Hints and Partial Solutions for *Functional Analysis, 2nd edition* by Walter Rudin

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May 21, 2022

Part I

General Theory

Chapter 1

Topological Vector Spaces

1. (a) The z we seek is $y - x$ (check that it works). If z' also satisfies the required property, then

$$z' = z' + 0 = z' + (x - x) = (z' + x) - x = y - x$$

so z is unique.

(b) $0x = (0 + 0)x = 0x + 0x$, and adding $-(0x)$ to both sides gives $0x = 0$. Similarly, $\alpha 0 = \alpha(0 + 0) = \alpha 0 + \alpha 0$.

(c) If $a \in A$, then $2a = (1 + 1)a = a + a \in A + A$. For an example where $2A \neq A + A$, take $X = \mathbb{R}$ and $A = \{0, 1\}$.

(d) Suppose A is convex, and $x \in sA + tA$. Then $x = sa_1 + ta_2$ for some $a_1, a_2 \in A$. Consider

$$\frac{1}{s+t}x = \frac{s}{s+t}a_1 + \frac{t}{s+t}a_2.$$

The right side of the above equation is in A since A is convex, and so $1/(s + t)x \in A$. Thus $x = (s + t)/(s + t)x \in (s + t)A$. The inclusion $(s + t)A \subseteq sA + tA$ is clear. If conversely $(s + t)A = sA + tA$, suppose $a_1, a_2 \in A$. Then if $0 < r < 1$, $ra_1 + (1 - r)a_2 \in rA + (1 - r)A = (r + (1 - r))A = A$.

(e) Suppose $\{B_\lambda\}$ is a collection of balanced sets, $|\alpha| \leq 1$, and $x \in \bigcup_\lambda B_\lambda$. Then $x \in B_{\lambda_0}$ for some λ_0, and thus $\alpha x \in B_{\lambda_0}$ because B_{λ_0} is balanced. It follows that $\alpha x \in \bigcup_\lambda B_\lambda$. If $x \in \bigcap_\lambda B_\lambda$, then $x \in B_\lambda$ for all λ. Hence $\alpha x \in B_\lambda$ for every λ because each is balanced, and thus $\alpha x \in \bigcap_\lambda B_\lambda$.

(f) Suppose $\{C_\lambda\}$ is a collection of convex sets, $0 < t < 1$, and that $x, y \in \bigcap_\lambda C_\lambda$. Then $x, y \in C_\lambda$ for each λ so $tx + (1 - t)y \in C_\lambda$ for each λ because each is convex. Thus $tx + (1 - t)y \in \bigcap_\lambda C_\lambda$.

(g) Suppose $\{C_\lambda\}$ is a totally ordered (by inclusion) collection of convex sets, $0 < t < 1$, and that $x, y \in \bigcup_\lambda C_\lambda$. Then $x \in C_{\lambda_1}$ and $y \in C_{\lambda_2}$ for some λ_1 and λ_2. Since $\{C_\lambda\}$ is totally ordered, either $C_{\lambda_1} \subseteq C_{\lambda_2}$ or $C_{\lambda_1} \subseteq C_{\lambda_1}$. Suppose without loss of generality that $C_{\lambda_1} \subseteq C_{\lambda_2}$. Then $x, y \in C_{\lambda_2}$ and thus $tx + (1 - t)y \in C_{\lambda_2}$ because C_{λ_2} is convex. Thus $tx + (1 - t)y \in \bigcup_\lambda C_\lambda$.

(h) Suppose $x, y \in A + B$ and $0 < t < 1$. We have $x = a + b$ and $y = a' + b'$ for $a, a' \in A$ and $b, b' \in B$. Now

$$tx + (1-t)y = t(a+b) + (1-t)(a'+b') = (ta + (1-t)a') + (tb + (1-t)b')$$

and we have $ta + (1-t)a' \in A$ and $tb + (1-t)b' \in B$ by convexity. So $tx + (1-t)y = (ta + (1-t)a') + (tb + (1-t)b') \in A + B$.

(i) Suppose $x \in A + B$ and $|\alpha| \leq 1$. $x = a + b$ for $a \in A$ and $b \in B$. Then $\alpha x = \alpha(a+b) = \alpha a + \alpha b$ and we have $\alpha a \in A$ and $\alpha b \in B$ since A and B are balanced. So $\alpha x = \alpha a + \alpha b \in A + B$.

(j) Tedious, but easy.

2. It is straightforward to show that the given set is convex. Let $\{C_\lambda\}$ be all convex sets that contain A. Because the given (convex) set contains A, we have the inclusion

$$\bigcap_\lambda C_\lambda \subseteq \left\{ \sum_{i=1}^n t_i x_i : x_i \in A, t_i \geq 0, \sum_{i=1}^n t_i = 1 \right\}.$$

To prove the reverse inclusion, we use the following proposition.

Proposition. C is convex if and only if $\sum_{i=1}^n t_i x_i \in C$ whenever $x_i \in C$, $t_i \geq 0$ and $\sum_{i=1}^n t_i = 1$.

Proof. By taking $n = 2$, the sufficieny condition is simply the definition of convexity. Conversely, if C is convex, the claim is true for $n = 2$, so suppose it is true for $n = k - 1$. Then

$$\begin{aligned} & t_1 x_1 + \cdots + t_k x_k \\ =& (t_1 + \cdots + t_{k-1}) \overbrace{\left(\frac{t_1}{t_1 + \cdots + t_{k-1}} x_1 + \cdots + \frac{t_{k-1}}{t_1 + \cdots + t_{k-1}} x_{k-1} \right)}^{(1)} + t_k x_k \end{aligned}$$

and the expression (1) is in C by the induction hypothesis. Then the whole sum is in C by the truth of the $n = 2$ case. $\square$

Now suppose $x_1, x_2, \ldots, x_n \in A$, $t_i \geq 0$ and $\sum_{i=1}^n t_i = 1$. Then $x_1, x_2, \ldots, x_n \in C_\lambda$ for all λ since $A \subseteq C_\lambda$ for all λ. Then $\sum_{i=1}^n t_i x_i \in C_\lambda$ for all λ by the proposition, and thus $\sum_{i=1}^n t_i x_i \in \bigcap_\lambda C_\lambda$.

3. (a) Let U_1 and U_2 be open. Then $U_1 + U_2$ is open, since $U_1 + U_2 = \bigcup_{x \in U_1}(x + U_2)$. By induction, $U_1 + U_2 + \cdots + U_n$ is open for any n. If $t \neq 0$, then tU is also open. The convex hull of an arbitrary open set U is

$$\bigcup \{t_1 U + t_2 U + \cdots t_n U : t_i \geq 0, \sum t_i = 1\},$$

which is open.

(b) Let V be a convex neighbourhood of 0. By assumption, there exists an $s > 0$ such that $E \subseteq tV$ when $t > s$. But tV is a convex set containing E, so $\text{conv}(E) \subseteq tV$ if $t > s$. If U is an arbitrary neighbourhood of 0, not necessarily convex, then choose $V \subseteq U$ to be convex, and upon choosing a suitable $s > 0$ corresponding to V, we have $\text{conv}(E) \subseteq tV \subseteq tU$ when $t > s$.

(c) Let U be a neighbourhood of 0. Choose a neighbourhood V of 0 such that $V + V \subseteq U$. Then there exist $s_1 > 0$ and $s_2 > 0$ such that $A \subseteq tV$ when $t > s_1$ and $B \subseteq tV$ when $t > s_2$. Let $s = \max\{s_1, s_2\}$. Then $A + B \subseteq tV + tV \subseteq tU$ when $t > s$.

(d) $A + B$ is the image of the compact set $A \times B$ under addition, which is continuous.

(e) Suppose $x \notin A + B$, or, equivalently, that $A \cap (x - B) = \varnothing$. By Theorem 1.10, there is a symmetric neighbourhood V of 0 such that

$$(A + V) \cap ((x - B) + V) = \varnothing.$$

If $(x + V) \cap (A + B)$ were not empty, then there would exist v in V, a in A and b in B such that $x + v = a + b$, or that $x - b + 0 = a - v$. This contradicts the fact that $(A + V) \cap ((x - B) + V) = \varnothing$, so $x + V$ is a neighbourhood of x that does not intersect $A + B$. It follows that the complement of $A + B$ is open, and hence that $A + B$ is closed.

(f) The classical example is to take the two closed sets $\mathbb{Z}$ and $\gamma\mathbb{Z} = \{\gamma n : n \in \mathbb{Z}\}$ in $\mathbb{R}$, where γ is an irrational number ($\mathbb{Z} + \gamma\mathbb{Z}$ is proper and dense, therefore not closed). I'll try to come up with another example that's more original.

4. If $0 < |\alpha| \leq 1$, we have $|\alpha z_1| \leq |\alpha z_2|$ if and only if $|\alpha||z_1| \leq |\alpha||z_2|$ if and only if $|z_1| \leq |z_2|$. If $\alpha = 0$, the result is clear. To see that the interior is not balanced, we show that 0 is not in the interior. Indeed, if U is a open ball around 0 of radius ε, then the point $(\varepsilon/2, 0)$ is in U, but $(\varepsilon/2, 0) \notin B$. Thus, by taking $\alpha = 0$, we have $\alpha B^\circ = \{0\} \not\subseteq B^\circ$.

5.

6. Suppose E is not bounded. Then there is a neighbourhood V of 0 such that for every $s > 0$, there exists $t > s$ such that $E \setminus tV \neq \varnothing$. By successively choosing $s = 1, 2, 3, \ldots$, find $n_1, n_2, n_3, \ldots$ with $n_k > k$ for each k and $E \setminus n_k V \neq \varnothing$. So we can choose $x_k \in E \setminus n_k V$ for each k. Then $\{x_k\}$ is a countable unbounded subset of E. Indeed, take V as before, and if $s > 0$ is arbitrary, choose k so that $n_k > s$, and then $x_k \in E \setminus n_k V$.

7. We first show that a sequence $\{f_n\}$ converges pointwise if and only if it converges in the topology given by the seminorms. Suppose that $f_n(x) \to 0$ for all $x \in [0, 1]$. Let $\bigcap_{i=1}^k V(p_{x_i}, n_i)$ be a neighbourhood of 0. Choose N large enough so that $p_{x_i}(f_N) = |f_N(x_i)| < 1/n_i$ for all i from 1 to k. This shows that f_n is eventually in every neighbourhood of 0. On the other hand, fix x and $\varepsilon > 0$. Choose n so that $1/n < \varepsilon$. Then $V(p_x, n)$ is a neighbourhood of 0, and so there exists N so that $f_M \in V(p_x, n)$ if $M > N$ or that $|f_M(x)| < 1/n < \varepsilon$.

   Take a bijection between $[0, 1]$ and the set of all sequences converging to 0, so that with each $x \in [0, 1]$ there is associated a single sequence $\{z_n^x\}$ with $z_n^x \to 0$ as $n \to \infty$. For a postive integer k, define $f_k(x) = z_k^x$. Then $f_k(x) \to 0$ for all $x \in [0, 1]$, but if $\gamma_n \to \infty$, then $1/\gamma_n \to 0$ and so this sequence is paired with a point from $[0, 1]$, say x'. Then $\gamma_k f_k(x') = \gamma_k(1/\gamma_k) = 1 \not\to 0$.

8.

9. Define $f(\pi(x)) = \Lambda x$. This is well defined: if $\pi(x) = \pi(y)$, then $x - y \in N$, so $\Lambda(x - y) = 0$ and $\Lambda x = \Lambda y$, which means $f(\pi(x)) = f(\pi(y))$. We have

$$f(\pi(x) + t\pi(y)) = f(\pi(x + ty)) = \Lambda(x + ty) = \Lambda x + t\Lambda y = f(\pi(x)) + tf(\pi(y))$$

so f is linear. If f is continuous, then $\Lambda = f \circ \pi$ is continuous, being the composition of continuous maps. Suppose Λ is continuous and U is open in Y. Then $f^{-1}(U) = \pi(\Lambda^{-1}(U))$, which is open since $\Lambda^{-1}(U)$ is open and π is an open map.

Chapter 2

Completeness

1. Let $\{M_n\}_{n\geq 1}$ be a countable collection of finite-dimensional subspaces of X such that $X = \bigcup_{n\geq 1} M_n$. Each M_n is closed by Theorem 1.21, so it suffices to show that each M_n has empty interior. Take any $n_0 \geq 1$, any basic neighbourhood U of 0 and any vector x not in M_{n_0}. Then $rx \to 0$ as $r \to 0$ and thus $r_0 x \in U$ for some r_0. But $r_0 x \notin M_{n_0}$, so $U \not\subseteq M_{n_0}$.

   Since F-spaces are of second category in themselves, they cannot be written as a countable union of finite-dimensional subspaces by the previous paragraph and hence cannot have a countable Hamel basis.

2. Let $\{r_n\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Fix $k \geq 1$, and consider the open sets

$$U_{n,k} = \left(r_n - \frac{1}{k2^{n+1}}, r_n + \frac{1}{k2^{n+1}} \right) \cap [0,1].$$

   Then $A_k = \bigcup_{n=1}^{\infty} U_{n,k}$ has Lebesgue measure at most $1/k$ and so $\bigcap_{k=1}^{\infty} A_k$ has Lebesgue measure zero. Each A_k is open and dense, so the complement of $\bigcap_{k=1}^{\infty} A_k$ is of the first category.

3.

4. (a) If f is in L^1 and $\int |f|^2 > n$, then the set of all g in L^1 such that $\|f - g\|_1 < \frac{1}{?}$ also satisfy $\int |g|^2 > n$, for

$$\int |g|^2 \geq \int |f|^2 - \int |f - g| > n - (small) > n$$

   (b) If f is in L^2, by Hölder's inequality we have

$$\left| \int f g_n \right| \leq \int |f||g_n| \leq \|f\|_2 \|g_n\|_2 = \frac{\|f\|_2}{n} \to 0.$$

   However, if

$$f(x) = \begin{cases} \frac{1}{x^{2/3}} & x > 0 \\ 0 & x = 0 \end{cases},$$

then f is in L^1 and

$$\int fg_n = 3$$

for all n.

(c) We have $\|\cdot\|_1 \leq \|\cdot\|_2$ so the inclusion map is continuous, but the function f from part (b) is in L^1 but not L^2. Now use the Open Mapping Theorem.

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8. Any infinite subset of K has 0 as a limit point. Indeed, such a subset must contain x_n for arbitrarily large n, and such x_n are arbitrarily close to 0. Thus K is compact.

   We compute

$$\Lambda_m x_n = \begin{cases} 0 & m < n \\ n & m \geq n \end{cases}$$

   so that $\Lambda_m x_n \leq n$ for all m, but $\Lambda_m x_m = m$, which is unbounded over all m.

Chapter 3

Convexity

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7. If $f = \chi_{[0,\frac{1}{2}]}$, then for any g such that $\|f - g\|_\infty < \frac{1}{2}$, $\lim_{x\to\frac{1}{2}} g(x)$ cannot exist, and so C is not dense in the norm topology of $L^\infty([0,1])$.

   Fix f in $L^\infty([0,1])$, $h_1, h_2, \ldots, h_n$ in $L^1([0,1])$, and $\varepsilon > 0$. We want to find a continuous function g such that
   $$\left|\int (f-g)h_i\right| < \varepsilon$$
   for all i; this will show that C is dense in the weak* topology. Note that $\int |f| < \infty$ since $[0,1]$ is a finite measure space. Thus f is, in fact, in $L^1([0,1])$, so we may choose a continuous function g such that $\|f-g\|_1 < \frac{\varepsilon}{M}$, where $M = \max \|h_i\|_1$. Then
   $$\left|\int (f-g)h_i\right| \leq \|h_i\|_1 \|f-g\|_1 < \varepsilon.$$

   C is closed in the norm topology of $L^\infty([0,1])$ (it is complete in the supremum norm), but not in the weak* topology (if it were closed *and* dense, it would be all of $L^\infty([0,1])$).

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12. Let $x_n = (1, 1, \ldots, 1, 0, 0, \ldots)$ (there are n 1's). Then $\{x_n\}$ has no cluster point, for suppose $x = (x_k)$ is one (and suppose $x_n \to x$ weakly by dropping to a subsequence if necessary). Then if we regard $y = (1, 0, 0, \ldots)$ as an element of ℓ^1,

$$x_1 = y(x) = \lim_n y(x_n) = 1.$$

Now if $y = (1, 1, 0, 0, \ldots)$,

$$x_1 + x_2 = y(x) = \lim_n y(x_n) = 2$$

so that $x_2 = 1$. Continuing, we obtain that $x_k = 1$ for all k, but $x = (1, 1, 1, \ldots)$ is not in c_0.

Chapter 4

Duality in Banach Spaces

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24. (a)

(b)

(c) If $y_n = x_{n-1} - x_\infty = 0$ for all $n \geq 2$ and $y_1 = x_\infty = 0$, then it must be that $x_n = 0$ for all $n \geq 1$. If (y_n) is in c_0, then

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Chapter 5

Some Applications

1.

Part II

Distributions and Fourier Transforms

Chapter 6

Test Functions and Distributions

1.

Chapter 7

Fourier Transforms

1.

Chapter 8

Applications to Differential Equations

1.

Chapter 9

Tauberian Theory

1.

Part III

Banach Algebras and Spectral Theory

Chapter 10

Banach Algebras

1.

Chapter 11

Commutative Banach Algebras

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Chapter 12

Bounded Operators on a Hilbert Space

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3. $\|x_n - y_n\|^2 = \langle x_n - y_n, x_n - y_n\rangle = \|x_n\|^2 + \|y_n\|^2 - 2\Re\langle x_n, y_n\rangle \leq 2 - 2\Re\langle x_n, y_n\rangle \to 0$ and $\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle = \|x_n\|^2 + \|x\|^2 - 2\Re\langle x_n, x\rangle \to 2\|x\|^2 - 2\langle x, x\rangle = 0$.

4. It is easy to check the properties of an inner product. Definiteness is because if $[x^*, x^*] = 0$, then $\psi x^* = 0$, whence $x^*(y) = \langle y, \psi x^*\rangle = \langle y, 0\rangle = 0$ for all y, so $x^* = 0$.

Chapter 13

Unbounded Operators

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