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A Family of Simple C^*-Algebras Related to Weighted Shift Operators

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In this paper we study a family of C^*-algebras which occurs naturally in the study of C^*-algebras generated by weighted shifts. We show that these algebras are simple modulo the compacts, and while they share many of the properties of uniformly hyperfinite C^*-algebras, they are not approximately finite dimensional.

INTRODUCTION

In this paper we study a family of C^*-algebras which occurs naturally in the study of C^*-algebras generated by weighted shifts [3]. For any separable infinite dimensional complex Hilbert space $\mathscr{H}$, let $\mathscr{B}(\mathscr{H})$ denote the set of all bounded linear operators on $\mathscr{H}$, let $\mathscr{K}$ denote the ideal of all compact operators, and let ν be the canonical quotient map from $\mathscr{B}(\mathscr{H})$ onto the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}$. For each strictly increasing sequence of integers $\{n_k\}$, with n_k dividing n_{k+1} for all k, let $\mathfrak{A}(n_k)$ be the C^*-algebra generated by all periodic weighted shifts of period n_k for some k. The algebra $\mathfrak{A}(n_k)$ contains $\mathscr{K}$ and in Section 1 we prove that $\nu(\mathfrak{A}(n_k))$ is a simple C^*-algebra. It is of interest to construct new examples of simple C^*-algebras because classifying simple C^*-algebras would be a first step in classifying all C^*-algebras.

In Section 2 we show that $\nu(\mathfrak{A}(n_k))$ is not a UHF C^*-algebra, although it shares many of the properties of UHF algebras. For example, the same condition on the sequence $\{n_k\}$ that classifies UHF algebras of type $\{n_k\}$ up to isomorphism [8] also classifies $\mathfrak{A}(n_k)$ and $\nu(\mathfrak{A}(n_k))$ up to isomorphism. The algebra $\nu(\mathfrak{A}(n_k))$ contains a UHF algebra $\nu(\mathscr{M}(n_k))$ and is contained in the Calkin algebra $\nu(\mathscr{B}(\mathscr{H}))$. In Section 3 we represent $\nu(\mathfrak{A}(2^k))$ as an algebra of multiplications and

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translations on $L^2[0, 1]$ and use this representation to show that the relative commutant of $\nu(\mathcal{M}(2^k))$ in $\nu(\mathfrak{A}(2^k))$ is the scalars. We are interested in relative commutants because of Dixmier's conjecture that the relative commutant of a simple C^*-algebra in a containing simple C^*-algebra is again simple [6]. We remark that this question is only of interest when the subalgebra in question has the same identity as the containing C^*-algebra, since if $\mathfrak{A} \subseteq \mathcal{B}$ with E the identity of $\mathfrak{A}$ and F the identity of $\mathcal{B}$, then $(F - E)\mathcal{B}(F - E)$ is a two-sided ideal of $\mathfrak{A}' \cap \mathcal{B}$. It is easy to give examples of simple C^*-algebras $\mathfrak{A}$ and $\mathcal{B}$ with $\mathfrak{A} \subseteq \mathcal{B}$ and the identity of $\mathfrak{A}$ not equal to the identity of $\mathcal{B}$. We are unable to determine whether the commutant of $\nu(\mathfrak{A}(n_k))$ in the (simple) Calkin algebra is simple or not, but in Section 3 we do show that the commutant of $\nu(\mathfrak{A}(2^k))$ in the Calkin algebra is not abelian. Finally in Section 4 we construct a representation of the canonical anticommutation relations [12, p. 4] as elements of $\mathfrak{A}(2^k)$. While this representation is unitarily equivalent to the Fock representation, it has the property that the operators are given directly as bounded operators on a familiar Hilbert space.

We now give some definitions and fix our notation. We fix a separable Hilbert space $\mathscr{H}$ and an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathscr{H}$. A bounded linear operator S on $\mathscr{H}$ is called a *weighted shift* with weights $\{\alpha_n\}_{n=1}^{\infty} \in l^{\infty}$ if $Se_n = \alpha_{n+1}e_{n+1}$ for all $n \geqslant 0$. We assume throughout that $\alpha_n \geqslant 0$. When $\alpha_n \equiv 1$, we obtain the unilateral shift U_+ defined by $U_+e_n = e_{n+1}$. A weighted shift with weights $\{\alpha_n\}$ is called periodic of period p if there exists a positive integer p such that $\alpha_n = \alpha_{n+p}$ for all n. If S is any weighted shift, then $S = U_+D$ where D is the diagonal operator given by $De_n = \alpha_{n+1}e_n$. For any bounded linear operator A on $\mathscr{H}$, we denote by $C^*(A)$ the smallest C^*-subalgebra of $\mathcal{B}(\mathscr{H})$ containing A and the identity I. Since we are assuming that $\alpha_n \geqslant 0$, note that $D = (S^*S)^{1/2} \in C^*(S)$, and that $S = U_+D$ is the polar decomposition of S if $\alpha_n > 0$ for all n. If the weights $\{\alpha_n\}$ are bounded below away from zero, then $U_+ = SD^{-1} \in C^*(S)$.

Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers such that n_k divides n_{k+1} for all k. Let $S(n_k)$ be the weighted shift with weights $\alpha_m = 1$ if $m = 1 + ln_k$, $l \geqslant 0$ and $\alpha_m = 1/2$ otherwise. Then $C^*(S(n_k))$ is the C^*-algebra generated by all weighted shifts of period n_k and $C^*(S(n_k)) \subseteq C^*(S(n_{k+1}))$. Let $\mathfrak{A}(n_k)$ be the norm closure of the union of all the $C^*(S(n_k))$. The algebras $\mathfrak{A}(n_k)$ were first studied in [3]. Since $\mathscr{K} \subseteq C^*(U_+)$, we have that $\mathscr{K} \subseteq C^*(S(n_k))$ for all k. It follows from [3, proof of Theor. 2.2] that $\nu(C^*(S(n_k)))$ is isomorphic to $M_{n_k}(C(T)) \equiv$ the algebra of all $n_k \times n_k$ matrices whose entries are continuous functions on the unit circle T. Under this isomorphism, if S is a shift of period n_k and weights $\alpha_1, \alpha_2, \ldots, \alpha_{n_k}$ then we have that $\nu(S)_{1,n_k}(z) = \alpha_{n_k} z$, $\nu(S)_{i+1,i}(z) = \alpha_i I$ for $1 \leqslant i \leqslant n_k - 1$, and $\nu(S)_{i,j} = 0$ for all other i, j.

For each k, let $P_j^{(k)}$, $1 \leqslant j \leqslant n_k$, be the diagonal projection with weights $\alpha_m = 1$ if $m = j + ln_k$, $l \geqslant 0$, and $\alpha_m = 0$ otherwise. For $1 \leqslant j \leqslant i \leqslant n_k$ let $E^{(k)}(i,j) = U_+^{(i-j)} P_j^{(k)}$ and for $1 \leqslant j \leqslant i \leqslant n_k$ let $E^{(k)}(i,j) = P_i^{(k)} U_+^{*(j-i)}$. Then the family $\{E^{(k)}(i,j)\colon 1 \leqslant i,j \leqslant n_k\}$ forms a system of $n_k \times n_k$ matrix units in $C^*(S(n_k))$. Let $N(n_k)$ denote the C^*-algebra generated by the family $\{E^{(k)}(i,j)\colon 1 \leqslant i,j \leqslant n_k\}$. Then $N(n_k) \subseteq N(n_{k+1})$ and if $\mathscr{M}(n^k)$ is the C^*-algebra generated by the union of all the $N(n_k)$, then $\mathscr{M}(n_k)$ is a uniformly hyperfinite (UHF) C^*-algebra of type $\{n_k\}$ [8]. The algebra $\mathfrak{A}(n_k)$ is the C^*-algebra generated by $\mathscr{M}(n_k)$ and U_+.

1. Simplicity of $\nu(\mathfrak{A}(n_k))$

The UHF algebra $\mathscr{M}(n_k)$ is simple (that is, has no nonzero proper two-sided ideals) because it is the norm closure of the union of an ascending sequence of simple C^*-algebras [8]. Since the C^*-algebras $\nu(C^*(S(n_k)))$ have nontrivial ideals, it is interesting that $\nu(\mathfrak{A}(n_k))$ is a simple C^*-algebra. In order to prove this we need the following (probably known) lemma.

Lemma 1. *Let f be a continuous function on the unit circle T. Then if n is any positive integer, there are continuous functions $f_0, f_1, \ldots, f_{n-1}$ on T such that $f(z) = \sum_{j=0}^{n-1} z^j f_j(z^n)$ for all z. The mapping f to f_j is linear and norm continuous for each j.*

Proof. Let $\rho = \exp(2\pi i/n)$ and let $g_j(z) = (\bar{z}^j/n) \sum_{k=0}^{n-1} \bar{\rho}^{jk} f(\rho^k z)$. Then $f(z) = \sum_{j=0}^{n-1} z^j g_j(z)$. It is easily seen that $g_j(z) = g_j(\rho^m z)$ for all $z \in T$ and all integers m. Hence if $y^n = z^n$ then $g_j(y) = g_j(z)$, and hence the definition $f_j(z) = g_j(z^{1/n})$ defines a well-defined continuous function on T. Then $f(z) = \sum_{j=0}^{n-1} z^j f_j(z^n)$ and the lemma follows.

Theorem 2. *The C^*-algebras $\nu(\mathfrak{A}(n_k))$ are simple.*

Proof. Since $\nu(\mathfrak{A}(n_k))$ is a Banach algebra with identity, it suffices to show that $\nu(\mathfrak{A}(n_k))$ has no nonzero proper closed two-sided ideals. Let J be a nonzero closed two-sided in $\nu(\mathfrak{A}(n_k))$. Since

$$\nu(\mathscr{A}(n_k)) = \overline{\bigcup_{k=1}^{\infty} \nu(C^*(S(n_k)))},$$

it is easily seen (for example, see the proof of Lemma 3.1 in [1]) that

$$J = \overline{\bigcup_{k=1}^{\infty} J_{n_k}},$$

where $J_{n_k} = \nu(C^*(S(n_k))) \cap J$. Since J is nonzero, some J_{n_j} must be a nonzero ideal of $\nu(C^*(S(n_j)))$. Since $\nu(C^*(S(n_j)))$ is isomorphic to $M_{n_j}(C(T))$, it follows that J_{n_j} can be identified with $M_{n_j}(Z(F_{n_j}))$, where $Z(F_{n_j})$ denotes the set of continuous functions on the unit circle which vanish on the closed set F_{n_j}. If J were a proper ideal, then J_{n_k} would be a proper ideal of $\nu(C^*(S(n_k)))$ for arbitrarily large k, and hence F_{n_k} would be nonempty for arbitrarily large k. Let $z_0 \in F_{n_k}$, where $k \geqslant j$. We will show that every (n_k/n_j)-root of z_0 is in F_{n_j}. Thus the closed set F_{n_j} would contain a dense subset of the unit circle, and we would have that $F_{n_j} = T$ and consequently $J_{n_j} = \{0\}$, which is a contradiction. The proof will thus be complete after we show that if $k \geqslant j$ and $z_0 \in F_{n_k}$ then every (n_k/n_j)-root of z_0 is in F_{n_j}. The proof of this depends on the embedding of $M_{n_k}(C(T))$ into $M_{n_{k+1}}(C(T))$ that results from identifying $M_{n_k}(C(T))$ with $\nu(C^*(S(n_k)))$.

For k a positive integer let $q_k = n_{k+1}/n_k$. Then in $\nu(C^*(S(n_k)))$, $A = (\nu(U_+))^{n_k}$ corresponds to the element of $M_{n_k}(C(T))$ with the identity function $(z \mapsto z)$ on each coordinate of the main diagonal and zeros elsewhere. But in $\nu(C^*(S(n_{k+1})))$, A corresponds to the element of $M_{n_{k+1}}(C(T))$ given by

$$A_{n_k+i,i}(z) = 1 \qquad \text{if} \quad 1 \leqslant i \leqslant n_{k+1} - n_k$$

and

$$A_{i,n_{k+1}-n_k+i}(z) = z \qquad \text{if} \quad 1 \leqslant i \leqslant n_k$$

and $A_{i,j} = 0$ otherwise. We then assert that if B is an element of $\nu(C^*(S(n_k)))$ whose matrix in $M_{n_k}(C(T))$ has a fixed continuous function f on each coordinate of the main diagonal and zeros elsewhere, then in $\nu(C^*(S(n_{k+1})))$, B corresponds to the element of $M_{n_{k+1}}(C(T))$ given by

$$B_{jn_k+i,i}(z) = f_j(z) \qquad \text{if} \quad 0 \leqslant j \leqslant q_k - 1, \qquad 1 \leqslant i \leqslant n_{k+1} - jn_k$$

and

$$B_{i,jn_k+i}(z) = zf_{q_k-j}(z) \qquad \text{if} \quad 1 \leqslant j \leqslant q_k - 1, \qquad 1 \leqslant i \leqslant n_{k+1} - jn_k,$$

and $B_{m,n} = 0$ elsewhere; here the functions f_j are the functions given by Lemma 1. In order to check this assertion, first check it for

$f(z) = z^n$ and $f(z) = \bar{z}^n$ (which is straightforward), and then use the linearity and norm continuity properties from Lemma 1.

Now suppose z_0 is such that $z_0^{q_j} \in F_{n_{j+1}}$ and let $f \in C(T)$ be such that f vanishes on F_{n_j}. Then let B be the element of J_{n_j} which corresponds to the matrix in $M_{n_j}(C(T))$ which has f on each coordinate of the main diagonal and zeros elsewhere. Then B also belongs to $J_{n_{j+1}}$, and so by the previous paragraph $f_l(z_0^{q_j}) = 0$, $0 \leqslant l \leqslant q_j - 1$. So by Lemma 1 we have that $f(z_0) = 0$. Hence $z_0 \in F_{n_j}$. Thus every (q_j)-root of a number in $F_{n_{j+1}}$ is in F_{n_j}. So if $k \geqslant j$ and $z_0 \in F_{n_k}$ then every (n_k/n_j)-root of z_0 is in F_{n_j}. This completes the proof.

2. UHF ALGEBRAS AND $\nu(\mathfrak{A}(n_k))$

In [3, Theor. 3.8] it was shown that $\mathfrak{A}(n_k)$ has a unique central state f, and that $\pi_f(\mathfrak{A}(n_k))'' = \pi_f(\mathcal{M}(n_k))''$, where π_f is the cyclic representation corresponding to f. Thus $\mathfrak{A}(n_k)$, and hence $\nu(\mathfrak{A}(n_k))$, has a representation as a hyperfinite II_1-factor. However $\nu(\mathfrak{A}(n_k))$ is not a UHF C^*-algebra, in fact, $\nu(\mathfrak{A}(n_k))$ is not approximately finite dimensional in the sense of Bratteli [1]. A C^*-algebra $\mathfrak{A}$ is approximately finite dimensional if $\mathfrak{A} = \overline{\bigcup \mathfrak{A}_n}$, where each $\mathfrak{A}_n$ is a finite dimensional C^*-algebra, and $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$. The proof that $\nu(\mathfrak{A}(n_k))$ is not approximately finite dimensional uses the recent concept of a quasitriangular operator [7, 9].

THEOREM 3. *The algebra $\nu(\mathfrak{A}(n_k))$ is not approximately finite dimensional.*

Proof. Suppose there exist finite dimensional C^*-algebras $\mathscr{B}_l$ such that $\mathscr{B}_l \subseteq \mathscr{B}_{l+1}$ and $\nu(\mathfrak{A}(n_k)) = \overline{\bigcup_{l=1}^{\infty} \mathscr{B}_l}$. Then for any $\epsilon > 0$ there is a $B \in \bigcup \mathscr{B}_l$ with $\| \nu(U_+) - B \| < \epsilon$. Let $A \in \mathfrak{A}(n_k)$ be such that $\nu(A) = B$. Then $\| \nu(U_+ - A)\| < \epsilon$. Hence there exists a compact operator $C \in \mathscr{K}$ with $\| U_+ - A - C \| < \epsilon$. But B is algebraic, so there is a polynomial p with $p(B) = p(\nu(A)) = 0$. Thus A is polynomially compact and hence quasitriangular [7, Theor. 6]. But then $A + C$ is also quasitriangular, and we obtain that U_+ is in the norm closure of the quasitriangular operators, and is thus itself quasitriangular [9]. But U_+ is not quasitriangular [9]. Thus $\nu(\mathfrak{A}(n_k))$ is not approximately finite dimensional.

We have not been able to determine whether $\nu(\mathcal{M}(n_k))$ is a maximal UHF subalgebra of $\nu(\mathfrak{A}(n_k))$. Nor have we been able to determine whether $\nu(\mathcal{M}(n_k))$ is a maximal quasitriangular C^*-subalgebra of

$\nu(\mathfrak{A}(n_k))$, where a subalgebra is called quasitriangular if each of its elements is quasitriangular.

We now show that Glimm's classification of the *-isomorphism classes of UHF algebras [8] carries over to our situation.

THEOREM 4. *The algebras $\nu(\mathfrak{A}(n_k))$ and $\nu(\mathfrak{A}(q_j))$ are *-isomorphic if and only if for every k there is a j such that n_k divides q_j, and for every j there is a k such that q_j divides n_k.*

Proof. The proof of Theorem 3.7 in [3] carries over with only slight changes.

We denote by a (A) the approximate point spectrum of an operator A and by $\mathrm{sp}(A)$ the spectrum of A. Since by [2, Prop. 5].

$$a(A) = \{\lambda \in \mathrm{sp}(A) \colon C^*(A)(A - \lambda I) \neq C^*(A)\},$$

the notion of approximate point spectrum is not really spatial and can be defined for elements of an abstractly given C^*-algebra.

LEMMA 5. *Let $\mathfrak{A}$ be a C^*-algebra. Then $\{B \in \mathfrak{A} \colon a(B) = \mathrm{sp}(B)\}$ is closed in $\mathfrak{A}$.*

Proof. By [5, 2.6.1] we may assume that $\mathfrak{A}$ is faithfully represented on a Hilbert space $\mathscr{H}$. Now let $A_n \in \mathscr{B}(\mathscr{H})$ be a sequence of operators with $a(A_n) = \mathrm{sp}(A_n)$ and $\| A_n - A \| < 1/n$. We will show that $a(A) = \mathrm{sp}(A)$. Let $\lambda \in \mathrm{sp}(A)$. If for all N these is an $n \geqslant N$ such that $A_n - \lambda I$ is invertible, then by [13, Theor. 1.5.4] $A - \lambda I$ is a left topological divisor of zero. So there exists a sequence of operators $C_n \in \mathscr{B}(\mathscr{H})$ such that $\| C_n \| = 1$ and $\|(A - \lambda I) C_n \|$ converges to zero. It is then clear that $\lambda \in a(A)$. Now if there exists an N such that $A_n - \lambda I$ is singular for all $n \geqslant N$, then we may assume that $\lambda \in \mathrm{sp}(A_n) = a(A_n)$ for all n. So for each n there exists a unit vector x_n such that $\|(A_n - \lambda I) x_n \| < 1/n$. Then $\|(A - \lambda I) x_n \| < 2/n$ and $\lambda \in a(A)$.

THEOREM 6. *If $\mathscr{M}$ is a* UHF *C^*-algebra, then $a(A) = \mathrm{sp}(A)$ for all $A \in \mathscr{M}$. Likewise if $A \in \nu(\mathfrak{A}(n_k))$ then $a(A) = \mathrm{sp}(A)$.*

Proof. Let $\mathscr{M} = \overline{\bigcup_{k=1}^{\infty} \mathscr{M}_{n_k}}$ where $\mathscr{M}_{n_k}$ is a $n_k \times n_k$ matrix algebra and $\mathscr{M}_{n_k} \subseteq \mathscr{M}_{n_{k+1}}$. Since each $\mathscr{M}_{n_k}$ can be faithfully represented on a finite dimensional Hilbert space, $a(A) = \mathrm{sp}(A)$ for all $A \in \bigcup_{k=1}^{\infty} \mathscr{M}_{n_k}$. Lemma 5 then implies that $a(A) = \mathrm{sp}(A)$ for all $A \in \mathscr{M}$. Now if $A \in \nu(C^*(S(n_k)))$ and $\lambda \notin a(A)$ then there exists a $B \in \nu(C^*(S(n_k)))$ with $B(A - \lambda I) = I$. But if π is an irreducible representation of $\nu(C(S(n_k)))$, then by [3, Theor. 2.2] $\mathscr{H}_\pi$ is finite dimensional, and $\pi(B)\, \pi(A - \lambda I) =$

$I = \pi(A - \lambda I)\,\pi(B)$, so $(A - \lambda I)B = I$ and $\lambda \in \mathrm{sp}(A)$. Thus $a(A) = \mathrm{sp}(A)$ for all $A \in \nu(C^*(S(n_k)))$ and all k. Lemma 5 then implies $a(A) = \mathrm{sp}(A)$ for all $A \in \nu(\mathfrak{A}(n_k))$.

3. Relative Commutants

We now construct a representation of $\nu(\mathfrak{A}(2^k))$ as an algebra of multiplications and translations on $L^2[0, 1]$. Let ϕ be defined from $[0, 1]$ into $[0, 1]$ as in Fig. 1. That is,

$$\phi(x) = \begin{cases} x + 1/2 & 0 \leqslant x < 1/2 \\ x - 1/4 & 1/2 \leqslant x < 3/4 \\ x - 5/8 & 3/4 \leqslant x < 7/8 \\ \vdots & \vdots \\ x - \dfrac{2^{n+1} - 3}{2^{n+1}} & \dfrac{2^n - 1}{2^n} \leqslant x < \dfrac{2^{n+1} - 1}{2^{n+1}} \\ \vdots & \vdots \\ 0 & x = 1. \end{cases}$$

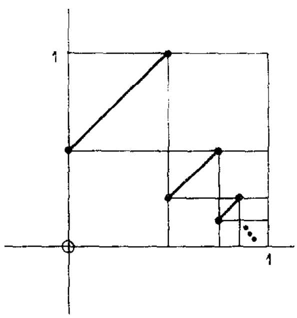

FIGURE 1

Then define T_ϕ on $L^2[0, 1]$ by $(T_\phi f)(x) = f(\phi(x))$. Then T_ϕ is a well defined unitary operator on $L^2[0, 1]$. Theorem 3 of [4] then implies that there is *-representation π of $C^*(U_+)$ onto $C^*(T_\phi)$ with $\pi(U_+) = T_\phi$. The representation π then induces a representation, also called π, from $\nu(C^*(U_+)) = C(T)$ onto $C^*(T_\phi)$ by $\pi(\nu(U_+)) = T_\phi$. We then have a representation

$$\pi^{(k)}: M_{2^k}(C(T)) \to M_{2^k}(C^*(T_\phi)),$$

which is just π acting on each matrix unit. We denote by $L^2[a, b]$ the projection of $L^2[0, 1]$ onto the subspace of functions vanishing off $[a, b]$.

Let $\mathscr{H}_j^{(k)}$ be the range of the projection $L^2(\phi^{-j+1}[1 - 1/2^k, 1])$, $1 \leqslant j \leqslant 2^k$. Then clearly $\mathscr{H}_j^{(k)}$ is the range of T_ϕ^{j-1} restricted to $\mathscr{H}_1^{(k)}$. Let $W_j^{(k)}$ be the natural unitary mapping of $\mathscr{H}_j^{(k)}$ onto $L^2[0, 1]$ (that is, if $\mathscr{H}_j^{(k)} = L^2[b_j^{(k)}/2^k, (b_j^{(k)} + 1)/2^k]$, $0 \leqslant b_j \leqslant 2^k - 1$, then $(W_j^{(k)}f)(x) = f(x/2^k + b_j^{(k)}/2^k))$. Then let $W^{(k)}$ be the unitary mapping of $L^2[0, 1]$ onto the direct sum of 2^k copies of $L^2[0, 1]$ with the property that $W^{(k)}$ takes $\mathscr{H}_j^{(k)}$ into the jth copy of $L^2[0, 1]$, and $W^{(k)} \mid \mathscr{H}_j^{(k)} = W_j^{(k)}$. Then let $\rho^{(k)}$ be the representation of $C^*(S(2^k))$ into $\mathscr{B}(L^2[0, 1])$ defined by

$$\rho^{(k)}(A) = (W^{(k)})^* \, \pi^{(k)}(\nu(A)) \, W^{(k)}.$$

We now show that $\rho^{(k)} \mid C^*(S(2^{k-1})) = \rho^{(k-1)}$.

Now

$$\begin{aligned} \rho^{(k)}(U_+) &= (W^{(k)})^* \, \pi^{(k)}(\nu(U_+)) \, W^{(k)} \\ &= (W^{(k)})^* \, N(I, I, \ldots, I, T_\phi) \, W^{(k)}, \end{aligned}$$

where $N(A_1, A_2, \ldots, A_p)$ is the $p \times p$ matrix defined by

$$N(A_1, A_2, \ldots, A_p)(x_1, x_2, \ldots, x_p) = (A_p x_p, A_1 x_1, \ldots, A_{p-1} x_{p-1}).$$

Using the facts

$$(1) \quad \phi(x)/2^k = \phi(x/2^k + (2^k - 1)/2^k) \qquad \text{for all } k \text{ and } 0 \leqslant x \leqslant 1,$$

and

$$\begin{aligned} (2) \quad & b_1^{(1)} = 1, \qquad b_2^{(1)} = 0. \\ & b_{2^k-j}^{(k)} = 2b_{2^{k-1}-j}^{(k-1)}, \qquad 0 \leqslant j < 2^{k-1}, \\ & b_{2^k-j}^{(k)} = b_{2^k-j+2^{k-1}}^{(k)} + 1, \qquad 2^{k-1} \leqslant j < 2^k - 1, \end{aligned}$$

we see that $\rho^{(k)}(U_+) = T_\phi$ for all $k \geqslant 1$. It is immediate that $\rho^{(k)}(P_1^{(k)}) = L^2[1 - 1/2^k, 1]$. The recursive relations (2) imply that

$$\begin{aligned} & L^2 \left[b_1^{(k)}/2^k, \frac{b_1^{(k)} + 1}{2^k} \right] + L^2 \left[\frac{b_{1+2^{k-1}}^{(k)}}{2^k}, \frac{b_{1+2^{k-1}}^{(k)} + 1}{2^k} \right] \\ & \quad = L^2 \left[\frac{b_1^{(k-1)}}{2^{k-1}}, \frac{b_1^{(k-1)} + 1}{2^{k-1}} \right]. \end{aligned}$$

So that we have

$$\rho^{(k)}(P_1^{(k-1)}) = \rho^{(k)}(P_1^{(k)} + P_{1+2^{k-1}}^{(k)})$$
$$= L^2[1 - (1/2^{k-1}), 1]$$
$$= \rho^{(k-1)}(P_1^{(k-1)}).$$

Hence $\rho^{(k)} \mid C^*(S(2^k)) = \rho^{(k-1)}$. But then there is a representation

$$\theta\colon \mathscr{A}(2^k) \to \mathscr{B}(L^2[0, 1])$$

defined by $\theta(A) = \rho^{(k)}(A)$ if $A \in C^*(S(2^k))$. Under this representation we have that

$$\theta(P_j^{(k)}) = L^2(\phi^{-j+1}[1 - 1/2^k, 1])$$

for all $k \geqslant 1$, $1 \leqslant j \leqslant 2^k$, and

$$\theta(U_+) = T_\phi\,.$$

Note that since $\nu(\mathfrak{A}(2^k))$ is simple, θ induces a $*$-isomorphism of $\nu(\mathfrak{A}(2^k))$ with $\theta(\mathfrak{A}(2^k))$.

THEOREM 7. *The representation*

$$\theta\colon \mathscr{M}(2^k) \to \mathscr{B}(L^2[0, 1]) \text{ is irreducible.}$$

Proof. Let $A \in (\theta(\mathscr{M}(2^k)))'$. Then A commutes with

$$L^2[i/2^k, (i+1)/2^k]$$

for all $k \geqslant 1$, and $0 \leqslant i \leqslant 2^k - 1$. Hence A commutes with M_f for all $f \in L^\infty[0, 1]$, where M_f denotes the multiplication operator on $L^2[0, 1]$ associated with f. Since the algebra $\{M_f\colon f \in L^\infty[0, 1]\}$ is maximal abelian, we have that $A = M_g$ for some $g \in L^\infty[0, 1]$. But A also commutes with $\theta(E^{(1)}(1, 2)) = \theta(P_1^{(1)}U_+{}^*) = L^2[(1/2), 1]\, T_\phi^{-1}$, hence $g(x) = g(\phi(x))$ for almost every $x \in [0, (1/2)]$. Likewise A commutes with $\theta(E^{(2)}(2, 3)) = \theta(P_2^{(2)}U_+{}^*) = L^2[(1/4), (1/2)]\, T_\phi^{-1}$, and hence $g(x) = g(\phi(x))$ for a.e., $x \in [(1/2), 3/4]$. Also A commutes with $\theta(E^{(3)}(4, 5)) = L^2[1/8, (1/4)]\, T_\phi^{-1}$ and hence $g(x) = g(\phi(x))$ for a.e., $x \in [3/4, 7/8]$. Continuing in this manner we obtain that $g(x) = g(\phi(x))$ for a.e., $x \in [0, 1]$. But this easily implies that $g(x) = g(x + p/2^k)$ for a.e., $x \in [0, 1]$, all $k \geqslant 1$, $0 \leqslant p < 2^k$, where addition is modulo one. But it is well known (see [11, Lemma 13.2.1]) that this implies that g

is constant a.e. Hence A is a scalar and θ restricted to $\mathscr{M}(2^k)$ is irreducible.

COROLLARY 8. *We have that* $\nu(\mathscr{M}(2^k))' \cap \nu(\mathfrak{A}(2^k))$ *consists of scalars. Eequivalently if* $A \in \mathfrak{A}(2^k)$ *is such that* $AB - BA \in \mathscr{K}$ *for all* $B \in \mathscr{M}(2^k)$, *then* $A = \lambda I + C$ *for some* $C \in \mathscr{K}$.

Proof. If $A \in \mathfrak{A}(2^k)$ is such that $AB - BA \in \mathscr{K}$ for all $B \in \mathscr{M}(2^k)$, then $\theta(A) \in \theta(\mathscr{M}(2^k))'$. So $\theta(A) = \lambda I$ for some complex number λ by Theorem 7. Hence $A = \lambda I + C$ for some compact C.

As mentioned in the introduction, we are interested in relative commutants because of Dixmier's conjecture that the relative commutant of a simple C^*-algebra in a simple C^*-algebra is again simple. We are unable to determine if the relative commutant of $\nu(\mathfrak{A}(2^k))$ in the Calkin algebra $\nu(\mathscr{B}(\mathscr{H}))$ is simple or not. However, we can exhibit an element of $\nu(\mathscr{B}(\mathscr{H}))$ which is not normal but which does commute with $\nu(\mathfrak{A}(2^k))$. Let $B^{(k)}$ be the $2^k \times 2^k$ matrix such that for $k \geqslant 1$

$$B^{(k)}_{j,j+2^{k-1}} = (1/2^{k-1}) + ((j-1)/2^{k-2}) \qquad \text{if} \quad 1 \leqslant j \leqslant 2^{k-2},$$

and

$$B^{(k)}_{j,j+2^{k-1}} = B^{(k)}_{2^{k-1}-j+1,2^k-j+1} \qquad \text{if} \quad 2^{k-2}+1 \leqslant j \leqslant 2^{k-1},$$

and $B^{(k)}_{i,j=0}$ for all other i, j. That is, $B^{(1)}_{1,2} = 1$; $B^{(2)}_{1,3} = 1/2 = B^{(2)}_{2,4}$; $B^{(3)}_{1,5} = 1/4$, $B^{(3)}_{2,6} = 3/4$, $B^{(3)}_{3,7} = 3/4$, $B^{(3)}_{4,8} = 1/4$; etc. Let $B = \sum_{k=1}^{\infty} \oplus B^{(k)}$. Then $B \in \mathscr{B}(\mathscr{H})$ and an easy calculation shows that $BU_+ - U_+B \in \mathscr{K}$ and that $BP_1^{(k)} - P_1^{(k)}B$ is finite rank for all $k \geqslant 1$. Hence $\nu(B)$ commutes with $\nu(S)$ for S any shift of period 2^k, $k \geqslant 1$. Hence $\nu(B) \in \nu(\mathfrak{A}(2^k))'$. But $BB^* - B^*B$ is not compact, so $\nu(B)$ is not normal.

Remark. We note that the discussion of this section concretely exhibits an irreducible operator $T \in \mathscr{B}(L^2[0, 1])$ such that $C^*(T)$ is simple and contains no nonzero compact operator. The existence of such an operator was proven in [15], but the operator was not explicitly described. Let $S = \sum_{k=1}^{\infty} (1/2^{k-1})\, S(2^k)$. Then by [3, Cor. 3.3] we have that $C^*(S) = \mathfrak{A}(2^k)$ so that $C^*(\theta(S)) = \theta(\mathfrak{A}(2^k))$ is a simple, irreducible C^*-algebra, which contains no nonzero compact operators. Using the above calculations, we see that

$$T = \theta(S) = T_\phi \left(I + \sum_{k=1}^{\infty} (1/2^k)\, M_{\chi[1-1/2^k,1]}\right)$$

is an operator with the asserted properties. Finally, we note that since $\theta(C^*(S))$ is irreducible and contains no nonzero compacts, the algebra $C^*(S)$ is not a GCR algebra [5, 9.1]. The fact that $C^*(S)$ is not GCR was proven by a different method in [3, Cor. 3.2].

4. A Representation of CAR

Let $\mathscr{H}$ be a Hilbert space. A representation of the canonical anticommutation relations (CAR) over $\mathscr{H}$ is a linear mapping a: $\mathscr{H} \to \mathscr{B}(\mathscr{H}')$, where $\mathscr{H}'$ is a Hilbert space, such that

$$a(f)\, a(g) + a(g)\, a(f) = 0,$$

and

$$a(f)^*\, a(g) + a(g)\, a(f)^* = (g, f)I$$

for all $f, g \in \mathscr{H}$. The usual construction of a representation of CAR, called the Fock representation, first defines the operators $a(f)$ on a prehilbert space and then uses the automatic boundedness of the operators to extend them to bounded operators on a Hilbert space [12, 1.3]. We construct a representation of CAR over $\mathscr{H}$ directly as bounded operators on l^2. This representation will be unitarily equivalent to the Fock representation. If a is any representation of CAR over $\mathscr{H}$, then the C^*-algebra generated by $\{a(f): f \in \mathscr{H}\}$ is a UHF algebra of type (2^k) [12, Sect. 3.2]. In our context this implies that there must be a representation of CAR as elements of $\mathscr{M}(2^k)$. We construct such a representation.

Let a and b be two symbols. By the Morse recurrent sequence on the symbols a and b we mean the following sequence of a's and b's:

$$a \mid b \mid ba \mid baab \mid baababba \mid \cdots .$$

"The rule here is that the block of terms between $\mid\ \mid$ is obtained from all that goes before by interchanging the a's and b's" [10, p. 198]. Let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$. Let V_n be the diagonal operator of period 2^n whose first 2^n terms are obtained as in the Morse recurrent sequence on the symbols 1 and -1 and define $P_j^{(k)}$ as in the introduction; that is, $P_j^{(k)}$ is the diagonal projection with weights $\alpha_m = 1$ if $m = j + l2^k$, $l \geqslant 0$ and $\alpha_m = 0$ otherwise. Let $V_0 = I$. Then define a on $\{f_n\}_{n=1}$ by

$$a(f_n) = V_{n-1} \left(\sum_{j=1}^{2^{n-1}} P_j^{(n)} \right) U_+^{*2^{n-1}}.$$

Some laborious calculations then show that

$$a(f_n)\, a(f_m) + a(f_m)\, a(f_n) = 0,$$
$$a(f_n)^*\, a(f_m) + a(f_m)\, a(f_n)^* = \delta_{n,m}\,.$$

We then extend a by linearity to the prehilbert space of finite linear combinations of the f_n. The automatic boundedness of a representation of CAR then allows us to extend a to all of $\mathscr{H}$ to obtain a representation of CAR over $\mathscr{H}$. In this representation $\mathscr{M}(2^k)$ is the C^*-algebra generated by the $a(f)$.

In this representation of CAR, the vector e_0 is a cyclic vector for the $a(f)$ and $a(f)\, e_0 = 0$ for all f. Hence by [12, Lemma 4.7] this representation of CAR is unitarily equivalent to the Fock representation. This representation makes more obvious some of the calculations in Størmer's proof [14] of the fact that the even CAR algebra is a UHF algebra.

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