Hints and Partial Solutions for *Topology, 2nd edition* by James R. Munkres

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Part I

General Topology

Chapter 1

Set Theory and Logic

1.1 Fundamental Concepts

1. We prove the second distributive law and the first of DeMorgan's laws.

   If x is in $A \cup (B \cap C)$ then it is either in A or $B \cap C$. If x is in A, then it is in both $A \cup B$ and $A \cup C$, hence it is in $(A \cup B) \cap (A \cup C)$. If it is in $B \cap C$, then it is in both B and C. This means, again, that it is in both $A \cup B$ and $A \cup C$, hence it is in $(A \cup B) \cap (A \cup C)$. Conversely, suppose x is in $(A \cup B) \cap (A \cup C)$. If x is in A, then it is in $A \cup (B \cap C)$, so suppose it is not in A. It is in both $A \cup B$ and $A \cup C$, but not in A, so it must be in both B and C. Hence x is in $B \cap C$ and thus in $A \cup (B \cap C)$.

   If x is in $A - (B \cup C)$ then x is in A and not in $B \cup C$. This means x is not in B and so x is in $A - B$. Similarly, x is not in C so x is also in $A - C$. It follows that x is in $(A - B) \cap (A - C)$. Conversely, if x is in $(A - B) \cap (A - C)$, then x is in A but not B, and also not in C. Thus x is in A but not $B \cup C$, so x is in $A - (B \cup C)$.

2. Let's do (m). If (x, y) is in $(A \times B) \cup (C \times D)$, then it's either in $A \times B$ or $C \times D$. If it's in $A \times B$, then x is in A (hence in $A \cup C$) and y is in B (hence in $B \cup D$). If (x, y) is in $C \times D$, the proof is similar. This shows that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

   The other inclusion may not be true. Take $X = Y = \{a, b\}$ and $A = B = \{a\}$ and $C = D = \{b\}$.

3. (a) The contrapositive is "If $x^2 - x \leq 0$, then $x \geq 0$." The converse is "If $x^2 - x > 0$, then $x < 0$." The original and contrapositive are true; the converse is false.

   (b) The contrapositive is "If $x^2 - x \leq 0$, then $x \leq 0$." The converse is "If $x^2 - x > 0$, then $x > 0$." All of them are false.

4. (a) For at least one $a \in A$, it is true that $a^2 \notin B$.

   (b) For every $a \in A$, it is true that $a^2 \notin B$.

   (c) For at least one $a \in A$, it is true that $a^2 \in B$.

(d) For every $a \notin A$, it is true that $a^2 \notin B$.

5. (a) True.
   (b) False.
   (c) True.
   (d) True.

6. (a) $x \notin A$ for all $A \in \mathcal{A}$ implies $x \notin \bigcup_{A\in\mathcal{A}} A$.
   (b) $x \notin A$ for at least one $A \in \mathcal{A}$ implies $x \notin \bigcup_{A\in\mathcal{A}} A$.
   (c) $x \notin A$ for all $A \in \mathcal{A}$ implies $x \notin \bigcap_{A\in\mathcal{A}} A$.
   (d) $x \notin A$ for at least one $A \in \mathcal{A}$ implies $x \notin \bigcap_{A\in\mathcal{A}} A$.

7. $D = A \cap (B \cup C)$, $E = (A \cap B) \cup C$, $F = (A - B) \cup (A \cap B \cap C)$.

8. If $A = \{a, b\}$, then

$$\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, A\}$$

so $\mathcal{P}(A)$ has four elements. If A has one element, $\mathcal{P}(A)$ has two elements. If A has three elements, $\mathcal{P}(A)$ has eight elements. $\mathcal{P}(\varnothing) = \{\varnothing\}$ so that $\mathcal{P}(\varnothing)$ has one element. $\mathcal{P}(A)$ is called the power set of A because $\mathcal{P}(A)$ has 2^n elements when A has n elements.

9. $X - \bigcup_{A\in\mathcal{A}} A = \bigcap_{A\in\mathcal{A}}(X - A)$ and $X - \bigcap_{A\in\mathcal{A}} A = \bigcup_{A\in\mathcal{A}}(X - A)$

10. (a) $\{(x, y) : x \text{ is an integer}\} = \mathbb{Z} \times \mathbb{R}$.
    (b) $\{(x, y) : 0 < y \leq 1\} = \mathbb{R} \times (0, 1]$.
    (c) Suppose $\{(x, y) : y > x\} = A \times B$ for two subsets A and B of $\mathbb{R}$. We have that both $(0, 1)$ and $(1, 2)$ are in $A \times B$, thus $1 \in A$ and $1 \in B$. It follows that $(1, 1) \in A \times B$, but $1 \not< 1$.
    (d) $\{(x, y) : x \text{ is not an integer and } y \text{ is an integer}\} = (\mathbb{R} - \mathbb{Z}) \times \mathbb{Z}$.
    (e) Suppose $\{(x, y) : x^2 + y^2 < 1\} = A \times B$ for two subsets A and B of $\mathbb{R}$. We have that both $(1/\sqrt{2}, 0)$ and $(0, 1/\sqrt{2})$ are in $A \times B$, thus $1/\sqrt{2} \in A$ and $1/\sqrt{2} \in B$. It follows that $(1/\sqrt{2}), 1/\sqrt{2}) \in A \times B$, but $(1/\sqrt{2})^2 + (1/\sqrt{2})^2 = 1 \not< 1$.

1.2 Functions

1. (a) One simply follows the definitions: if $a \in A_0$, then $f(a) \in f(A_0)$ and so $a \in f^{-1}(f(A_0))$. If $a \in f^{-1}(f(A_0))$, then $f(a) \in f(A_0)$ and a is the only preimage of $f(a)$ if f is injective.
   (b) If $b \in f(f^{-1}(B_0))$, then $b = f(a)$ for some $a \in f^{-1}(B_0)$. But $a \in f^{-1}(B_0)$ means that $b = f(a) \in B_0$. If $b \in B_0$, there exists some $a \in A$ such that $b = f(a)$ provided that f is surjective. Then $a \in f^{-1}(B_0)$ and so $b = f(a) \in f(f^{-1}(B_0))$.

2. (a) If $x \in f^{-1}(B_0)$, then $f(x) \in B_0$. Then $f(x) \in B_1$ by assumption, so $x \in f^{-1}(B_1)$.

3.

4.

5. (a) Suppose $f(x) = f(y)$. If there is a function $g : B \to A$ such that $g \circ f = i_A$, then

$$f(x) = f(y) \text{ implies } g(f(x)) = g(f(y)) \text{ implies } i_A(x) = i_A(y) \text{ implies } x = y.$$

Now suppose $b \in B$. If there exists a function $h : B \to A$ such that $f \circ h = i_B$, then the element $h(b) \in A$ is a preimage for b under f because

$$f(h(b)) = i_B(b) = b.$$

(b) Any function which is injective but not surjective.

(c) Any function which is surjective but not injective.

(d) Yes. Define $f : \mathbb{Z} \to \mathbb{Z} : n \mapsto 2n$. Then

$$g_1(n) = \begin{cases} n/2 & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases} \text{ and } g_2(n) = \begin{cases} n/2 & n \text{ is even} \\ 1 & n \text{ is odd} \end{cases}$$

are both left inverses for f. Now look at $g : \mathbb{Z} \to \{0\} : n \mapsto 0$. Two right inverses for g are $h_1 : \{0\} \to \mathbb{Z} : 0 \mapsto 0$ and $h_2 : \{0\} \to \mathbb{Z} : 0 \mapsto 1$.

(e) By part (a), f is both injective and surjective so it is bijective and has an inverse f^{-1}. We first show that $g = f^{-1}$. Take $b \in B$. Since f is surjective, there is an element $x \in A$ such that $f(x) = b$. Then

$$i_A(x) = i_A(x) \text{ implies } g(f(x)) = f^{-1}(f(x)) \text{ implies } g(b) = f^{-1}(b).$$

Now we show that $h = f^{-1}$. If $b \in B$ again,

$$i_B(b) = i_B(b) \text{ implies } f(h(b)) = f(f^{-1}(b)) \text{ implies } h(b) = f^{-1}(b)$$

because f is injective.

1.3 Relations

1.

2.

3. Take the empty relation on A, that is, $\varnothing \subseteq A \times A$. $\varnothing$ is symmetric and transitive by default, and is not reflexive since $(a, a) \notin \varnothing$ for any $a \in A$. Using this example, it is a small step to see the flaw in the proof provided: it assumes that there are elements in C to begin with.

4.

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9.

10.

11.

12. In (i), every element has an immediate predecessor ((m, n) has $(m, n-1)$), and the set does not have a smallest element.

13. Suppose A has the least upper bound property and A_0 is a nonempty subset of A which is bounded below. Let B be the set of all lower bounds for A_0. B is nonempty and bounded above (any element of A_0 is an upper bound) and the claim is that the least upper bound of B is the greatest lower bound of A_0. Let b be the least upper bound of B. Let us first check that b is a lower bound for A_0: if $a \in A_0$, then $c \leq a$ for all $c \in B$ and so a is an upper bound for B. Since b is the least such, we have $b \leq a$. Seeing that it is the greatest lower bound is easy: if b' is any other lower bound for A_0, then $b' \in B$ and thus $b' \leq b$ since b is an upper bound for B.

1.4 The Integers and the Real Numbers

1.

2.

3. (a) Clearly $1 \in \bigcap_{A \in \mathcal{A}} A$ since 1 is in each A. If $x \in \bigcap_{A \in \mathcal{A}} A$, then x is in each A, hence $x+1$ is in each A because they are all inductive. It follows that $x+1 \in \bigcap_{A \in \mathcal{A}} A$ and that $\bigcap_{A \in \mathcal{A}} A$ is inductive.

   (b) Property (1) follows from part (a). If A_0 is an inductive set of positive integers, it is in particular a set of positive integers and so $A_0 \subseteq \mathbb{Z}_+$. On the other hand, if $\mathcal{A}$ is the collection of all inductive subsets of $\mathbb{R}$, then

$$\mathbb{Z}_+ = \bigcap_{A \in \mathcal{A}} A \subseteq A_0$$

   since $A_0 \in \mathcal{A}$.

4.

5.

6.

7.

8. (a) This follows from Exercise 13 in Section 1.3.

   (b) The fact that 0 is a lower bound follows from $n > 0$ for all $n \in \mathbb{Z}_+$, and Exercise 2 (i). To see that 0 is the greatest lower bound, suppose some $x > 0$ satisfies $x \leq 1/n$ for all n. Then $n \leq 1/x$ for all n, contradicting the fact that $\mathbb{Z}_+$ is not bounded above.

(c) First observe that 0 is indeed a lower bound (use $a > 0$, Exercise 2 (b), and induction). Following the hint, we let $h = (1-a)/a$ and show by induction that $(1+h)^n \geq 1 + nh$ for all $n \in \mathbb{Z}_+$. It is clearly true for 1, so suppose it is true for an arbitrary n. Then

$$(1+h)^{n+1} \geq (1+nh)(1+h) = 1 + (n+1)h + nh^2 \geq 1 + (n+1)h.$$

since $nh^2 \geq 0$ (by Exercise 2 (f)). Now note that $(1+h)^n = 1/a^n$, $1+nh = n/a-(n-1)$, and so we have shown that

$$\frac{1}{n(\frac{1}{a}-1)+1} \geq a^n$$

for all $n \in \mathbb{Z}_+$. Now use part (b) in some clever way.

1.5 Cartesian Products

1. Define

$$\Phi : A \times B \to B \times A : (a, b) \mapsto (b, a).$$

Φ is injective: if $(b, a) = (b', a')$, then $a = a'$ and $b = b'$, so $(a, b) = (a', b')$. It is also surjective since (b, a) has preimage (a, b).

2.

3.

4. (a) Since X is nonempty, let $x \in X$. Then set

$$f : X^m \to X^n : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, \overbrace{x, \ldots, x}^{n-m}).$$

(b) Set

$$g : X^m \times X^n \to X^{m+n} : ((x_1, \ldots, x_m), (y_1, \ldots, y_n)) \mapsto ((x_1, \ldots, x_m, y_1, \ldots, y_n).$$

(c) Again let $x \in X$. Set

$$h : X^n \to X^\omega : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, x, x, x, \ldots).$$

(d) Set

$$k : X^n \times X^\omega \to X^\omega : ((x_1, \ldots, x_n), (y_1, y_2, \ldots)) \mapsto (x_1, \ldots, x_n, y_1, y_2, \ldots).$$

(e) Set

$$\ell : X^\omega \times X^\omega \to X^\omega : ((x_1, x_2, \ldots), (y_1, y_2, \ldots)) \mapsto (x_1, y_1, x_2, y_2, x_3, \ldots).$$

(f) If $\mathbf{x}_k = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots)$ is in A^ω for $k = 1, 2, \ldots n$, set

$$m : (A^\omega)^n \to B^\omega$$

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \mapsto (x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n)}, x_2^{(1)}, x_2^{(2)}, \ldots).$$

5. (a) $\mathbb{Z}^\omega$.

(b) $\prod_{i=1}^{\infty}[i, \infty)$.

(c) $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{Z} \times \mathbb{Z} \times \cdots$.

(d) Suppose $\{\mathbf{x} : x_2 = x_3\} = A_1 \times A_2 \times A_3 \times \cdots$ where each A_i is a subset of $\mathbb{R}$. Then the elements $(0, 0, 0, \ldots)$ and $(1, 1, 1, \ldots)$ are both in $\prod A_i$, which means 0 and 1 are both in A_i for all i. But then $(0, 1, 0, 0, 0, \ldots) \in \prod A_i = \{\mathbf{x} : x_2 = x_3\}$, which is a contradiction because $0 \neq 1$.

1.6 Finite Sets

1.

2. This is just the contrapositive of Corollary 6.6.

3. Set
$$\Phi : X^\omega \to X^\omega : (x_n)_{n\in\mathbb{Z}^+} \mapsto (x_{n+1})_{n\in\mathbb{Z}^+}.$$
The image of Φ is a proper subset of X^ω, and Φ is a bijection onto its image.

4.

5. Yes. If $b \in B$ is arbitrary, there is a bijection between A and $A \times \{b\}$. $A \times \{b\}$ is a proper subset of $A \times B$, so $A \times \{b\}$, and hence A, is finite. An identical argument applies to B.

6.

7. Every function $f : A \to B$ is a subset of $A \times B$, thus there can be only finitely many.

1.7 Countable and Uncountable Sets

1. The set $\mathbb{Z} \times \mathbb{Z}$ is countable since $\mathbb{Z}$ is countable, and the function
$$f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$$
defined by $f(r) = (m, n)$, where $r = m/n$ is in lowest terms and $n > 0$, is well-defined and injective.

2.

3. The bijection is given by $\mathcal{P}(\mathbb{Z}_+) \to X^\omega : A \mapsto \chi_A$, where χ_A is the characteristic function of $A \subseteq \mathbb{Z}_+$.

4.

5.

6. (a) Following the hint, we verify that $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \cdots$. Clearly $A_1 \supset B_1$, so suppose $A_n \supset B_n$; we will show that $A_{n+1} \supset B_{n+1}$. If $x \in B_{n+1} = f(B_n)$, then $x = f(y)$ for some $y \in B_n$. $y \in A_n$ by assumption, so $x \in f(A_n) = A_{n+1}$. That $A_n \subseteq B_{n-1}$ for all $n \geq 2$ is proven similarly.

Now take the function $h : A \to B$ as defined and suppose that $h(x) = h(y)$. To show that h is injective we consider three cases.

Case 1. $x \notin A_n - B_n$ for any n and $y \notin A_n - B_n$ for any n. Then $x = h(x) = h(y) = y$.

Case 2. $x \in A_n - B_n$ for some n and $y \in A_m - B_m$ for some m. Then $f(x) = h(x) = h(y) = f(y)$, so $x = y$ since f is injective.

Case 3. $x \in A_n - B_n$ for some n and $y \notin A_m - B_m$ for any m (we are assuming, then, that $x \neq y$). We have $f(x) = h(x) = h(y) = y$, but $y = f(x) \in f(A_n - B_n) = f(A_n) - f(B_n) = A_{n+1} - B_{n+1}$, a contradiction.

To see that h is surjective, let $b \in B$. If $b \in A_n - B_n$ for some n, then $b = f(x) = h(x)$ for some $x \in A_{n-1} - B_{n-1}$. If $b \notin A_n - B_n$ for any n, then $b = h(b)$.

(b) We have $g(C) \subseteq A$ and $g \circ f : A \to g(C)$ is an injection, so apply part (a).

1.8 The Principle of Recursive Definition

1. If $f : \{1, \ldots, m\} \to \mathbb{R}$, let $\rho(f) = f(m) + b_{m+1}$. Then there is a function $h : \mathbb{Z}_+ \to \mathbb{R}$ such that $h(1) = b_1$ and
$$h(i) = \rho(h|_{\{1,\ldots,i-1\}}) = h(i-1) + b_i.$$
Then $h(n) = \sum_{i=1}^{n} b_i$.

2. Similar to the previous problem, except let $\rho(f) = f(m)b_{m+1}$.

3. Take the sequences $\{a, a, a, \ldots\}$ and $\{1, 2, 3, \ldots\}$.

4. If $f : \{1, \ldots, m\} \to \mathbb{Z}_+$, let
$$\rho(f) = \begin{cases} f(m) + f(m-1) & m \geq 2 \\ 1 & m = 1 \end{cases}.$$
Then Theorem 8.4 gives a function h such that $h(1) = 1$,
$$h(2) = \rho(h|_{\{1\}}) = 1$$
and
$$h(i) = \rho(h|_{\{1,\ldots,i-1\}}) = h(i-1) + h(i-2)$$
when $i > 2$.

5. Set $\rho(f) = \sqrt{f(m) + 1}$.

6. (a) Iterating the formula a couple of times, we obtain $h(2) = \sqrt{2}$ and $h(3) = \sqrt{\sqrt{2} - 1}$, but then $h(4)$ is not well-defined since $\sqrt{\sqrt{2} - 1} < 1$.

   This does not violate the principle because ρ cannot be well-defined. If one were to try defining $\rho(f) = \sqrt{f(m) - 1}$, it cannot be guaranteed that $f(m) - 1 \geq 0$.

1.9 Infinite Sets and the Axiom of Choice

1. Setting $f(n) = (0, 0, \ldots, 0, \overset{n}{1}, 0, 0, \ldots)$ does not require choice.
2.
3.
4. In the proof of Theorem 7.5, an arbitrary function $f_n : \mathbb{Z}_+ \to A_n$ was found for each n. Each set
$$\mathcal{A}_n = \{\text{all surjective functions } \mathbb{Z}_+ \to A_n\}$$
is nonempty since each A_n is countable. Then use the choice axiom on the collection $\{\mathcal{A}_n\}$.

1.10 Well-Ordered Sets

1. It is equivalent to show that every well-ordered set has the greatest lower bound property. (Why?) If A is well-ordered and B is a nonempty subset, denote by a the least element of B. This is certainly a lower bound for B, and it is the greatest because it is an element of B.
2. (a) Given any element x which is not the largest element, let
$$A = \{y : x < y\}.$$
A is not empty since x is not the largest element, so A contains a smallest member. This member is the immediate successor of x.

   (b) Take $\mathbb{Z}$ with its usual order.
3. No. In $\{1, 2\} \times \mathbb{Z}_+$, the element $(2, 1)$ has no immediate predecessor, but every element of $\mathbb{Z}_+ \times \{1, 2\}$ has an immediate predecessor.
4. (a) Suppose A is not well-ordered, so there is a nonempty set $B \subseteq A$ such that B has no least element. Choose any b_{-1} in B; b_{-1} is not the smallest element of B (there is none) so there is an element b_{-2} such that $b_{-2} < b_{-1}$. Continuing, we obtain
$$b_{-1} > b_{-2} > b_{-3} > b_{-4} > \cdots$$
and
$$\mathbb{Z}_- \to B : n \mapsto b_n$$
is an order preserving injection into A. Conversely, if we have an order preserving injection of $\mathbb{Z}_-$ into A, its range is a subset of A which has no least element.

   (b) If A were not well-ordered, part (a) would give a sequence $\{b_n\}$ in A which is countable and not well-ordered.
5. Let $\mathcal{A}$ be a nonempty collection of sets. Put a well-ordering on each $A \in \mathcal{A}$. Then define $c(A)$ to be the least element of A.
6. (a) Suppose S_Ω has a largest element α. Then $S_\Omega = S_\alpha \cup \{\alpha\}$. But S_α is countable, hence S_Ω is countable, a contradiction.

(b) If not, S_Ω would be a finite union of countable sets, namely $S_\alpha \cup \{x : \alpha < x\} \cup \{\alpha\}$.

(c) Suppose X_0 is countable. By Theorem 10.3, X_0 has an upper bound, say α. Since every element β in S_Ω has an immediate successor $s(\beta)$, consider

$$\{\alpha, s(\alpha), s(s(\alpha)), \ldots\}.$$

The above set is countable and thus has a least upper bound γ. Then γ has no immediate predecessor. If δ was an immediate predecessor of γ, then there is some n such that $\delta = s^n(\alpha)$, but then $\delta < s^{n+1}(\alpha) < \gamma$. It follows that $\gamma \in X_0$, which cannot happen since γ is an upper bound for X_0 which is larger than α.

7. If $J_0 \neq J$, choose the smallest element α which is not in J_0. Then $S_\alpha \subseteq J_0$, whence $\alpha \in J_0$, a contradiction.

8.

9.

10. (a) Let J_0 be the set of all elements of J at which h and k agree. If $x \in J$ and $S_x \subseteq J_0$, then $h(S_x) = k(S_x)$ and

$$h(x) = \text{smallest}[C - h(S_x)] = \text{smallest}[C - k(S_x)] = k(x).$$

So by the principle of transfinite induction, $J_0 = J$ and $h = k$.

(b)

(c)

1.11 The Maximum Principle

1. $a - a = 0$ is not positive, and $c - a = (c - b) + (b - a) > 0$ is positive and rational if both $c - b$ and $b - a$ are. $\mathbb{Q}$ is a maximal simply ordered subset: if A is another which properly contains $\mathbb{Q}$, then any irrational number is not comparable with 0 (or any other rational number). The other maximal simply ordered subsets are $a + \mathbb{Q}$ for $a \in \mathbb{R}$ since they have the same order type as $\mathbb{Q}$.

2.

3.

4.

5. The union of elements in any such collection $\mathcal{B}$ is an upper bound for the elements of $\mathcal{B}$. Zorn's Lemma then implies that $\mathcal{A}$ has a maximal element, i.e., an element which is not properly contained in any other element.

6. Let $\mathcal{B}$ be a subcollection which is simply ordered by proper inclusion, and consider $\bigcup_{B \in \mathcal{B}} B$. If $F \subseteq \bigcup_{B \in \mathcal{B}} B$ is finite, then $F \subseteq B_0$ for some $B_0 \in \mathcal{B}$ since $\mathcal{B}$ is simply ordered. Then F belongs to $\mathcal{A}$ since it is a finite subset of B_0, which is in $\mathcal{A}$. Thus $\bigcup_{B \in \mathcal{B}} B$ is in $\mathcal{A}$, and Kuratowski's Lemma implies that there is a maximal element of $\mathcal{A}$.

7. Take a strict partial order on a set A. Following the hint, let $\mathcal{A}$ be the collection of all subsets of A which are simply ordered. We show that $\mathcal{A}$ is of finite type.

   Suppose $B \in \mathcal{A}$ and F is a finite subset of B. Clearly F is simply ordered. Conversely, if B is not simply ordered, choose two distinct elements x and y in B which are not comparable. Then $F = \{x, y\}$ is a finite subset of B which is not simply ordered.

   Applying the Tukey Lemma gives a maximal simply ordered subset of A.

8.

1.12 Supplementary Exercises: Well-Ordering

1.

Chapter 2

Topological Spaces and Continuous Functions

2.1 Topological Spaces

2.2 Basis for a Topology

1. For each $x \in A$, choose an open set U_x such that $x \in U_x$ and $U_x \subseteq A$. Then $A = \bigcup_{x \in A} U_x$ is a union of open sets.
2.
3.
4.
5.
6. $[0,1)$ is open in $\mathbb{R}_\ell$ but not in $\mathbb{R}_K$. Indeed, there is no set of the form (a,b) or $(a,b) - K$ that contains 0 and is contained in $[0,1)$. Conversely, the set $\mathbb{R} - K$ is open in $\mathbb{R}_K$, but if $a < b$ and $0 \in [a,b)$, then $[a,b) \cap K \neq \varnothing$. So $\mathbb{R} - K$ is not open in $\mathbb{R}_\ell$.

2.3 The Order Topology

None

2.4 The Product Topology on $X \times Y$

None

2.5 The Subspace Topology

1. If U is open in A (from either the topology from X or Y) and V is open in X, note that $U = A \cap V = A \cap (V \cap Y)$ since $A \subseteq Y$.
2. The subspace topology from $\mathcal{T}'$ is strictly finer, since any open set from $\mathcal{T}$ intersecting Y will be an open set from $\mathcal{T}'$.
3.
4. It suffices to check basis elements since maps preserve arbitrary unions. Let U and V be open in X and Y respectively. Then $\pi_1(U \times V) = U$ and $\pi_2(U \times V) = V$; both are open.
5.
6. This follows from Theorem 15.1 and Exercise 8(a) from Section 13.

2.6 Closed Sets and Limit Points

1. DeMorgan's Laws.
2. $A = C \cap Y$ where C is closed in X, and $C \cap Y$ is the intersection of two sets which are closed in X.
3. $X \times Y - A \times B = (X \times (Y - B)) \cup ((X - A) \times Y)$.
4. $U - A = U \cap (X - A)$ and $A - U = A \cap (X - U)$.
5. The inclusion holds because $[a, b]$ is a closed set which contains (a, b). For equality to hold, the points a and b must be limit points of (a, b). This occurs if and only if a does not have an immediate successor and b does not have an immediate predecessor. Indeed, if a has an immediate successor a_+, then $(-\infty, a_+)$ is a neighbourhood of a which does not intersect (a, b). If a does not have an immediate successor, then any basic neighbourhood of a, $(-\infty, c)$ where $a < c$, contains points of (a, b).
6. (a) If $A \subseteq B$, then $A \subseteq \overline{B}$. $\overline{B}$ is a closed set containing A, so $\overline{A} \subseteq \overline{B}$.
   (b) $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, so $A \cup B \subseteq \overline{A} \cup \overline{B}$. The set $\overline{A} \cup \overline{B}$ is closed, so $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. On the other hand, if $x \in \overline{A} \cup \overline{B}$, then every neighbourhood of x contains points of either A or B, hence points of $A \cup B$. It follows that $x \in \overline{A \cup B}$.
   (c) The proof of the inclusion is similar to the proof in (b). For the counterexample,
   $$\bigcup_{r \in \mathbb{Q}} \overline{\{r\}} = \bigcup_{r \in \mathbb{Q}} \{r\} = \mathbb{Q},$$
   but
   $$\overline{\bigcup_{r \in \mathbb{Q}} \{r\}} = \overline{\mathbb{Q}} = \mathbb{R}.$$
7. It is true that every neighbourhood U must intersect $\bigcup A_\alpha$, but different neighbourhoods may intersect different sets A_α.

8.

9.

10. If x and y are distinct points with $x < y$, then $(-\infty, y)$ and (x, ∞) are disjoint neighbourhoods.

11.

12. If x and y are distinct points in the subspace $A \subseteq X$, they have disjoint neighbourhoods U and V which are open in X. Then $U \cap A$ and $V \cap A$ are disjoint neighbourhoods in A.

13. If $x \neq y$ in X, then $x \times y$ has a basic neighbourhood $U \times V$ that does not intersect Δ. This implies that U and V are disjoint. Conversely, if X is Hausdorff, the complement of Δ is a union of neighbourhoods of the form $U \times V$ (where U and V separate x and y), hence is open.

14.

15.

16.

17.

18.

19. (a) g

    (b) If A is clopen, then $A = \overline{A}$ and $X - A = \overline{X - A}$, so $\overline{A} \cap \overline{(X - A)} = A \cap (X - A) = \varnothing$. If the boundary is empty, then the closure of A is equal to the interior of A by (a), hence A is clopen.

    (c) If U is open, then $\overline{X - U} = X - U$ and so $\overline{U} - U = \overline{U} \cap (X - U)$. Conversely, if $\overline{U} - U = \overline{U} \cap \overline{(X - U)}$,

    (d) It is not true: take $U = (-\infty, 0) \cup (0, \infty)$ in $\mathbb{R}$.

2.7 Continuous Functions

1. Let V be open in $\mathbb{R}$. If $f^{-1}(V) = \varnothing$ there is nothing to do, so assume $x \in f^{-1}(V)$. Then $f(x) \in V$ and thus there is an $\varepsilon > 0$ such that $(f(x) - \varepsilon, f(x) + \varepsilon) \subseteq V$ because V is open. By assumption, there is a $\delta > 0$ so that

$$y \in (x - \delta, x + \delta) \text{ implies } f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon)$$

which means that $(x - \delta, x + \delta) \subseteq f^{-1}(V)$. Thus $f^{-1}(V)$ is open.

2. No: let $f : \mathbb{R} \to \mathbb{R} : x \mapsto 0$. Then 0 is a limit point of $(0, 1)$, but the set $f((0, 1)) = \{0\}$ has no limit points at all.

3.

4.

5. Maps of the like

$$f : [0,1] \to [a,b] : x \mapsto (b-a)x + a$$

   do the trick.

6. The classical example is $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} .$$

7.

8.

2.8 The Product Topology

1. If $(x_\alpha) \in \prod X_\alpha$, choose, for each α, a basis element B_α with $x_\alpha \in B_\alpha$. Then $(x_\alpha) \in \prod B_\alpha$, so every point is contained in a basis element.

   If $\prod B_\alpha$ and $\prod C_\alpha$ are two basis elements with nonempty intersection, choose, for each α, a basis element $D_\alpha \subseteq B_\alpha \cap C_\alpha$. Then

$$\prod D_\alpha \subseteq \prod (B_\alpha \cap C_\alpha) = \left(\prod B_\alpha\right) \cap \left(\prod C_\alpha\right).$$

2.

3.

4.

5.

6.

7. The closure in the product topology is $\mathbb{R}^\omega$. The closure in the box topology is $\mathbb{R}^\infty$.

8.

9. An element of $\prod A_\alpha$ is a choice function, so the product being nonempty is the same as a choice function existing.

2.9 The Metric Topology

1.

2.

3.

4.

5. The closure of $\mathbb{R}^\infty$ in the uniform topology is all sequences which converge to 0. If $x_n \to 0$ and $\varepsilon > 0$, pick N large enough so that $|x_n| < \varepsilon/2$ for $n \geq N$. Then

$$(x_1, x_2, \ldots, x_N, 0, 0, \ldots)$$

is in the ε-neighbourhood of $\{x_n\}$.

6. (a) For simplicity let $\mathbf{x} = (0, 0, 0, \ldots)$. So

$$U = (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times \cdots .$$

The element

$$\mathbf{y} = \left(\frac{\varepsilon}{2}, \frac{2\varepsilon}{3}, \ldots, \frac{n\varepsilon}{n+1}, \ldots\right)$$

is in U but not in $B_{\overline{\rho}}(\mathbf{x}, \varepsilon)$ since $\overline{\rho}(\mathbf{x}, \mathbf{y}) = \varepsilon \not< \varepsilon$.

(b) The point $\mathbf{y}$ from part (a) has no neighbourhood ball contained in U.

(c) If $\overline{\rho}(\mathbf{x}, \mathbf{z}) < \varepsilon$, then there is a $\delta > 0$ so that $\overline{\rho}(\mathbf{x}, \mathbf{z}) < \delta < \varepsilon$. Then $\mathbf{z} \in U(\mathbf{x}, \delta)$. Conversely, if $\mathbf{z} \in U(\mathbf{x}, \delta)$ for some $\delta < \varepsilon$, then $\overline{\rho}(\mathbf{x}, \mathbf{z}) \leq \delta < \varepsilon$.

2.10 The Metric Topology (continued)

1.

2. If $x_n \to x$ in X, then $d(f(x_n), f(x)) = d(x_n, x) \to 0$, so $f(x_n) \to f(x)$ and f is continuous. f is injective because if $f(x) = f(y)$, then $d(x, y) = d(f(x), f(y)) = 0$ so $x = y$.

3.

4.

5.

6.

7.

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9.

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = xy$ and $g(x, y) = x^2 + y^2$. They are both continuous by Exercise 5, and $A = f^{-1}(\{1\})$, $S^1 = g^{-1}(\{1\})$, and $B^2 = g^{-1}([0, 1])$.

2.11 The Quotient Topology

1. This is more satisfying to do in one's head rather than writing it down. Pull back all possible subsets of $\{a, b, c\}$ to see if the preimages are open in $\mathbb{R}$.

2. (a) Let $U \subseteq Y$ such that $p^{-1}(U)$ is open in X. Then $U = f^{-1}(p^{-1}(U))$ is open since f is continuous.

   (b) Let $\iota : A \to X$ denote the inclusion. It is continuous and $r \circ \iota$ is the identity on A, so r is a quotient map by part (a).

3. The set $[0, \infty) \times (1, 2)$ is open in A, but $[0, \infty)$ is not open in $\mathbb{R}$. The set of points on the curve $y = 1/x$ for $x > 0$ is closed in A, but $(0, \infty)$ is not closed in $\mathbb{R}$.

   Note that saturated sets in A are unions of vertical lines in the right half plane and individual points in the left half plane. Let U be a saturated open set in A and consider a point $x_0 \in \pi_1(U)$. The only interesting case is when $x_0 = 0$, so I'll leave the cases $x_0 > 0$ and $x_0 < 0$ out. Since $0 \in \pi_1(U)$, we have that $\pi_1^{-1}(\{0\}) = 0 \times \mathbb{R} \subseteq U$ since U is saturated. Thus $0 \times 0 \in U$ so we may choose an open rectangle $V = (a, b) \times (c, d)$ centred at 0×0 such that $V \cap A \subseteq U$ (at this point it helps to draw a picture).

   Now the crux of the argument: because V is centred at 0×0 it must contain some members of the negative horizontal axis (which are in A). Then $\pi_1(V) = \pi_1(V \cap A) = (a, b)$ is open in $\mathbb{R}$, $0 \in \pi_1(V)$, and $\pi_1(V) \subseteq \pi_1(U)$. It follows that $\pi_1(U)$ is open.

2.12 Supplementary Exercises: Topological Groups

Chapter 3

Connectedness and Compactness

3.1 Connected Spaces

1. If $A \cup B$ is a separation of X in the coarser topology, then it is also a separation in the finer topology since A and B are both open in the finer topology.

2. Suppose $C \cup D$ is a separation of $\bigcup A_n$. Then each A_n lies entirely in C or entirely in D. Let
$$\mathcal{M} = \{n : A_n \subseteq C\} \text{ and } \mathcal{N} = \{n : A_n \subseteq D\}.$$
Use some well-ordering nonsense to show that there is a pair of consecutive integers k and $k+1$ such that $A_k \subseteq C$ and $A_{k+1} \subseteq D$ (or vice versa). But then $A_k \cap A_{k+1} = \varnothing$.

3. Suppose $C \cup D$ is a separation of $A \cup (\bigcup A_\alpha)$. A is connected so it lies entirely in C (or in D). D is nonempty, so it must contain an element of some A_{α_0}, hence all of A_{α_0} because A_{α_0} is connected. But then $A \cap A_{\alpha_0} = \varnothing$.

4. There cannot be a nonempty proper set which is open and closed since such a set would be simultaneously finite and cofinite.

5. If a subset of a discrete space has more than one point, any nontrivial partition gives a separation because all subsets are open.

6.

7.

8.

9.

10.

11. Suppose $C \cup D$ is a separation of X. Then each $p^{-1}(\{y\})$ lies entirely in C or D since each is connected. It follows that C and D are open and saturated, so $p(C)$ and $p(D)$ are open in Y. But then $p(C) \cup p(D)$ is a separation of Y.

12.

3.2 Connected Subspaces of the Real Line

1. (a) Suppose $f : (0,1] \to (0,1)$ is a homeomorphism. If we denote $x_0 = f(1)$, then $f|_{(0,1)} : (0,1) \to (0,x_0) \cup (x_0,1)$ is a homeomorphism. But $(0,1)$ is connected and $(0,x_0) \cup (x_0,1)$ is not. The other cases are similar.

   (b) $(0,1)$ can clearly be imbedded in $(0,1]$ (via the inclusion map) and $(0,1]$ can be imbedded into $(0,1)$ by $x \mapsto \frac{1}{2}x$.

   (c) Similar tactic as in part (a): remove a point from $\mathbb{R}$ and it becomes disconnected, but not $\mathbb{R}^n$.

2. The map g defined by $g(x) = f(x) - f(-x)$ is continuous. $g(1) = f(1) - f(-1)$ and $g(-1) = f(-1) - f(1)$, hence $g(1) = -g(-1)$. If $g(1) = 0$, then $f(1) = f(-1)$ and we are done. Otherwise, $g(1) > 0$ and $g(-1) < 0$ (or vice versa) hence there is a point $c \in S^1$ such that $g(c) = 0$ by the Intermediate Value Theorem. Then $f(c) = f(-c)$.

3. Same idea as the previous exercise with $g(x) = f(x) - x$. Take the function $f(x) = \frac{1}{2}(x+1)$ for the two counterexamples.

4. Suppose there are two elements x and y of X such that there is no z with $x < z < y$. Then $(-\infty, y) \cup (x, \infty)$ is a separation of X.

   Now suppose A is a subset of X which is bounded above but does not have a least upper bound. Let

   $$C = \bigcup\{(-\infty, x) : x \in A\} \text{ and } D = \bigcup\{(y, \infty) : y \text{ is an upper bound for } A\}.$$

   Then $C \cup D$ is a separation of X.

5.

6.

7.

8.

9. Let x and y be two points in $\mathbb{R}^2 - A$. There are uncountably many lines passing through each of x and y, so there is a pair of lines, one passing through x and one through y, which are not parallel and do not intersect A. These two lines give a path between x and y.

10.

11.

12.

3.3 Components and Local Connectedness

3.4 Compact Spaces

1. (a) If $\mathcal{T}'$ is compact, so is $\mathcal{T}$. A cover consisting of elements of $\mathcal{T}$ is also a cover consisting of elements of $\mathcal{T}'$, and thus has a finite subcover.

   (b) Suppose they are comparable, say $\mathcal{T} \subseteq \mathcal{T}'$. The identity map $(X, \mathcal{T}') \to (X, \mathcal{T}) : x \mapsto x$ is continuous since $\mathcal{T}' \supset \mathcal{T}$, and is bijective. Theorem 26.6 implies that it is a homeomorphism, and so $\mathcal{T} = \mathcal{T}'$.

2. (a) If $\{U_\alpha\}$ is an open cover, choose one U_{α_0} that contains all but finitely many points. Then choose finitely many members of the collection to cover the remaining points.

   (b) No, in fact, the only compact subsets of $\mathbb{R}$ in this topology are the finite ones. For suppose $A \subseteq \mathbb{R}$ is infinite and let $C = \{x_1, x_2, \ldots\}$ be a countably infinite subset of A. Then
   $$C_n = \{x_n, x_{n+1}, \ldots\}$$
   is closed in A, nonempty, and $C = C_1 \supseteq C_2 \supseteq \cdots$. But $\bigcap C_n = \varnothing$.

3. The union of finitely many collections of sets, each containing finitely many sets, again contains finitely many sets.

4. The first part of the proof of Theorem 27.3 carries over. For the counterexample, take any infinite set and put the discrete metric on it.

5. For each $x \in A$ use Lemma 26.4 to obtain disjoint open sets U_x and V_x with $x \in U_x$ and $B \subseteq V_x$. The sets U_x cover A, so choose finitely many
   $$U_{x_1}, \ldots U_{x_n}$$
   that cover A. Then the open sets
   $$\bigcup_{i=1}^{n} U_{x_i} \text{ and } \bigcap_{i=1}^{n} V_{x_i}$$
   are disjoint and respectively contain A and B.

6.

7. Suppose F is closed in $X \times Y$ and let x_0 be a point not in $\pi_1(F)$ (If $\pi_1(F) = X$ there is nothing to do). Then the slice $x_0 \times Y$ is disjoint from F. For every $y \in Y$, choose a neighbourhood U_y in $X \times Y$ of $x_0 \times y$ that is disjoint from F. Then $\bigcup U_y$ is a tube about $x_0 \times Y$, so apply the tube lemma to obtain a neighbourhood W of x_0 such that $W \times Y \subseteq \bigcup U_y$. Then W is disjoint from $\pi_1(F)$, hence $X - \pi_1(F)$ is open.

3.5 Compact Subspaces of the Real Line

1. Let $A \subseteq X$ be nonempty and bounded above. Consider the collection

$$\mathcal{A} = \{[a, b] \mid a \in A \text{ and } b \text{ is an upper bound for } A\}$$

Show that $\mathcal{A}$ has the finite intersection property, so that $C = \bigcap_{A \in \mathcal{A}} A$ is nonempty. Then show that if x is in C, x is the least upper bound for A.

2. (a) $d(x, A) = 0$ if and only if $d(x, A) < \varepsilon$ for every $\varepsilon > 0$, if and only if for every $\varepsilon > 0$ there is some a in A with $d(x, a) < \varepsilon$, if and only if $B(x, \varepsilon) \cap A \neq \varnothing$ for every $\varepsilon > 0$, if and only if $x \in \overline{A}$.

   (b) Fix x. We show that the map

$$A \to \mathbb{R} : a \mapsto d(x, a)$$

   is continuous. If $a_n \to a$, that is, $d(a_n, a) \to 0$, then

$$|d(x, a_n) - d(x, a)| \leq d(a_n, a) \to 0$$

   by the triangle inequality. Note that compactness was not used here. It is, however, necessary in applying Theorem 27.4 which says that the above map attains a minimum, which is what we wanted to show.

   (c) Fix $\varepsilon > 0$. We want to show that

$$U(A, \varepsilon) = \bigcup_{a \in A} B(a, \varepsilon).$$

   $x \in U(A, \varepsilon)$ if and only if $d(x, A) < \varepsilon$, if and only if $d(x, a) < \varepsilon$ for some a in A, if and only if $x \in B(a, \varepsilon)$ for some a in A.

   (d) For each a in A, there is some $\varepsilon_a > 0$ such that $B(a, 2\varepsilon_a) \subseteq U$. The open balls $B(a, \varepsilon_a)$ cover A, so there is a finite number $B(a_i, \varepsilon_{a_i})$ for $i = 1, 2, \ldots, n$ which cover A. Let $\varepsilon = \min\{\varepsilon_{a_i}\}$. Then $U(A, \varepsilon) \subseteq U$. Indeed, if $a \in A$, then $a \in B(a_i, \varepsilon_{a_i})$ for some i and thus

$$B(a, \varepsilon) \subseteq B(a_i, \varepsilon + \varepsilon_{a_i}) \subseteq B(a_i, 2\varepsilon_{a_i}) \subseteq U.$$

   (e) See Figure 26.3 for a hint.

3.

4.

3.6 Limit Point Compactness

1. $x_n = (0, 0, \ldots, 0, \overset{n}{1}, 0, \ldots)$.

2. The sequence $n/(n+1)$ has no limit point in $[0, 1]$ with the lower limit topology. The only candidate would be 1, but $\{1\}$ is a neighbourhood that contains no points of the sequence.

3.

4.

5.

6. If $f(x) = f(y)$, then $d(x, y) = d(f(x), f(y)) = 0$, hence $x = y$ and f is injective. If $a \notin f(X)$, we may choose an ε-neighbourhood of a which is disjoint from $f(X)$ because $f(X)$ is compact in X and hence closed. If we define a sequence in X by $x_n = f^{n-1}(a)$ for $n \geq 1$, then $d(x_m, x_n) \geq \varepsilon$ for all $m \neq n$. Indeed, suppose $m < n$. Then

$$d(x_m, x_n) = d(f^{m-1}(a), f^{n-1}(a)) = d(a, f^{n-m}(a)) \geq \varepsilon$$

where we obtain the second equality because f is an isometry. But then $\{x_n\}$ cannot have a convergent subsequence.

7. (a) First we show uniqueness. If x and y are fixed points of f, then

$$d(x, y) = d(f(x), f(y)) \leq \alpha d(x, y)$$

and since $\alpha < 1$, we must have $d(x, y) = 0$.

Now for existence. Suppose, by normalizing the metric if necessary, that the diameter of X is 1. Then $A_1 = f(X)$ has diameter at most α. Indeed,

$$d(f(x), f(y)) \leq \alpha d(x, y) \leq \alpha.$$

Similarly, $A_2 = f(f(X))$ has diameter at most α^2, and more generally $A_n = f^n(X)$ has diameter at most α^n. Since $\alpha^n \to 0$, the intersection $\bigcap A_n$ consists of at most one point. Since each A_n is closed and X is compact, $\bigcap A_n$ consists of exactly one point, say x. Then x is a fixed point of f. Indeed, since $x \in A_1 \cap A_2 \cap A_3 \cap \cdots$, we have $f(x) \in A_2 \cap A_3 \cap A_4 \cap \cdots$. But since this intersection consists of only one point, it must be that $f(x) = x$.

(b) Uniqueness is proven similarly as in (a), that is, if x and y are distinct fixed points, then

$$d(x, y) = d(f(x), f(y)) < d(x, y),$$

a contradiction.

For existence, consider the sets A_n from (a). $\bigcap A_n$ is nonempty as before, so we need only show that it consists of a single point. Suppose $x \in \bigcap A_n$ and for each n, choose x_n so that $x = f^{n+1}(x_n)$. X is compact so the sequence $y_n = f^n(x_n)$ has a limit point, say $y_{n_k} \to a$. Then

$$f(a) = \lim_{k\to\infty} f(y_{n_k}) = \lim_{k\to\infty} f(f^{n_k}(x_{n_k})) = \lim_{k\to\infty} f^{n_k+1}(x_{n_k}) = \lim_{k\to\infty} x = x$$

where we obtain the first equality because f is continuous. If $a \notin A_N$ for some N, then $d(y_n, a) \geq \varepsilon$ for some $\varepsilon > 0$ for $n > N$ because A_N is closed, but then $y_{n_k} \not\to a$. So $a \in \bigcap A_n$, and thus $A \subseteq f(A)$. The inclusion $f(A) \subseteq A$ is clear, so $A = f(A)$ and so the diameter of A is 0 because f is shrinking.

(c) $f(0) = 0$ and $f(1) = 1/2$, while $f'(x) = 1 - x \geq 0$. f is therefore increasing and so $f(X) \subseteq [0, 1]$. Suppose there was a number $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for all x and y in $[0, 1]$, or equivalently,

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \alpha$$

for all x and y in $[0, 1]$. Then, by taking $y = 0$,

$$\frac{f(x)}{x} \leq \alpha$$

for all x in $[0, 1]$. But $f(x)/x \to 1$ as $x \to 0$, which means there is an x_0 such that

$$\frac{f(x_0)}{x_0} > \alpha,$$

a contradiction. So f is not a contraction.

To see that it is shrinking, suppose $x > y$ and observe

$$\frac{f(x) - f(y)}{x - y} = \frac{(x - x^2/2) - (y - y^2/2)}{x - y} = 1 - \frac{x^2/2 - y^2/2}{x - y} < 1.$$

(d) Setting $f(x) = x$ yields $x^2/4 + 1 = 0$ after a little algebra, which is impossible. So f has no fixed point. We compute

$$f'(x) = \frac{1}{2} + \frac{x}{2(x^2 + 1)^{1/2}}$$

and observe that $f'(x) \to 1$ as $x \to \infty$. By the mean value theorem,

$$f(x + 1) - f(x) \to 1 \text{ as } x \to \infty,$$

so f is not a contraction, as in (c). To see that f is shrinking, apply the mean value theorem again to

$$\frac{f(x) - f(y)}{x - y}$$

when $x > y$ and observe that $f' < 1$.

3.7 Local Compactness

1. If U is open and C is compact in $\mathbb{Q}$ with $U \subseteq C$, then every sequence in C would have a convergent subsequence with its limit in $\mathbb{Q}$. But any sequence in $\mathbb{Q}$ converging to an irrational number in U does not have this property.

2. (a) To show that each X_α is locally compact, use the fact that the projection maps π_α are continuous and open. Let $U = \prod U_\alpha$ be any basic open set in $\prod X_\alpha$ which is contained in a compact set C. Then $U_\alpha = X_\alpha$ for all but finitely many α, so $X_\alpha = \pi_\alpha(U) \subseteq \pi_\alpha(C)$ is compact for all but finitely many α.

   (b) If (x_α) is in $\prod X_\alpha$, find open neighbourhoods U_α of x_α and compact $C_\alpha \supseteq U_\alpha$ for the non-compact X_α, and set $U_\alpha = C_\alpha = X_\alpha$ for the rest. Then $(x_\alpha) \in \prod U_\alpha \subseteq \prod C_\alpha$, the latter set being compact by the Tychonoff theorem.

3. The answer is no if f is only continuous. For example, take $f : \mathbb{Z} \to \mathbb{Q}$ to be any bijection; it is continuous because $\mathbb{Z}$ has the discrete topology, but $\mathbb{Z}$ is locally compact (any one-point set is a compact neighbourhood) and $\mathbb{Q}$ is not (Exercise 1).

   On the other hand, if f is continuous and open, take a point y in $f(X)$ and pick x in $f^{-1}(y)$. Find a neighbourhood U and a compact set C with $x \in U \subseteq C$. Then $f(U)$ is open, $f(C)$ is compact, and $y \in f(U) \subseteq f(C)$.

4. Take the point $(0, 0, 0, \ldots)$ and basic ε-neighbourhood

$$U = \bigcup_{\delta<\varepsilon} U(\mathbf{0}, \delta)$$

   (see Section 20, Exercise 6(c)). Then

$$\overline{U} = [-\varepsilon, \varepsilon]^\omega$$

   Define x_n in $\overline{U}$ by

$$x_n = (0, 0, 0, \ldots, 0, 0, \varepsilon, 0, 0, \ldots)$$

   with ε in the n-th spot. Then $\rho(x_m, x_n) = \varepsilon$ for $m \neq n$, so it does not have a convergent subsequence.

5. We define the extension $\tilde{f} : X_1 \cup \{\infty_1\} \to X_2 \cup \{\infty_2\}$ in the obvious way: $\tilde{f}(x) = f(x)$ for all x in X_1, and $\tilde{f}(\infty_1) = \infty_2$.

6. $\mathbb{R}$ is homeomorphic to $S^1 - \{(1, 0)\}$. Use Theorem 29.1.

7. Use Theorem 29.1.

8. $\mathbb{Z}_+$ is homeomorphic to $\{1/n \mid n \in \mathbb{Z}_+\}$. Use Theorem 29.1.

9. Use Exercise 5(c) from the supplementary exercises on topological groups, and Exercise 3 from this section.

10. Find a neighbourhood W of x and a compact set C with $W \subseteq C$. Then W is a locally compact Hausdorff space by Corollary 29.4. Apply Theorem 29.2 with W in place of X.

3.8 Supplementary Exercises: Nets

1.

2. If α and β are in K, choose γ in J with $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. Then use cofinality of K to choose ω in K such that $\gamma \preceq \omega$, and hence $\alpha \preceq \omega$ and $\beta \preceq \omega$.

Chapter 4

Countability and Separation Axioms

4.1 The Countability Axioms

1. (a) Let x be a point of X and $\{B_n\}$ be a countable base of neighbourhoods at x. Then $\bigcap B_n = \{x\}$ because if y is a point which is not x, the T_1 axiom implies that there is an open set U containing x but not y. Some B_N must be contained in U, and hence y cannot be in $\bigcap B_n$.

   (b) Consider $\mathbb{R}^\omega$ with the box topology. Every point is a G_δ set as

   $$\{(x_1, x_2, x_3, \ldots)\} = \bigcap_{n=1}^{\infty} \left[\left(x_1 - \frac{1}{n}, x_1 + \frac{1}{n} \right) \times \left(x_2 - \frac{1}{n}, x_2 + \frac{1}{n} \right) \times \cdots \right]$$

   but the space is not first-countable. Suppose $\{B_n\}$ is a countable base at $(0, 0, 0, \ldots)$. We may assume that each $B_n = \prod_{k=1}^{\infty} U_k^{(n)}$ where each $U_k^{(n)}$ is open in $\mathbb{R}$, see Exercise 2. For each k, choose an open set V_k in $\mathbb{R}$ that contains 0 and is properly contained in $U_k^{(k)}$. Then no B_n is contained in $\prod_{k=1}^{\infty} V_k$.

2. Following the hint, Let $C_{n,m} \in \mathcal{C}$ be such that $B_n \subseteq C_{n,m} \subseteq B_m$, whenever this is possible. Then the collection of all $C_{n,m}$ is a countable base. Indeed, if U is open and x is in U, choose B_m with $x \in B_m \subseteq U$. Then choose $C \in \mathcal{C}$ with $x \in C \subseteq B_m \subseteq U$. Then choose B_n with $x \in B_n \subseteq C \subseteq B_m \subseteq U$. Then replace C with $C_{n,m}$ so that $x \in C_{n,m} \subseteq U$.

3. Denote a countable base by $\mathcal{B} = \{B_n\}$ and let $E \subseteq A$ be all points of A that are not limit points. For each x in E, choose an integer n_x such that $x \in B_{n_x}$ and $B_{n_x} \cap A = \{x\}$. The map $E \to \mathcal{B} : x \mapsto B_{n_x}$ is injective, thus E is at most countable. Since $A = E \cup (A - E)$, A is uncountable, and E is at most countable, it follows that $A - E$ (all limit points of A) is uncountable.

4. For $n \geq 1$, the collection $\{B(x, \frac{1}{n}) \mid x \in X\}$ is a cover, so by compactness there is a finite subcover $\mathcal{A}_n$. The collection $\mathcal{B} = \bigcup_{n \geq 1} \mathcal{A}_n$ is a countable base.

5. (a) If D is a countable dense subset, the collection $\{B(x, \frac{1}{n}) \mid x \in D \text{ and } n \geq 1\}$ is a countable base.

   (b) For $n \geq 1$, the collection $\{B(x, \frac{1}{n}) \mid x \in X\}$ is a cover, so since the space is Lindelöf, there is a countable subcover $\mathcal{A}_n$. The collection $\mathcal{B} = \bigcup_{n \geq 1} \mathcal{A}_n$ is a countable base.

6. $\mathbb{R}_\ell$ has a countable dense subset by Example 3. If $\mathbb{R}_\ell$ were metrizable, it would have a countable base by Exercise 5(a), but it does not, by Example 3.

   I_o^2 is compact, so if it were metrizable, it would have a countable base by Exercise 4. Then the subset $A \subseteq I_o^2$ from Example 5 would have a countable base by Theorem 30.2. Then A would have a countable dense subset by Theorem 30.3(b). But A is the disjoint union of an uncountable collection of open subsets, and this contradicts Exercise 13 below.

7. If D is a countable subset of S_Ω, then it has an upper bound in S_Ω by Theorem 10.3, call it α. Then D is a subset of the closed proper subset $\{x \in S_\Omega \mid x \leq \alpha\}$, so D cannot be dense. It follows that S_Ω is not separable.

   S_Ω is not second-countable since it is not separable.

   Let x be in S_Ω and y be the least element in S_Ω such that $x < y$. The collection $\{(a, y) \mid a \in S_x\}$ is a countable base at x, therefore S_Ω is first-countable.

   Since S_Ω has no largest element, every element is contained in a section, so $\bigcup\{S_x \mid x \in S_\Omega\}$ is an open cover of S_Ω. If $\{S_{x_n} \mid n \geq 1\}$ is a countable subcollection, then $\bigcup_{n \geq 1} S_{x_n}$ would be a countable union of countable sets, hence countable, hence not all of S_Ω. It follows that S_Ω is not Lindelöf.

   If D is dense in $\overline{S}_\Omega$, then $D \cap S_\Omega$ is dense in S_Ω. Since S_Ω is not separable, D cannot be countable, hence $\overline{S}_\Omega$ is not separable.

   $\overline{S}_\Omega$ is not second-countable since it is not separable.

   Suppose $\{B_n\}$ is a countable base at Ω. By replacing B_n with $B_1 \cap B_2 \cap \cdots \cap B_n$, we may assume that $B_n \supseteq B_{n+1}$ for all n. For each n, pick x_n in B_n such that $x_n \neq \Omega$. Then $x_n \to \Omega$. $\{x_n\}$ is a countable subset of S_Ω, so it has an upper bound α in S_Ω. But then x_n cannot converge to Ω since $(\alpha, \Omega]$ is a neighbourhood of Ω that contains none of the terms x_n. Thus $\overline{S}_\Omega$ is not first-countable.

   $\overline{S}_\Omega$ is compact, so it is Lindelöf.

8. In the uniform topology, $\mathbb{R}^\omega$ is a metric space, so it is first-countable.

   The set of all $\mathbf{q} = (q_1, q_2, \ldots)$ in $\mathbb{R}^\omega$ such that q_j is rational for all j is a countable dense subset.

   By Exercise 5(a), $\mathbb{R}^\omega$ is second-countable.

   By Theorem 30.3(a), $\mathbb{R}^\omega$ is Lindelöf.

9. The proof that A is Lindelöf is analogous to the proof of Theorem 26.2.

   For the counterexample, consider the subset L of $\mathbb{R}_\ell^2$ from Example 4.

10. If D_n is a countable dense subset of X_n for all $n \geq 1$, then $\prod_{n=1}^\infty D_n$ is a countable dense subset of $\prod_{n=1}^\infty X_n$.

11. If $\mathcal{A}$ is an open cover of $f(X)$, then $\{f^{-1}(U) \mid U \in \mathcal{A}\}$ is an open cover of X, so extract a countable subcover and map back to $f(X)$. If D is dense in X, $f(D)$ is dense in $f(X)$ since $f(X) = f(\overline{D}) \subseteq \overline{f(D)}$, see Theorem 18.1.

12. More generally, a continuous open map takes bases to bases. Let $\mathcal{A}$ be a base for X and $V \subseteq f(X)$ be open. Then $f^{-1}(V)$ is open, hence $f^{-1}(V) = \bigcup_{\text{some } U\text{'s in } \mathcal{A}} U$ and $V = \bigcup_{\text{some } U\text{'s in } \mathcal{A}} f(U)$.

13. If $\mathcal{A}$ is a collection of disjoint open sets and D is countable and dense, pick a point x_U from $D \cap U$ for each U in $\mathcal{A}$. Then the map $\mathcal{A} \to D : U \mapsto x_U$ is injective, hence $\mathcal{A}$ is at most countable.

14. Let $\mathcal{A}$ be a cover of $X \times Y$. For each x in X, $\{x\} \times Y$ is compact, so it may be covered by finitely many elements $A_1, \dots, A_n$ of $\mathcal{A}$. Using Lemma 26.8 with $N = A_1 \cup \dots \cup A_n$, we obtain an open set $W_x \subseteq X$ with $x \in W_x \times Y \subseteq N$. The sets W_x cover X, so extract a countable subcover $\{W_{x_j}\}$. Each $W_{x_j} \times Y$ is covered by finitely many elements of $\mathcal{A}$, so all these elements taken together form a countable subcover of $X \times Y$.

15. Use the Stone-Weierstrass theorem; the countable dense subalgebra is all polynomials with rational coefficients.

16. (a) Let D be the countable set of all functions in $\mathbb{R}^I$ of the form

$$\sum_{j=1}^{m} r_j \chi_{[q_{j-1}, q_j]}$$

where r_j, q_j are in $\mathbb{Q}$ and $0 = q_0 < q_1 < \dots < q_m = 1$. Let U be a basic open subset of $\mathbb{R}^I$, which means there are finitely many elements $x_1, \dots, x_n$ in I and finitely many open sets $U_1, \dots, U_n$ in $\mathbb{R}$ such that

$$U = \{f \in \mathbb{R}^I \mid f(x_j) \in U_j, 1 \leq j \leq n\}$$

Assume without loss of generality that $x_1 < x_2 < \dots < x_n$. Choose rationals q_j such that

$$0 = q_0 \leq x_1 < q_1 < x_2 < q_2 < x_3 < q_3 < \dots < x_{n-1} < q_{n-1} < x_n \leq q_n = 1$$

and rationals r_j such that r_j is in U_j for $1 \leq j \leq n$. Then the function above is in U. It follows that D is dense.

(b) Following the hint, let $D \subseteq \mathbb{R}^J$ be dense, fix an interval (a, b) in $\mathbb{R}$, and define $f : J \to \mathcal{P}(D)$ by $f(\alpha) = D \cap \pi_\alpha^{-1}((a, b))$. $f(\alpha)$ is always nonempty because D is dense and $\pi_\alpha^{-1}((a, b))$ is open. If $\alpha_0 \neq \beta_0$ in J, pick (x_α) in $D \cap \pi_{\alpha_0}^{-1}((a, b)) \cap \pi_{\beta_0}^{-1}(\mathbb{R} - [a, b])$. Then the tuple (x_α) is in $D \cap \pi_{\alpha_0}^{-1}((a, b))$ but not $D \cap \pi_{\beta_0}^{-1}((a, b))$, so f is injective. If D were countable, there would be an injective function $g : \mathcal{P}(D) \to \mathcal{P}(\mathbb{Z}_+)$, which means $g \circ f : J \to \mathcal{P}(\mathbb{Z}_+)$ is injective, which contradicts J having greater cardinality than $\mathcal{P}(\mathbb{Z}_+)$.

17. The space $\mathbb{Q}^\infty$ is trivially Lindelöf and separable because the set itself is countable. It is not first-countable (use a similar argument as in Exercise 1(b)) and hence not second-countable.

18. If x is in G

4.2 The Separation Axioms

1. Use Lemma 31.1(a).
2. Use Lemma 31.1(b).
3. We use Lemma 31.1(a). We know that every order topology is Hausdorff, so one-point sets are closed. Let x be a point of X and U a neighbourhood of x. Find an open interval (a, b) with $a < x < b$ and $(a, b) \subseteq U$. If there are c and d with $a < c < x < d < b$, then $x \in (c, d) \subseteq \overline{(c, d)} \subseteq [c, d] \subseteq (a, b) \subseteq U$, so let $V = (c, d)$. If there are no such c and d, then $(a, b) = \{x\}$, and $\{x\}$ is closed and open, so take $V = \{x\}$. If there is no c but there is a d, take $V = [x, d)$, and similarly take $V = (c, x]$ if there is a c but no d.
4. A Hausdorff (regular, normal) space is also Hausdorff (regular, normal) in any finer topology since there are more open sets available to separate points and/or closed sets.
5. Let $A = \{x \mid f(x) = g(x)\}$ and choose y in $X - A$ (if $X - A = \varnothing$, then A is closed automatically). Then $f(y) \neq g(y)$, so, since Y is Hausdorff, there are disjoint open sets U and V in Y with $f(y)$ in U and $g(y)$ in V. Then $f^{-1}(U) \cap g^{-1}(V)$ is an open set in X that contains y. It is disjoint from A, for if there were some z in $A \cap f^{-1}(U) \cap g^{-1}(V)$, then

   $$U \ni f(z) = g(z) \in V$$

   contradicting $U \cap V = \varnothing$. Thus $X - A$ is open and A is closed.
6. We first verify the claim in the hint. Let y be in Y and U an open set with $p^{-1}(y) \subseteq U$. $X - U$ is closed, so $p(X - U)$ is closed and does not contain y. Find a neighbourhood W with y in W and $W \cap p(X - U) = \varnothing$. Then $p^{-1}(W) \cap (X - U) = \varnothing$, so $p^{-1}(W) \subseteq U$.

   Now let A and B be disjoint closed sets in Y, so $p^{-1}(A)$ and $p^{-1}(B)$ are disjoint closed sets in X. Find disjoint open sets U and V with $p^{-1}(A) \subseteq U$ and $p^{-1}(B) \subseteq V$. Now we may, by the claim above, find open sets $W_y \ni y$ for each y in A and $W_z \ni z$ for each z in B with $p^{-1}(W_y) \subseteq U$ and $p^{-1}(W_z) \subseteq V$. Then $A \subseteq \bigcup_{y \in A} W_y$ and $B \subseteq \bigcup_{z \in B} W_z$, and these unions are disjoint.
7. (a) Let x and y be distinct in Y. Since $p^{-1}(x)$ and $p^{-1}(y)$ are disjoint and compact in the Hausdorff space X, there are disjoint open sets U and V with $p^{-1}(x) \subseteq U$ and $p^{-1}(y) \subseteq V$. Use the hint from the previous exercise to obtain open sets W_x and W_y in Y with $p^{-1}(W_x) \subseteq U$ and $p^{-1}(W_y) \subseteq V$. Then W_x and W_y separate x and y.

   (b) Similar to (a).

   (c) Let y be in Y. Since $p^{-1}(y)$ is compact, we may cover it with finitely many compact neighbourhoods. This implies that there is an open set U and a compact set K with $p^{-1}(y) \subseteq U \subseteq K$. Use the hint from the previous exercise to obtain a neighbourhood W of y such that $p^{-1}(W) \subseteq U$. Then $y \in W \subseteq p(K)$.

4.3 Normal Spaces

1. Suppose X is normal and $A \subseteq X$ is closed. Take E and F to be two closed subsets of A. Then E and F are closed in X because A is closed. Find two open subsets U and V of X that separate E and F. Then $A \cap U$ and $A \cap V$ are open subsets of A that separate E and F.

4.4 The Urysohn Lemma

4.5 The Urysohn Metrization Theorem

4.6 The Tietze Extension Theorem

4.7 Imbeddings of Manifolds

4.8 Supplementary Exercises: Review of the Basics

Chapter 5

The Tychonoff Theorem

5.1 The Tychonoff Theorem

5.2 The Stone-Čech Compactification

Chapter 6

Metrization Theorems and Paracompactness

6.1 Local Finiteness

6.2 The Nagata-Smirnov Metrization Theorem

6.3 Paracompactness

6.4 The Smirnov Metrization Theorem

Chapter 7

Complete Metric Spaces and Function Spaces

7.1 Complete Metric Spaces

1. (a) Suppose $\varepsilon > 0$ satisfies the hypothesis and let $\{x_n\}$ be Cauchy. Choose N so that $d(x_m, x_n) < \varepsilon$ when $m, n \geq N$. Then x_n is in the ε-ball about x_N when $n \geq N$. Thus $\{x_n\}_{n \geq N}$ has a cluster point, and by Lemma 43.1 $\{x_n\}$ converges.
   (b) Take X to be an open interval in $\mathbb{R}$.
2. For $x \in \overline{A} - A$, take a sequence x_n in A with $x_n \to x$. Then x_n is Cauchy, and by uniform continuity, so is $f(x_n)$. Thus it converges to a point y. Define $g(x) = y$.
3.
4.
5.

7.2 A Space-Filling Curve

7.3 Compactness in Metric Spaces

1. Take $\varepsilon > 0$ and choose i_0 so that $1/i_0 < \varepsilon$. Also for each $n < i_0$ choose a finite subset $x_1^n, x_2^n, \ldots, x_{k_n}^n$ of X_n.
2.
3.
4. (a) At the point 1.
   (b)

7.4 Pointwise and Compact Convergence

1. If $f \in B_{C_1}(f_1, \varepsilon_1) \cap B_{C_2}(f_2, \varepsilon_2)$, then $B_{C_1 \cup C_2}(f, \min\{\varepsilon_1, \varepsilon_2\}) \subseteq B_{C_1}(f_1, \varepsilon_1) \cap B_{C_2}(f_2, \varepsilon_2)$.
2.
3.
4. It converges pointwise and compactly.

7.5 Ascoli's Theorem

Chapter 8

Baire Spaces and Dimension Theory

8.1 Baire Spaces

8.2 A Nowhere-Differentiable Function

8.3 Introduction to Dimension Theory

8.4 Supplementary Exercises: Locally Euclidean Spaces

Part II

Algebraic Topology

Chapter 9

The Fundamental Group

9.1 Homotopy of Paths

1. Let $F : X \times I \to Y$ be a homotopy between h and h' and $G : Y \times I \to Z$ a homotopy between k and k'. Then $H : X \times I \to Z$ defined by $H(x,t) = G(F(x,t),t)$ is a homotopy between $h \circ k$ and $h' \circ k'$.

2. This is taken care of by Exercise 3.

3. (a) The map $F : X \times I \to X$ defined by $F(x,t) = tx$ is a homotopy between i_X and the constant map $x \mapsto 0$ for both $X = I$ and $X = \mathbb{R}$.

   (b) Let $F : X \times I \to X$ be a homotopy with $F(x,0) = x_0$ and $F(x,1) = x$ for all x in X. Let x and y be points of X. Then $G : I \to X$ defined by

$$G(t) = \begin{cases} F(x, 1-2t) & 0 \leq t \leq \frac{1}{2} \\ F(y, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

   is a path from x to y.

   (c) Let $F : Y \times I \to Y$ be a homotopy with $F(y,0) = y_0$ and $F(y,1) = y$ for all y in Y. Let $f, f' : X \to Y$ be two continuous maps. Then $G : X \times I \to Y$ defined by

$$G(x,t) = \begin{cases} F(f(x), 1-2t) & 0 \leq t \leq \frac{1}{2} \\ F(f'(x), 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

   is a homotopy between f and f'.

   (d) Let $F : X \times I \to X$ be a homotopy with $F(x,0) = x_0$ and $F(x,1) = x$ for all x in X and $G : I \to Y$ a path with $G(0) = f(x_0)$ to $G(1) = f'(x_0)$. Let $f, f' : X \to Y$ be two continuous maps. Then $H : X \times I \to Y$ defined by

$$H(x,t) = \begin{cases} f(F(x, 1-3t)) & 0 \leq t \leq \frac{1}{3} \\ G(3t-1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ f'(F(x, 3t-2)) & \frac{2}{3} \leq t \leq 1 \end{cases}$$

   is a homotopy between f and f'.

9.2 The Fundamental Group

9.3 Covering Spaces

9.4 The Fundamental Group of the Circle

9.5 Retractions and Fixed Points

9.6 The Fundamental Theorem of Algebra

9.7 The Borsuk-Ulam Theorem

9.8 Deformation Retracts and Homotopy Type

9.9 The Fundamental Group of S^n

9.10 Fundamental Groups of Some Surfaces