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Skew Products of Relations and the Structure of Simple C^*-Algebras

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In this paper, we give a general construction which can be applied to give some interesting examples of simple C^*-algebras. The construction is described in two ways: as an inductive limit, reminiscent of the construction in [1] and [2], in Sect. 1; and as the C^*-algebra of a skew product of relations in Sect. 2. The second approach presents the C^*-algebras as groupoid C^*-algebras [21] with an easily described diagonal [16].

The specific examples constructed are simple C^*-algebras with a shortage of projections. The first algebra A_1 is a stable simple C^*-algebra with no nonzero projections. Earlier examples of such C^*-algebras were constructed in [1], [8], and [11; p. 110]; but the construction given here is perhaps the most elementary, requiring neither K-theory nor the theory of AF algebras – it really only uses the Dixmier-Douady classification of continuous trace C^*-algebras [13], along with a related result of Serre and Grothendieck [15]. The skew product description of this algebra owes much to Raeburn and Taylor [20].

The second example A_2 (and its variant A_3) has a more subtle lack of projections: although every hereditary C^*-subalgebra of $A_3 \otimes \mathscr{K}$ contains a nonzero projection, there are nonetheless distinct traces on $A_2 \otimes \mathscr{K}$ and $A_3 \otimes \mathscr{K}$ which agree on all projections. Thus, although A_3 has no minimal projections, it is in a sense "more projectionless" than the examples constructed in [2], which at least have enough projections in matrix algebras to distinguish traces.

As a related result in the spirit of Sect. 2, we show in Sect. 3 that every hereditary C^*-subalgebra of a Bunce-Deddens weighted shift algebra [7] has an approximate identity of projections.

Most of the work presented in this paper was actually done by one or the other author individually. However, the whole has turned out to be more than the sum of its parts, and it is clear to us that the overall results of the paper never would have been obtained without the benefit of the numerous scintillating conversations we have had together over the last several months. This work was done while we were both visitors at the Mathematics Institute, Universität Tübingen, supported by Forschungsstipendiums from the Alexander von Humboldt-Stiftung. We would like to thank the people of both institutions for their generous hospitality and support.

1. Inductive Limit Construction of the Examples

We first describe a functorial construction of a "twisted double embedding" θ_ρ: $B \to M_2(B)$ for any C^*-algebra B and $\rho \in \mathrm{Aut}(B)$, which can be iterated to give a (functorial) inductive limit construction.

Definition 1.1. Given a C^*-algebra B and $\rho \in \mathrm{Aut}(B)$, let $\theta_\rho(b) = \begin{bmatrix} b & 0 \\ 0 & \rho(b) \end{bmatrix} \in M_2(B)$. Let $M_\infty^\rho(B) = \varinjlim(M_{2^n}(B), \theta_\rho \otimes \mathrm{id})$.

There are of course many possible variations of this construction – the embedding multiplicities and automorphisms can be varied from step to step, for example.

Remark 1.2. We may alternately write $M_\infty^\rho(B)$ as a crossed product as in [22; 2.5(4)] as follows. Let $G = \bigoplus_{\mathbb{N}} \mathbb{Z}_2$ act on $A = B \otimes C(\Omega)$, where Ω is the Cantor set $\{0,1\}^{\mathbb{N}}$, by letting $g_n = (0, \ldots, 0, 1, 0, \ldots)$ act on $A = A_0^{(n)} \oplus A_1^{(n)}$ ($A_i^{(n)} = B \otimes \chi_i^{(n)} C(\Omega) \chi_i^{(n)}$, where $\chi_i^{(n)}$ is the characteristic function of $\{(e_k): e_n = i\} \subseteq \Omega$) by $\lambda(g_n)(b_0 \otimes f_0, b_1 \otimes f_1) = (\rho(b_1) \otimes f_1, \rho^{-1}(b_0) \otimes f_0)$. Then $M_\infty^\rho(B) \cong A \times_\lambda G$.

Proposition 1.3. *If $X = Prim(B)$ is a compact Hausdorff space and the homeomorphism of X induced by ρ is minimal, then $M_\infty^\rho(B)$ is simple.*

Proof. The argument is quite similar to the proof of [1; 3.2]. If $y \in X = \mathrm{Prim}(B)$, write $y^{(n)}$ for the primitive ideal of $M_{2^n}(B)$ consisting of matrices with all entries in y. Then

$$\begin{aligned} y^{(n+k)} \cap \theta_\rho^k(M_{2^n}(B)) &= \theta_\rho^k(\{b : b \in y^{(n)}, \rho(b) \in y^{(n)}, \ldots, \rho^k(b) \in y^{(n)}\}) \\ &= \theta_\rho^k(y^{(n)} \cap \rho^{-1}(y^{(n)}) \cap \ldots \cap \rho^{-k}(y^{(n)})). \end{aligned}$$

If I is a (closed two-sided) ideal of $M_{2^{n+k}}(B)$ and $I_0 = (\theta_\rho^k)^{-1}(I \cap \theta_\rho^k(M_{2^n}(B))$ is the corresponding ideal of $M_{2^n}(B)$, then I and I_0 correspond to closed subsets E and E_0 of X respectively. $I = \bigcap_{y \in E} y^{(n+k)}$,

$$\begin{aligned} I_0 = (\theta_\rho^k)^{-1}\Big(\bigcap_{y \in E} [y^{(n+k)} \cap \theta_\rho^k(M_{2^n}(B))]\Big) &= \bigcap_{y \in E} [y^{(n)} \cap \ldots \cap \rho^{-k}(y^{(n)})] \\ &= \bigcap \{y^{(n)} : y \in E \cup \rho^{-1}(E) \cup \ldots \cup \rho^{-k}(E)\}, \end{aligned}$$

so $E_0 = E \cup \rho^{-1}(E) \cup \ldots \cup \rho^{-k}(E)$. Now if J is a proper (closed two-sided) ideal of $M(B)$, let $J_n = (\theta_\rho^\infty)^{-1}[J \cap \theta_\rho^\infty(M_{2^n}(B)]$, and F_n the corresponding closed subset of X. For n sufficiently large, J_n is a proper ideal, so F_n is nonempty; and

$$F_n = F_{n+k} \cup \rho^{-1}(F_{n+k}) \cup \ldots \cup \rho^{-k}(F_{n+k})$$

for each k, since $J_n=(\theta_\rho^k)^{-1}[J_{n+k}\cap\theta_\rho^k(M_{2^n}(B))]$, so if $x\in F_n$, for any k we must have $\rho^r(x)\in F_{n+k}$ for some r, $0\leqq r\leqq k$, and hence $\{\rho^{-r}(x),\rho^{-r+1}(x),\dots,\rho^{-r+k}(x)\}\subseteq F_n$. Thus F_n contains an interval of the orbit of x, including x itself, of length k. Since k is arbitrary, we have either $\{\rho^k(x): k\geqq 0\}$ or $\{\rho^{-k}(x): k\geqq 0\}$ contained in F_n. Each of these sets is dense by the following lemma, so $F_n=X$, $J_n=0$. This is true for all n, so $J=0$. □

Lemma 1.4. *Let X be a compact Hausdorff space, ρ a minimal homeomorphism of X. Then, for any $x\in X$, $\Sigma_+=\{\rho^n(x): n\geqq 0\}$ and $\Sigma_-=\{\rho^{-n}(x): n\geqq 0\}$ are dense in X.*

Proof. Let $E_k=\{\rho^n(x): n\geqq k\}^-$. Then $\bigcap E_k$ is a nonempty closed subset of X which is invariant under ρ, so $\bigcap E_k=X$ by minimality; thus $(\Sigma_+)^-=E_0\supseteq\bigcap E_k=X$. The argument for Σ_- is similar. □

Example 1.5. (A_1) Let B be a stable continuous trace C^*-algebra with $\hat{B}=\mathbb{T}^3$, the 3-torus, with nontrivial Dixmier-Douady invariant ([13], [12; 10.7.14]). (Recall that $H^3(\mathbb{T}^3,\mathbb{Z})\cong\mathbb{Z}$.) Let ρ be the minimal homeomorphism of $\mathbb{T}^3$ given by coordinatewise rotation by three irrational angles θ_1, θ_2, θ_3 which are linearly independent over $\mathbb{Q}$. Since ρ is homotopic to the identity, the induced automorphism ρ^* of $H^3(\mathbb{T}^3,\mathbb{Z})$ is the identity. The stable continuous trace C^*-algebras with fixed base space X are parametrized up to isomorphism exactly by $H^3(X,\mathbb{Z})$ [12; 10.8.4], and so there is an automorphism of B (also denoted ρ) which induces the homeomorphism ρ on $\hat{B}$. We may assume $\rho\cong\gamma\otimes\mathrm{id}$ on $B\cong B\otimes\mathcal{K}$. A specific ρ with these properties is given in 2.5. By [15; 1.7], B contains no nonzero projections since the Dixmier-Douady obstruction is of infinite order. (If B contained a projection p, then pBp would be an Azumaya algebra with the same H^3 invariant as B.) The same is true of $M_{2^n}(B)\cong B$ and hence of $A_1=M_\infty^\rho(B)$. A_1 is simple by 1.3, and stable because a tensor product factor of $\mathcal{K}$ can be split off at each stage: $M_\infty^\rho(B)\cong M_\infty^\gamma(B)\otimes\mathcal{K}$.

The above construction would appear to depend strongly on the choice of the angles θ_1, θ_2, θ_3 (as well as ρ), but it is not clear how to find invariants which will distinguish between the algebras corresponding to different angles. For example, any two such algebras will have identical K-groups. In fact, any such algebra will have the same shape [5] as $B\otimes M_{2^\infty}$, where M_{2^∞} is the *UHF* algebra of type 2^∞.

Example 1.6. (A_2, A_3) Let X be a connected compact Hausdorff space, and ρ a minimal homeomorphism of X for which there is more than one invariant probability measure. For example, we can take $X=\mathbb{T}^2$, ρ the homeomorphism constructed in [14; 1.17]; there are uncountably many extremal invariant measures in this case. Let $B=C(X)$; we use ρ also to denote the corresponding automorphism of B. Then $A_2=M_\infty^\rho(B)$ is simple. A_2 has many tracial states (any invariant measure on X induces one), but all tracial states of A_2 agree on projections (it is enough to show this for $M_{2^n}(B)$, where it is obvious.) The same is true for any matrix algebra over A_2, and hence also for $A_3=A_2\otimes M_{2^\infty}$.

Recall the following properties ([19, 4]) for a C^*-algebra B.

(LP): The linear span of the projections in B is dense.

(FS): Every self-adjoint element of B can be approximated arbitrarily closely by a self-adjoint element of finite spectrum.

(HP): Every hereditary C^*-subalgebra of B has an approximate identity of projections.

(SP): Every nonzero hereditary C^*-subalgebra of B contains a nonzero projection.

We have (LP) $\Leftarrow$ (FS) $\Leftrightarrow$ (HP) $\Rightarrow$ (SP) and (LP) $\not\Rightarrow$ (FS) by the results of [19] and [4; 2.7]. It is clear that A_2 and A_3 do not have (LP) (all traces on A_2 or A_3 agree on linear combinations of projections), and hence (HP) also fails. However, A_3 does satisfy (SP). In fact, we have a more general result.

If $x \in B$, we let B_x denote the hereditary C^*-subalgebra $(x^* B x)^-$ of B. For $\delta > 0$, we let f_δ be the continuous function on $\mathbb{R}$ which is 0 on $(-\infty, \delta/2]$, 1 on $[\delta, \infty)$, and linear on $[\delta/2, \delta]$.

The proof of the next lemma essentially uses G. Elliott's argument from [19; Prop. 14].

Lemma 1.7. *Let B be a C^*-algebra, x_n, $x \in B_+$ with $x_n \to x$. Suppose for some fixed $\delta > 0$ there is a projection $p_n \in B_{f_\delta(x_n)}$ for all n. Let $\varepsilon > 0$. Then for all sufficiently large n, there is a projection $q_n \in B_{f_{\delta/2}(x)}$ with $\|p_n - q_n\| < \varepsilon$.*

Proof. Let $0 < \eta < 1/2$ be such that $2\eta + 4\eta(1+\sqrt{1-2\eta})^{-1} < \varepsilon$. We have $f_{\delta/2}(x_n) \to f_{\delta/2}(x)$; choose N so that $\|f_{\delta/2}(x_n) - f_{\delta/2}(x)\| < \eta$ for all $n \geqq N$. Since $f_{\delta/2}(x_n)$ is a unit for p_n, we have (for $n \geqq N$) $\|p_n f_{\delta/2}(x)^2 p_n - p_n\| < 2\eta < 1$; so $p_n f_{\delta/2}(x)^2 p_n$ and hence also $[p_n f_{\delta/2}(x)^2 p_n]^{1/2} = z_n$ is invertible in B_{p_n}, i.e. there is an $r_n \in (B_{p_n})_+$ with $r_n z_n = z_n r_n = p_n$. We have $\|r_n - p_n\| < 2\eta(1+\sqrt{1-2\eta})^{-1}$. Let $u = f_{\delta/2}(x) p_n r_n$. Then $u^* u = p_n$, and $q_n = u u^* \in B_{f_{\delta/2}(x)}$.

$$\begin{aligned}\|u - p_n\| &\leqq \|(f_{\delta/2}(x) p_n - p_n) p_n\| \\ &\quad + \|f_{\delta/2}(x)(r_n - p_n)\| < \eta + 2\eta(1+\sqrt{1-2\eta})^{-1},\end{aligned}$$

so

$$\begin{aligned}\|q_n - p_n\| &\leqq \|q_n - p_n u^*\| + \|p_n u^* - p_n\| \\ &\leqq 2\|u - p_n\| < 2\eta + 4\eta(1+\sqrt{1-2\eta})^{-1} < \varepsilon. \quad \square\end{aligned}$$

Lemma 1.8. *Let B be a C^*-algebra, S a dense subset of B_+.*

(a) *If, for every $x \in S$ and $\delta > 0$ with $f_\delta(x) \neq 0$, $B_{f_\delta(x)}$ contains a nonzero projection, then B has (SP).*

(b) *If each $B_{f_\delta(x)}$ has an approximate identity of projections, then B has (HP).*

Proof. We prove only (b); the other is similar but easier. Fix $\varepsilon > 0$ and $x \in B_+$, $\|x\| = 1$. Let $x_n \in S$ with $x_n \to x$; set $\delta = \varepsilon/5$, and let p_n be a projection in $B_{f_\delta(x_n)}$ with $\|f_\delta(x_n) - f_\delta(x_n) p_n\| < \varepsilon/5$. Choose n large enough that $\|f_\delta(x_n) - f_\delta(x)\| < \varepsilon/5$, and let $q_n \in B_{f_{\delta/2}(x)} \subseteq B_x$ with $\|p_n - q_n\| < \varepsilon/5$. Then we have

$\|f_\delta(x)-f_\delta(x)\,q_n\|<3\,\varepsilon/5$. Since $\|x-xf_\delta(x)\|<\delta<\varepsilon/5$, we have

$$\|x-xq_n\|<2\,\|x-xf_\delta(x)\|+\|xf_\delta(x)-xf_\delta(x)q_n\|<\varepsilon.$$

The general result now follows from [6; I.1.10]. □

Theorem 1.9. *Let B be a simple C^*-algebra and D a UHF algebra. If B contains a nonzero projection, then $A=B\otimes D$ has (SP).*

Proof. The argument is similar to that of [3; A2]. Write $D=\bigotimes_{k=1}^{\infty} D_k$, with D_k a matrix algebra. Let $A_n=B\otimes\left(\bigotimes_{k=1}^{n} D_k\right)\otimes 1\subseteq A$; then $A=[\bigcup A_n]^-$. Let $\delta>0$ and $x\in\left[B\otimes\left(\bigotimes_{k=1}^{n} D_k\right)\right]_+$, with $f_\delta(x)\neq 0$. If p is a nonzero projection in B, then $(p\otimes 1)\otimes 1$ is in the 2-sided ideal of A_n generated algebraically by $f_\delta(x)\otimes 1$. So by the argument of [10; 1.1 IV and 1.7], for sufficiently large m, we have $p\otimes 1\otimes q\otimes 1$ equivalent to a projection in the hereditary C^*-subalgebra of A_{n+m} generated by $f_\delta(x)\otimes 1\otimes 1$, where q is a one-dimensional projection in $\bigotimes_{k=n+1}^{n+m} D_k$. Thus A satisfies the hypotheses of 1.8(a). □

Corollary 1.10. *A_3 satisfies (SP).*

It follows that (SP) $\not\Rightarrow$ (LP), and *a fortiori* (SP) $\not\Rightarrow$ (HP). This is perhaps somewhat surprising, since one might hope to use some kind of exhaustion argument to deduce (HP) from (SP).

2. Skew Products of Relations

The twisted double embedding construction of Sect. 1 may be viewed as a C^*-cross-product where the group is replaced by a groupoid (the action is given by a 1-cocycle with values in the automorphism group of the C^*-algebra.) In this section we confine ourselves to introducing a notion of skew product of relations; here the action is given by a 1-cocycle defined on one relation taking values in the automorphism group of another (two kindred notions may be found in [21] under the names skew-product and semi-direct product.) The examples of Sect. 1 can then be exhibited as the C^*-algebras of skew products with appropriate 2-cocycles.

Recall that a principal discrete groupoid is a locally compact topological groupoid G with unit space X and source and range maps s, r: $G\to X$ which are local homeomorphisms, such that the map $\alpha\mapsto(r(\alpha),s(\alpha))$ from G to $X\times X$ is one-to-one. G may then be regarded as an equivalence relation on X. In the following, the phrase "R is a relation on X" means that R is a principal discrete groupoid with unit space X. The inverse is denoted by an asterisk (*).

Let R be a relation on X and Q a relation on Y; we consider continuous 1-cocycles on R with values in the automorphism group of Q.

Definition 2.1. *A* continuous function ϕ: $R\times Q\to Q$ is an $R-Q$ cocycle (written $(\alpha,\beta)\mapsto\phi(\alpha)\beta$) if:

(i) $\phi(\alpha_1\alpha_2)\beta=\phi(\alpha_1)\phi(\alpha_2)\beta \quad \alpha_1, \alpha_2\in R, \beta\in Q$

(ii) $\phi(\alpha)(\beta_1\beta_2)=(\phi(\alpha)\beta_1)(\phi(\alpha)\beta_2) \quad \alpha\in R, \beta_1, \beta_2\in Q$

(iii) $\phi(s(\alpha))\beta=\beta \quad \alpha\in R, \beta\in Q$.

Note. In order for (ii) to make sense $\phi(\alpha)$ must map composable pairs to composable pairs. It follows that each $\phi(\alpha)$ is a structure-preserving homeomorphism on Q (with inverse $\phi(\alpha^*)$), hence an automorphism.

Definition 2.2. Given an $R-Q$ cocycle ϕ, the skew product $R\underset{\phi}{\times}Q$ is the relation on $X\times Y$ which, as a topological space, is $R\times Q$, with groupoid structure given by the formulae:

i) $s(\alpha,\beta)=(s(\alpha),s(\beta))$

ii) $(\alpha,\beta)^*=(\alpha^*,\phi(\alpha)\beta^*)$

iii) $(\alpha_1,\phi(\alpha_2)\beta_1)(\alpha_2,\beta_2)=(\alpha_1\alpha_2,\beta_1\beta_2)$.

Proposition 2.3. *With the above operations, $R\underset{\phi}{\times}Q$ is a relation on $X\times Y$.*

Proof. Clearly, all the above operations are continuous. The range map is given by $r(\alpha,\beta)=(r(\alpha),r(\phi(\alpha)\beta))$. Then the image of $R\underset{\phi}{\times}Q$ under the map $r\times s$ constitutes an equivalence relation on $X\times Y$. □

Note that if ϕ: $R\times Q\to Q$ is trivial (i.e. $\phi(\alpha)\beta=\beta$), then the skew product is simply the direct product relation $R\times Q$. This is true more generally for suitably defined coboundaries. An $R-Q$ cocycle ϕ is said to be a coboundary if there is a continuous map h: $X\times Q\to Q$ with $h(x)\in\mathrm{Aut}(Q)$ for each $x\in X$, such that $\phi(\alpha)=h(r(\alpha))^{-1}h(s(\alpha))$. In this case, $R\underset{\phi}{\times}Q\cong R\times Q$ under the correspondence $(\alpha,\beta)\mapsto(\alpha,h(s(\alpha))\beta)$.

Example 2.4. Let R be the *CAR* relation on the Cantor set Ω and m: $R\to\mathbb{Z}$ be the number cocycle (i.e. $\Omega=\{(e_i)$: $e_i=0$ or 1, $i\geqq 0\}$ and $(e_i)\sim(f_i)$ if there is $n\geqq 0$ so that $e_i=f_i$ for all $i\geqq n$; $m(e,f)=\sum_i(e_i-f_i)$, cf. [21; pp. 129–130].) Let ρ be a homeomorphism on a locally compact space X. Then, viewing X as the trivial relation on itself, we define an $R-X$ cocycle ϕ by $\phi(\alpha)x=\rho^{m(\alpha)}(x)$. Note that if ρ is a minimal homeomorphism, then the skew product $R\underset{\phi}{\times}X$ is a minimal relation, i.e. there are no nontrivial invariant open subsets of $\Omega\times X$. Further, the associated C^*-algebra $C^*(R\underset{\phi}{\times}X)$ is isomorphic to $M_\infty^\rho(C_0(X))$. This may be seen as follows.

Topologically, $R\underset{\phi}{\times}X$ is a countable disjoint union of spaces homeomorphic to $\Omega\times X$, indexed by $G=\bigoplus_{k=1}^{\infty}\mathbb{Z}_2$, and $C^*(R\underset{\phi}{\times}X)$ is the completion of $C_c(R\underset{\phi}{\times}X)$ as described in [21]. If A_n is the set of functions which vanish outside $G_n=\{0,1\}^n\times\{0\}$ and whose values in each copy of $\Omega\times X$ depend only on the point in X and the first n coordinates in Ω, then A_n is a C^*-subalgebra of $C^*(R\underset{\phi}{\times}X)$ and $C^*(R\underset{\phi}{\times}X)=\varinjlim A_n$. Note that $A_n\cong C^*(R_n\underset{\phi_n}{\times}X)$, where R_n is the transitive relation on the discrete space $\Omega_n=\{0,1\}^n$, and ϕ_n the R_n-X cocycle defined in the same way as ϕ. We have $C^*(R_n \underset{\phi_n}{\times} X) \cong M_{2^n}(C_0(X))$, and the corresponding embedding of A_n into A_{n+1}, when identified as an embedding of $M_{2^n}(C_0(X))$ into $M_{2^{n+1}}(C_0(X))$, is easily seen to be θ_ρ. The algebra A_2 of Example 1.6 can thus be described as the C^*-algebra of a relation; A_3 can be described as the C^*-algebra of the product relation $R \times (R \underset{\phi}{\times} X)$.

Example 2.5. Using a technique of Raeburn and Taylor [20] for constructing a continuous trace algebra with prescribed Dixmier-Douady invariant, we describe the algebra A_1 of Example 1.4 as the C^*-algebra of a skew product relation with 2-cocycle (cf. [18]).

If T is a locally compact space, it is known that $H^3(T, \mathbb{Z})$ is canonically isomorphic to second cohomology with coefficients in the sheaf of germs of continuous circle-valued functions; the construction in [20] uses cocycles from the sheaf cohomology. Let ρ be the homeomorphism of $T = \mathbb{T}^3$ given in 1.5. Choose an open cover $\mathscr{U} = \{U_i : i \in I\}$ of T so that $\rho^n(U_i) \in \mathscr{U}$ for each n, i (write $U_{i'} = \rho(U_i)$), together with a nontrivial Čech 2-cocycle λ relative to $\mathscr{U}$ such that:

$$\lambda_{i'j'k'}(\rho(t)) = \lambda_{ijk}(t) \qquad \text{for } t \in U_i \cap U_j \cap U_k.$$

This may be done by choosing a finite cover $\{V_1, \ldots, V_m\}$ such that $V_i \cap \rho^n(V_j)$ is contractible for all i, j, n; then a nontrivial 2-cocycle relative to $\{V_i\}$ may be extended to the cover consisting of all translates under ρ.

Put $Y = \coprod_i U_i$ (the disjoint union) and let $\psi \colon Y \to T$ be the "reinclusion" map (i.e. $t \in U_i \mapsto t \in T$). Let $R(\psi) = \{(x, y) \in Y \times Y \colon \psi(x) = \psi(y)\}$ (cf. [17]) and note that $R(\psi)$ is homeomorphic to $\coprod_{i,j} U_i \cap U_j$. The groupoid of composable pairs in $R(\psi)$ is homeomorphic to $\coprod_{i,j,k} U_i \cap U_j \cap U_k$, so the Čech 2-cocycle λ defines a 2-cocycle $\tilde{\lambda}$ on $R(\psi)$. The resulting C^*-algebra, $C^*(R(\psi), \tilde{\lambda})$, is continuous trace with Dixmier-Douady invariant $[\lambda]$ [20].

Let $\hat{\rho} \in \mathrm{Aut}(R(\psi))$ be such that $\hat{\rho}(U_i \cap U_j) = U_{i'} \cap U_{j'}$. Observe that $\hat{\rho}$ leaves the 2-cocycle $\tilde{\lambda}$ invariant; thus $\hat{\rho}$ induces an automorphism ρ of $C^*(R(\psi), \tilde{\lambda})$.

We define an $R - R(\psi)$ cocycle (R is again the CAR relation):

$$\phi(\alpha)\beta = \hat{\rho}^{m(\alpha)}(\beta) \qquad \text{(where } m \text{ is the number cocycle).}$$

We define a 2-cocycle on the skew product by

$$\sigma((\alpha_1, \phi(\alpha_2)\beta_1), (\alpha_2, \beta_2)) = \tilde{\lambda}(\beta_1, \beta_2).$$

Then, as in 2.4, $C^*(R \underset{\phi}{\times} R(\psi), \sigma) \cong A_1$ (if the ρ of 1.5 is chosen to be the one constructed here).

3. The Bunce-Deddens Algebras

In this section, we will show that if A is one of the simple C^*-algebras constructed in [7], then $A \otimes \mathscr{K}$ has the (HP) property. The two constructions which will be relevant for our purposes are:

(1) $A \otimes \mathscr{K}$ is isomorphic to the crossed product $C_0(\mathbb{R}) \times_\tau G$, where G is a

dense subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$, acting by translation on $\mathbb{R}$. A itself is $C(\mathbb{T}) \times_\tau H$, where H is a dense torsion subgroup of $\mathbb{T}$; in fact, $H = G/\mathbb{Z}$ [23].

(2) $A = \varinjlim A_n$, where $A_n = M_{k_n}(C(\mathbb{T}))$, and the embedding of A_n into A_{n+1} is an "r_n-times-around" embedding. We may take $k_1 = 1$.

Viewed in either way, the canonical copy of $C(\mathbb{T})$ in A is a diagonal, and it is shown in [16; §8, 13] that if $g \in C(\mathbb{T})$, then A_g has an approximate identity of projections.

Our approach will be to use this fact, along with 1.8(b), to show that $A \otimes \mathcal{K}$ has (HP).

First note that if n is fixed, then $A \cong M_{k_n} \otimes B$ for some Bunce-Deddens algebra B, with $A_n \cong M_{k_n} \otimes C(\mathbb{T})$, where $C(\mathbb{T})$ is a diagonal in B as above. Thus, if g is any element of A_n which is diagonal under this decomposition, then A_g has an approximate identity of projections. (Such a g can be written as a sum of orthogonal elements each of which is of the form considered in [16].)

It is not quite true that every positive element g of $C(\mathbb{T}, M_k)$ is conjugate to a diagonal element. But a slightly weaker result holds, which is still strong enough to satisfy the hypotheses of 1.8:

Lemma 3.1. *Let* $g \in C(\mathbb{T}, M_k)_+$, $\delta > 0$. *Then there are elements* $u_1, \ldots, u_k \in C(\mathbb{T}, M_k)$ *such that:*

(1) $u_i^* u_i \perp u_j^* u_j$ *for* $i \neq j$;

(2) $\sum u_i^* u_i$ *is a unit for* $f_\delta(g)$;

(3) $u_i u_i^* \leqq e_{ii} \in C(\mathbb{T}, M_k)$, *where* $e_{ii} \in M_k$ *is the* ii'th *matrix unit, interpreted as a constant function.*

Proof. We construct the u_i inductively. Let $\mathcal{U}_1 = \{z \in \mathbb{T}\colon g(z) \neq 0\}$. For each $z \in \mathcal{U}_1$, choose a one-dimensional projection p_z in M_k orthogonal to the projection onto the null space of $g(z)$. Because $[g\,C(\mathbb{T}, M_k)g]^-$ is a continuous trace C^*-algebra, the corresponding continuous field satisfies Fell's condition [12; 10.5.7], and thus p_z can be continuously extended over a neighborhood of z to a one-dimensional projection orthogonal to the null space of g. Choosing a cover of $\mathcal{U}_1$ with such intervals so that only successive intervals overlap, the sections can be fitted together to get a global field $\{q_z\colon z \in \mathcal{U}_1\}$ of one-dimensional projections orthogonal to null(g). By [9; Lemma 7], we can then find a continuous field $\{v_z\colon z \in \mathcal{U}_1\}$ with $v_z^* v_z = q_z$, $v_z v_z^* = e_{11}$. Set $v_z = 0$ for $z \notin \mathcal{U}_1$. Then v is discontinuous, but $u_1 = v f_{\delta/2}(g)$ is continuous and has the right properties. (v is a multiplier of $C_0(\mathcal{U}_1, M_k)$.) Now replace g by

$$g_2 = (1 - v^* v)\, g (1 - v^* v) \in C(\mathbb{T}, M_k), \qquad \mathcal{U}_2 = \{z \in \mathbb{T}\colon g_2(z) = 0\}. \quad \square$$

Return to the case of $A \cong M_{k_n} \otimes B$, $A_n \cong M_{k_n} \otimes C(\mathbb{T})$. If g is an arbitrary positive element of A_n and $\delta > 0$, let $u_1, \ldots, u_{k_n}$ be as in Lemma 3.1. Set $u = \sum u_i$, and $h = \sum (u_i^* u_i) f_\delta(g)(u_i^* u_i)$. Then $A_h = A_{f_\delta(g)}$ (it is enough to check that $h(z)$ and $[f_\delta(g)](z)$ have the same null space for each $z \in \mathbb{T}$). Then $x \mapsto u x u^*$ is an isomorphism of A_h onto A_{uhu^*}, and uhu^* is diagonal; hence these hereditary algebras have approximate identities of projections. Therefore, by 1.8(b) A has (HP). The same argument shows that $M_r(A)$ has (HP) for all r, and so another application of 1.8(b) shows that $A \otimes \mathcal{K}$ has (HP) too.

This example shows that even a simple C^*-algebra with (HP) can contain a projectionless masa, in fact a projectionless diagonal. We conjecture that even *UHF* algebras contain projectionless masas, although it seems unlikely that such a masa could be a diagonal.

It would be of great interest to know whether the irrational rotation algebras also have (HP). The argument given here does not seem to work, since there is no known reasonable way of writing the irrational rotation algebras as inductive limits.

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