C^*-algebras Generated by Weighted Shifts II

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Introduction. In this paper we study a family of simple C^*-algebras which are C^*-algebras generated by weighted shifts modulo the compact operators. It has been proved by the first author that if T is a weighted shift whose sequence of weights is an almost-periodic, non-periodic sequence of positive numbers which are bounded below then $C^*(T)$ contains the compact operators as a maximal ideal. In the first section of this paper we show that with such an operator T we can associate a countably infinite subgroup Λ of the unit circle Γ so that the crossed-product $C(\Gamma) \times \Lambda$ of $C(\Gamma)$ with Λ (the group action being rotations of the circle through elements of Λ) is isomorphic to $C^*(T)/\mathscr{K}$. Conversely with every countably infinite subgroup Λ of Γ we can associate a weighted shift T so that $C^*(T)/\mathscr{K}$ is isomorphic to $C(\Gamma) \times \Lambda$. In the second section we exploit this correspondence to prove the existence of projections in $C^*(T)/\mathscr{K}$ and discuss the questions of isomorphism. We conjecture that Λ determines the isomorphism class of $C^*(T)/\mathscr{K}$. In the particular case that Λ is a torsion group, we compute the K_0 and K_1 groups of $C(\Gamma) \times \Lambda$ and show that in this case the isomorphism class is determined by Λ. We also show that $C(\Gamma) \times \Lambda$ and hence $C^*(T)/\mathscr{K}$ can be imbedded in a UHF algebra whose "generalized integer" is determined by Λ.

Notation. H denotes a separable Hilbert space with an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ and P_n denotes the orthogonal projection onto span of e_n. If $\{\alpha_n\}_{n=0}^{\infty}$ is a bounded sequence of positive numbers then T denotes the weighted shift with weights $\{\alpha_n\}_{n=0}^{\infty}$ i.e. $Te_n = \alpha_n e_{n+1}$ for all $n \geq 0$, D denotes the diagonal operator $De_n = \alpha_n e_n$ and S denotes the unweighted shift $Se_n = e_{n+1}$. $C^*(T)$ denotes the C^*-algebra generated by T and the identity operator I. If A is a C^*-algebra, G is a locally compact abelian group and θ is a continuous homomorphism of G into the group of automorphisms of A (Aut A) then $A \times G$ denotes the crossed-product of G with A by the action θ of G. In particular if $A = C(\Gamma)$, Λ is any subgroup of Γ and Λ acts on $C(\Gamma)$ by rotation of Γ, then we denote the corresponding crossed-product by A_Λ. Note that A_Λ can be identified with the C^*-algebra of operators acting on $L^2(\Gamma)$

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generated by the operators V and W_λ, $(\lambda \in \Lambda)$ where $(Vf)(z) = zf(z)$ and $(W_\lambda f)(z) = f(z\lambda)$ for f in $L^2(\Gamma)$.

Throughout this paper $\{\alpha_n\}_{n=0}^{\infty}$ denotes a bounded sequence of positive numbers which is almost periodic (see [6, page 173]) but which is not of the form $\{\beta_n\}_{n=0}^{\infty} + \{\gamma_n\}_{n=0}^{\infty}$ where $\{\beta_n\}_{n=0}^{\infty}$ is periodic and $\gamma_n \to 0$. We also assume that $\{\alpha_n\}$ is bounded below. Since $T = SD$, with this assumption, $C^*(T)$ contains the ideal $\mathscr{K}$ of compact operators. It has been proved in [4] that $C^*(T)/\mathscr{K}$ is a simple, unital C^*-algebra which possesses a unique (normalized) trace.

1. The main result of this section is the following:

Theorem 1.1. *Let* $\Lambda = \{\lambda \in \Gamma : \operatorname{diag}\{\lambda^n\}_{n=0}^{\infty} \in C^*(T)\}$. *Then*

(1) Λ *is a countably infinite subgroup of* Γ.

(2) $C^*(T)$ *is generated by* S *and* $\{\operatorname{diag}\{\lambda^n\}_{n=0}^{\infty} : \lambda \in \Lambda\}$.

(3) There is an isomorphism π *of* A_Λ *onto* $C^*(T)/\mathscr{K}$ *such that* $\pi(V) = S + \mathscr{K}$ *and* $\pi(W_\lambda) = \operatorname{diag}\{\lambda^n\}_{n=0}^{\infty} + \mathscr{K}$.

Conversely given a countably infinite subgroup Λ *of* Γ *there exists a weighted shift* T *whose sequence of weights is almost periodic and for which* $\Lambda = \{\lambda : \operatorname{diag}\{\lambda^n\}_{n=0}^{\infty} \in C^*(T)\}$.

The proof of this theorem is broken down into a number of lemmas. As in [8] we let ϕ denote the shift on l^∞, i.e. $\phi\{r_0, r_1, \ldots\} = \{r_1, r_2, \ldots\}$ and let $\mathscr{D} = C^*\{\phi^n(D), P_n, I, n \in \mathbf{Z}\}$. $\mathscr{D}$ may be identified with a subalgebra of l^∞ and it is obvious that $\mathscr{D}$ contains C_0, the algebra of sequences converging to zero. Since the almost periodic sequences form a subalgebra of l^∞ denoted by AP, we have $\mathscr{D} \subseteq AP$. ϕ defines an automorphism of AP/C_0 and preserves $\mathscr{D}/C_0$. It follows from [8, Theorem 1.4.1] that $\mathscr{D}/C_0 \times_\phi \mathbf{Z}$ is isomorphic to $C^*(T)/\mathscr{K}$.

First, we give another description of $\mathscr{D}$.

Lemma 1.2.

$$\mathscr{D} = \{\{\gamma_n\}_{n=0}^{\infty} : \operatorname{diag}\{\gamma_n\}_{n=0}^{\infty} \in C^*(T)\}.$$

Proof. For $z \in \Gamma$ and $A \in C^*(T)$ set $\beta_z(A + \mathscr{K}) = U_z A U_z^* + \mathscr{K}$ where $U_z = \operatorname{diag}\{z^n\}_{n=0}^{\infty}$. Then $\beta_z(S + \mathscr{K}) = z(S + \mathscr{K})$ and $\beta_z(D + \mathscr{K}) = D + \mathscr{K}$. Hence β_z is an automorphism of $C^*(T)/\mathscr{K}$ and $z \to \beta_z(A + \mathscr{K})$ is continuous for each $A \in C^*(T)$. Now if $A \in \mathscr{D}$ then $\int_\Gamma \beta_z(A + \mathscr{K})dz = A + \mathscr{K}$ and $\int_\Gamma \beta_z(S + \mathscr{K})dz = 0$. Hence for any $A \in C^*(T)$ $\int_\Gamma \beta_z(A + \mathscr{K})dz \in q(\mathscr{D})$ where q is the quotient map from $C^*(T)$ to $C^*(T)/\mathscr{K}$. In particular if $\operatorname{diag}\{\gamma_n\}_{n=0}^{\infty} \in C^*(T)$ then $\operatorname{diag}\{\gamma_n\}_{n=0}^{\infty} + \mathscr{K} = \operatorname{diag}\{\delta_n\}_{n=0}^{\infty} + \mathscr{K}$ for some choice of $\{\delta_n\}_{n=0}^{\infty}$ for which $\operatorname{diag}\{\delta_n\}_{n=0}^{\infty} \in \mathscr{D}$. Since $\mathscr{D}$ contains the projections $\{P_n, n \geq 0\}$ it contains $\operatorname{diag}\{\gamma_n - \delta_n\}_{n=0}^{\infty}$ and thus $\operatorname{diag}\{\gamma_n\}_{n=0}^{\infty} \in \mathscr{D}$.

Lemma 1.3. *(i) The action* $n \to \phi^n$ *of* $\mathbf{Z}$ *on* AP/C_0 *extends to* $b\mathbf{Z}$, *the Bohr compactification of* $\mathbf{Z}$. *Denote this extension by* $t(\in b\mathbf{Z}) \to \phi_t$.

(ii) There is an ismorphism κ of AP/C_0 with $C(b\mathbf{Z})$ which transforms ϕ_t to the shift by t on $C(b\mathbf{Z})$.

(iii) If $\langle\cdot,\cdot\rangle$ denotes the pairing between $b\mathbf{Z}$ and its dual group Γ (with the discrete topology), for $\lambda \in \Gamma$, let χ_λ be the character of $b\mathbf{Z}$ given by $\chi_\lambda(t) = \langle t,\lambda\rangle$. The ismorphism κ carries $\{\lambda^n\}_{n=0}^{\infty} + C_0$ to a multiple of χ_λ.

Proof. The group of automorphisms of AP/C_0, $\mathrm{Aut}(AP/C_0)$ is a topological group with respect to the topology of pointwise norm convergence. A simple topological argument shows that a closed subset E of $\mathrm{Aut}(AP/C_0)$ is compact if and only if $\{\beta(a):\beta \in E\}$ is compact for every a in AP/C_0. If $a \in AP/C_0$, it can be shown using an argument similar to the one in [7, page 171] that the closure of $\{\phi^n(a):n \in \mathbf{Z}\}$ is compact. Hence the closure of $\{\phi^n, n \in \mathbf{Z}\}$ in $\mathrm{Aut}(AP/C_0)$ is a compact topological group. By the universal property of the Bohr compactification it follows that the map $n \to \phi^n$ extends continuously to $t(\in b\mathbf{Z}) \to \phi_t$ (see [3, page 333]. Thus if $t \in b\mathbf{Z}$ then each ϕ_t induces a homeomorphism of the spectrum of AP/C_0 (which we denote by Y). As in [4, Lemma 2.2] the orbit of every point in Y under $\{\phi^n, n \in \mathbf{Z}\}$ is dense in Y and since $b\mathbf{Z}$ is a compact group, it follows that $\{\phi_t y_0, t \in b\mathbf{Z}\} = Y$ for every y_0 in Y. If t is an element of $b\mathbf{Z}$ which is different from the identity we can choose λ in Γ such that $\langle t,\lambda\rangle \neq 1$. Since $\{\lambda^n\}_{n=0}^{\infty}$ is almost periodic and $\phi_t(\{\lambda^n\}_{n=0}^{\infty} + C_0) = \langle t,\lambda\rangle(\{\lambda^n\}_{n=0}^{\infty} + C_0)$ it follows that ϕ_t is not the identity automorphism of AP/C_0. Thus Y is homeomorphic to $b\mathbf{Z}$ and the homeomorphism transforms ϕ_t to the translation by t. This proves (i) and (ii). (iii) is now obvious.

The following lemma may be well known. We include the statement and a short proof for the sake of completeness.

Lemma 1.4. *Let B be a C^*-subalgebra of $C(b\mathbf{Z})$ which is invariant under all translations ϕ_t, $t \in b\mathbf{Z}$. Set $H = \{t \in b\mathbf{Z}: \phi_t(f) = f \text{ for all } f \in B\}$ and $\Sigma = \{\lambda \in \Gamma: <\cdot,\lambda> \in B\}$. Then $B = C(b\mathbf{Z}/H)$, $H^\perp = \Sigma$ and B is generated by $\{\langle\cdot,\lambda\rangle : \lambda \in \Sigma\}$.*

Proof. Clearly $B \subset C(b\mathbf{Z}/H)$. To show the reverse inclusion it suffices to show that B separates the points of $b\mathbf{Z}/H$. If $s, t \in b\mathbf{Z}$ such that $s - t \notin H$, we claim that there exists a function f in B with $f(s) \neq f(t)$. Suppose not; then $f(s + \gamma) = f(t + \gamma)$ for all f in B and $\gamma \in b\mathbf{Z}$ i.e. $\phi_s f = \phi_t f$ for all f in B. Hence $\phi_{t-s} f = f$ for all f in B and $s - t \in H$. The rest of the lemma follows from the fact that the dual group of $b\mathbf{Z}/H$ is identified with $H^\perp$.

Proof of Theorem 1.1. By Lemma 1.2 $\Lambda = \{\lambda \in \Gamma: \{\lambda^n\}_{n=0}^{\infty} \in \mathscr{D}\}$ since $\mathscr{D}/C_0$ can be identified with a subalgebra of $C(b\mathbf{Z})$ which is invariant under all translations in $b\mathbf{Z}$. Lemma 1.3 (iii) and 1.4 now show that $\mathscr{D}/C_0$ is isomorphic to $C(b\mathbf{Z}/\Lambda^\perp)$ and $\mathscr{D}$ is generated by C_0 and $\{\{\lambda^n\}_0^{\infty}:\lambda \in \Lambda\}$. Thus in order to prove (i) we need only note that $C^*(T)$ and hence $C(b\mathbf{Z}/\Lambda^\perp)$ are separable C^*-algebras and since $\{\alpha_n\}_{n=0}^{\infty} \neq \{\beta_n\}_{n=0}^{\infty} + \{\gamma_n\}_{n=0}^{\infty}$ where $\{\beta_n\}_{n=0}^{\infty}$ is periodic and $\gamma_n \to 0$, ϕ is not periodic on $\mathscr{D}/C_0$. (ii) follows from the fact that $C^*(T)$ is generated by $\mathscr{D}$ and S.

To prove (iii) note that $(\operatorname{diag}\{\lambda^n\}_{n=0}^{\infty} + \mathscr{K})(S + \mathscr{K})(\operatorname{diag}\{\lambda^n\}_0^{\infty} + \mathscr{K})^* = \lambda(S + \mathscr{K})$ for each λ in Λ. Now the universal property of the crossed-product shows the existence of a homomorphism π satisfying the properties stated in (iii). Since A_Λ is simple, π is an isomorphism.

For the converse let $\mathscr{D}_1$ be the C^*-subalgebra of AP generated by C_0 and $\{\{\lambda^n\}_{n=0}^{\infty}:\lambda \in \Lambda\}$. We must show that $\Lambda = \{\lambda \in \Gamma : \{\lambda^n\}_0^{\infty} \in \mathscr{D}_1\}$ and that there is a positive invertible element D of $\mathscr{D}_1$ such that $\mathscr{D}_1$ is generated by C_0 and $\{\phi^n(D): n \in \mathbf{Z}\}$. As before Lemmas 1.3 and 1.4 show that $\mathscr{D}_1/C_0$ is isomorphic to $C(b\mathbf{Z}/\Lambda^{\perp})$ and that $\Lambda = \{\lambda \in \Gamma : \{\lambda^n\}_0^{\infty} \in \mathscr{D}_1\}$. It suffices to show that there is a positive function f on $b\mathbf{Z}/\Lambda^{\perp}$ such that $\{\phi_t(f): t \in b\mathbf{Z}\}$ generates $C(b\mathbf{Z}/\Lambda^{\perp})$. Since Λ is countable the group $b\mathbf{Z}/\Lambda^{\perp}$ is second countable and we can find a sequence $\{U_n\}_{n=1}^{\infty}$ of neighborhoods of 0 such that $\bigcap_{n=1}^{\infty} U_n = \{0\}$ and $U_1 \supseteq \bar{U}_2 \supseteq U_2 \supseteq \bar{U}_3 \ldots$. For each $n = 1, 2, \ldots$ choose a function f_n, in $C(b\mathbf{Z}/\Lambda^{\perp})$ such that $0 \le f_n \le 1$, $f(U_n) = \{1\}$ and the support of f_n is contained in $\bar{U}_n$. Let $f = \sum_{n=1}^{\infty} (1/2^{n+1}) f_n + (1/2)$. We claim that $\{\phi_t f : t \in b\mathbf{Z}\}$ separates points. Whenever x_1 and x_2 are two distinct points in $b\mathbf{Z}/\Lambda^{\perp}$ there exists t in $b\mathbf{Z}$ such that $\phi_t(x_1) = 0$, $\phi_t(x_2) \neq 0$. Hence $f(\phi_t x_1) = 1$ and $f(\phi_t x_2) \neq 1$. This proves the claim and completes the proof.

Remark 1.5. If Λ_n consists of all n^{th} roots of unity let D be the operator $\operatorname{diag}\{\alpha_m\}_{m=0}^{\infty}$ where $\alpha_m = 1$ if m is a multiple of n and $\alpha_m = 1/2$ otherwise. Then the C^*-subalgebra of AP/C_0 generated by $\{\{\lambda^k\}_{k=0}^{\infty} + C_0 : \lambda \in \Lambda_n\}$ is the same as the C^*-subalgebra generated by $\{\phi^k(D) + C_0 : k \in \mathbf{Z}\}$. In fact both of these algebras are equal to $\{\{\gamma_k\}_0^{\infty} + C_0, \phi^n(\{\gamma_k\}_0^{\infty} + C_0) = \{\gamma_k\}_0^{\infty} + C_0\}$. This can be seen by applying Lemma 1.4 to the group $\mathbf{Z}/n\mathbf{Z}$ instead of $b\mathbf{Z}$. Thus by taking an increasing sequence of positive integers $\{n_k\}$ so that $\Lambda_{n_k} \subseteq \Lambda_{n_{k+1}}$ we show that the Bunce-Deddens algebra described in [1] is singly generated and is isomorphic to A_Λ where $\Lambda = \{\lambda \in \Gamma, \lambda^{n_k} = 1$ for some $n_k\}$.

The conclusion of Theorem 1.1 for the Bunce-Deddens algebra was noted by P. Green [5, page 248]. We would like to acknowledge that the main idea for proving the converse in the proof of Theorem 1.1 was suggested to us by C. K. Fong.

2. Now that the study of $C^*(T)/\mathscr{K}$ has been reduced to the study $C(\Gamma) \times \Lambda$ where Λ is a countably infinite subgroup of Γ, questions regarding isomorphism and existence of projections in $C^*(T)/\mathscr{K}$ can be studied in terms of algebras of operators acting on $L^2(\Gamma)$.

Remark 2.1. If A_Λ is the algebra of operators on $L^2(\Gamma)$ generated by the shift V and the rotation operators W_λ then it is easy to check that there is a unique tracial state τ on A_Λ. M. A. Rieffel has shown that if $\Lambda = \{e^{2\pi i n\alpha}, n \in \mathbf{Z}\}$ where α is an irrational in (0,1), then the range of the trace τ on projections in A_Λ is exactly $(\mathbf{Z} + \mathbf{Z}\alpha) \cap [0,1]$. (See [10, Theorem

1.2].) Since the trace on A_Λ is unique, in this case the isomorphism class of A_Λ is determined by Λ.

As an immediate consequence of this remark and Theorem 1.1 we obtain the following result.

Proposition 2.2. *$C^*(T)/\mathscr{K}$ contains projections.*

For the rest of this section we assume that $\Lambda = \bigcup_{k=1}^{\infty} \Lambda_{n_k}$ where $\Lambda_{n_k} = \{e^{2\pi i m/n_k}, m \in \mathbf{Z}\}$ and $\{n_k\}$ is a strictly increasing sequence of positive integers such that n_k divides n_{k+1}; we denote $n_{k+1}/n_k = p_{k+1}$ for $k \geq 1$ and $n_1 = p_1$.

Lemma 2.3. *There is an imbedding of $C(\Gamma) \times \Lambda$ into the* UHF *algebra $\bigotimes_{k=1}^{\infty} M_{p_k}$ such that if π is a representation of $\bigotimes_{k=1}^{\infty} M_{p_k}$ (on separable Hilbert space) with no type I summand, then $\pi(C(\Gamma) \times \Lambda)'' = \pi\left(\bigotimes_{k=1}^{\infty} M_{p_k}\right)''$.*

Proof. Let X be the compact space $\prod_{k=1}^{\infty} \{0,1,\ldots,p_k - 1\}$ and let Y be the subset of finitely non-zero sequences in X. The map $\{x_k\} \in X \overset{\psi}{\leftrightarrow} \prod_{k=1}^{\infty} e^{2\pi i x_k/n_k}$ maps Y bijectively onto Λ and $X \setminus Y$ bijectively onto Γ. If $x = \{x_k\} \in X$ and $y = \{y_1, y_2, \ldots, y_n, 0, 0, \ldots\} \in Y$ then we may define $\sigma_y x = \{z_1, z_2, \ldots\}$ where $z_k = x_k$ if $k > n$, $z_n = x_n + y_n \bmod p_n$ and if $1 \leq k \leq n$, then $z_k = x_k + y_k \bmod p_k$ if $x_{k+1} + y_{k+1} \leq p_{k+1} - 1$ and $z_k = x_k + y_k + 1 \bmod p_k$ if $x_{k+1} + y_{k+1} \geq p_{k+1}$. It is easy to check that this action of Y on X is intertwined by ψ with the canonical action of Λ on Γ and hence we can extend the action of Λ to $C(X)$. We denote this action also by σ. Along with this identification of $C(\Gamma)$ as a subalgebra of $C(X)$ this lets us identify $C(\Gamma) \times \Lambda$ with a subalgebra of $C(X) \underset{\sigma}{\times} \Lambda$. We claim that $C(X) \underset{\sigma}{\times} \Lambda$ is isomorphic to $\bigotimes_{k=1}^{\infty} M_{p_k}$. To see this we note that $C(X) \underset{\sigma}{\times} \Lambda$ is generated by $C(X)$ and a unitary representation $\lambda \leftrightarrow U(\lambda)$ of Λ such that $U(\lambda) f\, U(\lambda)^* = \sigma_\lambda(f)$. For each k, let q_k be the projection in $C(X)$ defined by $q_k\{a_m\}_1^\infty = 1$ if $a_1 = a_2 = \ldots = a_k = 0$ and $q_k\{a_m\}_1^\infty = 0$ otherwise. Then the projections $\sigma_{e^{2\pi i m/n_k}}(q_k)$, $m = 0, \ldots, n_k - 1$ form a partition of unity in $C(X)$ and the C^*-subalgebra of $C(X) \underset{\sigma}{\times} \Lambda$ generated by q_k and $U(e^{2\pi i/n_k})$ is isomorphic to M_{n_k}. Since $\bigotimes_{k=1}^{\infty} M_{p_k}$ is isomorphic to $\overline{\bigcup_{k=1}^{\infty} M_{n_k}}$ this establishes our claim.

Next, if π is a representation of $C(X) \underset{\sigma}{\times} \Lambda$, a standard argument shows that there exists a Radon measure μ on X and $\pi | C(X)$ extends to a normal homomorphism of $\mathscr{L}^{\infty}(X,\mu)$ onto $\pi\ (C(X))''$. The homeomorphisms σ_{λ} leave μ quasi-invariant and $\pi(U(\lambda))\ \pi(f)\ \pi(U(\lambda))^* = \pi(f \circ \sigma_{\lambda})$ for all λ in Λ and all f in $L^{\infty}(X,\mu)$. It is easy to check that since π has no type I summands, μ has no atoms. Since X/Y is Borel isomorphic to Γ, it follows that $\pi(C(\Gamma) \times \Lambda)'' = \pi(C(X) \underset{\sigma}{\times} \Lambda)''$.

In the next proposition, we exhibit a short computation involving K_0 and K_1 groups of the algebra A_{Λ}. For precise definitions and general theory concerning K_0 and K_1 of a C^*-algebra, see [2]. Since the Bunce-Deddens algebras have been studied in detail by various people it is likely that the conclusions of the next proposition are common knowledge. However to the best of our knowledge they are unpublished. Hence we include a brief sketch of the proof and deduce some results on isomorphism as corollaries.

Proposition 2.4. $K_0(A_{\Lambda}) = \{m/n_k, m \in Z, k = 1,2,\ldots\}$ $K_1(A_{\Lambda}) = Z$.

Proof. Note that A_{Λ} is generated by unitaries $\{V, W_k, k \geq 1\}$ (W_k denotes $W_{e^{2\pi i/n_k}}$) satisfying the following relations: $W_k V W_k^* = e^{2\pi i/n_k} V$, $W_{k+1}^{n_{k+1}/n_k} = W_k$ and $W_k^{n_k} = 1$ for all k. (See the section on notation, following introduction.) Thus V^{n_k} commutes with W_k; $C^*(W_k, V) \subseteq C^*(W_{k+1}, V)$ and $A_{\Lambda} = \bigcup_{k=1}^{\infty} C^*(W_k, V)$. It follows that $K_0(A_{\Lambda}) = \varinjlim K_0(C^*(W_k, V))$ and $K_1(A_{\Lambda}) = \varinjlim K_1(C^*(W_k, V))$ where the bonding maps of the inductive limits arise from the inclusions $C^*(W_k, V) \subseteq C^*(W_{k+1}, V)$. If $W_k = \sum_{l=0}^{n_k - 1} e^{2\pi i l/n_k}\ q_l^k$ is the spectral resolution of W_k we have $V q_l^k V^* = q_{l+1}^k$ for each l (here $l + 1$ is taken mod n_k). Thus the partial isometries $V^l q_0^k$, $l = 0, \ldots n_k - 1$ induce an isomorphism of $C^*(W_k, V)$ with $C^*(V^{n_k} q_0^k) \otimes M_{n_k}$. Identifying $C^*(V^{n_k} q_0^k)$ with $C(\Gamma)$ so that $V^{n_k} q_0^k$ corresponds to the identity function V in $C(\Gamma)$ we obtain an isomorphism π of $C^*(W_k, V)$ with $C(\Gamma) \otimes M_{n_k}$ such that $\pi(V) = \begin{bmatrix} V & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$ and $\pi(q_0^k) = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$. Now $K_1(C(\Gamma) \otimes M_{n_k}) = Z$ and the equivalence class of $\begin{bmatrix} V & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ in K_1 is 1 (see [9, page 163]). Since the equivalence class of a scalar matrix in K_1 is 0, it follows that

$K_1(C^*(W_k,V)) = Z$ and the equivalance class of V in K_1 is 1 in Z. Hence all the bonding maps in the inductive limit reduce to the identity map from Z to $\mathbf{Z}$. Hence $K_1(A_\Lambda) = \mathbf{Z}$.

To compute K_0 note that $K_0(C(\Gamma) \otimes M_{n_k}) = Z$ and the equivalance class of $\pi(q_0^k)$ in K_0 is 1 in Z. Hence $K_0(C^*(W_k,V)) = Z$ and the equivalance class of q_0^k in K_0 is 1. Since $q_0^k = \sum_{l=0}^{p_{k+1}-1} q_{l_{n_k}}^{k+1}$ and the projections occurring in this decomposition of q_0^k are unitarily equivalent it follows that the bonding map at the k^{th} stage is $m \leftrightarrow m(n_{k+1}/n_k) = mp_{k+1}$. Therefore $K_0(A_\Lambda) = \{m/n_k, m \in Z, k = 1,2,\ldots\}$.

Corollary 2.5. *(i) The isomorphism class of A_Λ is determined by Λ.*

(ii) A_Λ and $\bigotimes_{k=1}^{\infty} M_{p_k}$ are not homotopy equivalent $\left(\text{even though } \bigotimes_{k=1}^{\infty} M_{p_k} \subseteq A_\Lambda\right.$ and A_Λ can be imbedded in $\left.\bigotimes_{k=1}^{\infty} M_{p_k}\right)$.

Proof. To prove (i) note that Λ can be retrieved from $K_0(A_\Lambda)$ by the map $t \leftrightarrow e^{2\pi i t}$.

(ii) follows from the fact that $K_1\left(\bigotimes_{k=1}^{\infty} M_{p_k}\right)$ is $\{0\}$.

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