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C^*-Algebras of Real Rank Zero

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The concept of real rank of a C^*-algebra is introduced as a non-commutative analogue of dimension. It is shown that real rank zero is equivalent to the previously defined conditions FS and HP, and that it is invariant under strong Morita equivalence, in particular under stable isomorphism. Real rank zero is also invariant under inductive limits and split extensions, and the class may well be regarded as the conceptual completion of the AF-algebras. In some cases, notably when the algebra is matroid, it is shown that the multiplier algebra also has real rank zero—although that is not true in general. By a result of G. J. Murphy, this implies a Weyl–von Neumann type result for self-adjoint multiplier elements in these cases. © 1991 Academic Press, Inc.

1. REAL RANK OF C^*-ALGEBRAS

For a unital C^*-algebra A we define the *real rank* of A to be the smallest integer, $\mathrm{RR}(A)$, such that for each $n$tuple $(x_1, ..., x_n)$ of self-adjoint elements in A, with $n \leqslant \mathrm{RR}(A)+1$, and every $\varepsilon > 0$, there is an n-tuple $(y_1, ..., y_n)$ in A_{sa} such that $\sum y_k^2$ is invertible and

$$\left\| \sum (x_k - y_k)^2 \right\| < \varepsilon. \tag{$*$}$$

Identifying each $n$tuple $(x_1, ..., x_n)$ with the matrix x in $\mathbf{M}_n(A)$ that has $x_1, ..., x_n$ as its first column and zeros elsewhere, the estimate in $(*)$ simply means that $\|x - y\| < \varepsilon$ in $\mathbf{M}_n(A)$. Moreover, the invertibility of $\sum y_k^2$ is

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equivalently expressed by the equation $1 = \sum z_k y_k$ for a suitable $n$tuple $(z_1, ..., z_n)$, or, in terms of left ideals, by the equality

$$A = Ay_1 + \cdots + Ay_n.$$

If A is non-unital, we define its real rank to be $\mathrm{RR}(\tilde{A})$, where $\tilde{A}$ as usual denotes the unital C^*-algebra $A \oplus \mathbb{C}$.

Our definition of real rank is identical with Rieffel's notion of topological stable rank [31], later identified with the Bass stable rank [21], except that we demand all elements to be self-adjoint. Especially for small values of the rank, this changes the concept fundamentally, as we shall see. Thus, borrowing shamelessly from Rieffel's ideas, we obtain an invariant which may be closer to non-commutative dimension for C^*-algebras than the Bass stable rank.

1.1. PROPOSITION. *If X is a compact Hausdorff space, then*

$$\mathrm{RR}(C(X)) = \dim X.$$

Proof. By [25, 3.3.2] the covering dimension of X is the smallest integer n such that every continuous function f from X into $\mathbb{R}^{n+1}$ can be approximated arbitrarily well by another such function g for which $0 \notin g(X)$. Identifying f and g with $n$tuples of real functions in $C(X)$, and noting that $0 \notin g(X)$ iff $\sum g_k(x)^2 > 0$ for every x in X, we see that the definitions of $\dim X$ and of $\mathrm{RR}(C(X))$ are identical. ∎

1.2. PROPOSITION. *If A is a C^*-algebra and* $\mathrm{tsr}(A)$ *denotes its topological stable rank* ($=$*Bass stable rank*), *then*

$$\mathrm{RR}(A) \leqslant 2\,\mathrm{tsr}(A) - 1.$$

Proof. Passing if necessary to $\tilde{A}$ we may assume that A is unital, and that $\mathrm{tsr}(A) = n < \infty$. Given $(x_1, x_2, ..., x_{2n})$ in A_{sa}, let $a_k = x_k + \mathrm{i}x_{k+n}$ for $1 \leqslant k \leqslant n$. By assumption there is, for each $\varepsilon > 0$, an $n$tuple $(b_1, ..., b_n)$ in A such that

$$\sum (a_k - b_k)^* (a_k - b_k) \leqslant \varepsilon \qquad (**)$$

$$\sum b_k^* b_k \geqslant \delta \qquad (***)$$

for some $\delta > 0$. Write $b_k = y_k + \mathrm{i}y_{k+n}$, with $y_1, y_2, ..., y_{2n}$ in A_{sa}. Then by $(***)$

$$2\sum_1^{2n} y_k^2 = 2\sum_1^n (y_k^2 + y_{k+n}^2)$$
$$= \sum_1^n (b_k^* b_k + b_k b_k^*) \geqslant \sum_1^n b_k^* b_k \geqslant \delta,$$

so that $\sum y_k^2$ is invertible. By (**) we know that $(a_k - b_k)^* (a_k - b_k) \leqslant \varepsilon$ for each k, whence

$$(a_k - b_k)(a_k - b_k)^* \leqslant \|a_k - b_k\|^2 \leqslant \varepsilon;$$

and thus

$$\sum (a_k - b_k)(a_k - b_k)^* \leqslant n\varepsilon.$$

Consequently

$$\begin{aligned} 2\sum_1^{2n} (x_k - y_k)^2 &= 2\sum_1^{n} ((x_k - y_k)^2 + (x_{k+n} - y_{k+n})^2) \\ &= \sum_1^{n} ((a_k - b_k)^* (a_k - b_k) + (a_k - b_k)(a_k - b_k)^*) \leqslant (n+1)\varepsilon, \end{aligned}$$

so that the $x$tuple is arbitrarily well approximated by the $y$tuple. Thus $\mathrm{RR}(A) \leqslant 2n - 1$, as desired. ∎

In the rest of this paper we are exclusively concerned with the case $\mathrm{RR}(A) = 0$, which seems at the moment to be the most tractable. By definition $\mathrm{RR}(A) = 0$ iff every self-adjoint element in A can be approximated by an invertible, self-adjoint element. As examples of such algebras we mention here the class that motivated our deviation from Rieffel's notion of stable rank.

1.3. PROPOSITION. *Every von Neumann algebra has real rank zero.*

Proof. If $x \in A_{sa}$, A a von Neumann algebra, and $\varepsilon > 0$ is given, let p denote the spectral projection of x corresponding to the interval $[-\varepsilon, \varepsilon]$. Then $y = (1 - p)x + \varepsilon p$ is invertible in A_{sa} and $\|x - y\| \leqslant 2\varepsilon$. ∎

2. REAL RANK ZERO

For a C^*-algebra A we denote by A_{sa} (respectively A_+) its self-adjoint (respectively positive) part. If $x \in A$ we write $|x| = (x^*x)^{1/2}$, and if further $x = x^*$ it has a unique decomposition $x = x_+ - x_-$ as a difference of orthogonal elements in A_+. Moreover, $|x| = x_+ + x_-$.

2.1. LEMMA. *If x and y are elements in A_+ with $\|xy\| \leqslant \varepsilon^2$, then with $z = x - y$ we have*

$$\| |z| - (x + y)\| \leqslant 2\varepsilon, \qquad \|z_+ - x\| \leqslant \varepsilon, \qquad \|z_- - y\| \leqslant \varepsilon.$$

Proof. We easily estimate

$$\|z^2-(x+y)^2\| = \|2(xy+yx)\| \leqslant 4\varepsilon^2.$$

Since the square root function is operator monotone [26, 1.3.8], and moreover subadditive on commuting elements in $\tilde{A}_+$, we obtain

$$x+y \leqslant (z^2+4\varepsilon^2)^{1/2} \leqslant |z|+2\varepsilon.$$

Similarly $|z| \leqslant x+y+2\varepsilon$, whence $\| |z|-(x+y)\| \leqslant 2\varepsilon$. Since

$$2(z_+-x) = |z|+z-2x = |z|-(x+y),$$

it follows that $\|z_+-x\| \leqslant \varepsilon$. Similarly $\|z_- - y\| \leqslant \varepsilon$. ∎

2.2. LEMMA. *If x and y are elements in A_{sa} with $\|x-y\| \leqslant \varepsilon$, then with $\delta^2 = (\|x\|+\|y\|)\varepsilon$ we have*

$$\| |x|-|y| \| \leqslant \delta, \qquad \|x_+-y_+\| \leqslant \tfrac{1}{2}(\delta+\varepsilon), \qquad \|x_- - y_-\| \leqslant \tfrac{1}{2}(\delta+\varepsilon).$$

Proof. The equation

$$x^2-y^2 = x(x-y)+(x-y)\,y$$

shows that $\|x^2-y^2\| \leqslant \delta^2$. As in Lemma 2.1 this implies that $\| |x|-|y| \| \leqslant \delta$. Furthermore,

$$2(x_+-y_+) = (|x|+x)-(|y|+y) = (|x|-|y|)+(x-y),$$

whence $\|x_+-y_+\| \leqslant \frac{1}{2}(\delta+\varepsilon)$. Similarly $\|x_- - y_-\| \leqslant \frac{1}{2}(\delta+\varepsilon)$. ∎

2.3. LEMMA (cf. [31, 3.4]). *Suppose that A is unital, p is a projection in A, and $x \in A$ such that the element $b=(1-p)\,x(1-p)$ is invertible in $(1-p)\,A(1-p)$. Then x is invertible in A if and only if $a-cb^{-1}d$ is invertible in pAp, where $a=pxp$, $c=px(1-p)$ and $d=(1-p)xp$.*

Proof. With the obvious matrix notation we have

$$x = \begin{pmatrix} a & c \\ d & b \end{pmatrix} = \begin{pmatrix} p & cb^{-1} \\ 0 & 1-p \end{pmatrix} \begin{pmatrix} a-cb^{-1}d & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} p & 0 \\ b^{-1}d & 1-p \end{pmatrix}.$$

Since the outer factors in this product are always invertible in A, the invertibility of x is equivalent to the invertibility of the factor in the middle, which is diagonal. The conclusion follows. ∎

2.4. *Remark.* Note that if $x=x^*$ in Lemma 2.3, then $d=c^*$, and a and b are self-adjoint (in pAp and $(1-p)\,A(1-p)$, respectively). Therefore also $a-cb^{-1}c^*$ is self-adjoint.

2.5. THEOREM. *If A is a C^*-algebra with* $\mathrm{RR}(A)=0$, *then* $\mathrm{RR}(pAp)=0$ *for every projection p in $M(A)$. Conversely, if* $\mathrm{RR}(pAp)=0$ *and* $\mathrm{RR}((1-p)A(1-p))=0$ *for some projection p in $M(A)$, then* $\mathrm{RR}(A)=0$.

Proof. Assume first that A is unital; whence $p\in A$. If $\mathrm{RR}(A)=0$ and $x\in(pAp)_{sa}$, we can by assumption for each $\varepsilon>0$ find an invertible element y in A_{sa} such that $\|x+1-p-y\|\leqslant\varepsilon$. With $b=(1-p)\,y(1-p)$ this means that $\|1-p-b\|\leqslant\varepsilon$. Assuming that $\varepsilon<1$ it follows that b is invertible in $(1-p)\,A(1-p)$. By Lemma 2.3 this implies that the self-adjoint element

$$z=pyp-py(1-p)\,b^{-1}(1-p)\,yp$$

is invertible in pAp. Estimating the Neumann series for b^{-1} we get $\|b^{-1}\|\leqslant(1-\varepsilon)^{-1}$; whence

$$\|py(1-p)\,b^{-1}(1-p)\,yp\|\leqslant(1-\varepsilon)^{-1}\,\|py(1-p)\|^2\leqslant(1-\varepsilon)^{-1}\,\varepsilon^2.$$

Thus

$$\|x-z\|\leqslant\|x-pyp\|+(1-\varepsilon)^{-1}\,\varepsilon^2\leqslant\varepsilon+(1-\varepsilon)^{-1}\,\varepsilon^2,$$

which shows that $\mathrm{RR}(pAp)=0$.

Conversely, if $\mathrm{RR}(pAp)=\mathrm{RR}((1-p)\,A(1-p))=0$ (and A is still unital) we take x in A_{sa} and write it as

$$x=\begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$$

as in the proof of Lemma 2.3. Given $\varepsilon>0$ we can find b_0 invertible in $(1-p)\,A(1-p)$ with $b_0=b_0^*$ and $\|b-b_0\|\leqslant\varepsilon$. Then, considering $a-cb_0^{-1}c^*$, we can find a_0 in pAp with $a_0=a_0^*$ and $\|a-a_0\|\leqslant\varepsilon$, such that $a_0-cb_0^{-1}c^*$ is invertible in pAp. By Lemma 2.3 this means that the self-adjoint element

$$x_0=\begin{pmatrix} a_0 & c \\ c^* & b_0 \end{pmatrix}$$

is invertible in A. Evidently $\|x-x_0\|\leqslant\varepsilon$, so $\mathrm{RR}(A)=0$.

In the case where A is non-unital, but $p\in A$ we consider $\tilde{A}$ and note that $((1-p)\,A(1-p))^\sim=(1-p)\,A(1-p)+\mathbb{C}(1-p)$; after which the arguments above can be used verbatim. Similarly we can dispense with the case where $1-p\in A$.

We are left with the case where A is non-unital and where $p\notin A$ and $1-p\notin A$. To show that $\mathrm{RR}(pAp)=0$, under the assumption that $\mathrm{RR}(A)=0$, identify $(pAp)^\sim$ with $pAp+\mathbb{C}p$. Given $x=x_0+\lambda p$ in $(pAp)^\sim_{sa}$

we may assume that $\lambda \neq 0$ (by approximation). Then $x_0 + \lambda \in \tilde{A}_{sa}$ and we can find an invertible element y in $\tilde{A}_{sa}$ such that $\|x_0 + \lambda - y\| < \varepsilon$, where $\varepsilon < \lambda$. As in the unital case this implies that $b = (1-p)\,y(1-p)$ is invertible in $((1-p)\,A(1-p))^{\sim}$ (identified with $(1-p)\,A(1-p) + \mathbb{C}(1-p)$) and that the element z, defined as before, is invertible in $(pAp)^{\sim}$ and close to x.

Finally, if $\mathrm{RR}(pAp) = \mathrm{RR}((1-p)\,A(1-p)) = 0$, consider an element

$$x = \begin{pmatrix} a+\lambda & c \\ c^* & b+\lambda \end{pmatrix}$$

in $\tilde{A}_{sa}$. Again we may assume that $\lambda \neq 0$. Choose an invertible element $b_0 + \lambda_0$ in $((1-p)\,A(1-p))^{\sim}_{sa}$ such that $\|b + \lambda - (b_0 + \lambda_0)\| \leqslant \varepsilon$ and $\varepsilon \ll |\lambda|$. Then choose $a_1 + \lambda_1$ in $(pAp)^{\sim}_{sa}$ with $\|a + \lambda - (a_1 + \lambda_1)\| \leqslant \varepsilon$ such that $a_1 + \lambda_1 - c(b_0 + \lambda_0)^{-1}\,c^*$ is invertible in $(pAp)^{\sim}$. Then the element

$$x_0 = \begin{pmatrix} a_1+\lambda_1 & c \\ c^* & b_0+\lambda_0 \end{pmatrix}$$

is self-adjoint and invertible in $A + \mathbb{C}p + \mathbb{C}(1-p)$, but $x_0 \notin \tilde{A}$ unless $\lambda_0 = \lambda_1$. However, $|\lambda - \lambda_0| \leqslant \varepsilon$, so with $t = \lambda_1 \lambda_0^{-1}$ we have

$$x_1 = \begin{pmatrix} 1 & 0 \\ 0 & t^{1/2} \end{pmatrix} x_0 \begin{pmatrix} 1 & 0 \\ 0 & t^{1/2} \end{pmatrix} = \begin{pmatrix} a_1+\lambda_1 & t^{1/2}c \\ t^{1/2}c^* & tb_0+\lambda_1 \end{pmatrix},$$

which is invertible in $\tilde{A}_{sa}$ and satisfies

$$\begin{aligned} \|x - x_1\| &= \left\| \begin{pmatrix} a+\lambda-(a_1+\lambda_1) & (1-t^{1/2})c \\ (1-t^{1/2})c^* & b+\lambda-(tb_0+\lambda_1) \end{pmatrix} \right\| \\ &\leqslant \varepsilon + 2\,|1-t^{1/2}|\;\|c\| + |1-t|\;\|b+\lambda\| + \varepsilon t. \end{aligned}$$

Since $|1-t| \leqslant 2\varepsilon\;(\varepsilon + |\lambda|)^{-1}$, we can approximate x arbitrarily well with x_1, as desired. ∎

2.6. THEOREM. *For a C^*-algebra A the following six conditions are equivalent*:

(i) $\mathrm{RR}(A) = 0$;

(ii) *the elements in A_{sa} with finite spectra are dense in A_{sa}*;

(iii) *every hereditary C^*-subalgebra of A has an approximate unit (not necessarily increasing) consisting of projections*;

(iv) *for each pair of orthogonal elements x, y in $\tilde{A}_+$ and $\varepsilon > 0$ there is a projection p in $\tilde{A}$ (thus $p \in A$ or $1-p \in A$), such that $\|(1-p)x\| \leqslant \varepsilon$ and $py = 0$*;

(v) *for each pair of orthogonal elements* x, y *in* $\tilde{A}_+$ *and* $\varepsilon > 0$ *there is a projection* p *in* $\tilde{A}$, *such that* $\|(1-p)x\| \leqslant \varepsilon$ *and* $\|py\| \leqslant \varepsilon$;

(vi) *for each pair* x, y *in* $\tilde{A}_+$ *and* $\varepsilon > 0$, *such that* $\|xy\| < \varepsilon^2$, *there is a projection* p *in* $\tilde{A}$, *such that* $\|(1-p)x\| < \varepsilon$ *and* $\|py\| < \varepsilon$.

Proof. We may evidently assume that A is unital.

(i) $\Rightarrow$ (vi) Take $\varepsilon_1 > 0$ such that

$$\|xy\|^{1/2} + \tfrac{1}{2}(((2\,\|x-y\| + \varepsilon_1)\varepsilon_1)^{1/2} + \varepsilon_1) < \varepsilon.$$

Then choose an invertible element z in A_{sa} such that $\|x-y-z\| \leqslant \varepsilon_1$. Since $0 \notin sp(z)$ there is by spectral theory a projection p in A such that $pz = z_+$ and $(1-p)z = z_-$. Using Lemmas 2.1 and 2.2 we have

$$\begin{aligned}\|x - z_+\| &\leqslant \|x-(x-y)_+\| + \|(x-y)_+ - z_+\| \\ &\leqslant \|xy\|^{1/2} + \tfrac{1}{2}(((\|x-y\| + \|z\|)\varepsilon_1)^{1/2} + \varepsilon_1) < \varepsilon.\end{aligned}$$

Similarly $\|y - z_-\| < \varepsilon$, and consequently

$$\|(1-p)x\| = \|(1-p)(x-z_+)\| < \varepsilon,$$

and similarly $\|py\| < \varepsilon$.

(iv) $\Rightarrow$ (v) is evident.

(v) $\Rightarrow$ (i) If $x \in A_{sa}$ and $\varepsilon > 0$, consider the orthogonal elements x_+ and x_- in A_+. Choose a projection p in A such that $\|(1-p)x_+\| \leqslant \varepsilon$ and $\|px_-\| \leqslant \varepsilon$. As in the proof of the implication (v) $\Rightarrow$ (ii) we get

$$\|x - (pxp + (1-p)\,x(1-p))\| \leqslant 2\varepsilon$$

$$-\varepsilon p \leqslant pxp,\ (1-p)\,x(1-p) \leqslant \varepsilon(1-p).$$

Thus the element

$$y = pxp + 2\varepsilon p + (1-p)\,x(1-p) - 2\varepsilon(1-p)$$

is invertible in A_{sa}, and $\|x-y\| \leqslant 4\varepsilon$.

(v) $\Rightarrow$ (ii) Given x in A_{sa} and real numbers r, t such that $r \leqslant x \leqslant t$, put $s = \frac{1}{2}(r+t)$ and for $\varepsilon > 0$ choose a projection p in A such that

$$\|(1-p)(x-s)_+\| < \varepsilon, \qquad \|p(x-s)_-\| < \varepsilon.$$

Then

$$\begin{aligned}&\|x - (pxp + (1-p)\,x(1-p))\| \\ &\quad = \|px(1-p) + (1-p)xp\| \\ &\quad = \|px(1-p)\| = \|p((x-s)_+ - (x-s)_-)(1-p)\| \leqslant 2\varepsilon.\end{aligned}$$

Moreover,

$$(s-\varepsilon)\,p \leqslant sp + p((x-s)_+ - \varepsilon)\,p \leqslant sp + p(x-s)\,p = pxp \leqslant tp.$$

Similarly,

$$r(1-p) \leqslant (1-p)\,x(1-p) \leqslant (s+\varepsilon)(1-p).$$

Since both $\mathrm{RR}(pAp)=0$ and $\mathrm{RR}((1-p)\,A(1-p))=0$ by Theorem 2.5, the argument may be repeated with pxp and $(1-p)\,x(1-p)$ in place of x. Thus by a simple induction argument, starting with $\|x\| \leqslant 1$ and taking $\varepsilon_n = 2^{-n}\varepsilon$, we arrive at the nth step at pairwise orthogonal projections p_k, $1 \leqslant k \leqslant 2^n$, with sum 1 such that

$$\left\| x - \sum p_k x p_k \right\| \leqslant 2(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n) \leqslant 2\varepsilon.$$

Moreover,

$$(k-1)2^{-n+1} - 1 - \varepsilon \leqslant p_k x p_k \leqslant k2^{-n+1} - 1 + \varepsilon$$

for every k. Put $x_n = \sum (k2^{-n+1} - 1)\,p_k$. Then x_n has finite spectrum (in fact all spectral points are dyadic rationals) and

$$\|x - x_n\| \leqslant 2\varepsilon + \left\| \sum p_k x p_k - (k2^{-n+1} - 1)\,p_k \right\| \leqslant 3\varepsilon + 2^{-n+1},$$

giving an approximation as close as we wish.

(ii) $\Rightarrow$ (iii) This is the implication (FS) $\Rightarrow$ (HP) proved in [27, Proposition 14].

(iii) $\Rightarrow$ (iv) If $xy = 0$ in A_+, let B denote the hereditary C^*-subalgebra of A generated by x, i.e., $B = (xAx)^=$. Since $x \in B$ there is by assumption a projection p in B such that $\|(1-p)x\| \leqslant \varepsilon$ for any given $\varepsilon > 0$. Since y annihilates B, $py = 0$ as desired. ∎

2.7. *Remarks.* The conditions (ii) and (iii) in Theorem 2.6 were labeled (FS) and (HP) in [27], and in [27, Proposition 16] there is an example, based on arguments by Blackadar and Elliott of a simple, separable C^*-algebra A satisfying (FS), i.e., $\mathrm{RR}(A)=0$, but not approximately finite-dimensional. Later Blackadar and Kumjian showed in [7] that the Bunce–Deddens algebras have real rank zero, and recently Choi and Elliott showed in [13] that for a dense set of irrational numbers, the corresponding irrational rotation algebras have real rank zero. It seems safe to predict that this is actually true for all irrational numbers.

2.8. COROLLARY. *If A is a C^*-algebra with* $\mathrm{RR}(A)=0$, *then* $\mathrm{RR}(B)=0$ *for every hereditary C^*-subalgebra B of A.*

Proof. Condition (iii) in Theorem 2.6 is clearly hereditary, even though we must verify it for $\tilde{B}$. ∎

2.9. PROPOSITION (cf. [23, Theorem 6]). *If A is a σ-unital C^*-algebra with* $\mathrm{RR}(A)=0$, *then it has an increasing sequence of projections, which form an approximate unit for A.*

Proof. Let h be a strictly positive element in A and put $\varepsilon_n = n^{-1}$. By 2.6(iii) there is a projection p_1 in A such that $\|(1-p_1)h\| \leqslant \varepsilon_1$. Applying 2.6(iii) to $(1-p_1)A(1-p_1)$, we obtain a projection p_2, orthogonal to p_1, such that

$$\|(1-p_2)(1-p_1)\,h^2(1-p_1)\| < \varepsilon_2^2,$$

which implies that

$$\|(1-(p_1+p_2))h\|^2 = \|(1-p_2)(1-p_1)\,h^2(1-p_1)(1-p_2)\| \leqslant \varepsilon_2^2.$$

Continuing by induction we obtain a sequence (p_n) of pairwise orthogonal projections in A, such that $\|(1-\sum_{k=1}^n p_k)h\| \leqslant \varepsilon_n$, which means that the partial sums of the p_k's form an approximate unit for A. ∎

2.10. THEOREM. *If A is a C^*-algebra with* $\mathrm{RR}(A)=0$, *then* $\mathrm{RR}(\mathbb{M}_n(A))=0$ *for every n.*

Proof. Since $M(\mathbb{M}_n(A)) = \mathbb{M}_n(M(A))$ we may assume that A is unital. By induction, suppose that $\mathrm{RR}(\mathbb{M}_k(A))=0$ for all $k \leqslant n$. Let $p = 1 \otimes e_n$ in $\mathbb{M}_{n+1}(A)$, where e_n denotes the projection in $\mathbb{M}_{n+1}(\mathbb{C})$ on the space spanned by the first n basis vectors. Then $p\mathbb{M}_{n+1}(A)p$ is isomorphic to $\mathbb{M}_n(A)$, and $(1-p)\,\mathbb{M}_{n+1}(A)(1-p)$ is isomorphic to A. By assumption both of these algebras have real rank zero, whence $\mathrm{RR}(\mathbb{M}_{n+1}(A))=0$ by Theorem 2.5. ∎

3. TENSOR PRODUCTS, MORITA EQUIVALENCE, AND EXTENSIONS

Determining the real rank of a C^*-algebra is, with our present knowledge, just as hard as finding the topological (=Bass) stable rank. However, a surprisingly large number of our stock in trade C^*-algebras have real rank zero for obvious reasons. As mentioned in Remark 2.7 other algebras, notably the Bunce–Deddens algebras and (some of) the irrational rotation algebras, have real rank zero for reasons that are not clear today.

3.1. PROPOSITION. *If a C^*-algebra A is the inductive limit of a net $(A_\lambda)_{\lambda\in\Lambda}$ of C^*-algebras with real rank zero, then* $\mathrm{RR}(A)=0$.

Proof. We may assume that A is unital, and that $1\in A_\lambda$ for all λ. If A_λ is non-unital this is accomplished by identifying $\tilde{A}_\lambda$ with $A_\lambda+\mathbb{C}1$; and if A_λ is unital with $1_\lambda\neq 1$, we just observe that $B_\lambda=A_\lambda+\mathbb{C}(1-1_\lambda)$ has real rank zero and that the inductive limit of (B_λ) is $\tilde{A}$.

Now, if $x\in A_{sa}$ and $\varepsilon>0$, we first find x_λ in $(A_\lambda)_{sa}$ such that $\|x-x_\lambda\|<\frac{1}{2}\varepsilon$; Then, since $\mathrm{RR}(A_\lambda)=0$, we find an invertible element y_λ in $(A_\lambda)_{sa}$, hence invertible in A_{sa}, such that $\|x_\lambda-y_\lambda\|<\frac{1}{2}$. Thus $\|x-y_\lambda\|<\varepsilon$ and $\mathrm{RR}(A)=0$. ∎

3.2. THEOREM. *If A is a C^*-algebra with* $\mathrm{RR}(A)=0$ *and B is an approximately finite-dimensional C^*-algebra, then* $\mathrm{RR}(A\otimes B)=0$.

Proof. We know that B is the inductive limit of a net $(B_\lambda)_{\lambda\in\Lambda}$ of finite-dimensional algebras, and we may assume that B is unital, and its unit contained in every B_λ. For each λ, $A\otimes B_\lambda$ is the direct sum of a finite number of algebras, each of the form $\mathbb{M}_k(A)$ $(=A\otimes\mathbb{M}_k(\mathbb{C}))$, for various values of k, and thus $\mathrm{RR}(A\otimes B_\lambda)=0$ by Theorem 2.10. Since $A\otimes B$ is the inductive limit of the net $(A\otimes B_\lambda)_{\lambda\in\Lambda}$, it follows from Proposition 3.1 that $\mathrm{RR}(A\otimes B)=0$. ∎

3.3. COROLLARY. *If $\mathscr{K}$ denotes the algebra of compact operators on ℓ^2 and A is a C^*-algebra with* $\mathrm{RR}(A)=0$, *then* $\mathrm{RR}(A\otimes\mathscr{K})=0$.

3.4. COROLLARY. *If A is a C^*-algebra with* $\mathrm{RR}(A)=0$ *and X is a locally compact Hausdorff space with* $\dim X=0$, *then* $\mathrm{RR}(C_0(X,A))=0$.

Proof. If $\dim X=0$, then $C_0(X)$ is approximately finite-dimensional. ∎

3.5. *Remark.* The results above naturally provoke the conjecture

$$\mathrm{RR}(A\otimes B)\leqslant\mathrm{RR}(A)+\mathrm{RR}(B) \qquad (*)$$

for arbitrary C^*-algebras A and B, in analogy with the commutative formula $\dim(X\times Y)\leqslant\dim X+\dim Y$. In the case $\dim X=0$ it is straightforward to establish the formula $\mathrm{RR}(C_0(X,A))\leqslant\mathrm{RR}(A)$; and presumably the formula $(*)$ can be verified without too much bloodshed, when one of the factors are commutative.

Consider now a C^*-dynamical system (A,G,α), and for simplicity assume that G is abelian. Viewing the crossed product $G\times_\alpha A$ as a "skew" tensor product between the C^*-algebras $C^*(G)$ $(=C_0(\hat{G}))$ and A, one is tempted to write down formulas like

$$\mathrm{RR}(G\times_\alpha A)\leqslant\dim\hat{G}+\mathrm{RR}(A). \qquad (**)$$

Reality is not so kind. Note first that each irrational rotation algebra A_θ has the form $\mathbb{Z} \times_\alpha C(\mathbb{T})$ (where $\alpha f(t) = f(\theta t)$ for f in $C(\mathbb{T})$). Since $C^*(\mathbb{Z}) = C(\mathbb{T})$ we feel that A_θ should be a "non-commutative torus of dimension 2." But as mentioned in Remark 2.7, the real rank of A_θ is always $\leqslant 1$, and actually $= 0$ for many (probably all) irrational numbers θ. This fact allowed T. Natsume to produce for us the following counterexample to the formula $(**)$.

3.6. PROPOSITION. *There exists a C^*-dynamical system $(A, \mathbb{T}, \beta)$ with* $\mathrm{RR}(A) = 0$, *such that* $\mathrm{RR}(\mathbb{T} \times_\beta A) = 1$.

Proof. Let $A = \mathbb{Z} \times_\alpha C(\mathbb{T})$, where α is an irrational rotation on $\mathbb{T}$ such that $\mathrm{RR}(A) = 0$; cf. [13]. Now take β to be the dual action of $\mathbb{T}$ on A, so that by the Takai–Takesaki duality theorem [26, 7.9.3] we obtain

$$\mathbb{T} \times_\beta A = C(\mathbb{T}) \otimes \mathcal{K}.$$

Since $\dim \mathbb{T} = 1$, the topological stable rank of $C(\mathbb{T})$ is 1, and thus by [31, Theorem 3.3], $\mathrm{tsr}(C(\mathbb{T}) \otimes \mathcal{K}) = 1$. By Proposition 1.2 this implies that $\mathrm{RR}(\mathbb{T} \times_\beta A) \leqslant 1$. It cannot be zero, because then $\mathrm{RR}(C(\mathbb{T})) = 0$ by Theorem 2.5, contradicting Proposition 1.1. Thus $\mathrm{RR}(\mathbb{T} \times_\beta A)) = 1$, as claimed. ∎

Despite the failure, in general, of the formula $(**)$ one might still hope for its validity in the purely combinatorial situation arising when G is finite. Although we now know that even when $G = \mathbb{Z}_2$, a crossed product with A an AF-algebra need not result in an AF-algebra (see [5]), all the examples up till now have had real rank zero. However, to prove that $\mathrm{RR}(\mathbb{Z}_2 \times_\alpha A) = 0$, whenever $\mathrm{RR}(A) = 0$, one must show that every element in A of the form $1 - x^*x$, where $x^* = \alpha(x)$, can be approximated by an invertible element of the same kind; and that, upon closer reflection, is far from obvious.

3.7. LEMMA. *If A and B are strongly Morita equivalent C^*-algebras and* $\mathrm{RR}(A) = 0$, *there is for every separable C^*-subalgebra B_0 of B a pair A_∞, B_∞ of separable, strongly Morita equivalent C^*-subalgebras of A and B, respectively, such that $B_0 \subset B_\infty$ and* $\mathrm{RR}(A_\infty) = 0$.

Proof. By assumption there is an $A - B$ imprimitivity bimodule X. This we may view as a closed, linear space of operators, such that the set

$$C = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

is a C^*-algebra under the obvious matrix operations, and such that A and

B are full hereditary C^*-subalgebras of C (i.e., neither is contained in any proper, closed ideal of C); cf. [11, Theorem 1.1].

Choose a separable subspace X_1 of X, such that the C^*-subalgebra B_1 of B generated by $X_1X_1^*$ contains B_0, and let A_0 be the (separable) C^*-subalgebra of A generated by $X_1^*X_1$. Since $\mathrm{RR}(A)=0$, there is by Theorem 2.6(ii) a countable set F_1 in A_{sa}, such that each element in F_1 has finite spectrum, and every element in $(A_0)_{sa}$ belongs to the closure of F_1. Let A_1 be the separable C^*-subalgebra of A generated by F_1 (and note that $A_0 \subset A_1$). Now choose a separable subspace X_2 of X, containing X_1, such that $C^*(X_2^*X_2) \supset A_1$ and $C^*(X_2X_2^*) \supset B_1$. Call the latter algebra B_2 and let A_2 be a separable C^*-subalgebra of A with enough self-adjoint elements with finite spectra to approximate every element in $C^*(X_2^*X_2)_{sa}$. By induction we obtain sequences of separable spaces/algebras (X_n), (A_n), and (B_n) and we let X_∞, A_∞, and B_∞ be the closures of their respective unions. Evidently X_∞ is an $A_\infty - B_\infty$ imprimitivity bimodule, so that A_∞ and B_∞ are strongly Morita equivalent. And since $(A_n)_{sa}$ is contained in the closure of the elements in $(A_{n+1})_{sa}$ with finite spectra, for every n, we see that $\mathrm{RR}(A_\infty)=0$. ∎

3.8. THEOREM. *If A and B are strongly Morita equivalent C^*-algebras and* $\mathrm{RR}(A)=0$, *then* $\mathrm{RR}(B)=0$.

Proof. Clearly it suffices to show that every self-adjoint element in B is contained in a separable C^*-subalgebra B_∞ with $\mathrm{RR}(B_\infty)=0$. By Lemma 3.7 this means that we may assume that both A and B are separable. But then, by [11, Theorem 1.2] we have an isomorphism

$$A \otimes \mathscr{K} = B \otimes \mathscr{K}.$$

From Corollary 3.3 we conclude that $\mathrm{RR}(B \otimes \mathscr{K})=0$, whence $\mathrm{RR}(B)=0$ by Corollary 2.8. ∎

Recall from [16] (see also [35]) that a C^*-algebra A is called purely infinite if every non-zero hereditary C^*-subalgebra of A contains an infinite projection, i.e., a projection which is Murray–von Neumann equivalent to a proper subprojection.

The following result was proved by Zhang in [36] (and again in [37]). We present here an alternative proof inspired by the methods in [32].

3.9. PROPOSITION. *If A is a simple, unital, purely infinite C^*-algebra, then* $\mathrm{RR}(A)=0$.

Proof. Consider an element x in A_{sa}. To approximate x with invertible elements we need only consider the case where $0 \in \mathrm{sp}(x)$. Replacing x with

$f(x)$, where f is a continuous function which is zero in a neighborhood of zero, but such that $f(t)=t$ outside a slightly larger neighborhood (so that $\|x-f(x)\|$ is small), we may assume that the annihilator B of x is a non-zero, hereditary C^*-subalgebra of A.

By assumption B contains an infinite projection p. Since A is simple and unital we can find a partial isometry v in A such that $vv^*=1-p$ and $v^*v=q\leqslant p$; cf. [15, 1.10 and 2.2]. Set $u=v+v^*+(p-q)$ and check by computation that u is a self-adjoint unitary (a symmetry) with $uqu=1-p$.

For $\varepsilon>0$, take $y=x+\varepsilon u$. Then $y\in A_{sa}$ and $\|x-y\|=\varepsilon$. Moreover,

$$y^2=x^2+\varepsilon^2+\varepsilon(xu+ux)=x^2+\varepsilon^2+\varepsilon(xv+v^*x).$$

In matrix notation, with respect to the decomposition of A given by p and $1-p$, we have

$$y^2=\begin{pmatrix}\varepsilon^2p & \varepsilon v^*x\\ \varepsilon xv & x^2+\varepsilon^2(1-p)\end{pmatrix}.$$

To prove that y^2 is invertible, we invoke the test given in Lemma 2.3 (with p replaced by $1-p$). Compute

$$\begin{aligned}&x^2+\varepsilon^2(1-p)-\varepsilon xv(\varepsilon^{-2}p)\varepsilon v^*x\\ &\quad=x^2+\varepsilon^2(1-p)-x(1-p)x=\varepsilon^2(1-p).\end{aligned}$$

Since this element is invertible in $(1-p)A(1-p)$, y^2 is invertible in A; whence y is invertible, as desired. ∎

3.10. COROLLARY. *The Cuntz algebras O_n and the Cuntz–Krieger algebras O_A all have real rank zero.*

Proof. These algebras (for irreducible A's) are simple and purely infinite; see [14; 17; 15, 1.6]. ∎

3.11. COROLLARY. *If A is a simple, unital C^*-algebra which is not stably finite, and B is a UHF (Glimm) algebra, then* $\mathrm{RR}(A\otimes B)=0$.

Proof. By [32, Theorem 6.9] these tensor products are purely infinite. ∎

3.12. *Remark.* As Rørdam shows in [32], if A is stably finite (simple and unital), then $\mathrm{GL}(A\otimes B)$ is dense in $A\otimes B$ for every Glimm algebra B. By Proposition 1.2 this means that

$$\mathrm{RR}(A\otimes B)\leqslant 1.$$

At this point it should be recalled that Blackadar has given us examples of simple C^*-algebras with no non-trivial projections [1; 2]. Evidently these cannot have real rank zero. Considering our relative ignorance of the species of simple C^*-algebras, there is little reason to believe that their real rank, or, for that matter, their Bass stable rank should be restricted.

For our last theorem (3.21) we need a number of results obtained recently by S. Zhang. For the readers convenience they are stated separately, and sometimes with slightly different proofs.

3.13. LEMMA (cf. [34, 2.5]). *If I is a closed ideal in a C^*-algebra A such that* $\mathrm{RR}(I)=0$, *and if B is a hereditary C^*-subalgebra of A, then every projection in $B/B\cap I$ $(=B+I/I)$ that lifts to a projection in A can be lifted to a projection in B.*

Proof. We are given a projection p in A, such that $p\in B+I$, and we must find a projection q in B, such that $p-q\in I$. Toward this end, write $p=b+x$ for some self-adjoint elements b in B and x in I, and consider the hereditary C^*-subalgebra pIp of I. By Theorem 2.6(iii) there is for each $\varepsilon>0$ a projection $r\leqslant p$ in I, such that $\|px^2p(1-r)\|<\varepsilon^2$; whence $\|x(p-r)\|<\varepsilon$. Now

$$p-r=b(p-r)+x(p-r), \tag{***}$$

so if we set $p_1=p-r$ and $b_1=b(p-r)b$, then by squaring the equation (***) we get $p_1=b_1+x_1$, where $x_1\in I$ and $\|x_1\|<2\,\|b\|\,\varepsilon+\varepsilon^2$. Choosing ε small enough this implies that $\|b_1-b_1^2\|<\frac{1}{4}$, so that the element b_1 has a gap around $\frac{1}{2}$ in its spectrum. Thus $q=f(b_1)$ is a projection for a suitable continuous function f on $\mathrm{sp}(b_1)$ – viz. $f(t)=0$ for $t<\frac{1}{2}$ and $f(t)=1$ for $t>\frac{1}{2}$. Since $b_1\in B$, $q\in B$, and evidently $p-q\in I$, as desired. ∎

3.14. THEOREM (cf. [36, 3.2]. *If I is a closed ideal in a C^*-algebra A, then* $\mathrm{RR}(A)=0$ *if and only if* $\mathrm{RR}(I)=\mathrm{RR}(A/I)=0$ *and every projection in A/I lifts to a projection in A.*

Proof. If $\mathrm{RR}(A)=0$, then evidently $\mathrm{RR}(A/I)=0$, and $\mathrm{RR}(I)=0$ by Corollary 2.8. Consider now an element x in A_{sa} whose image in A/I is a projection. Take $\varepsilon>0$ and find an element y in A_{sa} with finite spectrum, such that $\|x-y\|\leqslant\varepsilon^2$. If $y=\sum\lambda_k p_k$ is the spectral resolution of y, and π denotes the quotient map, we see that

$$\left\|\pi\left(\sum(\lambda_k-\lambda_k^2)\,p_k\right)\right\|=\|\pi(y-y^2-(x-x^2))\|\leqslant 3\varepsilon^2.$$

This means that $p_k \in I$ $(=\ker \pi)$ whenever $|\lambda_k - \lambda_k^2| \geqslant 4\varepsilon^2$. Let σ and μ denote the set of k's for which $|1-\lambda_k| < 2\varepsilon$ and $|\lambda_k| < 2\varepsilon$, respectively, and set $p = \sum_\sigma p_k$. Since $p_k \in I$ if $k \notin \sigma \cup \mu$ we see that

$$\|\pi(p-y)\| = \left\| \sum_\sigma p_k - \sum_{\sigma \cup \mu} \lambda_k p_k \right\| \leqslant 2\varepsilon.$$

It follows that $\|\pi(p-x)\| \leqslant 2\varepsilon + \varepsilon^2$, so for ε small enough we can find a self-adjoint element z in A with $z-x$ in I and $\|p-z\| < \frac{1}{12}$. This implies that

$$\|z - z^2\| = \|z - p - (z^2 - p^2)\| < \tfrac{1}{4},$$

so the spectrum of z has a gap around $\frac{1}{2}$. Consequently $q = f(z)$ is a projection in A for a suitable continuous function f on $\mathrm{sp}(z)$, and q is a lift of the projection $\pi(x)$.

Conversely, if the three conditions are satisfied, consider an orthogonal pair x, y in A_+. Let B denote the hereditary C^*-subalgebra of A generated by x, i.e., $B = (xAx)^-$, and with π the quotient map consider the hereditary C^*-subalgebra $\pi(B)$ of $\pi(A)$ (generated by $\pi(x)$; cf. [26, 1.5.11]). Since by assumption $\mathrm{RR}(\pi(A)) = 0$, there is by Theorem 2.6(iii) for each $\varepsilon > 0$ a projection q in $\pi(B)$ such that $\|(1-q)\,\pi(x)\| < \varepsilon$. We know that q can be lifted to a projection p in A, and it follows from Lemma 3.13 that we may assume that $p \in B$. Now consider the ideal $(1-p)(I \cap B)(1-p)$ in $(1-p)\,B(1-p)$, and again let π denote the quotient map (identifying the quotient with $(1-q)\,\pi(B)(1-q)$). Since $\mathrm{RR}(I) = 0$, also $\mathrm{RR}((1-p)(I \cap B)(1-p)) = 0$; and as $\|\pi((1-p)\,x^2(1-p))\| < \varepsilon^2$, there is by Theorem 2.6(iii) a projection r in $(1-p)(I \cap B)(1-p)$ such that $\|(1-r)(1-p)x\| < \varepsilon$. Since $r \leqslant 1-p$ and $r \in B$, $p_1 = p + r$ is a projection in B with $\|(1-p_1)x\| < \varepsilon$. By construction, $p_1 y = 0$; so that condition (iv) in Theorem 2.6 is satisfied. Consequently $\mathrm{RR}(A) = 0$. ∎

3.15. Proposition. *If I is a closed ideal in a C^*-algebra A such that* $\mathrm{RR}(I) = 0$, *and if the induced homomorphism from $K_0(A)$ to $K_0(A/I)$ is surjective, then every projection in A/I can be lifted to a projection in A.*

Proof. From the surjectivity of the K_0-map we conclude that every projection in A/I lifts to a projection in a matrix algebra $\mathbb{M}_n(A)$, for some n. But then Lemma 3.13 immediately shows that it can also be lifted to a projection in the hereditary C^*-subalgebra A of $\mathbb{M}_n(A)$. ∎

3.16. Corollary [34, Corollary 2.12]. *If I is a closed ideal in a C^*-algebra A, such that* $\mathrm{RR}(I) = 0$ *and* $K_1(I) = 0$, *then every projection in A/I can be lifted to a projection in A.*

Proof. From the six-term exact sequence in K-theory [4, 9.3.1], in particular the terms

$$K_0(A) \to K_0(A/I) \to K_1(I)(=0),$$

it follows that the K_0-map is surjective; and the result follows from Proposition 3.15. ∎

3.17. *Remark.* We are indebted to G. A. Elliott for the observation that 3.13 would give 3.15, and thus an elegant route to 3.16. The observant reader will immediately see the similarity between this argument and the argument that projections lift from quotient AF-algebras if the kernel is an AF-algebra as well. During the conversation with Elliott the following two results also arose. The first shows that split extensions of real rank zero algebras by other real rank zero algebras produces algebras of real rank zero. For general extensions this is no longer true: Every Bunce–Deddens algebra has a one-dimensional extension (determined by a nonliftable projection in the corona), with real rank one.

3.18. PROPOSITION. *If I is a closed ideal in a C^*-algebra A, and B is a C^*-subalgebra such that $A = B + I$, then* $\mathrm{RR}(A) = 0$ *provided that* $\mathrm{RR}(I) = \mathrm{RR}(B) = 0$.

Proof. Let π denote the quotient map and note that $\pi(A) = \pi(B)$. Since $\mathrm{RR}(B) = 0$, every projection in $\pi(A)$ lifts to a projection in A (indeed, one in B) by Theorem 3.14. But since $\mathrm{RR}(I) = \mathrm{RR}(\pi(A)) = 0$, it follows from the other half of Theorem 3.14 that $\mathrm{RR}(A) = 0$. ∎

3.19. PROPOSITION. *If (B_n) is a sequence of hereditary C^*-subalgebras of a separable C^*-algebra A, such that* $\mathrm{RR}(B_n) = 0$ *for every n, and $\bigcup B_n$ is not contained in any proper, closed ideal of A, then* $\mathrm{RR}(A) = 0$.

Proof. For each n, let I_n denote the closed ideal of A generated by B_n. Then I_n is stably isomorphic to B_n by [9, Theorem 2.8], so that $\mathrm{RR}(I_n) = 0$ by Theorem 3.8. If $J_n = I_1 + \cdots + I_n$, it follows by an inductive application of Proposition 3.18 that $\mathrm{RR}(J_n) = 0$ for every n. Finally, A is the inductive limit of the ideals (J_n), whence $\mathrm{RR}(A) = 0$ by Proposition 3.1. ∎

3.20. LEMMA. *If A is a simple, σ-unital C^*-algebra with real rank zero, then its corona algebra $M(A)/A$ is purely infinite and every non-zero projection in it is infinite. Moreover, every hereditary C^*-subalgebra of $M(A)/A$ is the closed linear span of its projections.*

Proof. This is [35, Theorems 1.1 and 1.3], and the proof cannot be improved, we think. ∎

3.21. THEOREM. *If A is a separable, matroid C^*-algebra, then its multiplier algebra has real rank zero.*

Proof. Assume first that A is finite (i.e., has a finite trace). Then $M(A)/A$ is simple by [19, Theorem 3.1]. Thus, combining Lemma 3.20 with Proposition 3.9, we see that $\mathrm{RR}(M(A)/A)=0$. Since A is approximately finite-dimensional, $K_1(A)=0$ and $\mathrm{RR}(A)=0$; so by combining Corollary 3.16 and Theorem 3.14 it follows that $\mathrm{RR}(M(A))=0$.

Now consider the case where A is infinite. Then $A=B\otimes\mathscr{K}$, where B is a finite, matroid C^*-algebra, and by [19, Theorem 3.2] there is a unique, non-trivial, closed ideal J such that $A\subset J\subset M(A)$. Choose a finite projection p in J such that $pAp=B$. Then $pM(A)p=M(B)$, whence $\mathrm{RR}(pM(A)p)=0$ by the first part of the proof. However, $pM(A)p$ is a full hereditary C^*-subalgebra of J (because J/A is simple), and thus $pM(A)p$ and J are strongly Morita equivalent by [11, Theorem 1.1]. By Theorem 3.8 this implies that $\mathrm{RR}(J)=0$. Since $K_1(M(B))=0$—in fact the unitary group is connected by [19, Theorem 2.4]—we have $K_1(pM(A)p)=0$; and since K_1 is stable under strong Morita equivalence it follows that $K_1(J)=0$. Thus every projection in $M(A)/J$ lifts to a projection in $M(A)$ by Corollary 3.16. To conclude from Theorem 3.14 that $\mathrm{RR}(M(A))=0$, we only need to show that $\mathrm{RR}(M(A)/J)=0$, and that by Proposition 3.9 holds if $M(A)/J$ is purely infinite. To prove that this is the case, note first that every non-trivial, hereditary C^*-subalgebra D of $M(A)/J$ contains a non-trivial projection p. Indeed, by Lemma 3.20, D is the closed linear span of images of projections from $M(A)/A$. Lifting p to a projection q in $M(A)\backslash J$, and noting that J is the closed ideal generated by the finite projections in $M(A)$, we see that q is infinite. In fact $q\sim 1$ as shown in the proof of [19, Theorem 3.2] and again in the proof of [35, Proposition 2.1]. It follows that the image p in $M(A)/J$ is infinite as well. ∎

3.22. *Remarks.* Our original proof (from the Summer of 1988) of the theorem above was based on Elliott's results in [19] and contained several ad hoc arguments. Since then Zhang has launched his massive attack on the problems concerning real rank zero, [33–40], and we have chosen to borrow some of his techniques. Evidently the proof applies (with minor modifications) to other AF-algebras, as long as the ideal structure of their multiplier algebras is not too complicated. Any natural assumption on A that guarantees that the ideal lattice of $M(A)$ has finite height would probably suffice. On the other hand, there seems to be no evident obstructions for the conjecture that

(i) $\mathrm{RR}(M(A))=0$ for every AF-algebra A.

A much more daring conjecture would be that

(ii) $\mathrm{RR}(M(A))=0$ for every C^*-algebra A for which $\mathrm{RR}(A)=0$ and $K_1(A)=0$.

The most desirable result one could hope for would bypass K-theory and be a general statement about corona algebras, viz.,

(iii) If A is a C^*-algebra with $\mathrm{RR}(A)=0$, then $\mathrm{RR}(M(A)/A)=0$.

The conjecture (iii) (which implies (ii) by Theorem 3.14 combined with Corollary 3.16) has been considered by Zhang in [36]. As we saw in Lemma 3.20 (rather, [35, Theorem 1.1]), $M(A)/A$ is well supplied with projections; which of course is not enough to ensure real rank zero. But we can show that every positive element in a hereditary C^*-subalgebra B of $M(A)/A$ can be decomposed as the sum of two commuting elements from B_+, each of which can be arbitrarily well approximated by elements in B_+ with finite spectra. Despite this promising lead, and the special characteristics that prevail in corona algebras, cf. [24; 29], the conjecture refuses to come around.

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