C^*-Algebras Generated by Weighted Shifts

JOHN W. BUNCE & JAMES A. DEDDENS

Communicated by the Editors

§1. Introduction. For A a bounded linear operator on a Hilbert space let $C^*(A)$ denote the C^*-algebra generated by A and the identity I. The operator A is called GCR, or postliminal, if $C^*(A)$ is a GCR C^*-algebra. In this paper we consider the question of which weighted shift operators are GCR. In §2 we show that periodic weighted shifts are GCR, while in §3 we introduce a general construction that shows not all weighted shifts are GCR; §4 contains some nontrivial examples of GCR weighted shifts and various remarks and questions.

Recall that a C^*-algebra $\mathfrak{A}$ is called CCR, or liminal, if for every irreducible representation π of $\mathfrak{A}$ on a Hilbert space and for every $A \in \mathfrak{A}$, $\pi(A)$ is compact [7, 4.2.1]. A C^*-algebra $\mathfrak{A}$ is called GCR if $\mathfrak{A}$ has an increasing family of closed two-sided ideals $(\mathcal{I}_\rho)_{0 \leqq \rho \leqq \alpha}$ satisfying $\mathcal{I}_0 = \{0\}$, $\mathcal{I}_\alpha = \mathfrak{A}$, if $\rho \leqq \alpha$ is a limit ordinal, then $\mathcal{I}_\rho$ is the uniform closure of $\cup \{\mathcal{I}_{\rho'} : \rho' < \rho\}$ and $\mathcal{I}_{\rho+1}/\mathcal{I}_\rho$ is CCR [7, 4.3.4]. Equivalently $\mathfrak{A}$ is GCR if every irreducible representation of $\mathfrak{A}$ contains a nonzero compact operator [20, 4.6.4]. It has also been shown that this is equivalent to requiring that for every representation π on a Hilbert space, $\pi(\mathfrak{A})$ generates a type I W^*-algebra [19]. The basic examples of CCR algebras are commutative C^*-algebras, the algebras M_n of all $n \times n$ complex matrices, and the algebra $\mathcal{K}$ of all compact operators on a Hilbert space $\mathcal{H}$. Also, C^*-subalgebras of GCR algebras are GCR [7, 4.3.5].

Recall that an operator A on a Hilbert space $\mathcal{H}$ is called *n-normal* [16] if

$$\sum \operatorname{sgn}(\sigma) A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(2n)} = 0$$

where $A_1, A_2, \cdots, A_{2n}$ are arbitrary elements of the C^*-algebra generated by A, and the summation is taken over all permutations σ of $(1, 2, \cdots, 2n)$. It is clear that if A is n-normal, then every operator in $C^*(A)$ is also n-normal. Every n-normal operator A can be written as the direct sum of $\{A_k\}_{k=1}^n$ where each A_k is a $k \times k$ operator-valued matrix whose entries belong to a commutative C^*-algebra [16]. Thus n-normal operators are CCR, since every irreducible representation is of dimension less than or equal to n.

Indiana University Mathematics Journal, ©, Vol. 23, No. 3 (1973)

Let $\mathcal{K}$ denote the ideal of compact operators in $\mathcal{B}(\mathcal{H})$, and let ν denote the canonical homomorphism for $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$. We call an operator A *essentially n-normal* if $\nu(A)$ is n-normal, or equivalently if

$$\sum \operatorname{sgn}(\sigma) A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(2n)} \in \mathcal{K}$$

where $A_1, A_2, \cdots, A_{2n}$ are arbitrary elements of the C^*-algebra generated by A. We remark that essentially n-normal operators are GCR since $C^*(A)/C^*(A) \cap \mathcal{K}$ is an n-normal algebra and hence CCR. A large class of essentially n-normal operators are those operators which can be written as direct sums of $k \times k$ operator-valued matrices ($k \leqq n$) with entries in a C^*-algebra $\mathcal{C}_k$ such that $\nu(\mathcal{C}_k)$ is commutative. As an important example we mention those $n \times n$ operator-valued matrices whose entries are Toeplitz operators with continuous symbol [9, p. 184].

§2. Periodic weighted shifts. We now fix a separable Hilbert space $\mathcal{H}$ and an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$. A bounded linear operator S on $\mathcal{H}$ is called a *weighted shift* with weights $\{\alpha_n\}_{n=1}^{\infty} \in \ell^{\infty}$ if $Se_n = \alpha_{n+1}e_{n+1}$ for $n = 0, 1, 2, \cdots$. Since the weighted shift with weights $\{\alpha_n\}_{n=1}^{\infty}$ is unitarily equivalent to the weighted shift with weights $\{|\alpha_n|\}_{n=1}^{\infty}$, we assume throughout that $\alpha_n \geqq 0$. When $\alpha_n \equiv 1$, we obtain the unilateral shift U_+ defined by $U_+e_n = e_{n+1}$. Notice that U_+ is a pure isometry that is essentially normal hence GCR. In fact $\mathcal{K} \subseteq C^*(U_+)$ and $C^*(U_+)/\mathcal{K}$ is $*$-isomorphic to $C(T)$, the continuous functions on the unit circle T [6]. If S is any weighted shift, then $S = U_+D$ where D is the diagonal operator, $D = \operatorname{Diag}(\alpha_1, \alpha_2, \alpha_3, \cdots)$, defined by $De_n = \alpha_{n+1}e_n$. Since we are assuming that $\alpha_n \geqq 0$, note that $D = (S^*S)^{1/2} \in C^*(S)$, and that $S = U_+D$ is the polar decomposition of S if $\alpha_n > 0$ for all n. We remark that S is irreducible if and only if $\alpha_n > 0$ for all n.

The following easy lemma is used repeatedly.

Lemma 2.1. *Let S be a weighted shift with weights $\{\alpha_n\}_{n=1}^{\infty}$. If there exists an $\alpha > 0$ such that $\alpha_n \geqq \alpha \geqq 0$ for all n, then $U_+ \in C^*(S)$ and hence $\mathcal{K} \subseteq C^*(S)$.*

Proof. Since $S = U_+D$ we have that $D \in C^*(S)$. By hypothesis D is invertible so that $D^{-1} \in C^*(S)$. Thus $U_+ = SD^{-1} \in C^*(S)$ and $\mathcal{K} \subseteq C^*(U_+) \subseteq C^*(S)$.

We remark that we have been unable to determine whether $\mathcal{K} \subseteq C^*(S)$ whenever S is irreducible. Of course an irreducible weighted shift S such that $\mathcal{K} \nsubseteq C^*(S)$ would not be GCR.

A weighted shift S with weights $\{\alpha_n\}_{n=1}^{\infty}$ is called *periodic* if there exists an integer p such that $\alpha_n = \alpha_{n+p}$ for all n. In this case S is said to be periodic of period p. If S is a periodic weighted shift of period p, then $S^p = \alpha_1\alpha_2 \cdots \alpha_p U_+^p$, so that S^p is essentially normal. We will show that S is actually essentially p-normal.

For convenience we introduce the following notation. If $A_1, A_2, \cdots, A_p$ are operators on a Hilbert space $\mathcal{H}$, then $N = N(A_1, A_2, \cdots, A_p)$ will denote the operator on $\sum_{i=1}^{p} \oplus \mathcal{H}$ defined by

$$N(x_1, x_2, \cdots, x_p) = (A_p x_p, A_1 x_1, A_2 x_2, \cdots, A_{p-1} x_{p-1}).$$

Theorem 2.2. *If S is a periodic weighted shift of period p, then S is essentially p-normal and hence* GCR. *Furthermore if S is irreducible and p is the smallest period of S, then any irreducible representation π of $C^*(S)$ is either unitarily equivalent to the identity representation or there exists a unique $0 \leqq \theta < 2\pi$ such that $\pi(S)$ is unitarily equivalent to $N(\alpha_1, \cdots, \alpha_{p-1}, e^{i\theta}\alpha_p) \in M_p$.*

Proof. Suppose S is a periodic weighted shift of period p. Define a unitary operator W from $\mathcal{H}$ onto $\sum_{i=1}^{p} \oplus \mathcal{H}_i$ (here $\mathcal{H}_i = \mathcal{H}$ for $i = 1, 2, \cdots, p$) by $We_n = e_k^{(i+1)}$ if $n = kp + i$, $0 \leqq i \leqq p - 1$; where $\{e_k^{(i)} : k = 0, 1, \cdots\}$ is the orthonormal basis for $\mathcal{H}_i$. For $n = kp + i$, $0 \leqq i \leqq p - 1$, we have that $WS\, e_n$ equals $\alpha_{n+1}e_k^{(i+2)}$ if $i < p - 1$ and equals $\alpha_{n+1}e_{k+1}^{(1)}$ if $i = p - 1$, while $N(\alpha_1 I, \cdots, \alpha_{p-1}I, \alpha_p U_+)We_n$ equals $\alpha_{i+1}e_k^{(i+2)}$ if $i < p - 1$ and equals $\alpha_p e_{k+1}^{(1)}$ if $i = p - 1$. Since S is periodic of period p, $\alpha_{n+1} = \alpha_{i+1}$ if $i < p - 1$ and $\alpha_{n+1} = \alpha_p$ if $i = p - 1$. Thus S is unitarily equivalent to $N(\alpha_1 I, \cdots, \alpha_{p-1}I, \alpha_p U_+)$. Hence S is essentially p-normal and hence GCR.

Now suppose that S is an irreducible periodic weighted shift. Then by Lemma 2.1 we have $\mathcal{K} \subseteq C^*(S)$. If π is an irreducible representation of $C^*(S)$, then either $\pi(\mathcal{K}) \neq \{0\}$ or $\pi(\mathcal{K}) = \{0\}$. In the first case, π is unitarily equivalent to the identity representation [7, 2.11.3, 4.1.5]. In the second case π factors through the Calkin algebra. But every irreducible representation of the C^*-algebra generated by

$$\nu(N(\alpha_1 I, \cdots, \alpha_{p-1}I, \alpha_p U_+)) = N(\alpha_1 I, \cdots, \alpha_{p-1}I, \alpha_p \nu(U_+))$$

is unitarily equivalent to a restriction of the representation $\hat{\rho}$ induced by a character ρ of $C^*(\alpha_p\nu(U_+), \alpha_i I) = C^*(\nu(U_+))$ [5, Proposition 2]. Since $\nu(U_+)$ is unitary with spectrum the unit circle, there exists a $0 \leqq \theta < 2\pi$ such that $\rho(I) = 1$ and $\rho(\gamma(U_+)) = e^{i\theta}$. Thus $\pi(S)$ is unitarily equivalent to

$$N(\alpha_1, \cdots, \alpha_{p-1}, e^{i\theta}\alpha_p) \mid \mathfrak{M}$$

where the subspace $\mathfrak{M}$ reduces $N_\theta = N(\alpha_1, \cdots, \alpha_{p-1}, e^{i\theta}\alpha_p)$. However, if S is irreducible and if p is the smallest period for S, then N_θ is irreducible for all $0 \leqq \theta < 2\pi$, and thus $\pi(S)$ is unitarily equivalent to N_θ. To see that N_θ is irreducible, suppose that $X = X^* = (x_i) \in M_p$ commutes with N_θ. Let $D \in M_p$ be the diagonal operator with diagonal $(\alpha_1, \alpha_2, \cdots, \alpha_p)$. Then

$$N_\theta = N(1, 1, \cdots, 1, e^{i\theta})D$$

is the polar decomposition of N_θ, and X must commute with both $N(1, \cdots, 1, e^{i\theta})$ and D. Since X commutes with $N(1, \cdots, 1, e^{i\theta})$, it follows that every diagonal of X is constant, $|x_{1,(k+1)}| = |x_{1,(p+1-k)}|$ for each $1 \leqq k < p$, and the i^{th} column of X is

$$(x_{1,i}, x_{1,(i-1)}, \cdots, x_{11}, \bar{x}_{12}, \cdots, \bar{x}_{1,(p+1-i)}).$$

If $\{f_i : 1 \leqq i \leqq p\}$ is the natural basis for $\mathbf{C}^p$, we must have $DXe_i = \alpha_i Xe_i$ for each i, $1 \leqq i \leqq p$. It follows that if $1 \leqq i \leqq p$ and $1 \leqq j \leqq i$, then

$$\alpha_j x_{1,i-(j-1)} = \alpha_i x_{1,i-(j-1)} ;$$

and if $1 \leqq i \leqq p$ and $i < j \leqq p$, then

$$\alpha_j \bar{x}_{1,(j-i)+1} = \alpha_i \bar{x}_{1,(j-i)+1} .$$

Now for each k, $1 \leqq k < p$, there is an i, $1 \leqq i \leqq p$, such that $\alpha_i \neq \alpha_{i+k}$; for otherwise p would not be the smallest period of S. If $i + k \leqq p$, then

$$\alpha_{i+k} \bar{x}_{1,(k+1)} = \alpha_i \bar{x}_{1,(k+1)} ,$$

so that $x_{1,(k+1)} = 0$. On the other hand, if $p < i + k$, then $\alpha_{i+k} = \alpha_{i+k-p} \neq \alpha_i$, so that $i + k - p < i$ and

$$\alpha_{i+k-p} x_{1,p+1-k} = \alpha_i x_{1,p+1-k} .$$

Hence $0 = |x_{1,p+1-k}| = |x_{1,(k+1)}|$; and in all cases, if $1 \leqq k < p$, then $x_{1,(k+1)} = 0$. But X has constant diagonals, thus $X = x_{11} I$ and N_θ is irreducible. Now if $0 \leqq \theta \neq \phi < 2\pi$, then N_θ is not unitarily equivalent to N_ϕ . The easiest way to see this is to notice the p distinct p^{th} roots of $\alpha_1 \alpha_2 \cdots \alpha_p e^{i\theta}$ are the eigenvalues for N_θ . So that if $0 \leqq \theta \neq \phi < 2\pi$, then N_θ and N_ϕ have different spectra.

Corollary 2.3. *If S is a weighted shift with weights $\{\alpha_n\}$ such that $\alpha_n - \beta_n \to 0$ as $n \to \infty$ for some periodic sequence $\{\beta_n\}$, then S is* GCR.

Proof. The hypothesis implies that $S = S_1 + K$ where S_1 is the periodic shift with weights $\{\beta_n\}$ and K is a compact operator. Since S_1 is essentially p-normal for some p, S is also essentially p-normal and hence GCR.

§3. Non-GCR weighted shifts. In this section we show that if $\mathcal{S}$ is any family of weighted shifts, then $C^*(\mathcal{S})$ has a representation π such that $\pi(C^*(\mathcal{S}))''$ is a finite W^*-factor. We can then exhibit a class of weighted shifts which are not GCR operators. We denote by ℓ^∞ the C^*-algebra of all bounded sequences of complex numbers. By a *Banach limit* on ℓ^∞, we mean a state f on ℓ^∞ such that $f(\alpha_1, \alpha_2, \alpha_3, \cdots) = f(\alpha_2, \alpha_3, \cdots)$ for all $\alpha = (\alpha_1, \alpha_2, \cdots) \in \ell^\infty$, and $f(\alpha_1, \alpha_2, \cdots) = a$ if the sequence $\{\alpha_n\}$ has a as limit.

For f a Banach limit on ℓ^∞, we define a state g on $\mathcal{B}(\mathcal{H})$ by $g(A) = f((Ae_n, e_n))$. Then if S is any weighted shift with weights $\{\alpha_i\}$, we have

$$g(SA) = f(0, \alpha_1(Ae_1, e_0), \quad \alpha_2(Ae_2, e_1), \cdots)$$

and

$$g(AS) = f(\alpha_1(Ae_1, e_0), \quad \alpha_2(Ae_2, e_1), \cdots).$$

Since f is a Banach limit we have that $g(AS) = g(SA)$ for any $A \in \mathcal{B}(\mathcal{H})$ and any weighted shift S. It is easily seen that, for any state g on $\mathcal{B}(\mathcal{H})$, the set $\{B \in \mathcal{B}(\mathcal{H}) : g(AB) = g(BA) \text{ for all } A \in \mathcal{B}(\mathcal{H})\}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.

Hence $g(AB) = g(BA)$ for all $A \in \mathcal{B}(\mathcal{H})$ and $B \in C^*(\mathcal{S})$, where $\mathcal{S}$ is the family of all weighted shifts. This construction appeared earlier as theorem 1.11 in the thesis of Joel Anderson (*Derivations, commutators, and the essential numerical range*, Indiana University, 1971).

We now let $\mathcal{S}$ be any nonempty subset of the family of all weighted shifts. We call a linear functional g on a C^*-algebra $\mathcal{A}$ *central* if $g(AB) = g(BA)$ for all $A, B \in \mathcal{A}$. The above paragraph shows that the weak*-compact convex set K consisting of all central states on $C^*(\mathcal{S})$ is nonempty. For $g \in K$ let π_g be the representation on Hilbert space $\mathcal{H}_g$ with cyclic vector 1_g given by the Gelfand–Naimark–Segal construction [7, §2]. It is known [7, §6] that the W^*-algebra $\pi_g(C^*(\mathcal{S}))''$ is a factor if and only if g is an extreme point of K. For completeness we include a proof that $\pi_g(C^*(\mathcal{S}))''$ is a factor if g is an extreme point of K. If $g \in K$, then for $A, B \in C^*(\mathcal{S})$

$$\begin{aligned} g(AB) &= (\pi_g(A)\pi_g(B)1_g\,,\,1_g) \\ &= g(BA) \\ &= (\pi_g(B)\pi_g(A)1_g\,,\,1_g); \end{aligned}$$

hence $(AB1_g\,,\,1_g) = (BA1_g\,,\,1_g)$ for all $A, B \in \pi_g(C^*(\mathcal{S}))''$. Let E be a projection in the center of $\pi_g(C^*(\mathcal{S}))''$ and define a positive linear functional f on $C^*(\mathcal{S})$ by $f(A) = (\pi_g(A)E1_g\,,\,1_g)$. Then f is a central linear functional on $C^*(\mathcal{S})$ and $0 \leqq f(I) \leqq 1$. It is easily seen that $f(I) = 0$ if and only if $E = 0$ and that $f(I) = 1$ if and only if $E = I$. Also, $0 \leqq f \leqq g$. Hence, if $E \neq 0, E \neq I$, we have that $(g - f)/(1 - f(I))$ and $f/f(I)$ are in K and

$$g = (1 - f(I))[(g - f)/(1 - f(I))] + f(I)[f/f(I)].$$

Also if $f = f(I)g$, then

$$(\pi_g(A)E1_g\,,\,1_g) = (\pi_g(A)f(I)1_g\,,\,1_g)$$

for all $A \in C^*(\mathcal{S})$, and since 1_g is a cyclic vector it is immediate that $E = f(I)I$ and $E = 0$ or I. Thus if g is an extreme point of K, then $\pi_g(C^*(\mathcal{S}))''$ must be a factor.

If g is an extremal central state on $C^*(\mathcal{S})$, then $\pi_g(C^*(\mathcal{S}))''$ is a finite factor and is a type II_1 factor if and only if $\pi_g(C^*(\mathcal{S}))$ is infinite dimensional. Let $K(g) = \{A \in C^*(\mathcal{S}) : g(A^*A) = 0\}$ be the left kernel of g. Since g is central, $K(g)$ is a two-sided ideal and $K(g) = \pi_g^{-1}(0)$ by [7, 2.4.10]. Let $\mathcal{D}$ be the C^*-algebra consisting of the diagonal operators in $C^*(\mathcal{S})$ and let $g_0 = g \mid \mathcal{D}$. Then $\mathcal{D}/K(g_0)$ has a natural injection into $\pi_g(C^*(\mathcal{S}))$ and hence if $\mathcal{D}/K(g_0)$ is infinite dimensional, then $\pi_g(C^*(\mathcal{S}))''$ is a type II_1 factor.

Theorem 3.1. *Let $\mathcal{S}$ be a family of weighted shifts which contains all periodic weighted shifts of period n for arbitrarily large n. Then $C^*(\mathcal{S})$ has a type II_1 factor representation.*

Proof. Let g be any extremal central state on $C^*(\mathcal{S})$. We show that the vector space $\mathfrak{D}/K(g_0)$ defined above is infinite dimensional. For arbitrarily large n, $\mathcal{S}$ contains the shift $S = U_+D$, where D is a diagonal operator with weights $\alpha_{1+kn} = 1$ for $k = 0, 1, 2, \cdots$, and all other $\alpha_i = 0$. We show that the family $\{U_+^iDU_+^{*i} : 0 \leqq i \leqq n-1\}$ maps into a linearly independent set in $\mathfrak{D}/K(g_0)$. Suppose $c_0, c_1, \cdots, c_{n-1}$ are complex numbers such that $\sum c_iU_+^iDU_+^{*i}$ is in $K(g_0)$. Then since the product of any two distinct elements of $\{U_+^iDU_+^{*i} : 0 \leqq i \leqq n-1\}$ is zero,

$$0 = g_0((\sum c_iU_+^iDU_+^{*i})^*(\sum c_iU_+^iDU_+^{*i}))$$
$$= g_0(\sum |c_i|^2\, U_+^iDU_+^{*i}).$$

But g is central, so $g(U_+^iDU_+^{*i}) = g(D)$ and $0 = \sum |c_i|^2\, g(D)$. Now $I = \sum U_+^iDU_+^{*i}$, so $g(D) \neq 0$, hence $c_i = 0$ for each i. Hence $\{U_+^iDU_+^{*i} + K(g_0): 0 \leqq i \leqq n-1\}$ is a linearly independent set in $\mathfrak{D}/K(g_0)$. Thus $\mathfrak{D}/K(g_0)$ is infinite dimensional and $\pi_g(C^*(\mathcal{S}))''$ is a II_1 factor.

We note that the weighted shift in the following corollary is closely related to Kakutani's shift [11, Solution 87].

Corollary 3.2. *Let S be the weighted shift with weights $\{\alpha_n\}$, where $\alpha_1 = 2$ and $\alpha_m = 2 - 1/2^k$ if $m \geqq 2$ and $m - 1 = 2^k(2\ell + 1)$. Then $C^*(S)$ contains every weighted shift of period 2^n for all n and $C^*(S)$ is not a* GCR *algebra.*

Proof. By Lemma 2.1 $C^*(S)$ contains the compact operators and $\mathfrak{D}$, the set of diagonal operators in $C^*(S)$, contains c_0, the space of sequences which converge to zero. Also $\mathfrak{D}$ contains $P_1 = U_+^*(2I - D)U_+$ which is a diagonal operator with weights (β_n), where $\beta_n = 1/2^k$ with $n = 2^k(2 + 1)$. Then the sequence of operators P_1^n converges uniformly to the diagonal Q_1 with periodic weights $(1, 0, 1, 0, \cdots)$. Let $P_2 = 2U_+^*(P_1 - Q_1)U_+$; then P_2^n converges uniformly to $Q_2 \in \mathfrak{D}$, where Q_2 has weights $Q_2(n) = 1$ if $n = 1 + 4k, 0 \leqq k$, $Q_2(n) = 0$ otherwise. Let $P_3 = 2U_+^{*2}(P_2 - Q_2)U_+^2$; then P_3^n converges uniformly to Q_3 where $Q_3(n) = 1$ if $n = 1 + 8k$, $0 \leqq k$, $Q_3(n) = 0$ otherwise. Inductively, it is clear that $\mathfrak{D}$ contains all periodic diagonals of period 2^n for all n. Hence $C^*(S)$ contains every weighted shift of period 2^n for all n and by the theorem $C^*(S)$ has a II_1 factor representation and is not GCR.

We remark that there exists a weighted shift S with only two distinct weights which is not GCR. In particular, consider the weighted shift S with weights $\{\alpha_n\}$ where $\alpha_n = 1$ if $n - 1 = 2^k(2\ell + 1)$ for k even, and $\alpha_n = 2$ if $n - 1 = 2^k(2\ell + 1)$ for k odd, $\alpha_1 = 1$. Then $S = U_+D = U_+(I + E)$, E a diagonal projection in $C^*(S)$. Now $E_1 = E \cdot U_+^{*4}EU_+^4$ is a periodic projection of period 4, and $E_2 = (E - E_1)U_+^{*16}(E - E_1)U_+^{16}$ is a periodic projection of perod 16; continuing in this manner we see that $C^*(S)$ contains all periodic weighted shifts of period 4^n for all n and hence is not GCR.

It is now easy for us to show that the set of GCR operators in $\mathcal{B}(\mathcal{H})$ is not norm closed. This answers a question in [3, p. 307].

Corollary 3.3. *The set of* GCR *weighted shifts is not norm closed in the set of weighted shifts.*

Proof. Let $S(k)$ be the weighted shift of period k with weights $\{\alpha_m\}$ where $\alpha_m = 1$ if $m = 1 + \ell k$, $0 \leqq \ell$, and $\alpha_m = 1/2$ otherwise. Then, if S is the weighted shift of Corollary 3.2, we have

$$S = \sum_{n=1}^{\infty} (1/2^{n-1})S(2^n).$$

But each finite partial sum $\sum_{i=1}^{N} (1/2^{n-1})S(2^n)$ is periodic of period 2^N and hence is GCR, while the norm limit, S, is not GCR.

We now need some definitions and results from [14]. For $\{\alpha_n\}$ a bounded sequence of real numbers let

$$q(\alpha) = \lim_p \left(\limsup_k (1/p) \sum_{i=0}^{p-1} \alpha_{k+i}\right),$$

$$q'(\alpha) = \lim_p \left(\liminf_k (1/p) \sum_{i=0}^{p-1} \alpha_{k+i}\right).$$

These definitions are formally different from the q and q' defined by Lorentz in [14, p. 168], but by [12, p. 88] they are actually the same. If $\{\alpha_n\}$ is a bounded sequence of real numbers, then there is a Banach limit f on ℓ^∞ such that $f(\alpha_n) = a$ if and only if $q'(\alpha) \leqq a \leqq q(\alpha)$ [14, p. 169]. Lorentz then defined a bounded complex sequence (α_n) to be *almost convergent to a* if $f(\alpha_n) = a$ for all Banach limits f on ℓ^∞.

By a *character* on a C^*-algebra $\mathfrak{A}$, we mean a state f on $\mathfrak{A}$ such that $f(AB) = f(A)f(B)$ for all $A, B \in \mathfrak{A}$. The construction at the beginning of this section allows us to prove that $C^*(S)$ has a character for certain weighted shifts S.

Theorem 3.4. *If S is a weighted shift satisfying either condition* (1) *or* (2), *then there is a character on $C^*(S)$.*

(1) *$S = U_+D$ where $D = aI + Z$ with $a > 0$ and the weights $\{z_n\}$ of Z a nonnegative sequence such that $q'(z) = 0$.*

(2) *$S = U_+D$ where the weights $\{d_n\}$ of D are such that $q'(b) = 0$, where $b_n = |d_n^2 - d_{n-1}^2|$, $d_0 = 0$.*

Proof. Suppose S satisfies condition (2). By the result of Lorentz mentioned above, there is a Banach limit f on ℓ^∞ with $f(b_n) = 0$. Let g be defined on $C^*(S)$ by $g(A) = f((Ae_n e_n))$. Then g is central on $C^*(S)$ and, if $\mathfrak{D}$ is the algebra of diagonal operators in $C^*(S)$, $g(\text{Diag }(\alpha_n)) = f(\alpha_n)$ for $\text{Diag }(\alpha_n) \in \mathfrak{D}$. Then we have that $\pi_g^{-1}(0) = K(g)$ and

$$\mathfrak{D} \cap K(g) = \{\text{Diag }(\alpha_n) \in \mathfrak{D} : f\,(|\alpha_n|) = 0\}.$$

But $|S^*S - SS^*| = \text{Diag }(b_n)$ is in $\mathfrak{D}$ and $f(b_n) = 0$, thus $|S^*S - SS^*| \in \pi_g^{-1}(0)$ and $\pi_g(S)^*\pi_g(S) = \pi_g(S)\pi_g(S)^*$. Hence $\pi_g(C^*(S))$ is abelian and nonzero. Hence, by the Gelfand theory for commutative C^*-algebras, $C^*(S)$ has a character.

If S satisfies (1), then there is a Banach limit f on ℓ^∞ with $f(z_n) = 0$. Let $g(A) = f(Ae_n, e_n)$ for all $A \in C^*(S)$. Then $Z \in \pi_g^{-1}(0)$, and since f is zero on c_0, π_g is zero on the compact operators and hence $\pi_g(U_+)$ is unitary. But $\pi_g(S) = a\pi_g(U_+)$ is then normal. Hence $\pi_g(C^*(S))$ is abelian and again $C^*(S)$ has a character.

We remark that the proof of Theorem 3.4 also shows that shifts S which satisfy either of the following conditions do not have a type II_1 factor representation.

(3) $S = U_+D$ where $D = aI + Z$ with $a > 0$ and the weights $\{z_n\}$ of Z a nonnegative sequence such that $q(z) = 0$.

(4) $S = U_+D$ where the weights $\{d_n\}$ of D are such that $q(b) = 0$, where $b_n = |d_n^2 - d_{n-1}^2|$, $d_0 = 0$.

However, by methods of this section, we are not able to determine if such shifts are GCR. In §4 we show that certain shifts satisfying condition (3) are GCR.

In analogy with the concept of an almost periodic function on the real line, Lorentz [14, p. 173] defined a bounded complex sequence $\{\alpha_n\}$ to be *almost periodic* if for every $\epsilon > 0$ there are two natural numbers N and ℓ such that every interval $(k, k + \ell)$ contains a p such that $|\alpha_{n+p} - \alpha_n| < \epsilon$ for all $n \geqq N$. As in [4] it can be shown that the set of almost periodic sequences forms a C^*-subalgebra of ℓ^∞ which contains c_0. Also, every almost periodic sequence is amost convergent. We also use the fact that if $\{\alpha_n\}$ is an almost periodic sequence of nonnegative reals which is almost convergent to zero, then $\{\alpha_n\}$ actually converges to zero. The proof of this is the same as the proof of the corresponding fact for almost periodic functions [4, p. 63].

Lemma 3.5. *Let $\mathfrak{D}$ be a C^*-algebra of ℓ^∞ which consists of almost periodic sequences and contains all convergent sequences. If f is a Banach limit on ℓ^∞, then any projection in $\mathfrak{D}/K(f)$ is the image of a projection in $\mathfrak{D}$.*

Proof. Let E be a projection in $\mathfrak{D}/K(f)$. Note that

$$K(f) = \{\{\alpha_n\} \in \mathfrak{D} : f(|\alpha_n|) = 0\}.$$

Let $q : \mathfrak{D} \to \mathfrak{D}/K(f)$ be the canonical map and choose $\alpha \in \mathfrak{D}$ with $q(\alpha) = E$. Then since $q((\alpha + \alpha^*)/2) = E$, we may assume $\alpha = \alpha^*$. Thus min $(\alpha, 1)$ given by min $(\alpha, 1) = (\alpha + 1 - |\alpha - 1|)/2$ is in $\mathfrak{D}$ and

$$q(\min(\alpha, 1)) = (E + I - |E - I|)/2 = E,$$

so we may assume that $\alpha \leqq 1$. Also $q(|\alpha|) = |q(\alpha)| = E$, so we may assume that $0 \leqq \alpha \leqq 1$. Then $q(\alpha^2 - \alpha) = 0$, so $\alpha - \alpha^2 \in K(f)$ and $f(\alpha_n(1 - \alpha_n)) = 0$. But $\mathfrak{D}$ consists of almost periodic sequences, so $\{\alpha_n(1 - \alpha_n)\}$ is a nonnegative almost periodic sequence which almost converges to zero. Hence, by the remarks before the lemma, $\{\alpha_n(1 - \alpha_n)\}$ actually converges to zero. Hence there exists an N such that $\alpha_n(1 - \alpha_n) < 1/16$ for all $n \geqq N$. It follows that α_n is not in the interval $(1/4, 3/4)$ if $n \geqq N$. Then by adding a finitely nonzero sequence to α, we may assume that α_n is never in $(1/4, 3/4)$. Now let $x = \min(\alpha + 1/4, 1)$. Then $x \in \mathfrak{D}$ and $x_n \in [0, 1/2] \cup \{1\}$ for all n. Hence the sequence of elements (x^n) in $\mathfrak{D}$ converges uniformly to a projection z in $\mathfrak{D}$. Also $|z_n - \alpha_n|/4 \leqq \alpha_n(1 - \alpha_n)$, so $f(|z_n - \alpha_n|) = 0$ and $z - \alpha \in K(f)$. Hence $q(z) = q(\alpha) = E$.

Theorem 3.6. *Let S be a weighted shift whose weights form a positive almost periodic sequence which is bounded away from zero. Then S is* GCR *if and only if $S = Q + Z$, where Q is a periodic weighted shift with strictly positive weights and Z is a weighted shift with real weights which converge to zero.*

Proof. Let $S = U_+D$, where D is a diagonal operator whose weights form a positive almost periodic sequence which is bounded away from zero. Let $\mathfrak{D}$ be the C^*-subalgebra of $C^*(S)$ consisting of those diagonal operators in $C^*(S)$ whose weights are almost periodic. We identify the diagonal operators in $\mathfrak{D}$ with the corresponding sequences in ℓ^∞. Let f be an extremal central state on $C^*(S)$. Then by the results at the beginning of this section, $\pi_f(C^*(S))''$ is a finite W^*-factor, so if S is GCR, then $\pi_f(C^*(S))$ must be finite-dimensional. But if $f_0 = f \mid \mathfrak{D}$, then $\mathfrak{D}/K(f_0)$ has a natural injection into $\pi_f(C^*(S))$ and hence $\mathfrak{D}/K(f_0)$ is a finite-dimensional C^*-algebra. Thus there exist orthogonal projections $E_1, E_2, \cdots, E_n$ in $\mathfrak{D}/K(f_0)$ which form a Hamel basis. Then by Lemma 3.5 there are projections $Q_1, Q_2, \cdots, Q_n$ in $\mathfrak{D}$ with $q(Q_i) = E_i$. Then $q(D) = \sum_{i=1}^n c_iE_i$, where $c_i > 0$ since each c_i is in the spectrum of D, and $D - \sum c_iQ_i \in K(f_0)$. If $Z_0 = D - \sum c_iQ_i$ and $Q_0 = \sum c_iQ_i$, then $D = Q_0 + Z_0$. The weights of each Q_i form an almost periodic sequence of zeroes and ones; hence it is clear that the weights of each Q_i are periodic from some point on. Thus the weights of Q_0 are periodic from some point on. By adding a finitely nonzero sequence to Q_0, and subtracting it from Z_0, we may assume that the weights of Q_0 are actually periodic. Also the weights of Z_0 form an almost periodic sequence which is absolutely almost convergent to zero; hence the weights of Z_0 converge to zero. If $Q = U_+Z_0$ and $Z = U_+Q_0$, then $S = Q + Z$ and Q and Z satisfy the properties of the theorem. The converse is true by Corollary 2.3.

Let $\{n_k\}$ be a sequence of positive integers such that n_k divides n_{k+1} for all k and $n_k \to \infty$. Let $S(n_k)$ be the weighted shift with weights $\alpha_m = 1$ if $m = 1 + \ell n_k$, $\ell \geqq 0$, and $\alpha_m = 1/2$ otherwise. Then $C^*(S(n_k))$ is the C^*-algebra generated by all weighted shifts of period n_k, and by the results of the first section the algebra $C^*(S(n_k))$ is essentially n_k-normal. We have that $C^*(S(n_k)) \subseteq C^*(S(n_{k+1}))$. Let $\mathfrak{A}(n_k)$ be the norm closure of the union of all the $C^*(S(n_k))$. We remark that if $\{p_1, p_2, \cdots\}$ is an enumeration of the primes, then $\mathfrak{A}(m_k)$ is the C^*-algebra generated by all periodic weighted shifts, where $m_k = (p_1p_2 \cdots p_k)^k$. The algebras $\mathfrak{A}(n_k)$ then have a structure similar to the structure of uniformly hyperfinite C^*-algebras.

For each k, let $P_j^{(k)}$, $1 \leqq j \leqq n_k$ be the diagonal projection with weights $\alpha_m = 1$ if $m = j + \ell n_k$, $\ell \geqq 0$, and $\alpha_m = 0$ otherwise. For $1 \leqq j \leqq i \leqq n_k$, let $E^{(k)}(i, j) = U_+^{(i-j)}P_j^{(k)}$ and for $1 \leqq i \leqq j \leqq n_k$ let $E^{(k)}(i, j) = P_i^{(k)}U_+^{*(j-i)}$. Then the $\{E^{(k)}(i, j) : 1 \leqq i \leqq n_k, 1 \leqq j \leqq n_k\}$ form a system of $n_k \times n_k$ matrix units in $C^*(S(n_k))$. Let $N(n_k)$ be the C^*-algebra generated by the family $\{E^{(k)}(i, j)\}$. Then $N(n_k) \subseteq N(n_{k+1})$ and if $\mathfrak{M}(n_k)$ is the C^*-algebra generated by the union of the $\{N(n_k) : 1 \leqq k\}$, then $\mathfrak{M}(n_k)$ is a uniformly hyperfinite C^*-algebra of type $\{n_k\}$ [10]. The algebra $\mathfrak{A}(n_k)$ is the C^*-algebra generated by $\mathfrak{M}(n_k)$ and U_+.

We now show that Glimm's classification of the *-isomorphism classes of uniformly hyperfinite C^*-algebras [10] carries over to this situation.

Theorem 3.7. *The algebras $\mathfrak{A}(n_i)$ and $\mathfrak{A}(q_j)$ defined above are *-isomorphic if and only if for all i there is a j such that n_i divides q_j and for all j there is an i such that q_j divides n_i. The algebras $\mathfrak{A}(n_i)$ and $\mathfrak{A}(q_j)$ are thus *-isomorphic if and only if they are equal.*

Proof. The proof is easy if we use a difficult technical lemma of Glimm [10, Lemma 1.10]. Let F be a *-isomorphism from $\mathfrak{A}(n_i)$ onto $\mathfrak{A}(q_j)$. For a given i, $\mathfrak{A}(n_i)$ contains matrix units $\{E^{(i)}(m, n) : 1 \leqq m, n \leqq n_i\}$. Let $\delta(1, n_i)$ be the positive number of [10, Lemma 1.10]. Then $F(E^{(i)}(m, n)) \in \mathfrak{A}(q_j)$ for all m, n, so there exist operators $A(m, n) \in \bigcup C^*(S(q_j))$ such that $\|F(E^{(i)}(m, n)) - A(m, n)\| < \delta(1, n_i)$ for all m, n. Thus there is a j such that $A(m, n) \in C^*(S(q_j))$ for all m, n. Then by Glimm's Lemma there are matrix units $\{D(m, n) : 1 \leqq m, n \leqq n_i\}$ in $C^*(S(q_j))$. Now by Theorem 2.2, $C^*(S(q_j))$ has an irreducible representation π such that $\pi(C^*(S(q_j)))$ is the $q_j \times q_j$ complex matrices. But the set $\{\pi(D(m, n))\}$ forms a $n_i \times n_i$ system of matrix units in $\pi(C^*(S(q_j)))$; hence n_i divides q_j. By symmetry for each j there exists an i such that q_j divides n_i. If n_i divides q_j, then $C^*(S(n_i))$ is contained in $C^*(S(q_j))$; hence the rest of the theorem is immediate.

In the general construction at the beginning of this section, we considered an extremal central state on $C^*(\mathcal{S})$, for $\mathcal{S}$ a family of weighted shifts. In the case of the algebras $\mathfrak{A}(n_k)$, there is a unique central state, just as is the case for uniformly hyperfinite C^*-algebras [10, Theorem 4.1].

Theorem 3.8. *Let f be a central state on $\mathfrak{A}(n_k)$, where $\{n_k\}$ is a sequence of integers such that n_k divides n_{k+1} and $n_k \to \infty$. Then $\pi_f(\mathfrak{A}(n_k))'' = \pi_f(\mathfrak{M}(n_k))''$ and is a hyperfinite II_1 factor. Hence $\mathfrak{A}(n_k)$ has an unique central state.*

Proof. Let π_f be the cyclic representation corresponding to the central state f. Then since $I = \sum_{i=1}^{n_k} E^{(k)}(i, i)$, $f(E^{(k)}(n_k, n_k)) = 1/n_k$ and $(\pi_f(E^{(k)}(n_k, n_k))1_f, 1_f) = 1/n_k$. The state g given by $g(A) = (A1_f, 1_f)$ is a normal state on $\pi_f(\mathfrak{A}(n_k))''$. Since a cyclic trace vector for a W^*-algebra is also cyclic for the commutant [18, p. 35], we have that the support of g which is the projection on the closed linear span of $\{A1_f : A \in \pi_f(\mathfrak{A}(n_k))'\}$ is all of $\mathcal{H}_f$ [8, p. 57]. Hence $\pi_f(E^{(k)}(n_k, n_k))$ converges strongly to zero by [8, Proposition 4, p. 58]. An easy computation shows that for each k,

$$U_+ = \sum_{j=1}^{n_k-1} E^{(k)}(j+1, j) + E^{(k)}(1, 1)U_+E^{(k)}(n_k, n_k).$$

Hence $\pi_f(U_+)$ belongs to $\pi_f(\mathfrak{M}(n_k))''$, and $\pi_f(\mathfrak{A}(n_k))'' = \pi_f(\mathfrak{M}(n_k))''$. But by [10, Theorem 4.1] $\pi_f(\mathfrak{M}(n_k))''$ is a factor. Thus every central state on $\mathfrak{A}(n_k)$ is an extremal central state, which immediately implies that $\mathfrak{A}(n_k)$ has an unique central state.

Now $\mathfrak{A}(n_k)$ is the norm closure of an increasing sequence of GCR C^*-algebras, so $\mathfrak{A}(n_k)$ is strongly amenable in the sense of Johnson [13, p. 70, p. 76, p. 78]. Hence the weak closure of any representation of $\mathfrak{A}(n_k)$ has Schwartz's Property P [13, p. 82]. Since $\mathfrak{A}(n_k)$ is not GCR, $\mathfrak{A}(n_k)$ has a type III factor representation [20, 4.6.4]. We have not been able to find a natural construction of a type III factor representation of $\mathfrak{A}(n_k)$. Since it is not clear that every type III factor representation of $\mathfrak{A}(n_k)$ is hyperfinite, there may exist a type III factor representation of $\mathfrak{A}(n_k)$ which is not hyperfinite but has Property P. As far as we know, no example of a nonhyperfinite factor with Property P has appeared in the literature.

§4. Additional examples and comments. In §3 it was established that not all weighted shifts, not even all weighted shifts with only finitely many weights, are GCR. In this section we introduce two constructions of additional examples of GCR weighted shifts. In particular, we show that certain GCR weighted shifts with only two weights have an interesting composition series. We have not been able to determine exactly which weighted shifts with only two weights are GCR.

The next theorem establishes that certain weighted shifts, which satisfy condition (3) of §3 but which are not of the form periodic weighted shift plus compact, are GCR.

Theorem 4.1. *Let $\{n_k : k = 1, 2, \cdots\}$ be a strictly increasing sequence of positive integers ($n_1 = 1$) such that for each $N \geqq 0$ there are only finitely many solutions to $n_k - n_\ell = N$, (for example, $n_k = k^2$). If S is the weighted shift with weights $\alpha_n = 2$ if $n = n_k$, $k \geqq 1$, and $\alpha_n = 1$ otherwise, then S is* GCR. *In fact, if $\mathfrak{C}$ denotes the commutator ideal of $C^*(S)$, then $C^*(S)$ has the composition series $\{0\} \subseteq \mathfrak{K} \subseteq \mathfrak{C} \subseteq C^*(S)$, with $C^*(S)/\mathfrak{C}$ commutative and $\mathfrak{C}/\mathfrak{K}$ isomorphic to $\mathfrak{K}$.*

Proof. If E is the diagonal projection operator with weights $\alpha_n = 1$ if $n = n_k$ and $\alpha_n = 0$ otherwise, then $S = U_+D = U_+(I + E)$. If E_0 denotes the projection onto the span of e_0, then $S^*S - SS^* = E_0 + 3E - 3U_+EU_+^*$, and the assumptions on the weights imply that

$$U_+^*(S^*S - SS^*)U_+(S^*S - SS^*) = K - 9E$$

for some compact K. By Lemma 2.1 $U_+ \in C^*(S)$ and $\mathfrak{K} \subseteq C^*(S)$, so that $\mathfrak{C}$, which is also the closed ideal generated by $S^*S - SS^*$ in $C^*(S)$ [1, Corollary 3.3.4], is equal to the ideal generated by E in $C^*(S)$. Now the hypothesis on $\{n_k\}$ imply that EU_+^NE is compact for $N > 0$. Indeed, $EU_+^NEU_+^{*N}$ is the diagonal projection operator with $\alpha_n = 1$ if and only if $n = n_k$ and $n = n_\ell + N$ for some k, ℓ. But then $n_k = n_\ell + N$ and $k > \ell$ so that there are only a finite number of solutions to $n_k = n_l + N$. Hence $EU_+^N EU_+^{*N}$ is a projection operator with finite dimensional range, and is thus compact. From this we conclude that $\{\nu(U_+)^n \nu(E)\nu(U_+^*)^n\}_{n=-\infty}^{\infty}$ are mutually orthogonal projections. In addition, since $S = U_+(I + E)$ we obtain for any words w, w' in two variables that

$$w(S, S^*)Ew'(S, S^*) = w(U_+, U_+^*)Ew'(U_+, U_+^*) + K$$

which equals either $U_+^n EU_+^{*k} + K'$ or $U_+^{*n} EU_+^k + K''$ for some integers n, k and compact operators K, K', and K''. Hence the closed ideal generated by E in $C^*(S)$ is equal to the C^*-algebra generated by $\{U_+^n EU_+^{*k}\}$, $\{U_+^* EU_+^{kn}\}$ and $\mathcal{K}$. Thus $\mathcal{C}/\mathcal{K}$ is isomorphic to the C^*-algebra generated by $\{\nu(U_+)^n \nu(E)\nu(U_+^*)^k\}_{n,k=-\infty}^{\infty}$ which is in turn isomorphic to $\mathcal{K}$, since $\nu(U_+)$ is unitary and $\{\nu(U_+^n)\nu(E)\nu(U_+^*)^n\}_{n=-\infty}^{\infty}$ are mutually orthogonal projections.

Remarks. (1) We note that these techniques allow us to derive the conclusions of Theorem 4.1 for other classes of weighted shifts with only two weights, for example the weighted shift with weights

$$(1, 2, 2, 1, 1, 2, 2, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2, 1, \cdots).$$

In fact, all that is needed is that the lengths of the blocks of consecutive 2's be bounded, while the lengths $\{p_i\}$ of the blocks of consecutive 1's satisfy, for every $N > 0$ there are only a finite number of solutions to $p_i = N$.

(2) Perhaps more interesting is that the same conclusions also hold for the weighted shift with weights

$$(1, 2, 1, 1, 2, 2, 1, 1, 1, 2, 2, 2, \cdots).$$

In fact, all that is needed is that the lengths $\{p_i\}$ and $\{q_j\}$ of the blocks of consecutive 1's and 2's both satisfy, for every $N > 0$ there are only a finite number of solutions to $p_i = N$ and $q_j = N$. We briefly indicate the proof. If S is such a weighted shift, then $S^*S - SS^* = E_0 + 3G - 3H$ where G, H are diagonal projections. Our assumptions imply that $\{\nu(U_+)^n \nu(G)\nu(U_+^*)^n\}_{n=-\infty}^{\infty}$ and $\{\nu(U_+)^n \nu(H)\nu(U_+^*)^n\}_{n=-\infty}^{\infty}$ together form a mutually orthogonal family of projections, and that $\mathcal{C}/\mathcal{K}$ is isomorphic to the C^*-algebra generated by

$$\{\nu(U_+)^n \nu(G)\nu(U_+^*)^k,\ \nu(U_+)^n \nu(H)\nu(U_+^*)^k\}_{n,k=-\infty}^{\infty}$$

where $\mathcal{C}$ is the commutator ideal in $C^*(S)$.

Since C^*-subalgebras of GCR algebras are GCR, Theorem 4.1 implies that the weighted shift S_0 with weights

$$(1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, \cdots)$$

is GCR. Observe that S_0 is unitarily equivalent to $\sum_{n=2}^{\infty} \oplus J(n)$ on $\sum_{n=2}^{\infty} \oplus \mathbf{C}^n$ where $J(n)(x_1, x_2, \cdots, x_n) = (0, x_1, x_2, \cdots, x_{n-1})$ is the Jordan block on $\mathbf{C}^n$. We now introduce a different method that establishes S_0 is GCR. In fact, suppose S is any weighted shift with weights

$$(\alpha_1, 0, \alpha_3, \alpha_4, 0, \alpha_6, \alpha_7, \alpha_8, 0, \alpha_{10}, \cdots),$$

that is, $\alpha_n = 0$ for $n = (k(k+1)/2) - 1$, $k = 2, 3, 4, \cdots$. It is easy to see that if $n \geqq 0$, then there are unique integers $k \geqq 0$ and i, $0 \leqq i \leqq k$, with $n = (k(k+1)/2) + i$. Define a unitary operator V from $\mathfrak{H}$ onto $\sum_{i=1} \oplus \mathfrak{H}_i$ (here $\mathfrak{H}_i = \mathfrak{H}$ for all i) by $Ve_n = e_k^{(1)}$ if $n = (k(k+1)/2) + k$ and $Ve_n = e_{k-i-1}^{(i+2)}$ if $n = (k(k+1)/2) + i$ with $0 \leqq i < k$, where $\{e_k^{(i)} : k = 0, 1, \cdots\}$ is the orthonormal basis for $\mathfrak{H}_i$. Then S is unitarily equivalent, via V, to the operator valued matrix

$$T = \begin{bmatrix} 0 & & & & \\ D_1 & 0 & & & 0 \\ & D_2U_+^* & 0 & & \\ & & D_3U_+^* & 0 & \\ & & & \ddots & \ddots \\ & 0 & & & \ddots \end{bmatrix}$$

where D_1 is the diagonal operator with weights $(\alpha_1, \alpha_3, \alpha_6, \alpha_{10}, \cdots)$ and D_k is the diagonal operator with weights

$$(\alpha_{\ell+k}, \alpha_{\ell+k+(k+1)}, \alpha_{\ell+k+(k+1)+(k+2)}, \cdots)$$

where $\ell = (k(k+1)/2) - 1$.

Thus by means of the unitary operator V we see that S_0 is unitarily equivalent to the operator valued matrix

$$T = \begin{bmatrix} 0 & & & & \\ I & 0 & & & 0 \\ & U_+^* & 0 & & \\ & & U_+^* & 0 & \\ & & & \ddots & \ddots \\ & 0 & & & \ddots \end{bmatrix}$$

But then $T \varepsilon C^*(U_+ \otimes I, I \otimes U_+) = C^*(U_+) \otimes C^*(U_+)$ which is GCR since $U_+ \otimes I$ and $I \otimes U_+$ doubly commute [15, Theorem 1]. Hence S_0 is GCR. Similarly, utilizing our knowledge that periodic weighted shifts are GCR, we can establish that other such weighted shifts are GCR.

Using this description of S_0 it can be seen using [7, 2.10.2; and 17, proof of Theorem 3] that if π is any irreducible representation of $C^*(S_0)$, then one of the following must occur:

(1) $\qquad \pi(S_0) \cong J(n)$ for some $n = 2, 3, \cdots$,

(2) $\qquad \pi(S_0) \cong U_+$,

$$\pi(S_0) \cong U_+^* , \quad \text{or} \tag{3}$$

$$\pi(S_0) = e^{i\theta} \quad \text{for some} \quad 0 \leqq \theta < 2\pi. \tag{4}$$

Thus the spectrum of $C^*(S_0)$ consists of a circle of one-dimensional irreducible representations, one n-dimensional irreducible representation for each $2 \leqq n < \infty$, and two infinite dimensional irreducible representations. The hull-kernel topology on the spectrum of $C^*(S_0)$ can be explicitly described. We remark that the operator S_0 is the usual example given to show that the spectrum of an infinite direct sum of operators need not be the union of the spectra [11, Solution 81]. Notice that $\sigma(J(n)) = \{0\}$ while $\sigma(S_0) = \{z : |z| \leqq 1\}$. It is interesting to note that both $\pi(S_0) = U_+$ and $\pi(S_0) = U_+^*$ induce representations of $C^*(S_0)$.

It is now possible to prove that any weighted shift with only zero and one weights is GCR. Indeed any such $0 - 1$ shift is easily seen to be unitarily equivalent to $\sum_{n \varepsilon L} \oplus J(n, k_n)$, where L is a subset of the natural numbers and $J(n, k_n)$ is the direct sum of k_n copies of $J(n)$, $0 \leqq k_n \leqq \infty$. Equivalently $J(n, k_n)$ is unitarily equivalent to the operator J defined on $\sum_{i=1}^{n} \oplus \mathcal{H}(k_n)$ by

$$J(x_1 , x_2 , \cdots , x_n) = (0, x_1 , \cdots , x_{n-1}),$$

where $\mathcal{H}(k_n)$ is a Hilbert space of dimension k_n. Clearly $C^*(J(n)) \cong C^*(J(n, k_n))$ for all n, k_n, and

$$C^*(\textstyle\sum_{n \varepsilon L} \oplus J(n)) \cong C^*(\textstyle\sum_{n \varepsilon L} \oplus J(n, k_n)).$$

It is easily verified that $A \oplus B$ is GCR if and only if A and B are GCR [15, Theorem 1]. Since $\sum_{n=2}^{\infty} \oplus J(n)$ is GCR, we conclude that any $0 - 1$ weighted shift is GCR.

We note that our results readily yield results concerning bilateral weighted shifts and finite dimensional weighted shifts. An operator B is called a *bilateral weighted shift* with weights $\{\beta_n\}_{n=-\infty}^{\infty}$ on a Hilbert space $\mathcal{H}$ with orthonormal basis $\{f_n\}_{n=-\infty}^{\infty}$ if $Bf_n = \beta_{n+1} f_{n+1}$ for all n. However, $B - \beta_0(f_0 \otimes f_{-1})$ is unitarily equivalent to $S_-^* \oplus S_+$ where S_+ and S_- are weighted shifts with weights $\{\beta_n\}_{n=1}^{\infty}$ and $\{\beta_{-n}\}_{n=1}^{\infty}$ respectively. If K is a compact operator, then $A + K$ is GCR if and only if A is GCR [2, Theorem 3]. Combining this with our previous remark about direct sums, we conclude that B is GCR if and only if both S_+ and S_- are GCR. Thus the study of bilateral weighted shifts is reduced to the study of unilateral weighted shifts. An operator S is called an *n-dinensional weighted shift* if S is unitarily equivalent to the operator T defined on $\sum_{i=0}^{\infty} \oplus \mathbf{C}^n$ by

$$T(x_0 , x_1 , \cdots) = (0, A_1 x_0 , A_2 x_1 , \cdots),$$

where $A_i \varepsilon M_n$ and the sequence $\{||A_i||\}$ is bounded. Many of our results could then be stated for n-dimensional weighted shifts. For example, if we define S to be periodic of period p if $A_{i+p} = A_i$ for all i, then the results of §2 yield that periodic n-dimensional weighted shifts are GCR. For in this case, S is merely an $n \times n$ operator valued matrix all of whose entries are periodic shifts of period p. Since $C^*(S(p))$ is GCR we have that $C^*(S(p)) \otimes M_n$ is GCR [15, Lemma 1] and hence S is GCR.

Added in proof. Donal O'Donovan in his 1973 Berkeley thesis "Weighted shifts and covariance algebras", among other things, completely characterizes those weighted shifts whose generated C^*-algebra contains the compacts. He also finds necessary and sufficient conditions for a weighted shift which is bounded below to be GCR.

References

1. William B. Arveson, *Subalgebras of C*-algebras*, Acta Math. **123** (1969), 141–224.
2. Horst Behncke, *Structure of certain nonnormal operators*, J. Math. Mech. **18**, (1968), 103–107.
3. Horst Behncke, *Structure of certain nonnormal operators II*, Indiana Univ. Math. J. **22** (1972), 301–308.
4. Harald Bohr, *Almost periodic functions*, Chelsea, New York, 1947.
5. J. W. Bunce & J. A. Deddens, *Irreducible representations of the C*-algebra generated by an n-normal operator*, Trans. Amer. Math. Soc. **171** (1972) 301–307.
6. L. A. Coburn, *The C*-algebra generated by an isometry II*, Trans. Amer. Math. Soc. **937** (1969), 211–217.
7. Jacques Dixmier, *Les C*-algèbres et Leurs Représentations*, Gauthier-Villars, Paris, 1964.
8. Jacques Dixmier, *Les Algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1969.
9. Ronald G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.
10. James G. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), 318–340.
11. P. R. Halmos, *A Hilbert space problem book*, van Nostrand, Princeton, 1967.
12. Meyer Jerison, *The set of all generalized limits of bounded sequences*, Canad, J. Math. **9** (1957), 79–89.
13. B. E. Johnson, *Cohomology in Banach algebras*, Memoir Amer. Math. Soc. Number 127, 1972.
14. G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.
15. T. Okayasu, *On GCR-operators*, Tôhoku Math. J. **21** (1969), 573–579.
16. C. Pearcy, *A complete set of unitary invariants for operators generating finite W*-algebras of type I*, Pacific J. Math. **12** (1962), 1405–1416.
17. M. Takesaki, *On the cross-norm of the direct product of C*-algebras*, Tôhoku Math. J. **16** (1964), 111–122.
18. David M. Topping, *Lectures on von Neumann algebras*, van Nostrand Reinhold, New York, 1971.
19. S. Sakai, *On a characterization of type I C*-algebras*, Bull. Amer. Math. Soc. **72** (1966), 508–512.
20. S. Sakai, *C*-algebras and W*-algebras*, Springer-Verlag, New York, 1971.

Both authors gratefully acknowledge the support of the University of Kansas General Research Fund.

University of Kansas

Date communicated: January 15, 1973