1 \otimes \text{id}_A) \rtimes \mathbb{Z} : A \rtimes_{\psi} \mathbb{Z} \rightarrow (\mathcal{C}(X) \otimes A) \rtimes_{\beta \otimes \psi} \mathbb{Z}.$$

Then one has

$$\dim_{\text{nuc}}^{+1}((1 \otimes \text{id}_A) \rtimes \mathbb{Z}) \leq 2 \cdot \dim_{\text{nuc}}^{+1}(A).$$

**Remark 16.2.3.** By identifying  $\iota : A \rtimes_{\psi} \mathbb{Z} \subset \mathcal{M}(\mathcal{K}(\ell^2(\mathbb{Z})) \otimes A)$ , we may view elements in the crossed product as infinite  $A$ -valued matrices indexed over  $\mathbb{Z}$ . More precisely, the identification goes as

$$a \mapsto \sum_{j \in \mathbb{Z}} e_{j,j} \otimes \psi^{-j}(a) = \begin{pmatrix} \ddots & & & & \\ & \psi(a) & & & \\ & & a & & \\ & & & \psi^{-1}(a) & \\ & & & & \ddots \end{pmatrix}$$

and

$$\lambda = \lambda_{\psi} \mapsto \sum_{j \in \mathbb{Z}} e_{j,j-1} = \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

The crossed product is then the norm-closure of the linear span of products of the form  $a\lambda_{\psi}^j$  for  $j \in \mathbb{Z}$ .

*Proof of the lemma.* Let  $\mathcal{F} \subset A \rtimes_{\psi} \mathbb{Z}$  be given. In order to show the claim, we will show that one can find some large  $N$ , a completely positive map  $\Psi : A \rtimes_{\psi} \mathbb{Z} \rightarrow M_{2N}(A)$  and  $*$ -homomorphisms  $\Sigma^{(1)}, \Sigma^{(2)} : M_{2N}(A) \rightarrow (\mathcal{C}(X) \otimes A) \rtimes_{\beta \otimes \psi} \mathbb{Z}$  such



that the following diagram approximately commutes on  $\mathcal{F}$ :

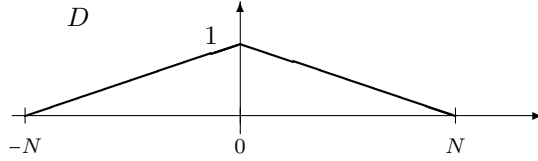
$$\begin{array}{ccc}
 A \rtimes_{\psi} \mathbb{Z} & \xrightarrow{(1 \otimes \text{id}_A) \rtimes \mathbb{Z}} & (\mathcal{C}(X) \otimes A) \rtimes_{\beta \otimes \psi} \mathbb{Z} \\
 \searrow \Psi & & \nearrow \Sigma^{(1)} + \Sigma^{(2)} \\
 & M_{2N}(A) &
 \end{array}$$

Since the crossed product is the closure of the linear span of products as in the above remark, let us assume for convenience that there exists some  $n \in \mathbb{N}$  such that

$$x \in \mathcal{F} \implies x = a\lambda^j \text{ for some } a \in A, \|a\| \leq 1 \text{ and } 0 \leq j \leq n.$$

Let us then choose a natural number  $N \gg n$ . Let us consider a *decay function*

$$D : \mathbb{Z} \rightarrow [0, 1], \quad D(k) = \begin{cases} 0 & , \quad |k| \geq N \\ 1 - \frac{|k|}{N} & , \quad |k| < N. \end{cases}$$



We may view  $D$  as a diagonal matrix over  $\mathbb{Z}$  with support in  $I = \{1 - N, \dots, N\}$ . We identify  $M_{2N}(A) \cong M_I(A)$  and then consider

$$\Psi : A \rtimes_{\psi} \mathbb{Z} \rightarrow M_I(A), \quad x \mapsto D^{1/2} \iota(x) D^{1/2} = D^{1/2} 1_I \iota(x) 1_I D^{1/2}.$$

By our choice of  $n$ , the matrices  $\iota(x)$  for  $x \in \mathcal{F}$  can only have values that are at most  $n$  off the diagonal. As  $N$  is much bigger than  $n$ , the function  $D$  is very flat compared to  $n$ , and hence one gets

$$\Psi(x) \approx D \cdot 1_I \iota(x) 1_I, \quad x \in \mathcal{F}.$$

More explicitly, one has

$$\Psi(a\lambda^j) \approx \sum_{k=1+j-N}^N D(k) \cdot e_{k, k-j} \otimes \psi^{-k}(a), \quad 0 \leq j \leq n.$$

Now let us construct the upward map. We consider the elementary dynamical system  $(\mathcal{C}(\mathbb{Z}/2N\mathbb{Z}), \sigma^{(2N)})$ , which by definition is a unital subsystem of  $(\mathcal{C}(X), \beta)$ . Let  $\{E_q\}_{q \in \mathbb{Z}/2N\mathbb{Z}}$  be the canonical characteristic functions in  $\mathcal{C}(\mathbb{Z}/2N\mathbb{Z})$ . We consider

$$\Sigma^{(i)} : M_I(A) \rightarrow (\mathcal{C}(\mathbb{Z}/2N\mathbb{Z}) \otimes A) \rtimes_{\sigma^{(2N)} \otimes \psi} \mathbb{Z}, \quad i = 1, 2$$

given by

$$\Sigma^{(i)}(e_{k_1, k_2} \otimes a) = \begin{cases} \lambda^{k_1}(E_{\bar{0}} \otimes a) \lambda^{-k_2}, & i = 1 \\ \lambda^{k_1}(E_{\bar{N}} \otimes a) \lambda^{-k_2}, & i = 2. \end{cases}$$

One can directly check that these define  $*$ -homomorphisms; this shall remain an exercise. For a contraction  $a \in A$  and  $0 \leq j \leq n$ , we compute

$$\begin{aligned} \Sigma^{(1)} \circ \Psi(a\lambda^j) &\approx \sum_{k=1+j-N}^N D(k) \cdot \Sigma^{(1)}(e_{k, k-j} \otimes \psi^{-k}(a)) \\ &= \sum_{k=1+j-N}^N D(k) \cdot \lambda^k(E_0 \otimes \psi^{-k}(a)) \lambda^{j-k} \\ &\approx \sum_{k=1-N}^N D(k) \cdot (E_{\bar{k}} \otimes a) \lambda^j. \end{aligned}$$

Analogously, we see that

$$\Sigma^{(2)} \circ \Psi(a\lambda^j) \approx \sum_{k=1-N}^N D(k) (E_{\overline{k+N}} \otimes a) \lambda^j.$$

Upon taking a closer look at the function  $D$ , we see that

$$\sum_{k=1-N}^N D(k) (E_{\bar{k}} + E_{\overline{k+N}}) = 1,$$

and thus

$$\Sigma^{(1)} \circ \Psi(a\lambda^j) + \Sigma^{(2)} \circ \Psi(a\lambda^j) \approx (1 \otimes a) \lambda^j.$$

This finishes the proof.  $\square$

## 16.3 Topological dynamics

Let us now consider dynamical systems coming from a classical setup, namely homeomorphisms  $T : X \rightarrow X$  on a compact metric space. The associated  $C^*$ -automorphism on  $\mathcal{C}(X)$  is given by  $\bar{T}(f) = f \circ T^{-1}$ .

**Question 16.3.1.** Is there a topological viewpoint to detect  $\dim_{\text{Rok}}(\bar{T}) < \infty$ ?

**Definition 16.3.2** (Winter). Let  $T : X \rightarrow X$  be a homeomorphism on a compact metric space. The topological Rokhlin dimension  $\dim_{\text{Rok}}^{\text{top}}(T)$  is the smallest natural number  $d$  satisfying the following property:

Given  $n \in \mathbb{N}$ , there is a collection of open sets  $\{U_j^{(l)}\}_{0 \leq l \leq d, 0 \leq j \leq n-1}$  in  $X$  such that

- $X = \bigcup_{l=0}^d \bigcup_{j=0}^{n-1} U_j^{(l)}$ ;
- $U_{j+1}^{(l)} = T(U_j^{(l)})$  for all  $j < n-1$ ;
- $\bar{U}_{j_1}^{(l)} \cap \bar{U}_{j_2}^{(l)} = \emptyset$  for  $j_1 \neq j_2$ .

**Lemma 16.3.3** (Winter). *One has  $\dim_{\text{Rok}}^{+1}(\bar{T}) \leq 2 \dim_{\text{Rok}}^{\text{top},+1}(T)$ .*

*Proof.* Set  $d := \dim_{\text{Rok}}^{\text{top}}(T)$ , which we may assume to be finite. Let  $n \in \mathbb{N}$ . Since we are in the abelian setting, it suffices to construct positive functions

$$\{f_j^{(l)}\}_{j=0,\dots,n-1}^{l=0,\dots,d} \subset A$$

satisfying

$$1 \approx \sum_{l=0}^d \sum_{j=0}^{n-1} f_j^{(l)}; \quad f_i^{(l)} f_j^{(l)} \approx \delta_{i,j} f_i^{(l)2}; \quad f_j^{(l)} \circ T^{-1} \approx f_{j+1}^{(l)} \pmod{n}$$

from Definition 16.1.1. Moreover, again because of commutativity, it suffices to replace the first of these conditions by

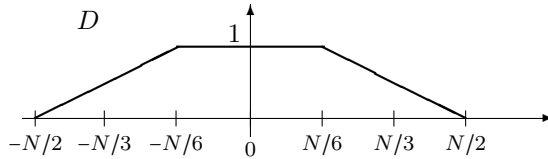
$$\sum_{l=0}^d \sum_{j=0}^{n-1} f_j^{(l)} \geq 1,$$

since we may just multiply the elements by the inverse of this sum if necessary.

Let  $N := 6Ln \gg n$  for some large number  $L \in \mathbb{N}$ . Using the definition of  $d$ , we may find a collection of open sets  $\{U_r^{(l)}\}_{0 \leq l \leq d, 0 \leq r \leq N-1}$  as in Definition 16.3.2. For each  $l = 0, \dots, d$ , there is an open neighbourhood  $V^{(l)} \supset \bar{U}_0^{(l)}$  with  $V^{(l)} \cap T^r(V^{(l)}) = \emptyset$  for  $r = 1, \dots, N-1$ . Using Urysohn–Tietze, choose some continuous function  $h^{(l)} : X \rightarrow [0, 1]$  with  $\text{supp}(h^{(l)}) \subseteq V^{(l)}$  and  $h^{(l)}|_{U_0^{(l)}} = 1$ . By these properties, we see that  $h^{(l)} \perp (h^{(l)} \circ T^{-r})$  for  $r = 0, \dots, N-1$ .

Let us consider a function

$$D : \mathbb{Z} \rightarrow [0, 1], \quad D(k) = \begin{cases} 0 & , \quad |k| \geq N/2 \\ 1 & , \quad |k| \leq N/6 \\ 1 - \frac{3(|k| - N/6)}{N} & , \quad N/6 < |k| < N/2. \end{cases}$$



For  $j = 1, \dots, n-1$ ,  $l = 0, \dots, d$  and  $i = \pm 1$ , we define

$$f_j^{(i,l)} = \sum_{r \in \mathbb{Z}} D(rn + j + iN/3) \cdot (h^{(l)} \circ T^{-(rn+j)}).$$

Then we easily see  $f_{j_1}^{(i,l)} \perp f_{j_2}^{(i,l)}$  for  $j_1 \neq j_2$ . from the properties of  $h^{(l)}$  and the fact that  $D$  is supported on an interval no longer than  $N$ . By the flatness of the

function  $D^2$  we may arrange  $D(r+j) \approx D(r)$  whenever  $j \leq n$ , and hence

$$\begin{aligned} f_j^{(i,l)} \circ T^{-1} &= \sum_{r \in \mathbb{Z}} D(rn + j + iN/3) \cdot (h^{(l)} \circ T^{-(rn+(j+1))}) \\ &\approx \sum_{r \in \mathbb{Z}} D(rn + (j+1) + iN/3) \cdot (h^{(l)} \circ T^{-(rn+(j+1))}) \\ &= f_{j+1}^{(i,l)} \end{aligned}$$

when  $j < n-1$ , and at the endpoint one has

$$\begin{aligned} f_{n-1}^{(i,l)} \circ T^{-1} &= \sum_{r \in \mathbb{Z}} D(rn + (n-1) + iN/3) \cdot (h^{(l)} \circ T^{-(r+(n+1))}) \\ &\approx \sum_{r \in \mathbb{Z}} D(r(n+1) + iN/3) \cdot (h^{(l)} \circ T^{-(r+(n+1))}) \\ &= f_0^{(i,l)}. \end{aligned}$$

Another property of the function  $D$  is that

$$D(r - N/3) + D(r + N/3) = 1$$

for all  $|r| \leq N/2$ . In particular, by the construction of  $h^{(l)}$ , we get that

$$\sum_{i=\pm 1}^d \sum_{l=0}^{n-1} \sum_{j=0}^{n-1} f_j^{(i,l)} \geq 1$$

on

$$\bigcup_{l=0}^d \bigcup_{r=0}^{N-1} T^{N/2}(U_r^{(l)}) \supseteq X.$$

This finishes the proof.  $\square$

**Theorem 16.3.4** (Hirshberg–Winter–Zacharias, Gutman). *Let  $T : X \rightarrow X$  be an aperiodic homeomorphism. Then  $\dim_{\text{Rok}}^{\text{top},+1}(T) \leq 2 \dim^+(X)$ .*

In the case of being the Cantor set, this follows easily from Theorem 13.3.1. For the general case, this follows from [18, Theorem 6.1] along with an inspection of its proof, or alternatively [42, Corollary 5.1]<sup>3</sup>.

**Corollary 16.3.5** (Toms–Winter, Hirshberg–Winter–Zacharias, S [47, 21, 41]). *For an aperiodic homeomorphism  $T : X \rightarrow X$  on a compact metric space, one has  $\dim_{\text{nuc}}(\mathcal{C}(X) \rtimes_{\bar{T}} \mathbb{Z}) < \infty$  if  $\dim(X) < \infty$ .*

<sup>2</sup>This comes from the largeness of  $L$ .

<sup>3</sup>A general and detailed treatment is unfortunately beyond the scope of these notes.

# Chapter 17

## Outlook: Generalizations

### 17.1 Residually finite groups

In analogy to how we defined Rokhlin dimension for  $\mathbb{Z}$  using its finite quotients, it is possible to define Rokhlin dimension for residually finite groups [42, 44]. Recall that a discrete group  $G$  is called residually finite, if its finite-index subgroups separate points, i.e.,  $\bigcap \{H \leq G \mid [G:H] < \infty\} = \{1_G\}$ .

Note that the general definition below should again be stated in terms of finite sets and small tolerances, but let us make life a bit easier by stating the (revisionist) definition in terms of the sequence algebra.

**Definition 17.1.1** (S–Wu–Zacharias). Let  $A$  be a separable, unital  $C^*$ -algebra. Let  $G$  be a residually finite group and  $\alpha : G \curvearrowright A$  an action. The Rokhlin dimension of  $\alpha$ , written  $\dim_{\text{Rok}}(\alpha)$ , is the smallest number  $d \geq 0$  such that for every finite-index subgroup  $H \leq G$ , there exist equivariant order zero maps

$$\varphi^{(0)}, \dots, \varphi^{(d)} : (\mathcal{C}(G/H), G\text{-shift}) \rightarrow (A_\infty \cap A', \alpha_\infty)$$

with  $\varphi^{(0)}(1) + \dots + \varphi^{(d)}(1) = 1$ .

The key difference to everything so far is that, while Proposition 16.1.3 generalizes easily, a clear-cut analog of Theorem 16.2.1 (or more precisely Lemma 16.2.2) does not always hold. If we take a close look at its proof, we might recall that we used the *decay functions* over  $\mathbb{Z}$  as a way to construct a crucial part of the approximation diagram. The existence or non-existence of such decay functions over a group  $G$  can be recast in a coarse-geometric invariant, namely the *asymptotic dimension of the box space*  $\square_s G$  over  $G$ . For more details, see [44, 12, 9].

**Theorem 17.1.2** (S–Wu–Zacharias). *For an action  $\alpha : G \curvearrowright A$  of a residually finite group on a  $C^*$ -algebra, one has*

$$\dim_{\text{nuc}}^{+1}(A \rtimes_\alpha G) \leq \text{asdim}^{+1}(\square_s G) \dim_{\text{Rok}}^{+1}(\alpha) \dim_{\text{nuc}}^{+1}(A).$$

**Example 17.1.3.** One has

- $\text{asdim}(\square_s G) = 0$  if  $G$  is finite;
- $\text{asdim}(\square_s \mathbb{Z}) = 1$ ;
- $\text{asdim}(\square_s \mathbb{Z}^m) = m$ ;
- $\text{asdim}(\square_s G) < \infty$  if  $G$  is finitely generated and virtually nilpotent. (S–Wu–Zacharias, Finn–Sell–Wu, Delabie–Tounton)
- $\text{asdim}(\square_s G) \geq \text{asdim}(G)$ , thus the value is infinite for many groups such as  $\mathbb{Z}^\infty$ .

The example class coming from topological dynamics also generalizes in a nice manner. The crucial topological observations leading to the result below were also independently discovered by Bartels [2] almost simultaneously.

**Theorem 17.1.4** (S, S–Wu–Zacharias, Bartels). *Let  $X$  be a compact metric space with  $\dim(X) < \infty$ . Let  $G$  be a finitely generated, virtually nilpotent group. If  $\alpha : G \curvearrowright X$  is a free action, then  $\dim_{\text{Rok}}(\bar{\alpha} : G \curvearrowright \mathcal{C}(X)) < \infty$ . In particular,  $\dim_{\text{nuc}}(\mathcal{C}(X) \rtimes_{\bar{\alpha}} G) < \infty$ .*

**Remark 17.1.5.** The initial paper [42] proved this theorem for  $\mathbb{Z}^m$ -actions. In that setup, the approach is almost identical to the one outlined in Section 5, but more clever ideas are needed in greater generality. This ties into the next subsection.

## 17.2 Groupoids

In parallel to many of these developments, Guentner–Willett–Yu [16, 17] have invented a useful dynamical notion of dimension for groupoids. Although their motivation came from entirely different areas, they discovered that their *dynamic asymptotic dimension* can also be used to estimate the nuclear dimension of crossed products and in fact groupoid  $C^*$ -algebras.

**Definition 17.2.1.** Let  $\mathcal{G}$  be an étale groupoid with unit space  $\mathcal{G}^{(0)}$ . Then  $\mathcal{G}$  has dynamic asymptotic dimension  $d \geq 0$ , written  $\text{dad}(\mathcal{G}) = d$ , if  $d$  is the smallest number with the following property: For every open relatively compact subset  $K \subseteq \mathcal{G}$ , there are open subsets  $U_0, \dots, U_d$  of  $\mathcal{G}^{(0)}$  that cover  $s(K) \cup r(K)$  such that for each  $i$ , the subgroupoid  $\mathcal{G}_i$  generated by the set

$$\{g \in K \mid s(g), r(g) \in U_i\}$$

is relatively compact in  $\mathcal{G}$ .

**Theorem 17.2.2** (Guentner–Willett–Yu). *Let  $\mathcal{G}$  be an étale principal groupoid with unit space  $\mathcal{G}^{(0)}$ . Then*

$$\dim_{\text{nuc}}^{+1}(C^*(\mathcal{G})) \leq \text{dad}^{+1}(\mathcal{G}) \dim^{+1}(\mathcal{G}^{(0)}).$$

**Remark 17.2.3.** The methods to prove this are a little different from those for Rokhlin dimension, and naturally the groupoid picture is crucially exploited. The key idea is that, if  $\mathcal{G}$  has finite dynamic asymptotic dimension, then  $C^*(\mathcal{G})$  is in a sense *covered* by finitely many groupoid  $C^*$ -subalgebras  $C^*(\mathcal{G}_i)$ . (In a way that is compatible with nuclear dimension.) The  $\mathcal{G}_i$  are as in the definition of dynamic asymptotic dimension. An abstract theorem then tells us that by openness and relative compactness,  $C^*(\mathcal{G}_i)$  is a recursive subhomogeneous  $C^*$ -algebra whose nuclear dimension is at most  $\dim(\mathcal{G}^{(0)})$ . Although the methods to obtain the results about nuclear dimension here are different, there is at least one way to compare this notion to Rokhlin dimension:

**Theorem 17.2.4** (S–Wu–Zacharias). *Let  $G$  be a residually finite group and  $\alpha : G \curvearrowright X$  a free action on a compact metric space. Then*

$$\mathrm{dad}^{+1}(X \rtimes_{\alpha} G) \leq \mathrm{asdim}^{+1}(\square_s G) \dim_{\mathrm{Rok}}^{+1}(\bar{\alpha}).$$

For groupoids it is clear that dynamic asymptotic dimension is the more natural and more powerful tool to use. An impressive demonstration of this can be found in the recent work of Deelay–Strung [8] on  $C^*$ -algebras associated to Smale spaces.

## 17.3 Flows

Much of the Rokhlin dimension theory discussed so far can be pushed beyond the discrete setting. Instead of residual finiteness, one would require that the group admits enough discrete cocompact subgroups. For example, for flows (i.e., continuous  $\mathbb{R}$ -actions) a notion of Rokhlin dimension has been introduced by Hirshberg–S–Winter–Wu [19], allowing for similar results as for single automorphisms. This generalizes the Rokhlin property for flows due to Kishimoto [30].<sup>1</sup>

**Definition 17.3.1.** Let  $\alpha : \mathbb{R} \curvearrowright A$  be a flow on a separable, unital  $C^*$ -algebra. The Rokhlin dimension of  $\alpha$  is the smallest number  $d \geq 0$  such that for every  $T > 0$  there exist equivariant order zero maps

$$\varphi^{(0)}, \dots, \varphi^{(d)} : (\mathcal{C}(\mathbb{R}/T\mathbb{Z}), \mathbb{R}\text{-shift}) \rightarrow (A_{\infty} \cap A', \alpha_{\infty})$$

with  $\varphi^{(0)}(1) + \dots + \varphi^{(d)}(1) = 1$ .

**Theorem 17.3.2** (Hirshberg–S–Winter–Wu). *Let  $\alpha : \mathbb{R} \curvearrowright A$  be a flow on a  $C^*$ -algebra. Then*

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_{\alpha} \mathbb{R}) \leq 2 \dim_{\mathrm{Rok}}^{+1}(\alpha) \dim_{\mathrm{nuc}}^{+1}(A).$$

<sup>1</sup>Note that the original definition might again be stated with a slightly different flavor, but we make life easier by giving the (equivalent revisionist) definition that most resembles what we already know.

In particular, the natural class of examples coming from topological dynamics is still covered, just as for single homeomorphisms.

**Theorem 17.3.3** (Hirshberg–S–Winter–Wu). *Let  $Y$  be a locally compact metric space with a topological flow  $\Phi : \mathbb{R} \curvearrowright Y$ . Denote by  $\alpha : \mathbb{R} \curvearrowright \mathcal{C}_0(Y)$  the associated flow given by  $\alpha(f) = f \circ \Phi_{-t}$ . Suppose that  $\Phi$  is free. If  $\dim(Y) < \infty$ , then  $\dim_{\text{Rok}}(\alpha) < \infty$ , and in particular  $\dim_{\text{nuc}}(\mathcal{C}_0(Y) \rtimes_{\alpha} \mathbb{R}) < \infty$ .*



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