Factor groupoids and prescribed K-theory

GPOTS 2021

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Motivation

Problem. Given some K-theory data, find a groupoid G such that the data is $K_*(C_r^*(G))$.

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Notable references:

Li, X. "Every classifiable simple C^*-algebra has a Cartan subalgebra". Invent. math. 219, 653–699 (2020).

Putnam, I.F. "Some classifiable groupoid C^*-algebras with prescribed K-theory". Math. Ann. 370, 1361–1387 (2018).

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Obtain an inclusion $C_r^*(G') \subseteq C_r^*(G)$ via $b \mapsto b \circ \pi$ (b in $C_c(G')$)

Bratteli diagrams

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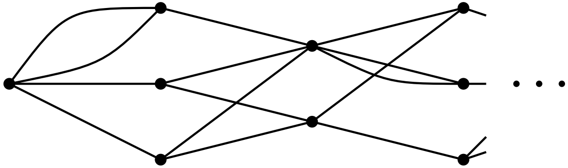

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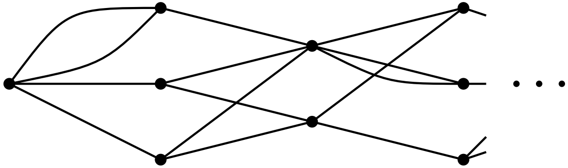

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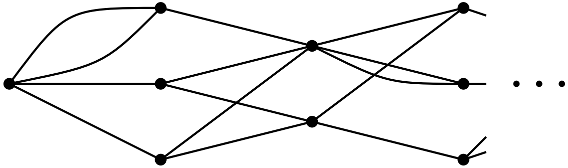

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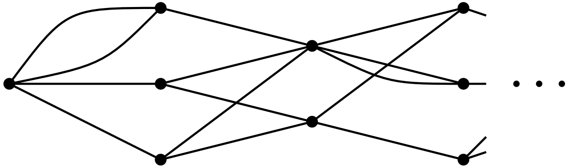

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Goal: make a factor groupoid of R_E.

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Equivalence relation $\sim_\xi$ on X_E:

$$(x_1, x_2, \ldots, x_{n_0-1}, x_{n_0}, \xi^0(z_{n_0+1}), \xi^0(z_{n_0+2}), \ldots) \tag{1}$$
$$\sim_\xi (x_1, x_2, \ldots, x_{n_0-1}, x'_{n_0}, \xi^1(z_{n_0+1}), \xi^1(z_{n_0+2}), \ldots) \tag{2}$$

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Facts:

1. X_ξ is a second-countable compact Hausdorff space,
2. the covering dimension of X_ξ is 1,
3. each connected component is either a single point or homeomorphic to S^1.

Examples

Example 1. We let (V, E) be the Bratteli diagram with one vertex at each level and two edges at each level. Identify X_E with $\{0, 1\}^\omega$.

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$$(x_1, x_2, \dots, x_n, 1, 0, 0, 0, 0, \dots) \tag{3}$$
$$\sim_\xi (x_1, x_2, \dots, x_n, 0, 1, 1, 1, 1, \dots) \tag{4}$$

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$$\{0,1\}^\omega \to S^1 : (x_n) \mapsto \exp\left(2\pi i \sum_{n=1}^{\infty} x_n 2^{-n}\right)$$

The fibres are precisely the $\sim_\xi$ equivalence classes, so X_ξ is homeomorphic to S^1.

Examples

Example 2. Let (V, E) have one vertex and three edges at each level. Identify X_E with $\{0, 1, 2\}^\omega$.

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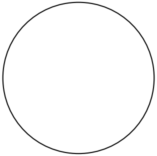
There is a nested sequence $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_E$ such that

$$X_E = \overline{\bigcup_{n=1}^{\infty} X_n}$$

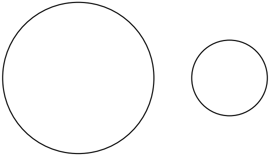
and each $\rho(X_n)$ is a disjoint union of finitely many circles.

Examples



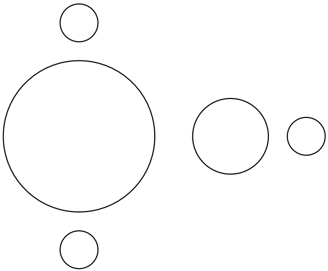
$\rho(X_1)$

Examples



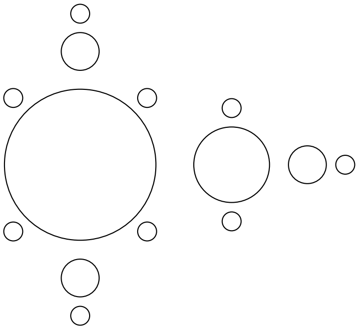
$\rho(X_2)$

Examples



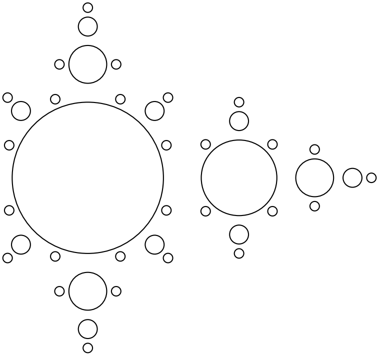
$\rho(X_3)$

Examples



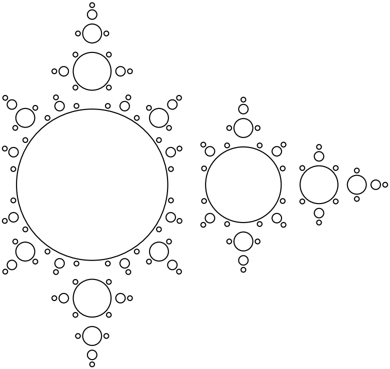
$\rho(X_4)$

Examples



$\rho(X_5)$

Examples



$\rho(X_6)$

The groupoid R_ξ

Let $R_\xi = \rho \times \rho(R_E)$.

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With the quotient topology, R_ξ is a second-countable locally compact Hausdorff étale groupoid, and a factor of R_E via $\rho \times \rho : R_E \to R_\xi$.

We want to analyze the K-theory of $C_r^*(R_\xi) \subseteq C_r^*(R_E)$.

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K_1(A) \arrow[r] & K_0(A';A) \arrow[r] & K_0(A') \arrow[d, "\iota_*"] \\
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Excision: (Putnam, 2020)

$$K_*(C_r^*(G'); C_r^*(G)) \cong K_*(C_r^*(H'); C_r^*(H))$$

where $H \subseteq G$ and $H' \subseteq G'$ are where π is not one-to-one.

Through the results on the previous slide, we obtain

$$K_0(C_r^*(R_\xi)) \cong K_0(C_r^*(R_E)) \qquad K_1(C_r^*(R_\xi)) \cong K_0(C_r^*(R_F))$$

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Through the set-up $\xi^0, \xi^1 : (W, F) \to (V, E)$, we can prescribe $K_*(C_r^*(R_\xi))$.

Further work

1. Increasing the number of embeddings ξ^j:
    1. More complicated connected components
    2. Still one-dimensional
    3. K_0 the same, K_1 more direct summands
2. (Based on work of Deeley, Putnam, Strung) $\pi : \tilde{X} \to X_E$ where $\pi^{-1}(x)$ is either a single point or homeomorphic to a fixed point set of an iterated function system

Thank you!

References

1. Deeley, R.J.; Putnam, I.F.; Strung, K.R. "Non-homogeneous extensions of Cantor minimal systems". to appear, Proc. A.M.S.
2. Haslehurst, M.J. "Relative K-theory for C^*-algebras". (in preparation)
3. Haslehurst, M.J. "Some examples of factor groupoids". (in preparation)
4. Li, X. "Every classifiable simple C^*-algebra has a Cartan subalgebra". Invent. math. 219, 653–699 (2020).
5. Putnam, I.F. "Some classifiable groupoid C^*-algebras with prescribed K-theory". Math. Ann. 370, 1361–1387 (2018).
6. Putnam, I.F. "An excision theorem for the K-theory of C^*-algebras, with applications to groupoid C^*-algebras". to appear, Munster Mathematics Journal.