FROM GROUPS TO GROUPOIDS: A BRIEF SURVEY

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1. *Introduction*

A groupoid should be thought of as a *group with many objects, or with many identities.* A precise definition is given below. A groupoid with one object is essentially just a group. So the notion of groupoid is an extension of that of groups. It gives an additional convenience, flexibility and range of applications, so that even for purely group-theoretical work, it can be useful to take a path through the world of groupoids.

A succinct definition is that a groupoid G is a small category in which every morphism is an isomorphism. Thus G has a set of morphisms, which we shall call just *elements* of G, a set $\mathrm{Ob}(G)$ of *objects* or *vertices*, together with functions $s, t\colon G \to \mathrm{Ob}(G)$, $i\colon \mathrm{Ob}(G) \to G$ such that $si = ti = 1$. The functions s, t are sometimes called the *source* and *target* maps respectively. If $a, b \in G$ and $ta = sb$, then a *product* or *composite* ab exists such that $s(ab) = sa$, $t(ab) = tb$. Further, this product is associative; the elements $ix, x \in \mathrm{Ob}(G)$, act as identities; and each element a has an inverse a^{-1} with $s(a^{-1}) = ta$, $t(a^{-1}) = sa$, $aa^{-1} = isa$, $a^{-1}a = ita$. An element a is often written as an arrow $a\colon sa \to ta$.

$$sa \xrightarrow{\;a\;} ta = sb \xrightarrow{\;b\;} tb \qquad\qquad sa \xrightarrow{\;ab\;} tb$$

Groupoids were introduced by Brandt in his 1926 paper [**11**], although he always used the extra condition that for all x, y in $\mathrm{Ob}(G)$ there is an a in G such that $sa = x$, $ta = y$—such a groupoid we nowadays call *connected* or *transitive*. Brandt's definition of groupoid arose out of his work for over thirteen years [**6–10**] on generalising to quaternary quadratic forms a composition of binary quadratic forms due to Gauss [**63**]. Brandt then saw how to use the notion of groupoid in generalising to the non-commutative case the arithmetic of ideals in rings of algebraic integers, replacing the classical finite abelian group by a finite groupoid [**12**]. This latter theory has been considerably generalised and refined by a number of writers—further references may be found in [**85, 104, 128**]. For a recent discussion of the quadratic form problem, see [**81, 91, 92**].

At about the same time as Brandt's work, Loewy [**98**] introduced similar 'compound groups' to describe isomorphisms between conjugate field extensions. His ideas were developed by Baer in [**4**]. The most recent account of the use of groupoids in classical Galois theory seems to be that by Michler in [**105**]. We say more later on the use of groupoids in the Galois theory of rings.

Received 13 February 1985; revised 4 August 1986 and 23 October 1986.

1980 *Mathematics Subject Classification* 20–02, 18B40, 01–02.

This is an expanded version of a talk given at several universities; at the Oberwolfach Category Meeting, 1983; at the Welsh Pure Mathematics Colloquium, Gregynog, 1984; and at the British Mathematical Colloquium, Cambridge, 1985.

Bull. London Math. Soc. 19 (1987) 113–134

The topic of groupoids continued to be known through further work on the ideal theory of non-commutative rings, and the notion of order, particularly by K. Asano (compare [85]). I have heard it remarked that Brandt's axioms for groupoids influenced Eilenberg and MacLane in their definition of a category [55]. As categories became generally accepted in the 1950s, interest in groupoids broadened, since the invertible elements of a small category form a groupoid. The use of groupoids was expanded greatly by Ehresmann from 1950 in various main areas, for example: in fibre bundle theory, with his groupoid EE^{-1} associated to a principal bundle E (this groupoid is also formulated below as $G(E)$); in differential geometry, with the use of the groupoid EE^{-1} for studying higher order connections; in foliation theory, with the groupoid of germs of a pseudo-group; in differential topology, with groupoids of jets; and in his use of groupoids of operators for discussing species of structures and of local structures. For an assessment of the contributions of these ideas to mathematics, the reader should turn to Ehresmann's *Oeuvres complètes et commentées* [54] and the commentaries and comments there. Groupoid techniques in foliation theory were developed by Haefliger [70], and then many others—see the survey [96], the bibliography [66]. This and other uses of topological groupoids are noted in the bibliographies to [21, 22]. Abstract groupoids were applied by Dedecker in a series of papers on non-abelian cohomology (see [47]).

The situation now is that groupoids have been used in a wide variety of areas of mathematics, from ergodic theory and functional analysis to homotopy theory, algebraic geometry, differential geometry, differential topology and group theory. However, this wide and considerable use is not so well-known, even to those using groupoids in their own speciality, and this has perhaps made it easier to form a dismissive attitude. It seems timely therefore to attempt some overall survey.

A complete account of the use of groupoids is out of the question, because the ramifications are so wide. This brief survey is written as something of a personal account, and reflects my own interests. But it does give an opportunity for various threads of uses of groupoids to be drawn together, so that they may be followed to entry points in the literature. I hope that this will give a starting point for readers to perceive and assess past and current uses of groupoids, and so help them to judge the potentiality of new applications.

2. *Examples*

There is always room for argument about whether and how to generalise an algebraic structure, while maintaining both the force of the original motivating examples and the character of the theory. For example, the theories of monoids or of semigroups are dissimilar in many ways to that of groups. We would want to justify the argument that the theory of groupoids does not differ widely in spirit and aims from the theory of groups.

In the theory of groups, two motivating examples are *symmetry groups*, that is, groups of automorphisms, and groups derived from paths in a space, that is, the Poincaré or *fundamental groups*. We find that these examples generalise to give '*symmetry groupoids*' and *fundamental groupoids*. In the latter case, it has been known for at least 40 years that the fundamental groupoid is convenient for handling change of base point for the fundamental group. The recognition of the utility of groupoids for handling ideas of 'variable symmetry' (see Example 4 below) is more recent. Both types of groupoids give gains in flexibility over the corresponding groups, and without any consequent loss.

The following give some of the basic ways in which groupoids arise.

EXAMPLE 1. A disjoint union $G = \amalg_\lambda G_\lambda$ of groups G_λ, $\lambda \in \Lambda$, is a groupoid: the product ab is defined if and only if a, b belong to the same G_λ, and ab is then just the product in the group G_λ. There is an identity 1_λ for each $\lambda \in \Lambda$. The maps s, t coincide and map G_λ to $\{\lambda\}$, $\lambda \in \Lambda$.

EXAMPLE 2. An equivalence relation R on X becomes a groupoid with $s, t\colon R \to X$ the two projections, and product

$$(x,y)(y,z) = (x,z).$$

whenever $(x,y), (y,z) \in R$. There is an identity, namely (x,x), for each $x \in X$. (This example is due to Croisot [**42**].) A special case of this groupoid is the *coarse groupoid* $X \times X$, which is obtained by taking $R = X \times X$. This apparently banal and foolish example is found to play a key role in the theory and applications. At the opposite extreme to the coarse groupoid $X \times X$ is the *fine groupoid* on X; this can be considered as the diagonal equivalence relation on X, or alternatively as the groupoid X consisting only of identities, namely the elements of X.

This consideration of an equivalence relation as a groupoid also suggests the utility of groupoids for studying quotienting constructions, particularly in cases where the quotient set X/R cannot carry the appropriate structure. For a discussion of this in the case of differential manifolds, see [**58**].

EXAMPLE 3. Let the group G operate on the set X on the right. All of us find it convenient to picture such an operation by the diagram

$$x \bullet\!\xrightarrow{\quad g \quad}\!\bullet\, x^g \qquad g \in G,\ x \in X.$$

The arrow here is based at x and so is more accurately labelled (x, g). This suggests defining a product

$$(x,g)(x^g, h) = (x, gh);$$

it is easily checked that this product gives a groupoid with object set X and $s\colon (x,g) \mapsto x$, $t\colon (x,g) \mapsto x^g$. There is no consistent terminology for this groupoid: I like the term *semi-direct product* groupoid, and so the notation $X \rtimes G$, because this groupoid is a special case of the semi-direct product groupoid obtained from an action of a group, or more generally groupoid, on another groupoid [**16**]. A term suggested by Pradines is *actor* groupoid. Note that for this example, there is an identity $(x, 1)$ for each $x \in X$. This construction is due to Ehresmann [**52**].

Thus we find that a set X, a group G, and an action of G on X, can all be considered as examples of groupoids. This common viewpoint is found to be convenient in a variety of areas of mathematics.

EXAMPLE 4. Groups occur naturally as automorphism groups, or symmetry groups, of various structures, and this is a fundamental observation behind Klein's famous Erlangen Programme: *study a geometry by means of its group of automor-*

phisms. It has more recently been found fruitful to consider not just one geometry, or one structure, but indexed families $E = \{E_x\}_{x \in B}$ of structures, often thought of as constituting a 'bundle' E over B, with projection $p: E \to B$, and with $E_x = p^{-1}(x)$. The 'symmetry' of such a gadget is appropriately expressed by the groupoid $G(E)$, with object set B, and with elements consisting of all isomorphisms $E_x \to E_y$ for all $x, y \in B$.

For x in B, the group $G(E_x)$ of automorphisms of E_x expresses the 'symmetry' of E_x. These 'varying symmetries' are encompassed in the groupoid $G(E)$. An isomorphism $E_x \to E_y$ allows one to define an isomorphism $G(E_x) \to G(E_y)$ of groups, and so gives 'transport of symmetry'. Perhaps $G(E)$ should be called a *symmetry groupoid.*

This idea is at the root of many applications of groupoids pioneered by Ehresmann in differential geometry **[54]**. If $p: E \to B$ is a principal bundle with group H, then $G(E)$ is to consist of the *admissible maps* $E_x \to E_y$, $x, y \in B$, that is, the homeomorphisms commuting with the action of the group H. If $p: E \to B$ is locally trivial, then the trivialisations determine a topology on $G(E)$, or even, in the differentiable case, a differential structure. For references to the literature in this area, see **[21, 22, 99]**.

The use of groupoids for studying order–disorder structures in crystals **[48, 59]** suggests further possibilities for the general analysis of 'variable symmetry'—see **[60]** for a recent article.

Another application occurs in the theory of formal groups and is due to Landweber **[95]**. For any augmented, supplemented, commutative algebra A over a field k, the set of isomorphisms of formal groups over A forms a groupoid FGL(A). This defines a functor FGL from the category $\mathbf{Alg}_k$ of such algebras over k to the category of groupoids. It is important that this functor is representable:

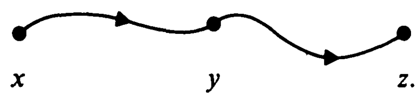
$$\mathrm{FGL}(A) \cong \mathbf{Alg}_k(P, A).$$

The fact that FGL(A) is a groupoid gives the algebra P the structure of *Hopf algebroid.* The term is due to H. Miller, and the book **[122]** gives a good account of the uses in stable homotopy theory, also developed by Morava **[108]**. In fact a more descriptive term would be 'Hopfoid algebra' since it is the diagonal map $A \to A \otimes A$ of the usual Hopf algebra which is generalised from a cogroup to a cogroupoid structure (**[122]**, pp. 306–307]). Further, their term 'algebroid' has also been used by B. Mitchell since 1972 for an 'algebra with many objects'—for a recent paper see **[106]**.

A recent application of groupoids is in combinatorics by Joyal **[86]** using *species* (French: *espèces*, German: *Gattungen*) *of structure*. The term is due to Bourbaki **[5]**; its aim is to give a general description of the kind of structures which occur in mathematics, so there are species of structure of order, of topology, of vector space, of complex analytic manifold of dimension n, and so on. In particular, if E is a set then there is a set $M(E)$ of structures of a given species M on E. An important property of species of structures is *transportability*—if $t: E \to F$ is a bijection of sets, then t induces a bijection $M(E) \to M(F)$ of the structures of a given species. This idea is abstracted by Ehresmann **[52]** using the notion of a category operating on a set. Joyal **[86]** follows the spirit of Ehresmann's work, but in less generality, and defines a *species* to be simply an endofunctor $M: B \to B$, where B is the groupoid of finite sets and bijections between them. He defines two generating series for a species, and relates constructions on such generating series such as product, sum and substitution, to categorical constructions on species. The point is that in combinatorics one often wants to compute the number of structures of species M on the standard set

$[n] = \{1, \ldots, n\}$. However, in carrying out an argument the standard set $[n]$ may appear in a non-standard way, for example as $X = \{x_1, \ldots, x_n\}$. A relation between $M(X)$ and $M([n])$ is determined by a non-canonical bijection $X \to [n]$. The use of species enables one to keep track of all these non-standard forms of the standard set, together with their labellings. For further work, see also [**109**].

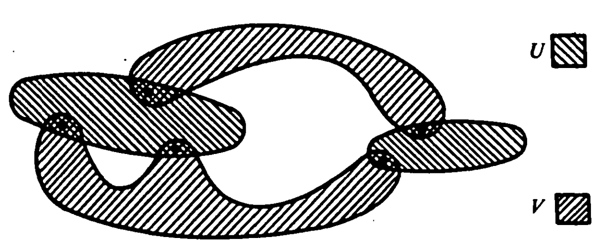
EXAMPLE 5. The fundamental, or Poincaré, group $\pi_1(X, x)$ of a space X with base point x is well-known. However, there are several pressures to replace the base point x by a set A of base points in X, where A could be X itself, and so obtain the *fundamental groupoid* $\pi_1(X, A)$ on the set A. The identities of this groupoid correspond to the elements of A, and an 'arrow' $x \to y$ is a homotopy class, relative to the end points, of paths $(I, 0, 1) \to (X, x, y)$, with product induced by the usual composition of paths:



3. *Applications of the fundamental groupoid*

My own introduction to the use of groupoids came with this last example, in 1965. I was writing a topology text [**14**], which was to include the Van Kampen Theorem on the fundamental group of a union of spaces. I wanted a version of this theorem which would imply the determination of the fundamental group of the circle, and was dissatisfied with the length and tedium of my then current exposition, using non-abelian cohomology. I came across the paper [**76**] of P. J. Higgins which defined presentations and also free products with amalgamation of groupoids. This suggested inserting an exercise on expressing the fundamental groupoid $\pi_1 X = \pi_1(X, X)$ as a free product of the groupoids $\pi_1 U$, $\pi_1 V$, amalgamated over $\pi_1 W$, when X is the union of open sets U, V with intersection W. It then seemed desirable to write out a solution to the exercise; to my surprise, the solution had the qualities of clarity and concision which I had hoped for, but had been unable to obtain, in my previous version!

The problem addressed by Van Kampen in 1935 [**143**] would be expressed in modern form as follows. The space X is given to be the union of open sets U, V with intersection W: *determine the fundamental group* $\pi_1(X, x)$ *in terms of information on* U, V, W *and the inclusions* $W \to U$, $W \to V$. Notice that Van Kampen did not assume, as did Seifert in an earlier result for simplicial complexes,that U, V, W are connected; so a typical diagram for the situation could be:



The difficulty is: where should we put the base point x? It seems reasonable to take x in W, but in which of the many components of W should x lie? One way of coping with such a problem of decision is to avoid it altogether. So we choose a set A of base points, one point in each component of W, and attempt to compute the fundamental groupoid $\pi_1(X, A)$. This strategy of avoiding decision turns out to be optimal: *the groupoid $\pi_1(X, A)$ is the free product of the groupoids $\pi_1(U, A)$ and $\pi_1(V, A)$ amalgamated over $\pi_1(W, A)$*. From this, one can in principle compute the group $\pi_1(X, x)$, by choosing trees in each component of $\pi_1(U, A)$ and of $\pi_1(V, A)$. These choices lead to the particular formulae written down by Van Kampen (compare [**13**; **14**, p. 289, Exercise 4]). Also, the proof [**14**] of the determination of $\pi_1(X, A)$ is simpler than previous proofs of the 'Van Kampen Theorem' for U, V, W connected. So one obtains a simpler proof of a more powerful theorem; which can't be all bad. The most general formulation to date of this theorem on the fundamental groupoid is in [**36**].

Other texts which have followed this approach are [**39, 78, 160**]. Somewhat earlier, Crowell and Fox in [**43**, p. 153] took the view that a few definitions 'like that of a group, or a topological space, have a fundamental importance for the whole of mathematics that can hardly be exaggerated. Others are more in the nature of convenient, and often highly specialised, labels which serve principally to pigeonhole ideas. As far as this book is concerned, the notions of category and groupoid belong in this latter class. It is an interesting curiosity that they provide a convenient systematisation of the ideas involved in developing the fundamental group.' It is this kind of viewpoint, emphasising the algebra we know rather than that which might evolve, which perhaps has led people to fail to see properly the advantages of an algebra which models the geometry more appropriately than the usual algebra of groups. However, the earliest use of the term 'fundamental groupoid' which I have been able to find is in Fox's paper [**61**].

The difficulty there can be in seeing that the groupoid approach provides not only a conceptual tool, but one which guides specific calculations, is shown in some remarks from A. Grothendieck's discursive venture towards a non-abelian cohomology theory, from which it is worth quoting at length ([**68**, p. 194–195]):

> From Y. who looked through a lot of literature on the subject, it strikes me (agreeably of course) that nobody yet hit upon 'the' natural presentation of the Teichmüller groupoids, which kind of imposes itself quite forcibly in the set-up I let myself be guided by. Technically speaking (and this will rejoice Ronnie Brown I'm sure!), I suspect one main reason why this is so, is that people are accustomed to work with fundamental groups and generators and relations for these and stick to it, even in contexts when this is wholly inadequate, namely when you get a clear description by generators and relations only when working simultaneously with a whole bunch of base-points chosen with care—or equivalently working in the algebraic context of *groupoids*, rather than groups. Choosing paths for connecting the basepoints natural to the situation to one among them, and reducing the groupoid to a single group, will then hopelessly destroy the structure and inner symmetries of the situation, and result in a mess of generators and relations no one dares to write down, because everyone feels they won't be of any use whatever, and just confuse the picture rather than clarify it. I have known such perplexity myself a long time ago, namely in Van Kampen type situations, whose only understandable formulation is in terms of (amalgamated sums of) groupoids. Still, standing habits of thought are very strong, and during the long march through Galois theory, two years ago, it took me weeks and months trying to formulate everything in terms of groups or

'exterior groups' (i.e. groups 'up to inner automorphism'), and finally learning the lesson and letting myself be convinced progressively, not to say reluctantly, that groupoids only would fit nicely. Another 'technical point' of course is the basic fact (and the wealth of intuitions accompanying it) that the Teichmüller groups are fundamental groups indeed,—a fact ignored it seems by most geometers, because the natural 'spaces' they are fundamental groups of are not topological spaces, but the modular 'multiplicities' $M_{g,\nu}$—namely *topoi*! The 'points' of these 'spaces' are just the structures being investigated (namely algebraic curves of type (g,ν)), and the (finite) automorphism groups of these 'points' enter into the picture in a very crucial way. They can be adequately chosen as part of the system of basic generators for the Teichmüller groupoid $T_{g,\nu}$. The latter of course is essentially (up to suitable restriction of base-points) just the fundamental groupoid of $M_{g,\nu}$. It is through this interpretation of the Teichmüller groups or groupoids that it became clear that the profinite Galois group $\mathrm{Gal}_{\bar{Q}/Q}$ operates on the profinite completion of these and of their various variants, and this (it turns out) in a way respecting the manifold structures and relationships tying them tightly together.

Another use of the groupoid $\pi_1(X, A)$ is due to P. J. Higgins and J. Taylor [**80**]. Let the discrete group G act on the space X. The problem is: *compute the fundamental group $\pi_1(X/G, x)$ of the orbit space X/G.* As an example, let X be the unit circle of complex numbers z with $|z| = 1$, let $G = \mathbb{Z}_2$, and let the action be reflection $z \mapsto \bar{z}$. Then the orbit space X/G is essentially a semicircle, so that $\pi_1(X/G, 1) = 1$. However, $\pi_1(X, 1) = \mathbb{Z}$, with action $n \mapsto -n$, so that the quotient of $\mathbb{Z}$ by this action is $\mathbb{Z}_2$, and not 1. Why has this approach given the wrong answer?

Notice that in this example we chose as base point the complex number 1. But the geometry of the action makes no distinction between $+1$ and -1, and these are the only fixed points. So we had better avoid a decision, and, using the induced action of $\mathbb{Z}_2$, consider the quotient not of the group $\pi_1(X, 1)$ but of the groupoid $\pi_1(X, \{\pm 1\})$. This quotient groupoid does have trivial vertex groups, as required.

More generally, for 'reasonable' actions of a group G on a CW-complex X, we have an isomorphism

$$\pi_1(X, A)/G \cong \pi_1(X/G, A/G),$$

provided that A is G-invariant, so that A/G is defined, and that A meets each component of the fixed point set of each element of G [**80**].

The 'orbit groupoid' $\pi_1(X, A)/G$ of the groupoid $\pi_1(X, A)$ with the action of the group G is, of course, obtained from the groupoid $\pi_1(X, A)$ by imposing the relations $\alpha g = \alpha$ for all $\alpha \in \pi_1(X, A)$, $g \in G$. This makes sense because presentations for groupoids can be defined in a similar manner to presentations of groups [**76**]. An explicit construction of this orbit groupoid is more complicated [**80**]—it is a quotient groupoid of the semi-direct product $\pi_1(X, A) \rtimes G$.

Another application to group actions is for a proof of a theorem of Macbeath and Swan giving an exact sequence of the form

$$1 \longrightarrow N \longrightarrow \pi_1(X, x) \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

Here G is a group acting on the space X, which is assumed to have a path-connected open set V containing x whose translates by G cover X (thus V is a *fundamental domain* for the action). The group Γ has generators $[g]$ for $g \in G$ such that $V \cap gV \neq \varnothing$, and relations $[gh] = [g][h]$ whenever $g, h \in G$ and $V \cap gV \cap hgV \neq \varnothing$. The papers **[1, 123]** show that this theorem is related to a description of the fundamental groupoids of the nerve and of the classifying space of the cover $\{gV: g \in G\}$ of X.

There is a subtle question of the description of an orbit space of a manifold under the action of a pseudo-group, and the definition of a suitable concept of 'geometric' fundamental group, different from the generally uninteresting topological one. This is discussed by van Est in **[58]**. The rôle of the fundamental groupoid in this situation is emphasised in **[58]** and **[118]**. The latter paper characterises the fundamental groupoid by a suitable universal property among the groupoids which are 'locally coarse', there called 'Galois groupoids'. Related work is in **[144]**.

4. *The category of groupoids*

It is now time to say more about the formal, algebraic properties of groupoids.

A *homomorphism* of groupoids G, H is essentially a *functor*, that is, it consists of a pair of functions $f: G \to H$, $\mathrm{Ob}(f): \mathrm{Ob}(G) \to \mathrm{Ob}(H)$, preserving all the structure. So one obtains a category **Gpd** of groupoids and homomorphisms.

If G is a groupoid, and $x, y \in \mathrm{Ob}(G)$, then we write $G(x, y)$ for the set of elements a in G with $sa = x$, $ta = y$, and we write $G(x)$ for $G(x, x)$. The product on G restricts to a group structure on $G(x)$, and this group is called the *object group* or *vertex group* of G at x. The groupoid G is *transitive* or *connected* if $G(x, y) \neq \varnothing$ for all x, y in $\mathrm{Ob}(G)$. In this case, the groups $G(x)$ are all isomorphic, and indeed are conjugate, in the obvious sense, in G. In order to emphasise the topological analogy, some authors like to write $\pi_1(G, x)$ for the vertex group $G(x)$.

The definition of *subgroupoid* presents no problem and we assume it is understood. A subgroupoid need not contain all objects and indeed may be empty. This contrasts with the usage for subgroups. The maximal, transitive subgroupoids of a groupoid G are called the *components* of G. The set of components of G is often written $\pi_0 G$. Note that in the case of the groupoid $X \rtimes G$ of Example 3, the vertex groups are essentially the groups of stability of the action, and the components are essentially the orbits of the action.

The classification of groupoids up to isomorphism was early on found to be reducible to the classification of groups. First, any groupoid G is clearly the disjoint union of its components. Second, if G is transitive, and $x \in \mathrm{Ob}(G)$, then there is a *non-canonical* isomorphism of G to the product of the group $G(x)$ and the coarse groupoid on $\mathrm{Ob}(G)$:

$$G \cong G(x) \times (\mathrm{Ob}(G) \times \mathrm{Ob}(G)). \qquad (*)$$

Such an isomorphism is obtained by choosing elements $\tau_y \in G(x, y)$ for all $y \in \mathrm{Ob}(G)$ and then sending

$$a \mapsto (\tau_y a \tau_z^{-1}, (y, z)), \quad y, z \in \mathrm{Ob}(G),\ a \in G(y, z).$$

The isomorphism (*) gives also a retraction $G \to G(x)$; the existence of (*) is a basic fact on groupoids.

The isomorphism (*) should not lead mathematicians to draw the moral that 'groupoids reduce to groups'. Indeed, it can be pointed out that we have long passed the day when the classification of objects up to isomorphism could be considered the sole purpose of mathematics. For example, finite dimensional, real vector spaces V are classified up to isomorphism by the number $n = \dim V$. This does not mean that the term 'real vector space' can be conveniently excised from the mathematical literature! As one illustration of this, the classification of real vector spaces with p endomorphisms is interesting for $p = 1$, difficult for $p = 2$, and unsolved for $p = 3$. (I am grateful to A. Heller for this trenchant expression of view.)

In a similar manner to the use of vector spaces, one finds that it is in studying morphisms of groupoids, and the relationships between various groupoids, that the theory of groupoids obtains its power and flexibility. One of the features of groupoids is the variety of types of homomorphisms. For groups, we have basically monomorphisms, epimorphisms, isomorphisms. For groupoid homomorphisms we have similar terminology to that for functors, namely faithful, full, representative, and also a variety of other types such as quotient, universal, covering [78], fibration, and discrete kernel [15]. See [45] for a discussion of congruences in groupoids. It may disturb people to learn that the first isomorphism theorem fails for groupoids. But in fact these apparent difficulties and complications lead to a theory richer than that of groups, and with wider uses.

To determine the fundamental group $\pi_1(X,x)$, $x \in A$, in the Van Kampen situation considered above, one has to use the isomorphism (*) on each component of $\pi_1(U,A)$, and of $\pi_1(V,A)$, and analyse the effect of all the choices that have been made. This technique of making various appropriate choices of isomorphisms of the type of (*) is a basic tool in P. J. Higgins's applications of groupoids to subgroup theorems in group theory [78]. (Similar methods were used earlier by Hasse [72].) For example, in the proof that a subgroup of a free group G on X is free, the isomorphism (*) is obtained from the choice of a maximal tree in a generating graph $\tilde{X}$ for a free groupoid $\tilde{G}$ covering G (see below); the choice of this tree in $\tilde{X}$ is equivalent to the classical choice of a Schreier transversal.

It seems fair to suggest that these methods give the first real applications of a *theory* of groupoids—the earlier applications to the arithmetic of ideals seem by contrast only descriptive, and once the groupoid of ideals has been obtained, not too much is done with it.

Consider again the coarse groupoid $X \times X$, and the particular case when $X = \{0, 1\}$. The groupoid $\mathscr{I} = \{0,1\} \times \{0,1\}$ has two objects 0 and 1, and non-identity arrows $\iota: 0 \to 1$ and $\iota^{-1}: 1 \to 0$, say. Its vertex groups are trivial. So we can think of $\mathscr{I}$ as consisting of two distinct but trivial groups, and the unique isomorphism ι between them! Note that if a is an element of a groupoid G, then there is a unique homomorphism $f: \mathscr{I} \to G$ of groupoids such that $f(\iota) = a$; so $\mathscr{I}$ plays for groupoids the role that the infinite cyclic group $\mathbb{Z}$ plays for groups. Homomorphisms $\mathscr{I} \to G$, for G a finite cyclic group, give easy examples of the failure of the usual isomorphism theorems of group theory.

Another feature of $\mathscr{I}$ is that, with the two inclusions $\{0\} \to \mathscr{I}$, $\{1\} \to \mathscr{I}$, it has properties analogous to the unit interval in the homotopy theory of spaces. So it is easy to write down a corresponding homotopy theory for groupoids, with notions of homotopy equivalence, covering morphism, fibration, exact sequence, and so on [**15**, **88**, **89**]. As an application, the basic results on covering spaces can be summarised as saying that for reasonable spaces X there are equivalences of categories (compare [**62**, Appendix 1; **15**])

$$\begin{aligned}\text{(covering spaces of } X) &\sim \text{(covering groupoids of } \pi_1 X)\\ &\sim \text{(operations of } \pi_1 X \text{ on sets)}\\ &\sim \text{(functors } \pi_1 X \to \text{(sets))}.\end{aligned}$$

(These last two equivalences are essentially due to Ehresmann—compare [**54**, Partie II-1 Comment 129.2].) Indeed the construction of covering spaces is nicely expressed in terms of the problem of topologising the object set of a covering groupoid G of the fundamental groupoid $\pi_1 X$ [**14**]. Fibrations of groupoids [**15**] occur naturally in a number of ways in group or group action theory; the resulting exact sequences give results on the original group theoretic situation [**74, 25, 26**]. We should also refer to the neglected paper by P. A. Smith [**137**] where a covering morphism is called a *regular* homomorphism.

One of the irritations of group theory is that the set $\operatorname{Hom}(H, K)$ of homomorphisms between groups H, K does not have a natural group structure. However, homotopies between homomorphisms of groupoids H, K may be composed to give a groupoid $\operatorname{HOM}(H, K)$ with object set $\operatorname{Hom}(H, K)$. It is easily checked that for any groupoids G, H, K, there is a natural bijection

$$\operatorname{Hom}(G \times H, K) \cong \operatorname{Hom}(G, \operatorname{HOM}(H, K)).$$

This bijection is part of a groupoid isomorphism

$$\operatorname{HOM}(G \times H, K) \cong \operatorname{HOM}(G, \operatorname{HOM}(H, K)).$$

This isomorphism is useful even when G, H, K are groups. It has a generalisation to the case of groupoids over a given groupoid [**40, 84**].

An application of this generalisation is pointed out in [**24**], as follows. Let $f\colon A \to B$ be an epimorphism of groups. Suppose B has a presentation $B = \operatorname{colim}_\lambda B_\lambda$ as a colimit over a connected diagram. Let $A_\lambda \to A$ be the pullback of the canonical map $B_\lambda \to B$ by f. Then the canonical map $\operatorname{colim}_\lambda A_\lambda \to A$ is an isomorphism. It is not easy to see how this result can be proved within the framework of group theory.

As another application of the groupoid $\mathscr{I}$, note that if A is a group, then the groupoid $A \times \mathscr{I}$ can be regarded as consisting of two copies of A and an isomorphism between them. An HNN-extension of groups $G*_\theta$, where θ is an isomorphism $A \to B$ of subgroups of G, can be described as an amalgamated sum (pushout) of groupoids

$$\begin{tikzcd}
A \times \{0, 1\} \arrow[r, "\psi"] \arrow[d] & G \arrow[d] \\
A \times \mathscr{I} \arrow[r] & G*_\theta,
\end{tikzcd}$$

where ψ sends $(a, 0) \mapsto a$, $(a, 1) \mapsto \theta a$.

Thus the groupoid $\mathscr{I}$, which at first sight seems unworthy of notice, plays a key role in the theory of groupoids, and in applications. A failure to extend group theory so as to include the use of $\mathscr{I}$, on the grounds that $\mathscr{I}$ is a trivial object of only formal interest, is analogous to failing to use the number 0 in arithmetic, a failure which in fact held back mathematics for centuries. Of course, if you allow $\mathscr{I}$, then in effect you allow all groupoids since any groupoid is a colimit of a diagram of copies of $\mathscr{I}$, in the same way as any group is a colimit of a diagram of copies of $\mathbb{Z}$.

5. *Some applications*

As explained above, special cases of groupoids are sets, groups, and equivalence relations. These have wide applications! It is not so well-known how widespread are the uses of groupoids over and above these examples. Here we indicate some of these uses; a wider impression is given in the references.

Groupoids were brought into the Galois theory of rings by Villamayor and Zelinsky [**145, 146**]. The following quotation is from [**145**, p. 722]: 'Our approach actually interposes between a subgroup and its associated fixed ring a certain groupoid composed of all the isomorphisms between components of S that can be induced by automorphisms in the subgroup. The standard group-to-algebra correspondence is split into the composite of a many-to-one correspondence from groups of automorphisms to groupoids of isomorphisms, followed by a one-to-one correspondence from groupoids to algebras. The correspondence group $\to$ groupoid is one-to-one exactly on the fat subgroups of the automorphism group.' See also [**101, 102**].

The semi-direct product groupoid $X \rtimes G$ associated to a group action of G on X arises in combinatorial group theory, particularly for subgroup theorems. If H is a subgroup of a group G, then in general the set G/H of right cosets has no canonical group structure. However, G operates on the set G/H and so the semi-direct product $\tilde{G} = (G/H) \rtimes G$ can be formed. Its vertex group $\tilde{G}(H)$ at the coset H is isomorphic to H. So the strategy for subgroup theorems is to lift a presentation of the group G to a presentation of the groupoid $\tilde{G}$, and then to choose a retraction $\tilde{G} \to G(H)$, in a manner appropriate to the presentation of $\tilde{G}$, to obtain a presentation of $\tilde{G}(H)$ and so of H [**44, 78**]. This strategy also gives results for topological groups [**23, 110**].

Another use of $X \rtimes G$ is in ergodic theory. G. W. Mackey describes in [**100**] his route to the use of groupoids. The starting point was the question: granted that a transitive action of a group G on a set X corresponds to a conjugacy class of subgroups of G, what then corresponds to an ergodic action of G on X? He invented the term 'virtual subgroup', and this concept or analogy was finally expressed in terms of the groupoid $X \rtimes G$ described above. In order for the action to be ergodic, G and X must have Borel and measure structures, and these structures are inherited by $X \rtimes G$. The idea of conjugacy class is expressed by a definition of equivalence in which homomorphisms defined almost everywhere are also allowed. This study now has an extensive literature of which [**119, 120, 121, 151**] is a selection. Mackey told me of his work after I had given a talk on groupoids at the British Mathematical Colloquium in 1967. This meeting suggested to me that the groupoid concept had much more to it than I had envisaged, and so was a spur to further work.

Topological groupoids have a theory of *Haar measure*, or transverse measure, which was considered by my student A. K. Seda in his 1974 Ph.D. thesis [**132**] and by a number of other writers (compare [**124, 125**]). There is for such groupoids a notion of convolution algebra, and the resulting C^*-algebras have been powerfully exploited by A. Connes and others [**41**]. For example, they lead to an index theorem for foliations, generalising the Atiyah–Singer index theorem. The Introduction to [**90**] gives a succinct summary of the uses of groupoids in Connes' theory. J. Renault writes [**127**] some comments on the history of convolution algebras: 'They seem to be as old as operator algebras themselves. Earlier examples by von Neumann included not only group algebras but also the group measure construction. In "Harmonic analysis on groupoids" ([**148**]) J. Westman makes reference to earlier examples by Dixmier and by Glimm. In fact A. Connes likes to say that Heisenberg discovered matrix algebra by staring at the Ritz combination principle for spectral maps—an example of groupoid composition law in contrast with the group law of harmonics.'

It is quite possible to have a topological groupoid G with a non-discrete topology but for which each vertex group has the discrete topology. This is common for example with groupoids of germs with the sheaf topology. Thus the 'variable symmetry' described by G with its topology is in no way encompassed by the family of vertex groups. Also, such a groupoid need not be topologically the sum of its abstract components.

The use of groupoids runs through much of the corpus of Grothendieck's work on algebraic geometry. See [**87**] for one aspect of this, the fact that an *étendue*, which is a kind of generalised space, may also be described in terms of actions of a groupoid. This is related to work of Magid [**101, 102**].

There are some crucial differences between the theory of topological groupoids and that of topological groups. As one example, for a group G, a topology on G making G a topological group is defined by a fundamental system of neighbourhoods of the identity, satisfying suitable conditions. The reason is that in a topological group left translation by an element maps open sets to open sets. This is no longer true in a topological groupoid G. As observed by Ehresmann [**54**], it is left translation by a *local section* of G which maps open sets to open sets, where a local section σ of G is a map $\sigma\colon U \to G$ where U is open in $\mathrm{Ob}(G)$, $s\sigma(x) = x$, $x \in U$, and $t\sigma$ maps U homeomorphically to an open set V of $\mathrm{Ob}(G)$. Pradines has observed (private communication) that it is this fact which leads to the holonomy groupoid of a differential piece of a groupoid (as announced in [**114**], which also includes results on monodromy groupoids). Diverse uses of differential groupoids have been surveyed by Pradines in [**115**] and an account of uses in differential geometry is given by Mackenzie in [**99**]. For some applications of groupoids in the framework of synthetic differential geometry, see [**93, 94**].

The papers [**21, 22**] give a bibliography of over 80 papers on topological and differential groupoids. A major area of application of topological groupoids is, following Ehresmann, to foliation theory, using either the holonomy groupoid, or the classifying space (see below) $B\Gamma$ where Γ is the topological groupoid of germs of elements of a pseudo-group. This area deserves a complete survey of its own, but here we mention also the Bibliography [**66**], [**144**] and the articles in [**116**], for example [**58, 71**].

6. *The classifying space of a topological groupoid*

The *nerve* of a small category C is the simplicial set NC such that $N_n C$ is the set of functors $\{0, 1, \dots, n\} \to C$, where $\{0, 1, \dots, n\}$ is regarded as the category of pairs (i,j) where $i \leqslant j$ and composition is $(i,j)(j,k) = (i,k)$. This definition is due to Grothendieck, who also characterised simplicial sets of the form NC. If G is a groupoid, then G is also a category, and so its nerve NG is defined. (In fact NG has more structure, namely it is a 'simplicial T-complex of rank 1', as shown by Dakin [**46**]. See also Ashley [**2**].)

The geometric realisation $|NG|$ of the nerve of the groupoid G is called the *classifying space* BG of the groupoid G. It is a CW-complex, with one vertex for each element of $\mathrm{Ob}(G)$, one component for each component of G, and the fundamental group $\pi_1(BG, x)$, $x \in \mathrm{Ob}(G)$ is isomorphic to the vertex group $G(x)$. Further,

$\pi_1(BG, x) = 0$, $i > 1$. It is well-known that if X is any CW-complex then there is a natural bijection

$$[X, BG] \cong [\pi_1 X, G]$$

between the set of (free) homotopy classes of maps $X \to BG$ and the conjugacy classes of homomorphisms of groupoids $\pi_1 X \to G$.

This formula allows for a neat proof of a result of Gottlieb [67]. Let Y be a finite CW-complex, and let $(BG)^Y$ denote the space of (unpointed) maps $Y \to BG$. Then for any CW-complex X there is a sequence of natural bijections

$$\begin{aligned}[X, (BG)^Y] &\cong [X \times Y, BG] \\ &\cong [\pi_1(X \times Y), G] \\ &\cong [\pi_1 X \times \pi_1 Y, G] \\ &\cong [\pi_1 X, \mathrm{HOM}(\pi_1 Y, G)] \\ &\cong [X, B(\mathrm{HOM}(\pi_1 Y, G))].\end{aligned}$$

It follows that $(BG)^Y$ is of the homotopy type of $B(\mathrm{HOM}(\pi_1 Y, G))$. Note that if Y is connected, G is a group and $f\colon Y \to BG$ is a pointed map, then the vertex group of $\mathrm{HOM}(\pi_1 Y, G)$ at f_* is the centraliser of $f_*(\pi_1(Y, y))$ in G, which is the result of [67].

If G is a topological groupoid, then its nerve NG becomes a simplicial space. The realisation $BG = |NG|$ is still defined, but is no more a CW-complex [136].

The applications of this classifying space are legion. In the case G is a topological group, BG classifies principal bundles with group G. We mention some uses of the groupoid cases. When G is the groupoid of germs arising from a pseudo-group Γ, BG then classifies Γ-structures (see [66, 70, 96]). Also, the cohomology of $B\Gamma$ gives rise to characteristic classes for foliations [66].

If $G = X \rtimes H$, the semi-direct product topological groupoid arising from an action of the topological group H on the topological space X, then BG is also known as the homotopy limit of the action [140]. It is known that BG is of the homotopy type of the space $X \times_H PH$, where $PH \to BH$ is a universal principal H-bundle. The *equivariant cohomology* of the H-space X is defined to be $H^*(X \times_H PH)$ [154], and is thus simply $H^*(BG)$ (compare [136, 153, 158]).

In [51] it is proved that if π is a finite p-group, and G is a compact Lie group, then $\mathrm{HOM}(\pi, G)$, with its structure as Lie groupoid, has the property that the natural map $B(\mathrm{HOM}(\pi, G)) \to (BG)^{B\pi}$ is a strong mod p equivalence. In [152] it is proved that B maps certain pushouts of topological groupoids or categories to homotopy pushouts, and this result includes some classical ones, such as descriptions of $\Omega\Sigma X$ in terms of free topological monoids or free topological groups.

7. *Structured groupoids*

We have already met groupoids with various additional elements of structure. In order to describe uniformly these various kinds of structured groupoid, it is convenient to define a *groupoid object G internal to a category $\mathscr{C}$*. The definition is analogous to that given above for a groupoid in the category of sets, except that G, $\mathrm{Ob}(G)$ are to be objects of $\mathscr{C}$, while s, t, i, the inverse map, and the multiplication, m, are all to be morphisms of $\mathscr{C}$, where m is defined on the pull-back of s, t, which is assumed to exist. The axioms for a groupoid are expressed in a standard way using diagrams in $\mathscr{C}$. For example, one finds in [3, 64] the definition of an *algebraic groupoid* as a groupoid object in the category of algebraic spaces. In fact the general notions of structured category and structured groupoid were defined and developed as long ago as 1963 by Ehresmann [**54**]. See also [**155, 156**].

The Examples 1, 2, 3 of § 1 will normally transfer to the structured situation. For example, a set (topological space, differential manifold, algebraic space) X gives rise to a coarse groupoid (topological groupoid, differential groupoid, algebraic groupoid) $X \times X$.

For a structured groupoid G there is no reason for there to be an isomorphism (*) which preserves the structure. If such exists, the structured groupoid is called *trivial*, in analogy with a trivial principal bundle. In the topological and differential case, there are useful notions of *local triviality* but there are also lots of good examples of differential groupoids without this property. It is usually the case that constructions for manifolds extend to locally trivial differential groupoids, and, although they may not extend to all differential groupoids, the attempt to do so usually leads to interesting questions.

A major source of examples of structured groupoids in Ehresmann's work was from topology and differential topology, so giving rise to topological groupoids and differential groupoids. This interest was paralleled in the Soviet Union, though at a later date, in the work of A. V. Vagner [**141, 142**]. They were both interested in the relations between groupoids and what the geometers called *pseudo-groups*—these were called *generalised groups* by Vagner, but among semigroup theorists these are called, following Petrich, *inverse semigroups*. An inverse semigroup defines in a natural way a groupoid with an additional partial order structure—these were called *inductive groupoids* by Ehresmann [**52**]. See [**129**] for an account of the relations between groupoids and inverse semigroups, and [**130**] for a survey of the relations between abstract inverse semigroups and those arising from sets of partial transformations.

Localic groupoids are central to topos theory. Here a localic groupoid is a groupoid object in the category of locales, a category which generalises the category of lattices of open sets of topological spaces and the maps f^{-1} induced by continuous maps f. For every topos $\mathscr{E}$ there is a localic groupoid G such that $\mathscr{E}$ is equivalent to the category of étale spaces E over $\mathrm{Ob}(G)$ together with a continuous action of G on E over $\mathrm{Ob}(G)$. If the topos has enough points (as do most of the toposes arising in algebraic geometry, for instance), G can in fact to be taken to be a topological group. Moerdijk shows in [**107**] that this representation of a topos can be extended to maps of toposes, which makes the category of toposes a category of fractions of a category of localic groupoids.

Ehresmann's work on structured categories and groupoids also led to notions of categories structured by categories, that is, to double categories, and so to n-tuple categories [**54**]. The existence of such definitions, and the basic example of a double category, namely the double category $\square\mathscr{C}$ of commuting squares in a category $\mathscr{C}$, were important to the writer in 1965–72 in contemplating the possibility of extending the Van Kampen Theorem to dimension 2.

It is clear that the definition of a groupoid object makes sense in any category with pullbacks, and this includes many standard categories of an algebraic character in the usual sense, for example such categories as those of groups, rings (without 1), Lie algebras, and many others. So we can consider a set with two compatible structures, one a groupoid structure, and the other, for example, a ring structure.

Here again we see a complete contrast between groups and groupoids. A group internal to the category of groups is just an abelian group—this is a well-known fact which leads to the suggestion that a 'higher dimensional group theory', based on intuitive ideas of composing squares or cubes instead of paths, cannot exist. Similarly, a group object internal to rings is a ring with zero multiplication. In general, group objects internal to the standard algebraic categories are 'abelian' in some way.

The situation is quite different for groupoid objects. A result published in [37], but known much earlier, is that a groupoid object internal to groups, which we call here a *cat¹-group*, is equivalent to a *crossed module*, which is a homomorphism $\mu\colon M \to P$ of groups, together with an action $(m,p) \mapsto m^p$ of P on M satisfying the two rules:

$$\text{(CM1)} \quad \mu(m^p) = p^{-1}(\mu m)p,$$

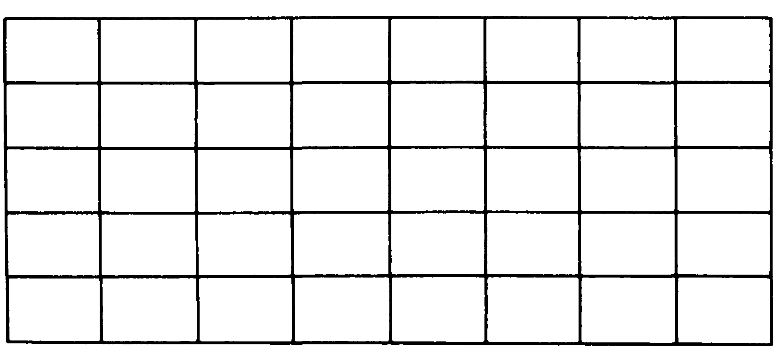
$$\text{(CM2)} \quad m^{-1}nm = n^{\mu m},$$

for all $m, n \in M, p \in P$. Examples of crossed modules are: an ordinary P-module, when $\mu = 0$; a normal subgroup, when μ is an inclusion; the inner automorphism map $\chi\colon M \to \operatorname{Aut} M$, for any group M; any epimorphism $\mu\colon M \to P$ with central kernel; and the map of fundamental groups $\pi_1 F \to \pi_1 E$ for any (pointed) fibration $F \to E \to B$. So there are lots of good examples of crossed modules. The notion is due to J. H. C. Whitehead [**149**], the name being used in [**150**]. Surveys of their use and relationships with classical notions of homotopy theory and homological algebra are given in [**17, 18, 30, 31**].

The equivalence between crossed modules and cat¹-groups is given as follows. Let $\mu\colon M \to P$ be a crossed module. Let $G = P \rtimes M$ be the semi-direct product group, using the action of P on M, and let $s, t\colon G \to P$ be the maps $(p, m) \mapsto p$, $(p, m) \mapsto p(\mu m)$ respectively. Condition (CM1) on a crossed module is equivalent to t being a homomorphism of groups. The formula $g \circ h = g(sh)^{-1}h$, when $tg = sh$, $g, h \in G$, defines a category structure on G with s, t as initial and final maps. This category structure is compatible with the group structure if and only if (CM1) and (CM2) hold. This compatibility condition is also equivalent to $[\operatorname{Ker} s, \operatorname{Ker} t] = 1$, as shown in [**97**]. Conversely, given the cat¹-group $s, t\colon G \to P$, then the restriction of t to $\operatorname{Ker} s \to P$ can be given the structure of crossed module. This procedure applies to other situations than groups—see [**113**] for a general discussion.

If $\mu\colon M \to P$ is a crossed module, then $\operatorname{Ker} \mu$ is an abelian group, that is, a group internal to the category of groups. Thus we see how the theory of groups with an algebraic structure is a pale shadow of a rich theory of algebraically structured groupoids.

Crossed modules are also equivalent to *double groupoids with connections* [**38**]. These model well the idea of using squares instead of paths, so that one can form compositions of the type



(**)

In this way double groupoids allow for 'an algebraic inverse to subdivision'. It turns out that in double groupoids with connection the composition (**) can be translated into a linear composition, but there will be several ways of doing this. The resulting algebra will be of a more familiar kind, but the geometry will be lost. I like to think that, for this reason, a general advance from 1-dimensional to 2- and n-dimensional algebra could become widely significant. This can be put in the more provocative way: *n-dimensional phenomena require for their description n-dimensional algebra.* Double categories with connection are applied to homotopy theory in **[138]**. One practical use of double groupoids is that they allow for a proof of a 2-dimensional Van Kampen type theorem for crossed modules **[27]** which yields some new homotopy computations in dimension 2 (see also **[19]**). Similar remarks apply to all dimensions, using ω-groupoids **[28, 29]**; these have interrelated structures in all dimensions, with n groupoid structures in dimension n, corresponding to the gluing of n-cubes in the n different directions. There is a *fundamental ω-groupoid functor ρ* on filtered spaces, and this satisfies a Van Kampen type theorem. The proof in **[29]** requires precisely the idea of having an algebra which appropriately models subdivision. As noted in **[29]**, this generalised Van Kampen theorem implies the relative Hurewicz theorem.

Just as a group of groupoid G has a classifying space BG **[136]**, so also does any ω-groupoid. This functor B: (ω-groupoids) $\rightarrow$ (spaces) gives ω-groupoids as algebraic models of certain homotopy types, and so allows a Van Kampen theorem for ω-groupoids to yield information on homotopy types. For a survey of this area, see **[18]**. One should mention here also the 'hypergroupoids' of Duskin and Lawvere, which are used in **[65]** to give realisations of general cohomology classes.

It turns out that ω-groupoids do not model all homotopy types. J.-L. Loday has introduced the notion of *cat^n-group* (originally, *n-cat-group* **[97]**), which can be defined inductively as a groupoid object internal to the category of cat^{n-1}-groups, where cat^1-groups are defined above. So a cat^n-group should be thought of as a group with n compatible groupoid structures. For such an object G, Loday has defined a classifying space BG such that $\pi_1(BG) = 0$ for $i > n+1$. To anyone familiar with simplicial sets, the definition of BG will seem the simplest possible: G has $n+1$ compatible groupoid structures, one of them in fact a group structure; taking the nerve for each structure gives an $(n+1)$-simplicial set; the geometric realisation of this is BG **[97]**. Loday and R. Steiner have also proved that if X is a pointed, connected CW-complex with $\pi_1 X = 0$ for $i > n+1$, then X is of the homotopy type of BG for some cat^n-group G **[97, 139]**. This demonstrates how complicated cat^n-groups can be. Guin-Waléry and Loday have given an equivalence between the categories of cat^2-groups and of crossed squares (see **[69]**). Ellis and Steiner **[57]** give an equivalence between the categories of cat^n-groups and of 'crossed n-cubes of groups', thus giving a subtle, n-fold version of crossed modules. Loday and I have proved a Van Kampen theorem for cat^n-groups **[32, 33]**, which generalises the major part of the Van Kampen theorem for ω-groupoids. The case $n = 2$ leads to some new algebraic constructions, such as a non-abelian tensor product $M \otimes N$ of groups M, N each of which acts on the other. The rather tight description of cat^n-groups as crossed n-cubes leads to some new computations in homotopy theory **[57]** and in algebraic K-theory **[56]**.

8. *Conclusion*

In this last section I would like to draw some wider morals and make some possibly outrageous speculations.

First, it seems that the transition from group to groupoid often leads to a more thoroughly non-abelian theory. This is seen in the von Neumann algebra of a measured groupoid, which has also been thought to be appropriate for quantisation in physics [90]. It is perhaps more clearly seen in the algebraically structured groupoids, as above. It is clearly the key aspect of Brandt's original examples. Another example is the non-abelian tensor product of groups, referred to above.

Second, the concept of groupoid is a long way from being recognised as a fundamental concept in our mathematical culture, but this reluctance is diminishing, as is shown by this survey. In due course, groupoid methods will seem as natural as, say, principal bundles, which in fact they often conveniently replace (compare [99]). At present, it has sometimes been recognised that groupoids form an interesting generalisation of groups. Perhaps in another decade it will be agreed that groups are interesting examples of groupoids! Indeed F. W. Lawvere has suggested in conversation that the word group should simply be extended to cover groupoids.

The speculations I would like to make concern the use of multiple groupoids. We have already seen that the use of the usual groupoids allows for a more flexible and powerful approach to both fundamental groups and ideas of symmetry. Also, higher dimensional groupoids have led in homotopy theory to new results and calculations which seem unobtainable by other means [19, 33, 34, 57]. In view of the fundamental nature of our ideas of symmetry, I expect that multiple groupoids will lead to a formulation of ideas of 'higher order symmetry', or 'symmetry of symmetries' and methods of calculation for these.

This is more a programme than a conjecture in the usual sense. It seems a little tricky, since several workers have thought about it in terms of generalising to dimension 2 the covering space approach to Van Kampen's Theorem [49], without coming up with a clear answer. This last problem is important because of the relationship of covering spaces to Galois theory and problems of descent in algebraic geometry [103]. I hope the description of 'higher-order symmetry' will not take anything like the 9 years that it took of staring, off and on, at diagram (**), before a 2-dimensional Van Kampen theorem was found! I find such attempts to bring concepts out of the dark, even without a clear idea of applications, an attractive occupation. Also, in view of the 'unreasonable success of mathematics', these conjectured higher order symmetries should prove fundamental to further progress in our understanding of nature, for example of some physical processes.

ACKNOWLEDGEMENTS. I would like to thank the following for helpful comments on earlier drafts and for useful references and information: Andreé C. Ehresmann, N. D. Gilbert, P. J. Higgins, K. H. Kamps, M. Kneser, P. S. Landweber, K. C. H. Mackenzie, A. Magid, J.-P. Meyer, H. Miller, I. Moerdijk, T. Porter, J. Pradines, J. Renault, J. C. Robson, B. Schein, J. Virsik and M. Zisman.

NOTES ADDED IN PROOF

§1. The groupoid EE^{-1} of a principal H-bundle is defined as the orbit groupoid $(E \times E)/H$, and is isomorphic to the groupoid $G(E)$ of §2, Example 4.

§2, Example 2. The terminology here is not standard, and X is also called the *null* or *discrete* groupoid on X, while $X \times X$ is also called *codiscrete*, *simplicial*, or a *tree*. It is important to have a terminology which is appropriate also for topological and Lie groupoids.

§2, Example 3. The groupoid $X \rtimes G$ is also called the *translation* groupoid. Another possible term is *affine* groupoid. The definition of $X \rtimes G$ was essentially given in 1932, for the transitive case, on p. 28 of [159].

§3. The path-groupoid is, in effect, defined on p. 107 of [159].

§6. J.-P. Meyer and M. Zisman have pointed out that $B(X \rtimes H)$ is homeomorphic to $X \times_H PH$. Zisman has supplied a proof, and Meyer notes that it follows from Corollary 4.4 of [157].

§7. J. Virsik has pointed out that the book [156] is a good source for many of Ehresmann's ideas—for example there are chapters on inductive groupoids, and on species of structure, as well as results on quotient groupoids and free categories and groupoids.

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