Hints and Partial Solutions for *Principles of Mathematical Analysis, 3rd edition* by Walter Rudin

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Chapter 1

The Real and Complex Number Systems

1. Suppose $r + x = \frac{p}{q}$ for some integer p and positive integer q. Then $x = \frac{p}{q} - r$, but $\frac{p}{q} - r$ is rational and thus x is rational; this is a contradiction. Similarly, if $rx = \frac{p}{q}$ in the same fashion then $x = \frac{p}{qr}$ and we get a contradiction again.

2. If there were a pair of positive coprime integers p and q such that $(p/q)^2 = 12$, then we would have $p^2 = 12q^2 = 6(2q^2)$. It follows that p^2 is divisible by 6, hence p is divisible by 6, so $p = 6k$ for some integer k. Then $p^2 = 36k^2 = 12q^2$ which means $q^2 = 3k^2$. So q^2 is divisible by 3 and thus q is divisible by 3. So p and q have a factor in common, namely 3, which is a contradiction.

3. We have

$$y = 1y = \left(\frac{1}{x}x\right)y = \frac{1}{x}(xy) = \frac{1}{x}(xz) = \left(\frac{1}{x}x\right)z = 1z = z$$

which shows (a). Take $z = 1$ in (a) to obtain (b). Take $z = 1/x$ in (a) to obtain (c). Finally, replace x with $1/x$ and y with x in (c) to obtain (d).

4. Suppose the opposite, that $\alpha > \beta$. E is nonempty, so there is an x in E and this x satisfies $x \leq \beta < \alpha \leq x$, which means $x < x$. But we also have $x = x$, which is a contradiction since only one of these can hold.

5. We have $\inf A \leq x$ for all $x \in A$, and so $-\inf A \geq -x$ for all $x \in A$. This means that $-\inf A$ is an upper bound for $-A$. To see that it is the smallest, suppose α is another upper bound for $-A$, so $\alpha \geq -x$ for all $x \in A$. This means that $-\alpha \leq x$ for all $x \in A$ so $-\alpha$ is a lower bound for A. But $\inf A$ is the greatest lower bound, so $-\alpha \leq \inf A$ and thus $\alpha \geq -\inf A$. It follows that $-\inf A = \sup(-A)$, which is equivalent to what was to be shown.

6. (a) It is an easy exercise to prove that $(b^m)^{1/n} = (b^{1/n})^m$ using Theorem 1.21. We will use this fact freely, as follows. We have

$$((b^m)^{1/n})^q = ((b^m)^q)^{1/n} = (b^{mq})^{1/n} = (b^{np})^{1/n} = ((b^p)^{1/n})^n = b^p$$

so by raising $(b^m)^{1/n}$ to the power q, we obtain b^p. But by Theorem 1.20, this number is unique, so it must be equal to $(b^p)^{1/q}$.

(b) So we now take $b^r = (b^m)^{1/n}$ to be the definition of a rational exponent. If $r = m/n$ and $s = p/q$, we have

$$b^{r+s} = b^{(mq+pn)/nq} = (b^{mq+pn})^{1/nq} = (b^{mq}b^{pn})^{1/nq} \overset{(1)}{=} b^{mq/nq}b^{pn/nq} = b^r b^s$$

(in (1) we used Theorem 1.21).

(c) We prove the following lemma.

Lemma. *r is a positive rational if and only if $b^r > 1$.*

Proof. Let $r = m/n$, where m and n are positive integers. Then, since $b > 1$, we have $b^m > 1$ and thus $b^r = (b^m)^{1/n} > 1$. If $r = 0$, then $b^r = 1$. If r is negative, then $r = m/n$ where m is a negative integer and n is a positive integer. Then $-m$ is positive so $b^{-m} = 1/b^m > 1$ and thus $b^m < 1$. It follows that $b^r = (b^m)^{1/n} < 1$. □

An immediate consequence of part (b) and the lemma is that if r and s are rational, then $r < s$ if and only if $b^r < b^s$. Now consider the set $B(r)$ for a rational r. Clearly b^r is an upper bound for $B(r)$ since if t is rational and $t \leq r$, then $b^t \leq b^r$ by the lemma. But $b^r \in B(r)$ since $r \leq r$, so b^r is the least upper bound (if an upper bound of a set is in the set, then it must be the supremum).

(d) We prove an extension of the previous lemma.

Lemma. *x is a positive real number if and only if $b^x > 1$.*

Proof. Choose a positive rational $r \leq x$. By the previous lemma, $b^r > 1$. Also, $b^r \in B(x)$, so $b^x = \sup B(x) \geq b^r > 1$. If $x = 0$, then $b^x = 1$. If x is negative, choose a negative rational $s \geq x$. Then b^s is an upper bound for $B(x)$ (if t is such that $t \leq x \leq s$, then $b^t \leq b^s$) and so $b^x = \sup B(x) \leq b^s < 1$. □

The set $B(x+y)$ is, by definition, the set of all b^t such that t is rational and $t \leq x+y$. Given such a t, choose rational r and s such that $r \leq x$, $s \leq y$, and $t \leq r+s$. Then

$$b^t \leq b^{r+s} = b^r b^s \leq b^x b^y$$

so $b^x b^y$ is an upper bound for $B(x+y)$, hence $b^{x+y} \leq b^x b^y$. On the other hand, if $z < b^x b^y$, choose r such that $r \leq x$ and $z/b^y < b^r$. Then choose s such that $s \leq y$ and $z/b^r < b^s$. We have

$$z < b^r b^s = b^{r+s}$$

and $r + s \leq x + y$, so z is not an upper bound for $B(x+y)$.

7. (a) By the geometric sum formula we have

$$\frac{b^n - 1}{b - 1} = b^{n-1} + b^{n-2} + \cdots + b + 1 \geq 1 + 1 + \cdots + 1 = n$$

(b) Replace b with $b^{1/n}$ in part (a) (if $b > 1$, $b^{1/n} > 1$).

(c) If we assume that $n(t-1) > b-1$, then by part (b), $n(t-1) > b-1 \geq n(b^{1/n}-1)$ and thus $n(t-1) > n(b^{1/n}-1)$. Simplifying yields $b^{1/n} < t$.

(d) Use part (c) with $t = yb^{-w}$.

(e) Use part (c) with $t = y^{-1}b^{w}$.

(f) Suppose that $b^x < y$. Then by part (d), there is an n such that $b^{x+(1/n)} < y$ and thus $x + 1/n \in A$, contradicting the fact that x is an upper bound for A. If $b^x > y$, apply part (e) to obtain an n such that $b^{x-(1/n)} > y$. Then $x - 1/n$ is an upper bound for A since $b^{x-(1/n)} > y > b^w$ means $w < x - 1/n$ (by the second lemma in Exercise 6). This contradicts the fact that x is the smallest upper bound.

(g) If $x \neq z$, then either $x < z$ or $x > z$. These cases yield, respectively, that $b^x < b^z$ and $b^x > b^z$ by the second lemma in Exercise 6.

8. If there were such an order, then $-1 = i^2 > 0$ by Proposition 1.18 (d). But also $1 = 1^2 > 0$. Then both $1 > 0$ and $-1 > 0$, which contradicts Proposition 1.18 (a).

9. Write $z = a+bi$ and $w = c+di$ and suppose that $z \neq w$. If $a < c$ then $z < w$, and if $c > a$ then $z > w$. If $a = c$, then we must have either $b < d$ or $b > d$, so that $z < w$ or $w > z$, respectively.

   Write $z = a + bi$, $w = c + di$, and $v = e + fi$ and suppose that $z < w$ and $w < v$. If either $a < c$ or $c < e$, we have $a < e$ and so $z < v$. If $a = c = e$, we must have $b < d$ and $d < f$, so $b < f$ and $z < v$.

   The set of all $z = a + bi$ such that $a \leq 0$ has no least upper bound.

10. We have

$$z^2 = (a+bi)^2 = (a^2 - b^2) + 2abi = \frac{|w|+u}{2} - \frac{|w|-u}{2} + 2i\frac{(|w|+u|)^{1/2}(|w|-u|)^{1/2}}{2}$$

   which simplifies to

$$u + i(|w|^2 - u^2)^{1/2} = u + i|v|$$

   Thus if $v \geq 0$, then $|v| = v$ and $z^2 = w$. If $v \leq 0$, then $|v| = -v$ so $z^2 = \overline{w}$, hence $(\overline{z})^2 = w$. Replacing z with $-z$ (or $\overline{z}$ with $-\overline{z}$) we obtain two square roots for each complex w, with the exception of $w = 0$.

11. If $z \neq 0$, take $r = |z|$ and $w = z/|z|$. They are unique in this case, for if $z = rw$, then $|z| = |rw| = |r||w| = r \cdot 1 = r$, and thus $w = z/r = z/|z|$. If $z = 0$, then $r = 0$ and w can be any complex number of modulus 1.

12. The case for $n = 2$ is given by Theorem 1.33(e). The general case can be proved by induction.

13. We have

$$|x| = |x - y + y| \leq |x - y| + |y|$$

   so

$$|x| - |y| \leq |x - y|.$$

   By interchanging the roles of x and y above, we also get $|y| - |x| \leq |y - x| = |x - y|$. Thus $||x| - |y|| \leq |x - y|$.

14. We compute

$$|1+z|^2 + |1-z|^2 = (1+z)(1+\overline{z}) + (1-z)(1-\overline{z}) = 2 + 2z\overline{z} = 4.$$

15. Examining the proof of the Schwarz inequality, we see that equality is achieved when $Ba_j = Cb_j$ for $j = 1, \ldots, n$. In other words,

$$\sum_{k=1}^{n} a_j |b_k|^2 = \sum_{k=1}^{n} b_j a_k \overline{b_k}$$

for $j = 1, \ldots, n$.

16. *Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:*

 (a) *If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ such that*

 $$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

 (b) If $\mathbf{z}$ satisfies

 $$2|\mathbf{z} - \mathbf{x}| = 2|\mathbf{z} - \mathbf{y}| = |\mathbf{x} - \mathbf{y}|$$

 (c) If there were,

 $$d = |\mathbf{x} - \mathbf{y}| \leq |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}| = 2r,$$

 a contradiction.

 How must these statements be modified if k is 2 or 1?

17. *Prove that*

 $$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

 if $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

 Solution.

18. If $\mathbf{x} = \mathbf{0}$, then any nonzero $\mathbf{y}$ will do. Otherwise, pick a nonzero x_i, let $y_j = 1$ for $j \neq i$, and set

 $$y_i = -\frac{1}{x_i} \sum_{j \neq i} x_j.$$

 This is not true for $k = 1$ by Proposition 1.16(b).

19. *Suppose $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ such that*

 $$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

 if and only if $|\mathbf{x} - \mathbf{c}| = r$. (Solution: $3\mathbf{c} = 4\mathbf{b} - a$, $3r = 2|\mathbf{b} - \mathbf{a}|$.)

 Solution.

20. *With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.*

 Solution.

Chapter 2

Basic Topology

1. If the opposite were true, then there would be a nonempty set A and an element x which is in the empty set but not in A. But the empty set has no elements.

2. This solution does not follow the hint, but rather uses the Fundamental Theorem of Algebra. The set $\bigcup_{n=1}^{\infty}(\mathbb{Z}^n \setminus (0, 0, \ldots, 0))$ is countable by Theorem 2.12 and Theorem 2.13. Define a function Ω from $\bigcup_{n=1}^{\infty}(\mathbb{Z}^n \setminus (0, 0, \ldots, 0))$ into $\mathcal{P}(\mathbb{C})$, the set of all subsets of the complex numbers (the power set), by setting, for $a = (a_0, a_1, \ldots, a_n)$,

$$\Omega(a) = \{z : a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0\}.$$

   In other words, map the tuple a to the set of roots of the polynomial with $a_0, a_1, \ldots, a_n$ as coefficients. Note that $\Omega(a) = \varnothing$ will occur for some a (indeed, for all nonzero constant polynomials). The algebraic numbers are precisely the union of all sets in the range of Ω, and each of these sets is at most finite by the Fundamental Theorem of Algebra. So, being a countable union of finite (or empty) sets, the algebraic numbers are at most countable. But every integer is algebraic, so the set of algebraic numbers is exactly countable.

3. If all real numbers were algebraic, the set of all real numbers would be contained in a countable set by the previous exercise, hence countable by Theorem 2.11.

4. No; if it were, then the real numbers would be a union of two countable sets, namely, the rational numbers and the irrational numbers, hence countable.

5. The set

$$E = \{m + 1/n : m \in \{0, 1, 2\} \text{ and } n = 2, 3, 4, \ldots\}$$

   has $E' = \{0, 1, 2\}$.

6. Let p be a limit point of E'. If U is a neighbourhood of p, then there is a point $q \in E'$ with $q \in U$. Since U is open, there is a neighbourhood V of q with $V \subseteq U$. Then, since q is a limit point of E, there is a point $r \in E$ with $r \in V \subseteq U$. This shows that every neighbourhood of p intersects E and hence $p \in E'$. It follows that E' is closed.

   If p is a limit point of E, then every neighbourhood of p intersects E and also $\overline{E}$ since

$E \subseteq \overline{E}$. This shows that the limit points of E are contained in the limit points of $\overline{E}$. On the other hand, if p is a limit point of $\overline{E}$ and U is a neighbourhood of p, then there is a point $q \in \overline{E}$ with $q \in U$. Then either $q \in E$, in which case we are done, or $q \in E'$, in which case one may, as above, find a neighbourhood V of q with $V \subseteq U$. Then, since q is a limit point of E, there is a point $r \in E$ with $r \in V \subseteq U$. Thus p is a limit point of E.

If E is the set given in the solution to Exercise 5, then $E' = \{0, 1, 2\}$ but $E'' = \varnothing$ by the corollary to Theorem 2.20. So E and E' do not always have the same limit points.

7. (a) We prove it for $n = 2$; the general case can be proved by induction. Suppose that $C = A \cup B$. If p is in $\overline{A} \cup \overline{B}$, suppose p is in $\overline{A}$ without loss of generality. If U is any neighbourhood of p then U intersects A and hence C. It follows that p is in $\overline{C}$. On the other hand, suppose that $p \notin \overline{A}$ and $p \notin \overline{B}$. Then there are neighbourhoods U and V of p such that $U \cap A = V \cap B = \varnothing$. Then $U \cap V$ is an open set containing p and hence there is a neighbourhood W of p such that $W \subseteq U \cap V$. Then W does not intersect C, so $p \notin \overline{C}$.

   (b) The proof of this is similar to the first direction in part (a).

   If we enumerate the rational numbers $\{r_i\}_{i=1}^{\infty}$ and let $A_i = \{r_i\}$, then $B = \bigcup_{i=1}^{\infty} A_i = \mathbb{Q}$ but $\overline{B} = \mathbb{R}$ and $\bigcup_{i=1}^{\infty} \overline{A}_i = \mathbb{Q}$.

8. The answer to the first question is yes. If E is open, then every point of E is an interior point. This means that if $p \in E$, then there is a neighbourhood U of p such that $U \subseteq E$. Let us see that p is a limit point: if V is any other neighbourhood of p, then either $U \subseteq V$ or $V \subseteq U$. In the first case, choose any $q \in U$ such that $q \neq p$ (possible since neighbourhoods in $\mathbb{R}^2$ are interiors of circles; such sets certainly contain points other than their centres). Then $q \in V$ and $q \in E$. In the second case, choose any $q \in V$ such that $q \neq p$. Then $q \in E$ also, and it follows that p is a limit point of E.

   The answer to the second question is no. Take any finite set (which is closed) and look at the corollary to Theorem 2.20.

9. (a) If $p \in E^\circ$, we may find a neighbourhood such that $U \subseteq E$. We show that, in fact, $U \subseteq E^\circ$, and this will prove that E° is open. For if $q \in U$, then, because U is open (Theorem 2.19), there is a neighbourhood V of q such that $V \subseteq U$. Thus we have found a neighbourhood V of q such that $V \subseteq E$, and this shows that q, like p, is also an interior point of E. It follows that $U \subseteq E^\circ$ and hence that E° is open.

   (b) This is simply a matter of examining definitions.

   (c) If $p \in G$, find a neighbourhood U of p with $U \subseteq G$. Then $U \subseteq E$, so that $p \in E^\circ$.

   (d) $p \notin E^\circ$ if and only if no neighbourhood of p is contained in E if and only if every neighbourhood of p intersects E^c if and only if p is a limit point of E^c. Thus $p \notin E^\circ$ implies that $p \in \overline{E^c}$ and that if p is a limit point of E^c, then $p \notin E^\circ$. If $p \in \overline{E^c}$ but p is not a limit point of E^c, then $p \in E^c$. But then $p \notin E$ which implies that $p \notin E^\circ$ (since $E^\circ \subseteq E$).

   (e) No: look at $E = \mathbb{Q}$ as a subset of $\mathbb{R}$.

   (f) No: look at $E = \mathbb{Q}$ as a subset of $\mathbb{R}$.

10. Parts (a) and (b) of Definition 2.15 are easy to check. Now we examine the statement

$$d(p,q) \le d(p,r) + d(r,q).$$

We see that if $p = q$ then the inequality is true since d is nonnegative. Suppose that $p \ne q$, that is, $d(p,q) = 1$. Then, since r cannot be equal to both p and q, at least one of $d(p,r)$ and $d(r,q)$ must be equal to 1. The inequality then becomes either $1 \le 1$ or $1 \le 2$, both of which are true. It follows that this is a metric.

Observe that $N_{\frac{1}{2}}(p) = \{p\}$ for any $p \in X$. This shows that every set consisting of one point is open in X, and hence that *every* subset of X is open, by Theorem 2.24(a). The corollary to Theorem 2.23 shows that every subset is also closed. If $E \subseteq X$ then $\{U_p\}_{p\in E}$ is an open cover of E, where $U_p = \{p\}$. If E is compact, there are finitely many $p_1, \dots, p_n$ such that

$$E \subseteq U_{p_1} \cup \cdots \cup U_{p_n} = \{p_1, \dots, p_n\}$$

which means E is finite. Thus only the finite subsets of X are compact.

11. d_2 and d_5 are metrics, while d_1, d_3, and d_4 are not. We have

$$d_1(1,-1) = 4 > 2 = d_1(1,0) + d_1(0,-1)$$

and

$$d_3(1,0) = 1 \ne 2 = d_3(0,1).$$

$d_4(1,0) \ne d_4(0,1)$ similarly.

It is easy to check parts (a) and (b) of Definition 2.15 for d_2 and d_5. To see that the triangle inequality holds for both, first observe that, for $r \ge 0$, the functions

$$f(r) = \sqrt{r} \ \ \& \ \ g(r) = \frac{r}{1+r}$$

are nondecreasing, concave, and satisfy $f(0) = g(0) = 0$. It follows that both are subadditive. Then observe that

$$d_2(x,y) = f(|x-y|) \ \ \& \ \ d_5(x,y) = g(|x-y|).$$

12. If $\{U_\alpha\}$ is a cover, one of the elements contains 0 and hence all but finitely many of the $1/n$. Use this to get a finite subcover.

13. Take the closure of $\{2^{-n} + 2^{-k} : n \ge 1, k \ge 2 \text{ and } k > n\}$.

14. $(0,1) = \bigcup_{x>1}(1/x, 1)$.

15. $\bigcap_{n=1}^{\infty}[n,\infty) = \varnothing$ and $\bigcap_{n=1}^{\infty}(0,1/n) = \varnothing$.

16. First show that

$$E = \Big[(-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})\Big] \cap \mathbb{Q}$$

which, by Theorem 2.30, shows that E is open in $\mathbb{Q}$. But then, since no rational number satisfies $p^2 = 2$ or $p^2 = 3$,

$$E^c = \Big[(-\infty, -\sqrt{3}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{3}, \infty)\Big] \cap \mathbb{Q}$$

which shows that E^c is also open in $\mathbb{Q}$. It follows from the corollary to Theorem 2.23 that E is closed. E is bounded since $|p| < \sqrt{3}$ for all $p \in E$. To see that E is not compact, look at the open cover $\{U_n\}_{n=1}^{\infty}$, where

$$U_n = \left[(-\sqrt{3}, -\sqrt{2}) \cup \left(\sqrt{2} + \frac{1}{n}, \sqrt{3}\right)\right] \cap \mathbb{Q}.$$

17. E is not countable; a diagonalization argument works to verify this. E is not dense since the largest number in E is

$$0.7777777\ldots = \frac{7}{9}$$

and so the interval $(7/9, 1]$ does not intersect E. To see that E is compact, try to write it as a countable intersection of closed and bounded sets (look at the construction of the Cantor set). To see that E is perfect, we need only show that every point of E is a limit point of E. Let $p \in E$ and $r > 0$. Choose n so that $10^{-n} < r$. Then let q be the point of E which has all the same digits as p except for the 10^{n+1} place. We have

$$|p - q| = 3 \cdot 10^{-n-1} < 10^{-n} < r$$

which shows that p is a limit point. It follows that E is perfect.

18. Let $\{r_n\}$ be the rational numbers in a closed interval with irrational endpoints. Start removing middle thirds as with the Cantor set, with a small modification: if r_n remains in the set after the nth step, remove a small open interval $(r_n - \delta, r_n + \delta)$ where $\delta > 0$ is irrational.

19. (a) A and B are closed, so $\overline{A} = A$ and $\overline{B} = B$. Thus

$$A \cap \overline{B} = \overline{A} \cap B = A \cap B = \varnothing.$$

(b) If A is open, then A^c is closed, and $B \subseteq A^c$. By Theorem 2.27(c), $\overline{B} \subseteq A^c$. Thus $A \cap \overline{B} = \varnothing$, and the same argument with A and B interchanged shows that $\overline{A} \cap B = \varnothing$.

(c) Both A and B are open, so use part (b).

(d) Suppose X is countable and let p_0 and q_0 be two points in X with $p_0 \neq q_0$. The set

$$E = \{d(p, q) : p, q \in X\} \subseteq [0, \infty)$$

is at most countable, and the segment $(0, d(p_0, q_0))$ is uncountable. Thus there is some δ in $(0, d(p_0, q_0))$ such that $d(p, q) \neq \delta$ for all p and q in X. Then X is the union of the two nonempty separated sets A and B given in part (c). This contradicts X being connected.

20. Closures, yes. Suppose that $A \cup B$ is a separation of $\overline{E}$. If $E \cap A \neq \varnothing$ and $E \cap B \neq \varnothing$, then these two sets separate E. If one of them is empty, say $E \cap A = \varnothing$, then $E \subseteq B$ and A consists only of limit points of E. But $E' \subseteq \overline{E} \subseteq \overline{B}$, which contradicts the fact that $\overline{B} \cap A = \varnothing$.

Interiors, no. Consider two disjoint closed disks with a line segment running between them.

21. *Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define*

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbb{R}$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

(a) Suppose there is some $t \in \overline{A}_0 \cap B_0$.

(b) *Prove that there exists* $t_0 \in (0,1)$ *such that* $\mathbf{p}(t_0) \notin A \cup B$.

(c) *Prove that every convex subset of* $\mathbb{R}^k$ *is connected.*

22. As per the hint, we look at $\mathbb{Q}^k$ as a subset of $\mathbb{R}^k$. If $\varepsilon > 0$ and $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ is any element in $\mathbb{R}^k$, we may choose, for each i, a rational number q_i such that $|x_i - q_i| < \frac{\varepsilon}{\sqrt{k}}$ because $\mathbb{Q}$ is dense in $\mathbb{R}$. Then, with $\mathbf{q} = (q_1, q_2, \ldots, q_k)$,

$$|\mathbf{x} - \mathbf{q}| = \sqrt{(x_1 - q_1)^2 + \cdots + (x_k - q_k)^2} < \varepsilon$$

so that every point of $\mathbb{R}^k$ is a limit point of $\mathbb{Q}^k$. This shows that $\mathbb{Q}^k$ is dense.

23. Let $\{p_n\}$ be a dense sequence. The collection $\{V_{n,k}\}$ is a countable base, where

$$V_{n,k} = \{x : d(x, p_n) < 1/k\}.$$

Indeed, if U is open, then there is some N such that $p_N \in U$. Then, since U is open, there is an $\varepsilon > 0$ such that

$$\{x : d(x, p_N) < \varepsilon\} \subseteq U.$$

Then choose K so that $1/K < \varepsilon$, and we have

$$V_{N,K} \subseteq \{x : d(x, p_N) < \varepsilon\} \subseteq U.$$

24. If the process described in the hint did not stop after a finite number of steps, we would obtain a sequence $\{x_n\}$ in X such that $d(x_m, x_n) \geq \delta$ for all $m \neq n$. Then the range of $\{x_n\}$ is an infinite subset of X with no limit point, for if x was a limit point, then there would be infinitely many terms of the sequence within $\delta/2$ distance from x. In particular, there would exist x_{n_1} and x_{n_2} such that $d(x, x_{n_i}) < \delta/2$. But then

$$d(x_{n_1}, x_{n_2}) \leq d(x, x_{n_1}) + d(x, x_{n_2}) < \delta/2 + \delta/2 = \delta,$$

a contradiction. This means that for every $\delta > 0$, there is a finite number of points $x_1, x_2, \ldots, x_j$ such that for every $x \in X$, there exists an $i = 1, 2, \ldots, j$ such that $d(x, x_i) < \delta$. If we sucessively choose $\delta = 1/n$ for $n = 1, 2, 3, \ldots$, we obtain, for each n, a finite number of points $x_1^n, x_2^n, \ldots, x_{j_n}^n$ that satisfy the aforementioned condition. I claim that

$$\{x_i^n : n = 1, 2, 3 \ldots \text{ and } i = 1, 2, \ldots, j_n\}.$$

is dense. Suppose U is an nonempty open set that does not contain any point of the form x_i^n. Pick $x \in U$ and choose $\varepsilon > 0$ so that $\{y : d(x, y) < \varepsilon\} \subseteq U$. Choosing N so that $1/N < \varepsilon$ means that there is a point x_i^N with $d(x, x_i^N) < 1/N < \varepsilon$, a contradiction.

25. First we show that the statement in the hint is true. Fix n. For each $p \in K$, let $U_{p,n}$ be the ball of radius $1/n$ with centre p. $\{U_{p,n}\}_{p \in K}$ covers K, so there is a finite subcover.

    Let $\mathcal{B}_n$ be a finite collection of neighbourhoods of radius $1/n$ whose union covers K. The union of all $\mathcal{B}_n$ is a countable base. Indeed, if $p \in K$ and $r > 0$, choose n large enough so that $1/n < r/2$. Then choose an element U of $\mathcal{B}_n$ that contains p, and $p \in U \subseteq N_r(p)$, for if $q \in U$, then $d(p, q) < 2/n < r$.

26. Picking up where the hint left off, let p be a limit point of E (which exists by assumption) and let p_n be the point of E chosen from F_n. Choose N such that $p \in G_N$ and choose a neighbourhood U of p such that $U \subseteq G_1 \cup \cdots \cup G_N$. In fact, choose U small enough so that $p_1, \cdots, p_{N-1}$ are not in U. Then U is a neighbourhood of p which contains no other point of E, since $p_m \in F_m \subseteq F_N \subseteq U^c$ when $m \geq N$. We have a contradiction.

27. Following the hint, let $\{V_n\}$ be a countable base for $\mathbb{R}^k$, let

$$\mathcal{A} = \{V_n : E \cap V_n \text{ is at most countable}\}$$

and let

$$W = \bigcup_{V_n \in \mathcal{A}} V_n.$$

We show that $P = W^c$. If $p \in W$, then $p \in V_n$ for some n and V_n is a neighbourhood of p such that $E \cap V_n$ is at most countable, so $p \notin P$. Conversely, if $p \notin P$ then there is a neighbourhood U of p such that $E \cap U$ is at most countable. Choose V_n such that $p \in V_n$ and $V_n \subseteq U$ (possible since $\{V_n\}$ is a base). Then, since $E \cap V_n \subseteq E \cap U$, $E \cap V_n$ is at most countable and thus $p \in W$. Now we may observe that

$$E \cap P^c = E \cap W = E \cap \bigcup_{V_n \in \mathcal{A}} V_n = \bigcup_{V_n \in \mathcal{A}} (E \cap V_n)$$

is an at most countable union of sets, each of which are at most countable, so the entire union is at most countable. It remains to show that P is perfect. It is closed since it is the complement of the open set W. Let $p \in P$ and suppose U is a neighbourhood of p that does not contain any other points of P. This means that $U \setminus \{p\} \subseteq P^c = W$ and so $U \cap E \subseteq (E \cap W) \cup \{p\}$. Thus an uncountable set is contained in an at most countable set; a contradiction.

28. Let E be closed and let P be the set of condensation points of E. E contains its limit points and hence its condensation points, so $P \subseteq E$. It follows that $E = P \cup (E \cap P^c)$. By the previous exercise, P is perfect and $E \cap P^c$ is at most countable.

29. Let U be an open set. For every $x \in U$ let $\{I_\alpha\}_{\alpha \in A_x}$ be the collection of all segments I_α such that $x \in I_\alpha \subseteq U$. This collection is nonempty since U is open. Let $I_x = \bigcup_{\alpha \in A_x} I_\alpha$. Clearly $I_x \subseteq U$, and I_x is a segment since it is a union of segments that all share a point, namely x. Now we show that if x and y are in U and $x \neq y$, we either have $I_x \cap I_y = \varnothing$ or $I_x = I_y$. To this end, suppose we have $z \in I_x \cap I_y$. Then $I_x \cup I_y$ is a segment (since I_x and I_y share a point) which contains x, so $I_x \cup I_y \subseteq I_\alpha$ for some $\alpha \in A_x$. This means $I_x \cup I_y \subseteq I_x$, and thus that $I_y \subseteq I_x$. Similarly we obtain $I_x \subseteq I_y$. So the collection $\{I_x\}_{x \in U}$ is pairwise disjoint; to see that it is countable, note that every I_x contains a rational number r_x by Exercise 22. Since the segments are disjoint, they must contain distinct rational numbers, and hence the collection is at most countable.

30. We prove the equivalent statement. Let V_1 be open in $\mathbb{R}^k$; we show that there is a point in V_1 which is also in $\bigcap_{n=1}^{\infty} G_n$. Since G_1 is dense, there is a point x_1 in $V_1 \cap G_1$. $V_1 \cap G_1$ is also open, so we can find a neighbourhood V_2 around x_1 so that $V_2 \subseteq V_1 \cap G_1$. In fact, by choosing the radius of V_2 small enough, we may assume that $\overline{V_2} \subseteq V_1 \cap G_1$. Now, since G_2

is dense, we can choose a point x_2 in $V_2 \cap G_2$ and also a neighbourhood V_3 of x_2 such that $\overline{V_3} \subseteq V_2 \cap G_2$. Continuing, we obtain a sequence

$$\overline{V_2} \supseteq \overline{V_3} \supseteq \overline{V_4} \supseteq \cdots$$

of nonempty ($x_n \in \overline{V_{n+1}}$ for all n), compact (closed and bounded) sets, so by Theorem 2.36 there is a point x in $\bigcap_{n=2}^{\infty} \overline{V_n}$. But $\overline{V_n} \subseteq V_{n-1} \cap G_{n-1} \subseteq G_{n-1}$ for all $n \geq 2$, so x is in $\bigcap_{n=1}^{\infty} G_n$. It follows that $\bigcap_{n=1}^{\infty} G_n$ is dense.

Chapter 3

Numerical Sequences and Series

1. Let $\varepsilon > 0$ be arbitrary. Choose N so that $n \geq N$ implies that $|s_n - s| < \varepsilon$. Then $n \geq N$ implies that $||s_n| - |s|| < \varepsilon$ also by Chapter 1, Exercise 13. The converse is not true, as the sequence $\{(-1)^n\}$ demonstrates.

2. $$\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+1/n}+1} \to \frac{1}{2}$$

3. $s_1 = \sqrt{2} < 2$, and if $s_n < 2$, then $s_{n+1} = \sqrt{2+s_n} < \sqrt{2+2} = 2$. It is just as easy to show that $s_n < s_{n+1}$ for all n by induction, and thus $\{s_n\}$ converges.

4. *Find the upper and lower limits of the sequence $\{s_n\}$ defined by*

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Solution. The sequence starts by

$$\left\{0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \ldots\right\}$$

and it appears that the even terms are tending to 1/2 and the odd terms are tending to 1. Let us prove that for $m \geq 1$,

$$s_{2m} = \frac{2^{m-1}-1}{2^m} \qquad s_{2m-1} = \frac{2^m - 1}{2^m}.$$

5. Let $\varepsilon > 0$. Then $a_n \leq \limsup a_n + \varepsilon$ for all but finitely many n, and the same is true for $\{b_n\}$. Thus the inequality

$$a_n + b_n \leq \limsup a_n + \limsup b_n + 2\varepsilon$$

holds for all but finitely many n and so

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we are done.

6. *Investigate the behaviour (convergence or divergence) of* $\sum a_n$ *if*

   (a) $a_n = \sqrt{n+1} - \sqrt{n}$;

   (b) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$;

   (c) $a_n = (\sqrt[n]{n} - 1)^n$;

   (d) $a_n = \dfrac{1}{1+z^n}$, *for complex values of* z.

   Solution. For (a), the nth partial sum is $\sqrt{n+1} - 1$, so $s_n \to \infty$. For (b),

   $$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} < \frac{1}{2n^{3/2}}$$

   so the series converges by Theorem 3.28. (c) converges by the Root test, and (d) converges when $|z| > 1$, for

   $$\left| \frac{1}{1+z^n} \right| \leq \frac{1}{|1 - |z|^n|}$$

7. *Prove that the convergence of* $\sum a_n$ *implies the convergence of*

   $$\sum \frac{\sqrt{a_n}}{n}$$

   if $a_n \geq 0$.

   Solution.

8. *If* $\sum a_n$ *converges, and if* b_n *is monotonic and bounded, prove that* $\sum a_n b_n$ *converges.*

   Solution. Assume that b_n is increasing, and that M is an upper bound. Let $s_n = a_1 + \cdots + a_n$ and $t_n = a_1 b_1 + \cdots + a_n b_n$. We have

   $$t_n = a_1 b_1 + \cdots + a_n b_n$$

9. (a) 1, (b) ∞, (c) 1/2, (d) 3.

10. If $|z| > 1$, then $\{a_n z^n\}$ is unbounded since infinitely many a_n are nonzero integers. Thus $a_n z^n \not\to 0$, so $\sum a_n z^n$ cannot converge.

11. *Suppose* $a_n > 0$, $s_n = a_1 + \cdots + a_n$, *and* $\sum a_n$ *diverges.*

    (a) If $a_n \not\to 0$, then $\frac{a_n}{1+a_n} \not\to 0$ and hence $\sum \frac{a_n}{1+a_n}$ diverges. If $a_n \to 0$, let $r = \max\{a_n\}$. Then

    $$\sum \frac{a_n}{1+a_n} \geq \frac{1}{1+r} \sum a_n.$$

    (b) Since $\{s_n\}$ increases, we have

    $$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}.$$

(c) Since $s_n^2 \geq s_n s_{n-1}$, we have

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{a_n}{s_n s_{n-1}} \geq \frac{a_n}{s_n^2}.$$

(d) *What can be said about*

$$\sum \frac{a_n}{1+na_n} \text{ and } \sum \frac{a_n}{1+n^2a_n}?$$

Solution.

12. (a) Since $\{r_n\}$ decreases, we have

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} \geq \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} = \frac{r_m - r_{n+1}}{r_m} = 1 - \frac{r_{n+1}}{r_m} > 1 - \frac{r_n}{r_m}$$

(b) We have

$$2(\sqrt{r_n} - \sqrt{r_{n+1}}) = 2\frac{r_n - r_{n+1}}{\sqrt{r_n} + \sqrt{r_{n+1}}} = 2\frac{a_n}{\sqrt{r_n} + \sqrt{r_{n+1}}} > 2\frac{a_n}{\sqrt{r_n} + \sqrt{r_n}} = \frac{a_n}{\sqrt{r_n}}$$

Solution.

13. *Prove that the Cauchy product of two absolutely convergent series converges absolutely.*

Solution.

14. *If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by*

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \quad (n = 0, 1, 2, \ldots).$$

(a) *If $\lim s_n = s$, prove that $\lim \sigma_n = s$.*

(b) *Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.*

(c) *Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?*

(d) *Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that*

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} ka_k.$$

Assume that $\lim(na_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $na_n \to 0$.]

(e) *Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \leq M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the given outline.*

Solution. For $\varepsilon > 0$, choose N so that $1/N < \varepsilon$ and $n \geq N$ implies that $|s_n - s| < \varepsilon$. Then if $n \geq N$,

$$
\begin{aligned}
|\sigma_n - s| &= \left| \frac{s_0 + s_1 + \cdots + s_n}{n+1} - s \right| && (3.1) \\
&= \frac{1}{n+1} \left| \sum_{k=1}^{n} (s_k - s) \right| && (3.2) \\
&= && (3.3)
\end{aligned}
$$

15. *Definition 3.21 can be extended to the case in which the a_n lie in some fixed $\mathbb{R}^k$. Absolute convergence is defined as convergence of $\sum |\mathbf{a}_n|$. Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs.)*

Solution.

16. *Fix a positive number α. Choose $x_1 > \alpha$, and define $x_2, x_3, x_4, \ldots,$ by the recursion formula*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(a) *Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.*

(b) *Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that*

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \qquad (n = 1, 2, 3, \ldots).$$

(c) *This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < \frac{1}{10}$ and that therefore*

$$\varepsilon_5 < 4 \cdot 10^{-16}, \quad \varepsilon_6 < 4 \cdot 10^{-32}.$$

Solution.

17. *Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define*

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

(a) *Prove that $x_1 > x_3 > x_5 > \cdots$.*

(b) *Prove that $x_2 < x_4 < x_6 < \cdots$.*

(c) *Prove that $\lim x_n = \sqrt{\alpha}$.*

(d) *Compare the rapidity of convergence of this process with the one described in Exercise 16.*

Solution.

18. *Replace the recursion formula of Exercise 16 by*

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1}$$

where p is a fixed positive integer, and describe the behaviour of the resulting sequences $\{x_n\}$.

Solution.

19. *Associate to each sequence $a = \{\alpha_n\}$, in which α_n is* 0 *or* 2, *the real number*

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is precisely the Cantor set described in Sec. 2.44.

Solution.

20. *Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.*

Solution.

21. Pick x_n in E_n for each n. Then $\{x_n\}$ is Cauchy since $d(x_m, x_n) < \text{diam } E_n$ when $m \leq n$. Since X is complete, x_n converges to some x. Since x_n is in E_m whenever $m \leq n$, x is in E_m because E_m is closed. This is true for all m, so x is in $\bigcap E_n$. If $\bigcap E_n$ consisted of two distinct points x and y, then both points are in each E_n, so diam $E_n \geq d(x, y) > 0$, contradicting the fact that diam $E_n \to 0$.

22. *Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. (In fact, it is dense in X.)* **Hint:** *Find a shrinking sequence of neighbourhoods E_n such that $\overline{E_n} \subseteq G_n$, and apply Exercise 21.*

Solution.

23. *Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.* **Hint:** *For any m, n,*

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.

Solution.

24. *Let X be a metric space.*

   (a) *Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X* equivalent *if*

   $$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

   Prove that this is an equivalence relation.

   (b) *Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define*

   $$\Delta(P,Q) = \lim_{n\to\infty} d(p_n, q_n);$$

   by Exercise 23, this limit exists. Show that the number $\Delta(P,Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function on X^*.

   (c) *Prove that the resulting metric space X^* is complete.*

   (d) *For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that*

   $$\Delta(P_p, P_q) = d(p,q)$$

   for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^*.

   (e) *Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the* completion *of X.*

   Solution. For (a), reflexivity and symmetry are easy. If $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{r_n\}$, then

   $$d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n) \to 0$$

   as $n \to \infty$. For (b), suppose $\{p_n\}, \{p'_n\} \in P$ and $\{q_n\}, \{q'_n\} \in Q$. Then

   $$d(p_n, q_n) \leq d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)$$

   for all n, and taking a limit shows $\lim_{n\to\infty} d(p_n, q_n) \leq \lim_{n\to\infty} d(p'_n, q'_n)$. Reversing the roles shows the other inequality. For (c), suppose $\{P_k\}$ is a Cauchy sequence of equivalence classes in the metric Δ. (d) is easy provided the constant sequences are taken as representatives.

25. The completion may be identified with the real numbers. To see this, we define a function Ω from the completion $\mathbb{Q}^*$ of the rational numbers into $\mathbb{R}$ as follows. Given an equivalence class P, choose a representative $\{p_n\}$ and let $\Omega(P) = \lim_{n\to\infty} p_n$. This is independent of the representative, and is a surjective isometry.

Chapter 4

Continuity

1. No; let $f(x) = 0$ if $x \neq 0$, and $f(0) = 1$. f is clearly discontinuous at 0, but if $x = 0$,

$$\lim_{h\to 0}[f(x+h) - f(x-h)] = \lim_{h\to 0}(0-0) = 0$$

and if $x \neq 0$, then once $|h| < |x|$, $f(x+h) = f(x-h) = 0$.

2. Choose $x \in \overline{E}$ so as to show that $f(x) \in \overline{f(E)}$. Let U be a neighbourhood of $f(x)$. Then $f^{-1}(U)$ is a neighbourhood of x since f is continuous. So there is a point $x' \in E$ such that $x' \in f^{-1}(U)$ and thus $f(x') \in U$.
Define f on $[0,\infty)$ on $\mathbb{R}$ by $f(x) = x/(x+1)$. Then $f([0,\infty)) = [0,1)$, but $f(\overline{[0,\infty)}) = f([0,\infty)) = [0,1) \neq \overline{[0,1)} = [0,1]$

3. $Z(f) = f^{-1}(\{0\})$; the result now follows from the corollary of Theorem 4.8.

4. By Exercise 2, $\overline{f(E)} \supseteq f(\overline{E}) = f(X)$.

5. *If f is a real continuous function defined on a closed set $E \subseteq \mathbb{R}$, prove that there exist continuous real functions g on $\mathbb{R}$ such that $g(x) = f(x)$ for all $x \in E$. (Such functions are called* continuous extensions *of f from E to $\mathbb{R}$.) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector-valued functions.* **Hint:** *Let the graph of g be a straight line on each of the segments which constitute the complement of E (compare Exercise 29, Chap. 2). The result remains true if $\mathbb{R}$ is replaced by any metric space, but the proof is not so simple.*

   Solution.

6. Suppose f is continuous and $(x_n, f(x_n))$ is a sequence in its graph. Since E is compact, there is a convergent subsequence $x_{n_k} \to x$ where $x \in E$. Then $f(x_{n_k}) \to f(x)$ by continuity, and so $(x_{n_k}, f(x_{n_k})) \to (x, f(x))$ in the graph. Conversely, suppose the graph is compact and $x_n \to x$ in E. The sequence $(x_n, f(x_n))$ has a convergent subsequence $(x_{n_k}, f(x_{n_k})) \to (y, f(y))$ where $y \in E$, and this means that $x_{n_k} \to y$. But $x_n \to x$ implies that $x = y$, and thus that $f(x_{n_k}) \to f(x)$. This shows that every convergent subsequence of $f(x_n)$ converges to $f(x)$ and thus $f(x_n) \to f(x)$.

7. *If $E \subseteq X$ and if f is a function defind on X, the restriction of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on $\mathbb{R}^2$ by: $f(0,0) = g(0,0) = 0$, $f(x,y) = xy^2/(x^2+y^4)$, $g(x,y) = xy^2/(x^2+y^6)$ if $(x,y) \neq (0,0)$. Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighbourhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous!*

   Solution.

8. Fix a point x_0 in E, choose $\delta > 0$ so that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$, and choose an integer $n \geq 0$ such that $E \subseteq [-n\delta, n\delta]$. We claim that f is bounded above by $2n + |f(x_0)|$. If x is any point in E, there is an integer $k \leq 2n$ and points $x_1, x_2, \ldots, x_k = x$ in E such that $|x_i - x_{i+1}| < \delta$ for $i = 0, 1, 2, \ldots, k-1$ (draw a picture to convince yourself of this). Then

$$\begin{aligned} |f(x)| &= |(f(x_0) - f(x_1)) + (f(x_1) - f(x_2)) + \cdots + (f(x_{k-1}) - f(x)) - f(x_0)| \\ &\leq |f(x_0) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_{k-1}) - f(x)| + |f(x_0)| \\ &\leq k + |f(x_0)| \\ &\leq 2n + |f(x_0)| \end{aligned}$$

   Simply take $f(x) = x$ and $E = \mathbb{R}$ for the counterexample.

9. Assume the original definition holds, and let $\varepsilon > 0$ be given. Choose $\delta > 0$ so that $d_X(p,q) < \delta$ implies that $d_Y(f(p), f(q)) < \varepsilon/2$. Then if E is such that diam $E < \delta$, then $d_X(p,q) < \delta$ for all p and q in E. Then $d_Y(f(p), f(q)) < \varepsilon/2$ for all p and q in E, so diam $f(E) \leq \varepsilon/2 < \varepsilon$. Conversely, if the definition in terms of diameter of sets holds, choose a suitable δ for a given ε. Then $d_X(p,q) < \delta/2$ implies that diam $E \leq \delta/2 < \delta$. Thus diam $f(E) < \varepsilon$ and so $d_Y(f(p), f(q)) < \varepsilon$.

10. *Complete the details of the following alternative proof of Theorem 4.19: If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$. Use Theorem 2.37 to obtain a contradiction.*

    Solution.

11. Let $\varepsilon > 0$ be given, and choose $\delta > 0$ so that $d_X(x,y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$. Because $\{x_n\}$ is Cauchy, there exists an N so that $m, n \geq N$ implies $d_X(x_m, x_n) < \delta$. Thus $m, n \geq N$ implies $d_Y(f(x_m), f(x_n)) < \varepsilon$, hence $\{f(x_n)\}$ is Cauchy.

12. *A uniformly continuous function of a uniformly continuous function is a uniformly continuous function. State this more precisely and prove it.*

    Solution.

13. *Let E be a dense subset of a metric space X, and let f be a uniformly continuous* real *function defined on E. Prove that f has a continuous extension from E to X (see Exercise 5 for terminology). (Uniqueness follows from Exercise 4.)* **Hint:** *For each $p \in X$ and each positve integer n, let $V_n(p)$ be the set of all $q \in E$ with $d(p,q) < 1/n$. Use Exercise 9 to show that the intersection of the closures of the sets $f(V_1(p))$, $f(V_2(p))$,$\cdots$, consists of a single point,*

say $g(p)$, of $\mathbb{R}$. Prove that the function so defined on X is the desired extension of f. Could the range space $\mathbb{R}$ be replaced by $\mathbb{R}^k$? By any compact metric space? By any complete metric space? By any metric space?

Solution.

14. Define g on $[0,1]$ by $g(x) = f(x) - x$. g is continuous, and we have that $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 1 - 1 = 0$. By the Intermediate Value Theorem, there is a point x in $[0,1]$ with $g(x) = 0$, so $f(x) = x$.

15. In fact, f must be *strictly* increasing (or decreasing). Suppose $a < b$ and $f(a) = f(b)$. Then (a,b) is open and thus $f((a,b))$ is open. Since f is continuous, it achieves a maximum and minimum value on $[a,b]$.

    Case 1. *f attains its maximum at some point c in (a,b).* Then $f(c) \geq f(x)$ for all x in (a,b), thus $f(c) + \varepsilon$ is not in $f((a,b))$ no matter how small $\varepsilon > 0$ is. Then $f(c)$ is not an interior point of $f((a,b))$, a contradiction.

    Case 2. *$f(a) = f(b) \geq f(x)$ for all x in (a,b).* Then the minimum value is attained at some point c in (a,b). Then $f(c) - \varepsilon$ is not in $f((a,b))$ for any $\varepsilon > 0$, and again we get a contradiction as in Case 1.

16. *Let $[x]$ denote the largest integer contained in x, that is, $[x]$ is the integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the functions $[x]$ and (x) have?*

    Solution.

17.

    Solution.

18.

    Solution.

19.

    Solution.

20.

    Solution.

21. Suppose that for each n, we may find $x_n \in F$ and $y_n \in K$ such that $d(x_n, y_n) < 1/n$. Since K is compact, $\{y_n\}$ has a convergent subsequence, say $y_{n_k} \to y$ for some point $y \in K$. Then

$$d(x_{n_k}, y) \leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, y) \to 0 \text{ as } k \to \infty,$$

so $x_{n_k} \to y$. But F is closed, so $y \in F$. This contradicts $F \cap K = \varnothing$.

22.

Solution.

23.

Solution.

24.

Solution.

25.

Solution.

26.

Solution.

Chapter 5

Differentiation

1. For any x, we have
$$\lim_{t\to x}\left|\frac{f(t)-f(x)}{t-x}\right| \le \lim_{t\to x}|t-x| = 0$$
so that $f'(x)=0$. Now apply Theorem 5.11(b).

2. That f is strictly increasing can be read off from the equation in the proof of Theorem 5.11. Hints for the second part: first show g is continuous (since f is the inverse of g, you may show that f is open; see Exercise 15 in Chapter 4). Then let $\varepsilon > 0$ be arbitrary and pick $\delta > 0$ small enough so that $|t-x| < \delta$ implies that $|f(t)-f(x)| < \varepsilon$ and
$$\left|\frac{t-x}{f(t)-f(x)} - \frac{1}{f'(x)}\right| < \varepsilon.$$
Use this to make the difference quotient for g small.

3. We have
$$f'(x) = 1 + \varepsilon g'(x) \ge 1 + \varepsilon(-M) > 0$$
as long as $\varepsilon < \frac{1}{M}$. Apply the previous exercise.

4. Set
$$f(x) = C_0 x + \frac{C_1 x^2}{2} + \cdots + \frac{C_{n-1}x^n}{n} + \frac{C_n x^{n+1}}{n+1}$$
so then $f(0)=f(1)=0$. Now use Rolle's Theorem (Theorem 5.10 with $f(a)=f(b)$).

5. Given $\varepsilon > 0$, choose $M > 0$ large enough so that $x > M$ implies $|f'(x)| < \varepsilon$. For such an x, by the Mean Value Theorem we may find c between x and $x+1$ such that
$$f(x+1) - f(x) = f'(c)$$
so
$$|g(x)| = |f(x+1)-f(x)| = |f'(c)| < \varepsilon.$$

6. g is differentiable for $x > 0$ and we have

$$g'(x) = \frac{f'(x)x - f(x)}{x^2} = \frac{f'(x) - g(x)}{x}$$

so $g'(x) \geq 0$ if and only if $f'(x) \geq g(x)$.

7.

Solution.

8.

Solution.

9.

Solution.

10.

Solution.

11.

Solution.

12. We compute

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{|t|^3 - |x|^3}{t - x} = \lim_{t \to x} \frac{(|t| - |x|)(t^2 + |tx| + x^2)}{t - x}.$$

If $x > 0$, then we get $f'(x) = 3x^2$; if $x < 0$, we get $f'(x) = -3x^2$; $f'(0) = 0$.

13.

Solution.

14.

Solution.

15.

Solution.

16.

Solution.

17.

Solution.

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Solution.

19.

Solution.

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Solution.

21.

Solution.

22.

Solution.

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Solution.

24.

Solution.

25.

Solution.

26.

Solution.

27.

Solution.

28.

Solution.

29.

Solution.

Chapter 6

The Riemann-Stieltjes Integral

1.

   Solution.

2. If $f(c) > 0$ for some c in $[a, b]$, then there is a $\delta > 0$ such that $f(x) \geq \frac{f(c)}{2}$ for all x in $(c-\delta, c+\delta)$. Then
$$\int_a^b f \geq \int_{c-\delta}^{c+\delta} f \geq 2\delta \frac{f(c)}{2} > 0,$$
   a contradiction.

3.

   Solution.

4.

   Solution.

5.

   Solution.

6.

   Solution.

7.

   Solution.

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   Solution.

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Solution.

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Solution.

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Solution.

12.

Solution.

13.

Solution.

14.

Solution.

15.

Solution.

16.

Solution.

17.

Solution.

18.

Solution.

19.

Solution.

Chapter 7

Sequences and Series of Functions

1.

   Solution.

2.

   Solution.

3.

   Solution.

4.

   Solution.

5.

   Solution.

6.

   Solution.

7. Clearly $f_n \to 0$ pointwise, and we have

$$|f_n(x)| = \frac{|x|}{1 + nx^2} \leq \frac{1}{2\sqrt{n}}$$

   for all x and all n, so $f_n \to 0$ uniformly by Theorem 7.9. Now

$$f_n'(x) = \frac{(1 + nx^2) - x(2nx)}{(1 + nx^2)^2} = \frac{1 - nx^2}{(1 + nx^2)^2}$$

$f_n'(0) = 1$ for all n, and if $x \neq 0$,

$$\frac{1 - nx^2}{(1+nx^2)^2} = \frac{\frac{1}{n^2} - \frac{x^2}{n}}{\frac{1}{n^2} + \frac{2x^2}{n} + x^4} \to 0$$

as $n \to \infty$.

8.

Solution.

9. Take $\varepsilon > 0$. We may choose N large enough so that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon/2$ for all x in E. Choose $\delta > 0$ so that $|y - x| < \delta$ implies that $|f(y) - f(x)| < \varepsilon/2$. Thus, by choosing $M \geq N$ such that $n \geq M$ implies that $|x_n - x| < \delta$, we have

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon$$

when $n \geq M$.

10.

Solution.

11.

Solution.

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Solution.

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Solution.

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Solution.

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Solution.

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Solution.

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Solution.

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Solution.

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Solution.

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Solution.

21.

Solution.

22.

Solution.

23.

Solution.

24. We have

$$|f_p(x)| = |d(x,p) - d(x,a)| \leq d(a,p)$$

by the triangle inequality. Hence $f_p \in \mathcal{C}(X)$ because

$$|f_p(x) - f_p(y)| = |(d(x,p) - d(x,a)) - (d(y,p) - d(y,a))| \leq 2d(x,y).$$

We have

$$|f_p(x) - f_q(x)| = |d(x,p) - d(x,q)| \leq d(p,q)$$

for every $x \in X$, and so

$$\|f_p - f_q\| = \sup_{x \in X} |f_p(x) - f_q(x)| \leq d(p,q).$$

On the other hand,

$$|f_p(p) - f_q(p)| = |d(p,p) - d(p,q)| = d(p,q)$$

so the value $d(p,q)$ is attained and

$$\|f_p - f_q\| = d(p,q).$$

Thus the map $\Phi : X \to \mathcal{C}(X) : p \mapsto f_p$ is an isometry. It remains to be shown that $\overline{\Phi(X)}$ is complete in $\mathcal{C}(X)$. But this is clear: each f_p is bounded and thus $\Phi(X) \subseteq \mathcal{C}_b(X) \subseteq \mathcal{C}(X)$. Since $\mathcal{C}_b(X)$ is complete, $\overline{\Phi(X)}$ is as well.

25.

Solution.

26.

Solution.

Chapter 8

Some Special Functions

1.

Solution.

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31.

Solution.

Chapter 9

Functions of Several Variables

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   Solution.

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   Solution.

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   Solution.

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   Solution.

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Solution.

Chapter 10

Integration of Differential Forms

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Chapter 11

The Lebesgue Theory

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Solution.