RELATIVE K-THEORY FOR C^*-ALGEBRAS

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Abstract. Given C*-algebras A and B and a *-homomorphism $\phi : A \to B$, we adopt the portrait of the relative K-theory $K_*(\phi)$ due to Karoubi using Banach categories and Banach functors. We show that the elements of the relative groups may be represented in a simple form. We prove the existence of two six-term exact sequences, and we use these sequences to deduce the fact that the relative theory is isomorphic, in a natural way, to the K-theory of the mapping cone of ϕ as an excision result.

1. Introduction

The notion of relative K-theory is not new in the subject of operator algebras. Indeed, references such as [1] and [4] contain a concise exposition under the assumptions that A is unital, B is a quotient of A by some ideal I, and ϕ is the quotient map. The relative group produced is usually denoted $K_0(A, I)$ (although the notation varies throughout the literature). These assumptions are quite reasonable, since $K_0(A, I)$ provides the noncommutative generalization of the relative group $K^0(X, Y)$ in topological K-theory, where X is a compact space and Y is a closed subset of X. The key feature of these groups in both cases, commutative or not, is that they satisfiy excision: they depend only on a smaller substructure in question, namely $X - Y$ in the topological case and I in the noncommutative case. Specifically, the group $K^0(X, Y)$ is isomorphic to the group $K^0(X/Y, \{y\})$, where $\{y\}$ is the set Y collapsed to a point, and the group $K_0(A, I)$ is isomorphic to the group $K_0(\tilde{I}, I)$, where $\tilde{I}$ is the unitization of I; see II.2.35 of [5] and 5.4.2 of [1].

To obtain a relative theory for a more general *-homomorphism, we appeal to a construction of Karoubi in [5]. The approach is to describe the elements of the relative groups using certain triples, sometimes referred to as "relative K-cycles". One may notice that this resembles the construction of $K^0(X, Y)$ via triples of the form (E, F, α), where E and F are vector bundles over X and $\alpha : E|_Y \to F|_Y$ is an isomorphism between the bundles E and F when restricted to Y. This approach may appear somewhat classical at first, particularly because it is at odds with the mapping cone, a shortcut seen in both topological and operator K-theory. Indeed, analogous versions of the two main results here (Theorem 2.1 and Theorem 2.2) are obtained very easily using standard methods if one uses $K_*(C_\phi)$ as the definition of relative K-theory, where C_ϕ denotes the mapping cone of $\phi : A \to B$. Moreover, it turns out that the two portraits are isomorphic in a natural way (Corollary 2.3). It is therefore a reasonable question to ask why one would employ an alternative portrait at all.

The answer has multiple parts. First, there is much more freedom in selecting the elements from the algebras that represent elements of the relative groups via the setup in [5], which makes viewing and working with the groups easier in many situations. Second, certain maps in the six-term exact sequences are easier to compute. All in all, the resulting presentation is much simpler to work with, and applications of a simpler portrait in the context of C*-algebras are beginning to make their way into the literature. Indeed, the proof of the excision theorem in [7], which is a significant generalization of its predecessor in [6] (where the mapping cone was used), rests heavily on the new presentation, particularly on the notions of "isomorphism" of triples and "elementary" triples, to address certain fine, technical details. A portrait using partial isometries is developed in [6], and in fact, the map κ constructed there is essentially the same as Δ_0 in Corollary 2.3 below, the difference being that Δ_0 is a functorially induced group isomorphism, while κ is a bijective map constructed concretely. Also worth mentioning is recent work on groupoid homology [2], where a portrait of the relative K_0-group using triples is presented in order to elucidate the connections between K-theory and homology. An isomorphism between the resulting monoid and the K_0-group of the mapping cone is constructed under the assumption that A and B are unital and ϕ is unit-preserving. The unital assumption can be done away with if ϕ is nondegenerate and A contains an approximate identity of projections. Here, the general description of the relative groups allows us to do away with these assumptions. To provide further justification, we elaborate on the K_0- and K_1-groups separately.

It is a standard fact that the three usual notions of equivalence of projections, Murray-von Neumann, unitary, and homotopy, are all stably equivalent; in other words, they are the same modulo passing to matrix algebras. In constructing a relative K_0-group, it is therefore necessary to select a notion of equivalence with which to build the elements. The mapping cone C_ϕ is made from paths of projections in B with one endpoint equal to a scalar projection and the other endpoint in the image of ϕ. Therefore, in effect, $K_0(C_\phi)$ catalogues projections arising from A that are homotopic when moved to B via ϕ. It is often more desirable to describe equivalences of projections using partial isometries (see Example 2.5), from which the newer portrait is built. It is also worth mentioning that, although both portraits often require unitizations of the algebras involved, the appended unit seems to be less of a hindrance in the newer portrait.

As for K_1, homotopy is a much more natural equivalence and hence the portrait of the relative K_1 group gets less of a makeover than that of K_0. In fact, the two portraits are more or less the same. However, we draw a useful property from [5] which deserves to be mentioned. It is possible to define (ordinary and relative) K_1-groups more generally using partial unitaries (elements which are partial isometries and normal) rather than unitaries alone. This more general representation of group elements is especially convenient if one of (or both) A and B are not unital but contain nontrivial projections, such as $\mathcal{K}$, the compact operators on a separable Hilbert space (see Example 2.6).

The goal of the paper is to develop a clear picture of relative K-theory for C*-algebras using the setup in [5], as an alternative to the mapping cone. We remark that a preliminary development of the picture may be found in [7], where the relative K_0-group of an inclusion $A' \subseteq A$ is described using this approach. We also remark that, although the intention in [5] is mainly to develop topological K-theory, the setup lends itself quite well to C*-algebras. In fact, one could pursue these results more generally using Banach algebras, but we will not conduct such a pursuit beyond the definitions and preliminary results.

The paper is organized as follows. In section 2 we state the theorems and discuss some examples. In section 3 we present the general definition of relative K-theory in the context of C*-algebras and show that the elements of the theory can be represented in a simple form. In section 4 we prove the results.

Acknowledgements

The content of this paper constitutes a portion of the research conducted for my PhD dissertation. I am grateful to my advisor Ian F. Putnam for suggesting the project to me, and for numerous helpful discussions.

2. Summary and examples

In order to properly state the main results, we provide a simplified picture of the relative groups $K_0(\phi)$ and $K_1(\phi)$. Both may be thought of informally as groups that catalogue things in A that become K-theoretically the same in B when they pass through the *-homomorphism ϕ. Before we begin, we state a rather important remark regarding notation.

Remark 1. *In* [8]*, the symbols $K_0(\phi)$ and $K_1(\phi)$ are used to denote the group homomorphisms $K_0(A) \to K_0(B)$ and $K_1(A) \to K_1(B)$ induced by ϕ. Throughout this paper, the symbols $K_0(\phi)$ and $K_1(\phi)$ will be used to denote the relative groups, not group homomorphisms. Induced maps will instead be denoted more classically as ϕ_*.*

The group $K_0(\phi)$ is made from triples (p, q, v), where p and q are projections in some matrix algebras over $\tilde{A}$, the unitization of A, and v is an element in a matrix algebra over $\tilde{B}$ such that $v^*v = \phi(p)$ and $vv^* = \phi(q)$, that is, $\phi(p)$ and $\phi(q)$ are Murray-von Neumann equivalent (if A and B are unital and $\phi(1) = 1$, we may ignore unitizations, see Proposition 3.11). The triples are sorted into equivalence classes, denoted $[p, q, v]$, and are given a well-defined group operation by the usual block diagonal sum,

$$[p, q, v] + [p', q', v'] = [p \oplus p', q \oplus q', v \oplus v']$$

For two triples (p, q, v) and (p', q', v') to yield the same equivalence class, p and p' must be (at least stably) Murray-von Neumann equivalent in $\tilde{A}$, as must be q and q'. Moreover, elements c and d implementing such equivalences must play well with v and v' in that we require $\phi(d)v = v'\phi(c)$. More generally, we may use idempotents instead of projections to represent the elements of $K_0(\phi)$.

$K_1(\phi)$ is made from triples (p, u, g), where p is a projection in $M_\infty(\tilde{A})$, u is a unitary in $pM_\infty(\tilde{A})p$, and g is a unitary in $C[0,1] \otimes \phi(p)M_\infty(\tilde{B})\phi(p)$ such that $g(0) = \phi(p)$ and $g(1) = \phi(u)$. The triples are sorted into equivalence classes, denoted $[p, u, g]$, and are given a well-defined group operation by diagonal sum as before, although we have the formula

$$[p, u, g] + [p', u', g'] = [p, uu', gg']$$

if $p = p'$. For two triples (p, u, g) and (p', u', g') to yield the same equivalence class, p and p' must be (at least stably) Murray-von Neumann equivalent in $\tilde{A}$, and a partial isometry v implementing such an equivalence must satisfy $vu = u'v$ and $\phi(v)g(s) = g'(s)\phi(v)$ for $0 \leq s \leq 1$. The equivalence may also be described as stable homotopy: u and u' must be (at least stably) homotopic, as must be g and g'. Moreover, such homotopies u_t and g_t must satisfy $g_t(1) = \phi(u_t)$ for $0 \leq t \leq 1$. In general, the projection p may be replaced by an idempotent, u with an invertible in $pM_n(\tilde{A})p$, and g with a path of invertibles through $\phi(p)M_n(\tilde{B})\phi(p)$.

We now state the first main result concerning the relative groups.

Theorem 2.1. *There is a six-term exact sequence*

$$\begin{array}{ccccc} K_1(B) & \xrightarrow{\mu_0} & K_0(\phi) & \xrightarrow{\nu_0} & K_0(A) \\ \uparrow{\scriptstyle \phi_*} & & & & \downarrow{\scriptstyle \phi_*} \\ K_1(A) & \xleftarrow{\nu_1} & K_1(\phi) & \xleftarrow{\mu_1} & K_0(B) \end{array}$$

If $\phi = 0$, then the sequence splits at $K_0(A)$ and $K_1(A)$, i.e., both ν_0 and ν_1 have a right inverse.

The maps ν_0 and ν_1 are given by the formulas

$$\nu_0([p, q, v]) = [p] - [q] \qquad \nu_1([p, u, g]) = [u + 1_n - p]$$

where p and q are projections in $M_n(\tilde{A})$. The maps μ_0 and μ_1 are given by the formulas

$$\mu_0([u]) = [1_n, 1_n, u] \qquad \mu_1([p] - [q]) = [1_n, 1_n, f_p f_q^*]$$

where u is a unitary in $M_n(\tilde{B})$, and $f_p(t) = e^{2\pi i t p}$ for a projection p.

Before we state the second theorem, it will be useful to have the following functorial property in mind (see Proposition 3.10 for a proof). If there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & C \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ B & \xrightarrow{\beta} & D \end{array} \tag{1}$$

of C*-algebras and *-homomorphisms, then there are well-defined group homomorphisms $\alpha_* : K_j(\phi) \to K_j(\psi)$ for $j = 0, 1$ that satisfy $\alpha_*([p, q, v]) = [\alpha(p), \alpha(q), \beta(v)]$ and $\alpha_*([p, u, g]) = [\alpha(p), \alpha(u), \beta(g)]$.

Theorem 2.2. *Suppose*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{\iota_A} & A & \xrightarrow{\pi_A} & A/I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & J & \xrightarrow{\iota_B} & B & \xrightarrow{\pi_B} & B/J & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows. Then there is a six-term exact sequence

$$\begin{tikzcd}
K_0(\psi) \arrow[r, "\iota_*"] & K_0(\phi) \arrow[r, "\pi_*"] & K_0(\gamma) \arrow[d, "\partial_0"] \\
K_1(\gamma) \arrow[u, "\partial_1"] & K_1(\phi) \arrow[l, "\pi_*"] & K_1(\psi) \arrow[l, "\iota_*"]
\end{tikzcd}$$

The boundary maps ∂_1 and ∂_0 are natural in a functorial sense, as well as in the sense that they behave well with the boundary maps associated with the two short exact sequences of C*-algebras in the given diagram. This will be stated more precisely in section 4.

Before we state the third result (which is essentially an excision result), we briefly outline how the first two results are obtained if we use the mapping cone picture for relative K-theory: there is a short exact sequence

$$0 \longrightarrow SB \longrightarrow C_\phi \longrightarrow A \longrightarrow 0 \tag{2}$$

Applying the standard six-term exact sequence to the above sequence, together with the natural isomorphisms $K_1 \cong K_0 S$ and $K_0 \cong K_1 S$, we obtain the exact sequence of Theorem 2.1. If we have the commutative diagram with exact rows in Theorem 2.2, we obtain the short exact sequence

$$0 \longrightarrow C_\psi \longrightarrow C_\phi \longrightarrow C_\gamma \longrightarrow 0 \tag{3}$$

and, applying the standard six-term exact sequence, we obtain the conclusion of Theorem 2.2.

Corollary 2.3. *There are group isomorphisms $\Delta_j : K_j(\phi) \to K_j(C_\phi)$ for $j = 0, 1$ that are natural in the sense that if (1) is commutative, then the following diagram is commutative*

$$\begin{tikzcd}
K_j(\phi) \arrow[r, "\alpha_*"] \arrow[d, "\Delta_j"] & K_j(\psi) \arrow[d, "\Delta_j"] \\
K_j(C_\phi) \arrow[r, "(\alpha \oplus C\beta)_*"] & K_j(C_\psi)
\end{tikzcd}$$

Moreover, the isomorphisms Δ_j intertwine the six-term exact sequence of Theorem 2.1 with the six-term exact sequence associated to (2), and they intertwine the six-term exact sequence of Theorem 2.2 with the six-term exact sequence associated to (3). Details are found in section 4.

We now discuss some examples to illustrate the utility of Theorem 2.1 and Theorem 2.2. When A is a C*-subalgebra of B and $\phi : A \to B$ is the inclusion map, we denote $K_*(\phi)$ by $K_*(A; B)$.

Example 2.4. The following four examples are rather standard.

(i) If $K_*(B) = 0$, the maps ν_j in Theorem 2.1 give isomorphisms $K_j(\phi) \cong K_j(A)$. In particular, if $A \subseteq \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is an infinite dimensional Hilbert space and

$\phi : A \to \mathcal{B}(\mathcal{H})$ is the inclusion map, the K-theory of A is the same as the relative K-theory as a subalgebra of $\mathcal{B}(\mathcal{H})$.

(ii) If $K_*(A) = 0$, the maps μ_j in Theorem 2.1 gives isomorphisms $K_j(\phi) \cong K_{1-j}(B)$.

(iii) If I is an ideal in A and $\phi : A \to A/I$ is the quotient map, then $K_j(\phi) \cong K_j(I)$. This is a standard excision result, see 5.4.2 of [1].

(iv) If I is an ideal in A and $\phi : I \to A$ is the inclusion map, then $K_j(I; A) \cong K_{1-j}(A/I)$. This less standard excision result requires Bott periodicity; see [6] for a proof.

Example 2.5. Let D be any C*-algebra, and let A be the subalgebra of $B = M_2(D)$ consisting of the diagonal matrices. Since $K_*(A) \cong K_*(D) \oplus K_*(D)$ and $K_*(B) \cong K_*(D)$, we may write the six-term exact sequence of Theorem 2.1 as

$$\begin{tikzcd}
K_1(D) \arrow[r] & K_0(A;B) \arrow[r] & K_0(D) \oplus K_0(D) \arrow[d, "\phi_*"] \\
K_1(D) \oplus K_1(D) \arrow[u, "\phi_*"] & K_1(A;B) \arrow[l] & K_0(D) \arrow[l]
\end{tikzcd}$$

The vertical maps are both $\phi_*(g, h) = g + h$. Exactness implies that $K_*(A; B) \cong \ker \phi_* \cong K_*(D)$.

As a special case of interest, let $\mathcal{H}$ be a separable Hilbert space of dimension at least 2, and $\mathcal{M}$ a closed subspace such that $\mathcal{M} \neq \{0\}$ and $\mathcal{M} \neq \mathcal{H}$. Let $A = \mathcal{K}(\mathcal{M}) \oplus \mathcal{K}(\mathcal{M}^\perp)$, where $\mathcal{K}(\mathcal{M})$ is the C*-algebra of compact operators on $\mathcal{M}$, regarded as a subalgebra of $B = \mathcal{K}(\mathcal{H})$ as operators that leave $\mathcal{M}$ and $\mathcal{M}^\perp$ invariant. Then $K_0(A; B) \cong \mathbb{Z}$ and $K_1(A; B) = 0$. If we fix a unit vector ξ in $\mathcal{M}$, a unit vector η in $\mathcal{M}^\perp$, and a partial isometry v in B with source subspace $\operatorname{span}\{\xi\}$ and range subspace $\operatorname{span}\{\eta\}$, the group $K_0(A; B)$ is generated by the class of the triple (v^*v, vv^*, v).

Example 2.6. Let D be any C*-algebra and consider $A = D$ as a subalgebra of $B = D \oplus D$ via the embedding $d \mapsto (d, d)$. The six-term exact sequence of Theorem 2.1 becomes

$$\begin{tikzcd}
K_1(D) \oplus K_1(D) \arrow[r] & K_0(A;B) \arrow[r] & K_0(D) \arrow[d, "\phi_*"] \\
K_1(D) \arrow[u, "\phi_*"] & K_1(A;B) \arrow[l] & K_0(D) \oplus K_0(D) \arrow[l]
\end{tikzcd}$$

This time the vertical maps are $\phi_*(g) = (g, g)$, which are injective, whence exactness implies $K_0(A; B) \cong K_1(D)$ and $K_1(A; B) \cong K_0(D)$. In the case that $D = \mathcal{K}$, the group $K_1(A; B) \cong \mathbb{Z}$ is generated by the class of the triple (p, p, g), where p is a rank one projection in $\mathcal{K}$ and $g(s) = (e^{2\pi i s}p, p)$. Observe that we do not need to consider the unit in the unitization $\tilde{\mathcal{K}}$ to describe the group $K_1(A; B)$.

The final two examples illustrate that the boundary maps in Theorem 2.2 need not be trivial.

Example 2.7. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & C_0(\mathbb{R}^2) \arrow[r] \arrow[d, "\psi"] & C(\mathbb{D}) \arrow[r] \arrow[d, "\phi"] & C(S^1) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & C[0,1] \arrow[r, equal] & C[0,1] \arrow[r] & 0 \end{tikzcd}$$

where $C_0(\mathbb{R}^2)$ is identified with functions that vanish on the boundary of $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$. The algebra $C(S^1)$ is viewed as all f in $C[0, 1]$ with $f(0) = f(1)$, and ϕ is the composition of the restriction to the boundary $C(\mathbb{D}) \to C(S^1)$ with the inclusion $C(S^1) \hookrightarrow C[0, 1]$. We have $K_0(\phi) = K_1(\phi) = 0$ since $K_1(C[0, 1]) = K_1(C(\mathbb{D})) = 0$ (see Corollary 4.4) and the induced map $\phi_* : K_0(C(\mathbb{D})) \to K_0(C[0, 1])$ is an isomorphism. The six-term exact sequence of Theorem 2.2 becomes

$$\begin{tikzcd} K_0(C_0(\mathbb{R}^2)) \arrow[r] & 0 \arrow[r] & K_0(C(S^1); C[0,1]) \arrow[d, "\partial_0"] \\ K_1(C(S^1); C[0,1]) \arrow[u, "\partial_1"] & 0 \arrow[l] & K_1(C_0(\mathbb{R}^2)) \arrow[l] \end{tikzcd}$$

(we identify $K_*(\psi)$ with $K_*(C_0(\mathbb{R}^2))$ since $\psi = 0$). It can be shown that $K_1(C(S^1); C[0, 1]) \cong \mathbb{Z}$ is generated by the class of $(1, z, g)$ where z is the function $z \mapsto z$ on S^1 and $g(t) = f_t$, where $f_t(s) = e^{2\pi ist}$. Using the notation in Definition 4.6, let $l = 1$,

$$w = \begin{bmatrix} z & -(1-|z|^2)^{1/2} \\ (1-|z|^2)^{1/2} & \overline{z} \end{bmatrix}$$

and $h = g$. Then

$$\partial_1([1, z, g]) = \left[\begin{bmatrix} |z|^2 & z(1-|z|^2)^{1/2} \\ \overline{z}(1-|z|^2)^{1/2} & 1-|z|^2 \end{bmatrix}\right] - \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right]$$

Example 2.8. Let D be the diagonal matrices in $M_2(\mathbb{C})$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & C_0(\mathbb{R}) \arrow[r] \arrow[d] & C[0,1] \arrow[r, "\pi"] \arrow[d, "\phi"] & D \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & M_2(\mathbb{C}) \arrow[r, equal] & M_2(\mathbb{C}) \arrow[r] & 0 \end{tikzcd}$$

Where $C_0(\mathbb{R})$ is identified with all functions in $C[0, 1]$ that vanish at the endpoints, $\pi(f) = \begin{bmatrix} f(0) & 0 \\ 0 & f(1) \end{bmatrix}$, and ϕ is the composition of π with the diagonal inclusion. We have $K_0(C_0(\mathbb{R})) = 0$ and $K_1(D; M_2(\mathbb{C})) = 0$, the latter by Example 2.5. The six-term exact sequence of Theorem 2.2 becomes

$$\begin{tikzcd} 0 \arrow[r] & K_0(\phi) \arrow[r] & K_0(D; M_2(\mathbb{C})) \arrow[d, "\partial_0"] \\ 0 \arrow[u] & K_1(\phi) \arrow[l] & K_1(C_0(\mathbb{R})) \arrow[l] \end{tikzcd}$$

Using Theorem 2.1, it can be shown that $K_0(\phi) = 0$ and $K_1(\phi) \cong \mathbb{Z}/2\mathbb{Z}$, with the nontrivial element in $K_1(\phi)$ given by the class of the triple $(1, 1, g)$, where

$$g(s) = \begin{bmatrix} e^{2\pi i s} & 0 \\ 0 & 1 \end{bmatrix}$$

The map ∂_0 is therefore injective and takes a generator of $K_0(D; M_2(\mathbb{C})) \cong \mathbb{Z}$ to twice a generator of $K_1(C_0(\mathbb{R})) \cong \mathbb{Z}$. More concretely,

$$\partial_0 \left(\left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right] \right) = -[z^2]$$

where z is the function $z \mapsto z$ on $C(S^1) \cong \widetilde{C_0(\mathbb{R})}$.

3. Definitions and a portrait of $K_*(\phi)$

We begin by establishing some notation and terminology. If A is a C*-algebra, we let $\tilde{A}$ denote its unitization. If a is in $\tilde{A}$, let $\dot{a}$ denote the scalar part of a. Let $M_n(A)$ denote the $n \times n$ matrices with entries in A, regarded as a C*-algebra in the usual way. Let $M_\infty(\tilde{A})$ be the union $\bigcup_{n=1}^\infty M_n(\tilde{A})$, which may be regarded as an increasing union by means of the inclusions $M_n(\tilde{A}) \subseteq M_{n+1}(\tilde{A})$, $a \mapsto \operatorname{diag}(a, 0)$. If a and b are in $M_\infty(\tilde{A})$ we define

$$a \oplus b = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Admittedly, there is some ambiguity in the above definition of $a \oplus b$ since a and b may be regarded as matrices of arbitrarily large size. However, since K-theory doesn't distinguish elements that are "moved down the diagonal", there will be negligible harm done by ignoring this technical issue. We denote by 1_n the identity matrix in $M_n(\tilde{A})$, or the matrix in $M_\infty(\tilde{A})$ with n consecutive occurences of 1 down the diagonal, and 0 elsewhere. Two projections p and q in $M_\infty(\tilde{A})$ are called Murray-von Neumann equivalent if there is a partial isometry v in $M_\infty(\tilde{A})$ such that $v^*v = p$ and $vv^* = q$. The group $K_0(\tilde{A})$ is the Grothendieck completion of the semigroup of Murray-von Neumann classes of projections in $M_\infty(\tilde{A})$ with the operation $[p] + [q] = [p \oplus q]$. The group $K_0(A)$ is the kernel of the map $K_0(\tilde{A}) \to \mathbb{Z}$ induced by the scalar map $\tilde{A} \to \mathbb{C}$. The group $K_1(A)$ is the group of stable homotopy classes of unitaries over $\tilde{A}$ with the operation $[u] + [v] = [uv]$ (regard u and v as elements of the same matrix algebra so that the product is well-defined). Every element of $K_0(A)$ may be represented by a formal difference $[p] - [q]$ of classes such that $\dot{p} = 1_n$ and $q = 1_n$ for some n (6.2.7 of [9]). Every element of $K_1(A)$ may be represented by a class $[u]$ such that u is in $M_n(\tilde{A})$ and $\dot{u} = 1_n$.

We denote the compact operators on a separable Hilbert space by $\mathcal{K}$. We write SA for $C_0(0, 1) \otimes A = C_0((0, 1), A)$, the suspension of A, and CA for $C_0(0, 1] \otimes A = C_0((0, 1], A)$, the cone of A. If $\phi : A \to B$ is a *-homomorphism, we commit the usual notation abuse and denote the obvious induced maps $\tilde{A} \to \tilde{B}$, $SA \to SB$, $CA \to CB$, $M_n(A) \to M_n(B)$ (or any combination of these) by ϕ. Clarity will sometimes be needed for the first three, in which case they will be denoted by $\tilde{\phi}$, $S\phi$, and $C\phi$ respectively. The mapping cone of ϕ is defined to be the pullback of ϕ and the map $\pi_B : CB \to B$, $\pi_B(f) = f(1)$. That is,

$$C_\phi = \{(a, f) \mid f(1) = \phi(a)\} \subseteq A \oplus CB$$

We denote the induced maps $K_j(A) \to K_j(B)$ by ϕ_* for both $j = 0, 1$. We denote the natural isomorphism $K_1(A) \to K_0(SA)$ by θ_A and the Bott map $K_0(A) \to K_1(SA)$ by β_A. If

$$0 \longrightarrow I \hookrightarrow A \xrightarrow{\pi} A/I \longrightarrow 0$$

is a short exact sequence of C*-algebras, we denote the index maps $K_n(A/I) \to K_{n-1}(I)$ by δ_n for $n \geq 1$ and the exponential map $K_0(A/I) \to K_1(I)$ by δ_0. We refer the reader to [1], [8], or [9] for more details on K-theory for C*-algebras.

We now establish terminology introduced by Karoubi. We will assume that A and B are C*-algebras throughout, but the initial construction of the relative groups $K_j(\phi)$ may be done more generally for Banach algebras. Given idempotents e and f in $M_\infty(\tilde{A})$ (that is, $e^2 = e$ and $f^2 = f$), we call an element a of $M_\infty(\tilde{A})$ such that $a = fae$ a morphism from e to f. In this way, the idempotents in $M_\infty(\tilde{A})$ are the objects of an additive category $\mathcal{C}_A$ with morphisms just described. Composition of morphisms is given by multiplication, and the identity morphism from e to e is simply e. Clearly a morphism is invertible if there is a morphism b from f to e such that $ba = e$ and $ab = f$, which is the usual notion of algebraic equivalence of idempotents. At the risk of possible confusion with invertible elements of a C*-algebra, we will denote the inverse of the morphism a by a^{-1}.

If e is in $M_m(\tilde{A})$ and f is in $M_n(\tilde{A})$, then a morphism from e to f may be regarded as an element of $fM_{n,m}(\tilde{A})e$. This has an obvious structure of a Banach space whose linear and norm structure is compatible with morphism composition, making $\mathcal{C}_A$ a Banach category in the sense of Karoubi, see II.2.1 of [5].

If $\phi : A \to B$ is a *-homomorphism, then it preserves addition and multiplication, hence gives rise to an additive functor from $\mathcal{C}_A$ to $\mathcal{C}_B$. It is linear and continuous with respect to the Banach space structure on the collections of morphisms, hence is a Banach functor in the sense of Karoubi, see II.2.6 of [5]. It is quasi-surjective: given an idempotent f in $M_\infty(\tilde{B})$, we can find idempotents e in $M_\infty(\tilde{A})$ and g in $M_\infty(\tilde{B})$ such that $\phi(e)$ is isomorphic to $f \oplus g$. By choosing n large enough so that f is in $M_n(\tilde{B})$, the elements $e = 1_n$ and $g = 1_n - f$ suffice, by the equivalence of 1_n and $f \oplus (1_n - f)$.

We now define the group $K_0(\phi)$ and discuss some of its properties. Since ϕ may be regarded as a quasi-surjective Banach functor, the setup in II.2.13 of [5] applies to the situation at hand. As such, the definition of $K_0(\phi)$ may be found there, but we record the construction here in our own context. Denote by $\Gamma_0(\phi)$ the set of all triples (e, f, b) where e and f are idempotents in $M_\infty(\tilde{A})$ and b is an invertible morphism from $\phi(e)$ to $\phi(f)$. For brevity, we will often denote these triples by the symbols σ and τ. Define a direct sum operation on $\Gamma_0(\phi)$ by

$$(e, f, b) \oplus (e', f', b') = (e \oplus e', f \oplus f', b \oplus b').$$

We say that two such triples (e, f, b) and (e', f', b') are isomorphic, written $(e, f, b) \cong (e', f', b')$, if there exist invertible morphisms c and d from e to e' and from f to f', respectively, that intertwine b and b', that is, $\phi(d)b = b'\phi(c)$. A triple (e, f, b) is called elementary if $e = f$ and there is a continuous path b_t for $0 \le t \le 1$ such that $b_0 = \phi(e)$ and $b_1 = b$, and b_t is an invertible morphism from $\phi(e)$ to $\phi(e)$ for all t. Say that two triples σ and σ' in $\Gamma_0(\phi)$ are equivalent, written $\sigma \sim \sigma'$, if there exist elementary triples τ and τ' such that $\sigma \oplus \tau \cong \sigma \oplus \tau'$. Denote by $[\sigma]$, or $[e, f, b]$, the equivalence class of the triple $\sigma = (e, f, b)$ via the relation $\sim$.

Definition 3.1. *$K_0(\phi)$ is defined to be the quotient of $\Gamma_0(\phi)$ by the relation $\sim$, that is,*

$$\{[\sigma] \mid \sigma \in \Gamma_0(\phi)\} = \Gamma_0(\phi)/\sim$$

We make some simple observations. First, the notions of isomorphism and elementary for triples behave well with respect to the direct sum operation: if $\sigma_1 \cong \sigma_2$ and $\sigma_3 \cong \sigma_4$, then $\sigma_1 \oplus \sigma_3 \cong \sigma_2 \oplus \sigma_4$, and for any two triples σ and σ', we have $\sigma \oplus \sigma' \cong \sigma' \oplus \sigma$. Moreover, if σ and σ' are elementary, then so is $\sigma \oplus \sigma'$. Second, all elementary triples are equivalent to each other, and two isomorphic triples are equivalent. Third, if (e, f, b) is any triple in $\Gamma_0(\phi)$, then the triple

$$\left(\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \begin{bmatrix} 0 & b^{-1} \\ -b & 0 \end{bmatrix} \right)$$

is elementary because of the homotopy

$$b_t = \begin{bmatrix} \phi(e) & tb^{-1} \\ 0 & \phi(f) \end{bmatrix} \begin{bmatrix} \phi(e) & 0 \\ -tb & \phi(f) \end{bmatrix} \begin{bmatrix} \phi(e) & tb^{-1} \\ 0 & \phi(f) \end{bmatrix}$$

Proposition 3.2. *$K_0(\phi)$ is an abelian group when equipped with the binary operation*

$$[\sigma] + [\sigma'] = [\sigma \oplus \sigma']$$

where the identity element is given by $[0, 0, 0]$ and the inverse of $[e, f, b]$ is given by $[f, e, b^{-1}]$.

Proof. That $K_0(\phi)$ is an abelian group follows quite readily from the observations above. That $(e, f, b) \oplus (0, 0, 0) \cong (e, f, b)$ follows from the equation

$$\begin{bmatrix} \phi(f) & 0 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} = b \begin{bmatrix} \phi(e) & 0 \end{bmatrix}$$

and hence that $[0, 0, 0]$ is the identity element of the group. To prove the last statement, note that

$$[e, f, b] + [f, e, b^{-1}] = [e \oplus f, f \oplus e, b \oplus b^{-1}]$$

and the triple $(e \oplus f, f \oplus e, b \oplus b^{-1})$ is isomorphic to the triple

$$\left(\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \begin{bmatrix} 0 & -b^{-1} \\ b & 0 \end{bmatrix} \right)$$

since

$$\begin{bmatrix} 0 & -\phi(f) \\ \phi(e) & 0 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} = \begin{bmatrix} 0 & -b^{-1} \\ b & 0 \end{bmatrix} \begin{bmatrix} \phi(e) & 0 \\ 0 & \phi(f) \end{bmatrix}$$

As previously noted, the latter triple is elementary. □

We collect some useful properties of the elements of $K_0(\phi)$.

Proposition 3.3. *We have the following.*

(i) Given two invertible morphisms b and b' from $\phi(e)$ to $\phi(f)$, if b and b' are homotopic throughout the invertible morphisms from $\phi(e)$ to $\phi(f)$, then $[e,f,b] = [e,f,b']$.

(ii) For two triples (e,f,b) and (e',f',b') in $\Gamma_0(\phi)$, if $f = e'$ then we have

$$[e,f,b] + [e',f',b'] = [e,f',b'b].$$

(iii) Let (e,f,b) and (e',f',b') be two triples in $\Gamma_0(\phi)$. If $ee' = e'e = 0$, then

$$(e,f,b) \oplus (e',f',b') \cong \left(e+e', f \oplus f', \begin{bmatrix} b & 0 \\ b' & 0 \end{bmatrix}\right)$$

If $ff' = f'f = 0$, then

$$(e,f,b) \oplus (e',f',b') \cong \left(e \oplus e', f + f', \begin{bmatrix} b & b' \\ 0 & 0 \end{bmatrix}\right)$$

If $ee' = e'e = ff' = f'f = 0$, then

$$(e,f,b) \oplus (e',f',b') \cong (e+e', f+f', b+b')$$

(iv) $[e,f,b] = 0$ if and only if there exist idempotents g and h in $M_\infty(\tilde{A})$ and invertible morphisms x and y in $M_\infty(\tilde{A})$ from $e \oplus g$ to h and $f \oplus g$ to h, respectively, such that $\phi(y)(b \oplus \phi(g))\phi(x^{-1})$ is homotopic to $\phi(h)$ through the invertible morphisms of $\phi(h)$.

Proof. (i) This is essentially II.2.15 of [5]; we repeat the proof here. We compute

$$[e,f,b] - [e,f,b'] = [e \oplus f, f \oplus e, b \oplus b'^{-1}]$$

and the triple $(e \oplus f, f \oplus e, b \oplus b'^{-1})$ is isomorphic to

$$\left(\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \begin{bmatrix} 0 & -b'^{-1} \\ b & 0 \end{bmatrix}\right)$$

similarly as in the proof of Proposition 3.2. Since b is homotopic to b', the two matrices

$$\begin{bmatrix} 0 & -b'^{-1} \\ b & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -b^{-1} \\ b & 0 \end{bmatrix}$$

are homotopic. It follows that the latter triple is elementary.

(ii) This is essentially II.2.16 of [5]; we repeat the proof here. We compute

$$[e,f,b] + [f,f',b'] = [e \oplus f, f \oplus f', b \oplus b']$$

and observe that the triple $(e \oplus f, f \oplus f', b \oplus b')$ is isomorphic to the triple

$$\left(\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \begin{bmatrix} f' & 0 \\ 0 & f \end{bmatrix}, \begin{bmatrix} 0 & -b' \\ b & 0 \end{bmatrix}\right).$$

since

$$\begin{bmatrix} 0 & -\phi(f') \\ \phi(f) & 0 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b' \end{bmatrix} = \begin{bmatrix} 0 & -b' \\ b & 0 \end{bmatrix} \begin{bmatrix} \phi(e) & 0 \\ 0 & \phi(f) \end{bmatrix}$$

We also have that $[e, f', b'b] = [e \oplus f, f' \oplus f, b'b \oplus \phi(f)]$ since $(f, f, \phi(f))$ is elementary. Now

$$\begin{bmatrix} 0 & -b' \\ b & 0 \end{bmatrix} \begin{bmatrix} b^{-1}b'^{-1} & 0 \\ 0 & \phi(f) \end{bmatrix} = \begin{bmatrix} 0 & -b' \\ b'^{-1} & 0 \end{bmatrix}$$

and the matrix on the right is homotopic to $\phi(f') \oplus \phi(f)$. It follows that the two matrices

$$\begin{bmatrix} 0 & -b' \\ b & 0 \end{bmatrix} \qquad \begin{bmatrix} b'b & 0 \\ 0 & \phi(f) \end{bmatrix}$$

are homotopic, and hence, by part (i), the triples are equivalent.

(iii) If $ee' = e'e = 0$, then $b\phi(e') = b\phi(e)\phi(e') = 0$, and similarly $b'\phi(e) = 0$. Thus we have

$$\begin{bmatrix} \phi(f) & 0 \\ 0 & \phi(f') \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b' \end{bmatrix} = \begin{bmatrix} b & 0 \\ b' & 0 \end{bmatrix} \begin{bmatrix} \phi(e) & \phi(e') \\ 0 & 0 \end{bmatrix}$$

so the triples are isomorphic. The other two claims are similar.

(iv) It is a direct consequence of the definitions that $[e, f, b] = 0$ if and only if there are elementary triples (g, g, c) and (h, h, d) such that

$$(e, f, b) \oplus (g, g, c) \cong (h, h, d)$$

This is true if and only if there are invertible morphisms x and y in $M_\infty(\tilde{A})$ such $d\phi(x) = \phi(y)(b \oplus c)$. Then $d = \phi(y)(b \oplus c)\phi(x^{-1})$, and since d is homotopic to $\phi(h)$ and c is homotopic to $\phi(g)$, we have the conclusion.

□

In II.3.3 of [5], Karoubi introduces a definition of the K_1-group, there denoted $K^{-1}(\mathcal{C})$ for a Banach category $\mathcal{C}$, that gives an equivalent but slightly more general description. We provide the definition in order to motivate the definition of the relative K_1-group. Consider the set $\Gamma_1(A)$ of all pairs (e, a) such that e is an idempotent in $M_\infty(\tilde{A})$ and a is an element of $M_\infty(\tilde{A})$ that is an invertible morphism from e to e (equivalently, a is an invertible element of the Banach algebra $eM_\infty(\tilde{A})e$). Define the direct sum $(e, a) \oplus (e', a') = (e \oplus e', a \oplus a')$, as usual. Say that two pairs (e, a) and (e', a') are isomorphic, written $(e, a) \cong (e', a')$, if there is an invertible morphism b from e to e' such that $ba = a'b$. We say a pair (e, a) is elementary if there is a path from a to e through the invertibles in $eM_\infty(\tilde{A})e$. We say that two pairs σ and σ' in $\Gamma_1(A)$ are equivalent, written $\sigma \sim \sigma'$, if there exist elementary pairs τ and τ' such that $\sigma \oplus \tau \cong \sigma \oplus \tau'$. Denote by $[\sigma]$, or $[e, a]$, the equivalence class of the pair $\sigma = (e, a)$ via the relation $\sim$. $K^{-1}(\mathcal{C}_A)$ is defined to be the quotient of $\Gamma_1(A)$ by the relation $\sim$. It is an abelian group with $[0, 0] = 0$ and $-[e, a] = [e, a^{-1}]$.

The proof of the following result uses similar, but simpler, techniques to those in Proposition 3.3. For this reason, we omit the proof.

Proposition 3.4. *The map $\Omega_A : K_1(A) \to K^{-1}(\mathcal{C}_A)$ defined by $\Omega_A([u]) = [1_n, u]$ (for a unitary or invertible u in $M_n(\tilde{A})$) is a natural isomorphism.*

Now we construct $K_1(\phi)$. Consider the set $\Gamma_1(\phi)$ of all triples (e, a, g) such that e is an idempotent in $M_\infty(\tilde{A})$, a is an element of $M_\infty(\tilde{A})$ that is an invertible morphism from e to e, and g is an element of

$$C[0,1] \otimes \phi(e) M_\infty(\tilde{B}) \phi(e)$$

such that $g(0) = \phi(e)$, $g(1) = \phi(a)$, and $g(s)$ is an invertible morphism from $\phi(e)$ to $\phi(e)$ for $0 \leq s \leq 1$. Define

$$(e, a, g) \oplus (e', a', g') = (e \oplus e', a \oplus a', g \oplus g').$$

We say that two triples (e, a, g) and (e', a', g') are isomorphic if there is an invertible morphism b in $M_\infty(\tilde{A})$ from e to e' such that $ba = a'b$ and $\phi(b)g(s) = g'(s)\phi(b)$ for $0 \leq s \leq 1$. A triple (e, a, g) is called elementary if there are continuous paths (a_t) and (g_t) such that $a_1 = a$, $g_1 = g$, $a_0 = e$, $g_0(s) = \phi(e)$ for $0 \leq s \leq 1$, and (e, a_t, g_t) is in $\Gamma_1(\phi)$ for $0 \leq t \leq 1$. Say that two triples σ and σ' in $\Gamma_1(\phi)$ are equivalent, written $\sigma \sim \sigma'$, if there exist elementary triples τ and τ' such that $\sigma \oplus \tau \cong \sigma \oplus \tau'$. Denote by $[\sigma]$, or $[e, a, g]$, the equivalence class of the triple $\sigma = (e, a, g)$ via the relation $\sim$.

Definition 3.5. *$K_1(\phi)$ is defined to be the set*

$$\{[\sigma] \mid \sigma \in \Gamma_1(\phi)\} = \Gamma_1(\phi)/\sim$$

It is easily checked that, like $\Gamma_0(\phi)$, the direct sum operation of triples in $\Gamma_1(\phi)$ behaves well with respect to the notions of isomorphism and elementary.

Proposition 3.6. *$K_1(\phi)$ is an abelian group when equipped with the binary operation*

$$[\sigma] + [\sigma] = [\sigma \oplus \sigma']$$

where the identity element is given by $[0, 0, 0]$ and the inverse of $[e, a, g]$ is given by $[e, a^{-1}, g^{-1}]$.

Proof. We verify the last claim. We have

$$[e, a, g] + [e, a^{-1}, g^{-1}] = [e \oplus e, a \oplus a^{-1}, g \oplus g^{-1}].$$

A concrete path a_t of invertibles from $a \oplus a^{-1}$ to $e \oplus e$ is

$$a_t = \begin{bmatrix} e & -ta \\ 0 & e \end{bmatrix} \begin{bmatrix} e & 0 \\ ta^{-1} & e \end{bmatrix} \begin{bmatrix} e & -ta \\ 0 & e \end{bmatrix} \begin{bmatrix} e & -te \\ 0 & e \end{bmatrix} \begin{bmatrix} e & 0 \\ te & e \end{bmatrix} \begin{bmatrix} e & -te \\ 0 & e \end{bmatrix}$$

For every s and t, let $g_t(s)$ be

$$\begin{bmatrix} \phi(e) & -tg(s) \\ 0 & \phi(e) \end{bmatrix} \begin{bmatrix} \phi(e) & 0 \\ tg(s)^{-1} & \phi(e) \end{bmatrix} \begin{bmatrix} \phi(e) & -tg(s) \\ 0 & \phi(e) \end{bmatrix} \begin{bmatrix} \phi(e) & -t\phi(e) \\ 0 & \phi(e) \end{bmatrix} \begin{bmatrix} \phi(e) & 0 \\ t\phi(e) & \phi(e) \end{bmatrix} \begin{bmatrix} \phi(e) & -t\phi(e) \\ 0 & \phi(e) \end{bmatrix}$$

Then (g_t) is a path with $g_t(1) = \phi(a_t)$ for all t, $g_1 = g \oplus g^{-1}$ and $g_0(s) = \phi(e) \oplus \phi(e)$ for all s. Thus $(e \oplus e, a \oplus a^{-1}, g \oplus g^{-1})$ is elementary. ☐

The following result is similar to Proposition 3.3, so we omit the proof.

Proposition 3.7. *We have the following.*

(i) *Suppose we have two triples (e, a, g) and (e', a', g') and that $e = e'$. If e is in $M_n(\tilde{A})$ and (a_t) is a path of invertibles from a to a' in $eM_n(\tilde{A})e$ and (g_t) is a path of invertibles from g to g' in $C[0,1] \otimes \phi(e) M_n(\tilde{B}) \phi(e)$ such that $g_t(1) = \phi(a_t)$ for all t, then $[e, a, g] = [e, a', g']$.*

(ii) *If $e = e'$, we have*

$$[e, a, g] + [e', a', g'] = [e, aa', gg'] = [e, a'a, g'g].$$

(iii) *If (e, a, g) and (e', a', g') are two triples in $\Gamma_1(\phi)$ such that $ee' = e'e = 0$, then $(e, a, g) \oplus (e', a', g') \cong (e + e', a + a', g + g')$.*

(iv) *If e is in $M_n(\tilde{A})$, $[e, a, g] = 0$ if and only if there is an integer $k \geq 1$, an idempotent e' in $M_k(\tilde{A})$ and paths of invertibles a_t in $(e \oplus e')M_{n+k}(\tilde{A})(e \oplus e')$ and g_t in $C[0, 1] \otimes (\phi(e) \oplus \phi(e'))M_{n+k}(\tilde{B})(\phi(e) \oplus \phi(e'))$ such that $a_0 = e \oplus e'$, $a_1 = a \oplus e'$, $g_0 = e \oplus \phi(e')$, $g_1 = g \oplus \phi(e')$, and $g_t(1) = \phi(a_t)$ for all t.*

We now collect some properties that hold for both relative groups.

Proposition 3.8. *Suppose that G is an abelian group and $\nu : \Gamma_j(\phi) \to G$ is a map that satisfies*

(i) *$\nu(\sigma \oplus \tau) = \nu(\sigma) + \nu(\tau)$,*
(ii) *$\nu(\sigma) = 0$ if σ is elementary, and*
(iii) *if $\sigma \cong \tau$, then $\nu(\sigma) = \nu(\tau)$.*

Then ν factors to a unique group homomorphism $\alpha : K_j(\phi) \to G$.

Proof. If $\sigma \sim \sigma'$, find elementary triples τ and τ' such that $\sigma \oplus \tau \cong \sigma' \oplus \tau'$. Then

$$\nu(\sigma) = \nu(\sigma) + \nu(\tau) = \nu(\sigma \oplus \tau) = \nu(\sigma' \oplus \tau') = \nu(\sigma') + \nu(\tau') = \nu(\sigma')$$

So the map $\alpha([\sigma]) := \nu(\sigma)$ is well-defined. It is a group homomorphism by property (i). □

If $\phi : A \to B$ and $\psi : C \to D$ are *-homomorphisms, we denote by $\phi \oplus \psi$ the component-wise *-homomorphism $A \oplus C \to B \oplus D$.

Proposition 3.9. *Suppose $\phi : A \to B$ and $\psi : C \to D$ are *-homomorphisms. Then the map $K_0(\phi \oplus \psi) \to K_0(\phi) \oplus K_0(\psi)$ defined by $[(e, e'), (f, f'), (b, b')] \mapsto ([e, f, b], [e', f', b'])$ is a group isomorphism. An analogous statement holds for K_1.*

Proof. For a triple $((e, e'), (f, f'), (b, b'))$ in $\Gamma_0(\phi \oplus \psi)$, define

$$\nu((e, e'), (f, f'), (b, b')) = ([e, f, b], [e', f', b']).$$

It is straightforward to check that ν satisfies the hypotheses of Proposition 3.8, so we get a well-defined group homomorphism that factors ν. The fact that the group homomorphism is surjective is clear, and injectivity follows from a simple application of part (iv) of Proposition 3.3. The proof is similar for K_1. □

Proposition 3.10. *Suppose that*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \xrightarrow{\psi} & D \end{array}$$

is a commutative diagram of C-algebras. Then there are well-defined group homomorphisms $\alpha_* : K_j(\phi) \to K_j(\psi)$ that satisfy $\alpha_*([e, f, b]) = [\alpha(e), \alpha(f), \beta(b)]$ for K_0 and $\alpha_*([e, a, g]) = [\alpha(e), \alpha(a), \beta(g)]$ for K_1. If α and β are *-isomorphisms, then α_* is a group isomorphism.*

Proof. For a triple (e, f, b) in $\Gamma_0(\phi)$, set $\nu(e, f, b) = [\alpha(e), \alpha(f), \beta(b)]$. Again, the hypotheses of Proposition 3.8 are easy to check, so ν factors to a group homomorphism α_*. If α and β are *-isomorphisms, then the diagram

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[d, "\alpha^{-1}"'] & D \arrow[d, "\beta^{-1}"] \\ A \arrow[r, "\phi"] & B \end{tikzcd}$$

is commutative and the same argument works to obtain the group homomorphism $(\alpha^{-1})_*$, which is easily seen to be the inverse of α_*. The proof is again similar for K_1. □

As an application of the above results, we will show that if A and B are unital and $\phi(1) = 1$, one may define $K_0(\phi)$ without unitizations while remaining consistent with the results above. To verify this, let $K_0^u(\phi)$ be the group defined in the same way as $K_0(\phi)$, but avoid unitizing A and B and use the units already present. Notice $K_0(\phi)$ and $K_0^u(\tilde{\phi})$ are precisely the same objects, and all preceding results about $K_0(\phi)$ remain true for $K_0^u(\phi)$ with appropriate modifications.

Proposition 3.11. *If A and B are unital and $\phi(1) = 1$, then $K_j(\phi)$ and $K_j^u(\phi)$ are isomorphic as groups.*

Proof. The map $\nu_A : A \oplus \mathbb{C} \to \tilde{A}$ defined by $\nu_A(a, \lambda) = a + \lambda(1_{\tilde{A}} - 1_A)$ is a *-isomorphism and the diagram

$$\begin{tikzcd} A \oplus \mathbb{C} \arrow[r, "\phi\oplus\mathrm{id}_{\mathbb{C}}"] \arrow[d, "\nu_A"] & B \oplus \mathbb{C} \arrow[d, "\nu_B"] \\ \tilde{A} \arrow[r, "\tilde{\phi}"] & \tilde{B} \end{tikzcd}$$

is commutative. Therefore $K_j(\phi) = K_j^u(\tilde{\phi})$ is isomorphic to $K_j^u(\phi \oplus \mathrm{id}_{\mathbb{C}})$ by Proposition 3.10. Then

$$K_j(\phi) = K_j^u(\tilde{\phi}) \cong K_j^u(\phi \oplus \mathrm{id}_{\mathbb{C}}) \cong K_j^u(\phi) \oplus K_j^u(\mathrm{id}_{\mathbb{C}}) \cong K_j^u(\phi)$$

where the third isomorphism is due to Proposition 3.9. The fact that $K_j^u(\mathrm{id}_{\mathbb{C}}) = 0$ is rather clear, but the skeptical reader is referred to part (ii) of Corollary 4.4. □

We now focus on simplifying the portrait of $K_*(\phi)$ using the underlying C*-algebra structure. When one is working with C*-algebra K-theory, it is convenient to work with projections, unitaries, and partial isometries instead of idempotents and invertible morphisms. To this end, we introduce a more specific picture using a refined definition of triples. Define $\Gamma_0^*(\phi)$ to be the subset of $\Gamma_0(\phi)$ consisting of all triples (p, q, v), where p and q are projections in $M_\infty(\tilde{A})$ and v is a partial isometry in $M_\infty(\tilde{B})$ with $v^*v = \phi(p)$, $vv^* = \phi(q)$. We define a refined equivalence relation $\sim_*$ on $\Gamma_0^*(\phi)$ as follows. Say that (p, q, v) and (p', q', v') are $*$-isomorphic, written $(p, q, v) \cong_* (p', q', v')$, if there are partial isometries c and d in $M_\infty(\tilde{A})$ with $c^*c = p$, $cc^* = p'$, $d^*d = q$, $dd^* = q'$, and $\phi(d)v = v'\phi(c)$. A triple (p, q, v) is called $*$-elementary if $p = q$ and there is a continuous path (v_t) of partial isometries such that $v_0 = \phi(p)$ and $v_1 = v$, and $v_t^* v_t = v_t v_t^* = \phi(p)$ for all t. Say that two triples σ and σ' in $\Gamma_0^*(\phi)$ are $*$-equivalent, written $\sigma \sim_* \sigma'$, if there exist $*$-elementary triples τ and τ' such that $\sigma \oplus \tau \cong_* \sigma \oplus \tau'$. Denote by $[\sigma]_*$, or $[p, q, v]_*$, the equivalence class of the triple $\sigma = (p, q, v)$ via the relation $\sim_*$.

It is straightforward to verify that the set of equivalence classes forms an abelian group in the same way as for $K_0(\phi)$. To see that $-[p, q, v]_* = [q, p, v^*]_*$, notice that if two unitaries are homotopic as invertibles, they are homotopic as unitaries. The analogue of Proposition 3.8 also holds. We denote this new group by $K_0^*(\phi)$. The notation is not desirable, but it is temporary, as we now aim to show that $K_0^*(\phi)$ and $K_0(\phi)$ are isomorphic in a natural way.

Lemma 3.12. *We have the following.*

(i) For every triple σ in $\Gamma_0(\phi)$, there is a triple τ in $\Gamma_0^*(\phi)$ with $\sigma \sim \tau$. Moreover, such a triple $\tau = (p, q, v)$ in $\Gamma_0^*(\phi)$ may be chosen so that one of p or q is equal to 1_n for some $n \geq 1$, and $\dot{p} = \dot{q} = \dot{v} = 1_n$.

(ii) Suppose $m \geq n$, p is in $M_m(\tilde{A})$, $(p, 1_n, v)$ is in $\Gamma_0^*(\phi)$, and $\dot{p} = \dot{v} = 1_n$. Then $[p, 1_n, v] = 0$ in $K_0(\phi)$ if and only if there exist $k \geq 0$ and a partial isometry w in $M_{m+k}(\tilde{A})$ with $w^* w = \dot{w} = 1_n \oplus 0_{m-n} \oplus 1_k$ and $ww^* = p \oplus 1_k$ such that $(v \oplus 1_k)\phi(w)$ is a unitary in $(1_n \oplus 0_{m-n} \oplus 1_k) M_{m+k}(\tilde{B})(1_n \oplus 0_{m-n} \oplus 1_k)$ homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$.

Proof. (i) First, if e is an idempotent in $\tilde{A}$,

$$\rho(e) = ee^*(1 + (e - e^*)(e^* - e))^{-1}$$

is a projection, and $e\rho(e) = \rho(e)$ and $\rho(e)e = e$ (see 4.6.2 of [1]). Thus, if (e, f, b) is a triple in $\Gamma_0(\phi)$, we have $(e, f, b) \cong (\rho(e), \rho(f), \phi(f) b \phi(\rho(e)))$. Set $p = \rho(e)$, $q = \rho(f)$, and $b_1 = \phi(f) b \phi(\rho(e))$.

Next, notice that $b_1 b_1^*$ is an invertible element of the C*-algebra $\phi(q) M_\infty(\tilde{B}) \phi(q)$ with inverse $(b_1^{-1})^* b_1^{-1}$. Now $(b_1 b_1^*)^{-t/2} b_1$ for $0 \leq t \leq 1$ is a homotopy from b_1 to $(b_1 b_1^*)^{-1/2} b_1$, and each $(b_1 b_1^*)^{-t/2} b_1$ is an invertible morphism from $\phi(p)$ to $\phi(q)$ with inverse $b_1^{-1}((b_1^{-1})^* b_1^{-1})^{-t/2}$. Set $v = (b_1 b_1^*)^{-1/2} b_1$, which is a partial isometry with $v^* v = \phi(p)$ and $vv^* = \phi(q)$. By part (i) of Proposition 3.3, $(p, q, b_1) \sim (p, q, v)$.

Now choose n such that p and q are in $M_n(\tilde{A})$ and v is in $M_n(\tilde{B})$. By adding the elementary triple $(1_n - p, 1_n - p, 1_n - \phi(p))$ and using part (iii) of Proposition 3.3,

$$(p, q, v) \sim \left(\begin{bmatrix} 1_n & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} q & 0 \\ 0 & 1_n - p \end{bmatrix}, \begin{bmatrix} v & 0 \\ 1_n - \phi(p) & 0 \end{bmatrix} \right)$$

Set $q_1 = q \oplus (1_n - p)$ and $v_1 = \begin{bmatrix} v & 0 \\ 1_n - \phi(p) & 0 \end{bmatrix}$. Adding $1_n - q$ instead of $1_n - p$ would replace q with 1_n.

We have $\dot{v}_1^* \dot{v}_1 = 1_n$ and $\dot{v}_1 \dot{v}_1^* = \dot{q}_1$, so choose $m \geq n$ and a unitary u in $M_m(\mathbb{C})$ such that $u \dot{q}_1 u^* = 1_n$. Then $(1_n, q_1, v_1) \cong (1_n, u q_1 u^*, u v_1)$ since $u v_1 \phi(1_n) = \phi(u q_1) v_1$. The scalar part of $u q_1 u^*$ is 1_n and we now set $q_2 = u q_1 u^*$ and $v_2 = u v_1$. Lastly, $\dot{v}_2$ may then be regarded as a unitary in $M_n(\mathbb{C})$, so choose a homotopy v_t from $\dot{v}_2$ to 1_n and observe that each $v_2 v_t^*$ is a partial isometry from 1_n to $\phi(q_2)$.

By part (i) of Proposition 3.3, $(1_n, q_2, v_2) \sim (1_n, q_2, v_2\dot{v}_2^*)$, and the third element of the latter triple has scalar part 1_n.

(ii) For appropriate $m \geq n$, obtain elementary triples (r, r, c) and (s, s, d) such that

$$(p, 1_n \oplus 0_{m-n}, v) \oplus (r, r, c) \cong (s, s, d)$$

as in part (iv) of Proposition 3.3. As we saw in part (i), an elementary triple in $\Gamma_0(\phi)$ is isomorphic to an elementary triple consisting of projections, not just idempotents, so we may assume that r and s are projections. If $k \geq 0$ is large enough such that r is in $M_k(\tilde{A})$, then

$$(r, r, c) \oplus (1_k - r, 1_k - r, 1_k - \phi(r)) \cong (1_k, 1_k, c + 1_k - \phi(r))$$

by part (iii) of Proposition 3.3, so

$$(p, 1_n \oplus 0_{m-n}, v) \oplus (1_k, 1_k, c + 1_k - \phi(r)) \cong (s, s, d) \oplus (1_k - r, 1_k - r, 1_k - \phi(r))$$

and so we may assume that $r = 1_k$ for some $k \geq 0$. Obtain x and y as in part (iv) of Proposition 3.3, so that $d\phi(x) = \phi(y)(v \oplus c)$, hence $\phi(y^{-1})d\phi(y) = (v \oplus c)\phi(x^{-1}y)$. Since d is homotopic to $\phi(s)$, $\phi(y^{-1})d\phi(y)$ is homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$ and also c is homotopic to 1_k. Lastly, $\phi(x^{-1}y)$ is homotopic to $\phi((x^{-1}x^{-1*})^{-1/2}x^{-1}(yy^*)^{-1/2}y)$ as seen in part (i), and

$$(v \oplus 1_k)\phi((x^{-1}x^{-1*})^{-1/2}x^{-1}(yy^*)^{-1/2}y)$$

is then a unitary in $(1_n \oplus 0_{m-n} \oplus 1_k)M_{m+k}(\tilde{B})(1_n \oplus 0_{m-n} \oplus 1_k)$. It is still homotopic to the identity through invertibles, and it is standard that two unitaries homotopic through invertibles are homotopic through unitaries. Upon naming $w_1 = (x^{-1}x^{-1*})^{-1/2}x^{-1}(yy^*)^{-1/2}y$ and $w = w_1\dot{w}_1^*$, we have the conclusion.

□

The following analogous result holds in the K_1 case. Let $\Gamma_1^*(\phi)$ denote the subset of $\Gamma_1(\phi)$ consisting of triples $(1_n, u, g)$, where $n \geq 1$, u is a unitary in $M_n(\tilde{A})$, and g is a unitary in $C[0,1] \otimes M_n(\tilde{B})$ with $g(0) = 1_n$ and $g(1) = \phi(u)$.

Lemma 3.13. *We have the following.*

(i) For every triple σ in $\Gamma_1(\phi)$, there is a triple τ in $\Gamma_1^*(\phi)$ with $\sigma \sim \tau$. Moreover, such a triple $\tau = (1_n, u, g)$ in $\Gamma_1^*(\phi)$ may be chosen so that $\dot{u} = \dot{g}(s) = 1_n$ for all s.

(ii) $[1_n, u, g] = 0$ in $K_1(\phi)$ if and only if there exist $k \geq 0$, a path u_t of unitaries in $M_{n+k}(\tilde{A})$ and a path g_t of unitaries in $C[0,1] \otimes M_{n+k}(\tilde{B})$ such that $u_0 = 1_{n+k}$, $u_1 = u \oplus 1_k$, $g_0 = 1_{n+k}$, $g_1 = g \oplus 1_k$, and $g_t(1) = \phi(u_t)$ for all t.

Proof. The techniques in the proof are similar to those seen before, so we merely sketch the proof.

(i) Take a triple (e, a, g), add the elementary triple $(1_n - e, 1_n - e, 1_n - \phi(e))$, and use part (iii) of Proposition 3.7 to obtain 1_n in place of e. Use the paths $(aa^*)^{-t/2}a$ and $(gg^*)^{-t/2}g$ and part (i) of Proposition 3.7 to obtain unitaries in place of invertibles. Replace u with $\dot{u}^*u$ and g with $\dot{g}^*g$ and use the fact that $\dot{u}$ and $\dot{g}$ are homotopic to the identity to obtain the last claim.

(ii) If $[1_n, u, g] = 0$, use part (iv) of Proposition 3.7 to find an integer $k \geq 0$, an idempotent e' in $M_k(\tilde{A})$ and paths of invertibles a_t in $(1_n \oplus e')M_{n+k}(\tilde{A})(1_n \oplus e')$ and g_t in $C[0,1] \otimes (1_n \oplus \phi(e'))M_{n+k}(\tilde{B})(1_n \oplus \phi(e'))$ such that $a_0 = 1_n \oplus e'$, $a_1 = u \oplus e'$, $g_0 = 1_n \oplus \phi(e')$, $g_1 = g \oplus \phi(e')$, and $g_t(1) = \phi(a_t)$ for all t. This path takes place in $(1_n \oplus \phi(e'))M_{n+k}(\tilde{B})(1_n \oplus \phi(e'))$, so by adding $0_n \oplus (1_k - \phi(e'))$ to everything, we obtain paths of invertibles in M_{n+k}, and using the usual polar decomposition trick, we obtain paths of unitaries.

□

The previous results combine to give the following proposition.

Proposition 3.14. *The inclusion $\Gamma_j^*(\phi) \hookrightarrow \Gamma_j(\phi)$ induces a group isomorphism $\Omega_\phi : K_j^*(\phi) \to K_j(\phi)$. Moreover, given the commutative diagram (1), we have the commutative diagram*

$$\begin{tikzcd} K_j^*(\phi) \arrow[r, "\alpha_*"] \arrow[d, "\Omega_\phi"] & K_j^*(\psi) \arrow[d, "\Omega_\psi"] \\ K_j(\phi) \arrow[r, "\alpha_*"] & K_j(\psi) \end{tikzcd}$$

Thus the groups $K_j^*(\phi)$ and $K_j(\phi)$ are, for all intents and purposes, identical when A and B are C*-algebras, and we will identify the two hereon, denoting them by $K_j(\phi)$.

4. Proofs

4.1. **Proof of Theorem 2.1.** The exaxt sequence of Proposition 4.1 below is essentially that of Theorem 3.22 in [5]. We provide a proof using our refined portrait.

Define the map $\mu_0 : K_1(B) \to K_0(\phi)$ by $\mu_0([u]) = [1_n, 1_n, u]$, where u is a unitary in $M_n(\tilde{B})$. By part (i) of Proposition 3.3, μ_0 is well-defined, and clearly it is a group homomorphism.

Define a map $\nu : \Gamma_0(\phi) \to K_0(A)$ by $\nu(p, q, v) = [p] - [q]$. Observe that the image of ν is indeed in $K_0(A)$ (not just $K_0(\tilde{A})$) since $\dot{v}^*\dot{v} = \dot{p}$ and $\dot{v}\dot{v}^* = \dot{q}$, hence $[\dot{p}] = [\dot{q}]$. It is easy to check that ν satisfies the hypotheses of Proposition 3.8, hence factors to a well-defined group homomorphism $\nu_0 : K_0(\phi) \to K_0(A)$.

Proposition 4.1. *The sequence*

$$K_1(A) \xrightarrow{\phi_*} K_1(B) \xrightarrow{\mu_0} K_0(\phi) \xrightarrow{\nu_0} K_0(A) \xrightarrow{\phi_*} K_0(B)$$

is exact.

Proof. It is quite clear that all compositions are zero. If $\phi_*([p]-[q]) = [\phi(p)]-[\phi(q)] = 0$, choose $k \geq 0$ and w in $M_\infty(\tilde{B})$ such that $w^*w = \phi(p) \oplus 1_k$ and $ww^* = \phi(q) \oplus 1_k$. Then

$$[p] - [q] = \nu_0([p \oplus 1_k, q \oplus 1_k, w])$$

If $(p, 1_n, v)$ is such that $\nu_0([p, 1_n, v]) = [p] - [1_n] = 0$, choose $k \geq 0$ and w in $M_\infty(\tilde{A})$ such that $w^*w = p \oplus 1_k$ and $ww^* = 1_n \oplus 0_{m-n} \oplus 1_k$. Then

$$(p \oplus 1_k, 1_n \oplus 0_{m-n} \oplus 1_k, v \oplus 1_k) \cong (1_n \oplus 0_{m-n} \oplus 1_k, 1_n \oplus 0_{m-n} \oplus 1_k, (v \oplus 1_k)\phi(w^*))$$

and hence

$$[p, 1_n, v] = \mu_0([(v \oplus 1_k)\phi(w^*) + 0_n \oplus 1_{m-n} \oplus 0_k])$$

Finally, if $\mu_0([u]) = [1_n, 1_n, u] = 0$, use part (ii) of Lemma 3.12 to find $k \geq 0$ and a partial isometry w such that $\phi(w)(u \oplus 1_k)$ is a unitary and homotopic to 1_{n+k} in $M_{n+k}(\tilde{B})$. Since $u \oplus 1_k$ is a unitary, so is w and $u \oplus 1_k$ is homotopic to $\phi(w^*)$. Thus

$$[u] = [u \oplus 1_k] = [\phi(w^*)] = \phi_*([w^*])$$

which completes the proof. □

For a unitary g in $C[0,1] \otimes M_n(\tilde{B})$ with $g(0) = g(1) = \dot{g} = 1_n$, set $\mu_1([g]) = [1_n, 1_n, g]$. By part (i) of Proposition 3.7, this is a well-defined group homomorphism $\mu_1 : K_1(SB) \to K_1(\phi)$. For a triple (e, a, g) in $\Gamma_1(\phi)$, define $\nu(e, a, g) = [e, a]$. The hypotheses of Proposition 3.8 are satisfied, so we get a group homomorphism $\nu_1 : K_1(\phi) \to K_1(A)$ such that $\nu_1([e, a, g]) = [e, a]$ (here we use the picture of K_1 described before Proposition 3.4). With the unitary picture of $K_1(\phi)$, the formula is more simply $\nu_1([1_n, u, g]) = [u]$.

Proposition 4.2. *The sequence*

$$K_1(SA) \xrightarrow{(S\phi)_*} K_1(SB) \xrightarrow{\mu_1} K_1(\phi) \xrightarrow{\nu_1} K_1(A) \xrightarrow{\phi_*} K_1(B)$$

is exact.

Proof. Again, all compositions are clearly zero. If $\phi_*([u]) = 0$, we may find $k \geq 0$ and a unitary g in $C[0,1] \otimes M_{n+k}(\tilde{B})$ such that $g(1) = \phi(u) \oplus 1_k$ and $g(0) = 1_{n+k}$. Then

$$[u] = \nu_1([1_{n+k}, u \oplus 1_k, g])$$

If $\nu_1([1_n, u, g]) = [u] = 0$, find $k \geq 0$ and a unitary f in $C[0,1] \otimes M_{n+k}(\tilde{A})$ such that $f(0) = 1_{n+k}$ and $f(1) = u \oplus 1_k$. Set

$$\tilde{g}(s) = \begin{cases} g(2s) \oplus 1_k & 0 \leq s \leq 1/2 \\ \phi(f(2-2s)) & 1/2 \leq s \leq 1 \end{cases}$$

Then $\tilde{g}$ is a unitary in $C[0,1] \otimes M_{n+k}(\tilde{B})$ and $\tilde{g}(0) = \tilde{g}(1) = 1_{n+k}$. Now for a fixed t in $[0,1]$, the function g_t defined by

$$g_t(s) = \begin{cases} g(s(1-\frac{1}{2}t)^{-1}) \oplus 1_k & 0 \leq s \leq 1 - \frac{1}{2}t \\ \phi(f(3-2s-t)) & 1 - \frac{1}{2}t \leq s \leq 1 \end{cases}$$

satisfies $g_0 = g \oplus 1_k$, $g_1 = \tilde{g}$, and $g_t(1) = \phi(f(1-t))$, and so

$$[1_n, u, g] = [1_{n+k}, u \oplus 1_k, g \oplus 1_k] = [1_{n+k}, 1_{n+k}, \tilde{g}] = \mu_1([\tilde{g}]).$$

Finally, if $\mu_1([g]) = [1_n, 1_n, g] = 0$, use part (ii) of Lemma 3.13 to find an integer k and paths (u_t) and (g_t) such that $u_0 = u_1 = 1_{n+k}$, $g_1 = g \oplus 1_k$, $g_0 = 1_{n+k}$, and $g_t(1) = \phi(u_t)$ for all t. Write $f(t) = u_t$ and set

$$\tilde{g}_t(s) = \begin{cases} g_t(2s) & 0 \leq s \leq 1/2 \\ \phi(f((2-2t)s + 2t - 1)) & 1/2 \leq s \leq 1 \end{cases}$$

Then $\tilde{g}_t(0) = \tilde{g}_t(1) = 1_{n+k}$ for all t and

$$\tilde{g}_1(s) = \begin{cases} g(2s) \oplus 1_k & 0 \leq s \leq 1/2 \\ 1_{n+k} & 1/2 \leq s \leq 1 \end{cases}$$

and

$$\tilde{g}_0(s) = \begin{cases} 1_{n+k} & 0 \leq s \leq 1/2 \\ \phi(f(2s-1)) & 1/2 \leq s \leq 1 \end{cases}$$

Which are homotopic to $g \oplus 1_k$ and $S\phi(f)$, respectively. Thus

$$[g] = [g \oplus 1_k] = [S\phi(f)] = (S\phi)_*([f])$$

which completes the proof. □

Proposition 4.3. *If $\phi = 0$, then the sequence in Theorem 2.1 splits at $K_0(A)$ and $K_1(A)$. In other words, for each $j = 0, 1$ there is a group homomorphism $\lambda_j : K_j(A) \to K_j(\phi)$ such that $\nu_j \circ \lambda_j$ is the identity map on $K_j(A)$.*

Proof. If p and q are two projections in $M_\infty(\tilde{A})$ with $[\dot{p}] = [\dot{q}]$, let v be a partial isometry in $M_\infty(\mathbb{C})$ such that $v^*v = \dot{p}$ and $vv^* = \dot{q}$. If u is a unitary in $M_n(\tilde{A})$, let g be any unitary in $C[0,1] \otimes M_n(\mathbb{C})$ such that $g(0) = 1_n$ and $g(1) = \dot{u}$. Define

$$\lambda_0([p] - [q]) = [p, q, v] \qquad \lambda_1([u]) = [1_n, u, g]$$

For both $j = 0, 1$, it is straightforward to check that λ_j is well-defined, additive, independent of the choices of v and g, and that $\nu_j \circ \lambda_j$ is the identity. □

By combining all results in this subsection, we obtain Theorem 2.1. The map μ_1 in Theorem 2.1 is (by abuse of notation) the composition of the Bott map β_B and μ_1 from Proposition 4.2. It may therefore be written as $\mu_1([p] - [q]) = [1_n, 1_n, f_p f_q^*]$, where f_p is the projection loop $f_p(s) = e^{2\pi i s p}$ in $C[0,1] \otimes M_n(\tilde{B})$. Since the Bott map is natural, this does not affect exactness.

We also record the following immediate, but useful, consequences of Theorem 2.1.

Corollary 4.4. *We have the following.*

(i) If $K_*(A) = K_*(B) = 0$, then $K_*(\phi) = 0$.
(ii) If $\phi : A \to B$ is a *-isomorphism, then $K_*(\phi) = 0$.

4.2. **Proof of Theorem 2.2.** Throughout this subsection, we will assume that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{\iota_A} & A & \xrightarrow{\pi_A} & A/I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & J & \xrightarrow{\iota_B} & B & \xrightarrow{\pi_B} & B/J & \longrightarrow & 0 \end{array} \tag{4}$$

is a commutative diagram with exact rows.

Proposition 4.5. *The sequence*

$$K_0(\psi) \xrightarrow{\iota_*} K_0(\phi) \xrightarrow{\pi_*} K_0(\gamma)$$

is exact. If $\lambda_A : A/I \to A$ and $\lambda_B : B/J \to B$ are splittings of the rows in (4) that keep the diagram commutative, then the sequence

$$0 \longrightarrow K_0(\psi) \xrightarrow{\iota_*} K_0(\phi) \underset{\lambda_*}{\overset{\pi_*}{\rightleftarrows}} K_0(\gamma) \longrightarrow 0$$

is split exact.

Proof. It is clear that the composition is zero. Conversely, suppose that $[1_n, q, v]$ is in the kernel of π_*, so $[1_n, \pi_A(q), \pi_B(v)] = 0$. Find (in order):

(i) an integer $m \geq n$ so that q is in $M_m(\tilde{A})$,

(ii) an integer $k \geq 0$ and a partial isometry w in $M_{m+k}(\widetilde{A/I})$ such that $w^*w = \pi_A(q) \oplus 1_k$ and $ww^* = 1_n \oplus 0_{m-n} \oplus 1_k$ and $\gamma(w)(\pi_B(v) \oplus 1_k)$ is homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$ (use Lemma 3.12),

(iii) an integer $l \geq 0$ and a unitary z homotopic to 1_{m+k+l} in $M_{m+k+l}(\widetilde{A/I})$ such that $z(\pi_A(q) \oplus 1_k \oplus 0_l)z^* = 1_n \oplus 0_{m-n} \oplus 1_k \oplus 0_l$ and $\gamma(z)(\pi_B(v) \oplus 1_k \oplus 0_l) = (\gamma(w)(\pi_B(v) \oplus 1_k)) \oplus 0_l$. For example, one may take $l = m + k$ and

$$z = \begin{bmatrix} w & 1_{m+k} - ww^* \\ 1_{m+k} - w^*w & w^* \end{bmatrix}$$

see the discussion following Definition 4.12.

(iv) a unitary U in $M_{m+k+l}(\tilde{A})$ such that $\pi_A(U) = z$ (use (iii)),

(v) a unitary V in $(1_n \oplus 0_{m-n} \oplus 1_k) M_{m+k}(\tilde{B})(1_n \oplus 0_{m-n} \oplus 1_k)$ homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$ such that $\pi_B(V) = \gamma(w)(\pi_B(v) \oplus 1_k)$ (use (ii)).

Then

$$\begin{aligned}
[1_n, q, v] &= \left[\begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}, \begin{bmatrix} q & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix}, \begin{bmatrix} v & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} \right] \\
&= \left[\begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}, U \begin{bmatrix} q & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} U^*, \phi(U) \begin{bmatrix} v & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} \right] \\
&= \left[\begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}, U \begin{bmatrix} q & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} U^*, \phi(U) \begin{bmatrix} v & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} \right] \\
&\quad + \left[\begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}, \begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}, \begin{bmatrix} V^* & 0 \\ 0 & 0_l \end{bmatrix} \right] \\
&= \left[\begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}, U \begin{bmatrix} q & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} U^*, \phi(U) \begin{bmatrix} v & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & 0_l \end{bmatrix} \right]
\end{aligned}$$

To get the first equality above, we added an elementary scalar triple. To get the second, notice that the two triples are isomorphic via the unitary U. In the third equality, the new triple being added is elementary because V is homotopic to the identity. The fourth equality follows from part (ii) of Proposition 3.3. Regarding the elements of the latter triple, we have

$$\pi_A\left(U\begin{bmatrix} q & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix}U^*\right) = \pi_B\left(\phi(U)\begin{bmatrix} v & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & 0_l \end{bmatrix}\right) = \begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}$$

from which it follows that $[1_n, q, v]$ is in the image of ι_*.

For the split exact sequence, it is clear that λ_* is a right inverse for π_*, so we need only show that ι_* is injective. Suppose that $(1_n, q, v)$ is a triple in $\Gamma_0(\psi)$ with $\dot{q} = \dot{v} = 1_n$ and $[1_n, q, v] = 0$ in $K_0(\phi)$. Choose $m \geq n$ so that $1_n \oplus 0_{m-n}$ and q are in $M_m(\tilde{I})$ and v is in $M_m(\tilde{J})$. Use Lemma 3.12 to find an integer $k \geq 0$ and a partial isometry w in $M_{m+k}(\tilde{A})$ with $w^*w = q \oplus 1_k$ and $ww^* = 1_n \oplus 0_{m-n} \oplus 1_k$ and $\phi(w)(v \oplus 1_k)$ is homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$. Let y_t be such a homotopy, that is, $y_0 = \dot{y}_t = 1_n \oplus 0_{m-n} \oplus 1_k$ for all t and $y_1 = \phi(w)(v \oplus 1_k)$. Set $x = \lambda_A(\pi_A(w^*))w$. Then $\pi_A(x) = 1_n \oplus 0_{m-n} \oplus 1_k$ so that x is in $M_{m+k}(\tilde{I})$. We have $x^*x = q \oplus 1_k$ and $xx^* = 1_n \oplus 0_{m-n} \oplus 1_k$ and, since $\pi_B(v \oplus 1_k) = 1_n \oplus 0_{m-n} \oplus 1_k$,

$$\begin{aligned}
\psi(x)\begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix} &= \psi(\lambda_A(\pi_A(w^*))w)\begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix} \\
&= \lambda_B(\pi_B(\phi(w^*)))\phi(w)\begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix} \\
&= \lambda_B\left(\pi_B\left(\begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix}\phi(w^*)\right)\right)\phi(w)\begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix} \\
&= \lambda_B\left(\pi_B\left(\phi(w)\begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix}\right)^*\right)\phi(w)\begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix}
\end{aligned}$$

is homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$ through $M_{m+k}(\tilde{J})$ via $\lambda_B(\pi_B(y_t^*))y_t$. It follows that $[1_n, q, v] = 0$ in $K_0(\psi)$. □

Now we associate an index map $\partial_1 : K_1(\gamma) \to K_0(\psi)$ to the diagram (4).

Definition 4.6. *The index map $\partial_1 : K_1(\gamma) \to K_0(\psi)$ in relative K-theory is given by*

$$\partial_1([1_n, u, g]) = \left[w\begin{bmatrix} 1_n & 0 \\ 0 & 0_l \end{bmatrix}w^*, \begin{bmatrix} 1_n & 0 \\ 0 & 0_l \end{bmatrix}, \begin{bmatrix} h(1) & 0 \\ 0 & 0_l \end{bmatrix}\phi(w^*)\right]$$

where $l \geq 0$, w in $M_{n+l}(\tilde{A})$ is a unitary such that $\pi_A(w)(1_n \oplus 0_l) = u \oplus 0_l$, and h in $M_n(\widetilde{CB})$ is a unitary such that $\pi_B(h) = g$.

Observe that such elements l, w, and h always exist: one may take $l = n$, w to be a lift of $u \oplus u^*$, and h exists because $C(B/J)$ is contractible. It is straightforward to verify that ∂_1 is independent of these choices, and depends only on the class of the triple $(1_n, u, g)$.

The map ∂_1 is natural in the following sense. Suppose that

$$\begin{tikzcd}
& 0 \arrow[rr] & & I' \arrow[rr] \arrow[dd, "\psi'"] & & A' \arrow[rr] \arrow[dd, "\phi'"] & & A'/I' \arrow[rr] \arrow[dd, "\gamma'"] & & 0 \\
0 \arrow[rr] & & I \arrow[rr] \arrow[ur, "\sigma"] \arrow[dd, "\psi"] & & A \arrow[rr] \arrow[ur] \arrow[dd, "\phi"] & & A/I \arrow[rr] \arrow[ur, "\tau"] \arrow[dd, "\gamma"] & & 0 & \\
& 0 \arrow[rr] & & J' \arrow[rr] & & B' \arrow[rr] & & B'/J' \arrow[rr] & & 0 \\
0 \arrow[rr] & & J \arrow[rr] \arrow[ur, "\sigma'"] & & B \arrow[rr] \arrow[ur] & & B/J \arrow[rr] \arrow[ur, "\tau'"] & & 0 &
\end{tikzcd}$$

is a commutative diagram with exact rows. Then the diagram

$$\begin{tikzcd}
K_1(\gamma) \arrow[r, "\partial_1"] \arrow[d, "\tau_*"] & K_0(\psi) \arrow[d, "\sigma_*"] \\
K_1(\gamma') \arrow[r, "\partial_1'"] & K_0(\psi')
\end{tikzcd}$$

is commutative, where ∂_1' is the index map associated to the diagram in the back. We leave the straighforward proof to the reader.

Proposition 4.7. *The sequence*

$$K_1(\phi) \xrightarrow{\pi_*} K_1(\gamma) \xrightarrow{\partial_1} K_0(\psi) \xrightarrow{\iota_*} K_0(\phi)$$

is exact and the diagram

$$\begin{tikzcd}
K_1(S(B/J)) \arrow[r, "\mu_1"] \arrow[d, "\theta_J^{-1}\circ\delta_2"] & K_1(\gamma) \arrow[r, "\nu_1"] \arrow[d, "\partial_1"] & K_1(A/I) \arrow[d, "\delta_1"] \\
K_1(J) \arrow[r, "\mu_0"] & K_0(\psi) \arrow[r, "\nu_0"] & K_0(I)
\end{tikzcd}$$

is commutative.

Proof. For ease of notation we will denote

$$p = w \begin{bmatrix} 1_n & 0 \\ 0 & 0_l \end{bmatrix} w^* \qquad v = \begin{bmatrix} h(1) & 0 \\ 0 & 0_l \end{bmatrix} \phi(w^*)$$

It is a simple calculation to see that the right square in the diagram is commutative. For the left square, take $[f]$ in $K_1(S(B/J))$, where f is in $M_n(\widetilde{S(B/J)})$, and find h in $M_n(\widetilde{CB})$ such that $\pi_B(h) = f$. Then

$$\partial_1(\mu_1([f])) = \partial_1([1_n, 1_n, f]) = [1_n, 1_n, h(1)]$$

Now find g in $M_{2n}(\widetilde{SB})$ such that $\pi_B(g) = f \oplus f^*$, and let

$$\tilde{g}(t) = \begin{cases} g(2t) & 0 \le t \le 1/2 \\ h(2t-1) \oplus h(2t-1)^* & 1/2 \le t \le 1 \end{cases}$$

Then $[g(1_n \oplus 0_n)g^*] - [1_n \oplus 0_n] = [\tilde{g}(1_n \oplus 0_n)\tilde{g}^*] - [1_n \oplus 0_n]$ in $K_0(SJ)$, the latter being equal to $\theta_J([h(1)])$ since $\tilde{g}(1) = h(1) \oplus h(1)^*$. All in all, we have

$$\mu_0(\theta_J^{-1}(\delta_2([f]))) = \mu_0(\theta_J^{-1}([\tilde{g}(1_n \oplus 0_n)\tilde{g}^*] - [1_n \oplus 0_n])) = \mu_0([h(1)]) = [1_n, 1_n, h(1)]$$

which shows commutativity of the left square.

The composition $\partial_1 \circ \pi_*$ is clearly zero since everything has a unitary lift. We also have $\iota_* \circ \partial_1$ zero since

$$[p, 1_n \oplus 0_l, v] = [p, 1_n \oplus 0_l, v] + [1_n \oplus 0_l, 1_n \oplus 0_l, h(1)^* \oplus 0_l] = [p, 1_n \oplus 0_l, (1_n \oplus 0_l)\phi(w^*)]$$

Because $(1_n \oplus 0_l, 1_n \oplus 0_l, h(1)^* \oplus 0_l)$ is elementary in $\Gamma_0(\phi)$ and $(p, 1_n \oplus 0_l, (1_n \oplus 0_l)\phi(w^*)) \cong (1_n \oplus 0_l, 1_n \oplus 0_l, 1_n \oplus 0_l)$.

Now suppose that

$$\partial_1([1_n, u, g]) = [p, 1_n \oplus 0_l, v] = [w(1_n \oplus 0_l)w^*, 1_n \oplus 0_l, (h(1) \oplus 0_l)\phi(w^*)] = 0$$

Find $k \geq 1$ and a partial isometry x in $M_{n+l+k}(\tilde{I})$ with $xx^* = p \oplus 1_k$ and $\dot{x} = x^*x = 1_n \oplus 0_l \oplus 1_k$, and such that $(v \oplus 1_k)\psi(x)$ is homotopic to $1_n \oplus 0_l \oplus 1_k$. Let y_t be such a homotopy, with $\dot{y}_t = y_0 = 1_n \oplus 0_l \oplus 1_k$ for all t and $y_1 = (v \oplus 1_k)\psi(x)$. Set

$$z = \begin{bmatrix} 1_n & 0 & 0 \\ 0 & 0_l & 0 \\ 0 & 0 & 1_k \end{bmatrix} \begin{bmatrix} w^* & 0 \\ 0 & 1_k \end{bmatrix} x$$

and

$$h'(t) = \begin{cases} y_{2t} & 0 \leq t \leq 1/2 \\ (h(2t-1)^* \oplus 0_l \oplus 1_k)(v \oplus 1_k)\psi(x) & 1/2 \leq t \leq 1 \end{cases}$$

Then $\pi_A(z) = u \oplus 0_l \oplus 1_k$ and

$$\pi_B(h'(t)) = \begin{cases} 1_n \oplus 0_l \oplus 1_k & 0 \leq t \leq 1/2 \\ g(2t-1) \oplus 0_l \oplus 1_k & 1/2 \leq t \leq 1 \end{cases}$$

which is clearly homotopic to $g \oplus 0_l \oplus 1_k$. Moreover, $h'(1) = \phi(z)$. It follows that

$$[1_n, u, g] = [1_n \oplus 0_l \oplus 1_k, u \oplus 0_l \oplus 1_k, g \oplus 0_l \oplus 1_k] = \pi_*([1_n \oplus 0_l \oplus 1_k, z, h'])$$

Now suppose that $(p, 1_n, v)$ is a triple in $\Gamma_0(\psi)$ with $[p, 1_n, v] = 0$ in $K_0(\phi)$. Choose $m \geq n$ such that $1_n \oplus 0_{m-n}$ and p are in $M_m(\tilde{I})$ and v is in $M_m(\tilde{J})$. Find $k \geq 0$ and a partial isometry x in $M_{m+k}(\tilde{A})$ with $xx^* = p \oplus 1_k$ and $\dot{x} = x^*x = 1_n \oplus 0_{m-n} \oplus 1_k$, and such that $(v \oplus 1_k)\phi(x)$ is homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$. Find a unitary U in $M_{m+k}(\mathbb{C})$ such that

$$U(1_n \oplus 0_{m-n} \oplus 1_k)U^* = 1_{n+k} \oplus 0_{m-n}$$

and let $p' = U(p \oplus 1_k)U^*$, $v' = U(v \oplus 1_k)U^*$, and $x' = UxU^*$. Clearly $(p, 1_n, v) \oplus (1_k, 1_k, 1_k) \cong (p', 1_{n+k}, v')$, $x'x'^* = p'$, $x'^*x' = 1_{n+k} \oplus 0_{m-n}$, and that $v'\phi(x')$ is homotopic to $1_{n+k} \oplus 0_{m-n}$. Let y_t be such a homotopy, with $\dot{y}_t = y_0 = 1_{n+k} \oplus 0_{m-n}$ for all t and $y_1 = v'\phi(x')$. Notice that $\pi_A(x') = (1_{n+k} \oplus 0_{m-n})\pi_A(x')(1_{n+k} \oplus 0_{m-n})$, so we may regard $\pi_A(x')$ as a unitary in $M_{n+k}(\widetilde{A/I})$, and similarly we may regard y_t as a path of unitaries in $M_{n+k}(\tilde{B})$. Set $g(t) = \pi_B(y_t)$ and notice that

$$g(1) = \pi_B(\phi(x'^*))\pi_B(v'^*) = \gamma(\pi_A(x'^*))$$

so that $(1_{n+k}, \pi_A(x'^*), g)$ is a triple in $\Gamma_1(\gamma)$. Moreover, we see that its image under ∂_1 is $[p, 1_n, v]$ by using $l = 2m + k - n$,

$$w = \begin{bmatrix} x' & 1_{m+k} - x'x'^* \\ 1_{m+k} - x'^*x' & x'^* \end{bmatrix}$$

in $M_{2(m+k)}(\tilde{A})$ and $h(t) = y_t$. ☐

Corollary 4.8. *There is an isomorphism $\theta_\phi : K_1(\phi) \to K_0(S\phi)$. Moreover, the diagram*

$$\begin{tikzcd} K_1(SB) \arrow[r, "\mu_1"] \arrow[d, equal] & K_1(\phi) \arrow[r, "\nu_1"] \arrow[d, "\theta_\phi"] & K_1(A) \arrow[d, "\theta_A"] \\ K_1(SB) \arrow[r, "\mu_0"] & K_0(S\phi) \arrow[r, "\nu_0"] & K_0(SA) \end{tikzcd}$$

is commutative.

Proof. The map θ_ϕ is the index map ∂_1 associated to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & SA \arrow[r] \arrow[d, "S\phi"] & CA \arrow[r] \arrow[d, "C\phi"] & A \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & SB \arrow[r] & CB \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

CA and CB are contractible, hence the relative groups $K_0(C\phi)$ and $K_1(C\phi)$ are trivial by Corollary 4.4. It follows that θ_ϕ is an isomorphism. ☐

An explicit description of θ_ϕ is as follows. Let $(1_n, u, g)$ be a triple in $\Gamma_1(\phi)$, and let w be a unitary in $C[0, 1] \otimes M_{2n}(\tilde{A})$ with $w(0) = 1_{2n}$ and $w(1) = u \oplus u^*$. Then

$$\theta_\phi([1_n, u, g]) = \left[w \begin{bmatrix} 1_n & 0 \\ 0 & 0_n \end{bmatrix} w^*, \begin{bmatrix} 1_n & 0 \\ 0 & 0_n \end{bmatrix}, \begin{bmatrix} g & 0 \\ 0 & 0_n \end{bmatrix} \phi(w^*) \right]$$

Corollary 4.9. *The sequence*

$$K_1(\psi) \xrightarrow{\iota_*} K_1(\phi) \xrightarrow{\pi_*} K_1(\gamma)$$

is exact. If $\lambda_A : A/I \to A$ and $\lambda_B : B/J \to B$ are splittings of the rows in (4) that keep the diagram commutative, then the sequence

$$\begin{tikzcd} 0 \arrow[r] & K_1(\psi) \arrow[r, "\iota_*"] & K_1(\phi) \arrow[r, shift left, "\pi_*"] & K_1(\gamma) \arrow[l, shift left, "\lambda_*"] \arrow[r] & 0 \end{tikzcd}$$

is split exact.

Proof. The map θ_ϕ is natural, so we have the commutative diagram

$$\begin{tikzcd} K_1(\psi) \arrow[r, "\iota_*"] \arrow[d, "\theta_\psi"] & K_1(\phi) \arrow[r, "\pi_*"] \arrow[d, "\theta_\phi"] & K_1(\gamma) \arrow[d, "\theta_\gamma"] \\ K_0(S\psi) \arrow[r, "\iota_*"] & K_0(S\phi) \arrow[r, "\pi_*"] & K_0(S\gamma) \end{tikzcd}$$

in which, by Proposition 4.5, the bottom row is exact. It follows that the top row is exact as well. The proof for split exactness is similar. □

At this point we may unambiguously define higher relative groups $K_j(\phi)$ by $K_0(S^j\phi)$ and higher index maps $\partial_j : K_j(\gamma) \to K_{j-1}(\psi)$ to obtain a long exact sequence. We proceed to prove that Bott periodicity holds so that the long exact sequence collapses to the six-term exact sequence in Theorem 2.2.

For Bott periodicity we will follow the original proof in [3] that uses the Toeplitz algebra. Recall that the Toeplitz algebra $\mathcal{T}$ is the universal C*-algebra generated by an isometry. Let $\pi : \mathcal{T} \to C(S^1)$ be the *-homomorphism that sends the generating isometry to the function z on S^1. The kernel of π is isomorphic to $\mathcal{K}$, and by identifying $C_0(0,1)$ with elements in $C(S^1)$ that vanish at 1 and letting $\mathcal{T}_0 = \pi^{-1}(C_0(0,1))$, we obtain the reduced Toeplitz extension

$$0 \longrightarrow \mathcal{K} \hookrightarrow \mathcal{T}_0 \xrightarrow{\pi} C_0(0,1) \longrightarrow 0$$

We will assume the nontrivial fact that $K_*(\mathcal{T}_0) = 0$; we refer the reader to [3] for the original proof.

Lemma 4.10. *If C is in the bootstrap category (22.3.4 of* [1]*) and $K_*(C) = 0$, then $K_*(\phi \otimes \mathrm{id}_C) = 0$. In particular, $K_*(\phi \otimes \mathrm{id}_{\mathcal{T}_0}) = 0$.*

Proof. By the Künneth Theorem for tensor products (see 23.1.3 of [1]), we have $K_*(A \otimes C) = K_*(B \otimes C) = 0$. The conclusion follows from Corollary 4.4. □

Lemma 4.11. $K_j(\phi \otimes \mathrm{id}_{\mathcal{K}}) \cong K_j(\phi)$ for $j = 0, 1$.

Proof. For either $j = 0, 1$, we have a commutative diagram with exact rows

$$\begin{tikzcd}
K_{1-j}(A) \arrow[r] \arrow[d] & K_{1-j}(B) \arrow[r] \arrow[d] & K_j(\phi) \arrow[r] \arrow[d] & K_j(A) \arrow[r] \arrow[d] & K_j(B) \arrow[d] \\
K_{1-j}(A \otimes \mathcal{K}) \arrow[r] & K_{1-j}(B \otimes \mathcal{K}) \arrow[r] & K_j(\phi \otimes \mathrm{id}_{\mathcal{K}}) \arrow[r] & K_j(A \otimes \mathcal{K}) \arrow[r] & K_j(B \otimes \mathcal{K})
\end{tikzcd}$$

where all the vertical maps are induced by the embedding $a \mapsto a \otimes p$, where p is any rank one projection in $\mathcal{K}$. All vertical maps except for the middle one are known to be isomorphisms. The five lemma then shows that the middle vertical arrow is an isomorphism. □

We now produce the Bott map. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \otimes \mathcal{K} \arrow[r] \arrow[d, "\phi\otimes\mathrm{id}_{\mathcal{K}}"] & A \otimes \mathcal{T}_0 \arrow[r] \arrow[d, "\phi\otimes\mathrm{id}_{\mathcal{T}_0}"] & SA \arrow[r] \arrow[d, "S\phi"] & 0 \\
0 \arrow[r] & B \otimes \mathcal{K} \arrow[r] & B \otimes \mathcal{T}_0 \arrow[r] & SB \arrow[r] & 0
\end{tikzcd}$$

Proposition 4.7 implies that

$$K_1(\phi \otimes \mathrm{id}_{\mathcal{T}_0}) \longrightarrow K_1(S\phi) \longrightarrow K_0(\phi \otimes \mathrm{id}_{\mathcal{K}}) \longrightarrow K_0(\phi \otimes \mathrm{id}_{\mathcal{T}_0})$$

is exact, and Lemma 4.10 and Lemma 4.11 together give an isomorphism $K_0(\phi) \cong K_1(S\phi)$. We let $\beta_\phi : K_0(\phi) \to K_1(S\phi)$ denote this isomorphism. By Proposition 4.7 we have the commutative diagram

$$\begin{tikzcd} K_1(B) \arrow[r, "\mu_0"] \arrow[d, "\beta_{SB}\circ\theta_B"] & K_0(\phi) \arrow[r, "\nu_0"] \arrow[d, "\beta_\phi"] & K_0(A) \arrow[d, "\beta_A"] \\ K_1(S^2B) \arrow[r, "\mu_1"] & K_1(S\phi) \arrow[r, "\nu_1"] & K_1(SA) \end{tikzcd}$$

We introduce a useful piece of notation before giving an explicit description of β_ϕ.

Definition 4.12. *For a triple $(p, 1_n, v)$ in $\Gamma_0(\phi)$, choose $m \geq n$ such that p is in $M_m(\tilde{A})$, and let*

$$p_v(s) = w(s)^* \begin{bmatrix} 1_n & 0 \\ 0 & 0_{2m-n} \end{bmatrix} w(s)$$

where w is a path of unitaries in $M_{2m}(\tilde{B})$ with $w(0) = 1_{2m}$ and

$$w(1) = \begin{bmatrix} v & 1_m - vv^* \\ 1_m - v^*v & v^* \end{bmatrix}$$

Note that such a path w exists since

$$\begin{bmatrix} v & 1_m - vv^* \\ 1_m - v^*v & v^* \end{bmatrix} = \begin{bmatrix} 0 & 1_m \\ 1_m & 0 \end{bmatrix} \begin{bmatrix} 1_m - v^*v & v^* \\ v & 1_m - vv^* \end{bmatrix}$$

and the two unitaries on the right are self-adjoint, hence homotopic to 1_{2m} by spectral theory. The choice of w will not be relevant for its use, but to make the definition more concrete, if u is a self-adjoint unitary then $u = \exp(i\pi(1-u)/2)$ [9], so one may take the path $\exp(i\pi t(1-u)/2)$ for $0 \leq t \leq 1$.

We then have $\beta_\phi([p, 1_n, v]) = [1_{2m}, u, g]$ where

$$u(t) = \exp\left(2\pi i t \begin{bmatrix} p & 0 \\ 0 & 0_m \end{bmatrix}\right) \exp\left(-2\pi i t \begin{bmatrix} 1_n & 0 \\ 0 & 0_{2m-n} \end{bmatrix}\right)$$

and

$$g(s,t) = \exp(2\pi i t p_v(s)) \exp\left(-2\pi i t \begin{bmatrix} 1_n & 0 \\ 0 & 0_{2m-n} \end{bmatrix}\right)$$

Now we complete the six-term exact sequence in Theorem 2.2. We define the exponential map $\partial_0 : K_0(\gamma) \to K_1(\psi)$ to be the group homomorphism that makes the diagram

$$\begin{tikzcd} K_0(\gamma) \arrow[r, dashed, "\partial_0"] \arrow[d, "\beta_\gamma"] & K_1(\psi) \arrow[d, "\theta_\psi"] \\ K_1(S\gamma) \arrow[r, "\partial_2"] & K_0(S\psi) \end{tikzcd}$$

commutative. All maps in the above diagram are natural, so the sequence in Theorem 2.2 is exact everywhere.

An explicit description of ∂_0 is as follows. Given a triple $(p, 1_n, v)$ in $\Gamma_0(\gamma)$, choose m and p_v as in Definition 4.12. Let a in $M_m(\tilde{A})$ be such that $a = a^*$, $\pi_A(a) = p$, and let f in $M_{2m}(\widetilde{CB})$ be such that $f(t) = f(t)^*$ for all t, $\pi_B(f) = p_v$, and $f(1) = \phi(a) \oplus 0_m$. Then we have

$$\partial_0([p, 1_n, v]) = -[1_{2m}, \exp(2\pi i(a \oplus 0_m)), \exp(2\pi i f)]$$

Remark 2. *It is interesting to note that split exactness was not necessary to prove Theorem 2.2. An examination of the proof of Bott periodicity in* [3] *reveals that split exactness is crucial in deducing that $K_*(\mathcal{T}_0) = 0$ from the isomorphism $K_*(\mathcal{T}) \cong K_*(\mathbb{C})$. Here we were able to sneak around this difficulty using Corollary 4.4.*

4.3. **Proof of Corollary 2.3.** Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & SB \arrow[r, "\iota_A"] \arrow[d, equal] & C_\phi \arrow[r, "\pi_A"] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & SB \arrow[r, hook] & CB \arrow[r, "\pi_B"] & B \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where $\iota_A(f) = (0, f)$, $\pi_A(a, f) = a$, $\pi_B(f) = f(1)$, and $\sigma(a, f) = f$. Since $K_*(CB) = 0$ because CB is contractible, we have by Example 2.4(i) that $\nu_j : K_j(\sigma) \to K_j(C_\phi)$ is an isomorphism for $j = 0, 1$. By Theorem 2.2 and Corollary 4.4,

$$\begin{tikzcd}
0 \arrow[r] & K_0(\sigma) \arrow[r, "\pi_*"] & K_0(\phi) \arrow[d] \\
K_1(\phi) \arrow[u] & K_1(\sigma) \arrow[l, "\pi_*"'] & 0 \arrow[l]
\end{tikzcd}$$

is exact and hence $\pi_* : K_j(\sigma) \to K_j(\phi)$ is an isomorphism for $j = 0, 1$. Define

$$\Delta_j = \nu_j \circ \pi_*^{-1}$$

for $j = 0, 1$. By naturality of the index map ∂_1, we have the commutative diagram

$$\begin{tikzcd}
K_1(\gamma) \arrow[r, "\partial_1"] \arrow[d, "\Delta_1"] & K_0(\psi) \arrow[d, "\Delta_0"] \\
K_1(C_\gamma) \arrow[r, "\delta_1"] & K_0(C_\psi)
\end{tikzcd} \tag{5}$$

where δ_1 is the index map associated to the sequence

$$0 \longrightarrow C_\psi \longrightarrow C_\phi \longrightarrow C_\gamma \longrightarrow 0$$

and the *-homomorphisms ψ, ϕ, and γ are as in (4). It follows that similar commutative diagrams exist for the Bott map and the exponential map, though one needs to employ the canonical identification $SC_\phi \cong C_{S\phi}$.

We provide a description of the maps Δ_j. For simplicity, we assume that A and B are unital and that $\phi(1) = 1$. Given a triple $(p, 1_n, v)$ in $\Gamma_0(\phi)$, we have

$$\Delta_0([p, 1_n, v]) = [(p \oplus 0_m, p_v)] - [1_n \oplus 0_{2m-n}]$$

where m and p_v are as in Definition 4.12. Given a triple $(1_n, u, g)$ in $\Gamma_1(\phi)$, we have

$$\Delta_1([1_n, u, g]) = [(u, g)]$$

These formulas, together with (5), may be used to verify the formulas for the Bott map and the exponential map given above.

The proof that the diagrams given in Corollary 2.3 are commutative is straightforward and is left to the reader. This fact, together with (5), proves the intertwining result for Theorem 2.2. We conclude by verifying the intertwining result for Theorem 2.1

Proposition 4.13. *The diagram*

$$\begin{tikzcd}
& K_0(\phi) \arrow[d, "\Delta_0"] \arrow[dr, "\nu_0"] & \\
K_1(B) \arrow[ur, "\mu_0"] \arrow[r, "\iota_*\circ\theta_B"'] & K_0(C_\phi) \arrow[r, "\pi_*"'] & K_0(A) \arrow[d, "\phi_*"] \\
K_1(A) \arrow[u, "\phi_*"] & K_1(C_\phi) \arrow[l, "\pi_*"'] & K_0(B) \arrow[l, "\iota_*\circ\beta_B"'] \arrow[dl, "\mu_1"] \\
& K_1(\phi) \arrow[u, "\Delta_1"] \arrow[ul, "\nu_1"] &
\end{tikzcd}$$

is commutative up to sign, where the maps ι and π come from the short exact sequence $0 \longrightarrow SB \xrightarrow{\iota} C_\phi \xrightarrow{\pi} A \longrightarrow 0$.

Proof. The bottom two triangles are fairly straightforward. For the top right triangle, we have $\Delta_0([p, 1_n, v]) = [p_v] - [1_n \oplus 0_{2m-n}]$, and evaluating at 1 gives $[p] - [1_n] = \nu_0([p, 1_n, v])$. For the top left triangle, when we compute $\Delta_0([1_n, 1_n, u])$ for a unitary u in $M_n(\tilde{B})$, the path p_v is the same path produced when computing $\theta_B([u])$. □

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