Lectures on K-Theory and Operator Algebras
Math 582
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Lecture 1
Introduction

Organizational Details

- Main center of the course is our *Piazza* site here. Piazza is a highly collaborative discussion forum and communication platform.
- Lecture notes form the course "textbook". They will be posted on Piazza, usually in advance of the lecture time. My expectation is that you will read and think through the notes carefully **before** the lecture time—not afterwards!
- If you have a question or problem, *post it on Piazza*. Responses to questions can be collaboratively edited, wiki style. I will also post some questions and expect y'all to work on the answers. Do not think that something has to be perfect for you to post it.
- This is an advanced graduate course—grading is not a big deal. Students who remain fully engaged in the course until the end will receive an "A". One way in which I will assess whether you are "fully engaged" is your level of participation on Piazza.
- If you **must** have a book, get *An introduction to K-theory for C^*-algebras* by Rørdam et al.
- Personal stuff about the course.

History

This is a series of lectures presenting a perspective on **K-Theory**. K-theory originated in the 1950s from Grothendieck's work on the Riemann-Roch theorem; he conceived of the idea of organizing *equivalence classes* (German *Klasse*, hence *K-theory*) of coherent sheaves on a smooth variety V into a group $K(V)$, and the functorial properties of this group were key elements in the proof of his R–R theorem. A couple of years later (around 1958), Atiyah and Hirzebruch noticed that one could apply the same idea to (complex) *vector bundles* over a topological space. Vital to their approach was the then-brand-new *Bott periodicity theorem* about the homotopy groups of the infinite unitary group. (Nowadays, this theorem is regarded as an intrinsic part of K-theory itself, but Bott's original proof depended on Morse theory applied to homogeneous spaces of Lie groups: the last part of Milnor's book *Morse theory* gives an exposition of this proof.) Atiyah and his collaborators used K-theory in the development and proof of the *index theorem* for elliptic operators, which in a sense is a "smooth manifold" version of Grothendieck's R–R theorem.

As Atiyah and Hirzebruch's "topological" K-theory was developed, it became clear that it had an alternative description in terms of purely algebraic notions: namely, the K-theory of a (compact) space X can be described in terms of *projective modules* over the ring $C(X)$ of continuous, complex-valued functions on X. At roughly the same time, developments in geometric topology (Whitehead torsion, Wall finiteness obstruction) made it clear that important ideas in the topology of manifolds and cell complexes could be described in a similar way, but where the modules concerned were over the *noncommutative* ring $\mathbb{Z}[\Gamma]$, the group ring of $\Gamma = \pi_1(Y)$, the fundamental group of a space Y being investigated. This leads to the subject of *algebraic* K-theory of an arbitrary, potentially noncommutative ring R. (There are also connections to number theory here but I don't know enough to say anything useful about them.)

In this course our major though not exclusive focus will be on a form of K-theory that combines the potentially noncommutative input of algebraic K-theory with the topological ideas of Atiyah-Hirzebruch, namely the K-theory of *C^*-algebras*. Bott periodicity is valid in this context, and indeed attains its most elegant form here; but the study of the K-theory of noncommutative C^*-algebras such as group or groupoid algebras opens up the whole realm of higher index theory and *noncommutative topology*, as was perceived by Alain Connes. In addition to this index-theoretic application, it has turned out that the K-theory of C^*-algebras, considered now with an additional *order* structure, provides invariants which can characterize the algebras themselves up to *isomorphism* within a broad class. This is the *Elliott classification program*. We will make a small step along this road by interpreting the classification of UHF (and maybe AF?) algebras within the framework of K-theory.

Background on C^*-algebras may be found in my notes from Fall 2015, which will be referred to as "2015 notes". These can be downloaded here. You will also need some acquaintance with the basic ideas of algebraic topology. Occasionally I may head off on sidelines that require additional background also.

Modules

We will begin our study of K-theory in the *purely algebraic* context. Let R be a ring—it need not be (in fact probably will not be) commutative, but for the present discussion it must have a unit. (When we get to K-theory for C^*-algebras we will have to deal more seriously with what happens when there is no unit. We won't consider that possibility for now, however.)

A *module* over R is a "vector space with R as scalars". In other words, it is an abelian group together with a "scalar multiplication" operation that puts together an element of R and an element of M to obtain another element of M. Because R is not assumed to be commutative, though, there are two essentially different ways to do this:

- M is a *left module* if there is given a "multiplication" $R \times M \to M$ such that $(r_1 + r_2)\xi = r_1\xi + r_2\xi$, $r(\xi_1 + \xi_2) = r\xi_1 + r\xi_2$, $0\xi = 0$, $1\xi = \xi$, and $(r_1r_2)\xi = r_1(r_2\xi)$.
- Dually, M is a *right module* if we are given a "multiplication $M \times R \to M$ such that $\xi(r_1 + r_2) = \xi r_1 + \xi r_2$, $(\xi_1 + \xi_2)r = \xi_1 r + \xi_2 r$, $\xi 0 = 0$, $\xi 1 = \xi$, and $(\xi r_1)r_2 = \xi(r_1r_2)$.

I'll try to be specific about what kind of modules we are talking about in any situation, but the default assumption will be that "module" means *right* module.

We will also have occasion to consider *bimodules*. Let R and S be rings. An (R, S)-*bimodule* M is simultaneously a left R-module and a right S-module, and obeys the additional associativity law

$$(r\xi)s = r(\xi s)$$

for all $r \in R$, $s \in S$, $\xi \in M$.

Example 1.1. Suppose that R is commutative. Then any left (or right) R-module M is automatically an (R, R)-bimodule, if we define $r_1 \cdot \xi \cdot r_2 = (r_1 r_2)\xi$. Thus all these module concepts reduce to the same thing for a commutative ring, which explains why they don't usually get distinguished in a first introduction to multilinear algebra.

Example 1.2. A (two-sided) ideal in R is an (R, R)-bimodule, with the usual multiplication.

Example 1.3. Suppose that $\alpha\colon R \to S$ is a homomorphism. Then the ring S becomes an (R, S)-bimodule with law $r \cdot \xi \cdot s = \alpha(r)\xi s$. This example suggests why we may think of bimodules as "generalized morphisms".

Example 1.4. Consider $R^n = R \oplus \cdots \oplus R$ (n factors). This is a left (and right) R-module. On the other hand, if we think of it as made up of row vectors, it is also a right $M_n(R)$-module under the usual law of matrix multiplication. Thus it becomes an $(R, M_n(R))$ bimodule. Of course the space of n-dimensional *column* vectors is an $(M_n(R), R)$ bimodule. We will have occasion to think of these as mutually inverse "generalized morphisms", giving a *Morita equivalence* (see later) between R and $M_n(R)$. Exercise: show that there is no $(M_n(R), M_n(R))$ bimodule structure on R^n, even when R is a field.

Definition 1.5. Let X be a set. An R-module M (right, left, or two-sided) is *free* on X if there is given a map $X \to M$ and, for every map $X \to M'$ (where M' is another module of the same type) there is one and one only module map making the diagram

$$\begin{tikzcd} X \arrow[d] \arrow[dr] & \\ M \arrow[r, dotted, "\exists!"] & M' \end{tikzcd}$$

commute.

Proposition 1.6. *Free modules exist, and are unique up to canonical isomorphism.*

Proof. The collection of finitely supported functions $X \to R$ forms a free module over X in an obvious way. Uniqueness follows from abstract nonsense. $\square$

Remark 1.7. Let M be any module at all and let F be the free module generated by M (as a set). Then the identity map $M \to M$ extends to a surjective module-homomorphism $F \to M$. Thus, *every module is a quotient of a free module.* This construction is not very economical: often, one can use a much smaller generating set. In particular, if M is a homomorphic image of R^n for some n (i.e., a free module on a *finite* set) it is called *finitely generated*.

Remark 1.8. For many familiar examples (e.g. principal ideal domains) the *rank* of a finitely generated free module is an invariant: that is, if $R^m \cong R^n$ as right R-modules, them $m = n$. This is **not true** in full generality. For example, let R be the ring of *column-finite* but infinite (parameterized by $\mathbb{N}$)) matrices over a field k (column-finite means that each column has only finitely many nonzero entries). Considered as an R-module, then, R is the module of set maps from $\mathbb{N}$ to k^∞, the of finitely-nonzero column vectors. Thus, clearly, R^n and R^m are isomorphic for any $n, m \in \mathbb{N}$.

Next, a very important definition.

Definition 1.9. An R-module M (again, right, left or two-sided) is *projective* if every surjective module-homomorphism $\alpha\colon N \to M$ splits: that is, there is a module-homomorphism $\beta\colon M \to N$ with $\alpha \circ \beta = 1_M$.

There is a dual definition of *injective* module, which involves splitting monomorphisms out of M rather than epimorphisms into M, but we most likely won't need it.

Proposition 1.10. *A module M is projective if and only if it is a direct summand in a free module.*

Proof. Suppose that M is projective. Write M as a quotient of a free module, that is, find an epimorphism $\alpha\colon F \to M$ with F free. Let β be a splitting of α and let $M' = \mathrm{Ker}(\alpha)$. Then

$$\xi \mapsto (\alpha(\xi), \xi - \beta \circ \alpha(\xi))$$

defines an isomorphism $F \to M \oplus M'$.

Conversely, if $M \oplus M' = F$, where F is free on X, and $\alpha\colon N \to M$ is an epimorphism, then $\alpha \oplus 1\colon N \oplus M' \to M \oplus M' \cong F$ is also an epimorphism. For each $x \in X$ pick $\gamma(x) \in N \oplus M'$ mapping onto x, and extend γ using freeness to a morphism $F \to M \oplus N'$. Note that $(\alpha \oplus 1)\gamma$ is the identity on X, hence is the identity on all of F. Now define $\beta\colon M \to N$ to be the composite

$$M \hookrightarrow M \oplus M' \cong F \xrightarrow{\gamma} N \oplus M' \twoheadrightarrow N.$$

Then β is the desired splitting of α. $\square$

The following is the most important motivation for K-theory.

Example 1.11. Let X be a compact Hausdorff space and $R = C(X)$ the C^*-algebra of continuous complex-valued functions on X. Let E be a vector bundle over X. We will denote by $\Gamma(E)$ the space of continuous sections of E. It is a module over R (one can multiply a section by a function, and the result is another section). If E is a n-dimensional trivial bundle, then $\Gamma(E) \cong R^n$ is a free module.

What is more, if $\alpha\colon E \to F$ is a map of vector bundles over X, then the induced map $\alpha_*\colon \Gamma(E) \to \Gamma(F)$ is a homomorphism of $C(X)$-modules. Moreover, every $C(X)$-module homomorphism between spaces of sections of vector bundles arises in this way. (This can be proved by using partitions of unity and local frames to reduce to the statement: every $C(X)$-module map from $C(X)$ to $C(X)$ is given by pointwise multiplication by some continuous function.)

Proposition 1.12. *For any vector bundle E over a compact Hausdorff space X, the module of sections $\Gamma(E)$ is finitely generated and projective. Moreover, every finitely generated projective $C(X)$-module arises in this way.*

Proof. Bearing in mind the above discussion, all we have to do is to show that every vector bundle over X is a direct summand in a finite-dimensional trivial bundle. Let $E \to X$ be a k-dimensional bundle and (using compactness) find a finite open cover $U_1, \ldots, U_N$ of X trivializing E. For each U_i let $\varphi_i^1, \ldots, \varphi_i^k$ be sections of $E^*_{|U_i}$ establishing an isomorphism between $E_{|U_i}$ and the trivial bundle $\mathbb{C}^k$ over U_i. Let $\{\psi_i\}$ be a partition of unity subordinate to $\{U_i\}$. Then

$$\Phi = (\psi_1\varphi_1^1.\psi_1\varphi_1^2, \ldots, \psi_n\varphi_n^k)\colon E \to \mathbb{C}^{nk}$$

is a well-defined bundle map embedding E as a subbundle of a trivial bundle $\mathbb{C}^{nk}$. Such a subbundle is always complemented (take the orthogonal complement with respect to the standard Hermitian metric). Thus $\Gamma(E)$ is projective. $\square$

Lecture 2
The definition of K_0

Now we will discuss the functoriality of modules. Let M be a (right) R-module. If $\beta\colon S \to R$ is a ring homomorphism then M becomes an S-module in a natural way (define $\xi \cdot s = \xi\beta(s)$). But this is not a “good” construction for our purposes: even if M is finitely generated or projective as an R-module, the resulting S-module need *not* have either of these properties. (Exercise: Give examples!)

The “correct” functoriality goes in the opposite direction, and makes use of tensor products.

Definition 2.1. Let M_R be a right R-module and let ${}_RN$ be a left R-module. By definition, the *tensor product* $M \otimes_R N$ is the abelian group generated by symbols $m \otimes n$ ($m \in M$, $n \in N$) with relations

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, \quad m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2,$$
$$(mr) \otimes n = m \otimes (rn)$$

It is the universal object for *mesolinear*[1] maps from $M \times N$ to an abelian group, that is maps μ such that $\mu(mr, n) = \mu(m, rn)$.

Remark 2.2. The tensor product has no module structure in the above situation, it is just an abelian group. However, it is clearly functorial. That tells us, for example, that if N is an (R, S) bimodule then $M \otimes_R N$ is a right S-module; if M is a (Q, R) bimodule than $M \otimes_R N$ is a left Q-module; and so on. To see these things one need only observe that a (R, S) bimodule N (for instance) is the same as a left R-module equipped with a homomorphism $S^{\mathrm{op}} \to \mathrm{End}_R(N)$; and then apply functoriality of the tensor product.

Revert now to the functoriality of modules.

Definition 2.3. Let $\alpha\colon R \to S$ be a homomorphism of rings and let M be a right R-module. Then $\alpha_* M$ denotes the right S-module $M \otimes_R S$, where S is considered as an (R, S) bimodule as in Example 1.3 above.

It is easy to check from the definition that $\alpha_*(R) = S$. Consequently (by functoriality of the tensor product) if M is finitely generated, free, or projective over R, then $\alpha_*(M)$ has the corresponding property over S.

Example 2.4. Let X and Y be compact Hausdorff spaces, let E be a vector bundle over X, and let $f\colon Y \to X$ be a continuous map. let $\alpha\colon C(X) \to C(Y)$ be the induced homomorphism. Then we have

$$\alpha_*(\Gamma(E)) = \Gamma(f^* E),$$

where f^*E is the pull-back of the vector bundle E. Exercise: prove this.

Now we can give the definition of K-theory. Let R be a ring (with unit).

[1] I just made this word up.

Definition 2.5. The *K-theory* group $K_0(R)$ is the abelian group with one generator $[P]$ for each isomorphism class of finitely generated projective modules P over R, and with one relation

$$[P] + [Q] = [P \oplus Q]$$

for each pair P, Q of such modules.

From the discussion above, K-theory is functorial: if $\alpha\colon R \to S$ is a homomorphism, then there is an induced homomorphism of abelian groups $\alpha_*\colon K_0(R) \to K_0(S)$, sending P to $\alpha_* P = P \otimes_\alpha S$.

The restriction to finitely-generated modules is important. If we left it out, we would always get the zero group, by the following argument: given P, let $Q = \oplus^\infty P$ (which is still projective). Then $Q \cong P \oplus Q$ and thus

$$[Q] + [P] = [Q]$$

and hence $[P] = 0$. This is called the *Eilenberg swindle*.

Example 2.6. Suppose that R is a field. Then a finitely generated projective R-module is just a finite-dimensional vector space, and such are classified up to isomorphism by their dimension, a natural number. The assignment $[P] \mapsto \dim P$ then extends to an isomorphism $K_0(R) \to \mathbb{Z}$. Using the classification of finitely generated modules over principal ideal domains one can see that the same result holds if R is a PID (every such module is the direct sum of a free part and a torsion part, and is projective if and only if the torsion part is zero.)

Remark 2.7. Let X be a compact Hausdorff space. Then, by definition, $K^0(X) = K_0(C(X))$. By Proposition 1.12, we may describe $K^0(X)$ as the abelian group with one generator for each isomorphism class of vector bundles over X and one relation

$$[V] + [W] = [V \oplus W]$$

for each pair of such vector bundles. This is the original definition of Atiyah and Hirzebruch. It is a *contravariant* functor of X.

Definition 2.8. Let P_1 and P_2 be fg projective R-modules. They are *stably isomorphic* if there exists a fg projective (or free) R-module Q such that $P_1 \oplus Q \cong P_2 \oplus Q$. (Note that since every projective module is a direct summand in a free module, there is no difference between the two versions of the definition).

Lemma 2.9. *Let P_1 and P_2 be fg projective R-modules. Then $[P_1] = [P_2]$ in $K_0(R)$ if and only if P_1 and P_2 are stably isomorphic.*

Proof. Let $\mathfrak{F}$ be the free abelian group with one generator $\langle P \rangle$ for each isomorphism class of finitely generated projectives P. Since $\mathfrak{F}$ is free, if two sums $\sum m_i \langle M_i \rangle$ and $\sum n_j \langle N_j \rangle$ (with $m_, n_j \in \mathbb{N}$) agree in $\mathfrak{F}$, then they differ only by permutation of factors and in particular the corresponding direct sums of modules

$$\bigoplus M_i^{m_i}, \quad \bigoplus N_j^{n_j}$$

are isomorphic.

Now to say that $[P_1] = [P_2]$ in $K_0(M)$ is to say that $\langle P_1\rangle - \langle P_2\rangle$ is a linear combination of relators in $\mathfrak{F}$, that is,

$$\langle P_1\rangle - \langle P_2\rangle = \sum_{k=1}^{m} \ell_k \left(\langle M_k\rangle + \langle N_k\rangle - \langle M_k \oplus N_k\rangle\right),$$

with $\ell_k \in \mathbb{Z}$. Rearranging to get rid of the minus signs and using the remark above gets us $P_1 \oplus Q \cong P_2 \oplus Q$, where $Q \cong \bigoplus (M_k \oplus N_k)^{|\ell_k|}$. $\square$

Exercise 2.10. Show that every element of $K_0(A)$ can be expressed as a difference $[P] - [Q]$ of two finitely generated projective modules. Show further that *one* of the two (conventionally Q) can even be assumed to be free.

Remark 2.11. I want to talk a bit about products on K-theory. Suppose that R is a ring. Is there a natural "cap product" (so-called by analogy with cohomology theory) making R into a ring? In the commutative case the answer is yes, by an obvious construction. Namely, if E_1 and E_2 are vector bundles over X, then so is their tensor product $E_1 \otimes E_2$. Tensoring passes to isomorphism classes and so defines a product map

$$K_0(X) \otimes K_0(X) \to K_0(X)$$

which is distributive with respect to $+$ (because of the natural isomorphism $E \otimes (F_1 \oplus F_2) \cong (E \otimes F_1) \oplus (E \otimes F_2)$ and so makes $K_0(X)$ into a ring.

The natural thing to do in the non-commutative case, if we wanted a ring structure on K-theory, would be to consider the tensor product of *modules* instead of of vector bundles. But this doesn't work: although, as remarked above, we can tensor a right R-module and a left R-module (obtaining an abelian group), there is *no way* to tensor two right R-modules. So, $K_0(R)$ for R noncommutative has *no ring structure*.

The above discussion does suggest, though, a sort of replacement for the missing ring structure: if we consider a fixed (R, S)-*bimodule* N, then tensoring with N will convert R-modules to S-modules. If this process preserves projectivity and finite generation, then it will give a homomorphism $K_0(R) \to K_0(S)$. It is natural (in retrospect) to think of organizing such *bimodules* into a group "$K_0(R, S)$" and thereby obtaining a replacement for the missing ring structure via "bivariant K-theory". This is the fundamental idea of Kasparov.

Remark 2.12. Another thing that does not pass over from the commutative to the general case is the existence of *operations*. Let E be a vector bundle over X. We can form its k'the exterior power $\bigwedge^k(E)$, and this process passes to a natural abelian group homomorphism $K(X) \to K(X)$ for any X, i.e., a *cohomology operation* on K-theory. These operations, or more exactly the so-called λ-operations which are algebraically derived from them, play an important part in Adams' work on elements of Hopf invariant one, vector fields on spheres, etc. But, again, they only exist in the commutative case.

The idempotent picture of K-theory

Let P be a finitely generated projective module over R. Finite generation gives us a surjection $\alpha\colon R^n \to P$ (for some n) and projectivity gives a splitting $\beta\colon P \to R^n$ ("splitting" means $\alpha\beta$ is the identity endomorphism of P). This means that $e = \beta\alpha \in \mathrm{End}(R^n)$ is an *idempotent*[2], that is, $e^2 = e$. The ring $\mathrm{End}(R^n)$, where R^n is considered as a *right* R-module, is $M_n(R)$ (matrices acting by *left* multiplication on column vectors) and so e is an idempotent matrix in $M_n(R)$. The projective module P can be recovered (up to isomorphism) from e: it is the (right) submodule

$$P = \{v \in \mathbb{R}^n : e(v) = v\},$$

that is, the range of e.

It follows that the K-theory group $K_0(R)$ can be thought of as generated by equivalence classes of idempotents e in rings of matrices $M_n(R)$ (or one can even combine them all into one ring $M_\infty(R)$ of finitely-nonzero matrices), the equivalence relation corresponding to isomorphism of the ranges considered as projective modules. What is this equivalence relation more explicitly?

Lemma 2.13. *Let $e, f \in M_n(R)$ be idempotents. The ranges of e and f are isomorphic (as projective modules) if and only if there exist $x, y \in M_n(R)$ with $xy = e$ and $yx = f$. If this is the case, we can always find x, y satisfying the additional conditions $xyx = x$ and $yxy = y$.*

Proof. Let $\alpha\colon \mathrm{Im}(e) \to \mathrm{Im}(f)$ be an isomorphism. The composite

$$R^n \xrightarrow{e} \mathrm{Im}(e) \xrightarrow{\alpha} \mathrm{Im}(f) \xrightarrow{\subseteq} R^n$$

is an endomorphism of R^n; let y be the matrix of this endomorphism. Similarly, let x be the matrix of

$$R^n \xrightarrow{f} \mathrm{Im}(f) \xrightarrow{\alpha^{-1}} \mathrm{Im}(e) \xrightarrow{\subseteq} R^n\ .$$

Then x and y have the desired properties. Conversely, if such x and y exist, the identity $ye = fy$ shows that y maps $\mathrm{Im}(e)$ into $\mathrm{Im}(f)$; similarly x maps $\mathrm{Im}(f)$ to $\mathrm{Im}(e)$ and they are mutually inverse isomorphisms.

For the final statement let x, y have $xy = e$ and $yx = f$. Define $x' = xyx = xf = ex$ and $y' = yxy = ye = fy$. Then $x'y' = exye = e^3 = e$ and similarly $y'x' = f$; moreover, $x'y'x' = ex' = e^2x = ex = x'$ and similarly $y'x'y' = y'$, as required. ◻

We say that two idempotents $e, f \in M_n(R)$ are *equivalent* if there exists $x, y \in M_n(R)$ such that $xy = e$ and $yx = f$. (In the context of selfadjoint idempotents and C^*-algebras, a related notion is called *Murray-von Neumann* equivalence.) Idempotents in $M_\infty(R)$ are *equivalent* if there is some $M_n(R) \subseteq M_\infty(R)$ in which they are equivalent. From the above discussion we may then reformulate the definition of K-theory as follows.

[2] Also often called a *projection* — hence, "projective module" — but in this course the word "projection" is reserved for a *self-adjoint* idempotent, $e = e^2 = e^*$, in the context of C^*-algebras

Definition 2.14 (Idempotent Picture)**.** Let R be a ring. Then $K_0(R)$ is the abelian group with one generator for each equivalence class of idempotents in $M_\infty(R)$ and with one relator of the form

$$[e] + [f] = \left[\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \right]$$

for each pair of idempotents e, f.

Remark 2.15. Standard examples (e.g. the unilateral shift) show that the equivalence relation on idempotents in $M_n(R)$ defined above is *not* the same as the relation of similarity (conjugation by an invertible) in $M_n(R)$. However, after *stabilization* by passing to larger matrices these two relations do coincide. Specifically, suppose that e and f are equivalent in $M_n(R)$ via elements x, y satisfying all the conditions of Lemma 2.13. Then $u = \left(\begin{smallmatrix} y & 1-f \\ 1-e & x \end{smallmatrix}\right) \in M_{2n}(R)$ is invertible with inverse $u^{-1} = \left(\begin{smallmatrix} x & 1-e \\ 1-f & y \end{smallmatrix}\right)$, and computation shows that

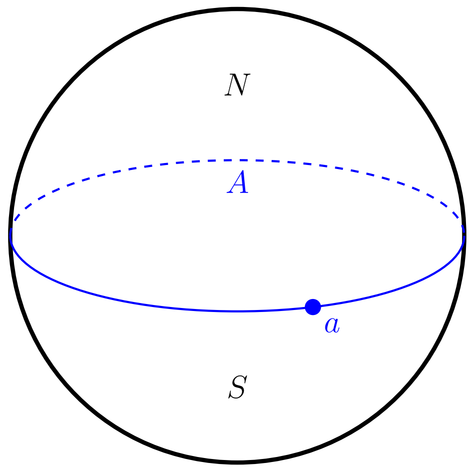
$$u \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} u^{-1} = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix},$$

so that e and f become similar after taking the direct sum with a zero matrix. In particular, we could modify the idempotent picture of K-theory by saying that two idempotents are equivalent if they are similar in some $M_n(R)$, and we would obtain the same result. We will see several similar "two by two matrix tricks" as we proceed.

Exercise: Did you check the algebra here? It is important to do so! (After all, I might screw up.)

Exercise 2.16. Show that every element of $K_0(A)$ can be represented in the idempotent picture by a formal difference $[e] - [f]$ of two idempotents, where one of the idempotents (conventionally f) may be taken to be of the block form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. (This is the matrix counterpart to Exercise 2.10.)

Lecture 3
Clutching projective modules and K_1



In topology, one often constructs vector bundles over a compact space by "gluing" (aka "clutching") together vector bundles over two closed subspaces whose union is the whole space. (Compare Atiyah, *K-Theory*, pages 20–23.) An important example is illustrated in the figure above. The *canonical bundle* H over the unit sphere $S^2 = \mathbb{C} \cup \{\infty\}$ is constructed by identifying two copies of the 1-dimensional trivial (complex) line bundle, one over the southern hemisphere $S = \{z : |z| \leqslant 1\}$ and one over the northern hemisphere $N = \{z : |z| \geqslant 1\} \cup \{\infty\}$, by using the isomorphism "multiplication by z" over points of the intersection $A = N \cap S = \{z : |z| = 1\}$. We will see later in the course that this bundle H is the generator of the *Bott periodicity* isomorphism in (complex) K-theory.

Let us describe the construction more precisely. Let $X = X_1 \cup X_2$ where all of X, X_1, X_2 are compact Hausdorff. Let E_1, E_2 be bundles over X_1, X_2 respectively and let $\varphi\colon E_{1|A} \to E_{2|A}$ be an isomorphism, where A denotes the intersection $X_1 \cap X_2$. Define $E = E_1 \cup_\varphi E_2$ to be quotient of the disjoint union $E_1 \sqcup E_2$ by the equivalence relation identifying each fiber of E_1 over $a \in A$ with the fiber of E_2 over a via the isomorphism φ. There is then an obvious map $E \to X$ whose fibers are vector spaces, and I claim that E is in fact a vector bundle over X. [It is important to realize that this is not completely obvious: the question is how to establish local triviality of E near a point like a in the figure, which is not interior either to X_1 or to X_2. To answer this, since the question is local, we may assume that E_1 and E_2 are trivial so that φ becomes a map $A \to GL(n, \mathbb{C})$. Using the Tietze extension theorem we may extend φ to a map $X \to M_n(\mathbb{C})$, and by openness of $GL(n, \mathbb{C})$ in $M_n(\mathbb{C})$ there is a neighborhood U of a on which the extended φ still takes values in the invertible matrices. It is now easy to use this extended φ to define a trivialization of $E_{|U}$.]

We want to carry out the analogous clutching construction in algebra.

Definition 3.1. A commutative diagram of rings and homomorphisms

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_1 \arrow[d, "\alpha_1"] \\ R_2 \arrow[r, "\alpha_2"] & R_0 \end{tikzcd}$$

is a *pullback square* if R is isomorphic to the ring

$$\{(r_1, r_2) \in R_1 \times R_2 : \alpha_1(r_1) = \alpha_2(r_2)\},$$

equipped with the obvious homomorphisms (coordinate projections) to R_1 and R_2.

The pullback can be characterized by a universal property (briefly stated, it is the limit of the diagram $R_1 \to R_0 \leftarrow R_2$), but I don't think we'll need that. We say that R is "the pullback of R_1 and R_2 over R_0".

Example 3.2. Let X be compact Hausdorff and let X_1, X_2 be closed subspaces with $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = A$. Then $C(X)$ is the pullback of $C(X_1)$ and $C(X_2)$ over $C(A)$. (Exercise: prove this!)

Suppose that we have a pullback of rings as in Definition 3.1. Suppose that M_1, M_2 are (right) modules over R_1, R_2 and that φ is an isomorphism of R_0-modules from $\alpha_{1*}(M_1)$ to $\alpha_{2*}(M_2)$. Then put

$$M = \{(m_1, m_2) \in M_1 \times M_2 : \varphi(\alpha_{1*}(m_1)) = \alpha_{2*}(m_2)\}.$$

A priori, M is an abelian group. However, it can be made into a right R-module in the obvious way, namely $(m_1, m_2) \cdot (r_1, r_2) = (m_1 r_1, m_2 r_2)$. We call this module the *pullback* of M_1 and M_2 over φ; notation, $M_1 \oplus_\varphi M_2$.

Proposition 3.3. *Suppose that we have a pullback square of rings, as above, in which at least one of the homomorphisms α_1, α_2 is surjective. Suppose that M_i is a finitely generated projective R_i-module (for $i = 1, 2$). If φ is an isomorphism of R_0-modules from $\alpha_{1*}(M_1)$ to $\alpha_{2*}(M_2)$, the pullback $M = M_1 \oplus_\varphi M_2$ is a finitely generated projective R-module.*

Remark 3.4. In Example 3.2 above, the surjectivity of both α_1 and α_2 follows from the Tietze extension theorem; you will remember that we needed that theorem, above, to show that topological "clutching" produces a locally trivial bundle.

In order to prove Proposition 3.3, we need to use some facts about lifting invertibles which are frequently important in K-theory. Let $\alpha\colon R \to S$ be an epimorphism of rings. If $x \in S$ is invertible, then surjectivity gives $y \in R$ such that $\alpha(y) = x$; but there may not be an *invertible* y with this property, as easy examples show. If there is, we say that y is *liftable*. The fact that we need is the following.

Lemma 3.5. *Suppose that $\alpha\colon R \to S$ is a surjection and that x is an invertible element of S (or more generally an invertible matrix over S). Then the matrix*

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

is liftable to an invertible matrix over R.

Proof. Note two obvious facts: (a) a product of liftable matrices is liftable, and (b) an upper or lower triangular matrix with 1's down the diagonal is liftable. Given these, the desired result follows from the matrix identity

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(Compare Milnor, *Algebraic K-Theory*, Lemma 2.5.) The argument works fine even for non-square invertible matrices x, which can exist over general rings (Remark 1.8). $\square$

This is, of course, another "two by two matrix trick", sometimes referred to as the *Whitehead trick*.

Proof of Proposition 3.3. We'll begin by establishing a very special case. Suppose that M_1 and M_2 are free, say $M_1 \cong R_1^{n_1}$ and $M_2 \cong R_2^{n_2}$. Then the isomorphism φ is given by a (possibly non-square) invertible matrix x_{ij} with elements in R_0. Suppose further that $x_{ij} = \alpha_1(y_{ij})$ where y_{ij} is an invertible matrix with entries in R_1. Use this matrix to establish an isomorphism between M_1 and $R_1^{n_2}$. If we write things in terms of this new basis, we have $M_i = R_i^{n_2}$ for $i = 1, 2$ and φ becomes the $n_2 \times n_2$ identity matrix acting on $R_0^{n_2}$. It is then evident that $M = M_1 \oplus_\varphi M_2$ is isomorphic to R^{n_2}, i.e., it is a finitely generated free module.

Next, consider the case where $M_1 = R_1^{n_1}$ and $M_2 = R_2^{n_2}$ are still free but we no longer assume that the matrix $X = (x_{ij})$ with entries in R_0 lifts to an invertible matrix with entries in R_1. We may assume however (by the hypothesis of Proposition 3.3) that α_1 is surjective. Then $\begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix}$ is an invertible $(n_1 + n_2) \times (n_1 + n_2)$ matrix over R representing an endomorphism of $R^{n_1+n_2}$. By Lemma 3.5, this matrix is liftable and therefore, by the special case we have already studied, the corresponding module M' over R is free. But M is clearly a direct summand in M', so M is projective.

Finally, in the general case where M_1 and M_2 are projective, choose complementary modules N_1 and N_2 so that $M_i \oplus N_i$ is free over R_i, say isomorphic to R_i^n (by adding extra free summands we can assume that the same n is used for $M_1 \oplus N_1$ and $M_2 \oplus N_2$). I claim that we can also arrange (possibly after increasing n) that $\alpha_{1*}(N_1) \cong \alpha_{2*}(N_2)$. Indeed, we have

$$\alpha_{1*}(M_1) \oplus \alpha_{1*}(N_1) \cong R_0^n \cong \alpha_{2*}(M_1) \oplus \alpha_{2*}(N_1), \quad \alpha_{1*}(M_1) \cong \alpha_{2*}(M_2)$$

as R_0-modules. These give

$$\alpha_{1*}(N_1) \oplus R_0^n \cong R_0^n \oplus \alpha_{2*}(N_2).$$

Thus we can achieve the desired isomorphism $\psi \colon \alpha_{1*}(N_1) \cong \alpha_{2*}(N_2)$ if we replace the original N_i by $N_i \oplus R_i^n$. But now the module

$$(M_1 \oplus N_1) \oplus_{\varphi \oplus \psi} (M_2 \oplus N_2)$$

is projective by the previous case of the proof, and $M = M_1 \oplus_\varphi M_2$ is a direct summand in it and is therefore projective also. $\square$

The discussion above reveals that in order to systematize the construction of projective modules over R by this "gluing" process, we need to think about some structure that classifies *automorphisms* of projective R-modules up to an appropriate notion of equivalence, i.e., invertible linear maps $P \to P$. That structure is $K_1(R)$, sometimes called the *Whitehead group*.

Definition 3.6. Let R be a ring.

(i) $GL(n,R)$ is the group of invertibles in $M_n(R)$. Because of our assumption that R is unital, there is an injection $GL(n,R) \to GL(n+1,R)$ by taking the direct sum with R. The direct limit $\lim\limits_{\to} GL(n,R)$ is called $GL(\infty,R)$, or just $GL(R)$ if I am feeling lazy.

(ii) $E(n,R)$ is the subgroup of $GL(n,R)$ generated by the elementary matrices, i.e. those with 1's down the diagonal and at most a single nonzero entry elsewhere. Similar definition for $E(\infty,R) = \lim\limits_{\to} E(n,R)$.

Lemma 3.7. *The subgroup $E(\infty,R)$ is the commutator subgroup of $GL(\infty,R)$. In particular, it is normal.*

Proof. Every elementary matrix is a commutator of two other elementary matrices (Exercise for the student! There are several cases that need to be checked.) Also, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is a product of elementary matrices. But now suppose that $A, B \in GL(n,R)$. By the calculation in the proof of lemma 3.5, the matrices

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix}, \quad \begin{pmatrix} A^{-1}B^{-1} & 0 \\ 0 & BA \end{pmatrix}$$

belong to $E(2n,R)$ and hence so does their product (in the given order), which is

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

This completes the proof. $\square$

Definition 3.8. The abelian group $K_1(R)$ is defined as the quotient $GL(\infty,R)/E(\infty,R)$.

Remark 3.9. We can regard $E(n,R)$ as those matrices that are invertible for "stupid reasons" (or "elementary reasons" if you want to be polite). Thus $K_1(R)$ measures which invertible matrices are invertible for "stably non-stupid" reasons. The basic theory of linear algebra over a field (Math 220) says that there is a homomorphism (the determinant) $GL(n,k) \to k^\times$ (the nonzero elements of k) and that all matrices with determinant 1 are invertible for stupid reasons; that is, the determinant establishes an isomorphism between $K_1(k)$ and $k^\times$.

Example 3.10. The determinant (again) establishes an isomorphism $K_1(\mathbb{Z}) \to \{\pm 1\}$, because the Smith normal form theorem (applicable to a Euclidean domain such as $\mathbb{Z}$) tells us that every invertible matrix with determinant 1 is a product of elementary matrices.

Lemma 3.11. *Let P be any finitely generated projective R-module and let Q be a complement, i.e. an R-module such that $P \oplus Q \cong R^n$ is free. For any automorphism $\alpha\colon P \to P$, the invertible*

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \in GL(n, R)$$

gives a well-defined element of $K_1(R)$ (i.e. independent of the choice of Q, the choice of n, and the choice of isomorphism $P \oplus Q \to R^n$). Hence we obtain a homomorphism $\mathrm{Aut}(P) \to K_1(R)$.

Proof. Exercise, or see Milnor, p.28 $\square$

Suppose now that we have a pullback square of rings as in Definition 3.1 with at least one homomorphism surjective. Then we can construct a homomorphism $\partial\colon K_1(R_0) \to K_0(R)$ by the following device. Represent an element $u \in K_1(R_0)$ by an invertible matrix $U \in GL(n, R_0)$ and consider U as an isomorphism between the free modules $M_1 = \alpha_{1*}(R_1^n)$ and $M_2 = \alpha_{2*}(R_2^n)$. Using Proposition 3.3, form the finitely generated projective module $M = M_1 \oplus_\varphi M_2$ over R. We define the K-theory class of the difference $[M] - [R^n]$ in $K_0(R)$ to be $\partial[u]$. Of course it is necessary to verify that this process is well defined, but that is not difficult: changing u by an elementary matrix multiplies it by an invertible which is liftable to an automorphism of M_1, and the argument in the proof of Proposition 3.3 shows that this does not change the isomorphism class of the resulting projective module.

Definition 3.12. The homomorphism $\partial\colon K_1(R_0) \to K_0(R)$ defined above is called the *Mayer-Vietoris boundary map* associated to the pullback square of rings.

Proposition 3.13. *Associated to a pullback square of rings (with one arrow surjective) there is an exact Mayer-Vietoris sequence*

$$\begin{tikzcd}
K_1(R) \arrow[r] & K_1(R_1) \oplus K_1(R_2) \arrow[r] & K_1(R_0) \arrow[dll, "\partial"'] \\
K_0(R) \arrow[r] & K_0(R_1) \oplus K_0(R_2) \arrow[r] & K_0(R_0)
\end{tikzcd}$$

in which the maps $K_*(R) \to K_*(R_1) \oplus K_*(R_2)$ and $K_*(R_1) \oplus K_*(R_2) \to K_*(R_0)$ are respectively the sum and difference of induced homomorphisms. $\square$

Proof. This is an exercise, and is best handled by talking extremely fast. $\square$

Lecture 4
More algebraic ideas

In ordinary homology, there are two more or less equivalent ways to understand the long exact sequence: one in terms of Mayer-Vietoris, the other in terms of relative groups associated to a pair (a space and a subspace). We have seen the algebraic analog of Mayer-Vietoris. The algebraic analog of a pair is an *ideal* in a ring.

Let R be a ring (unital, as usual) and let I be a (two-sided) ideal in R. Then the *double* of R along I is defined to be the pullback $D = D(R, I)$ appearing in the square

$$\begin{tikzcd} D \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & R \arrow[d] \\ R \arrow[r] & R/I \end{tikzcd}$$

where the two maps to R/I are quotient maps. Explicitly what this means is that $D = \{(r_1, r_2) : r_1, r_2 \in R,\ r_1 - r_2 \in I\}$, with pointwise operations; the homomorphisms π_i take $(r_1, r_2) \in D$ to $r_i \in R$.

Definition 4.1. The (relative) K-theory groups of I in R are defined as

$$K_j(R, I) = \mathrm{Ker}\big(\pi_{1*} \colon K_j(D) \to K_j(R)\big).$$

Notice that since π_1 is split by the diagonal homomorphism $r \mapsto (r, r)$, there is in fact a direct sum decomposition $K_j(D) \cong K_j(R, I) \oplus K_j(R)$.)

Working algebraically with the Mayer-Vietoris sequence for the pullback square for D, we can extract the exact sequence

$$\begin{tikzcd} K_1(R,I) \arrow[r] & K_1(R) \arrow[r] & K_1(R/I) \arrow[dll, "\partial"'] \\ K_0(R,I) \arrow[r] & K_0(R) \arrow[r] & K_0(R/I) \end{tikzcd}$$

for the relative groups.

The *excision problem* is to determine to what extent $K_j(R, I)$ depends on R and not only on I. We will not talk about this problem in the general algebraic context, but we will later address it for the (topological) K-theory of C^*-algebras, where it has a strong positive answer: $K_*^{\mathrm{top}}(R, I)$ in the C^*-algebra case depends on I only.

Algebra versus topology

Remark 4.2. Although we have introduced a group we are calling K_1, you'll notice that we did not apply it (as in Atiyah-Hirzebruch) to the ring $C(X)$ of continuous functions to produce[3] $K^{-1}(X) = K_1(C(X))$. There is a reason for this. Remember

[3] People in topology often write $K^1(X)$, which involves an implicit invocation of Bott periodicity; $K^{-1}(X)$ is correct (or would be if we were talking about topological K-theory rather than algebraic).

that elements of $K_1(C(Y))$ are supposed to correspond to "clutchings" of (trivial) vector bundles over Y (look back at the figure at the beginning of Lecture 3). Now it is true that clutching two trivial bundles by a map to elementary matrices gives a trivial bundle again (why? because each such map is linearly homotopic to the identity through elementary matrices), so that an element of K_1 as we have defined it does yield a well-defined "clutched" bundle. But the equivalence relation we have used (quotienting by elementary matrices) is *too weak* to be useful in the context of vector bundles and topology. What we should be doing if we are topologists is quotienting not by the *elementary* matrices in $GL(\infty, R)$ but by the *component of the identity* in $GL(\infty, R)$ (considered as a topological group — this presumes a topology on R of course). In other words, we have *two* different though related invariants:

- *Algebraic* K-theory, $K_1^{\mathrm{alg}}(R)$, defined as in the discussion above (Definition 3.8), and
- *Topological* K-theory, $K_1^{\mathrm{top}}(R)$, defined for a topological ring (e.g. a Banach algebra) R, which is the quotient of $GL(\infty, R)$ (considered as a topological group with the direct limit topology) by the connected component of the identity, or more briefly
$$K_1^{\mathrm{top}}(R) = \pi_0(GL(\infty, R)) = \lim_{n\to\infty} \pi_0(GL(n, R)).$$

Since elementary matrices are connected to the identity, there is a natural homomorphism $K_1^{\mathrm{alg}}(R) \to K_1^{\mathrm{top}}(R)$.

Of course we could do the same thing with K_0, defining K_0^{top} by varying the definition of equivalence in the idempotent picture (Definition 2.14): we say two idempotents are *topologically equivalent* if they are connected by a path of idempotents. The surprising fact is that this often does not make any difference:

Proposition 4.3. *Let A be a (unital) Banach algebra. If two idempotents in A are connected by a path of idempotents, then they are conjugate. In fact, any path of idempotents e_t is of the form $e_t = u_t^{-1} e_0 u_t$ where u_t is a path of invertibles,*

Proof. It suffices to show that for any idempotent e there exists $\varepsilon > 0$ such that, if f is idempotent and $\|f - e\| < \varepsilon$, then f and e are conjugate. Let
$$u = ef + (1-e)(1-f);$$
then, clearly, $eu = ef = uf$ so it suffices to show u is invertible. A little algebra gives
$$u = 1 + (1-2e)(e-f);$$
thus, if $\|e - f\| < \varepsilon = \|1-2e\|^{-1}$, we find that u is invertible because of the well-known fact that the ball of radius 1 about the identity, in a Banach algebra, consists of invertible elements. $\square$

Thus, for Banach algebras at least, K_0^{top} and K_0^{alg} are the same thing and we need not bother distinguishing them.

The main focus of this course will be on topological K-theory, specifically for C^*-algebras (which introduces a new element, the involution, into the mix). But we are not going to move away from pure algebra just yet.

Morita equivalence

Let R and S be unital rings.

Definition 4.4. We say that R and S are *Morita equivalent* if there exist an (R, S)-bimodule P and an (S, R)-bimodule Q such that $P \otimes_S Q \cong R$ (as an (R, R)-bimodule) and $Q \otimes_R P \cong S$ (as an (S, S)-bimodule).

Example 1.4 shows that, for any ring R, there is a Morita equivalence between R and $M_n(R)$; the bimodules implementing the equivalence are both R^n, one interpreted as row vectors and one as column vectors.

Example 4.5. (This contains the previous example.) Let R be a ring containing an idempotent e which is *full* in the sense that $ReR = R$. Let $S = eRe$ (the "corner" defined by R). Then S and R are Morita equivalent; the equivalence bimodules are Re and eR.

Remark 4.6. Suppose R and S are Morita equivalent as above. Then for every right R-module M we can functorially obtain a right S-module $M \otimes_R P$, and *vice versa*: the categories of right R-modules and right S-modules are equivalent. Conversely, a theorem of Watts (*Intrinsic characterizations of some additive functors*, Proc. AMS **11** (1960), 5–8; the proof is easy) says that any functor between categories of modules that is right exact and commutes with direct sums—in particular, any equivalence of such categories—is given by tensoring with an appropriate bimodule. Thus, R and S are Morita equivalent iff they have equivalent categories of right (or left) modules, and this is often given as the definition. The bimodule approach is more appropriate when we later come to generalize to C^*-algebras.

Remark 4.7. Suppose that R and S are Morita equivalent, with the equivalence implemented by a bimodule Q. Let $E = \mathrm{End}_{\mathbb{Z}}(Q)$ where we are considering the endomorphisms of Q as an abelian group. The bimodule structure embeds both S^{op} and R as subrings of E which *commute* with each other, i.e., using notation familiar from von Neumann algebra theory, $S^{\mathrm{op}} \subseteq R'$ and $R \subseteq (S^{\mathrm{op}})'$. In fact, I claim that in the case of Morita equivalence, R and S are full commutants of each other, i.e. $S^{\mathrm{op}} = R'$ and vice versa. For the equivalence of categories tells us that S, as an S-module, must be for the form $M \otimes_R Q$, where M is some R-module; and anything of this sort is an R'-module by definition. Thus $S \subseteq R'$ is an R'-module and hence $S = R'$.

In particular it follows that *if R, S are commutative and Morita equivalent, they are isomorphic*. Thus Morita equivalence is a "purely non commutative" phenomenon.

The point of discussing Morita equivalence at all is of course the following theorem.

Proposition 4.8. *A Morita equivalence between rings R and S induces a natural isomorphism $K_j(R) \cong K_j(S)$, $j = 0, 1$, between their (algebraic) K-theories.*

Proof. Let's think about K_0 first. A Morita equivalence from R to S induces an equivalence of categories from R-modules to S-modules. Any property defined "purely in terms of category theory" will be preserved by such an equivalence. Projectivity

(Definition 1.9) is such a property and therefore the equivalence will take projective R-modules to projective S-modules.

We want to refine this statement, to show that it will take *finitely generated* projective R-modules to the same kind of S-modules. To characterize finite generation of projective modules "purely in terms of category theory", observe that a projective R-module M is finitely generated if and only if the functor $\mathrm{Hom}_R(M, -)$ distributes over arbitrary direct sums (think about the proof for a free module first). Thus the Morita equivalence maps fg projective R-modules to fg projective S-modules, and hence (by definition) maps $K_0(R)$ to $K_0(S)$.

In the case of K_1, suppose that $u \in GL(n, R)$ defines an element of $K_1(R)$. Thus, u is an automorphism of the free (hence, projective) R-module R^n. The Morita equivalence turns R^n into a projective S-module N, and turns u into an automorphism $u \otimes 1$ of N. Via the construction of Lemma 3.11 this gives an element of $K_1(S)$. Thus we get a homomorphism $K_1(R) \to K_1(S)$, which has an inverse and so is an isomorphism. $\square$

Lecture 5

Whitehead torsion and the s-cobordism theorem

Before getting deeply into the K-theory of C^*-algebras I want to explain one of the geometric applications of *algebraic* K-theory (there are several). I've chosen to talk about the *s-cobordism theorem*, a crucial part (though only a part) of the structure theory of high-dimensional manifolds which was developed in the 1960s by people like Smale, Stallings, Browder, Novikov, Wall and many others.

Exercise 5.1. Let R be a ring, and let $U \in GL(n, R)$. If there exists a decomposition of R^n (as an R-module) into a direct sum $P_1 \oplus \cdots \oplus P_k$, with the property that $U = I + V$ where

$$V(P_1) = 0, \qquad V\left(\bigoplus_{i=1}^{m} P_i\right) \subseteq \bigoplus_{i=1}^{m-1} P_i \text{ for } m = 2, \ldots, k,$$

then we say that U is *triangularizable*. Prove that if U is triangularizable, then $[U] \in K_1(R)$ is the identity element.

Definition 5.2. Let Γ be a group. The *group ring* $\mathbb{Z}[\Gamma]$ is the free abelian group generated by elements of Γ, with multiplication defined as the unique linear extension of multiplication in Γ. In other words, the elements of $\mathbb{Z}[\Gamma]$ are finite formal linear combinations $\sum n_i[g_i]$, with $n_i \in \mathbb{Z}$ and $g_i \in \Gamma$, and such combinations are added and multiplied in the hopefully obvious way.

The ring $\mathbb{Z}[\Gamma]$ will be noncommutative if Γ is. We are interested in the algebraic K-theory of $\mathbb{Z}[\Gamma]$, or rather of a certain quotient called the *Whitehead group*, defined as follows. Each group element $g \in \Gamma$ is invertible in $\mathbb{Z}[\Gamma]$ and so defines an element of $K_1(\mathbb{Z}[\Gamma])$, and the same goes for $-g$. The elements $\{\pm g\}$ form a subgroup of $K_1(\mathbb{Z}[\Gamma])$ and we define

Definition 5.3. The *Whitehead group of* Γ, $\mathrm{Wh}(\Gamma)$, is the quotient

$$K_1^{\mathrm{alg}}(\mathbb{Z}[\Gamma])/\{\pm g : g \in \Gamma\}.$$

At the moment this is just a bare definition; later on we will see the geometric sense behind it.

Example 5.4. The Whitehead group of the trivial group is trivial; this follows from Example 3.10.

A much harder fact is the following

Example 5.5. The Whitehead group of a free abelian group is trivial. (H. Bass, A. Heller and R. G. Swan, The Whitehead group of a polynomial extension, Inst. Hautes Études Sci. Pubi. Math. No. **22** (1964), 61–79)

Example 5.6. The Whitehead group of any torsionfree hyperbolic group is trivial. (Bartels, Lück and Reich, The K-theoretic Farrell-Jones conjecture for hyperbolic groups, Invent. Math. **172** (2008), no. 1, 29-70.)

These results place the calculation of the Whitehead group firmly in the realm of the *Novikov conjecture* and similar things which operator algebraists know about (sometimes without knowing why. My own study of geometric topology was largely motivated by the desire to stop pretending that I knew why one should be interested in the Novikov conjecture.)

By contrast, Whitehead groups of finite or torsion groups are often nontrivial (and sometimes computable):

Example 5.7. (exercise?) Let Γ be the cyclic group of order 5 and t a generator. Show that $t - 1 - t^{-1}$ is invertible in $\mathbb{Z}[\Gamma]$ and hence defines an element in $\mathrm{Wh}(\Gamma)$. To show this is a nontrivial element, consider the following device. For every matrix over $\mathbb{Z}[\Gamma]$ one obtains a matrix over $\mathbb{C}$ by sending t to the complex number $e^{2\pi i/5}$, and this defines a homomorphism. Show that by taking the absolute value of the determinant of the corresponding matrix we get a homomorphism from $K_1(\mathbb{Z}[\Gamma])$ to the multiplicative group of positive real numbers. Use this trick to show that $t-1-t^{-1}$ is nontrivial in the Whitehead group (in fact, it generates an infinite cyclic subgroup).

Whitehead torsion arises from situations where we have a *contractible* chain complex of $R = \mathbb{Z}[\Gamma]$-modules where the chain modules are *free* and (in some sense) *based*, that is, given a preferred basis. The simplest situation is a chain complex of length two

$$0 \longrightarrow C_k \xrightarrow{d} C_{k-1} \longrightarrow 0\ .$$

We make the assumption that bases are given for C_{k-1} and C_k, making them isomorphic to R^a and R^b respectively. We additionally assume that $a = b$ (notice that this does not follow from the assumption that the complex is contractible, see Remark 1.8). Contractibility now tells us that the map d is an isomorphism, i.e. with our assumptions an element of $GL(a, R)$, and therefore gives an element of $K_1(R)$.

Definition 5.8. The *Whitehead torsion* of the based short complex above is the element of $\mathrm{Wh}(\Gamma) = K_1(R)/\{\pm\Gamma\}$ corresponding to d.

Why do we take the quotient? The modules that arise in practical applications are not *based* in the precise sense that we have assumed here (namely, that they are endowed with a specific choice of basis). Specifically, notice that R, considered as a right R-module, is provided with a number of automorphisms coming from *left* multiplication by elements of Γ.

Definition 5.9. Let's say that a *quasibasis* of a fg free R-module M is a *equivalence class* $\mathfrak{B}$ of (unordered) bases of M, all having the same cardinality[4] denoted $\dim \mathfrak{B}$, where two bases $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ are considered equivalent if there exist a permutation $\sigma \in S_n$, signs $s_1, \ldots, s_n \in \{-1, +1\}$, and group elements $g_1, \ldots, g_n \in \Gamma$ such that

$$x_i = g_i \cdot s_i y_{\sigma(i)} \quad \text{for } i = 1, \ldots, n.$$

[4]We need to specify this explicitly because of Remark 1.8 again.

In the geometric applications we have in mind, our modules are provided with a quasibasis (rather than a basis). Now if we change the basis for C_{k-1} or C_k used in the calculation (Definition 5.8) above, the element of $K_1(\mathbb{Z}[\Gamma])$ that results is changed by multiplying by various g's and their signs and also potentially by an additional sign (the sign of the permutation σ). Considered as an element of $\mathrm{Wh}(\Gamma)$, therefore, the torsion is unaffected by this change of basis. Now we will give the general definition.

Definition 5.10. Let $\mathcal{C} : 0 \to C_n \to \cdots \to C_0 \to 0$ be a finite chain complex of free $\mathbb{Z}[\Gamma]$-modules (the boundary map is denoted d as usual). Assume

(a) Each chain module C_k is provided with a quasibasis $\mathfrak{B}_k$.
(b) The "Euler characteristic" $\sum(-1)^k \dim \mathfrak{B}_k$ is equal to zero.
(c) There is given a chain contraction h of $\mathcal{C}$, in other words a family of module maps $h\colon C_k \to C_{k+1}$ such that $dh + hd = 1$.

Then $V = d + h$, considered as a map from $C_e := \bigoplus_{k \text{ even}} C_k$ to $C_o := \bigoplus_{k \text{ odd}} C_k$, is invertible. If we choose bases for C_e and C_o from $\biguplus_{k \text{ even}} \mathfrak{B}_k$ and $\biguplus_{k \text{ odd}} \mathfrak{B}_k$ respectively, then, V defines an element of $K_1(\mathbb{Z}[\Gamma])$. The equivalence class of this element in $\mathrm{Wh}(\Gamma)$ is independent of the choice of basis (from the given quasibasis) and of the choice of contacting homotopy h. It is called the *Whitehead torsion* $\tau(\mathcal{C})$ of the quasibased contractible complex $\mathcal{C}$.

Proof. This is a definition but it also includes various assertions that have to be proved. Specifically:

$d + h$ is invertible. By calculation,

$$(d + h)^2 = d^2 + (dh + hd) + h^2 = 1 + h^2.$$

Since h^2 shifts degree (by 2) in a finite chain complex, it is nilpotent, so $1 + h^2$ is invertible. It follows that $d + h$ is invertible.

The definition of $\tau(\mathcal{C}) \in \mathrm{Wh}(\Gamma)$ is independent of the choice of basis. Notice that the assumption of zero Euler characteristic means that the chosen bases of C_e and C_o must have the same cardinality. The rest of the argument is the same as the discussion of the special case following Definition 5.8.

The definition of $\tau(\mathcal{C}) \in \mathrm{Wh}(\Gamma)$ is independent of the choice of contracting homotopy h. Suppose that h' is another contracting homotopy. Then $h' = h + r$, where $r\colon C_k \to C_{k+1}$ is a family of maps such that $dr + rd = 0$. Now compute

$$U = (d + h + r)(d + h)^{-1} = 1 + r(d + h)^{-1} = \\ 1 + r(d + h)(1 - h^2 + h^4 - \cdots) = 1 + rd + T,$$

where T contains only terms of positive degree. Now since we have a finite acyclic complex of free $\mathbb{Z}[\Gamma]$-modules, the submodules $Z_k = \mathrm{Ker}(d_k) = \mathrm{Im}(d_{k+1})$ are projective and complemented, say $C_k = Z_k \oplus Y_k$. (Proof by induction on k, starting with $Z_0 = C_0$ which is free.) Looking at the operator $1 + dr + T$, we note that $dr = rd$ maps $Z_k \oplus Y_k$ to Z_k and maps Z_k to zero. Thus, the operator U is triangularizable (Exercise 5.1) and so gives the identity element in $K_1(\mathbb{Z}[\Gamma])$. □

Let us now consider the geometric context in which these ideas will be applied. We work in the context of smooth manifolds. A *cobordism* is an oriented manifold W whose boundary ∂W is divided into two connected components, W^+ and W^- (either one may be empty); we call it a cobordism *between* W^+ and W^-, and we say that W^+ and W^- are *cobordant*. A simple gluing construction shows that "being cobordant" is an equivalence relation. A *product cobordism* is one of the form $M \times [0,1]$ (in this case the "top" and "bottom" boundaries are the same, namely M. An *h-cobordism* is a cobordism for which the inclusions $W^\pm \hookrightarrow W$ are homotopy equivalences. Clearly, a product cobordism is an h-cobordism.

The notion of h-cobordism was introduced by Thom (circa 1957) into the theory of exotic spheres, as a substitute for the relation of diffeomorphism, which seemed at that time to be inaccessible to algebraic study. This changed a few years later when Steve Smale proved the following result:

Theorem 5.11 (h-cobordism theorem)**.** *Let W be a* simply-connected *h-cobordism of dimension at least 6. Then W is diffeomorphic to a product cobordism.*

Reference: S. Smale, *On the structure of manifolds* Amer. J. Math., 84 (1962) pp. 387-399. When Smale discovered this result he was an NSF postdoc working at IMPA in Brazil, and in fact he wrote that the inspiration for the h-cobordism theorem came to him "on the beaches of Rio". When he later became a professor in Berkeley, Smale was a committed antiwar activist and was subpoena'd by the House Un-American Activities Committee while he was on his way to Moscow to accept the Fields Medal at the International Congress of Mathematicians in 1966. To quote Smale, "The subsequent press conference I held in Moscow attacking US policies in the Vietnam War (as well as Russian intervention in Hungary) created a long lasting furore in Washington, DC." During this "furore", it was discovered that Smale had spent US taxpayer's money thinking about (apparently) incomprehensible mathematics on Copacabana Beach, which led President Johnson's science adviser to denounce him publicly in the pages of the journal *Science*. (See S. Smale. *The story of the higher dimensional Poincare conjecture: what actually happened on the beaches of Rio.* Mathematical Intelligencer, **12**:44–51, 1990. Applications to the contemporary context may be drawn at the reader's discretion.)

Let's talk a bit about the proof of Theorem 5.11. The first step is to use Morse Theory to provide a *handle decomposition* of the h-cobordism W (of dimension $n+1$, say). We equip W with a Morse function h (thought of as the "height" above the bottom end W^-). Then W is obtained from the trivial cobordism $W^- \times [0,1]$ by adding "handles" corresponding to the critical points of the Morse function h (the handle corresponding to a critical point of index i is a product $D^i \times D^{n+1-i}$ of disks. Associated to this decomposition there is a chain complex, the *handle complex*, that computes the relative homology of (W, W^-). The k'th chain group has one generator for each critical point of index k, and the matrix entry for the boundary map between a given k-cell and a given $(k-1)$-cell is the number of flow lines of the Morse function that start at the k-cell and end at the $(k-1)$-cell.

So, we choose a Morse function, and we obtain a handle decomposition whose associated chain complex computes the relative homology of (W, W^-), which is *trivial* because of the h-cobordism hypothesis. And remember, this is a chain complex over the integers. Every invertible matrix over the integers is invertible for stupid reasons (cf Remark3.9), and generalizing this a bit we can say that every acyclic chain complex over the integers is acyclic for stupid reasons—more exactly, there is a sequence of elementary algebraic operations that will reduce it to the trivial complex. Smale's idea was this: if we can show that each of these elementary algebraic operations can be implemented by a topological operation on (W, h) (the cobordism together with the decomposition provided by its Morse function), then in the end we will show that W is obtained from W^- by adding no handles at all, i.e. it is a product[5].

Example 5.12. (Handle cancelation). I'll talk and draw pictures about this. . .

Though the idea is easy to state, the details are long and difficult. And in particular the fundamental group $\Gamma = \pi_1(W)$ is involved in a way that the discussion above does not make very clear. There is a crucial point in the argument where it is necessary to use the fact that the coefficients in the boundary matrices—which are basically intersection numbers, the algebraic count of the number of intersection points of certain submanifolds of complementary dimensions— can be "realized geometrically". For instance, in the case of handle cancelation, we want to be sure that a coefficient of 1 really does represent exactly *one* intersection point or flow line, and not say three, two positively oriented and one negatively oriented. There is a standard device called the *Whitney trick* for arranging this in high dimensions, but it depends critically on being able to span any embedded circle by an embedded disc, i.e., on simple connectivity. When we are in the non simply connected world, we can use the Whitney trick only if we count intersection numbers over $\mathbb{Z}[\Gamma]$, not over $\mathbb{Z}$. And this brings torsion into the picture.

Let W be an h-cobordism with fundamental group Γ. Equip it with a handle decomposition, then lift the decomposition to the universal cover $\widetilde{W}$. If we think about the corresponding chain complex, it can now be regarded as a chain complex of free $\mathbb{Z}[\Gamma]$ modules. To give a basis, one has to choose a lift in $\widetilde{W}$ for each handle of W. These lifts are arbitrary up to translation by an element of Γ and choice of orientation, so what we have is exactly a *quasibasis* for the chain modules in the sense of Definition 5.9. The other hypotheses of Definition 5.10 also apply to this chain complex of $\mathbb{Z}[\Gamma]$-modules, so we end up with a well-defined *torsion* $\tau(W) \in \mathrm{Wh}(\Gamma)$. Moreover, we know that if the torsion is nonzero, Smale's approach to the h-cobordism theorem will not suffice for a proof, because we need to use something non-stupid to show the contractibility of the handle complex (on the universal cover).

[5]Smale works with topological operations on the handles; Milnor's presentation, in his lectures, involves modifying the Morse function h. These amount to the same thing. I find Milnor's approach more appealing, but it is technically harder to implement.

Theorem 5.13 (s-cobordism theorem: M.A. Kervaire, *Le théorème de Barden-Mazur-Stallings.* Comment. Math. Helv., **40**:31-42, 1965.)**.** *In dimensions* $\geqslant 6$*, a possibly non-simply-connected h-cobordsim is a product if and only if its torsion vanishes.*

Lecture 6

K-theory for C^*-algebras: beginnings

From this point on we are going to focuss our attention on K-theory for *C^*-algebras.* By definition, a C^*-algebra is an involutive Banach algebra which is isometrically $*$-isomorphic to an algebra of bounded operators on a Hilbert space. In most contexts (e.g. in the 2015 notes) "C^*-algebra" means by default a *complex* C^*-algebra, i.e., a complex Banach algebra represented on a complex Hilbert space. Such algebras can be abstractly characterized as complex involutive Banach algebras satisfying the *C^*-axiom* $\|x^*x\| = \|x\|^2$. We may later have occasion to discuss *real* C^*-algebras — these are important in some applications of index theory. The abstract characterization of such algebras is more complicated (to put it another way, a real involutive Banach algebra satisfying the C^*-axiom need *not* be a real C^*-algebra.)

A C^*-algebra is *unital* if it has a unit element 1. Non unital C^*-algebras are important and we will need to extend the definition of K-theory to them. For now though let us think about unital C^*-algebras. The K_0 and K_1 groups for C^*-algebras are by definition *topological* K-theory groups (see Remark 4.2): $K_1^{\rm top}(A)$ is $GL(\infty; A)$ modulo the subgroup of connected components, and $K_0^{\rm top}(A)$ is generated by idempotents in $M_\infty(A)$, with direct sum as addition and "homotopy" (being in the same connected component) as the equivalence relation. Notice that these definitions make no mention of the involution. A useful preliminary result for C^*-algebras is that we can always arrange for K-theory elements to respect the involution in an appropriate sense. Recall

Definition 6.1. A *projection* in a C^*-algebra A is a self-adjoint idempotent, i.e. an element p such that $p = p^* = p^2$. A *unitary* in a unital C^*-algebra is an invertible whose inverse is its adjoint, i.e. an element u such that $uu^* = u^8u = 1$.

Proposition 6.2. *Let A be a unital C^*-algebra.*

(a) The group $K_0(A)$ is generated by equivalence classes of projections in $M_\infty(A) = \lim_{n\to\infty} M_n(A)$, with two projections being considered equivalent if they are connected by a path of projections in some $M_n(A)$, and with the relations $[p] + [q] = [p \oplus q]$;

(b) The group $K_1(A)$ is the group of equivalence classes of unitaries in $GL_\infty(A) = \lim_{n\to\infty} GL_n(A)$, with two unitaries being considered as equivalent if they are connected by a path of unitaries in some $GL(n, A)$.

The direct limits here are taken in a purely algebraic sense, as in our earlier discussions. We will talked about *completed* direct limits (that is, direct limits in the category of C^*-algebras) later.

Proof. Let e be an idempotent in a unital C^*-algebra A and put $x = 1+(e-e^*)(e^*-e)$. Then x is of the form $1 +$ something positive, so it is invertible, and $ex = ee^*e = xe$ so both e and e^* commute with x. Let $p = ee^*x^{-1}$, which is selfadjoint, and note that since $ee^*x = e(e^*ee^*) = (ee^*)^2$ we have

$$p = (ee^*)x^{-1} = (ee^*x)x^{-2} = (ee^*)^2x^{-2} = (ee^*x^{-1})^2 = p^2;$$

that is, p is a projection. Moreover, $ep = p$ and $pe = ee^*ex^{-1} = ex \cdot x^{-1} = e$, so p and e are equivalent in the sense of Lemma 2.13 and so define the same K-theory class. Notice also that if e already *was* a projection then $p = e$.

The process we described for passing from idempotents to projections (in A or in a matrix algebra over it) uses only the functional calculus, so by the continuity of the functional calculus (2015 notes, Proposition 12.9) a continuous path of idempotents will be transformed into a continuous path of projections. Thus we have shown that every K_0 class is represented by a projection and a continuous path through idempotents between two given projections can be transformed into a continuous path through projections. This completes the proof of part (a) of the proposition.

As for part (b), this employs a similar device but a different functional calculus formula. If v is an invertible in A then $u = v(v^*v)^{-1/2}$ is a unitary (in fact, it is the unitary part in the polar decomposition of v, see 2015 notes, Definition 7.3). As the parameter t runs from 0 to 1 the path

$$v_t = v(v^*v)^{-t/2}$$

gives a homotopy through invertibles between v and the unitary u. The rest of the argument now proceeds as before. $\square$

Our homotopical picture of K-theory naturally leads to an emphasis on *lifting* properties.

Lemma 6.3. *Let $\pi\colon A \to B$ be a surjective homomorphism of unital C^*-algebras.*

(a) Let $P(A)$ and $P(B)$ denote the spaces of projections in A and B respectively. Then $\pi\colon P(A) \to P(B)$ has the path-lifting property: given a path $\gamma\colon [0,1] \to B$ and an element $a_0 \in A$ with $\pi(a_0) = \gamma(0)$, there exists a path $\tilde{\gamma}\colon [0,1] \to A$ with $\tilde{\gamma}(0) = a_0$.

(b) Let $U(A)$ and $U(B)$ denote the unitary groups in A and B respectively. Then $\pi\colon U(A) \to U(B)$ has the path-lifting property.

Proof. We'll do part (b) first. By a compactness argument, it's enough to establish that a small path segment can be lifted, say the segment from $t = 0$ to $t = \varepsilon$, and without loss of generality (since the unitaries form a group) we may take $\gamma(0) = 1$. Then there is $\varepsilon > 0$ so small that -1 does not belong to the spectrum of $\gamma(t))$ for all $t \in [0, \varepsilon]$, hence $\gamma(t) = \exp(ix_t)$ for some continuous path of self-adjoints $x_t = \log\gamma(t)$ (choosing a branch of the logarithm on $\mathbb{C} \setminus \mathbb{R}^-$ and using the continuity of the functional calculus). But there is no problem in lifting a path of selfadjoints in B to a path of selfadjoints in A—choose any continuous lift[6] y_t and if it is not selfadjoint replace it by $\frac{1}{2}(y_t + y_t^*)$. Now $\tilde{\gamma}(t) = \exp(iy_t)$ defines the desired lift of γ.

Part (a) actually follows from this. Recall 4.3 that any path of idempotents is the conjugate of a fixed idempotent by a path of invertibles. Using the functional calculus as in Proposition 6.2(b), it is easy to check (exercise!) that any path of *projections*

[6]A continuous lift always exists by the Bartle-Graves selection theorem, but one does not need to deploy such heavy machinery here. Consider the induced map $C([0,1], A) \to C([0,1], B)$; it is a $*$-homomorphism between C^*-algebras, and it obviously has dense range, so it is surjective.

is the conjugate of a fixed projection by a path of *unitaries*. Thus, path-lifting for projections follows from path-lifting for unitaries. $\square$

Continuing to develop the technology of C^*-algebra K-theory, suppose we have a short exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0\ ,$$

where for the moment the C^*-algebra A is unital. Then the technology of Lecture 4 gives us a "$5\frac{1}{2}$-term" exact sequence

$$\begin{tikzcd}
K_1(A,I) \arrow[r] & K_1(A) \arrow[r] & K_1(A/I) \arrow[dll, "\partial"'] \ , \\
K_0(A,I) \arrow[r] & K_0(A) \arrow[r] & K_0(A/I)
\end{tikzcd}$$

where the relative groups are defined (Definition 4.1) using the dual algebra $D = \{(a_1, a_2) \in A : a_1 - a_2 \in I\}$. We are going to show that the relative groups depend on the ideal I only; in fact, we'll give a definition of K-theory for any non-unital C^*-algebra, and the relative groups will just be the the K-theory of I according to this definition.

Exercise 6.4. (Splitting the exact sequence) Suppose that the short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & A \arrow[r] & A/I \arrow[r] \arrow[l, bend left, dotted] & 0\ ,
\end{tikzcd}$$

is *split* by a $*$-homomorphism. Show that both rows of the $5\frac{1}{2}$-term exact sequence above are in fact split short exact sequences.

Recall that given any non-unital C^*-algebra, such as I, there exists a unital C^*-algebra $\tilde{I}$, called the *unitalization* of I, that contains I as a codimension-1 ideal (2015 notes, Lemma 6.2); that is, there is a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & \tilde{I} \arrow[r] & \mathbb{C} \arrow[r] \arrow[l, bend left, dotted] & 0.
\end{tikzcd} \tag{6.5}$$

This sequence is split by the homomorphism that sends $1 \in \mathbb{C}$ to the unit in $\tilde{I}$. The corresponding homomorphism $\tilde{I} \to \tilde{I}$ that takes each element to its "unit part" will sometimes be denoted by υ.

Definition 6.6. Let I be a non-unital C^*-algebra. We define $K_j(I)$ to be the kernel of the induced homomorphism $K_j(\tilde{I}) \to K_j(\mathbb{C})$.

Exercise 6.7. Show that this definition is functorial for all $*$-homomorphisms, and in the unital case it is canonically isomorphic to the old definition. (Notice that if you put these together you get the slightly non-trivial fact that the K-theory of *unital* C^*-algebras is functorial for *non-unital* homomorphisms.)

Lecture 7
Properties of C^*-algebra K-theory

If I is an ideal in the unital algebra A, then $\tilde{I}$ is a subalgebra of A in a natural way. We can define a $*$-homomorphism $\delta\colon \tilde{I} \to D(A,I)$ by $x \mapsto (\upsilon(x), x)$, where υ is the "unit part" homomorphism as mentioned earlier. By definition of relative K-theory (Definition 4.1), the induced homomorphism δ_* maps $K_j(I) \subseteq K_j(\tilde{I})$ to $K_j(A,I) \subseteq K_j(D(A,I))$.

Proposition 7.1 (Excision Theorem)**.** *The homomorphism*

$$\delta_*\colon K_j(I) \to K_j(A,I),$$

defined above, is an isomorphism.

Proof. We'll prove this in two steps. *Step 1:* prove the result when $A = \tilde{I}$. *Step 2:* show that the choice of A doesn't matter, i.e., that the natural inclusion of algebras $\tilde{I} \to A$ induces an isomorphism $K_j(\tilde{I}, I) \to K_J(A,I)$, whatever A is.

Step 1: Since the short exact sequence 6.5 is split, Exercise 6.4 shows that

$$K_j(\tilde{I}, I) \cong \operatorname{Ker}(K_j(\tilde{I}) \to K_j(\mathbb{C})),$$

and the result follows.

Step 2: Let A be any unital algebra containing I as an ideal, and consider the diagram

$$\begin{tikzcd} D(A,I) \arrow[r, "\beta_1"] \arrow[d, "\beta_2"] & A \arrow[d, "\alpha_1"] \\ A \arrow[r, "\alpha_2"] & A/I \end{tikzcd}$$

An element x of $K_0(D(A,I))$ can be written as a formal difference of projections in some $M_n(D(A,I))$, $x = [(p_1, p_2)] - [(q_1, q_2)]$, where we may assume that the q's represent a free submodule, i.e. they are of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (see Exercise 2.16). Suppose that x belongs to the relative group $K_0(A,I)$. Then, by definition, $\beta_{1*}(x) = [p_1] - [q_1]$ represents the zero element of $K_0(A)$ and thus (possibly after additional stabilization) there is a path of projections joining p_1 to q_1. Using the path-lifting property for projections, applied to the surjective $*$-homomorphism β_1, it follows that there is a path of projections in $M_n(D(A,I))$ from (p_1, p_2) to (p_1', p_2'), where $\beta((p_1', p_2') = p_1' = q_1$. By definition of $D(I,A)$, $p_2' - p_1 \in M_n(I)$, which implies $p_2' \in M_n(\tilde{I})$. Thus x is represented by a projection in $D(I, \tilde{I})$ and hence the map $K_0(\tilde{I}, I) \to K_0(A,I)$ is surjective. An analogous argument applies to K_1.

Now we consider injectivity, and for a change I will discuss the case of K_1. Suppose that $x \in K_1(\tilde{I}, I)$ maps to zero in $K_1(A,I)$. That is to say, x is represented by a pair of unitaries $(u, u') \in M_n(D(\tilde{I}, I))$, and (possibly after stabilization) there is a path w_t of unitaries in $M_n(\tilde{I})$ from $w_0 = u$ to $w_1 = 1$ Moreover, because x maps to zero, there is another path of unitaries (again after suitable stabilization) $(u_t, u'_t) \in M_n(D(A,I))$ from $(u_0, u'_0) = (u, u')$ to $(u_1, u'_1) = (1,1)$. But now consider the path of unitaries (w_t, w'_t) where we define

$$w'_t = w_t u_t^* u'_t.$$

Note that $w_t - w'_t = w_t u_t^*(u_t - v_t) \in M_n(I)$, and $w'_t \in M_n(\tilde{I})$, so (w_t, w'_t) is a path of unitaries in $M_n(D(\tilde{I}, I))$ from (u, u') to $(1,1)$, thus showing that x represents the zero element of $K_1(\tilde{I}, I)$. The K_0 case can be handled in a similar way using the fact that every path of projections is the conjugate of a fixed projection by a path of unitaries, compare the way that part (a) of Lemma 6.3 follows from part (b). □

Remark 7.2. From the excision theorem we obtain a different expression of the exact sequence of K-theory: if

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

is a short exact sequence of C^*-algebras, there is a $5\frac{1}{2}$-term exact sequence

$$\begin{tikzcd} K_1(I) \arrow[r] & K_1(A) \arrow[r] & K_1(A/I) \arrow[d] \\ K_0(A/I) & K_0(A) \arrow[l] & K_0(I) \arrow[l] \end{tikzcd}$$

of K-groups.

Remark 7.3. Writers on complex K-theory conventionally represent the sequence this way with the bottom row "backwards". This is because, as we shall see, Bott periodicity allows us to complete the diagram by putting in a left-hand vertical map going up and so obtaining a six-term cyclic exact sequence. I don't like this convention because it does not work for real K-theory, where one would need a cyclic 24-gon to express the same idea, but it is so common that I felt I ought to use it at least once.

Exercise 7.4. In our discussion of the exact sequence we have assumed that A (and hence A/I) are *unital* algebras. Show that the sequence still works even for non-unital algebras.

Exercise 7.5. The vertical map in the six term exact sequence is called the *index map*. Prove that it can be described in the following way (we assume that A is unital): for each unitary $u \in M_n(A/I)$ one can find a unitary $v \in M_{2n}(A)$ that lifts $\left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$. Let $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_{2n}(\tilde{I})$. Then $q = u^* p u$ also belongs to $M_{2n}(\tilde{I})$ and the formal difference of projections $[p] - [q]$ defines a class in $K_0(I)$. The process sending $[u] \in K_1(A/I)$ to $[p] - [q] \in K_0(I)$ is the index map[7].

Exercise 7.6. (Partial isometry picture of the index map) Suppose (as above) that A is unital and that u is unitary in $M_n(A/I)$. Show that $\left(\begin{smallmatrix} u & 0 \\ 0 & 0 \end{smallmatrix}\right)$ can always be lifted to a *partial isometry* $v \in M_{2n}(A)$. Show that $1 - v^*v$ and $1 - vv^*$ are projections in $\tilde{I}$ and that their formal difference defines an element of $K_0(I)$ which is the index of $[u]$.

[7]At least up to sign—I don't promise I got the signs right here.

Definition 7.7. Let A, B be C^*-algebras and $\alpha_0, \alpha_1 \colon A \to B$ be $*$-homomorphisms. A *homotopy* between α_0 and α_1 is a "continuous path of $*$-homomorphisms connecting them". More precisely, it is a $*$-homomorphism $\alpha \colon A \to C([0,1]; B)$ with $\mathrm{ev}_i \circ \alpha = \alpha_i$ for $i = 0, 1$.

It's often confusing at first that the $[0,1]$ appears at the "right hand" rather than the "left hand" of the homotopy, but this is a manifestation of the contravariance of the functor $X \mapsto C(X)$.

Proposition 7.8. *Homotopic $*$-homomorphisms induce the same map on K-theory.*

Proof. Obvious from the definitions in the unital case; the non-unital case is a simple exercise (from a homotopy $A \to C([0,1]; B)$, with A, B possibly non-unital, produce a homotopy $A^+ \to C([0,1]; B^+)$ between unitalizations). □

Remark 7.9. Let A be a C^*-algebra. If the identity map and the zero map $A \to A$ are homotopic (remember we can now work with non-unital $*$-homomorphisms!), A is said to be *contractible*. For example, the non-unital algebra $C_0[0,1)$ is contractible. From the proposition, it follows that a contractible C^*-algebra has zero K-theory. This is important for many computations later.

Suppose that $A_1 \to A_2 \to A_3 \to \ldots$ is a sequence of C^*-algebras and injective $*$-homomorphisms α_i. Let $\mathcal{A}$ denote the (algebraic) inductive limit of this sequence. (Reminder about what this means: Think of unions. More formally, the elements of $\mathcal{A}$ are equivalence classes of sequences $\{a_i\}$, $a_i \in A_i$, which are required to satisfy $\alpha_i(a_i) = a_{i+1}$ for all but finitely many i, and where two sequences are considered to be equivalent if they differ only in finitely many places. These may be added, multiplied, normed (remember that the α_i are isometric inclusions!), and so on, by pointwise operations.) The algebra $\mathcal{A}$ is a pre-C^*-algebra; that is, a normed algebra which satisfies all the C^*-axioms except that it need not be complete. Its completion is a C^*-algebra which is called the *inductive limit* or *direct limit* (in the category of C^*-algebras) of the given sequence. UHF algebras provide standard examples: see 2015 notes, Lecture 34.

Remark 7.10. One can define C^*-algebraic direct limits without the assumption that the homomorphisms in the sequence are injective, but we won't need them. On the other hand, when the homomorphisms *are* injective we may omit mentioning them (i.e. assume that each A is a subalgebra of all the subsequent ones) if it simplifies notation.

Proposition 7.11. *Suppose that $A_1 \to A_2 \to \ldots$ is a direct sequence of C^*-algebras and injective $*$-homomorphisms, as above, with direct limit A. Then*

$$K_j(A) = \lim_{i\to\infty} K_j(A_i), \qquad (j = 0, 1),$$

with the direct limit on the RHS being taken in the category of abelian groups.

Proof. We didn't say that either the algebras or the homomorphisms have to be unital. However, the non-unital case follows from the unital one by unitalizing everything (exercise!!) so let's assume unitality.

Consider $K_0(A)$. There is a natural map $\varphi\colon \lim_i (K_0(A_i)) \to K_0(A)$ which we will show is an isomorphism. Surjectivity first: let $p \in M_n(A)$ define a class $[p] \in K_0(A)$ (it suffices to show that all such classes belong to $\mathrm{Im}(\varphi)$). By definition of the inductive limit, given any $\varepsilon > 0$ there exists an i, and $a \in M_n(A_i)$ with $a = a^*$ and $\|a - p\| < \varepsilon$. Take ε small enough that $\|a^2 - a\| < \frac{1}{4}$. Then $\frac{1}{2} \notin \mathrm{Spectrum}(a)$; in fact $\mathrm{Spectrum}(a) \subseteq (-\frac{1}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$. let $q \in M_n(A_i)$ be the spectral projection of a corresponding to the interval $(\frac{1}{2}, \frac{3}{2})$. Then $\|q - p\| < \frac{1}{2}$ so q and p define the same element of $K_0(A)$ (see proof of Proposition 4.3) and we have shown that $[p] = \varphi([q])$.

For injectivity, suppose that $p_0, p_1 \in M_n(A_i)$ are projections and that $\varphi([p_0] - [p_1]) = 0$. Then, possibly after stabilization (which we assume done), there exists a path p_t of projections in $M_n(A)$ joining p_0 and p_1. Apply the construction in the previous paragraph to (p_t) to get a path (continuous because of the continuity of functional calculus) of projections q_t in $M_n(A_{i'})$ for some $i' \geqslant i$, where we may choose $a_0 = p_0$ so then $q_0 = p_0$, and similarly $q_1 = p_1$. But this path shows that $([p_0] - [p_1]) = ([q_0] - [q_1])$ vanishes in the algebraic direct limit $\lim_i (K_0(A_i))$; thus φ is injective.

The proof for K_1 (similar) is left as an exercise. $\square$

Example 7.12. The most important example is the sequence of matrix algebras

$$\mathbb{C} \to M_2(\mathbb{C}) \to M_3(\mathbb{C}) \to \dots,$$

where each homomorphism is the (non-unital!) top left corner inclusion $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$. The algebras are all (algebraically) Morita equivalent and the inclusions all induce isomorphisms on K-theory (see Lecture 4, or you can prove this directly). The direct limit is the non-unital C^*-algebra $\mathfrak{K}$ of compact operators. Thus we find that $K_0(\mathfrak{K}) \cong \mathbb{Z}$ and $K_1(\mathfrak{K}) \cong 0$.

Exercise 7.13. Let H be a Hilbert space, $A = \mathfrak{B}(H)$ and $I = \mathfrak{K}(H)$. Show that the index map $\partial\colon K_1(\mathfrak{Q}(H)) \to K_0(\mathfrak{K}(H)) = \mathbb{Z}$ actually computes the Fredholm index of an essentially unitary Fredholm operator (i.e. a unitary element of the Calkin algebra).

Example 7.14. More generally one can consider the sequence

$$A \to M_2(A) \to M_3(A) \to \dots,$$

for *any* C^*-algebra A. The maps still induce isomorphisms on K-theory, and the direct limit is $A \otimes \mathfrak{K}$, so we get the *stability property* of C^*-algebra K-theory: the homomorphism $a \mapsto a \otimes p$, where p is a (fixed) rank one projection, induces an isomorphism

$$K_j(A) \to K_j(A \otimes \mathfrak{K})$$

for any C^*-algebra A. (Rieffel's theory of *strong Morita equivalence* allows us to consider this fact too as following from some kind of Morita equivalence, but we have not studied that yet.)

Lecture 8
Bott periodicity

Our discussion up to this point has shown the following properties of the functors K_j , ($j = 0, 1$), for C^*-algebras:

- Homotopy invariance: homotopic $*$-homomorphisms induce the same map on K-theory. (Proposition 7.8).
- Half exactness: a short exact sequence
$$0 \to I \to A \to A/I \to 0$$
induces $K_j(I) \to K_j(A) \to K_j(A/I)$ which is exact at the middle term (this is a part of the content of Remark 7.2, though by no means the full force of it).
- Stability: tensoring with a rank one projection induces an isomorphism $K_j(A) \to K_j(A \otimes \mathfrak{K})$. (Example 7.14).

We're now going to give Cuntz' remarkable proof that *any* functor E on complex C^*-algebras having these three properties must satisfy Bott periodicity. First, we need to explain a bit what that means.

Definition 8.1. Let A be a C^*-algebra. The *suspension* of A is the C^*-algebra $C_0(0,1) \otimes A$, i.e. the algebra of continuous functions $f\colon [0,1] \to A$ such that $f(0) = f(1) = 0$.

We will be using C^*-algebra tensor products a bit in this proof, and we have the first example above. All the tensor products we work with will have at least one factor nuclear (e.g. $C_0(0,1)$ is commutative and hence nuclear), so it will not matter whether we work with the "minimal" or "maximal" tensor product.

Proposition 8.2. *Let A be any C^*-algebra. There is a natural isomorphism $K_1(A) \cong K_0(SA)$.*

Proof. Consider the short exact sequence
$$0 \to C_0(0,1) \otimes A \to C_0[0,1) \otimes A \to A \to 0.$$
The middle algebra is contractible (Remark 7.9) and thus has zero K-theory. Now the result follows from the $5\frac{1}{2}$-term exact sequence (Remark 7.2). The isomorphism is the index map associated to the short exact sequence. □

Definition 8.3. Let E be any functor from C^*-algebras to abelian groups which is homotopy invariant and half exact. The notation $E_k(A)$ (for $k \in \mathbb{N}$) denotes $E(S^k A)$. (This notation is suggested by Proposition 8.2.)

Proposition 8.4 (Barratt-Puppe sequence)**.** *Let E be any functor from C^*-algebras to abelian groups which is homotopy invariant and half exact. Let*
$$0 \to I \to A \to A/I \to 0$$
be a short exact sequence of C^*-algebras. There is a long exact sequence
$$\ldots \to E_2(A/J) \to E_1(J) \to E_1(A) \to E_1(A/J) \to E(J) \to E(A) \to E(A/J)$$

extending indefinitely to the left.

Proof. The key to this proof is the *mapping cone construction.* Given a short exact sequence of C^*-algebras

$$0 \longrightarrow J \xrightarrow{\iota_1} B \xrightarrow{\pi_1} B/J \longrightarrow 0\ ,$$

this construction produces another short exact sequence

$$0 \longrightarrow S(B/J) \xrightarrow{\iota_2} C(\pi_1) \xrightarrow{\pi_2} B \longrightarrow 0$$

(where $S(B/J)$ is the suspension that we defined in 8.1) with the following properties:

(i) There is a homomorphism $\alpha_1\colon J \to C(\pi_1)$ which induces an isomorphism on E_n for all n.

(ii) The composite $\pi_2 \circ \alpha_1\colon J \to B$ induces the same map on E_n as the inclusion ι_1.

Granted that such a construction exists, we apply it inductively to get a series of short exact sequences $\mathfrak{S}_m$, where $\mathfrak{S}_1$ is the original exact sequence $0 \to I \to A \to A/I \to 0$ and $\mathfrak{S}_{m+1}$ is obtained from $\mathfrak{S}_m$ by the mapping cone construction. Half-exactness associates to each of these exact sequences a half-exact sequence of E-groups:

(1) $E(I) \to E(A) \to E(A/I)$;
(2) $E(S(A/I)) \to E(I) \to E(A)$ (using property (i));
(3) $E(S(A)) \to E(S(A/I)) \to E(I)$;
(4) $E(S(I)) \to E(S(A)) \to E(S(A/I))$;

and so on. Each of these exact sequences has one arrow in common with the one before (e.g. both (1) and (2) contain an arrow $E(I) \to E(A)$); property (ii) says that these arrows represent the same homomorphism. Thus, all these little half-exact sequences can be sewn together to obtain the long exact sequence of the proposition.

It remains to prove the existence of the mapping cone with the properties stated. By definition

$$C(\pi_1) = \{(b,f) : b \in B,\ f\colon [0,1] \to B/J,\ f(0) = 0,\ f(1) = \pi_1(b)\}.$$

The homomorphism α_1 sends $j \in J$ to $(j, 0)$. Clearly, then, there is a short exact sequence

$$0 \longrightarrow J \xrightarrow{\alpha_1} C(\pi_1) \longrightarrow C_0(0,1] \otimes (B/J) \longrightarrow 0$$

The right hand algebra here is contractible, so half-exactness tells us that $E(J) \to E(C(\pi_1))$ is *injective.* (The corresponding result for the higher E-groups is also true and follows by tensoring the above exact sequence with $C_0((0,1)^n)$.) To see that this map is also *surjective* we look at a different exact sequence. Let Q be the C^*-algebra of continuous functions $f\colon [0,1] \to A$ such that $f(0) \in J$. It is easy to see (exercise) that Q and J are homotopy equivalent. But there is clearly a short exact sequence

$$0 \longrightarrow C_0(0,1] \otimes J \longrightarrow Q \longrightarrow C(\pi_1) \longrightarrow 0\ ,$$

and here the *left* hand algebra is contractible. We deduce, from half exactness again, that $E_n(J) \to E_n(C(\pi_1))$ is surjective. This completes the proof. $\square$

Exercise 8.5. Show that if $0 \to I \to A \to A/I \to 0$ is any *split* short exact sequence, then $E_n(A) \cong E_n(I) \oplus E_n(A/I)$.

The following property is useful for K-theory, but in fact is a consequence simply of our axioms.

Lemma 8.6. *Let E be any half-exact and homotopy invariant functor from C^*-algebras to abelian groups. If $\alpha, \beta \colon A \to B$ are* orthogonal *$*$-homomorphisms (meaning that* $\mathrm{Im}(\alpha) \cdot \mathrm{Im}(\beta) = 0$*) then their sum $\alpha + \beta$ is also a $*$-homomorphism and*

$$(\alpha + \beta)_* = \alpha_* + \beta_* \colon E(A) \to E(B).$$

Proof. It's easy to check that orthogonality implies that $(\alpha+\beta)$ is a $*$-homomorphism, and also that $\mathrm{Im}(\alpha+\beta) \subseteq \mathrm{Im}(\alpha) \oplus \mathrm{Im}(\beta)$. We may replace B by $\mathrm{Im}(\alpha) \oplus \mathrm{Im}(\beta)$, so that by Exercise 8.5, $E(B) \cong E(\mathrm{Im}(\alpha)) \oplus E(\mathrm{Im}(\beta))$. Since $(\alpha+\beta)_*$ and $\alpha_* + \beta_*$ both have the same components with respect to this direct sum decomposition, they are equal. $\square$

The key player in Cuntz' proof (and, more or less explicitly, in all proofs of Bott periodicity) is the *Toeplitz algebra* $\mathfrak{T}$. This was discussed in Lecture 20 of the 2015 notes; it is the universal C^*-algebra generated by a single isometry V, and it fits into a short exact sequence

$$0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{T} \xrightarrow{\ \sigma\ } C(S^1) \longrightarrow 0\ .$$

This exact sequence can be explicitly realized in terms of operators on the so-called *Hardy space*, that is the subspace $H^2(S^1)$ of $L^2(S^1)$ spanned by nonnegative powers of $z = e^{i\theta}$. The *Toeplitz operator* with *symbol* $f \in C(S^1)$ is the operator

$$T_f = PM_f \colon H^2(S^1) \to H^2(S^1),$$

where M_f is pointwise multiplication by f and P is the orthogonal projection $L^2(S^1) \to H^2(S^1)$; the Toeplitz algebra is then generated by Toeplitz operators, with the *symbol map* σ sending T_f to f. In this representation, the generating isometry V corresponds to T_z.

Remark 8.7. An *isometry*, let's recall, is any element x of a unital C^*-algebra B such that $x^*x = 1$. The universal property of $\mathfrak{T}$ asserts that there is then a unique unital $*$-homomorphism $\mathfrak{T} \to B$ sending V to x. There is also a non-unital version of this property. A *partial isometry* in B (unital or not) is an element x such that $x^*x = p$ is a projection. Given a partial isometry x there is a unique (non unital) $*$-homomorphism $\mathfrak{T} \to B$ taking V to x. The range of this $*$-homomorphism is contained in the corner pBp.

Since we are going to be working with tensor products we take a moment to explicitly register the fact that the tensor product of the Toeplitz extension with any C^*-algebra A is again an extension (i.e., is a short exact sequence); this follows from Corollary 3.7.4 in Nate and Taka's book.

We are going to compute $E(\mathfrak{T})$, and more generally $E(\mathfrak{T} \otimes A)$, for any functor E satisfying the axioms listed above. Let $\iota \colon \mathbb{C} \to \mathfrak{T}$ be the inclusion of the unit, and let $\sigma_1 \colon \mathfrak{T} \to \mathbb{C}$ be the composite of the symbol map σ with evaluation at $1 \in S^1$ (equivalently, σ_1 is the homomorphism defined by the universal property which takes the generating isometry V to 1).

Proposition 8.8. *The $*$-homomorphisms ι and σ_1 induce mutually inverse isomorphisms $E(\mathbb{C}) \cong E(\mathfrak{T})$. More generally, for any C^*-algebra A, they induce mutually inverse isomorphisms $E(A) \cong E(\mathfrak{T} \otimes A)$.*

Before giving the proof let us see why it gives us the Bott periodicity theorem. Let $\mathfrak{T}_0 = \ker(\sigma_1)$. Then for any A there is a short exact sequence

$$0 \longrightarrow \mathfrak{K} \otimes A \longrightarrow \mathfrak{T}_0 \otimes A \xrightarrow{\ \sigma\ } SA \longrightarrow 0\ ,$$

(identifying $C_0(0,1)$ with the continuous functions on S^1 vanishing at 1) and moreover from Proposition 8.8 it follows that $E_n(\mathfrak{T}_0 \otimes A) = 0$ for all n and A. Thus from the Barratt-Puppe sequence 8.4, the connecting map gives us an isomorphism

$$K_2(A) \equiv K_1(SA) \to K_0(\mathfrak{K} \otimes A) \cong K_0(A),$$

using the stability axiom in the last step. Thus we get

Theorem 8.9 (Bott periodicity)**.** *Let E be any homotopy invariant, half-exact, stable functor from (complex) C^*-algebras to abelian groups. Then there is a natural isomorphism $E_2(A) \cong E(A)$.* $\square$

Proof of Proposition 8.8. Clearly σ_1 is a left inverse of ι, so $E(A)$ is a direct summand in $E(\mathfrak{T} \otimes A)$. The converse direction is what's hard. To avoid too much notation I will write out the proof just when $A = \mathbb{C}$; you can verify afterwards that the entire argument can be "tensored with A".

Consider the following diagram

$$\begin{tikzcd}
 & \mathfrak{T} \arrow[d, "\varphi"] & & & \\
0 \arrow[r] & \mathfrak{K} \otimes \mathfrak{T} \arrow[r, "\alpha"] \arrow[d, "="] & \mathfrak{A} \arrow[r, "\pi"] \arrow[d, "\rho"] & \mathfrak{T} \arrow[r] \arrow[d, "\gamma"] \arrow[l, dotted, bend right, "\beta"'] & 0 \\
0 \arrow[r] & \mathfrak{K} \otimes \mathfrak{T} \arrow[r] & \mathfrak{T} \otimes \mathfrak{T} \arrow[r, "\sigma\otimes 1"] & C(S^1) \otimes \mathfrak{T} \arrow[r] & 0
\end{tikzcd}$$

which uses more Fraktur letters than I have ever used in one place before, and whose constituents I will now attempt to explain. The bottom row is gotten by tensoring the standard Toeplitz extension with another copy of $\mathfrak{T}$. The vertical map γ on the right is defined in terms of the universal property of $\mathfrak{T}$: it sends the generating isometry V to the *unitary* $z \otimes 1 \in C(S^1) \otimes \mathfrak{T}$. The middle algebra $\mathfrak{A}$ is a pullback, as we discussed in Lecture 3 (in other words it is the subset of $(\mathfrak{T} \otimes \mathfrak{T}) \oplus \mathfrak{T}$ consisting of pairs (x, y) with $(\sigma \otimes 1)(x) = \gamma(y)$). This gives us the whole commutative diagram except for the splitting β of the top row, which is defined as follows. Let $e \in \mathfrak{K}$ be the rank one projection $1 - VV^*$ and let $\beta(V) = W := (V(1-e) \otimes 1, V)$ Notice that $V(1-e)$ is a *partial isometry* in $\mathfrak{T}$ and that

$$(\sigma \otimes 1)(V(1-e) \otimes 1) = z \otimes 1 = \gamma(V),$$

Thus W is a partial isometry in $\mathfrak{A}$ and thus β is a non-unital $*$-homomorphism $\mathfrak{T} \to \mathfrak{A}$. This gives a splitting of the top sequence (by definition, $\pi \circ \beta(V)$ is the second component of W, that is V, so $\pi \circ \beta = \mathrm{id}$), which shows in particular that α_* is injective.

The vertical map φ at top left is the stabilization homomorphism $a \mapsto e \otimes a$. By the stability axiom, it induces an isomorphism on E. Remember, what we want to do is to show that the identity (ψ_0) and the homomorphism $\psi_1 = \iota \circ \sigma_1 \colon \mathfrak{T} \to \mathfrak{T}$ induce the same homomorphism on $E(\mathfrak{T})$. We use the following trick. Consider the maps

$$\chi_i = \alpha \circ \varphi \circ \psi_i + \beta \colon \mathfrak{T} \to \mathfrak{A}, \quad (i = 0, 1).$$

Since α and β have orthogonal ranges, these χ_i are $*$-homomorphisms. Suppose we can prove that the χ_i induce the same map on $E(\cdot)$. Then by lemma 8.6 we will have

$$\alpha_* \circ \varphi_* \circ \psi_{0*} + \beta_* = \alpha_* \circ \varphi_* \circ \psi_{1*} + \beta_*.$$

But we have already noted that α_* is injective and φ_* is an isomorphism. It will thus follow that $\psi_{0*} = \psi_{1*}$, the result we want.

The rest of the proof is an Eilenberg swindle type argument to show that $\chi_{0*} = \chi_{1*}$. The second components of χ_{0*} and χ_{1*} are the identity map $\mathfrak{T} \to \mathfrak{T}$, so it suffices to show that their first components, namely the $*$-homomorphisms $\theta_0, \theta_1 \colon \mathfrak{T} \to \mathfrak{T} \otimes \mathfrak{T}$ defined by

$$\theta_0(V) = V(1-e) \otimes 1 + e \otimes V, \quad \theta_1(V) = V(1-e) \otimes 1 + e \otimes 1,$$

are joined by a homotopy θ_t such that $(\sigma \otimes 1)(\theta_t(V)) = z \otimes 1 = \gamma(V)$. It is helpful to represent these two isometries in terms of the standard basis of the $H^2 \otimes H^2$, namely ξ_{ij}, $i, j \geqslant 0$. We then have the following representations: $\theta_0(V)(\xi_{ij}) = \xi_{i,j+1}$ if $i = 0$, and $= \xi_{i+1,j}$ if $i > 0$, and $\theta_1(V)(\xi_{ij}) = \xi_{ij}$ if $i = 0$, and $= \xi_{i+1,j}$ if $i > 0$. Thus we may write

$$\theta_0(V) = F_0(V \otimes 1), \qquad \theta_1(V) = F_1(V \otimes 1),$$

where F_0 and F_1 are *symmetries* (self-adjoint unitaries) defined by

$$F_0(\xi_{ij}) = \begin{cases} \xi_{ij} & (i = j = 0) \\ \xi_{i+1,j-1} & (i = 0, j > 0) \\ \xi_{i-1,j+1} & (i = 1) \\ \xi_{ij} & (i \geqslant 2) \end{cases} \qquad F_1(\xi_{ij}) = \begin{cases} \xi_{i+1,j} & (i = 0) \\ \xi_{i-1,j} & (i = 1) \\ \xi_{ij} & (i \geqslant 2) \end{cases}$$

(Notice that the product of two isometries—in particular, the product of a unitary and an isometry—is an isometry.) We may write

$$F_0 = V(1-e)V^* \otimes 1 + eV^* \otimes V + Ve \otimes V^* + e \otimes e,$$
$$F_1 = V(1-e)V^* \otimes 1 + eV^* \otimes 1 + Ve \otimes 1.$$

These presentations make it clear that $F_0, F_1 \in \mathfrak{T} \otimes \mathfrak{T}$ and that both operators differ from the identity by something belonging to $\mathfrak{K} \otimes \mathfrak{T}$. Now, using the functional calculus, any symmetry in a (complex) unital C^*-algebra can be linked by a path of unitaries to the identity operator; and if the (-1)-eigenprojection of the symmetry belongs to some ideal, the entire path can be taken to consist of unitaries differing from 1 by members of that ideal. In the present case, then, F_0 and F_1 can be linked by a path F_t of unitaries in $(\mathfrak{K} \otimes \mathfrak{T})^+$, with each F_t differing from the identity by an element of $\mathfrak{K} \otimes \mathfrak{T}$. Thus, V_0 and V_1 can be linked by a path $V_t = F_t(V \otimes 1)$ of isometries, each of which maps under $\sigma \otimes 1$ to $z \otimes 1$. Consequently, θ_0 and θ_1 are homotopic through $*$-homomorphisms. As we observed above, this is sufficient to complete the proof. $\square$

Lecture 9
Hilbert Modules

In this lecture and the next few we are going to work towards Rieffel's definition of strong Morita equivalence for C^*-algebras. This will lead us to another picture of K-theory, (for both unital and non unital algebras), which depends on the notion of *Hilbert module* which was discussed in Lecture 10 of the 2015 notes. The definition is as follows:

Let A be a C^*-algebra. A *Hilbert module* over A is a right A-module M equipped with an A-valued $\mathbb{C}$-sesquilinear 'inner product'

$$\langle \cdot, \cdot \rangle \colon M \times M \to A$$

satisfying the following axioms analogous to the usual ones for a Hilbert space:

(i) $\langle x, ya + y'a' \rangle = \langle x, y\rangle a + \langle x, y'\rangle a'$, for all $x, y, y' \in M$ and $a, a' \in A$;
(ii) $\langle x, y \rangle = \langle y, x\rangle^*$;
(iii) $\langle x, x \rangle \geqslant 0$ (the inequality in terms of the order on A_{sa});
(iv) If $\langle x, x\rangle = 0$ then $x = 0$;
(v) M is complete in the norm $\|x\|_M = \|\langle x, x\rangle\|_A^{\frac{1}{2}}$ (one can check that, given (i) through (iv), this really is a norm—this follows from the Cauchy-Schwarz inequality

$$\langle x, y\rangle^* \langle x, y\rangle \leqslant \|\langle x, x\rangle\| \langle y, y\rangle,$$

with respect to the ordering on the positive cone of A.).

Example 9.1. An important example of a Hilbert A-module is the 'universal' module H_A which is comprised of sequences $\{a_n\}$ of elements of A such that $\sum a_n^* a_n$ converges in A. Using the Cauchy-Schwarz inequality it is not hard to show that this is a Hilbert module. Note the convergence condition carefully however: it is not equivalent to say that $\sum \|a_n\|^2$ converges (a series of positive elements of a C^*-algebra can converge in norm without converging absolutely, for instance the series $\sum \frac{1}{n} e_n$, where e_n is the orthogonal projection onto the n'th basis vector, converges in norm but not absolutely in $\mathfrak{K}(\ell^2)$).

Definition 9.2. Let M and N be Hilbert A-modules. An *adjointable map* from M to N is a linear map $T \colon M \to N$ for which there exists an *adjoint* $T^* \colon N \to M$, necessarily unique, satisfying

$$\langle Tx, y\rangle = \langle x, T^* y\rangle, \quad x \in M,\ y \in N.$$

The set of adjointable maps will be denoted $\mathfrak{B}(M, N)$, or $\mathfrak{B}(M)$ if $M = N$.

Proposition 9.3. *Let M be a Hilbert A-module. Then $\mathfrak{B}(M)$ is a C^*-algebra under the operator norm*

$$\|T\| = \sup\{\|T\xi\| : \xi \in E,\ \|\xi\| \leqslant 1\}.$$

Proof. 2015 notes, Proposition 10.7 □

Let E and F be Hilbert A-modules. A *rank one operator* from E to F is a linear map $E \to F$ of the form

$$\theta_{x,y}(z) = x\langle y, z\rangle, \quad x \in F, y \in E. \tag{9.4}$$

It is adjointable, with adjoint $\theta_{y,x}$. The closed linear span of rank one operators is denoted $\mathfrak{K}(E, F)$ and called the space of *compact* operators from E to F. (Warning: They need not be compact in the sense of Banach space theory!) If $E = F$, the subspace $\mathfrak{K}(E)$ of compact operators in $\mathfrak{B}(E)$ is a C^*-ideal, in fact an *essential* ideal (that is, it has trivial annihilator: if $S \in \mathfrak{B}(E)$ has $ST = 0$ for all $T \in \mathfrak{K}(E)$, then $S = 0$.)

Exercise 9.5. Let A be a C^*-algebra. Show that $\mathfrak{K}(H_A) \cong A \otimes \mathfrak{K}$, where H_A is the Hilbert module of Example 9.1.

Lemma 9.6. *Let E be a Hilbert A-module. Then $\mathfrak{B}(E)$ is the* multiplier algebra *of $\mathfrak{K}(E)$; that is to say, it contains $\mathfrak{K}(E)$ as an essential ideal and any $*$-isomorphism $J \to \mathfrak{K}(E)$, where J is an essential ideal in B, extends uniquely to an injective $*$-homomorphism $B \to \mathfrak{B}(E)$.* $\square$

Proof. We covered the case $E = A$ in the 2015 notes; indeed we used it to *construct* multiplier algebras. In the general case we already know that $\mathfrak{K}(E)$ is an essential ideal in $\mathfrak{B}(E)$ and we must show that any D containing $\mathfrak{K}(E)$ as an essential ideal embeds in $\mathfrak{B}(E)$.

Let (u_j) be an approximate unit for $\mathfrak{K}(E)$. For $d \in D$, $k_1, \ldots k_n \in \mathfrak{K}(E)$ and $\xi_1, \ldots, \xi_n \in E$ we have

$$\left\| \sum_{i=1}^{n} (dk_i)\xi_i \right\| = \lim_j \left\| \sum_i (du_j k_i)\xi_i \right\|$$
$$= \lim_j \left\| (du_j) \sum_i k_i \xi_i \right\| \leqslant \|d\| \left\| \sum_i k_i \xi_i \right\|.$$

Thus, the map $\sum k_i \xi_i \mapsto d \sum k_i \xi_i$ is well-defined and continuous. Since vectors of the form $\sum k_i \xi_i$, that is members of $E\langle E, E\rangle$, are dense in E (2015 notes, Lemma 10.5) it follows that multiplication by d extends to a linear map $E \to E$, which is adjointable (its adjoint is multiplication by d^*). Thus we get a map $D \to \mathfrak{B}(E)$, which is injective because of our assumption that $\mathfrak{K}(E)$ is essential in D. We have shown that $\mathfrak{B}(E)$ contains any C^*-algebra having $\mathfrak{K}(E)$ as an essential ideal; so by the universal property of the multiplier algebra, $\mathfrak{B}(E)$ is the multiplier algebra of $\mathfrak{K}(E)$. $\square$

We are going to develop some of the operator theory associated to Hilbert modules (main reference, which we follow closely: Chris Lance, *Hilbert C^*-modules: a toolkit for operator algebraists.*). The reason is that this material is sometimes not familiar to those who want to work on K-theory, it is vital to Kasparov's approach, and it abounds in pitfalls if you are too willing to be guided by the analogy between Hilbert modules and Hilbert *spaces.* Most of these pitfalls arise from the following

Definition 9.7. A closed submodule F of a Hilbert A-module E is *complemented* if $E = F \oplus F^{\perp}$.

Of course, in a Hilbert space every closed subspace is complemented. This is *not true* in the Hilbert module case (easy exercise: find some examples!)

Lemma 9.8. *Let E be a Hilbert A-module and suppose that $T \in \mathfrak{B}(A)$ is selfadjoint and has*

$$\|T\xi\| \geqslant \varepsilon\|\xi\| \quad (\forall \xi \in E),$$

for some constant $\varepsilon > 0$. Then T is invertible.

Proof. Just use the functional calculus. Suppose for contradiction that 0 belongs to the spectrum of T. Let $S = f(T)$ where f is a bump function equal to 1 at 0 (its maximum value) and vanishing outside $(-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon)$. Then $\|S\| = 1$ and $\|TS\| < \frac{1}{2}\varepsilon$. There exists a unit vector $\xi \in E$ with $\|S\xi\| > \frac{1}{2}$; then if we put $\eta = S\xi$ we get $\|T\eta\| = \|TS\xi\| \leqslant \frac{1}{2}\varepsilon < \varepsilon\|\eta\|$, a contradiction. $\square$

Proposition 9.9 (Mischenko's Theorem)**.** *Let E, F be Hilbert A-modules and suppose that $T \in \mathfrak{B}(E, F)$ has closed range. Then* $\ker(T)$ *and* $\operatorname{Im}(T)$ *are complemented (in E and F, respectively). Moreover, T^* also has closed range.*

Proof. Start by considering $\ker(T)$. We may assume wlog (by the closed range hypothesis) that T is surjective. A surjective bounded linear map between Banach spaces is open (the open mapping theorem), which translates into the following estimate: there is a constant C such that for every $\eta \in F$ there exists $\xi \in E$ with $T\xi = \eta$ and $\|\xi\| \leqslant C\|\eta\|$. Suppose now that ξ, η are as described and observe

$$\begin{aligned} \|\eta\|^2 = \|\langle T\xi, \eta\rangle\| = \|\langle \xi, T^*\eta\rangle\| \leqslant \|\xi\|\|T^*\eta\| \\ = \|\xi\|\|\langle \eta, TT^*\eta\rangle\|^{\frac{1}{2}} \leqslant C\|\eta\|^{\frac{3}{2}}\|TT^*\eta\|^{\frac{1}{2}}. \end{aligned}$$

Thus $\|TT^*\eta\| \geqslant C^{-2}\|\eta\|$ for any $\eta \in F$. It follows from Lemma 9.8 that TT^* is invertible. Now for any $\xi \in E$ we may write

$$\xi = (\xi - T^*(TT^*)^{-1}T\xi) + T^*(TT^*)^{-1}T\xi.$$

The first summand belongs to $\ker(T)$ and the second to $\operatorname{Im}(T^*)$. Thus these two spaces, which are obviously orthogonal, are in fact complements of each other. In particular, $\ker(T)$ is complemented.

It looks as though we also proved that $\operatorname{Im}(T^*)$ is closed, but have a care! At the beginning we said that wlog we can assume T is surjective, so what we have really shown at this point is that $\operatorname{Im}(T^*)$ is closed if T is surjective. We need to check that if T_0 denotes the mapping T considered in $\mathfrak{B}(E, \operatorname{Im}(T))$, then $\operatorname{Im}(T_0^*) = \operatorname{Im}(T^*)$. But this is not hard:

$$\operatorname{Im}(T_0^*) \subseteq \operatorname{Im}(T^*) \subseteq \ker(T)^{\perp} = \operatorname{Im}(T_0^*);$$

the first inclusion is because the restriction of T^* to $\operatorname{Im}(T)$ is T_0^*, the second by standard orthogonality, the last equality is what we just proved. Conclusion: $\operatorname{Im}(T^*) = \operatorname{Im}(T_0^*)$ is closed and is the orthogonal complement of $\operatorname{Ker}(T)$. Having got this far, apply everything we have done to T^* (which we now know has closed range) to deduce that $\mathrm{Im}(T)$ is complemented by $\ker(T^*)$. □

Corollary 9.10. *Given any idempotent in $\mathfrak{B}(E)$, there is a projection with the same range.*

Proof. $\mathrm{Im}(e) = \mathrm{Ker}(1-e)$ is closed, so the previous result applies to show that $E = \mathrm{Im}(e) \oplus \mathrm{Im}(e)^{\perp}$,giving the required orthogonal projection. □

As usual, an operator $U \in \mathfrak{B}(E,F)$ (where E, F are Hilbert A-modules) is called *unitary* if $UU^* = 1_F$, $U^*U = 1_E$. Clearly, a unitary is a surjective, isometric A-module map. In the Hilbert space case, these conditions *characterize* unitaries. What about the module case? It turns out that things work the same way there too.

Proposition 9.11. *If $U\colon E \to F$ is a surjective, isometric A-module map, then it is unitary.*

Proof. Recall the A-valued "absolute value" on a Hilbert A-module: $|\xi| = (\langle \xi, \xi \rangle)^{\frac{1}{2}} \in A$. Given U as above, for all $a \in A$,

$$\begin{aligned} \| |U\xi| a \| = \| a^* \langle U\xi, U\xi \rangle a \|^{\frac{1}{2}} &= \| \langle U(\xi a), U(\xi a) \rangle \|^{\frac{1}{2}} \\ &= \| U(\xi a) \| = \| \xi a \| = \| a^* \langle \xi, \xi \rangle a \|^{\frac{1}{2}} = \| |\xi| a \|. \end{aligned}$$

We'd like to infer from this that $|U\xi| = |\xi|$ (as elements of A) for all ξ. If we could do that, then we'd get by squaring and polarizing

$$\langle U\xi_1, U\xi_2 \rangle = \langle \xi_1, \xi_2 \rangle \in A$$

for all $\xi_1, \xi_2 \in E$; and since the assumptions tell us that U is invertible (as a bounded linear map) this shows us that U is adjointable with $U^* = U^{-1}$.

The key to the proof is therefore the following *Claim: If x_1, x_2 are positive elements of A and if $\|x_1 a\| = \|x_2 a\|$ for all $a \in A$, then $x_1 = x_2$.* This is a purely C^*-algebraic statement whose proof is an **Exercise** to the reader who likes such things. □

We're working towards an important criterion for unitary equivalence of Hilbert modules. (There's nothing really like this in the Hilbert space case because all interesting Hilbert spaces are unitarily equivalent anyhow.)

Lemma 9.12. *Let E, F be Hilbert A-modules and let $T \in \mathfrak{B}(E,F)$. Suppose that T has dense range. Then TT^* has dense range also.*

This is obvious in the Hilbert space case because both ranges have the same orthogonal complement, $\ker(T^*)$. Absent the machinery of complements we have to proceed more indirectly.

Proof. Let J and K be the closed one-sided (right) ideals in $\mathfrak{B}(F)$ defined by

$$J = \overline{TT^*\mathfrak{B}(F)}, \qquad K = \overline{T\mathfrak{B}(F,E)}.$$

It will suffice to show that JF is dense in F.

I claim however that KF is dense in F. Indeed, let (u_i) be an approximate unit for the closure of $\langle F, F\rangle$ and let $\xi \in E$. Then $T\xi = \lim_i (T\xi)u_i$ and

$$T(\xi u_i) \in T(E\overline{\langle F, F\rangle}) \subseteq \overline{T(\mathfrak{K}(F, E)F)} \subseteq \overline{KF},$$

giving the required density. It suffices therefore to show that $J = K$. Clearly $J \subseteq K$. For the reverse, consider a state σ on $\mathfrak{B}(F)$ that vanishes on J. Then $\sigma(TT^*) = 0$. Consider the semi inner product on $\mathfrak{B}(E, F)$ defined by

$$(X, Y) \mapsto \sigma(XY^*).$$

By the Cauchy-Schwarz inequality

$$|\sigma(TY)|^2 \leqslant \sigma(TT^*)\sigma(YY^*) = 0.$$

Thus σ vanishes on K. By the Hahn-Banach theorem for states and right ideals[8], $K \subseteq J$. $\square$

Theorem 9.13 (Equivalence theorem)**.** *Let E and F be Hilbert A-modules. Suppose that there exists $T \in \mathfrak{B}(E, F)$ such that T and T^* both have dense range. Then E and F are unitarily equivalent.*

Proof. By Lemma 9.12, T^*T has dense range, and thus so does $|T| = (T^*T)^{\frac{1}{2}}$. Define a map U from $\mathrm{Im}(T)$ to $\mathrm{Im}(|T|)$ by $T\xi \mapsto |T|\xi$ (this is well-defined and A-linear!). For $\xi_1, \xi_2 \in E$,

$$\langle UT\xi_1, UT\xi_2\rangle = \langle |T|\xi_1, |T|\xi_2\rangle = \langle \xi_1, T^*T\xi_2\rangle = \langle T\xi_1, T\xi_2\rangle.$$

Thus U is isometric. Since both T and $|T|$ have dense range, U extends by continuity to an isometric A-linear bijection from F to E. Such a map is a unitary by Proposition 9.11. $\square$

[8] Dixmier, C^*-algbras, 2.9.4. We didn't prove this exact result in the 2015 course, though we got close, especially in lecture 18.

Lecture 10

Hilbert modules (continued)

A Hilbert module is said to be *countably generated* if it has a countable subset that is contained in no proper closed submodule.

Theorem 10.1 (Kasparov stabilization theorem)**.** *Let A be a C^*-algebra and let E be any countably generated Hilbert A-module. Then $E \oplus H_A$ is unitarily equivalent to H_A.*

Proof. One may assume without loss of generality that A is unital. Choose a countable generating set S for E made up of unit vectors, and let (ξ_n) be a sequence of unit vectors in E in which each element of S occurs infinitely often. Let (η_n) be the standard generating set for H_A. Define $T \in \mathfrak{K}(H_A, E \oplus H_A)$ by

$$T = \sum_{n=1}^{\infty} 2^{-n}\theta_{\xi_n,\eta_n} + 4^{-n}\theta_{\eta_n,\eta_n}.$$

For each $s \in S$ there are infinitely many n for which $\xi_n = s$, and for these n,

$$T(2^n\eta_n) = s + 2^{-n}\eta_n;$$

hence s is in the closure of $\mathrm{Im}(T)$; and then since each $\xi_n \in \overline{\mathrm{Im}(T)}$ and

$$\eta_n = T(4^n\eta_n) - 2^n\xi_n$$

it follows that $\eta_n \in \overline{\mathrm{Im}(T)}$ also. Thus, $\mathrm{Im}(T)$ is dense in $E \oplus H_A$. Now consider T^*: we have

$$T^* = \sum_{n=1}^{\infty} 2^{-n}\theta_{\eta_n,\xi_n} + 4^{-n}\theta_{\eta_n,\eta_n},$$

so that $T^*(4^n\eta_n) = \eta_n$ and thus $\mathrm{Im}(T^*)$ is dense in H_A. The result now follows from Theorem 9.13. $\square$

We investigate "finite dimensionality" for Hilbert modules.

Proposition 10.2. *Let E be a Hilbert A-module. Then the following are equivalent:*

(i) E is algebraically *finitely generated (that is, fg as a right A-module, ignoring topology);*

(ii) $\mathfrak{K}(E)$ is unital.

When either of these conditions is satisfied and A is unital, E is projective.

Proof. Suppose that $\mathfrak{K}(E)$ is unital. Let $\mathfrak{F}(E)$ denote the collection of "finite rank" operators, i.e., finite linear combinations of the operators $\theta_{\xi,\eta}$ defined by Equation 9.4. Then $\mathfrak{F}$ is a dense ideal in $\mathfrak{K}(E)$. But a dense ideal in a unital Banach algebra must be the whole thing. Thus the identity operator has "finite rank", and it follows that E is finitely generated.

Conversely, suppose that E is algebraically finitely generated. Then there is a surjective A-module map $T\colon A^n \to E$, of the form $T(a_1, \ldots, a_n) = \xi_1 a_1 + \cdots + \xi_n a_n$, and this map is adjointable with $T^*(\eta) = (\langle \xi_i, \eta\rangle)_{i=1}^n$. By Mischenko's theorem 9.9,

$\operatorname{Ker}(T)$ is complemented in A^n. The restriction of T to $(\operatorname{Ker} T)^\perp$ is an adjointable operator which is a bijection, hence (by Theorem 9.13, easy case) E is unitarily equivalent to pA^n where $p \in M_n(A)$ is an orthogonal projection. (In particular, this tells us that E is projective if A is unital.) Since p is a finite matrix over A it is an A-compact operator, but it is the identity of pA^n, so $\mathfrak{K}(pA^n) \cong \mathfrak{K}(E)$ is unital. □

Remark 10.3. There exist *topologically* finitely generated Hilbert A-modules which are not algebraically finitely generated. However, if a Hilbert module is topologically finitely generated and projective, it is automatically algebraically finitely generated.

The following factorization lemma will be needed in a moment. You may already be familiar with the corresponding result for C^*-algebras: it is "the best shot we can have at a polar decomposition without falling outside the algebra". Remember that if E is a Hilbert A-module and $\xi \in E$, $|\xi|$ denotes the element $\langle \xi, \xi \rangle^{\frac{1}{2}}$ of A.

Lemma 10.4. *For $\xi \in E$ a Hilbert A-module and $0 < \rho < 1$, there exists $\xi' \in E$ such that $\xi = \xi' |\xi|^\rho$.*

Proof. Exercise. (We want to write $\xi' = \xi |x|^{-\rho}$. Construct a sequence of truncations of the unbounded function of $|\xi|$ on the right giving a Cauchy sequence that converges to the desired ξ'.) □

We'll also need some information about matrices of inner products.

Lemma 10.5. *Let E be a Hilbert A-module and let $\xi_1, \ldots, \xi_n \in E$. Then the matrix X of inner products $\langle \xi_i, \xi_j \rangle$ is positive in $M_n(A)$. Moreover, if $T \in \mathfrak{B}(E)$ and W denotes the matrix of inner products $\langle T\xi_i, T\xi_j \rangle$ then $W \leqslant \|T\|^2 X$ in $M_n(A)$.*

Remarks about a proof. The first assertion is Lemma 28.4 of 2015 notes for the case $E = A$, and this case clearly implies the result for $E = H_A$ and thus for any submodule of H_A. By Theorem 10.1 this includes all countably generated E, enough for our purposes. But to get the second statement as well (and to avoid the unnecessary hypothesis about countable generation), I think one has to go back to the direct proof in Lance, lemma 4.2. □

We are going to talk about tensor products of Hilbert modules. There is more than one construction that can bear that name: the one we need is distinguished as the "internal tensor product". The data for this is as follows:

- A Hilbert A-module, E,
- A Hilbert B-module, F,
- A $*$-homomorphism $\varphi \colon A \to \mathfrak{B}_B(F)$.

Out of this data we shall construct a Hilbert B-module $E \otimes_\varphi F$. The basic idea is simple. First, we form the *algebraic* (i.e. no damn functional analysis) tensor product $E \odot F$, thought of as a B-module in the obvious way, and we equip this with the B-valued sesquilinear form defined (on simple tensors) by

$$\langle \xi_1 \odot \eta_1, \xi_2 \odot \eta_2 \rangle = \langle \eta_1, \varphi(\langle \xi_1, \xi_2 \rangle_A) \eta_2 \rangle_B.$$

I claim that

(a) this form is positive semidefinite, and
(b) its kernel $N = \{\zeta \in E \odot F : \langle \zeta, \zeta \rangle = 0\}$ is exactly the subspace of $E \odot F$ spanned by the differences $\xi a \odot \eta - \xi \odot \varphi(a)\eta$ for $\xi \in E$, $\eta \in F$, and $a \in A$.

Once these claims are proved, we know that $(E \odot F)/N$ is an inner product B-module, and its completion is a Hilbert B-module which we denote $E \otimes_\varphi F$ and call the *internal tensor product* of E and F over φ.

Proof of Claim (a). Let $\zeta = \sum_{i=1}^n \xi_i \odot \eta_i \in E \odot F$ and consider $\langle \zeta, \zeta \rangle$. Let X denote the matrix of inner products $\langle \xi_i, \xi_j \rangle$. Then X is a positive element of $M_n(A)$ by Lemma 10.5. Now

$$\langle \zeta, \zeta \rangle = \sum_{i,j} \langle \eta_i, \varphi(\langle \xi_i, \xi_j \rangle)\eta_j \rangle = \langle \hat\eta, \varphi^{(n)}(X)\hat\eta \rangle,$$

where $\hat\eta$ is the vector $(\eta_1, \ldots, \eta_n) \in F^n$ and $\varphi^{(n)}$ is the n-fold inflation of φ. Since φ is a $*$-homomorphism, it is completely positive, so the right hand side is $\geqslant 0$ as required. $\square$

Proof of Claim (b). It is easy to check (exercise!) that all difference elements (of the form $\xi a \odot \eta - \xi \odot \varphi(a)\eta$) belong to N. Conversely suppose $\zeta = \sum_{i=1}^n \xi_i \odot \eta_i \in E \odot F$ belongs to N. Then, in the notation of the previous proof,

$$\langle \hat\eta, \varphi^{(n)}(X)\hat\eta \rangle = 0.$$

Let $T = \varphi^{(n)}(X) \in \mathfrak{B}(F^n)$; then $T \geqslant 0$ and $T^{\frac12}\hat\eta = 0$. Since $|T^{\frac14}\hat\eta|^2 = \langle \hat\eta, T^{\frac12}\hat\eta \rangle$ it follows that $T^{\frac14}\hat\eta = 0$ also.

Now let $\hat\xi = (\xi_1, \ldots, \xi_n) \in E^n$ which we regard as a Hilbert $M_n(A)$-module (the fact that we can so regard it comes from the positivity property of matrices of inner products, Lemma 10.5). In this Hilbert module structure $|\hat\xi| = X^{\frac12}$. By the factorization lemma 10.4 above, then, there exists $\theta = (\theta_1, \ldots, \theta_n) \in E^n$ with $\theta X^{\frac14} = \hat\xi$. Write (a_{ij}) for the matrix elements of $X^{\frac14}$, so that $(\varphi(a_{ij}))$ are the matrix elements of $T^{\frac14}$. We have now

$$\xi_j = \sum_i \theta_i a_{ij}, \quad \sum_j \varphi(a_{ij})y_j = 0,$$

whence

$$\zeta = \sum_{i,j} \left(\theta_i a_{ij} \odot \eta_j - \theta_i \odot \varphi(a_{ij})\eta_j\right),$$

which is a sum of difference elements as required. $\square$

Of course this is a functorial construction: it does right by operators as well as by spaces. Let data be as above and let $T \in \mathfrak{B}_A(E)$. The map defined on simple tensors by $\xi \odot \eta \mapsto T\xi \odot \eta$ extends to a linear map on $E \odot F$. A computation using the second part of Lemma 10.5 shows that this map is bounded (with norm bounded by $\|T\|$) and thus it extends to a bounded B-linear map on $E \otimes_\varphi F$, which is evidently adjointable. Thus we have obtained a unital $*$-homomorphism $T \mapsto T \otimes 1$ from $\mathfrak{B}_A(E) \to \mathfrak{B}_B(E \otimes_\varphi F)$. (We won't need to worry about tensoring with operators on

F other than the identity; if we did try this, they would of course have to respect the left A-module structure coming from φ.)

Lecture 11

Morita equivalence, and graded stuff

Let E be a Hilbert A-module. It is a theorem that $E\langle E,E\rangle$ is dense in E (2015 notes, Lemma 10.5). However, $\langle E,E\rangle$ need *not* be dense in A (easy examples).

Definition 11.1. When $\langle E,E\rangle$ is dense in A, we call the Hilbert module E *full*.

Definition 11.2 (Rieffel)**.** Two C^*-algebras A,B are called *strongly Morita equivalent* if there is a full Hilbert A-module E such that $B\cong\mathfrak{K}(E)$.

Of course, it is not obvious at the moment that this is an equivalence relation, let alone what it has to do with the algebraic Morita equivalence we discussed before. Let's begin elucidating that. Clearly the relation of strong Morita equivalence is reflexive ($A=\mathfrak{K}_A(A)$).

Proposition 11.3. *The relation of strong Morita equivalence is symmetric.*

Proof. Suppose that $B=\mathfrak{K}_A(E)$ where E is a full Hilbert A-module, and let $F=\mathfrak{K}_A(E,A)$. This is a Hilbert B-module with the inner product $\langle x,y\rangle=x^*y\in B$. If $x\in F$ (so that $x\colon E\to A$ is a compact map of Hilbert A-modules), and $a\in A$, then L_ax (where L_a denotes left multiplication by a) belongs to F also, and $x\mapsto L_ax$ is an adjointable B-linear map $F\to F$ with adjoint $x\mapsto L_{a^*}x$. Thus we have defined a $*$-homomorphism $\alpha\colon A\to\mathfrak{B}_B(F)$.

To see that α is injective, suppose that $a\in\ker(\alpha)$. Then $L_ax=0$ for all $x\in F$ and in particular $L_a\theta_{a',\xi}=0$ for all $\xi\in E$, $a'\in A$. Thus $aa'\langle\xi,\xi'\rangle=0$ for all $a'\in A$ and $\xi,\xi'\in E$; since E is full, this implies $a=0$. Thus α is injective.

To see that the range of α is $\mathfrak{K}_B(F)$, let $a,a'\in A$ and $\xi,\xi'\in E$. Let $x=\theta_{a,\xi}\in F$ and define x' similarly. We compute

$$\theta_{x,x'}=\alpha(a\langle\xi,\xi'\rangle a')\in\mathfrak{K}_B(F);$$

to see this, evaluate the result of applying both sides to $(x'')(\xi'')$ where $x''\in F$ and $\xi''\in E$. Since the $\theta_{x,x'}$ generate $\mathfrak{K}(F)$,we see that $\operatorname{Im}(\alpha)\supseteq\mathfrak{K}(F)$. On the other hand, since E is full, the expressions to which α is applied here are dense in A, so that $\operatorname{Im}(\alpha)\subseteq\mathfrak{K}(F)$. Thus $\operatorname{Im}(\alpha)=\mathfrak{K}(F)$ as required.

To complete the proof it remains to show that F is full as a B-module; this is an **exercise**. □

Proposition 11.4. *The relation of strong Morita equivalence is transitive.*

Proof. Suppose that $B=\mathfrak{K}_A(E)$ and that $C=\mathfrak{K}_B(F)$ where E and F are full Hilbert modules over A and B respectively. Let $\varphi\colon B\to\mathfrak{B}_A(E)$ be the inclusion of the compacts in all the adjointable operators and let $G=F\otimes_\varphi E$, using the internal tensor product that we discussed above. We defined there a map $\beta\colon T\mapsto T\otimes 1$ from $\mathfrak{B}_B(F)$ to $\mathfrak{B}_A(G)$. I claim that it restricts to an *isomorphism* from $\mathfrak{K}_B(F)$ to $\mathfrak{K}_A(G)$. Once we know that G is full (another **exercise**) this will finish the proof.

Suppose for a moment that $\eta\in F$ is given. The map $\theta_\eta\colon E\to F\otimes_\varphi E$ given by $\xi\mapsto\eta\otimes\xi$ is bounded and adjointable, with norm less than or equal to $\|\eta\|$; its adjoint is defined on elementary tensors by $\theta_\eta^*(\eta' \odot \xi) = \langle \eta, \eta' \rangle \xi$. Now for all $b \in B$, and $\eta, \eta' \in F$ one computes

$$\beta(\theta_{\eta b, \eta'}) = \theta_\eta b \theta_{\eta'}^*,$$

which is a compact operator on G since b is a compact operator on F. Letting b run over an approximate unit for B we deduce that β maps $\mathfrak{K}_B(F)$ into $\mathfrak{K}_A(G)$. It is easy to see that β is injective (because φ is injective). Finally we consider surjectivity. For $\xi, \xi' \in E$ let $b = \theta_{\xi, \xi'} \in B$ and let $\eta, \eta' \in F$. Computation gives

$$\theta_{\eta\otimes\xi, \eta'\otimes\xi'} = \beta(\theta_{\eta b, \eta'}).$$

So all $\theta_{\eta\otimes\xi, \eta'\otimes\xi'}$ belong to $\mathrm{Im}(\beta)$ and since the closed linear span of these operators is $\mathfrak{K}(G)$ it follows that β is surjective as required. □

We are going to need the terminology of ($\mathbb{Z}/2$) graded or "super" objects eventually; now is as good a time to introduce it as any.

Definition 11.5. A *graded* or *super* C^*-algebra A (the same definition can apply to any ring) is a C^*-algebra A provided with a direct sum decomposition $A = A^{\mathrm{ev}} \oplus A^{\mathrm{odd}}$, where A^{ev} is a subalgebra, A^{odd} is a linear subspace which is a bimodule over A^{ev}, and $A^{\mathrm{odd}} \cdot A^{\mathrm{odd}} \subseteq A^{\mathrm{ev}}$. Equivalently, A is provided with an involutive automorphism (the *grading automorphism*) which acts as the identity on A^{ev} and reverses sign on A^{odd}. If the grading automorphism is inner, then A is called *inner graded* (sometimes this terminology is extended, in the case of non-unital A, to say that A is *inner graded* if the grading arises from an inner automorphism of the multiplier algebra of A).

Example 11.6. Any C^*-algebra can be considered as *trivially* graded with $A^{\mathrm{odd}} = 0$.

Example 11.7. Let A be any ungraded C^*-algebra. The direct sum $B = A \oplus A$ can be graded by saying that B^{ev} consists of elements (a, a) and B^{odd} consists of elements $(a, -a)$; this grading is *not* inner. It is called the *standard grading* on the direct sum. $\mathbb{C} \oplus \mathbb{C}$ with the standard grading is the *Clifford algebra* $\mathbb{C}_1$.

Example 11.8. Let A be any ungraded C^*-algebra. The algebra of 2×2 matrices $M_2(A)$ can be graded by declaring that even and odd subspaces consist of matrices of the following form:

$$\text{even}: \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \qquad \text{odd}: \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

This grading *is* inner if A is unital (or in general, if we use the extended definition of inner grading involving multiplier algebras). It is called the *standard grading* on the 2×2 matrices. $M_2(\mathbb{C})$ with the standard grading is the Clifford algebra $\mathbb{C}_2$.

Example 11.9. The algebra $C_0(\mathbb{R})$ can be graded by declaring that the even and odd subspaces are those functions that are even and odd in the sense of elementary calculus ($f(x) = f(-x)$ for even, $f(x) = -f(-x)$ for odd). Equipped with this grading, the algebra will be denoted $\mathcal{S}$: it will play an important role later.

There is an obvious notion of *graded homomorphism* (i.e. a homomorphism that respects the grading) between graded C^*-algebras. If A is a graded C^*-algebra the elements of $(A^{\mathrm{ev}} \cup A^{\mathrm{odd}}) \setminus \{0\}$ are called *homogeneous* of *degree* 0 (for A^{ev}) or 1 (for A^{odd}). The degree of a homogeneous a will be denoted $\deg a$.

It is natural to define linear operations involving the grading by defining on homogeneous elements first and then extending by linearity. For example, the *graded commutator* is defined on homogeneous elements by

$$[a, b] = ab - (-1)^{\deg(a)\deg(b)} ba.$$

So if a is odd, $[a, a] = 2a$. Graded commutators satisfy graded versions of the usual identities like the Liebniz rule

$$[x, yz] = [x, y]z + (-1)^{\deg(x)\deg(y)} y[x, z]$$

and the Jacobi identity

$$(-1)^{\deg(x)\deg(z)}[[x, y], z] + (-1)^{\deg(y)\deg(x)}[[y, z], x] \\ + (-1)^{\deg(z)\deg(y)}[[z, x], y] = 0$$

for homogeneous x, y and z.

Definition 11.10. A *graded Hilbert module* over a graded C^*-algebra A is a Hilbert A-module E provided with a direct sum decomposition $E = E^- \oplus E^+$, where E^- and E^+ are Hilbert A^{ev}-submodules and $E^{\pm} \cdot A^{\mathrm{odd}} \subseteq E^{\mp}$. (In particular, the inner product of two elements of E^+ or E^- has values in A^{ev}, while the inner product of an element of E^+ and an element of E^- has values in A^{odd}). The order of the summands is part of the data, so that (E^-, E^+) defines a different grading (the *opposite* grading) from (E^+, E^-). We denote E with the opposite grading by E^{op}.

Example 11.11. Suppose that B is a graded C^*-algebra; then it is a graded Hilbert module over itself. More generally, so is H_B, the space of "square summable" sequences from B (Example 9.1). By $\widehat{H}_B$ we will denote the direct sum $H_B \oplus H_B^{\mathrm{op}}$; this is the *universal graded Hilbert module* over B. Note in particular what this gives when B is a trivially graded C^*-algebra.

Definition 11.12. If E is a graded module over a graded algebra A, then the algebras $\mathfrak{B}(E)$ and $\mathfrak{K}(E)$ are also graded: an adjointable operator $T\colon E \to E$ is even if $T(E^{\pm}) \subseteq E^{\pm}$, and is odd if $T(E^{\pm}) \subseteq E^{\mp}$.

Example 11.13. Let A be a trivially graded C^*-algebra. Then $\mathfrak{B}(\widehat{H}_A) \cong M_2(\mathfrak{B}(H_A))$ and $\mathfrak{K}(\widehat{H}_A) \cong M_2(\mathfrak{K}(H_A))$ where the algebras of 2×2 matrices have the standard grading (Example 11.8).

When dealing with tensor products between graded objects, we need the notion of *graded tensor product*, denoted $\widehat{\otimes}$. There is a "graded" version corresponding to every kind of ordinary tensor product, but the common idea is that the objects which ought to *commute* in the ordinary tensor product should *gradedly commute* (that is, have zero graded commutator) in the graded tensor product. So for example if A, B are graded algebras and $a, a' \in A$, $b, b' \in B$ are homogeneous elements, one has

$$(a \widehat{\otimes} b)(a' \widehat{\otimes} b') = (-1)^{\deg(b)\deg(a')}(aa' \widehat{\otimes} bb'),$$

because the a' and the b have been "moved across" one another in the formula. The graded tensor product is then also a graded algebra, with even part $(A^{\mathrm{ev}} \otimes B^{\mathrm{ev}}) \oplus (A^{\mathrm{odd}} \oplus B^{\mathrm{odd}})$ and odd part $(A^{\mathrm{ev}} \otimes B^{\mathrm{odd}}) \oplus (A^{\mathrm{odd}} \oplus B^{\mathrm{ev}})$.

Exercise 11.14. Let $A = \mathbb{C} \oplus \mathbb{C}$ with its standard grading. Show that $A \widehat{\otimes} A \cong M_2(\mathbb{C})$ with *its* standard grading.

Lecture 12
The Fredholm Picture of K-Theory

What we want to do now, in part, is to show that the K-theory of C^*-algebras is invariant under the relation of strong Morita equivalence that we have introduced at the end of the last lecture. For unital algebras this is true by (essentially) the algebraic proof that we gave in lecture 4. Indeed, a Morita equivalence of this sort gives an equivalence between the categories of "finite dimensional" (cf. Proposition 10.2) Hilbert modules over A and B, and we showed that these are just the (algebraically) finite rank projectives, so $K_0(A) \cong K_0(B)$.

But the whole point of Rieffel's definition is to encompass the non unital case, most immediately seen in the Morita equivalence of $\mathfrak{K}$ and $\mathbb{C}$ (via the Hilbert module H, an infinite dimensional Hilbert space). And it is *not* the case that if two algebras are strong Morita equivalent so are their unitalizations (**Exercise:** $\mathfrak{K}$ and $\mathbb{C}$ provide a counterexample). So we can't just "pass to unitalization" to prove the desired result. What it is instead asking us to do is to look at K-theory a different way. This alternative perspective goes back to Atiyah (again!) and generalizes two ideas both of which can be found in his book *K-theory*:

(i) The picture of K-theory for non-compact spaces in terms of chain complexes of vector bundles which are exact outside a compact subset, and
(ii) The fact that the space of Fredholm operators on a Hilbert space is a classifying space for K-theory (i.e., has the homotopy type of $\mathbb{Z} \times BU$).

Let A be a C^*-algebra (possibly graded, but the trivially graded case is the one which will connect up with the previous discussion). Let E be a graded Hilbert module over A.

Definition 12.1. A *Fredholm operator* F on the graded Hilbert module E is an odd operator $F \in \mathfrak{B}(E)$ such that $F - F^*$ and $F^2 - 1$ belong to $\mathfrak{K}(E)$.

Example 12.2. Let $A = \mathbb{C}$ so that $E = E^- \oplus E^+$ is the sum of two ordinary Hilbert spaces. Then F has the form

$$F = \begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix},$$

and from the conditions on F we get $U \sim V^*$, $UV \sim 1$, $VU \sim 1$, where $\sim$ denotes equality modulo the compacts. Thus U is an essentially unitary Fredholm operator from E^- to E^+. (We could think about more general Fredholm operators in the Hilbert module case too—corresponding to "essentially invertible" rather than "essentially unitary"—but these are the only ones we will need.)

Remark 12.3. In the 2015 notes we talked about *Fredholm modules* over a C^*-algebra A (lecture 27). These are *not* the same as the Fredholm operators on Hilbert modules defined above. Reaching for a common generalization of both concepts will give us Kasparov's KK-theory.

We're going to organize the *Fredholm pairs* (F, E), where E is a (countably generated) graded Hilbert module over A and F is a Fredholm operator on it, into a group. A few observations/definitions.

(i) There is an obvious notion of *direct sum* for Fredholm pairs (take the direct sum of the modules and the operators).

(ii) There is also an obvious notion of (unitary) equivalence: two pairs (F, E) and (F', E') are unitary equivalent if there is an even unitary $U\colon E \to E'$ with $UFU^* = F'$.

(iii) There is a covariant functoriality for Fredholm pairs. Let $\alpha\colon A \to B$ be a $*$-homomorphism and (F, E) a Fredholm pair over A. Then $(F \otimes 1, E \otimes_\alpha B)$ is a Fredholm pair over B, where the tensor product is the interior tensor product of Hilbert modules we discussed in the last section, with B considered as a Hilbert module over itself and given a left A-action via α. Note the analogy with the functoriality of algebraic K-theory, Definition 2.3. [If A, B are graded we need to use a graded tensor product here.]

(iv) Finally, there is a notion of *homotopy* of Fredholm pairs: the pairs (F_0, E_0) and (F_1, E_1) over A are *homotopic* if there is a pair over $A \otimes C[0, 1]$ whose images (in the sense of (iii) above) under the evaluation homomorphisms $e_0, e_1\colon A \otimes C[0, 1] \to A$ are unitarily equivalent to the given pairs (F_0, E_0) and (F_1, E_1).

Definition 12.4. Now given a C^*-algebra A, let us form an abelian semigroup $\mathcal{K}(A)$ as follows.

- The elements of $\mathcal{K}(A)$ are homotopy classes of countably generated Fredholm pairs over A (note that, by the way (iv) above is worded, homotopy includes unitary equivalence).
- The semigroup operation is direct sum.

(In Kasparov's theory, this object would be denoted $KK(\mathbb{C}, A)$.)

Notice that this semigroup has a zero element: the pair consisting of the zero module and zero operator [observe that $1 = 0$ as operators on the zero module]. We are going to prove first that $\mathcal{K}(A)$ is a group, and second that, if A is ungraded, then $\mathcal{K}(A)$ is naturally isomorphic to what we earlier called $K_0(A)$, whether A is unital or not. (It was shown by Ruy Exel that the restriction to countably generated modules is unnecessary, but it is required in the path of argument that we will use, as well as in other parts of KK-theory.)

Definition 12.5. A Fredholm pair (F, E) is called *degenerate* if it satisfies $F^2 = 1$, $F = F^*$. A graded module E is called *balanced* if it admits at least one degenerate Fredholm pair.

Lemma 12.6. *Every degenerate Fredholm pair is homotopic to the zero pair (and thus defines the zero element of* $\mathcal{K}(A)$*).*

Proof. Let (E, F) be a degenerate Fredholm pair. Let $E \otimes C[0,1)$ denote[9] the Hilbert module over $A \otimes C[0,1]$ consisting of those continuous functions $[0,1] \to E$ that vanish at 1. Let $F \otimes 1$ denote the obvious extension of F to a (bounded, adjointable) operator on $E \otimes C[0,1)$. Then the pair $(F \otimes 1, E \otimes C[0,1))$ is a homotopy between (F, E) and the zero module. N.B. Be sure you understand **where the hypothesis of degeneracy** is used in this proof! $\square$

Lemma 12.7. *Every Fredholm pair is homotopic to a selfadjoint Fredholm pair (one where $F = F^*$).*

Proof. A linear homotopy $F_t = \frac{1}{2}((2-t)F + tF^*)$ (on the module $E \otimes C[0,1]$) does the job. $\square$

Lemma 12.8. *The semigroup $\mathcal{K}(A)$ is a group. The inverse of (F, E) is $(-F, E^{op})$.*

Proof. Consider the pair $(F \oplus (-F), E \oplus E^{\mathrm{op}})$. Let V be the obvious odd unitary equivalence $E \to E^{\mathrm{op}}$. The homotopy

$$[0, \tfrac{1}{2}\pi] \ni \theta \mapsto \begin{pmatrix} F\cos\theta & V^*\sin\theta \\ V\sin\theta & -F\cos\theta \end{pmatrix}$$

shows that $(F \oplus (-F), E \oplus E^{\mathrm{op}})$ is homotopic to the degenerate pair $(\left(\begin{smallmatrix} 0 & V^* \\ V & 0 \end{smallmatrix}\right), E \oplus E^{\mathrm{op}})$. $\square$

Lemma 12.9. *Let B be a unital C^*-algebra and let w be an isometry in B (that is to say, $w^*w = 1$). Then the homomorphism $\mathrm{Ad}_w \colon B \to B$ defined by $\mathrm{Ad}_w(b) = wbw^*$ induces the identity map on K-theory.*

Proof. If the isometry is unitary, this is immediate from the definition of K-theory. To reduce to the unitary case, let

$$u = \begin{pmatrix} w & 1 - ww^* \\ ww^* - 1 & w^* \end{pmatrix}.$$

This is a unitary in $M_2(B)$. The desired result follows from the commutativity of the diagram

$$\begin{array}{ccc} B & \longrightarrow & M_2(B) \\ \downarrow{\scriptstyle Ad_w} & & \downarrow{\scriptstyle Ad_u} \\ B & \longrightarrow & M_2(B) \end{array},$$

where the horizontal maps are the top left corner inclusions, together with the fact that these horizontal maps induce isomorphisms on K-theory. $\square$

Remark 12.10. In fact, a more general statement is true, though we don't need it here: Suppose that A is a *non*-unital C^*-algebra. if w is an isometry (or unitary) in any unital C^*-algebra B that contains A as an ideal, then $\mathrm{Ad}_w \colon A \to A$ induces the

[9]This is an example of an external tensor product of Hilbert modules, but we can just define it "by hand".

identity map on K-theory. To see this, one can reduce to the unitary case as above, and then observe that there is an explicit path of unitaries connecting $v = \left(\begin{smallmatrix} w & 0 \\ 0 & w^* \end{smallmatrix}\right)$ to the identity, and therefore that Ad_v induces the identity on $M_2(A)$. Now argue as in the proof of Lemma 12.9 with the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "Ad_w"'] & M_2(A) \arrow[d, "Ad_v"] \\ A \arrow[r] & M_2(A) \end{tikzcd}$$

Now we set about proving that, if A is ungraded, $\mathcal{K}(A)$ is the same as the more familiar group $K(A)$ we have already encountered.

Lemma 12.11. *Let A be an ungraded C^*-algebra. Every Fredholm pair over A is homotopic to a balanced Fredholm pair (i.e., a pair in which the module E is balanced). If two balanced pairs are homotopic, the pair which implements the homotopy may also be assumed to be balanced.*

Proof. Exercise. (You can knock this out in one line by appealing to Theorem 10.1, but it is also interesting to give a construction that does not use the stabilization theorem.) $\square$

The point of the lemma is that we would make no change to the definition of $\mathcal{K}(A)$ if we insisted from the start that all Fredholm pairs (and derivative concepts, such as homotopy) had to be balanced.

Suppose now that (F, E) is a balanced Fredholm pair. Then E^- and E^+ are unitarily equivalent: choose a unitary $U \in \mathfrak{B}(E^-, E^+)$. Let V be the component of F that maps from E^+ to E^-. Then $UV \in \mathfrak{B}(E^+)$ is unitary modulo $\mathfrak{K}(E^+)$, and so defines an element $[UV] \in K_1(\mathfrak{Q}(E^+))$, where the "generalized Calkin algebra" $\mathfrak{Q}(E^+)$ is by definition equal to $\mathfrak{B}(E^+)/\mathfrak{K}(E^+)$. That is, there is a short exact sequence

$$0 \longrightarrow \mathfrak{K}(E^+) \longrightarrow \mathfrak{B}(E^+) \longrightarrow \mathfrak{Q}(E^+) \longrightarrow 0\,. \tag{12.12}$$

Now by the stabilization theorem 10.1, there is a unitary $E^+ \oplus H_A \cong H_A$, and thus there is an "inclusion by zero" homomorphism $\mathfrak{K}(E^+) \to \mathfrak{K}(H_A)$. This induces a map on K-theory,

$$\iota\colon K_0(\mathfrak{K}(E^+)) \to K_0(\mathfrak{K}(H_A)) \cong K_0(A),$$

where the latter isomorphism comes from 9.5 and 7.14. (The inclusion $\mathfrak{K}(E^+) \to \mathfrak{K}(H_A)$ is well-defined only up to conjugation by a unitary in $\mathfrak{B}(H_A)$, but such conjugation does not affect the induced map on K-theory by Remark 12.10.)

Definition 12.13. The element $\iota \circ \partial[UV]$, where $\partial\colon K_1(\mathfrak{Q}(E^+)) \to K_0(\mathfrak{K}(E^+))$ is the connecting homomorphism, is called the *index* of the Fredholm pair (F, E), written $\mathrm{Index}(F, E)$.

Proposition 12.14. *The formation of the index gives a well-defined homomorphism $\mathcal{K}(A) \to K_0(A)$.*

Proof. First let's check that the choice of "balancing" U makes no difference. Indeed, a change in this choice corresponds to multiplying by an element of $K_1(\mathfrak{B}(E^+))$. By the exactness of the K-theory sequence, this has no effect once we reach $K_0(\mathfrak{K}(E^+))$. The same argument shows that unitarily equivalent Fredholm pairs have the same index.

The direct sum of Fredholm pairs corresponds to direct sum in $K_1(\mathfrak{Q}(E^+))$, which is one way of representing the addition operation in K_1.

Finally, if two pairs are homotopic, the homotopy between them is a pair over $C[0,1] \otimes A$. Taking the index of the homotopy H gives an element $\mathrm{Index}(H)$ of $K_0(C[0,1] \otimes A)$ which maps to the indices of the two given pairs (in $K_0(A)$) under the evaluation maps at 0 and 1. But by the homotopy invariance of ordinary K-theory, these two images of $\mathrm{Index}(H)$ are the same. $\square$

Now we are going to prove that this homomorphism is an *isomorphism.*

Proposition 12.15. *Let A be a C^*-algebra. Then $K_i(\mathfrak{B}(H_A)) = 0$ for $i = 0, 1$.*

(This is a weak form of *Mingo's theorem*[10] which states that the unitary group of $\mathfrak{B}(H_A)$ is contractible.)

Proof. Write $E_A \equiv H_A \oplus H_A \oplus H_A \oplus$ (infinitely many summands; of course this is isomorphic to H_A itself but it is helpful to make a notational difference). Let $\alpha\colon \mathfrak{B}(H_A) \to \mathfrak{B}(E_A)$ map T to $(T, 0, 0, 0, \ldots)$ and let $\beta\colon \mathfrak{B}(H_A) \to \mathfrak{B}(E_A)$ map T to $(0, T, T, T, \ldots)$; these are $*$-homomorphisms with orthogonal ranges (Lemma 8.6) and thus $(\alpha + \beta)$ is also a homomorphism (sending T to $(T, T, T, T, \ldots)$) and $(\alpha + \beta)_* = \alpha_* + \beta_*$.

Let $W\colon E_A \to E_A$ be the "unilateral shift" mapping each H_A summand to the next one. It is an isometry. Moreover, $\mathrm{Ad}_W \circ (\alpha + \beta) = \beta$. By Lemma 12.9, $\alpha_* + \beta_* = \beta_*$, whence $\alpha_* = 0$. But α_* is an isomorphism (it's the top left corner inclusion of $\mathfrak{B}(H_A)$ into $M_2(\mathfrak{B}(H_A))$ and it follows that $K_i(\mathfrak{B}(H_A)) = 0$ as asserted. $\square$

Theorem 12.16. *Let A be an ungraded C^*-algebra. The index as defined in Proposition 12.14 gives an* isomorphism $\mathcal{K}(A) \to K_0(A)$.

Proof. Consider the exact sequence of Equation 12.12 and the associated long exact sequence of K-theory in the case of the module H_A. Since, as we have just proved, the middle algebra has zero K-theory, the connecting map $\partial\colon K_1(\mathfrak{Q}(H_A)) \to K_0(\mathfrak{K}(H_A)) = K_0(A)$ is an isomorphism. On the other hand, given any element of $K_1(\mathfrak{Q}(H_A))$, that is a unitary u in $\mathfrak{Q}(H_A)$ (or a finite matrix algebra over it—but the finite matrices over $\mathfrak{Q}(H_A)$ are isomorphic to $\mathfrak{Q}(H_A)$ again) we may lift it to an element $V \in \mathfrak{B}(H_A)$ and then form

$$F = \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix}$$

on $H_A \oplus H_A^{\mathrm{op}} = \widehat{H}_A$. Then $(F, \widehat{H}_A)$ is a Fredholm pair whose index, by construction, equals $\partial[u]$. This process defines a homomorphism $\gamma\colon K_1(\mathfrak{Q}(H_A)) \to \mathcal{K}(A)$. Thus we

[10]Mingo gave the proof when A is unital. The general case is due to Cuntz and Higson.

have obtained a commutative diagram

$$\begin{tikzcd} & K_1(\mathfrak{Q}(H_A)) \arrow[dl, "\gamma"'] \arrow[dr, "\partial"] & \\ \mathcal{K}(A) \arrow[rr, "\text{Index}"] & & K_0(A) \end{tikzcd}$$

where ∂ is an isomorphism, and to complete the proof it suffices to show that γ is surjective. But this is an easy consequence of the stabilization theorem (10.1): if (F, E) is any Fredholm pair, its direct sum with the degenerate pair $(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \widehat{H}_A)$ is a pair whose underlying module is unitarily equivalent to $\widehat{H}_A$, hence belongs to the image of γ. □

Let's use this picture of K-theory to show its invariance under strong Morita equivalence.

Proposition 12.17. *Let A and B be strongly Morita equivalent (ungraded) C^*-algebras. Then $K_i(A) \cong K_i(B)$ for $i = 0, 1$.*

With suitable definitions the result is true in the graded case also.

Proof. By using suspensions it suffices to prove the result for K_0, that is, for $\mathcal{K}$. By definition of Morita equivalence, there is a Hilbert B-module M such that $A \cong \mathfrak{K}_B(M)$, say via an injective $*$-homomorphism $\varphi\colon A \to \mathfrak{B}_B(M)$. Suppose that (F, E) is a Fredholm pair over A. I claim that $(F \otimes 1, E \otimes_\varphi M)$ is a Fredholm pair[11] over B. Indeed, let $\beta\colon \mathfrak{B}_A(E) \to \mathfrak{B}_B(E \otimes_\varphi M)$ be the $*$-homomorphism $T \mapsto T \otimes 1$. As shown in the proof of Proposition 11.4, β maps $\mathfrak{K}(E)$ to $\mathfrak{K}(E \otimes_\varphi M)$. Therefore, $\beta(F)^2 - 1 = \beta(F^2 - 1)$ and $\beta(F) - \beta(F)^* = \beta(F - F^*)$ are compact, so $\beta(F) = F \otimes 1$ defines a Fredholm pair as asserted.

The same process converts a homotopy of Fredholm pairs over A to a homotopy of Fredholm pairs over B. Thus we get a homomorphism $\mathcal{K}(A) \to \mathcal{K}(B)$. Its inverse is given by tensoring with the inverse equivalence module (see Proposition 11.3) and so we have an isomorphism. □

[11] E is graded but M, under our conditions, is not; in this easy case there is no problem in defining $E \otimes_\varphi M$ as a graded Hilbert B-module.

Lecture 13
The Spectral Picture

We discuss one special case when the relationship between the Fredholm and "classical" pictures of K-theory can be expressed simply.

Lemma 13.1. *Suppose that A is a unital C^*-algebra and that $E = E^- \oplus E^+$ is a graded Hilbert A-module which is "finite dimensional" in the sense of Proposition 10.2. Then $E^\pm$ are finitely generated projective modules and the index of* any *Fredholm pair (F, E) is equal to $[E^+] - [E^-] \in K_0(A)$.*

Proof. Proposition 10.2 gives us that $E^\pm$ are finitely generated projective and also that $1 \in \mathfrak{K}(E)$, whence all Fredholm pairs are homotopic; so we might as well take $F = 0$. To compute the index of this pair according to our machinery, we must first make it balanced by adding a degenerate Fredholm pair (compare Lemma 12.11): we do this by adding the obvious degenerate pair $((\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}), \widehat{H}_A = H_A \oplus H_A)$ and then appealing to the stabilization theorem 10.1 to see that $E \oplus \widehat{H}_A = \widehat{H}_A$ is balanced. From the construction in the previous lecture, we must now take the element of the generalized Calkin algebra $\mathfrak{Q}(H_A)$ given by the operator

$$W: \; H_A \xrightarrow{\cong} E^+ \oplus H_A \xrightarrow{(0,\mathrm{id})} E^- \oplus H_A \xrightarrow{\cong} H_A \;,$$

and then form its index. But W is clearly a partial isometry on H_A and therefore by Exercise 7.6 its index is the difference of its kernel and cokernel projections, which are the projections onto E^+ and E^- respectively. $\square$

Closely related to the Fredholm picture is what Nigel calls the *spectral picture* of K-theory. This is a version of K-theory which naturally arises in the formulation of "higher index theorems", and we will give an example in this section. Recall from Example 11.9 that $\mathcal{S}$ denotes the C^*-algebra $C_0(\mathbb{R})$, considered as graded by "even and odd functions".

Definition 13.2. Let A be a C^*-algebra. A *spectral pair* over A is a pair (α, E), where E is a graded Hilbert module over A and $\alpha\colon \mathcal{S} \to \mathfrak{K}(E)$ is a graded $*$-homomorphism.

Let's talk briefly about what this definition entails. If E is a Hilbert *space* (not module), and we neglect the grading for the moment, then every $*$-homomorphism $C_0(\mathbb{R}) \to \mathfrak{K}(E)$ is of the form $f \mapsto f(D)$, where D is an *unbounded* selfadjoint operator with *compact resolvent* (which is to say, D has a complete set of eigenvectors with eigenvalues tending to ∞ in absolute value). Putting the grading back in to the picture, a spectral pair over $\mathbb{C}$ corresponds to an unbounded selfadjoint operator on $E = E^- \oplus E^+$ of the form

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix};$$

each of D^-, D^+ is the adjoint of the other. The reader will recognize that this is exactly the information provided by the basic analysis of *generalized Dirac operators* on (even dimensional) compact manifolds. We expect, therefore, that spectral pairs over more compelx Hilbert modules will arise from Dirac operators in more complicated geometric situations: we will deal with an example in a moment.

Remark 13.3. There is a theory of unbounded operators on Hilbert modules (more difficult to construct without orthogonality, but it can be done; see the final chapters of Lance's book). In terms of this theory one can express any spectral pair in the form $f \mapsto f(D)$ used above. We will not need this, however.

Definition 13.4. A *chopping function* χ is an odd, real-valued element of $C[-\infty,\infty]$ which tends to ± 1 at $\pm\infty$.

Let (α, E) be a spectral pair over A. Since $\mathcal{S}$ is an essential ideal in $C[-\infty,\infty]$, and $\mathfrak{B}(E)$ is the multiplier algebra of $\mathfrak{K}(E)$ (Lemma 9.6), it follows that α extends uniquely to a $*$-homomorphism (also denoted α) from $C[-\infty,\infty]$ to $\mathfrak{B}(E)$. Let χ be a chopping function and $F = \alpha(\chi) \in \mathfrak{B}(E)$. Then F is odd, self-adjoint, and $F^2 - 1 \in \mathfrak{K}(E)$; thus, (F, E) is a Fredholm pair. Moreover, the choice of chopping function only affects F by a compact perturbation, so the class of (F, E) in $\mathcal{K}(A)$ is well-determined.

Definition 13.5. We call $(\alpha(\chi), E) \in \mathcal{K}(A) = K_0(A)$, where χ is any chopping function, the *index* of the spectral pair (α, E), written $\mathrm{Index}(\alpha, E)$.

Remark 13.6. We've shown that every spectral pair gives rise to a K-theory class. Of course one can go on and show that every K-theory class arises from a spectral pair, and that in fact K-theory can be *defined* in terms of spectral pairs modulo a suitable equivalence relation. Rather than go through the details of this, let's just state the final result: for any C^*-algebra A there is an isomorphism

$$K_0(A) = [\![\mathcal{S}, \mathfrak{K}(\widehat{H}_A)]\!],$$

where the notation $[\![B, C]\!]$ denotes the space of *homotopy classes of graded $*$-homomorphisms* from B to C. This definition (due to Guentner and Higson) is called the *spectral picture* of K-theory.

For the rest of this section we will consider the following extended example; we assume the basic language of elliptic operator theory as found, for example, in my red book (*Elliptic operators, topology and asymptotic methods*). Suppose that M is a compact, oriented, even-dimensional Riemannian manifold and that $S = S^- \oplus S^+$ is a graded bundle of Clifford modules on M. Associated to the Clifford module structure there is a *Dirac operator* D which is a first order, formally self-adjoint elliptic operator acting on sections of S; the operator D is odd relative to the grading, which is to say that D has the matrix form

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

that appeared above, relative to the decomposuition of $S = S^- \oplus S^+$. The basic analytical results about elliptic operators tell us that D has a unique extension to an unbounded self-adjoint operator on $L^2(S)$, and that this operator has compact resolvent. The functional calculus for (this extension of) D then gives a graded $*$-homomorphism $\mathcal{S} \to \mathfrak{K}(L^2(S))$, that is, a spectral pair for $\mathbb{C}$.

Proposition 13.7. *The element of $K_0(\mathbb{C}) = \mathbb{Z}$ corresponding to the above spectral pair is the usual Fredholm index of D.*

Proof. Choose a chopping function χ and then consider the homotopy $\chi_s(t) = \chi(s^{-1}t)$, $s \in [0,1]$ (**think carefully** about why we have continuity at $s = 0$). This provides a homotopy between the Fredholm module $(\chi(D), E)$ and the Fredholm module $(0, PE)$ where P is the (even) projection onto $\mathrm{Ker}(D)$). Now apply Lemma 13.1. □

So far this is all "lower" index theory. "Higher" index theory concerns itself with what happens on the universal covering $X = \widetilde{M}$ of M. Let $\Gamma = \pi_1(M)$, so that X is equipped with a free and proper action of Γ, which preserves the Lebesgue measure μ. Let $\tilde{S}$ denote the pullback of the spinor bundle S to X.

The Hilbert space $L^2(X, \tilde{S})$ is obtained by completing the space $C_c(X; \tilde{S})$ of continuous and compactly supported sections on X relative to the norm defined by the complex-valued inner product

$$\langle f, g \rangle = \int (f(x), g(x))\, d\mu(x).$$

(Here the symbol $(\cdot,\cdot)$ denotes the Hermitian inner product on the fibers of $\tilde{S}$.) But we can also equip $C_c(X; \tilde{S})$ with a $\mathbb{C}\Gamma$-valued inner product:

$$\langle f, g \rangle_{C^*_r(\Gamma)} = \sum_{\gamma \in \Gamma} \langle f^\gamma, g \rangle [\gamma]$$

where the right Γ-action on sections on X is defined by the formula

$$f^\gamma(x) = f(\gamma x).$$

Lemma 13.8. *The formula above defines a $\mathbb{C}\Gamma$-valued inner product on $C_c(X; \tilde{S})$.*

Proof. The right action of Γ on $C_c(X; \tilde{S})$ makes $C_c(X; \tilde{S})$ a right $\mathbb{C}\Gamma$-module. It is easy to verify that the inner product that we have defined is linear (in the second variable) with respect to this action, and that the inner product is conjugate-symmetric. Finally we must check positivity, which is to say we must check that for every $f \in C_c(X; \tilde{S})$ the element $\langle f, f \rangle$ is positive in $C^*_r(\Gamma)$. Let

$$T\colon \ell^2(\Gamma) \to L^2(X; \tilde{S})$$

be the bounded linear operator defined by left convolution with f; that is, $T[\gamma] = f^\gamma$. Then T^*T is right $\mathbb{C}\Gamma$-linear, and the calculation

$$\langle [\gamma], T^*T[1] \rangle = \langle T[\gamma], T[1] \rangle = \langle f^\gamma, f \rangle$$

shows that T^*T is given by the action of $\langle f, f \rangle_{C^*_r(\Gamma)}$ on $\ell^2(\Gamma)$ through the regular representation. Thus $\langle f, f \rangle_{C^*_r(\Gamma)}$ is a positive operator, and it vanishes if and only if $T = 0$, which is to say $f = 0$. □

We will use the notation $L^2_\Gamma(X; \tilde{S})$ for the Hilbert $C^*_r(\Gamma)$-module obtained by the process of simultaneous completion of $\mathbb{C}\Gamma$ to $C^*_r(\Gamma)$ and of $C_c(X; \tilde{S})$ to a Hilbert module over $C^*_r(\Gamma)$. Notice that if τ denotes the canonical trace on $C^*_r(\Gamma)$, then $\tau(\langle f, g \rangle_\Gamma) = \langle f, g \rangle$. Consequently, the identity map on $C_c(X)$ extends to a contractive map of Banach spaces from $L^2_\Gamma(X;\tilde{S})$ to $L^2(X;\tilde{S})$. More precisely one can use the left regular representation λ of $C^*_r(\Gamma)$ on $\ell^2(\Gamma)$ to write

$$L^2(X;\tilde{S}) = L^2_\Gamma(X;\tilde{S}) \otimes_\lambda \ell^2(\Gamma)$$

where the tensor product on the right is our interior tensor product, so that

$$\langle f_1 \odot v_1, f_2 \odot v_2\rangle = \langle v_1, \lambda(\langle f_1, f_2\rangle_{C^*_r(\Gamma)})v_2\rangle.$$

Lemma 13.9. *The map* $\alpha\colon \mathfrak{B}(L^2_\Gamma(X;\tilde{S})) \to \mathfrak{B}(L^2(X;\tilde{S}))$*, defined by* $T \mapsto T \otimes 1$*, is an isometric* $*$*-homomorphism.*

Proof. The map is certainly a $*$-homomorphism of C^*-algebras, so according to C^*-algebra theory it will be isometric if it is injective. Injectivity follows easily from the fact that the left regular representation λ is faithful, that is, is injective as a $*$-homomorphism $C^*_r(\Gamma) \to \mathfrak{B}(\ell^2(\Gamma))$. □

In terms of this discussion we can give a very simple characterization of the algebra $\mathfrak{K}(L^2_\Gamma(X;\tilde{S}))$. Let $C^*_\Gamma(X;\tilde{S})$ denote the (so-called) *equivariant Roe algebra* of $(X,\tilde{S})$. That is, it is the C^*-subalgebra of $\mathfrak{B}(L^2(X;\tilde{S}))$ generated by those operators T which are

- *Locally compact*: for every $f \in C_0(X)$, the operators TM_f and M_fT are compact.
- *Finite propagation*: there is a $d > 0$ (depending on T) such that if $f, g \in C_c(X)$ with $\operatorname{dist}(\operatorname{Support}(f), \operatorname{Support}(g)) > d$, then $M_fTM_g = 0$.
- *Equivariant*: for all $s \in L^2(X;\tilde{S})$ and all $\gamma \in \Gamma$, $(Ts)^\gamma = T(s^\gamma)$.

Lemma 13.10. *The* $*$*-homomorphism* α *described above restricts to an isometric* $*$*-isomorphism from* $\mathfrak{K}(L^2_\Gamma(X;\tilde{S}))$ *onto* $C^*_\Gamma(X;\tilde{S})$*.*

Proof. Recall that $\mathfrak{K}(L^2_\Gamma(X;\tilde{S}))$ is generated by the 'rank one' operators

$$\theta_{g,h}(f) = g\langle h, f\rangle_{C^*_r(\Gamma)}$$

where $g, h \in C_c(X;\tilde{S})$. A simple translation of the definitions shows that the operator $\alpha(\theta_{g,h})$ is defined on $L^2(X;\tilde{S})$ by

$$f \mapsto \sum_\gamma g^\gamma \langle h^\gamma, f\rangle,$$

which is a Γ-invariant integral operator of $L^2(X;\tilde{S})$ with continuous kernel $\sum g^\gamma(x)\bar{h}^\gamma(y)$. It is therefore a locally compact, finite propagation, Γ-invariant operator, thus a member of $C^*_\Gamma(X;\tilde{S})$; we deduce that $\alpha(\mathfrak{K}(L^2_\Gamma(X;\tilde{S}))) \subseteq C^*_\Gamma(X;\tilde{S})$. Moreover, using an orthonormal basis of $L^2(X;\tilde{S})$ made up of continuous functions with uniformly bounded compact supports, we can show that integral operators of the kind considered above are dense in $C^*_\Gamma(X;\tilde{S})$, and it follows that α maps $\mathfrak{K}(L^2_\Gamma(X;\tilde{S}))$ isomorphically *onto* $C^*_\Gamma(X;\tilde{S})$, as asserted. □

Now let $\tilde{D}$ be the Dirac operator on X, acting on sections of $\tilde{S}$. Note that X is a complete, but potentially non compact, Riemannian manifold.

Proposition 13.11. *The operator $\tilde{D}$ is essentially self-adjoint on $H = L^2(X; \tilde{S})$. The associated functional calculus map $f \mapsto f(\tilde{D}) \in \mathfrak{B}(H)$ maps $C_0(\mathbb{R})$ into the equivariant Roe algebra $C^*_\Gamma(X; \tilde{S})$.*

Proof. This is a special case of Proposition 10.5.6 in *Analytic K-Homology*. Here is a brief review of the proof. One starts with the *wave operators* $e^{it\tilde{D}} \colon C_c^\infty(X; \tilde{S}) \to C_c^\infty(X; \tilde{S})$, which are defined as the solution operators to the hyperbolic partial differential equation $\partial s/\partial t = i\tilde{D}s$. The theory of symmetric hyperbolic PDE tells us that these operators are well defined, unitary (with respect to the L^2 inner product), and that

$$\text{Support}(e^{it\tilde{D}}s) \subseteq N(\text{Support}(s); |t|). \tag{13.12}$$

An argument due to Chernoff shows that, for any symmetric unbounded operator T, if T generates a 1-parameter group of unitaries preserving a dense subspace of its domain, then T is essentially self-adjoint. That applies to our operator $\tilde{D}$ and proves its self-adjointness. Finally, let $f \in C_0(\mathbb{R})$ have compactly supported Fourier transform $\hat{f}$ (such functions are dense in $C_0(\mathbb{R})$). Then we may write

$$f(\tilde{D}) = \frac{1}{2\pi} \int \hat{f}(t) e^{it\tilde{D}}\, dt,$$

and the unit propagation speed property expressed by Equation 13.12 shows that $f(\tilde{D})$ has finite propagation. It is clearly equivariant since $\tilde{D}$ is, and the local compactness follows from standard elliptic estimates (Rellich's theorem). $\square$

Now we refer back to Lemma 13.10 which identifies $C^*_\Gamma(X, \tilde{S})$ with the C^*-algebra of compact operators on a certain Hilbert $C^*_r(\Gamma)$-module, namely $L^2_\Gamma(X; \tilde{S})$. Thus (taking appropriate account of the grading) the result of Proposition 13.11 can be expressed in the following way: the functional calculus for $\tilde{D}$ gives a spectral pair over $C^*_r(\Gamma)$. This spectral pair has (of course) an index which lies in $K_0(C^*_r(\Gamma))$.

Definition 13.13. The index defined above is called the *higher index* of D (or sometimes of $\tilde{D}$) lying in $K_0(C^*_r(\Gamma))$.

It is a natural question then how to *compute* this higher index, or some part of it; or even to compute the group $K(C^*_r(\Gamma))$ to which it belongs. We'll take these questions up next time.

Remark 13.14. (The ungraded case) Suppose that a spectral pair (or a Fredholm pair) *does not have* a grading, as would result if we applied the above construction to a Dirac operator on an *odd* dimensional manifold. The operator F now gives a *self-adjoint involution* in $\mathfrak{Q}(E)$, which after stabilization becomes a *projection*, thus a class in $K_0(\mathfrak{Q}(E))$. Its index then is a class in $K_1(\mathfrak{K}(E))$ which maps to $K_1(A)$. in this way we obtain a "Fredholm picture" of $K_1(A)$ as the group $\mathcal{K}_1(A)$ of ungraded Fredholm pairs over A; in particular, a Dirac operator an an *odd* dimensional manifold has a higher index in $K_1(C^*_r(\Gamma))$.

Notice that the Bott periodicity theorem is involved here (via the boundary map from $K_0(\mathfrak{Q})$ to $K_1(\mathfrak{K})$). One should really avoid slipping Bott periodicity in like this, especially in geometric applications which often involve real structures; but that will have to wait for later (if at all) when we discuss Clifford algebras and "reality"

Lecture 14
Traces and K_0

In this and the next lecture we are going to look at the simplest numerical invariants that can be associated to K-theory. The basic idea can be stated in a couple of lines, but there are several subtleties that arise when we want to apply it.

We'll begin by going right back to the algebraic context of lectures 1–3. Let A be a unital algebra over a field $\mathbb{F}$. By definition, a *trace* on A is a linear functional $\tau\colon A \to \mathbb{F}$ such that $\tau(aa') = \tau(a'a)$ for all $a, a' \in A$; it is *normalized* if $\tau(1) = 1$. Matrix algebras $M_n(\mathbb{F})$ have the standard trace $\mathrm{tr}\colon M_n(\mathbb{F}) \to \mathbb{F}$ given by summing the diagonal elements. More generally

Lemma 14.1. *Let A be an algebra with a trace τ. For any $n \in \mathbb{N}$, the linear functional $\tau_n\colon M_n(A) \to \mathbb{F}$ defined by*

$$\tau_n \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \sum_{j=1}^{n} \tau\left(a_{jj}\right)$$

is a trace. □

We'll write τ_∞ for τ_n if n isn't important. Now consider the idempotent picture of $K_0(A)$ (Definition 2.14). Two idempotents e, f are equivalent if and only if there exists x, y with $xy = e$ and $yx = f$ (Lemma 2.13). Thus, $\tau_\infty(e) = \tau_\infty(f)$ for any trace τ on A. Whence we obtain

Proposition 14.2. *Let A be an algebra over $\mathbb{F}$, with $\tau\colon A \to \mathbb{F}$ a trace. Then the formula $[e] \mapsto \tau_\infty(e)$ defines a homomorphism* $\dim_\tau$ *from $K_0(A)$ to the additive group of $\mathbb{F}$.* □

The quantity $\tau_\infty(e)$ is called the *generalized dimension* or *τ-dimension* of the projective module represented by e.

Remark 14.3. Suppose that A is a *non*-unital algebra over $\mathbb{F}$. The unitalization $\tilde{A}$ can be defined purely algebraically as $A \oplus \mathbb{F}$ with the multiplication $(a, \lambda) \cdot (a', \lambda') = (aa' + \lambda a' + \lambda' a, \lambda\lambda')$, and a trace τ on A extends to a (non-normalized) trace on the unitalization by setting $\tilde{\tau}(a, \lambda) = \tau(a)$. If now we choose to define $K_0(A)$ in accordance with Definition 6.6, we find that a trace τ on A induces a dimension function on $K_0(A)$ even in the non-unital case.

If now A is a unital C^*-algebra, a *tracial state* on A is a state $\tau\colon A \to \mathbb{C}$ (that is, a positive linear functional of norm 1) which is also a trace. Positivity has the trivial but important implication that the τ-dimension of any projective module is a positive real number, and therefore that the range of the homomorphism $K_0(A) \to \mathbb{C}$ is contained in the $\mathbb{R}$ (since any idempotent is equivalent to a self-adjoint projection (Proposition 6.2) which is a positive operator).

Remark 14.4. An important special case is the canonical trace τ on $C^*_r(\Gamma)$ defined by $\tau(T) = \langle \delta_e, T\delta_e \rangle$ (remember that $C^*_r(\Gamma)$ is represented on $\ell^2(\Gamma)$ which has a canonical basis $\{\delta_g\}_{g\in\Gamma}$). In the previous lecture we showed how the higher index of a Dirac operator on an (even dimensional) manifold with fundamental group Γ can be viewed as lying in $K_0(C^*_r(\Gamma))$. Computing $\dim_\tau(\operatorname{Index} D)$ is the first nontrivial question in higher index theory. It was solved by the L^2 *index theorem* of Atiyah and Singer (worked out in the early 1970s and published in 1976). We'll state and prove this after some more discussion of traces.

From the perspective of noncommutative geometry, a trace on an algebra A is a *cyclic 0-cocycle* on A. A fundamental issue that arises in relating cyclic cohomology to K-theory of C^*-algebras, already with traces as soon as we move away from the unital case and still more acutely with higher cocyles, is that the *domain of definition* of such a functional may not be the whole of A but just some dense subalgebra. (Remember that, in the commutative case, a C^*-algebra consists of all *continuous* functions on a space X whereas cyclic cohomology relates to de Rham homology and thus to *differentiation* and *smooth* functions.) We therefore find ourselves asking the following question of "noncommutative smoothing theory": given a C^*-algebra A, under what conditions will a dense subalgebra $\mathcal{A}$ have the same K-theory as A?

Recall that a *Fréchet algebra* is an algebra over $\mathbb{C}$ (or $\mathbb{R}$) which also has a Fréchet topology (a complete vector topology induced by countably many seminorms) for which multiplication is jointly continuous.

Theorem 14.5. *Let A be a C^*-algebra and let $\mathcal{A} \subseteq A$ be a dense subalgebra such that*

(a) $\mathcal{A}$ is equipped with a Fréchet algebra topology stronger than the topology it inherits from A.

(b) $\mathcal{A}$ is inverse closed *in A (that is to say, if $a \in \mathcal{A}$ is invertible in A, then $a^{-1} \in \mathcal{A}$. If A is not unital, the condition applies to the unitalizations of A and $\mathcal{A}$.)*

Then the natural maps induced by inclusion, $K_i(\mathcal{A}) \to K_i(A)$, $i = 0, 1$, are isomorphisms.

In this theorem, the group $K_1(\mathcal{A})$ should be interpreted as K_1^{top} in the sense of Remark 4.2. The group $K_0(\mathcal{A})$ may be interpreted in the purely algebraic sense, though in fact it will follow from the proof that it can also be interpreted as K_0^{top} in the sense of Remark 4.2.

Remark 14.6. Theorem 14.5 is one of the most general results of its type and requires a bit of effort to prove (Lecture 16). Before going over the proof, therefore, we'll see how the result helps us with the study of traces. For this we need only the following corollary.

Corollary 14.7. *Let A be a C^*-algebra and let $\mathcal{A} \subseteq A$ be a dense $*$-subalgebra which becomes a* Banach *algebra in some norm $|||\cdot|||$ greater than or equal to the norm $\|\cdot\|$ of A. Suppose also that there is a constant C such that*

$$(*) \qquad |||xy||| \leqslant C\,(\|x\|\,|||y||| + |||x|||\,\|y\|)$$

for all $x, y \in \mathcal{A}$. Then the natural maps induced by inclusion, $K_i(\mathcal{A}) \to K_i(A)$, $i = 0, 1$, are isomorphisms.

Proof. We may take it, without loss of generality, that $\mathcal{A}$ and A are unital (with the same unit). If we need to adjoin units, the norm on $\widetilde{\mathcal{A}}$ is

$$|||a + \lambda 1||| := |||a||| + |\lambda|;$$

it is then easily checked that the inclusion $\widetilde{\mathcal{A}} \subseteq \tilde{A}$ satisfies the conditions of the theorem.

Assuming Theorem 14.5, we simply need to prove inverse closure. First apply the inequality $(*)$ to $x = y = a^n$, where $a \in \mathcal{A}$, to get

$$\left|\left|\left|a^{2n}\right|\right|\right| \leqslant 2C|||a^n|||\,\|a^n\|.$$

Take n'th roots, let $n \to \infty$ and use the spectral radius formula to get $\rho_{\mathcal{A}}(a) \leqslant \rho_A(a)$, where ρ denotes the spectral radius. Since the opposite inequality is obvious, the spectral radius of a is the same in $\mathcal{A}$ and in A.

Now suppose that $b \in \mathcal{A}$ is invertible in A. Then b^*b is invertible in A, which implies that for all sufficiently large $\lambda \in \mathbb{R}^+$, the spectral radius of $\lambda 1 - b^*b$ (in A) is strictly less than λ. (Compare 2015 notes, Proposition 7.5) But then it follows that the spectral radius of $\lambda 1 - b^*b$ (in $\mathcal{A}$) is strictly less than λ, whence 0 does not belong to the spectrum (in $\mathcal{A}$) of b^*b; that is, b^*b is invertible there. Since $b^* \in \mathcal{A}$ because $\mathcal{A}$ is a $*$-subalgebra, b is left invertible in $\mathcal{A}$. A similar argument with bb^* shows that b is right invertible also, completing the proof. □

Our application of this result will be to the domain of an *unbounded trace* on a C^*-algebra.

Definition 14.8. Let A be a C^*-algebra and denote by A^+ the set of positive elements in A. A *tracial weight* on A is a function $\tau\colon A^+ \to [0, \infty]$ such that

$$\tau(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1 \tau(a_1) + \lambda_2 \tau(a_2)$$

for all $\lambda_1, \lambda_2 \geqslant 0$ and all $a_1, a_2 \in A^+$, and such that $\tau(aa^*) = \tau(a^*a)$ for all $a \in A$. A tracial weight τ is *(lower) semicontinuous* if

$$\tau(\lim_{n\to\infty} a_n) \leqslant \liminf_{n\to\infty} \tau(a_n)$$

for all norm-convergent sequences $\{a_n\}$ in A^+. It is *densely defined* if the set $\{a \in A^+ : \tau(a) < \infty\}$ is dense in A^+. For short, we will call a densely defined, lower semicontinuous tracial weight an *unbounded trace*.

Remark 14.9. The standard example showing that unbounded traces occur naturally for non-unital C^*-algebras is the linear functional on (a dense subspace of) $C_0(\mathbb{R})$ given by integration with respect to Lebesgue measure. In this situation, lower semicontinuity corresponds to Fatou's lemma. A noncommutative example is furnished by the "usual" trace on compact operators (whose eigenvalues decay sufficiently fast).

Lemma 14.10. *Let τ be an unbounded trace on a C^*-algebra A. Let $I^+ = \{a \in A^+ \colon \tau(a) < \infty\}$ and let I be the linear span of I^+. Then*

(a) I is a dense $*$-ideal in A and τ extends uniquely to a linear functional on I which is real on self-adjoint elements.

(b) *The ideal I is* hereditary *(that is, $0 \leqslant a \leqslant a'$ and $a' \in I$ imply $a \in I$).*
(c) *The (extended) linear functional τ has the property that $\tau(xy) = \tau(yx)$ whenever $x \in I$ and $y \in A$. In particular, it is a trace (in the algebraic sense) on I.*
(d) *If $a \in I$ then also $|a| \in I$, where $|a| = (a^*a)^{1/2}$ is defined by the functional calculus. For all $a \in I$, $b \in A$ we have*

$$|\tau(ab)| \leqslant \|b\|\tau(|a|).$$

(e) *The inequality*

$$\tau(|a+b|) \leqslant \tau(|a|) + \tau(|b|)$$

holds for all $a, b \in I$.
(f) *For $a \in A$, if $|a| \in I$, then $a \in I$.*
(g) *The formula*

$$|\!|\!|a|\!|\!| = \|a\| + \tau(|a|)$$

defines a norm on I with respect to which it becomes a (non-unital) Banach algebra satisfying the condition $(*)$ of Corollary 14.7.

The ideal I defined above is called the domain *of τ and written* $\mathrm{dom}(\tau)$.

Proof. (a) It is clear that I is dense and that τ extends to a linear functional on it. A self-adjoint element is a real linear combination of two positive elements, so τ applied to it is real. We need to show that I is an ideal. We proceed indirectly: first let $J = \{x \in A : \tau(x^*x) < \infty\}$. (For the classical trace Tr on the compacts, I will be the trace class operators and J will be the Hilbert-Schmidt operators.) The inequality

$$(x+y)^*(x+y) \leqslant (x+y)^*(x+y) + (x-y)^*(x-y) = 2(x^*x + y^*y)$$

shows that J is closed under addition. Moreover, if $x \in J$ and $a \in A$ then $x^*a^*ax \leqslant \|a\|^2 x^*x$ so $ax \in J$; thus J is a left ideal. But since $\tau(xx^*) = \tau(x^*x)$, J is closed under $*$; therefore it is a two-sided ideal. Note the implication $\|ax\|_{HS_\tau} \leqslant \|a\|\|x\|_{HS_\tau}$, where the "$HS_\tau$" norm is associated to the inner product

$$\langle x, y\rangle_{HS} = \tau(x^*y), \quad x, y \in J;$$

the polarization identity

$$(\ddagger) \qquad \tau(x^*y) = \frac{1}{4}\left(\sum_{n=0}^{3} i^n \tau(z_n^* z_n)\right), \quad \text{where } z_n = i^n x + y$$

shows that this inner product is finite on J. It follows from this observation that $J^2 \subseteq I$. I claim that in fact $I = J^2$; then I will be a $*$-ideal also. Indeed, if a is positive and $\tau(a) < \infty$, then $a = a^{1/2}a^{1/2} \in J^2$. Since such elements generate I, it follows that $I \subseteq J^2$. This completes the proof of (a).

Statement (b) is obvious: if $0 \leqslant a \leqslant a'$ and $\tau(a') < \infty$, then $\tau(a) < \infty$ also.

For statement (c), consider first the case where x and y belong to the "Hilbert-Schmidt" ideal J. We may assume without loss of generality that they are self adjoint. Refer again to the polarization identity ($\ddagger$) above. Since $\tau(z_n^* z_n) = \tau(z_n z_n^*)$ for all n

we find $\tau(xy) = \tau(x^*y) = \tau(y^*x) = \tau(yx)$. Now if $x \in I$ is positive and $y \in A$, then $x^{1/2} \in J$ and we may apply the previous result twice:

$$\tau(xy) = \tau(x^{1/2}x^{1/2}y) = \tau(x^{1/2}yx^{1/2}) = \tau(yx^{1/2}x^{1/2}) = \tau(yx),$$

using that J is an ideal. Since positive elements span I this completes the proof of (c).

Now for (d), the natural idea is to use polar decomposition (2015 notes, Proposition 7.4) to write $|a| = V^*a$ where V is a partial isometry. Unfortunately the partial isometry part of the polar decomposition may not be in the C^*-algebra (cf Lemma 10.4) and thus we must use semicontinuity and an approximation argument. Observe first that for any *positive* $a \in I$,

$$|\tau(ab)| = \langle a^{1/2}, a^{1/2}b\rangle_{HS} \leqslant \|a^{1/2}\|_{HS}\|a^{1/2}b\|_{HS} \leqslant \|a^{1/2}\|_{HS}^2\|b\| = \tau(a)\|b\|.$$

Note therefore, since any $a \in I$ is a linear combination of positive elements, this tells us that for any $a \in I$ there is *some* constant C such that

$$|\tau(ab)| = |\tau(ba)| \leqslant C\|b\|.$$

Now let $a \in I$ be arbitrary and write $w_n = a((1/n) + a^*a)^{-1/2}$. It is easy to see that $w_n \in I$ and $\|w_n\| \leqslant 1$ (consider $w_n^*w_n$). Now (w_n^*a) is a sequence of positive elements converging in norm to $|a|$. By the inequality above, the sequence $\tau(w_n^*a)$ is bounded. By lower semicontinuity, then, $\tau(|a|) < \infty$, that is, $|a| \in I$. Now consider $\tau(ab)$ again and suppose temporarily that $b \in I$ also. The sequence $(w_n|a|)$ converges in norm to a. Thus

$$|\tau(ab)| = \lim_{n\to\infty} |\tau(w_n|a|b)| = \lim_{n\to\infty} |\tau(|a|bw_n)| \leqslant \tau(|a|)\sup_n \|bw_n\| \leqslant \tau(|a|)\|b\|.$$

This is the desired inequality for $b \in I$. But both sides a norm continuous in b, and I is dense in A, so the inequality holds for all $b \in A$.

Now for (e), observe that

$$\tau(|a|) = \sup\{|\tau(ax)| : \|x\| \leqslant 1\};$$

indeed, the final inequality of (d) shows that the LHS is greater than or equal to the RHS, while in the reverse direction $\tau(|a|) = \lim \tau(w_n^*a)$ (with w_n as in (d)) by the usual argument that Fatou's lemma implies the monotone convergence theorem. The inequality

$$\tau(|a+b|) \leqslant \tau(|a|) + \tau(|b|), \qquad a, b \in I$$

follows easily. We also obtain the inequality

$$(\dagger) \qquad \tau(|ab|) \leqslant \tau(|a|)\|b\|$$

which we will need in a moment.

For (f), note first that if $|a| \in I$ and a is *self adjoint*, then $a \in I$. Indeed, a self-adjoint $a = a^+ - a^-$, where $A^\pm$ are positive operators and $|a| = a^+ + a^-$, which easily gives the result. To reduce to this case, write a general $a = x + iy$ where x, y are self adjoint. Then

$$2\tau(|x|) \leqslant \tau(|x+iy|) + \tau(|x-iy|) = \tau(|a|) + \tau(|a^*|) < \infty$$

using (e). Thus $x \in I$ and similarly $y \in I$.

Finally for (g), it is clear from (e) that $|||a||| = \|a\| + \tau(|a|)$ satisfies the triangle inequality. The other axioms for a norm are obvious (we could even leave out adding $\|a\|$ if τ were "faithful" in the natural sense), and the inequality of Corollary 14.7 follows from the inequality (†) displayed above. We must now prove completeness so suppose that (a_n) is a sequence Cauchy in $|||\cdot|||$. Take a "fast" subsequence so that $|||a_{n+1} - a_n||| < 2^{-n}$. It has a limit a in the usual norm, we must show that $a \in I$ and that $\tau(|a_n - a|) \to 0$. By (e), $\{\tau(|a_n|\}$ is bounded. By semicontinuity we have

$$\tau(|a|) \leqslant \liminf_{n\to\infty} \tau(|a_n|) < \infty,$$

so $a \in I$ by (f). Given ε, choose N so large that $2^{-N} < \varepsilon$. Then for $n > N$, $|\tau(|a_N - a_n|) < 2\varepsilon$ and so by semicontinuity $\tau(|a_N - a|) \leqslant 2\varepsilon$. Since ε was arbitrary this gives the desired convergence. □

From the preceding lengthy discussion we conclude

Proposition 14.11. *An unbounded trace τ on a C^*-algebra A defines a dimension homomorphism* $\dim_\tau \colon K(A) \to \mathbb{R}$. □

Proof. τ defines a dimension homomorphism on its domain $\mathrm{dom}(\tau)$ (denoted I in the proof above) in a purely algebraic fashion as in Proposition 14.2. The inclusion $K_0(I) \to K_0(A)$ is an isomorphism on K-theory by part (g) of Lemma 14.10 together with Corollary 14.7. □

Example 14.12. Let $A = \mathfrak{K}$ and let τ be the usual operator trace defined by $\tau(T) = \sum \langle Te_i, e_i\rangle$ for $T \geqslant 0$, where $\{e_i\}$ is an orthonormal basis. This is an unbounded trace in our sense (**Exercise:** Prove this!) and the associated ideal $I = \mathrm{dom}(\tau)$ is the ideal of *trace class operators*. The dimension homomorphism associated to τ assigns the rank of p to any finite rank projection p. It therefore implements the canonical isomorphism from $K(\mathfrak{K})$ to $\mathbb{Z}$.

Exercise 14.13. Show that there are no non-trivial dense ideals in a unital C^*-algebra. Deduce that the only unbounded traces (in our sense) on a unital C^*-algebra are in fact bounded (that is, a positive scalar multiple of a tracial state).

Exercise 14.14. Let Γ be a non-amenable group. Define a map τ from the dense subset $\mathbb{C}\Gamma$ of $C^*_r(\Gamma)$ to $\mathbb{C}$ by

$$\tau\left(\sum \lambda_g[g]\right) = \sum \lambda_g.$$

Then τ is tracial ($\tau(aa') = \tau(a'a)$ on $\mathbb{C}\Gamma$) because it is in fact a homomorphism. Show, however, that the restriction of τ to $\mathbb{C}\Gamma \cap (C^*_r(\Gamma))^+$ does *not* define an unbounded trace in our sense.

Lecture 15
Traces and the Index

Let A be a C^*-algebra equipped with an unbounded trace τ in the sense of Definition 14.8. Let H_A be the universal Hilbert module over A (Example 9.1). For each $n \in \mathbb{N}$ there are "n'th coordinate" module maps $i_n\colon A \to H_A$ and $p_n\colon H_A \to A$, and if $T \in \mathfrak{K}(H_A)$ then $T_n = p_n \circ T \circ i_n$ is a compact operator $A \to A$, that is, a member of A.

Lemma 15.1. *In the above situation the functional* $(\tau \otimes \mathrm{Tr})\colon \mathfrak{K}(H_A)^+ \to \mathbb{R}^+ \cup \{+\infty\}$, *defined by*

$$(\tau \otimes \mathrm{Tr})(T) = \sum_n \tau(T_n),$$

is an unbounded trace on $\mathfrak{K}(H_A)$.

Proof. Exercise. The notation $\tau \otimes \mathrm{Tr}$ comes from the isomorphism $\mathfrak{K}(H_A) \cong A \otimes \mathfrak{K}$ (Exercise 9.5); our trace is a "tensor product" of the given trace τ on A with the canonical trace Tr on $\mathfrak{K}$, to which it reduces if $A = \mathbb{C}$. □

Remark 15.2. If A is *unital* then H_A has an "orthonormal basis" consisting of the vectors $\xi_n = (0, \ldots, 0, 1, 0, \ldots)$, with 1 in the n'th slot. In this case the trace defined above can be written in the more familiar-looking form

$$(\tau \otimes \mathrm{Tr})(T) = \sum_n \tau\left(\langle \xi_n, T\xi_n \rangle\right).$$

Lemma 15.3. *Let A be a C^*-algebra equipped with an unbounded trace τ. The following diagram is commutative*

$$\begin{tikzcd}
K_0(\mathfrak{K}(H_A)) \arrow[d, "\cong"'] \arrow[dr, "\dim_{\tau\otimes\mathrm{Tr}}"] & \\
K_0(A) \arrow[r, "\dim_\tau"'] & \mathbb{R}
\end{tikzcd}$$

Here the vertical arrow is the isomorphism given by Morita equivalence.

Proof. Since $\mathfrak{K}(H_A) \cong A \otimes \mathfrak{K} = \lim_{n\to\infty} M_n(A)$, it suffices to prove the result with $\mathfrak{K}(H_A)$ replaced by $M_n(A)$; the functional $\tau \otimes \mathrm{Tr}$ is defined on $M_n(\mathrm{dom}(\tau)) \subseteq M_n(A)$ by

$$\tau \otimes \mathrm{Tr} = \tau_n \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \sum_{j=1}^{n} \tau\left(a_{jj}\right)$$

as in Lemma 14.1. Now make the purely algebraic observation that for any unital $\mathbb{C}$-algebra R with a trace σ the diagram

$$\begin{tikzcd} K_0(M_n(R))) \arrow[d, "\cong"'] \arrow[dr, "\dim_{\sigma_n}"] & \\ K_0(R) \arrow[r, "\dim_\sigma"'] & \mathbb{C} \end{tikzcd}$$

commutes, and apply this where R is the (algebraic) unitalization of $\mathrm{dom}(\tau)$ and σ is the extension of τ by zero (Remark 14.3). $\square$

Suppose that E is any (countably generated) Hilbert A-module. By the stabilization theorem, $E \oplus H_A \cong H_A$. We may define an unbounded trace on $\mathfrak{K}(E)$ by the composite

$$\mathfrak{K}(E)^+ \xrightarrow{(\mathrm{id},0)} \mathfrak{K}(E \oplus H_A)^+ \longrightarrow \mathfrak{K}(H_A)^+ \xrightarrow{\tau\otimes\mathrm{Tr}} \mathbb{R}^+ \cup \{+\infty\} ;$$

the trace property ensures that this doesn't depend on the choice of unitary equivalence $E \oplus H_A \cong H_A$. By abuse of notation, let's denote by τ *all* of the traces on different A-modules that are derived from τ in this way.

Proposition 15.4. *Let A be a C^*-algebra equipped with an unbounded trace τ. Let (F, E) be a Fredholm pair over A and suppose that*

(a) F is self-adjoint ($F = F^*$)
(b) $\|F\| \leqslant 1$
(c) $(1 - F^2)^{1/2}$ (which is well-defined by (a) and (b)) belongs to the domain of the trace τ on $\mathfrak{K}(E)$.

Then

$$\dim_\tau(\mathrm{Index}(F, E)) = \tau(\varepsilon(1 - F^2)),$$

where $\varepsilon \in \mathfrak{B}(E)$ is the grading operator*, equal to ± 1 on $E^\pm$ respectively.*

Note that any Fredholm pair is homotopic to one that satisfies (a) and (b) (easy exercise). The key condition is (c). Since, given (a) and (b),

$$0 \leqslant (1 - F^2) \leqslant (1 - F^2)^{1/2},$$

condition (c) implies that $(1-F^2)$ belongs to the hereditary ideal $\mathrm{dom}(\tau)$, so that the index formula is well-defined. Note also that the τ on the right of the index formula is the trace on $\mathfrak{K}(E)$ defined by embedding E in H_A; on the left, the index lies in $K_0(\mathfrak{K}(H_A)) \cong K_0(A)$, and by lemma 15.3 it makes no difference whether we view the dimension function as defined by the original trace τ on A or the "induced" trace $\tau \otimes \mathrm{Tr}$ on $\mathfrak{K}(H_A)$.

Proof. By stabilization we may assume that $E = \widehat{H}_A = H_A \oplus H_A$. Then by (a), F has the form $\left(\begin{smallmatrix} 0 & u^* \\ u & 0 \end{smallmatrix}\right)$, where $u \in \mathfrak{B}(H_A)$ and $1 - uu^*$ and $1 - u^*u$ belong to $\mathfrak{K}(H_A)$. Then $\mathrm{Index}(F, E)$ is, by definition, the element of $K_0(\mathfrak{K}(H_A)) \cong K_0(A)$ which is the image of $[u] \in K_1(\mathfrak{Q}(H_A))$ under the boundary map. From Exercise 7.5 we can extract a description of this boundary map: namely, we should find a unitary matrix $w \in M_2(\mathfrak{B}(H_A))$ that is equal modulo compacts to $\left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$. Let $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_{2n}(\widetilde{\mathfrak{K}}(H_A))$. Then $q = w^*pw$ also belongs to $M_{2n}(\widetilde{\mathfrak{K}}(H_A)))$ and the formal difference of projections $[p] - [q]$ defines a class in $K_0(\widetilde{\mathfrak{K}}(H_A)))$.

Under the assumption (b), we may take

$$w = \begin{pmatrix} u & (1 - uu^*)^{1/2} \\ -(1 - u^*u)^{1/2} & u^* \end{pmatrix},$$

and then by calculation

$$q = \begin{pmatrix} u^*u & u^*(1 - uu^*)^{1/2} \\ (1 - uu^*)^{1/2}u & 1 - uu^* \end{pmatrix}.$$

Then q belongs to $M_2((\mathrm{dom}(\tau))^\sim)$ and thus $\dim_\tau([p] - [q]) = \tau(1 - u^*u) - \tau(1 - uu^*)$. But this is exactly $\tau(\varepsilon(1 - F^2))$ as asserted. $\square$

We want to relate this calculation to the spectral picture (Definition 13.2), where an element of K-theory is defined by a graded $*$-homomorphism $\mathcal{S} \to \mathfrak{K}(E)$. The following lemma is very important.

Lemma 15.5. *Let A be a C^*-algebra and let $\psi\colon C_0(\mathbb{R}) \to A$ be a $*$-homomorphism. Let I be a dense hereditary ideal in A. If h is a compactly supported continuous function on $\mathbb{R}$, then $\psi(h) \in I$.*

Proof. We may assume that $h \geqslant 0$. Let $a = \psi(h^{1/2})$. There is a positive element $b \in A$ such that $ab = a$. For example if $g \in C_0(\mathbb{R})$ is nonnegative and equal to 1 on the support of h, then $b = \psi(g)$ will do the job. By density, there is a positive element $c \in I$ such that $\|b - c\| < \frac{1}{2}$. Let us write

$$a^2 = aba = a(b - c)a + aca.$$

From the inequality $\|b - c\| < \frac{1}{2}$ it follows that

$$a(b - c)a \leqslant \tfrac{1}{2}a^2.$$

As a result, $a^2 \leqslant 2aca$. But $aca \in I$ since I is an ideal, so $a^2 = \psi(h) \in I$ since I is hereditary. $\square$

Proposition 15.6. *Let A be a C^*-algebra equipped with an unbounded trace τ. Let (α, E) be a spectral pair over A. Then $\dim_\tau(\mathrm{Index}(\alpha, E)) = \tau(\varepsilon\alpha(h))$, where ε is the grading operator and h is any compactly supported even function on $\mathbb{R}$ having $h(0) = 1$.*

Notice that $\alpha(h) \in \mathrm{dom}(\tau)$ by lemma 15.5.

Proof. First we establish that for any even, compactly supported function h, $\tau(\varepsilon\alpha(h))$ depends only on $h(0)$. To do so, it is of course enough to show that $\tau(\varepsilon\alpha(h)) = 0$ for $h(0) = 0$. But if $h(0) = 0$ we may write $h = g^2$ for some *odd*, compactly supported and continuous g. Now

$$\tau(\varepsilon\alpha(g^2)) = \tau(\varepsilon\alpha(g)\alpha(g)) = -\tau(\alpha(g)\varepsilon\alpha(g)) = -\tau(\varepsilon\alpha(g)\alpha(g)) = -\tau(\varepsilon\alpha(g^2)),$$

so $\tau(\varepsilon\alpha(g^2)) = 0$ as asserted.

Suppose again now that $h(0) = 1$. By what we just proved, we may assume without loss of generality that h is real-valued and $0 \leqslant h \leqslant 1$. Then let

$$\chi(t) = \pm(1 - h(t))^{1/2},$$

with the sign being the same as that of t. Then χ is a chopping function (Definition 13.4). By definition 13.5, the index of the spectral pair is the index of the Fredholm pair (F, E), where $F = \alpha(\chi)$. But now $1 - F^2 = \alpha(h)$ and the result follows from Proposition 15.4. $\square$

Remark 15.7. Of course, for particular spectral pairs, there may be a larger class[12] of functions than $C_c(\mathbb{R})$ that is mapped by α into the domain of τ; then Proposition 15.6 will hold for h belonging to this larger class. For instance, consider the spectral pair $\mathbb{S} \to \mathfrak{K}(L^2(S))$ associated to a Dirac operator on a compact, even-dimensional manifold (Proposition 13.7). Standard elliptic estimates show that if f is of rapid decay, then $f(D)$ maps $L^2(S)$ to all the Sobolev spaces $W^k(S)$. For k sufficiently large, the inclusion $W^k(S) \to L^2(S)$ is a trace-class operator (the quantitative Rellich theorem) and therefore $f(D)$ belongs to the domain of the canonical trace Tr on the compacts. Combining this observation with Proposition 13.7 and applying to $h(s) = \exp(-ts^2)$ gives the so-called *McKean-Singer* formula for the index of an elliptic operator.

Remark 15.8. The same elliptic estimates (which are local) show that if we consider the spectral pair $f \mapsto f(\tilde{D}) \in \mathfrak{K}(L^2_\Gamma(X; \tilde{S}))$ which defines the *higher* index as an element of $K_0(C^*_r\Gamma)$ (Definition 13.13), and if τ denotes the canonical trace on $C^*_r(\Gamma)$ (Remark 14.4), then $f(\tilde{D})$ belongs to the domain of $\tau \otimes \mathrm{Tr}$ for all f of rapid decay.

Theorem 15.9 (Atiyah L^2 index theorem)**.** *Let M be a compact even-dimensional manifold, with associated bundle of Clifford modules S and Dirac operator D. Let $\Gamma = \pi_1(M)$, let $X = \widetilde{M}$ be the universal cover of M, with lifted bundle and Dirac operator $\tilde{S}$ and $\tilde{D}$. Let τ be the canonical trace on $C^*_r(\Gamma)$ and let $\mathrm{Index}(\tilde{D}) \in K_0(C^*_r(\Gamma))$ be the higher index of $\tilde{D}$ (according to Definition 13.13). Then*

$$\dim_\tau(\mathrm{Index}(\tilde{D})) = \mathrm{Index}(D) \in \mathbb{Z}.$$

Proof. There exists $\rho > 0$ such that $d(x, gx) \geqslant 3\rho$ for all non-identity $g \in \Gamma$. Choose a good fundamental domain for the action of Γ on X, in the following sense: an open subset U such that $U \cap gU = \emptyset$ for all non-identity $g \in \Gamma$, the boundary ∂U has measure zero, and the Γ-translates of the closure $\overline{U}$ cover X. Choose a finite cover of $\overline{U}$ by open balls B_j of radius ρ and let $U_j = B_j \cap U$. Let $V_j = \pi(U_j) \subseteq M$, where $\pi\colon X \to M$ is the covering map; note that the restriction of π to each U_j is an isometry $U_j \to V_j$.

Choose an orthonormal basis $\{\xi_k\}$ of $L^2(M; S)$ each of whose members has support in some one of the sets V_j, and let $\tilde{\xi}_k = \mathcal{L}_j(\xi_k)$ be the lift of ξ_k via the isometry

[12]Specifically, a subalgebra closed under the formation of square roots.

$\pi\colon U_j \to V_j$; in other words, it is the section of $\tilde{S}$ defined by

$$\mathcal{L}_j\xi_k(x) = \begin{cases} \xi_k(\pi(x)) & \text{using the isomorphism } \tilde{S}_x = S_{\pi(x)}, \text{ if } x \in U_j \\ 0 & \text{otherwise.} \end{cases}$$

Then the L^2 inner product of $\tilde{\xi}_k$ and $\tilde{\xi}_\ell$ is $\delta_{k\ell}$, and the L^2 inner product of $\tilde{\xi}_k^g$ and $\tilde{\xi}_\ell$ is zero for all non-identity $g \in \Gamma$. Thus the $\tilde{\xi}_k$ form an orthonormal basis[13] of $L^2_\Gamma(X;\tilde{S})$ as a $C^*_r(\Gamma)$-module (that is to say, they implement an explicit isomorphism between $L^2_\Gamma(X;\tilde{S})$ and $H_{C^*_r(\Gamma))}$).

Now we use a finite propagation speed argument. Let X_j be the ball in X with the same center as U_j and radius 2ρ, and let $W_j = \pi(X_j) \subseteq M$; the restriction of π to each X_j is an isometry $X_j \to W_j$ (and is covered by an isometry between the fibers of $\tilde{S}$ to the fibers of S). Because of this and the finite propagation speed property for the wave operator (see equation 13.12), we have

$$(*) \qquad \mathcal{L}_j\left(e^{itD}s\right) = e^{it\tilde{D}}\mathcal{L}_j(s)$$

for all s supported in V_j and all $|t| < \rho$; moreover, the quantity $(*)$ is supported within X_j. In particular, for any automorphism ε of S (our interest is in the grading automorphism, hence the notation)

$$(\dagger) \qquad \langle e^{it\tilde{D}}\tilde{\xi}_k, \varepsilon\tilde{\xi}_k\rangle_{C^*_r(\Gamma)} = \langle e^{itD}\xi_k, \varepsilon\xi_k\rangle[e],$$

where $[e]$ is the class of the identity element in $C^*_r(\Gamma)$ (note that $gX_j \cap U_j = \emptyset$ for all non-identity $g \in \Gamma$).

Now choose a function h of rapid decay with $h(0) = 1$ and such that the Fourier transform $\hat{h}$ has support in $[-\varepsilon, \varepsilon]$. By the generalized McKean-Singer formula

$$\dim_{\mathrm{Tr}}(\mathrm{Index}(D)) = \mathrm{Tr}(\varepsilon h(D)) = \frac{1}{2\pi}\sum_k \int_{-\infty}^{\infty} \hat{h}(t)\langle e^{itD}\xi_k, \varepsilon\xi_k\rangle\, dt,$$

using Fourier analysis and the definition of Tr. We notice that since Tr is the ordinary trace on the compacts, $\dim_{\mathrm{Tr}}(\mathrm{Index}\, D) = \mathrm{Index}\, D$ (Example 14.12). The same formula applied on the covering, using Remark 15.2, gives

$$\dim_\tau(\mathrm{Index}(\tilde{D})) = (\tau \otimes \mathrm{Tr})(\varepsilon h(\tilde{D})) = \frac{1}{2\pi}\sum_k \int_{-\infty}^{\infty} \hat{h}(t)\tau\left(\langle e^{it\tilde{D}}\tilde{\xi}_k, \varepsilon\tilde{\xi}_k\rangle\right)\, dt.$$

But by equation $(\dagger)$ the expressions under the integral signs are the same. This completes the proof. $\square$

[13] See Remark 15.2. Such bases don't always have to exist. But they do in this case.

Lecture 16
Proof of Theorem 14.5

Let's recall the statement of the theorem: Let A be a C^*-algebra and let $\mathcal{A} \subseteq A$ be a dense subalgebra such that

(a) $\mathcal{A}$ is equipped with a Fréchet algebra topology stronger than the topology it inherits from A.
(b) $\mathcal{A}$ is *inverse closed* in A (that is to say, if $a \in \mathcal{A}$ is invertible in A, then $a^{-1} \in \mathcal{A}$. If A is not unital, the condition applies to the unitalizations of A and $\mathcal{A}$.)

Then the natural maps induced by inclusion, $K_i(\mathcal{A}) \to K_i(A)$, $i = 0, 1$, are isomorphisms.

Remark 16.1. We may assume without loss of generality that $\mathcal{A}$ and A are unital (with the same unit). We are going to give our proof under the additional hypothesis that $\mathcal{A}$ is closed under the involution, i.e. is a $*$-subalgebra. Most of the proof given below works without this hypothesis, but at one point we would need to replace a simple calculation from Lecture 6 of the 2015 notes with a more involved argument using the theory of holomorphic functions in Banach algebras. Because I haven't developed that material, I choose to avoid using it here by adding the $*$-closure hypothesis. But it is truly NBD.

We are going to study the inverse closure property by looking at representations of the algebras (that is to say, modules). To motivate, consider the classic example of a dense subalgebra that is *not* inverse closed, the polynomial functions $\mathcal{A}$ inside $A = C[0, 1]$. The action of $\mathcal{A}$ on $\mathbb{C}$ by $\lambda \cdot p = p(2)\lambda$ makes $\mathbb{C}$ into an irreducible $\mathcal{A}$-module N. But clearly this $\mathcal{A}$-module cannot be extended to an A-module: there is no A-module M at all (irreducible or not) such that N is an $\mathcal{A}$-submodule of the restriction of M to $\mathcal{A}$. Why? Because the kernel of N,

$$\mathfrak{n} = \{a \in \mathcal{A} : na = 0 \ \forall n \in N\}, \tag{16.2}$$

is *dense* in A and in particular contains an element a' invertible in A. But now, since a' is invertible in A, it cannot annihilate any nonzero element of M, which contradicts the supposition that N is extended to M.

Formalizing this argument we get

Lemma 16.3. *Let $\mathcal{A}$ be a dense subalgebra of the Banach algebra A (both unital). The following are equivalent:*

(i) $\mathcal{A}$ is inverse closed in A.
(ii) Every maximal right ideal $\mathfrak{n}$ of $\mathcal{A}$ is a relatively closed subset (that is, $\mathfrak{n} = \mathcal{A} \cap \overline{\mathfrak{n}}$).
(iii) Every irreducible $\mathcal{A}$-module extends to an A-module (in the sense described above).

Proof. **(i) $\Rightarrow$ (ii):** Suppose $\mathfrak{n} \subseteq \mathcal{A}$ is a maximal right ideal. By hypothesis (i), $\mathfrak{n}$ does not intersect the open set of invertibles in A. Thus its closure $\overline{\mathfrak{n}}$ is a proper right ideal in A, so $\mathcal{A} \cap \overline{\mathfrak{n}}$ is a proper right ideal in $\mathcal{A}$. By maximality $\mathfrak{n} = \mathcal{A} \cap \overline{\mathfrak{n}}$ as asserted.

(ii) $\Rightarrow$ (iii): Suppose that N is an irreducible $\mathcal{A}$-module and let $\mathfrak{n}$ be the kernel of N as in equation 16.2 above. Choosing any nonzero $n_0 \in N$ gives an isomorphism of $\mathcal{A}$-modules between N and $\mathfrak{n}\backslash\mathcal{A}$; moreover, irreducibility of N implies that $\mathfrak{n}$ is maximal. Let $\mathfrak{m} = \overline{\mathfrak{n}}$ which is a right ideal in A, and let $M = \mathfrak{m}\backslash A$. By assumption (ii),

$$N \cong (\mathfrak{m} \cap \mathcal{A})\backslash\mathcal{A} \cong \mathfrak{m}\backslash(\mathfrak{m} + \mathcal{A}) \subseteq \mathfrak{m}\backslash A = M,$$

where the inclusion is one of $\mathcal{A}$-modules. This proves (iii).

(iii) $\Rightarrow$ (i): We prove the contrapositive. Suppose that (i) fails, and let $a \in \mathcal{A}$ be invertible in A but not in $\mathcal{A}$. Notice that a cannot even be one-sided invertible in $\mathcal{A}$; thus it is contained in a maximal right ideal $\mathfrak{n} \subseteq \mathcal{A}$ Then $N = \mathfrak{n}\backslash\mathcal{A}$ is an irreducible $\mathcal{A}$-module, and a annihilates it. But it cannot be contained in any A-module M because, as we argued above, the invertibility of a implies that for all nonzero $m \in M$, $(ma)a^{-1} = m \neq 0$ whence $ma \neq 0$: that is, a does not annihilate any nonzero element of M. Thus we have contradicted (iii). ☐

Corollary 16.4. *Let $\mathcal{A}$ be a dense subalgebra of the Banach algebra A (both unital). If $\mathcal{A}$ is inverse closed in A, then $M_n(\mathcal{A})$ is inverse closed in $M_n(A)$ for any n.*

Proof. It is a standard fact that every module over a matrix algebra $M_n(B)$ is of the form $\mathbb{C}^n \otimes W = W \oplus \cdots \oplus W$ (n factors) for some module W over B, with $\mathbb{C}^n \otimes W$ being irreducible if and only if W is. It easily follows that every irreducible $M_n(\mathcal{A})$ module extends to a $M_n(A)$ module if and only if every irreducible $\mathcal{A}$ module extends to an A module. By the previous lemma, this gives our result. ☐

Our discussion so far did not make use of the Fréchet hypothesis on $\mathcal{A}$. That is needed so we can use the following automatic continuity result.

Proposition 16.5. *Let G be a group with a complete metric topology, and assume that multiplication is (jointly) continuous $G \times G \to G$. Then inversion is also continuous as a map $G \to G$: that is to say, G is a topological group.*

Proof. (Pfister) We must show that for every neighborhood U of the identity e there exists another neighborhood V of e with $V^{-1} \subseteq U$. Suppose not and let U be a neighborhood of e for which there is no such V. Without loss of generality we may take U to be an open ball around e. Construct inductively a sequence $\{U_n\}$ of open balls about e with radius tending to zero, such that $\overline{U}_{n+1}^2 \subseteq U_n$ for $n = 0, 1, 2, \ldots$ and $U = U_0$; the existence of such a sequence follows from the continuity of multiplication. Choose $x_n \in U_n$ such that $x_n^{-1} \notin U$. Clearly $x_n \to e$ and thus $xx_n \to x$ for every $x \in G$. By induction, we can find a subsequence $z_k = x_{n_k}$ such that, if $y_k = z_1 \cdots z_k$, we have $d(y_k, y_{k+1}) < 2^{-k}$ and thus $y_k \to y$ for some y.

Since yU_1 is a neighborhood of y, there exists $k \geqslant 2$ such that $y_{k-1} \in yU_1$. Then

$$z_k^{-1} = y_k^{-1} y_{k-1} \in y_k^{-1} y U_1.$$

But

$$y_k^{-1} y = \lim_{j\to\infty} y_k^{-1} y_j = \lim_{j\to\infty} (z_{k+1} \cdots z_j).$$

The expression under the limit belongs to $U_{k+1}\cdots U_j \subseteq U_k$, so the limit itself belongs to $\overline{U_k} \subseteq U_{k-1} \subseteq U_1$. Thus $z_k^{-1} \subseteq y_k^{-1} y U_1 \subseteq U_1^2 \subseteq U$ which is a contradiction. $\square$

Exercise 16.6. Use the same idea to prove automatic continuity of the inverse when G is a *locally compact* group.

Remark 16.7. Now under the hypotheses of Theorem 14.5, the invertibles in $\mathcal{A}$ are an open subset of $\mathcal{A}$ (because they are the intersection of $\mathcal{A}$ with the invertibles of A). Hence, their topology is given by a complete metric (an open subset of a complete metric space has a completely metrizable topology). The conditions of Proposition 16.5 are satisfied and we conclude that the inversion operation is continuous on (the invertibles of) $\mathcal{A}$. By Corollary 16.4 the same conclusions apply to matrix algebras over $\mathcal{A}$.

Proof of Theorem 14.5 for K_0. Let us prove first that the natural map $K_0(\mathcal{A}) \to K_0(A)$ is *surjective*. It suffices to prove that if $p \in M_n(A)$ is a projection, there is an projection $q \in M_n(\mathcal{A})$ that is equivalent (in A) to p. Let $x \in M_n(\mathcal{A})$ have $\|x-p\| < \frac{1}{4}$. We may assume that x is self-adjoint[14] by replacing it by $\frac{1}{2}(x+x^*)$. Then the spectrum of x lies in $[-\frac{1}{4},\frac{1}{4}] \cup [\frac{3}{4},\frac{5}{4}]$. Let q be the spectral projection of x corresponding to the function that is 0 on the first of these intervals and 1 on the second. Then $\|q-x\| \leqslant \frac{1}{4}$ and thus $\|q-p\| < \frac{1}{2}$ whence q and p are equivalent (Proposition 4.3). It remains to show that $q \in M_n(\mathcal{A})$. This follows from the Cauchy integral formula

$$q = \frac{1}{2\pi i}\int_\Gamma (z1-x)^{-1}\,dz, \tag{$\sharp$}$$

where Γ is a contour encircling the interval $[\frac{3}{4},\frac{5}{4}]$ but not the interval $[-\frac{1}{4},\frac{1}{4}]$, for example the circle center 1 and radius $\frac{1}{2}$. It is proved in 2015 notes, Remark 6.3, that this integral converges in the norm of A to q. But now, by inverse closure, $(z1-x)^{-1}$ belongs to $M_n(\mathcal{A})$ for all $z \in \Gamma^*$ and it is a continuous function of z *in the Fréchet topology of* $\mathcal{A}$ by Remark 16.7. The integral therefore converges in the topology of $\mathcal{A}$ and thus $q \in \mathcal{A}$.

[If we do not assume that $\mathcal{A}$ is $*$-closed, then we can no longer assume that x, above, is self-adjoint. The spectrum of x is contained in the union of two discs of radius $\frac{1}{4}$, one about 0 and one about 1. Now we must use the formula ($\sharp$) to *define* q via the *holomorphic functional calculus*. This calculus shows that q is an idempotent (not necessarily a projection) and that $\|q-x\| < \frac{1}{4}$. If this is granted, the rest of the argument proceeds as before.]

To show that the natural map $K_0(\mathcal{A}) \to K_0(A)$ is *injective*, it suffices to show that if p_0 and p_1 are projections in $M_n(\mathcal{A})$ that are conjugate in $M_n(A)$, then they are conjugate in $M_n(\mathcal{A})$. Let $u \in GL_n(A)$ with $u^{-1}p_0u = p_1$. By density and inverse closure, there exists $v \in GL_n(\mathcal{A})$ as close as we please (in the norm of A) to u; so there exists $q = v^{-1}p_0v \in M_n(\mathcal{A})$ as close as we please (in the norm of A) to p_1. It suffices then to show that if two idempotents e, f in $M_n(\mathcal{A})$ are sufficiently close in the norm

[14] Here is where the $*$-closure hypothesis is used!

of A then they are conjugate in $M_n(\mathcal{A})$. Refer back to the proof of Proposition 4.3: if we put

$$w = ef + (1-e)(1-f) = 1 + (1-2e)(e-f),$$

then $w \in M_n(\mathcal{A})$, $ew = wf$, and w is invertible (in $M_n(\mathcal{A})$, by inverse closure) if $\|e-f\|$ is sufficiently small. This completes the proof. $\square$

Proof of Theorem 14.5 for K_1. Let us prove first that the natural map $K_1(\mathcal{A}) \to K_1(A)$ is *surjective*. It suffices to show that if $u \in GL_n(A)$, then there exists $v \in GL_n(\mathcal{A})$ and a path in $GL_n(A)$ connecting u to v. Let $\varepsilon = \|u^{-1}\|^{-1}$. Then all members of $B(u;\varepsilon) \subseteq M_n(A)$ are invertible (by the usual Neumann series argument). This ball must meet $GL_n(\mathcal{A})$ by density and inverse closure; let $v \in GL_n(\mathcal{A}) \cap B(u;\varepsilon)$. The straight line path from v to u then consists of elements of $GL_n(A)$, as required.

Now let us prove that the natural map $K_1(\mathcal{A}) \to K_1(A)$ is *injective*. Here it suffices to show that if $v \in GL_n(\mathcal{A})$ is connected to the identity by a path in $GL_n(A)$, then it is also connected to the identity by a path $\gamma\colon [0,1] \to GL_n(\mathcal{A})$. A compactness argument shows that there exists $\varepsilon > 0$ such that any ball $B(\gamma(t);\varepsilon)$, $t \in [0,1]$, consists of invertible matrices. Choose by another compactness argument a sequence $t_0 = 0, t_1, \ldots, t_n = 1$ such that $\|\gamma(t_j) - \gamma(t_{j+1})\| < \varepsilon/3$. For each j, choose (using density and inverse closure) an element v_j of $GL_n(\mathcal{A})$ lying in $B(\gamma(t_j);\varepsilon/3)$, making sure to choose $v_0 = v$ and $v_n = 1$. Then the straight line path from v_j to v_{j+1} lies in $M_n(\mathcal{A}) \cap B(\gamma(t_j);\varepsilon)$. The latter ball is a subset of $GL_n(A)$, so the straight line segment lies in $GL_n(\mathcal{A})$ by inverse closure. Concatenating these line segments gives the desired path. $\square$

Remark 16.8. For later reference, I need to make an observation about the $*$-algebra case (where $\mathcal{A}$ is a $*$-subalgebra of A). We've shown that the K-theory class of any projection in A (or a matrix algebra over it, but we'll stick with A for notational simplicity) is represented be a projection (i.e. self-adjoint) in $\mathcal{A}$, and we've shown that an equivalence (actually conjugacy) in A between such projections translates into an equivalence in $\mathcal{A}$. For completeness in the $*$-case we should show that we can get a *unitary* equivalence in $\mathcal{A}$. This can be accomplished easily as follows: suppose that $ve_1 = e_2 v$ where e_1, e_2 are projections and v is invertible (all in $\mathcal{A}$). Then $e_1 v^* = v^* e_2$ and it follows that e_1 commutes with v^*v. Hence it commutes with any function of v^*v, in particular with $(v^*v)^{-1/2}$. Now $u = v(v^*v)^{-1/2}$ is unitary and

$$ue_1 = v(v^*v)^{-1/2}e_1 = ve_1(v^*v)^{-1/2} = e_2 v(v^*v)^{-1/2} = e_2 y.$$

It remains to observe that

$$(v^*v)^{-1/2} = \frac{1}{2\pi i}\int_\Gamma z^{-1/2}(z1 - v^*v)^{-1}\,dz,$$

where the contour Γ lies in the right half plane and encircles the spectrum of v^*v, and $z^{-1/2}$ is defined on the plane cut along the negative real axis; thus $(v^*v)^{-1/2} \in \mathcal{A}$ (arguing as above) and hence $u \in \mathcal{A}$.

Lecture 17
K-theory and Classification

[The written portion of this is a mini-lecture. Nate Brown will take it from there. . .]

In 2015 we studied the classification of UHF (uniformly hyperfinite) C^*-algebras, also known as *Glimm algebras*. A UHF algebra is the inductive limit of a sequence of matrix algebras and unital $*$-homomorphisms.

Lemma 17.1. *Let $M_m(\mathbb{C})$ and $M_n(\mathbb{C})$ be matrix algebras. There exists a unital $*$-homomorphism $M_m(\mathbb{C}) \to M_n(\mathbb{C})$ if and only if $m|n$; and, when this condition is satisfied, all such unital $*$-homomorphisms are unitarily equivalent to that given by*

$$T \mapsto \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

with T repeated n/m times down the diagonal and zeroes elsewhere. $\square$

Suppose now that $k_1|k_2|k_3|\cdots$ is an increasing sequence of natural numbers, each of which divides the next. By the lemma, there is associated to this sequence a unique (up to unitary equivalence) sequence of matrix algebras and unital $*$-homomorphisms

$$M_{k_1}(\mathbb{C}) \xrightarrow{\alpha_1} M_{k_2}(\mathbb{C}) \xrightarrow{\alpha_2} M_{k_3}(\mathbb{C}) \longrightarrow \cdots.$$

The inductive limit of this sequence is a UHF algebra, and all UHF algebras arise in this way.

For each prime p there is a natural number or infinity n_p defined to be the supremum of the number of times p divides k_i, as $i \to \infty$. The formal product $\prod p_r^{n_{p_r}}$ is called the *supernatural number* associated to the UHF algebra. We proved in 2015 the classification theorem for UHF algebras, due to Glimm:

Theorem 17.2. *Two UHF algebras are isomorphic if and only if they have the same supernatural number.* $\square$

This discussion made no mention of K-theory. However, we can reformulate the result in K-theoretic terms. Let $\mathfrak{k}$ be a supernatural number. Associate to it the subgroup $\mathbb{Q}(\mathfrak{k})$ of the additive group of the rationals, comprised of all those fractions m/n such that n divides $\mathfrak{k}$ in the obvious sense (that is, each prime p appears in n at most as many times as it appears in $\mathfrak{k}$).

Recall that a UHF algebra has a unique tracial state τ, the limit of the normalized traces on the matrix algebras from which it is constructed.

Proposition 17.3. *Let A be a UHF algebra with supernatural number $\mathfrak{k}$ and canonical trace τ. Then the dimension homomorphism*

$$\dim_\tau \colon K_0(A) \to \mathbb{R}$$

maps $K_0(A)$ isomorphically onto the group $\mathbb{Q}(\mathfrak{k}) \leqslant \mathbb{R}$. Under this isomorphism, the K-theory class $[1]$ of the unit is mapped to $1 \in \mathbb{Q}(\mathfrak{k})$.

Proof. Use Proposition 7.11 to compute the K-theory group as the limit of the sequence

$$\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \dots,$$

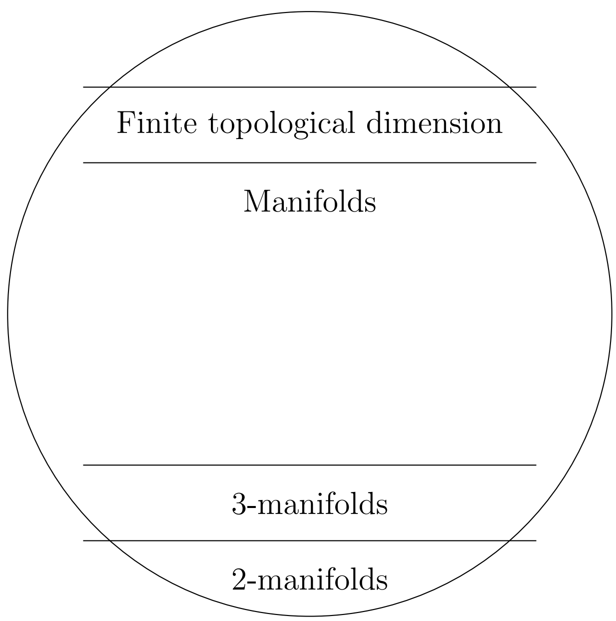
where the successive homomorphisms are multiplication by k_{j+1}/k_j and the normalized trace τ on the j'th copy of $\mathbb{Z}$ is multiplication by $1/k_j$. $\square$

Now it is easy to see that two additive groups $\mathbb{Q}(\mathfrak{k})$ and $\mathbb{Q}(\mathfrak{k}')$ are isomorphic, by an isomorphism sending 1 to 1, if and only if $\mathfrak{k} = \mathfrak{k}'$. Thus we recover Glimm's classification theorem from K-theory. In fact, if you will go back and look at the proof of Theorem 17.2 we gave in 2015, you'll see that it uses the same ideas as the proof of Proposition 7.11 which we relied on in the argument above.

More relevantly, though, the point is that this class of algebras A are completely determined, up to isomorphism, by the pair $(K_0(A), [1_A])$, that is "K-theory plus a little extra information". The audacious idea that this statement might be significantly generalizable is the root of the Elliott classification program.

Over to you, Nate. (What follows is my rather scrappy notes of Nate's lecture.)

Picture of the universe of topological spaces.



What can we classify? 2-manifolds (by fundamental group); 3-manifolds (Poincaré conjecture); Cantor set classification (perfect etc).

What about classifying noncommutataive topological spaces, aka C^*-algebras?

Many basic examples are *simple* (no ideals) e.g. UHF algebras, irrational rotation algebras, Cuntz algebras, $C^*_r(F_n)$.

Classification of "simple" objects is possible in other contexts e.g. finite simple groups, simple Lie groups, simple injective von Neumann algebras, etc.

Try to find C^* analogue of injectivity... possible analogs are AF, nuclear. AF means direct limits of finite dimensional algebras.

Theorem 17.4. *(Elliott 1976) Two AF algebras A and B are isomorphic if and only if their K_0 groups (considered as ordered groups with unit) are isomorphic.*

In the mid 1980s Elliott generalized this to $A\mathbb{T}$ algebras. An $A\mathbb{T}$ algebra is an inductive limit of C^*-algebras of the form $F \otimes C(\mathbb{T})$, F finite dimensional.

Theorem 17.5. *(Elliott 1989) If A, B are simple, unital $A\mathbb{T}$ with lots of projections and $K_*(A) \cong K_*(B)$ (with order and unit on K_0) then $A \cong B$.*

This applies to the irrational rotation algebra in particular.

Motivated by these few examples, Elliott conjectured that for simple unital *nuclear* C^*-algebras, the so-called *Elliott invariant* $\mathrm{Ell}(A)$ (K-theory plus information about traces) is a complete isomorphism invariant.

Kirchberg-Phillips showed the conjecture true when there are *no traces* (e.g. Cuntz algebras). But in the 1990s counterexamples of various kinds to the conjecture started to appear, e.g. in the world of AH algebras (inductive limits of $F_n \otimes C(X_n)$ or dynamical systems ($C(X) \rtimes \mathbb{Z}$). But some infinite dimensionality appears, e.g. $\dim X_n \to \infty$, or $\dim X = \infty$ in the second example.

At this point in the early 2000's, no-one knew what they were doing. Can the slowly growing family of special examples make contact with the general theory of nuclear C^*-algebras?

In 2010 Wintner and Zacharias came up with the notion of "noncommutative topological dimension" or nuclear dimension, $\dim_{nuc}(A)$ (a positive integer or $+\infty$). In 2012 we have

Theorem 17.6. *If a simple unital C^*=algebra A has finite nuclear dimension, then $A \cong A \otimes \mathfrak{Z}$, where $\mathfrak{Z}$ is the Jiang-Su algebra (a special infinite-dimensional C^*-algebra that is KK-equivalent to $\mathbb{C}$).*

This led to a "$\otimes UHF$-classification", i.e. that if $A \otimes U \cong B \otimes U$, where U ia a UHF algebra and the isomorphism is of a specific "fiber preserving" kind, then $A \otimes \mathfrak{Z} \cong B \otimes \mathfrak{Z}$. At this point the general theory of nuclear algebras made contact with the developing family of special cases (TA1 algebras). Final result

Theorem 17.7. *Suppose A, B are simple, unital, have finite nuclear dimension, and satisfy UCT (possibly redundant), then $A \cong B$ iff* $\mathrm{Ell}(A) = \mathrm{Ell}(B)$.

Some of the special cases that had to be developed.

- TAF algebras: an algebra A is TAF i for every finite subset $\mathcal{F}$ of A and every $\varepsilon > 0$, there is a finite dimensional subalgebra B of A with unit e such that
  - (i) $\|[e, x]\| < \varepsilon$ for all $x \in \mathcal{F}$;
  - (ii) $d(exe, B) < \varepsilon$;
  - (iii) $\tau(e) > 1 - \varepsilon$ (for canonical trace which exists).
- TA1 algebras — same except that B is now a subhomogenous algebra of dimension 1, i.e. $B \subseteq F \otimes C[0, 1]$.
- (Elliott, Gong, Lin, Niu, 2015) If $dim_{nuc}(A) < \infty$ and A satisfies UCT and $T(A) = T(A)_{QD}$ (quasidiagonal traces) then $A \otimes UHF \in TA1$.

- (Tikuisis-White-Wintner 2016) can remove the quasidiagonality assumption from the above.

Lecture 18

Crossed products

My aim for (probably) the rest of the course is to understand a profound paper by Connes, *An analogue of the Thom Isomorphism for Crossed Products of a C^*-Algebra by an Action of* $\mathbb{R}$, Advances in Math 39(1981), 31–55. This is one of the foundational papers of noncommutative geometry. The classical Bott periodicity theorem tells you about the K-theory of $X \times \mathbb{R}$ (or $X \times \mathbb{R}^n$) in terms of the K-theory of X. Replace the trivial vector bundle $X \times \mathbb{R}^n$ by a nontrivial (but appropriately oriented) vector bundle and you have the *Thom isomorphism theorem* in K-theory. Connes wants to do something similar, but noncommutative: replace the suspension $A \otimes C_0(\mathbb{R})$ of a C^*-algebra A by a "twisted" suspension, a crossed product by an action of $\mathbb{R}$.

In this lecture we will review results about crossed products, I want to start with some basic stuff about Fourier analysis. Let G be a locally compact abelian topological group. Then $\widehat{G}$ denotes $\mathrm{Hom}(G, \mathbb{T})$, the homomorphisms from G to the circle group $\mathbb{T}$. It is also a locally compact abelian group (under the obvious pointwise product of homomorphisms): it is compact if G is discrete, and vice versa. It is called the (Pontrjagin) *dual group* of G. Moreover, the process of forming the dual is involutive: the dual of $\widehat{G}$ is G again. We sketched this theory, very briefly, in lecture 23 of the 2015 notes.

We don't need the general theory for what we are doing here: I just bring it up to establish notation. The classical cases are

- The dual group of $\mathbb{T}$ is $\mathbb{Z}$ (and vice versa): every homomorphism $\mathbb{T} \to \mathbb{T}$ is of the form $z \mapsto z^n$, where n is an integer, and every homomorphism $\mathbb{Z} \to \mathbb{T}$ is of the form $n \mapsto e^{in\theta}$, where $e^{i\theta} \in \mathbb{T}$.
- The dual group of $\mathbb{R}$ is $\mathbb{R}$ again (and vice versa, which hardly needs saying in this case): every homomorphism $\mathbb{R} \to \mathbb{T}$ is of the form $t \mapsto e^{ist}$, where $s \in \mathbb{R}$. Sometimes it is helpful to distinguish between $\mathbb{R}$ and $\widehat{\mathbb{R}}$, so that we might write the above pairing as

$$(s, t) \mapsto e^{ist}, \quad s \in \widehat{\mathbb{R}},\ t \in \mathbb{R}.$$

Definition 18.1. Let G be a locally compact group (we will mostly be interested in $\mathbb{R}$). An *action* of G on a C^*-algebra A is a group homomorphism $G \to \mathrm{Aut}(A)$ such that, for each $a \in A$, the map $G \to A$ defined by $g \mapsto \alpha_g(a)$ is continuous.

Examples

(i) The trivial action of any group on $\mathbb{C}$.

(ii) Any group G acts by translation on $C_0(G)$. Using notation appropriate for abelian groups (because our main interest is $\mathbb{R}$), $\alpha_s f(x) = f(x - s)$.

(iii) If X is a compact Hausdorff space and φ is a *flow* on X, that is a one-parameter group of homeomorphisms, then $\alpha_s f(x) = f \circ \varphi_{-s}(x)$ defines an action of $\mathbb{R}$ on $C(X)$ (similarly for $C_0(X)$ if X is only locally compact). Exercise: Verify why point-norm continuity is true and is the appropriate continuity condition in this and the previous case.

Suppose now that G, which we will assume is abelian (for notation), acts on A via α. We are going to construct a *crossed product* C^*-algebra $A \rtimes_\alpha G$. As a first step, consider $C_c(G; A)$ and equip it with the following product and involution:

$$(f * g)(t) = \int_G f(s)\alpha_s(g(t-s))ds, \tag{18.2}$$
$$f^*(t) = \alpha_t\left(f(-t)^*\right).$$

The integration is with respect to Haar measure on G (Lebesgue measure if $G = \mathbb{R}$). Then $C_c(G; A)$ becomes a $*$-algebra (check!). Define an L^1 norm on $C_c(G; A)$ in the obvious way as $\|f\|_1 = \int \|f(t)\|dt$ and complete to obtain a Banach algebra $L^1(G; A)$ (not unital unless G is discrete and A is unital).

Now we want to complete to obtain a C^*-algebra. Define the Hilbert A-module $\mathfrak{H} = L^2(G; A)$ to be the completion of $C_c(G; A)$ with respect to the A-valued inner product

$$\langle \xi, \eta \rangle = \int_G \xi(t)^* \eta(t)\, dt;$$

this can be regarded as the measurable functions $\xi\colon G \to A$ such that

$$\int_G \xi(t)^* \xi(t) dt < \infty$$

where the integral takes values in $A^+ \cup \{\infty\}$. There are natural representations $\pi\colon A \to \mathfrak{B}(\mathfrak{H})$ and $\rho\colon G \to U(\mathfrak{H})$ given by

$$(\pi(a)\xi)(t) = \alpha_{-t}(a)\xi(t), \quad (\rho(s)\xi)(t) = \xi(t-s), \tag{18.3}$$

and these satisfy the *covariance condition*

$$\pi(\alpha_t(a)) = \rho(t)\pi(a)\rho(t)^*. \tag{18.4}$$

By construction, any pair of representations satisfying the covariance condition gives rise to a $*$-representation of $L^1(G; A)$, call it ψ, via

$$\psi(f) = \int_G \pi(f(t))\rho(t)\, dt.$$

Definition 18.5. The representation of $L^1(G; A)$ on $\mathfrak{B}(\mathfrak{H}) = \mathfrak{B}(L^2(G; A))$ obtained in this way is called the *regular representation* of $L^1(G; A)$. This representation is faithful, and the C^*-algebra obtained by completion of $L^1(G; A)$ in the norm arising from the regular representation is called the *reduced crossed product* $A \rtimes_\alpha G$.

Remark 18.6. The reduced crossed product is usually defined without mentioning Hilbert modules! Instead, one considers a faithful, essential representation π_0 of A on a Hilbert space H_0 and builds a representation of A on $H = L^2(G; H_0)$ (a Hilbert *space*) using the same formulas as above. Then one has to show independence of π_0. Related to our formulation,

$$H = \mathfrak{H} \otimes_{\pi_0} H_0$$

using the interior tensor product of Hilbert modules defined in Lecture 10, and the "conventional" regular representation is the image of ours under the map $\mathfrak{B}(\mathfrak{H}) \to$

$\mathfrak{B}(H)$, $T \mapsto T \otimes 1$; since this map is isometric (for π_0 faithful) all the "conventional" regular representations have the same norm as ours (and in particular the same norm as each other).

Remark 18.7. One can also define a "full" crossed product by completing in the maximal norm (compare group C^*-algebras), and usually the notation $\rtimes$ is reserved for the full crossed product, with $\rtimes_r$ for the reduced. But we are only interested in abelian groups for which the two concepts coincide anyhow, so I won't sweat the notation.

A couple of examples with $\mathbb{R}$.

Example 18.8. $\mathbb{C} \rtimes \mathbb{R}$ is the group C^*-algebra of $\mathbb{R}$, which is $C_0(\widehat{\mathbb{R}})$. Let's remind ourselves how the proof goes: the Plancherel theorem identifies $L^2(\mathbb{R})$ with $L^2(\widehat{\mathbb{R}})$, and moreover identifies the action of an element $f \in L^1(\mathbb{R})$ on $L^2(\mathbb{R})$ by convolution with the action of its Fourier transform $\hat{f}$ (which is an element of $C_0(\widehat{\mathbb{R}})$ by the Riemann-Lebesgue lemma) on $L^2(\widehat{\mathbb{R}})$ by pointwise multiplication. The transforms of L^1 functions form a dense subset of $C_0(\widehat{\mathbb{R}})$ and therefore, on completing in the norm of $\mathfrak{B}(L^2(\mathbb{R})) = \mathfrak{B}(L^2(\widehat{\mathbb{R}}))$, we obtain the whole of $C_0(\widehat{\mathbb{R}})$.

Example 18.9. Consider $C_0(\mathbb{R}) \rtimes_\alpha \mathbb{R}$, where $\mathbb{R}$ acts on $C_0(\mathbb{R})$ by translation as in example (ii) above, $\alpha_s f(t) = f(t-s)$. An element of $C_c(G; A)$ is then a function on $\mathbb{R} \times \mathbb{R}$; if we take the dense subset of $A = C_0(\mathbb{R})$ consisting of compactly supported functions, we are considering elements of $C_c(G; A)$ that are functions $f(t)(x)$ compactly supported in both variables. The composition law above gives for $h = f * g$,

$$h(t)(x) = \int f(s)(x) g(t-s)(x-s)\, ds.$$

Make a change of variables so that $f(t)(x) = F(x, x-t)$. Then the composition law becomes

$$H(x, x-t) = \int F(x, x-s) G(x-s, x-t)\, ds$$

or

$$H(x, z) = \int F(x, y) G(y, z)\, dy.$$

This is the ordinary rule for composition of (compactly supported) integral kernels on $\mathbb{R} \times \mathbb{R}$. Moreover, the regular representation on $L^2(\mathbb{R} \times \mathbb{R})$ becomes

$$(F\xi)(x, z) = \int F(x, y) \xi(y, z)\, dy;$$

that is the tensor product of the usual representation of smoothing operators on $L^2(\mathbb{R})$ with a trivial representation on $L^2(\mathbb{R})$. The norm induced by the regular representation is therefore the same as the usual operator norm on $L^2(\mathbb{R})$, and the completion in this norm is the compact operators. That is, $C_0(\mathbb{R}) \rtimes_\alpha \mathbb{R} \cong \mathfrak{K}$.

Suppose that G is an abelian group and that $\widehat{G}$ is its dual group. Let $A \rtimes_\alpha G$ be some crossed product with G. There is an action $\hat{\alpha}$ of $\widehat{G}$ on this crossed product defined for $f \in C_c(G; A) \subseteq A \rtimes_\alpha G$ by

$$(\hat{\alpha}_s f)(t) = (s|t) f(t).$$

Here we've used $(s|t)$ to denote the pairing between G and $\widehat{G}$. In the case of $G = \mathbb{R}$ which is our main interest, $\widehat{G}$ is isomorphic to $\mathbb{R}$ also and the definition of this *dual action* becomes

$$(\hat{\alpha}_s f)(t) = e^{its} f(t).$$

Remark 18.10. The construction of the crossed product is *functorial* in an obvious sense: if (A, α) and (B, β) are two $\mathbb{R}$-C^*-algebras and $\psi\colon A \to B$ is an $\mathbb{R}$-equivariant $*$-homomorphism, then ψ induces a $*$-homomorphism

$$\psi \rtimes \mathbb{R}\colon A \rtimes_\alpha \mathbb{R} \to B \rtimes_\beta \mathbb{R}$$

in the (hopefully) "obvious" way. Moreover, this induced homomorphism is itself functorial for the dual action.

Theorem 18.11 (Takai duality theorem)**.** *Let A be equipped with an $\mathbb{R}$-action α. There is an isomorphism*

$$(A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}} \cong A \otimes \mathfrak{K}.$$

(The corresponding theorem is true for any LCA group but we give the proof only for $\mathbb{R}$.)

Remark 18.12. Suppose that the action α were *trivial*. Then $A \rtimes_\alpha \mathbb{R}$ is equal to $A \otimes C^*(\mathbb{R}) = A \otimes C_0(\widehat{\mathbb{R}})$. The dual action of $\widehat{\mathbb{R}}$ on $C_0(\widehat{\mathbb{R}})$ is by translations, so the result follows from the Example 18.9. The general proof uses a kind of "untwisting" to get back to this special case.

Proof. We consider two C^*-dynamical systems

(a) $(A \rtimes_\alpha \mathbb{R})$ with the dual $\widehat{\mathbb{R}}$ action.

(b) $(A \rtimes_\iota \widehat{\mathbb{R}})$ with the dual $\mathbb{R}$ action (here ι denotes the *trivial* action).

We claim that these two systems give isomorphic crossed products. Since the second system gives $A \otimes \mathfrak{K}$ (by the argument outlined in the preceding remark) this will complete the proof.

Look at the regular representations. Represent A on a Hilbert space H. Then $A \rtimes_\alpha \mathbb{R}$ is represented on $L^2(\mathbb{R}; H)$ and $(A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}}$ is represented on $L^2(\widehat{\mathbb{R}} \times \mathbb{R}; H)$. For $f \in C_c(\widehat{\mathbb{R}} \times \mathbb{R}; A) \subseteq (A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}}$ and $\xi \in L^2(\widehat{\mathbb{R}} \times \mathbb{R}; H)$ the formula for the action π_1 is

$$\pi_1(f)\xi(s,t) = \iint \alpha_{-t}(f(x,y))\xi(s-x, t-y)e^{-isy}\, dxdy.$$

Similarly for each $g \in C_c(\mathbb{R} \times \widehat{\mathbb{R}}; A) \subseteq (A \rtimes_\iota \widehat{\mathbb{R}}) \rtimes_{\hat{\iota}} \mathbb{R}$ and $\eta \in L^2(\mathbb{R} \times \widehat{\mathbb{R}}; H)$ the formula for the action π_2 is

$$\pi_2(g)\eta(t,s) = \iint \alpha_{-t}(g(y,x))\eta(s-x, t-y)e^{itx} dxdy.$$

Now if we define an isometry $\Phi\colon C_c(\mathbb{R}\times\widehat{\mathbb{R}};A)\to C_c(\widehat{\mathbb{R}}\times\mathbb{R};A)$ by

$$(\Phi f)(s,t)=e^{-ist}f(t,s)$$

and a unitary $W\colon L^2(\mathbb{R}\times\widehat{\mathbb{R}};H)\to L^2(\widehat{\mathbb{R}}\times\mathbb{R};H)$ by

$$W\eta(s,t)=e^{-ist}\eta(t,s),\quad W^*\xi(t,s)=e^{its}\xi(s,t),$$

then

$$\pi_1(\Phi(f))W\eta=W\pi_2(f)\eta.$$

Thus (W,Φ) conjugate the representation π_1 of $C_c(\widehat{\mathbb{R}}\times\mathbb{R};A)$ to the representation π_2. Since these functions are dense in both crossed products, it follows that the crossed products are isomorphic. □

Consider now any covariant pair of representations (π,ρ) of $(A,\mathbb{R})$ on a Hilbert space H (see equation 18.4). The representation ρ corresponds to an infinitesimal generator X, an unbounded self-adjoint operator on H, by Stone's theorem, and we have

$$\pi(\delta(a))=i[X,\pi(a)];$$

in other words, the derivation δ passes to the commutator with the unbounded operator X, which is an "unbounded normalizer" for A in this representation (and an "unbounded multiplier" for the crossed product).

Remark 18.13. Notice that A also embeds into the multipliers of the crossed product algebra (it would be an actual subalgebra if G was discrete). Connes develops an abstract theory of "unbounded multipliers" to provide a place for the operator I've called X above to live. However, I think the concrete representation will be enough. . . we'll see.

Definition 18.14. Two actions α,α' of $\mathbb{R}$ on a C^*-algebra A are *inner equivalent* if there is a strictly continuous map $t\mapsto u_t$, $\mathbb{R}\to U(\mathfrak{M}(A))$ (the unitaries in the multiplier algebra of A) which is a *cocycle* for α in the sense that

$$u_{s+t}=u_s\alpha_s(u_t)$$

and which *conjugates* α to α' in the sense that

$$\alpha'_t(a)=u_t\alpha_t(a)u_t^*\quad\forall\ a\in A,\ t\in\mathbb{R}.$$

[The cocycle condition ensures that α'_t, defined by the conjugation formula, is a 1-parameter group of automorphisms.]

Reminder — *Strictly continuous* means that for each fixed $a\in A$, the maps $t\mapsto u_ta$ and $t\mapsto u_t^*a$ are norm continuous — in this case the second condition follows from the first, of course.

Proposition 18.15. *If α,α' are inner equivalent then the crossed products $A\rtimes_\alpha\mathbb{R}$ and $A\rtimes_{\alpha'}\mathbb{R}$ are isomorphic. Moreover, this isomorphism is equivariant for the dual actions.*

Proof. We consider the regular representations of $A \rtimes_\alpha \mathbb{R}$ and $A \rtimes_{\alpha'} \mathbb{R}$ on the same Hilbert module $\mathfrak{H} = L^2(\mathbb{R}; A)$ (see Definition 18.5). Remember that the multiplier algebra of A is just the algebra $\mathfrak{B}(A)$ where A is considered as a Hilbert A-module. Thus the cocycle $t \mapsto u_t$ defines (by pointwise multiplication) a unitary operator U on $\mathfrak{H}$, and this unitary clearly conjugates the two regular reprsentations, which are therefore isomorphic.

The fact that U is a pointwise-multiplication operator shows it commutes with the action of $\widehat{\mathbb{R}}$, which is a pointwise multiplication by something in the center of $\mathfrak{B}(A)$. Thus, the isomorphism that we have described is $\widehat{\mathbb{R}}$-equivariant. $\square$

Lecture 19

Connes' Thom isomorphism

Before we start it will be helpful to know the explicit form of the suspension isomorphisms σ_i between $K_i(A)$ and $K_{i\pm 1}(SA)$, where $SA = A \otimes C_0(\mathbb{R})$ is the suspension.

Lemma 19.1. *Let A be a unital C^*-algebra.*

- *The isomorphism $\sigma_0 \colon K_0(A) \to K_1(SA)$ is described as follows: Let $u \colon \mathbb{R} \to \mathbb{T}$ be any function such that $u(t) \to 1$ as $t \to \pm\infty$ and such that the induced map from the 1-point compactification of $\mathbb{R}$ to $\mathbb{T}$ has winding number -1; for example, $u(t) = -\exp(-i\pi\chi(t))$ where χ is a normalizing function. Let e be a projection in $M_n(A)$ describing a class in $K_0(A)$. Then $t \mapsto (1-e) + eu(t)$ is a loop of unitaries in $\widetilde{SA}$, mapping to $0 \in K_1(A)$ under evaluation at 1 and thus describing an element of $K_1(SA)$. This is $\sigma_0[e]$.*
- *The isomorphism $\sigma_1 \colon K_1(A) \to K_0(SA)$ is described as follows: Let $u \in M_n(A)$ be a unitary defining a class in $K_1(A)$. Let $t \mapsto R(t)$ be a continuous map from $\mathbb{R}$ into $SO(2n)$ such that $R(t) \to \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ as $t \to -\infty$ and $R(t) \to \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ as $t \to +\infty$; for example*

$$R(t) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix}, \quad \theta(t) = \frac{\pi(1+\chi(t))}{4}, \quad \chi \text{ normalizing.}$$

Then the formal difference of projections $[e] - [e_0]$, where

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e = W(t)e_0W(t)^*, \quad W(t) = R(t)\begin{pmatrix} u^* & 0 \\ 0 & 0 \end{pmatrix}R(t)^*,$$

defines a class in $K_0(SA)$. This class is $\sigma_1[u]$.

Proof. (a) According to our description, the isomorphism $K_1(SA) \to K_0(A)$ (i.e. the inverse of σ_0) is described by the index map coming from the Toeplitz extension tensored with A; see the discussion after Proposition 8.8. The claimed description of σ_0 therefore follows from the Toeplitz index theorem (the index of a Toeplitz operator with symbol u is minus the winding number of u, see 2015 notes, Theorem 20.6).

(b) This is a consequence of the description of the boundary map from K_1 to K_0 given in Exercise 7.5, together with the fact that, with $W(t)$ as in the statement, $W(t)\left(\begin{smallmatrix} u^* & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is a path joining $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$. ☐

Connes' Thom isomorphism is a noncommutative version of these suspension isomorphisms. Rather like Cuntz' proof of Bott periodicity (lecture 7), the Connes Thom map is introduced by axioms.

Before stating Connes' theorem let's make a bit of a notational change. If A is an $\mathbb{R}$-C^*-algebra, we'll denote the crossed product $A \rtimes \mathbb{R}$ simply by $\widehat{A}$. The dual action now tells us that $\widehat{A}$ is a $\widehat{\mathbb{R}}$-C^*-algebra, and Takai's duality theorem simply says $\widehat{\widehat{A}} \cong A \otimes \mathfrak{K}$. Similarly we'll write $\hat{\psi}$ for the homomorphism $\widehat{A} \to \widehat{B}$ induced by an equivariant homomorphism $\psi \colon A \to B$, and so on.

Connes' theorem states

Theorem 19.2. *There is a unique way of associating, to every $\mathbb{R}$-C^*-algebra A, a pair of homomorphisms $\varphi_0\colon K_0(A) \to K_1(\widehat{A})$ and $\varphi_1\colon K_1(A) \to K_0(\widehat{A})$ having the following properties:*

*(i) (**Normalization**) If $A = \mathbb{C}$, the map φ_0 sends the standard generator of $K_0(\mathbb{C}) = \mathbb{Z}$ to the standard generator of $K_1(C^*(\mathbb{R})) = K_1(C_0(\widehat{\mathbb{R}})) = \mathbb{Z}$.*

*(ii) (**Naturality**) The maps φ_0, φ_1 are natural for $\mathbb{R}$-equivariant homomorphisms $\psi\colon A \to B$, i.e., the diagrams*

$$\begin{tikzcd} K_0(A) \arrow[r, "\varphi_0^A"] \arrow[d, "\psi_*"] & K_1(\widehat{A}) \arrow[d, "\hat{\psi}_*"] \\ K_0(B) \arrow[r, "\varphi_0^B"] & K_1(\widehat{B}) \end{tikzcd}, \qquad \begin{tikzcd} K_1(A) \arrow[r, "\varphi_1^A"] \arrow[d, "\psi_*"] & K_0(\widehat{A}) \arrow[d, "\hat{\psi}_*"] \\ K_1(B) \arrow[r, "\varphi_1^B"] & K_0(\widehat{B}) \end{tikzcd}$$

commute.

*(iii) (**Suspension**) The maps φ_0, φ_1 commute with suspensions, i.e. the diagrams*

$$\begin{tikzcd} K_0(A) \arrow[r, "\varphi_0^A"] \arrow[d, "\sigma_0"] & K_1(\widehat{A}) \arrow[d, "\sigma_1"] \\ K_1(SA) \arrow[r, "\varphi_1^{SA}"] & K_0(S\widehat{A}) \end{tikzcd}, \qquad \begin{tikzcd} K_1(A) \arrow[r, "\varphi_1^A"] \arrow[d, "\sigma_1"] & K_0(\widehat{A}) \arrow[d, "\sigma_0"] \\ K_0(SA) \arrow[r, "\varphi_1^{SA}"] & K_1(S\widehat{A}) \end{tikzcd}$$

commute. (Note that $\widehat{SA} = S\widehat{A}$.)

Moreover, these uniquely defined maps are in fact isomorphisms.

We'll get to the proof. Let me remind you now, however, of some basic results on Hilbert space

Theorem 19.3 (Stone's theorem)**.** *Let ρ be a unitary representation of $\mathbb{R}$, that is, a homomorphism from $\mathbb{R}$ to the unitary group of H that is continuous in the point-norm sense that*

$$\lim_{t\to 0} \|\rho(t)\xi - \xi\| = 0 \quad \forall\, \xi \in H.$$

Then there is an unbounded self-adjoint operator X on H such that $\rho(t) = e^{itX}$. Moreover, we can define X by

$$X\xi = \lim_{t\to 0} \frac{\rho(t)\xi - \xi}{t},$$

with domain $\mathrm{dom}(X)$ *being the set of those ξ for which this limit exists.*

Proof. The $\{\rho(t)\}$ generate a commutative von Neumann algebra of operators. Thus by spectral theory we can reduce to the case where $H = L^2(X, \mu)$ and $\rho(t)$ is multiplication by a $\mathbb{T}$-valued function f_t. The point-norm continuity tells us that $f_t(x) \to 1$

as $t \to 0$ for almost all x. Moreover the identity

$$\rho(2^{-n})^2 = \rho(2^{-n+1})$$

translates to say that for almost all x, $f_{2^{-n}}(x)^2 = f_{2^{-n+1}}(x)$. Thus for almost all $x \in X$, the sequence $z_n = f_{2^{-n}}(x) \in \mathbb{T}$ satisfies $z_n = z_{n+1}^2$ and $z_n \to 1$. It is easy to see that any such sequence must be given for sufficiently large n by

$$z_n = \exp(i2^{-n}\theta), \quad \text{where } \theta = \lim_{n\to\infty} i^{-1}2^n\left(z_n - 1\right).$$

Approximating by dyadic rationals and using the group law, we find that for almost all x, the limit

$$g(x) = \lim_{n\to\infty} i^{-1}2^n\left(f_{2^{-n}}(x) - 1\right)$$

exists, is real, and $f_t(x) = e^{itg(x)}$ for all t. Letting X be the unbounded self-adjoint operator corresponding to multiplication by g we get Stone's theorem as stated. □

We will also want the perturbation theory for these one-parameter groups.

Proposition 19.4 (Duhamel formula). *Let X be unbounded and self-adjoint, generating a 1-parameter group e^{itX} as above. Let Q be* bounded *and self-adjoint. Then $Y = X + Q$ is also self-adjoint (this is a standard fact) and the 1-parameter groups e^{itX} and e^{itY} are related by the formula $e^{itY} = U(t)e^{itX}$, where $U(t)$ is given by the norm convergent series*

$$(\heartsuit) \qquad U(t) = \sum_{n=0}^{\infty} i^n \int_{0\leqslant s_1\leqslant\cdots\leqslant s_n\leqslant t} Q_{s_1}\cdots Q_{s_n}\,ds_1\cdots ds_n,$$

where $Q_s = e^{isX}Qe^{-isX}$. In particular, $U(t)$ belongs to the C^*-algebra generated by the Q_s.

Note that if Q *commutes* with X then $Q_s = Q$ for all s and the formula ($\heartsuit$) boils down to

$$\sum_{n=0}^{\infty} \frac{(it)^n}{n!} Q^n = e^{itQ},$$

as expected.

Proof. The series for $U(t)$ indeed converges in norm (remember that the volume of the unit n-simplex is $1/n!$). Moreover, it can be differentiated term by term. If we do that, we get

$$\dot U(t) = \sum_{n=1}^{\infty} i^n \int_{0\leqslant s_1\leqslant\cdots\leqslant s_{n-1}\leqslant t} Q_{s_1}\cdots Q_{s_{n-1}}Q_t\,ds_1\cdots ds_n = iU(t)Q_t.$$

Thus, for a smooth vector ξ,

$$\begin{aligned}\frac{d}{dt}\left(U(t)e^{itX}e^{-itY}\xi\right) &= \\ & i\left(U(t)Q_te^{itX} + U(t)Xe^{itX}\right)e^{-itY}\xi - iU(t)e^{itX}Ye^{-itY}\xi = \\ & \qquad iU(t)e^{itX}(P + X - Y)e^{-itY}\xi = 0.\end{aligned}$$

It follows that $U(t)e^{itX}e^{-itY} \equiv 1$, whence the result. $\square$

Remark 19.5. From the group laws for $t \mapsto e^{itX}$ and $t \mapsto e^{itY}$ we see that U satisfies the cocycle condition

$$U(s+t) = U(s)\left(e^{isX}U(t)e^{isX}\right);$$

compare Definition 18.14.

Proposition 19.6. *Let A be a unital C^*-algebra equipped with an $\mathbb{R}$-action α. Define the "smooth subalgebra" $\mathcal{A}$ (also known as A^∞) to be*

$$\mathcal{A} = \{a \in A : t \mapsto \alpha_t(a) \textit{is a } \text{smooth } \textit{map } \mathbb{R} \to A\}.$$

Then $\mathcal{A}$ is a dense, inverse closed Fréchet subalgebra of A, i.e. it satisfies the conditions of Theorem 14.5.

Proof. Let h be a smooth bump function on $\mathbb{R}$ (compactly supported, positive, total mass 1) and let $h_\varepsilon(t) = \varepsilon^{-1}h(\varepsilon^{-1}t)$. For any $a \in A$, the element

$$a_\varepsilon = \int h_\varepsilon(t)\alpha_t(a)\,dt$$

belongs to $\mathcal{A}$, since we can integrate by parts:

$$\frac{\alpha_s(a_\varepsilon) - a_\varepsilon}{s} = \int \frac{h_\varepsilon(t-s) - h_\varepsilon(t)}{s}\alpha_t(a)\,dt \to -\int h'_\varepsilon(t)\alpha_t(a)\,dt,$$

and similarly for higher derivatives. But $a_\varepsilon \to a$ as $\varepsilon \to 0$; hence $\mathcal{A}$ is dense. Fréchet seminorms are provided by the norms of the derivatives of $s \mapsto \alpha_s(a)$ at 0. For inverse closure, suppose that $a \in \mathcal{A}$; then $\alpha_t(a^{-1}) = \alpha_t(a)^{-1}$ is a smooth function of t because inversion is a smooth function, from the open subset of A consisting of invertibles, to A itself. $\square$

Suppose now that $a \in A^\infty$ as above. Then we may define

$$\delta(a) = \lim_{t\to 0} \frac{\alpha_t(a) - a}{t} = \left[\frac{d}{dt}\alpha_t(a)\right]_{t=0}.$$

The notation is supposed to bring out that this is a sort of C^*-analog of the "unbounded generator" of a unitary group on Hilbert space, that appears in Stone's theorem. δ is a densely defined *unbounded derivation* of A; in other words, on its domain $\mathcal{A}$ it satisfies

$$\delta(ab) = a\delta(b) + \delta(a)b,$$

as follows from the ordinary product rule for differentiation.

Consider now the regular representation of the crossed product $A \rtimes \mathbb{R}$ on $L^2(\mathbb{R}; A)$. The unitaries $U_t\xi(s) = \xi(s-t)$ do not belong to the regular representation of $A \rtimes \mathbb{R}$, but they are *multipliers* of it; and A also lies inside the multipliers of $A \rtimes \mathbb{R}$ by identifying an element $a \in A$ with the "constant" function $s \mapsto \alpha_{-s}(a)$. Moreover, inside the multiplier algebra $M(A \rtimes \mathbb{R})$ we have the crucial relation

$$\alpha_t(a) = U_t a U_t^*,$$

which characterizes the crossed product. By Stone's theorem, then, we have that given any faithful Hilbert space representation H of the crossed product, there is an unbounded self-adjoint operator X such that $e^{itX} \in M(A \rtimes \mathbb{R})$ for all t (this is what we refer to as an *unbounded multiplier* of $A \rtimes \mathbb{R}$ in this representation) and such that the unitary group e^{itX} implements the action of $\mathbb{R}$ on A in the sense that

$$\alpha_t(a) = \exp(itX) a \exp(-itX).$$

Of course, in the regular representation, $X = -id/dt$. Note that if $a \in \mathcal{A}$ (the domain of δ as defined in Proposition 19.6), then

$$\delta(a) = \left[\frac{d}{dt} \exp(itX) a \exp(-itX)\right]_{t=0} = iXa - iaX = i[X, a];$$

this commutator extends to a bounded operator on H which is an element of $A \subseteq M(A \rtimes \mathbb{R})$.

Remark 19.7. Our implementation of the notion "unbounded multiplier" goes through a specific representation. In Appendix 5 to the paper, Connes sets up a representation-independent notion of "unbounded multiplier". This is a nice thing to have but does not seem to be necessary for the proofs. In fact, one can avoid talking about unbounded multipliers at all, but then some formulas that are used (see later) seem to be invented by magic—the unbounded multiplier theory answers the question, "where the `!%^&*` did that come from?"

Here is the crucial lemma in proving the uniqueness of the Thom maps.

Lemma 19.8. *Let A and $\mathcal{A}$ be as in Proposition 19.6. Suppose that $e \in \mathcal{A}$ is a projection. Then there is another action α' of $\mathbb{R}$ on A, inner equivalent to the originally given one, that fixes e.*

Proof. Represent the crossed product $A \rtimes_\alpha \mathbb{R}$ faithfully on a Hilbert space H. As remarked above, there is an unbounded multiplier X such that the unitary group $\exp(itX)$ implements the action α_t of $\mathbb{R}$ on A. We want to change to a different group of automorphisms that fixes e; this can be achieved if we replace the generator X with a new generator Y that *commutes* with e.

Put

$$\begin{aligned} Y = eXe + (1-e)X(1-e) = eX - e[X,e] + (1-e)X + (1-e)[X,e] = \\ X + (1-2e)[X,e] = X + [[X,e],e] = X + Q, \end{aligned}$$

where $Q = -i[\delta(e), e]$ is a self-adjoint element of A. In particular Y is a bounded perturbation of X and hence a self-adjoint operator (no domain/deficiency issues). (The next to last equality above uses that e is a projection: differentiating $e^2 = e$ gives $[X,e] = e[X,e] + [X,e]e$. whence $(1-2e)[X,e] = -e[X,e] + [X,e]e = [[X,e],e]$.) From Proposition 19.4 above we find $\exp(itY) = U(t)\exp(itX)$, where $U(t)$ is given by the Duhamel formula (♡)

$$U(t) = \sum_{n=0}^{\infty} i^n \int_{0 \leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant t} \alpha_{s_1}(Q) \cdots \alpha_{s_n}(Q)\, ds_1 \cdots ds_n,$$

which converges in norm and so defines a one-parameter family of unitaries in $M(A)$ (in A if A is unital). To put that another way, $U(t)$ is a cocycle (Remark 19.5) that defines an inner equivalence between the originally given group of automorphisms $\alpha_t(a) = \exp(itX)a\exp(-itX)$ and a new group $\alpha'_t(a) = \exp(itY)a\exp(-itY)$ which, by construction, fixes e. □

Proof of uniqueness in theorem 19.2. Let's sketch out why this lemma tells us that the Connes Thom maps are unique (if they exist). By the suspension axiom it is sufficient to show that φ_0 is unique since φ_1 is related to it by the suspension isomorphisms.

One needs to establish a couple of preliminary points (from the axioms):

(a) We can reduce to the case A unital without loss of generality.

(b) We can pass to matrices: the action $\mathbb{R}$ on A extends in an obvious way to an action on $M_n(A)$, and the "top left corner inclusion" $\rho\colon A \to M_n(A)$ is equivariant. If we have a projection e in $M_n(A)$ we can think of it as giving a K-theory class either for $M_n(A)$ or for A, and these classes are related by $\rho_*\colon K_0(A) \cong K_0(M_n(A))$. Naturality gives $\hat\rho_*\varphi_0^A([e]) = \varphi_0^{M_n(A)}\rho_*([e])$. But $\hat\rho_*$ is an isomorphism (as ρ_* is), since $M_n(A) \rtimes \mathbb{R} = M_n(A \rtimes \mathbb{R})$: so it will suffice to consider K-theory classes defined by projections in A.

(c) If α and α' are inner equivalent actions of $\mathbb{R}$ on A, then the diagram

$$\begin{tikzcd} K_0(A) \arrow[r, "\varphi_0^\alpha"] \arrow[dr, "\varphi_0^{\alpha'}"'] & K_1(A \rtimes_\alpha \mathbb{R}) \arrow[d, "\cong"] \\ & K_1(A \rtimes_{\alpha'} \mathbb{R}) \end{tikzcd}$$

commutes, where the vertical isomorphism is induced by the isomorphism of Proposition 18.15.

Suppose that these (easy) preliminaries are taken care of, then the uniqueness follows from Lemma 19.8. Suppose given a K-theory class for A. By passing to matrices, if needed, we assume that it's given by a projection e in A. By smoothing theory, Theorem 14.5, we may assume in fact that the projection e belongs to the smooth subalgebra $\mathcal{A}$. By changing the action to an inner equivalent one, Lemma 19.8, we can now assume that the projection e is actually *invariant* under the action. But this means that e is the image of $1 \in \mathbb{C}$ by an *equivariant* (though non-unital) homomorphism $\psi\colon \mathbb{C} \to A$. Since the action of the Thom isomorphism on $K(\mathbb{C})$ is determined by the normalization axiom, the naturality axiom now tells us that $\varphi_0([e])$ must equal $\hat\psi_*([z])$, where z denotes the canonical generator of $K_1(C_0(\widehat{\mathbb{R}}))$. □

Remark 19.9. I should say a few words about "easy preliminary" (c) above. Given the set-up for an inner equivalence, we can make $\mathbb{R}$ act on $M_2(A)$ by

$$\beta_t\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha_t(a_{11}) & \alpha_t(a_{12})u_t^* \\ u_t\alpha_t(a_{21}) & u_t\alpha_t(a_{22})u_t^* = \alpha'_t(a_{22}) \end{pmatrix}$$

The top left corner inclusion gives an isomorphism (Morita equivalence) between the K-theory of $A \rtimes_\alpha \mathbb{R}$ and $M_2(A) \rtimes_\beta \mathbb{R}$; similarly the bottom right corner gives an isomorphism between the K-theory of $A \rtimes_{\alpha'} \mathbb{R}$ and $M_2(A) \rtimes_\beta \mathbb{R}$.

Lecture 20
Construction of the Thom map

As far as the proof of Theorem 19.2 goes, so far we have shown the *uniqueness* of a Thom map satisfying the axioms, but we have not shown its *existence*, nor that it is an isomorphism. However the uniqueness proof that we have just completed also tells us how to construct the map, if we think about it. Rephrase the uniqueness proof: it follows from the naturality and normalization axioms that if the action of $\mathbb{R}$ on A is trivial, then the Thom isomorphism from $K_0(A)$ to $K_1(\widehat{A}) = K_1(A \otimes C_0(\widehat{\mathbb{R}}))$ is just the suspension map, for which we gave the explicit formula in Lemma 19.1. But we also showed that given a single K-theory class $[e]$, we can change the action by an inner equivalence to make it trivial on the subalgebra generated by e. On that subalgebra, the Thom map is therefore the suspension map: in particular, we know what the Thom map must do to $[e]$.

Let's fill in the details here. Let notation be as at the end of the last lecture: A is a unital $\mathbb{R}$-C^*-algebra, X is an unbounded multiplier of $A \rtimes \mathbb{R}$ that generates the $\mathbb{R}$-action, $e \in \mathcal{A}$ is a projection. In earlier calculations we considered $Y = eXe + (1-e)X(1-e)$; now let us just look at one summand $Z = eXe = eYe = eY = Ye$. Since Y is self-adjoint and commutes with e, Z is self-adjoint (on the domain $\{\xi : e\xi \in \mathrm{dom}(Y) = \mathrm{dom}(X)\}$). We have

$$\exp(itZ) = e\exp(itY) + (1-e) = eU(t)\exp(itX) + (1-e),$$

where $U(t)$ is the cocycle given by the Duhamel formula ($\heartsuit$) as in the previous lecture, that provides the inner equivalence to an action fixing e. Explicitly $\alpha_t(e) = U(t)^* e U(t)$ and $U(s+t) = U(s)\alpha_s(U(t))$.

Suppose that $f \in \mathcal{S}(\widehat{\mathbb{R}})$ is a function belonging to the Schwartz class. Then the operator $f(Z)$ can be written

$$f(Z) = \frac{1}{2\pi}\int \hat{f}(t)\exp(itZ)\,dt = \frac{1}{2\pi}\int \hat{f}(t)eU(t)\exp(itX)\,dt + f(0)(1-e).$$

In particular

$$ef(Z) = \frac{1}{2\pi}\int \hat{f}(t)eU(t)\exp(itX)\,dt;$$

this is the representation of an element of $A \rtimes \mathbb{R}$ (in fact of $L^1(\mathbb{R}; A)$) given by the function

$$t \mapsto (2\pi)^{-1}\hat{f}(t)eU(t). \tag{20.1}$$

Let us now consider what the uniqueness proof is telling us about how to construct the Thom map. If e is a projection in $\mathcal{A}$ we can perturb the original action α of $\mathbb{R}$ to an inner equivalent one, α', that fixes e by replacing the original unbounded generator X with $Y = eXe + (1-e)X(1-e)$. Now the inclusion $\rho\colon \mathbb{C} \to A$ defined by $\lambda \mapsto \lambda e$ becomes α'-equivariant. Consider the induced map $\hat{\rho}\colon \mathbb{C} \times \mathbb{R} = C_0(\widehat{\mathbb{R}}) \to A \rtimes \mathbb{R}$. By construction, for $f \in C_0(\widehat{\mathbb{R}})$,

$$\hat{\rho}(f) = \frac{1}{2\pi} \int \hat{f}(t) e \exp(itY)\, dt = ef(Y) = ef(eXe).$$

Notice that both $\mathbb{C} \rtimes \mathbb{R}$ and $A \rtimes \mathbb{R}$ are non-unital algebras. To obtain the image of $[e]$ under the Thom map we must first unitalize $\hat{\rho}$, and then apply the unitalized map to the appropriate generator of $K_1(C(S^1))$, regarding S^1 as the 1-point compactification of $\widehat{\mathbb{R}}$. This tells us that $\varphi_0[e]$ should be given by the unitary

$$(\diamondsuit) \qquad 1 + ef(Y) = 1 + ef(eXe) \in (A \rtimes \mathbb{R})^{\sim}$$

where $f \in \mathcal{S}(\mathbb{R})$ and $\hat{f}+1$ is a map from the 1-point compactification of $\mathbb{R}$ to the circle having winding number -1. Notice that $ef(Z)$ can be expressed, using formula 20.1, as an explicit element of $A \rtimes \mathbb{R}$. We must check that $(\diamondsuit)$ gives a well-defined element of $K_1(A \rtimes \mathbb{R})$ (independent of the choices made) and that the process that assigns to e this element gives a well-defined map on K-theory (equivalent projections give the same element of $K_1(A \rtimes \mathbb{R})$) and is a homomorphism of abelian groups.

Remark 20.2. We have said that formula $(\diamondsuit)$ defines an explicit element of $A \rtimes \mathbb{R}$, but we have not said whether we are referring to $A \rtimes_{\alpha'} \mathbb{R}$ or $A \rtimes_{\alpha} \mathbb{R}$. In fact, the discussion leading to $(\diamondsuit)$ would lead us to interpret the formula as an explicit element of $L^1(\mathbb{R}; A) \subseteq A \rtimes \alpha'\mathbb{R}$. However, being an explicit element of $L^1(\mathbb{R}; A)$, the formula can also be interpreted as an element of $A \rtimes_{\alpha} \mathbb{R}$, and when so interpreted it defines the same K-theory class. This is because the isomorphism between these algebras is given by conjugation by $U(t)$, a unitary in the multiplier algebra, and such conjugation induces the identity on K-theory (Remark 12.10).

Lemma 20.3. *The class in $K_1(\widehat{A})$ defined by $(\diamondsuit)$ does not depend on the choice of f (satisfying the winding number condition). In fact, we can even add an arbitrary self-adjoint bounded perturbation B to $Z = eXe$, considering $[1 + ef(eXe + B)]$, and we will still obtain the same K-theory element.*

Proof. Any two choices of f are connected by a (sup norm continuous) homotopy f_t, so the corresponding path of unitary operators $t \mapsto 1 + ef_t(eXe)$ is norm continuous and thus defines a constant K-theory class. We can apply the same argument to the bounded perturbation B, using the homotopy

$$[0,1] \ni t \mapsto 1 + ef(eXe + tB);$$

the continuity of this homotopy follows from the Duhamel perturbation formula (Proposition 19.4). □

Lemma 20.4. *Suppose that e_1 and e_2 are equivalent projections in $\mathcal{A}$. Then the formula $(\diamondsuit)$ produces the same K-theory class when applied to the projection e_1 as it does when applied to the projection e_2.*

Notice that we are using the full force of Theorem 14.5 here: the equivalence, as well as the projections themselves, may be assumed to take place inside $\mathcal{A}$.

Proof. According to Remark 16.8 we may take the equivalence in the form $e_2 = ue_1u^*$ with $u \in \mathcal{A}$ unitary. Then

$$e_2Xe_2 = u(e_1u^*Xue_1)u^* = u(e_1Xe_1 + ie_1\delta(u^*)ue_1)u^*,$$

and thus

$$e_2f(e_2Xe_2) = ue_1f(e_1Xe_1 + B)u^*,$$

where $B = ie_1\delta(u^*)ue_1$ is bounded and self-adjoint. Since u is a unitary in A it belongs to $M(A \rtimes \mathbb{R})$, whence conjugation by u induces the identity on $K_1(A \rtimes \mathbb{R})$ by Remark 12.10. Thus

$$[1 + e_2f(e_2Xe_2)] = [1 + e_1f(e_1Xe_1 + B)] = [1 + e_1f(e_1Xe_1)]$$

in $K_1(A \rtimes \mathbb{R})$, using lemma 20.3 for the last step. □

Lemma 20.5. *Suppose that $e_1, e_2 \in \mathcal{A}$ are mutually orthogonal projections ($e_1e_2 = 0$), which implies that e_1+e_2 is a projection also. Then the image of e_1+e_2 under formula ($\Diamond$) is the sum (in $K_1(\widehat{A})$) of the images under formula ($\Diamond$) of e_1 and of e_2.*

Proof. We need to consider $f((e_1 + e_2)X(e_1 + e_2))$. By orthogonality

$$\begin{aligned}(e_1 + e_2)X(e_1 + e_2) = e_1Xe_1 + e_2Xe_2 + i\delta(e_2)e_1 + i\delta(e_1)e_2 \\ = e_1Xe_1 + e_2Xe_2 + B,\end{aligned}$$

where B is bounded and self-adjoint. According to Lemma 20.3, then, the K-theory class of $1 + (e_1 + e_2)f((e_1 + e_2)X(e_1 + e_2))$ is the same as the K-theory class of $1 + (e_1 + e_2)f(e_1Xe_1 + e_2Xe_2)$. Since the two summands here are orthogonal, *this* K-theory class is just $1 + e_1f(e_1Xe_1) + e_2f(e_2Xe_2)$ where again the two summands are orthogonal. The result follows, by observing that

$$(1 + e_1f(e_1Xe_1))(1 + e_2f(e_2Xe_2)) = 1 + e_1f(e_1Xe_1) + e_2f(e_2Xe_2)$$

in the unitary group of $(A \rtimes \mathbb{R})^{\sim}$. □

Now Lemmas 20.3, 20.4 and 20.5 together establish that formula ($\Diamond$) gives a well-defined homomorphism from $K_0(A) = K_0(\mathcal{A})$ to $K_1(\widehat{A})$. It is clear from its construction that it satisfies the Naturality and Normalization axioms (for unital A). Moreover, using these axioms we can remove the unitality assumption by defining φ_0 for non-unital algebras by restricting the Thom map on the unitalization. Finally, we can define φ_1 by forcing the Suspension axiom to be true. that is, defining

$$\varphi_1^A = \left(\sigma_0^{\widehat{SA}}\right)^{-1} \circ \varphi_0^{SA} \circ \sigma_1^A.$$

This completes the construction of Thom maps obeying the axioms. The next task is to show that they are *isomorphisms.*

The basic idea is to use Takai duality to make the inverse of the Thom homomorphism be *itself* applied to the dual action. We need one more piece of information (a normalization) to get this started.

Lemma 20.6. *Consider $C_0(\mathbb{R})$ with the action of $\mathbb{R}$ by translation (example 18.9). Then the Thom map*

$$\varphi_1 \colon \mathbb{Z} = K_1(C_0(\mathbb{R})) \to K_0(C_0(\mathbb{R}) \rtimes \mathbb{R}) = K_0(\mathfrak{K}) = \mathbb{Z}$$

is an isomorphism.

Connes' proof of this lemma is very interesting and computational; we'll discuss it later (next lecture?), possibly along with an alternate proof. Right now I want to show that with this information we can prove that the Thom map is an isomorphism in general.

Let A be an $\mathbb{R}$-C^*-algebra with action α. We'll use t_A or t_α to denote the isomorphism provided by Takai duality between $(A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}}$ and $A \otimes \mathfrak{K}(L^2(\mathbb{R}))$.

Lemma 20.7. *Suppose that α and α' are inner equivalent actions. Let ι denote the resulting $\widehat{\mathbb{R}}$-equivariant isomorphism $\iota \colon A \rtimes_\alpha \mathbb{R} \to A \rtimes_{\alpha'} \mathbb{R}$ (Proposition 18.15) and $\hat{\iota}$ the induced isomorphism*

$$\hat{\iota} \colon (A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}} \to (A \rtimes_{\alpha'} \mathbb{R}) \rtimes_{\hat{\alpha}'} \widehat{\mathbb{R}}.$$

Then there is a unitary $W \in \mathfrak{M}(A \otimes \mathfrak{K})$ such that

$$\mathrm{Ad}_W \circ t_{\alpha'} \circ \hat{\iota} = t_\alpha \colon (A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}} \to A \otimes \mathfrak{K}(L^2(\mathbb{R})).$$

The notation $\mathfrak{M}$ denotes the multiplier algebra: I switched from using straight M because there are also matrices in this proof.

Proof. Consider the action β of $\mathbb{R}$ on $B = M_2(A)$ constructed in Remark 19.9. There are then canonical isomorphisms

$$\rho \colon (A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}} \to e_{11} \left[(B \rtimes_\beta \mathbb{R}) \rtimes_{\hat{\beta}} \widehat{\mathbb{R}} \right] e_{11}$$

and

$$\rho' \colon (A \rtimes'_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}'} \widehat{\mathbb{R}} \to e_{22} \left[(B \rtimes_\beta \mathbb{R}) \rtimes_{\hat{\beta}} \widehat{\mathbb{R}} \right] e_{22}$$

where the e's are the usual matrix units. These extend to homomorphisms of the multiplier algebras, which include a lot of stuff, like A itself, and $A \rtimes \mathbb{R}$, and the unitaries U_t, V_s that implement the $\mathbb{R}$-action and the $\widehat{\mathbb{R}}$-action respectively, and the matrix units $e_{ij} \in \mathfrak{M}((B \rtimes_\beta \mathbb{R}) \rtimes_{\hat{\beta}} \widehat{\mathbb{R}}$.

We get the relation

$$e_{21} \rho(x) e_{12} = \rho' \left(\hat{\iota}(x) \right) \quad \forall\, x \in (A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}},$$

by checking separately on elements of A, generators of the $\mathbb{R}$-action, and generators of the $\widehat{\mathbb{R}}$-action (all in the multiplier algebra). Or you can check the whole thing directly in the double regular representation (see proof of Theorem 18.11), if you prefer.

Note that the matrix units e_{11}, e_{22} are fixed by β, whereas the off-diagonal matrix units e_{12}, e_{21} are not. This implies that $t_\beta(e_{jj}) = e_{jj} \otimes I \in \mathfrak{M}(M_2(A) \otimes \mathfrak{K})$. In turn, this implies by simple calculations that

$$t_\beta(e_{21}) = \begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix},$$

where $W \in U(\mathfrak{M}(A))$.

From the definitions

$$t_\beta(\rho(x)) = \begin{pmatrix} t_\alpha(x) & 0 \\ 0 & 0 \end{pmatrix} \quad \forall\, x \in (A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}},$$

and similarly

$$t_\beta(\rho'(x')) = \begin{pmatrix} 0 & 0 \\ 0 & t_{\alpha'}(x') \end{pmatrix} \quad \forall\, x' \in (A \rtimes_{\alpha'} \mathbb{R}) \rtimes_{\hat{\alpha}'} \widehat{\mathbb{R}}.$$

Now for $x \in (A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}}$,

$$\begin{pmatrix} 0 & 0 \\ 0 & t_{\alpha'}(\hat{\imath}(x)) \end{pmatrix} = t_\beta(\rho'(\hat{\imath}(x)) = t_\beta(e_{21}\rho(x)e_{12}) =$$

$$= t_\beta(e_{21})t_\beta(\rho(x))t_\beta(e_{12}) = \begin{pmatrix} 0 & 0 \\ 0 & Wt_\alpha(x)W^* \end{pmatrix}$$

giving the result. □

Proposition 20.8. *Let A be an $\mathbb{R}$-C^*-algebra (with action α). The composite map*

$$(t_\alpha)_* \circ \varphi_{i\pm 1}^{\hat{\alpha}} \circ \varphi_i^\alpha \colon K_i(A) \to K_i(A \otimes \mathfrak{K}) = K_i(A)$$

is the identity. Consequently, the Thom maps themselves are isomorphisms.

Proof. Let $\Psi_i \colon K_i(A) \to K_i(A)$ be the map in the statement of the theorem (that is, $(t_\alpha)_* \circ \varphi_{i\pm 1}^{\hat{\alpha}} \circ \varphi_i^\alpha$). We want to show that Ψ_i is the identity for all A. By using suspensions it is enough to consider Ψ_0. Note that Ψ_0 is natural for equivariant homomorphisms. Moreover, Lemma 20.7 together with Remark 12.10 tell us that $\Psi_0^\alpha = \Psi_0^{\alpha'}$ if α and α' are inner equivalent. The same reduction we made in the uniqueness part of the proof (using stability, consider projections in $\mathcal{A}$ or matrix algebras over it, and then change the action to an inner equivalent one fixing the given projection) now tells us that we need only consider the case $A = \mathbb{C}$ with trivial action. This case follows from the Normalization axiom together with Lemma 20.6. □

Just as a reminder, we have not proved Lemma 20.6 yet; that is deferred until next time.

Lecture 21
The Thom map and K_1

We have completed the proof that Connes' Thom map is (up to sign) its own inverse—and therefore is an isomorphism—modulo checking one special case, namely the case where $A = C_0(\mathbb{R})$ with $\mathbb{R}$-action by translation, and thus $A \rtimes \mathbb{R} = \mathfrak{K}(L^2(\mathbb{R}))$ (see Lemma 20.6). In this case the domain and codomain of

$$\varphi_1 \colon K_1(A) \to K_0(A \rtimes \mathbb{R})$$

are both $\mathbb{Z}$; we have seen that *if* we know that φ_1 is an isomorphism in this case, we will get the isomorphism property (Proposition 20.8) in general.

First I am going to discuss a direct proof of this special case. (The ideas here appear in Rieffel, *Connes' analog for crossed products of the Thom isomorphism*, Contemporary Math 10 (1982), 143–154, though Rieffel develops them into a somewhat independent proof.) We consider the extension of C^*-algebras

$$0 \longrightarrow C_0(-\infty,\infty) \longrightarrow C_0(-\infty,\infty] \longrightarrow \mathbb{C} \longrightarrow 0\ ,$$

which naturally appears when discussing suspensions (compare Proposition 8.2). This can be considered to be a sequence of $\mathbb{R}$-C^*-algebras, in fact, with $\mathbb{R}$ acting on $\mathbb{R} = (-\infty,\infty)$ by translation, and fixing the point ∞. Let us form the crossed product with $\mathbb{R}$, obtaining the exact sequence

$$(\spadesuit) \qquad 0 \longrightarrow C_0(-\infty,\infty)\rtimes\mathbb{R} \longrightarrow C_0(-\infty,\infty]\rtimes\mathbb{R} \longrightarrow \mathbb{C}\rtimes\mathbb{R} \longrightarrow 0\ ,$$

[*Side Remark*: The sequence of crossed products is exact because $\mathbb{R}$ is an amenable group and amenable groups are exact (see Nate and Taka's book). Alternatively, one may prove that $\rtimes\mathbb{R}$ is an exact functor directly using Takai duality, as Connes does: a failure of exactness applied to a sequence $0 \to A \to B \to C \to 0$ would mean that the image of $\widehat{A}$ would be a *proper* subset of the kernel of $\widehat{B} \to \widehat{C}$; taking the crossed product with the dual, we'd then find that $A \otimes \mathfrak{K}$ is a proper subset of the kernel of $B \otimes \mathfrak{K} \to C \otimes \mathfrak{K}$, contradicting the exactness of tensoring with the compacts.]

We are interested especially in the middle algebra in the sequence ($\spadesuit$) above.

Lemma 21.1. *Consider the algebra $C_0(-\infty,\infty]$ with the $\mathbb{R}$-action by translations ($-\infty$ being a fixed point). Then $K_i(C_0(-\infty,\infty]\rtimes\mathbb{R}) = 0$ for $i = 0, 1$.*

Of course $C(-\infty,\infty]$ is a *contractible* C^*-algebra (hence has zero K-theory) and so the result would be obvious if we already knew that the Thom map was an isomorphism. Sadly, that's what we're trying to prove.

Proof. Consider the six term exact sequence associated to the sequence of crossed products above.

$$\begin{tikzcd}
K_1(C_0(-\infty,\infty)\rtimes\mathbb{R})=0 \arrow[r] & K_1(C_0(-\infty,\infty]\rtimes\mathbb{R})=? \arrow[r] & K_1(\mathbb{C}\rtimes\mathbb{R})=\mathbb{Z} \arrow[d] \\
K_0(\mathbb{C}\rtimes\mathbb{R})=0 \arrow[u] & K_0(C_0(-\infty,\infty]\rtimes\mathbb{R})=? \arrow[l] & K_0(C_0(-\infty,\infty)\rtimes\mathbb{R})=\mathbb{Z} \arrow[l]
\end{tikzcd}$$

I have included the groups we know from the isomorphisms $\mathbb{C}\rtimes\mathbb{R}\cong C_0(\widehat{\mathbb{R}})$ and $C_0(\mathbb{R})\rtimes\mathbb{R}\cong\mathfrak{K}$.

It will be enough if we can prove that the right hand vertical map is surjective (a surjective homomorphism $\mathbb{Z}\to\mathbb{Z}$ is automatically injective). For that, it suffices to show that the unitalization of $C_0(-\infty,\infty]\rtimes\mathbb{R}$ contains a Fredholm operator of index one. We will now do this by explicit calculation, following Lemma 6 of Green, *C^*-algebras of transformation groups with smooth orbit space*, Pacific J of Math **72**(1977), 71–97).

As in the example 18.9, $A=C_0(-\infty,\infty]\rtimes\mathbb{R}$ can be described as the norm completion of the C^*-algebra of finite propagation, continuous kernels k on $\mathbb{R}\times\mathbb{R}$ that have the following property: along every diagonal they tend to zero at $-\infty$ and to some limit, but not necessarily 0, at $+\infty$; that is to say, for fixed s, the function $t\mapsto k(t,s+t)$ belongs to $C_0(-\infty,\infty]$. An example of an element of A is the operator T defined by the kernel

$$G(x,y)=\begin{cases} e^{-x/2}e^{y/2}=e^{(y-x)/2} & (0\leqslant y\leqslant x)\\ 0 & \text{otherwise}\end{cases}$$

(even though this kernel is not continuous and of finite propagation, it is easy to check by simple estimates that the operator it defines is the limit of operators that *are* defined by such kernels.) I claim that $I-T$ is a Fredholm operator of index -1. It is clear that $I-T$ respects the decomposition of $L^2(\mathbb{R})=L^2(\mathbb{R}^-)\oplus L^2(\mathbb{R}^+)$, and acts as the identity on the first summand; thus only the $L^2(\mathbb{R}^+)$ summand need be considered.

A convenient orthonormal basis for $L^2(\mathbb{R})$ is provided by the functions $\varphi_n=L_n(t)e^{-t/2}$, where the L_n are the *Laguerre polynomials*

$$L_n(t)=\sum_{j=0}^{n}\binom{n}{j}\frac{(-t)^j}{j!}.$$

Computing in this basis

$$(1-T)\varphi_n(x)=e^{-x/2}\left(L_n(x)-\int_0^x L_n(t)dt\right)=$$
$$e^{-x/2}\left(\sum_{j=0}^{n}\binom{n}{i}\frac{(-x)^j}{j!}+\sum_{j=1}^{n+1}\binom{n}{j-1}\frac{(-x)^j}{j!}\right)=\varphi_{n+1}(x).$$

Thus $I-T$ acts as the unilateral shift relative to the basis $\{\varphi_n\}$, giving the result. $\square$

Proof of lemma 20.6. It follows from the Barratt-Puppe construction (compare Proposition 8.4) that a pair of maps on K-theory which satisfies the Naturality and Suspension axioms is also natural with respect to boundary maps. In other words, if $0 \to A \to B \to C \to 0$ is a short exact sequence of $\mathbb{R}$-C^*-algebras, and we consider the exact sequence of crossed products $0 \to \widehat{A} \to \widehat{B} \to \widehat{C} \to 0$, then the diagram

$$\begin{tikzcd} K_0(C) \arrow[r, "\partial"] \arrow[d, "\varphi_0^C"] & K_1(A) \arrow[d, "\varphi_1^A"] \\ K_1(\widehat{C}) \arrow[r, "\partial"] & K_0(\widehat{A}) \end{tikzcd}$$

commutes (as does the corresponding diagram with the roles of K_1 and K_0 reversed, but this is the one we will need.)

Consider now the suspension exact sequence discussed above (see (♠)). By the previous paragraph, there is a commutative diagram

$$\begin{tikzcd} K_0(\mathbb{C}) \arrow[r, "\partial"] \arrow[d, "\varphi_0"] & K_1(C_0(\mathbb{R})) \arrow[d, "\varphi_1"] \\ K_1(\mathbb{C} \rtimes \mathbb{R} = C_0(\widehat{\mathbb{R}})) \arrow[r, "\partial"] & K_0(C_0(\mathbb{R}) \rtimes \mathbb{R} = \mathfrak{K}) \end{tikzcd}$$

The boundary maps are isomorphisms because of the contractibility of $C_0(-\infty, \infty]$ and its equivariant counterpart, Lemma 21.1. The left hand vertical map is an isomorphism by the normalization axiom. Hence the right hand vertical map is an isomorphism too. $\square$

Now we are going to talk about how Connes addresses the crucial normalization fact, Lemma 20.6 (and proves a lot of other interesting things along the way). One of the things that makes this hard to work with is that it involves φ_1, which is only defined indirectly from φ_0, via suspension. He calculates using traces to (partly) get around this.

Suppose that A is equipped with both an $\mathbb{R}$-action and a *trace* τ that is invariant under the $\mathbb{R}$-action. (If A is unital, this means a tracial state; if A is not unital, τ can be any unbounded trace in the sense of our Definition 14.8.) Then there is a *dual trace* $\hat{\tau}$ defined as follows.

Definition 21.2. Let τ be an α-invariant trace on A. The *dual trace* on $A \rtimes_\alpha \mathbb{R}$ is the functional defined as follows: for $f \in C_c(\mathbb{R}; A) \subseteq A \rtimes \mathbb{R}$, having values in $\mathrm{dom}(\tau)$, set $\hat{\tau}(f) = \tau(f(0))$. This functional has the tracial property on $C_c(\mathbb{R}; A)$, and it extends to an unbounded trace on $A \rtimes \mathbb{R}$ in the sense of Definition 14.8.

Even if τ is bounded, $\hat{\tau}$ will not be: for instance, if $A = \mathbb{C}$ then $\hat{\tau}$ on $C^*(\mathbb{R}) = C_0(\widehat{\mathbb{R}})$ is integration with respect to Lebesgue measure on $\widehat{\mathbb{R}}$.

Proof. Let us check the tracial property using the definition of the product in Equation 18.2. We have

$$\hat{\tau}(f * g) = \tau(f * g)(0) = \int \tau\left(f(s)\alpha_s(g(-s))\right) ds.$$

Meanwhile

$$\begin{aligned}\hat{\tau}(g * f) = \tau(g * f)(0) &= \int \tau\left(g(s)\alpha_s(f(-s))\right) ds \\ &= \int \tau\left(\alpha_s(f(-s))g(s)\right) ds \\ &= \int \tau\left(\alpha_{-s}(f(s))g(-s)\right) ds \\ &= \int \tau\left(f(s)\alpha_s(g(-s))\right) ds\end{aligned}$$

using the trace property, substitution, and α-invariance in successive lines. To see that $\hat{\tau}$ is a positive linear functional, observe

$$\hat{\tau}(f^* * f) = \int \tau\left(f(-s)^* f(-s)\right) ds;$$

the integrand is positive, and the set of f for which the integral is finite (e.g. continuous functions having values in $\operatorname{dom}(\tau)$) are dense. $\square$

Let A be equipped with an (unbounded) trace τ and an $\mathbb{R}$-action. Let me use $\mathcal{A}$ now to denote the common domain both for τ and δ: i.e., if $a \in \mathcal{A}$, then $t \mapsto \alpha_t(a)$ is smooth and this function and all its derivatives belong to $\operatorname{dom}(\tau)$. (Still a dense Frechet subalgebra which is inverse closed!) Connes proved the following formula

Proposition 21.3 (Connes trace formula)**.** *Let u be a unitary in $\widetilde{A}$ defining a class in $K_1(A)$, of the form $u = 1 + z$ where $z \in \mathcal{A}$. Then*

$$\dim_{\hat{\tau}}\left(\varphi_1[u]\right) = \frac{1}{2\pi i}\tau\left(\delta(u)u^*\right).$$

(We will discuss the proof later.)

Let us look at how this could be used to give an alternate proof of the key Lemma 20.6. Indeed, if we let $A = C_0(\widehat{\mathbb{R}})$ with trace given by integration with respect to Lebesgue measure, then $\delta(u) = du/dt$ and the right hand side is simply the usual formula for the winding number

$$\frac{1}{2\pi i}\oint \frac{du}{u}, \quad u \in \mathbb{T}.$$

On the other hand, on the left side the dual trace $\hat{\tau}$ is given by integrating the kernel $k(x, y)$ of a smoothing operator along the diagonal, which is to say that it is just the usual trace on $\mathfrak{K}$. By Example 14.12 the trace here induces an isomorphism from $K_0(\mathfrak{K})$ to $\mathbb{Z} \subseteq \mathbb{R}$. In particular, the generator u with winding number ± 1 passes to the generator of $K_0(\mathfrak{K})$, as asserted by Lemma 20.6.

Lecture 22

Applications to non commutative geometry

Let V be a compact oriented smooth manifold (say of dimension n), and let X be a vector field on V. By integration, X generates a flow on V and therefor a one-parameter group α of automorphisms of $C(V)$. We are interested in the crossed product algebra $C(V)\rtimes\mathbb{R}$ obtained from this flow. If X is nowhere vanishing, this is a (simple) example of the *foliation algebras* of Connes, corresponding to a 1-dimensional foliation.

In the above situation, let μ be an α-invariant probability measure on V. Then $\tau(f)=\int f d\mu$ is an invariant trace on $C(V)$. The *Ruelle-Sullivan current* C associated to this data is the 1-current (that is, linear functional on 1-forms) defined by

$$C(\omega)=\int_V \omega(X)d\mu.$$

It is a *closed current*, that is, $C(df)=0$, because

$$C(df)=\int df(X)\,d\mu=\int (X.f)\,d\mu=\lim_{t\to 0}\int \frac{\alpha_t f-f}{t}d\mu=0,$$

since the measure μ is invariant. The dual trace $\hat{\tau}$ on the crossed product is an example of the Connes trace on a foliation algebra coming from an invariant transverse measure. All of these notions have appropriate higher-dimensional analogs.

Proposition 22.1. *In the above situation, the subgroup*

$$\dim_{\hat{\tau}}(K_0(C(V)\rtimes_\alpha\mathbb{R}))\ \subseteq\ \mathbb{R}$$

is exactly the image of $H^1(V;\mathbb{Z})$ under pairing with $[C]\in H_1(V;\mathbb{R})$.

Proof. Consider an element of $\dim_{\hat{\tau}}(K_0(C(V)\rtimes_\alpha\mathbb{R}))$. Because of the Thom isomorphism and the trace formula (Proposition 21.3) it is of the form

$$\frac{1}{2\pi i}\int_V \mathrm{tr}\,(\delta(u)u^*)\ d\mu, \tag{$\clubsuit$}$$

where $u\colon V\to U(n)$ is some smooth map and δ denotes differentiation along X. But for any such u, the formula

$$\omega=\mathrm{tr}\,((du)u^*)$$

defines a closed 1-form on V, namely, the pull-back of the fundamental class of $\mathbb{T}=U(1)$ via the composite map

$$V\xrightarrow{\ u\ }U(n)\xrightarrow{\ \det\ }U(1)=\mathbb{T}$$

Since that fundamental class belongs to $H^1(\mathbb{T};\mathbb{Z})$, it follows that $[\omega]\in H^1(V;\mathbb{Z})$. The formula ($\clubsuit$) is simply the pairing $\langle\omega,C\rangle$. Thus

$$\dim_{\hat{\tau}}(K_0(C(V)\rtimes_\alpha\mathbb{R}))\subseteq\langle H^1(V;\mathbb{Z}),[C]\rangle.$$

The other inclusion is apparent since any element of $H^1(V;\mathbb{Z})$ can be realized by a map to the circle. $\square$

Example 22.2. Consider the irrational slope foliation on the two-torus $\mathbb{T}^2$. In other words, X is the translation invariant vector field $\mathbf{i} + \theta\mathbf{j}$ on $\mathbb{R}^2$ whose orbits are the straight lines $y = \theta x + c$, with θ irrational (and we take $0 < \theta < 1$). Taking the quotient by $\mathbb{Z}^2$, X passes to a vector field on $V = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ whose orbits are copies of $\mathbb{R}$ wrapped densely around the torus. Let $A = C(V) \rtimes \mathbb{R}$ be the corresponding crossed product algebra by the $\mathbb{R}$-action α given by flowing along X.

The Lebesgue measure λ on $\mathbb{R}^2$ passes to a translation invariant probability measure on V, which is (in particular) an α-invariant trace. The associated Ruelle-Sullivan current C has $\langle C, dx\rangle = 1$ and $\langle C, dy\rangle = \theta$. Since the classes $m[dx] + n[dy]$ describe the integer lattice $H^1(V;\mathbb{Z})$ in $H^1(V;\mathbb{R})$, by Proposition 22.1 the image $\dim_{\hat\lambda}(K_0(A))$ is the dense subgroup $\mathbb{Z} + \theta\mathbb{Z} \subseteq \mathbb{R}$.

We will see in a moment that the crossed product algebra A above is Morita equivalent to the *irrational rotation algebra* A_θ (2015 notes, Lecture 9) generated by two unitaries U and V with $UV = e^{2\pi i\theta}VU$. Moreover, under this Morita equivalence the trace $\hat\lambda$ corresponds to the unique (canonical) trace τ on the irrational rotation algebra which sends all monomials U^aV^b to zero except when $a = b = 0$ when it gives 1. It was thought for a time that the irrational rotation algebras might be examples of simple unital projectionless C^*-algebras, but this was shown to be false when Powers and Rieffel constructed a nontrivial projection in A_θ. We'll recall the construction of this projection and see that it corresponds exactly to the "missing generator" of the lattice $\mathbb{Z} + \theta\mathbb{Z}$ of traces (the other generator, 1, obviously corresponds to the unit 1 thought of as a projection).

We can regard A_θ as the crossed product $C(S^1) \rtimes \mathbb{Z}$, where $C(S^1)$ is the subalgebra generated by the unitary V (say) and the action of $\mathbb{Z}$ is implemented by the unitary U. This means that (a dense subalgebra of) A_θ consists of finite sums

$$\sum_{n=-M}^{M} U^n f_n(V), \tag{22.3}$$

where the functions f are smooth functions on the circle. We will choose to regard these as 1-periodic smooth functions on the line: this means that the notation $f(V)$ is not strictly accurate, it should read something like

$$\exp\left[2\pi i f\big((2\pi i)^{-1}\log V\big)\right],$$

but the simpler notation is conventional and doesn't seem to cause any confusion. With this convention the product law becomes

$$f(V)U = Ug(V), \quad \text{where } g(x) = f(x-\theta),$$

and the unique tracial state is

$$\tau\left(\sum_{n=-M}^{M} U^n f_n(V)\right) = \int_0^1 f_0(x)\,dx.$$

Powers and Rieffel asked themselves what it would take for an expression of the form (22.3) to be a projection. You can write down the necessary equations:

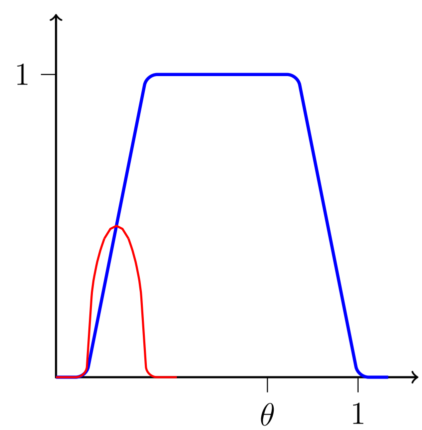
$$f_k(x) = \overline{f_{-k}(x+k\theta)},$$
$$f_k(x) = \sum_{n=-\infty}^{\infty} f_{k-n}(x+n\theta) f_n(x)$$

where by convention $f_k \equiv 0$ for $k < -M$ or $k > M$. The first equation tells us the negative f's in terms of the positive ones, and all but finitely many cases of the second equation just say $0 = 0$, so there are only a finite number of equations to solve here.

It is natural to begin solving with small values of M. For $M = 0$ we simply find that f_0 is real and $f_0 = f_0^2$, which gives us the two obvious projections, 0 and 1. For $M = 1$ things are more interesting. We'll assume that both functions f_0 and f_1 are real valued, and f_{-1} is determined by f_1. There are then three values of k (0,1,2) for which the second equation is relevant, giving the three equations

$$f_0(x) = f_0(x)^2 + f_1(x-\theta)^2 + f_1(x)^2$$
$$f_1(x) = f_1(x)(f_0(x) + f_0(x+\theta))$$
$$0 = f_1(x) f_1(x+\theta)$$

Powers and Rieffel realized that these equations have a non-trivial solution where f_0 is a smooth "bump" of width θ and steep sides, symmetrical in the sense that $f_0(x) + f_0(x+\theta) = 1$ for x belonging to the "left side" of the bump; this makes the second equation true automatically. If the left and right sides of the bump are steep enough, we can use the first equation to determine f_1 on the left side of the bump; the symmetry ensures that equation will also be satisfied on the right side of the bump, and the third equation will be automatic. This is all easier to graph than to describe.



Blue curve is graph of f_0; red curve is graph of f_1.

Because of the symmetry of the "bump" we compute the trace of the corresponding projection easily as

$$\tau(e) = \int_0^1 f_0(x)\,dx = \theta.$$

Thus this projection realizes the K-theory class giving the other generator of the lattice of traces.

If you are very energetic you can solve the equations for higher values of M and obtain additional projections in A_θ (all with traces determined by the Connes K-theory calculation, of course). See M. Eckstein, *On Projections in the Noncommutative 2-Torus Algebra*, SIGMA **10**(2014), 29–43.

We introduce the notion of mapping torus, which underlies some of the discussion above.

Definition 22.4. Let B be a C^*-algebra with an automorphism β. The *mapping torus* of (B, β) is the subset of $C([0;1], B))$ consisting of those functions f for which $f(1) = \beta(f(0))\}$. (Equivalently, it is the functions $g\colon \mathbb{R} \to B$ such that $g(t+1) = \beta(g(t))$.) The mapping torus is an $\mathbb{R}$-C^*-algebra, as is most easily seen in the second formulation.

In the previous example of an irrational rotation acting on the circle, the mapping torus $V = \mathbb{T}^2$ and the $\mathbb{R}$-action is exactly the one that gives the irrational slope foliation described above. Thus the assertion that we made above that $C(V) \rtimes \mathbb{R}$ is Morita equivalent to A_θ is a special case of the following proposition.

Proposition 22.5. *Let B be a C^*-algebra with automorphism β and let A be the corresponding mapping torus with $\mathbb{R}$-action α. Then the crossed products $A \rtimes_\alpha \mathbb{R}$ and $B \rtimes_\beta \mathbb{Z}$ are Morita equivalent. Moreover, suppose that τ_B is an invariant trace on B. Then*

$$\tau_A(f) = \int_0^1 \tau_B(f(t))\, dt$$

is an invariant trace on the mapping torus A. These traces correspond under the Morita equivalence in the sense that

$$\dim_{\tau_A}(\varphi[e]) = \dim_{\tau_B}([e]),$$

where $\varphi\colon K_0(B) \to K_0(A)$ is the isomorphism provided by the Morita equivalence.

Sketch Proof. Look at the proof of Lemma 13.8. We are doing basically the same thing, but with B-valued functions, considering the covering $\mathbb{R} \to S^1$ with fundamental group $\mathbb{Z}$.

Specifically, consider the continuous, compactly supported functions $C_c(\mathbb{R}; B)$. These can be made into a pre-Hilbert $B \rtimes \mathbb{Z}$-module: we let B act by pointwise multiplication, $\mathbb{Z}$ act by translation followed by the automorphism β, and the inner product is

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} \int \bar{f}(x-n)\beta^n(g(x))\, dx$$

(there may be a few signs off here). Complete it to a Hilbert $B \rtimes \mathbb{Z}$-module $\mathfrak{H}$. Then $A \rtimes \mathbb{R}$ acts on this module by compact operators: an element of A, thought of as a function $g\colon \mathbb{R} \to B$ such that $g(t+1) = \beta(g(t))$, acts by pointwise multiplication, and $\mathbb{R}$ acts by translation. Show that every $\theta_{f,g}$, where $f, g \in C_c(\mathbb{R}; B) \subseteq \mathfrak{H}$, is a convolution with an element of $A \rtimes \mathbb{R}$ via the action described above: this tells us that $A \rtimes \mathbb{R} \supseteq \mathfrak{K}_{B\rtimes\mathbb{Z}}(\mathfrak{H})$. Conversely, show that every convolution with a trigonometric polynomial in $C_c(\mathbb{R}; A)$ is a linear combination of θ's. Together these will give the desired result. □

Some more deductions from this.

Proposition 22.6 (Pimnser-Voiculescu exact sequence)**.** *Let B be a C^*-algebra with automorphism β. There is a 6-term exact sequence*

$$\begin{tikzcd} K_0(B) \arrow[r] & K_0(B) \arrow[r] & K_0(B \rtimes \mathbb{Z}) \arrow[d] \\ K_1(B \rtimes \mathbb{Z}) \arrow[u] & K_1(B) \arrow[l] & K_1(B) \arrow[l] \end{tikzcd}$$

where the maps $K_*(B) \to K_*(B)$ are $1 - \beta_*$.

Proof. Consider the short exact sequence $0 \to SB \to A \to B \to 0$ where A is the mapping torus. The PV sequence is the six term exact sequence for this together with the Thom isomorphism for $K_i(A \rtimes \mathbb{R}) = K_i(B \rtimes \mathbb{Z})$. □

Proposition 22.7. *Let V be a compact oriented manifold with $H^1(V; \mathbb{Z}) = 0$. Let φ be a minimal diffeomorphism. Then $C(V) \rtimes_\varphi \mathbb{Z}$ is a simple unital projectionless C^*-algebra (compare 2015 notes for another example in Lecture 27).*

Proof. We take it as known that the crossed product by a minimal diffeomorphism is simple. We can always choose an invariant measure μ, and then $\hat{\mu}$ will be a trace on $A = C(V) \rtimes \mathbb{Z}$ which must be *faithful* by the simplicity of the algebra. As in the example in 2015 ($C^*_r(F_2)$) it is now enough to show that the range of $\dim_{\hat{\mu}}$ on $K_0(A)$ is the integers. According to Proposition 22.5 it suffices to consider the range of $\dim_{\hat{\tau}}$ on $K_0(A)$, where A is the mapping cylinder, which in this case is the algebra corresponding to a flow on $W = V \times [0, 1]/\sim$, where $(0, x) \sim (1, \varphi(x))$. By proposition 22.1 this is the same as the range of the Ruelle Sullivan current on $H^1(W; \mathbb{Z})$. Since V has no first cohomology, the only cohomology classes for W are those pulled back by the obvious map to the circle, and they clearly pair integrally with the Ruelle-Sullivan current as required. □

In the proof of Proposition 22.5, such as it was, we found ourselves reaching back to the index theoretic ideas of Lectures 13 and following. I'd like to talk some more about this, which once again takes us back to the roots of Connes' NCG.

Let V be a compact smooth manifold and F a *foliation* of it: the simplest example is the collection of orbits of a flow given by a *nowhere vanishing* vector field X. Connes associates to this data a C^*-algebra $C^*(V, F)$, called the *foliation C^*-algebra*; in the case of a flow, $C^*(V; F)$ is just the crossed product $C(V) \rtimes \mathbb{R}$ that we have been studying. The general picture of $C^*(V; F)$ can be inferred from this special case. Indeed, let's think about what an element of $C(V) \rtimes \mathbb{R}$ looks like: a dense subalgebra $C_c(\mathbb{R}; C(V))$ consists of functions from $\mathbb{R}$ to $C(V)$, which we may think of as smooth functions k defined on the *subset of $V \times V$ comprised of points (x, y) related by the $\mathbb{R}$-action*, otherwise known as the *graph* of the foliation. This graph, suitably defined, is a manifold G of dimension $2p + q$, where p is the dimension of F and q the codimension, equipped with two maps $s, t\colon G \to V$ (the "source" and "target"); in the case of a flow, $G = V \times \mathbb{R}$, $s(x, u) = x$, $t(x, u) = \alpha_u(x)$. The generators of the C^*-algebra are smooth, compactly supported functions on G and the C^*-algebra multiplication becomes the operation of "leafwise convolution", that is

$$(k_1 * k_2)(\xi) = \int_{\xi_1 \circ \xi_2 = \xi} k_1(\xi_1) k_2(\xi_2).$$

[*Commentary:* The relation $\xi_1 \circ \xi_2 = \xi$ over which the integration takes place implies $s(\xi_1) = s(\xi)$, $t(\xi_1) = s(\xi_2)$, $t(\xi_2) = t(\xi)$ and for a flow is equivalent to these; at a first read-through, just take it to mean this. The question of what measure to integrate with respect to is resolved canonically be using appropriate half-density bundles, this looks really complicated but is actually NBD.]

Anyhow, the point of this set-up is the following. Suppose that D is a differential operator on the manifold V. It is said to be a *leafwise* operator if it only involves differentiation in the directions of the leaves of F; or, to put it another way, if $Ds(x)$ depends only on the restriction of s to a neighborhood of x in the leaf through x. A leafwise operator is *leafwise elliptic* if restricts to an elliptic operator on each leaf. Now, here is the basic tie-up to the index theory we studied before.

Lemma 22.8. *Let (V, F) be a compact foliated manifold and let D be a self-adjoint leafwise elliptic operator acting on sections of a bundle S (possibly with grading, as discussed in earlier sections). Then*

(a) The sections of S can be completed to a Hilbert $C^*(V; F)$-module $\mathfrak{S}$.

(b) There is a spectral pair (Definition 13.2) $(\alpha, \mathfrak{S})$, where α is (appropriately interpreted) the functional calculus $f \mapsto f(D)$ associated to the leafwise elliptic operator D.

Consequently there is defined $\mathrm{Index}(D) \in K_i(C^*(V, F))$ *($i = p \pmod 2$).*

The simplest derivation of this, IMO, is in my paper *Finite propagation speed and Connes' foliation algebra*, MPCPS *102*(1987), 459–466.

Now though I want to mention something that we could also have talked about before (in our earlier discussion of higher index theory). We can introduce *coefficients* into this whole business. Suppose that E is a vector bundle over V and that the other data are as in the preceding lemma. From the (leafwise) elliptic operator D on S we can generate another operator D_E on $S \otimes E$; the basic formula is that $D_E = D \otimes 1$— this does not make specific sense, as any attempt to define it involves choices of connections of partitions of unity or something, but all choices for what it might mean have the same principal symbol and differ only by lower order terms. From that one can infer that all these choices have the same index, so the map $E \mapsto \mathrm{Index}(D_E)$ is well defined. Indeed, this image depends only on the K-theory class defined by E so what we really have here is a map

$$K_0(V) \to K_i(C^*(V, F)), \quad [E] \mapsto \mathrm{Index}(D_E)$$

defined by the leafwise elliptic operator D. With a bit more energy we could also construct a map $K_1(V) \to K_{i\pm1}(C^*(V,F))$.

Now for the kicker. Go back to the case of flows. Then the generating vector field iX, itself, is a selfadjoint leafwise elliptic operator (you might prefer to call it "id/dt" if you think of the $\mathbb{R}$-action as evolution in time). According to the above discussion its index theory therefore gives rise to maps

$$\mathrm{Index}_X\colon K_i(V) \to K_{i\pm1}(C(V) \rtimes \mathbb{R}).$$

What are they?

Theorem 22.9. *The index maps described above are exactly the Thom isomorphisms of Connes.*

Lecture 23
On bivariant theories

This is mostly an introduction to KK-theory but I was challenged first to give an account of the alternative proof of the Connes trace formula, Proposition 21.3, that I hinted at in an earlier lecture. Let me first remind you of the statement: *Let A be a C^*-algebra with an $\mathbb{R}$-action and an invariant trace τ. Let u be a unitary in $\widetilde{A}$ defining a class in $K_1(A)$, of the form $u = 1 + z$ where $z \in \mathcal{A}$. Then*

$$\dim_{\hat{\tau}}(\varphi_1[u]) = \frac{1}{2\pi i}\tau(\delta(u)u^*).$$

where δ is the derivation that generates the $\mathbb{R}$-action on A.

Remark 23.1. It is easy to see that the RHS defines a functional on K-theory (of the smooth algebra, hence of A by Theorem 14.5). Indeed, it sends 1 to 0 and plays right by matrices, so we need only show the invariance under homotopies, and we may assume these homotopies to be smooth. Let $t \mapsto u_t$ be a one parameter smooth family of unitaries, and let us compute the time derivative $d/dt(\tau(\delta(u)u^*)) = \tau(\delta(\dot{u})u^*) + \tau(\delta(u)\dot{u}^*)$. By the invariance of the trace

$$0 = \tau(\delta(\dot{u}u^*)) = \tau(\delta(\dot{u})u^*) + \tau(\dot{u}\delta(u^*)).$$

Differentiating the defining identity $uu^* = 1$ we get, though,

$$\delta(u^*) = -u^*\delta(u)u^*, \quad \dot{u}^* = -u^*\dot{u}u^*,$$

and applying these and the trace identity to the second term on the right

$$\tau(\dot{u}\delta u^*) = -\tau(\dot{u}u^*\delta(u)u^*) = -\tau(\delta(u)u^*\dot{u}u^*) = \tau(\delta(u)\dot{u}^*),$$

yielding the desired result $d/dt(\tau(\delta(u)u^*)) = 0$.

I'm going to prove a much easier "dual" statement and then argue that it is equivalent to the original one. Here is the dual statement.

Lemma 23.2. *Let B be a C^*-algebra (say unital) with an $\mathbb{R}$-action β, and let e be a projection in B (or a matrix algebra over it) defining a K-theory class. Suppose τ is an invariant trace on B with $\hat{\tau}$ the dual trace on $B \rtimes_\beta \mathbb{R}$. Then*

$$\dim_\tau([e]) = \frac{1}{2\pi i}\hat{\tau}(u^*\delta(u)),$$

where u is a unitary in the smooth subalgebra of $(B \rtimes \mathbb{R})^\sim$ representing $\varphi_0[e]$ and δ is the derivation that generates the dual $\widehat{\mathbb{R}}$-action on $B \rtimes \mathbb{R}$.

Proof. Applying the remark above, it does not matter *which* smooth unitary we choose to represent the class $\varphi_0[e]$, the RHS will be the same for all. So we can use a convenient representative, for which we just have to remind ourselves how φ_0 is defined. First we may assume without loss of generality that e is smooth; then we may replace the action of $\mathbb{R}$ by an inner equivalent one that fixes e; once this is done, the unitary u that represents $\varphi_0[e]$ is exactly of the form

$$1 + ef(t), \quad \text{where } ef(t) \in L^1(\mathbb{R}; A),$$

f being a Schwartz class function whose Fourier transform $g = \hat{f}$ has the property that $1 + \hat{f}$ has winding number -1 about 0. If we restrict attention to the unital subalgebra generated by e, say $B_0 \subseteq B$, then $B_0 \rtimes \mathbb{R} = B_0 \otimes C_0(\widehat{\mathbb{R}})$ and under this isomorphism

$$u = 1 + eg(s) = (1 - e) + e(g(s) + 1).$$

In this dual picture the dual trace $\hat{\tau}$ is given by

$$\hat{\tau}(h) = \int_{-\infty}^{\infty} \tau(h(s))\, ds,$$

and the derivation δ is $-d/ds$. Thus

$$\frac{1}{2\pi i}\hat{\tau}(u^*\delta(u)) = \int_{-\infty}^{\infty} \tau(e)(-g'(s))(1 + g(s))^{-1}\, ds = \tau(e)$$

by the standard integral formula for the winding number. We may well be astray by a minus sign overall owing to my using different conventions from Connes in some places.

This proof was given under the assumption that we have changed the $\mathbb{R}$ action by an inner equivalence so as to fix e. However, the dual trace is not changed by inner equivalence (to put this more precisely, if we have two inner equivalent actions β, β' of $\mathbb{R}$ on B, one and hence both fixing τ, the isomorphism $B \rtimes_\beta \mathbb{R} \to B \rtimes_{\beta'} \mathbb{R}$ carries the dual trace on one to the dual trace on the other) so that is enough for the proof in general. □

So now let's see how to derive Connes' actual trace formula from this low-rent version. We have by now proved, independent of the trace formula, that the Thom isomorphisms are their own inverses (up to the standard Morita equivalence isomorphisms coming from tensoring with $\mathfrak{K}$, and also potentially up to some signs that I was not too careful about). So, in the situation of the original CTF, let's put $B = A \rtimes \mathbb{R}$ and equip it with the dual action of $\widehat{\mathbb{R}}$, so that A is Morita equivalent to $B \rtimes \widehat{\mathbb{R}}$. We may assume (because the Thom isomorphism is an isomorphism) that the K_1-class $[u]$ which is the input to the CTF is actually of the form $\varphi_0[e]$, where $[e]$ is a projection in B and $[e] = [\varphi_1(u)] \in K_0(B)]$. But now the CTF reduces to its dual form proved in Lemma 23.2 above.

Bivariant theories

Anyhow, that was a digression. What I really want to begin a discussion of today is the phenomenon noticed at the end of the last lecture: an elliptic operator (in that context we were talking about leafwise elliptic operators on foliated manifolds) is not just "a thing that has an index" (in that context an element of $K_p(C^*(V, F))$) but "a thing that defines a map on K-theory" (in that context a homomorphism $K_i(V) \to K_{i+p}(C^*(V, F))$). Even in the primordial case of standard elliptic operators on compact manifolds, the observation that such an operator defines not just an *index* (an integer) but a *homomorphism* $K_0(V) \to \mathbb{Z}$ is one of the basic ideas in the proof of the Atiyah-Singer index theorem. We are going to formalize this in the language of

C^*-algebras. The plan for doing so is to smoosh together the ideas of *Fredholm pair* (from Lecture 12) and *Fredholm module* (2015 notes, Lecture 27) to get a common notion that we could refer to as a *Fredholm bimodule*.

Let us assume that all C^*-algebras mentioned are countably generated, except those that obviously aren't (e.g. $\mathfrak{B}(H)$).

Definition 23.3. Let A and B be C^*-algebras. A *Fredholm* (A, B)*-bimodule* (also known as a *Kasparov cycle* or KK*-cycle* for (A, B)) consists of the following data:

(a) A graded Hilbert module E over B (thus E is a *right* B-module);
(b) A $*$-homomorphism $A \to \mathfrak{B}_B(E)$ which respects the grading (thus making E a *left* A-module, and indeed a bimodule);
(c) An odd operator $F \in \mathfrak{B}_B(E)$ (like in the definition of Fredholm pair);

such that $(F^2 - 1)a$, $(F - F^*)a$, and the commutator $[F, a]$ are *compact* Hilbert B-module operators on E.

Various remarks.

(i) We didn't use a symbol for the homomorphism $A \to \mathfrak{B}(E)$ in point (b) of the definition. If we had given it a name, say φ, then the "such that" conditions would more correctly be written "$[F, \varphi(a)]$" is compact and so on. There is no requirement that φ be injective.
(ii) If $A = \mathbb{C}$ and φ is the unique unital $*$-homomorphism, then this is exactly the definition of a Fredholm pair over B (12.1).
(iii) If $B = \mathbb{C}$ this gives the definition of *Fredholm module* over A (2015 notes, lecture 27; actually we made some small extra normalizations there, but they don't make a significant difference);
(iv) A $*$-homomorphism $f\colon A \to B$ gives a Fredholm bimodule: take $E = B \oplus 0$, the A-action given by f, and take F to be the zero operator (notice that the action of A is by elements of $B = \mathfrak{K}_B(B)$, so this is a Fredholm bimodule);
(v) We can take A and/or B to be graded C^*-algebras if we want. In this case φ should be a homomorphism of graded C^*-algebras, and $[F, \varphi(a)]$ should be taken as a graded commutator (that's why I used commutator notation rather than writing $Fa - aF$ which is correct in the ungraded case).
(vi) The four "obvious notes" mentioned for Fredholm pairs in lecture 12 (direct sum, unitary equivalence, covariant functoriality in B, and homotopy as an object over $B \otimes C[0, 1]$) all pass to Fredholm bimodules. There is also an obvious contravariant functoriality in A.
(vii) If A and B are Morita equivalent, then $A = \mathfrak{K}(E_0)$ for some (ungraded) full Hilbert B-module E_0. Then putting $E = E_0 \oplus 0$ (as a graded module) and $F = 0$ we obtain a Fredholm (A, B)-bimodule. Because Morita equivalence is an equivalence relation, we also obtain a Fredholm (B, A)-bimodule from the equivalence. These two objects should be mutually inverse in the calculus we are going to develop.
(viii) In lecture 12 we proved that the collection of homotopy classes of Fredholm pairs over B is an *abelian group* with direct sum as the group operation. (We

called this group $\mathcal{K}(A)$ and worked out a proof that it is isomorphic to $K_0(A)$ defined in the elementary way.) The same argument shows that the collection of homotopy classes of Fredholm (A, B)-bimodules is an abelian group. This is the *Kasparov group* denoted $KK(A, B)$. A special case is the *K-homology group* $K^0(A) = KK(A, \mathbb{C})$ made up of homotopy classes of Fredholm modules.

(ix) There are "odd" versions of all these constructions but I won't sweat these because I want to talk about real K-theory in a short while where one cannot avoid the introduction of Clifford algebras to organize the 8 (rather than 2) different cases. However it's worth remarking that $KK^1(A, \mathbb{C}) = \mathrm{Ext}(A)$, the Brown Douglas Fillmore group from 2015 lectures 30–33, provided that A is *nuclear* (needed to apply Stinespring's theorem, compare Theorem 32.4 from 2015).

Theorem 23.4. *An element of the group $KK(A, B)$ (defined above) gives rise to group homomorphisms $K_i(A) \to K_i(B)$. In fact, there is a pairing*

$$K_i(A) \otimes KK(A, B) \to K_i(B).$$

Proof. I will discuss this for $i = 0$, and I will assume that A is unital (the standard reduction gets us to this case). Suppose that we have an element of $K_0(A)$, represented by a projection e in some $M_n(A)$. We can "inflate" a Fredholm bimodule over (A, B) to one over $(M_n(A), B)$ in an obvious way (replace E by $E \oplus \cdots \oplus E$ and F by $F \oplus \cdots \oplus F$, with n factors in each case) so by assuming this is done I will take $n = 1$. Then $E' = eE$ is also a Hilbert B-module, $F' = eFe$ is an odd operator on E', and (F', E') is a *Fredholm pair* over B. For example, let us check that $((F')^2 - 1) \in \mathfrak{K}(E')$. Considered as operators on E,

$$((F')^2 - 1)e = (eFeFe - 1)e = e(F^2 - 1)e + e[F, e]Fe \in \mathfrak{K}(E);$$

but this exactly tells us that $((F')^2 - 1) \in \mathfrak{K}(E')$. A homotopy of projections e gives us a homotopy of Fredholm pairs, so we have defined a map $K_0(A) \to \mathcal{K}(B) = K_0(B)$. A similar device may be used to define a map on K_1, using the "ungraded" version of Fredholm pairs briefly mentioned at Remark 13.14. □

Remark 23.5. You might correctly feel that there is some cheating going on here, as the "model" of K-theory used on the "input" side of this picture is quite different from the model used on the "output" side. If we have an element of $\mathcal{K}(A)$ (a Fredholm pair), is there a direct way to feed it in to this mapping machinery? That is a good question whose answer leads to the Kasparov product. Before addressing that, though, we should be sure that elliptic operators provide examples of the structures that we have been discussing.

Proposition 23.6 (Saad Baaj and Pierre Julg)**.** *Let (α, E) be a spectral pair over B (Definition 13.2), where we write α as $D \mapsto f(D)$. Suppose that E is equipped with a homomorphism $A \to \mathfrak{B}_B(E)$ (as in (b) od Definition 23.3 above) which has the following property:*

- *there is a dense set of $a \in A$ for which the commutator $[D, a]$ is a* bounded *operator. [*Side Remark: *It is easy to work out a way of expressing what* *this condition means in terms of the homomorphism α and its extension to multipliers, but this way of putting it is most intuitive.]*

Then the Fredholm pair ("index") associated to (α, E) (see Definition 13.5) is in fact a Fredholm bimodule in the sense of Definition 23.3

Proof. We need to prove that $[a, \chi(D)]$ is compact for some (and hence every) chopping function χ, and all a; it is enough to consider the dense set of $a \in A$ for which the bulleted condition above is true.

One natural way to make sense of the bulleted condition in terms of the homomorphism α only is this: α extends to a $*$-homomorphism from $C_b(\mathbb{R})$ (the bounded continuous functions on $\mathbb{R}$) to $\mathfrak{B}(E)$ (the multiplier algebra of $\mathfrak{K}(E)$); indeed, this extension is what is needed to define the Fredholm pair $F = \chi(D) = \alpha(\chi)$ where χ is a normalizing function. This extension also allows us to define the 1-parameter unitary group e^{itD}; the bulleted condition can be interpreted as saying that there is a dense set of a for which

$$t \mapsto W_t(a) = e^{itD} a e^{-itD}$$

is a C^1 (operator-valued) function of t.

We can choose our normalizing function χ so that $\chi' = g$ is smooth and has compactly supported Fourier transform; let's do that. Then the (distributional) Fourier transforms involved satisfy $is\hat{\chi}(s) = \hat{g}(s)$, so $\hat{\chi}$ has a singularity at 0, as expected. We can write the commutator

$$[\hat{\chi}(D), a] = PV \int \frac{\hat{g}(s)}{is} W_s(a) e^{isD}\, ds,$$

where the PV indicates that we will resolve the singularity by taking the Cauchy principal value, i.e. look at $\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}$ and take the limit as $\varepsilon \to 0$. Putting $G(s) = -i\hat{g}(s) W_s(a) e^{isD}$, a C^1 operator-valued function of s, we can integrate by parts to obtain

$$\int_{-\infty}^{\infty} i \log |s| G'(s) ds,$$

which is a convergent integral showing that the commutator is compact. (See my old functional analysis notes, Example 9.10, for this regularization. Baaj and Julg gave an entirely different argument, using the specific normalizing function $\chi(t) = t(1+t^2)^{-1/2}$ and contour integration to estimate the resulting commutators. In the end though the idea is the same, use some integral transforms (Cauchy integral formula for B+J, Fourier for me) to reduce the commutator with $\chi(D)$ to simpler commutators that we can estimate.) $\square$

The condition of Proposition 23.6 is very natural for differential operators: if we have a differential operator D on a compact manifold V, operating on the sections of some bundle (perhaps promoted to a Hilbert module in some way), and functions $a \in A = C(V)$ operate on such sections by multiplication, then $[D, a]$ can be written as some combination of derivatives of a and thus is *bounded* whenever a belongs to the dense set $C^\infty(V) \subseteq A$. The basic example therefore is that a Dirac type operator D on a graded bundle over an (even dimensional) manifold M gives us a class

$$[D] \in KK(C(M), \mathbb{C}), \tag{23.7}$$

called the *K homology class* of D.

Example 23.8. More generally, on a foliated manifold our construction shows that a leafwise elliptic operator gives an element of $KK(C(V), C^*(V,F))$ and thus a homomorphism $K(V) \to K(C^*(V,F))$, which is the same one that we defined in a more pedestrian way at the end of the last lecture. For a compact V with fundamental group Γ, an elliptic operator gives an element of $KK(C(V), C^*_r(\Gamma))$ giving the higher index of Lecture 13. Many other examples can be developed.

Now I ought to face the fact (see Remark 23.5) that we have made things really a bit too easy for ourselves with the above construction. I should have shown that, if you have an element of $KK(A,B)$, you can take a *Fredholm pair* for A to a Fredholm pair for B. More ambitiously, I should produce for you a bilinear product map

$$KK(A,B) \otimes KK(B,C) \to KK(A,C).$$

This object, called the *Kasparov product*, does indeed exist and it has many beautiful properties and applications. Its construction has (probably rightly) acquired a reputation as somewhat intimidating. I'm going to try to demystify it, probably without success, but here goes.

Suppose that we have an (A,B) Fredholm bimodule (φ_1, F_1, E_1) (I'm using $\varphi\colon A \to \mathfrak{B}(E)$ to denote the left action for now) and a (B,C) Fredholm bimodule (φ_2, F_2, E_2). The internal tensor product machinery of lecture 10 (suitably twiddled to deal with the graded case) manufactures for us a Hilbert C-module $E = E_1 \widehat{\otimes}_{\varphi_2} E_2$, and this has a left action of A given by $\varphi = \varphi_1 \widehat{\otimes} 1$. The module E is also equipped with an operator $F_1 \widehat{\otimes} 1$, but there is no *natural* notion of a "lift" of F_2 to E. Even if there was, there is a question how to put such a "lift" together with $F_1 \widehat{\otimes} 1$ to obtain an appropriate operator F on E.

The first question (lifting) is dealt with by observing that though there is no unique notion of a lift of F_2, there *is* a notion of "lift up to compacts". We already encountered this idea in considering elliptic operators with coefficients (end of last lecture) so it should come as no surprise. Specifically, for $\xi_1 \in E_1$ let $\theta_{\xi_1}\colon E_2 \to E$ be the bounded, adjointable operator $\xi_2 \mapsto \xi_1 \widehat{\otimes} \xi_2$ (which already appears in the proof of Proposition 11.4). An operator $F \in \mathfrak{B}(E)$ is called a *connection* for F_2 if for all $\xi_1 \in E_1$ the operators

$$\theta_{\xi_1} F_2 \mp F \theta_{\xi_1}, \quad F_2 \theta^*_{\xi_1} \mp \theta^*_{\xi_1} F$$

are compact (from E_2 to E or from E to E_2 respectively); the sign goes according to the grading of a homogeneous ξ, minus if ξ is even and plus if ξ is odd. The terminology *connection* is due to Skandalis: the analogy is that a connection on a vector bundle is a lift of differentiation "up to lower order terms", and this kind of connection is a lift of F up to lower order (that is, compact) terms. It is easy to show that connections always exist and that the space of connections is affine (in particular, connected): one uses the stabilization theorem to reduce to the trivial case in much the same way as the fact that every vector bundle is a summand in a trivial bundle shows that every vector bundle has a connection.

Now let me introduce the concept of *alignment*.

Definition 23.9. Let (F, φ, E) be a Fredholm (A, C) bimodule. Another odd operator $F' \in \mathfrak{B}(E)$, self-adjoint modulo compacts, is *aligned* with the given bimodule if for all $a \in A$, the operator $\varphi(a)[F, F']\varphi(a)^*$ is positive modulo compacts (i.e., is the sum of a positive and a compact operator, or equivalently, projects to a positive element in the Calkin algebra $\mathfrak{Q}(E)$).

Notice that the commutator here is a *graded* commutator, which makes it $[F, F'] = FF' + F'F$ in this case. The relevance of alignment is demonstrated by the following simple lemma.

Lemma 23.10. *If (F, φ, E) and (F', φ, E) are Fredholm bimodules such that F, F' are aligned, then they represent the same element of $KK(A, B)$.*

Proof. The path

$$t \mapsto F_t = \cos(t)F + \sin(t)F', \quad t \in [0, \tfrac{\pi}{2}],$$

provides a homotopy joining F to F'. □

Definition 23.11. Suppose that we have an (A, B) Fredholm bimodule (φ_1, F_1, E_1) and a (B, C) Fredholm bimodule (φ_2, F_2, E_2), as above. Let $E = E_1 \widehat{\otimes}_{\varphi_2} E_2$. A *Kasparov product* for the two given bimodules is an (A, C)-bimodule (φ, F, E), with $\varphi = \varphi_1 \widehat{\otimes} 1$, where F satisfies

(a) F is aligned with $F_1 \widehat{\otimes} 1$.
(b) F is a connection for F_2.

Theorem 23.12. *Kasparov products exist, and they are unique up to (operator) homotopy. The Kasparov product gives an associative, bilinear pairing*

$$KK(A, B) \otimes KK(B, C) \to KK(A, C).$$

(In fact, there is a more general product

$$KK(A_1, B_1 \widehat{\otimes} D) \otimes KK(D \widehat{\otimes} A_2, B_2) \to KK(A_1 \widehat{\otimes} A_2, B_1 \widehat{\otimes} B_2),$$

which can be got from the version we define above and the obvious induction map (tensor with 1) $KK(A, B) \to KK(A \widehat{\otimes} C, B \widehat{\otimes} C)$.) I will use the notation $\boxtimes$ for the Kasparov product. □

It is this theorem whose proof requires the mysterious *Kasparov Technical Theorem*. Actually, even the KTT is not as mysterious as it was when Kasparov first proved it, since Nigel (in his thesis) showed that it was a consequence of the existence of quasicentral approximate units plus an induction argument. However, it is just the thing to skip over at the end of a course! The point is that Definition 23.11 tells you how to *recognize* a Kasparov product, now matter what method you used to construct it; often this is enough, as in the following exercises.

Exercise 23.13. Let $f\colon A \to B$ and $g\colon B \to C$ be $*$-homomorphisms with corresponding classes $[f] \in KK(A, B)$ and $[g] \in KK(B, C]$ (as in (iv) above). Show that $\boxtimes g\colon KK(A, B) \to KK(A, C)$ is the map functorially induced by g and that $f\boxtimes\colon KK(B, C) \to KK(A, C)$ is the map functorially induced by f.

Exercise 23.14. Show that $KK(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$ as a *ring*.

Exercise 23.15. Let D_1 and D_2 be (graded) Dirac type operators on bundles S_1 and S_2 over compact manifolds M_1 and M_2. Let $[D_j] \in KK(C(M_j), \mathbb{C})$, for $j = 1, 2$, be the K-homology class of the elliptic operator D_j (see 23.7). Show that

$$[D_1] \boxtimes [D_2] = [D] \in KK(C(M_1 \times M_2), \mathbb{C}),$$

where D is the Dirac operator over $M = M_1 \times M_2$ on the graded tensor product bundle $S = (\pi_1^* S_1)\widehat{\otimes}(\pi_2^* S_2)$; that is to say, $D = D_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} D_2$ on $L^2(S_1)\widehat{\otimes}L^2(S_2)$.

Exercise 23.16. Suppose that A and B are Morita equivalent. Use the Morita equivalence to construct elements $x \in KK(A, B)$ and $y \in KK(B, A)$, as in (vii) above. Show that $x \boxtimes y = 1_A \in KK(A, A)$ (where 1_A is the KK-class of the identity map $A \to A$, which of course acts as the identity of the ring $KK(A, A)$) and that $y \boxtimes x = 1_B \in KK(B, B)$.

If there exist elements x, y as in the preceding exercise, then the algebras A and B are said to be KK-*equivalent*; in particular, they have the same K-theory and K-homology groups. Morita equivalent algebras are KK-equivalent (that is what the exercise says), but the converse is false. We will see an important example of this next time.

Lecture 24
In Which We Make Some Contact With Reality

What is a real C^*-algebra? Of course, the natural definition is that it is a norm-closed algebra of operators on a real Hilbert space. Unlike in the complex case, however, the C^*-identity $\|a^*a\| = \|a\|^2$ does not suffice to characterize real C^*-algebras (example: consider $\mathbb{C}$ as a real Banach algebra with *trivial* involution: it clearly satisfies the C^*-identity but it contains a self-adjoint element with square -1. There are more elaborate versions of the C^*-identity which will do the job but it seems simpler to understand real C^*-algebras by looking at their *complexifications*. If A is a real C^*-algebra (as I defined it above), then $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ is a complex C^*-algebra which is equipped with an *antilinear* involution (coming from complex conjugation on $\mathbb{C}$). Of course, given $A_{\mathbb{C}}$ and the involution we can recover A (as the fixed point set). Thus we make the following definition.

Definition 24.1. A *real C^*-algebra* is a pair consisting of a complex C^*-algebra A and an *antilinear* involution $\tau\colon A \to A$ that is a $*$-automorphism: that is,

$$\tau(\lambda a) = \bar{\lambda}\tau(a), \quad \tau(a+b) = \tau(a) + \tau(b),$$
$$\tau(ab) = \tau(a)\tau(b), \quad \tau(a^*) = \tau(a)^*.$$

Usually we denote τ simply as complex conjugation, i.e. $\tau(a) = \bar{a}$.

Sometimes the objects defined by this definition are called “Real” C^*-algebras and the ones in our original sense (the fixed point subalgebras $\{a : a = \tau(a)\}$) are called “real”. To my mind this gnerates more confusion than it solves, I prefer to rely on context to indicate which version of the definition we're using.

Remark 24.2. Instead of considering τ which is an *antilinear involutive automorphism* we could consider $\sigma(a) = \tau(a^*)$ which is a *linear involutive antiautomorphism.* The existence of these two possible approaches does make for some possible confusion. Whichever, though, note the difference between a *real* C^*-algebra and a *graded* C^*-algebra, which is equipped with an action of $\mathbb{Z}/2$, i.e. a *linear involutive automorphism.* Of course there is nothing to stop us considering algebras which are both real and graded, and in fact we will need to do so.

Example 24.3. (The commutative case) For a *commutative* algebra the distinction between automorphisms and antiautomorphisms disappears. Thus the most general commutative real C^*-algebra is given by the “real-valued functions” on a locally compact Hausdorff space (X, τ) equipped with an involution (a homeomorphism $\tau\colon X \to X$ such that $\tau^2 = 1$). By definition, the “real valued functions” are those functions $f\colon X \to \mathbb{C}$ such that $f(\tau(x)) = \overline{f(x)}$. Only if τ is trivial are these “real valued” in the old-fashioned sense.

An important special case is $C_0(\mathbb{R}, \tau)$, where $\tau(x) = -x$. This is the (completion of the) algebra of Fourier transforms of real-valued functions on $\widehat{\mathbb{R}}$, in other words it is the real group C^*-algebra of $\widehat{\mathbb{R}}$. As Atiyah realized, because of the ubiquity of Fourier transforms in the theory of elliptic operators (specifically in the relation between an operator and its symbol) this means that the K-theory of spaces with non trivial involution will play an unavoidable role in the real version of the index theorem.

More generally we can think of $\mathbb{R}^{p,q}$, which means $\mathbb{R}^{p+q}$ with the involution that reflects the first p factors and leaves the last q factors alone; often written $(i\mathbb{R})^p \oplus \mathbb{R}^q$, thinking of the involution as complex conjugation. In particular $\mathbb{R}^{1,1} = \mathbb{C}$ (as spaces with involution). We use $S^{p,q}$ for the unit sphere in $\mathbb{R}^{p,q}$ (this has dimension $p+q-1$, of course, but there is no convenient way to reflect that in the notation.)

Remark 24.4. The K_0 groups for real C^*-algebras can be defined in the standard algebraic way (lecture 2). One can check that this corresponds to taking the Grothendieck group of "real vector bundles" in the following sense: such a bundle over a space X with involution is a complex vector bundle $E \to X$, itself equipped with an involution which is compatible with that on X, and such that for each $x \in X$ the diagram

$$\begin{tikzcd} \mathbb{C} \times E_x \arrow[r] \arrow[d] & E_x \arrow[d] \\ \mathbb{C} \times E_{\bar{x}} \arrow[r] & E_{\bar{x}} \end{tikzcd}$$

commutes, where the horizontal maps are scalar multiplication on fibers and the vertical maps are the involutions (including $\lambda \mapsto \bar{\lambda}$ on $\mathbb{C}$).

There are *two* kinds of "suspension" in this theory: tensoring by $C_0(\mathbb{R}^{0,1})$ (the usual notion of suspension) and tensoring by $C_0(\mathbb{R}^{1,0})$ (suspension using the nontrivial involution). We will use the notation

$$K_{p,q}(A) = K(A \otimes C_0(\mathbb{R}^{p,q})).$$

There is a Barratt-Puppe sequence in the q variable (just as in lecture 8) but what about the p variable? The answer is partly contained in the $(1,1)$ *periodicity theorem:*

Theorem 24.5. *There is a natural isomorphism $K_0(A) \to K_0(A \otimes C_0(\mathbb{R}^{1,1}))$; thus $K_{p,q}(A) \cong K_{p+1,q+1}(A)$ for all A.*

Proof. The Cuntz proof of Bott periodicity works once we realize that in the *real* version of the Toeplitz extension

$$0 \longrightarrow \mathfrak{K}(H_{\mathbb{R}}) \longrightarrow \mathfrak{T}(H_{\mathbb{R}}) \longrightarrow C(S^1) \longrightarrow 0\ ,$$

the symbols $C(S^1)$ should be taken as the "real" functions on S^1 with complex conjugation as the involution (this being the *real* group C^*-algebra of $\mathbb{Z}$, or equivalently the real C^*-algebra generated by a single unitary; this is the Fourier series counterpart of the phenomenon we discussed above for the Fourier transform.) $\square$

From this (1,1) periodicity theorem one can get the 8-fold periodicity of real K-theory by various "elementary" tricks (see Atiyah, *On K-theory and reality*). But these tricks involve working with Clifford algebras and once we are going to introduce all that machinery anyhow we may as well present the "right" proof of periodicity which involves the famous Dirac and dual Dirac constructions, together with the Kasparov product.

Let V be a real vector space equipped with a symmetric bilinear form b. The *Clifford algebra* $\mathrm{Cliff}(V)$ is defined to be the quotient of the complex tensor algebra $\mathcal{T}(V)$ by the ideal generated by all elements $v \otimes v - b(v,v)1$. (If $b = 0$ this is the exterior algebra but we are more interested in the case where the bilinear form is non degenerate.) It is not hard to see that this is a graded C^*-algebra of dimension 2^n, where $n = \dim V$. Moreover, $\mathrm{Cliff}(V_1 \oplus V_2) \cong \mathrm{Cliff}(V_1) \widehat{\otimes} \mathrm{Cliff}(V_2)$, so that for nondegenerate forms everything is built up using graded tensor products from the 1-dimensional case $\mathrm{Cliff}(\mathbb{R}) = \mathbb{C} \oplus \mathbb{C}$ with the grading in Example 11.7. As you probably know, $\mathrm{Cliff}(\mathbb{R}^2)$ is the 2×2 matrices and now the tensor rpoduct formula ensures that there are only two basic examples: matrix algebras of dimension 2^n (for $V = \mathbb{R}^{2n}$) and sums of two such (for $V = \mathbb{R}^{2n+1}$).

Things get more interesting though if we allow a space $V = \mathbb{R}^{p,q}$ with *nontrivial* involution. Composing the involution on V with complex conjugation induces an involutory antilinear automorphism of the Clifford algebra, i.e. a Real structure. There are now two basic building blocks instead of one: in terms of their real forms (the fixed points of the involution) they are

$$\mathrm{Cliff}(\mathbb{R}^{0,1}) = \mathbb{R} \oplus \mathbb{R}. \quad \mathrm{Cliff}(\mathbb{R}^{1,0}) = \mathbb{C}.$$

In the second case the (real) grading automorphism is *complex conjugation* (thus 1 is even and i is odd).

Now we should say that the Kasparov formulation discussed last time extends to real C^*-algebras with no more headaches than can be resolved by a few shots of espresso. Let V be one of the spaces $\mathbb{R}^{p,q}$ and let $\mathcal{C}(V)$ denote the tensor product $C_0(V) \otimes \mathrm{Cliff}(V)$ (this is a graded algebra, but all of the grading comes from the Clifford algebra side, so it is legit to use the ordinary tensor product symbol here). Suppose that x is a point of V. By c_x we denote the operation of (left) *Clifford multiplication by* x: this is an odd endomorphism of $\mathrm{Cliff}(V)$, of (operator) norm equal to $\|x\|$. Let $\mathfrak{S}$ denote the *graded* algebra $C_0(\mathbb{R})$ which has already made its appearance in the spectral picture of K-theory. Then for any $f \in \mathfrak{S}$, the function

$$f(X) : x \mapsto f(c_x)$$

is an element of $\mathcal{C}(V) = C_0(V; \mathrm{Cliff}(V))$, and the assignment $\alpha\colon f \mapsto f(X)$ is a graded $*$-homomorphism. In other words, $(\alpha, \mathcal{C}(V))$ is a *spectral pair* for $\mathcal{C}(V)$ (definition 13.2).

Definition 24.6. The KK-class defined by this spectral pair is called the *dual Dirac class* or *Bott class* in $\beta_V \in KK(\mathbb{R}, \mathcal{C}(V))$.

Clearly, if there is a dual Dirac class, there should also be a Dirac class, right? In fact we have the machinery to define that too. We can consider V itself to be a (non compact) manifold, and $\mathrm{Cliff}(V)$ to be the fibers of a trivial bundle over V (which is indeed a real vector bundle in the sense of Remark **??**). As a manifold equipped with a bundle of Clifford modules, V has a Dirac operator. Since the manifold is *complete* in its natural metric, the functional calculus $D \mapsto f(D)$ for this operator is still defined. It is not true that $f(D)$ is compact any more, but $f(D)a$ *is* compact for any $a \in C_0(V)$; in other words $f(D)$ is "locally compact". This is enough for us to follow the Baaj-Julg argument (Proposition 23.6) to obtain a K-homology class from D:

Definition 24.7. The KK-class defined from the Dirac operator by the Baaj-Julg construction is called the *Dirac class* in $\alpha_V \in KK(\mathcal{C}(V), \mathbb{R})$.

Remark 24.8. We don't actually need completeness to define a K-homology class from an elliptic operator. We only need something that is "locally aligned with $\chi(D)$", a notion that can actually be defined without $\chi(D)$ making global sense. See *Analytic K-Homology* for details on this.

Now here is the (hardly surprising, after the build-up) punchline.

Theorem 24.9 (Bott-Kasparov periodicity)**.** *The Dirac and dual Dirac classes are KK-inverses, i.e.,* $\alpha_V \boxtimes \beta_V = 1_{\mathcal{C}(V)}$ *and* $\beta_V \boxtimes \alpha_V = 1_{\mathbb{R}}$*. Thus for any C^*-algebra A, and any V,* $K(A) \cong K(A \widehat{\otimes} \mathcal{C}(V))$.

The algebraic structure of the Clifford algebras (over $\mathbb{R}$ or $\mathbb{C}$) is well known, however: in the complex case, $\mathrm{Cliff}(\mathbb{R}^2)) = M_2(\mathbb{C})$ and in the real case $\mathrm{Cliff}(\mathbb{R}^8) = M_{16}(\mathbb{R})$. Thus shifting the Clifford parameter by 2 (complex case) or 8 (real case) just means taking a large matrix algebra over what we had before. From Bott-Kasparov periodicity it follows that (in the real case) we have 8-fold periodicity:

$$K_{p,q}(A) \cong K_{p,q+8}(A) \cong K_{p+8,q}(A).$$

Notice that the (1,1) periodicity shows that these are isomorphic to $K_{p+4,q+4}(A)$ but it requires some extra trickery, as in Atiyah's paper referenced, to move all the Clifford generators to one side or the other.

We can go a bit further, following the famous paper of Atiyah, Bott and Shapiro, *Clifford modules*, and calculate the *coefficient ring* for real K-theory: in other words (if you will) the graded ring $KKR_*(\mathbb{R}, \mathbb{R})$ for $* > 0$. (Ordinary Bott periodicity tells us the answer in the complex case: $KK_*(\mathbb{C}, \mathbb{C}) = \mathbb{Z}[b]$ where b is the Bott generator, thought of as an element of grading 2).

First we need the actual tabulation of the real Clifford algebras $C_p = \mathrm{Cliff}(\mathbb{R}^{p,0})$:

p	0	1	2	3	4	5	6	7
C_p	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$

From then on the algebras repeat according to the periodicity $C_{p+8} = M_{16}(C_p)$, as already mentioned. All of these algebras are graded algebras, and the even part of C_p is canonically isomorphic to the whole of C_{p-1} (considered as an ungraded algebra).

Since these algebras are matrix algebras over fields or skew-fields it is easy to classify all their finite-dimensional representations (that is modules), and all of these are automatically Hilbert modules. Moreover, because these are finite-dimensional unital algebras, their KK-theory can be completely described by considering finite-dimensional (graded) modules and the zero operator for F. Let $M(C_p)$ denote the

Grothendieck group of finite-dimensional (graded) C_p-modules; per the discussion preceding, this is the same as the Grothendieck group of *ungraded* C_{p-1} modules. From the table above,

$$M(C_p) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & (\text{if } 4|p \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

This does not complete the computation of the KK groups however. There is an obvious restriction map $i^*\colon M(C_{p+1}) \to M(C_p)$. If we consider a module in the image of i^* as a KK-class, it can be equipped with an operator $F = \varepsilon e_{p+1}$ (where e_{p+1} is the extra generator of C_{p+1} and ε is the grading operator) which is odd, self-adjoint, C_p-linear and has square 1: in other words, it makes the corresponding Fredholm pair *degenerate*. (Definition 12.5). Thus the module defines the zero element of KK-theory. It follows that what is really relevant here is the cokernel

$$A_p = M(C_p)/i^*(C_{p+1}).$$

Proposition 24.10. *The above construction defines an isomorphism $A_p \to KK_p(\mathbb{R}, \mathbb{R})$. Moreover, the family of these isomorphisms respects the ring structure (coming from tensor product of modules).*

Now we have to compute the groups A_p. This is elementary (see Atiyah, Bott and Shapiro again)

$$\begin{array}{c|cccccccc} \text{p} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ A_p = KR_p(\text{point}) & \mathbb{R} & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 & \mathbb{Z} & 0 & 0 & 0 \end{array}$$

and thereafter repeating with period 8. The ring structure is as follows: let ξ be the 1-dimensional generator, μ the 4-dimensional generator, and λ the 8-dimensional generator of Bott periodicity. Then

$$2\xi = 0, \quad \xi^3 = 0, \quad \mu^2 = 4\lambda$$

and $A_* = KO_*(\text{point})$ is generated by ξ, μ, λ subject to these relations.

Let's talk about Hitchin's application to positive scalar curvature on exotic spheres (generalized using higher index theory by Gromov-Lawson-Rosenberg to a conjectural *necessary and sufficient condition* for a compact spin manifold to admit a positive scalar curvature metric, many cases of which were subsequently proved true by Stolz and others).

Remember that if M is a compact spin manifold it possesses a canonical elliptic operator, the *Dirac operator* D, which was shown by Lichnerowicz to satisfy the crucial *Bochner-Lichnerowicz identity*

$$D^2 = \nabla^*\nabla + \tfrac{1}{4}\kappa,$$

where κ is (the operator of multiplication by) the scalar curvature. Consequently if $\kappa > 0$ (and is therefore bounded below by a positive constant), the operator D^2 is a *strictly positive* unbounded operator and therefore D has zero kernel ($\mathbb{Z}/2$ graded) and therefore zero index. In dimensions $4p$ the index is determined by the Atiyah Singer index theorem as $\widehat{\mathcal{A}}(M)$, a certain topological invariant (Pontrjagin number) of M, and thus one gets Lichnerowicz' theorem (1963): a compact spin $4p$-manifold with $\widehat{\mathcal{A}}(M) \neq 0$ cannot admit a metric of positive scalar curvature.

Let us think about this from the perspective of (real) K-homology. The Dirac operator on M^p defines a class in the K-homology group $KO_p(M) = KK_{p,0}(C_{\mathbb{R}}(M), \mathbb{R})$, using the Baaj Julg construction 23.6 as we have already discussed. The Bochner-Lichnerowicz argument does *not* prove that this homology class is zero (in fact it isn't) but it *does* prove that its image under the canonical map $M \to \text{point}$ is zero (using the rescaling trick from Proposition 13.7). In dimensions congruent to 0 mod 4 this is essentially the usual index and we recover Lichnerowicz' original theorem. However in dimensions congruent to 1 or 2 mod 8 there is a new invariant in $KO_p(\text{point}) = \mathbb{Z}/2$ and this invariant must also vanish for positive scalar curvature.

A few words now about the classification of exotic spheres. The collection Θ_p of diffeomorphism classes of smooth p-manifolds homeomorphic to the p-sphere is an abelian group under connected sum (with the standard smooth sphere being the identity element). Milnor and Kervaire (also 1963) computed this group (for $p \geqslant 6$) by identifying a subgroup bP_{p+1} of those spheres that bound *parallelizable* manifolds and then identifying Θ_p/bP_{p+1} with the cokernel of the so-called J-homomorphism (which is also related to real Bott periodicity, but that's another story). It follows from this work that in dimensions congruent to 1 and 2 mod 8, exactly half of the exotic spheres fail to bound a spin manifold: moreover, the $KO_p(\text{point})$-valued index of the Dirac operator (which is an invariant of spin bordism) exactly detects this phenomenon. Thus, as Hitchin found (1975), half the exotic spheres in these dimensions do not support any metric of positive scalar curvature. Mentally we associate "positive curvature", in whatever sense, with being "round", so one can say that in dimensions 9 and 10, for instance, half of the possible smooth structures on the sphere are about as far from round as it is possible to get.