Hints and Partial Solutions for *Real and Complex Analysis, 3rd edition* by Walter Rudin

Mitch Haslehurst

May 21, 2022

Chapter 1

Abstract Integration

1. No; to prove this we need a lemma first.

   Lemma. *If $\mathcal{M}$ is an infinite σ-algebra, there exists a countable collection $\{F_n\}_{n\geq 1}$ of nonempty, pairwise disjoint sets in $\mathcal{M}$.*

   Proof. Let $\{E_n\}_{n\geq 1}$ be any countable collection of nonempty, distinct elements in $\mathcal{M}$. The idea is to "chop" up the sets into pieces which are disjoint.

   First we look at E_1 and E_2 to get the idea. If $E_1 \subsetneq E_2$ (or vice versa), then set $F_1 = E_1$ and $F_2 = E_2 \setminus E_1$ (or vice versa, accordingly). Otherwise, set $F_1 = E_1 \setminus E_2$ and $F_2 = E_2 \setminus E_1$. In any case, the sets F_1 and F_2 are nonempty and pairwise disjoint sets in $\mathcal{M}$.

   Now suppose $F_1, F_2, \ldots, F_n$ have been constructed from $\{E_n\}_{n\geq 1}$ and E_N is an element which has not been used yet. If E_N is equal to a union of elements from the list $F_1, F_2, \ldots, F_n$, replace E_N with an element which does not have this property (possible because $\{E_n\}$ is infinite and there are finitely many F_k's) and has not been used yet, and proceed as follows. If $E_N \subsetneq F_1 \cup F_2 \cup \cdots \cup F_n$, find an element F_k such that $E_N \cap F_k \neq \varnothing$ and $F_k \setminus E_N \neq \varnothing$. Set $F_{n+1} = E_N \cap F_k$ and replace F_k with $F_k \setminus E_N$ in this case. If at least part of E_N does not meet any of the F_k's, set $F_{n+1} = E_N \setminus (F_1 \cup F_2 \cup \cdots \cup F_n)$. ◇

   Now obtain $\{F_n\}_{n\geq 1}$ as in the lemma, and let $\mathcal{P}(\mathbb{N})$ be the set of all subsets of $\mathbb{N}$. The map

   $$\mathcal{P}(\mathbb{N}) \to \mathcal{M} : A \mapsto \bigcup_{n\in A} F_n$$

   is injective since $\{F_n\}_{n\geq 1}$ is pairwise disjoint. $\mathcal{P}(\mathbb{N})$ is uncountable, hence $\mathcal{M}$ is uncountable.

2. The proof for n functions is almost identical, except replace every occurrence of "open rectangle" with "open n-dimensional box"

3. Let α be a real number and let $\{r_n\}$ be a sequence of rationals with $r_n > \alpha$ for all n and

$r_n \to \alpha$. The conclusion follows from the fact that

$$\{x \mid f(x) \geq \alpha\} = \bigcup_{n \geq 1} \{x \mid f(x) \geq r_n\}.$$

4. Let $\varepsilon > 0$. Then $x_n \leq \limsup x_n + \varepsilon$ for all but finitely many n, and the same is true for $\{y_n\}$. Thus the inequality

$$x_n + y_n \leq \limsup x_n + \limsup y_n + 2\varepsilon$$

holds for all but finitely many n and so

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we are done.

5. (a) The function $f - g$ is measurable, and the given sets are $(f-g)^{-1}([-\infty, 0))$ and $(f-g)^{-1}(\{0\})$, respectively.

   (b) If $\{f_n\}$ is the sequence, then

$$\limsup f_n \text{ and } \liminf f_n$$

   are measurable by Theorem 1.14. The set of all said points is exactly the set on which the two above functions are equal, so apply part (a).

6. Clearly (i) and (ii) hold in Definition 1.3(a). If $A = \bigcup_{n=1}^{\infty} A_n$ with each A_n at most countable, then A is at most countable. Otherwise, choose A_N such that A_N^c is at most countable. Then

$$A^c = \left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c \subseteq A_N^c$$

so that A^c is at most countable.

Suppose $\{A_n\}$ is a countable collection of disjoint sets in $\mathfrak{M}$ and put $A = \bigcup_{n=1}^{\infty} A_n$. Observe that at most one of the A_n can have A_n^c be at most countable: if there were two such A_m and A_n, then $A_m \cap A_n = \varnothing$ implies that $A_m^c \cup A_n^c = X$ is countable, which is not the case. Thus either all A_n are at most countable (which means A is) and

$$\mu(A) = 0 = \sum_{n=1}^{\infty} 0 = \sum_{n=1}^{\infty} \mu(A_n)$$

or (only) one A_n has A_n^c at most countable (which means A^c is at most countable) and

$$\mu(A) = 1 = 0 + 0 + \cdots + 0 + 1 + 0 + \cdots = \sum_{n=1}^{\infty} \mu(A_n)$$

7.

8. It gives a strict inequality. For a solid example, take $X = \{0, 1\}$ and $E = \{0\}$ with counting measure.

Chapter 2

Positive Borel Measures

1. I love nets, so I'll use the following useful proposition:

   Proposition. *Let X be a topological space and $f : X \to \mathbb{R}$ be a function. f is upper semicontinuous if and only if whenever $x_\lambda \to x_0$ and $\varepsilon > 0$, there is a λ_0 such that $\lambda \geq \lambda_0$ implies $f(x_0) > f(x_\lambda) - \varepsilon$. f is lower semicontinuous if and only if when $x_\lambda \to x_0$ and $\varepsilon > 0$, there is a λ_0 such that $\lambda \geq \lambda_0$ implies $f(x_0) < f(x_\lambda) + \varepsilon$.*

   Proof. If f is upper semicontinuous, then $\{x : f(x) < f(x_0) + \varepsilon\}$ is open and contains x_0. Then, since $x_\lambda \to x_0$, x_λ is eventually in that set. Conversely, suppose $E = \{x : f(x) < \alpha\}$ is not open for some α. Then there is some x_0 in E with a net (x_λ) in E^c converging to x_0. But then $f(x_0) < \alpha \leq f(x_\lambda)$ for all λ, so the assumed condition does not hold if one chooses $\varepsilon = \alpha - f(x_0)$. □

   Now suppose f_1 and f_2 are upper semicontinuous, $x_\lambda \to x_0$ and $\varepsilon > 0$. Choose λ_0 so that $\lambda \geq \lambda_0$ implies that $f_j(x_0) > f_j(x_\lambda) - \frac{\varepsilon}{2}$ for $j = 1, 2$. Then

   $$f_1(x_0) + f_2(x_0) > f_1(x_\lambda) + f_2(x_\lambda) - \varepsilon$$

   so $f_1 + f_2$ is upper semicontinuous. Lower semicontinuity is similar.

   $$\sum_{n=1}^{\infty} f_n(x_0) > \sum_{n=1}^{\infty} f_n(x_\lambda) - \varepsilon$$

2.

3.

4.

5. E can be written as $\bigcap_{n\geq 1} E_n$ where E_n is a disjoint union of 2^n closed intervals of length 3^{-n}. Using Theorem 1.19(e), we have

   $$m(E) = m\left(\bigcap_{n\geq 1} E_n\right) = \lim_{n\to\infty} m(E_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

6. Try the fat Cantor set to obtain K. Suppose v is lower semicontinuous and $v \le \chi_K$, and suppose that there is a number α such that $0 < \alpha \le 1$ and $v(x) = \alpha$. Then $\{x : v(x) > \frac{\alpha}{2}\}$ is open and nonempty in $\mathbb{R}$, and χ_K must be 1 on that set since it dominates v. This means K contains a nonempty open set in $\mathbb{R}$, contradicting total disconnectedness.

7. Let $\{r_n\}_{n\ge 1}$ be an enumeration of the rationals in $[0,1]$ and let U_n be an open interval in $[0,1]$ containing r_n with length at most $\varepsilon 2^{-n}$. Set $E = \bigcup_{n\ge 1} U_n$.

8.

9.

10.

11.

12.

13.

14.

15.

16. If $m(Y) > 0$,

17.

18.

19.

20.

21. The sets $\{x : f(x) < \alpha\}$ form an open cover of X as α ranges over $\mathbb{R}$, thus by compactness there is some α_0 such that $X = \{x : f(x) < \alpha_0\}$. This shows that f is bounded above. Let $\beta = \sup_{x\in X} f(x)$. The sets $\{x : f(x) \ge \beta - \frac{1}{n}\}$ are closed, nonempty, and nested in X. By Theorem 2.4 they are all compact and by Theorem 2.6 there is a point in all of them, say y. Then $\beta - \frac{1}{n} \le f(y) \le \beta$ for all n, so $f(y) = \beta$.

22.

23.

24.

25.

Chapter 3

L^p-Spaces

Chapter 4

Elementary Hilbert Space Theory

1. If x is in M and y is in $M^\perp$, then $(x,y)=0$, so x is in $(M^\perp)^\perp$. Thus $M \subseteq (M^\perp)^\perp$, even if M is not closed. If M is closed and x is in $(M^\perp)^\perp$, use Theorem 4.11 to write $x = Px + Qx$ where Px is in M and Qx is in $M^\perp$. Then

   $$x - Px = Qx.$$

   The left side is in $(M^\perp)^\perp$ since $M \subseteq (M^\perp)^\perp$, and the right side is in $M^\perp$, thus both sides are 0. It follows that $x = Px$ and x is in M.

   In fact, we have $\overline{M} = (M^\perp)^\perp$ for *any* subspace M. To show this, first prove that $M^\perp = (\overline{M})^\perp$, then use the above proof.

2.

3.

4. If $\{u_\alpha\}$ is an uncountable orthonormal system, then $\|u_\alpha - u_\beta\| = \sqrt{2}$ when $\alpha \neq \beta$, so H cannot be separable. Conversely, if $\{u_\alpha\}$ is a countable orthonormal basis, then the set P of finite linear combinations of members of $\{u_\alpha\}$ is dense in H by Theorem 4.18. But restricting to all linear combinations with *rational* coefficients gives a dense subset of P. This latter set is countable.

5.

6. Let $A = \{u_1, u_2, \ldots\}$. A is clearly bounded since $\|u_n\| = 1$ for all n. Since $\|u_m - u_n\| = \sqrt{2}$ for all $m \neq n$, every Cauchy sequence of elements from A must be eventually constant, hence convergent to an element of A. This implies that A is closed. By taking balls of radius 1 centred at each u_n, we see that we cannot extract a finite subcover and hence A is not compact.

   We prove that S is compact if and only if $\sum_{n=1}^{\infty} \delta_n^2 < \infty$; compactness of the Hilbert cube Q will follow from this. Suppose first that the given sum is finite.

Chapter 5

Examples of Banach Space Techniques

1.

2.

3.

4. If f is in M,

$$\int_0^{1/2} (\lambda f(t) + (1-\lambda) f(t))\, dt - \int_{1/2}^1 (\lambda f(t) + (1-\lambda) f(t))\, dt$$

$$= \lambda \left(\int_0^{1/2} f(t)\, dt - \int_{1/2}^1 f(t)\, dt \right) + (1-\lambda) \left(\int_0^{1/2} f(t)\, dt - \int_{1/2}^1 f(t)\, dt \right)$$

$$= \lambda + (1-\lambda) = 1$$

so M is convex. If f_n is in M for all n and $f_n \to f$ uniformly, then $\{f_n\}$ is bounded, so by Lebesgue's Dominated Convergence Theorem we have

$$\int_0^{1/2} f(t)\, dt - \int_{1/2}^1 f(t)\, dt = \lim_{n\to\infty} \left(\int_0^{1/2} f_n(t)\, dt - \int_{1/2}^1 f_n(t)\, dt \right) = \lim_{n\to\infty} 1 = 1$$

so M is closed. Now we proceed to show that M does not contain any f with $\|f\|_\infty = 1$, but there are functions in M with norm arbitrarily close to 1.

If f is in M, then

$$1 = \left| \int_0^{1/2} f(t)\, dt - \int_{1/2}^1 f(t)\, dt \right| \leq \int_0^{1/2} |f(t)|\, dt + \int_{1/2}^1 |f(t)|\, dt = \int_0^1 |f(t)|\, dt \leq \|f\|_\infty$$

Suppose to the contrary that f is in M and $\|f\|_\infty = 1$. Then all the above inequalities become equalities, thus

$$\int_0^1 |f(t)|\, dt = 1$$

Applying Exercise 2 from *Principles* to $1 - |f|$, we must have that $|f(t)| = 1$ for all t. But this contradicts the condition $\int_0^{1/2} f(t)\, dt - \int_{1/2}^1 f(t)\, dt = 1$. Now if we define $f_n(t) = 1 + \frac{1}{n}$ on the interval $[0, \frac{n-1}{2(n+1)}]$, $f_n(t) = -1 - \frac{1}{n}$ on $[1 - \frac{n-1}{2(n+1)}, 1]$, and linear in between, we have f_n in M and $\|f_n\|_\infty = 1 + \frac{1}{n}$ for all n.

5.

6.

7.

8.

9. (a) We have

$$\sum |\xi_i \eta_i| = \sum |\xi_i||\eta_i| \leq \|x\|_\infty \sum |\eta_i| \leq \|x\|_\infty \|y\|_1 < \infty$$

so that Λ is well-defined. Obviously Λ is linear. By the above estimate we also have

$$|\Lambda x| = \left| \sum \xi_i \eta_i \right| \leq \sum |\xi_i \eta_i| \leq \|x\|_\infty \|y\|_1$$

which implies $\|\Lambda\| \leq \|y\|_1$. On the other hand, let $\varepsilon > 0$ and choose N so that $\|y\|_1 - \varepsilon \leq \sum_{i=1}^N |\eta_i|$. For each i in $\{1, 2, \ldots, N\}$, let α_i be a complex number with $|\alpha_i| = 1$ and $\alpha_i \eta_i = |\eta_i|$. Let $x_N = (\alpha_1, \alpha_2, \ldots, \alpha_N, 0, 0, \ldots)$, which is in c_0 and $\|x_N\|_\infty = 1$. Then

$$|\Lambda x_N| = \left| \sum_{i=1}^N \alpha_i \eta_i \right| = \left| \sum_{i=1}^N |\eta_i| \right| = \sum_{i=1}^N |\eta_i| \geq \|y\|_1 - \varepsilon$$

which implies $\|\Lambda\| \geq \|y\|_1$. Now let Λ be any functional in $(c_0)^*$. Let e_i denote the sequence in c_0 which is 1 in the i-th place and 0 elsewhere. Letting α_i and x_N be as above (with η_i replaced with Λe_i), we have

$$\sum_{i=1}^N |\Lambda e_i| = \sum_{i=1}^N \alpha_i \Lambda e_i = \Lambda \left(\sum_{i=1}^N \alpha_i e_i \right) = \Lambda x_N \leq \|\Lambda\|$$

for every N, which shows that $\{\Lambda e_i\}$ is in ℓ^1, and

$$\Lambda x = \Lambda \left(\lim_{N \to \infty} \sum_{i=1}^N \xi_i e_i \right) = \lim_{N \to \infty} \sum_{i=1}^N \xi_i \Lambda e_i = \sum_{i=1}^\infty \xi_i \Lambda e_i$$

(b)

(c)

(d) Finite linear combinations of $\{e_i\}$ with rational coefficients are dense in both c_0 and ℓ^1. The collection $\{\chi_A\}_{A \subseteq \{1,2,3,\ldots\}}$ is an uncountable subset of ℓ^∞ and $\|\chi_A - \chi_B\|_\infty = 1$ when $A \neq B$. Thus ℓ^∞ cannot be separable.

10. This follows from (a) in the previous exercise.