Hints and Partial Solutions for ***An Introduction to K-Theory for C^*-Algebras*** by Mikael Rørdam, Flemming Larsen & Niels Jakob Laustsen

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Chapter 1

C^*-Algebra Theory

1. Consider the diagram below. The top part (with φ and ψ) is assumed to be exact. We want to show ν (with $\nu \circ \varphi(a) = a$ for all a in A) exists if and only if the *-isomorphism θ (which makes the diagram commute) exists.

$$\begin{tikzcd}
& & E \arrow[dr, "\psi"] \arrow[dl, dashed, bend left, "\nu"'] \arrow[dd, dashed, "\theta"] & & \\
0 \arrow[r] & A \arrow[ur, bend left, "\varphi"] \arrow[dr, "\iota_A"'] & & B \arrow[r] & 0 \\
& & A \oplus B \arrow[ur, "\pi_B"'] & &
\end{tikzcd}$$

First suppose that the *-isomorphism θ exists. Let $\nu = \pi_A \circ \theta$. Then, for a in A,

$$\nu \circ \varphi(a) = \pi_A \circ \theta \circ \varphi(a) = \pi_A \circ \iota_A(a) = a.$$

Conversely, suppose that there is a *-homomorphism $\nu : E \to A$ such that $\nu \circ \varphi = \mathrm{id}_A$. Let

$$\theta(e) = (\nu(e), \psi(e)).$$

If $\theta(e) = 0$, then $\psi(e) = 0$ and thus e is in the image of φ by exactness. Write $e = \varphi(a)$ for some a in A, and so then

$$0 = \nu(e) = \nu \circ \varphi(a) = a$$

and thus $e = \varphi(a) = \varphi(0) = 0$. Thus θ is injective.

If a is in A, then

$$\theta(\varphi(a)) = (\nu \circ \varphi(a), \psi \circ \varphi(a)) = (a, 0).$$

If b is in B, find some e in E with $\psi(e) = b$. Then

$$\theta(e - \varphi \circ \nu(e)) = (\nu(e), \psi(e)) - (\nu(e), 0) = (0, b).$$

Thus all $(a, 0)$ and all $(0, b)$ are in the image of θ and θ is surjective.

2. Here we want to show that the sequence below is exact, and that a supposed homomorphism indicated by the dotted arrow does not exist.

$$\begin{tikzcd} 0 \arrow[r] & C_0(0,1) \arrow[r, "\iota"] & C[0,1] \arrow[r, bend left, "\psi"] & \mathbb{C}\oplus\mathbb{C} \arrow[l, dashed, bend left, "\nexists"] \arrow[r] & 0 \end{tikzcd}$$

We first show that ι is injective. If $\iota(f)$ is the zero function on the closed interval $[0,1]$, then it is certainly zero on the open interval $(0,1)$. It follows that $f=0$.

To show that ψ is surjective, let $(w_1,w_2)\in\mathbb{C}\oplus\mathbb{C}$. Set $g(z)=w_2z+w_1(1-z)$. Then $\psi(g)=(w_1,w_2)$.

Lastly, we show that $\mathrm{im}(\iota)=\ker(\psi)$. If $h\in C_0((0,1))$, then $\iota(h)$ vanishes at the endpoints of $[0,1]$, so $\iota(h)\in\ker(\psi)$. On the other hand, if h vanishes at the endpoints of $[0,1]$, then $h|_{(0,1)}\in C_0((0,1))$ and $h\in\mathrm{im}(\iota)$.

To see that the sequence does not split, suppose λ is a right inverse for ψ. The element $(0,1)$ is a projection in $\mathbb{C}\oplus\mathbb{C}$, so $\lambda[(0,1)]$ is a projection in $C[0,1]$ and hence $\lambda[(0,1)]=0$ or $\lambda[(0,1)]=1$. Thus either $\psi\circ\lambda[(0,1)]=(0,0)$ or $\psi\circ\lambda[(0,1)]=(1,1)$, a contradiction.

3. (i) These are straightforward calculations.

(ii) Since $|\pi(a)|=0$, we have

$$\begin{aligned}\|a\|_{\widetilde{A}}=\|\!|a|\!\|_{\widetilde{A}}&=\sup\{\|ba\|_A : b\in A \text{ and } \|b\|_A\le 1\}\\ &\le \sup\{\|b\|_A\|a\|_A : b\in A \text{ and } \|b\|_A\le 1\}\\ &=\|a\|_A\end{aligned}$$

so that $\|\!|a|\!\|_{\widetilde{A}}\le\|a\|_A$. Also, $\|a^*/\|a\|_A\|_A=1$, so

$$\|\!|a|\!\|_{\widetilde{A}}=\sup\{\|ba\|_A : b\in A \text{ and } \|b\|_A\le 1\}\ge\|(a^*/\|a\|_A)a\|_A=\|a\|_A.$$

(iii) If $\|x\|_{\widetilde{A}}=0$, then $\pi(x)=0$ and $x\in A$. But then $0=\|x\|_{\widetilde{A}}=\|x\|_A$ by (ii), so $x=0$.

(iv) Positive definiteness of $\|\cdot\|_{\widetilde{A}}$ is shown in (iii). We show the triangle inequality and leave the other similar computations to the reader.

$$\begin{aligned}\|x+y\|_{\widetilde{A}}&=\max\{\sup_b\{\|b(x+y)\|_A\},|\pi(x+y)|\}\\ &\le\max\{\sup_b\{\|bx\|_A+\|by\|_A\},|\pi(x)|+|\pi(y)|\}\\ &\le\max\{\sup_b\{\|bx\|_A\},|\pi(x)|\}+\max\{\sup_b\{\|by\|_A\},|\pi(y)|\}\\ &=\|x\|_{\widetilde{A}}+\|y\|_{\widetilde{A}}.\end{aligned}$$

(v) It is clear that ι is injective and that π is surjective. We have $\pi(\iota(a))=\pi(a,0)=0$, and if $\pi(a,\alpha)=0$, then $\alpha=0$ and thus $(a,\alpha)=(a,0)=\iota(a)$. Also, $\pi(\lambda(\alpha))=\pi(0,\alpha)=\alpha$ which shows that the sequence splits. $\iota(A)$ (which may be identified with A) is an ideal in $\widetilde{A}$.

(vi) If A is not unital, then $A \oplus \mathbb{C}$ is not unital and so cannot be isomorphic to $\widetilde{A}$. If A is unital, let $f = 1_{\widetilde{A}} - 1_A$. The map $A \oplus \mathbb{C} \to \widetilde{A} : (a, \alpha) \mapsto a + \alpha f$ is a *-isomorphism:

$$(a+b, \alpha+\beta) \mapsto (a+b) + (\alpha+\beta)f = (a+\alpha f) + (b+\beta f)$$

$$(ab, \alpha\beta) \mapsto ab + \alpha\beta f = ab + \overset{=0}{\beta a f} + \overset{=0}{\alpha b f} + \alpha\beta f = (a+\alpha f)(b+\beta f)$$

$$(a^*, \overline{\alpha}) \mapsto a^* + \overline{\alpha} f = (a+\alpha f)^*.$$

4. Let p be normal and let ι be the identity map $z \mapsto z$ on $\operatorname{sp}(p)$. Apply the functional calculus to p; then $p = p^2 = p^*$ if and only if $\iota = \iota^2 = \iota^*$ if and only if $z = z^2 = \overline{z}$ for all $z \in \operatorname{sp}(p)$ if and only if $\operatorname{sp}(p) \subseteq \{0, 1\}$. This establishes (i) and (ii). (iii) and (iv) are similar, with unitaries in place of projections and $\mathbb{T}$ in place of $\{0, 1\}$.

5. a is normal, so apply the functional calculus. Because $\operatorname{sp}(a)$ is disconnected, there is a continuous function f on $\operatorname{sp}(a)$ which takes only the values 0 and 1. Such a function yields a projection $f(a)$ in A.

   If A is not unital, consider the unitization $\widetilde{A}$. Then we obtain a function f as before such that $f(a)$ is a projection in $\widetilde{A}$. Write $f(a) = b + \alpha 1$ and note

$$b + \alpha 1 = (b + \alpha 1)^2 = b^2 + 2\alpha b + \alpha^2 1.$$

   We must have $\alpha = \alpha^2$, so α is 0 or 1. If $\alpha = 0$, then $f(a)$ is the projection we are looking for. If $\alpha = 1$, then $b^2 = -b$. Thus b^2 is a nontrivial projection in A.

6. If a is invertible, then $(aa^*)^{-1} = (a^{-1})^* a^{-1}$ and $(a^*a)^{-1} = a^{-1}(a^{-1})^*$. If both a^*a and aa^* are invertible, then

$$(a^*a)^{-1}a^* = (a^*a)^{-1}a^* 1 = \overbrace{(a^*a)^{-1}a^*(a}^{=1}\, a^*)(aa^*)^{-1} = a^*(aa^*)^{-1}$$

   and it is easy to check that a^{-1} is equal to the above; this shows (i).

   For (ii), if b is invertible then $0 \notin \operatorname{sp}(b)$, and so the function $f(z) = 1/z$ is continuous on $\operatorname{sp}(b)$. Then $b^{-1} = f(b)$.

   (iii) follows from (ii) and that $f(a) \in C^*(a)$ for every normal element a.

7. Let $a = (x + x^*)/2$ and $b = (x - x^*)/2i$.

8. If $\lambda \notin \operatorname{sp}(a)$, then $a - \lambda 1$ is invertible. Thus $\varphi(a - \lambda 1) = \varphi(a) - \lambda 1$ is invertible in B and $\lambda \notin \operatorname{sp}(\varphi(a))$. If φ is injective, it has an inverse defined on its range, say, ψ. Now if $\lambda \notin \operatorname{sp}(\varphi(a))$, $\varphi(a) - \lambda 1 = \varphi(a - \lambda 1)$ is invertible and thus $\psi(\varphi(a) - \lambda 1) = a - \lambda 1$ is invertible. This establishes (i).

   For (ii), note that $r(a) = \|a\|$ if a is self-adjoint, and

$$\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = r(\varphi(a^*a)) \le r(a^*a) = \|a^*a\| = \|a\|^2.$$

   The inequality holds because $\operatorname{sp}(\varphi(a)) \subseteq \operatorname{sp}(a)$. If φ is injective, then equality holds because $\operatorname{sp}(\varphi(a)) = \operatorname{sp}(a)$.

9. Since f is continuous, its supremum norm is equal to its essential supremum norm in $L^\infty(X, \mu)$, and so

$$\|M_f\xi\|_2^2 = \int_X |f|^2|\xi|^2 d\mu \leq \|f\|_\infty^2 \int_X |\xi|^2 d\mu = \|f\|_\infty^2\|\xi\|_2^2.$$

The requested conclusions in (i) are clear.

(ii) is straightforward.

For (iii), if f is in $C(X)$ and $f \neq 0$, then it is nonzero on a nonempty open subset U of X (by taking $-f$ if necessary, assume f is positive). Choose a Borel set $E \subseteq U$ with $0 < \mu(E) < \infty$. Then $M_f\chi_E = f\chi_E$, and since E has positive measure and f is nonzero on E, $M_f\chi_E \neq 0$ in $L^2(X, \mu)$ and thus $M_f \neq 0$. It follows that π is injective.

For (iv), let $\{x_n : n \geq 1\}$ be a dense sequence in X. Define

$$\mu = \sum_{n\geq 1} 2^{-n}\delta_{x_n}$$

Where δ_x is the point-mass measure at the point x. Then μ is a finite measure and if U is open and nonempty, it contains some x_{n_0}. Then

$$\mu(\{x_{n_0}\}) = 2^{-n_0} > 0.$$

10. Suppose first that $C_0(X)$ separates points. If $x_1 \neq x_2$, choose $f \in C_0(X)$ such that $f(x_1) \neq f(x_2)$. $\mathbb{C}$ is Hausdorff, so choose disjoint neighbourhoods of U_1 and U_2 of $f(x_1)$ and $f(x_2)$ respectively, and then $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are disjoint neighbourhoods of x_1 and x_2. Now take x_0 in X and f in $C_0(X)$ such that $f(x_0) \neq 0$. Choose an open subset U of $\mathbb{C}$ with $f(x_0) \in U$ and such that U does not intersect some open disk centred at 0 with radius ε. Then $x_0 \in f^{-1}(U) \subseteq \{x \in X \mid |f(x)| \geq \varepsilon\}$. The latter set is compact, so X is locally compact.

Suppose now that X is a locally compact Hausdorff space and x_1 and x_2 are distinct points. Choose an open set V with x_1 in V and x_2 not in V. Then use Urysohn's Lemma to see that there is a continuous function f with support contained in V (therefore $f(x_2) = 0$) and such that $f(x_1) = 1$.

11. The properties are obtained analogously to the case $C_0(X)$, including the C^*-identity:

$$\|f^*f\| = \sup \|f(x)^*f(x)\| = \sup \|f(x)\|^2 = (\sup \|f(x)\|)^2 = \|f\|^2.$$

12. Let

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix};$$

then

$$x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

and these two matrices are equal only if $b = c = 0$. The sufficiency condition is clear. If $x = \mathrm{diag}(a, b)$ is unitary, then

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = x^*x = \begin{bmatrix} a^* & 0 \\ 0 & b^* \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^*a & 0 \\ 0 & b^*b \end{bmatrix}$$

which shows $a^*a = b^*b = 1$. A similar computation with xx^* shows that $aa^* = bb^* = 1$.

13.

14. The inverse of $a = (a_{ij})$ is

$$\begin{bmatrix} a_{11}^{-1} & -a_{11}^{-1}a_{12}a_{22}^{-1} & a_{11}^{-1}a_{12}a_{22}^{-1}a_{23}a_{33}^{-1} - a_{11}^{-1}a_{13}a_{33}^{-1} & \cdots & b_{1n} \\ 0 & a_{22}^{-1} & -a_{22}^{-1}a_{23}a_{33}^{-1} & \cdots & b_{2n} \\ 0 & 0 & a_{33}^{-1} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{-1} \end{bmatrix}.$$

15. The map

$$A \oplus A \to M_2(A) : (a, b) \mapsto \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

is clearly an injective homomorphism. By Exercise 1.8, it is an isometry.

Chapter 2

Projections and Unitary Elements

1. Let p and q be projections in A. $p-q$ is self-adjoint, so we may apply Lemma 2.2.3 to it with $p-q$ in place of a. Note that $\delta = \|p-(p-q)\| = \|q\| \le 1$ so that

$$\mathrm{sp}(p-q) \subseteq [-\delta, \delta] \cup [1-\delta, 1+\delta] \subseteq [-1, 2].$$

   A similar argument shows that $\mathrm{sp}(q-p) \subseteq [-1, 2]$. But the spectral mapping theorem implies that $\mathrm{sp}(p-q) \subseteq [-2, 1]$, so $\mathrm{sp}(p-q) \subseteq [-1, 1]$. It follows that $\|p-q\| = r(p-q) \le 1$.

   Let u and v be unitary elements in A. Then

$$\|u-v\| \le \|u\| + \|v\| = 1 + 1 = 2.$$

2. We check that

$$(a - i\sqrt{1-a^2})(a + i\sqrt{1-a^2}) = a^2 + 1 - a^2 = 1;$$

   the other equalities have analogous proofs. This, together with Exercise 1.7, makes the conclusion that every element is a linear combination of four unitaries easy to obtain.
   It is not the case that every element in a C^*-algebra may be written as a linear combination of projections. Take, for example, $C[0,1]$: the only projections are 0 and 1, and linear combinations of these produce only constant functions, which certainly does not encompass all of $C[0,1]$.

3. The map

$$t \mapsto a_t = \begin{bmatrix} 1 & (1-t)a_{12} & (1-t)a_{13} & \cdots & (1-t)a_{1n} \\ 0 & 1 & (1-t)a_{23} & \cdots & (1-t)a_{2n} \\ 0 & 0 & 1 & \cdots & (1-t)a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

   is a continuous path from a to 1 in $GL(M_n(A))$. Each a_t is invertible by Exercise 1.14, and continuity follows from the estimate

$$\|a_s - a_t\| \le |s-t| \sum_{i<j} \|a_{ij}\|$$

   which is established in Exercise 1.13.

4. (i) $\Longrightarrow$ (ii) If $pq = 0$, then $qp = q^*p^* = (pq)^* = 0^* = 0$ as well. Then
$$(p+q)^2 = p^2 + pq + qp + q^2 = p + 0 + 0 + q = p + q.$$
(ii) $\Longrightarrow$ (iii) If $p + q$ is a projection, then so is $1 - (p + q)$. Projections are positive, so $1 - (p + q) \geq 0$.

   (iii) $\Longrightarrow$ (i) Since $p + q \leq 1$, we may multiply both sides of this "inequality" (for lack of a better term) on the left and right by the self-adjoint element p to obtain $p(p + q)p \leq p$. Expanding gives $-pqp \geq 0$. But since $q \geq 0$, we also have $pqp \geq 0$ and thus $pqp = 0$. Finally,
$$0 = pqp = pqqp = pqq^*p^* = pq(pq)^*$$
which implies that $pq = 0$, by the C^* identity.

   This can be extended inductively for projections $p_1, p_2, \ldots, p_n$ satisfying (i), (ii), and (iii).

5. Let $z = v - vv^*v$. Then
$$z^*z = v^*(1 - vv^*)(1 - vv^*)v = v^*v - v^*vv^*v - v^*vv^*v + v^*vv^*vv^*v = 2v^*v - 2v^*v = 0$$
and thus $z = 0$ by the C^* identity. Putting $p = v^*v$ and $q = vv^*$ yields
$$vp = v(v^*v) = v = (vv^*)v = qv \text{ and } qvp = (vv^*)v(v^*v) = v(v^*v) = v$$

6. Each $v_j^*v_j$ and $v_jv_j^*$ is a projection, and so the assumption is equivalent to the mutual orthogonality of the families $\{v_j^*v_j\}$ and $\{v_jv_j^*\}$ by Exercise 2.4. So, using Exercise 2.5,
$$\begin{aligned}(v_1^* + v_2^* + \cdots + v_n^*)(v_1 + v_2 + \cdots + v_n) &= \sum_{j=1}^n v_j^*v_j + \sum_{i \neq j} v_i^*v_j \\ &= 1 + \sum_{i \neq j} v_i^* \overbrace{(v_iv_i^*v_jv_j^*)}^{=0} v_j \\ &= 1\end{aligned}$$
and similarly for $(v_1 + v_2 + \cdots + v_n)(v_1^* + v_2^* + \cdots + v_n^*)$.

7. (Yet to be finished) First suppose a is self-adjoint and that $\varepsilon < 1/2$. If f is the real function $f(x) = |x - x^2|$, choose $\delta > 0$ so that $f(x) < \delta$ implies that $x \in [-\varepsilon, \varepsilon] \cup [1 - \varepsilon, 1 + \varepsilon]$ (draw a picture!). Then if we denote by ι the identity map $z \mapsto z$ in $\mathbb{C}$ and apply the functional calculus to a, we have that $\|a - a^2\| = \|\iota - \iota^2\|_\infty < \delta$ implies that the spectrum of a is contained in $[-\varepsilon, \varepsilon] \cup [1 - \varepsilon, 1 + \varepsilon]$. For $x \in \text{sp}(a)$, let
$$g(x) = \begin{cases} 0 & \text{if } x \in [-\varepsilon, \varepsilon] \\ 1 & \text{if } x \in [1 - \varepsilon, 1 + \varepsilon] \end{cases}$$
and set $p = g(a)$. Then $\|a - p\| = \|\iota - g\|_\infty \leq \varepsilon$.
If a is not self-adjoint, let $b = (a + a^*)/2$ and apply the previous argument to obtain a $\delta > 0$ so that $\|b - b^2\| < \delta$ implies that there is a projection p so that $\|b - p\| \leq \varepsilon$.

8. (Yet to be finished) First, we may stipulate that $\delta < 1$ so that a^*a and aa^* are invertible. Using the triangle inequality, we obtain

$$\|a^*a\| \leq \|a^*a - 1\| + 1 < 2$$

so that $\|a\| = \sqrt{\|a^*a\|} < \sqrt{2}$. Let $f(x) = x^{-1/2}$. Choose $\delta > 0$ so that $\sigma(a^*a) \subseteq [1-\delta, 1+\delta]$ implies $\|f - 1\|_\infty < \varepsilon/\sqrt{2}$ on $\sigma(a^*a)$. If ι is the continuous function $\iota(x) = x$, observe that $\|\iota - 1\|_\infty < \delta$ on $\sigma(a^*a)$ implies $\sigma(a^*a) \subseteq [1-\delta, 1+\delta]$. Then

$$\|a(a^*a)^{-1/2} - a\| \leq \|a\|\|(a^*a)^{-1/2} - 1\| \leq \sqrt{2}\|f-1\|_\infty \leq \varepsilon$$

and $a(a^*a)^{-1/2}$ is unitary.

9. (i) $\Longrightarrow$ (ii) If $p = v^*v$ and $q = vv^*$, we have $\mathrm{Tr}(p) = \mathrm{Tr}(v^*v) = \mathrm{Tr}(vv^*) = \mathrm{Tr}(q)$.

   (ii) $\Longrightarrow$ (iii) $\mathrm{Tr}(p) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of p. Since p is a projection, $\lambda_i \in \{0, 1\}$ for all i. The algebraic multiplicity of the eigenvalue 0 must be equal to the dimension of the kernel of p by the Rank-Nullity Theorem, and thus $\mathrm{Tr}(p)$ is a positive integer which counts the dimension of the range of p, i.e., $\mathrm{Tr}(p) = \mathrm{rank}(p)$. The conclusion follows.

   (iii) $\Longrightarrow$ (i) Say p and q have rank k. Choose orthonormal bases $\{e_1, e_2, \ldots, e_k\}$ of $p(\mathbb{C}^n)$ and $\{f_1, f_2, \ldots, f_k\}$ of $q(\mathbb{C}^n)$ respectively, and extend them both to orthonormal bases of $\mathbb{C}^n$, say, $\{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_n\}$ and
   $\{f_1, f_2, \ldots, f_k, f_{k+1}, \ldots, f_n\}$. Define

$$v : \mathbb{C}^n \to \mathbb{C}^n : e_i \mapsto \left\{ \begin{array}{ll} f_i & i \leq k \\ 0 & i > k \end{array} \right. .$$

   Then if $x = \sum_{i=1}^n \alpha_i e_i$,

$$v^*v(x) = v^*v\left(\sum_{i=1}^n \alpha_i e_i\right) = v^*\left(\sum_{i=1}^k \alpha_i f_i\right) = \sum_{i=1}^k \alpha_i e_i = p(x)$$

   so that $p = v^*v$, and similarly, $q = vv^*$.

   To show that $\mathcal{D}(\mathbb{C}) \cong \mathbb{Z}^+$, define the map

$$\dim : \mathcal{D}(\mathbb{C}) \to \mathbb{Z}^+ : [p]_\mathcal{D} \mapsto \mathrm{rank}(p).$$

   dim is well-defined and injective by the equivalence of (i) and (iii): $[p]_\mathcal{D} = [q]_\mathcal{D}$ if and only if $p \sim_0 q$ if and only if $\mathrm{rank}(p) = \mathrm{rank}(q)$. Since $\dim([1_n]_\mathcal{D}) = n$ for all $n \in \mathbb{Z}^+$, dim is surjective. Finally, $\dim([p]_\mathcal{D} + [q]_\mathcal{D}) = \dim([p \oplus q]_\mathcal{D}) = \mathrm{rank}(p \oplus q) = \mathrm{rank}(p) + \mathrm{rank}(q)$, so it is an isomorphism.

   If $p \sim q$, then they have the same rank, and by the Rank-Nullity theorem, their kernels have the same dimension. Since the kernel of p is the range of $1-p$, the projections $1-p$ and $1-q$ have the same rank, and hence $1 - p \sim 1 - q$. Proposition 2.2.2 implies that $p \sim_u q$.

   If $p \sim q$, then $p = uqu^*$ for some unitary u in $M_n(\mathbb{C})$ by the above argument and by the equivalence of (i) and (ii) in Proposition 2.2.2. By Corollary 2.1.4, u is homotopic to 1_n, and hence $p \sim_h q$ by Proposition 2.2.6.

10. (Yet to be finished)

11. Notice that there is an obvious way to identify $M_n(\mathbb{C}\oplus\mathbb{C})$ with $M_n(\mathbb{C})\oplus M_n(\mathbb{C})$ for all $n \geq 1$. Define the map

$$\dim : \mathcal{D}(\mathbb{C}\oplus\mathbb{C}) \to \mathbb{Z}^+ \oplus \mathbb{Z}^+ : [(p,q)]_{\mathcal{D}} \mapsto (\mathrm{rank}(p), \mathrm{rank}(q))$$

and use Exercise 2.9.

12. (i) follows from the fact that the spectrum of an element in $C(\mathbb{T})$ is equal to its range, and $v(\mathbb{T}) = \mathbb{T}$; use part (iv) of Exercise 1.4.

For (ii), suppose that there is such a unitary u. To be a lift of v which is a unitary, u must fix $\mathbb{T}$ and have a range contained in $\mathbb{T}$. Define $w : \mathbb{D} \to \mathbb{D} : z \mapsto e^{i\pi}z$. Then $w \circ u$ is a continuous function from $\mathbb{D}$ to $\mathbb{D}$ that has no fixed point. Indeed, if $z \in \mathbb{T}$, then $w \circ u(z) = e^{i\pi}z \neq z$, and if $z \notin \mathbb{T}$, then $w \circ u(z) = e^{i\pi}u(z) \in \mathbb{T}$, so $e^{i\pi}u(z) \neq z$. This contradicts Brouwer's Fixed Point Theorem.

An alternative, slightly slicker proof of (ii) is as follows: if there did exist a u as above, this would imply that $\mathbb{T}$ is a retract of $\mathbb{D}$. This would imply that the induced homomorphism between fundamental groups $\pi_1(\mathbb{T}) \to \pi_1(\mathbb{D})$ is injective. But $\pi_1(\mathbb{T}) \cong \mathbb{Z}$ and $\pi_1(\mathbb{D}) = 0$, so this is impossible.

For (iii), $v \notin \psi(\mathcal{U}_0(C(\mathbb{D}))) = \mathcal{U}_0(C(\mathbb{T}))$ by Lemma 2.1.7 (i). Thus $v \not\sim_h 1$ and one may take $v_1 = v$ and $v_2 = 1$. Apply Proposition 2.1.6 (iii) to obtain the final conclusion.

Chapter 3

The K_0-Group of a Unital C^*-Algebra

1. If A is separable, then so is $M_n(A)$ for every n. Suppose $K_0(A)$ is uncountable. Choose one projection p_α from each equivalence class. Then p_α is not homotopic to $p_{\alpha'}$ if $\alpha \neq \alpha'$ by Proposition 3.1.7 (iii), and thus $\|p_\alpha - p_{\alpha'}\| = 1$ by Proposition 2.2.4. So there is an uncountable set of elements in $\bigcup M_n(A)$ that are pairwise at a distance of 1 from each other, which is impossible in a separable space.

2. (i) $\implies$ (ii) If $p \sim_u q$, then $p \oplus 0 \sim_h q \oplus 0$ and so

$$\nu(p) = \nu(p) + \nu(0) = \nu(p \oplus 0) = \nu(q \oplus 0) = \nu(q) + \nu(0) = \nu(q).$$

   (ii) $\implies$ (iii) Suppose $p \in \mathcal{P}_m(A)$, $q \in \mathcal{P}_n(A)$ and that $m \leq n$. Then $p \oplus 0_{n-m} \sim q$. Now $(p \oplus 0_{n-m}) \oplus 0 \sim_u q \oplus 0$ and so

$$\nu(p) = \nu(p) + \nu(0) = \nu(p \oplus 0_{n-m+1}) = \nu(q \oplus 0) = \nu(q) + \nu(0) = \nu(q).$$

   (iii) $\implies$ (iv)

3. Use the homotopy $[0,1] \times X \to X : (t,x) \mapsto tx$ for both $X = [0,1]$ and $X = \mathbb{D}$.

4. We do the usual identification of $M_n(C(X))$ with $C(X, M_n(\mathbb{C}))$.

   (i) As pointed out in Example 3.3.5, for p in $\mathcal{P}_\infty(C(X))$, the function φ_p defined by $\varphi_p(x) = \mathrm{Tr}(p(x))$ is in $C(X, \mathbb{Z})$. Moreover, the map

$$\mathcal{P}_\infty(C(X)) \to C(X, \mathbb{Z}) : p \mapsto \varphi_p$$

   satisfies the assumptions of Proposition 3.1.8, so we get a group homomorphism $\dim : K_0(C(X)) \to C(X, \mathbb{Z})$ satisfying $\dim([p]_0) = \varphi_p$, or $\dim([p]_0)(x) = \mathrm{Tr}(p(x))$. To see that dim is surjective, let f be any element of $C(X, \mathbb{Z})$. Since X is compact, $f(X)$ is finite; denote $f(X) = \{n_1, n_2, \ldots, n_k\}$. For $j = 1, 2, \ldots, k$, let $X_j = f^{-1}(n_j)$; then $X_1, X_2, \ldots, X_k$ is a partition of X into clopen sets. Let $p_j = \chi_{X_j}$, the characteristic function of X_j, which is a projection in $C(X)$. Finally, let

$$p = p_1^{\oplus n_1} \oplus p_2^{\oplus n_2} \oplus \cdots \oplus p_k^{\oplus n_k}$$

where

$$p_j^{\oplus n_j} = \begin{bmatrix} p_j & 0 & 0 & \cdots & 0 \\ 0 & p_j & 0 & \cdots & 0 \\ 0 & 0 & p_j & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_j \end{bmatrix}$$

with p_j appearing n_j times. Then p is in $\mathcal{P}_\infty(C(X))$ and $\dim([p]_0) = \varphi_p = f$.

(ii) We have $\dim([p]_0) = \dim([q]_0)$ if and only if $\mathrm{Tr}(p(x)) = \mathrm{Tr}(q(x))$ for all x in X. By Exercise 2.9, this occurs if and only if for each x there is some v_x in $M_{m,n}(\mathbb{C})$ with $v_x^* v_x = p(x)$ and $v_x v_x^* = q(x)$. (I'm pretty sure $M_{m,n}(C(X))$ is a typo, because otherwise the notation doesn't make sense.)

(iii) Assume $\dim([p]_0) = \dim([q]_0)$; by adding zeros down the diagonal if necessary, we may assume that p and q are the same size n. By part (ii), for each x there is v_x in $M_n(\mathbb{C})$ such that $v_x^* v_x = p(x)$ and $v_x v_x^* = q(x)$. Use uniform continuity to find a partition $X_1, X_2, \ldots, X_r$ of X into clopen sets such that, if x and y are in X_j, then $\|p(x) - p(y)\| < 1$ and $\|q(x) - q(y)\| < 1$. Pick any point x_j in X_j for all $j = 1, 2, \ldots, r$ and set $v_j = v_{x_j}$. Regard each v_j as an element of $C(X, M_n(\mathbb{C}))$ which takes the constant value v_j on X_j and zero elsewhere. Then if x is in X_j, we have $\|v_j^* v_j(x) - p(x)\| = \|p(x_j) - p(x)\| < 1$ and $\|v_j v_j^*(x) - q(x)\| = \|q(x_j) - q(x)\| < 1$. By Proposition 2.2.4,

$$p|_{X_j} \sim_h v_j^* v_j \sim v_j v_j^* \sim_h q|_{X_j}$$

and since the v_j's are pairwise orthogonal, $v = \sum_{j=1}^r v_j$ is a partial isometry in $C(X, M_n(\mathbb{C}))$ with $v^* v \sim_h p$ and $vv^* \sim_h q$, hence $p \sim_0 q$.

5. (i) Let $a = [a_{ij}]$ and $b = [b_{ij}]$ be in $M_n(A)$. The following calculation is similar to the one for $M_n(\mathbb{C})$, although τ plays a vital role.

$$\tau_n(ab) = \sum_{i=1}^n \tau((ab)_{ii}) = \sum_{i=1}^n \tau\left(\sum_{j=1}^n a_{ij} b_{ji}\right) = \sum_{i=1}^n \sum_{j=1}^n \tau(a_{ij} b_{ji})$$

Then use the fact that $\tau(a_{ij} b_{ij}) = \tau(b_{ij} a_{ij})$, and work backwards.

(ii) The observations that $e_{ij} e_{jk} = e_{ik}$, that $e_{ij} e_{kl} = 0$ when $j \neq k$, and that $\sum_{i=1}^n e_{ii} = 1_n$ are pretty clear. The matrix $e_{1i} b e_{j1}$ is the matrix with b_{ij} in the $(1,1)$-entry and zeros elsewhere. Therefore,

$$\rho(e_{1i} b e_{j1}) = \rho(\mathrm{diag}(b_{ij}, 0, \ldots, 0)) = \tau(b_{ij})$$

and

$$\tau_n(e_{1i} b e_{j1}) = \tau_n(\mathrm{diag}(b_{ij}, 0, \ldots, 0)) = \tau(b_{ij})$$

so that $\rho(e_{1i} b e_{j1}) = \tau_n(e_{1i} b e_{j1})$. Now because ρ is a trace,

$$\rho(e_{ii} b e_{ii}) = \rho(e_{ii} e_{ii} b) = \rho(e_{ii} b)$$

and

$$\rho(e_{1i} b e_{i1}) = \rho(e_{i1} e_{1i} b) = \rho(e_{ii} b)$$

so that $\rho(e_{ii}be_{ii}) = \rho(e_{1i}be_{i1})$. Similarly,

$$\rho(e_{ii}be_{jj}) = \rho(e_{jj}e_{ii}b) = \rho(0) = 0.$$

Putting this all together,

$$\begin{aligned}
\rho(b) &= \rho\left(\sum_{i,j=1}^{n} e_{ii}be_{jj}\right) && (3.1)\\
&= \sum_{i,j=1}^{n} \rho(e_{ii}be_{jj}) && (3.2)\\
&= \sum_{i=1}^{n} \rho(e_{ii}be_{ii}) && (3.3)\\
&= \sum_{i=1}^{n} \rho(e_{1i}be_{i1}) && (3.4)\\
&= \sum_{i=1}^{n} \tau(b_{ii}) && (3.5)\\
&= \tau_n(b). && (3.6)
\end{aligned}$$

(iii) This follows easily.

6. (i) $\Longrightarrow$ (ii) This is trivial.

   (ii) $\Longrightarrow$ (iii) Write $a = c^2$ where c is self-adjoint. Then

$$\tau(uc^2u^*) = \tau(uc(uc)^*) = \tau((uc)^*uc) = \tau(cu^*uc) = \tau(c^2).$$

   (iii) $\Longrightarrow$ (i) First let u be a unitary and x be arbitrary. We may write $xu = \sum t_i x_i$, a finite linear combination of positive elements x_i. Then

$$\tau(ux) = \tau(uxuu^*) = \tau\left(u\sum t_i x_i u^*\right) = \tau\left(\sum t_i x_i\right) = \tau(xu).$$

   To show that $\tau(xy) = \tau(yx)$ for all x and y in A, write y as a finite linear combination of unitaries and apply the first proof.

7. Suppose a, b are such that the maps

$$t \mapsto \varphi_t(a) \text{ and } t \mapsto \varphi_t(b)$$

   are continuous. If $t_n \to t$ in $[0, 1]$, then

$$\varphi_{t_n}(a + \lambda b) = \varphi_{t_n}(a) + \lambda\varphi_{t_n}(b) \to \varphi_t(a) + \lambda\varphi_t(b) = \varphi_t(a + \lambda b)$$

   and

$$\varphi_{t_n}(ab) = \varphi_{t_n}(a)\varphi_{t_n}(b) \to \varphi_t(a)\varphi_t(b) = \varphi_t(ab)$$

since the algebraic operations are all continuous.

Let a be in A and $t_n \to t$ in $[0,1]$. We want to show that $\lim_{n\to\infty} \varphi_{t_n}(a) = \varphi_t(a)$. Let $\varepsilon > 0$ be arbitrary, and choose f in F with $\|a - f\| < \frac{\varepsilon}{3}$. Then we have $\|\varphi(a) - \varphi(f)\| < \frac{\varepsilon}{3}$ for any *-homomorphism (because they satisfy $\|\varphi\| \leq 1$). Choose N so that $n \geq N$ implies $\|\varphi_{t_n}(f) - \varphi_t(f)\| < \frac{\varepsilon}{3}$. Then for such n,

$$\begin{aligned}
\|\varphi_{t_n}(a) - \varphi_t(a)\| &= \|\varphi_{t_n}(a) - \varphi_{t_n}(f) + \varphi_{t_n}(f) - \varphi_t(f) + \varphi_t(f) - \varphi_t(a)\| \\
&\leq \|\varphi_{t_n}(a) - \varphi_{t_n}(f)\| + \|\varphi_{t_n}(f) - \varphi_t(f)\| + \|\varphi_t(f) - \varphi_t(a)\| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

which completes the proof.

8. Suppose p is of the given form. A computation shows that $p = p^2 = p^*$. Then observe that the trace of p is 1, and use Exercise 2.9.

Now suppose p is a one-dimensional projection. Write

$$p = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix}.$$

Thus $a = \overline{a}$ and $d = \overline{d}$ so a and d are real. Since the trace of p is 1 (use Exercise 2.9 again), we have $a + d = 1$, so set $a = t$ and hence $d = 1 - t$. Now write $b = \omega|b|$ in its polar form and observe that, because $a = a^2 + bc$,

$$|b|^2 = b\overline{b} = bc = a - a^2 = t - t^2 = t(1-t)$$

so that

$$|b| = \sqrt{t(1-t)}.$$

The rest now follows easily.

The map

$$[0,1] \times \mathbb{T} \to G_{2,1} : (t, \omega) \mapsto \begin{bmatrix} t & \omega\sqrt{t(1-t)} \\ \overline{\omega}\sqrt{t(1-t)} & 1-t \end{bmatrix}$$

is a continuous surjective map from a compact space to a Hausdorff space, therefore it is a quotient map. It is injective on $(0,1) \times \mathbb{T}$ and constant when $t = 0$ and when $t = 1$. So it drops to a homeomorphism on the quotient space $[0,1] \times \mathbb{T}/\sim$ where $\{0\} \times \mathbb{T}$ is collapsed to a point, as is $\{1\} \times \mathbb{T}$. But this space is precisely S^2. (Think of gluing the ends of a paper towel tube shut.)

9. (i) Easy.

(ii) By definition, $\mathrm{Mor}(N, N')$ and $\mathrm{Mor}(N', N)$ contain only one element each, denoted $0_{N,N'}$ and $0_{N',N}$, respectively. For the same reason, $\mathrm{Mor}(N, N)$ and $\mathrm{Mor}(N', N')$ also contain only one element each, and since they respectively contain id_N and $\mathrm{id}_{N'}$, these must be the unique elements. But $0_{N,N'} \circ 0_{N',N}$ is in $\mathrm{Mor}(N, N)$, so we must have $0_{N,N'} \circ 0_{N',N} = \mathrm{id}_N$. Similarly, we must have $0_{N',N} \circ 0_{N,N'} = \mathrm{id}_{N'}$, therefore $N \cong N'$.

(iii) If N' is another zero object, we have

$$0_{B,A} = 0_{B,N} \circ 0_{N,A} = 0_{B,N} \circ \mathrm{id}_N \circ 0_{N,A} = 0_{B,N} \circ 0_{N,N'} \circ 0_{N',N} \circ 0_{N,A} = 0_{B,N'} \circ 0_{N',A}$$

since $0_{B,N} \circ 0_{N,N'}$ is in $\mathrm{Mor}(B, N')$, which consists of only one element, so we must have $0_{B,N} \circ 0_{N,N'} = 0_{B,N'}$. Similarly, $0_{N',N} \circ 0_{N,A} = 0_{N',A}$.

10.

11. (i) Let $x = e - e^*$ so that $h = 1 + x^*x$. Since x^*x commutes with 1,

$$\sigma(h) = \sigma(1 + x^*x) \subseteq \sigma(1) + \sigma(x^*x) \subseteq [1, \infty)$$

therefore h is invertible. It is straightforward to check that $eh = ee^*e = he$ and $e^*h = e^*ee^* = he^*$. This implies that h commutes with ee^*, hence $ee^*h^{-1} = h^{-1}ee^*$. Thus $p = ee^*h^{-1}$ is self-adjoint. Also,

$$p^2 = ee^*h^{-1}ee^*h^{-1} = h^{-1}(ee^*e)e^*h^{-1} = h^{-1}(he)e^*h^{-1} = ee^*h^{-1} = p$$

Finally, we compute

$$pe = ee^*h^{-1}e = h^{-1}ee^*e = h^{-1}he = e$$

and

$$ep = eee^*h^{-1} = ee^*h^{-1} = p$$

(ii) If $ba = p$ and $ab = q$, take $c = (aa^*)^{-1/2}a$.

12.

13. Observe that the function $\beta : [0, 1] \times X \to [0, \infty)$ defined by $\beta(t, x) = |f(\alpha(t, x)) - f(\alpha(t_0, x))|$ is continuous, and $W = \beta^{-1}([0, \varepsilon))$, so the set W is open. It also contains $\{t_0\} \times X$, so the proposed $\delta > 0$ exists by the Tube Lemma. Lastly,

$$\|\varphi_t(f) - \varphi_{t_0}(f)\| = \sup_{x \in X} |\varphi_t(f)(x) - \varphi_{t_0}(f)(x)| = \sup_{x \in X} |f(\alpha(t, x)) - f(\alpha(t_0, x))| \leq \varepsilon$$

Chapter 4

The Functor K_0

1. If A is separable, so is $\widetilde{A}$. Then $K_0(A)$ is a subgroup of $K_0(\widetilde{A})$, the latter of which is countable by Exercise 3.1.

2. If X is disconnected with a separation $X_1 \cup X_2$, define the map

$$\Phi : C_0(X) \to C_0(X_1) \oplus C_0(X_2) : f \mapsto (f|_{X_1}, f|_{X_2}).$$

Since $\|f\| = \max\{\|f|_{X_1}\|, \|f|_{X_2}\|\}$, Φ is an isometry. Suppose $g_i \in C_0(X_i)$ for $i = 1, 2$ and define $g \in C_0(X)$ by $g(x) = g_i(x)$ if $x \in X_i$ for $i = 1, 2$. g is continuous since $X_1 \cup X_2$ is a separation of X, and $\Phi(g) = (g_1, g_2)$, so Φ is surjective.

$\widetilde{C_0(\mathbb{R})}$ can be identified with $C(\mathbb{T})$ since $\mathbb{T}$ is the one-point compactification of $\mathbb{R}$. Taking for granted that $K_0(C(\mathbb{T})) \cong \mathbb{Z}$, note that the sequence

$$0 \longrightarrow C_0(\mathbb{R}) \xrightarrow{\ \iota\ } C(\mathbb{T}) \xrightarrow{\ \pi\ } \mathbb{C} \longrightarrow 0$$

is split exact, and so

$$0 \longrightarrow K_0(C_0(\mathbb{R})) \xrightarrow{\ K_0(\iota)\ } K_0(C(\mathbb{T})) \xrightarrow{\ K_0(\pi)\ } K_0(\mathbb{C}) \longrightarrow 0$$

is also split exact. But $K_0(C(\mathbb{T})) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ and any surjective homomorphism from $\mathbb{Z}$ to itself is necessarily injective. Thus $K_0(\pi)$ is injective and it follows that $K_0(C_0(\mathbb{R})) = \ker(K_0(\pi)) = 0$. We also have $K_0(C_0((0, 1])) = 0$ since $C_0(0, 1]$ is contractible, and hence

$$K_0(C_0(U)) \cong \mathbb{Z} \oplus 0 \oplus 0 \oplus 0 \oplus \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

3. It is straightforward to show that μ is an endomorphism. If $s_n = \operatorname{diag}(s, s, \ldots, s)$ and $a = (a_{ij}) \in M_n(A)$, then

$$\mu_n(a) = (s a_{ij} s^*) = s_n a s_n^*.$$

If p is a projection in $M_n(A)$, note that

$$\mu_n(p) = s_n p s_n^* = s_n p p^* s_n^* = s_n p (s_n p)^* \sim (s_n p)^* s_n p = p^* s_n^* s_n p = p$$

and so

$$K_0(\mu)([p]_0) = [\mu(p)]_0 = [p]_0 = [\mathrm{id}(p)]_0 = K_0(\mathrm{id})([p]_0).$$

4. The second statement is a direct result of the equation

$$s(p) - \mathrm{diag}(1_n, 0_n) = s(p - \mathrm{diag}(1_n, 0_n))$$

Let us now prove the first statement. Let g be an element of $K_0(A)$. Write $g = [q]_0 - [s(q)]_0$ for some $q \in \mathcal{P}_n(\widetilde{A})$. Then

$$\begin{aligned}
[q]_0 - [s(q)]_0 &= [q]_0 - [s(q)]_0 + 0 \\
&= [q]_0 - [s(q)]_0 + ([1_n - s(q)]_0 - [1_n - s(q)]_0) \\
&= ([q]_0 + [1_n - s(q)]_0) - ([s(q)]_0 + [1_n - s(q)]_0) \\
&= \left[\begin{pmatrix} q & 0 \\ 0 & 1_n - s(q) \end{pmatrix}\right]_0 - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}\right]_0
\end{aligned}$$

By Exercise 2.9 we may find a unitary u in $\mathcal{U}_{2n}(\mathbb{C})$ such that

$$u \begin{pmatrix} s(q) & 0 \\ 0 & 1_n - s(q) \end{pmatrix} u^* = \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}$$

and let

$$p = u \begin{pmatrix} q & 0 \\ 0 & 1_n - s(q) \end{pmatrix} u^*$$

Then $g = [p]_0 - [\mathrm{diag}(1_n, 0)]_0$ and $s(p) = \mathrm{diag}(1_n, 0)$.

5. For convenience, we will work with the Hilbert space l^2 with its standard orthonormal basis $e_m = (0, 0, \ldots, 0, \overset{m}{1}, 0, \ldots)$.

   (i) For $j = 1, 2, \ldots, n$ and $m \geq 1$, define s_j by $s_j e_m = e_{mn+j}$. The adjoint s_j^* is then $s_j^* e_m = e_{\frac{m-j}{n}}$ if n divides $m - j$, and is zero otherwise.

   (ii) Notice that us_j is an isometry for $j = 1, 2, \ldots, n$, so use the universal property of $\mathcal{O}_n$ to get φ_u. Also,

   $$\sum_{j=1}^n \varphi_u(s_j)s_j^* = \sum_{j=1}^n us_js_j^* = u\left(\sum_{j=1}^n s_js_j^*\right) = u1 = u$$

   (iii) Let

   $$u = \sum_{j=1}^n \varphi(s_j)s_j^*$$

   Then u is a unitary, and $us_k = (\sum_j \varphi(s_j)s_j^*)s_k = \varphi(s_k)s_k^*s_k = \varphi(s_k)$, and since φ_u is the unique endomorphism that does this, we must have $\varphi = \varphi_u$.

   (iv) If p is a projection in $\mathcal{P}_\infty(\mathcal{O}_n)$, then the $s_jps_j^*$ are pairwise orthogonal projections in $\mathcal{O}_n$, and

   $$s_jps_j^* = s_jp(s_jp)^* \sim (s_jp)^*s_jp = p$$

so

$$K_0(\lambda)([p]_0) = [\lambda(p)]_0 = \left[\sum_{j=1}^{n} s_j p s_j^*\right]_0 = \sum_{j=1}^{n}[s_j p s_j^*]_0 = \sum_{j=1}^{n}[p]_0 = n[p]_0$$

(v) Using the same trick in (iii),

$$u = \sum_{j=1}^{n} \lambda(s_j)s_j^* = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} s_k s_j s_k^*\right) s_j^* = \sum_{j,k=1}^{n} s_k s_j s_k^* s_j^*$$

which is a self-adjoint unitary, so $\sigma(u) \subseteq \mathbb{R} \cap \mathbb{T} = \{1, -1\}$ and hence $u \sim_h 1$ by part (ii) of Lemma 2.1.3. It follows that $\lambda \sim_h \mathrm{id}$, so $K_0(\lambda) = \mathrm{id}$ by part (i) of Proposition 3.2.6.

(vi) Combining parts (iv) and (v), we see that $K_0(\lambda)(g)$ is equal to both g and ng, so $(n-1)g = 0$ in $K_0(\mathcal{O}_n)$. In the particular case when $n = 2$, we get $K_0(\mathcal{O}_2) = 0$.

6. (i) Find two projections p and q in A with $pq = 0$ and $p \sim q \sim 1$. Let s_1 be such that $s_1^* s_1 = 1$ and $s_1 s_1^* = p$, and s_2 be such that $s_2^* s_2 = 1$ and $s_2 s_2^* = q$. Then s_1 and s_2 are isometries, and $s_1 s_1^* \perp s_2 s_2^*$.

(ii) Use the hint. Remember that products of isometries are isometries.

(iii) Easy.

(iv) We have $(v_n p v_n^*)^2 = v_n p v_n^* v_n p v_n^* = v_n p 1_n p v_n^* = v_n p^2 v_n^* = v_n p v_n^*$, and it is easy to see it's self-adjoint. Let $w = v_n p$. Then $w^* w = p$ and $ww^* = v_n p v_n^*$, so $p \sim_0 v_n p v_n^*$.

(v) Notice that $t_1^* t_2 = t_1^* t_1 t_1^* t_2 t_2^* t_2 = 0$, and similarly for $t_2^* t_3$ and $t_1^* t_3$. Therefore by orthogonality and part (iv),

$$\begin{aligned}
[r]_0 &= [t_1 p t_1^* + t_2(1-q)t_2^* + t_3(1 - t_1 t - 1^* - t_2 t_2^*)t_3^*]_0 && (4.1)\\
&= [t_1 p t_1^*]_0 + [t_2(1-q)t_2^*]_0 + [t_3(1 - t_1 t_1^* - t_2 t_2^*)t_3^*]_0 && (4.2)\\
&= [p]_0 + [1-q]_0 + [1 - t_1 t_1^* - t_2 t_2^*]_0 && (4.3)\\
&= [p]_0 + [1-q]_0 + ([1] - ([t_1 t_1^*]_0 + [t_2 t_2^*]_0)) && (4.4)\\
&= [p]_0 + [1-q]_0 - [1]_0 && (4.5)\\
&= [p]_0 - [q]_0 && (4.6)
\end{aligned}$$

(vi) Easy.

(vii) For $\mathcal{O}_n$, take the projections $s_1 s_1^*$ and $s_2 s_2^*$. If H is infinite dimensional, partition an orthonormal basis into two subsets of equal cardinality with the entire basis. Then take the projections on to the spans of these respective subsets.

Chapter 5

The Ordered Abelian Group $K_0(A)$

1. Let b be a left inverse for a. By the proof of Lemma 5.1.2, we have $a^*a - \|b\|^{-2}1 \geq 0$, so $\sigma(a^*a) \subseteq [\|b\|^{-2}, \infty)$ and a^*a is invertible. If a^*a is invertible, then it is easily checked that $(a^*a)^{-1}a^*$ is a left inverse for a. The proof involving aa^* is similar.

2. (i) If $x = (x_{ij}) \in M_n(A)$, the kth diagonal entry of x^*x is $a_k := x_{1k}^*x_{1k} + x_{2k}^*x_{2k} + \cdots + x_{nk}^*x_{nk}$. Thus

$$\tau_n(x^*x) = \sum_{k=1}^{n} \tau(a_k) = \sum_{k=1}^{n} \tau\left(\sum_{i=1}^{n} x_{ik}^*x_{ik}\right) = \sum_{i,j=1}^{n} \tau(x_{ij}^*x_{ij}).$$

   (ii) This is clear.

   (iii) Let $x \in M_n(A)$ be nonzero. Then at least one entry x_{ij} is nonzero, hence $\tau(x_{ij}^*x_{ij}) > 0$. It follows that $\tau_n(x^*x) > 0$.

   (iv) It is enough to show that A is finite, since a faithful positive trace on A produces one on $M_n(A)$ for every n. Let s be an isometry; then $\tau(1) = \tau(s^*s) = \tau(ss^*)$. Hence $\tau(1 - ss^*) = 0$ and, because $1 - ss^* \geq 0$ and τ is faithful, $ss^* = 1$. Every isometry is thus a unitary and so A is finite by Lemma 5.1.2.

3. (Yet to be finished)

4. We take the same isomorphism as before, $\dim : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$.

Chapter 6

Inductive Limit C^*-algebras

1. Let $(G_1, \{\mu_n\})$ be the inductive limit of the first sequence. Define

$$\lambda_n : \mathbb{Z} \to \mathbb{Q} : k \mapsto \frac{k}{(n-1)!}$$

for all n. Then $\lambda_n = \lambda_{n+1} \circ n$ for all n and so $(\mathbb{Q}, \{\lambda_n\})$ satisfies part (ii) of Definition 6.2.2. Thus there is a unique group homomorphism $\lambda : G_1 \to \mathbb{Q}$ such that $\lambda_n = \lambda \circ \mu_n$ for all n. If $p/q \in \mathbb{Q}$, we have

$$\frac{p}{q} = \frac{p(q-1)!}{q!} = \lambda_{q+1}(p(q-1)!) = \lambda \circ \mu_{q+1}(p(q-1)!),$$

so λ is surjective. Now suppose λ is not injective. Then by Proposition 6.2.5 there exists an n_0 such that $\ker(\mu_{n_0}) \neq \ker(\lambda_{n_0}) = \{0\}$, i.e., μ_{n_0} is not injective. But then $\lambda_{n_0} = \lambda \circ \mu_{n_0}$ is not injective, a contradiction. Thus λ is an isomorphism and $G_1 \cong \mathbb{Q}$.

Let $(G_2, \{\mu_n\})$ be the inductive limit of the second sequence. Define

$$\lambda_n : \mathbb{Z} \to \mathbb{Q} : k \mapsto \frac{k}{2^n}$$

for all n. Then $\lambda_n = \lambda_{n+1} \circ 2$ for all n and so $(\mathbb{Q}, \{\lambda_n\})$ satisfies part (ii) of Definition 6.2.2. Thus there is a unique group homomorphism $\lambda : G_2 \to \mathbb{Q}$ such that $\lambda_n = \lambda \circ \mu_n$ for all n. Since each λ_n is injective we obtain, as before, that λ is injective. Thus λ is an isomorphism onto its range, and by Proposition 6.2.5,

$$G_2 \cong \lambda(G_2) = \bigcup_{n=1}^{\infty} \lambda_n(\mathbb{Z}) = \bigcup_{n=1}^{\infty} \frac{1}{2^n}\mathbb{Z} = \mathbb{Z}\left[\frac{1}{2}\right],$$

the additive group of dyadic rational numbers.

2.

3.

4.

5.

6.

7. (i) We have

$$\mu_{n_{k+1}} \circ \psi_k = \mu_{n_{k+1}} \circ (\varphi_{n_{k+1}-1} \circ \cdots \circ \varphi_{n_k+1} \circ \varphi_{n_k}) = \mu_{n_k}$$

(ii) For each n, define $\psi_n(a + \ker \mu_n) = \varphi_n(a) + \ker \mu_{n+1}$. Since $\mu_n = \mu_{n+1} \circ \varphi_n$, we have $a - b \in \ker \mu_n$ if and only if $\varphi_n(a) - \varphi_n(b) \in \ker \mu_{n+1}$ which shows that ψ_n is well-defined and injective.

π exists by a diagram chase. If $a \in \ker(\lambda_n \circ \pi_n)$. Then $\lambda_n(a + \ker \mu_n) = 0$, and thus $a \in \ker \mu_1$ (λ_n is injective since ψ_n is injective, use part (iii)) and so π is injective. We also have

$$\overline{\bigcup_{n=1}^{\infty} (\lambda_n \circ \pi_n)(A_n)} = \overline{\bigcup_{n=1}^{\infty} \lambda_n(B_n)} = \varinjlim B_n$$

so π is surjective.

(iii) Suppose $\mu_n(a) = 0$. Then

$$0 = \|\mu_n(a)\| = \lim_{m \to \infty} \|\varphi_{m,n}(a)\| = \|a\|$$

since $\varphi_{m,n}$ is injective for all $m \geq n$. So $a = 0$ and each μ_n is injective, hence an isometry.

If A contains 1_A, pick a self-adjoint element y in $\bigcup_{n=1}^{\infty} \mu_n(A_n)$ (so in $\mu_{n_0}(A_{n_0})$ for some n_0) with $\|1_A - y\| < 1$. Then y is invertible in A. Now since $0 \notin \mathrm{sp}(y)$, the function $f(z) = 1/z$ is continuous on $\mathrm{sp}(y)$ and moreover may be approximated uniformly by polynomials which vanish at 0. Let p_n be a sequence of polynomials which vanish at 0 and $p_n \to f$ uniformly on $\mathrm{sp}(y)$. Then $p_n(y) \in \mu_{n_0}(A_{n_0})$ for all n, and

$$y^{-1} = f(y) = \lim_{n \to \infty} p_n(y) \in \mu_{n_0}(A_{n_0})$$

since $\mu_{n_0}(A_{n_0})$ is closed. Thus $1_A = yy^{-1} \in \mu_{n_0}(A_{n_0})$ and since μ_{n_0} is an isomorphism onto its image, $\mu_{n_0}^{-1}(1)$ is a unit in A_{n_0}.

8. A diagram chase shows that there is a *-homomorphism $\alpha : A \to B$ as well as another $\beta : B \to A$. Uniqueness gives that $\alpha \circ \beta = \mathrm{id}_B$ and $\beta \circ \alpha = \mathrm{id}_A$.

Chapter 7

Classification of AF-algebras

1. The first is
$$\mathbb{C} \longrightarrow \mathbb{C} \oplus \mathbb{C} \longrightarrow \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \longrightarrow \cdots$$
where the first map is
$$z \mapsto (z, z),$$
the second is
$$(w, z) \mapsto (w, \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}, z),$$
and so forth. The second is
$$\mathbb{C} \longrightarrow M_2(\mathbb{C}) \longrightarrow M_4(\mathbb{C}) \longrightarrow M_8(\mathbb{C}) \longrightarrow \cdots$$
where each map is
$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

2. Suppose (G, G^+) is a countable ordered Abelian dimension group. Let $x \in G$ be such that $nx \geq 0$ for some $n \geq 1$. Choose some $k \geq 1$ and some $y \in \mathbb{Z}^{n_k}$ with $\beta_k(y) = x$. Then $\beta_k(ny) = nx \geq 0$ so, assuming β_k is injective (without loss of generality), $ny \geq 0$. Then $y \geq 0$ since $\mathbb{Z}^{n_k}$ is unperforated, and hence $x \geq 0$.

   Now let $x_i \leq y_j$ in G for $i, j = 1, 2$. Choose x_i' and y_j' so that $\beta_k(x_i') = x_i$ and $\beta_k(y_j') = y_j$. By Exercise 6.10 there is some $\ell \geq k$ such that $\alpha_{\ell,k}(x_i') \leq \alpha_{\ell,k}(y_j')$. These elements are in $\mathbb{Z}^{n_\ell}$, so choose z with $\alpha_{\ell,k}(x_i') \leq z \leq \alpha_{\ell,k}(y_j')$. Then $x_i \leq \beta_\ell(z) \leq y_j$.

3.

4. Suppose A and B have the cancellation property. Let (p_1, p_2) and (q_1, q_2) be projections in $\mathcal{P}_\infty(A \oplus B)$ with $[(p_1, p_2)]_0 = [(q_1, q_2)]_0$. Then $[p_i]_0 = [q_i]_0$ for $i = 1, 2$ by Proposition 4.34 and hence $p_i \sim_0 q_i$ for $i = 1, 2$. Find elements v_i so that $p_i = v_i^* v_i$ and $q_i = v_i v_i^*$ and then $(p_1, p_2) = (v_1^* v_1, v_2^* v_2) = (v_1, v_2)^*(v_1, v_2) \sim_0 (v_1, v_2)(v_1, v_2)^* = (v_1 v_1^*, v_2 v_2^*) = (q_1, q_2)$.

   Now suppose A_n has the cancellation property for each $n \geq 1$. Let

5.

6.

7.

8.

9. (G, G^+) is the inductive limit of

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{3} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \cdots$$

where

$$(\mathbb{Z} \oplus \mathbb{Z})^+ = \{(x, y) : x > 0, y > 0\} \cup \{(0, 0)\}.$$

If $(x, y) \in G^+$ is nonzero, then for every $(x', y') \in G$, $(-nx, -ny) \leq (x', y') \leq (nx, ny)$ for n large enough.

By Proposition 7.2.8, there is an AF-algebra A such that (G, G^+) is isomorphic to $(K_0(A), K_0(A)^+)$. To show that $\mathrm{Aut}(K_0(A))$ contains two elements, we may show that there are only two elements in $\mathrm{Aut}(\mathbb{Q} \oplus \mathbb{Q})$. Clearly the identity and the map $(x, y) \mapsto (y, x)$ are in $\mathrm{Aut}(\mathbb{Q} \oplus \mathbb{Q})$, and the fact that these are the only two follows from some linear algebra nonsense. If every automorphism on A was approximately inner, $\iota : \overline{\mathrm{Inn}}(A) \to \mathrm{Aut}(A)$ would be an isomorphism, hence $K_0 : \mathrm{Aut}(A) \to \mathrm{Aut}(K_0(A))$ is trivial, which contradicts surjectivity of K_0.

Chapter 8

The Functor K_1

1. If $K_1(A)$ is uncountable, there are uncountably many pairwise nonhomotopic unitaries u_α in $\bigcup M_n(A)$, so $\alpha \neq \alpha'$ implies that $\|u_\alpha - u_{\alpha'}\| = 2$. This is impossible if A is separable.

2. $C(X)$ is homotopy equivalent to $\mathbb{C}$ since X is contractible, so $K_1(C(X)) \cong K_1(\mathbb{C}) = 0$.

3. CA is contractible.

4. The sequence

$$0 \longrightarrow C_0(\mathbb{R}) \xrightarrow{\iota} C(\mathbb{T}) \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

is split exact, and so

$$0 \longrightarrow K_1(C_0(\mathbb{R})) \xrightarrow{K_1(\iota)} K_1(C(\mathbb{T})) \xrightarrow{K_1(\pi)} K_1(\mathbb{C}) \longrightarrow 0$$

is also split exact. Since $K_1(\mathbb{C}) = 0$, $K_1(\iota)$ is an isomorphism and thus $K_1(C_0(\mathbb{R})) \cong \mathbb{Z}$. We also have $K_1(C_0((0,1])) = 0$ since $C_0(0,1]$ is contractible, and hence

$$K_1(C_0(U)) \cong 0 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus 0 \oplus 0 \cong \mathbb{Z} \oplus \mathbb{Z}.$$

5. We have $\rho([u + f1_n]_1) = [u]_1$, where $f = 1_{\widetilde{A}} - 1_A$ and ρ is as in Proposition 8.1.6. Notice that $\widetilde{\varphi}(f) = 1_{\widetilde{A}} - \varphi(1_A)$, thus

$$K_1(\varphi)([u + f1_n]_1) = [\widetilde{\varphi}(u + f1_n)]_1 = [\varphi(u) + 1_n - \varphi(1_n)]_1$$

6. φ is unital since it is surjective. Suppose there is some $[v]_1$ such that $[u]_1 = K_1(\varphi)([v]_1) = [\widetilde{\varphi}(v)]_1$. We have $[\widetilde{\varphi}(v)]_1 = [\varphi(v)]_1$ by Exercise 8.5. Then $u \sim_1 \varphi(v)$ and so there is some $k \geq \max\{m, n\}$ so that $u \oplus 1_{k-n} \sim_h \varphi(v) \oplus 1_{k-m}$. So set $v' = v \oplus 1_{k-m} \in \mathcal{U}_k(A)$. Conversely, if there exists such a v, then $K_1(\varphi)([v]_1) = [u]_1$ since $\varphi(v) \sim_1 u$.

7. The K_1-group of any finite-dimensional C*-algebra is zero by combining Example 8.1.8, Proposition 8.2.6, and Proposition 8.2.8. Therefore, if A is an AF-algebra, $K_1(A) = 0$ is immediate from Proposition 8.2.7. We can also prove it directly by observing that $M_n(\widetilde{A})$ is also an AF-algebra for every n, and for any unitary u in $M_n(\widetilde{A})$, we can use part (i) of Proposition 6.2.4 and Exercise 2.8 to find a unitary v with finite spectrum such that $\|u - v\| < 2$, so $u \sim_h v \sim_h 1_n$.

8. (i) Let $\alpha(a) = vav^*$ and $v_n = \operatorname{diag}(v, v, \ldots, v)$ in $\mathcal{U}_n(\widetilde{A})$. Then if u is in $\mathcal{U}_n(\widetilde{A})$, we have $K_1(\alpha)([u]_1) = [\alpha(u)]_1 = [v_n u v_n^*]_1 = [v_n]_1 + [u]_1 - [v_n]_1 = [u]_1$.
   (ii) Let α be approximately inner and u in $\mathcal{U}_n(\widetilde{A})$. Choose an inner automorphism β with $\|\alpha(u) - \beta(u)\| < 2$. Then $\alpha(u) \sim_1 \beta(u)$, so $K_1(\alpha)([u]_1) = [\alpha(u)]_1 = [\beta(u)]_1 = K_1(\beta)([u]_1) = [u]_1$ by part (i).
   (iii) Easy.
   (iv) Easy.
   (v) The automorphism α of $C(\mathbb{T}) \oplus C(\mathbb{T})$ defined by $\alpha(f, g) = (g, f)$ works.

9. (i) A computation shows that $sus^* + (1 - ss^*)$ is unitary. Setting

$$v = \begin{pmatrix} s & 1 - ss^* \\ 0 & s^* \end{pmatrix}$$

   (which is unitary), we get $v\operatorname{diag}(u, 1)v^* = \operatorname{diag}(sus^* + (1 - ss^*), 1)$. Thus

$$sus^* + (1 - ss^*) \sim_1 \operatorname{diag}(sus^* + (1 - ss^*), 1) = v\operatorname{diag}(u, 1)v^* \sim_1 \operatorname{diag}(u, 1) \sim_1 u.$$

   (ii) We have $u = \prod_{k=1}^{n}(s_k u_k s_k^* - (1 - s_k s_k^*))$, and hence u is unitary. Then

$$[u]_1 = \left[\prod_{k=1}^{n}(s_k u_k s_k^* - (1 - s_k s_k^*))\right]_1 = \sum_{k=1}^{n}[s_k u_k s_k^* - (1 - s_k s_k^*)]_1 = \sum_{k=1}^{n}[u_k]_1$$

   by (i).
   (iii) It is easy to check that t is an isometry. Let $u = (a_{ij})_{i,j=1}^{n}$ be a unitary in $M_n(A)$. A very hefty computation shows that $tut^* + (1_n - tt^*)$ is equal to

$$\left(\left(\sum_{1 \leq i,j \leq n} s_i a_{ij} s_j^*\right) + (1 - s_1 s_1^* - \cdots - s_n s_n^*)\right) \oplus 1_{n-1}.$$

   Exercise 2.6 shows that $v_j = \sum_{k=1}^{n} a_{jk}$ is unitary in A for all $j = 1, \ldots, n$, and apply (ii) to see that the top left corner of the above matrix is unitary.
   (iv) Since A is properly infinite, by Exercise 4.6 there is a sequence of isometries $\{s_n\}$ such that their range projections are mutually orthogonal. If u is a unitary in $M_n(A)$, take t as in (iii) with the isometries so obtained. Then $u \sim_1 tut^* + (1 - tt^*) \sim_1 v$ for some unitary v in A by parts (i) and (iii). This shows that

$$K_1(A) = \{[u]_1 : u \in \mathcal{U}(A)\} = \{\omega(\langle u \rangle) : u \in \mathcal{U}(A)\}$$

   and thus ω is surjective.

10.

11. (i) Since $u_0^* u_0 = 1 - p$, multiplying on the left and right by p gives $0 = pu_0^* u_0 p = (u_0 p)^* u_0 p$, hence $u_0 p = 0$. Thus we have

$$u^* u = 1 + pu_0 + u_0^* p = 1$$

and similarly for uu^*.

(ii) If $[u]_1 = 0$, then there exists some n so that $u \oplus 1_n \sim_h 1_{n+1}$. Let $t \mapsto w_t$ be a path in $\mathcal{U}_{n+1}(A)$ such that $w_0 = 1_{n+1}$ and $w_1 = u \oplus 1_n$. Now note that $p \oplus 1_n$ is a projection in $M_{n+1}(A)$, but since p is properly infinite and full, by Exercise 4.9 (i) there is a projection q in A such that $p \oplus 1_n \sim_0 q \leq p$. Hence find $v_0 \in M_{1,n+1}(A)$ such that $v_0^* v_0 = p \oplus 1_n$ and $v_0 v_0^* \leq p$. Then by letting

$$v = \begin{pmatrix} 1 - p & 0 & \cdots & 0 \end{pmatrix} + v_0$$

and $z_t = v w_t v^* + (1 - vv^*)$, we have $z_0 = 1$, $z_1 = u$.

12.

13.

14. The same proof used for $B(H)$ in Example 8.1.8 works because von Neumann algebras are closed under Borel functional calculus.

Chapter 9

The Index Map

1. f

2. f

3. (i) We have $a(a^*a)^n = (aa^*)^n a$ for all n, and so the first claim is true if f is a polynomial. By density, it is true for all continuous functions, in particular the function $f(x) = x^{-1/2}$. With this fact and a simple computation, it is easy to show that v is unitary.

   (ii) Since u is unitary, $\pi((1 - a^*a)^{-1/2}) = (1 - u^*u)^{-1/2} = 0$.

   (iii) Identify $\widetilde{I}$ with $I + \mathbb{C}1_A$. We calculate

$$v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^* = \begin{pmatrix} aa^* & -a(1 - a^*a)^{1/2} \\ -(1 - a^*a)^{1/2}a^* & 1 - a^*a \end{pmatrix}$$

   and

4. (i) $(T + K) + \mathcal{K}(\mathcal{H}) = T + \mathcal{K}(\mathcal{H})$.

   (ii) If T is normal, it may be written as $D + K$ where D is a diagonal operator and K is compact.

Chapter 10

The Higher K-Functors

1. (i) We may identify SA with $C_0(\mathbb{T} \smallsetminus \{1\}, A)$ and take

$$0 \longrightarrow SA \xrightarrow{\iota} \mathbb{T}A \xrightarrow{\psi} A \longrightarrow 0$$

where ι is inclusion and $\psi(f) = f(1)$. Also let $\lambda : A \to \mathbb{T}A$ take an element $a \in A$ to the constant function in $C(\mathbb{T}, A)$ which takes only the value a.

(ii) Use the splitting lemma together with the fact that $K_n(SA) = K_{n+1}(A)$.

(iii) If X and Y are compact and Hausdorff, the map $\Phi : C(X \times Y) \to C(X, C(Y))$ defined by $[\Phi(f)(x)](y) = f(x, y)$ is an isomorphism. Thus

$$\mathbb{T}^2\mathbb{C} = \mathbb{T}(\mathbb{T}\mathbb{C}) = C(\mathbb{T}, C(\mathbb{T})) \cong C(\mathbb{T}^2),$$

and induction does the rest.

Chapter 11

Bott Periodicity

1. (i) $\alpha_0 = \theta_{SA} \circ \beta_A$ and $\alpha_1 = \beta_{SA} \circ \theta_A$.

   (ii) This is essentially due to the fact that the transformation θ needs to double matrix sizes in general, while the Bott map keeps matrices the same size.

   (iii) Similar to (ii).

2.

3.

4.

5.

6. Since

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for each $z \in \Omega$ and $a^2 = 0$, we have

$$f(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} a^n = f(0)1 + f'(0)a = \begin{pmatrix} f(0) & f'(0) \\ 0 & f(0) \end{pmatrix}.$$

Chapter 12

The Six-Term Exact Sequence

1.

2.

3. Let x_0 be the one and only cutpoint in Z_n. Then $U := Z_n \smallsetminus \{x_0\}$ is homeomorphic to a disjoint union of n open intervals in $\mathbb{R}$. Then we have an exact sequence

$$0 \longrightarrow C_0(U) \xrightarrow{\iota} C(Z_n) \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

where $\iota : C_0(U) \to C(Z_n)$ is the canonical inclusion and $\pi : C(Z_n) \to \mathbb{C}$ is evaluation at x_0. Then we have the six-term exact sequence

$$\begin{tikzcd}
K_0(C_0(U)) \arrow[r, "K_0(\iota)"] & K_0(C(Z_n)) \arrow[r, "K_0(\pi)"] & K_0(\mathbb{C}) \arrow[d, "\delta_0"] \\
K_1(\mathbb{C}) \arrow[u, "\delta_1"] & K_1(C(Z_n)) \arrow[l, "K_1(\pi)"] & K_1(C_0(U)) \arrow[l, "K_1(\iota)"]
\end{tikzcd}$$

We know that

$$K_0(\mathbb{C}) \cong \mathbb{Z} \text{ and } K_1(C_0(U)) \cong \mathbb{Z}^n \text{ and } K_0(C_0(U)) = K_1(\mathbb{C}) = 0$$

So, up to isomorphism, the diagram can be redrawn to

$$\begin{tikzcd}
0 \arrow[r, "K_0(\iota)"] & K_0(C(Z_n)) \arrow[r, "K_0(\pi)"] & \mathbb{Z} \arrow[d, "\delta_0"] \\
0 \arrow[u, "\delta_1"] & K_1(C(Z_n)) \arrow[l, "K_1(\pi)"] & \mathbb{Z}^n \arrow[l, "K_1(\iota)"]
\end{tikzcd}$$

Thus $K_0(\pi)$ is injective and $K_1(\iota)$ is surjective. Showing that $\delta_0 = 0$ will prove that both $K_0(\pi)$ and $K_1(\iota)$ are isomorphisms.

It is enough to show that $\delta_0([1]_0) = 0$ since $K_0(\mathbb{C})$ is generated by the class containing $1 \in \mathbb{C}$. Note that $\pi(1_{C(Z_n)}) = 1$ and since $\exp(2\pi i 1_{C(Z_n)}) = 1_{C(Z_n)}$, $1_{\widetilde{C_0(U)}}$ is the unique unitary in $\widetilde{C_0(U)}$ such that $\bar{\iota}(1_{\widetilde{C_0(U)}}) = \exp(2\pi i 1_{C(Z_n)})$. It follows that

$$\delta_0([1]_0) = -[1_{\widetilde{C_0(U)}}]_1 = 0.$$

4. (i) The six-term exact sequence is

$$\begin{tikzcd}
K_0(C_0(0,1)) \arrow[r, "K_0(\iota)"] & K_0(C[0,1]) \arrow[r, "K_0(\psi)"] & K_0(\mathbb{C} \oplus \mathbb{C}) \arrow[d, "\delta_0"] \\
K_1(\mathbb{C} \oplus \mathbb{C}) \arrow[u, "\delta_1"] & K_1(C[0,1]) \arrow[l, "K_1(\psi)"] & K_1(C_0(0,1)) \arrow[l, "K_1(\iota)"]
\end{tikzcd}$$

We know that

$$K_0(C_0(0,1)) = K_1(\mathbb{C} \oplus \mathbb{C}) = K_1(C[0,1]) = 0,$$
$$K_0(C[0,1]) \cong K_1(C_0(0,1)) \cong \mathbb{Z},$$

and

$$K_0(\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Let us redraw the diagram to see where everything is

$$\begin{tikzcd}
0 \arrow[r, "K_0(\iota)"] & \mathbb{Z} \arrow[r, "K_0(\psi)"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, "\delta_0"] \\
0 \arrow[u, "\delta_1"] & 0 \arrow[l, "K_1(\psi)"] & \mathbb{Z} \arrow[l, "K_1(\iota)"]
\end{tikzcd}$$

This entails that $K_0(\iota) = \delta_1 = K_1(\psi) = K_1(\iota) = 0$.

$K_0(\mathbb{C} \oplus \mathbb{C})$ is generated by $[(1,0)]_0$ and $[(0,1)]_0$, so it suffices to compute $\delta_0([(1,0)]_0)$ and $\delta_0([(0,1)]_0)$ to describe δ_0. The function $\mathrm{id}(x) = x$ in $C[0,1]$ is a self-adjoint element which satisfies $\psi(\mathrm{id}) = (0,1)$. The unitary $u(x) = \exp(2\pi i x)$ in $\widetilde{C_0(0,1)}$ then satisfies $\delta_0([(0,1)]_0) = -[u]_1$. Similarly, the function $f(x) = 1 - x$ in $C[0,1]$ satisfies $\psi(f) = (1,0)$, and the unitary $u_2(x) = \exp(-2\pi i x)$ satisfies $\delta_0([(1,0)]_0) = -[u_2]_1$.

$K_0(C[0,1])$ is generated by $[1]_0$, and $K_0(\psi)([1]_0) = [\psi(1)]_0 = [(1,1)]_0$, so $K_0(\psi)$ maps $K_0(C[0,1])$ onto the diagonal $\langle [(1,1)]_0 \rangle$.

(ii) The six-term exact sequence is

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{K_0(\iota)} & K_0(\widetilde{A}) & \xrightarrow{K_0(\pi)} & K_0(\mathbb{C}) \\
\uparrow \delta_1 & & & & \downarrow \delta_0 \\
K_1(\mathbb{C}) & \xleftarrow[K_1(\pi)]{} & K_1(\widetilde{A}) & \xleftarrow[K_1(\iota)]{} & K_1(A)
\end{array}$$

We know both K-groups of $\mathbb{C}$, so redraw to see that

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{K_0(\iota)} & K_0(\widetilde{A}) & \xrightarrow{K_0(\pi)} & \mathbb{Z} \\
\uparrow \delta_1 & & & & \downarrow \delta_0 \\
0 & \xleftarrow[K_1(\pi)]{} & K_1(\widetilde{A}) & \xleftarrow[K_1(\iota)]{} & K_1(A)
\end{array}$$

So $K_1(\pi) = \delta_1 = 0$. Let's see what δ_0 does. A self-adjoint lift of $1 \in \mathbb{C}$ is $1_{\widetilde{A}}$ and a unitary in $\widetilde{A}$ which lifts $1_{\widetilde{A}}$ is clearly $1_{\widetilde{A}}$. So $\delta_0([1]_0) = -[1_{\widetilde{A}}]_1 = 0$ and thus $\delta_0 = 0$.

$K_1(\iota)$ is then seen to be an isomorphism. We also see that $K_0(\iota)$ is injective and that $K_0(\pi)$ is surjective, although we already knew that because the original given sequence is split exact.

5. If $\iota(f) = (f, 0) = 0$, then $f = 0$, so ι is injective. If a is in A, then $\pi(f, a) = a$ where f is the straight line path from 0 to $\varphi(a)$. Clearly $\pi \circ \iota = 0$ and if $a = 0$, then (f, a) in E_φ must have $f(0) = f(1) = 0$ so that f is in SB. Thus the sequence is exact.

   Using Proposition 12.2.2 and the definition of the Bott map, check that both $\delta_0([p] - [s(p)])$ and $-\beta_B \circ K_0(\varphi)([p] - [s(p)])$ are equal to $-[\exp(2\pi i f)]$, where $f(t) = t\varphi(p)$.

   For a unitary u in $M_n(\widetilde{A})$, let

$$f(t) = \begin{bmatrix} t\varphi(u) & 0 \\ (1 - t^2)^{1/2} 1_{M_n(\widetilde{B})} & 0 \end{bmatrix}$$

   so that $v = (f, \mathrm{diag}(u, 0))$ is a partial isometry lift of $\mathrm{diag}(u, 0)$. Using Proposition 9.2.2 and the proof of Theorem 10.1.3 (remember that θ_B is, in fact, an index map), check that both $\delta_1([u])$ and $\theta_B \circ K_1(\varphi)([u])$ are equal to $[1 - f^* f] - [1 - f f^*]$.