

DYNAMIC ASYMPTOTIC DIMENSION AND MATUI'S HK CONJECTURE

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ABSTRACT. We prove that the homology groups of a principal ample groupoid vanish in dimensions greater than the dynamic asymptotic dimension of the groupoid. As a consequence, the K-theory of the C^* -algebras associated with groupoids of finite dynamic asymptotic dimension can be computed from the homology of the underlying groupoid. In particular, principal ample groupoids with dynamic asymptotic dimension at most two satisfy Matui's HK-conjecture.

We also construct explicit maps from the groupoid homology groups to the K-theory groups of their C^* -algebras in degrees zero and one, and investigate their properties.

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1. INTRODUCTION

Dynamic asymptotic dimension is a notion of dimension for actions of discrete groups on locally compact spaces, and more generally, for locally compact étale groupoids introduced by the last named author with Guentner and Yu in [21]. It is inspired by Gromov's theory of asymptotic dimension. At the same time it is strongly connected to other existing dimension theories for dynamical systems, for example the conditions introduced by Bartels, Lück and Reich [3] or Kerr's tower dimension [24].

The original article [21] focused on the fine structure of C^* -algebras associated with étale groupoids of finite dynamic asymptotic dimension, while later work by the same set of authors in [20] presented some consequences to K-theory and topology.

In the present work we aim to explore the implications of dynamic asymptotic dimension for groupoid homology and its relation to the K-theory of groupoid C^* -algebras. A homology theory for étale groupoids was introduced by Crainic and Moerdijk in [9]. More recently, groupoid homology attracted a considerable amount of interest from the topological dynamics and operator algebras communities following the work of Matui [29]. The main contribution of this article is the following:

Theorem A. Let G be a locally compact, Hausdorff, étale, principal, σ -compact, ample groupoid with dynamic asymptotic dimension at most d . Then $H_n(G) = 0$ for $n > d$ and $H_d(G)$ is a free group.

Our proof of Theorem A goes via a description of groupoid homology in terms of semi-simplicial spaces equipped with a G -action. As a byproduct this leads to a description of these homology groups in terms of the classical Tor groups of homological algebra, quite analogous to the well-known case of the homology of a discrete group. While this may be known to experts, it seems worthwhile recording it as we are not aware of its appearance in the literature.

Theorem B. Let G be a locally compact, Hausdorff, étale, ample groupoid with σ -compact base space. There is a canonical isomorphism $H_*(G) \cong \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$.

Our result allows us to draw some significant consequences to the following conjecture formulated by Matui in [31].

Conjecture (Matui). *For a minimal, essentially principal, ample groupoid G there are isomorphisms*

$$K_i(C_r^*(G)) \cong \bigoplus_{n \geq 0} H_{2n+i}(G), \quad i = 0, 1$$

The conjecture as stated has been disproved by Scarpato [43]. The main reason Scarpato's example fails to satisfy the conjecture is the existence of torsion in its isotropy groups. For ample groupoids with exclusively torsion free isotropy groups the conjecture has been confirmed in several interesting cases [15, 31, 33, 50] even beyond the originally postulated minimal setting.

Combining Theorem A with a spectral sequence recently constructed by Proietti and Yamashita [37], we can confirm the HK-conjecture for a large class of low-dimensional groupoids:

Corollary C. Let G be a locally compact, Hausdorff, étale, principal, second countable, ample groupoid with dynamic asymptotic dimension at most 2. Then the HK-conjecture holds for G , i.e.

$$K_0(C_r^*(G)) \cong H_0(G) \oplus H_2(G), \quad K_1(C_r^*(G)) \cong H_1(G).$$

Note that having finite dynamic asymptotic dimension forces all the isotropy groups to be locally finite. Consequently, it is natural to restrict our attention

to the class of principal groupoids to avoid the trouble caused by torsion in the isotropy groups.

The second theme of this work is an attempt to make the HK-conjecture more explicit. It is well-known and easy to see that there is a canonical homomorphism

$$\mu_0 : H_0(G) \rightarrow K_0(C_r^*(G)).$$

In general, not much seems to be known about this map, and there are only partial results on the existence of maps in higher dimensions. To formulate our progress in this direction, for an ample groupoid G we denote by $[[G]]$ its topological full group. Moreover, Matui constructs in [29] an *index map* $I : [[G]] \rightarrow H_1(G)$.

Theorem D. Let G be a locally compact, Hausdorff, étale, ample groupoid. Then there exists a homomorphism

$$\mu_1 : H_1(G) \rightarrow K_1(C_r^*(G))$$

which factors the canonical map $[[G]] \rightarrow K_1(C_r^*(G))$ via the index map $I : [[G]] \rightarrow H_1(G)$.

If moreover G is σ -compact, principal, and has dynamic asymptotic dimension at most 1, then μ_0 is an isomorphism and μ_1 is surjective.

The construction of μ_1 is straightforward if the index map I is surjective, and under further structural assumptions on G , Matui was already able to prove this. Our construction of the map μ_1 is completely general and uses a variation on ideas of Putnam [38]. Note also that the proof of the second part of the theorem is independent of the results mentioned above and instead uses an alternative approach via a Mayer-Vietoris argument in quantitative K-theory.

It was shown in [21] that the dynamic asymptotic dimension yields an upper bound for the nuclear dimension of reduced groupoid C^* -algebras. In particular, if G is a second countable, principal, minimal ample groupoid with finite dynamic asymptotic dimension, then $C_r^*(G)$ is classifiable.¹ Our result allows us to completely determine the classifying invariant (usually called the Elliott invariant) in the 1-dimensional case:

Corollary E. Let G be a locally compact, Hausdorff, étale, σ -compact, principal, ample groupoid with compact base space and with dynamic asymptotic dimension at most 1. Then

$$\text{Ell}(C_r^*(G)) = (H_0(G), H_0(G)^+, [1_{G^{(0)}}], H_1(G), M(G), p).$$

In light of these results one is tempted to formulate a stronger version of the HK-conjecture in low dimensions, by asking that for ample principal groupoids with $H_n(G) = 0$ for $n \geq 2$ the canonical maps μ_i are isomorphisms. To see that this cannot be the case, we construct a counterexample using groupoids with topological property (T) introduced in [13]. The example is based on the construction of counterexamples to the Baum-Connes conjecture by Higson, Lafforgue and Skandalis [22] and work of Alekseev and Finn-Sell [1].

Theorem F. There exists a locally compact, Hausdorff, étale, second countable, principal, ample groupoid G with $H_n(G) = 0$ for all $n \geq 2$ such that $\mu_0 : H_0(G) \rightarrow K_0(C_r^*(G))$ is not surjective.

As an application of our results, we study a geometric class of examples. Given a metric space X with bounded geometry, Skandalis, Tu and Yu construct an ample groupoid $G(X)$ which encodes many coarse geometric properties of the underlying

¹This is due to Kirchberg and Phillips in the purely infinite case [25], [34], and due to many hands in the finite case, including Elliott, Gong, Lin, and Niu [18], [19], [14], and Tikuisis, White, and Winter [47] (see [7] for an alternative proof of classification in the finite case).

space X [44]. The following result adds to this list of connections. It might be known to experts but does not seem to appear in the literature so far (except for degree zero, which has been treated in [2]).

Theorem G. Let X be a bounded geometry metric space and $G(X)$ be the associated coarse groupoid. Then there is a canonical isomorphism

$$H_*(G(X)) \cong H_*^{\text{uf}}(X)$$

between the groupoid homology of the coarse groupoid and the uniformly finite homology of X in the sense of Block and Weinberger [6].

As the dynamic asymptotic dimension of the coarse groupoid equals the asymptotic dimension of the underlying metric space (in the sense of Gromov), a combination of Theorems A and G yields the following purely geometric corollary.

Corollary H. Let X be a bounded geometry metric space.

(1) If $\text{asdim}(X) \leq 2$, then

$$K_0(C_u^*(X)) \cong H_0^{\text{uf}}(X) \oplus H_2^{\text{uf}}(X), \quad K_1(C_u^*(X)) \cong H_1^{\text{uf}}(X).$$

(2) If $\text{asdim}(X) \leq 3$ and X is non-amenable, then

$$K_0(C_u^*(X)) \cong H_2^{\text{uf}}(X), \quad K_1(C_u^*(X)) \cong H_1^{\text{uf}}(X) \oplus H_3^{\text{uf}}(X).$$

Outline of the paper. In Section 2 we give a new picture for Crainic and Moerdijk homology by defining a G -equivariant homology theory for an appropriate notion of semi-simplicial G -spaces. This is done in section 2.2, by describing the homology groups as the left derived functor of the coinvariants in the sense of classical homological algebra. The necessary background is given in section 2.1. It turns out that the groups $H_*(G)$ will be naturally isomorphic to the equivariant homology of the semi-simplicial G -space EG_* .

The central piece of the vanishing result is tackled in Section 3. There we define a *colouring* of G , which will induce an appropriate cover of G . The *nerve* of this cover in the sense of section 3.3 is a semi-simplicial G -space, and that defines the homology of the colouring. The central idea is to define the anti-Čech sequence of G as a sequence of colourings with induced covers that are bigger and bigger, in the spirit of anti-Čech covers in coarse geometry, introduced by Roe (see [41, chapter 5]). We show that the inductive limit of an anti-Čech cover is well defined and that it converges to the Crainic-Moerdijk homology groups for principal σ -compact ample groupoids in theorem 3.29.

Section 4 is dedicated to the construction of the map

$$H_1(G) \rightarrow K_1(C_r^*(G))$$

and the proof of Theorem B.

Finally, in section 5 we interpret our results in various different settings, including coarse geometry and Smale spaces, and present the negative results regarding a stronger form of the HK conjecture announced in Theorem D.

2. MODELS FOR GROUPOID HOMOLOGY

2.1. The category of G -modules and the coinvariant functor. Let us first fix our notations. A groupoid is a (small) category in which all arrows are invertible. We will denote G and G^0 for the set of arrows and objects respectively. The range and source maps are denoted by $r, s : G \rightarrow G^0$, and the corresponding fibres by $G^x := r^{-1}(x)$ and $G_x := s^{-1}(x)$. A pair $(g, h) \in G \times G$ is composable if $s(g) = r(h)$, in which case the product is written gh . The identity at $x \in G^0$ is written $e_x \in G$. For subsets A, B of G we write $AB = \{gh \mid g \in A, h \in B, r(h) = s(g)\}$, and we write gA for $\{g\}A$.

We work exclusively with topological groupoids: G and G^0 carry locally compact Hausdorff topologies that are compatible, and all structure maps are continuous. A bisection is a subset B of G such that the restrictions $r|_B$ and $s|_B$ are homeomorphisms onto their images; a groupoid G is étale if the open bisections form a basis for its topology, and is ample if the compact open bisections form a basis for the topology.

We will be focusing throughout on (étale) ample groupoids, or equivalently étale groupoids with a totally disconnected base space. Examples include discrete groups and totally disconnected spaces, but also action groupoids of discrete groups acting on Cantor sets by homeomorphisms, coarse groupoids, and more examples associated to k -graphs, etc.

If G is an ample groupoid, let $\mathbb{Z}[G]$ denote the set of compactly supported continuous functions with integer values,

$$\mathbb{Z}[G] = C_c(G, \mathbb{Z}),$$

with the ring structure given by pointwise addition and convolution. This is a ring (in general without identity) with local units, i.e. for any $f_1, \dots, f_n \in \mathbb{Z}[G]$ there is an idempotent $e \in \mathbb{Z}[G]$ such that $ef_j = f_j e = f_j$ for all $j = 1, \dots, n$. The element e can always be picked in $\mathbb{Z}[G^0] \subseteq \mathbb{Z}[G]$ as the characteristic function on a compact open subset of G^0 . Define the augmentation map $\varepsilon : \mathbb{Z}[G] \rightarrow \mathbb{Z}[G^0]$ by

$$\varepsilon(f)(x) = \sum_{g \in G_x} f(g).$$

We will denote the category of (left, non-degenerate²) $\mathbb{Z}[G]$ -modules with $\mathbb{Z}[G]$ -linear maps as morphisms by $G\text{-}\mathbf{mod}$. We will often use the term G -module as shorthand for $\mathbb{Z}[G]$ -module.

Definition 2.1. Let M be a $\mathbb{Z}[G]$ -module. Define M_0 to be the submodule generated by the elements of the form

$$fm - \varepsilon(f)m, \quad f \in \mathbb{Z}[G], \quad m \in M.$$

The group of *coinvariants* of M is the abelian group

$$M_G = M/M_0.$$

This naturally defines the coinvariant functor

$$\text{Coinv} : G\text{-}\mathbf{mod} \rightarrow \mathbf{Ab}.$$

As it may make the above more conceptual, and as we will need it later, we give a different description of the coinvariants functor. Define a right action of $\mathbb{Z}[G]$ on $\mathbb{Z}[G^0]$ by $a \cdot f := \varepsilon(af)$ for $a \in \mathbb{Z}[G^0]$ and $f \in \mathbb{Z}[G]$, i.e.

$$(a \cdot f)(x) = \sum_{g \in G_x} a(r(g))f(g) \quad \forall f \in \mathbb{Z}[G], a \in \mathbb{Z}[G^0], x \in G^0.$$

Lemma 2.2. *For any (left, non-degenerate) G -module M , there is a canonical isomorphism $M_G \cong \mathbb{Z}[G^0] \otimes_{\mathbb{Z}[G]} M$.*

Proof. We leave it to the reader to check that the map

$$\mathbb{Z}[G^0] \otimes_{\mathbb{Z}[G]} M \rightarrow M_G, \quad a \otimes m \mapsto am + M_0$$

is an isomorphism of abelian groups. The inverse is given as follows: for $m + M_0 \in M_G$ pick $a \in \mathbb{Z}[G^0] \subseteq \mathbb{Z}[G]$ such that $am = m$. The inverse map is given by $m + M_0 \mapsto a \otimes m$. \square

²If R is a ring with local units, a left R -module M is called *non-degenerate* whenever $RM = M$. All of our modules will be non-degenerate by assumption.

We are now going to present a natural source of G -modules: spaces with a topological action of G .

Recall first that if $p : Y \rightarrow X$ and $q : Z \rightarrow X$ are continuous maps, the fibered product is the space

$$(1) \quad Y_{p \times q} Z := \{(y, z) : p(y) = q(z)\}.$$

equipped with the subspace topology it inherits from $Y \times Z$.

Recall that a left G -space is the data of a topological space X together with:

- a continuous map $p : X \rightarrow G^0$, called the *anchor map*,
- a continuous map $\alpha : G_s \times_p X \rightarrow X$, called the *moment map*, satisfying $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for every $(g, h, x) \in G_s \times_r G_s \times_p X$ and $\alpha(e_{p(x)}, x) = x$ for every $x \in X$.

A left G -space is *étale* if the anchor map is étale, i.e. a local homeomorphism. We will typically suppress α from the notation, for example writing ' gx ' instead of ' $\alpha(g, x)$ '. A left G -space is *free* if $gx = x$ forces $g = e_{p(x)}$, and is *proper* if the map $\alpha \times pr_2 : G_s \times_p X \rightarrow X \times X$ is proper. The quotient of a G -space X is written $G \backslash X$, and is the quotient of X by the equivalence relation $x \sim gx$, equipped with the quotient topology. If the action of G on X is free and proper, then $G \backslash X$ is locally compact and Hausdorff, and the quotient map $q : X \rightarrow G \backslash X$ is étale.

Definition 2.3. Let \mathbf{Top}_G be the category of left G -spaces which are locally compact, Hausdorff and étale, with morphisms being the G -equivariant étale maps.

We denote by $\mathbb{Z}[X]$ the abelian group of compactly supported integer valued functions on X . It is a non-degenerate G -module with respect to the action defined by

$$(fa)(x) = \sum_{g \in G^x} f(g)a(g^{-1}x) \quad \forall f \in \mathbb{Z}[G], a \in \mathbb{Z}[X], x \in X.$$

If $f : X \rightarrow Y$ is G -equivariant and étale, it induces a $\mathbb{Z}[G]$ -linear map $f_* : \mathbb{Z}[X] \rightarrow \mathbb{Z}[Y]$ by the formula

$$f_*(a)(y) := \sum_{x \in f^{-1}(y)} a(x) \quad \forall y \in Y.$$

The sum is finite since $a \in \mathbb{Z}[X]$ is compactly supported, and $f_*(a)$ is continuous as f is étale.

Lemma 2.4. *The assignments $X \mapsto \mathbb{Z}[X]$ and $f \mapsto f_*$ define a functor, called the functor of global sections,*

$$\mathbb{Z}[_] : \mathbf{Top}_G \rightarrow G\text{-}\mathbf{mod}.$$

The composition of the global section functor and the coinvariant functor is denoted by $\mathbb{Z}[_]_G$.

Lemma 2.5. *If X is a free and proper space in \mathbf{Top}_G then there is an isomorphism of abelian groups*

$$\mathbb{Z}[X]_G \cong \mathbb{Z}[G \backslash X].$$

Proof. Denote by $[x] \in G \backslash X$ the class of $x \in X$, and define a $\mathbb{Z}[G]$ -linear map $\tilde{\varepsilon} : \mathbb{Z}[X] \rightarrow \mathbb{Z}[G \backslash X]$ by

$$\tilde{\varepsilon}(f)([x]) = \sum_{g \in G_{p(x)}} f(g \cdot x) \quad \forall x \in X.$$

This obviously factors through $\mathbb{Z}[X]_G$, giving a map which we still denote by

$$\tilde{\varepsilon} : \mathbb{Z}[X]_G \rightarrow \mathbb{Z}[G \backslash X].$$

Let us build an inverse to $\tilde{\varepsilon}$. Since the action of G on X is proper, the quotient is locally compact Hausdorff, and hence a partition of unity argument shows that the family of functions $f \in \mathbb{Z}[G \backslash X]$ such that there exists a compact open subset $V \subseteq X$ with $\text{supp}(f) = q(V)$ and such that $q|_V : V \rightarrow q(V)$ is a homeomorphism, generates $\mathbb{Z}[G \backslash X]$ as an abelian group. Now given such a function f , we define $f_V := f \circ q|_V \in \mathbb{Z}[X]$ and our first goal is to show that the class of f_V in $\mathbb{Z}[X]_G$ does not depend on the choice of V . So let $V' \subseteq X$ be another compact open set with $\text{supp}(f) = q(V')$ and such that $q|_{V'} : V' \rightarrow q(V')$ is a homeomorphism. Then $q(V) = q(V')$ and hence every element $x \in V$ can be written as $x = gy$ for some $y \in V'$, $g \in G$. Using that G is étale and compactness of V we can decompose $V = \bigcup_i V_i$ and $V' = \bigcup_i V'_i$ such that there exist bisections $S_i \subseteq G$ implementing a homeomorphism $\alpha_i : V_i \rightarrow V'_i$. Hence doing another partition of unity argument, we may assume that the sets V and V' themselves are related in this way, i.e. there exists a bisection $S \subseteq G$ which induces a homeomorphism $\alpha_S : V \rightarrow V'$ by $\alpha_S(x) = hx$ where $h \in S$ is the unique element in $S \cap G^{p(x)}$. But then

$$f_V(x) = f(q_V(x)) = f(q_{V'}(\alpha_S(x))) = f_{V'}(\alpha_S(x)) \quad \forall x \in V.$$

This relation can be rewritten as $f_V = \chi_{S^{-1}} \cdot f_{V'}$, and hence we have $[f_V] = [f_{V'}]$ in $\mathbb{Z}[X]_G$ as desired. Let $\psi : \mathbb{Z}[G \backslash X] \rightarrow \mathbb{Z}[X]_G$ be the map given by $\psi(f) = [f_V]$. If $V \subseteq X$ is open such that $q|_V : V \rightarrow q(V)$ is a homeomorphism, then for a given $x \in X$ there is at most one $g \in G_{p(x)}$ such that $gx \in V$ since the action is free. It follows that for any function $f \in \mathbb{Z}[G \backslash X]$ supported in such a V we have

$$\tilde{\varepsilon}[f_V]([x]) = \sum_{g \in G_{p(x)}} f(q_V(gx)) = f([x])$$

and hence $\tilde{\varepsilon} \circ \psi = \text{id}$.

Conversely, if $f \in \mathbb{Z}[X]$ is a function supported in a set of the form $S \cdot V$ for a bisection $S \subseteq G$ and $V \subseteq X$ is open such that $q|_V$ is a homeomorphism onto its image, then one easily checks that $\tilde{\varepsilon}[f]$ is supported in $q(V)$. Hence $[\psi(\tilde{\varepsilon}[f])] = [\tilde{\varepsilon}[f] \circ q|_V]$. Using freeness again, one checks that the latter class equals $[\chi_S \cdot f] = [\varepsilon(\chi_{S^{-1}}) \cdot f] = [f]$. Since functions f as above generate $\mathbb{Z}[X]$ we are done. \square

2.2. Semi-simplicial G -spaces and homology. For n a nonnegative integer, denote by $[n]$ the interval $\{0, \dots, n\}$. Recall (see for example [49, Chapter 8]) the definition of the semi-simplicial category Δ_s : its objects are the nonnegative integers, and $\text{Hom}_{\Delta_s}(m, n)$ consists of the increasing maps $f : [m] \rightarrow [n]$. A semi-simplicial object in a category \mathcal{C} is a contravariant functor from Δ_s to \mathcal{C} . The collection of all semi-simplicial objects in a category is itself a category, with morphisms given by natural transformations.

Let $\varepsilon_i^n : [n-1] \rightarrow [n]$ be the only increasing map whose image misses i . We will omit the superscript n if it does not cause confusion. Any increasing map $f : [m] \rightarrow [n]$ has a unique factorization $f = \varepsilon_{i_1} \varepsilon_{i_2} \dots \varepsilon_{i_k}$ with $0 \leq i_k \leq \dots \leq i_1 \leq n$ (see Lemma 8.1.2 in [49]). Thus, any semi-simplicial object is the data, for all n , of an object C_n of \mathcal{C} , together with arrows $\varepsilon_i^n : C_n \rightarrow C_{n-1}$ in \mathcal{C} , called face maps, satisfying the (semi-)simplicial identities $\varepsilon_j^{n-1} \varepsilon_i^n = \varepsilon_i^{n-1} \varepsilon_{j-1}^n$ if $i < j$. Similarly, if C_* and D_* are semi-simplicial objects in \mathcal{C} , a morphism between them is a collection of morphisms $f_n : C_n \rightarrow D_n$ in \mathcal{C} that are compatible with the face maps in the natural sense.

A semi-simplicial object in the category of locally compact Hausdorff topological spaces will be called a *semi-simplicial topological space*.

Definition 2.6. A *semi-simplicial G -space* is a semi-simplicial object in the category \mathbf{Top}_G .

As an example, define EG_* to be the semi-simplicial G -space

$$EG_n = G_{r \times r} G_{r \times r} \dots_{r \times r} G \quad (n+1 \text{ times})$$

with

- anchor map $p : EG_n \rightarrow G^0$ given by the common range of the tuple, $p(\gamma_0, \dots, \gamma_n) = r(\gamma_0) = \dots = r(\gamma_n)$ for $\gamma \in EG_n$,
- left action given by left multiplication by G on all factors,
- if $f : [m] \rightarrow [n]$, then $EG(f) : G_m \rightarrow G_n$ is defined by

$$(\gamma_0, \dots, \gamma_n) \mapsto (\gamma_{f(0)}, \dots, \gamma_{f(m)})$$

(note that the face maps are given by

$$\partial_i^{n+1} : (\gamma_0, \dots, \gamma_n) \mapsto (\gamma_0, \dots, \hat{\gamma}_i, \dots, \gamma_n)$$

where the hat means that the entry is omitted).

One checks that the moment and the face maps are G -equivariant and étale.

On the other hand, define the *classifying space* of G , denoted by BG_* , to be the semi-simplicial topological space defined by

$$BG_n = \{(g_1, \dots, g_n) \in G^n \mid r(g_i) = s(g_{i-1}) \text{ for all } i\}$$

with face maps

$$\epsilon_i^n : (g_1, \dots, g_n) \mapsto \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0, \\ (g_1, \dots, g_{i-1}, g_i g_{i+1}, \dots, g_n) & \text{if } 1 \leq i < n, \\ (g_1, \dots, g_{n-1}) & \text{if } i = n. \end{cases}$$

The G -action on EG is free and proper and hence $G \backslash EG_*$ is a semi-simplicial topological space. The maps $(g_1, \dots, g_n) \mapsto [r(g_1), g_1, g_1 g_2, \dots, g_1 \cdots g_n]$ and $[\gamma_0, \dots, \gamma_n] \mapsto (\gamma_0^{-1} \gamma_1, \dots, \gamma_{n-1}^{-1} \gamma_n)$ define maps of semi-simplicial topological spaces between BG_* and $G \backslash EG_*$ that are mutually inverse.

A semi-simplicial G -space $(X_*, \{\epsilon_i^*\}_i)$ naturally induces a chain complex of $\mathbb{Z}[G]$ -modules $(\mathbb{Z}[X_*], \tilde{\partial}_*)$ by composing by the $\mathbb{Z}[?]$ functor, where the boundary maps are

$$\tilde{\partial}_n = \sum_{i=0}^n (-1)^i (\partial_i^n)_*$$

and thus a chain complex of abelian groups $(\mathbb{Z}[X_*]_G, \tilde{\partial}_*)$ by composing by the coinvariant functor.

Definition 2.7. Let $(X_*, \{\partial_i^*\}_i)$ be a semi-simplicial G -space. We define the equivariant homology group $H_*^G(X)$ to be the homology of the chain complex of abelian groups $(\mathbb{Z}[X_*]_G, \tilde{\partial}_*)$.

The relationship with the homology of Matui and Crainic-Moerdijk is now a consequence of Lemma 2.5. Namely, Matui introduced a chain complex of abelian groups to compute Crainic-Moerdijk homology groups in the special case of an ample groupoid. This chain complex is none other than $(\mathbb{Z}[BG_*], \tilde{\epsilon}_*)$. As EG_* is a free and proper semi-simplicial G -space with $BG_* \cong G \backslash EG_*$, this homology is isomorphic to $H_*^G(EG) \cong H_*(BG)$. In other words, we have proved the following.

Proposition 2.8. *Let G be an ample groupoid. The complex introduced by Matui to compute $H_*(G)$ identifies canonically with the homology of the complex $(\mathbb{Z}[EG_*]_G, \tilde{\partial}_*)$. \square*

2.3. Projective resolutions and Tor. The aim of this section is to identify $H_*(G)$ with one of the standard objects studied in homological algebra. These results will not be used in the rest of the paper; we include them as they can be derived without too much difficulty from our other methods, and seem interesting.

Remark 2.9. If R is a non-unital ring, then free R -modules are not necessarily projective. However, for any idempotent $e \in R$ it is easily seen that $\text{Hom}_R(Re, M) \cong eM$ naturally for any R -module M . If $N \twoheadrightarrow M$ is an epimorphism then elements in eM lift to eN , so it follows that Re is projective. Consequently, if R is a ring with enough idempotents, i.e. it contains a family $(e_i)_{i \in I}$ of mutually orthogonal idempotents such that $R \cong \bigoplus_{i \in I} Re_i \cong \bigoplus_{i \in I} e_i R$, then free R -modules are projective. More generally, a (non-degenerate) R -module is projective exactly when it is a direct summand of a free R -module.

We will restrict to ample groupoids G with σ -compact base space G^0 . The main reason is that it implies that G^0 (resp. G) can be written as a disjoint union of compact open sets (resp. compact open bisections).³ If $G^0 = \bigsqcup_{i \in I} U_i$ with U_i compact and open, then $\mathbb{Z}[G] \cong \bigoplus_i \mathbb{Z}[G]\chi_{U_i} \cong \bigoplus_i \chi_{U_i}\mathbb{Z}[G]$. By the above remark it follows that free G -modules are projective, and similarly for $\mathbb{Z}[G^0]$.

To state the main result of this subsection, note that G^0 admits left and right actions of G (with the anchor map being the identity) defined by

$$gx := r(g) \quad \text{and} \quad xg := s(g).$$

These actions make $\mathbb{Z}[G^0]$ into both a left and a right G -module. Thus the Tor groups $\text{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$ (see for example [49, Definition 2.6.4]) of homological algebra make sense.

Theorem 2.10. *Let G be an ample groupoid with σ -compact base space. There is a canonical isomorphism $H_*(G) \cong \text{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$.*

The rest of this section will be spent proving this theorem, which will proceed by a sequence of lemmas. To give the idea of the proof, recall (see for example [49, Definition 2.6.4 and Theorem 2.7.2]) that $\text{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$ can be defined by starting with an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z}[G^0] \longrightarrow 0$$

of $\mathbb{Z}[G]$ -modules, where each P_n is projective. The group $\text{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$ is then by definition the n^{th} homology group of the complex

$$\cdots \xrightarrow{\text{id} \otimes \partial} \mathbb{Z}[G^0] \otimes_{\mathbb{Z}[G]} P_2 \xrightarrow{\text{id} \otimes \partial} \mathbb{Z}[G^0] \otimes_{\mathbb{Z}[G]} P_1 \xrightarrow{\text{id} \otimes \partial} \mathbb{Z}[G^0] \otimes_{\mathbb{Z}[G]} P_0$$

of abelian groups. We will prove Theorem 2.10 by showing that each $\mathbb{Z}[G]$ -module $\mathbb{Z}[EG_n]$ is projective, and that we have an exact sequence

$$\cdots \xrightarrow{\partial} \mathbb{Z}[EG_1] \xrightarrow{\partial} \mathbb{Z}[EG_0] \xrightarrow{\partial} \mathbb{Z}[G^0] \longrightarrow 0$$

where the boundary maps are the alternating sums of the face maps. As

$$\mathbb{Z}[G^0] \otimes_{\mathbb{Z}[G]} M = M_G$$

for any left non-degenerate G -module M by Lemma 2.2, Proposition 2.8 completes the proof.

³In general, this fails for locally compact, Hausdorff, totally disconnected spaces, such as Counterexample 65 from [46] (\mathbb{R} equipped with a rational sequence topology). One can also show that if X is this particular example, then $\mathbb{Z}[X]$ is not projective as a $\mathbb{Z}[X]$ -module, so free $\mathbb{Z}[X]$ -modules are not projective.

We now embark on the details of the proof. Recall that G^0 is equipped with a left G -action defined by stipulating that the anchor map is the identity and defining the moment map by $gx := r(g)$, and that $\mathbb{Z}[G^0]$ is a left G -module with the induced structure. Define $\partial : \mathbb{Z}[EG_0] \rightarrow \mathbb{Z}[G^0]$ to be the map induced on functions by the étale map $r : G \rightarrow G^0$.

Lemma 2.11. *The sequence*

$$\cdots \xrightarrow{\partial} \mathbb{Z}[EG_1] \xrightarrow{\partial} \mathbb{Z}[EG_0] \xrightarrow{\partial} \mathbb{Z}[G^0] \longrightarrow 0.$$

is exact.

Proof. For $n \geq 0$, define

$$h : EG_n \rightarrow EG_{n+1}, \quad (g_0, \dots, g_n) \mapsto (r(g_0), g_0, \dots, g_n).$$

Then one computes on the spatial level that $\partial^i h = h\partial^{i-1}$ for $1 \leq i \leq n$, and $\partial^0 h$ is the identity map. Hence by functoriality

$$\begin{aligned} \partial h_* + h_* \partial &= \sum_{i=0}^n (-1)^i (\partial^i h)_* + \sum_{i=0}^{n-1} (-1)^i (h\partial^i)_* \\ &= (\partial^0 h)_* + \sum_{i=1}^n (-1)^i ((\partial^i h)_* - (h\partial^{i-1})_*), \end{aligned}$$

and this is the identity. Define also $h : G^0 \rightarrow EG_0$ by $h(x) = x$. Then the map $rh : G^0 \rightarrow G^0$ is the identity, and so ∂h_* is the identity map on $\mathbb{Z}[G^0]$.

To summarise, h_* is a chain homotopy (as in for example [49, Definition 1.4.4]) between the identity map and the zero map. This implies that the complex has trivial homology (see for example [49, Lemma 1.4.5]), or, equivalently, is exact. \square

For the next lemma, let us define a right action of $\mathbb{Z}[G^0]$ on $\mathbb{Z}[G]$ via

$$(f\phi)(g) := f(g)\phi(s(g)) \quad \forall f \in \mathbb{Z}[G], \phi \in \mathbb{Z}[G^0]$$

and similarly a left action of $\mathbb{Z}[G^0]$ on $\mathbb{Z}[EG_n]$ via $(\phi a)(g_0, \dots, g_n) := \phi(r(g_0))a(g_0, \dots, g_n)$. Note that this right action of $\mathbb{Z}[G^0]$ on $\mathbb{Z}[G]$ commutes with the canonical left action of $\mathbb{Z}[G]$ on itself: indeed, the action of $\mathbb{Z}[G^0]$ is just the action by right-multiplication of the submodule $\mathbb{Z}[G^0]$ of $\mathbb{Z}[G]$, so this commutativity statement is associativity of multiplication. For notational convenience, we define $EG_{-1} := G^0$.

Lemma 2.12. *With notation as above, for any $n \geq -1$ there is a canonical isomorphism of $\mathbb{Z}[G]$ -modules.*

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[G^0]} \mathbb{Z}[EG_n] \cong \mathbb{Z}[EG_{n+1}].$$

Proof. For $f \in \mathbb{Z}[G]$ and $a \in \mathbb{Z}[EG_n]$, we define

$$(f, a) : EG_{n+1} \rightarrow \mathbb{Z}, \quad (g_0, \dots, g_{n+1}) \mapsto f(g_0)a(g_0^{-1}g_1, \dots, g_0^{-1}g_{n+1}).$$

Note that (f, a) is in $\mathbb{Z}[EG_{n+1}]$ by the support conditions defining $\mathbb{Z}[G]$ and $\mathbb{Z}[EG_n]$. Note also that if $\phi \in \mathbb{Z}[G^0]$, then

$$(f\phi, a) : (g_0, \dots, g_{n+1}) \mapsto f(g_0)\phi(s(g_0))a(g_0^{-1}g_1, \dots, g_0^{-1}g_n)$$

and

$$(f, \phi a) : (g_0, \dots, g_{n+1}) \mapsto f(g_0)\phi(r(g_0^{-1}g_1))a(g_0^{-1}g_1, \dots, g_0^{-1}g_n);$$

these are the same however, as for any $(g_0, \dots, g_{n+1}) \in EG_{n+1}$, $r(g_0^{-1}g_1) = s(g_0)$. Hence by the universal property of the balanced tensor product, we have a well-defined map

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[G^0]} \mathbb{Z}[EG_n] \rightarrow \mathbb{Z}[EG_{n+1}].$$

We claim that this map is an isomorphism.

For injectivity, say we have an element $\sum_{i=1}^n f_i \otimes a_i$ that goes to zero. Splitting up the sum further, we may assume that each f_i is the characteristic function χ_{B_i} of a compact open bisection B_i such that $B_i \cap B_j = \emptyset$ for $i \neq j$. For each i , let ϕ_i be the characteristic function of $s(B_i)$. As $\chi_{B_i} = \chi_{B_i} \phi_i$, we have that $\chi_{B_i} \otimes a_i = \chi_{B_i} \otimes \phi_i a_i$, we may further assume that each a_i is supported in $\{(g_0, \dots, g_n) \in EG_n \mid r(g_0) \in s(B_i)\}$. Now, we are assuming that $\sum_{i=1}^n (\chi_{B_i}, a_i) = 0$. As $B_i \cap B_j = \emptyset$ for $i \neq j$, the functions (χ_{B_i}, a_i) have disjoint supports, and therefore we have that $(\chi_{B_i}, a_i) = 0$ for each i . Assume for contradiction that $a_i \neq 0$, so there exists $(h_0, \dots, h_n) \in EG_n$ with $a(h_0, \dots, h_n) \neq 0$; our assumptions force $r(h_0) \in s(B_i)$. Let $g \in B_i$ be such that $s(g) = r(h_0)$. Then $(g, gh_0, \dots, gh_n) \in EG_{n+1}$ and (χ_{B_i}, a) evaluates to $a(h_0, \dots, h_n) \neq 0$ at this point, giving the contradiction and completing the proof of injectivity.

For surjectivity, as any element of $\mathbb{Z}[EG_{n+1}]$ is a finite \mathbb{Z} -linear combination of characteristic functions of subsets of the form $B_0 \times \dots \times B_{n+1} \cap EG_{n+1}$ with each B_i a compact open bisection in G , it suffices to show that any such characteristic function is in B . Set $f \in \mathbb{Z}[G]$ to be the characteristic function of B_0 , and $a \in \mathbb{Z}[EG_n]$ to be the characteristic function of $B_0^{-1}B_1 \times \dots \times B_0^{-1}B_n \cap EG_n$. We leave it to the reader to check that $f \otimes a$ maps to the function we want. \square

For the next lemma, we consider $\mathbb{Z}[G]$ as a left $\mathbb{Z}[G^0]$ -module via the left-multiplication action induced by the inclusion $\mathbb{Z}[G^0] \subseteq \mathbb{Z}[G]$.

Lemma 2.13. *If G^0 is σ -compact, then, considered as a (left) module over $\mathbb{Z}[G^0]$, $\mathbb{Z}[G]$ is projective.*

In many interesting cases $\mathbb{Z}[G]$ is actually free over $\mathbb{Z}[G^0]$: for example, this happens for transformation groupoids associated to actions of discrete groups. However, freeness does not seem to be true in general.

Proof. For a compact open bisection U , let $\chi_U \in \mathbb{Z}[G]$ denote the characteristic function of that bisection. As G^0 is σ -compact we may choose a covering $G = \bigsqcup_{i \in I} U_i$ of G by disjoint compact open bisections, and for each i , let $\mathbb{Z}[G^0]\chi_{U_i}$ denote the $\mathbb{Z}[G^0]$ -submodule of $\mathbb{Z}[G]$ generated by χ_{U_i} . Then as $\mathbb{Z}[G^0]$ -modules, we have

$$\mathbb{Z}[G] \cong \bigoplus_{i \in I} \mathbb{Z}[G^0]\chi_{U_i}.$$

It thus suffices to prove that each $\mathbb{Z}[G^0]\chi_{U_i}$ is projective. For this, one checks that $\mathbb{Z}[G^0]\chi_{U_i}$ is isomorphic as a $\mathbb{Z}[G^0]$ -module to $\mathbb{Z}[G^0]\chi_{r(U_i)}$, so projective by Remark 2.9, and we are done. \square

Corollary 2.14. *If G^0 is σ -compact, the G -module $\mathbb{Z}[EG_n]$ is projective for $n \geq 0$.*

Proof. We proceed by induction on n . For $n = 0$, $\mathbb{Z}[EG_0]$ identifies with $\mathbb{Z}[G]$ as a left $\mathbb{Z}[G]$ -module, so free, and thus projective by Remark 2.9. Now assume we have the result for $\mathbb{Z}[EG_n]$. By Remark 2.9, $\mathbb{Z}[EG_n]$ is a direct summand in a free $\mathbb{Z}[G]$ -module, say $\bigoplus_J \mathbb{Z}[G]$. Hence, as $\mathbb{Z}[G^0]$ -modules, $\mathbb{Z}[EG_n]$ is a direct summand in $\bigoplus_J \mathbb{Z}[G]$ which is projective by Lemma 2.13. Therefore, $\mathbb{Z}[EG_n]$ is projective as a $\mathbb{Z}[G^0]$ -module. Let then N be a $\mathbb{Z}[G^0]$ -module such that $\mathbb{Z}[EG_n] \oplus N$ is isomorphic as a $\mathbb{Z}[G^0]$ -module to $\bigoplus_{i \in I} \mathbb{Z}[G^0]$ for some index set I . It follows that

as $\mathbb{Z}[G]$ -modules

$$\begin{aligned} \left(\mathbb{Z}[G] \otimes_{\mathbb{Z}[G^0]} \mathbb{Z}[EG_n] \right) \oplus \left(\mathbb{Z}[G] \otimes_{\mathbb{Z}[G^0]} N \right) &\cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[G^0]} (M \oplus N) \\ &\cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[G^0]} \left(\bigoplus_{i \in I} \mathbb{Z}[G^0] \right) \\ &\cong \bigoplus_{i \in I} \mathbb{Z}[G]. \end{aligned}$$

Hence $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G^0]} \mathbb{Z}[EG_n]$ is isomorphic to a direct summand of a free $\mathbb{Z}[G]$ -module, so projective as a $\mathbb{Z}[G]$ -module. Using Lemma 2.12, we are done. \square

Proof of Theorem 2.10. Lemma 2.11 and Corollary 2.14 together imply that

$$\cdots \xrightarrow{\partial} \mathbb{Z}[EG_1] \xrightarrow{\partial} \mathbb{Z}[EG_0] \xrightarrow{\partial} \mathbb{Z}[G^0] \longrightarrow 0.$$

is a resolution of $\mathbb{Z}[G^0]$ by projective modules. The groups $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}[G^0], \mathbb{Z}[G^0])$ are therefore by definition the homology groups of the complex

$$\cdots \xrightarrow{\mathrm{id} \otimes \partial} \mathbb{Z}[G^0] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[EG_1] \xrightarrow{\mathrm{id} \otimes \partial} \mathbb{Z}[G^0] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[EG_0] \longrightarrow 0.$$

However, using Lemma 2.2, this is the same as the complex

$$\cdots \xrightarrow{\partial} \mathbb{Z}[EG_1]_G \xrightarrow{\partial} \mathbb{Z}[EG_0]_G \longrightarrow 0,$$

and we have already seen that the homology of this is the same as the homology $H_*(G)$. \square

3. COLOURINGS AND HOMOLOGY

3.1. Colourings and nerves. The goal of this section is to use what we shall call a *colouring* of a groupoid G to produce an associated nerve space, and prove that the nerve is a semi-simplicial G -space. This lets us define the homology of a colouring in terms of the homology of the associated semi-simplicial G -space. Throughout almost the entirety of this section we assume that G is an ample groupoid with compact base space G^0 : the exception is the main result—Theorem 3.36—at the end, where we drop the compact base space assumption.

Definition 3.1. A *colouring* for G is a finite ordered collection $\mathcal{C} = (G_0, \dots, G_d)$ of compact open subgroupoids of G such that the collection $\{G_0^0, \dots, G_d^0\}$ of unit spaces covers G^0 .

Elements of the set $\{0, \dots, d\}$ are called the *colours* of the colouring, and the colour of G_i is i .

We will associate a cover of G to a colouring. Let $\mathcal{P}(G)$ denote the collection of subsets of G .

Definition 3.2. Let G_0, \dots, G_d be a colouring of G . The *cover* associated to the colouring is

$$\mathcal{U} := \{(gG_i^{s(g)}, i) \in \mathcal{P}(G) \times \{0, \dots, d\} \mid g \in G, G_i^{s(g)} \neq \emptyset\}.$$

Typically, we will just write U for an element of \mathcal{U} , and treat U as a (non-empty) subset of G . In particular, when considering intersections $U_0 \cap U_1$ for $U_0, U_1 \in \mathcal{U}$ we just mean the intersection of the corresponding subsets of G (ignoring the colour!). The precise definition however calls for pairs as we want each element of \mathcal{U} to have a well-defined colour in $\{0, \dots, d\}$. Note that an element $U \in \mathcal{U}$ could be equal to $(gG_i^{s(g)}, i)$ and $(hG_i^{s(h)}, i)$ for $g \neq h$ in G , i.e. representations of elements of \mathcal{U} of

the form $gG_i^{s(g)}$ need not be unique. Note that (ignoring colours), the fact that G_0^0, \dots, G_d^0 covers G^0 implies that \mathcal{U} is a cover of G , but typically not by open sets: indeed, an element $gG_i^{s(g)}$ of \mathcal{U} is contained in the single range fibre $G^{r(g)}$.

Definition 3.3. Let G_0, \dots, G_d be a colouring of G with associated cover \mathcal{U} . For $n \geq 0$, set

$$\mathcal{N}_n := \left\{ (U_0, \dots, U_n) \in \mathcal{U}^{n+1} \mid \bigcap_{i=0}^n U_i \neq \emptyset \right\}.$$

The sequence $(\mathcal{N}_n)_{n=0}^\infty$ is denoted \mathcal{N}_* , and called the *nerve* of the colouring.

As noted before, the intersection in the definition above is to be interpreted as the intersection of the corresponding subsets of G (ignoring the colour). In particular we allow distinct U_i appearing in a tuple as above to have different colours.

Our next goal is to give \mathcal{N}_* the structure of a semi-simplicial G -space.

Definition 3.4. For each n , the anchor map $r : \mathcal{N}_n \rightarrow G^0$ takes (U_0, \dots, U_n) to the unique $x \in G^0$ such that U_0 is a subset of G^x . Let $G_{s \times r} \mathcal{N}_n$ be the fibered product as in line (1) above, and define the moment map by

$$G_{s \times r} \mathcal{N}_n \rightarrow \mathcal{N}_n, \quad (g, (U_0, \dots, U_n)) \mapsto (gU_0, \dots, gU_n).$$

Direct checks show that the action of G on \mathcal{N}_n is well-defined, and satisfies the algebraic rules for a groupoid action on a set. Our next goal is to introduce a topology on \mathcal{N}_n , and prove that the action is continuous.

Definition 3.5. Let $i \in \{0, \dots, d\}$ and let V be an open subset of G such that $G_i^{s(g)}$ is non-empty for all $g \in V$. Define

$$U_{V,i} := \{(gG_i^{s(g)}, i) \mid g \in V\}.$$

It is straightforward to check that $U_{V,i} \cap U_{W,i} = U_{V \cap W, i}$ and $U_{V,i} \cap U_{W,j} = \emptyset$ for $i \neq j$. Hence the collection of sets $U_{V,i}$ as V ranges over the compact open bisections of G and $0 \leq i \leq d$ form the basis for a topology on \mathcal{N}_0 . For each $n \geq 0$, equip \mathcal{N}_0^{n+1} with the product topology, and give $\mathcal{N}_n \subseteq \mathcal{N}_0^{n+1}$ the subspace topology.

Lemma 3.6. *For each n , the topology on \mathcal{N}_n is locally compact, Hausdorff, and totally disconnected. Moreover, the action defined in Definition 3.4 above is continuous.*

Proof. We first look at the case $n = 0$. Given a compact open bisection $V \subseteq G$ with $s(V) \subseteq G_i^0$ one readily verifies that the mapping $g \mapsto (gG_i^{s(g)}, i)$ defines a homeomorphism $V \rightarrow U_{V,i}$. In particular, the sets $U_{V,i}$ for such compact open bisections V are themselves compact open Hausdorff subsets of \mathcal{N}_0 . Consequently, \mathcal{N}_0 is locally compact and locally Hausdorff. To check that it is Hausdorff, it suffices to check that if $(U_j)_j$ is a net in \mathcal{N}_0 that converges to both U and V , then $U = V$. Equivalently, say we have a net $(g_jG_{i_j}^{s(g_j)}, i_j)_j$ in \mathcal{N}_0 such that there are h_j in G with $g_jG_{i_j}^{s(g_j)} = h_jG_{i_j}^{s(h_j)}$ for all j , and so that $g_j \rightarrow g$ and $h_j \rightarrow h$. First note that since the net converges, i_j will be eventually constant so we may as well assume that $i_j = i$ for all j ; we need to check that $gG_i^{s(g)} = hG_i^{s(h)}$.

Say then that gk is in $gG_i^{s(g)}$ with $k \in G_i^{s(g)}$. By symmetry, it suffices to check that gk is in $hG_i^{s(h)}$. As G_i is open in G , it is étale, whence for all suitably large j , we can find $k_j \in G_i^{s(g_j)}$ such that $k_j \rightarrow k$. As $g_jG_{i_j}^{s(g_j)} = h_jG_{i_j}^{s(h_j)}$ for all j , we can also find $l_j \in G_i^{s(h_j)}$ for all j so that $g_jk_j = h_jl_j$. Using that G_i is compact, we

⁴and therefore all the U_j are subsets of

may pass to a subnet and so assume that (l_j) converges to some $l \in G_i$, which is necessarily in $G_i^{s(h)}$. Hence $gk = \lim g_j k_j = \lim h_j l_j = hl$, and so gk is in $hG_i^{s(h)}$ as required.

The first paragraph of the proof implies that \mathcal{N}_0 admits a basis of compact open subsets and hence it is totally disconnected.

Continuity of the action follows on observing that if $(g_j, h_j G_i^{s(h)})$ is a convergent net in $G_{s \times r} \mathcal{N}_0$, then $(g_j h_j G_i^{s(g_j h)})$ is a convergent net in \mathcal{N}_0 by continuity of the multiplication in G .

We now look at the case of general n . The facts that \mathcal{N}_n is Hausdorff and totally disconnected, as well as the continuity of the G -action, all follow directly from the corresponding properties for \mathcal{N}_0 .

We claim that \mathcal{N}_n is closed in \mathcal{N}_0^{n+1} ; as closed subsets of locally compact spaces are locally compact, this will suffice to complete the proof. To check closedness, for each $j \in \{0, \dots, n\}$, say $(g_j^k)_{k \in K}$ is a net such that $g_j^k \rightarrow g_j$ as $k \rightarrow \infty$, and such that $(g_0^k G_i^{s(g_0^k)}, \dots, g_n^k G_i^{s(g_n^k)})$ is in \mathcal{N}_n for all k . We need to show that $(g_0 G_i^{s(g_0)}, \dots, g_n G_i^{s(g_n)})$ is also in \mathcal{N}_n . Indeed, as $(g_0^k G_i^{s(g_0^k)}, \dots, g_n^k G_i^{s(g_n^k)})$ is in \mathcal{N}_n , there exist $h_0^k, \dots, h_n^k \in G_i$ such that

$$g_0^k h_0^k = g_1^k h_1^k = \dots = g_n^k h_n^k.$$

As G_i is compact, we may assume that each net $(h_j^k)_{k \in K}$ converges to some h_j in G_i . Hence

$$g_0 h_0 = g_1 h_1 = \dots = g_n h_n$$

is a point in $g_0 G_i^{s(g_0)} \cap \dots \cap g_n G_i^{s(g_n)}$, which is thus non-empty. Hence

$$(g_0 G_i^{s(g_0)}, \dots, g_n G_i^{s(g_n)})$$

is in \mathcal{N}_n as required. \square

We thus have shown that each of the spaces \mathcal{N}_n is in the category \mathbf{Top}_G of Definition 2.3. To show that \mathcal{N}_* is a semi-simplicial G -space, it remains to build the face maps and prove that they are equivariant and étale.

Definition 3.7. For each $n \geq 1$ and each $j \in \{0, \dots, n\}$, define the j^{th} face map to be the function

$$\partial_j^n : \mathcal{N}_n \rightarrow \mathcal{N}_{n-1}, \quad (U_0, \dots, U_n) \mapsto (U_0, \dots, \widehat{U_j}, \dots, U_n),$$

where the hat “ $\widehat{\cdot}$ ” means to omit the corresponding element.

Lemma 3.8. For each j , ∂_j as in Definition 3.7 is equivariant and étale, and moreover

$$\partial_j^{n-1} \partial_i^n = \partial_i^{n-1} \partial_{j-1}^n \text{ if } i < j.$$

Proof. Let (U_0, \dots, U_n) be a point in \mathcal{N}_n , and write $U_k = g_k G_{i_k}^{s(g_k)}$ for each $k \in \{0, \dots, n\}$. For each k , let B_k be an open bisection containing g_k . Then the set

$$W := \left\{ (h_0 G_{i_0}^{s(h_0)}, \dots, h_n G_{i_n}^{s(h_n)}) \mid h_k \in B_k \text{ for each } k \text{ and } \bigcap_{k=0}^n h_k G_{i_k}^{s(h_k)} \neq \emptyset \right\}$$

is an open neighbourhood of (U_0, \dots, U_n) in \mathcal{N}_n . We claim that ∂^j restricts to a homeomorphism on W . Indeed, let W_j be the image of W under ∂^j . Then an inverse is defined by sending a point (V_0, \dots, V_{n-1}) in W_j to the point $(V_0, \dots, h G_{i_j}^{s(h)}, \dots, V_{n-1})$ in W , where $h G_{i_j}^{s(h)}$ occurs in the j^{th} entry, and where h is the unique point in B_j so that $r(h) = p(V_0, \dots, V_{n-1})$.

The G -equivariance and the claimed relations between the face maps are straightforward. \square

Definition 3.9. Let \mathcal{C} be a colouring of G . The *homology of the colouring*, denoted $H_*(\mathcal{C})$, is the homology $H_*^G(\mathcal{N}_*)$ of the semi-simplicial G -space \mathcal{N}_* as in Definition 2.7.

The homology groups $H_*(\mathcal{C})$ depend strongly on the colouring. For example, \mathcal{C} could just consist of a partition of G^0 by compact open subsets, in which case one can check that the groups $H_n(\mathcal{C})$ are zero for $n > 0$. We will, however, eventually show that an appropriate limit of the homologies $H_*(\mathcal{C})$ as the colourings vary recovers the Cranic-Moerdijk-Matui homology $H_*(G)$ for principal and σ -compact G .

3.2. Homology vanishing. Our goal in this section is to prove that if

$$\mathcal{C} = \{G_0, \dots, G_d\}$$

is a colouring of G , then $H_n(\mathcal{C}) = 0$ for $n > d$. We will actually prove something a little more precise than this, as it will be useful later. The computations in this section are inspired by classical results in sheaf cohomology: see for example [17, Section 3.8]. More specifically, the precise formulas we use are adapted from [45, Tag 01FG].

Throughout this section, we fix an ample groupoid G with compact unit space G^0 , a colouring $\mathcal{C} = \{G_0, \dots, G_d\}$ as in Definition 3.1, and associated nerve space \mathcal{N}_* as in Definition 3.3.

Lemma 3.10. *Say \mathcal{U} is the cover associated to the colouring \mathcal{C} as in Definition 3.2. Then any two elements of \mathcal{U} that are the same colour and intersect non-trivially are the same. In particular, if (U_0, \dots, U_n) is a point of some \mathcal{N}_n , then any two elements of the same colour are actually the same.*

Proof. Say $gG_i^{s(g)} \cap hG_i^{s(h)} \neq \emptyset$; we need to show that $gG_i^{s(g)} = hG_i^{s(h)}$. Indeed, there are $k_g \in G_i^{s(g)}$ and $k_h \in G_i^{s(h)}$ with $gk_g = hk_h$. It follows that $h^{-1}g = k_h k_g^{-1}$, so $h^{-1}g$ is in G_i , as G_i is a subgroupoid. Hence whenever gk is in $gG_i^{s(g)}$, we have that $gk = h(h^{-1}gk)$ is also in $hG_i^{s(h)}$, and so $gG_i^{s(g)} \subseteq hG_i^{s(h)}$. Hence by symmetry, $gG_i^{s(g)} = hG_i^{s(h)}$. \square

Definition 3.11. For each n , define the *colour map*

$$c : \mathcal{N}_n \rightarrow \{0, \dots, d\}^{n+1}, \quad U \mapsto (\text{colour of } U_0, \dots, \text{colour of } U_n).$$

We leave it to the reader to check that c is continuous and invariant under the action of G .

Throughout we identify the symmetric group S_{n+1} with the permutations of $\{0, \dots, n\}$. For $\sigma \in S_{n+1}$, let σ_0 be the identity. For each $a \in \{1, \dots, n\}$, let σ_a be the unique permutation of $\{0, \dots, n\}$ such that:

- (i) the restriction of σ_a to $\{0, \dots, a-1\}$ agrees with the restriction of σ to this set;
- (ii) the restriction of σ_a to $\{a, \dots, n\}$ is the unique order-preserving bijection

$$\{a, \dots, n\} \rightarrow \{0, \dots, n\} \setminus \{\sigma(0), \dots, \sigma(a-1)\}.$$

For a point $x = (i_0, \dots, i_n) \in \{0, \dots, d\}^{n+1}$, let $\sigma^x \in S_{n+1}$ be the unique permutation determined by the conditions below:

- (i) $i_{\sigma^x(0)} \leq i_{\sigma^x(1)} \leq \dots \leq i_{\sigma^x(n)}$;
- (ii) σ^x is order-preserving when restricted to each subset S of $\{0, \dots, n\}$ such that the elements $\{i_j \mid j \in S\}$ all have the same colour (i.e. so that the set $\{i_j \mid j \in S\}$ consists of a single element of $\{0, \dots, d\}$).

Definition 3.12. For each n , let $c : \mathcal{N}_n \rightarrow \{0, \dots, d\}^{n+1}$ be the colour map of Definition 3.11 and define

$$\mathcal{N}_n^> := c^{-1}(\{(i_0, \dots, i_n) \mid i_0 < \dots < i_n\}).$$

Similarly, define

$$\mathcal{N}_n^{\geq} := c^{-1}(\{(i_0, \dots, i_n) \mid i_0 \leq \dots \leq i_n\}).$$

We note that each $\mathcal{N}_*^>$ and \mathcal{N}_*^{\geq} is a semi-simplicial G -space with the restricted structures from \mathcal{N}_* : indeed, each $\mathcal{N}_n^>$ and \mathcal{N}_n^{\geq} is a closed, open and G -invariant subset of \mathcal{N}_n as c is continuous and G -invariant, and the face maps of Definition 3.8 clearly restrict to maps $\mathcal{N}_n^> \rightarrow \mathcal{N}_n^>$ and $\mathcal{N}_n^{\geq} \rightarrow \mathcal{N}_n^{\geq}$.

For each $a \in \{0, \dots, n\}$ define now $h_a : \mathcal{N}_n \rightarrow \mathcal{N}_{n+1}$ by the formula

$$h_a(U) := (U_{\sigma_a^{c(U)}(0)}, \dots, U_{\sigma_a^{c(U)}(a-1)}, U_{\sigma_a^{c(U)}(a)}, U_{\sigma_a^{c(U)}(a)}, U_{\sigma_a^{c(U)}(a+1)}, \dots, U_{\sigma_a^{c(U)}(n)})$$

(in words, we use $\sigma_a^{c(U)}$ to rearrange the order of the components of U , but also insert $U_{\sigma_a^{c(U)}(a)}$ into the a^{th} position).

It is not too difficult to see that each h_a is an equivariant étale map, and so induces a pushforward map $(h_a)_* : \mathbb{Z}[\mathcal{N}_n] \rightarrow \mathbb{Z}[\mathcal{N}_{n+1}]$ of $\mathbb{Z}[G]$ -modules. We define

$$h : \mathbb{Z}[\mathcal{N}_n] \rightarrow \mathbb{Z}[\mathcal{N}_{n+1}]$$

by stipulating that for each $x \in \{0, \dots, d\}^{n+1}$, its restriction to each subset $\mathbb{Z}[c^{-1}(x)]$ equals

$$\sum_{a=0}^n (-1)^a \text{sign}(\sigma_a^x) (h_a)_*.$$

On the other hand, define

$$p : \mathbb{Z}[\mathcal{N}_n] \rightarrow \mathbb{Z}[\mathcal{N}_n^{\geq}]$$

by stipulating that for each $x \in \{0, \dots, d\}^{n+1}$, its restriction to each subset $\mathbb{Z}[c^{-1}(x)]$ equals $\text{sign}(\sigma^x) \sigma_*^x$. Finally, let

$$i : \mathbb{Z}[\mathcal{N}_*^{\geq}] \rightarrow \mathbb{Z}[\mathcal{N}_*]$$

be the canonical inclusion.

Lemma 3.13. *Let ∂ be the boundary map on $\mathbb{Z}[\mathcal{N}_*]$. Then*

$$\partial h + h \partial = \text{identity} - i \circ p.$$

Proof. We look at the restriction of ∂h to $\mathbb{Z}[c^{-1}(x)]$ for some x ; it suffices to prove the given identity for such restrictions. For notational simplicity, let $\sigma := \sigma^x$. We have then for this restriction that

$$\partial h = \sum_{i=0}^{n+1} \sum_{a=0}^n (-1)^{i+a} \text{sign}(\sigma_a) \partial_*^i (h_a)_*$$

We split the terms into three types.

- (i) Terms of the form $\partial_*^i (h_a)_*$ where $i < a$.
- (ii) Terms of the form $\partial_*^{i+1} (h_a)_*$ where $i \geq a$ and $\sigma_a(i) \neq \sigma(a)$.
- (iii) Terms of the form $\partial_*^a (h_a)_*$, or of the form $\partial_*^{i+1} (h_a)_*$ where $\sigma_a(i) = \sigma(a)$.

We look at each type in turn.

- (i) We leave it to the reader to compute that as maps on the spatial level, $\partial^i h_a = h_{a-1} \partial^{\sigma_a(i)}$. Moreover, $(\partial^i h_a)_*$ occurs in the sum defining ∂h with sign $(-1)^i (-1)^a \text{sign}(\sigma_a)$, and $(h_{a-1} \partial^{\sigma_a(i)})_*$ occurs in the sum defining $h \partial$ with the

$\text{sign}(-1)^{a-1}(-1)^{\sigma_a(i)}\text{sign}(\sigma'_{a-1})$, where σ'_{a-1} is the permutation defined as the composition

$$\begin{array}{ccc} \{0, \dots, n-1\} & \xrightarrow{f} & \{0, \dots, \widehat{i}, \dots, n\} \xrightarrow{\sigma_a} \{0, \dots, \widehat{\sigma_a(i)}, \dots, n\} \\ & & \downarrow g \\ & & \{0, \dots, n-1\} \end{array}$$

with f and g the unique order preserving bijections. One can compute that $\text{sign}(\sigma_a) = (-1)^{\sigma_a(i)-i}\text{sign}(\sigma'_{a-1})$, essentially as σ_a can be built from the same transpositions as used to construct σ'_a (conjugated by f and g) together with a cycle of length $|\sigma_a(i) - i| + 1$, which has sign $(-1)^{\sigma_a(i)-i}$. In conclusion, the term

$$(-1)^i(-1)^a\text{sign}(\sigma_a)(\partial^i h_a)_*$$

appearing in the sum defining ∂h is matched by the term

$$\begin{aligned} & (-1)^{a-1}(-1)^{\sigma_a(i)}\text{sign}(\sigma'_{a-1})(h_{a-1}\partial^{\sigma_a(i)})_* \\ & = (-1)^{a-1}(-1)^i\text{sign}(\sigma_a)(h_{a-1}\partial^{\sigma_a(i)})_* \end{aligned}$$

appearing in the sum defining $h\partial$; as these precisely match other than having opposing signs, they cancel.

- (ii) We compute that as maps on the spatial level $\partial^{i+1}h_a = h_a\partial^{\sigma_a(i)}$. Moreover, $(\partial^{i+1}h_a)_*$ occurs in the sum defining ∂h with sign $(-1)^{i+1}(-1)^a\text{sign}(\sigma_a)$, and $(h_a\partial^{\sigma_a(i)})_*$ occurs in the sum defining $h\partial$ with the sign $(-1)^a(-1)^{\sigma_a(i)}\text{sign}(\sigma'_a)$, where σ'_a is the permutation defined as the composition

$$\begin{array}{ccc} \{0, \dots, n-1\} & \xrightarrow{f} & \{0, \dots, \widehat{i}, \dots, n\} \xrightarrow{\sigma_a} \{0, \dots, \widehat{\sigma_a(i)}, \dots, n\} \\ & & \downarrow g \\ & & \{0, \dots, n-1\} \end{array}$$

with f and g the unique order preserving bijections. Much as in case (i), we have that $\text{sign}(\sigma_a) = (-1)^{\sigma_a(i)-i}\text{sign}(\sigma'_{a-1})$, and can conclude that the term

$$(-1)^{i+1}(-1)^a\text{sign}(\sigma_a)(\partial^{i+1}h_a)_*$$

appearing in the sum defining ∂h is matched by the term

$$(-1)^a(-1)^{\sigma_a(i)}\text{sign}(\sigma'_a)(h_a\partial^{\sigma_a(i)})_* = (-1)^a(-1)^i\text{sign}(\sigma_a)h_a\partial^{\sigma_a(i)}_*$$

appearing in the sum defining $h\partial$; as these precisely match other than having opposite signs, they cancel.

At this point, one can check that we have canceled *all* the terms appearing in the sum defining $h\partial$ using terms from ∂h of types (i) and (ii). It remains to consider terms of type (iii), and show that the sum of these equals

$$\text{identity} - i \circ p$$

as claimed. For each a , let i_a be the index such that $\sigma_a(i_a) = \sigma(a)$. The totality of terms of type (iii) looks like

$$\begin{aligned} & (\partial^0 h_0)_* \\ & + \sum_{a=0}^{n-1} ((-1)^{i_a+1}(-1)^a\text{sign}(\sigma_a)(\partial^{i_a+1}h_a)_* + (-1)^{a+1}(-1)^{a+1}\text{sign}(\sigma_{a+1})(\partial^{a+1}h_{a+1})_*) \\ & + (-1)^n(-1)^n\text{sign}(\sigma_n)(\partial^{i_n}h_n)_*. \end{aligned}$$

The first term is the identity, and the last term is $i \circ p$ (note that $\sigma_n = \sigma$). Hence it suffices to show that each term

$$(-1)^{i_a+1}(-1)^a \text{sign}(\sigma_a)(\partial^{i_a+1}h_a)_* + (-1)^{a+1}(-1)^{a+1} \text{sign}(\sigma_{a+1})(\partial^{a+1}h_{a+1})_*$$

in the sum in the middle is zero. Now, one computes on the spatial level that $\partial^{a+1}h_{a+1} = \partial^{i_a+1}h_a$ (this uses Lemma 3.10 to conclude that two coordinates with the same colour are actually the same), and so it suffices to prove that

$$(-1)^{i_a+1}(-1)^a \text{sign}(\sigma_a) + (-1)^{a+1}(-1)^{a+1} \text{sign}(\sigma_{a+1}) = 0,$$

or having simplified slightly, that $(-1)^{i_a+a+1} \text{sign}(\sigma_a) + \text{sign}(\sigma_{a+1}) = 0$. Indeed, one checks that σ_{a+1} differs from σ_a by a cycle moving the element in the i_a^{th} position to the a^{th} (and keeping all other elements in the same order), and such a cycle has sign $(-1)^{a-i_a}$. Hence $\text{sign}(\sigma_a) = \text{sign}(\sigma_{a+1})(-1)^{i_a-a}$ and we are done. \square

Corollary 3.14. *The natural inclusion $i : \mathbb{Z}[\mathcal{N}_*^{\geq}] \rightarrow \mathbb{Z}[\mathcal{N}_*]$ is a chain homotopy equivalence.*

Proof. Lemma 3.13 implies in particular that the map $i \circ p$ is a chain map; as i is an injective chain map, this implies that $p : \mathbb{Z}[\mathcal{N}_n] \rightarrow \mathbb{Z}[\mathcal{N}_n^{\geq}]$ is a chain map too. Lemma 3.13 implies that $i \circ p$ is chain-homotopic to the identity, while $p \circ i$ just is the identity. Hence p provides an inverse to i on the level of chain homotopies. \square

Our next goal is to show that the natural inclusion $j : \mathbb{Z}[\mathcal{N}_*^>] \rightarrow \mathbb{Z}[\mathcal{N}_*^{\geq}]$ is again a chain homotopy equivalence. For each $a \in \{0, \dots, n-1\}$, let us write D_a^n for the subset of $\{0, \dots, d\}^{n+1}$ consisting of those tuples (i_0, \dots, i_n) such that $i_0 < i_1 < \dots < i_a = i_{a+1} \leq i_{a+2} \leq \dots \leq i_n$. For $x \in D_a^n$, define $k^x : \mathbb{Z}[c^{-1}(x)] \rightarrow \mathbb{Z}[\mathcal{N}_{n+1}]$ by

$$k^x : (U_0, \dots, U_n) \mapsto (U_0, U_1, \dots, U_a, U_a, U_{a+1}, U_{a+2}, \dots, U_n).$$

Now define $k : \mathbb{Z}[\mathcal{N}_n^{\geq}] \rightarrow \mathbb{Z}[\mathcal{N}_{n+1}^{\geq}]$ by stipulating that for each $x \in \{0, \dots, d\}^{n+1}$ the restriction of k to $\mathbb{Z}[c^{-1}(x)]$ is given by $(-1)^a k_*^x$ if x is in D_a^n for some $a \in \{0, \dots, n-1\}$, and by zero otherwise. On the other hand, let

$$q : \mathbb{Z}[\mathcal{N}_n^{\geq}] \rightarrow \mathbb{Z}[\mathcal{N}_n^>]$$

be the natural projection that acts as the identity on each $\mathbb{Z}[c^{-1}(x)]$ with $c^{-1}(x) \subseteq \mathcal{N}_*^>$, and as 0 otherwise.

Lemma 3.15. *Let ∂ be the boundary map on $\mathbb{Z}[\mathcal{N}_*^{\geq}]$ and let $j : \mathbb{Z}[\mathcal{N}_*^>] \rightarrow \mathbb{Z}[\mathcal{N}_*^{\geq}]$ be the canonical inclusion. Then*

$$\partial k + k \partial = \text{identity} - j \circ q.$$

Proof. It suffices to check the formula for each restriction to a submodule of the form $\mathbb{Z}[c^{-1}(x)]$.

Say first that $x = (i_0, \dots, i_n)$ satisfies $i_0 < \dots < i_n$. Then $j \circ q$ acts as the identity on $\mathbb{Z}[c^{-1}(x)]$, whence the right hand side is zero. Note that k restricts to zero on $\mathbb{Z}[c^{-1}(x)]$, whence ∂k is zero. On the other hand, the image of $\mathbb{Z}[c^{-1}(x)]$ is contained in a direct sum of subgroups of the form $\mathbb{Z}[c^{-1}(y)]$ where $y = (j_0, \dots, j_{n-1}) \in \{0, \dots, d\}^n$ satisfies $j_0 < \dots < j_{n-1}$. Hence $k \partial = 0$ too, and we are done with this case.

Say then that $x = (i_0, \dots, i_n)$ does not satisfy $i_0 < \dots < i_n$. Then x is in D_a^n for some a . We then compute using the assumption that x is in D_a^n that $\partial^i k^x$ is the

identity map for $i \in \{a, a+1, a+2\}$, and therefore that

$$\begin{aligned}
 \partial k &= \sum_{i=0}^{n+1} (-1)^i \partial_*^i k \\
 &= \sum_{i=0}^{a-1} (-1)^a (-1)^i \partial_*^i k_*^x + (-1)^a ((-1)^a + (-1)^{a+1} + (-1)^{a+2}) \text{identity} \\
 &\quad + \sum_{i=a+3}^{n+1} (-1)^a (-1)^i \partial_*^i k_*^x \\
 (2) \quad &= \sum_{i=0}^{a-1} (-1)^a (-1)^i \partial_*^i k_*^x + \text{identity} + \sum_{i=a+3}^{n+1} (-1)^a (-1)^i \partial_*^i k_*^x.
 \end{aligned}$$

Looking instead at $k\partial$, note first that as x is in D_a , we have that $\partial^a = \partial^{a+1}$ when restricted to $c^{-1}(x)$, and so

$$(3) \quad k\partial = k \left(\sum_{i=0}^n (-1)^i \partial_*^i \right) = \sum_{i=0}^{a-1} (-1)^i k \partial_*^i + \sum_{i=a+2}^n (-1)^i k \partial_*^i.$$

Now, for $y \in \{0, \dots, d\}^{n+1}$, let $\partial^i y$ denote y with the i^{th} component removed. Then for $0 \leq i \leq a-1$, $\partial^i x \in D_{a-1}^{n-1}$, and so $k\partial_*^i = (-1)^{a-1} k_*^y \partial_*^i$. It is not difficult to prove that for such y , $k_*^y \partial_*^i = \partial_*^i k_*^x$. On the other hand, for $a+2 \leq i \leq n$, $\partial^i x$ is in D_a^{n-1} , and so $k\partial_*^i = (-1)^a k_*^y \partial_*^i$; one computes moreover that for such i , $k_*^y \partial_*^i = \partial_*^{i+1} k_*^x$. Putting this discussion together with the formula in line (3) gives that

$$\begin{aligned}
 k\partial &= \sum_{i=0}^{a-1} (-1)^i (-1)^{a-1} \partial_*^i k_*^x + \sum_{i=a+2}^n (-1)^i (-1)^a \partial_*^{i+1} k_*^x \\
 &= \sum_{i=0}^{a-1} (-1)^i (-1)^{a-1} \partial_*^i k_*^x + \sum_{i=a+3}^{n+1} (-1)^{i-1} (-1)^a \partial_*^i k_*^x.
 \end{aligned}$$

Comparing this with the formula in line (2), it follows that $k\partial + \partial k$ restricts to the identity on this summand $\mathbb{Z}[c^{-1}(x)]$. On the other hand, the same is true for the right hand side ‘identity $- j \circ q$ ’, so we are done. \square

Corollary 3.16. *The inclusion $j : \mathbb{Z}[\mathcal{N}_*^>] \rightarrow \mathbb{Z}[\mathcal{N}_*^{\geq}]$ is a chain homotopy equivalence.*

Proof. Lemma 3.15 implies that the natural projection q is an inverse on the level of chain homotopies. \square

Theorem 3.17. *Let \mathcal{N}_* be a nerve complex built from a colouring $\mathcal{C} = (G_0, \dots, G_d)$. Then the canonical inclusion $\mathcal{N}_*^> \rightarrow \mathcal{N}_*$ induces an isomorphism $H_*(\mathcal{C}) \cong H(\mathcal{N}_*^>)$. In particular, $H_n(\mathcal{C}) = 0$ for $n > d$.*

Proof. Corollaries 3.14 and 3.16 imply that the inclusion $\mathbb{Z}[\mathcal{N}_*^>] \rightarrow \mathbb{Z}[\mathcal{N}_*]$ is a chain homotopy equivalence. Functoriality of taking coinvariants then implies that the naturally induced map $\mathbb{Z}[\mathcal{N}_*^>]_G \rightarrow \mathbb{Z}[\mathcal{N}_*]_G$ is a chain homotopy equivalence, so in particular induces an isomorphism on homology. The remaining statement follows as $\mathcal{N}_n^>$ is empty for $n > d$. \square

3.3. Maps between nerves and G . In this technical section, we build maps between our nerve spaces \mathcal{N}_0 and G with good properties.

Definition 3.18. Let G_0, \dots, G_d be a colouring of G as in Definition 3.1, and let K be a subset of G . We say that the colouring G_0, \dots, G_d is:

- (i) *K-bounded* if every G_i is contained in K ;

- (ii) *K-Lebesgue* if for every $x \in G^0$, there exists G_i such that $G^x \cap K$ is contained in G_i .

For $n \geq 0$, recall that $EG_n = \{(g_0, \dots, g_n) \in G^{n+1} \mid r(g_0) = \dots = r(g_n)\}$, equipped with the subspace topology that it inherits from G^{n+1} . For $n \geq 1$ and any subset K of G , let EG_n^K denote the subspace of EG_n consisting of those tuples (g_0, \dots, g_n) such that $g_i^{-1}g_j \in K$ for all i, j . Let EG_0^K be just $EG_0 = G$, whatever K is.

Lemma 3.19. *Say K is a compact open subset of G , and that G_0, \dots, G_d is a K -Lebesgue colouring of G with associated nerve \mathcal{N}_* . Then there exists an equivariant étale map $\Phi_0 : G \rightarrow \mathcal{N}_0$ such that:*

- (i) $\Phi_0(g) \supseteq gK$ for all $g \in G$;
(ii) for all n , the map

$$\Phi_n : EG_n^K \rightarrow \mathcal{N}_n, \quad (g_0, \dots, g_n) \mapsto (\Phi_0(g_0), \dots, \Phi_0(g_n))$$

is a well-defined, equivariant local homeomorphism.

Proof. For each $i \in \{0, \dots, d\}$, define $V_i := \{x \in G^0 \mid G^x \cap K \subseteq G_i\}$. We claim that each V_i is open. Indeed, let $x \in V_i$. As the set $G^x \cap K$ is compact and discrete, it is finite. Write g_1, \dots, g_n for the elements of this set, which are all in G_i by assumption that x is in V_i . For each $j \in \{1, \dots, n\}$, let $W_j \subseteq G_i$ be a compact open bisection containing g_j . We may assume the W_j are disjoint by shrinking them if necessary. Define $W := \bigcap_{j=1}^n r(W_j)$, which is a compact open neighbourhood of x . We may write the compact open set $r^{-1}(W) \cap K$ as a finite disjoint union of compact open bisections of the form $W_1 \cap r^{-1}(W), \dots, W_n \cap r^{-1}(W), B_1, \dots, B_m$. Note that no B_j can intersect $G^x \cap K$, whence none of the sets $r(B_j)$ can contain x . Define

$$V := W \setminus \left(\bigcup_{j=1}^m B_j \right).$$

This is an open set containing x . Moreover, $r^{-1}(V)$ is contained in $W_1 \cup \dots \cup W_n$, and therefore in G_i . Hence V is an open neighborhood of x contained in V_i , so V_i is open as claimed.

Note now that V_0, \dots, V_d covers G^0 by the assumption that the underlying colouring is K -Lebesgue. As G^0 has a basis of compact open sets and is compact, there is a finite cover, say \mathcal{U} of G^0 consisting of disjoint compact open sets, and such that each $U \in \mathcal{U}$ is contained in some V_i . Define E_i to be the union of those $U \in \mathcal{U}$ such that i is the smallest element of $\{0, \dots, d\}$ with U contained in V_i . Then the sets E_0, \dots, E_d are a partition of G^0 by compact open subsets, and each E_i is contained in V_i .

Now, for each $g \in G$, let $i(g) \in \{0, \dots, d\}$ be the unique i such that $s(g)$ is in E_i ; as the partition $G^0 = \bigsqcup_{i=0}^d E_i$ is into clopen sets, the map $i : G \rightarrow \{0, \dots, d\}$ this defines is continuous. Define

$$\Phi_0 : G \rightarrow \mathcal{N}_0, \quad g \mapsto gG_{i(g)}^{s(g)}.$$

We claim this has the right properties. First, note that for $g \in G$, $G^{s(g)} \cap K \subseteq G_{i(g)}$ by definition of $i(g)$ and the cover V_0, \dots, V_d . Hence

$$gK = g(G^{s(g)} \cap K) \subseteq gG_{i(g)} = gG_{i(g)}^{s(g)} = \Phi_0(g).$$

We now show that Φ_0 is étale. Continuity of the restriction of Φ_0 to each set $s^{-1}(E_i)$ follows from the definition of the topology of \mathcal{N}_0 , and continuity of Φ_0 on all of G follows from this as the sets $s^{-1}(E_0), \dots, s^{-1}(E_d)$ are a closed partition of G . Let now $g \in G$, and let B be a clopen bisection containing g such that the map $i : G \rightarrow \{0, \dots, d\}$ is constant on B (such exists as i is continuous). Let $C = \Phi_0(B)$. Then the map $C \rightarrow B$ defined by sending U to the unique element of $B \cap r(U)$ is

a well-defined continuous inverse to the restriction $\Phi_0|_B$, completing the proof that Φ_0 is étale.

Equivariance of Φ_0 follows as if $s(g) = r(h)$, then $s(gh) = s(h)$ and $i(gh) = i(h)$, whence

$$\Phi_0(gh) = ghG_{i(gh)}^{s(gh)} = ghG_{i(h)}^{s(h)} = g\Phi_0(h).$$

To see that Φ_n is well-defined, note that if $(g_0, \dots, g_n) \in EG_n^K$, then $g_i^{-1}g_j$ is in K for all i, j . Hence in particular g_0 is in $g_iK \subseteq \Phi_0(g_i)$ for each i , and so g_0 is in $\Phi_0(g_0) \cap \dots \cap \Phi_0(g_n)$, and so this set is non-empty. Hence $(\Phi_0(g_0), \dots, \Phi_0(g_n))$ is a well-defined element of \mathcal{N}_n . Equivariance of Φ_n and the fact that it is étale are straightforward from the corresponding properties for Φ_0 , so we are done. \square

Lemma 3.20. *Assume that G is principal, and K is a compact open subset of G that contains G^0 . Let G_0, \dots, G_d be a K -bounded colouring with associated nerve \mathcal{N}_* . Then there exists an equivariant étale map $\Psi_0 : \mathcal{N}_0 \rightarrow G$ with the following properties:*

- (i) $\Psi_0(U) \in U$ for all $U \in \mathcal{N}_0$;
- (ii) for all n , the map

$$\Psi_n : \mathcal{N}_n \rightarrow EG_n^{KK^{-1}}, \quad (U_0, \dots, U_n) \mapsto (\Psi_0(U_0), \dots, \Psi_0(U_n))$$

is a well-defined, equivariant local homeomorphism.

To prove this, we need an ancillary lemma, which is based on the following structural result from [16, Lemma 3.4].

Lemma 3.21. *Say H is a compact, ample, principal groupoid. Then there are $m \in \mathbb{N}$ and*

- (i) *disjoint clopen subgroupoids H_1, \dots, H_m of H ,*
- (ii) *clopen subsets X_1, \dots, X_m of H^0 (equipped with the induced, i.e. trivial, groupoid structure), and*
- (iii) *finite pair groupoids P_1, \dots, P_m ,*

such that H identifies with the disjoint union $H = \bigsqcup_{i=1}^m H_i$ as a topological groupoid, and such that each H_i is isomorphic as a topological groupoid to $X_i \times P_i$. \square

Corollary 3.22. *Say H is a compact, ample, principal groupoid, and that H^0/H is the quotient space of H^0 by the equivalence relation induced by H : precisely $x \sim y$ if there is $h \in H$ with $s(h) = x$ and $r(h) = y$. Then H^0/H is Hausdorff and there are étale maps $\sigma : H^0/H \rightarrow H^0$ and $\tau : H^0 \rightarrow H$ such that:*

- (i) σ splits the quotient map $\pi : H^0 \rightarrow H^0/H$ (so in particular, π is étale);
- (ii) $r \circ \tau = \text{identity}$ and $s \circ \tau = \sigma \circ \pi$.

Proof. Assume first that $H = X \times P$, where X is a compact trivial groupoid and P is the pair groupoid on some finite set $\{0, \dots, n\}$. Then H^0/H identifies homeomorphically with X (so in particular is Hausdorff) via the map

$$H^0 \rightarrow X, \quad (x, (i, i)) \mapsto x.$$

Making this identification, we may define $\sigma(x) = (x, (0, 0))$ and $\tau(x, (i, i)) = (x, (i, 0))$. These maps have the right properties when $H = X \times P$.

In the general case, Lemma 3.21 gives a decomposition of H into groupoids of the form $X \times P$ as above, and we may build σ and τ on each separately using the method above. \square

Proof of Lemma 3.20. Let π_i, σ_i and τ_i be as in Corollary 3.22 for $H = G_i$. Define

$$\Psi_0 : \mathcal{N}_0 \rightarrow G, \quad gG_i^{s(g)} \mapsto g\tau_i(s(g)).$$

We first check that this is well-defined. Indeed, if $hG_i^{s(h)} = gG_i^{s(g)}$, then $r(h) = r(g)$ and $h^{-1}g \in G_i$. Hence

$$\pi_i(s(g)) = \pi_i(s(h^{-1}g)) = \pi_i(r(h^{-1}g)) = \pi_i(s(h)),$$

and so $\sigma_i(\pi_i(s(g)))$ and $\sigma_i(\pi_i(s(h)))$ are the same. As $\tau_i(x)$ has source $\sigma_i(\pi_i(x))$ for all $x \in G_i^{(0)}$, this implies that both $\tau_i(s(g))$ and $\tau_i(s(h))$ have the same source. As moreover g and h have the same range, the elements $g\tau_i(s(g))$ and $h\tau_i(s(h))$ of G have the same source and range and are therefore the same as G is principal. Having seen that Ψ_0 is well-defined, equivariance of Ψ_0 is straightforward. The fact that $\Psi_0(U) \in U$ for all $U \in \mathcal{N}_0$ follows as if we write $U = gG_i^{s(g)}$, then $\Psi_0(U) = g\tau_i(s(g))$, and $g\tau_i(s(g))$ is in $G_i^{s(g)}$, as $\tau_i(s(g))$ is in $G_i^{s(g)}$.

To see that Ψ_0 is étale, let $gG_i^{s(g)}$ be an element of \mathcal{N}_0 , and let B be a clopen bisection of g in G such that the set $\{hG_i^{s(h)} \mid h \in B\}$ is a clopen neighbourhood of $gG_i^{s(g)}$ in \mathcal{N}_0 ; in particular, this implies that $s(h) \in G_i^0$ for all $h \in B$. Using that both s and τ_i are étale, we have that $s(\tau_i(B))$ is open, and therefore that $\Psi_0(B)$ is open. We claim that the map

$$\kappa : \Psi_0(B) \rightarrow \mathcal{N}_0, \quad h \mapsto hG_i^{s(h)}$$

is a local inverse to Ψ_0 ; as it is continuous, this will suffice to complete the proof. Indeed, for any $h \in B$,

$$\kappa(\Psi_0(hG_i^{s(h)})) = h\tau_i(s(h))G_i^{s(\tau_i(s(h)))} = hG_i^{s(h)}.$$

On the other hand, for $h \in \Psi_0(B)$, as h is in the image of $\Psi_0(B)$, we have that $s(h)$ is in the image of σ_i , and therefore that $\sigma_i(\pi_i(s(h))) = s(h)$, and so $\tau_i(s(h)) = s(h)$. Hence

$$\Psi_0(\kappa(h)) = h\tau_i(s(h)) = h$$

and we are done with showing that Ψ_0 is étale.

To see that Ψ_n is well-defined, we need to check that if (U_0, \dots, U_n) is in \mathcal{N}_n , then $(\Psi_0(U_0), \dots, \Psi_0(U_n))$ is in $EG_n^{KK^{-1}}$. Write $g_j := \Psi_0(U_j)$ for notational simplicity, so g_j is in U_j by the properties of Ψ_0 . Let h be an element of $U_0 \cap \dots \cap U_n$. Then for all j , the fact that the colouring is K -bounded implies that $g_j^{-1}h$ is in K for each j . Hence for any i, j , $g_i^{-1}g_j = g_i^{-1}hg_j^{-1}h \in KK^{-1}$, completing the proof that Ψ_n is well-defined. The facts that Ψ_n is étale and equivariant follow from the corresponding properties for Ψ_0 , so we are done. \square

3.4. Anti-Čech homology. In this section, we build a model for the Crainic-Moerdijk-Matui homology groups $H_*(G)$ using colourings, at least in the case that G is principal and σ -compact.

The key definition is as follows.

Definition 3.23. An *anti-Čech sequence* for G consists of the following data:

- (i) a sequence $\mathcal{C}_m := (G_0^{(m)}, \dots, G_{d_m}^{(m)})_{m=0}^\infty$ of colourings of G with associated sequence of nerves $\mathcal{N}_*^{(m)}$;
- (ii) for each m a morphism $\iota^{(m)} : \mathcal{N}_*^{(m-1)} \rightarrow \mathcal{N}_*^{(m)}$ of semi-simplicial G -spaces such that for all $U \in \mathcal{N}_0^{(m-1)}$ we have that $\iota^{(m)}(U) \supseteq U$, and moreover so that for any compact open subset K of G , there exists m_K such that for all $m \geq m_K$ and all $U \in \mathcal{N}_0^{(m-1)}$, we have that $\iota^{(m)}(U) \supseteq UK$.

Definition 3.24. Let $\mathcal{A} = (\mathcal{C}_m)_{m=1}^\infty$ be an anti-Čech sequence for G , with associated sequence of morphisms $\iota^{(m)} : \mathcal{N}_*^{(m-1)} \rightarrow \mathcal{N}_*^{(m)}$. We define the *homology* of \mathcal{A} , denoted $H_*(\mathcal{A})$, to be the corresponding direct limit of the sequence of maps $(\iota_*^{(m)} : H_*(\mathcal{C}_{m-1}) \rightarrow H_*(\mathcal{C}_m))_{m=1}^\infty$.

Anti-Čech sequences always exist under the assumptions that G is principal and σ -compact. This follows from the next three lemmas.

Lemma 3.25. *Say that G is principal, and that K is a compact open subset of G that contains G^0 and that satisfies $K = K^{-1}$. Say G_0, \dots, G_d and H_0, \dots, H_e are colourings of G with associated nerves \mathcal{N}_* and \mathcal{M}_* respectively. Assume moreover that the colouring G_0, \dots, G_d is K -bounded, and that the colouring H_0, \dots, H_e is K^3 -Lebesgue.*

Then there exists an equivariant étale map $\iota_0 : \mathcal{N}_0 \rightarrow \mathcal{M}_0$ with the following properties:

- (i) $\iota_0(U) \supseteq UK$ for all $U \in \mathcal{N}_0$;
- (ii) for all n , the map

$$\iota_n : \mathcal{N}_n \rightarrow \mathcal{M}_n, \quad (U_0, \dots, U_n) \mapsto (\iota_0(U_0), \dots, \iota_0(U_n))$$

is, well-defined, equivariant, and étale.

Proof. Using Lemma 3.20, there is an equivariant étale map $\Psi_0 : \mathcal{N}_0 \rightarrow G$ such that $\Psi_0(U) \in U$ for all $U \in \mathcal{N}_0$, and such that for all n , the prescription

$$\Psi_n : \mathcal{N}_n \rightarrow EG_{n+1}^{K^3}, \quad (U_0, \dots, U_n) \mapsto (\Psi_0(U_0), \dots, \Psi_0(U_n))$$

gives a well-defined equivariant étale map (Lemma 3.20 has KK^{-1} in place of K^3 , but note that our assumptions imply that K^3 contains KK^{-1}). Using Lemma 3.19, there is an equivariant local homeomorphism $\Phi_0 : G \rightarrow \mathcal{M}_0$ such that $gK^3 \subseteq \Phi_0(g)$ for all $g \in G$, and so that the prescription

$$\Phi_n : EG_n^{K^3} \rightarrow \mathcal{N}_n, \quad (g_0, \dots, g_n) \mapsto (\Phi_0(g_0), \dots, \Phi_0(g_n))$$

is a well-defined equivariant local homeomorphism.

Define then $\iota_0 := \Phi_0 \circ \Psi_0$, and note that $\iota_n := \Phi_n \circ \Psi_n$ for all n . Each ι_n is then a well-defined equivariant étale map. Moreover, fix $U \in \mathcal{N}_0$ and write $U = gG_i^{s(g)}$. As $\Psi_0(U) \in U$, we may write $\Psi_0(U) = gh$ for some $h \in G_i^{s(g)}$. Then h^{-1} is in G_i , and so in K as each G_i is a subset of K . Hence for an arbitrary element gk of $gG_i^{s(g)}$ with $k \in G_i \subseteq K^{-1}$, we have that $gk = gh h^{-1}k \in \Psi_0(U)K^2$. As gk was an arbitrary element of U , this gives that $U \subseteq \Psi_0(U)K^2$, and so $UK \subseteq \Psi_0(U)K^3$. Using the properties of Ψ_0 and Φ_0 , we thus get that

$$UK \subseteq \Psi_0(U)K^3 \subseteq \Phi_0(\Psi_0(U)) = \iota_0(U)$$

and are done. \square

Lemma 3.26. *Say G is principal and K is a compact open subset of G that contains G^0 . Then for any $x \in G^0$ there is a compact open subset B_0 of G^0 containing x and open bisections B_1, \dots, B_n such that:*

- (i) $r^{-1}(B_0) \cap K = \bigsqcup_{i=0}^n B_i$;
- (ii) for each i , $r|_{B_i} : B_i \rightarrow B_0$ is a homeomorphism;
- (iii) for each $i \neq j$, $s(B_i) \cap s(B_j) = \emptyset$.

Proof. Say the elements of $r^{-1}(x) \cap K$ are $g_0 = x, g_1, \dots, g_n$. As G is principal, we have $s(g_i) \neq s(g_j)$ for all $i \neq j$. Hence for each g_i , we may choose a clopen bisection D_i containing g_i such that $s(D_i) \cap s(D_j) = \emptyset$ for $i \neq j$, so that $r|_{D_i}$ is a homeomorphism, and so that D_0 is contained in G^0 .

Set $C_0 := \bigcap_{i=0}^n r(B_i)$, which is a clopen set containing x , and for each i , set $C_i := B_i \cap r^{-1}(C_0)$, which is a clopen set containing g_i . The set $r^{-1}(C_0) \cap K$ is compact. We may thus write it as

$$\bigsqcup_{i=0}^n C_i \sqcup \bigsqcup_{j=0}^m E_j,$$

where each E_j is a clopen bisection such that $r|_{E_j}$ is a homeomorphism, and so that each E_j does not intersect $r^{-1}(x_0)$. Set

$$B_0 := C_0 \setminus \bigcup_{j=0}^m r(E_j),$$

and set $B_i := r^{-1}(B_0) \cap C_i$. These sets have the right properties. \square

Lemma 3.27. *Let G be principal, and say that K is a compact open subset of G that contains G^0 . Then there exists a K -Lebesgue colouring G_0, \dots, G_d for G .*

Proof. Fix $x \in X$, and let B_0, \dots, B_n be a collection of sets with the properties in Lemma 3.26. For each i , let $\rho_i : B_0 \rightarrow B_i$ be the inverse of $r|_{B_i}$. Let $P = \{0, \dots, n\}^2$ be the pair groupoid on the set $\{0, \dots, n\}$, and define

$$f : B_0 \times P \rightarrow G, \quad (x, (i, j)) \mapsto \rho_i(x)\rho_j(x)^{-1}.$$

It is not difficult to check that f is a homeomorphism onto its image, which is a compact open subgroupoid of G . Write G_x for the image of f . Moreover, by construction we have that for every $y \in G_x^0$, the set $r^{-1}(y) \cap K$ is contained in G_x .

The collection $\{G_x^0 \mid x \in G^0\}$ is an open cover of G^0 , and thus has a finite subcover. Let G_0, \dots, G_d be the collection of compact open subgroupoids of G whose base spaces appear in this subcover. This collection has the right properties. \square

Corollary 3.28. *For any σ -compact principal G with compact base space, an anti-Čech sequence always exists.*

Proof. As G is σ -compact, there is a sequence $L_0 \subseteq L_1 \subseteq \dots$ of compact open subsets of G such that each L_n equals L_n^{-1} and contains G^0 , and such that any compact subset of G is eventually contained in all of the L_n . Set $K_0 = L_0$. Lemma 3.27 implies that there is a K_0^3 -Lebesgue collection of compact open subgroupoids of G , say G_0, \dots, G_d , with associated nerve space $\mathcal{N}_*^{(0)}$. As this collection is finite, there exists some compact open subset M_0 of G that contains all of G_0, \dots, G_d , and that satisfies $M_0 = M_0^{-1}$. Set $K_1 := K_0 \cup L_1 \cup M_0$. Lemma 3.27 gives a new colouring that is K_1^3 -Lebesgue with associated nerve $\mathcal{N}_*^{(1)}$, and Lemma 3.25 gives a morphism of semi-simplicial étale G -spaces $\iota^{(1)} : \mathcal{N}_*^{(0)} \rightarrow \mathcal{N}_*^{(1)}$ with the properties there. Now let M_1 be a compact open subset of G such that $M_1 = M_1^{-1}$, and that contains all the groupoids from this new colouring. Set $K_2 := K_1 \cup L_2 \cup M_1$ and use Lemma 3.27 to build a K_2^3 -Lebesgue covering, and Lemma 3.25 to build a map $\iota^{(2)}$ from $\mathcal{N}_*^{(1)}$ to the associated nerve $\mathcal{N}_*^{(2)}$ with the properties in that lemma. Iterating this process builds an anti-Čech sequence as desired. \square

Our main goal in this section is to prove the following theorem.

Theorem 3.29. *Let G be principal and σ -compact. Let \mathcal{A} be an anti-Čech sequence for G . Then the homology groups $H_*(\mathcal{A})$ and $H_*(G)$ are isomorphic.*

The proof will proceed by some lemmas. First, we need a definition.

Definition 3.30. Let C_* be a semi-simplicial G -space and D_* be either the nerve of a colouring \mathcal{N}_* , or one of the EG_*^L for some compact open subset L of G . Let $\alpha, \beta : C_* \rightarrow D_*$ be two morphisms of semi-simplicial G -spaces. Then α, β are *close* if there exists a compact open subset K of G , either:

- (i) D_* is a nerve \mathcal{N}_* , and for all $x \in C_0$ there exists $g \in G$ such that $\alpha(x)$ and $\beta(x)$ are both subsets of gK ;
- (ii) D_* is of the form EG_*^L , and for all $x \in C_0$, $\alpha(x)^{-1}\beta(x)$ is in K .

Lemma 3.31. *Say $(\mathcal{N}_*^{(m)}, \iota^{(m)})$ is an anti-Čech sequence, and let $\alpha, \beta : C_* \rightarrow \mathcal{N}_*^{(m)}$ be close morphisms for some m . Let*

$$\iota^{(\infty)} : H_*(\mathcal{N}^{(m)}) \rightarrow \varinjlim H_*(\mathcal{N}^{(l)})$$

be the natural map to the direct limit, i.e. to the homology of the anti-Čech sequence. Then the compositions

$$\iota^{(\infty)} \circ \alpha_* : H_*(C) \rightarrow \varinjlim H_*(\mathcal{N}^{(l)}) \quad \text{and} \quad \iota^{(\infty)} \circ \beta_* : H_*(C) \rightarrow \varinjlim H_*(\mathcal{N}^{(l)})$$

are the same.

Proof. Let K be as in the definition of closeness for α and β , and assume also that K is so large that the colouring underlying $\mathcal{N}^{(m)}$ is K -bounded. Let $l \geq m$ be large enough so that if $\iota : \mathcal{N}^{(m)} \rightarrow \mathcal{N}^{(l)}$ is the composition of the morphisms in the definition of the anti-Čech sequence, then for all $U \in \mathcal{N}^{(m)}$, we have that $\iota(U) \supseteq UK^2$ (such an l exists by definition of an anti-Čech sequence). It will suffice to show that $\iota \circ \alpha$ and $\iota \circ \beta$ induce the same map $H_*(C) \rightarrow H_*(\mathcal{N}^{(l)})$.

Let now x be a point in C_n for some n . For each $j \in \{0, \dots, n\}$ let $\pi_j : C_n \rightarrow C_0$ be the map corresponding under the semi-simplicial structure to the map $\{0\} \rightarrow \{0, \dots, n\}$ that sends 0 to j (see Section 2.2 for notation). Define $x_j := \pi_j(x)$. We claim that the intersection

$$(4) \quad \bigcap_{j=0}^n \iota(\alpha(x_j)) \cap \bigcap_{j=0}^n \iota(\beta(x_j))$$

is non-empty. Indeed, $(\alpha(x_0), \dots, \alpha(x_n))$ is a point of $\mathcal{N}_n^{(l)}$, whence there is some g_α in the intersection $\bigcap_{j=0}^n \alpha(x_j)$, and similarly for g_β with β replacing α . As α and β are close with respect to K , we have that $\alpha(x_0)$ and $\beta(x_0)$ are both subsets of gK for some $g \in G$, whence there are k_α and k_β in K such that $g_\alpha = gk_\alpha$ and $g_\beta = gk_\beta$. Hence $g_\alpha = g_\beta k_\beta k_\alpha^{-1}$, so in particular g_α is in $g_\beta K^2$. Now, by choice of ι , $\iota(\beta(x_j)) \supseteq \beta(x_j)K^2$ for all j , whence g_α is in $\iota(\beta(x_j))$ for all j . Moreover, $g_\alpha \in \alpha(x_j) \subseteq \iota(\alpha(x_j))$ for all j , so g_α is a point in the claimed intersection.

For each n and each $i \in \{0, \dots, n\}$, we define a map

$$h^i : C_n \rightarrow \mathcal{N}_{n+1}^{(l)}$$

by the formula

$$x \mapsto (\iota \circ \alpha(\pi_0(x)), \dots, \iota \circ \alpha(\pi_i(x)), \iota \circ \beta(\pi_i(x)), \dots, \iota \circ \beta(\pi_n(x))),$$

which is well-defined by the claim. It is moreover an equivariant local homeomorphism as ι , α and β all have these properties. Hence h^i induces a map $h_*^i : \mathbb{Z}[C_n] \rightarrow \mathbb{Z}[\mathcal{N}_{n+1}^{(l)}]$ in the usual way. We define

$$h := \sum_{i=0}^n (-1)^i h_*^i : \mathbb{Z}[C_n] \rightarrow \mathbb{Z}[\mathcal{N}_{n+1}^{(l)}].$$

Direct checks show that h (and the map induced on coinvariants by h) is a chain homotopy between the maps induced by $\iota \circ \alpha$ and $\iota \circ \beta$. Hence $\iota \circ \alpha$ and $\iota \circ \beta$ indeed induce the same map on homology as claimed. \square

For the next lemma, let $K_0 \subseteq K_1 \subseteq K_2 \dots$ be a sequence of compact open subsets of G , all of which contain G^0 , and whose union is G . We then get a sequence $(EG_*^{K_m})_{m=0}^\infty$ of spaces. Note moreover that the corresponding limit $\varinjlim H_*(G^{K_n})$ canonically identifies with $H_*(G)$: indeed, this follows directly from the observation that for each n , EG_n is the increasing union of the $EG_n^{K_m}$, and the fact that taking homology groups commutes with direct limits.

Lemma 3.32. Say $(G_*^{K_m})_{m=0}^\infty$ is a sequence of spaces associated to a nested sequence of compact open subsets of G as above. Let $\alpha, \beta : C_* \rightarrow G_*^{(K_m)}$ be close morphisms for some m . Let

$$\kappa : H_*(G^{K_m}) \rightarrow \varinjlim H_*(G^{K_m})$$

be the natural map to the direct limit, i.e. to the homology $H_*(G)$ of G . Then the compositions

$$\kappa \circ \alpha_* : H_*(C) \rightarrow H_*(G) \quad \text{and} \quad \kappa \circ \beta_* : H_*(C) \rightarrow H_*(G)$$

are the same.

Proof. The proof is very similar to that of Lemma 3.31. We leave the details to the reader. \square

Proof of Theorem 3.29. Let \mathcal{A} be the given anti-Čech sequence with associated nerves and morphisms $\iota_*^{(m)} : \mathcal{N}_*^{(m-1)} \rightarrow \mathcal{N}_*^{(m)}$.

Let $m_1 = 1$. Then the colouring underlying $\mathcal{N}_*^{(1)}$ is K -bounded for some compact open subset K of G , which we may assume contains G^0 , and that satisfies $K = K^{-1}$. Set $K_1 := K^2$. Then Lemma 3.20 gives a morphism $\Psi^{(1)} : \mathcal{N}_*^{(m_1)} \rightarrow G_*^{K_1}$. On the other hand, by definition of an anti-Čech sequence there is $m_2 > m_1$ such that $\mathcal{N}^{(m_2)}$ is K_1 -Lebesgue, whence Lemma 3.19 gives a morphism $\Phi^{(1)} : G_*^{K_1} \rightarrow \mathcal{N}_*^{(m_2)}$. Continuing, the colouring underlying $\mathcal{N}^{(m_2)}$ is K -bounded for some compact open subset K of G that we may assume contains K_1 and satisfies $K = K^{-1}$. Set $K_2 := K^2$, so Lemma 3.20 gives a morphism $\Phi^{(2)} : \mathcal{N}_*^{(m_2)} \rightarrow G_*^{K_2}$.

Continuing in this way, we get sequences $1 = m_1 < m_2 < m_3 < \dots$ of natural numbers and $K_1 \subseteq K_2 \subseteq \dots$ of compact open subsets of G together with morphisms

$$\begin{array}{ccccccc} \mathcal{N}_*^{(m_1)} & & \mathcal{N}_*^{(m_2)} & & \mathcal{N}_*^{(m_3)} & & \dots \\ & \searrow \Psi^{(1)} & \nearrow \Phi^{(1)} & \searrow \Psi^{(2)} & \nearrow \Phi^{(2)} & \searrow \Psi^{(3)} & \\ & EG_*^{K_1} & & EG_*^{K_2} & & & \dots \end{array}$$

We may fill in horizontal arrows in the diagram: on the top row, these should be appropriate compositions of the morphisms $\iota_*^{(m)}$ coming from the definition of an anti-Čech sequence, while on the bottom row they should be induced by the canonical inclusions $EG_n^{K_k} \rightarrow EG_n^{K_{k+1}}$ coming from the fact that $K_k \subseteq K_{k+1}$ for all k . We thus get a (non-commutative!) diagram

$$(5) \quad \begin{array}{ccccccc} \mathcal{N}_*^{(m_1)} & \xrightarrow{\quad} & \mathcal{N}_*^{(m_2)} & \xrightarrow{\quad} & \mathcal{N}_*^{(m_3)} & \xrightarrow{\quad} & \dots \\ & \searrow \Psi^{(1)} & \nearrow \Phi^{(1)} & \searrow \Psi^{(2)} & \nearrow \Phi^{(2)} & \searrow \Psi^{(3)} & \\ & EG_*^{K_1} & \xrightarrow{\quad} & EG_*^{K_2} & \xrightarrow{\quad} & & \dots \end{array}$$

Notice that the limit of the horizontal maps in the top row is $H_*(\mathcal{A})$. Moreover, the definition of an anti-Čech sequence and the construction of the sequence (K_k) forces $G = \bigcup K_k$, so the limit of the horizontal maps on the bottom row is $H_*(G)$.

Now, consider the compositions

$$(6) \quad H_*(EG^{K_k}) \xrightarrow{\Phi_*^{(k)}} H_*(\mathcal{N}^{(m_k)}) \longrightarrow H_*(\mathcal{A}).$$

where the second arrow is the canonical one that exists by definition of the direct limit. Any two morphisms into any $\mathcal{N}^{(m_k)}$ are close using that all the colourings

are bounded. It follows therefore from Lemma 3.31 that for any k , the diagram

$$\begin{array}{ccc} H_*(\mathcal{A}) & \xlongequal{\quad} & H_*(\mathcal{A}) \\ \uparrow & & \uparrow \\ H_*(EG_*^{K_k}) & \longrightarrow & H_*(EG_*^{K_{k+1}}) \end{array}$$

commutes; here the vertical maps are the ones in line (6), and the bottom horizontal line is induced by the canonical inclusion $EG_*^{K_k} \rightarrow EG_*^{K_{k+1}}$. Taking the limit in k of the maps in line (6), we thus get a well-defined homomorphism

$$\Phi : H_*(G) \rightarrow H_*(\mathcal{A}).$$

Precisely analogously, using Lemma 3.32 in place of Lemma 3.31, we get a homomorphism

$$\Psi : H_*(\mathcal{A}) \rightarrow H_*(G).$$

We claim that Φ and Ψ are mutually inverse, which will complete the proof. Indeed, for any k , the triangles

$$\begin{array}{ccc} \mathcal{N}_*^{(m_k)} & \xrightarrow{\quad} & \mathcal{N}_*^{(m_{k+1})} \\ & \searrow \Psi^{(k)} & \nearrow \Phi^{(k)} \\ & EG_*^{K_k} & \end{array}$$

and

$$\begin{array}{ccc} & \mathcal{N}_*^{(m_{k+1})} & \\ & \nearrow \Phi^{(k)} & \searrow \Psi^{(k+1)} \\ EG_*^{K_k} & \xrightarrow{\quad} & EG_*^{K_{k+1}} \end{array}$$

appearing in line (5) commute up to closeness, whence the claim follows directly from Lemmas 3.31 and 3.32, so we are done. \square

3.5. Dynamic asymptotic dimension. In this section, we prove that the homology of an ample σ -compact principal groupoid with compact base space vanishes above its dynamic asymptotic dimension, and also that the top-dimensional homology group is free.

The following definition is [21, Definition 5.1].

Definition 3.33. Let $d \in \mathbb{N}$. A (locally compact, Hausdorff, étale) groupoid has *dynamic asymptotic dimension at most d* if for any relatively compact subset K of G there are open subsets U_0, \dots, U_d of G^0 that cover $r(K) \cup s(K)$ and such that for each i , the set $\{g \in K \mid s(g), r(g) \in U_i\}$ is contained in a relatively compact subgroupoid of G .

We record the some basic facts about products of subsets of a groupoid. See for example [21, Lemma 5.2] for a proof.

Lemma 3.34. Say G is a groupoid, and H and K are subsets of G . Then if H and K are open (respectively, compact, or relatively compact), the product HK is open (respectively, compact, or relatively compact).

Moreover, if K is open then the subgroupoid generated by K is also open. \square

Here is our key use of dynamic asymptotic dimension.

Lemma 3.35. Say G is an ample groupoid with compact unit space which has dynamic asymptotic dimension at most d . Then for any compact open subset $K \subseteq G$ there exists a K -Lebesgue colouring of G with at most $d + 1$ elements.

Proof. Say G has dynamic asymptotic dimension at most d , and let $K \subseteq G$ be compact and open. As making K larger only makes the problem more difficult, we may assume that $K = K^{-1}$, and that K contains G^0 . Note that K^3 is still compact and open by Lemma 3.34. The definition of dynamic asymptotic dimension at most d therefore gives an open cover U_0, \dots, U_d of G^0 such that for each i , the subgroupoid generated by $\{g \in K^3 \mid r(g), s(g) \in U_i\}$ is relatively compact. As G^0 is compact and as its topology has a basis of compact open sets, there is a cover of G^0 consisting of compact open sets V_0, \dots, V_d such that each V_i is contained in U_i . For each i , let $W_i = s(r^{-1}(V_i) \cap K)$. Note that $r^{-1}(V_i) \cap K$ is compact and open, whence W_i is also compact and open as s is an open continuous map. Moreover, W_i contains V_i as K contains G^0 . Let now G_i be the subgroupoid of G generated by $\{g \in K \mid r(g), s(g) \in W_i\}$. We claim that G_0, \dots, G_d is a K -Lebesgue colouring of G .

Indeed, each G_i is open by Lemma 3.34, as it is generated by an open set. For compactness, we first claim that G_i has compact closure. Indeed, say $k_1 \dots k_n$ is an element of G_i , where each k_j is in $\{g \in K \mid s(g), r(g) \in W_i\}$. Then by definition of W_i , for each $j \in \{1, \dots, n\}$ there is an element h_j of K such that $h_{j-1}k_jh_j^{-1}$ has range and source in U_i . Hence for each $j \in \{1, \dots, n\}$, $h_{j-1}k_jh_j^{-1}$ is in $\{g \in K^3 \mid r(g), s(g) \in U_i\}$. Let H_i be the subgroupoid of G generated by $\{g \in K^3 \mid r(g), s(g) \in U_i\}$, so H_i has compact closure by choice of the cover U_0, \dots, U_d . Moreover,

$$k_1 \dots k_n = h_0^{-1} \left(\prod_{j=1}^n h_{j-1}k_jh_j^{-1} \right) h_n \in KH_iK.$$

We now have that G_i is contained in KH_iK . However, KH_iK is relatively compact by Lemma 3.34, so we see that G_i is also relatively compact.

We next claim that there is $N \in \mathbb{N}$ such that any element of G_i can be written as a product of at most N elements of $\{g \in K \mid s(g), r(g) \in W_i\}$. Indeed, from [21, Lemma 8.10] there exists $N \in \mathbb{N}$ such that each range fibre G_i^x has at most N elements. Now, let $g = k_1 \dots k_n$ be an element of G , where each k_j is in K . Consider the ‘path’ $k_1, k_1k_2, \dots, k_1 \dots k_n$. If this path contains any repetitions, we may shorten it by just omitting all the elements in between. Hence the length n of this path can be assumed to be at most the number of elements of $G^{r(g)}$, which is N as claimed.

To complete the proof that G_i is compact, it now suffices to show that it is closed. For this, let (g_j) be a net of elements of G_i . Using the previous claim, we may write $g_j = k_1^j \dots k_N^j$, where each k_i^j is in $\{g \in K \mid r(g), s(g) \in W_i\}$ (possibly identity elements). Note that the latter set is compact, whence up to passing to subnets, we may assume that each net (k_i^j) converges to some $k_i \in \{g \in K \mid r(g), s(g) \in W_i\}$. Hence $g = k_1 \dots k_N$ is in G_i , so G_i is indeed compact.

Finally, we check that the colouring is K -Lebesgue. Note that for any $x \in G^0$, x is contained in V_i for some i , whence $s(r^{-1}(x) \cap K)$ is contained in W_i by definition of W_i . Hence G_i contains $r^{-1}(x) \cap K$, giving the K -Lebesgue condition. \square

Theorem 3.36. *Let G be a σ -compact ample groupoid. If G is principal and has dynamic asymptotic dimension at most d , then $H_n(G) = 0$ for $n > d$, and $H_d(G)$ is free.*

Proof. We first prove the result under the additional assumption that G^0 is compact. In that case Lemma 3.35 lets us build an anti-Čech sequence \mathcal{A} for G consisting of colourings $(\mathcal{C}^{(n)})$ each of which only has $d+1$ colours. As $H_*(\mathcal{A}) = \lim_{n \rightarrow \infty} H_*(\mathcal{C}^{(n)})$ and as Theorem 3.17 implies that $H_n(\mathcal{C}^{(m)}) = 0$ for all m and all $n > d$, we see that $H_n(\mathcal{A}) = 0$ for all $n > d$. Theorem 3.29 now gives the vanishing result.

For the freeness result, keep notation as above, and let us write $\mathcal{N}_*^{(n)}$ for the nerve space associated to $\mathcal{C}^{(n)}$. Then we have

$$\begin{aligned} H_d(\mathcal{A}) &= \lim_{n \rightarrow \infty} H_d(\mathcal{C}^{(n)}) = \lim_{n \rightarrow \infty} H_d(\mathbb{Z}[\mathcal{N}_*^{(n)}]_G) = \lim_{n \rightarrow \infty} H_d(\mathbb{Z}[(\mathcal{N}_*^{(n)})^>]_G) \\ &= H_d(\lim_{n \rightarrow \infty} \mathbb{Z}[(\mathcal{N}_*^{(n)})^>]_G), \end{aligned}$$

where the first equality is just by definition of $H_d(\mathcal{A})$, the second is by definition of $H_d(\mathcal{C}^{(n)})$, the third is Theorem 3.17, and the fourth follows as taking homology commutes with direct limits. Now, using Lemma 2.5, $\mathbb{Z}[(\mathcal{N}_d^{(n)})^>]_G$ naturally identifies with $\mathbb{Z}[G \setminus (\mathcal{N}_d^{(n)})^>]$. Each $\mathbb{Z}[G \setminus (\mathcal{N}_d^{(n)})^>]$ can be regarded as a commutative ring (with pointwise multiplication) that is generated by idempotents, and the maps $\mathbb{Z}[G \setminus (\mathcal{N}_d^{(n)})^>] \rightarrow \mathbb{Z}[G \setminus (\mathcal{N}_d^{(n+1)})^>]$ are spatially implemented and thus ring homomorphisms. Hence the limit $\lim_{n \rightarrow \infty} \mathbb{Z}[G \setminus (\mathcal{N}_d^{(n)})^>]$ is a ring generated by idempotents, and so the underlying abelian group of $\lim_{n \rightarrow \infty} \mathbb{Z}[G \setminus (\mathcal{N}_d^{(n)})^>]$ is free abelian by [5, Theorem 1.1]. On the other hand, $\lim_{n \rightarrow \infty} \mathbb{Z}[(\mathcal{N}_m^{(n)})^>] = 0$ for $m > d$. Hence $H_d(\mathcal{A})$ identifies with a subgroup of $\lim_{n \rightarrow \infty} \mathbb{Z}[G \setminus (\mathcal{N}_d^{(n)})^>]$, and so is free abelian.

Let us now assume that G^0 is only locally compact. In this case use σ -compactness to write G^0 as an increasing union $G^0 = \bigcup_{m \in \mathbb{N}} K_m$ of compact open subsets. Let $G_m := \{g \in G \mid s(g), r(g) \in K_m\}$ denote the restriction of G to K_m . Note that each G_m is a σ -compact principal ample groupoid in its own right and it is straightforward to check from the definition (see [21, Definition 5.1]) that the dynamic asymptotic dimension of each G_m is dominated by the dynamic asymptotic dimension of the ambient groupoid G . Then $G = \bigcup_{m \in \mathbb{N}} G_m$ and the inclusion maps $G_m \hookrightarrow G$ induce isomorphisms $H_n(G) \cong \lim_{n \rightarrow \infty} H_n(G_m)$ for each n (see for example [15, Proposition 4.7]). Applying the compact unit space case from above we obtain the vanishing result.

For the freeness result, we employ the same trick as above. To this end we need to find anti-Čech sequences for the approximating groupoids which are compatible with each other. We proceed as follows: As in Corollary 3.28 use σ -compactness of G to find a sequence $L_0 \subseteq L_1 \subseteq L_2 \subseteq \dots$ of compact open subsets of G that cover G . We may assume that $L_i = L_i^{-1}$ and that $s(L_i) \cup r(L_i) \subseteq L_i$. Now let $K_0 := L_0$. Lemma 3.35 implies that we can find a K_0^3 -Lebesgue colouring H_0, \dots, H_d of $G_0 := G|_{K_0 \cap G^0}$. Let $\mathcal{N}_{*,0}^{(0)}$ be the associated nerve space (a semi-simplicial étale G_0 -space). Now find $n_1 \in \mathbb{N}$ such that L_{n_1} contains H_0, \dots, H_d and L_1 . Let $K_1 := L_{n_1}$ for brevity. Then we can use Lemma 3.35 again to find a K_1^3 -Lebesgue colouring of $G_1 := G|_{K_1 \cap G^0}$. Let $\mathcal{N}_{*,1}^{(1)}$ be the associated nerve space (a semi-simplicial G_1 -space). Intersecting this colouring with G_0 gives a colouring of G_0 with the same properties and hence an associated semi-simplicial G_0 -space $\mathcal{N}_{*,0}^{(1)}$. Lemma 3.25 then gives a morphism of semi-simplicial étale G_0 -spaces $\mathcal{N}_{*,0}^{(0)} \rightarrow \mathcal{N}_{*,0}^{(1)}$ and by construction we also get a canonical map of semi-simplicial spaces $\mathcal{N}_{*,0}^{(1)} \rightarrow \mathcal{N}_{*,1}^{(1)}$. Continuing this process we construct a nested exhausting sequence of clopen subgroupoids $G_0 \subseteq G_1 \subseteq \dots \subseteq G$, each with a compact unit space, and a commutative diagram of semi-simplicial spaces

In the diagram, the m -th row defines an anti-Čech sequence for G_m and the vertical arrows are given by intersecting the colouring with the respective subgroupoids. After applying the $\mathbb{Z}[\cdot]$ -functor and the coinvariant functor we get an analogous diagram of abelian groups. Using these anti-Čech sequences for each G_m we can

$$\begin{array}{ccccccc}
\mathcal{N}_{*,0}^{(0)} & \longrightarrow & \mathcal{N}_{*,0}^{(1)} & \longrightarrow & \mathcal{N}_{*,0}^{(2)} & \longrightarrow & \mathcal{N}_{*,0}^{(3)} \cdots \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \mathcal{N}_{*,1}^{(1)} & \longrightarrow & \mathcal{N}_{*,1}^{(2)} & \longrightarrow & \mathcal{N}_{*,1}^{(3)} \cdots \\
& & & & \downarrow & & \downarrow \\
& & & & \mathcal{N}_{*,2}^{(2)} & \longrightarrow & \mathcal{N}_{*,2}^{(3)} \cdots \\
& & & & & & \vdots
\end{array}$$

compute the homology of G as

$$H_*(G) = \lim_{m \rightarrow \infty} H_*(G_m) = H_*(\lim_{m \rightarrow \infty} \lim_{l \geq m} \mathbb{Z}[G_m \setminus (\mathcal{N}_{*,m}^{(l)})^>]).$$

As the diagram above commutes, it is enough to take the limit along the diagonal, so

$$H_*(G) = H_*(\lim_{m \rightarrow \infty} \mathbb{Z}[G_m \setminus (\mathcal{N}_{*,m}^{(m)})^>])$$

Then as before, as all the maps are spatially implemented and hence ring homomorphisms. The same argument as in the first part of this proof shows that $H_d(G)$ must be free. \square

4. THE HK CONJECTURE

In this section we will present several results concerning the connections between groupoid homology and the K-theory of reduced groupoid C^* -algebras. In particular we describe our positive results regarding Matui's HK conjecture.

4.1. The one-dimensional comparison map. Our first goal is the construction of a canonical comparison map

$$\mu_1 : H_1(G) \rightarrow K_1(C_r^*(G))$$

for an arbitrary ample groupoid G , extending earlier constructions of Matui under additional restrictions on the structure of G (see [29, Corollary 7.15] and [30, Theorem 5.2]). Our construction is based on two ingredients: a suitable description of the mapping cone of the inclusion $C(G^0) \hookrightarrow C_r^*(G)$, and a relative version of groupoid homology with respect to an open subgroupoid.

Let us start with our description of the K_0 -group of a mapping cone, which is different from, but inspired by, results of Putnam in [38].

Given a $*$ -homomorphism $\phi : A \rightarrow B$ between C^* -algebras let $C(\phi)$ denote the mapping cone

$$(7) \quad C(\phi) := \{(a, f) \in A \oplus IB : f(0) = \phi(a), f(1) = 0\},$$

where $IB := C([0, 1], B)$. We let $c : B \rightarrow IB$ denote the constant $*$ -homomorphism.

Definition 4.1. Given a $*$ -homomorphism of C^* -algebras $\phi : A \rightarrow B$, we define the *relative K-theory groups* with respect to ϕ by

$$(8) \quad K_*(A, B; \phi) := K_*(C(\phi)).$$

Sometimes, when there is no risk of confusion, we simply write $K_*(A, B)$.

There is an obvious surjection $C(\phi) \rightarrow A$ with kernel $SB := C_0((0, 1), B)$. This induces a six-term exact sequence in K -theory

$$(9) \quad \begin{array}{ccccc} K_1(B) & \longrightarrow & K_0(A, B; \phi) & \longrightarrow & K_0(A) \\ \uparrow \phi_1 & & & & \downarrow \phi_0 \\ K_1(A) & \longleftarrow & K_1(A, B; \phi) & \longleftarrow & K_0(B). \end{array}$$

We will give a different picture of the relative K -group $K_0(A, B; \phi)$. For the construction we will need to unitise ϕ unless it was already unital. By Proposition 4.9 this is not necessary if ϕ is non-degenerate and A contains an approximate identity of projections, as is the case in our applications.

Remark 4.2 (Unital $*$ -homomorphisms). Let $\phi: A \rightarrow B$ be a $*$ -homomorphism. Let A^\dagger and B^\dagger denote the forced unitisations of A and B (i.e. we add a unit regardless if A and/or B are already unital), and let $\phi^\dagger: A^\dagger \rightarrow B^\dagger$ denote the induced unital $*$ -homomorphism. Then $C(\phi)$ is (canonically isomorphic to) a two-sided, closed ideal in $C(\phi^\dagger)$ with $C(\phi^\dagger)/C(\phi) \cong C_0([0, 1])$. Hence $K_*(A, B; \phi) \cong K_*(A^\dagger, B^\dagger; \phi^\dagger)$ canonically. Moreover, the canonical map $K_*(A^\dagger, B^\dagger; \phi^\dagger) \rightarrow K_*(A^\dagger)$ takes values in $\ker(K_*(A^\dagger) \rightarrow K_*(\mathbb{C})) = K_*(A)$, so the isomorphism $K_*(A, B; \phi) \cong K_*(A^\dagger, B^\dagger; \phi^\dagger)$ is well-behaved with respect to the six-term exact sequence (9).

Hence it is essentially no loss of generality to only develop the theory for unital $*$ -homomorphisms.

We will give another description of $K_0(A, B; \phi)$, which is a more natural way to think of the relative K -groups. It is based on the following observation.

Remark 4.3 (A general setup). Let $\phi: A \rightarrow B$ be a $*$ -homomorphism and define

$$(10) \quad D(\phi) := \{(a_0, f, a_1) \in A \oplus IB \oplus A : f(0) = \phi(a_0), f(1) = \phi(a_1)\}.$$

We obtain a split extension

$$(11) \quad 0 \rightarrow C(\phi) \xrightarrow{\iota} D(\phi) \xrightarrow{\pi^{(1)}} A \rightarrow 0$$

where $\iota(a, f) = (a, f, 0)$ and $\pi^{(1)}(a_0, f, a_1) = a_1$. There is a canonical splitting $*$ -homomorphism $\sigma: A \rightarrow D(\phi)$ given by $\sigma(a) = (a, \mathbf{c}(\phi(a)), a)$. Hence the composition

$$(12) \quad K_*(A, B; \phi) \xrightarrow{\iota_*} K_*(D(\phi)) \rightarrow K_*(D(\phi))/\sigma_*(K_*(A))$$

is an isomorphism.

If $\pi^{(i)}: D(\phi) \rightarrow A$ is given by $\pi^{(i)}(a_0, f, a_1) = a_i$ for $i \in \{0, 1\}$, then $\pi_*^{(0)} - \pi_*^{(1)}: K_*(D(\phi)) \rightarrow K_*(A)$ descends to $K_*(D(\phi))/\sigma_*(K_*(A)) \rightarrow K_*(A)$. The isomorphism (12) composed with $\pi_*^{(0)} - \pi_*^{(1)}$ is exactly the canonical map $K_*(A, B; \phi) \rightarrow K_*(A)$.

In what follows, when $\phi: A \rightarrow B$ is a $*$ -homomorphism, we abuse notation by letting ϕ also denote the induced homomorphism $M_n(A) \rightarrow M_n(B)$ for $n \in \mathbb{N}$. We let $U_n(A)$ denote the unitary group of $M_n(A)$ for a unital C^* -algebra A , and let $V(A)$ denote the semigroup of equivalence classes of projections in $\bigcup_{n \in \mathbb{N}} M_n(A)$, so that $K_0(A)$ is the Grothendieck group of $V(A)$ whenever A is unital.

Remark 4.4. Let $\phi: A \rightarrow B$ be a $*$ -homomorphism between C^* -algebras. Let $V_n(A, B; \phi)$ for $n \in \mathbb{N}$ denote the set of triples (p, v, q) where $p, q \in M_n(A)$ are projections and $vv^* = \phi(p)$, and $v^*v = \phi(q)$. Let $V_\infty(A, B; \phi) = \bigcup_{n \in \mathbb{N}} V_n(A, B; \phi)$ be equipped with the inductive limit topology, let \sim_h denote homotopy in $V_\infty(A, B; \phi)$,⁵

⁵A homotopy in $V_\infty(A, B; \phi)$ is automatically contained in $V_n(A, B; \phi)$ for some $n \in \mathbb{N}$, so one can also take this as the definition of homotopy.

and let \approx be the equivalence relation on $V_\infty(A, B; \phi)$ given as follows: $(p, v, q) \approx (p', v', q')$ exactly when there are projections $r, r' \in M_n(A)$ such that

$$(13) \quad (p, v, q) \oplus (r, \phi(r), r) \sim_h (p', v', q') \oplus (r', \phi(r'), r').$$

We will be considering the monoid $V_\infty(A, B; \phi)/\approx$ (with diagonal sums) and show that it is naturally isomorphic to $K_0(A, B; \phi)$ when ϕ is unital, or more generally, when ϕ is non-degenerate and A contains an approximate identity of projections.

In the special case where ϕ is an inclusion of a C^* -subalgebra, we simply write $V_\infty(A, B)$ instead of $V_\infty(A, B; \phi)$. Moreover, if $(p, v, q) \in V_\infty(A, B)$ then $p = vv^*$ and $q = v^*v$. Hence we will simply denote the elements of $V_\infty(A, B)$ by v .

The following lemma collects several elementary properties of $V_\infty(A, B; \phi)/\approx$, and in particular, it follows that it is a group (as opposed to a semigroup), with the inverse of (p, v, q) being (q, v^*, p) . Note also that a standard rotation trick as in the proof of part (1) below shows that $V_\infty(A, B; \phi)/\approx$ is abelian.

Lemma 4.5. *Let $\phi: A \rightarrow B$ be a unital $*$ -homomorphism.*

- (1) *If $(p, u, q), (q, v, r) \in V_n(A, B; \phi)$, then $(p, uv, r) \in V_n(A, B; \phi)$ and $(p, u, q) \oplus (q, v, r) \approx (p, uv, r)$.*
- (2) *If $(p, v, q) \in V_n(A, B; \phi)$ and $u, w \in M_n(A)$ are unitaries, then $(u^*pu, \phi(u)^*v\phi(w), w^*qw) \in V_n(A, B; \phi)$ and $(u^*pu, \phi(u)^*v\phi(w), w^*qw) \approx (p, v, q)$.*
- (3) *If $v \in M_n(A)$ is a partial isometry, then $(vv^*, \phi(v), v^*v) \approx (0, 0, 0)$.*
- (4) *If $(p, v, q) \in V_n(A, B; \phi)$ and $u, w \in M_n(A)$ are partial isometries with $u^*u = p$ and $q = ww^*$, then $(uu^*, \phi(u)v\phi(w), w^*w) \in V_n(A, B; \phi)$ and $(uu^*, \phi(u)v\phi(w), w^*w) \approx (p, v, q)$.*
- (5) *If $(p_0, v, p_1), (q_0, w, q_1) \in V_n(A, B; \phi)$ such that $p_j \perp q_j$ for $j = 0, 1$, then $(p_0 + q_0, v + w, p_1 + q_1) \in V_n(A, B; \phi)$ and $(p_0 + q_0, v + w, p_1 + q_1) \approx (p_0, v, p_1) \oplus (q_0, w, q_1)$.*
- (6) *If $(p, v, q), (p', v', q') \in V_n(A, B; \phi)$ are such that $\|p - p'\| < 1/8$, $\|q - q'\| < 1/8$ and $\|v - v'\| < 1$, then $(p, v, q) \approx (p', v', q')$.*

Proof. (1): For $t \in [0, 1]$ let R_t be the rotation matrix

$$R_t = \begin{pmatrix} \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}$$

Furthermore, define for each $t \in [0, 1]$ an element

$$w_t := \begin{pmatrix} u & 0 \\ 0 & \phi(q) \end{pmatrix} R_t \begin{pmatrix} v & 0 \\ 0 & \phi(q) \end{pmatrix} R_t^* \in M_{2n}(B).$$

Then one easily checks, that $w_t w_t^* = \phi(p \oplus q)$ and similarly $w_t^* w_t = \phi(R_t(r \oplus q)R_t^*)$. Hence $(p \oplus q, w_t, R_t(r \oplus q)R_t^*)_{t \in [0, 1]}$ is a homotopy in $V_{2n}(A, B; \phi)$ from $(p, uv, r) \oplus (q, \phi(q), q)$ to $(p, u, q) \oplus (q, v, r)$, so $(p, u, q) \oplus (q, v, r) \approx (p, uv, r)$.

(2): We first show that $(u^*pu, \phi(u^*p), p) \approx (0, 0, 0)$. By enlarging n and possibly replacing u by $u \oplus u^*$, we may assume that there is a unitary path $(u_t)_{t \in [0, 1]}$ from u to 1. Hence $t \mapsto (u_t^*pu_t, \phi(u_t^*p), p)$ is a homotopy in $V_\infty(A, B; \phi)$ from $(u^*pu, \phi(u^*p), p)$ to $(p, \phi(p), p) \approx (0, 0, 0)$. So $(u^*pu, \phi(u^*p), p) \approx (0, 0, 0)$ and similarly $(q, \phi(qw), w^*qw) \approx (0, 0, 0)$. Now apply (1) to $(u^*pu, \phi(u^*p), p)$, (p, v, q) , and $(q, \phi(qw), w^*qw)$ to obtain (2).

(3): Let $u = \begin{pmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{pmatrix} \in U_{2n}(A)$. Then

$$(vv^*, \phi(v), v^*v) \approx (vv^*, \phi(v), v^*v) \oplus (0, 0, 0) = (u(v^*v \oplus 0)u^*, \phi(u(v^*v \oplus 0)), v^*v \oplus 0)$$

Apply part (2) to get

$$(u(v^*v \oplus 0)u^*, \phi(u(v^*v \oplus 0)), v^*v \oplus 0) \approx (v^*v, \phi(v^*v), v^*v) \approx (0, 0, 0).$$

(4): Apply (1) to get

$$(uu^*, \phi(u)v\phi(w), w^*w) \approx (uu^*, \phi(u), p) \oplus (p, v, q) \oplus (q, \phi(w), w^*w).$$

The result then follows from part (3).

(5): Let $u_0 = \begin{pmatrix} p_0 & 0 \\ q_0 & 0 \end{pmatrix}$ and $u_1 = \begin{pmatrix} p_1 & q_1 \\ 0 & 0 \end{pmatrix}$ which are partial isometries in $M_{2n}(A)$ with $u_0^*u_0 = (p_0 + q_0) \oplus 0$ and $u_1u_1^* = (p_1 + q_1) \oplus 0$. Note that

$$u_0u_0^* = p_0 \oplus q_0, \quad u_1^*u_1 = p_1 \oplus q_1, \quad \phi(u_0)((v+w) \oplus 0)\phi(u_1) = v \oplus w.$$

Hence, using part (4) to $(p_0 + q_0, v + w, p_1 + q_1) \oplus (0, 0, 0)$, u_0 and u_1 , we get

$$(p_0 + q_0, v + w, p_1 + q_1) \approx (p_0, v, q_0) \oplus (p_1, w, q_1).$$

(6): Let $\delta = \|p - p'\|$ and $z = pp' + (1 - p)(1 - p')$. Then $\|z - 1\| \leq 2\delta$ (see [42, Proof of Proposition 2.2.6]) so in particular z is invertible and $\|z\| \in [1 - 2\delta, 1 + 2\delta]$. Let $u = z|z|^{-1}$. Then u is a unitary such that $p = up'u^*$. Also, $\|u|z| - 1\| \leq 2\delta$ and $\|z|z| - 1\| \leq 2\delta$, so

$$\|u - 1\| = \|u^* - 1\| \leq \|u^* - |z|\| + \||z| - 1\| \leq 4\delta < 1/2.$$

Similarly, we may find a unitary w such that $w^*q'w = q$ and $\|w - 1\| < 1/2$. Then $(p', v', q') \approx (p, \phi(u)v'\phi(w), q)$ by part (2) so it suffices to show that (p, v, q) and $(p, \phi(u)v'\phi(w), q)$ are homotopic.

Now $\phi(u)v'\phi(w)v^*$ is a unitary in $\phi(p)M_n(B)\phi(p)$ and

$$\|\phi(u)v'\phi(w)v^* - \phi(p)\| = \|\phi(u)v'\phi(w) - v\| < 1 + \|v' - v\| < 2,$$

where the first inequality follows since $\|u - 1\| < 1/2$ and $\|w - 1\| < 1/2$. Hence there is a continuous path $(U_t)_{t \in [0,1]}$ with $U_t^*U_t = U_tU_t^* = \phi(p) = U_0$ and $U_1 = \phi(u)v'\phi(w)v^*$. Then $((p, U_tv, q))_{t \in [0,1]}$ is a homotopy in $V_n(A, B; \phi)$ from (p, v, q) to $(p, \phi(u)v'\phi(w), q)$ as desired. \square

The proof of the following elementary lemma is left for the reader. In the following, $\text{Gr}(N)$ denotes the Grothendieck construction applied to N , and the order \leq is the usual algebraic order, i.e. $x \leq z$ means that there exists x' such that $x + x' = z$. The lemma will be used to identify $K_0(D(\phi))/\sigma_0(K_0(A))$ as in Remark 4.3 with $V(D(\phi))/\simeq$.

Lemma 4.6. *Let M and N be abelian semigroups, and suppose that $\rho: M \rightarrow N$ is a homomorphism such that for every $x \in N$ there exists $z \in M$ for which $x \leq \rho(z)$. Let \simeq be the equivalence relation on N given by $x \simeq y$ if and only if there exist $z, w \in M$ such that $x + \rho(z) = y + \rho(w)$. Then N/\simeq is a group canonically isomorphic to $\text{Gr}(N)/\text{Gr}(\rho)(\text{Gr}(M))$. \square*

Proposition 4.7. *For any unital $*$ -homomorphism $\phi: A \rightarrow B$ there is a natural isomorphism $\eta: K_0(A, B; \phi) \xrightarrow{\cong} V_\infty(A, B; \phi)/\approx$ such that the composition*

$$(14) \quad K_1(B) \rightarrow K_0(A, B; \phi) \xrightarrow{\eta} V_\infty(A, B; \phi)/\approx$$

is given by mapping $u \in U_n(B)$ to $(1_{M_n(A)}, u, 1_{M_n(A)})$, and the composition

$$(15) \quad V_\infty(A, B; \phi)/\approx \xrightarrow{\eta^{-1}} K_0(A, B; \phi) \rightarrow K_0(A)$$

is given by $[p, v, q] \mapsto [p]_0 - [q]_0$.

Proof. We adopt the notation from Remark 4.3 and use the observation that $K_0(A, B; \phi) \cong K_0(D(\phi))/\sigma_0(K_0(A))$. By Lemma 4.6, it follows that $K_0(D(\phi))/\sigma_0(K_0(A))$ is canonically isomorphic to $V(D(\phi))/\simeq$ (with $M = V(A)$, $N = V(D(\phi))$ and $\rho = V(\sigma)$). We will construct a natural isomorphism from $V(D(\phi))/\simeq$ to $V_\infty(A, B; \phi)/\approx$.

We construct a semigroup homomorphism $V(D(\phi)) \rightarrow V_\infty(A, B; \phi)$ as follows: let $(p_0, p, p_1) \in M_n(D(\phi))$ be a projection, i.e. $p_0, p_1 \in M_n(A)$ are projections and

$\mathbf{p} \in C([0, 1], M_n(B))$ is a homotopy of projections with $\mathbf{p}_0 = \phi(p_0)$ and $\mathbf{p}_1 = \phi(p_1)$. Using for example [23, Corollary 4.1.8], there is a continuous path of unitaries $(u_t)_{t \in [0, 1]}$ in $M_n(B)$ such that

$$\mathbf{p}_t = u_t^* \mathbf{p}_0 u_t = u_t^* \phi(p_0) u_t, \quad \text{for } t \in [0, 1], \quad \text{and} \quad u_0 = 1.$$

The map is given by mapping (p_0, \mathbf{p}, p_1) to $(p_0, \phi(p)u_1, p_1)$ in $V_\infty(A, B; \phi)/\approx$. If $(w_t)_{t \in [0, 1]}$ is any other such unitary path, then $t \mapsto (p_0, \phi(p_0)w_t u_t^* u_1, p_1)$ is a homotopy in $V_\infty(A, B; \phi)$ from $(p_0, \phi(p_0)u_1, p_1)$ to $(p_0, \phi(p_0)w_1, p_1)$, so it does not depend on the choice of $(u_t)_{t \in [0, 1]}$. To see that the map is well-defined, let (p_0, \mathbf{p}, p_1) and (q_0, \mathbf{q}, q_1) be equivalent projections in $M_n(D(\phi))$. By enlarging n we may assume that these are unitarily equivalent, so let $(v_0, \mathbf{v}, v_1) \in M_n(D(\phi))$ be a unitary such that

$$(v_0^* p_0 v_0, \mathbf{v}^* \mathbf{p} \mathbf{v}, v_1^* p_1 v_1) = (q_0, \mathbf{q}, q_1).$$

Let $(u_t)_{t \in [0, 1]}$ be a unitary path with $u_0 = 1$ and $\mathbf{p}_t = u_t^* \phi(p_0) u_t$, so that our map takes (p_0, \mathbf{p}, p_1) to $(p_0, \phi(p_0)u_1, p_1)$. Let $w_t := v_0^* u_t v_t$ for $t \in [0, 1]$. Then $(w_t)_{t \in [0, 1]}$ is a unitary path with $w_0 = 1$ and $\mathbf{q}_t = w_t^* \phi(q_0) w_t$ for all $t \in [0, 1]$, so (q_0, \mathbf{q}, q_1) is mapped to $(q_0, \phi(q_0)w_1, q_1)$. Using that $w_1 = \phi(v_0)^* u_1 \phi(v_1)$ and that $\phi(q_0)\phi(v_0^*) = \phi(v_0^*)\phi(p_0)$, it follows from Lemma 4.5 that

$$(q_0, \phi(q_0)w_1, q_1) = (v_0^* p_0 v_0, \phi(v_0)^* \phi(p_0) u_1 \phi(v_1), v_1^* p_1 v_1) \approx (p_0, \phi(p_0)u_1, p_1).$$

Hence the map $V(D(\phi)) \rightarrow V_\infty(A, B; \phi)/\approx$ described above is well-defined.

Hence this gives a well-defined homomorphism $V(D(\phi)) \rightarrow V_\infty(A, B; \phi)/\approx$, and as it clearly vanishes on projections $\sigma(p) = (p, c(\phi(p)), p)$ (it maps such projections to $(p, \phi(p), p) \in V_\infty(A, B; \phi)$), it descends to a group homomorphism $\frac{V(D(\phi))}{\approx} \rightarrow \frac{V_\infty(A, B; \phi)}{\approx}$. We denote the composition $K_0(A, B; \phi) \cong \frac{V(D(\phi))}{\approx} \rightarrow \frac{V_\infty(A, B; \phi)}{\approx}$ by η . We will show that η is an isomorphism.

Before proving that this is an isomorphism, we will show the last part of the proposition. Note that there is a homomorphism $\frac{V_\infty(A, B; \phi)}{\approx} \rightarrow K_0(A)$ given by $[p, v, q] \mapsto [p]_0 - [q]_0$, and that by Remark 4.3 the map $K_0(A, B; \phi) \rightarrow K_0(A)$ from (9) is exactly the composition $K_0(A, B; \phi) \rightarrow \frac{V_\infty(A, B; \phi)}{\approx} \rightarrow K_0(A)$. Hence the last part of the proposition follows (assuming η is an isomorphism).

Similarly, there is a homomorphism $K_1(B) \rightarrow \frac{V_\infty(A, B; \phi)}{\approx}$ given by $[u]_1 \mapsto [1_n, u, 1_n]$. We will show that it is the composition of η with $K_1(B) \rightarrow K_0(A, B; \phi)$ from (9). Let $u \in U_n(B)$. The canonical isomorphism $K_1(B) \cong K_0(SB)$ takes $[u]_1$ to $x := [(w_t^*(1_n \oplus 0_n)w_t)_{t \in [0, 1]}]_0 - [c(1_n \oplus 0_n)]_0$, where $(w_t)_{t \in [0, 1]}$ is a homotopy in $U_{2n}(B)$ from 1 to $u \oplus u^*$ (see for instance [42, Theorem 10.1.3]). The image of x in $V(D(\phi))/\approx$ is $[1_n \oplus 0, (w_t^*(1_n \oplus 0_n)w_t)_{t \in [0, 1]}, 1_n \oplus 0_n]$, and this element is mapped to $[1_n \oplus 0, w_1^*(1_n \oplus 0_n)w_1, 1_n \oplus 0_n]$ in $V_\infty(A, B; \phi)/\approx$ via η . This element is clearly $[1_n, u, 1_n]$ as desired.

For surjectivity of η , let $(p_0, v, p_1) \in V_n(A, B; \phi)$. By possibly increasing n , we may find a unitary $u \in U_n(B)$ in the connected component of 1 such that $\phi(p_0)u = v$. Let $(u_t)_{t \in [0, 1]}$ be a unitary path from 1 to u . Letting $\mathbf{p}_t = u_t^* \phi(p_0) u_t$ we get a path of projections such that (p_0, \mathbf{p}, p_1) is a projection in $M_n(D(\phi))$ which is mapped to (p_0, v, p_1) by our homomorphism $\frac{V(D(\phi))}{\approx} \rightarrow \frac{V_\infty(A, B; \phi)}{\approx}$. Hence η is surjective.

For injectivity of η , let $x \in K_0(A, B; \phi)$ be such that $\eta(x) = 0$. As the map $K_0(A, B; \phi) \rightarrow K_0(A)$ from (9) factors through η (which we proved above) the element x vanishes in $K_0(A)$. By exactness, there is $u \in U_n(B)$ so that $[u]_1 \in K_1(B)$ is mapped to x . We will show that $[u]_1$ is in the image of $\phi_1: K_1(A) \rightarrow K_1(B)$, and thus $x = 0$ by exactness.

The image of $[u]_1$ in $\frac{V_\infty(A, B; \phi)}{\approx}$ is $[1_n, u, 1_n]$, which is zero as $\eta(x) = 0$. Hence we may find projections $r \in M_k(A)$ and $p \in M_{n+k}(A)$ such that $(1_n \oplus r, u \oplus \phi(r), 1_n \oplus r) \sim_h (p, \phi(p), p)$. By adding $(1_k - r, 1_k - \phi(r), 1_k - r)$ we may assume that $r = 1_k$,

so by replacing n with $n + k$ and u with $u \oplus 1_k$, we may assume without loss of generality that $(1_n, u, 1_n) \sim_h (p, \phi(p), p)$. Let $(q_t, v_t, r_t)_{t \in [0,1]}$ be a homotopy in $V_m(A, B; \phi)$ for some (large enough) $m \geq n$ from $(1_n, u, 1_n)$ to $(p, \phi(p), p)$. As (q_t) and (r_t) are homotopies of projections, we may find continuous unitary paths $(w_t)_{t \in [0,1]}, (w'_t)_{t \in [0,1]}$ in $U_m(A)$ such that $w_0 = w'_0 = 1_m$ and $w_t^* 1_n w_t = q_t$ and $w'_t 1_n (w'_t)^* = r_t$ for all $t \in [0,1]$. It follows that $U_t := \phi(w_t) v_t \phi(w'_t)$ defines a continuous path with $U_t^* U_t = U_t U_t^* = 1_n \in M_m(B)$, and thus canonically induces a unitary path in $U_n(B)$. As $U_0 = v_0 = u$ and $U_1 = \phi(w_1) v_1 \phi(w'_1) = \phi(w_1 p w'_1)$, and as $w_1 p w'_1$ defines a unitary in $U_n(A)$, it follows that $[u]_1 = \phi_1([w_1 p w'_1]_1)$. Hence $x = 0$ by the exact sequence (9), so η is injective. \square

We will need the following perturbation lemma. In the following, $\chi_{[1/2, 3/2]}$ denotes the characteristic function on the interval $[1/2, 3/2]$. Note that when p_0 is self-adjoint with $\|p_0 - p_0^2\| < \delta < 1/2$, then $p = \chi_{[1/2, 3/2]}(p_0)$ is a projection such that $\|p - p_0\| < \delta$.

Lemma 4.8. *Let $0 < \delta < 1/8$, let $p_0, q_0 \in M_n(A)$ be self-adjoint, and $v_0 \in M_n(B)$ be such that*

$$\|p_0 - p_0^2\| < \delta, \quad \|q_0 - q_0^2\| < \delta, \quad \|v_0 v_0^* - \phi(p_0)\| < \delta, \quad \|v_0^* v_0 - \phi(q_0)\| < \delta.$$

Let $p = \chi_{[1/2, 3/2]}(p_0)$ and $q = \chi_{[1/2, 3/2]}(q_0)$. There exists $v \in M_n(B)$ such that $(p, v, q) \in V_n(A, B; \phi)$ and $\|v - v_0\| < 32\delta + \sqrt{2\delta}$.

Proof. As $\|p - p_0\| < \delta$ and $\|q - q_0\| < \delta$ we have $\|v_0 v_0^* - \phi(p)\| < 2\delta$ and $\|v_0^* v_0 - \phi(q)\| < 2\delta$. Hence the spectrum of $v_0^* v_0$ and $v_0 v_0^*$ is contained in $[-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta]$. Let $P = \chi_{[1/2, 3/2]}(v_0 v_0^*)$ and $Q = \chi_{[1/2, 3/2]}(v_0^* v_0)$ so that $\|P - \phi(p)\| < 4\delta$ and $\|Q - \phi(q)\| < 4\delta$. Hence, as in the proof of Lemma 4.5(6), there are unitaries $u_p, u_q \in M_n(B)^\dagger$ such that $\|u_p - 1\| < 16\delta$, $\|u_q - 1\| < 16\delta$, $u_p P u_p^* = \phi(p)$ and $u_q^* Q u_q = \phi(q)$.

Let $v_0 = w(v_0^* v_0)^{1/2}$ be the polar decomposition in $M_n(B)^{**}$. Then $w_0 := wQ = Pw$ is an element of $M_n(B)$ (as opposed to the bidual) such that $w_0^* w_0 = Q$ and $w_0 w_0^* = P$. Let $v := u_p w_0 u_q \in M_n(B)$. Then $vv^* = u_p w_0 w_0^* u_p^* = \phi(p)$ and similarly $v^* v = \phi(q)$, so $(p, v, q) \in V_n(A, B; \phi)$. Moreover,

$$\|v - v_0\| < 32\delta + \|w_0 - v_0\| \leq 32\delta + \|w(Q - (v_0^* v_0)^{1/2})\| \leq 32\delta + \sqrt{2\delta}. \quad \square$$

The following result allows us to avoid passing to unitisations when applying the above results.

Proposition 4.9. *Let $\phi: A \rightarrow B$ be a non-degenerate $*$ -homomorphism (i.e. $\overline{\phi(A)B} = B$) and suppose that A has an approximate identity of projections. Then the natural inclusions induce an isomorphism $\frac{V_\infty(A, B; \phi)}{\approx} \xrightarrow{\cong} \frac{V(A^\dagger, B^\dagger; \phi^\dagger)}{\approx}$.*

Proof. Let $(e_\lambda)_\lambda$ be an approximate identity of projections in A . As ϕ is non-degenerate, $(\phi(e_\lambda))_\lambda$ is an approximate identity of projections in B . We let $e_\lambda^{(n)} \in M_n(A)$ denote the diagonal matrix with e_λ down the diagonal.

Note that $V_\infty(A, B; \phi)/\approx$ is a priori only a monoid (as ϕ is not necessarily unital), but the map $\frac{V_\infty(A, B; \phi)}{\approx} \rightarrow \frac{V(A^\dagger, B^\dagger; \phi^\dagger)}{\approx}$ induced by inclusions is clearly a monoid homomorphism, so it still suffices to show that this monoid homomorphism is bijective.

For surjectivity, let $(p, v, q) \in V_n(A^\dagger, B^\dagger; \phi^\dagger)$. Let $(\bar{p}, \bar{v}, \bar{q}) \in V_n(A^\dagger, B^\dagger; \phi^\dagger)$ be the scalar part of (p, v, q) . Note that $e_\lambda^{(n)}$ commutes with \bar{p} and \bar{q} , and that $\phi(e_\lambda^{(n)})$ commutes with \bar{v} . As $\|p - (e_\lambda^{(n)} p e_\lambda^{(n)} + (1 - e_\lambda^{(n)}) \bar{p})\| \rightarrow 0$, and similarly for q and v , we may use Lemma 4.8 to pick $e = e_\lambda$ and $(p_0, v_0, q_0) \in V_n(eAe, \phi(e)B\phi(e); \phi)$ such

that $\|p - (p_0 + (1 - e^{(n)})\bar{p})\| < 1/8$, $\|q - (q_0 + (1 - e^{(n)})\bar{q})\| < 1/8$ and $\|v - (v_0 + (1 - \phi(e)^{(n)})\bar{v})\| < 1$. Hence by Lemma 4.5 it follows that

$$(p, v, q) \approx (p_0, v_0, q_0) \oplus ((1 - e^{(n)})\bar{p}, (1 - \phi(e)^{(n)})\bar{v}, (1 - e^{(n)})\bar{q}).$$

By considering \bar{v} (which is a scalar-valued matrix) as an element in $M_n(A^\dagger)$, it follows from Lemma 4.5(3) that the latter summand above is equivalent to $(0, 0, 0)$, so $(p, v, q) \approx (p_0, v_0, q_0) \in V_n(A, B; \phi)$. Hence our map $\frac{V_\infty(A, B; \phi)}{\approx} \xrightarrow{\cong} \frac{V(A^\dagger, B^\dagger; \phi^\dagger)}{\approx}$ is surjective.

For injectivity of the map, let $(p, v, q) \in V_n(A, B; \phi)$ such that $(p, v, q) \approx (0, 0, 0)$ in $V_\infty(A^\dagger, B^\dagger; \phi^\dagger)$. Let $r \in M_k(A^\dagger)$ and $r' \in M_{n+k}(A^\dagger)$ be projections such that $(p, v, q) \oplus (r, \phi^\dagger(r), r)$ is homotopic to $(r', \phi^\dagger(r'), r')$. By applying the same method as above, we may cut such a homotopy into a piece in $V_{n+k}(A, B; \phi)$ and a scalar matrix valued homotopy multiplied by $(1 - e_\lambda^{(n+k)})$, with endpoints close to $(p, v, q) \oplus (r_0, \phi(r_0), r_0)$. The piece of the homotopy in $V_\infty(A, B; \phi)$ induces $(p, v, q) \approx (0, 0, 0)$ by an application of Lemma 4.5. The exact details are left for the reader. \square

Remark 4.10. Using Proposition 4.9, Lemma 4.5 also holds for non-degenerate $*$ -homomorphisms if A has an approximate identity of projections.

The following theorem sums up the above results.

Theorem 4.11. *Let A and B be C^* -algebras and suppose $\phi: A \rightarrow B$ is a non-degenerate $*$ -homomorphism (i.e. $\overline{\phi(A)B} = B$). Suppose that A has an approximate identity of projections. Then there is an exact sequence*

$$K_1(A) \xrightarrow{\phi_1} K_1(B) \rightarrow \frac{V_\infty(A, B; \phi)}{\approx} \rightarrow K_0(A) \xrightarrow{\phi_0} K_0(B)$$

where the map $K_1(B) \rightarrow \frac{V_\infty(A, B; \phi)}{\approx}$ is given by $[u]_1 \mapsto [p \otimes 1_{M_n}, u, p \otimes 1_{M_n}]$ for $u \in U_n(\phi(p)B\phi(p))$ (where $p \in A$ is a projection); and $\frac{V_\infty(A, B; \phi)}{\approx} \rightarrow K_0(A)$ is given by $[p, v, q] \mapsto [p]_0 - [q]_0$.

Let us now turn to the second ingredient in the construction of the map $\mu_1: H_1(G) \rightarrow K_1(C_r^*(G))$: *relative groupoid homology*. To define it, we return to Matui's picture of groupoid homology as it allows for an elementary description. For an open subgroupoid $H \subseteq G$ the canonical inclusion induces for each $n \geq 0$ a short exact sequence of abelian groups

$$(16) \quad 0 \rightarrow C_c(H^{(n)}, \mathbb{Z}) \rightarrow C_c(G^{(n)}, \mathbb{Z}) \rightarrow C_c(G^{(n)}, \mathbb{Z})/C_c(H^{(n)}, \mathbb{Z}) \rightarrow 0$$

Let $C_n(G, H)$ denote the quotient group $C_c(G^{(n)}, \mathbb{Z})/C_c(H^{(n)}, \mathbb{Z})$. One easily checks that $C_c(H^{(n)}, \mathbb{Z})$ is invariant under the boundary maps ∂_n for G , and hence we obtain induced maps ∂'_n turning $(C_n(G, H), \partial'_n)$ into a chain complex such that the sequence (16) is an exact sequence of chain complexes.

Definition 4.12. Let G be an ample groupoid. The *relative homology* of G with respect to an open subgroupoid H is defined as the homology of the chain complex $(C_n(G, H), \partial'_n)$, i.e.

$$H_n(G, H) := \ker(\partial'_n)/\text{im}(\partial'_{n+1}).$$

From the short exact sequence (16) we obtain a long exact sequence of homology groups

$$\cdots \rightarrow H_2(G, H) \rightarrow H_1(H) \rightarrow H_1(G) \rightarrow H_1(G, H) \rightarrow H_0(H) \rightarrow H_0(G) \rightarrow H_0(G, H)$$

Important for us is the special case where $H = G^0$. In that case one easily checks that $H_0(G, G^0) = 0$, and $H_1(G, G^0) \cong C_c(G, \mathbb{Z})/\text{im}(\partial_2)$.

We are now ready to put everything together and construct the map $\mu_1: H_1(G) \rightarrow K_1(C_r^*(G))$.

Lemma 4.13. *Let G be an ample groupoid. Then there exists a well-defined homomorphism*

$$\tilde{\rho} : C_c(G, \mathbb{Z}) \rightarrow V_\infty(C_0(G^0), C_r^*(G))/\approx$$

determined by $\tilde{\rho}(1_W) = [1_W]$ and $\tilde{\rho}(-1_W) = [(1_W)^]$ for every compact open bisection $W \subseteq G$.*

Moreover, $\text{im}(\partial_2) \subseteq \ker(\tilde{\rho})$ and hence $\tilde{\rho}$ factors through a well-defined homomorphism

$$\rho : H_1(G, G^0) \cong C_c(G, \mathbb{Z})/\text{im}(\partial_2) \rightarrow V_\infty(C(G^0), C_r^*(G))/\approx.$$

Proof. Since G is an ample groupoid, then every function $f \in C_c(G, \mathbb{Z})$ can be written (non-uniquely) as a linear combination $f = \sum \lambda_i 1_{W_i}$ where $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ and W_1, \dots, W_n are pairwise disjoint compact open bisections of G . We have to show, that the resulting class of the partial isometry

$$\bigoplus_{\{i|\lambda_i \geq 0\}} \lambda_i 1_{W_i} \oplus \bigoplus_{\{i|\lambda_i < 0\}} |\lambda_i| 1_{W_i^{-1}}$$

in $V_\infty(C_0(G^0), C_r^*(G))/\approx$ only depends on f and not on the particular choice of the bisections W_i . Let us first note the following: If W is a compact open bisection of G such that $W = U \sqcup V$ for compact open subsets $U, V \subseteq W$, then $1_W = 1_U + 1_V \approx 1_U \oplus 1_V$ by Lemma 4.5 (and Remark 4.10 to handle the non-unital case). With this in mind we can easily prove the lemma: Let $f \in C_c(G, \mathbb{Z})$. We may assume that $f \geq 0$. Suppose we have $f = \sum \lambda_i 1_{U_i} = \sum \mu_j 1_{V_j}$ for $\lambda_i, \mu_j \in \mathbb{N}$ and two families $(U_i)_i$ and $(V_j)_j$ of pairwise disjoint compact open bisections of G . Since G is ample we may choose a common refinement of these two families by compact open bisections $(W_k)_k$. Let η_k be equal to λ_i if $W_k \subseteq U_i$ and equal to μ_j if $W_k \subseteq V_j$. Then we have

$$\bigoplus_k \eta_k 1_{W_k} \approx \bigoplus_j \bigoplus_{\{k|W_k \subseteq V_j\}} \mu_j 1_{W_k} \approx \bigoplus_j \mu_j \bigoplus_{\{k|W_k \subseteq V_j\}} 1_{W_k} \approx \bigoplus_j \mu_j 1_{V_j}.$$

Similarly, we obtain $\bigoplus_k \eta_k 1_{W_k} \approx \bigoplus_i \lambda_i 1_{U_i}$, from which we conclude that $\tilde{\rho}$ is indeed well-defined. It is a group homomorphism by construction.

For the second part let U and V be compact open bisections of G such that $s(U) = r(V)$. Then we have $\tilde{\rho}(\partial_2(1_{(U \times V) \cap G^{(2)}})) = \tilde{\rho}(1_U - 1_{UV} + 1_V) = 0$ since $1_U \oplus 1_V \approx 1_U 1_V = 1_{UV}$ by Lemma 4.5. \square

Theorem 4.14. *Let G be an ample groupoid. Then there exists a canonical homomorphism*

$$\mu_1 : H_1(G) \rightarrow K_1(C_r^*(G)).$$

Proof. Let $\iota : C_0(G^0) \rightarrow C_r^*(G)$ denote the canonical inclusion, which is non-degenerate. As G is ample, $C_0(G^0)$ contains an approximate identity of projections. By Theorem 4.11 and Lemma 4.13 we obtain a commutative diagram with exact rows

(17)

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_1(G) & \longrightarrow & C_c(G, \mathbb{Z})/\text{im}(\partial_2) & \longrightarrow & C_c(G^0, \mathbb{Z}) \longrightarrow H_0(G) \\ & & \downarrow \mu_1 & & \downarrow \rho & & \downarrow \cong \\ 0 & \longrightarrow & K_1(C_r^*(G)) & \longrightarrow & V_\infty(C_0(G^0), C_r^*(G))/\approx & \longrightarrow & K_0(C_0(G^0)) \xrightarrow{\iota_0} K_0(C_r^*(G)) \end{array},$$

where the top row is part of the long exact sequence for the pair (G, G^0) . By exactness, there exists a unique homomorphism $\mu_1 : H_1(G) \rightarrow K_1(C_r^*(G))$ filling in the dashed arrow such that the diagram commutes. \square

4.2. Dynamic asymptotic dimension one. In this section, we study ample groupoids with dynamic asymptotic dimension at most one. The main result shows that under this assumption the canonical map $\mu_0 : H_0(G) \rightarrow K_0(C_r^*(G))$ is an isomorphism, and the canonical map $\mu_1 : H_1(G) \rightarrow K_1(C_r^*(G))$ is surjective. The proof is based on [26, Section 5], which in turn borrows its technical ingredients from [20].

We will need the following structural result for compact ample groupoids, which follows directly from [16, Lemma 3.4].

Proposition 4.15. *Let G be a compact ample principal groupoid. For each n , let X_n be the subset of G^0 consisting of elements with n elements in their orbit, and let Y_n be X_n modulo the orbit equivalence relation. Then each X_n is a compact open subset of G^0 (possibly empty), and there exists N and a $*$ -isomorphism*

$$C_r^*(G) \cong \bigoplus_{n=1}^N M_n(C(Y_n)).$$

Moreover, this $*$ -isomorphism can be chosen so that the inclusion $C(G^0) \subseteq C_r^*(G)$ corresponds to the inclusion

$$\bigoplus_{n=1}^N C(Y_n) \otimes D_n \subseteq \bigoplus_{n=1}^N C(Y_n) \otimes M_n,$$

where $D_n \subseteq M_n$ is the standard inclusion of the diagonal subalgebra. \square

Our first aim is to prove the following result.

Theorem 4.16. *Let G be an ample groupoid with dynamic asymptotic dimension at most one. Then the inclusion*

$$\iota : C_0(G^0) \rightarrow C_r^*(G)$$

induces a surjection on K_0 -groups.

The proof of Theorem 4.16 uses some controlled K -theory machinery from [20, 12, 11].

Definition 4.17. Let G be a principal ample groupoid, and let \mathcal{C} denote the collection of compact open subsets C of G that satisfy $C = C^{-1}$. We equip \mathcal{C} with the partial order given by inclusion, and with the (commutative) monoid operation defined by

$$C_1 \circ C_2 := C_1 \cup C_2 \cup C_1 C_2 \cup C_2 C_1,$$

where the products on the right are the usual products for subsets of a groupoid, e.g.

$$C_1 C_2 = \{g \in G \mid \text{there are } c_1 \in C_1, c_2 \in C_2 \text{ with } c_1 c_2 = g\}.$$

A C^* -algebra A is *filtered* by \mathcal{C} if there are a collection of self-adjoint subspaces $(A_C)_{C \in \mathcal{C}}$ of A satisfying the following conditions:

- (i) $A_{C_1} \subseteq A_{C_2}$ whenever $C_1 \subseteq C_2$;
- (ii) $A_{C_1} A_{C_2} \subseteq A_{C_1 \circ C_2}$;
- (iii) $\bigcup_{C \in \mathcal{C}} A_C$ is dense in A .

The basic example is when $A = C_r^*(G)$ and A_C consists of all elements of $C_c(G)$ with support in C . However, we will need to use a slightly different filtration below.

If A is a C^* -algebra filtered by \mathcal{C} , the idea of the controlled K -theory group $K_0^{C, \epsilon}(A)$ is to build the K -theory group in the usual sort of way, but only allowing projections and homotopies from matrices over A_C ; as matrices over A_C may contain very few projections, to make good sense of this one allows projections-up-to- ϵ -error. The group $K_1^{C, \epsilon}(A)$ is defined similarly, but working with unitaries-up-to- ϵ -error. As usual, we write $K_*^{C, \epsilon}(A)$ for the graded group $K_0^{C, \epsilon}(A) \oplus K_1^{C, \epsilon}(A)$.

We will not need the precise definitions here, but refer the reader [12, Section 2.1] for details. Following [20], we usually just fix $\epsilon = 1/8$ in what follows and use the words *quasi-projection* or *quasi-unitary* for operators that are projections or unitaries up to $1/8$ -error in the appropriate sense.

The following is a slight variation on [20, Definitions 7.2 and 7.3]: the only difference is that we work here with filtrations indexed by the set \mathcal{C} of Definition 4.17, rather than the real numbers.

Definition 4.18. Let $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$, and A be a C^* -algebra. Let $A \otimes \mathcal{K}$ denote the spatial tensor product of A and \mathcal{K} ; using the canonical orthonormal basis on $\ell^2(\mathbb{N})$, we identify elements of $A \otimes \mathcal{K}$ with \mathbb{N} -by- \mathbb{N} matrices with entries from A . For a subspace S of A , let $S \otimes \mathcal{K}$ denote the subspace of $A \otimes \mathcal{K}$ consisting of matrices with all elements in S . Note that a filtration on A by \mathcal{C} induces a filtration on $A \otimes \mathcal{K}$ by defining $(A \otimes \mathcal{K})_C := A_C \otimes \mathcal{K}$.

Let $(A^\omega)_{\omega \in \Omega}$ be a collection of \mathcal{C} -filtered C^* -algebras, and filter each stabilization $A^\omega \otimes \mathcal{K}$ as above. For each $\omega \in \Omega$, let I^ω and J^ω be C^* -ideals in A^ω , which we assume are filtered by the subspaces $I_C^\omega := A_C^\omega \cap I^\omega$ and $J_C^\omega := A_C^\omega \cap J^\omega$. Equip the stabilizations $I^\omega \otimes \mathcal{K}$ and $J^\omega \otimes \mathcal{K}$ with the filtrations defined above.

The collection $(I^\omega, J^\omega)_{\omega \in \Omega}$ of pairs of ideals is *uniformly excisive* if for any $C_0 \in \mathcal{C}$, and $m_0 \geq 0$ and $\epsilon > 0$, there are $C \supseteq C_0$, $m \geq 0$, and $\delta > 0$ such that:

- (i) for any $\omega \in \Omega$ and any $a \in (A^\omega \otimes \mathcal{K})_{C_0}$ of norm at most m_0 , there exist elements $b \in (I^\omega \otimes \mathcal{K})_C$ and $c \in (J^\omega \otimes \mathcal{K})_C$ of norm at most m such that $\|a - (b + c)\| < \epsilon$;
- (ii) for any $\omega \in \Omega$ and any $a \in I^\omega \otimes \mathcal{K} \cap J^\omega \otimes \mathcal{K}$ such that

$$d(a, (I^\omega \otimes \mathcal{K})_{C_0}) < \delta \quad \text{and} \quad d(a, (J^\omega \otimes \mathcal{K})_{C_0}) < \delta$$

there exists $b \in (I^\omega \otimes \mathcal{K})_C \cap (J^\omega \otimes \mathcal{K})_C$ such that $\|a - b\| < \epsilon$.

The point of this definition is that it enables one to derive the following ‘‘Mayer-Vietoris principle’’. This is a slight variation on [20, Proposition 7.6]: the only real difference in the statement is that here we work with filtrations indexed by the set \mathcal{C} of Definition 4.17 rather than the real numbers, and this makes no difference to the proof.

Proposition 4.19. *Say $(A^\omega)_{\omega \in \Omega}$ is a collection of non-unital filtered C^* -algebras and $(I^\omega, J^\omega)_{\omega \in \Omega}$ a uniformly excisive collection of pairs of non-unital ideals. Then for any $C_0 \in \mathcal{C}$ there are $C_1, C_2 \supseteq C_0$ with the following property. For each ω , each $i \in \{0, 1\}$ and each $x \in K_i^{C_0, 1/8}(A^\omega)$ there is an element*

$$\partial_c(x) \in K_{i+1}^{C_1, 1/8}(I^\omega \cap J^\omega)$$

(here ‘ $i + 1$ ’ is to be understood mod 2) such that if $\partial_c(x) = 0$ then there exist

$$y \in K_i^{C_2, 1/8}(I^\omega) \quad \text{and} \quad z \in K_i^{C_2, 1/8}(J^\omega)$$

such that

$$x = y + z \quad \text{in} \quad K_i^{C_2, 1/8}(A^\omega)$$

(here we have abused notation, conflating y and z with the images under the canonical maps $K_i^{C_2, 1/8}(I^\omega) \rightarrow K_i^{C_2, 1/8}(A^\omega)$ and $K_i^{C_2, 1/8}(J^\omega) \rightarrow K_i^{C_2, 1/8}(A^\omega)$ induced by the inclusions of I^ω and J^ω in A^ω ; we will frequently abuse notation in this way below). \square

Let now G be an arbitrary ample groupoid. Using the canonical linear injective map $C_r^*(G) \rightarrow C_0(G)$ (see for example [40, Proposition 2.3.20]), we think of elements of $C_r^*(G)$ as functions $G \rightarrow \mathbb{C}$; it thus makes sense to speak of their support. We extend the definitions of support to $C_r^*(G) \otimes \mathcal{K}$ by considering elements as $\mathbb{N} \times \mathbb{N}$

matrices over $C_r^*(G)$, and defining the support of an element a to be the union of the supports of all its matrix entries.

For a subset U of G^0 and $C \in \mathcal{C}$, define

$$U^{(C)} := s(r^{-1}(U) \cap C) \cup r(s^{-1}(U) \cap C).$$

Let Ω be the set of all pairs U, V of compact open subsets of G^0 . Given such a pair and a compact subset C of G , define subspaces of $C_r^*(G)$ by

$$B(U)_C := \left\{ a \in C_r^*(G) \otimes \mathcal{K} \mid r(\text{supp}(a)) \cup s(\text{supp}(a)) \subseteq U^{(C)} \text{ and } \text{supp}(a) \subseteq C \right\},$$

and similarly for $B(V)_C$. Define also $B(U \cap V)_C$ to be the C^* -subalgebra

$$C^*\left(\left\{ a \in C_r^*(G) \otimes \mathcal{K} \mid r(\text{supp}(a)) \cup s(\text{supp}(a)) \subseteq U^{(C)} \cap V^{(C)}, \text{supp}(a) \subseteq C \right\} \right)$$

of $C_r^*(G) \otimes \mathcal{K}$. Define further

$$A_C^\omega := B(U)_C + B(V)_C + B(U \cap V)_C$$

and

$$I_C^\omega := B(U)_C + B(U \cap V)_C, \quad J_C^\omega := B(V)_C + B(U \cap V)_C$$

Define $A^\omega := \overline{\bigcup_{C \in \mathcal{C}} A_C^\omega}$, $I^\omega := \overline{\bigcup_{C \in \mathcal{C}} I_C^\omega}$ and $J^\omega := \overline{\bigcup_{C \in \mathcal{C}} J_C^\omega}$.

Lemma 4.20. *Equipped with the structures above, each A^ω is a C^* -algebra filtered by the subspaces A_C^ω . Moreover, I^ω and J^ω are ideals in A^ω , filtered by the subspaces I_C^ω and J_C^ω respectively.*

Proof. For any C, D , it is straightforward to check the following inclusions of subspaces:

$$B(U)_C \cdot B(U)_D \subseteq B(U)_{C \circ D}, \quad B(V)_C \cdot B(V)_D \subseteq B(V)_{C \circ D}$$

and

$$B(U \cap V)_C \cdot B(U \cap V)_D \subseteq B(U \cap V)_{C \circ D};$$

direct checks also give

$$B(U)_C \cdot B(V)_D \subseteq B(U \cap V)_{C \circ D}$$

and

$$B(U)_C \cdot B(U \cap V)_D \subseteq B(U \cap V)_{C \circ D}, \quad B(V)_C \cdot B(U \cap V)_D \subseteq B(U \cap V)_{C \circ D}.$$

Combining all these inclusions gives the statement of the lemma. \square

Note that each A^ω is just a copy of $C_r^*(G) \otimes \mathcal{K}$ as a C^* -algebra, but with a different filtration from the usual one.

Lemma 4.21. *With notation as above, the collection $(I^\omega, J^\omega)_{\omega \in \Omega}$ is uniformly excisive.*

Proof. For notational simplicity, let us ignore the tensored-on copies of \mathcal{K} ; as the reader can verify, this makes no difference to the proof. For a compact open subset U of G^0 , let 1_U denote the characteristic function of U , considered as an element of $C_0(G^0)$.

Let first $C_0 \in \mathcal{C}$, and let a be an element of $A_{C_0}^\omega$. Then we have that

$$a = 1_U a + 1_{V \setminus U} a$$

It is straightforward to check that the first term is in $I_{C_0}^\omega$ and the second in $J_{C_0}^\omega$ so this implies the condition (i) from Definition 4.18.

Now say we are given $\epsilon > 0$ and let $\delta = \epsilon/3$. Assume $a \in I^\omega \cap J^\omega$ satisfies $d(a, I_{C_0}^\omega) < \delta$ and $d(a, J_{C_0}^\omega) < \delta$, so there are $a_U \in I_{C_0}^\omega$ and $a_V \in J_{C_0}^\omega$ with $\|a - a_U\| < \delta$ and $\|a - a_V\| < \delta$. Let $1_{U^{(C_0)}}$ be the characteristic function of $U^{(C_0)}$. Then

$a_U 1_{U(C_0)} = a_U$, and it is not difficult to check that $a_V 1_{U(C_0)}$ is in $I_{C_0^{\circ 2}}^\omega \cap J_{C_0^{\circ 2}}^\omega$, where $C_0^{\circ 2} := C_0 \circ C_0$. Moreover,

$$\|a - a_V 1_{U(C_0)}\| \leq \|a - a_U\| + \|a_U 1_{U(C_0)} - a_V 1_{U(C_0)}\| < \epsilon,$$

completing the proof. \square

We are now ready for the proof of Theorem 4.16.

Proof of Theorem 4.16. It suffices to show that if p is a projection in $M_n(C_r^*(G))$ for some n , then the class $[p] \in K_0(C_r^*(G))$ is in the image of ι_* . Let q be a quasi-projection in $M_n(C_r^*(G))$ that is supported in C_0 for some compact open symmetric subset C_0 of G , and that approximates p well enough so that the comparison map defined analogously to [20, Definition 4.8] takes $[q] \in K_0^{C,1/8}(C_r^*(G))$ to $[p]$ (here $C_r^*(G)$ is equipped with the usual filtration by support).

Now, for the uniformly excisive family from Lemma 4.21, let C_1 and C_2 be associated to this C_0 as in Proposition 4.19. Using dynamic asymptotic dimension at most one, there exist compact open sets U and V such that if $C = (C_1 \cup C_2 \cup C_3)^{\circ 3}$ then $\{U, V\}$ covers $r(C) \cup s(C)$ and such that the subgroupoid G_U of G generated by

$$\{g \in C \mid r(g) \in U\}$$

is compact and open, and similarly for G_V with V replacing U . Consider the element

$$\delta_c[q] \in K_1^{C_1,1/8}(I^\omega \cap J^\omega) = K_1^{1/8}(B(U \cap V)_{C_1}).$$

If $g \in C_1$ satisfies $r(g), s(g) \in U^{(C_1)}$, then $g \in G_U$. In fact, there exists $h \in s^{-1}(U) \cap C_1$ such that $s(g) = r(g)$, so $s(gh) = s(h) \in U$ and $gh \in C_1 \circ C_1 \subseteq C$ which implies $gh \in G_U$. As $h^{-1} \in G_U$ it follows that $g = (gh)h^{-1} \in G_U$. Thus $B(U \cap V)_{C_1}$ is actually the stabilisation of the C^* -algebra of the ample groupoid generated by

$$\{g \in C_1 \mid r(g), s(g) \in U^{(C_1)} \cap V^{(C_1)}\}.$$

This is contained in G_U and is thus compact. Proposition 4.15 hence gives that $B(U \cap V)_{C_1}$ is stably isomorphic to a finite direct sum $\bigoplus_{n=1}^N M_n(C(Y_n))$ of matrix algebras over zero-dimensional spaces. Hence by the natural analogue in our context of [20, Proposition 4.9]

$$K_1^{C_1,1/8}(B(U \cap V)_{C_1}) = K_1\left(\bigoplus_{n=1}^N M_n(C(Y_n))\right) = 0,$$

and so in particular, $\partial_c[q] = 0$.

It follows from Proposition 4.19 that there are elements $y \in K_0^{C_2,1/8}(I^\omega)$, $z \in K_0^{C_2,1/8}(J^\omega)$ with $[q] = y + z$ in $K_0^{C_2,1/8}(A^\omega)$. Passing through the canonical maps

$$K_0^{C_2,1/8}(A^\omega) \rightarrow K_0(A^\omega) \cong K_0(C_r^*(G)),$$

it follows that $[p] = y + z$ in $K_0(C_r^*(G))$, where we have abused notation by conflating y, z , and their images under the canonical comparison maps (see [20, Definition 4.8])

$$K_0^{C_2,1/8}(I^\omega) \rightarrow K_0(C_r^*(G)) \quad \text{and} \quad K_0^{C_2,1/8}(J^\omega) \rightarrow K_0(C_r^*(G)).$$

On the other hand, by choice of C_2 again, there is a factorization of the first of these maps as

$$K_0^{C_2,1/8}(I^\omega) \rightarrow K_0(C_r^*(G_U)) \rightarrow K_0(C_r^*(G)),$$

and similarly for J and V replacing I and U . As the groupoid G_U is ample and compact, we have that again it is isomorphic to a finite direct sum

$$C_r^*(G_U) \cong \bigoplus_{m=1}^M M_m(C(Z_m))$$

for zero dimensional spaces Z_m . The diagonal inclusion corresponds to

$$\bigoplus_{m=1}^M C(Z_m) \otimes D_m \rightarrow \bigoplus_{m=1}^M C(Z_m) \otimes M_m,$$

and is thus surjective on K -theory. This suffices to complete the proof. \square

Corollary 4.22. *If G has dynamic asymptotic dimension at most one, then $\mu_0 : H_0(G) \rightarrow K_0(C_r^*(G))$ is surjective.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} C(G^0, \mathbb{Z}) & \longrightarrow & H_0(G) \\ \downarrow \cong & & \downarrow \mu_0 \\ K_0(C(G^0)) & \xrightarrow{\iota_0} & K_0(C_r^*(G)). \end{array}$$

Theorem 4.16 implies that ι_0 is surjective, whence by commutativity μ_0 is surjective too. \square

We will now turn our attention to the map μ_1 constructed earlier and prove an analogue of Theorem 4.16 in dimension one.

Lemma 4.23. *Let G be a compact ample principal groupoid. Then*

$$\rho : C(G, \mathbb{Z})/im(\partial_2) \rightarrow K_0(C(\iota))$$

is an isomorphism, and $K_1(C(\iota)) = 0$.

Proof. Since G is compact ample and principal we have $H_1(G) = 0 = K_1(C^*(G))$. Using the canonical isomorphism $K_0(C(\iota)) \cong V_\infty(C(G^0), C_r^*(G))/\approx$ from Proposition 4.7, the diagram from the proof of Theorem 4.14 degenerates to a homomorphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & C(G, \mathbb{Z})/im(\partial_2) & \longrightarrow & C(G^0, \mathbb{Z}) & \longrightarrow & H_0(G) \longrightarrow 0 \\ & & \downarrow \rho & & \downarrow \cong & & \downarrow \mu_0 \\ 0 & \longrightarrow & K_0(C(\iota)) & \longrightarrow & K_0(C(G^0)) & \longrightarrow & K_0(C_r^*(G)) \longrightarrow 0 \end{array}$$

Since μ_0 is known to be an isomorphism for elementary groupoids the five lemma completes the proof of the first assertion.

That $K_1(C(\iota)) = 0$ follows from considering the six-term exact sequence (9) for ι , using that $K_1(C(G^0)) = 0$ and $K_0(C(G^0)) \rightarrow K_0(C_r^*(G))$ is surjective. \square

We now want to use controlled K -theory methods on the mapping cone.

Lemma 4.24. *Let A be filtered by \mathcal{C} and $D \subseteq A$ be a C^* -subalgebra also filtered by \mathcal{C} such that the inclusion map $\iota : D \hookrightarrow A$ is a filtered map (i.e. $\iota(D_C) \subseteq A_C$ for all $C \in \mathcal{C}$). Then the mapping cone $C(\iota) = \{f \in C_0((0, 1], A) \mid f(1) \in D\}$ is filtered by the subspaces*

$$C(\iota)_C := \{f \in C_0((0, 1], A_C) \mid f(1) \in D_C\}.$$

Proof. If $C_1, C_2 \in \mathcal{C}$ with $C_1 \subseteq C_2$, then $C(\iota)_{C_1} \subseteq C(\iota)_{C_2}$ since $A_{C_1} \subseteq A_{C_2}$ and $D_{C_1} \subseteq D_{C_2}$.

Similarly, if $f_1 \in C(\iota)_{C_1}$ and $f_2 \in C(\iota)_{C_2}$, then $(f_1 f_2)(t) = f_1(t) f_2(t) \in A_{C_1} A_{C_2} \subseteq A_{C_1 C_2}$ and $(f_1 f_2)(1) = f_1(1) f_2(1) \in D_{C_1} D_{C_2} \subseteq D_{C_1 C_2}$ and hence $f_1 f_2 \in C(\iota)_{C_1 C_2}$.

It remains to show that $\bigcup_{C \in \mathcal{C}} C(\iota)_C$ is dense in $C(\iota)$. To this end note that the evaluation homomorphism $ev_1 : C(\iota) \rightarrow D$ admits a completely positive contractive linear cross-section $j : D \rightarrow C(\iota)$. Let $f \in C(\iota)$ and $\varepsilon > 0$ be given. Then $f - j(f(1)) \in SA$ and since $\bigcup A_C$ is dense in A , $\bigcup C(0, 1) \otimes A_C$ is dense in the suspension SA of A . Hence we can find $f_1 \in C(0, 1) \otimes A_C$ such that $\|(f - j(f(1))) - f_1\| < \frac{\varepsilon}{2}$. Moreover, since $\bigcup D_C$ is dense in D we can find $d \in D$ such that $\|f(1) - d\| < \frac{\varepsilon}{2}$. Then $f_1 + j(d) \in C(\iota)_C$ and combining the above inequalities we obtain

$$\|f - (f_1 + j(d))\| \leq \|f - j(f(1)) - f_1\| + \|j(f(1)) - j(d)\| < \varepsilon$$

□

Now let $A = C_r^*(G) \otimes \mathcal{K}(\ell^2(\mathbb{N}))$ and $D = C_0(G^0) \otimes c_0(\mathbb{N})$. For each pair $\omega = (U, V)$ of compact open subspaces of G^0 let A^ω be A with the filtration $A_C^\omega = B(U)_C + B(V)_C + B(U \cap V)_C$ as before. Filter $D^\omega := C_0(G^0)$ accordingly by letting $D_C^\omega := D \cap A_C^\omega$. Then we get an induced filtration of $C(\iota)^\omega$. Similarly, we get ideals $\tilde{I}^\omega, \tilde{J}^\omega \subseteq C(\iota)^\omega$, corresponding to the ideals I^ω and J^ω , with associated filtrations.

Lemma 4.25. *With notation as above, the collection $(\tilde{I}^\omega, \tilde{J}^\omega)_{\omega \in \Omega}$ is uniformly excisive.*

Proof. Let $C_0 \in \mathcal{C}$, $m_0 \geq 0$ and $\varepsilon > 0$ be given. Set $m := m_0$, $C = C_0 \cdot C_0 \cup C_0$ and $\delta := \frac{\varepsilon}{3}$. Then for any $\omega \in \Omega$ and $f \in C(\iota)_{C_0}^\omega$ with $\|f\| \leq m_0$ we let $f_1(t) := 1_{U(C)} f(t)$ and $f_2(t) := 1_{V(C) \setminus U(C)} f(t)$. Then $f_1 \in \tilde{I}_C^\omega$, $f_2 \in \tilde{J}_C^\omega$ with $\|f_1\| \leq m_0$ and $\|f_2\| \leq m_0$ and $f = f_1 + f_2$, which implies condition (i).

Now if $f \in \tilde{I}^\omega \cap \tilde{J}^\omega$ such that there exist $f_1 \in \tilde{I}_{C_0}^\omega$ and $f_2 \in \tilde{J}_{C_0}^\omega$ with $\|f - f_1\| < \delta$ and $\|f - f_2\| < \delta$, define $\tilde{f}(t) = f_2(t) 1_{U(C_0)}$. Then $\tilde{f} \in \tilde{I}_C^\omega \cap \tilde{J}_C^\omega$. Moreover, since $f_1(t) 1_{U(C_0)} = f_1(t)$ for all $t \in (0, 1]$ we have

$$\|f - \tilde{f}\| \leq \sup \|f(t) - f_1(t)\| + \|f_1(t) 1_{U(C_0)} - f_2(t) 1_{U(C_0)}\| < \varepsilon,$$

which completes the proof. □

We are now ready to prove the following result:

Theorem 4.26. Let G be an ample principal groupoid with dynamic asymptotic dimension at most one. Then the canonical map

$$\tilde{\rho} : C_c(G, \mathbb{Z}) \rightarrow K_0(C(\iota)),$$

constructed in Lemma 4.13 is surjective.

Proof. Let $x \in K_0(C(\iota))$. Consider $C(\iota)$ as a filtered C^* -algebra, where the filtration is the one induced by the usual filtration of $C_r^*(G)$ by support. Let C_0 be a compact open symmetric subset of G and let $y \in K_0^{C_0, \frac{1}{8}}(C(\iota))$ such that the canonical map $K_0^{C_0, \frac{1}{8}}(C(\iota)) \rightarrow K_0(C(\iota))$ takes y to x . Choose compact open subsets $C_1, C_1 \supseteq C_0$ as in Proposition 4.19. Apply the assumption that G has dynamic asymptotic dimension at most one to the compact open set $C := (C_0 \cup C_1 \cup C_2)^3$ to obtain $U_1, U_2 \subseteq G^0$ such that $s(C) \cup r(C) \subseteq U_1 \cup U_2$ and for $i = 1, 2$ the subgroupoid H_i of G generated by $\{g \in C \mid s(g), r(g) \in U_i\}$ is compact open in G . Moreover, let H be the open subgroupoid of G generated by $\{g \in C_1 \mid r(g), s(g) \in U_1^{(C_1)} \cap U_2^{(C_1)}\}$. By our choice of C , the groupoid H is compact as well. For $i = 1, 2$ let \tilde{I}_i^ω denote the ideal of $C(\iota)^\omega$ as defined prior to Lemma 4.25. Now by definition we have

$(\tilde{I}_1^\omega \cap \tilde{I}_2^\omega)_{C_1} = \{f \in C(\iota) \mid f(t) \in B(U_1 \cap U_2)\}_{C_1}$. Since $B(U_1 \cap U_2)_{C_1} = C_r^*(H)$, the latter is just the cone $C(\iota_H)$ of the canonical inclusion $\iota_H : C(H^{(0)}) \hookrightarrow C_r^*(H)$. Hence we have $K_1^{C_1, \frac{1}{8}}(\tilde{I}_1^\omega \cap \tilde{I}_2^\omega) = K_1^{C_1, \frac{1}{8}}(C(\iota_H)) = 0$ by Lemma 4.23. Consequently, we must have $\partial_c(y) = 0$ and by our choice of C_1 and C_2 there must exist elements $y_i \in K_0^{C_2, \frac{1}{8}}(\tilde{I}_i^\omega)$ such that $y = y_1 + y_2$. If x_1 and x_2 denote their respective images in $K_0(C(\iota))$ under the canonical maps we get $x = x_1 + x_2$. Now for $i = 1, 2$ these canonical maps factor as

$$K_0^{C_2, \frac{1}{8}}(\tilde{I}_i^\omega) \rightarrow K_0(C(\iota_i)) \rightarrow K_0(C(\iota)).$$

Since both H_1 and H_2 are compact, principal and ample groupoids the result follows from Lemma 4.23. \square

Corollary 4.27. Let G be an ample principal groupoid with dynamic asymptotic dimension at most one. Then the canonical map

$$\mu_1 : H_1(G) \rightarrow K_1(C_r^*(G))$$

is surjective and the canonical map

$$\mu_0 : H_0(G) \rightarrow K_0(C_r^*(G))$$

is an isomorphism.

Proof. Considering the diagram (17) again, the surjectivity of μ_1 clearly follows from the surjectivity of ρ . Furthermore, an application of the four-lemma then implies that μ_0 is injective. \square

Remark 4.28. In Corollary 4.29 below, we will see that under the assumptions of Corollary 4.27 (and a second countability assumption) there are abstract isomorphisms $H_0(G) \cong K_0(C_r^*(G))$ and $H_1(G) \cong K_1(C_r^*(G))$. It is natural to guess that these isomorphisms are given by μ_0 and μ_1 , but we were unable to establish this. The main interest of Corollary 4.27 over Corollary 4.29 is the explicit description of the maps involved.

4.3. The spectral sequence of Proietti and Yamashita. A recent result of Proietti and Yamashita in [37] established the existence of a convergent spectral sequence

$$(18) \quad E_{p,q}^2 = H_p(G, K_q(A)) \Rightarrow K_{p+q}(A \rtimes_r G)$$

for any G -algebra A , provided that G is a second countable ample groupoid with torsion free isotropy, which satisfies the strong Baum-Connes conjecture.

Combining this with our results from the previous section we immediately obtain more general isomorphism results at the expense of loosing an explicit description of the maps involved.

Corollary 4.29. Let G be a second countable principal ample groupoid with dynamic asymptotic dimension at most 2. Then the HK-conjecture holds for G , i.e.

$$K_0(C_r^*(G)) \cong H_0(G) \oplus H_2(G) \text{ and } K_1(C_r^*(G)) \cong H_1(G).$$

Proof. First of all we can apply the spectral sequence (18) since G is principal and any groupoid with finite dynamic asymptotic dimension is in particular amenable (this follows from the proof of [20, Theorem A.9]), and hence satisfies the strong Baum-Connes conjecture by the main result of [48]. Since $H_n(G) = 0$ for all $n \geq 3$ by Theorem 3.36 the spectral sequence (18) collapses on the second page and we conclude that $K_1(C_r^*(G)) \cong H_1(G)$ and that $K_0(C_r^*(G))$ fits into a short exact sequence

$$0 \rightarrow H_0(G) \rightarrow K_0(C_r^*(G)) \rightarrow H_2(G) \rightarrow 0.$$

We further know that $H_2(G)$ is free abelian which implies that the sequence above splits. The result follows. \square

Combining this abstract result in degree 1 and our explicit isomorphism μ_0 we have the following consequence.

Corollary 4.30. *Let G be a second countable, principal, ample groupoid with compact base space and dynamic asymptotic dimension at most 1. Then*

$$\text{Ell}(C_r^*(G)) = (H_0(G), H_0(G)^+, [1_{G^0}], H_1(G), M(G), p),$$

where $M(G)$ is the set of all G -invariant probability measures on G^0 and $p : M(G) \times H_0(G) \rightarrow \mathbb{R}$ is the pairing given by $p(\mu, [f]_0) = \int f d\mu$.

Proof. We have an abstract isomorphism $H_1(G) \cong K_1(C_r^*(G))$ from the previous corollary, and we know that the canonical map $\mu_0 : H_0(G) \rightarrow K_0(C_r^*(G))$ is an isomorphism from Theorem 4.27. It clearly extends to an isomorphism of ordered groups respecting the position of the unit. Since G is principal, there is an affine homeomorphism between the set $M(G)$ of G -invariant probability measures on G^0 and the tracial state space $T(C_r^*(G))$ of the reduced groupoid C^* -algebra (see, for instance, [27, Section 4.1]). Finally, from the definition of μ_0 it is clear that the pairings are compatible. \square

5. EXAMPLES AND APPLICATIONS

In this final section we discuss several applications of our results for specific classes of groupoids and exhibit some interesting examples.

5.1. Free actions on totally disconnected spaces. In [8, Theorem 1.3], Conley et. al. show that for a large class of countable groups Γ , any free action on a second countable, locally compact, zero-dimensional space has dynamic asymptotic dimension at most the asymptotic dimension of Γ , and that the latter is finite. If X is compact, for example a Cantor set, then the dynamic asymptotic dimension will therefore be exactly equal to the asymptotic dimension of Γ by [21, Theorem 6.5]. The class described by the authors of [8] is technical and we refer there for details; suffice to say that it includes many interesting examples such as all polycyclic groups, all virtually nilpotent groups, the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$, and the Baumslag-Solitar group $BS(1, 2)$.

Hence for such actions, Theorem 3.36 implies that $H_n(\Gamma, C_c(X, \mathbb{Z})) = 0$ for all $n > \text{asdim}(\Gamma)$. For actions of polycyclic groups on the Cantor set, we can make this quite explicit: combining results of Bell and Dranishnikov, and of Dranishnikov and Smith (see [4, Theorem 66]), the asymptotic dimension of a finitely generated polycyclic group is equal to its Hirsch length $h(\Gamma)$. Hence for a free action of a finitely generated polycyclic group on a Cantor set

$$\text{dad}(\Gamma \ltimes X) = \text{asdim}(\Gamma) = h(\Gamma).$$

In particular, if $h(\Gamma) \leq 2$, Corollary 4.29 gives isomorphisms

$$K_0(C(X) \rtimes \Gamma) \cong H_0(\Gamma, C(X, \mathbb{Z})) \oplus H_2(\Gamma, C(X, \mathbb{Z})), \text{ and}$$

$$K_1(C(X) \rtimes \Gamma) \cong H_1(\Gamma, C(X, \mathbb{Z})).$$

5.2. Smale spaces with totally disconnected (un)stable sets. A Smale space consists of a self-homeomorphism $\varphi : X \rightarrow X$ of a compact metric space X , such that the space can be locally decomposed into the product of a coordinate whose points get closer together as φ is iteratively applied, and a coordinate whose points

get farther apart under the map φ . We refer to [39] for basic definitions and details. Given a Smale space (X, φ) one can define two equivalence relations on X as follows:

$$x \sim_s y \text{ if and only if } \lim_{n \rightarrow \infty} d(\varphi^n(x), \varphi^n(y)) = 0, \text{ and}$$

$$x \sim_u y \text{ if and only if } \lim_{n \rightarrow \infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0$$

Let $X^s(x)$ and $X^u(x)$ denote the stable and unstable equivalence classes of a point $x \in X$ respectively. Upon choosing a finite set P of φ -periodic points one can construct étale principal groupoids $G^u(X, P)$ and $G^s(X, P)$ with unit space $X^s(P) = \bigcup_{x \in P} X^s(x)$ and $X^u(P) = \bigcup_{x \in P} X^u(x)$, respectively. In particular, if these unit spaces are totally disconnected, the groupoids are ample. Note further, that for an irreducible Smale space (X, φ) , the groupoids $G^u(X, P)$ and $G^s(X, P)$ only depend on P up to equivalence. In particular, the choice of P is irrelevant when computing their homology. Deeley and Strung prove in [10] that for an irreducible Smale space one has the estimate

$$\text{dad}(G^u(X, P)) \leq \dim X.$$

Combining this with our Theorem 3.36 and Corollary 4.29, and also [37, Theorem 4.15] we can compute the K-theory of the resulting C^* -algebras from Putnam's homology for Smale spaces.

Corollary 5.1. *Let (X, φ) be an irreducible Smale space with totally disconnected stable sets. Then $H_n^s(X, \varphi) = 0$ for all $n > \dim(X)$; and if $\dim(X) \leq 2$ then*

$$K_0(C^*(G^u(X, P))) \cong H_0^s(X, \varphi) \oplus H_2^s(X, \varphi) \text{ and } K_1(C^*(G^u(X, P))) \cong H_1^s(X, \varphi).$$

This result includes most of the previously known examples (see e.g. [50]) that were based on separate computations of the K-theory and homology, and hence provides a more conceptual explanation.

5.3. Bounded geometry metric spaces. A metric space X has *bounded geometry* if for each $r > 0$ there is a uniform bound on the cardinalities of all r -balls in X ; important examples come from groups with word metric, or from suitable discretisations of Riemannian manifolds.

Skandalis, Tu, and Yu [44] construct an ample groupoid $G(X)$ which captures the coarse geometric information of X . In particular, the reduced groupoid C^* -algebra $C_r^*(G(X))$ can be canonically identified with the uniform Roe algebra $C_u^*(X)$. Let us briefly recall the construction. Let βX denote the Stone-Čech compactification of X , i.e. the maximal ideal space of $\ell^\infty(X)$. For each $r > 0$, let E_r be the closure of $\{(x, y) \in X \times X \mid d(x, y) \leq r\}$ inside $\beta X \times \beta X$,⁶ which is a compact open set. Then the *coarse groupoid* $G(X)$ of X has as underlying set $\bigcup_{r=0}^\infty E_r$. The operations are the restriction of the pair groupoid operations from $\beta X \times \beta X$, and the topology is the weak topology coming from the union $\bigcup_{r=0}^\infty E_r$, i.e. a subset U of G is open if and only if $U \cap E_r$ is open for each r . Then $G(X)$ is a principal, ample, σ -compact groupoid with compact base space homeomorphic to βX , see [41, Theorem 10.20].

Our first goal is to identify the groupoid homology $H_*(G(X))$ with the uniformly finite homology of X introduced by Block and Weinberger in [6, Section 2]. We begin by recalling the relevant definitions. Let $C_n(X)$ denote the collection of all bounded functions $c : X^{n+1} \rightarrow \mathbb{Z}$ such that there exists $r > 0$ such that if $c(x_0, \dots, x_n) \neq 0$, then the diameter of the set $\{x_0, \dots, x_n\}$ is at most r . For each

⁶The authors of [44] take the closure in $\beta(X \times X)$ instead of in $\beta X \times \beta X$, but by [41, Proposition 10.15] these closures are canonically homeomorphic, so it does not matter which of them one uses.

$i \in \{0, \dots, n\}$, let $\partial^i : X^{n+1} \rightarrow X^n$ be defined by $\partial^i(x_0, \dots, x_n) := (x_0, \dots, \hat{x}_i, \dots, x_n)$. Define $\partial_*^i : C_n(X) \rightarrow C_{n-1}(X)$ by

$$(\partial_*^i c) = \sum_{\partial^i y = x} c(x)$$

and define $\partial : C_n(X) \rightarrow C_{n-1}(X)$ by $\partial := \sum_{i=0}^n (-1)^i \partial_*^i$. Then we have $\partial \circ \partial = 0$, so we get a chain complex. The *uniformly finite homology* of X , denoted $H_*^{\text{uf}}(X)$, is by definition the associated homology of this complex.

Having introduced all the main actors we can now prove the following theorem.

Theorem 5.2. *Let $G(X)$ be the coarse groupoid associated to a bounded geometry metric space X . Then there is a canonical isomorphism $H_*(G(X)) \cong H_*^{\text{uf}}(X)$.*

Proof. For brevity let us denote the coarse groupoid by G throughout the proof. Now, for $a \in \mathbb{Z}[EG_n]$, define $\bar{a} \in \mathbb{Z}[EG_n]$ by

$$\bar{a}(g_0, \dots, g_n) := \begin{cases} \sum_{h \in G_x} a(h, hg_1, \dots, hg_n) & g_0 = x \in G^0 \\ 0 & g_0 \notin G^0 \end{cases}.$$

One can check (we leave this to the reader) that the equivalence classes $[\bar{a}]$ and $[a]$ in $\mathbb{Z}[EG_n]_G$ of \bar{a} and a respectively are the same, and moreover that \bar{a} is the unique element of $[a]$ that is supported on $\{(g_0, \dots, g_n) \in EG_n \mid g_0 \in G^0\}$.

We define maps $\alpha : \mathbb{Z}[EG_n]_G \rightarrow C_n(X)$ and $\beta : C_n(X) \rightarrow \mathbb{Z}[EG_n]_G$ as follows. First,

$$(\alpha[a])(x_0, \dots, x_n) := \bar{a}((x_0, x_0), \dots, (x_0, x_n)).$$

This makes sense using that G contains the pair groupoid $X \times X$. We note that $\alpha[a]$ is bounded as \bar{a} is. Moreover, the fact that \bar{a} has compact support implies that it is supported in a set of the form $E_0 \times E_{r_1} \times \dots \times E_{r_n} \cap EG_n$ for compact open sets E_{r_i} as in the definition of G . It follows that $\alpha[a]$ is supported on the set of tuples with diameter at most $2 \max\{r_1, \dots, r_n\}$ and is thus a well-defined element of $C_n(X)$.

To define β , let first $((x, x_0), (x, x_1), \dots, (x, x_n)) \in EG_n$ where each pair (x, x_i) is in the pair groupoid. For $c \in C_n(X)$, define

$$(\beta c)((x, x_0), (x, x_1), \dots, (x, x_n)) := \begin{cases} c(x_0, \dots, x_n) & x = x_0 \\ 0 & x \neq x_0 \end{cases}.$$

Due to the support condition on elements of $C_n(X)$, there exists $r > 0$ such that c is supported in the set $\{(x_0, \dots, x_n) \in X^{n+1} \mid d(x_i, x_j) \leq r \text{ for all } i, j\}$. One can check using the bounded geometry condition that this implies that the closure of the support S of βc in $EG_n \cap (X \times X)^{n+1}$ canonically identifies with the Stone-Ćech compactification of S , and thus that βc extends uniquely to a function on the compact open set \bar{S} as it is bounded, and so a function on EG_n by setting it to be zero outside \bar{S} . We also denote βc the corresponding class in $\mathbb{Z}[EG_n]_G$.

Now, having built the maps α and β , note that both define maps of complexes as the face maps in both cases are given by omitting the i^{th} element in a tuple. To see that they are mutually inverse isomorphisms, one computes directly that $\alpha(\beta(c)) = c$, and that $\beta(\alpha[a]) = [\bar{a}]$; we leave this to the reader. The result follows. \square

Having identified the homology groups of the coarse groupoid with a more classical object, we would now like to apply our main results and draw some consequences for the computation of the K-theory groups of uniform Roe algebras $C_u^*(X)$ which can be canonically identified with $C_r^*(G(X))$. Since the spectral sequence (18) is only available in the case of second countable groupoids we need to do some additional work. To this end it is useful to consider a slightly different construction of the coarse groupoid.

Following [44, Section 2.2], let Γ_X denote the collection of all subsets $A \subseteq X \times X$ such that the first coordinate map $r : X \times X \rightarrow X$ and second coordinate maps $s : X \times X \rightarrow X$ are both injective when restricted to A , and such that $\sup_{(x,y) \in A} d(x,y) < \infty$. As in [44, Section 3.1], every $A \in \Gamma_X$ defines a bijection $t_A : s(A) \rightarrow r(A)$ with the property that $\sup_{x \in s(A)} d(x, t_A(x)) < \infty$. Every such bijection extends to a homeomorphism $\phi_A : \overline{s(A)} \rightarrow \overline{r(A)}$ between the respective closures in βX . As in [44, Definition 3.1], we write $\mathcal{G}(X)$ for the collection $\{\phi_A \mid A \in \Gamma_X\}$, which is a pseudogroup, i.e. closed under compositions and inverses. As in [44, Section 3.2], the coarse groupoid $G(X)$ of X can be realized as the groupoid of germs associated to this pseudogroup (see [44, Section 2.6] for the construction of the groupoid of germs associated to a pseudogroup and [44, Proposition 3.2] for the identification of the two constructions).

Now, as in [44, Section 3.3], let us say that a sub-pseudogroup \mathcal{A} of $\mathcal{G}(X)$ is *admissible* if $\bigcup_{\phi_A \in \mathcal{A}} \overline{A} = G(X)$. Define $G_{\mathcal{A}}$ to be the spectrum of the C^* -subalgebra of $C_0(G(X))$ generated by $\{\chi_{\overline{A}} \mid \phi_A \in \mathcal{A}\}$, and let $X_{\mathcal{A}}$ be the spectrum of the C^* -subalgebra of $C(\beta X) = \ell^\infty(X)$ generated $\{\chi_{s(\overline{A})} \mid \phi_A \in \mathcal{A}\}$. The following comes from [44, Lemma 3.3] and its proof.

Lemma 5.3. *Let \mathcal{A} be an admissible sub-pseudogroup of $\mathcal{G}(X)$. Then the groupoid operations naturally factor through the canonical quotient maps $G(X) \rightarrow G_{\mathcal{A}}$ and $G(X)^{(0)} \rightarrow X_{\mathcal{A}}$, making $G_{\mathcal{A}}$ an étale, locally compact, Hausdorff groupoid with base space $X_{\mathcal{A}}$, which is moreover second countable if \mathcal{A} is countable.*

Moreover, the quotient map $p : \beta X = G(X)^{(0)} \rightarrow X_{\mathcal{A}}$ gives rise to an action of $G_{\mathcal{A}}$ on βX , and there is a canonical isomorphism of topological groupoids $G(X) \cong \beta X \rtimes G_{\mathcal{A}}$.

Proof. The only part not explicitly in [44, Lemma 3.3] or its proof is the second countability statement. This follows as if \mathcal{A} is countable, then the C^* -subalgebra of $C_0(G(X))$ generated by $\{\chi_{\overline{A}} \mid \phi_A \in \mathcal{A}\}$ is separable. \square

Lemma 5.4. *Let X be a bounded geometry metric space. Then there exists a countable admissible sub-pseudogroup \mathcal{A} of $\mathcal{G}(X)$ such that $G_{\mathcal{A}}$ is principal.*

Proof. First, choose a countable admissible sub-pseudogroup \mathcal{A}' of $\mathcal{G}(X)$ as follows. For each $n \in \mathbb{N}$, a greedy algorithm based on bounded geometry (compare the discussion in [44, Section 2.2, part (a)]) gives a finite decomposition of $\{(x,y) \in X \times X \mid d(x,y) \leq n\}$ such that

$$\{(x,y) \in X \times X \mid d(x,y) \leq n\} = \bigsqcup_{i=1}^{m_n} A_i^{(n)}$$

and so that each $A_i^{(n)}$ is in Γ_X . Let \mathcal{A}' be the sub-pseudogroup of $\mathcal{G}(X)$ generated by all the $A_i^{(n)}$. It is countable (as generated by a countable set) and it is admissible by construction. Given $\phi \in \mathcal{A}'$ we can apply [36, Proposition 2.7] to decompose its domain $\text{dom}(\phi) = A_{0,\phi} \sqcup A_{1,\phi} \sqcup A_{2,\phi} \sqcup A_{3,\phi}$ into disjoint clopen sets, where $A_{0,\phi}$ is the set of fixed points of ϕ and $\phi(A_{i,\phi}) \cap A_{i,\phi} = \emptyset$ for $i = 1, 2, 3$. Let \mathcal{A} be the sub-pseudogroup generated by \mathcal{A}' and $\{\text{id}_{A_{i,\phi}} \mid \phi \in \mathcal{A}', 0 \leq i \leq 4\}$. Then \mathcal{A} is still countable and admissible. We claim that $G_{\mathcal{A}}$ is principal. So let $[\phi, \omega] \in G_{\mathcal{A}}$ such that its source and range are equal, i.e. $\phi(\omega) = \omega$. We may assume that $\phi \in \mathcal{A}'$ as there is nothing to show if ϕ was already the identity function on some clopen set. We then have $\phi|_{A_{0,\phi}} = \text{id}_{A_{0,\phi}}$ and hence $[\phi, \omega] = [\text{id}, \omega]$ as desired. \square

Let us now assume that X has asymptotic dimension at most d , or equivalently (see [21, Theorem 6.4]) that $G(X)$ has dynamic asymptotic dimension at most d . For each $n \in \mathbb{N}$, let $E_n := \overline{\{(x,y) \in X \times X \mid d(x,y) \leq n\}}$, where the closure is

taken in $\beta X \times \beta X$. Then E_n is a compact open subset of $G(X)$. Hence using the assumption that the dynamic asymptotic dimension of $G(X)$ is at most d , there exists a decomposition $\beta X = U_0^{(n)} \sqcup \dots \sqcup U_d^{(n)}$ of βX into compact open subsets such that for each $i \in \{0, \dots, d\}$ and each $n \in \mathbb{N}$, the subgroupoid of $G(X)$ generated by $\{g \in E_n \mid r(g), s(g) \in U_i^{(n)}\}$ is compact and open. Note that as each $U_i^{(n)}$ is clopen in βX , each $U_i^{(n)}$ is the closure of $V_i^{(n)} := U_i^{(n)} \cap X$ (this follows as clopen sets in βX are in one-one correspondence with arbitrary subsets of X in this way). Let \mathcal{B} denote the sub-pseudogroup of $\mathcal{G}(X)$ generated by \mathcal{A} as in Lemma 5.4, and by $\{\text{id}_{V_i^{(n)}} \mid n \in \mathbb{N}, i \in \{0, \dots, d\}\}$.

Then we have the following result.

Lemma 5.5. *Let X be a bounded geometry metric space with asymptotic dimension at most d . Then there is a second countable, étale, locally compact, Hausdorff principal groupoid G with dynamic asymptotic dimension at most d , and such that G acts on βX giving rise to a canonical isomorphism $\beta X \rtimes G \cong G(X)$.*

Proof. We claim that $G = G_{\mathcal{B}}$ works. Note that \mathcal{B} is countable (as generated by a countable set) and admissible (as it contains \mathcal{A} , which is admissible). Hence most of the statement follows from Lemma 5.3. As we are only adding further identity functions in the passage from \mathcal{A} to \mathcal{B} , we also retain principality by the same proof as in Lemma 5.4. We only need to show that the dynamic asymptotic dimension of $G_{\mathcal{B}}$ is at most d .

For each $n \in \mathbb{N}$, let us write $[E_n]$ for the image of $E_n \subseteq G(X)$ under the quotient map $G(X) \rightarrow G_{\mathcal{B}}$. Then $[E_n]$ is compact and open: indeed, with notation as in the construction of \mathcal{A} its characteristic function is equal to

$$\sum_{i=1}^{m_n} \chi_{\overline{A_i^{(n)}}},$$

and so in $C_0(G_{\mathcal{B}}) \subseteq C_0(G(X))$. Moreover, $G_{\mathcal{B}}$ is the union of the $[E_n]$ (as $G(X)$ is the union of the E_n). Hence to show that $G_{\mathcal{B}}$ has dynamic asymptotic dimension at most d , it suffices to show that for each n we can find an open cover W_0, \dots, W_d of $X_{\mathcal{B}}$ such that for each i the subgroupoid G_i of $G_{\mathcal{B}}$ generated by

$$\{g \in [E_n] \mid s(g), r(g) \in W_i\}$$

has compact closure. For this, let us take $W_i = U_i^{(n)}$, noting that each $U_i^{(n)}$ makes sense as a clopen subset of $X_{\mathcal{B}}$ by construction of \mathcal{B} . Then we have a decomposition

$$X_{\mathcal{B}} = \bigsqcup_{i=0}^d U_i^{(n)}$$

coming from the corresponding decomposition of βX . Finally, note that each G_i is contained in the image of the subgroupoid \widetilde{G}_i of $G(X)$ generated by

$$\{g \in E_n \mid s(g), r(g) \in U_i^{(n)}\}$$

under the canonical quotient map $G(X) \rightarrow G_{\mathcal{B}}$. As \widetilde{G}_i is compact (by choice of the $U_i^{(n)}$), and as this quotient map is continuous, we are done. \square

We can now use this observation to deduce the existence of a convergent spectral sequence also for the (non-second countable) coarse groupoid:

Proposition 5.6. *Let X be a discrete metric space with bounded geometry and finite asymptotic dimension. Then there exists a convergent spectral sequence*

$$E_{p,q}^2 = H_p^{\text{uf}}(X) \otimes K_q(\mathbb{C}) \Rightarrow K_{p+q}(C_u^*(X)).$$

Proof. By Lemma 5.5 we can write $G(X) = G \ltimes \beta X$ for a principal second countable ample groupoid G with finite dynamic asymptotic dimension. Further, we can write βX as an inverse limit $\beta X = \varprojlim Y_i$ of G -invariant second countable spaces Y_i . Since G is in particular amenable, it satisfies the strong Baum-Connes conjecture. Hence (18), for each $i \in I$, provides a convergent spectral sequence

$$E_{p,q}^2(i) = H_p(G, K_q(C(Y_i))) \Rightarrow K_{p+q}(C(Y_i) \rtimes_r G).$$

The spectral sequence (18) is a special case of the ABC spectral sequence constructed by Meyer in [32, Theorem 4.3], and hence it is functorial in the coefficient variable. Consequently, the abelian groups $E_{p,q}^d(i)$ together with the differential maps form a directed system of spectral sequences. Hence we obtain a spectral sequence with $E_{p,q}^d := \varinjlim_i E_{p,q}^d(i)$ in the limit. Each of the spectral sequences $(E_{p,q}^d(i))$ converges by [32, Theorem 5.1] and as explained on page 172 of [32], the associated filtrations are functorial in the appropriate sense. Hence taking limits again, we obtain an induced filtration of $\varinjlim K_{p+q}(C(Y_i) \rtimes_r G)$. Now since $E_{p,q}^2(i) = 0$ for all $p > \text{asdim}(X)$ and all $i \in I$ by Theorem 3.36, the induced filtration of $\varinjlim K_{p+q}(C(Y_i) \rtimes_r G)$ is finite and hence we obtain a convergent spectral sequence in the limit:

$$(19) \quad E_{p,q}^2 = \varinjlim_i H_p(G, K_q(C(Y_i))) \Rightarrow \varinjlim_i K_{p+q}(C(Y_i) \rtimes G).$$

There are canonical identifications $\varinjlim_i H_n(G, C(Y_i, \mathbb{Z})) \cong \varinjlim_i H_n(G \ltimes Y_i) = H_n(G(X))$ and by Theorem 5.2 we can identify the latter group with $H_n^{\text{uf}}(X)$. On the right hand-side we have $\varinjlim_i K_*(C(Y_i) \rtimes_r G) = K_*(C_r^*(G(X))) = K_*(C_u^*(X))$ and hence we are done. \square

Corollary 5.7. *Let X be a bounded geometry metric space with asymptotic dimension d . Then $H_n^{\text{uf}}(X) = 0$ for all $n > d$ and $H_d^{\text{uf}}(X)$ is free. Moreover,*

(1) *if $\text{asdim}(X) \leq 2$, then*

$$K_0(C_u^*(X)) \cong H_0^{\text{uf}}(X) \oplus H_2^{\text{uf}}(X), \text{ and } K_1(C_u^*(X)) \cong H_1^{\text{uf}}(X),$$

(2) *if $\text{asdim}(X) \leq 3$ and X is non-amenable, then*

$$K_0(C_u^*(X)) \cong H_2^{\text{uf}}(X), \text{ and } K_1(C_u^*(X)) \cong H_1^{\text{uf}}(X) \oplus H_3^{\text{uf}}(X).$$

Proof. The first case follows from Theorem 3.36 and Corollary 4.29 as in our earlier examples. For (2) note that if $\text{asdim}(X) \leq 3$ then the only possibly non-zero differentials on the E^3 -page are the maps $d_{3,2l}^3 : H_3^{\text{uf}}(X) \rightarrow H_0^{\text{uf}}(X)$ for $l \geq 0$. The sequence converges on the E^4 -page and hence there are short exact sequences

$$0 \rightarrow \text{coker}(d_{3,0}^3) \rightarrow K_0(C_u^*(X)) \rightarrow H_2^{\text{uf}}(X) \rightarrow 0, \text{ and}$$

$$0 \rightarrow H_1^{\text{uf}}(X) \rightarrow K_1(C_u^*(X)) \rightarrow \ker(d_{3,0}^3) \rightarrow 0$$

For a non-amenable space X the group $H_0^{\text{uf}}(X)$ vanishes by [6, Theorem 3.1]. Since $H_3^{\text{uf}}(X)$ is free the result follows. \square

5.4. Examples with topological property (T). The HK-conjecture asserts that for a principal ample groupoid we have abstract isomorphisms $\bigoplus_{j \in \mathbb{N}} H_{2j+i}(G) \cong K_i(C_r^*(G))$. If G has homological dimension 1 one might be tempted to strengthen this conjecture and ask for the canonical maps μ_0 and μ_1 to be isomorphisms. Here, we show that this strong version of the conjecture fails.

In order to exhibit the examples we need some preliminary facts about topological property (T). Topological property (T) for groupoids was introduced in [13, Definition 3.6], and we refer the reader there for the definition. Let $\mathcal{R}_G = \{\pi_x \mid x \in G^0\}$

denote the family of regular representations of G , i.e.

$$\pi_x : C_c(G) \rightarrow B(\ell^2(G_x)), \quad \pi_x(f)\delta_g = \sum_{h \in G_r(g)} f(h)\delta_{hg}.$$

Then we have the following generalization of [13, Proposition 4.19].

Lemma 5.8. *Let G be an ample groupoid with compact unit space acting on a compact space X . If G has property (T) with respect to \mathcal{R}_G then $G \ltimes X$ has property (T) with respect to $\mathcal{R}_{G \ltimes X}$.*

Proof. Let $p : X \rightarrow G^0$ denote the anchor map of the action. Let (K, c) be a Kazhdan pair for \mathcal{R}_G . Now let

$$L := \{(g, x) \in G \ltimes X \mid g \in K\} = (G \ltimes X) \cap (K \times X),$$

which is compact. We claim that L is a Kazhdan set for $\mathcal{R}_{G \ltimes X}$. Indeed consider the regular representation $\pi_x^{G \ltimes X} : C_c(G \ltimes X) \rightarrow B(\ell^2(G_{p(x)}))$ associated with an arbitrary point $x \in X$ and let $\xi \in \ell^2(G_{p(x)})$ be a unit vector. Since (K, c) is Kazhdan for \mathcal{R}_G , there exists a function $f \in C_c(G)$ with support in K such that $\|f\|_I \leq 1$ and

$$\|\pi_{p(x)}^G(f)\xi - \pi_{p(x)}^G(\Psi(f))\xi\| \geq c,$$

where $\Psi : C_c(G) \rightarrow C(G^0)$ is given by $\Psi(f)(x) = \sum_{g \in G^x} f(g)$. Since K is compact we can cover it with finitely many compact open bisections V_1, \dots, V_n and using a partition of unity argument, we can write $f = \sum f_i$ where $\text{supp}(f_i) \subseteq V_i$. Then there must be some $1 \leq i \leq n$ such that

$$\|\pi_{p(x)}^G(f_i)\xi - \pi_{p(x)}^G(\Psi(f_i))\xi\| \geq \frac{c}{n}.$$

Using that f_i is supported in a bisection one directly verifies that

$$\pi_{p(x)}^G(f_i) = \pi_{p(x)}^G(\Psi(f_i))\pi_{p(x)}^G(1_{V_i}) \text{ and } \Psi(f_i) = \Psi(f_i)1_{r(V_i)}$$

and combining this with the previous observation, we conclude that there exists an i such that

$$\|\pi_{p(x)}^G(1_{V_i})\xi - \pi_{p(x)}^G(1_{r(V_i)})\xi\| \geq \frac{c}{n}.$$

Now let $V_i \ltimes X$ denote the compact open set $(V_i \times X) \cap G \ltimes X$ and let $f' := 1_{V_i \ltimes X}$. Then f' is clearly supported in L with $\|f'\|_I \leq 1$ and since $\pi_x^{G \ltimes X}(f') = \pi_{p(x)}^G(1_{V_i})$ and $\pi_x^{G \ltimes X}(\Psi(f')) = \pi_{p(x)}^G(1_{r(V_i)})$ we conclude that

$$\|\pi_x^{G \ltimes X}(f')\xi - \pi_x^{G \ltimes X}(\Psi(f'))\xi\| \geq \frac{c}{n}.$$

Hence $(L, \frac{c}{n})$ is a Kazhdan pair for $\mathcal{R}_{G \ltimes X}$. \square

Now let Γ be a residually finite group and $\mathcal{L} = (N_i)_i$ a sequence of finite index normal subgroups. Let N_∞ denote the trivial subgroup of Γ and let $\pi_i : \Gamma \rightarrow \Gamma/N_i$ be the quotient map. We denote by $G_{\mathcal{L}}$ the associated HLS groupoid, i.e. the group bundle $\bigsqcup_{i \in \mathbb{N} \cup \{\infty\}} \Gamma/N_i$ equipped with the topology generated by the singleton sets $\{(i, \gamma)\}$ for $i \in \mathbb{N}$ and $\gamma \in \Gamma$, and the tails $\{(i, \pi_i(\gamma)) \mid i > N\}$ for each fixed $\gamma \in \Gamma$ and $N \in \mathbb{N}$. It is well-known that this groupoid is Hausdorff if and only if for each $\gamma \in \Gamma \setminus \{e\}$ the set $\{i \in \mathbb{N} \mid \gamma \in N_i\}$ is finite. This is in particular the case if the sequence is nested and has trivial intersection.

Following a construction of Alekseev and Finn-Sell in [1] we associate a principal groupoid to this data as follows. Let $X := \bigsqcup_{i \in \mathbb{N} \cup \{\infty\}} \Gamma/N_i$. Then X carries a canonical action of the HLS groupoid $G_{\mathcal{L}}$ given by left multiplication. For $\gamma N_i \in \Gamma/N_i$ let

$$Sh(\gamma N_i) = \bigcup_{j \geq i} \pi_{i,j}^{-1}(\gamma N_i)$$

be the *shadow* of γN_i in X . Now let \hat{X} be the spectrum of the smallest $G_{\mathcal{L}}$ -invariant C^* -subalgebra $B \subseteq \ell^\infty(X)$ containing

$$\{\delta_x \mid x \in X\} \cup \{1_{Sh(\gamma N_i)} \mid \gamma \in \Gamma, i \in \mathbb{N}\}.$$

Since B is $G_{\mathcal{L}}$ -invariant, \hat{X} also carries an action of $G_{\mathcal{L}}$ and we can form the transformation groupoid $\mathcal{G} := G_{\mathcal{L}} \ltimes \hat{X}$. As explained just after Remark 2.2 in [1], this groupoid is principal. Moreover, $X \subseteq \hat{X}$ is a dense open $G_{\mathcal{L}}$ -invariant subset with complement $\hat{X} \setminus X \cong \hat{\Gamma}_{\mathcal{L}} = \varprojlim \Gamma/N_i$. Hence we obtain isomorphisms

$$\mathcal{G}|_X \cong \bigsqcup_{i \in \mathbb{N} \cup \{\infty\}} \Gamma/N_i \ltimes \Gamma/N_i \quad \text{and} \quad \mathcal{G}|_{\hat{X} \setminus X} \cong \Gamma \ltimes \hat{\Gamma}_{\mathcal{L}}.$$

The following result relies on property (τ) as defined in [28, Definition 4.3.1].

Proposition 5.9. Suppose Γ is a finitely generated, residually finite group and $\mathcal{L} = (N_i)_i$ is a sequence of finite index normal subgroups with property (τ) . Then the following hold:

- (1) \mathcal{G} has topological property (T) with respect to the family of regular representations in the sense of [13].
- (2) The sequence

$$K_0(C_r^*(\mathcal{G}|_X)) \rightarrow K_0(C_r^*(\mathcal{G})) \rightarrow K_0(C_r^*(\mathcal{G}|_{\hat{X} \setminus X}))$$

is not exact in the middle.

Proof. Since the regular representations of \mathcal{G} extend to $C_r^*(\mathcal{G})$ by definition of the reduced groupoid C^* -algebra, the result follows from [13, Proposition 4.15] and Lemma 5.8. Part (2) follows from (1) and [13, Proposition 7.14]. \square

We can now provide some concrete principal ample groupoids where μ_0 is not surjective.

Let $\Gamma = \mathbb{F}_2$ and choose a nested sequence $(N_i)_i$ of finite index normal subgroups in \mathbb{F}_2 with property (τ) such that the associated HLS groupoid is Hausdorff.

Example 5.10. To have a concrete example of such a sequence in mind consider the nested family $(L_i)_i$ of finite index normal subgroups $L_i := \ker(SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z}/5^i))$ of $SL_2(\mathbb{Z})$. Embed \mathbb{F}_2 in $SL_2(\mathbb{Z})$ as a finite index normal subgroup and let $N_i := L_i \cap \mathbb{F}_2$. Then $(N_i)_i$ is a nested family of finite index normal subgroups of \mathbb{F}_2 with trivial intersection. Since \mathbb{F}_2 has finite index in $SL_2(\mathbb{Z})$ and $SL_2(\mathbb{Z})$ has property (τ) with respect to the family $(L_i)_i$ we conclude that \mathbb{F}_2 has (τ) with respect to the family $(N_i)_i$.

Let us first compute the homology of the associated groupoid \mathcal{G} . Consider the long exact sequence in homology

$$\cdots \rightarrow H_n(\mathcal{G}|_X) \rightarrow H_n(\mathcal{G}) \rightarrow H_n(\mathcal{G}|_{\hat{X} \setminus X}) \rightarrow H_{n-1}(\mathcal{G}|_X) \rightarrow \cdots \rightarrow H_0(\mathcal{G}|_{\hat{X} \setminus X})$$

corresponding to the decomposition $\hat{X} = X \sqcup \hat{X} \setminus X$. Since $\mathcal{G}|_X$ is a disjoint union of principal and proper groupoids, we have

$$H_n(\mathcal{G}|_X) = \bigoplus_{i \in \mathbb{N} \cup \{\infty\}} H_n(\Gamma/N_i \ltimes \Gamma/N_i) = 0 \text{ for all } n \geq 1, \text{ and}$$

$$H_0(\mathcal{G}|_X) = \bigoplus_{i \in \mathbb{N} \cup \{\infty\}} H_0(\Gamma/N_i \ltimes \Gamma/N_i) = \bigoplus_{i \in \mathbb{N} \cup \{\infty\}} \mathbb{Z}$$

From the long exact sequence we conclude that for all $n \geq 2$ the restriction to the boundary induces isomorphisms

$$H_n(\mathcal{G}) \cong H_n(\mathbb{F}_2 \ltimes \hat{\mathbb{F}}_{2\mathcal{L}}).$$

It is a well-known fact that $H_n(\mathbb{F}_2 \ltimes \widehat{\mathbb{F}_{2\mathcal{L}}}) \cong H_n(\mathbb{F}_2, C(\widehat{\mathbb{F}_{2\mathcal{L}}}, \mathbb{Z}))$ and the homology of the free group \mathbb{F}_2 is well-known to be trivial for all $n \geq 2$. Hence $H_n(\mathcal{G}) = 0$ for all $n \geq 2$.

Now by construction $H_0(\mathbb{F}_2 \ltimes \widehat{\mathbb{F}_{2\mathcal{L}}}) = C(\widehat{\mathbb{F}_{2\mathcal{L}}}, \mathbb{Z}) / \langle f - \gamma.f \mid \gamma \in \mathbb{F}_2 \rangle$ and from the Pimsner-Voiculescu exact sequence for actions of free groups from [35, Theorem 3.5] we obtain $K_0(C(\widehat{\mathbb{F}_{2\mathcal{L}}}) \rtimes_r \mathbb{F}_2) \cong C(\widehat{\mathbb{F}_{2\mathcal{L}}}, \mathbb{Z}) / \text{im}(\beta)$ where

$$\beta : C(\widehat{\mathbb{F}_{2\mathcal{L}}}, \mathbb{Z})^2 \rightarrow C(\widehat{\mathbb{F}_{2\mathcal{L}}}, \mathbb{Z}), (f_1, f_2) \mapsto f_1 - a^{-1}.f_1 + f_2 - b^{-1}.f_2$$

We clearly have $\text{im}(\beta) = \langle f - \gamma.f \mid \gamma \in \mathbb{F}_2 \rangle$ so that after the identification above, μ_0 is the identity.

Now consider the commutative diagram

$$\begin{array}{ccccc} H_0(\mathcal{G}|_X) & \xrightarrow{i} & H_0(\mathcal{G}) & \xrightarrow{p} & H_0(\mathcal{G}|_{\widehat{X} \setminus X}) \\ \downarrow \mu_0 & & \downarrow \mu_0^{\mathcal{G}} & & \downarrow \mu_0 \\ K_0(C_r^*(\mathcal{G}|_X)) & \xrightarrow{i_0} & K_0(C_r^*(\mathcal{G})) & \longrightarrow & K_0(C_r^*(\mathcal{G}|_{\widehat{X} \setminus X})) \end{array}$$

The top row is exact in the middle, as it is part of the long exact sequence in homology corresponding to the open invariant subset $X \subseteq \widehat{X}$. The bottom row however is not exact in the middle by Proposition 5.9. The map on the right hand side is an isomorphism by our reasoning above.

We claim that the map $\mu_0^{\mathcal{G}}$ is not surjective. Suppose for contradiction that it was. Let $x \in K_0(C_r^*(\mathcal{G}))$ be an element which maps to zero in $K_0(C_r^*(\mathcal{G}|_{\widehat{X} \setminus X}))$ but is not in the image of i_0 . Since $\mu_0^{\mathcal{G}}$ is surjective, we can find an element $y \in H_0(\mathcal{G})$ such that $\mu_0^{\mathcal{G}}(y) = x$. But then by commutativity of the right hand square we have $\mu_0(p(y)) = 0$ and since μ_0 is injective we conclude that $p(y)$ is zero and hence $y = i(z)$ for some $z \in H_0(\mathcal{G}|_X)$. Moreover, by commutativity of the left square we have $x = \mu_0^{\mathcal{G}}(y) = i_0(\mu_0(z))$, which contradicts our assumption that $x \notin \text{im}(i_0)$.

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REFERENCES

- [1] Vadim Alekseev and Martin Finn-Sell. Non-amenable principal groupoids with weak containment. *Int. Math. Res. Not.*, 2018(8):2332–2340, 2018.
- [2] Pere Ara, Christian Bönicke, Joan Bosa, and Kang Li. The type semigroup, comparison and almost finiteness for ample groupoids. [arXiv:2001.00376v2](https://arxiv.org/abs/2001.00376), 2020.
- [3] Arthur Bartels, Wolfgang Lück, and Holger Reich. Equivariant covers for hyperbolic groups. *Geom. Topol.*, 12(3):1799–1882, 2008.
- [4] Gregory Bell and Alexander Dranishnikov. On asymptotic dimension of groups. *Algebr. Geom. Topol.*, 1:57–71, 2001.
- [5] George Bergman. Boolean Rings of Projection Maps. *Journal of the London Mathematical Society*, 2:593–598, 05 1972.
- [6] Jonathan Block and Shmuel Weinberger. Aperiodic tilings, positive scalar curvature, and amenability of spaces. *J. Am. Math. Soc.*, 5(4):907–918, 1992.
- [7] Jose Carrión, James Gabe, Christopher Schafhauser, Aaron Tikuisis, and Stuart White. Classification of $*$ -homomorphisms I: simple nuclear C^* -algebras. In preparation.
- [8] Clinton Conley, Steve Jackson, Andrew Marks, Brandon Seward, and Robin Tucker-Drob. Borel asymptotic dimension and hyperfinite equivalence relations. [arXiv:2009.06721](https://arxiv.org/abs/2009.06721), 2020.

- [9] Marius Crainic and Ieke Moerdijk. A homology theory for étale groupoids. *J. Reine Angew. Math.*, 521:25–46, 2000.
- [10] Robin Deeley and Karen Strung. Nuclear dimension and classification of C^* -algebras associated to Smale spaces. *Trans. Am. Math. Soc.*, 370(5):3467–3485, 2018.
- [11] Clément Dell'Aiera. *Controlled K -theory for groupoids and applications*. PhD thesis, Université de Lorraine, 2017.
- [12] Clément Dell'Aiera. Controlled K -theory for groupoids & applications to Coarse Geometry. *J. Funct. Anal.*, 275(7):1756–1807, 2018.
- [13] Clément Dell'Aiera and Rufus Willett. Topological property (T) for groupoids. [arXiv:1811.07085v2](https://arxiv.org/abs/1811.07085).
- [14] George Elliott, Guihua Gong, Huaxin Lin, and Zhuang Niu. The classification of simple separable unital \mathcal{Z} -stable locally ASH algebras. *J. Funct. Anal.*, 272(12):5307–5359, 2017.
- [15] Carla Farsi, Alex Kumjian, David Pask, and Aidan Sims. Ample groupoids: Equivalence, homology and Matui's HK conjecture. *Münster J. of Math.*, 12:411–451, 2019.
- [16] Thierry Giordano, Ian Putnam, and Christian Skau. Affable equivalence relations and orbit structure of Cantor dynamical systems. *Ergodic Theory Dynam. Systems*, 24(2):441–475, 2004.
- [17] Roger Godement. *Topologie algébrique et théorie des faisceaux*. Hermann, 1958.
- [18] Guihua Gong, Huaxin Lin, and Zhuang Niu. A classification of finite simple amenable \mathcal{Z} -stable C^* -algebras, I: C^* -algebras with generalized tracial rank one. *C. R. Math. Acad. Sci. Soc. R. Canada*, 42(3):63–450, 2020.
- [19] Guihua Gong, Huaxin Lin, and Zhuang Niu. Classification of finite simple amenable \mathcal{Z} -stable C^* -algebras, II: C^* -algebras with rational generalized tracial rank one. *C. R. Math. Acad. Sci. Soc. R. Canada*, 42(4):451–539, 2020.
- [20] Erik Guentner, Rufus Willett, and Guoliang Yu. Finite dynamical complexity and controlled operator K -theory. [arXiv:1609.02093](https://arxiv.org/abs/1609.02093), 2016.
- [21] Erik Guentner, Rufus Willett, and Guoliang Yu. Dynamic asymptotic dimension: relation to dynamics, topology, coarse geometry, and C^* -algebras. *Math. Ann.*, 367(1-2):785–829, 2017.
- [22] Nigel Higson, Vincent Lafforgue, and Georges Skandalis. Counterexamples to the Baum-Connes conjecture. *Geom. Funct. Anal.*, 12(2):330–354, 2002.
- [23] Nigel Higson and John Roe. *Analytic K -homology*. Oxford University Press, 2000.
- [24] David Kerr. Dimension, comparison, and almost finiteness. *J. Eur. Math. Soc.*, (to appear in print), 2020.
- [25] Eberhard Kirchberg. Exact C^* -algebras, tensor products, and the classification of purely infinite algebras. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 943–954. Birkhäuser, Basel, 1995.
- [26] Kang Li and Rufus Willett. Low-dimensional properties of uniform Roe algebras. *J. London Math. Soc.*, 97:98–124, 2018.
- [27] Xin Li and Jean Renault. Cartan subalgebras in C^* -algebras. Existence and uniqueness. *Trans. Amer. Math. Soc.*, 372(3):1985–2010, 2019.
- [28] Alexander Lubotzky. *Discrete groups, expanding graphs and invariant measures. Appendix by Jonathan D. Rogawski*, volume 125. Basel: Birkhäuser, 1994.
- [29] Hiroki Matui. Homology and topological full groups of étale groupoids on totally disconnected spaces. *Proc. Lond. Math. Soc. (3)*, 104(1):27–56, 2012.
- [30] Hiroki Matui. Topological full groups of one-sided shifts of finite type. *J. Reine Angew. Math.*, 705:35–84, 2015.
- [31] Hiroki Matui. Étale groupoids arising from products of shifts of finite type. *Adv. Math.*, 303:502–548, 2016.
- [32] Ralf Meyer. Homological algebra in bivariant K -theory and other triangulated categories. II. *Tbil. Math. J.*, 1:165–210, 2008.
- [33] Eduard Ortega. Homology of the Katsura-Exel-Pardo groupoid. *J. Noncommut. Geom.*, 14:1–23, 2020.
- [34] N. Christopher Phillips. A classification theorem for nuclear purely infinite simple C^* -algebras. *Doc. Math.*, 5:49–114 (electronic), 2000.
- [35] Mihai Pimsner and Dan-Virgil Voiculescu. K -groups of reduced crossed products by free groups. *J. Oper. Theory*, 8:131–156, 1982.
- [36] David Pitts. Structure for regular inclusions. I. *J. Oper. Theory*, 78(2):357–416, 2017.
- [37] Valerio Proietti and Makoto Yamashita. Homology and K -theory of torsion-free ample groupoids and Smale spaces. [arXiv:2006.08028v1](https://arxiv.org/abs/2006.08028), 2020.
- [38] Ian Putnam. An excision theorem for the K -theory of C^* -algebras. *J. Oper. Theory*, 38(1):151–171, 1997.
- [39] Ian Putnam. A homology theory for Smale spaces, *Mem. Amer. Math. Soc.*, 232(1094):viii+122, 2014.

- [40] Jean Renault. *C*-algebras and dynamical systems*. Publicações Matemáticas do IMPA, 27^o Colóquio Brasileiro de Matemática. Instituto Nacional de Matemática Pura e Aplicada, 2009.
- [41] John Roe. *Lectures on coarse geometry*, volume 31. Providence, RI: American Mathematical Society (AMS), 2003.
- [42] Mikael Rørdam, Flemming Larsen, and Niels Laustsen. *An introduction to K-theory for C*-algebras*, volume 49 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2000.
- [43] Eduardo Scarparo. Homology of odometers. *Ergodic Theory Dyn. Syst.*, 40(9):2541–2551, 2020.
- [44] Georges Skandalis, Jean-Louis Tu, and Guoliang Yu. The coarse Baum-Connes conjecture and groupoids. *Topology*, 41(4):807–834, 2002.
- [45] The Stacks Project Authors. *Stacks Project*. <https://stacks.math.columbia.edu>, 2020.
- [46] Lynn Arthur Steen and J. Arthur Seebach, Jr. *Counterexamples in topology*. Springer-Verlag, New York-Heidelberg, second edition, 1978.
- [47] Aaron Tikuisis, Stuart White, and Wilhelm Winter. Quasidiagonality of nuclear C*-algebras. *Ann. of Math. (2)*, 185(1):229–284, 2017.
- [48] Jean-Louis Tu. La conjecture de Baum-Connes pour les feuilletages moyennables. *K-theory*, 17:215–264, 1999.
- [49] Charles Weibel. *An introduction to homological algebra*, volume 38. Cambridge: Cambridge University Press, 1994.
- [50] Inhyeop Yi. Homology and Matui's HK conjecture for groupoids on one-dimensional solenoids. *Bull. Aust. Math. Soc.*, 101(1):105–117, 2020.

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