An introduction to C*-algebras and Noncommutative Geometry

Heath Emerson

Dedicated to Juliette.

Contents

CHAPTER 1

Introduction

The study of C*-algebras was mainly intiated by the mathematician Gelfand, in connection with representation theory of locally compact groups. If one has two compactly supported, continuous functions on such a group, then their convolution product is the function on the group given by

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g) dh, \tag{0.1}$$

where dh denotes Haar measure on the group. Using this multiplication, one can construct a C*-algebra $C^*(G)$ out of any group, and the interest in the construction was that the representation theories of the group, and of the C*-algebra, are in a completely natural one-to-one correspondence.

A C*-algebra can be defined to be any subalgebra A of the operators $\mathbb{B}(H)$ of all operators on a Hilbert space, which is closed under operator adjoint and topologically closed in the operator norm topology. Gelfand formulated an abstract definition of C*-algebra, not depending on any such 'representation' on a Hilbert space, and proved a beautiful theorem, now bearing his name. If X is any locally compact space, then the collection $C_0(X)$ of continuous functions $f\colon X \to \mathbb{C}$ vanishing at infinity, forms a C*-algebra where the vector space structure is obvious, and the algebra multiplication operation is pointwise multiplication

$$(f_1 \cdot f_2)(x) = f_1(x) f_2(x).$$

What Gelfand proved was that *any* abstract commutative C*-algebra has this form: he describes a natural bijection between the categories of compact Hausdorff topological spaces, and the category of *commutative* unital C*-algebras – that is, for which the algebra multiplication is commutative

$$ab = ba \; \forall a, b \in A.$$

Gelfand's theorem, telling us that the category of *commutative* C*-algebras is essentially identical to the category of topological spaces, led to the idea that a general, possibly noncommutative C*-algebra, may be thought of as a kind of 'noncommutative space.'

The need for such a concept had, of course, already been felt in subatomic particles physics. Observables in sub-atomic systems had a strange 'noncommutative behavior' that made it seem better to model their behavior not as usual in physics, in terms of functions on appropriate spaces, but by, for example, the 2-by-2 matrices $M_2(\mathbb{C})$ with complex entries (a C*-algebra under matrix addition, multiplication, and conjugate transpose.) The famous Heisenberg Uncertainty Principal articulates mathematically that a certain C*-algebra, generated by a pair of 'observables,' that is, a pair of operators on a HIlbert space, is noncommutative.

In the 1950's and 1960's, seminal results in topology and analysis: the Bott Periodicity and Atiyah-Singer Index theorems, whose proofs and statements were very rapidly realized were best understood in the language of C*-algebras, and C*-algebra K-theory, vitalized interest in C*-algebra theory from a broader, and more geometric point of view. In the 1980's, the Field's

Medallist Alain Connes proposed a new field, in a sense, that of 'Noncommutative Geometry', an attempt to translate into C*-algebra language the key concepts of Index theory and Riemannian geometry, and thus, in a sense, extend them, from the 'commutative' case to the more general, possibly noncommutative one (see Connes' classic book [**5**]). Concurrently, the Russian mathematician Genadi Kasparov was developing KK-theory, a bivariant generalization of C*-algebra K-theory, and using it to prove cases of the Novikov Conjecture [**13**]. New and striking connections between the Novikov conjecture, group C*-algebras, and 'higher index theory' resulted, and led to a sort of sub-field of Noncommutative Geometry, generally referred to as 'coarse geometry,' and stimulated further interest in C*-algebras associated to groupoids attached to various types of dynamical systems. By associating a C*-algebra to a groupoid, one can then study the groupoid C*-algebraically, its K-theory invariants computed, representation-theoretic dual, primitive ideal space, tracial state space, *etc* computed. The use of this strategy has produced interesting results in connection with representation theory of locally compact groups, hyperbolic dynamical systems, Gromov hyperbolic groups, Penrose tilings, foliations, classification of minimal homeomorphisms of Cantor sets, systems and C*-algebras associated to number fields, to name a few, from recent years.

The field of Noncommutative geometry, at present, offers what seems to be a very enticing program. It is the goal of this book to bring the relative novice up to speed on the basic ideas of C*-algebra theory, K-theory, and to offer a glimpse onto the next level of this fascinating subject, concerning analytic index theory of elliptic operators, KK-theory, and so on. Thus, the book might better be titled 'An invitation to C*-algebras, *etc*', as that is the real intent: to invite the reader to pursue the subject further.

We will now describe the contents and arrangements of this book.

Chapters 2-4

The first three chapters deal with what one might call the 'elementary' part of the subject of C*-algebras – that which does not concern K-theory. The exception in a sense is the Toeplitz Index Theorem, which is highlighted as important for a number of reasons, in Chapter 2, which is mainly concerned with building a collection of examples of C*-algebras: from spaces, groups, and group actions and analyzing them to the extent that is possible without more general theory – *i.e.* spectral theory and the basic points of representation theory of C*-algebras (the GNS theorem), which is covered in Chapter 3. The audience I have in mind for Chapter 2 has at least the mathematical background of a third or fourth-year honours math undergraduate student in the Canadian or American system, and I have attempted to cover as many as possible of the most interesting ideas in C*-algebra theory (to me) while keeping almost entirely away from any more advanced ideas from analysis than basic measure and Hilbert space theory – this and a solid knowledge of basic linear algebra, group theory, a course in point-set topology, should be sufficient prerequisites. There is a fairly extensive discussion of group C*-algebras, especially of the integers and the circle, the Fourier transform and what it means C*-algebraically. We also discuss Toeplitz operators and Hardy spaces, the compact operators and the Calkin algebra, in preparation for the Toeplitz Index Theorem. Crossed-products and inductive limits are fairly briefly introduced in this chapter as well, and a discussion of the representation theory of finite groups, which explains the structure of their C*-algebras.

In this spirit of accumulating a good collection of examples first, we put off discussion of ideals and quotients in C*-algebras, to Chapter 3, and tensor products are also discussed here.We revert to the examples in order to analyze the (ideal) structure of the class of 'orbifold' C*-algebras associated to proper actions of discrete groups on locally compact spaces. The interpretation of these C*-algebras as appropriate section algebras of certain bundles, gives a starting point for analyzing their K-theory, discussed later, and these examples also give good intuition for 'noncommutative spaces,' since their ideal spectra involve a mixture of commutative and noncommutative data.

Chapter 4 begins to deal with more advanced topics, mainly in preparation for K-theory and KK-theory, with strong Morita equivalence being an important component. The chapter treats general C*-module (or Hilbert module) theory, that is, modules over C*-algebras, especially finitely generated projective modules, which K-theory classifies. We introduce vector bundles first, prove Swan's Theorem, which gives a certain equivalence between finitely generated projective modules over *commutative* *-algebras, and vector bundles (over their spectra.) Hilbert modules are introduced, strong Morita equivalence, and various examples from groups and group actions are studied. We include an extensive discussion of how to construct finitely generated projective modules over the irrational rotation algebra A_θ, based on a reduction-to-a-transversal procedure in groupoid theory.

Chapters 5-6: K-theory

Chapters 5 and 6 develop the kind of algebraic-topological tool used in Noncommutative Geometry to study noncommutative spaces, or C*-algebras – K-theory. We start with a description of *topological* K-theory, that is, K-theory for commutative C*-algebras, with a lot of emphasis on concrete problems with vector bundles, in order to build some geometric intuition, hopefully, about what K-theory classifies. We proceed to lay out the general basic results of topological K-theory: ring module structures, the long exact sequence of a pair, Bott Periodicity and the Thom Isomorphisms, although we do not prove the latter at this stage. We show how to use them to do computations (*e.g.* for projective spaces, *etc*).

Although our discussion of topological K-theory results in some duplication later, we have decided to give such a detailed development of C*-algebra K-theory in the commutative case for several reasons. The first is that, obviously, one would hope it helps to build some geometric intuition about what K-theory classifies before getting into K-theory for general C*-algebras. The second reason is that the Atiayh-Singer Index Theorem relates an analytic Fredholm index, to a certain K-theory datum, and this latter 'topological index' is defined in terms of various constructions in topological K-theory which are highly geometric (involving normal bundles, Thom isomorphisms, *etc*) and do not exist in C*-algebra K-theory, so they are not generally treated in C*-algebra K-theory books, for example. As one of the goals of this book is to state the Atiyah-Singer Index Theorem (although we do not prove it), a solid introduction to wrong-way maps and topological correspondences seemed appropriate. The third reason is that the theory of correspondences allows us to introduce a kind of 'baby' version of Kasparov theory, or KK-theory, which works well for nice commutative C*-algebras, like continuous functions on manifolds. In this 'geometric' KK-theory, one can compose morphisms by transversality, which makes it possible to do many KK-computations quite easily, in stark contrast to their counterpart calculations in analytic KK. The author hopes that working a bit with geometric KK, where calculations are relatively easy to do, might help the student get used to the basic mechanics of the theory, before proceeding to Kasparov's KK.

Chapter 6 gives a fairly standard treatment of C*-algebra K-theory, following more or less the efficient treatment of Higson and Roe [**14**]. The most important theorem of the subject, perhaps of Noncommutative Geometry as a whole, the Bott Periodicity theorem, is proved using an idea of Atiyah, based on the Toeplitz Index Theorem of Chapter 2. As application, of the 6-term exact sequence arising from Bott Periodicity, and our earlier analysis on the structure of crossed-products $C_0(X) \rtimes G$ by *proper* actions of discrete groups (including, thus, finite group actions), we compute the K-theory groups of some of these.

Chapters 7: Index theory, KK-theory

Further progress in computing K-theory groups of, say, crossed-products, by *non-proper* actions, like the action of $\mathbb{Z}$ by irrational rotations on the circle $\mathbb{T}$, requires significant use of Kasparov's KK-theory, and, to a significant extent, direct use of elliptic operator methods, methods arising, of course, from the seminal work of Atiyah and Singer, and thereby motivating Kasparov's work and the subsequent work on the Baum-Connes conjecture.

The last chapter of the book accordingly provides an introduction to this this analytic KK-theory, for which we have laid the groundwork already to some considerable extent. Kasparov's theory is somewhat notoriously difficult for beginners to learn. There are various reasons for this, but in any case the basic definitions of KK-theory are rather simple, given some basic acquaintance with Hilbert modules and Fredholm operators. Our development of K-theory used finitely generated projective modules extensively, so we hope the transition to 'Fredholm bimodules' might seem natural. The most important thing about KK-theory, practically speaking, is that the composition operation, or 'intersection product,' can be described axiomatically (see [**15**]). This means that one can compute intersection products by clever guess-work, and verification of the axioms, and for computational purposes, proofs of the general features of KK-theory (like the existence theorems for the Kasparov product) are somewhat beside the point. We do not prove the Stabilization Theorem for Hilbert modules, or the Kasparov Technical Theorem, partly for this reason (and partly because these theorems are easily accessible in other books), as our introduction of KK-theory has a practical purpose: to prove $\mathbb{Z}$-equivariant Bott Periodicity Theorem. The latter delivers a KK_1-equivalence between any crossed-product $A \rtimes \mathbb{Z}$ by the integers, and the crossed-product $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$ which is the key step in computing the K-theory of crossed-products $A \rtimes \mathbb{Z}$ in general. It is rather clear to see why this result is useful by taking $A = C_0(X)$ for a space X carrying an action of the integers $\mathbb{Z}$ by homeomorphisms. The $\mathbb{Z}$-space $\mathbb{R} \times X$ with the diagonal $\mathbb{Z}$-action, is, of course unlike X itself, in general, a free and proper $\mathbb{Z}$-space, and hence the crossed-product $C_0(X, A) \rtimes \mathbb{Z}$ is strongly Morita equivalent to $C_0(\mathbb{R} \times_{\mathbb{Z}} X)$, continuous functions on the mapping cylinder of the action. Hence

$$\mathrm{K}_*\big(C_0(X) \rtimes \mathbb{Z}\big) \cong \mathrm{K}^{*+1}(\mathbb{R} \times_{\mathbb{Z}} X).$$

Therefore, $\mathbb{Z}$-equivariant Bott Periodicity, an analytic exercise, reduces computation of the K-theory groups of crossed-products to computation of K-groups of mapping cylinders, which is rather easy, at least in principal, it being purely a matter of topological K-theory – more or less cohomology – of ordinary compact Hausdorff spaces.

Notably, this proof also works equivariantly with respect to any *closed subgroup* of $\mathbb{R}$, not just the integers – and in particular it works for $\mathbb{R}$ itself, giving the Connes-Thom isomorhpism (see [**10**]).

The class of groups G (even locally compact groups) for which, roughly speaking, this procedure can be carried out, is the issue of the Baum-Connes Conjecture, which initially suggested that this might be the class of all locally compact groups. The Baum-Connes conjecture is now known to be false, even for the class of discrete groups, although counter-examples at the present time are defined in a probabilistic way and are not very likely to be met in everyday life. The Baum-Connes conjecture is true, or true enough, for a huge class of examples – see for example [**12**] for what is perhaps the strongest positive result on it to date, and [**16**] for an important extension of it.

Victoria, August, 2018.

CHAPTER 2

An introduction to C*-algebras

1. The definition of C*-algebra

An (associative) *algebra* over the complex numbers is a complex vector space A equipped with an associative, bilinear (linear in each variable separately) multiplication operation $A \times A \to A$, $(a, b) \mapsto ab$. An algebra is unital if it contains an element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$.

The zero algebra $\{0\}$ is an algebra, albeit an extremely uninteresting one, as is the complex numbers itself $\mathbb{C}$. Both are unital. There are of course many, many other examples everywhere: polynomial algebras $\mathbb{C}[x_1, \ldots, x_n]$ and their subrings, the quaterion algebra, matrix algebras $M_n(\mathbb{C})$, algebras of operators on a Hilbert space, group algebras, *etc.*

For our purposes, it is extremely interesting and important that every compact Hausdorff space X has a (commutative) algebra canonically associated to it: namely the algebra $C(X)$ of continuous complex-valued functions on X. To get the algebra structure we apply the algebra structure of $\mathbb{C}$ pointwise: thus

$$(\lambda f)(x) := \lambda f(x), \ (f+g)(x) := f(x) + g(x), \ (fg)(x) := f(x)g(x).$$

The constant function 1 is the unit. The Hausdorff condition is to ensure an adequate supply of continuous functions to produce something meaningful from the construction.

The algebra $C(X)$ has further structure, which turns out to be important. Firstly, complex conjugation on the complex numbers $\mathbb{C}$ gives rise to an 'involution' on functions $f \in C(X)$ by setting $f^*(x) := \overline{f(x)}$. Secondly, any continuous function on a compact space is bounded. We set

$$\|f\| := \sup_{x \in X} |f(x)|, \text{ for } f \in C(X), \tag{1.1}$$

and call it the norm of f; it satisfies the standard set of conditions for a norm on a linear space:

$$\|\lambda f\| = |\lambda| \|f\|, \ \ \|f + g\| \leq \|f\| + \|g\|, \ \ \|f\| = 0 \iff f = 0.$$

So $C(X)$ gains a topology from the metric $d(f, g) = \|f - g\|$. It is a quite easy enough exercise, given completeness of the complex numbers, to prove that $C(X)$ is also complete with respect to this metric (thus is a Banach space). Furthermore, the conditions

$$\|fg\| \leq \|f\| \|g\|, \ \ \|f^*\| = \|f\|,$$

hold, for all f, g, which makes all the algebra operations continuous.

Exercise 1.1. Prove that if X is compact Hausdorff, (f_n) is a sequence of continuous functions on X which is Cauchy with respect to the norm (1.1), then (f_n) converges in the same norm, to a continuous function f. That is, show that $C(X)$ is complete with respect to the norm (1.1).

Generally, in working with infinite-dimensional algebras, one wants them to have a norm, as in this example, with respect to which the algebra operations are continuouus, and to be complete with respect to the norm. This leads, for example, to the study of *Banach algebras*, which are mentioned again in the next section. But amongst more general normed algebras, even amongst general Banach algebras, the normed algebras $C(X)$ have a very special property.

If A is any unital algebra, and $a \in A$, a complex number λ is in the *spectrum* of a if $\lambda - a$ is *not* invertible in A. The spectrum of an element, therefore, is a purely algebraic invariant of a (it doesn't make any reference to norms, nor, in fact, to adjoints.)

We let $\mathrm{Spec}(a) \subset \mathbb{C}$ be the spectrum of a.

EXERCISE 1.2. If A is a unital algebra, $u \in A$ is an invertible element, and $a \in A$, then the spectrum of uau^{-1} is the same as the spectrum of a.

EXERCISE 1.3. Verify the formula $(\lambda - ba)^{-1} = \lambda^{-1} + \lambda^{-1}b(\lambda - ab)^{-1}a$ for any elements a, b of a unital algebra. Deduce that $\mathrm{Spec}(ab) - \{0\} = \mathrm{Spec}(ba) - \{0\}$.

If $A = C(X)$, for X compact Hausdorff as above, then a function $f \in C(X)$ is invertible if and only if it does not vanish on X. We deduce that the spectrum of such f is precisely it's range. But the norm $\|f\|$ of f is precisely the modulus of the largest complex number in it's range. Hence we get

$$\|f\| = \sup_{\lambda \in \mathrm{Spec}(f)} |\lambda|. \tag{1.2}$$

This equation relates the *topology* and the *algebra* in a very tight way.

For example, we easily deduce the following.

PROPOSITION 1.4. *Let X and Y be compact Hausdorff, $A = C(X)$ and $B = C(Y)$. Then if $\alpha\colon A \to B$ is a unital algebra homomorphism, then α is automatically continuous with respect to the topologies on A, B determined by their C^*-norms.*

PROOF. Firstly, α is an algebra homomorphism. Hence it maps invertibles in A to invertibles in B. It follows that $\mathrm{Spec}\big(\alpha(f)\big) \subset \mathrm{Spec}(f)$ for any $f \in A$. Since the norm is the diameter of the spectrum, we deduce that $\|\alpha(f)\| \leq \|f\|$ as claimed.

∎

The algebra of n-by-n matrices has a similar property to the one appearing in (1.2). Firstly, the correct adjoint to use on this algebra is the conjugate transpose A^* of a matrix: thus $A^*_{ij} := \overline{A_{ji}}$. For the norm, we use the *operator norm*, defined for $A \in M_n(\mathbb{C})$ by

$$\|A\| := \sup_{\xi \in \mathbb{C}^n, \|\xi\|_{\mathbb{C}^n} = 1} \|A\xi\|_{\mathbb{C}^n}, \tag{1.3}$$

with $\|\cdot\|_{\mathbb{C}^n}$ the Hilbert space norm on $\mathbb{C}^n$. This is a submultiplicative norm $\|AB\| \leq \|A\|\|B\|$, and $\|A^*\| = \|A\|$, as with the sup norm on functions.

It does indeed turn out that the operator norm only depends on the algebra, in the following sense.

Firstly, the operator norm is easily checked first of all to be invariant under unitary conjugation $\|UAU^*\| = \|A\|$ for all $A \in M_n(\mathbb{C})$. Secondly, if A is any matrix, $\|A\|^2 = \|A^*A\|$ (this is more challenging to prove, and is done in the next section), which reduces the problem of finding $\|A\|$ to finding the norm of A^*A. Now A^*A is unitarily diagonalizable. So if it is unitarily conjugate to $\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A^*A (repeated according to multiplicity.) Hence $\|A^*A\|$ equals the norm of the diagonal matrix. But this is exactly the supremum $\max\{|\lambda_1|, \dots, |\lambda_n|\}$ of the entries of the diagonal matrix. We thus have that $\|A\|$ (for any A) can be characterized as

$$\|A\| = \sup_{\lambda \in \mathrm{Spec}(A^*A)} \sqrt{|\lambda|},$$

where $\mathrm{Spec}(A^*A)$ is the set of eigenvalues of A^*A. And this is exactly the same as in the case of $C(X)$, since we can write (1.2) in the fancier but equivalent way

$$\|f\| = \sup_{\lambda \in \mathrm{Spec}(f^*f)} \sqrt{|\lambda|}.$$

Definition 1.5. A *C*-algebra* A is an (associative) algebra over the complex numbers equipped with a map $*\colon A \to A$ (usually called the *adjoint*) and a norm $\|\cdot\|\colon A \to [0, \infty)$ satisfying

a) The map $*$ is a conjugate-linear, involutative anti-homomorphism, *i.e.* satisfies
   · $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$ for all $\lambda \in \mathbb{C}$, $a, b \in A$,
   · $(ab)^* = b^* a^*$ for all $a, b \in A$, and
   · $(a^*)^* = a$ for all $a \in A$.
b) With the metric assigning distance $\|a - b\|$ from a to b, A is complete, *i.e.* $(A, \|\cdot\|)$ is a Banach space.
c) $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.
d) $\|a^*a\| = \|a\|^2$ for all $a \in A$.

A is *unital* if there exists an element $1 \in A$ acting as the identity under multiplication.

The condition d) in Definition 1.5 is often called the C*-condition.

As an easy exercise in the definitions, note that $\|a^*\| = \|a\|$ for all $a \in A$, for using c) above, we have that $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$. The claim follows by switching the roles of a and a^*.

The following exercise is also routine, it follows from the uniqueness of the unit .

Exercise 1.6. The unit 1 in a unital C*-algebra satisfies $1^* = 1$.

As a matter of terminology, a **-algebra* is an associative complex algebra with an involution satisfying the condition a) above. A *Banach algebra* is an algebra (without necessarily an involution operation) together with a norm with respect to which it is complete, and satisfying $\|ab\| \leq \|a\|\|b\|$. A *Banach *-algebra* is a Banach algebra with an involution $*\colon A \to A$ making it also a *-algebra, and as well, the requirement that $\|a^*\| = \|a\|$ – a weakening of the C*-condition, as explained above. This weakening has a substantial effect on the resulting different theories.

Exercise 1.7. Let A be a C*-algebra (not necessarily unital). Prove that if $xa = 0$ for all $x \in A$ then $a = 0$, and that if a is an element of a C*-algebra A and $a^*a = 0$, then $a = 0$.

A **-homomorphism* $\varphi\colon A \to B$ between two C*-algebras is an algebra homomorphism such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. The notion of isomorphism of C*-algebras is then the obvious one: there must be two *-homomorphisms which compose to the identity. As we show when discussing Spectral Theory in more detail in Chapter 2, a *-homomorphism is automatically continuous.

A C*-algebra is *commutative* if, of course, its multiplication is commutative, $ab = ba$ for all $a, b \in A$. It is clear from the preceding discussion that $C(X)$ is a commutative C*-algebra for any compact Hausdorff spaces X.

We will finish in the next section the proof that the *-algebra of n-by-n matrices $M_n(\mathbb{C})$, with the operator norm, is also a C*-algebra, but it is evidently not commutative.

The mathematical principal of Quantum Mechanics is roughly as follows. In classical (Newtonian) physics, one deals with points in an appropriate space. If a particle moves through space time, its positions with regard to a fixed set of – say, three – coordinate axes as time changes, all of these constitute the points of a 4-dimensional space X. In the language of physics, continuous functions on X are 'observables.' For example, if one has a system $(x_1, \dots, x_n)$ of coordinates on X (valued in $\mathbb{R}^n$) then each $x_i \colon X \to \mathbb{R}$ is an observable: at a given point in space, and at a given time, one observes the x_ith coordinate of the particle.

At any rate, *observables* stripped down to their mathematical essentials, are continuous functions.

However, it was shown experimentally that when one attempts to study electrons within an atom, certain different measurements, namely position and momentum, interfere with each other in such a way as to make the *simultaneous* measurement of them impossible. Heisenberg postulated that the mathematics describing quantum physics should be the mathematics, not of functions on a space, but of linear operators on a Hilbert space, which, taken as an algebra, behaves, algebraically, much like the algebra of continuous functions on a space, but is not commutative. And the experimental fact noted with position and momentum of electrons would correspond to the failure of two specific operators to commute with each other. The operators are each self-adjoint and diagonalizable, and thus each, in its own right, is, up to unitary equivalence, just a (real-valued) function on a space (the space being the spectrum of the operator, the function being the inclusion of the spectrum in $\mathbb{R}$), but one cannot consider them both *simultaneously* as functions, because they cannot be simultaneously diagonalized, because they do not commute.

EXERCISE 1.8. Let A be a C*-algebra and X is a locally compact Hausdorff space. Consider $C(X, A)$, the collection of continuous functions $f \colon X \to A$. Endow $C(X, A)$ with the algebra operations

$$(f_1 + \lambda f_2)(x) := f_1(x) + \lambda f_2(x), \quad (f_1 f_2)(x) := f_1(x) f_2(x),$$

adjoint $(f^*)(x) = f(x)^*$ and norm $\|f\| := \sup_{x \in X} \|f(x)\|$;

Prove that $C(X, A)$ is a C*-algebra.

EXAMPLE 1.9. An important example of a Banach algebra which is not a C*-algebra is the *disk* algebra $\mathcal{A}(\mathbb{D})$, consisting of all continuous functions $f \in C(\overline{\mathbb{D}})$ on the closed disk, which are analytic in the open disk $\mathbb{D}$.

The norm on $\mathcal{A}(\mathbb{D})$ may be take to be the supremum norm on the closed disk, or, equivalently,

$$\|f\|_{\mathcal{A}(\mathbb{D})} = \sup_{z \in \partial \overline{\mathbb{D}} = \mathbb{T}} |f(z)| = \|f|_{\mathbb{T}}\|_{C(\mathbb{T})}.$$

Hence, $\mathcal{A}(\mathbb{D})$ can be regarded as a closed Banach subalgebra of the C*-algebra $C(\mathbb{T})$ (that is, the norm on $\mathcal{A}(\mathbb{D})$ is the restriction of the norm on $C(\mathbb{T})$. In particular it is a Banach algebra. It cannot naturally be given the structure of a C*-subalgebra of $C(\mathbb{T})$, however, because the $f^*(z) := \overline{f(z)}$ is not analytic even if f is analytic.

EXERCISE 1.10. Let A and B be two C*-algebras. Their *direct sum* $A \oplus B$ is defined to be the direct sum of A and B as vector spaces, with the algebra structure $(a, b) \cdot (c, d) := (ac, bd)$, adjoint $(a, b)^* := (a^*, b^*)$, and norm $\|(a, b)\| := \max\{\|a\|, \|b\|\}$.

Prove that $A \oplus B$ is a C*-algebra and that the two projection maps $\pi_1 \colon A \oplus B \to A$ and $\pi_2 \colon A \oplus B \to B$ are *-homomorphisms.

Similarly one defines the direct sum $A_1 \oplus \cdots \oplus A_n$ of finitely many C*-algebras, or even infinitely many. If I is an index set, and $\{A_i\}_{i \in I}$ is a family of C*-algebras, we let $\oplus_{i \in I} A_i$ be the collection of I-tuples $(a_i)_{i \in I}$ with

$$\|(a_i)_{i \in I}\| := \sup_{i \in I} \|a_i\|$$

finite. This is a C*-algebra, and the coordinate projections

$$\pi_j \colon \oplus_{i \in I} A_i \to A_j$$

are surjective *-homomorphisms.

We close with a basic but important C*-algebraic construction.

Let A be any C*-algebra, possibly non-unital. Let $A^+ = A \oplus \mathbb{C}$ as a vector space. Equip this vector space with the multiplication and adjoint

$$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu), \quad (a, \lambda)^* := (a^*, \bar{\lambda}).$$

If we let $1 = (0, 1) \in A^+$, we can write elements of A^+ in the form $a + \lambda 1$, or just $a + \lambda$. Then the multiplication becomes the 'obvious' one on such symbols

$$(a + \lambda)(b + \mu) = ab + \lambda b + \mu a + \lambda\mu.$$

Also, $A \subset A^+$, in the obvious way, and as such, A is a *-subalgebra of A^+. In fact if $a + \lambda \in A^+$ and $b \in A$, then $(a + \lambda)b \in A$. Hence A is an *ideal* in A^+, which is also clearly closed under adjoint.

For the norm, we set

$$\|a + \lambda\| := \max\{ \sup_{\|b\| \leq 1} \|(a + \lambda)b\|, \ |\lambda|\}.$$

EXERCISE 1.11. A^+ with the given norm, is a unital C*-algebra. Moreover, if A is already unital, then $A^+ \cong A \oplus \mathbb{C}$, where the direct sum refers to Exercise 1.10.

The C*-algebra A^+ so defined, is called the *unitization* of A.

EXERCISE 1.12. Let X be a locally compact Hausdorff space. A continuous function on X *vanishes at infinity* if for all $\epsilon > 0$ there exists a compact subset $K \subset X$ such that $|f(x)| < \epsilon$ for all $x \in X \setminus K$ Prove that the collection of continuous functions f on X which vanish at infinity, with the supremum norm, is a C*-algebra. It is denoted $C_0(X)$. (It is non-unitla).

EXERCISE 1.13. If X is a C*-algebra, $C_0(X, A)$ denotes continuous functions on X valued in A, which 'vanish at infinity.' Define carefully and prove it is a C*-algebra.

EXERCISE 1.14. If X is a locally compact Hausdorff space, then the 1-point compactification X^+ of X is the union $X \sqcup \{\infty\}$ and an additional point, labelled ∞, topologized with open sets the open subsets of X, together with the sets $U_K \cup \{\infty\}$, where $K \subset X$ is a compact subset and U_K its complement in X.

Thus, X^+ contains X as an open subset.

a) Prove that X^+ is compact Hausdorff. Notice how the *locally compact* assumption on X is used here.
b) Prove that if X is already compact, then $\{\infty\}$ is an isolated point in X^+.
c) Prove that a continuous map $f\colon X \to Y$ between locally compact Hausdorff spaces, extends continuously to a map $X^+ \to Y^+$ mapping the points of infinity to each other, if and only if the map $f\colon X \to Y$ is *proper*, *i.e.* if and only if $f^{-1}(K)$ is compact in X for every compact subset $K \subset Y$.
d) Prove that the C*-algebra $C(X^+)$ and $C_0(X)^+$ are canonically isomorphic.

Thus, unitization is the noncommutative version of the 1-point compactification.

A *partial isometry* is an element $s \in A$ such that s^*s and ss^* are projections.

EXAMPLE 1.15. If A is unital, then a *unitary* is an element $u \in A$ such that $u^*u = uu^* = 1$. If $A = C(\mathbb{T})$ is the C*-algebra of continuous functions on the circle $\mathbb{T}$, then the function $u(z) = z$ is a unitary in $C(\mathbb{T})$. More generally, if X is a compact Hausdorff space, then any map $u\colon X \to \mathbb{T}$ defines a unitary element of $C(X)$.

EXERCISE 1.16. A *projection* $p \in A$ in a C*-algebra is an element such that $p^2 = p$ and $p = p^*$. Thus, it is an idempotent, $p^2 = p$, but is also required to be self-adjoint. Projections will play an important role in C*-algebra theory, particularly in K-theory.

Let A be the C*-algebra $C(X)$, where X is compact Hausdorff. Show that then projections in $C(X)$ correspond to connected components of X. (Thus, a Cantor set has many projections).

2. The C*-algebra of bounded operators on a Hilbert space

A *pre-Hilbert* space is a complex vector space H equipped with a map, called an *inner product*

$$\langle \cdot, \cdot \rangle \colon H \times H \to \mathbb{C}$$

which is linear in the second variable, and the conditions

$$\begin{aligned} \langle \xi, \xi \rangle &\geq 0, \\ \langle \xi, \eta \rangle &= \overline{\langle \eta, \xi \rangle}, \\ \langle \xi, \xi \rangle = 0 &\Rightarrow \xi = 0, \end{aligned}$$

for any $\xi \in H$.

It follows that $\langle \cdot, \cdot \rangle$ is *conjugate linear* in the first variable.

We set

$$\|\xi\| := \sqrt{\langle \xi, \xi \rangle}.$$

The Cauchy-Schwartz inequality asserts that

$$|\langle \xi, \eta \rangle| \leq \|\xi\| \|\eta\|. \tag{2.1}$$

EXERCISE 2.1. Use the Cauchy-Schwartz inequality (2.1) to prove that

$$\|\xi + \eta\| \leq \|\xi\| + \|\eta\|.$$

By the exercise, a pre-Hilbert space is a special case of a normed linear space, and the formula

$$d(\xi, \eta) := \|\xi - \eta\|.$$

defines a *metric* on H.

A *Hilbert space* is a pre-Hilbert space which is also *complete* with respect to the metric d.

EXAMPLE 2.2. Of course the simplest example of a Hilbert space is $\mathbb{C}^n$ with inner product

$$\langle (z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle := \sum_{i=1}^{n} \overline{z_i} w_i,$$

it is a finite-dimensional Hilbert space.

For an infinite-dimensional example, take

$$l^2(\mathbb{N}) := \{(a_n)_{n=0}^{\infty} \mid \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

with inner product

$$\langle (z_n), (w_n) \rangle := \sum_{n=0}^{\infty} \overline{z_n} w_n.$$

We leave it to the reader to check that $l^2(\mathbb{N})$ is complete.

Let H, K be a pair of Hilbert spaces. A linear operator $T\colon H \to K$ is *bounded* if

$$\sup_{\xi \in H, \|\xi\|_H = 1} \|T\xi\|_K \tag{2.2}$$

is finite, where we have (exceptionally) subscripted the norms with the Hilbert space to which they are attached. If (2.2) is finite, then we write $\|T\|$ for the supremum. It is called the *operator norm* of T. We let $\mathbb{B}(H, K)$ denote all bounded linear operators $H \to K$.

If $H = K$, we just write $\mathbb{B}(H)$.

EXERCISE 2.3. Prove that if $T\colon H \to K$ is a linear map, such that

$$|\langle T\xi, \eta\rangle| \leq c\,\|\xi\| \cdot \|\eta\| \tag{2.3}$$

holds for a dense set of vectors $\xi, \eta \in H$, and some constant $c \geq 0$, then T is bounded, and

$$\|T\| \leq c.$$

Verify the slightly more precise statement that

$$\|T\| = \sup\{|\langle T\xi, \eta\rangle| \mid \xi, \eta \in H,\ \|\xi\| = \|\eta\| = 1\}.$$

EXERCISE 2.4. Prove the following about the operator norm.

a) Prove that the operator norm on $\mathbb{B}(H, K)$ is a norm, *i.e.* that
$$\|\lambda T\| = |\lambda|\|T\|,\quad \|S + T\| \leq \|S\| + \|T\|,\quad \forall \lambda \in \mathbb{C}, S, T \in \mathbb{B}(H, K)$$
and $\|T\| = 0$ if and only if $T = 0$.
b) Prove that $\|TS\| \leq \|T\|\|S\|$ for any $S \in \mathbb{B}(H, K)$ and $T \in \mathbb{B}(K, L)$.
c) Prove that $\mathbb{B}(H, K)$ is complete in the operator norm.
d) Prove that if H and K are *finite-dimensional* Hilbert spaces, then any linear operator $T\colon H \to K$ is automatically bounded.

EXERCISE 2.5. Show that if $H = K = \mathbb{C}^n$ and T is a diagonal bounded operator

$$T = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

with respect to the standard orthonormal basis, then $\|T\| = \sup_n |\lambda_n|$.

Deduce from the Spectral Theorem for self-adjoint matrices, that if T is self-adjoint, then $\|T\| = \sup\{|\lambda| \mid \lambda \text{ is an eigenvalue of } T\}$.

EXERCISE 2.6. If T is the operator on $\mathbb{C}^2$ with matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then $\|T\| = \sqrt{\frac{3+\sqrt{5}}{2}}$, and the maximal value of $\frac{\|T\xi\|^2}{\|\xi\|^2}$ is achieved any any nonzero point along the line spanned by $\xi = \begin{bmatrix} 1 + \sqrt{5} \\ 2 \end{bmatrix}$.

EXERCISE 2.7. Let $(P_n)_{n=1}^{\infty}$ be a sequence of projections in $\mathbb{B}(H)$ with the property that $\lim_{n\to\infty} P_n\xi = \xi$ for all $\xi \in H$. Prove that

$$\|T\| = \sup_n \|P_n T P_n\|$$

for all $T \in \mathbb{B}(H)$.

A *bounded linear functional* on a Hilbert space H is (by definition) a bounded linear operator $L\colon H \to \mathbb{C}$. For an example of such a functional, let $\xi \in H$ be a vector. Let $L_\xi\colon H \to \mathbb{C}$ be defined $L_\xi(\eta) = \langle \eta, \xi \rangle$. By the Cauchy Schwartz inequality,

$$|L_\xi(\eta)| = |\langle \eta, \xi\rangle| \le \|\eta\|\|\xi\|,$$

whence L_ξ is a bounded operator. Moreover, $\|L_\xi\| = \|\xi\|$, as is easily checked.

The *Riesz representation theorem* asserts that if $L\colon H \to \mathbb{C}$ is a bounded linear functional on a Hilbert space, then there is a unique vector $\xi \in H$ such that $L = L_\xi$ (as it is part of standard Hilbert space material, we will not prove it, but see [Conway].)

If $T\colon H \to K$ where $H = K$ we usually just speak of a 'bounded linear operator on H.' The bounded operators on H is an algebra under composition of operators, and is denoted $\mathbb{B}(H)$.

LEMMA 2.8. *For any bounded operator $T\colon H \to K$ between two Hilbert spaces H, K, there is a unique bounded operator $T^*\colon K \to H$ such that*

$$\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$$

holds for all $\xi \in H, \eta \in K$.

EXERCISE 2.9. Lemma 2.8 is proved (below) using the Riesz Representation Theorem. Show that, conversely, the Lemma immediately implies the Riesz Representation Theorem.

PROOF. Let T be as in the statement. Let $\eta \in H$ and $T_\eta\colon H \to \mathbb{C}$ be defined $T_\eta(\xi) = \langle T\xi, \eta\rangle$. Clearly T_η is linear. By the Cauchy-Schwartz inequality T_η is bounded. So by the Riesz Representation Theorem there is a unique vector $T^*(\eta)$ such that $T_\eta(\xi) = \langle \xi, T^*(\eta)\rangle$. Thus

$$\langle T\xi, \eta\rangle = \langle \xi, T^*(\eta)\rangle.$$

That T^* is linear and bounded is left as an exercise.

□

EXERCISE 2.10. Show the following properties of the operator adjoint.

- $*$ is conjugate linear.
- $(TS)^* = S^*T^*$.
- $(T^*)^* = T$.
- If $H = \mathbb{C}^n$, $\mathbb{B}(H) \cong M_n(\mathbb{C})$; under this identification, the adjoint T^* of an operator defined above corresponds to the conjugate transpose of a matrix: $(a^*)_{ij} := \overline{a}_{ji}$.

Hence $\mathbb{B}(H)$ is a *-algebra. We next show that when equipped with the operator norm, it is a C*-algebra.

THEOREM 2.11. *$\mathbb{B}(H)$ is complete in the operator norm, and $\|T^*T\| = \|T\|^2$ for any bounded operator T. Hence $\mathbb{B}(H)$ is a C*-algebra.*

PROOF. We just verify the C*-identity; the other requirements to be a C*-algebra are checked in Exercise 2.4. If $T \in \mathbb{B}(H)$, then

$$\|T\xi\|^2 = \langle T\xi, T\xi\rangle = \langle T^*T\xi, \xi\rangle \le \|T^*T\xi\|\|\xi\| \le \|T^*T\|\|\xi\|^2 \le \|T^*\|\|T\|\|\xi\|^2 \tag{2.4}$$

So

$$\|T\|^2 \le \|T^*T\| \le \|T^*\|\|T\| \tag{2.5}$$

Ignoring for the moment the middle term of the inequality (2.5), we see immediately that $\|T\| \le \|T^*\|$, and by interchanging the roles of T and T^* we get that $\|T\| = \|T^*\|$.

Now, returning to (2.5), since $\|T\| = \|T^*\|$, we now have $\|T\|^2 \le \|T^*T\| \le \|T\|^2$ so 'equality holds as required.

□

It is obvious, that a closed C*-subalgebra of $\mathbb{B}(H)$ (or more generally, of any C*-algebra), is in its own right a C*-algebra. Thus, we can find a lot of other examples of C*-algebras based on the fact that $\mathbb{B}(H)$ is one, since, for example, whenever $T \in \mathbb{B}(H)$ is a single bounded linear operator, it generates a C*-algebra.

More generally:

DEFINITION 2.12. The C*-algebra *generated* by a family $\{T_\lambda\}$ of operators on a Hilbert space H, is the smallest C*-algebra containing all the T_λ's.

Since $\mathbb{B}(H)$ is such a C*-algebra, there is at least one, and since the intersection of (unital) C*-subalgebras of $\mathbb{B}(H)$ is also a C*-subalgebra, the C*-algebra generated by a family of operators is the intersection of all C*-subalgebras of $\mathbb{B}(H)$ containing them.

REMARK 2.13. It may or may not be the case that the C*-algebra generated by a family of operators, or even a single operator T, contains the unit $1 \in \mathbb{B}(H)$. Nor are such C*-algebras unital in an intrinsic sense. The C*-algebra generated by a projection $p \in \mathbb{B}(H)$ is all scalar multiples of p. This C*-algebra has a unit, namely p, but the C*-algebra generated by p does not contain the unit of $\mathbb{B}(H)$. On the other hand, if T is the operator M_f on $L^2(\mathbb{R})$ of multiplication by $f(x) = \frac{1}{x+i}$, then the C*-algebra generated by M_f is isomorphic to $C_0(\mathbb{R})$ (by the Stone-Weierstrass Theorem) which is not unital at all.

We will generally be specific about it if we want a C*-algebra defined by generators to be unital or not.

EXAMPLE 2.14. Let $T \in \mathbb{B}(H)$ be a *self-operator*, $T^* = T$. We denote by $C^*(T)$ the C*-algebra generated by $\{T\}$, and denote by $C^*(T, 1)$ the (unital) C*-algebra generated by T and the unit $1 \in \mathbb{B}(H)$. It is clear that $C^*(1, T)$ is the closure in $\mathbb{B}(H)$ of the *-algebra of 'polynomials'

$$\sum_{k=0}^{n} \lambda_k T^k$$

in T, with complex coefficients, where by T^0 we understand the unit $1 \in \mathbb{B}(H)$. It is commutative and unital. And $C^*(T)$ is the closure of the polynomials $\sum_{k=0}^{n} \lambda_k T^k$ where $\lambda_0 = 0$.

The *Spectral Theorem* gives a complete analysis of $C^*(T)$ and $C^*(1, T)$, purely in terms of the spectrum of T. In finite-dimensions, this boils down to the standard results on diagonalization.

Since T is self-adjoint, all the eigenvalues of T are real, and there is an orthonormal basis of H consisting of eigenvectors of T. If the eigenvalues of T are $\lambda_1, \ldots, \lambda_n$, we may write T as a diagonal matrix organized into (diagonal) blocks

$$T = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

where each λ_i is understood to mean the corresponding multiple of the identity operator on the λ_i-eigenspace $\ker(\lambda_i - T)$. The spectral information is more than enough to determine completely the isomorphism class of $C^*(1, T)$. To see this, observe that $\prod_{j \neq i}(T - \lambda_j)$ is in $C^*(1, T)$ (remember that we have by definition included the unit $1 \in \mathbb{B}(H)$ in $C^*(1, T)$, so $\lambda_j := \lambda_j \cdot 1$ is in $C^*(T)$) and is the diagonal matrix with the nonzero scalar $\mu_i := \prod_{j \neq i}(\lambda_i - \lambda_j)$ in the ith block. This shows us that $P_i := \frac{1}{\mu_i} \prod_{j \neq i}(T - \lambda_j)$, the orthogonal projection onto the ith eigenspace, is in $C^*(T)$. Each P_i generates a one-dimensional *-subalgebra of $C^*(1, T)$ (namely, consisting of all scalar multiples of P_i) and $C^*(1, T)$ is the direct sum of these one-dimensional *-subalgebras, each of which, of course, is isomorphic to $\mathbb{C}$.

Hence $C^*(1, T) \cong \mathbb{C}^n$ where n is the number of distinct eigenvalues of T.

EXERCISE 2.15. Prove that, in the above notation, $C^*(T)$ – as opposed to $C^*(1,T)$ – is isomorphic to the C*-algebra of diagonal matrices

$$T = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

where $\lambda_1 = 0$, and hence it is isomorphic to $\mathbb{C}^{n-1}$ as C*-algebras. In particular, $C^*(T)$ is unital (since $\mathbb{C}^{n-1}$ is) but its unit, as an element of $\mathbb{B}(\mathbb{C}^n)$, is $1 - P_1$, which is $\neq 1$ unless T is *invertible.*

For an infinite-dimensional example, where T has continuous spectrum, let $T \in \mathbb{B}(L^2([0,1])$ be multiplication by $f(t) = t$. Then $C^*(T) \cong C_0\big((0,1]\big)$, while $C^*(1,T) \cong C([0,1])$, as we will see by the Spectral Theorem. In particular $C^*(T)$ is non-unital (has no unit).

EXERCISE 2.16. Two operators $T_1 \in \mathbb{B}(H_1)$ and $T_2 \in \mathbb{B}(H_2)$ on Hilbert spaces H_1 and H_2 are *unitarily conjugate* if there is a unitary operator $u\colon H_1 \to H_2$ such that $\mathrm{Ad}_u(T) := uT_1u^* = T_2$.

Show that if T_1 and T_2 are unitarily conjugate then $C^*(1,T_1) \cong C^*(1,T_2)$ by an isomorphism Ad_u taking T_1 to T_2 and taking $C^*(T_1) \subset C^*(1,T_1)$ to $C^*(T_2)$.

EXERCISE 2.17. Let T_1 and T_2 be self-adjoint operators on a pair of finite-dimensional Hilbert spaces, with respective eigenvalue sets $\mathrm{Spec}(T_1)$ and $\mathrm{Spec}(T_2)$.

a) $C^*(T_1) \cong C^*(T_2)$ if and only if T_1 and T_2 have the same *number* of distinct eigenvalues.
b) $C^*(T_1) \cong C^*(T_2)$ by an isomorphism sending T_1 to T_2 if and only if T_1 and T_2 have the *same* (sets of) eigenvalues.
b) $C^*(T_1)$ and $C^*(T_2)$ are isomorphic by a unitary conjugacy with $uT_1u^* = T_2$ if and only if $\mathrm{Spec}(T_1) = \mathrm{Spec}(T_2)$ and if $\ker(\lambda - T_1)$ and $\ker(\lambda - T_2)$ have the same dimension for all $\lambda \in \mathrm{Spec}(T_1) = \mathrm{Spec}(T_2)$.

(*Hint.* The reader is advised to first prove the analogous assertions for the unital versions $C^*(1,T_i)$, and deduce the result from that.)

We close this chapter with some remarks on self-adjoint operators.

A bounded operator T is *self-adjoint* if $T = T^*$.

EXERCISE 2.18. Show that if T is self-adjoint then $\langle T\xi, \xi\rangle \in \mathbb{R}$ for all vectors ξ.

LEMMA 2.19. *If T is a bounded, self-adjoint operator on a Hilbert space then*

$$\|T\| = \sup_{\xi \in H, \|\xi\|=1} |\langle T\xi, \xi\rangle|.$$

PROOF. By the Cauchy-Schwartz inequality,

$$|\langle T\xi, \xi\rangle| \leq \|T\|\|\xi\|^2 \tag{2.6}$$

for any vector $\xi \in H$, so if we let

$$M := \sup_{\xi \in H, \|\xi\|=1} |\langle T\xi, \xi\rangle|$$

then $M \leq \|T\|$ and we need to show that equality holds.

Observe first that for T self-adjoint, $\langle T\xi, \xi\rangle$ is real. If now ξ and η are two unit vectors in H then it follows from a little algebra that

$$\langle T(\xi \pm \eta), \xi \pm \eta\rangle = \langle T\xi, \xi\rangle \pm 2\langle A\xi, \eta\rangle + \langle A\eta, \eta\rangle.$$

Subtracting one of these equations from the other and using the fact that the equalities only involve real numbers (By Exercise 2.18), gives

$$\langle T(\xi+\eta), \xi+\eta\rangle - \langle T(\xi-\eta), \xi-\eta\rangle = \mathrm{Re}\,\langle T(\xi+\eta), \xi+\eta\rangle - \langle T(\xi-\eta), \xi-\eta\rangle = 4\mathrm{Re}\langle T\xi, \eta\rangle.$$

Now by the ordinary triangle inequality and the Cauchy-Schwartz inequality we get

$$\begin{aligned}|\langle T(\xi+\eta),\xi+\eta\rangle - \langle T(\xi-\eta),\xi-\eta\rangle| &\leq |\langle T(\xi+\eta),\xi+\eta\rangle| + |\langle T(\xi-\eta),\xi-\eta\rangle| \\ &\leq M\|\xi+\eta\|^2 + M\|\xi-\eta\|^2.\end{aligned}$$

The last inequality using (2.6). *any* ζ (not just unit vectors).

By the Parallelogram Law, $\|\xi+\eta\|^2+\|\xi-\eta\|^2 = 2\|\xi\|^2+2\|\eta\|^2$. Since ξ and η were assumed unit vectors, this gives, putting everything together, that $4|\mathrm{Re}\,\langle T\xi,\eta\rangle| \leq 4M$, so

$$|\mathrm{Re}\,\langle T\xi,\eta\rangle| \leq M \tag{2.7}$$

for any pair of unit vectors $\xi,\eta \in H$.

Now for a suitable complex number, $e^{i\theta}$, we have $e^{i\theta}\langle T\xi,\eta\rangle$ is real and positive. This equals $\langle T\xi, e^{-i\theta}\eta\rangle$. Since (2.7) holds for all unit vectors η and in particular for $e^{i\theta}\eta$ for any η, $|\langle T\xi,\eta\rangle| \leq M$ holds for any unit vectors ξ,η. If in this expression we put $\eta = \frac{T\xi}{\|T\xi\|}$ for an arbitrary unit vector ξ we get that $\|T\xi\| \leq M$, whence, taking sup over ξ we get $\|T\| \leq M$ as required.

□

EXERCISE 2.20. This exercise addresses *polar decompositions* of operators $T \in \mathbb{B}(H)$ on a Hilbert space.

a) A *partial isometry* $u : H \to H$ is a bounded operator which is an isometry from $\ker(u)^\perp$ to its range. Prove that u is a partial isometry if and only if $p := uu^*$ and $q := u^*u$ are projections (that is, $p^2 = p$ and $p = p^*$, and similarly for q.)
b) Prove that if u is a partial isometry then $\mathrm{ran}(u) = \mathrm{ran}(uu^*)$ and $\ker(u) = \ker(u^*u)$.
c) If T is a bounded operator on a Hilbert space, let $|T| := (T^*T)^{\frac{1}{2}}$. The existence of such 'operator square roots' requires some spectral theory, but for purposes of the exercise, the reader may assume for the moment merely that $|T|$ is an operator such that $|T|^2 = T^*T$. Prove that $\|T\xi\| = \||T|\xi\|$ for any $\xi \in H$. (In particular, $\ker(|T|) = \ker(T)$ follows.)
d) Show that the restriction of $|T|$ to $\ker(T)^\perp$ maps into $\ker(T)^\perp$, and that the range of this restricted operator is dense in $\ker(T)^\perp$. (*Hint.* Note first that $\mathrm{ran}(T^*T) \subset \mathrm{ran}((|T|) = \ker(|T|)^\perp$, since $|T|^2 = T^*T$. On the other hand $\mathrm{ran}(|T|) = \ker(|T|)^\perp = \ker(T)^\perp$.)
e) Define $u\colon \ker(T)^\perp \to \mathrm{ran}(T)$ by defining it on the dense subspace $\mathrm{ran}(|T|) \subset \ker(T)^\perp$ in the way it has to be defined, *i.e.* so that $T\xi = U|T|\xi$. Show that this densely defined operator is *isometric.*
f) Complete the proof of the Polar Decomposition theorem: that if $T \in \mathbb{B}(H)$ is any bounded operator on H, the there exists a partial isometry u with initial space $\ker(T)^\perp$ and final space $\overline{\mathrm{ran}(T)}$ such that $T = U|T|$.

C-algebras of equivalence relations on finite sets*

When one multiplies two matrices, one does by the formula

$$(A\cdot B)_{ij} = \sum_k A_{ik}B_{kj}.$$

If one considers A and B to be functions on the finite set $X = \{1,\ldots,n\}$, we can write this in the form

$$(A\cdot B)(x,y) = \sum_z A(x,z)B(z,y).$$

The adjoint of a matrix in this notation is given by

$$A^*(x,y) = \overline{A(y,x)}.$$

Now let $\sim$ be any equivalence relation on a finite set X, with graph $G := \{(x,y) \mid x \sim y\}$. We show that G has a natural (finite-dimensional) C*-algebra associated with it.

Indeed, consider the collection of 'matrices', or functions on $X \times X$, which are supported on G. It forms a *-algebra by simply restricting the formula for matrix multiplication to G:

$$(A \cdot B)(x,y) = \sum_{z \sim x} A(x,z)B(z,y).$$

The formula makes sense as $\sim$ is transitive, so that since $(x,y) \in G$, $(x,z) \in G$ implies $(z,y) \in G$, so that any (z,y) appearing in the sum is in G, and hence in the domain of B.

The adjoint is given by $A^*(x,y) = \overline{A(y,x)}$, it makes sense since $\sim$ is symmetric.

The *-algebra just defined is a *-subalgebra of $\mathbb{B}\big(l^2(X)\big)$, in an obvious way (by extending functions on $G \subset X \times X$ by zero to functions on $X \times X$.) Since we are in a finite-dimensional situation, this *-algebra is automatically complete.

So we obtain a C*-algebra, the C*-algebra $C_r^*(G)$ of the equivalence relation G.

Data such as the number of equivalence classes, and the size of the equivalence classes, can be clearly read off the structure of $C_r^*(G)$ due to the following simple Proposition.

PROPOSITION 2.21. *In the above notation,*

$$C_r^*(G) \cong \bigoplus_{[x] \in X/\sim} M_{n(x)}(\mathbb{C}),$$

where $n(x)$ is the cardinality of the equivalence class $[x]$.

.

We leave the easy proof to the reader. One of the most important themes of this book is the association of a C*-algebra to various kinds of equivalence relations, groups, group actions, and groupoids. The above is the simplest example of this process.

EXERCISE 2.22. Suppose that k is a measurable function on a measure space (X,μ) such that for some constant C, and almost all $x \in X$, the functions $y \mapsto k(x,y)$ are in $L^1(X)$, and have L^1-norm $\leq C$, and, similarly for a.e. y. Prove that the integral operator I_k is well-defined and bounded and has norm $\leq C$.

3. Group C*-algebras

In this section, we explain an important construction that associates a C*-algebra to a *group*. The C*-algebra's structure will eventually be shown to reflect in a delicate way the representation theory of the group.

In the second part of this section, we discuss the examples where the group is the integers and the circle, respectively. We will defer a discussion of the group $\mathbb{R}$ of real numbers to Section 11.

The construction applies to second countable, locally compact, topological groups, for example $G = \mathbb{R}$, or $G = \mathbb{T}$, or $G = \mathbb{Z}$. In this book, we will be focusing on basic examples, rather than giving an account of the general theory. But some of the key constructions are rather general. Any locally compact group has an essentially unique Borel measure μ which is left-translation-invariant, in the sense that the left translation maps $G \to G$ elements of G, are measure-preserving.

Let f be a continuous function on G of compact support on G. Then f determines a *convolution operator* $\lambda(f)\colon L^2(G)\to L^2(G)$,

$$\big(\lambda(f)\xi\big)(g) := \int_G f(h)\xi(h^{-1}g)d\mu(h) \tag{3.1}$$

with μ Haar measure on G, a Borel measure on G which is invariant under the left translation action of G. Haar measure is guaranteed to exist by the locally compact assumption.

Note the integrand of (3.1) is a compactly supported function, for each g, so the integral converges absolutely. Furthermore, if ξ is compactly supported, then $\lambda(f)\xi$ is also compactly supported, so in particular $\lambda(f)\xi\in L^2(G)$ for any compactly supported, continuous ξ on G.

Proposition 3.1. *$\lambda(f)$ extends continuously to is a bounded operator $L^2(G)\to L^2(G)$, and $\|(\lambda(f)\|\le \|f\|_{L^1(G)}$.*

Proof. Let ξ,η be a pair of compactly supported continuous functions on G. Then by the definitions

$$\langle\lambda(f)\xi,\eta\rangle = \int_G\left(\int_G \overline{f(h)\xi(h^{-1}g)}\;dh\right)\cdot\eta(g)\;dg \tag{3.2}$$

and we can switch the order of integration by Fubini's Theorem, giving

$$= \int_G \overline{f(h)}\;\left(\int_G\overline{\xi(h^{-1}g)}\eta(g)dg\right)dh \tag{3.3}$$

Set $U_h\xi(g) := \xi(h^{-1}g)$. Then $U_h\xi\in L^2(G)$ and $\|U_g\xi\| = \|\xi\|$. For each $h\in G$, the Cauchy-Schwartz inequality gives

$$|\int_G\overline{\xi(h^{-1}g)}\eta(g)dg| = |\langle U_h\xi,\eta\rangle|\le\|U_h\xi\|\cdot\|\eta\| = \|\xi\|\cdot\|\eta\|.$$

Putting everything together we get

$$\begin{aligned}|\langle\lambda(f)\xi,\eta\rangle| &\le \int_G|f(h)|\cdot|\int_G\xi(h^{-1}g)\eta(g)dg|\;dh \\ &\le \|\xi\|\cdot\|\eta\|\cdot\int_G|f(h)|dh = \|f\|_{L^1(G)}\cdot\|\xi\|\cdot\|\eta\|.\end{aligned} \tag{3.4}$$

Now set

$$\eta := \lambda(f)\xi$$

to derive

$$\|\lambda(f)\xi\|^2 = \langle\lambda(f)\xi,\lambda(f)\xi\rangle\le\|f\|_{L^1(G)}\cdot\|\xi\|\cdot\|\lambda(f)\xi\|, \tag{3.5}$$

divide both sides by $\|\lambda(f)\xi\|$ to get

$$\|\lambda(f)\xi\|\le\|f\|_{L^1(G)}\cdot\|\xi\|. \tag{3.6}$$

Since compactly supported ξ are dense in $L^2(G)$, (3.6) implies that $\lambda(f)$ extends continuously a linear map $L^2(G)\to L^2(G)$, with operator norm $\le\|f\|_{L^1(G)}$, as claimed.

□

Definition 3.2. The C*-algebra generated by the convolution operators $\lambda(f)$, as f ranges over $C_c(G)$, is the *reduced C*-algebra of* G and is written $C^*(G)$.

The reduced C*-algebra of a group G is a completion of an intrinsically defined *-algebra, namely of $C_c(G)$ under *convolution* of functions (not pointwise multiplication). Convolution is defined for $f_1,f_2\in C_c(G)$, by

$$(f_1*f_2)(g) := \int_G f_1(h)f_2(h^{-1}g)\,d\mu(h). \tag{3.7}$$

It is clear that $f_1 * f_2 \in C_c(G)$ if $f_1, f_2 \in C_c(G)$. It is easy to check that convolution is an associative bilinear operation.

The *modular function* on a locally compact group G is the function $\delta\colon G \to \mathbb{R}^*_+$ defined by the equation

$$\mu(Ag) = \delta(g^{-1}) \cdot \mu(A)$$

for any $A \subset G$ measureable, with μ Haar measure on G. If $f\colon G \to \mathbb{C}$is an integrable function then

$$\delta(g) \cdot \int_G f(gh)\, d\mu(g) = \int_G f(g)\, d\mu(g).$$

The map δ is a group homomorphism. A group is *uni-modular* if $\delta = 1$. Almost all the groups we will consider in this book are uni-modular: for example, discrete groups, compact groups, and abelian groups, are all uni-modular.

The reason we bring up the modular function, is that the operator adjoint on $f \in C_c(G) \subset \mathbb{B}\big(L^2(G)\big)$ corresponds to the following adjoint, intrinsically defined on $C_c(G)$:

$$f^*(g) = \delta(g^{-1}) \cdot \overline{f(g^{-1})}. \tag{3.8}$$

Exercise 3.3. Check that the convolution formula (3.7) matches composition of operators on $L^2(G)$, *i.e.* that

$$\lambda(f_1 * f_2) = \lambda(f_1) \cdot \lambda(f_2) \in \mathbb{B}\big(L^2(G)\big).$$

Check as well that the adjoint (3.8) agrees with the operator adjoint in $\mathbb{B}\big(L^2(G)\big)$.

The following exercise gives an example of a non-unimodular group.

Exercise 3.4. Let G be the upper triangular group

$$G := \{g = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \mid a, b \in \mathbb{R},\ a > 0\} \subset \mathrm{SL}_2(\mathbb{R})$$

of matrices in $\mathrm{SL}_2(\mathbb{R})$. It has evident coordinates a, b putting it into bijective correspondence with $\mathbb{R}^*_+ \times \mathbb{R}$. Compute the Haar measure on G in these coordinates and compute the modular function

$$\delta\colon G \to \mathbb{R}^*_+.$$

Proceeding with the general theory, the regular representation λ can be viewed as a *-algebra map $C_c(G) \to \mathbb{B}\big(L^2(G)\big)$. Exercise 3.9 below shows that this is an injection. Hence $C^*_r(G)$ can be considered the completion of the *-algebra $C_c(G)$ (that is, with its intrinsically defined convolution multiplication, and adjoint), with an appropriate norm.

In the important case where G is a *discrete* group, we can represent an element of $C_c(G)$, often alternatively denoted $\mathbb{C}[G]$, and called the *complex group algebra* of G, in the form

$$f = \sum_{g \in G} a_g [g],$$

where the sum is finite, each $a_g \in \mathbb{C}$, and where $[g]$ means point mass at g: the function, often written δ_g, which is 1 at g and 0 otherwise.

Thus, f is a function on G whose value at g is a_g.

If we are considering f as an operator on $l^2(G)$, by the regular representation λ, then $\lambda([g])$ (or $\lambda(g)$), means the unitary operator of left translation by g, $\lambda([g])\xi(h) = \xi(g^{-1}h)$.

It is easy to check that convolution multiplication in this notation becomes the 'obvious' multiplication of such expressions, using the rules that

$$[g] * [h] = [gh] \quad [g]^* = [g^{-1}].$$

Thus, for example

$$(\sum_{g\in G} a_g[g]) * (\sum_g b_g[g]) = \sum_{g,h} a_g b_h [gh],$$

and re-arranging the sum gives $= \sum_{g\in G}\big(\sum_{h\in G} a_h b_{h^{-1}g}\big)[g]$, which re-produces the convolution formula (3.7).

Similarly, the adjoint is given by $(\sum_{g\in G} a_g[g])^* = \sum_{g\in G} \overline{a_g}[g^{-1}]$.

EXAMPLE 3.5. Let G be the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$.

Then the reduced C*-algebra $C^*(G) = \mathbb{C}[G]$ is finite dimensional, since $\mathbb{C}[G]$ is, and consists of all combinations $\sum_{k=0}^{n-1} \lambda_k [k + n\mathbb{Z}]$.

The regular representation $\lambda\colon C^*_r(G) \to \mathbb{B}(l^2G) \cong \mathbb{B}(\mathbb{C}^n) \cong M_n(\mathbb{C})$ maps the generator $1 + n\mathbb{Z}$ to the shift matrix

$$U := \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & 1 & 0 \end{bmatrix}.$$

Thus, as a *-subalgebra of $M_n(\mathbb{C})$, $C^*_r(G)$ is generated by the shift, and consists of all operators of the form $\sum_{k=0}^{n-1} \lambda_k U^k$.

To understand the structure of this C*-algebra, note that the shift U acting on $\mathbb{C}^n$ can be diagonalized. Its eigenvalues are precisely the nth roots of unity, $1, \omega, \omega^2, \dots \omega^{n-1}$, and the eigenvector v_k corresponding to the eigenvalue ω^k is given by

$$v_k = (\omega^{-k}, \omega^{-2k}, \omega^{-3k}, \dots, \omega^{-nk}) \in \mathbb{C}^n.$$

Let χ_k be the *character*, that is, group homomorphism to the circle group $\mathbb{T}$, given by

$$\chi_k\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{T}, \quad \chi_k(m + n\mathbb{Z}) := \omega^{km}.$$

Since χ_k is a function on the group, it defines an element of $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}] = C^*(\mathbb{Z}/n\mathbb{Z})$. In group-algebra notation

$$\chi_k = \sum_m \omega^{km} \cdot [m] \in \mathbb{C}[\mathbb{Z}/n\mathbb{Z}].$$

EXERCISE 3.6. Show that $\lambda(\chi_k)$ acts on $l^2(\mathbb{Z}/n\mathbb{Z})$ by the matrix

$$p_k = \begin{bmatrix} 1 & \omega^{-k} & & & \ddots \\ \omega^k & 1 & \omega^{-k} & & \\ \omega^{2k} & \omega^k & 1 & \omega^{-k} & \\ & \omega^{2k} & \ddots & & \\ & \ddots & & & \omega^{-k} \\ & & \omega^{2k} & \omega^k & 1 \end{bmatrix},$$

(where the ellipses indicate a similar pattern; the matex has ‘constant diagonals’) and check that the matrix

$$p_k := \frac{1}{n} \cdot \lambda(\chi_k)$$

is orthogonal projection to the eigenspace spanned by v_k.

Thus, the group C*-algebra $C^*_r(\mathbb{Z}/n)$ contains n orthogonal projections $p_0, p_1, \ldots, p_{n-1}$ which sum to the identity operator. And in this notation $U = \sum_{k=0}^{n-1} \omega^k p_k$ expresses U as a diagonal operator with respect to the basis $v_0, \ldots, v_{n-1}$: with respect to this basis, U is the diagonal matrix

$$U = \begin{bmatrix} 1 & & & \\ & \omega & & \\ & & \omega^2 & \\ & & & \omega^{n-1} \end{bmatrix}. \tag{3.9}$$

This makes it clear that the C*-algebra $C^*(U) = C^*(\mathbb{Z}/nZ)$ generated by U consists of diagonal matrices. We have shown that this C*-algebra contains, as well, the projections to the elements of this basis, and hence it is isomorphic to the C*-algebra of n-by-n diagonal matrices – *i.e.* is isomorphic to $\mathbb{C} \oplus \cdots \oplus \mathbb{C} = \mathbb{C}^n$ as C*-algebras.

The point of this example is to illustrate how the representation theory of the group $\mathbb{Z}/n\mathbb{Z}$, in this case, is reflected in the structure of the group C*-algebra $C^*(\mathbb{Z}/n\mathbb{Z})$.

Exercise 3.7. Let $G = \mathbb{T}$ be the circle.

a) Show that the characters $\chi_n(z) := z^n$, viewed as elements of $C^*(\mathbb{T})$, are projections: $\chi_n = \chi_n^*$ and $\chi_n * \chi_n = \chi_n$, and that $\chi_n * \chi_m = 0$ unless $n = m$.
b) Prove that the infinite sum $\sum_{n\in\mathbb{Z}} \chi_n$ does *not* converge in the norm in $C^*(\mathbb{T})$, but that $\sum_{n\in\mathbb{Z}} f * \chi_n$ *does* converge (in norm) in $C^*(\mathbb{T})$, for any $f \in C^*(\mathbb{T})$, with $f * \chi_n$ convolution multiplication (the multiplication in $C^*(\mathbb{T})$.)
c) Let $\mathbb{C}[x, x^{-1}]$ be the algebra of Laurent polynomials in x. Prove that the group algebra $\mathbb{C}[\mathbb{Z}]$ of the integers satisfies $\mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[x, x^{-1}]$ as *-algebras, with the adjoint defined on $\mathbb{C}[x, x^{-1}]$ by setting
$$(\sum_{n=-N}^{N} a_n x^n)^* := \sum_{n=-N}^{N} \overline{a}_n x^{-n}.$$
d) Inside the algebra $\mathbb{C}[x]$ of polynomials in one variable, with complex coefficients, consider the ideal $\langle x^n - 1\rangle$ generated by $x^n - 1$. Then $\mathbb{C}[x]/\langle x^n - 1\rangle$ is an algebra with coset multiplication. Check that if we define, for a polynomail $\sum a_k x^k$,
$$(\sum_k a_k x^k)^* := \sum_k \overline{a}_k x^{k(n-1)},$$
then the ideal $\langle x^n - 1\rangle$ is closed under $*$ and so $*$ descends to an 'adjoint' on the $\mathbb{C}[x]/\langle x^n - 1\rangle$. Prove that $C^*(\mathbb{Z}/n\mathbb{Z})$ is isomorphic, as a *-algebra, to $\mathbb{C}[x]/\langle x^n - 1\rangle$ (by mapping x to U.)

Exercise 3.8. Let G be a finite group and $f \in \mathbb{C}[G] = C^*(G)$ a function which is constant on conjugacy classes in G. (Such functions are called *class functions.*) Prove that f is in the centre of $C^*(G)$, that is f commutes with all other elements of $C^*(G)$.

Exercise 3.9. Let G be any discrete group.

a) In $l^2(G)$, let $\tau\colon \mathbb{C}[G] \to \mathbb{C}$ be the map $\tau(\sum \lambda_[g]) := \lambda_e$. Prove that τ extends to a continuous linear functional $C^*(G) \to \mathbb{C}$, and that τ is a *trace*: $\tau(ab) = \tau(ba)$ for all $a \in A$. (*Hint.* Let $e_0 \in l^2(G)$ be the point mass at the identity of the group, check that $\tau(T) = \langle \lambda(T)e_0, e_0\rangle$ for all $T \in \mathbb{C}[G]$.)
b) Prove that if $T \neq 0$ is an element of $C^*_r(G)$ of the form $T = S^*S$, for some $S \in C^*_r G$, then $\tau(T) > 0$.
c) Prove that the map $\lambda\colon \mathbb{C}[G] \to C^*(G)$ is an injective map of *-algebras.

d) Suppose that $H \subset G$ is a finite subgroup. Let
$$p := \frac{1}{|H|} \sum_{h \in H} [h].$$
Show that p is a projection in C^*G), and that $\tau(p) = \frac{1}{|H|}$.

e) Show that p_H is projection onto the subspace of $l^2(G)$ consisting of functions $\xi \in l^2(G)$ which are constant on each H-coset in G (so for example if G is finite and $H = G$ then p_G is projection to the constant functions in $l^2(G)$).

It is a conjecture of Kadison and Kaplansky that if G has no torsion, then τ only takes integral values on projections. The problem is as yet unsolved, one way or another, with some of the strongest results being based on cases of the Baum-Connes conjecture (discussed in the last section of the book.)

Exercise 3.10. If H is a Hilbert space and $A \subset \mathbb{B}(H)$ is any self-adjoint set of bounded operators on H, then the *commutant* A' of A is the collection of bounded operators T on H which commute with all elements of A.

a) Prove that A' is a norm-closed *-subalgebra of $\mathbb{B}(H)$, and hence is a C*-algebra.

b) Check that if G is finite and
$$\rho(g)\colon l^2(G) \to l^2(G), \;\; \rho(g)(e_h) := e_{hg^{-1}},$$
is the right regular representation, then $C^*_r(G) = \{\rho(g) \mid g \in G\}'$.

c) Prove that if A is any C*-subalgebra of $\mathbb{B}H)$, where H is a finite-dimensional Hilbert space, then
$$A'' = A.$$

Remark 3.11. If G is infinite, the commutant $W^*(G) = \{\rho(g) \mid g \in G\}'$ is called the *group von Neumann algebra of* G. It contains $C^*(G)$ but is closed in the weak operator topology, which we will not discuss much in this book.

Exercise 3.12. Prove that a bounded operator $T \in \mathbb{B}\big(l^2G)\big)$ is in the group von Neumann algebra $L(G)$ if and only if it's matrix representation in the canonical basis $\{e_g \mid g \in G\}$ has constant 'diagonals' (the 'diagonals' are the vectors $(T_{k,gk})_{k\in G}$, one for each g, where $T_{k,h} := \langle T(e_h), e_k\rangle$ as usual.

Exercise 3.13. Let G be any locally compact group. An alternative to forming the C*-algebra of G, is to form the *Banach algebra* $L^1(G)$.

Use Fubini's Theorem to prove that if $f_1, f_2 \in C_c(G)$ then $f_1 * f_2 \in L^1(G)$ and
$$\|f_1 * f_2\|_{L^1(G)} \leq \|f_1\|_{L^1(G)} \|f_2\|_{L^1(G)}.$$
Deduce that convolution extends continuously to a multiplication on $L^1(G)$, and that with this multiplication and the L^1-norm, $L^1(G)$ is a Banach algebra.

Actually, $L^1(G)$ has a natural adjoint as well, and it is easy to check that $\|f^*\|_{L^1(G)} = \|f\|_{L^1(G)}$. Hence $L^1(G)$ has the structure of a Banach *-algebra. The regular representation λ extends continuously to a contractive map $L^1(G) \to C^*(G)$ of Banach *-algebras.

The C*-algebras $C^*(G)$, although much less concrete, will turn out to have a number of much nicer properties, however, than the Banach *-algebra $L^1(G)$.

4. C*-algebras of the integers and the circle

The structure of the C*-algebras of the groups of the integers and the group consisting of the points of the unit circle $\mathbb{T}$ of the complex plane, is completely elucidated by the classical *Fourier transform*, which we describe in slightly more general terms as follows.

If G is a locally compact, second countable, abelian group, its *Pontryagin dual* is the group $\widehat{G}$ of continuous group homomorphisms $\chi\colon G \to \mathbb{T}$ (they are called *characters*). The group structure on $\widehat{G}$ is by pointwise multiplication of characters $(\chi_1 \cdot \chi_2)(g) = \chi_1(g)\chi_2(g) \in \mathbb{T}$. The identity element is the trivial character, which we will usually denote $\epsilon\colon G \to \mathbb{T}$, $\epsilon(g) = 1 \in \mathbb{T}$ for all g. If χ is a character, then $\chi^*(g) := \overline{\chi(g)}$ is also a character, since complex conjugation on the circle is a (continuous) group homomorphism $\mathbb{T} \to \mathbb{T}$. And $\overline{\chi} \cdot \chi = \epsilon$. Hence characters from a group. It is clearly abelian. We topologize $\widehat{G}$ with the *compact-open* topology, with basis $U(K, \epsilon, \chi_0) := \{\chi \in \widehat{G} \mid |\chi(g) - \chi_0(g)| < \epsilon \ \forall g \in K\}$, as K range over compact subsets of G, $\chi_0 \in \widehat{G}$, and $\epsilon > 0$.

Lemma 4.1. *If G is compact, then $\widehat{G}$ is discrete. If G is discrete, then $\widehat{G}$ is compact.*

Proof. If G is compact then $\{\chi \in \widehat{G} \ \ |\chi(g) - 1| < \frac{1}{2}, \ \forall g \in G\}$ is a neighbourhood of the trivial character $\epsilon \in \widehat{G}$ containing only one point, namely ϵ itself, since the image of any character is a subgroup of the circle, and there are no subgroups of $\mathbb{T}$ which lie entirely within $\frac{1}{2}$ of 1. This shows that $\widehat{G}$ is discrete if G is compact.

On the other hand, if G is discrete, then the continuity requirement on a character becomes trivial, and it is then easy to check that $\widehat{G}$ embeds continuously as a closed subset of $\prod_G \mathbb{T}$, which is compact by Tychonoff's Theorem. Hence $\widehat{G}$ is compact if G is discrete.

$\square$

Example 4.2. Let $G = \mathbb{T}$, the circle. To each integer $n \in \mathbb{Z}$ we associate the character $\chi_n(z) := z^n$. This gives an isomorphism $\widehat{G} \cong \mathbb{Z}$.

If G is locally compact abelian, the *Fourier transform* for G involves the following construction, which can be applied to various classes of functions, with various results. Suppose that $f \in C_c(G)$. We let

$$\hat{f}(\chi) := \int_G f(g)\overline{\chi(g)}d\mu(g), \tag{4.1}$$

the integral with respect to Haar measure on G. Then it is immediate that

$$|\hat{f}(\chi)| \leq \|f\|_{L^1(G)}.$$

And if $|\chi(g) - \chi_0(g)| < \epsilon$ on $\mathrm{supp}(f)$, then

$$|\hat{f}(\chi) - \hat{f}(\chi_0)| < \|f\|_{L^1(G)}\epsilon.$$

Hence $\hat{f}$ is continuous on $\widehat{G}$ if $f \in C_c(G)$, and more generally, if $f \in L^1(G)$.

Now if G is compact, any two distinct characters χ_1, χ_2 of G, viewed as vectors in $L^2(G)$, are *orthogonal*. Indeed, if $h \in G$, then by invariance of Haar measure

$$\begin{aligned}\langle \chi_1, \chi_2 \rangle = \int_G \chi_1(g)\overline{\chi_2(g)}d\mu(g) &= \int_G \chi_1(hg)\overline{\chi_2(hg)}d\mu(g) \\ &= \chi_1(h)\chi_2(h)^{-1}\int_G \chi_1(g)\overline{\chi_2(g)}d\mu(g) = \chi_1(h)\chi_2(h)^{-1}\langle \chi_1, \chi_2 \rangle,\end{aligned}$$

which implies that either $\chi_1 = \chi_2$ or the product is zero.

Consequently: if $f \in L^2(G)$, since (4.1) implies instantly that $\hat{f}(\chi) = \langle f, \chi \rangle$, we get that

$$\sum_{\chi \in \widehat{G}} |\hat{f}(\chi)|^2 \leq \|f\|_{L^2(G)}. \tag{4.2}$$

from Bessel's inequality in Hilbert space theory. And in particular, $\hat{f}(\chi) \to 0$ as $\chi \to \infty$ in the discrete space $\widehat{G}$. This shows that if $f \in C_c(G)$ then $\hat{f} \in C_0(\widehat{G})$, when G is compact.

Furthermore, as $\|\hat{f}\|_{C_0(\widehat{G})} \leq \|f\|_{L^1(G)}$, the map $f \mapsto \hat{f}$ extends to a contractive map $L^1(G) \to C_0(\widehat{G})$. The following is an easy exercise left to the reader.

EXERCISE 4.3. If G is any locally compact abelian group and $f_1, f_2 \in L^1(G)$ then $\widehat{f_1 * f_2} = \hat{f}_1\hat{f}_2$.

Hence $f \mapsto \hat{f}$ is a contractive homomorphism of Banach algebras $L^1(G) \to C_0(\widehat{G})$, when G is compact.

If G is not compact, but is discrete, then $\widehat{G}$ is compact, and the above arguments show that if $f \in L^1(G)$ then $\hat{f} \in C(\widehat{G})$, and that $\|\hat{f}\|_{C(\widehat{G})} \leq \|f\|_{L^1(G)}$. Therefore, putting things together:

PROPOSITION 4.4. *If G is compact or discrete, the Fourier transform defines a contractive homomorphism of Banach *-algebras $L^1(G) \to C_0(\widehat{G})$.*

We will see shortly that the Fourier transofrm extends continuously to an *C^*-algebra isomorphism* $C^*(G) \to C_0(\widehat{G})$.

EXAMPLE 4.5. If $G = \mathbb{T}$ then $\widehat{\mathbb{T}} \cong \mathbb{Z}$ with the integer n corresponding to the character $\chi_n(z) := z^n$ of $\mathbb{T}$. The Fourier transform in this notation is

$$\hat{f}(n) = \int_{\mathbb{T}} f(z)z^{-n}d\mu(z), \quad f \in L^1(\mathbb{T}) \subset C^*_r(\mathbb{T}).$$

If $G = \mathbb{Z}$, then $\widehat{\mathbb{Z}} \cong \mathbb{T}$ with $z \in \mathbb{T}$ corresponding to the character $\chi_z(n) := z^n$. The Fourier transform for the integers is given by

$$\hat{f}(z) = \sum_{n\in\mathbb{Z}} f(n)z^{-n}, \quad f \in l^1(\mathbb{Z}) \subset C^*_r(\mathbb{Z}).$$

If G is a *compact* (abelian) group, then the characters $\{\chi\}_{\chi\in\widehat{G}}$ form an orthonormal *basis* for $L^2(G)$. Hence the inequality in (4.2) is actually an equality. So when G is compact, the restriction of the Fourier transform to $L^2(G) \subset L^1(G)$ determines an isometry between Hilbert spaces which we denote by

$$F_G\colon L^2(G) \to l^2(\widehat{G}).$$

For example,

$$F_{\mathbb{T}}\colon L^2(\mathbb{T}) \to L^2(\widehat{\mathbb{T}}) \cong l^2(\mathbb{Z})$$

maps a function in $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$ to the bi-infinite sequence $(\hat{f}(n))_{n\in\mathbb{Z}}$ of its Fourier coefficients with respect to the standard orthonormal basis $\{z^n\}_{n\in\mathbb{Z}}$ for $L^2(\mathbb{T})$.

When G is discrete (abelian), $\hat{f}(\chi) = \sum_{g\in G} f(g)\chi(g)$ for $f \in C_c(G) = \mathbb{C}[G]$, $\chi \in \widehat{G}$. Thus $\hat{f} = \sum_{g\in G} f(g)\check{g}$ where $\check{g}\colon \widehat{G} \to \mathbb{T}$ is the function $\check{g}(\chi) := \chi(g)$. Since $\check{g}$ so defined is a character of the compact dual group $\widehat{G}$, and all characters appear this way, and make an orthonormal basis of $L^2(\widehat{G})$, we again obtain that

$$\|\hat{f}\|_{L^2(\widehat{G})} = \sum_{g\in G} |f(g)|^2 = \|f\|_{l^2(G)}.$$

Thus, again, we see that F_G induces, now by extension by continuity from $l^1(\mathbb{Z}) \subset l^2(\mathbb{Z})$, a unitary isomorphism $F_G\colon l^2(G) \to L^2(\widehat{G})$, for any *discrete* G. So in either case, Fourier transform induces a unitary isomorphism.

For example, if $G = \mathbb{Z}$, then $F_G\colon l^2(\mathbb{Z}) \to L^2(\mathbb{T})$ maps a sequence $(a_n)_{n\in\mathbb{Z}}$ to the L^2-function $\sum_{n\in\mathbb{Z}} a_n z^n$.

Although one might expect that $F_{\widehat{G}}$ inverts F_G, this *almost* happens, but not quite. The statement is that

$$F_{\widehat{G}} \circ F_G = S_G.$$

where $S_G\colon L^2(G) \to L^2(G)$ is the self-adjoint unitary $(S_G f)(g) = f(g^{-1})$.

Similarly, still assuming G is compact, $F_G \circ F_{\widehat{G}} = S_{\widehat{G}}$. We conclude that if G is either compact or discrete, then since S_G is a unitary isomorphism, $F_G\colon L^2(G) \to L^2(\widehat{G})$ is a unitary isomorphism as well, and $F_G^* = F_G^{-1} = F_{\widehat{G}} \circ S_{\widehat{G}}$.

Lemma 4.6. *If $\lambda(f) \in C^*_r(G) \subset \mathbb{B}\big(L^2(G)\big)$ then $F_G\lambda(f)F_G^* = M_{\hat{f}}\colon L^2(\widehat{G}) \to L^2(\widehat{G})$, with $M_{\hat{f}}$ the multiplication operator by $\hat{f} \in C_0(\widehat{G})$.*

Theorem 4.7. *If G is any compact or discrete abelian group, then the Fourier transform $L^1(G) \to C_0(\widehat{G})$ of Proposition 4.4, extends continuously to a C*-algebra isomorphism $C^*_r(G) \to C_0(\widehat{G})$; moreover, this C*-algebra isomorphism is impiemented by unitary conjugation with the Fourier transform, as a unitary $L^2(G) \to L^2(\widehat{G})$.*

The statement contains the important fact that

$$\|\lambda(f)\| = \sup_{\chi \in \widehat{G}} |\hat{f}(\chi)|,$$

for all $\lambda(f) \in C^*_r(G)$ (in particular for $f \in C_c(G)$, for example.)

Remark 4.8. Although the space $\widehat{G}$ of characters of G, group homomorphisms $G \to \mathbb{T}$, may be rather trivial, and hence not a useful thing to consider, when G is not abelian, so that $C_0(\widehat{G})$ no longer is a very useful thing to think about, $C^*_r(G)$ always is defined, and, of course, agrees (by above) with $C_0(\widehat{G})$ in the commutative case. So this is an example where (noncommutative) C*-algebras may be of use – to study representation theory of nonabelian groups.

Exercise 4.9. If $f \in l^2(\mathbb{Z})$ then $(f * \xi)(n) := \sum_{n\in\mathbb{Z}} f(m)\xi(n-m)$ is absolutely convergent for all $\xi \in l^2(\mathbb{Z})$, and, moreover, $f * \xi \in l^2(\mathbb{Z})$.

Denote by $\lambda(f)$ the convolution operator $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$, $\lambda(f)\xi := f*\xi$. It is thus well-defined and linear. But it is not necessarily bounded. Show that this occurs if $f(n) = \frac{1}{n}$.

In particular, f is in $l^2(\mathbb{Z})$ but $\lambda(f) \notin C^*_r(\mathbb{Z})$. (*Hint.* It might be helpful to note that $\hat{f}(z) = -\log(1-z)$, which is Borel on the circle, but not continuous, and not bounded, so multiplication by it does not define a bounded operator.)

Exercise 4.10. Let T be a bounded operator on $l^2(\mathbb{Z})$ which is an element of $C^*(\mathbb{Z})$. Represent T as a $\mathbb{Z}$-by-$\mathbb{Z}$ matrix in the usual way with $T_{n,m} := \langle T(e_m), e_n\rangle$. Let $f(n) := T_{n,0}$.

a) Prove that $T_{n,m} = f(n-m)$. (T has constant ‘diagonals.’)
b) $f \in l^2(\mathbb{Z})$ and $T = \lambda(f)$.

So for every element $T \in C^*_r(\mathbb{Z})$ there is a (unique) $f \in l^2(\mathbb{Z})$ such that $T = \lambda(f)$. (Exercise 4.9 shows that not every $f \in l^2(\mathbb{Z})$ corresponds to a bounded operator T, however.)

c) Give an example of $f \in l^2(\mathbb{Z})$ such that $\lambda(f)$ is bounded, but $\lambda(f) \notin C^*_r(\mathbb{Z})$.

The previous exercises show that whether or not an operator $\lambda(f) \in C^*(\mathbb{Z})$, for given $f \in l^2(\mathbb{Z})$, depends delicately not just on the *set of values of* f, but on how precisely $\lambda(f)$ *acts* as an operator on $l^2(\mathbb{Z})$ (which is a question of how exactly those values are distributed).

Exercise 4.11. Suppose that $f \in L^2(\mathbb{T})$, so $f = \sum_{n\in\mathbb{Z}} f(n)z^{-n}$. an L^2-convergent series. Even if $f \in C(\mathbb{T})$, it is not necessarily true that the partial sums of the Fourier series of f converge uniformly to f. Give an example.

Exercise 4.12. Prove that the von-Neumann algebra $C_r^*(\mathbb{Z})''$ of the integers is isomorphic by Fourier transform to $L^\infty(\mathbb{T})$. (This shows that there is a plentiful supply of convolution operators $\lambda(f)\colon l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$, for $f \in l^2(\mathbb{Z})$, which are bounded operators, but are not in $C_r^*(\mathbb{Z})$).

Exercise 4.13. One quite strong, but sufficient, condition, *on the set of values* of $f \in l^2(\mathbb{Z})$, that $\hat{f} \in C(\mathbb{T})$, equivalently, that $\lambda(f) \in C^*(\mathbb{Z})$, is that $\forall k = 0, 1, 2, \ldots$, there exists $C_k \geq 0$ such that $|f(n)| \leq C_k |n|^{-k}$ for for all $n \in \mathbb{Z}$.

Such functions f are said to be of *rapid decay*. Prove that if f has rapid decay, then $\hat{f} \in C^\infty(\mathbb{T})$ (and in particular $\hat{f}$ is continuous on $\mathbb{T}$), and $\lambda(f) \in C_r^*(\mathbb{Z})$.

5. C*-algebras of finite groups

Let G be a finite (or any discrete) group. A unitary *representation* of G is a group homomorphism $\pi\colon G \to \mathbf{U}(H)$, where $\mathbf{U}(H)$ is the unitary group of a Hilbert space H. We will normally only deal with unitary representations in this book; we will refer to them usually as just 'reprsentations.'

A representation is *irreducible* if it has no G-invariant subspace. Two representations ρ_1, ρ_2 of G on H_1, H_2 are *equivalent* (or unitarily equivalent) if there is a unitary $U\colon H_1 \to H_2$ intertwining them: *i.e.*

$$U\rho_1(g)U^* = \rho_2(g), \quad \forall g \in G.$$

In this section we are going to briefly describe some general features of the representation theory of finite groups, and show how it ties in with the structure of their group C*-algebras.

Exercise 5.1. Let $\pi\colon G \to \mathbf{U}(H)$ be a unitary representation of G on a Hilbert space H.

a) Prove that if π is irreducible, then H is finite-dimensional. (*Hint.* Any orbit of G acting on H is finite, and so spans a finite- dimensional, invariant subspace.)

b) Prove that if π is any finite-dimensional unitary representation of G, then π is completely reducible: there exists a decomposition $H = H_1 \oplus \cdots H_n$ and irreducible representations $\pi_i\colon G \to \mathbf{U}(H_i)$, such that π is unitarily equivalent to the direct sum representation $\pi_1 \oplus \cdots \oplus \pi_n$.

Lemma 5.2. *(Schur's Lemma). If (π, H) is an irreducible representation of G finite, then the only operators T on H commuting with $\pi(G)$ are scalar multiples of the identity operator 1_H.*

Proof. Suppose that T commutes with G, that is, $\pi(g)T = T\pi(g)$ for all $g \in G$. Since $\ker(T)$ is then a G-invariant subspace, it must be either $\{0\}$ or H, since G acts irreducibly. So assume T has no kernel, (or else it's the zero operator), so T must be invertible. Now T^*T and TT^* each commute with G as well, and are self-adjoint operators on H and hence can be diagonalized. Since any eigenspace H_λ of – say, T^*T – corresponding to eigenvalue λ must be left invariant under G, T^*T can have at most one eigenvalue, and hence is a multiple $\lambda \cdot 1_H$ of the identity operator 1_H. A similar argument applies to TT^* and gets that $TT^* = \mu 1$ for some μ. Since we are assuming $T \neq 0$, neither λ nor μ are zero, and must be equal, since the nonzero eigenvalues of AB are always the same as those of BA, for any operators A, B. Replacing T by a scalar multiple then makes it unitary. Now unitary operators can also be diagonalized, and the same argument as above shows that a unitary which commutes with G must be a multiple of the identity. Hence T is a multiple of the identity, as claimed.

$\square$

Lemma 5.3. *Let (π, H) be a finite-dimensional irreducible unitary representation of G. Then*

$$\frac{1}{|G|}\sum_{g \in G} \pi(g)u\pi(g)^* = \frac{\mathrm{Trace}(u)}{\dim H} \cdot 1, \tag{5.1}$$

for any $u \in \mathbb{B}(H)$, where $1 \in \mathbb{B}(H)$ is the identity operator.

PROOF. The group G acts by unitary conjugation on the C*-algebra $\mathbb{B}(H)$, and the content of Schur's Lemma is that the fixed-point *-subalgebra of this action is one-dimensional, spanned by the identity operator. Clearly the operator on the left hand side of (5.1) is G-fixed in this sense. Hence it is a scalar multiple $\lambda(u)$ of 1_H.

Taking the trace of each side of the equation

$$\frac{1}{|G|}\sum_{g\in G}\pi(g)u\pi(g)^* = \lambda(u)\cdot 1_H, \tag{5.2}$$

then gives

$$\mathrm{Trace}(u) = \lambda(u)\cdot \dim H$$

as required. □

The statement can be generalized to deal with *pairs* of representations.

Suppose that if $\pi_i\colon G \to \mathbf{U}(H_i)$ are irreducible representations. And let $u\colon H_1 \to H_2$ be any linear map. Define a linear map $T_u\colon H_1 \to H_2$ by

$$T_u := \frac{1}{|G|}\sum_{g\in G}\pi_2(g)u\pi_1(g)^* \tag{5.3}$$

LEMMA 5.4. *The operator T_u of (5.3) is zero if π_1 and π_2 are inequivalent representations, for any $u\colon H_1 \to H_2$.*

PROOF. Clearly $\pi_2(g)T_u = T_u\pi_1(g)$. If T_u were invertible, it would give rise to a unitary conjugacy between the representations. Hence T_u must be non-invertible for any u. On the other hand, $\ker(T_u)$ is a G-invariant subspace of H_1, and $\ker(T_u^*)$ is likewise G-invariant subspace of H_2. Assuming T_u is not the zero operator, this makes it invertible. This contradicts the assumption that $T_u \neq 0$. □

In the notation of the Lemma, let $\xi \in H_2, \eta \in H_1$. Define the rank-one operator

$$T_{\xi,\eta}\colon H_1 \to H_2, \quad T_{\xi,\eta}(\zeta) := \langle \eta, \zeta\rangle\, \xi. \tag{5.4}$$

Note that

$$\langle T_{\xi,\eta}(\eta'), \xi'\rangle = \overline{\langle \eta, \eta'\rangle}\cdot\langle \xi, \xi'\rangle. \tag{5.5}$$

Furthermore,

$$\pi_2(g)T_{\xi,\eta}\pi_1(g)^*(\zeta) = \langle \eta, \pi_1(g)^*\zeta\rangle\cdot\pi_2(g)\xi = \langle \pi_1(g)\eta, \zeta\rangle\cdot\pi_2(g)\xi,$$

hence

$$\pi_2(g)T_{\xi,\eta}\pi_1(g)^* = T_{\pi_2(g)\xi,\pi_1(g)\eta}.$$

Hence, applying (5.3) to $u := T_{\xi,\eta}$ as in (5.4) gives the operator equation

$$\frac{1}{|G|}\sum_{g\in G}T_{\pi_2(g)\xi,\pi_1(g)\eta} = 0. \tag{5.6}$$

Now evaluate the operator on the left-hand-side at a vector $\eta' \in H_1$, and take the inner product of the result with ξ'. With the above remarks (and recall that our inner products are conjugate linear in the first variable), we obtain

$$\frac{1}{|G|}\sum_{g\in G}\overline{\langle \pi_2(g)\xi, \xi'\rangle}\cdot\langle \pi_1(g)\eta, \eta'\rangle = 0. \tag{5.7}$$

which is an orthogonality statement for two functions on G, thought of as vectors in $l^2(G)$. Such functions, of the form

$$f^\pi_{\xi,\xi'}\colon G \to \mathbb{C}, \quad f^\pi_{\xi,\xi'}(g) = \langle \pi(g)\xi, \xi'\rangle$$

are called *matrix coefficients* (of the representation.) Our computations have shown that

$$\langle f^{\pi_2}_{\xi,\xi'}, f^{\pi_1}_{\eta,\eta'}\rangle = 0 \tag{5.8}$$

for any pair of inequivalent, irreducible representations π_1, π_2, and any vectors ξ, ξ', η, η', where the inner product is in the Hilbert space $l^2(G)$.

Finally, we return to the case $H_1 = H_2 = H$ and $\pi_1 = \pi_2 = \pi$. Set $u = T_{\xi,\eta}$ in Lemma 5.3. It is an easy exercise to check that $\mathrm{Trace}(T_{\xi,\eta}) = \langle \eta, \xi\rangle$. Hence

$$\frac{1}{|G|}\sum_{g\in G} T_{\pi(g)\xi,\pi(g)\eta} = \frac{\langle \eta,\xi\rangle}{\dim H}\cdot 1_H \tag{5.9}$$

with 1_H the identity operator on H. Applying this operator equation to a vector η', and taking product with a vector ξ', and proceeding as above in the case of two representations, we get the identity

$$\frac{1}{|G|}\sum_{g\in G} \overline{f^\pi_{\xi,\xi'}(g)}\cdot f^\pi_{\eta,\eta'}(g) = \frac{\langle \eta,\xi\rangle\langle \xi',\eta'\rangle}{\dim H} \tag{5.10}$$

We summarize:

Proposition 5.5. *Let G be a finite group.*

a) *Matrix coefficients of any two inequivalent irreducible representations of G are orthogonal to each other as vectors in $l^2(G)$.*
b) *If $\pi\colon G \to \mathbb{B}(H)$ is an irreducible representation and ξ, ξ', η, η' are vectors in H, then*

$$\langle f^\pi_{\xi,\xi'}, f^\pi_{\eta,\eta'}\rangle = \frac{|G|}{\dim H}\cdot \langle \eta,\xi\rangle \cdot \langle \xi',\eta'\rangle, \tag{5.11}$$

where the $f^\pi_{\xi,\xi'}$ etc. are the corresponding matrix coefficients, regarded as elements of $l^2(G)$.

The *character* of a finite-dimensional representation (π, H) is the conjugation-invariant function

$$\chi\colon G \to \mathbb{C}, \quad \chi_\pi(g) := \mathrm{Trace}\big(\pi(g)\big).$$

If $\xi_1, \ldots, \xi_n$ is an orthonormal basis for H then

$$\chi_\pi(g) = \sum_i f^\pi_{\xi_i,\xi_i}$$

from the definitions.

If (π_1, H_1) and (π_2, H_2) are two inequivalent representations, it follows from (5.8) that

$$\langle \chi_{\pi_1}, \chi_{\pi_2}\rangle = 0$$

by setting $\xi = \xi_i = \xi_i'$ and $\eta = \eta_j = \eta'$, for orthonormal bases $\xi_1, \ldots, \xi_n$ of H_2 and $\eta_1, \ldots \eta_m$ of H_1, and summing over i, j,

Similarly, for a single representation apply (5.10) to $\xi = \xi_i = \xi_i'$, and $\eta = \eta_j = \eta_j'$, for two indices i, j, and then sum over i, j to get:

PROPOSITION 5.6. *If (π_1, H_1) and (π_2, H_2) are inequivalent irreducible representations with characters χ_{π_1} and χ_{π_2}, viewed as vectors in $l^2(G)$, then*

$$\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = 0.$$

If (π, H) is an irreducible representation, then

$$\|\chi_\pi\|^2_{l^2(G)} = |G|.$$

Every equivalence class of irreducible representation has a uniquely defined character, since trace is invariant under conjugation. The results above show that the set $\{\chi_\pi \mid [\pi] \in \widehat{G}\}$ forms an orthonormal set of vectors in $l^2(G)$, and hence there are only finitely many of them. In particular, we conclude that a finite group has only finitely many equivalence classes of irreducible representation, *i.e.* $\widehat{G}$ has only finitely many points.

Suppose they are $[\pi_1[, \ldots, [\pi_n]$. Let (H, π) be any finite-dimensional representation. Since it is completely reducible, it can be written as a direct sum

$$\pi \cong \oplus_i n_i \, \pi_i.$$

of the irreducibles. The integers n_i determine π up to isomorphism. Now $\chi_\pi = \sum_i n_i \, \chi_{\pi_i}$. Since $\langle \chi_{\pi_i}, \chi_{\pi_j} \rangle = |G| \, \cdot \delta_{ij}$,

$$\langle \chi_\pi, \chi_{\pi_i} \rangle = n_i \, \cdot |G|$$

so that the character χ_π determines the multiplicities n_i and hence the representation, up to isomorphism.

The basic example is the regular representation $(\lambda, l^2(G))$. Note that its character χ_λ is supported at the identity of the group, and has the value $|G|$ there.

PROPOSITION 5.7. *A finite group has only finitely many unitary equivalence classes of irreducible unitary representations.*

If the irreducible representations of G are $[\pi_1], [\pi_2] \ldots$, then

$$\lambda \cong \oplus_i \dim(\pi_i) \, \cdot \pi_i.$$

That is, the multiplicity of π_i in λ is $\dim(\pi_i)$.

Finally,

$$C^*(G) \cong \oplus_i \mathbb{B}(H_{\pi_i}),$$

as C-algebras, by taking the direct sum of the maps $\pi_i \colon \mathbb{C}[G] \to \mathbb{B}(H_{\pi_i})$.*

PROOF. The multiplicities are given by

$$n_i = \frac{1}{|G|} \cdot \langle \chi_\lambda, \chi_{\pi_i} \rangle = \frac{1}{|G|} \cdot \sum_{g \in G} \chi_\lambda(g) \chi_{\pi_i}(g).$$

Since χ_λ is supported at the identity $e \in G$ and $\chi_\lambda(e) = |G|$, $\chi_{\pi_i}(e) = \dim(\pi_i)$, the result follows.

For the last statement, since π_i is irreducible, and by Schur's Lemma, the image C*-algebra $\pi_i(\mathbb{C}[G]) \subset \mathbb{B}(H_{\pi_i})$ satisfies

$$\pi_i(\mathbb{C}[G])' = \mathbb{C} \cdot 1,$$

where A', for $A \subset \mathbb{B}(H)$, A' is the commutant. Since $(\mathbb{C} \cdot 1)' = \mathbb{B}(H_{\pi_i})$, we get

$$\pi_i(\mathbb{C}[G])'' = \mathbb{B}(H_{\pi_i}).$$

It follows from Exercise 3.10 that

$$\pi_i(\mathbb{C}[G]) = \mathbb{B}(H_{\pi_i}).$$

We leave it now as an exercise to show that the direct sum $\pi_1 \oplus \pi_2 \oplus \cdots$ of the representations defines an injective *-homomorphism $\mathbb{C}[G] \to \mathbb{B}(H_{\pi_1}) \oplus \mathbb{B}(H_{\pi_2}) \oplus \cdots \oplus \cdots)$. The result follows. $\square$

These simple observations lead to the following fact. Since we have shown $C^*G) \cong \oplus_i \mathbb{B}(H_{\pi_i})$, there must be, for each irreducible representation $\pi \in \widehat{G}$, a projection $e_\pi \in C^*(G)$, which is mapped by this isomorphism to the projection $(0, \dots 0, 1, 0, \dots 0)$, with the identity on the π-th block. Note that e_π commutes with all elements of $C^*(G)$, *i.e.* it is in the centre of $C^*(G)$.

By the definitions, $\pi = \pi_j$, then the projection e_π has the property that

$$\pi_i(e_\pi) = \delta_{ij}.$$

If χ is a character of G let $\chi^*(g) := \overline{\chi(g)}$.

PROPOSITION 5.8. *If π is an irreducible representation of G, then the induced C*-algebra representation $C^*(G) \to \mathbb{B}(H_\pi)$ maps the function $\frac{\dim(H_\pi)}{|G|} \cdot \chi_\pi^*$ on G to the projection e_π.*

In particular, $e_\pi = \frac{\dim(H_\pi)}{|G|} \cdot \chi_\pi^* \in \mathbb{C}[G]$ is a projection in the centre of $C^*(G)$, and

$$e_\pi \cdot C^*(G) \cong \mathbb{B}(H_\pi).$$

PROOF. Let $e_\pi = \sum_{g \in G} a_g [g]$; we solve for a_g, by the following arguments.

First note that any character $\chi \colon G \to \mathbb{C}$ extends linearly to a map $\mathbb{C}[G] \to \mathbb{C}$. In the case of the character of the regular representation, $\chi_\lambda(\sum_g b_g[g]) = |G| \cdot b_e$, for any $\sum b_g[g] \in \mathbb{C}[G]$, since $\mathrm{Trace}\big(\lambda(g)\big) = 0$ if $g \neq e$. Thus, in particular,

$$\chi_\lambda(e_\pi) = |G| \cdot a_e$$

and similarly, since multiplication by a group element $[g] \subset \mathbb{C}[G]$ just shifts the indices of e_π,

$$\chi_\lambda([g^{-1}] \cdot e_\pi) = |G| \cdot a_g. \tag{5.12}$$

On the other hand, if, say $\pi = \pi_j$, the projection e_π acts as zero on H_{π_i} unless $i = j$. Hence the same is true of the element $[g^{-1}] \cdot e_\pi \in \mathbb{C}[G]$. It acts on H_{π_j} by $\pi_j(g^{-1})$ since e_π acts as the identity on H_{π_j}. Finally, since $\chi_\lambda = \sum_i \dim(H_{\pi_i}) \cdot \chi_{\pi_i}$, we get

$$a_g = \frac{1}{|G|} \cdot \chi_\lambda([g^{-1}] \cdot e_\pi) = \frac{1}{|G|} \cdot \sum_i \dim(H_{\pi_i}) \cdot \chi_{\pi_i}([g^{-1}] \cdot e_\pi) = \frac{\dim(H_{\pi_j})}{|G|} \cdot \chi_j(g)^* \tag{5.13}$$

We conclude that

$$e_\pi = \frac{\dim(H_\pi)}{|G|} \cdot \chi_\pi^*$$

as functions on G.

□

We summarize what has been proved about C*-algebras of finite groups.

THEOREM 5.9. *Let G be a finite group, $\widehat{G}$ its set of equivalence classes of irreducible representations. Let $[\pi] \in \widehat{G}$, and χ_π its character. Let $e_\pi = \frac{\dim H_\pi}{|G|} \sum_{g \in G} \overline{\chi_\pi(g)}[g] \in C^*(G)$. Then:*

a) *e_π is a central projection in $C^*(G)$.*
b) *$\pi(e_\pi) = 1$, the identity in $\mathbb{B}(H_\pi)$, and if $[\rho] \in \widehat{G}$ and $[\rho] \neq [\pi]$ then $\rho(e_\pi) = 0$.*
c) *Summing the representations π for $[\pi] \in \widehat{G}$ gives an isomorphism*

$$C^*(G) \cong \oplus_{[\pi] \in \widehat{G}} e_\pi C^*(G), \quad \text{and } e_\pi C^*(G) \cong \mathbb{B}(H_\pi).$$

d) *If $\tau \colon C^*(G) \to \mathbb{C}$ is the trace $\tau(\sum_g a_g[g]) := a_e$ of Exercise 3.9 a), then*

$$\tau(e_\pi) = \frac{\dim H_\pi^2}{|G|}$$

for all $[\pi] \in \widehat{G}$.

6. The compact operators

In Noncommutative Geometry, 'compact' (operator) tends to suggest 'small' in some sense, as in, a 'small perturbation.' Sometimes they are argued to be the quantum physical (or non-commutative) analogue of the 'infinitesimals' one meets in calculus, or differential geometry.

One reason that compact operators should be thought of as infinitesimals is that they have infinitesimally 'small' ranges.

DEFINITION 6.1. A bounded linear operator $T \in \mathbb{B}(H,K)$ between Hilbert spaces H, K is a *compact* operator if the image $T(B_H)$ of the unit ball $B_H := \{\xi \in H \mid \|\xi\| \leq 1\}$ in H is pre-compact, that is, if its closure is compact.

REMARK 6.2. A standard result from point-set topology is that a subspace $A \subset X$ of a complete metric space is pre-compact if and only if it is totally bounded. Hence, a bounded operator $T\colon H \to K$ is compact if and only if for all $\epsilon > 0$ there exist vectors $\xi_1, \ldots, \xi_n \in B_H$ such that $T(B_H) \subset \bigcup_i B_\epsilon(T\xi_i)$.

If H is an infinite-dimensional Hilbert space then the closed unit ball B_H is non-compact since any infinite orthonormal set in B_H provides a net with no convergent subnet. Hence the identity operator on H is not compact, whence certainly not all operators are compact.

More generally:

EXERCISE 6.3. If $T\colon H \to K$ is bounded below on the unit ball of an infinite-dimensional subspace of H, then T is not compact.

EXAMPLE 6.4. Let T be a bounded operator on a Hilbert space whose matrix with respect to an orthonormal basis $e_1, e_2, \ldots$ is

$$T = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \end{bmatrix}$$

Then T is compact if and only if $\lim_{n\to\infty}|\lambda_n| = 0$. Indeed, if T is compact, then Exercise 6.3 implies that for all $\epsilon > 0$ the set $\{n \mid |\lambda_n| \geq \epsilon\}$ is finite. Hence $\lambda_n \to 0$ as claimed.

Conversely, suppose that $\lim_{n\to\infty}|\lambda_n| = 0$. Choose $\epsilon > 0$. Let N such that $|\lambda_n| < \frac{\epsilon}{2}$ if $n \geq N$. Any vector in $T(B_H)$ can be written in the form $T\xi + T\eta$ where ξ is a linear combination of $e_1, \ldots, e_N$, η is in the closed span of $e_{N+1}, e_{N+2}, \ldots$, with $\|\xi\| \leq 1$ and $\|\eta\| \leq 1$.

By choice of N and Exercise 2.5, $\|T\eta\| < \frac{\epsilon}{2}$, because the restriction of T to the closed span of $e_{N+1}, e_{N+2}, \ldots$ is diagonal, with entries bounded by $\frac{\epsilon}{2}$.

Hence every vector in $T(B_H)$ is at distance at most ϵ to a vector in $T(B_{H'})$ where H' is the subspace $\mathrm{span}(e_1, \ldots, e_N)$.

Now since H' is finite-dimensional, $B_{H'}$ is compact, so $T(B_{H'})$ is also compact, whence it is totally bounded, so there are finitely many vectors $\xi_1, \ldots, \xi_k$ in $B_{H'}$ such that $T(B_{H'}) \subset \bigcup_{i=1}^k B_{\frac{\epsilon}{2}}\big(T(\xi_i)\big)$.

Putting these together gives that $T(B_H) \subset \bigcup_{i=1}^k B_\epsilon\big(T(\xi_i)\big)$, as required.

As observed the previous proof, an operator $T \in \mathbb{B}(H,K)$ with finite-dimensional range maps B_H into a bounded subset of a finite-dimensional subspace of K. Such operators are said to have *finite rank*. Since a bounded and closed subset of a finite-dimensional Hilbert space is compact, we get the following basic result.

PROPOSITION 6.5. *Any finite rank operator is compact.*

EXERCISE 6.6. Let H be a separable Hilbert space. Fix an orthonormal basis $e_1, e_2, \ldots$ for H and represent operators by their matrices in the usual way with $T = (T_{ij})$, $T_{ij} = \langle T(e_j), e_i \rangle$. An operator has a *finitely supported matrix* (with respect to the given orthonormal basis) if it has only finitely many nonzero entries. Prove that

a) If T has finite-rank, there is a unitary u and a finitely supported operator S such that $uTu^* = S$.
b) If T is a finite-rank operator and $\epsilon > 0$ then there exists a finitely supported operator S such that $\|S - T\| < \epsilon$.

The following theorem gives the basic tool in showing that operators are compact.

THEOREM 6.7. *A bounded operator $T \colon H \to K$ between Hilbert spaces is compact if and only if it is a norm limit of finite-rank operators.*

The proof of Theorem 6.7 amounts to the following two Lemmas.

LEMMA 6.8. *The set of compact operators $\mathcal{K}(H, K)$ from H to K is a closed subset of $\mathbb{B}(H, K)$ in the operator norm topology. That is, any operator norm limit of compact operators is a compact operator.*

PROOF. See Remark 6.2. Let T be a limit point of the set of compact operators from H to K. Choose $\epsilon > 0$. There exists a compact operator S such that $\|S - T\| < \frac{\epsilon}{3}$. Since S is compact, there exist finitely many vectors $\xi_1, \ldots, \xi_n \in B_H$ such that $S(B_H) \subset \bigcup_{i=1}^n B_{\frac{\epsilon}{3}}(S\xi_i)$. Now we claim that $T(B_H) \subset \bigcup_{i=1}^n B_\epsilon(T\xi_i)$. For if $\xi \in B_H$, choose ξ_i so that $S\xi \in B_{\frac{\epsilon}{3}}(\xi_i)$, then by the triangle inequality

$$\|T\xi - T\xi_i\| \leq \|T\xi - S\xi\| + \|S\xi - S\xi_i\| + \|S\xi_i - T\xi_i\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

proving the claim.

□

LEMMA 6.9. *The finite-rank operators from H to K are dense in $\mathcal{K}(H, K)$.*

PROOF. We show that if $T \colon H \to K$ is a compact operator then there exists a sequence (T_n) of finite-rank operators from H to K such that $T_n \to T$ in operator norm.

Since $T(B_H)$ has compact closure, it's closure is separable, *i.e.* has a countable dense subset. It follows (exercise) that the closure of the range of T is a separable Hilbert space and so has a countable orthonormal basis $e_1, e_2, \ldots$.

Let P_n be the (orthogonal) projection to the span of $e_1, \ldots, e_n$:

$$P_n\xi = \sum_{i=1}^n \langle \xi, e_n \rangle e_n.$$

By standard Hilbert space theory $P_n\eta \to \eta$ for all $\eta \in L$. In particular, $P_nT\xi \to T\xi$ for all $\xi \in H$.

We claim that this convergence is actually uniform over B_H, *i.e.* that $P_nT \to T$ in operator norm, and thus provides the required approximation of T by finite-rank operators.

Choose $\epsilon > 0$. Choose a finite set of vectors $\xi_1, \ldots, \xi_n$ in B_H such that $T(B_H) \subset \bigcup_i B_\epsilon(T\xi_i)$. Choose any $\xi \in B_H$. Then $\|T\xi - T\xi_i\| < \epsilon$ for some i. Since $P_nT \to T$ pointwise as observed above, there exists N such that if $n \geq N$ then $\|P_nT\xi_i - T\xi_i\| < \epsilon$ for $i = 1, \ldots, n$. So we argue

$$\|P_nT\xi - T\xi\| \leq \|P_nT\xi - P_nT\xi_i\| + \|P_nT\xi_i - T\xi_i\| + \|T\xi_i - T\xi\|.$$

The first term is bounded by $\|P_n\|\|T\xi - T\xi_i\| = \|T\xi - T\xi_i\| < \epsilon$, and this is why the third term is also $< \epsilon$. The second term is also $< \epsilon$ if $n \geq N$.

□

COROLLARY 6.10. *If $T \in \mathbb{B}(H,K)$ is a compact operator, $S \in \mathbb{B}(M,H)$ and $R \in \mathbb{B}(K,L)$ then TS and RT are compact operators $M \to K$ and $H \to L$ respectively. The adjoint $T^*\colon K \to H$ of a compact operator $H \to K$ is compact.*

PROOF. All of these statements are clear for finite-rank operators, and they follow by taking norm limits for compact operators, by the density result Lemma 6.9. □

COROLLARY 6.11. *The collection $\mathcal{K}(H)$ of compact operators $H \to H$, is a C*-subalgebra of $\mathbb{B}(H)$ for any Hilbert space H, and in particular, is a C*-algebra, for any Hilbert space H.*

Moreover, $\mathcal{K}(H)$ is an ideal of $\mathbb{B}(H)$: if $T \in \mathbb{B}(H)$ and $S \in \mathcal{K}(H)$ then $ST, TS \in \mathcal{K}(H)$.

EXERCISE 6.12. Let H be a separable Hilbert space and $T \in \mathcal{K}(H)$ is any compact operator on H. Show that $\|Te_n\| \to 0$ as $n \to \infty$ for any infinite orthonormal set $e_0, e_1, \ldots$ of vectors in H. However, show that there exist operators with this property which are not compact.

(*Hint.* For the second question, let P be a countable infinite direct sum of the n-by-n (projection) matrices $\begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n}. \end{bmatrix}$.)

EXERCISE 6.13. Let T be a compact operator on a Hilbert space H and $\lambda \neq 0$ a nonzero complex number. Prove:

a) The subspace $H_\lambda := \{\xi \in H \mid T\xi = \lambda\xi\}$ of T is finite-dimensional. (*Hint.* Else there is an infinite dimensional closed subspace W of H such that $T|_W = \lambda \cdot \mathrm{id}_W$.)
b) The range of $\lambda - T$ is closed. (*Hint.* Show that there exists a constant $C \geq 0$ such that $\|(\lambda - T)\xi\| \geq C\|\xi\|$ for all $\xi \in B_H$, using pre-compactness of $T(B_H)$.)

EXERCISE 6.14. If $T\colon H \to K$ is a compact operator, then there exists $\xi \in B_H$ such that $\|T\xi\| = \|T\|$.

Hint. By pre-compactness of $T(B_H)$ and the definition of norm, there is a sequence $\xi_1, \xi_2, \ldots$ of unit vectors in H such that $T\xi_n \to \eta$, and $\|\eta\| = \|T\|$.

By the parallelogram law ,

$$\|\xi_n - \xi_m\|^2 = 4 - \|\xi_n + \xi_m\|^2 \leq 4 - \frac{\|T\xi_n + T\xi_m\|^2}{\|T\|^2} \to 0$$

as $n, m \to \infty$, since $T\xi_n \to \eta$. Hence (ξ_n) is a Cauchy sequence. The result follows.

We conclude the general discussion with a Spectral Theorem for compact operators.

THEOREM 6.15. *Let T be any self-adjoint compact operator on a separable Hilbert space H and $\Lambda \subset \mathbb{R}$ denote the set of eigenvalues of T. If $\lambda \in \Lambda$ let $H_\lambda = \ker(\lambda - T)$ be the corresponding eigenspace.*

Then Λ is at most a countable set, H_λ is finite-dimensional for all nonzero λ, H_λ is orthogonal to $H_{\lambda'}$ if $\lambda \neq \lambda'$, and $H = \oplus_{\lambda \in \Lambda} H_\lambda$.

PROOF. The proofs that T self-adjoint implies that all eigenvalues are real and that eigenspaces for distinct eigenvalues are orthogonal, are left to the reader − they are identical to the arguments used to prove these facts in finite-dimensional linear algebra.

Let $H' = \left(\oplus_{\lambda \in \Lambda} H_\lambda\right)^\perp$, and T' be the restriction of T to T', then since the restriction of a compact operator is compact, T' is both compact and injective, so H' is finite-dimensional. Furthermore, T' clearly has no eigenvalues, as they would also be eigenvalues for T, whence $H' = 0$.

□

Examples of compact integral operators

Let X be a locally compact Hausdorff space and μ a Borel measure on X. For example, $X = \mathbb{R}$, μ Lebesgue measure, or $X = \mathbb{T}$ with normalized Lebesgue measure.

Let $k \in L^2(X \times X)$. If $\xi \in L^2(X, \mu)$, set

$$I_k\xi\,(x) := \int_X k(x,y)\xi(y)d\mu(y). \tag{6.1}$$

The integral is the same as the inner product $\langle k(x,\cdot), \xi\rangle$ of $k(x,\cdot)$, viewed as a vector in $L^2(X)$, with ξ (it is a consequence of Fubini's theorem that $k(x,\cdot)$ is in $L^2(X)$ for a.e. $x \in X$.)

By the Cauchy-Swartz inequality we thus get that $|I_k\xi\,(x)|^2 \leq \|k(x,\cdot)\|^2_{L^2(X)}\|\xi\|^2_{L^2(X)}$. Integrating this over X gives that I_k is a bounded operator and $\|I_k\| \leq \|k\|_{L^2(X\times X)}$.

Proposition 6.16. *I_k is a compact operator for all $k \in L^2(X \times X)$.*

Proof. It is a fact (proved later in this book) that measurable functions of the form $r(x,y) = \sum_i f_i(x)g_i(y)$, where the sum is finite, and the f_i's and g_i's are in $L^2(X)$, are dense in $L^2(X \times X)$.

For such a function r, by the definitions,

$$I_r\xi\; = \sum_i \left[\int_X g_j(y)\xi(y)d\mu(y)\right] f_i.$$

In particular $\mathrm{ran}(I_k) \subset \mathrm{span}\{f_i\}$ which is finite-dimensional. Hence I_r has finite-rank.

Now if $k \in L^2(X \times X)$ let $k_n \to k$ with k_n of the form of r above. Then $\|I_{k_n} - I_k\| \leq \|k_n - k\|_{L^2(X\times X)} \to 0$ so I_k is a compact operator.

$\square$

Example 6.17. If G is a compact group and $f \in C(G)$, $\lambda(f)$ the corresponding convolution operator on $L^2(G)$, then $\lambda(f)$ is a special case of a compact integral operator, for we can write

$$\lambda(f)\xi\;(g) = \int_G f(h)\xi(h^{-1}g)d\mu(h) = \int_G f(gh^{-1})\xi(h)d\mu(h) = I_k\xi\;(g)$$

where I_k is the integral operator with kernel $k(g,h) = f(gh^{-1})$.

Compactness of G is needed here to ensure that $k \in L^2(G \times G)$; it is rather obvious, even for G discrete, that the kernel $k(g,h) = f(gh^{-1})$ is never square summable over $G \times G$, unless f is zero or G is finite.

In particular, convolution operators $\lambda(f)$ with $f \in C(G)$ are compact operators. Since it is an immediate consequence of Theorem 6.7 that operator norm limits of compact operators are also compact, it follows, moreover, that the reduced C*-algebra $C^*_r(G)$ of G consists entirely of compact operators on $L^2(G)$.

It is not true that $C^*_r(G) = \mathcal{K}\big(L^2(G)\big)$, but rather $C^*_r(G)$ is a C*-subalgebra of $\mathcal{K}$. It's structure is much more delicate that that of $\mathcal{K}$, and reflects the representation theory of G.

Exercise 6.18. If $G = \mathbb{T}$ and $f(z) = \sum_{k=-n}^m a_k z^k$ is a trigonometric polynomial in $C(\mathbb{T})$, then the convolution operator $\lambda(f)\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ has rank at most $n+m$.

Exercise 6.19. Let G be compact abelian with Pontryagin dual $\widehat{G}$, and let $f \in \mathbb{C}[\widehat{G}]$ be any finite linear combination of characters $\chi\colon G \to \mathbb{T}$ of G. Prove that $\lambda(f)$ is a finite-rank operator on $L^2(G)$.

Exercise 6.20. Let G be a locally compact group, $f \in C_c(G)$ and $h \in C_c(G)$, and let $\lambda(f) \in \mathbb{B}\big(L^2(G)\big)$ be convolution with f and M_h be multiplication by h. We have already noted that $\lambda(f)$ is compact if G is compact; it is clear that M_h is compact if G is discrete.

Prove that $\lambda(f)M_h$ is a compact operator *for any locally compact group* G. (*Hint.* Show that $\lambda(f)M_h = I_k$, an integral operator, with appropriate *compactly supported* kernel.)

7. The Toeplitz algebra, Cuntz algebras

The Toeplitz algebra is an important C*-algebra connected with analytic function theory in the disk. It eventually will be shown to play a big role in K-theory and the Index Theorem.

Let $H = L^2(\mathbb{T})$, with its standard orthonormal basis $\{z^n\}_{n\in\mathbb{Z}}$ of characters. The *Szegö projection* (or *Toeplitz projection*) $P_+ \in \mathbb{B}(H)$ is the orthogonal projection onto the closed subspace $\mathbf{H}^2 := \operatorname{span}\{z^n \mid n \geq 0\} \cong l^2(\mathbb{N})$ of H. Explicitly:

$$(P_+f)(z) = \sum_{n=0}^{\infty}\Big(\int_{\mathbb{T}} f(w)\overline{w}^n d\mu(w)\Big)z^n = \sum_{n=0}^{\infty}\int_{\mathbb{T}} f(w)(\overline{w}z)^n d\mu(w). \tag{7.1}$$

The sum is a convergent series of vectors in $L^2(\mathbb{T})$ for every $f \in L^2(\mathbb{T})$. Hence it converges for a.e. $z \in \mathbb{T}$.

Definition 7.1. If $f \in C(\mathbb{T})$, the *Toeplitz operator with symbol* f is by definition the operator $T_f := P_+M_f$ acting on the subspace $\mathbf{H}^2 \subset L^2(\mathbb{T})$.

The C*-algebra generated by the Toeplitz operators on $L^2(\mathbb{T})$ is called the *Toeplitz algebra* and will be denoted $\mathcal{T}$.

Operators in $\mathcal{T}$ will be called *pseudo-Toeplitz operators.*

Using the standard orthonormal basis $1, z, z^2, \dots$ for $\mathbf{H}^2$, we can expand a Toeplitz operator T_f into an infinite matrix, it's Fourier transform as an operator. The m, nth entry is by definition

$$\langle T_f e_n, e_m\rangle = \langle P_+(fz^n), z^m\rangle = \sum_{k\geq -m}\hat{f}(k)\langle z^{m+k}, z^n\rangle = \hat{f}(n-m).$$

Thus,

$$\widehat{T}_f = \begin{bmatrix} \hat{f}(0) & \hat{f}(-1) & \hat{f}(-2) & \ddots & \ddots \\ \hat{f}(1) & \hat{f}(0) & \hat{f}(-1) & \hat{f}(-2) & \ddots \\ \hat{f}(2) & \hat{f}(1) & \hat{f}(0) & \hat{f}(-1) & \ddots \\ \ddots & \hat{f}(2) & \hat{f}(1) & \hat{f}(0) & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{7.2}$$

so that along the various diagonals of the matrix, we see the values of $\hat{f}$; the negative values appear above the diagonal and the positive ones below.

Note also that if $f \in \mathbb{C}[z,\bar{z}]$ then $\widehat{T}_f$ has only finitely many diagonals, that is, the support of $\widehat{T}_f$ is contained in a neighbourhood of the diagonal: there exists a constant $C \geq 0$ such that

$$(\widehat{T}_f)_{n,m} \neq 0 \Rightarrow |n-m| \leq C.$$

(Such matrices are said to have *finite propagation* – See Exercise 7.7 below.)

For example, $\widehat{T}_z = S$ the shift, $S(e_n) = e_{n+1}$, $n = 0, 1, 2, \dots$

$$S = \begin{bmatrix} 0 & 0 & 0 & \ddots & \ddots \\ 1 & 0 & 0 & 0 & \ddots \\ 0 & 1 & 0 & 0 & \ddots \\ \ddots & 0 & 1 & 0 & \ddots \\ \ddots & \ddots & \ddots & 1 & \ddots\ddots \end{bmatrix}$$

EXERCISE 7.2. If S is the shift on $l^2\mathbb{N}$ then prove that

a) S is an isometry $l^2(\mathbb{N}) \to l^2(\mathbb{N})$, that is, $S^*S = 1$. Furthermore, $1 - SS^* = P_0$ where P_0 is orthogonal projection to $\mathbb{C}e_0$.
b) $1 - S^n(S^n)^* = \sum_{k=0}^{n-1} P_k$, with P_k the rank-one projection onto $\mathbb{C}e_k$.
c) $S^i P_0 (S^j)^* = E_{i,j}$ for all $i, j \in \mathbb{N}$, where $E_{i.j}$ is the rank-one operator whose matrix has a 1 in the i, jth spot and zeros everywhere else

Exercise 7.2 c) implies that the C*-algebra generated by S, and hence the Toeplitz algebra, contains all operators on $l^2(\mathbb{N})$ whose matrix representations contain only finitely many nonzero entries. In the next section, the closure of this *-algebra will be shown to be the C*-algebra of *compact operators* on $l^2(\mathbb{N})$.

We next observe that the Toeplitz algebra is actually generated by a single operator.

PROPOSITION 7.3. *The Toeplitz algebra is generated as a C*-algebra by $S := T_z$.*

PROOF. First note that $C^*(S)$ contains all Toeplitz operators T_f where $f \in \mathbb{C}[z, \bar{z}]$. For example, if $f(z) = 2\bar{z} + 1 + 3z + 4z^2$ then

$$\widehat{T}_f = \begin{bmatrix} 1 & 2 & 0 & 0 & \\ 3 & 1 & 2 & 0 & \ddots \\ 4 & 3 & 1 & 2 & \\ 0 & 4 & 3 & 1 & \\ & \ddots & & & \ddots \end{bmatrix} = 2S^* + 1 + 3S + 4S^2. \tag{7.3}$$

Furthermore, since $T_f = P_+ M_f|_{\mathbf{H}^2}$, it is clear that $\|T_f\| \le \|M_f\| = \|f\|$, so if $f_n \in \mathbb{C}[z, \bar{z}]$ is a uniformly convergent sequence, converging to $f \in C(\mathbb{T})$, then $T_{f_n} \to T_f$.

Consequently $C^*(S)$ contains all T_f's with $f \in C(\mathbb{T})$, and hence it contains the generators of $\mathcal{T}$. So it contains $\mathcal{T}$. Obviously $C^*(S) \subset \mathcal{T}$ since $S \in \mathcal{T}$. Hence $C^*(S) = \mathcal{T}$ as claimed.

$\square$

EXERCISE 7.4. (*Cuntz algebras*). Given any two intervals $[a, b]$, $[c, d]$ of $\mathbb{R}$ there is a canonical 'affine' map

$$\phi(x) = \left(\frac{b-a}{d-c}\right)x + ad - bc,$$

from $[c, d]$ to $[a, b]$, with constant derivative $\phi'(x) = \frac{b-a}{d-c}$. We define a linear operator

$$s \colon L^2([a, b]) \to L^2([c, d]), \quad (s\xi)(x) = \phi'(x)^{\frac{1}{2}} \cdot \xi\big(\phi(x)\big) = \sqrt{\frac{b-a}{d-c}} \cdot \xi\big(\phi(x)\big).$$

a Hilbert space map between the two corresponding L^2-spaces.

a) s is an isometry.

b) Now divide the unit interval $[0,1]$ into n subintervals $I_1, \dots I_k$ of equal length, and we regard $L^2(I_k)$ as a closed subspace of $L^2([0,1])$ in the obvious way by extending functions by zero. Let

$$s_k\colon L^2([0,1]) \to L^2(I_k) \to L^2([0,1]).$$

be the composition, the first being one of the interval isometries described above, and the second map the inclusion.

Prove that $s_1, \dots, s_k \in \mathbb{B}\big(L^2([0,1])$ are isometries with orthogonal ranges, compute the range projections $s_i s_i^*$, as Hilbert space projections on $L^2([0,1])$, and verify that

$$\sum_{i=1}^{n} s_i s_i^* = 1.$$

The C*-algebra generated by $s_1, \dots, s_n$ is the *Cuntz algebra* O_n.

Remark 7.5. The Cuntz algebra O_n turns out to be structurally unique in the sense of Coburn's Theorem: *any two C*-algebras generated by n isometries $s_1, \dots, s_n$, and $t_1, \dots, t_n$, with orthogonal ranges summing to the identity, are isomorphic by a map sending s_i to t_i.*

Exercise 7.6. Let $s_1, \dots, s_n$ be isometries of a Hilbert space H such that $\sum_{i=1}^n s_i s_i^* = 1$.

a) Prove that the s_i have orthogonal ranges.
b) Prove that the linear span of the elements $s_i s_j^*$, $i, j = 1, 2, \dots n$ form a *-algebra isomorphic to $M_n(\mathbb{C})$.
c) What about the *-algebra (or C*-algebra) generated by $s_i s_j s_k^* s_l^*$?

Exercise 7.7. Let T be an $\mathbb{N}$-by-$\mathbb{N}$ matrix of finite propagation – there exists $R > 0$ such that $T_{nm} = 0$ if $|n-m| > R$ – with uniformly bounded entries $T_{n,m}$. Prove that T is a bounded operator.

Exercise 7.8. Prove the following.

a) If T is a Toeplitz operator then $T_z^* T T_z = T$. What about the converse?
b) Show that T_f Toeplitz implies T_f^* is Toeplitz, and that $T_f^* = T_{f^*}$.
c) Let $\lambda\colon \mathbb{T} \to \mathbf{U}\big(L^2(\mathbb{T})\big)$ be the regular representation. It leaves the Hardy space $\mathbf{H}^2$ invariant so we will consider for the moment λ as a representation of $\mathbb{T}$ on $\mathbf{H}^2$. Show that $\lambda(w)^* T_z \lambda(w) = w T_z$ for any $w \in \mathbb{T}$. What does this say about the spectrum of T_z as a subset of the complex plane?
d) Prove that if $\pi\colon \mathbb{T} \to \mathbf{U}(H)$ is a *finite-dimensional* representation of $\mathbb{T}$ and $T \in \mathbb{B}(H)$ such that $\pi(z) T \pi(z)^* = zT$ for all $z \in \mathbb{T}$, then the spectrum of T is $\{0\}$. Then give an example of an operator T on $\mathbb{C}^2$ and a unitary representation of $\mathbb{T}$ on $\mathbb{C}^2$ such that $\pi(z) T \pi(z)^* = zT$ for all $z \in \mathbb{T}$, but T is not the zero operator.

The previous exercise shows that the Toeplitz algebra $\mathcal{T}$ carries an action of the group $\mathbb{T}$ as C*-algebra automorphisms: the automorphism of $\mathcal{T}$ corresponding to $z \in \mathbb{T}$ is given by $\alpha_z(T) := \lambda(z) T \lambda(z)^*$.

Remarks on the Toeplitz algebra and complex function theory

It is an interesting and useful exercise to try to produce a formula for the Szegö projection P_+ involving a direct, geometric formula in terms of the *points* of the circle, rather than a rather easy construction in the Fourier transformed picture of $\mathbb{T}$. Said another way – problem: find a 'coordinate-free' description of the Szegö projection, that is, one which does not use the standard orthonormal basis of $L^2(\mathbb{T})$.

To this end, going back to the definition (7.1) of the Szegö projection, interchange the integral and the sum, and setting aside, for the moment, the issue of whether or not this is legitimate, one obtains an *ansatz*

$$(P_+f)(z) = \int_{\mathbb{T}} \frac{f(w)}{1-\overline{w}z} d\mu(w) \tag{7.4}$$

by invoking, somewhat formally, the standard formula for summing a geometric series

$$\sum_{n=0}^{\infty} (\bar{w}z)^n = \frac{1}{1-\bar{w}z}.$$

The integral in (7.4) converges absolutely when $|z| < 1$, and when $|z| = 1$, it is a sort of 'singular integral,' in the sense that it can be naturally regularized – since it gives the boundary values of an analytic extension of f to the closed disk. To see this, note that (7.4) can be interpreted as a *contour* (or 'line') integral, of the kind met in basic complex analysis. We recall the definition. If $\gamma\colon [a,b] \to \mathbb{C}$ is a smooth curve in the complex plane and g is, say continuous, then the contour integral of g over γ is

$$\oint_\gamma g(z)\, dz := \int_a^b g\big(\gamma(t)\big) \cdot \gamma'(t)\, dt.$$

The Cauchy-Integral formula asserts that

$$g(z) = \frac{1}{2\pi} \oint_\gamma \frac{g(w)}{w-z}\, dw$$

if γ is closed, positively oriented with respect to its bounded component, g holonomorphic on the interior of the bounded component, continuous on the closure of the component, and z is a point in the interior of the bounded component.

In this situation, let $\oint_{\mathbb{T}}$ denote contour integration over the unit circle positively oriented, then by the definitions the integral (7.4) can be re-written

$$\int_{\mathbb{T}} \frac{f(w)}{1-\overline{w}z}\, d\mu(w) \quad = \quad \int_{\mathbb{T}} \frac{f(w)}{\overline{w}(w-z)}\, d\mu(w) \quad = \quad \int_0^{2\pi} \frac{f(e^{it})\, e^{it}}{e^{it}-z} \quad dt \quad = \quad \oint_{\mathbb{T}} \frac{f(w)}{w-z}\, dw, \tag{7.5}$$

for a continuous function f on $\mathbb{T}$.

In particular, if f happens to extend to a *holomorphic* function $\tilde{f}$ on the (open) disk $\mathbb{D}$, then by the Cauchy Integral formula

$$\int_{\mathbb{T}} \frac{f(w)}{1-\overline{w}z}\, d\mu(w) = \tilde{f}(z) \tag{7.6}$$

for all $|z| < 1$. Conversely, the left hand side of (7.6) defines a holomorphic $\tilde{f}$ on the disk, whose boundary values are the values of f, for any given $f \in C(\mathbb{T})$.

We introduce some notation. If $f \in C(\mathbb{T})$, let $\overline{P}_+ f := \tilde{f}$ be the function on $\mathbb{D}$ defined by (7.6).

Lemma 7.9. *Let $f \in C(\mathbb{T})$. Then*

1) *$\overline{P}_+ f$ is analytic in $\mathbb{D}$ and extends continuously to $\overline{\mathbb{D}}$. Thus , $\overline{P}_+ f \in \mathcal{A}(\mathbb{D})$ if $f \in C(\mathbb{T})$.*
2) *$\|\overline{P}_+ f\|_{\mathcal{A}(\mathbb{D})} \le \|f\|_{C(\mathbb{T})}$, and $\overline{P}_+ f = f$ if $f \in \mathcal{A}(\mathbb{D})$.*

3) *The restriction of the Szegö projection* P_+ *to* $C(\mathbb{T}) \subset L^2(\mathbb{T})$ *factors through the map* $\overline{P}_+ \colon C(\mathbb{T}) \to \mathcal{A}(\mathbb{D})$ *and the inclusion* $\mathcal{A}(\mathbb{D}) \to \mathbf{H}^2$, i.e. *the diagram*

$$\begin{tikzcd} C(\mathbb{T}) \arrow[r, "\overline{P}_+"] \arrow[dr, "P_+"'] & \mathcal{A}(\mathbb{D}) \arrow[d, "\mathrm{incl}"] \\ & \mathbf{H}^2 \end{tikzcd} .$$

commutes.

4) *If* $f \in C(\mathbb{T})$ *is* C^k, *then* $P_+ f$ *is* C^k.

5) *In explicit terms of the points of the circle, the Szegö projection* P_+ *is given on continuous functions by*

$$(P_+ f)(z) = \lim_{\epsilon \to 1} \int_{\mathbb{T}} \frac{f(w)}{1 - \epsilon \bar{w} z} d\mu(w). \tag{7.7}$$

The limit converges uniformly in z *as* $\epsilon \to 0$.

Moreover, the same formula holds pointwise for a.e. $z \in \mathbb{T}$, *if merely* $f \in L^2(\mathbb{T})$.

Proof. For $n = 0, 1, 2, \ldots$, the function z^n on $\mathbb{T}$ obviously extends to a continuous function on $\mathbb{D}$ analytic in $\mathbb{D}$, whence $\overline{P}_+ z^n = z^n$ for all $z \in \mathbb{D}$, by the Cauchy integral formula and the remarks preceding the proof. We leave it to the reader to use the Residue Formula to check that $\overline{P}_+ z^{-n} = 0$, for $n > 0$. Hence $\overline{P}_+(\sum_{k=-m}^{n} a_k z^k) = \sum_{k=0}^{n} a_k z^k$, where this polynomial is understood to be a function on $\mathbb{D}$. Such polynomials obviously extend continuously to $\overline{\mathbb{D}}$. Thus, $\overline{P}_+ f$ extends continuously to $\overline{\mathbb{D}}$ for all Laurent polynomials f. We may thus regard $\overline{P}_+$ as a linear map $\mathbb{C}[z, \bar{z}] \to C(\overline{\mathbb{D}})$.

By the Maximum Modulus Theorem.

$$\sup_{z \in \overline{\mathbb{D}}} |\tilde{f}(z)| \leq \sup_{z \in \mathbb{T}} |\tilde{f}(z)|,$$

for any $f \in C(\overline{\mathbb{D}})$ which is analytic on $\mathbb{D}$. In particular this holds for polynomials. Hence $\|\overline{P}_+ f\|_{C(\overline{\mathbb{D}})} \leq \|f\|_{C(\mathbb{T})}$ for all Laurent polynomials f. By density of Laurent polynomials in $C(\mathbb{T})$, and the fact that $C(\overline{\mathbb{D}})$ is complete, it follows that $\overline{P}_+$ maps $C(\mathbb{T})$ into $C(\overline{\mathbb{D}})$. By a standard exercise in the Dominated Convergence Theorem, $\overline{P}_+ f$ is analytic in $\mathbb{D}$. Hence $\overline{P}_+ f \in \mathcal{A}(\mathbb{D})$ for all $f \in C(\mathbb{T})$. That $\overline{P}_+ f = f$ if $f \in \mathcal{A}(\mathbb{D})$ follows from the Cauchy Integral formula.

The factorization of P_+ as described in the Theorem follows immediately from the above arguments. By the definitions, if $f \in C(\mathbb{T})$ then $\overline{P}_+ f \in \mathcal{A}(\mathbb{D}) \subset C(\mathbb{T})$ viewed as an element of $L^2(\mathbb{T})$ is exactly $P_+ f$. In particular, the natural inclusion $C(\mathbb{T}) \to L^2(\mathbb{T})$ maps $\mathcal{A}(\mathbb{D})$ into $\mathbf{H}^2$. ☐

We close with a remark about the Szegö projection and the group C*-algebra of $\mathbb{T}$.

Remark 7.10. The Szegö projection involves the singular integral expression

$$(P_+ f)(z) = \int_{\mathbb{T}} \frac{f(w)}{1 - \bar{w} z} d\mu(w). \tag{7.8}$$

which is a *convolution*, in the sense of convolution of two elements of a group C*-algebra, with the group being the circle $\mathbb{T}$: one is convolving the function f with the function $\chi(z) = \frac{1}{1-w}$:

$$P_+ f\,(z) = \int_{\mathbb{T}} \frac{f(w)}{1 - \overline{w} z}\, dw = \int_{\mathbb{T}} f(w) \chi(w^{-1} z) d\mu(z) = (f * \chi)\,(z), \tag{7.9}$$

$$\text{where } \chi(w) = \frac{1}{1 - w}.$$

If χ were continuous on $\mathbb{T}$, which it obviously is not, this would realize P_+ as an element of $C^*(\mathbb{T})$, since $C^*(\mathbb{T})$ is by definition, the closure of the *-algebra of convolution operators by a continuous function. As it happens, P_+ is 'almost' in $C^*(\mathbb{T})$: it is a *multiplier* of $C^*(\mathbb{T})$. We will discuss multipliers in Chapter 4.

EXERCISE 7.11. Use the Residue Formula to check that $\overline{P}_+ z^{-n} = 0$, for $n > 0$.

Commutativity of the Toeplitz algebra mod compact operators

Commutators are often very important in Noncommutative Geometry. The Toeplitz algebra offers a good example of how these arise, and what their geometric meaning is.

Let $f \in C(\mathbb{T})$. We compute the commutator $[f, P_+]$ by the (singular) integral formula P_+ developed above: we get

$$[P_+, f]\,\xi(z) = \int_{\mathbb{T}} \left(\frac{f(w) - f(z)}{1 - \overline{w}z}\right) \cdot \xi(w)\, d\mu(w) = \int_{\mathbb{T}} w \cdot \frac{\big(f(w) - f(z)\big)}{w - z} \cdot \xi(w)\, d\mu(w), \tag{7.10}$$

and the singular integral has become an ordinary, convergent one, if f is complex differentiable at z. More, if we let

$$k_f\,(z, w) = w \cdot \left(\frac{f(w) - f(z)}{w - z}\right),$$

defined initially on the complement of the diagonal $z = w$ in $\mathbb{T} \times \mathbb{T}$, and if the complex derivative $f'(z)$ exists everywhere on $\mathbb{T}$ then we can define k_f along the diagonal by

$$k_f(z, z) := z \cdot \frac{\partial f}{\partial z}$$

and this will give a continuous extension of k_f to $\mathbb{T} \times \mathbb{T}$ for which we may write

$$[P_+, f]\,\xi(z) = \int_{\mathbb{T}} k_f(z, w)\xi(w) d\mu(w),$$

realizing the operator commutator as an integral operator (and hence a compact operator).

Actually, $z \cdot \frac{\partial f}{\partial z}$ only depends on the *angular* differentiability of f. To see this, write f as a function of θ with $z = e^{i\theta}$, we have

$$\begin{aligned} \frac{\partial f}{\partial z}(e^{i\theta}) &= \lim_{\alpha \to 0} \frac{f(e^{i\theta}) - f(e^{i(\theta+\alpha)})}{e^{i\theta} - e^{i(\theta+\alpha)})} \\ &= e^{-i\theta} \cdot \lim_{\alpha \to 0} \left(\frac{f(e^{i\theta}) - f(e^{i(\theta+\alpha)})}{\alpha}\right) \cdot \left(\frac{\alpha}{1 - e^{i\alpha}}\right) \\ &= ie^{-i\theta} \cdot \frac{\partial f}{\partial \theta}(e^{i\theta}), \end{aligned} \tag{7.11}$$

since $\lim_{\alpha \to 0} \frac{\alpha}{1 - e^{i\alpha}} = i$. Thus,

$$z \cdot \frac{\partial f}{\partial z}(e^{i\theta}) = i \cdot \frac{\partial f}{\partial \theta}(e^{i\theta})$$

In particular, k_f is continuous on $\mathbb{T} \times \mathbb{T}$ for f merely smooth as a function on the 1-dimensional manifold $\mathbb{T}$, and therefore, $[P_+, f] = I_{k_f}$ is a compact operator on $L^2(\mathbb{T})$.

We have computed the following:

Proposition 7.12. *If $f \in C^\infty(\mathbb{T})$, define, in, polar coordinates θ, θ' on $\mathbb{T}$,*

$$k_f(\theta, \theta') = \begin{cases} \left(\frac{f(e^{i\theta}) - f(e^{i\theta'})}{e^{i\theta} - e^{i\theta'}}\right) & \text{if } \theta \neq \theta' \\ -i \cdot \frac{\partial f}{\partial \theta}(e^{i\theta}) & \text{if } \theta = \theta' \end{cases} \tag{7.12}$$

Then k_f is smooth on $\mathbb{T} \times \mathbb{T}$, and in particular continuous, and hence the integral operator I_{k_f} with kernel k is compact.

Let

$$\delta \colon C(\mathbb{T}) \to \mathbb{B}\big(L^2(\mathbb{T})\big), \ \ \delta(f) := [P_+, f],$$

be the derivation defined by commutator with the Szegö projection P_+.

Then δ maps $C(\mathbb{T})$ into $\mathcal{K}\big(L^2(\mathbb{T})\big)$, and if $f \in C^\infty(\mathbb{T})$, then $\delta(f) = I_{k_f}$.

In particular, the commutator $[P_+ f]$ is a compact operator on $L^2(\mathbb{T})$ for every $f \in C(\mathbb{T})$.

Proof. For the last statement, let $f_n \to f$ uniformly with f_n smooth. Then $P_+, f_n]$ converges to $[P_+ f]$ in operator norm. Since $[P_+, f_n]$ is an integral operator with smooth kernel for f smooth, it is a compact operator. Hence $[P_+, f]$ is compact.

□

Exercise 7.13. Compute the commutators $[P_+, z^n]$ for $n \in \mathbb{Z}$. Deduce the last statement of Proposition 7.12 by this (admittedly much more simple) method.

Now, let T_f and T_g be two Toeplitz operators. Then as operators on $\mathbf{H}^2$,

$$T_f T_g = P_+ M_f P_+ M_g = P_+ M_f (M_g + [P_+, M_g]) = T_{fg} + P_+ M_f [P_+, M_g], \tag{7.13}$$

By the previous Proposition, $P_+ M_f [P_+, M_g]$ is a compact operator. Similar arguments show that $T_f^* - T_{f^*}$ is a compact operator.

Thus, we have the following.

Corollary 7.14. *If T_f and T_g are two Toeplitz operators, then*

$$T_f T_g - T_{fg}, \quad T_f^* - T_{f^*},$$

are compact operators.

In particular, up to 'compact operator' error, the Toeplitz algebra is commutative. We will show shortly that the C*-algebra quotient $\mathcal{T}/\mathcal{K}$ is isomorphic to $C(\mathbb{T})$.

Exercise 7.15. For any complex number s let $|D|^s$ be the (unbounded) diagonal operator on $l^2(\mathbb{N})$ with matrix

$$|D|^s = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & \ddots \end{bmatrix}.$$

The domain of $|D|$ may be taken initially to be the dense subspace of finitely supported sequences in $l^2(\mathbb{N})$.

a) Prove that if $T \in \mathbb{B}\big(l^2(\mathbb{N})\big)$ is bounded and if $s \in \mathbb{C}$ with $\Re(s) > 1$ then $T|D|^{-s}$ is trace-class.
b) Prove that if $T \in \mathcal{T}$ then the function $\mathrm{Trace}(T|D|^{-s})$ extends to a meromorphic function on the complex plane with isolated finite poles.

c) Prove that the map

$$\Phi(T) := \mathrm{Res}_{s=1}\mathrm{Trace}(T|D|^{-s}) \tag{7.14}$$

defines a continuous linear functional $\mathcal{T} \to \mathbb{C}$ on the Toeplitz algebra, vanishing on compact operators, and that if $T = T_f$ is the Toeplitz operator with symbol $f \in C(\mathbb{T})$, then

$$\Phi(T) = \int_{\mathbb{T}} f(\theta)\, d\mu(\theta).$$

Some of the calculations above are important starting points for the development of 'Noncommutative Geometry.'

The ideal of compact operators in the Toeplitz algebra, Coburn's Theorem

Recall that an *ideal* J in an algebra A is a subalgebra such that ab and ba are in J if $a \in A$ and $b \in J$. Ideals will be discussed systematically in Chapter 3. Here we are interested, for ulterior motives, in one particularly important example of an ideal: the ideal of compact operators inside the Toeplitz algebra.

THEOREM 7.16. *The Toeplitz algebra contains $\mathcal{K}\big(L^2(\mathbb{T})\big)$ as a closed *-subalgebra and an ideal. Furthermore,*

a) *The equalities*

$$\mathrm{dist}(T_f, \mathcal{K}) := \inf\{\|T_f - S\| \mid S \in \mathcal{K}\big(L^2(\mathbb{T})\big)\} = \|T_f\| = \|f\|_{C(\mathbb{T})}$$

hold for any $f \in C(\mathbb{T})$.

b) *Every pseudo-Toeplitz operator $T \in \mathcal{T}$ can be written uniquely in the form $T = T_f + S$ where $f \in C(\mathbb{T})$ and $S \in \mathcal{K}$.*

c) *If $\mathcal{T}/\mathcal{K}$ is given the quotient norm $\|T + \mathcal{K}\| := \mathrm{dist}(T, \mathcal{K})$, and quotient *-algebra structure, then $\mathcal{T}/\mathcal{K}$ is a C*-algebra, and, moreover, it is isomorphic to $C(\mathbb{T})$, by the map*

$$\sigma\colon \mathcal{T}/\mathcal{K} \to C(\mathbb{T}), \quad \sigma(T_f + \mathcal{K}) := f,$$

where $T_f + \mathcal{K}$ denotes the coset of T_f in $\mathcal{T}/\mathcal{K}$.

If $T = T_f + S$ is a pseudo-Toeplitz operator, then we call f the *symbol* of T. That the symbol is uniquely defined is a consequence of the work done above.

PROOF. Statement a) follows from (7.13).

That $\mathcal{T}$ contains the compact operators follows from Exercise 7.2, which shows that $\mathcal{T}$ contains all operators whose associated matrix has only finitely many nonzero entries. The *-algebra of operators on $\mathbf{H}^2$ whose matrices with respect to the standard basis are finitely supported, are dense in $\mathcal{K}\big(L^2(\mathbb{T})\big)$, by Exercise 6.6, and Lemma 6.9. Since $\mathcal{T}$ is closed in the operator norm topology, $\mathcal{T}$ contains $\mathcal{K}$ as claimed.

Consider the matrix

$$T_f = \left[\begin{array}{cc|cccc} a_0 & a_{-1} & a_{-2} & a_{-3} & \dots & \\ a_1 & a_0 & a_{-1} & a_{-2} & \dots & \\ \hline a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \\ \dots a_3 & a_2 & a_1 & a_0 & a_{-1} & \\ & \dots & a_2 & a_1 & a_0 & \\ & & & \dots & & \dots \end{array}\right] \tag{7.15}$$

of a Toeplitz operator T_f. We have indicated a partition of the matrix. For each n let P_n be projection to the closed span of $e_n, e_{n+1}, \ldots$. Truncating the matrix as shown to the bottom right hand corner amounts to replacing T_f by $P_2 T_f P_2$, which has matrix

$$P_2 T_f P_2 = \left[\begin{array}{cc|cccc} 0 & 0 & 0 & 0 & \ldots & \\ 0 & 0 & 0 & 0 & \ldots & \\ \hline 0 & 0 & a_0 & a_{-1} & a_{-2} & \\ 0 & 0 & a_1 & a_0 & a_{-1} & \\ \ldots & \ldots & a_2 & a_1 & a_0 & \\ & & & \ldots & & \ldots \end{array}\right] \tag{7.16}$$

It is obvious from looking at the two corresponding matrices, that T_f and $P_2 T_f P_2$ have the same operator norm. Thus, we prove inductively the interesting fact that truncation is isometric when applied to Toeplitz operators:

$$\|P_n T_f P_n\| = \|T_f\|, \ n = 0, 1, \ldots. \tag{7.17}$$

Now let S be a finitely supported matrix. Then its truncations $P_n S P_n$ are zero for n large enough, and hence

$$\|S - T_f\| \geq \|P_n(S - T_f)P_n\| = \|P_n T_f P_n\| = \|T_f\|.$$

This shows that

$$\operatorname{dist}(T_f, F) \geq \|T_f\|$$

where F is the *-algebra of operators with matrices of finite support. Since F is dense in $\mathcal{K}$, it follows that

$$\operatorname{dist}(T_f, \mathcal{K}) \geq \|T_f\|.$$

On the other hand, $\operatorname{dist}(T_f, \mathcal{K}) \leq \|T_f\|$ is clear. So

$$\operatorname{dist}(T_f, \mathcal{K}) = \|T_f\|.$$

To prove that $\|T_f\| = \|f\|$, given the obvious inequality $\|T_f\| \leq \|f\|$, we use truncation ideas again. As above, but now for n any integer, let P_n be projection to the closed span of $e_n, e_{n+1}, \ldots$. Suppose $\xi \in \mathbb{C}[z, \bar{z}] \subset L^2(\mathbb{T})$ is a finite linear combination of the standard basis elements. ...

$\square$

Using the above result, we can show that the Toeplitz algebra strong and rather remarkable uniqueness property.

THEOREM 7.17. *(Coburn's Theorem) If W is any non-unitary isometry of a Hilbert space, then there is a unique C^*-algebra isomorphism $\mathcal{T} \to C^*(W)$ mapping S to W.*

In particular, any two C^*-algebras generated by non-unitary isometries are canonically isomorphic.

REMARK 7.18. If the caveat *non-unitary* is dropped in the statement, then the theorem obviously fails, since the unitary $1 \in \mathbb{C}$ generates the C*-algebra $\mathbb{C}$ and the unitary $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ generates $\mathbb{C} \oplus \mathbb{C}$.

PROOF. Note that W and all its positive powers have closed range, since they are isometries. So $H \supset W(H) \supset W^2(H) \supset \cdots$ is an infinite descending chain of closed subspaces of H.

Define $H_0 = H \ominus W(H)$, $H_1 := W(H) \ominus W^2(H)$, and so on, $H_{n+1} = H_n \ominus W(H_n)$. Then $H_0, H_1, \ldots$ are pairwise orthogonal subspaces of H and W maps H_i isometrically onto H_{i+1}. Set $H_\infty = \cap_{n=0}^{\infty} W^k(H)$, this is a closed subspace of H, and H decomposes orthogonally as a direct sum

$$H = \oplus_n H_n \ \oplus \ H_\infty.$$

The isometry W leaves each summand invariant, and is *unitary* on H_∞; denote by $U\colon H_\infty \to H_\infty$ this unitary, and W' the restriction of W to $\oplus_n H_n$.

Let $\mathcal{E}$ be any orthonormal basis for H_0. Then $W'(\mathcal{E})$ is an orthonormal basis for H_1, $(W')^2(\mathcal{E})$ is an orthonormal basis for H_2, and so on. So we obtain an orthonormal basis $\bigcup_{n=0}^\infty S^n(\mathcal{E})$ for $\oplus_n H$. With respect to this orthonormal basis, W' has the block matrix representation

$$W' = \begin{bmatrix} 0 & 0 & 0 & 0 & \ddots \\ I & 0 & 0 & 0 & \ddots \\ 0 & I & 0 & 0 & \ddots \\ 0 & 0 & I & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Each I is the identity matrix of size $\dim(H_0)$. If H_0 is 1-dimensional, this shows that W' is unitarily conjugate to the unilateral shift S, and then certainly $C^*(S)$ and $C^*(W')$ are isomorphic. In general, if $n = \dim(H_0)$, W' is unitarily conjugate to the direct sum $\oplus_i S$ of a certain number n of coples of the unilateral shift (where $n = \infty$ is possible.) We define the required isomorphism

$$C^*(S) \cong C^*(\oplus_i S)$$

between the C*-algebras generated by S and the direct sum of any number of copies $\oplus_i S$ of S, mapping S to $\oplus_i S$, by mapping an arbitrary element of the Toeplitz algebra $\mathcal{T}$, say $T_f + S$, to the corresponding direct sum $\oplus_i T_f + S$. It is clear that this is a C*-algebra isomorphism between $C^*(S)$ and $C^*(\oplus_i S)$. Therefore, $C^*(S)$ and $C^*(W')$ are isomorphic, as required.

If the unitary U is non-trivial, we need to use simple spectral theory – as we have not yet developed the Spectral Theorem, we refer the reader to Exercise 3.16 for the additional argument.

$\square$

Exercise 7.19. Let P_n be projection to the closed span of $e_n, e_{n+1}, \ldots$ in $l^2(\mathbb{N})$, as in the proof of Theorem 7.16. Prove that $\lim_{n\to\infty}\|P_n S\| = 0$ for any compact operator S.

8. Inductive limits of C*-algebras

We start by discussing the category-theoretic idea of a direct limit. Let $\mathcal{C}$ be a category and I a directed set: a set I equipped with a reflexive and transitive relation such that any two elements of I have an upper bound in I.

A *directed system* of objects of $\mathcal{C}$ is a family $\{A_i \mid i \in I\}$ of objects of the category, and a family of morphisms $\varphi_{ji}\colon A_i \to A_j$ for all $i \le j$ such that $\varphi_{ii} = \mathrm{id}_{A_i}$, the identity morphism $A_i \to A_i$, for all i, and such that

$$\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}, \quad \forall i \le j \le k.$$

Remark 8.1. If the directed set is just the natural numbers $1, 2, \ldots$ with its usual ordering, then the directed system is often written simply in the form

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \cdots$$

with $\varphi_n\colon A_n \to A_{n+1}$ the maps between the adjacent objects of the system.

Definition 8.2. A *direct limit* of a directed system $\{A_i \mid i \in I\}, \{\varphi_{ij}\colon A_i \to A_j \mid i \le j\}$ in a category $\mathcal{C}$ is an object A of $\mathcal{C}$ together with a family $\varphi_i\colon A_i \to A$ of morphisms, that satisfies the following universal property.

If B is any object of $\mathcal{C}$ and $\{\psi_i\colon A_i \to B \mid i \in I\}$ is a family of morphisms which is coherent in the sense that $\psi_j \circ \varphi_{ij} = \psi_i$ for all $i \le j$, then there is a unique morphism $\psi\colon A \to B$ such that $\psi \circ \varphi_i = \psi_i$ for all $i \in I$.

In general, direct limits may or may not exist in a category. In many familiar categories, however, like the categories of groups, rings, modules over a ring, and so on, they exist and are defined by a construction similar to the case of groups, which we explain first.

Let $\{G_i \mid i \in I\}$, $\{\varphi_{ij} \mid i \le j\}$ be a directed family of groups. On the disjoint union $\sqcup G_i$ let $\sim$ be the equivalence relation generated by $g_j \sim \varphi_{ji}(g_i)$ if $i \le j$. Thus, if $g \in G_i$, we identify g with any of its images $\varphi_{ji}(g) \in G_j$.

We endow $\sqcup G_i/\sim$ with the following group operation: if $g \in G_i$ and $h \in G_j$ then choose any $k \ge i, j$, push g and h into the same G_k using the structure maps φ_{ki} and φ_{kj} respectively, and multiply them in G_k. Thus, if $[g]$ denotes the equivalence class of $g \in G_i$ in $\sqcup G_i/\sim$, and similarly for h, then $[g] \cdot [h] := [\varphi_{ki}(g)\varphi_{kj}(h)]$. This is easily seen to be a group operation.

The morphisms $\varphi_i\colon G_i \to \varinjlim G_i$ are the evident maps; G_i embeds firstly into the disjoint union, as a set, and then by the quotient map into the direct limit, and this is clearly a group homomorphism. We leave it to the reader to check the universal property.

The following exercise gives practise in dealing with inductive limits (of abelian groups). These examples appear later as K_0-groups of certain inductive limit C*-algebras.

Example 8.3. With the usual ordering on the natural numbers, making it a directed set, for $n \le m$, let $\varphi_{n,m} : \mathbb{Z} \to \mathbb{Z}$ be the group homomorphism of multiplication by 2^{m-n}.

The corresponding direct limit G of groups is $\mathbb{Z}[\frac{1}{2}]$, the subgroup of $\mathbb{Z}$ generated by $\mathbb{Z}$ and the numbers $\frac{1}{2^n} \in \mathbb{Q}$, $n = 1, 2, \ldots$.

To see this, note that a typical element of the inductive limit is the equivalence class $[(n, m)]$ of a pair (n, m). The equivalence relation is that $(n, m) \sim (n + k, 2^k m)$ for $k = 1, 2, \ldots$. The group operation at the level of pairs is

$$[(n, m)] + [(r, s)] := [(n + r, 2^r m + 2^n s)].$$

The map $\phi\colon \varinjlim \mathbb{Z} \to \mathbb{Z}[\frac{1}{2}]$ by

$$\phi([(n, m)]) := \frac{m}{2^n}.$$

is a well-defined group isomorphism.

The same works for any positive integer d, not just $d = 2$.

Exercise 8.4. Let $\mathbb{N}$ be made into a directed set by letting $n \le m$ if $n|m$.

For each $n|m$ let $\varphi_{nm}\colon \mathbb{Z} \to \mathbb{Z}$ be the group homomorphism of multiplication by $\frac{m}{n}$ (an integer).

Show that $\varinjlim \mathbb{Z} \cong \mathbb{Q}$. *Hint.* We denote elements of the direct limit as classes $[(n, m)]$ of pairs $(n, m) \in \mathbb{N} \times \mathbb{Z}$ of integers, where the equivalence relation is that $(n, m) \sim (nk, mk)$. Identify $[(n, m)]$ with the fraction $\frac{n}{m}$.

Proposition 8.5. *Direct limits exist in the category of C*-algebras and *-homomorphisms.*

For the proof, which we will give only in the situation where the structure maps φ_{ij} of the system are injective, we will need the following Lemma; the (easy) result is the content of Corollary 1.17 of Chapter 3 and we refer the interested reader to the proof presented there.

Lemma 8.6. *If A and B are C*-algebras and $\varphi\colon A \to B$ is a *-homomorphism, then φ is norm contractive:*

$$\|\varphi(a)\| \le \|a\|, \quad \forall a \in A.$$

If φ is injective, then it is isometric.

$$\|\varphi(a)\| = \|a\|, \quad \forall a \in A.$$

PROOF. (Of Proposition 8.5). We will assume for simplicity that the structure maps $\varphi_{ij}\colon A_j \to A_i$ are all injective. See Example 5.36 for the general case.

As in the example of a direct limit of groups, we start by defining the algebra direct limit as $\mathcal{A} := \sqcup A_i/\sim$ where $\sim$ is the equivalence relation generated by identifying $a \in A_i$ with $\varphi_{ji}(a_i) \in A_j$ for any $i \leq j$. Note that this results in the zero elements $0 \in A_i$ all being identified (similarly the identity elements of the groups G_i are all identified in the construction of the direct limit of groups). Let $\varphi_i\colon A_i \to \mathcal{A}$ be the evident maps of A_i into $\mathcal{A}$.

As in the case of groups, we endow $\mathcal{A}$ with the structure of a *-algebra. If we wish to multiply $a \in A_i$ and $b \in A_j$ (or more precisely, if we want to multiply $\varphi_i(a)$ and $\varphi_j(b)$ in $\mathcal{A}$, we we instead choose $k \geq i, j$ and define the product to be the class in $\mathcal{A}$ of

$$\varphi_{ki}\big(\varphi_i(a)\big) \cdot \varphi_{kj}\big((\varphi_j(b)).$$

Similarly, we define the sum of two elements. The adjoint may be defined in the obvious way. We obtain a *-algebra $\mathcal{A}$. If $a \in A_i$, we set $\|\varphi_i(a)\| := \lim_{j\to\infty}\|\varphi_{ji}(a)\|$. The limit exists because it is a decreasing net of positive real numbers, because *-homomorphisms are automatically contractive, by Lemma 8.6. Now, if the φ_{ij} are all injective, then by the Lemma, they are isometric, and the norm defined above on $\mathcal{A}$ is actually a norm. In this case we can $\varinjlim A_i$ to be the completion of $\mathcal{A}$ with respect to this norm. It is easy to see that this results in a C*-algebra.

In the general case (if the φ_{ij} are not all injective), we obtain a pre-C*-algebra $(\mathcal{A}, \|\cdot\|)$ in the sense of Definition 5.24 of Chapter 3, and we define $\varinjlim A_i$ to be its completion in the sense of completions of pre-C*-algebras as discussed in Chapter 3 – we set aside the point for the present, as we are mainly interested, for the moment, in examples where the structure maps are injective.

We leave it as an exercise to check the universal property. □

EXERCISE 8.7. Show that if the structure maps $\varphi_{ij}\colon A_j \to A_i$, $i \geq j$ of a directed system are all injective, then the induced inclusions $\varphi_j\colon A_i \to \varinjlim A_i$ into the direct limit, are also injective, so that case $\varinjlim A_i$ is effectively the closure of the *union* of the A_i's in this case.

EXAMPLE 8.8. Infinite direct sums $\oplus_{i\in I}A_i$ of a family $\{A_i\}_{i\in I}$ of C*-algebras, are inductive limits of finite direct sums.

Indeed, let $\mathcal{F}$ be the directed set of finite subsets of I under inclusion. If $F \in \mathcal{F}$, let $A_F := \oplus_{i\in F}A_i$. If $F_1 \leq F_2$ there is an associated injective C*-algebra homomorphism $\varphi_{F_1,F_2}\colon A_{F_1} \to A_{F_2}$ by adding zeros to an F_1-tuple until one gets an F_2-tuple. We obtain a directed system of C*-algebras.

For each finite subset F, let $\psi_F\colon A_F \to B := \oplus_{i\in I}A_i$ be the map which adds zeros to an F-tuple to get an I-tuple. Evidently if $F_1 \leq F_2$ then $\psi_{F_2} \circ \varphi_{F_1,F_2} = \psi_{F_1}$, so we obtain a *-homomorphism

$$\varinjlim A_F \to \oplus_{i\in I}A_i.$$

Thus, we recover infinite direct sums by combining inductive limits and finite sums.

EXAMPLE 8.9. There is a natural C*-algebraic analogue of the inductive system of Example 8.3. The directed system is $\mathbb{N}$ with its usual ordering.

For $n = 1, 2, \ldots$ we set $A_n := M_{2^n}(\mathbb{C})$. If $m \geq n$, the map $\varphi_{n,m}\colon M_{2^n}(\mathbb{C}) \to M_{2^m}(\mathbb{C})$ places 2^{m-n} copies of a matrix $A \in M_{2^n}(\mathbb{C})$ along the diagonal, to make a matrix in $M_{2^m}(\mathbb{C})$.

For example,

$$\varphi_{1,2}(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}) = \begin{bmatrix}1 & 2 & 0 & 0\\ 3 & 4 & 0 & 0\\ 0 & 0 & 1 & 2\\ 0 & 0 & 3 & 4\end{bmatrix}$$

The inductive limit is a special kind, called a UHF algebra, and this particular one is usually denoted $U(2^\infty)$ in the literature.

Example 8.10. The inductive system of groups of 8.4 also has a kind of C*-algebraic analogue. Here the directed system is the natural numbers with the relation $n \leq m$ if and only if $n|m$.

If A is a k-by-k matrix then we can place l copies of A along the diagonal of a kl-by-kl matrix

$$\varphi_{kl,k}(A) = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix}.$$

This procedure define a *-homomorphism $\varphi_{n,m} \colon M_l(\mathbb{C}) \to M_{kl}(\mathbb{C})$; put otherwise, if n and m are positive integers and $n|m$, then by the procedure just explained we obtain a canonical unital *-homomorphism $M_n(\mathbb{C}) \to M_m(\mathbb{C})$.

We set

$$\mathcal{N} = \varinjlim_n M_n(\mathbb{C})$$

to be the corresponding direct limit.

Example 8.11. The *Bunce-Deddens* algebra is defined by the following inductive system. For each natural number n let $\phi_{n+1,n} \colon C\big(S^1, M_{2^n}(\mathbb{C})\big) \to C\big(S^1, M_{2^{n+1}}(\mathbb{C})\big)$ be the *-homomorphism $\phi_{n+1,n}(f)(z) := \begin{bmatrix} f(z^2) & 0 \\ 0 & f(z^2) \end{bmatrix}$. The Bunce-Deddens algebra B^{2^∞} is defined by

$$B_{2^\infty} := \varinjlim C\big(S^1, M_{2^n}(\mathbb{C})\big).$$

One similarly can define B_{d^∞} for any natural number d.

Exercise 8.12. Let X be a locally compact Hausdorff topological space. Let $\mathcal{U}$ be the directed set of all pre-compact open subsets of X, under the inclusion relation.

If $U \subset V$ and $f \in C_0(U)$, then by extending f to zero on $V \setminus U$, we obtain a continuous function $\varphi_{V,U}(f) \in C_0(V)$. Check that this describes an inductive system $\{\varphi_{V,U} \colon C_0(U) \to C_0(V) \mid U \subset V\}$, and prove that the associated direct limit satisfies $C_0(X) \cong \varinjlim C_0(U)$.

9. Crossed-product C*-algebras

In dynamical systems, one is generally interested in the long-term, or asymptotic properties of a group action, such as, for instance, the action of the group of integers induced by a single homeomorphism $\varphi \colon X \to X$ of (usually) a compact space.

We might be interested, for example, in the case of such an integer action, in the orbit of a single point: how the orbit $\{\varphi^n(x)\}_{n\in\mathbb{Z}}$ wanders around the space.

One might also be interested in parameterizing the *set* of all the orbits. In fact, this set, the quotient of X by the equivalence relation $x \sim y$ if $y = \varphi^n(x)$ for some $n \in \mathbb{Z}$, has a natural topology: the quotient topology. One might therefore hope that one could study the *space* of orbits, and acquire information about the dynamics for instance, by computing the standard invariants of algebraic topology of this space.

However, as the following example shows, in examples which are interesting from a dynamical systems point of view, the quotient space of a space by an interesting action, is rarely any good as a topological space.

EXAMPLE 9.1. Let $\omega = e^{2\pi i\theta} \in \mathbb{T}$ with θ irrational and let $R_\theta\colon \mathbb{T} \to \mathbb{T}$ be group multiplication by ω, (or rotation by θ.) Then every orbit of R_θ is dense in $\mathbb{T}$, and the only open sets in $\mathbb{Z}\backslash\mathbb{T}$ are $\emptyset$ and $\mathbb{Z}\backslash\mathbb{T}$. That is, the quotient topology on $\mathbb{Z}\backslash\mathbb{T}$ is trivial. This is because there are no nonempty proper open subsets of $\mathbb{T}$ invariant under R_θ.

The crossed-product construction makes a C*-algebra which is a substitute in a certain sense for the (C*-algebra of continuous functions on the) space of orbits of the action, but which is *noncommutative*. This crossed-product C*-algebra, carries a great amount of interesting information about the action – unlike, in general, the quotient space. The C*-algebra $C(\mathbb{Z}\backslash\mathbb{T})$ of continuous functions on the quotient space of Example 9.1, with the quotient topology, is simply isomorphic to $\mathbb{C}$, the constant functions on $\mathbb{Z}\backslash\mathbb{T}$. But the C*-algebraic crossed-product $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ associated to the action of the integers generated by group translation by θ, called the *irrational rotation algebra* is an extremely interesting, noncommutative C*-algebra, containing an amazing amount of finite geometric information about arithmetic properties of $\theta \in \mathbb{R}/\mathbb{Z}$, and the dynamics of the action.

Let G be a discrete group.

An *action* of G on a locally compact Hausdorff space X is a group homomorphism $G \to \mathrm{Homeo}(X)$ of G into the group of homeomorphisms of X. We say that X is a *G-space*.

If G acts on X, the G acts by C*-algebra automorphisms of the C*-algebra $C_0(X)$ by $f \mapsto f \circ g^{-1}$. More generally, a *group action on a C*-algebra* A is a group homomorphism $G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of *-automorphisms of A.

We call A a *G-C*-algebra*.

DEFINITION 9.2. Let G be a discrete group and A be a G-C*-algebra. The *twisted group algebra* $A[G]$ is the vector space $C_c(G, A)$ of finitely supported functions from G to A.

We write an element of $C_c(G, A)$ in the form $\sum_{g\in G} a_g[g]$, with $a_g \in A$ being the value of the function at g, and where, it is understood, that $a_g = 0$ for all but finitely many $g \in G$. With this convenient notation, we equip $C_c(G, A)$ with an algebra multiplication and an involution:

$$(\sum_{g\in G} a_g[g]) * (\sum_{g\in G} b_g[g]) = \sum_{g,h} a_g g(b_h)[gh], \quad (\sum_{g\in G} a_g[g])^* := \sum_{g\in G} g^{-1}(a_g^*)[g^{-1}] \tag{9.1}$$

EXERCISE 9.3. If $A = \mathbb{C}$, then $A[G]$ with twisted convolution is the same as the group algebra $\mathbb{C}[G]$ with convolution, as in (3.7). In particular, $\mathbb{C}[G]$ is a *-subalgebra of $A[G]$ if A is *unital*.

The algebra multiplication (9.1) is a kind of twisted version of the convolution operation (3.7) on scalar-valued functions on groups. It can be re-written

$$(f_1 \star f_2)(g) = \sum_{h\in G} f_1(h)h\left[f_2(h^{-1}g)\right], \quad f_1, f_2 \in C_c(G, A). \tag{9.2}$$

We generally prefer the group algebra notation, as it involves fewer brackets and seems more transparent.

EXERCISE 9.4. The reader familiar with tensor products, which are discussed later, might wish to prove that if G acts *trivially* on A, *i.e.* if every group element acts by the identity automorphism, then $A[G] \cong A \otimes \mathbb{C}[G]$, the (algebraic) tensor product of A and $\mathbb{C}[G]$ in the category of *-algebras.

EXERCISE 9.5. Answer the following questions about the twisted group algebra construction.

a) Prove that the map $A \to A[G]$, $a \mapsto a[e]$, with $e \in G$ the identity, is a *-homomorphism of *-algebras. Thus, A can be viewed as a *-subalgebra of $A[G]$, consisting of functions supported at the identity of the group.

b) Prove that if A is unital, then the elements $[g] \in A[G]$ are unitaries: $[g]^* = [g^{-1}] = [g]^{-1}$, that the resulting copy of the group G as unitaries in $A[G]$ normalizes the copy $A \subset A[G]$, and that $[g]a[g]^* = g(a)$ for all $g \in G$ and $a \in A$.

Thus, $A[G]$ is a larger algebra than A, in which the original action of G becomes one by *inner automorphisms*.

c) If G acts on C*-algebras A and B, and if there is a G-equivariant *-isomorphism $\alpha\colon A \to B$, then α induces a canonical *-homomorphism of *-algebras $A[G] \to B[G]$.

d) More generally, let A and B be C*-algebras, G and G' be two discrete groups, with G acting on A and G' acting on A'. Suppose that $\alpha\colon A \to A'$ is a *-homomorphism, $\varphi\colon G \to G'$ is a group homomorphism, and that

$$\varphi(g)\big(\alpha(a)\big) = \alpha\big(g(a)\big). \tag{9.3}$$

Then check that the map $A[G] \to A'[G']$ mapping $\sum_{g\in G} a_g[g]$ to $\sum_{g\in G} \alpha(a_g)[\varphi(g)]$ is a *-algebra homomorphism.

e) Suppose that B is a unital C*-algebra, $\alpha\colon A \to B$ is a *-homomorphism, and $\varphi\colon G \to \mathbf{U}(B)$ is a group homomorphism from G into the group of unitaries in B, such that the *covariance condition*

$$\varphi(g)\alpha(a)\varphi(g)^* = \alpha\big(g(a)\big) \tag{9.4}$$

holds for all $a \in A$, $g \in G$. Then α and φ combine to make a *-homomorphism $A[G] \to B$, mapping $\sum_{g\in G} a_g[g]$ to $\sum_{g\in G} \alpha(a_g)\varphi(g)$.

The pair (α, φ) is called a *covariant pair*. Prove that any unital $*$-homomorphism $A[G] \to B$ to a unital C*-algebra, arises from a covariant pair.

Example 9.6. Let $X = \{1, 2\}$ be the 2-point space, $A = C(X) \cong \mathbb{C} \oplus \mathbb{C}$, and $G = \mathbb{Z}/2$ act with the generator u flipping the points. Since the group has two elements every element of $C(X)[G]$ can be written $f + g[u]$. Associate to $f + g[u]$ the matrix $\begin{bmatrix} f(1) & g(1) \\ g(2) & f(2) \end{bmatrix}$.

Under twisted convolution we have

$$(f + g[u])(f' + g'[u]) = ff' + gu(g') + \big(gu(f') + fg'\big)[u],$$

which is easily checked to correspond to the product of the two matrices.

The adjoint: $(f + g[u])^* = f^* + u(g^*)[u]$ (since $u = u^{-1}$) corresponds to the adjoint on 2-by-2 matrices.

Hence $C(\{1, 2\})[\mathbb{Z}/2] \cong M_2(\mathbb{C})$ as *-algebras.

Exercise 9.7. Generalize the above and prove that $C(X)[\mathbb{Z}/n] \cong M_n(\mathbb{C})$ if X is the n-point space $\{1, 2, \ldots, n\}$ and $\mathbb{Z}/n$ acts on X by shifting.

In this section, we are going to give a provisional definition of the crossed-product C*-algebras $A \rtimes G$, which is a completion of $A[G]$. The definition we give here is fairly concrete, but is not very well adapted to proving general things. We return to the issue of completions later.

Definition 9.8. Let the discrete group G act on the C*-algebra A by automorphisms, and suppose that $\pi\colon A \to \mathbb{B}(H)$ is an injective representation of A on a Hilbert space H. Let $l^2(G, H)$ be the Hilbert space of L^2-functions on G valued in H, that is, the completion of $C_c(G, H)$ under the inner product $\langle f_1, f_2\rangle := \sum_{g\in G}\langle f_1(g), f_2(g)\rangle$.

Define a *covariant pair*, in the sense of Exercise 9.5, e), and induced *-homomorphism

$$\mathrm{Ind}(\pi)\colon A[G] \to \mathbb{B}\big(l^2(G, H)\big) \tag{9.5}$$

by

$$(g\xi)(h) = \xi(g^{-1}h), \quad (a \cdot \xi)(h) = \pi\big(h^{-1}(a)\big)(h).$$

The *reduced crossed-product* $A \rtimes G$ is the closure of $A[G]$ in the norm

$$\|\sum a_g[g]\| := \|\pi(\sum a_g[g])\|,$$

the norm being the operator norm on $l^2(G, H)$.

By definition $A \rtimes G$ comes equipped with a natural, injective representation, which we continue to denote by $\mathrm{Ind}(\pi)\colon A \rtimes G \to \mathbb{B}\big(l^2(G, H)\big)$, by extending the homomorphism (9.5).

Of course, if G is a finite group, acting on X, then $A \rtimes G = A[G] = C(X)[G]$, since the latter C*-algebra is already complete with respect to the norm defined in Definition 9.8, for any choice of π (exercise).

Furthermore, any group G acts trivially on the C*-algebra $\mathbb{C}$, it is essentially completely obvious from the definitions that $\mathbb{C} \rtimes_r G$ is exactly the same thing as the reduced C*-algebra $C_r^*(G)$ of Definition 3.2, where, of course, for an injective representation of $\mathbb{C}$ we use the identity map.

Example 9.9. Let G act on X locally compact Hausdorff by homeomorphisms, with induced action by C*-algebra automorphisms on $C_0(X)$ by $g(f) := f \circ g^{-1}$. Let μ be a Borel measure on X of full support, so that $f \in C_c(X)$ $f \neq 0$ and $f \geq 0$, implies $\int f d\mu > 0$. and $\pi_\mu\colon C_0(X) \to \mathbb{B}\big(L^2(X, \mu)\big)$ the representation by multiplication operators.

Then $\mathrm{Ind}(\pi_\mu) =: \lambda_\mu$ is an injective representation of $C_0(X) \rtimes G$ on $l^2\big(G, L^2(X, \mu)\big)$. It is injective because π_μ is, because μ has full support.

The formulas for the covariant pair are given by

$$(g\xi)(h) = \xi(g^{-1}h), \quad (f \cdot \xi)(h) = h^{-1}(f) \cdot \xi(h).$$

It is a good exercise to check that this really is a covariant pair. Note that there is no implied G action on X, and that when X is a point, we recover the left regular representation of $C_r^*(G)$.

Example 9.10. Let $G = \mathbb{Z}/2$ acting on $X := [-1, 1]$ by letting the generator u of G act by $u(x) = -x$. Then

$$C_0(X) \rtimes G \cong \{f\colon [0, 1] \to M_2(\mathbb{C}) \mid f \text{ is continuous, and } f(0) \text{ is a diagonal matrix}\}.$$

As G is finite, $C(X) \rtimes_r G = C(X)[G]$, *i.e.* no completion is involved in forming the crossed-product. So there is no need to locate an injective representation of $C(X)$. Instead, to understand the C*-algebra better, define a covariant pair and induced *-homomorphism $C(X)[G] \to M_2(\mathbb{C})$ by letting the group generator u map to the constant (unitary) matrix-valued function $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ on $[0, 1]$. If $f \in C([-1, 1])$, we map f to the matrix-valued function on $[0, 1]$ (note the change of domain) given by

$$\tilde{f}(x) = \begin{bmatrix} f(x) & 0 \\ 0 & f(-x) \end{bmatrix}.$$

This is clearly a covariant pair (exercise). It is injective on functions because if both $f(x)$ and $f(-x)$ vanish on $[0, 1]$ then f vanishes on $[-1, 1]$. An element $f + g[u] \in C(X)[G]$ is mapped to the function T with

$$T(x) = \begin{bmatrix} f(x) & g(x) \\ g(-x) & f(-x) \end{bmatrix}$$

at $x \in [0, 1]$. Fix any $x \neq 0$. The collection of matrices $T(x)$ obtained from some f, g is then $M_2(\mathbb{C})$. To see this, note that the range is automatically a *-subalgebra of $M_2(\mathbb{C})$. Since we can find a continuous function f with value 1 at x and value 0 at $-x$, and setting $g = 0$, we obtain the matrix

$$T = \begin{bmatrix} f(x) & 0 \\ 0 & f(-x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since the image under our *-homorphism is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and these two matrices generate $M_2(\mathbb{C})$ as an algebra, we obtain that the possible set of values of $T(x)$ is all of $M_2(\mathbb{C})$, if $x \neq 0$.

If $x = 0$, consider the collection of matrices of the form

$$\begin{bmatrix} f(0) & g(0) \\ g(0) & f(0) \end{bmatrix}.$$

for some continuous f, g on $[-1, 1]$. This is just the *-algebra of matrices

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

and this collection is simultaneously diagonalizable using the eigenvectors $\frac{1}{\sqrt{2}}(1, 1)$ and $\frac{1}{\sqrt{2}}(1, -1)$. From this we conclude, as is easily checked by the definitions, that the algebra obtained at $x = 0$ is isomorphic to $\mathbb{C} \oplus \mathbb{C}$ by the map

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \mapsto (a + b, a - b).$$

We obtain therefore a sort of 'picture' of this C*-algebra, as the space of sections of a continuous field of C*-algebras over $[0, 1]$ equal to $M_2(\mathbb{C})$ for $x \neq 0$, and equal to $\mathbb{C} \oplus \mathbb{C}$ at $= 0$.

As we will see later, the contribution at $x = 0$ is the representation ring $\mathrm{Rep}(\mathbb{Z}/2)$ of the isotropy group at that point. The relation of $[0, 1]$ to the original space $[-1, 1]$ on which the group acts, is that $[0, 1]$ is naturally homeomorphic to the quotient space of $[-1, 1]$ by the $\mathbb{Z}/2$-action.

EXERCISE 9.11. Prove that $C([-1, 1]) \rtimes \mathbb{Z}/2$ is isomorphic to the C*-algebra of continuous functions $f\colon [0, 1] \to M_2(\mathbb{C})$ such that $f(0)$ is a diagonal matrix.

EXAMPLE 9.12. (*Rotation by an irrational angle.*) Let $\omega = e^{2\pi i\theta} \in \mathbb{T}$ with θ irrational. The corresponding homeomorphism $R_\theta\colon \mathbb{T} \to \mathbb{T}$ of group multiplication by ω, or rotation by θ, has infinite order, since ω has infinite order (an easy exercise). We obtain an action of the integers $\mathbb{Z}$ on $\mathbb{T}$ and on $C(\mathbb{T})$ with the integer n acting by $f \mapsto f \circ R_\theta^{-n}$.

The corresponding C*-algebra crossed-product, which we frequently denote in the form $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ is called the *irrational rotation algebra* and is denoted A_θ.

It has some remarkable properties, among which is its *simplicity*: it has no proper, nonzero closed ideals.

As an injective representation of $C(\mathbb{T})$ to build the crossed-product, we can take $H := L^2(\mathbb{T})$ with $C(\mathbb{T})$ acting by multiplication operators. We then complete the group algebra $C(\mathbb{T})[\mathbb{Z}]$ by letting it act as operators on $l^2\big(\mathbb{Z}, L^2(\mathbb{T})\big)$ in the usual way. The group algebra $C(\mathbb{T})[\mathbb{Z}]$ consists of all finite sums $\sum_{n=-N}^{N} f_n[n]$ with $f_n \in C(\mathbb{T})$. If we require f to be a trigonometric polynomial we obtain a smaller *-subalgebra consisting of all finite double sums $\sum a_{n,m} z^n[m]$, and these act on the Hilbert space $l^2\big(\mathbb{Z}, L^2(\mathbb{T})\big)$ by operators of the form

$$\sum a_{n,m} u^n v^m, \tag{9.6}$$

where u is the operator on $l^2\big(\mathbb{Z}, L^2(\mathbb{T})\big)$ corresponding to the function $f(z) = z \in C(\mathbb{T}) \subset C(\mathbb{T})[\mathbb{Z}]$ and v the operator corresponding to the group generator $[1] \in \mathbb{C}[\mathbb{Z}] \subset C(\mathbb{T})[\mathbb{Z}]$.

If we use the standard identification of $L^2(\mathbb{T})$ as $l^2(\mathbb{Z})$, then we can identify $l^2\big(\mathbb{Z}, L^2(\mathbb{T})\big)$ with $l^2(\mathbb{Z} \oplus \mathbb{Z})$ with orthonormal basis $e_{n,m}$, with $e_{n,m}$ corresponding to $z^n[m] \in l^2\big(\mathbb{Z}, L^2(\mathbb{T})\big)$.

EXERCISE 9.13. In this notation,

$$u(e_{n,m}) = \omega^{-(n-1)} e_{n,m-1}, \quad v(e_{n,m}) = e_{n,m-1},$$

so that v acts as a 'horizontal' shift, and u by a vertical, weighted shift.

Verify that the basic relation

$$uv = \omega vu \tag{9.7}$$

for this pair of unitary operators. Of course this follows from a similar relation in the group algebra $C(\mathbb{T})[\mathbb{Z}]$. If $f(z) = z$ then $(f \circ R_\theta^{-1})(z) = \bar{\omega}z$. Hence in group algebra notation,

$$[1]z[1]^* = \bar{\omega}z.$$

REMARK 9.14. It turns out that A_θ is the unique C*-algebra, up to canonical isomorphism, generated by a pair u, v of unitaries satisfying $uv = \omega vu$, *i.e.* satisfying (9.7).

EXERCISE 9.15. By taking an orbit $\mathbb{Z}x_0$ of the irrational rotation action by $\mathbb{Z}$ on $\mathbb{T}$, we obtain in particular a (dense) subset of the circle. Any continuous function on $\mathbb{T}$ restricts to a bounded function on this subset, and so we obtain a representation $C(\mathbb{T}) \to \mathbb{B}(l^2\mathbb{Z})$. Inducing it in the manner explained above we obtain a representation

$$\pi_{x_0} \colon C(\mathbb{T}) \rtimes \mathbb{Z} = A_\theta \to \mathbb{B}\big(l^2(\mathbb{Z}^2)\big).$$

a) Show that the representations π_{x_0} are all injective.
b) Show that if x_0 and x_1 are in the same orbit then the representations π_{x_0} and π_{x_1} are unitarily equivalent.

EXAMPLE 9.16. (*Action of a free group on its boundary*). Let $G = \mathbb{F}_2$ be the free group on 2-generators a, b. Elements of $\mathbb{F}_2$ may be written uniquely as reduced words $s_1 \cdots s_n$ with s_i in the generating set $S := \{a, a^{-1}, b, b^{-1}\}$, where a word is of course reduced if it contains no occurrence of an $s_i s_i^{-1}$.

The collection of *infinite* reduced words $s_1 s_2 \cdots$ may be considered in an obvious way as a subspace of the Cantor space $\prod_{n=1}^\infty S$. It consists of those sequences for which no term s_i is followed by s_i^{-1}, for any $i = 1, ..4$. It is an easy exercise to prove that this is a closed subspace. We let $\partial\mathbb{F}_2$ be the collection of infinite such words, with the subspace topology. It is a Cantor set.

The left translation action of $\mathbb{F}_2$ on itself extends in a rather obvious way to an action of $\mathbb{F}_2$ on $\partial\mathbb{F}_2$ by left group multiplication on infinite reduced words. For example the group element $g = ab^{-2} \in \mathbb{F}_2$ maps the boundary point $\xi = baba^{-1}bbbb\cdots \in \partial\mathbb{F}_2$ to $g(\xi) = ab^{-1}aba^{-1}bbb\cdots$.

The dynamics of this group action is very interesting, and it's C*-algebra has some special properties not possessed by the irrational rotation algebra A_θ.

EXERCISE 9.17. Let $i \in S$ be a generator, and $U_i \subset \partial\mathbb{F}_2$ the clopen subset of all infinite reduced words beginning in i. Let $\chi_i := \chi_{U_i} \in C(\partial\mathbb{F}_2)$ and

$$s_i := \chi_s[i] \in C(\partial\mathbb{F}_2)[\mathbb{F}_2] \subset C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2.$$

a) Prove that the s_i's, for $i \in S$ are partial isometries, and that
$$\sum_{i \in S} s_i s_i^* = 1, \quad \text{and} \quad s_j^* s_j = \sum_{i \neq j^{-1}} s_i s_i^*.$$
b) If $g = i_1 \cdots i_k$ is a reduced word in $\mathbb{F}_2$, let $s_g := s_{i_1} \cdots s_{i_k}$. Prove that
$$s_g s_g^* = \chi_{U_g}$$
where U_g is all infinite reduced words in $\partial\mathbb{F}_2$ which begin with g.
c) Prove that $\{s_i \mid i \in S\}$ generates $C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2$. In fact, this is an example of a *Cuntz-Krieger algebra*. Such algebras occur in connection with topological Markov chains, as we discuss later.

The essential features of this example generalize in a significant way – the group $\mathbb{F}_2$ can be replaced by a general *Gromov hyperbolic group* G, for which a natural geometrically defined boundary ∂G can be defined, giving a compactification $\overline{G} = G \cup \partial G$, all of which is invariant under the group action. The dynamics of G on its boundary has many interesting and important features (see [**11**]).

EXAMPLE 9.18. Let $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{\pm\infty\}$ the usual 2-point compactification of the integers. Let the group of integers act on this set by translation, fixing the points at $\pm\infty$. This induces an action of $\mathbb{Z}$ on the C*-algebra $C(\overline{\mathbb{Z}})$.

The C*-algebra $C(\overline{\mathbb{Z}})$ is represented on $l^2(\mathbb{Z})$ by multiplication operators. Together with the left-regular representation of $\mathbb{Z}$ on the same Hilbert space, we obtain a covariant pair and *-homomorphism

$$C(\overline{\mathbb{Z}})[\mathbb{Z}] \to \mathbb{B}\big(l^2(\mathbb{Z})\big).$$

For the following exercise, the reader may assume the fact (which is not obvious) that this representation extends continuously to an injective representation of $C(\overline{\mathbb{Z}}) \rtimes \mathbb{Z}$ on $l^2(\mathbb{Z})$.

EXERCISE 9.19. A *corner* of a C*-algebra is a sub-C*-algebra of the form pAp, where $p \in A$ is a projection. A corner is *full* if ApA (the ideal generated by p) is dense in A. Prove that the Toeplitz algebra is a corner of the crossed-product C*-algebra $C(\overline{\mathbb{Z}}) \rtimes_r \mathbb{Z}$, but that it is not full.

The C*-algebra $C(\overline{\mathbb{Z}}) \rtimes_r \mathbb{Z}$ may be identified using Fourier transform, with the C*-algebra closure of the *-algebra $\Psi(\mathbb{T})$ of zero order *pseudodifferential* operators on the circle.

Show that the residue trace (7.15) extends to a trace on $C(\overline{\mathbb{Z}}) \rtimes_r \mathbb{Z}$. Show that elements of $C(\overline{\mathbb{Z}}) \rtimes_r \mathbb{Z}$ have sort of 'symbols,' generalizing the symbols of Toeplitz operators.

EXAMPLE 9.20. (*Rational rotations of the circle*). Let $\mathbb{Z}/n$, the cyclic group of order n, be realized as the corresponding subgroup of roots of unity in the circle $\mathbb{T}$. This subgroup then acts by group multiplication on $\mathbb{T}$. Taking a primitive nth root of unity $\omega = \exp(\frac{2\pi i}{n}) \in \mathbb{T}$ representing the generator of $\mathbb{Z}/n$, its action on $\mathbb{T}$ is rotation by $\frac{2\pi}{n}$ radians.

For a faithful representation of $A = C(\mathbb{T})$ we use the representation by multiplication operators on $L^2(\mathbb{T})$. The Hilbert space $l^2\big(\mathbb{Z}/n, L^2(S^1)\big)$ may be identified with $L^2(\mathbb{T}, \mathbb{C}^n)$. Let $f \in C(\mathbb{T})$, then by the definitions

- f acts by the $M_n(\mathbb{C})$-valued function
$$\tilde{f}(z) = \begin{bmatrix} f(z) & & & \\ & f(\omega z) & & \\ & & \cdots & \\ & & & f(\omega^{n-1} z) \end{bmatrix},$$
where we let such matrix-valued functions act on $L^2(\mathbb{T}, \mathbb{C}^n)$ in the obvious way.
- The group $\mathbb{Z}/n$ acts on $L^2(\mathbb{T}, \mathbb{C}^n)$ by the unitary representation implemented by sending the generator ω to the shift
$$U := \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

A change of coordinates makes things more clear. The Hilbert space $\mathbb{C}^n$ with $\mathbb{Z}/n$ acting by the shift is the regular representation of $\mathbb{Z}/n$, and it decomposes into character spaces $V_k = \{v \in \mathbb{C}^n \mid Uv = \omega^k v\}$ for the set of characters of $\mathbb{Z}/n$, which may be identified with the set of nth roots of unity, *i.e.* the points $1, \omega, \omega^2, \ldots, \omega^{n-1}$, (each power ω^k determines a character of $\mathbb{Z}/n$ by sending the generator ω to ω^k). So we have another orthonormal basis $v_0 \dots, v_{n-1}$ for $\mathbb{C}^n$with, explicitly, v_k the vector $\frac{1}{\sqrt{n}}(\omega^{-k}, \omega^{-2k}, \omega^{-3k}, \dots, \omega^{-nk})$. With respect to this orthonormal basis, U acts by the (constant) diagonal matrix-valued function

$$U(z) = \begin{bmatrix} 1 & & & \\ & \omega & & \\ & & \dots & \\ & & & \omega^{n-1} \end{bmatrix}.$$

while the function $f(z) = z$ on the circle acts by the weighted shift

$$V(z) = z \begin{bmatrix} 0 & 0 & \cdots & \omega^{n-1} \\ \omega & 0 & \cdots & 0 \\ 0 & \omega^2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Let P_0 be the diagonal matrix $\begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \dots & \\ & & & 0 \end{bmatrix}$. It is projection onto the span of the trivial character 1 (it is the matrix representation in this basis of the element $\frac{1}{n}\sum_{k=0}^{n-1} \omega^k \in C^*_r(\mathbb{Z}/n) \subset C(\mathbb{T}) \rtimes \mathbb{Z}/n$.) An easy computation shows that $V(z)^i P_0 V(z)^{-j} = z^{i-j}\omega^{i-j}E_{ij}$ where E_{ij} is the matrix with a 1 in the (i,j)th entry and has zeros in all other entries.

As the crossed-product also contains a copy of $C(\mathbb{T})$, identified with continuous functions valued in scalar multiples of the identity operator on $\mathbb{C}^n$, it follows that $C(\mathbb{T}) \rtimes_r \mathbb{Z}/n$ contains all matrix-valued functions of the form $f(z)E_{ij}$, for any $f \in C(\mathbb{T})$. Therefore it contains every element of $C\big(\mathbb{T}, M_n(\mathbb{C})\big)$.

This argument shows therefore that

PROPOSITION 9.21. *$C(\mathbb{T}) \rtimes \mathbb{Z}/n \cong C\big(\mathbb{T}, M_n(\mathbb{C})\big)$.*

EXERCISE 9.22. Prove that the *-algebras $A_\theta := C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ and $A_{-\theta} := C(\mathbb{T}) \rtimes_{-\theta} \mathbb{Z}$ are isomorphic, for any $\theta \in \mathbb{R}$ irrational, by constructing an appropriate covariant pair (see Exercise 9.5).

EXERCISE 9.23. Inside $\mathbb{Q}/\mathbb{Z}$ consider the group G generated by the cosets $\frac{1}{2^n} + \mathbb{Z}$. Let an element $r \in G$ act by group translation on S^1 by $\exp(2\pi i r)$. Prove that the resulting crossed-product $C(S^1) \rtimes_r G$ is isomorphic to the Bunce-Dedden's algebra B_{2^∞} defined above. See Example 9.20.

10. Fredholm index theory and the Toeplitz Index Theorem

In our discussion of the Toeplitz algebra in the previous section, several important points emerged. A primary one was that a (pseudo-)Toeplitz operator T, *when considered up to compact perturbation*, is exactly equivalent to a continuous function on the circle, its 'symbol.'

More precisely, the quotient Banach *-algebra $\mathcal{T}/\mathcal{K}$, with the quotient norm, is a C*-algebra isomorphic to $C(\mathbb{T})$, and the map $C(\mathbb{T}) \to \mathcal{T}/\mathcal{K}$, $f \mapsto T_f + \mathcal{K}$ is the isomorphism.

The Toeplitz Index Theorem, a special case of the Atiyah-Singer Index Theorem, and fundamental to the subject of operator algebras and Noncommutative Geometry, asserts the equality of two integer invariants of a pseudo-Toeplitz operator T with non-vanishing symbol.

The first comes from basic topology, and counts the number of times the symbol of T, a map $\mathbb{T} \to \mathbb{C}^*$, winds around the origin.

The second integer, called the Fredholm index, is is analytic in nature: it is by definition the difference in dimensions of the solutions spaces to the equations $T\xi = 0$ and to $T^*\xi = 0$.

It is not obvious that our assumption that f is non-vanishing, implies that these dimensions are even finite. This will fall out of the general theory of Fredholm operators.

In any case, the Toeplitz Index Theorem represents, therefore, an interesting bridge between analysis, on the one hand, and topology, on the other.

The Calkin algebra

By Corollary 6.10, for any Hilbert space H, the C*-algebra $\mathcal{K}(H)$ of compact operators on H is a closed *-ideal in $\mathbb{B}(H)$. By standard algebra, the quotient (vector space) $\mathbb{B}(H)/\mathcal{K}(H)$ is an algebra, under multiplication of cosets. Since $\mathcal{K}(H)$ is invariant under adjoint, $\mathbb{B}(H)/\mathcal{K}(H)$ inherits an adjoint operation as well.

Also, if we give the quotient $\mathbb{B}(H)/\mathcal{K}(H)$ the quotient norm

$$\|T + \mathcal{K}\| := \operatorname{dist}(T, \mathcal{K}) := \inf\{\|T + S\|; \mid S \in \mathcal{K}\},$$

then we obtain a Banach *-algebra. When we consider the smaller quotient $\mathcal{T}/\mathcal{K}$ in the previous section, we verified the C*-identity for the quotient norm. Similar arguments prove that actually the C*-identity holds for $\mathcal{Q}(H) := \mathbb{B}(H)/\mathcal{K}(H)$ as well, as we show.

Let $e_1, e_2, \ldots$ be an orthonormal basis for H and P_n as in the previous chapter be projection to the span of $e_n, e_{n+1}, \ldots$.

Lemma 10.1. *If* $T \in \mathbb{B}(H)$ *then*

$$\operatorname{dist}(T, \mathcal{K}) = \lim_{n\to\infty} \|P_n T\|.$$

Furthermore, $\lim_{n\to\infty}\|TP_n\|$*, and* $\lim_{n\to\infty}\|P_n T P_n\|$ *all equal* $\operatorname{dist}(T, \mathcal{K})$ *as well.*

The reader should compare to (7.17); that is, we have already verified the Lemma for Toeplitz operators T_f (indeed, it is obvious for them, since the sequence $\|P_1 T_f\|, \|P_2 T_f\|, \ldots$ was observed there to be *constant*.)

Proof. It is clear that $\operatorname{dist}(T, \mathcal{K}) \leq \|T - T(1 - P_n)\| = \|TP_n\|$, since $1 - P_n$ is compact. Since this is true for all n, we get

$$\operatorname{dist}(T, \mathcal{K}) \leq \liminf_{n\to\infty} \|TP_n\|.$$

On the other hand, if $S \in \mathcal{K}$ then

$$\|TP_n\| = \|TP_n + SP_n - SP_n\| = \|(T + S)P_n - SP_n\| \leq \|T + S\| + \|SP_n\|$$

and so, since $\lim_{n\to\infty}\|SP_n\| = 0$ by Exercise 7.19, we get

$$\limsup_{n\to\infty} \|P_n T\| \leq \operatorname{dist}(T, \mathcal{K}).$$

Putting these two results together completes the proof.

Exactly the same arguments show that the distance is equally computed by $\lim_{n\to\infty}\|TP_n\|$ or by $\lim_{n\to\infty}\|P_n T P_n\|$.

□

Corollary 10.2. $\mathcal{Q}(H) := \mathbb{B}(H)/\mathcal{K}(H)$ *endowed with the quotient norm* $\|T + \mathcal{K}\| := \operatorname{dist}(T, \mathcal{K})$ *is a C*-algebra.*

PROOF. The distance norm is always submultiplicative and is complete for for Banach algebras (see Exercise 567).

The proposed norm satisfies the C*-identity because

$$\operatorname{dist}(T,\mathcal{K})^2 = \lim_{n\to\infty}\|P_nT\|^2 = \lim_{n\to\infty}\|P_nTT^*P_n\| = \lim_{n\to\infty}\|P_nTT^*\| = \operatorname{dist}(TT^*,\mathcal{K}).$$

□

DEFINITION 10.3. The C*-algebra of cosets $T+\mathcal{K}(H)$, for $T\in\mathbb{B}(H)$, is called the *Calkin algebra* of H and will be denoted $\mathcal{Q}(H)$.

The quotient map from $\mathbb{B}(H)$ to $\mathcal{Q}(H)$ is usually denoted $\pi\colon \mathbb{B}(H)\to\mathcal{Q}(H)$. It is a surjective *-homomorphism with kernel $\mathcal{K}$.

The Toeplitz idea produces an interesting realization of $C(\mathbb{T})$ as a *-subalgebra of $\mathcal{Q}$; we summarize this below.

PROPOSITION 10.4. *The map*

$$\tau\colon C(\mathbb{T})\to\mathcal{Q}(\mathbf{H}^2),\quad \tau(f):=\pi(T_f),$$

*is an injective, unital *-homomorphism.*

In particular, if $f\in C(\mathbb{T})$ does not vanish anywhere on the circle, then $T_f+\mathcal{K}$ is an invertible in the C-algebra $\mathcal{Q}(\mathbf{H}^2)$.*

PROOF. By the definitions, $\mathcal{T}$ is a C*-subalgebra of $\mathbb{B}(\mathbf{H}^2)$, that is, there is an injective *-homomorphism $\mathcal{T}\to\mathbb{B}(\mathbf{H}^2)$. This *-homomorphism maps the ideal $\mathcal{K}(\mathbf{H}^2)$ to itself, and hence induces a C*-algebra homomorphism $\mathcal{T}/\mathcal{K}\to\mathbb{B}/\mathcal{K}$, which is injective by basic considerations of algebra.

Now since $C(\mathbb{T})\to\mathcal{T}/\mathcal{K}$ has already been shown to be a C*-algebra isomorphism in Theorem 7.16 d), the result follows.

□

Fredholm operators

We have already noted that a Toeplitz operator T_f with $f\in C(\mathbb{T})$ not vanishing anywhere, is invertible when considered as an element of the Calkin C*-algebra $\mathcal{Q}(\mathbf{H}^2)$. That is, T_f is 'invertible mod compacts,' when f is non-vanishing.

There is a general theory of such operators, which we now sketch.

LEMMA 10.5. *The following conditions are equivalent for a bounded operator $T\colon H\to K$ between two Hilbert spaces, and such T is called* Fredholm *if it satisfies them.*

1) *There exist bounded operators $Q,Q'\colon K\to H$ such that $QT-\mathrm{id}_H$ and $TQ'-\mathrm{id}_K$ are finite rank operators.*
2) *There exist bounded operators $Q,Q'\colon K\to H$ such that $QT-\mathrm{id}_H$ and $TQ'-\mathrm{id}_K$ are compact operators.*
3) $\ker(T)$ *and* $\operatorname{coker}(T):=H/\operatorname{ran}(T)$ *are each finite-dimensional vector spaces.*

Moreover, the range of an operator satisfying any of these equivalent conditions, is automatically closed.

If T is Fredholm, the index *of T is defined to be*

$$\operatorname{index}(T):=\dim\ker(T)-\dim\operatorname{coker}(T). \tag{10.1}$$

REMARK 10.6. Note that if T is Fredholm, then as T has closed range, $\operatorname{coker}(T) := H/\operatorname{ran}(T) \cong \operatorname{ran}(T)^\perp = \ker(T^*)$, so

$$\operatorname{index}(T) = \dim\ker(T) - \dim\ker(T^*). \tag{10.2}$$

In particular, $\operatorname{index}(T) = 0$ for any self-adjoint Fredholm operator.

The statement about T having closed range follows from the following Lemma. Actually, the conclusion above having closed range only requires T to have finite-dimensional cokernel.

LEMMA 10.7. *If W is a linear subspace of H, such that H/W is finite-dimensional, then W is closed.*

PROOF. For if $\dim(H/W) = n$, we can find n linearly independent vectors $\xi_1, \ldots, \xi_n \in H$ such that their cosets $\xi_1 + W, \ldots, \xi_n + W$ span H/W (exercise). Now the vectors $\operatorname{proj}_{W^\perp}(\xi_1) + \overline{W}, \ldots, \operatorname{proj}_{W^\perp}(\xi_n) + \overline{W}$ are linearly independent in $H/\overline{W}$ by an easy exercise (note that $\operatorname{proj}_{W^\perp}$ is the same as $\operatorname{proj}_{\overline{W}^\perp}$ because $W^\perp = \overline{W}^\perp$) whence in H/W too. Since H/W has dimension n, they form a basis for H/W. Since the map $H/W \to H/\overline{W}$ sends them to a linearly independent set of vectors, the map $H/W \to H/\overline{W}$ is injective. Since it is obviously surjective, it is bijective, and it follows that $\overline{W} = W$, *i.e.* W is closed.

□

PROOF. (Of Lemma 10.5). We just need to show 1) is equivalent to 3). Assume 3). By the Lemma, T has closed range. As a linear map $\ker(T)^\perp \to \operatorname{ran}(T)$, T is an invertible bounded linear map between two Hilbert spaces, so there is a (unique) bounded linear map $S\colon \operatorname{ran}(T) \to \ker(T)^\perp$ such that $ST = \mathrm{id}_{\operatorname{rank}(T)}$. We can extend S to H by setting it equal to zero on $\operatorname{ran}(T)^\perp$. The extension is now a bounded linear map $Q\colon K \to H$ such that QT is the identity operator on $\ker(T)^\perp$, and is zero on $\ker(T)$. Thus, $1 - QT$ is the orthogonal projection operator onto $\ker(T)$, a finite-rank, (whence compact) operator.

Now $\ker(T^*) = \operatorname{ran}(T)^\perp$ and $\operatorname{ran}(T^*)^\perp = \ker(T)$ so T^* also satisfies condition 3). Hence we can find bounded Q' such that $Q'T^* - 1$ is of finite rank. The adjoint of a finite rank operator is also of finite rank (exercise). Hence $T(Q')^* = 1$ modulo operators of finite rank. Putting $R := (Q')^*$ gives the result.

Conversely, let T be a bounded operator for which there exist bounded operators Q, Q' such that $QT - 1$ and $TQ' - 1$ are compact operators. Let $A := QT - 1$, $B = TQ' - 1$. Clearly the kernel of T is contained in the -1-eigenspace of A. But all eigenspaces of a compact operator corresponding to nonzero eigenvectors are finite-dimensional (Exercise 6.13 a).) So the kernel of T is finite-dimensional.

Next, since $\operatorname{ran}(TQ') \subset \operatorname{ran}(T)$, $\dim\operatorname{coker}(T) \leq \dim\operatorname{coker}(TQ')$. On the other hand, since $TQ' = 1 + B$, where B is compact, the range of TQ is closed, by Exercise 6.13 b). Hence

$$\operatorname{coker}(TQ') = H/\operatorname{ran}(TQ') \cong \operatorname{ran}(TQ)^\perp = \ker((TQ')^*) = \ker((Q')^*T^*) = \ker(1 + B^*),$$

which is finite-dimensional, again by Exercise 6.13 a). This shows that $\operatorname{coker}(TQ')$ is finite-dimensional, whence so is $\operatorname{coker}(T)$.

□

REMARK 10.8. We will sometimes refer to the operators Q, Q' constructed above as *parametrices* (plural of 'parametrix') of T – the term comes from pseudodifferential operator theory, which is one of the primary sources of interesting examples of Fredholm operators.

EXAMPLE 10.9. If $f \in C(\mathbb{T})$ is non-vanishing on the circle, then the Toeplitz operator $T_f := P_+M_f \in \mathbb{B}(\mathbf{H}^2)$ is Fredholm and if $g = \frac{1}{f}$ then T_g is a (2-sided) parametrix for T_f by (7.13).

EXERCISE 10.10. A *smoothing operator* on the circle $\mathbb{T}$ is any integral operator of the form $I_k \in \mathcal{K}\big(L^2(\mathbb{T})\big)$, where $k \in C^\infty(\mathbb{T}\times\mathbb{T})$ is a smooth kernel.

a) Prove that if f is a smooth function on $\mathbb{T}$ and I_k is a smoothing operator on $L^2(\mathbb{T})$, then the restriction of $T_f I_k$ to $\mathbf{H}$ is the restriction to $\mathbf{H}$ of a smoothing operator.
a) Prove that if T_f is a Toeplitz operator with smooth non-vanishing symbol $f \in C^\infty(\mathbb{T})$, then it's parametrix $Q := T_{\frac{1}{f}}$ inverts T_f modulo *smoothing operators*. (Use (7.13) (*Hint*. It follows from Lemma **??**).

In the case of Fredholm operators $T\colon H \to H$, we can succinctly express the Fredholm condition as follows:

COROLLARY 10.11. *A bounded operator $T \in \mathbb{B}(H)$ is a Fredholm operator if and only if it its image $\pi(T) \in \mathcal{Q}(H) := \mathbb{B}(H)/\mathcal{K}(H)$ is invertible, where $\pi\colon \mathbb{B}(H) \to \mathcal{Q}(H)$ is the quotient map.*

PROOF. Let T be Fredholm. By Lemma 10.5 that $\pi(T) \in \mathcal{Q}(H)$ is both left and right-invertible. Hence it is invertible.

Conversely, if T has invertible image in the Calkin algebra, let $S \in \mathbb{B}(H)$ such that $\pi(S) = \pi(T)^{-1}$, *i.e.* let S be a pre-image of its inverse. Then $ST-1$ and $TS-1$ are both compact operators. So T is Fredholm. ☐

Note that it is immediate from Corollary 10.11 that the product of two Fredholm operators is Fredholm, the adjoint of a Fredholm operator is Fredholm, and that any parametrix for a Fredholm operator is Fredholm.

EXERCISE 10.12. The *rank-nullity theorem* from basic linear algebra asserts that if $T\colon \mathbb{C}^n \to \mathbb{C}^n$ is a linear transformation then $\dim \operatorname{ran}(T) + \dim\ker(T) = n$. Since $n - \dim\operatorname{ran}(T) = \dim\operatorname{coker}(T)$, the statement is equivalent to $\dim\ker(T) - \dim\operatorname{coker}(T) = 0$, or, $\dim\ker(T) = \dim\ker(T^*)$, or, in our current language, that $\operatorname{index}(T) = 0$.

This statement of course has the advantage of making sense in infinite dimensions (for Fredholm operators), while the rank-nullity theorem does not.

Prove the rank-nullity theorem. That is, prove that $\operatorname{index}(T) = 0$ for any linear transformation of a finite-dimensional Hilbert space.

EXERCISE 10.13. Deduce from the rank-nullity theorem that if T is a *finite-rank* operator on an arbitrary Hilbert space H, then

$$\dim\ker(\lambda - T) = \dim\ker(\bar{\lambda} - T^*)$$

for every $\lambda \in \mathbb{C}$. (*Note*. The continuity properties of the Fredholm index will imply that this equality holds even for compact operators, if $\lambda \neq 0$. See below..)

Hint. By Exercise 6.6 or Exercise **??**, we can decompose the Hilbert space so that $T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, where the first summand is a finite-dimensional subspace V of H invariant under both A and A^*. It is routine to check that $\ker(\lambda - T) = \ker(\lambda - A)$ for all nonzero λ and $\ker(\bar{\lambda} - T^*) = \ker(\bar{\lambda} - A^*)$. The result follows from the Rank-Nullity Theorem.

The corresponding statement for $\lambda = 0$ is trivial since both have infinite-dimensional kernel.

THEOREM 10.14. *Let H be a Hilbert space and* $\operatorname{Fred}(H)$ *denote the set of Fredholm operators on H.*

a) *If S and T are Fredholm on H then so is ST and T^*, and* $\operatorname{index}(ST) = \operatorname{index}(S) + \operatorname{index}(T)$, *while* $\operatorname{index}(T^*) = -\operatorname{index}(T)$.
b) *If T is Fredholm and S is a compact operator then* $\operatorname{index}(T+S) = \operatorname{index}(T)$.
c) *The subspace* $\operatorname{Fred}(H) \subset \mathbb{B}(H)$ *is a open.*

d) *The function* index: $\mathrm{Fred}(H) \to \mathbb{Z}$ *is continuous.*

REMARK 10.15. The theorem may be summarized by saying that the Fredholm index induces a continuous group homomorphism

$$\text{index}\colon \mathbf{GL}(\mathcal{Q}) \to \mathbb{Z},$$

from the topological group of invertibles in the Calkin algebra, under mutiplication, to the group of integers under addition.

We will require several lemmas, the first of which is a matter of elementary linear algebra. A sequence of vector spaces and vector space maps

$$0 \to V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} V_n \to 0$$

is said to be *exact* if $\mathrm{ran}(f_i) = \ker(f_{i+1})$ for $i = 0, 2, \ldots, n$ (where f_0 is understood to be the inclusion of the zero subspace, and f_n the map *to* the zero subspace.)

In particular, f_1 is injective and f_{n-1} is surjective.

LEMMA 10.16. *Let*

$$0 \to V_1 \to V_2 \to \cdots \to V_n \to 0$$

be an exact sequence of vector spaces. Then $\sum_{k=1}^{n} (-1)^k \dim(V_k) = 0$.

PROOF. By induction. If $n = 3$ the result follows from $V_2/V_1 \cong V_3$.

If the result holds for all sequences of length $n \geq 2$ and if $n > 2$ and we are given an exact sequence

$$0 \to V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} V_n \xrightarrow{f_n} V_{n+1} \to 0,$$

then observe that the sequences

$$0 \to V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} \mathrm{ran}(f_{n-1}) \to 0$$

and

$$0 \to \mathrm{ran}(f_{n-1}) \to V_n \xrightarrow{f_n} V_{n+1} \to 0$$

are exact; now use the inductive hypothesis, the case $n = 2$, and a small amount of algebra. ☐

LEMMA 10.17. *Let* $T\colon H_1 \to H_2$ *be a bounded linear operator with closed range. Then there exists* $\epsilon > 0$ *such that if* $\|S - T\| < \epsilon$ *then* $\ker(S)$ *injects in* $\ker(T)$. *In particular,* $\dim\ker S \leq \dim\ker(T)$.

PROOF. Since T has closed range, its restriction to $\ker(T)^\perp$ is bijective onto a closed subspace of a Hilbert space. So there exists $C > 0$ such that $\|T\xi\| \geq C$ for all unit vectors ξ in $\ker(T)^\perp$.

Choose $\epsilon = \frac{C}{2}$. Let $P \in \mathbb{B}(H)$ be projection onto $\ker(T)^\perp$. Of course then $1-P$ is projection onto the kernel of T. Let ξ be a unit vector in $\ker(S)$. Then

$$\epsilon > \|(S - T)\xi\| = \|T\xi\| = \|T(P\xi + (1 - P)\xi)\| = \|TP\xi\| \geq C\|P\xi\|,$$

so $\|P\xi\| < \frac{1}{2}$. Hence $\|(1 - P)\xi\|^2 = \|\xi\|^2 - \|P\xi\|^2 = 1 - \|P\xi\|^2 > \frac{3}{4}$ for every unit vector ξ in $\ker(S)$, which shows that the restriction of $1 - P$ to $\ker(S)$ is injective.

☐

PROOF. (of Theorem 10.14). From Corollary 10.11 it is clear that the product of two Fredholm operators is Fredholm, and that the topological subspace $\mathrm{Fred}(H) \subset \mathbb{B}(H)$ of Fredholm operators on H, is open, since $\mathrm{Fred}(H) = \pi^{-1}\big(\mathbf{GL}(\mathcal{Q})\big)$, π is continuous, and the group $\mathbf{GL}(\mathcal{Q})$ of invertibles in $\mathcal{Q}$ is open, because the invertibles in any C*-algebra are open. Thus, c) is proved, and part of a).

Now let T_1 and T_2 be Fredholm. The sequence of finite-dimensional vector spaces

$$0 \to \ker(T_2) \to \ker(T_1T_2) \xrightarrow{T_2} \ker(T_1) \to H/\operatorname{ran}(T_2) \xrightarrow{T_1} H/\operatorname{ran}(T_1T_2) \to H/\operatorname{ran}(T_1) \to 0 \tag{10.3}$$

is routinely checked to be exact. An application of Lemma 10.16 to this sequence yields the result. This proves the remainder of a): the additivity of the index.

Next, we prove that the index is invariant under perturbation by *finite-rank* operators. Indeed, from Exercise 10.13, $\dim\ker(\lambda - F) = \dim\ker(\bar{\lambda} - F^*)$ for every finite-rank operator F and every $\lambda \in \mathbb{C}$. Applying this to the Fredholm operator $\lambda - F$ gives

$$\operatorname{index}(\lambda - F) = \dim\ker(\lambda - F) - \dim\ker(\bar{\lambda} - F^*) = 0.$$

Hence, any finite-rank perturbation of the identity operator (or of a nonzero multiple λ) has zero index.

Now if T is Fredholm and F has finite-rank, let Q be a parameterix for T with $QT = 1 + F'$, F' finite-rank. We get that $\operatorname{index}(Q) + \operatorname{index}(T) = \operatorname{index}(QT) = \operatorname{index}(1 + F) = 0$ by the result just proved. So $\operatorname{index}(Q) = -\operatorname{index}(T)$. Furthermore, as $F' + SF$ also has finite-rank, $\operatorname{index}(1 + F' + QF) = 0$. As $Q(T + F) = QT + F = 1 + F' + QF$, we deduce $\operatorname{index}(Q(T + F) = 0$. Since this equals $\operatorname{index}(Q) + \operatorname{index}(T + F) = -\operatorname{index}(T) + \operatorname{index}(T + F)$, we get $\operatorname{index}(T + F) = \operatorname{index}(T)$ as claimed, proving b).

Next, we show continuity of the index. Coupled with its invariance under finite-rank perturbation, this will imply invariance under *compact* perturbation, and conclude the proof of the Theorem.

We will show:

Claim. If T is Fredholm, then there exists $\epsilon > 0$ such that $\|S - T\| < \epsilon$ then S is Fredholm and $\operatorname{index}(S) \geq \operatorname{index}(T)$.

Once the claim is proved, the equality $\operatorname{index}(S) = \operatorname{index}(T)$ follows, since we can replace S by S^* and T by T^* in the claim without changing their distance apart, and the index changes sign when we take an adjoint.

To clarify things, we will use the decomposition $H = \ker(T)^\perp \oplus \ker(T)$. With respect to this decomposition we can write $T = \begin{bmatrix} T_0 & 0 \\ T_1 & 0 \end{bmatrix}$, with $T_0\colon \ker(T)^\perp \to \ker(T)^\perp$, $T_1\colon \ker(T)^\perp \to \ker(T)$. Since T_1 has finite-rank, $\operatorname{index}(T) = \operatorname{index}(T_0)$. Furthermore, if S as an operator on H, then $S - \begin{bmatrix} 0 & 0 \\ T_1 & 0 \end{bmatrix}$ is a finite-rank perturbation of S at the same distance from $T = \begin{bmatrix} T_0 & 0 \\ 0 & 0 \end{bmatrix}$ as the distance from T to S, and the index of S is the same as the index of $S - \begin{bmatrix} 0 & 0 \\ T_1 & 0 \end{bmatrix}$. These observations show that if we can prove the result for Fredholm operators T in whose matrix form $T = \begin{bmatrix} T_0 & 0 \\ T_1 & 0 \end{bmatrix}$, the term T_1 is zero, then we will be done in general.

So assume $T = \begin{bmatrix} T_0 & 0 \\ 0 & 0 \end{bmatrix}$, that is, that T maps $\ker(T)^\perp$ into itself. Note that $T_0\colon \ker(T)^\perp \to \ker(T)^\perp$ has trivial kernel closed range (since it is Fredholm.) Also, T_0^* is also Fredholm so has closed range as well.

By Lemma 10.17 we can choose

- $\epsilon_1 > 0$ such that if $A \in \mathbb{B}(\ker(T)^\perp)$ and $\|A - T_0\| < \epsilon_1$, then $\dim\ker(A) \leq \dim\ker(T_0) = 0$ (making A injective).
- $\epsilon_2 > 0$ that that if $A' \in \mathbb{B}(\ker(T)^\perp)$ and $\|A' - T_0^*\| < \epsilon_2$ then $\dim\ker(S) \leq \dim\ker(T_0^*)$.

Now let S be a bounded operator on H at distance $< \epsilon$ to T.

Write $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. We have

$$\|A - T_0\| = \|PSP - PTP\| = \|P(S-T)P\| \leq \|S - T\| < \epsilon.$$

Therefore, A is injective by choice of $\epsilon < \epsilon_1$.

Since $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ is a finite-rank perturbation of S (exercise), $\operatorname{index}(S) = \operatorname{index}(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix})$, and the latter is easily checked to equal $\operatorname{index}(A)$, which, since A is injective, equals $-\dim\operatorname{coker}(A)$.

Thus, we've shown that $\operatorname{index}(T) = \operatorname{index}(T_0) = -\dim\ker(T_0^*)$ and that $\operatorname{index}(S) = \operatorname{index}(A) = -\dim\ker(A^*)$. On the other hand, A^* and T_0^* are at distance at most ϵ as well, and hence by our choice of $\epsilon < \epsilon_2$, $\dim\ker(A^*) \leq \dim\ker(T_0^*)$. Now putting everything together gives

$$\operatorname{index}(S) = -\dim\ker(A^*) \geq \dim\ker(T_0^*) = \operatorname{index}(T)$$

for all S with $\|S - T\| < \epsilon$.

Finally, continuity of the index compled with density of finite-rank operators in the compact operators, implies the invariance of $\operatorname{index}(T)$ under compact, not just finite-rank, perturbations, and completes the proof.

□

Exercise 10.18. Prove that if $T\colon H \to K$ is a Fredholm operator between two (possibly different) Hilbert spaces then there exists a Fredholm operator $G\colon K \to H$ such that

$$1 - GT = \mathrm{pr}_{\ker T}, \qquad 1 - TG = \mathrm{pr}_{\ker(T^*)}. \tag{10.4}$$

where $\mathrm{pr}_{\ker T}$ is the projection to the kernel of T, *etc.* (G is for Green's operator!)

(*Hint.* Remember that T has closed range.)

Exercise 10.19. Let $T\colon H \to K$ be a Fredholm operator between two (possibly different) Hilbert spaces. If Q is a bounded operator $K \to H$ such that $1 - TQ$ and $1 - QT$ are each *trace-class* operators (such Q exists for any Fredholm operator, by Lemma 10.5), prove that

$$\operatorname{index}(T) = \operatorname{Trace}(1 - QT) - \operatorname{Trace}(1 - TQ). \tag{10.5}$$

This is a useful result in connection with index theory of elliptic operators, where one can arrange Q with the special and convenient property of having Schwartz kernel supported near the diagonal.

(*Hint.* Let $S := 1 - QT$. Let G be as in Exercise 10.18. Then

$$\begin{aligned} \operatorname{Trace}(T(1-QT)G) = \operatorname{Trace}\big(GT(1-QT)\big) &= \operatorname{Trace}\big(1 - p_{\ker T})(1-QT)\big) \\ &= \operatorname{Trace}(1-QT) - \operatorname{Trace}\big(p_{\ker T}(1-QT)\end{aligned} \tag{10.6}$$

using the tracial property twice. Now $T\mathrm{pr}_{\ker T} = 0$ of course and it follows that $\operatorname{Trace}\big(p_{\ker T}(1 - QT) = \operatorname{Trace}(p_{\ker T}) = \dim\ker T$. We obtain the formula

$$\operatorname{Trace}(T(1-QT)G) = \operatorname{Trace}(1-QT) - \dim\ker T.$$

A similar argument proves that

$$\operatorname{Trace}\big((1-TQ)TG\big) = \operatorname{Trace}(1-TQ) - \dim\ker T^*.$$

The result follows.)

The Toeplitz Index Theorem

We can now state and prove the Toeplitz Index Theorem.

It is a basic but non-trivial result of topology that the fundamental group of the circle is isomorphic to the group of integers. If $\varphi\colon \mathbb{T} \to \mathbb{T}$ is a loop in the compact space $\mathbb{T}$, with $\varphi(1) = 1$, then the integer corresponding to the class $[\gamma] \in \pi_1(\mathbb{T})$ is called the *winding number* of γ. It is an invariant of the homotopy-class of φ amongst maps $\mathbb{T} \to \mathbb{T}$.

Intuitively, it is the number of times (possibly negative) it wraps the circle around itself.

More generally, since $\mathbb{T} \subset \mathbb{C}^*$ is a deformation retract, for any non-vanishing continuous function $f\colon \mathbb{T} \to \mathbb{C}^*$ the composition

$$\mathbb{T} \xrightarrow{f} \mathbb{C}^* \xrightarrow{r} \mathbb{T}$$

defines an element of $\pi_1(\mathbb{T})$ and we denote the corresponding integer by $\mathrm{wind}_f(0)$.

REMARK 10.20. If $f \in C^1(\mathbb{T})$ then the winding number can be expressed as a contour integral

$$\mathrm{wind}_f(0) = \frac{1}{2\pi i}\oint_f \frac{dz}{z}, \tag{10.7}$$

where in the formula, f is understood to be the closed curve, *i.e.* closed contour, $[0, 2\pi] \xrightarrow{\exp} \mathbb{T} \xrightarrow{f} \mathbb{C}^*$ in the complex plane. To be slightly more explicit, if we view f as a function of the argument parameter $\theta \in [0, 2\pi]$, then

$$\mathrm{wind}_f(0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(\theta)}{f(\theta)}d\theta.$$

THEOREM 10.21. *(The Toeplitz Index Theorem). Let $f\colon \mathbb{T} \to \mathbb{C}^*$ be a non-vanishing continuous function on the circle. Then*

$$\mathrm{index}(T_f) = -\mathrm{wind}_f(0). \tag{10.8}$$

In particular, if f is C^1 then the index has the integral formula $\mathrm{index}(T_f) = \int_0^{2\pi} \frac{f'(\theta)}{f(\theta)}d\theta$.

PROOF. By the definitions, $\mathrm{wind}_f(0) = \mathrm{wind}_{r\circ f}(0)$ where $r\colon \mathbb{C}^* \to \mathbb{T}$ is the retraction $r(z) := \frac{z}{|z|}$, so we are reduced to considering only circle-valued functions, so we may as well assume that f itself is circle-valued. Since $\pi_1(\mathbb{T}) \cong \mathbb{Z}$, f is homotopic amongst maps $\mathbb{T} \to \mathbb{T}$ to the map $z^{\mathrm{wind}_f(0)}$, since a homotopy of maps f determines a corresponding homotopy of Fredholm operators T_f, the result now follows by checking that $\mathrm{index}(T_{z^n}) = -n$ for all integers n.

$\square$

EXAMPLE 10.22. Let $f(z) = z^{-3}$, then as a function of θ, $f(\theta) = e^{-6\pi i\theta}$, $\frac{f'(\theta)}{f(\theta)} = -6\pi i$ and $\mathrm{wind}_{z^{-3}}(0) = -3$. On the other hand, in terms of the standard orthonormal basis $1, z, z^2, \ldots$ for $\mathbf{H}^2$, $T_{z^{-3}}$ shifts sequences by 3 units to the left: $T_{z^{-3}}(z^k) = z^{k-3}$ if $k \geq 3$, and $T_{z^{-3}}(z^k) = 0$ if $k < 3$. Hence $\ker(T_{z^{-3}})$ is the span of $1, z, z^2$, it is 3-dimensional. And $\mathrm{coker}(T_{z^{-3}}) = 0$ so $\mathrm{index}(T_{z^{-3}}) = 3 = -\mathrm{wind}_{z^{-3}}(0)$.

EXERCISE 10.23. Prove that if $f\colon \mathbb{T} \to \mathbb{T}$ is not surjective, then $\mathrm{wind}_f(0) = 0$. Hence, for a non-surjective map $\mathbb{T} \to \mathbb{T}$, the corresponding Toeplitz operator T_f has zero index.

EXERCISE 10.24. Prove that if f is a polynomial $f(z) = \sum_{k=0}^n a_k z^k$ of degree n then $\mathrm{wind}_f(0) = n$. Generalize this result to trigonometric polynomials $f(z) = \sum_{k=-m}^n a_k z^k$.

EXERCISE 10.25. Prove that $[P_+, M_f]$ is *trace-class* for $f \in C^1(\mathbb{T})$ and that

$$\mathrm{Trace}(M_f^{-1}[P_+, M_f]) = \int_{\mathbb{T}} \frac{f'}{f}.$$

Use and Exercise 10.19 to give an alternative proof of the Toeplitz Index Theorem.

To conclude this section, the Toeplitz Index theorem expresses the Fredholm index of a Toeplitz operator $T = T_f \in \mathbb{B}(\mathbf{H}^2)$ in terms of a topological (homotopy) invariant of its symbol $f\colon \mathbb{T} \to \mathbb{C}^*$, namely, its winding number.

In a similar way, the Atiyah-Singer Index Theorem, one of the main theorems described in this book, computes the Fredholm index index(D) of an e*lliptic differential operator*

$$D\colon C^\infty(X, E) \subset L^2(X, E) \to L^2(X, F),$$

with symbol σ_D, between smooth vector bundles over a smooth, compact manifold X, in terms of an appropriate topological invariant of its symbol – a kind of generalized winding number. In fact, this invariant, which is of course is more complicated than a winding number, since manifolds have more complicated topology than the circle, in general, turns out to be be described extremely conveniently using K-theory.

11. The C*-algebra of the real line

In this section, we specialize our general treatment of group C*-algebras to the group $\mathbb{R}$, equipped of course with its standard topology.

Let $f \in C_c(\mathbb{R})$, $\lambda(f)\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the operator of convolution by f:

$$(\lambda(f)u)(x) := \int_{\mathbb{R}} f(y)u(x-y)dy. \tag{11.1}$$

The integral converges absolutely, by the Cauchy-Schwartz inequality.

Exercise 11.1. Prove that the integral defining $f_1 * f_2$ converges absolutely if merely $f \in L^1(\mathbb{R})$ and f_2 is bounded and measurable. Prove, in addition, that if u is smooth, then so is $f * u$, and that $(f * u)' = f * u'$. This shows that convolution of $f \in L^1(\mathbb{R})$ maps smooth functions on $\mathbb{R}$ to smooth functions on $\mathbb{R}$ (even if f is not smooth.)

The C*-algebra generated by the $\lambda(f)$ as f ranges over $C_c(\mathbb{R})$ is the (reduced) C*-algebra $C^*_r(\mathbb{R})$ of the group $\mathbb{R}$, a non-unital, commutative C*-algebra containing $L^1(\mathbb{R})$ as a *-subalgebra (also, $L^1(\mathbb{R})$ is a Banach algebra in its own right).

As we discussed in our the general overview of locally compact group C*-algebras, or the other examples $\mathbb{Z}$ and $\mathbb{T}$, $C^*_r(\mathbb{R})$ may be regarded as the closure in the operator norm topology, of the *-algebra $C_c(\mathbb{R})$, with *-algebra operations

$$(f_1 * f_2)(x) := \int_{\mathbb{R}} f_1(y)f_2(x-y)dy, \quad f^*(x) := \overline{f(-x)}.$$

In this notation, the action of $f \in C_c(\mathbb{R})$ on $L^2(\mathbb{R})$ by convolution in (11.1) can be written $\lambda(f)u = f * u$.

The following heuristic is sometimes helpful, if one wants to remember how convolution works.

Let $U_t\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by the unitary translation operator by t, $U_tf(x) = f(x-t)$. Then, as an operator,

$$\lambda(f) = \int f(t)U_t dt \in C^*_r(\mathbb{R}), \tag{11.2}$$

which is an integral version of the group-algebra notation $\sum a_g[g]$ we were using before, for discrete group C*-algebras, with the symbol $[t]$ (for t in the group $\mathbb{R}$ having been replaced by U_t.

If one multiplies the integral (11.2) somewhat formally, against an L^2-function u, and evaluate at $x \in \mathbb{R}$, one gets the formula (11.1).

Exercise 11.2. Prove that

a) The map $\mathbb{R} \to \mathbf{U}\big(L^2(\mathbb{R})\big)$, $t \mapsto U_t$, is continuous with respect to the strong operator topology.
b) The integral (11.2) converges converges absolutely if $f \in L^1(\mathbb{R}) \subset C_r^*(\mathbb{R})$.
c) The integral (11.2) converges in the strong operator topology for every $f \in C_r^*(\mathbb{R})$.
d) Prove that every element $T \in C_r^*(\mathbb{R})$ is 'represented' by some measurable function f on $\mathbb{R}$ in the sense that f is measurable and the integral

$$\int f(t)U_t dt \in C_r^*(\mathbb{R}),$$

converges to T in the strong operator topology. Such f need not be continuous, however!

If $\xi \in \mathbb{R}$ is a real number, then $\chi_\xi(t) := e^{it\xi}$ is a character $\chi_\xi \colon \mathbb{R} \to \mathbb{T}$ of the group of real numbers, and, conversely, all characters arise in this way. Thus, $\mathbb{R}$ may be canonically identified with its own Pontryagin dual $\widehat{\mathbb{R}}$. With this identification, the Fourier transform as defined in (4.1) has the form

$$\hat{u}(\xi) := (F_{\mathbb{R}}u)(\xi) := \int u(x)e^{-ix\xi}dx, \tag{11.3}$$

where for initial purposes, we can take $u \in C_c^\infty(\mathbb{R})$, but the integral clearly converges absolutely if merely $u \in L^1(\mathbb{R})$.

The following two features of the Fourier transform are key: let u be any measurable function on $\mathbb{R}$.

- If u has *rapid decay*, *i.e.* if $p(x)u(x)$ is bounded for every polynomial $p(x)$ (so that in particular $u \in L^1(\mathbb{R})$, and its Fourier transform makes sense as an absolutely convergent integral) then $\hat{u}$ is infinitely differentiable.
- If u is infinitely differentiable with all derivatives in $L^1(\mathbb{R})$, then $\hat{u}$ has rapid decay, *i.e.* $\hat{u}(\xi)p(\xi)$ is bounded for every polynomial $p(\xi)$.

To prove the first statement, use the Dominated Convergence theorem to prove that $\hat{u}$ is differentiable everywhere with

$$\hat{u}'(\xi) = (-i)\int u(x)xe^{-ix}dx,$$

i.e., differentiate under the integral sign. Thus,

$$\hat{u}'(\xi) = \widehat{xu}(\xi) \tag{11.4}$$

where by xu we mean the function $xu(x)$.

Smoothness is proved inductively using this idea.

For the other statement, we write

$$\xi\hat{u}(\xi) = \int u(x)\xi e^{-ix\xi}dx = i\int u(x)\frac{d}{dx}(e^{-ix\xi})dx = -i\int u'(x)e^{-ix\xi}dx$$

where the last step is by integration by parts.

$$\widehat{u'}(\xi) = \xi\hat{u}(\xi). \tag{11.5}$$

The statement is proved by an obvious inductive argument.

Because Fourier transforms exchanges differentiation and multiplication, a natural domain for it is the *Schwartz algebra* $\mathcal{S}(\mathbb{R})$ of the real line, defined as follows.

Definition 11.3. The Schwartz algebra $\mathcal{S}(\mathbb{R})$ is

$$\mathcal{S}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) \mid \forall m, n \geq 0,\ \sup_{x\in\mathbb{R}} |(1+|x|)^m\, \partial^n f(x)| < \infty\},$$

where $\partial^n f$ the n-th derivative of f.

The Schwartz space $\mathcal{S}$ has a natural structure of topological vector space with seminorms $p_{n,m}(f) := \sup_{x\in\mathbb{R}, 0\leq k\leq m}(1+|x|)^n\, |\partial^k f\,(x)|$.

Exercise 11.4. Prove that the convolution $f * g$ of two Schwartz functions $f, g \in \mathcal{S}$, is again in $\mathcal{S}$.

(*Hint.* The inequality $1+|x| \leq (1+|x-y|)(1+|y|)$ is helpful for this.)

Furthermore, the convolution of two Schwartz functions is again Schwartz, and with $F_{\mathbb{R}}$ the Fourier transform, $F_{\mathbb{R}} u \in \mathcal{S}$ if $u \in \mathcal{S}$. So $F_{\mathbb{R}} \colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, and is as such, continuous.

Theorem 11.5. *Let $F_{\mathbb{R}} \colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ be the Fourier transform.*

- *If $u \in \mathcal{S}(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\hat{u} \in L^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$ and $\|\hat{u}\|_{L^2(\mathbb{R})} = \|u\|_{L^2(\mathbb{R})}$.*
- *If $u, v \in \mathcal{S}(\mathbb{R})$ then $\widehat{uv} = \hat{u} * \hat{v}$ and $\widehat{u * v} = \hat{u}\hat{v}$.*
- *If $u \in \mathcal{S}(\mathbb{R})$ then $F_{\mathbb{R}}\hat{u}(x) = u(-x)$. Equivalently (the Fourier inversion formula)*

$$u(x) = \frac{1}{2\pi}\int \hat{u}(\xi) e^{ix\xi} d\xi. \tag{11.6}$$

holds for all Schwartz functions u.

Remark 11.6. The *inverse* Fourier transform is given thus by

$$F_{\mathbb{R}}^{-1} u\,(\xi) := \hat{u}(-\xi).$$

Note that the identity

$$\int \hat{u}v = \int u\check{v}$$

also holds, for all $u, v \in L^2(\mathbb{R})$, where $\check{v}(x) := \hat{v}(-x)$.

Corollary 11.7. *The Fourier transform $F_{\mathbb{R}} \colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ extends to a unitary isomorphism $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ conjugating $\lambda(f)$ to $M_{\hat{f}}$ and conjugating M_f to $\lambda(\hat{f})$, for any $f \in \mathcal{S}$.*

In particular

$$C_r^*(\mathbb{R}) \cong C_0(\mathbb{R})$$

by a C-algebra isomorphism sending the operator $\lambda(f)$ to the function $\hat{f} \in C_0(\mathbb{R})$, for $f \in \mathcal{S}(\mathbb{R}) \subset C^*(\mathbb{R})$.*

Example 11.8. Let $f(x) = \frac{1}{1+x^2}$. Then $f \in L^1(\mathbb{R})$ and its Fourier transform converges absolutely to $\hat{f}(\xi) = \pi e^{-|\xi|}$.

Proof. The proof uses some standard methods from basic complex analysis.

Let C_R' be the upper half of the circle of radius R in the complex plane, oriented counterclockwise, and let C_R be the closed contour in $\mathbb{C}$ consisting of the segment $[-R, R]$ joined to C_R'. Suppose $\xi < 0$. Observe that $\int_{C_R'} \frac{e^{-i\xi z} dz}{1+z^2} \to 0$ as $R \to \infty$. On the other hand, $\int_{C_R} \frac{e^{-i\xi z}}{1+z^2} dz = \int_{C_R} \frac{g(z)}{z-i} dz$ where $g(z) = \frac{e^{-i\xi z}}{z+i}$, an analytic function on the upper half plane, so by the Cauchy Integral formula $\int_{C_R} \frac{e^{-i\xi z}}{1+z^2} dz = 2\pi i g(i) = \pi e^{\xi}$.

Thus,

$$\hat{f}(\xi) = \pi e^{\xi}$$

when $\xi < 0$. On the other hand, since $f(x) = \frac{1}{1+x^2}$ is real-valued, $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$, so that if $\xi > 0$, by the computation above, $\hat{f}(-\xi) = \pi e^{-\xi}$ and hence $\hat{f}(\xi) = \overline{\pi e^{-\xi}} = \pi e^{-\xi}$.

Therefore,

$$f(x) = \frac{1}{1+x^2} \Rightarrow \hat{f}(\xi) = \pi e^{-|\xi|}, \ \ \xi \in \mathbb{R}$$

as claimed.

Notice that f is infinitely smooth, corresponding to the fact that it's Fourier transform has rapid decay (decays faster than any polynomial). However, f is *not* of rapid decay. Correspondingly, $\hat{f}$ is not differentiable (at the origin.)

Distributions

The vector space $C_c^\infty(\mathbb{R})$ is the union

$$C_c^\infty(\mathbb{R}) = \cup_K C_c^\infty(K),$$

where K runs over all pre-compact subsets of $\mathbb{R}$. Each $C_c^\infty(K)$ carries seminorms

$$p_{K,m}(f) := \sup_{x\in K}\{|\partial^\alpha f(x)| \mid x \in K, \, \alpha \leq m\}.$$

We equip $C_c^\infty(\mathbb{R})$ with the inductive limit topology: a subset is open if and only if its is contained in, and open in, one of the $C_c^\infty(K)$. A sequence, or net (f_i) of smooth, compactly supported functions, converges in $C_c^\infty(\mathbb{R})$ if and only if all but finitely many elements of it are in $C_c^\infty(K)$ for some K, and (f_i) converges in $C_c^\infty(K)$.

A *distribution* is an element of the dual space $C_c^\infty(\mathbb{R})'$. It is a linear functional

$$L \colon C_c^\infty(\mathbb{R}) \to \mathbb{C}$$

with the property that it restricts, for each $K \subset \mathbb{R}$ pre-compact, to a continuous functional $C_c^\infty(K) \to \mathbb{C}$. The space of distributions is denoted $\mathcal{D}'$. Following widespread convention, we often denote the value of a distribution L at a smooth function ϕ by $\langle L, \phi\rangle$.

EXERCISE 11.9.

a) Prove that any Radon measure on $\mathbb{R}$, or any locally integrable function, determines a distribution on $\mathbb{R}$, by $\langle \mu, u\rangle := \int u d\mu$ in the first case, and $\langle f, u\rangle := \int f(x)u(x)dx$ in the second case.

b) Let ∂ denote differentiation on the line. Prove that the map $\partial \colon C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$ is continuous.

c) Let f be any smooth function on $\mathbb{R}$ (unbounded is fine.) Prove that multiplication $M_f \colon C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$ is continuous.

It is common to identify a (locally integrable) function ϕ on $\mathbb{R}$ with its associated distribution as in part a) of the Exercise. In particular, elements of $L^2(\mathbb{R})$, or elements of $C^\infty(\mathbb{R})$, all define distributions (since both classes consist of locally integrable functions.)

Of course not all distributions are given by functions. The Dirac delta mass

$$\langle \delta_0, u\rangle := u(0)$$

at $0 \in \mathbb{R}$ is an example of a distribution not given by a function. Nonetheless, by part b) of the Exercise above, differentiation $\partial \colon C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$ being continuous, it induces a continuous map $\mathcal{D}' \to \mathcal{D}'$ by dualizing: we define the derivative ∂L of a distribution by

$$\langle \partial L, u\rangle := -\langle L, \partial u\rangle.$$

It follows from integration by parts that this agrees, when the distribution is given by a smooth function (or just a function with continuous first derivative), with the usual derivative (of a function).

Thus, with these definitions, every distribution $L \in \mathcal{D}'$ has derivatives $\partial^k L$ of all orders, and $\partial\colon \mathcal{D}' \to \mathcal{D}'$ is continuous. As a particular case, any L^2-function u has a distribution derivative ∂u (which is not, however, going to be an L^2-function anymore, without further hypotheses.)

EXERCISE 11.10. Prove that the distributional derivative of $\chi_{[0,\infty)}$ (a locally integrable function) is the Dirac distribution δ_0.

A distribution is *tempered* if it extends continuously to $\mathcal{S}$. The space of tempered distributions is denoted $\mathcal{S}'$. A distribution is thus tempered if and only if there exists $C \geq 0$ and k, m such that

$$|\langle L, u\rangle| \leq \sup_{x\in\mathbb{R}} (1+|x|)^k\, |\partial^m u(x)|, \quad \forall u \in \mathcal{S}.$$

Since the Fourier transform maps $\mathcal{S} \to \mathcal{S}$ continuously, it also maps, by dualizing, $\mathcal{S}' \to \mathcal{S}'$ continuously. Hence, the Fourier transform $\widehat{L}$ of any tempered distribution is defined, by the formula

$$\langle \widehat{L}, u\rangle := \langle L, \hat{u}\rangle.$$

EXERCISE 11.11. A measurable function f on $\mathbb{R}$ is *slowly growing* if it grows at most polynomially: that is, there exists m such that $|f(\xi)| \leq C|\xi|^m$ for all ξ.

a) Prove that any slowly increasing, continuous function f defines a tempered distribution with

$$\langle f, \phi\rangle := \int f(x)\phi(x)dx.$$

b) Suppose f is a slowly increasing function. Prove that multiplication by f determines a continuous linear map of topological vector spaces $\mathcal{S} \to \mathcal{S}$. Deduce that the linear map of multiplication by f extends continuously from $\mathcal{S}$ to $\mathcal{S}'$, by the formula

$$\langle f \cdot L, u\rangle := \langle L, f \cdot u\rangle.$$

Note that part a) of the Exercise shows that any slowly growing function has a Fourier transform, since it defines a tempered distribution. In particular, any polynomial has a Fourier transform in this sense.

REMARK 11.12. The definition of the Fourier transform of a slowly growing smooth function f, regarded as a distribution, was defined

$$\langle \hat{f}, u\rangle := \langle f, \hat{u}\rangle := \int\int f(x)u(\xi)e^{-ix\xi}d\xi dx. \tag{11.7}$$

The integral (11.7) is called an *oscillatory integral.* As written, as a double integral, it is not absolutely convergent; that is, the function $\psi(x,\xi) = f(x)u(\xi)e^{-ix\xi}$ is not absolutely integrable over $\mathbb{R} \times \widehat{\mathbb{R}}$. The meaning of the (11.7) is that the integration must be first carried out with respect to ξ, then x; the other order makes no sense.

However, we can re-write it (or regularize) the oscillatory integral as follows. Notice that $\partial_\xi^k(e^{-ix\xi}) = (-ix)^k$. Hence $(-ix)^{-k}\partial^k(e^{-ix\xi}) = e^{-ix\xi}$. Inserting this into the integral gives

$$= \int\int f(x)u(\xi)(-ix)^{-k}\partial_\xi^k(e^{-ix\xi})\, d\xi dx.$$

Now, neglecting the singularity at $x = 0$, the function $f(x)(-ix)^{-k}$ is, for sufficiently large k (since f is slowly growing), in $L^1(\mathbb{R})$. Using integration by parts in the ξ variable gives

$$= \int\int f(x)(-ix)^{-k}\partial_\xi^k u(\xi)e^{-ix\xi}\, d\xi dx$$

which is absolutely convergent.

EXERCISE 11.13. The argument given above is not complete, as it doesn't address the singularity of x^{-k} at $x = 0$. By using the operator $(1 + \partial_\xi^2)^k$ in place of ∂_ξ^k (sufficiently large k), complete the argument.

EXERCISE 11.14. Prove that the linear span of the Fourier transforms of the Dirac distribution δ_0 and its derivatives $\partial^k \delta_0$, are exactly the polynomials $\sum_{k=0}^n a_k \xi^k$, viewed as distributions.

EXERCISE 11.15. Let $f(x) = \frac{1}{x+i} \in C_0(\mathbb{R})$. Of course f is not in $L^1(\mathbb{R})$, so it's Fourier transform is not defined by an absolutely convergent integral. However, being a slowly-growing smooth function, it has a distributional Fourier transform, which is in fact a function. Show that

$$\hat{f}(x) = (1 - 2\chi)e^{-|\xi|}, \tag{11.8}$$

where χ is the characteristic function of $(-\infty, 0]$.

(*Hint.* To compute the principal value, use the technique of the previous example, and Jordan's Inequality, which states that

$$\int_0^\pi e^{-R\sin\theta} d\theta < \frac{\pi}{R}$$

for every $R > 0$. Use this to prove that if $f(x) = \frac{1}{x+i}$ then $\hat{f}(\xi) = 2\pi i e^{-\xi}$ for $\xi > 0$. Now if $g(x) = \frac{1}{x-i}$, then $\hat{f}(-\xi) = \hat{g}(\xi)$.) Now combine these two observations and Exercise 11.8 to complete the proof by using the identity $f(x) - g(x) = \frac{2i}{1+x^2}$.)

EXAMPLE 11.16. Another interesting example of a Fourier transform involves the function

$$\varphi(\xi) := \int_0^\xi \frac{\sin t}{t} dt, \quad \xi > 0,$$

where the function is defined for negative ξ either so as to produce an odd function, or by using the same formula, interpreting it as a 'signed' integral. By standard tricks of complex analysis, one can compute that $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$, so the function

$$\varphi(\xi) := \frac{2}{\pi} \int_0^\xi \frac{\sin t}{t} dt \tag{11.9}$$

is a smooth *normalizing function*:

a) φ is odd, and $-1 \le \varphi(\xi) \le 1$ for all $\xi \in \mathbb{R}$,
b) $\lim_{\xi \to \pm\infty} \varphi(\xi) = \pm 1$.

Since φ is slowly growing, it has a distributional Fourier transform, which turns out to be

$$\hat{\varphi}(\xi) = \frac{1}{\xi} \chi_{[-1,1]},$$

suitably interpreted (as a principal value).

More precisely, let L be the distribution defined by the principal value

$$\langle L, u \rangle := \text{PV} \int_{-1}^1 \frac{u(\xi)}{\xi} d\xi := \lim_{\epsilon \to 0} \int_{\epsilon \le |\xi| \le 1} \frac{u(\xi)}{\xi} d\xi.$$

I claim L is a tempered distribution. Indeed, if $u \in \mathcal{S}$ is a test function, then

$$\text{PV} \int_{-1}^1 \frac{u(\xi)}{\xi} d\xi = \int_{-1}^1 \frac{u(\xi) - u(0)}{\xi} d\xi$$

since $\frac{u(0)}{\xi}$ is an odd function, and an easy argument. Since

$$\sup_{|x|\leq 1}|\frac{u(\xi)-u(0)}{\xi}| \leq \sup_{x\in\mathbb{R}}|\partial u(x)|,$$

L is continuous on $\mathcal{S}$, *i.e.* is a tempered distribution.

To show that $\hat{L}=\varphi$, we compute

$$\langle \hat{L}, u\rangle := \langle L, \hat{u}\rangle = \lim_{\epsilon\to 0}\int_{\epsilon\leq|\xi|\leq 1}\frac{\hat{u}(\xi)}{\xi}d\xi \tag{11.10}$$

$$= \lim_{\epsilon\to 0}\int_{\epsilon\leq|\xi|\leq 1}\int\frac{u(x)e^{-ix\xi}}{\xi}dxd\xi = \lim_{\epsilon\to 0}\int u(x)\int_{\epsilon\leq|\xi|\leq 1}\int\frac{e^{-ix\xi}}{\xi}dxd\xi = \int u(x)\int_{-1}^{1}\frac{\sin x\xi}{\xi}\,d\xi dx$$

$$= \int u(x)\varphi(x)dx = \langle u, \varphi\rangle$$

as claimed.

Distributions as (convolution) operators

If $t\in\mathbb{R}$, the linear map $\lambda(t)\colon\mathcal{S}\to\mathcal{S}$, $\lambda(t)u(x) := u(t-x)$ dualizes to a linear map on $\mathcal{S}'$,

$$\lambda(t)L := L\circ\lambda(t).$$

If f is a function, say, slowly increasing, and thought of as a distribution, L, then calculating in terms of the definitions gives

$$\langle\lambda(t)L, u\rangle = \langle L, \lambda(t)u\rangle = \int f(\xi)u(t-\xi)d\xi = (f*u)(t).$$

Because of this, we use the notation

$$L*u\,(t) := \langle\lambda(t)L, u\rangle.$$

What kind of function is $L*u$?

Example 11.17. Let $u\in\mathcal{S}$.

(i) Let $L=\delta_0$. Then $L*u=u$ for all u.
(ii) $(\partial\delta_0)*u=u'$; that is, $\partial\cdot\delta_0=\partial$ as operators on $\mathcal{S}$.
(iii) Let $L=1$, then $L*u=c$ for a constant (function) c.

We derive the interesting formula

$$\sum a_k\partial^k u = (\sum a_k\partial^k\delta_0)*u,$$

an interpretation of the action of constant coefficient differential operators on the real line, acting on $\mathcal{S}$, as an action by convolution operators (in a distributional sense), by the corresponding linear combination of derivatives of the Dirac delta function δ_0.

However, the last example shows that convolution with $L\in\mathcal{S}'$ does not necessarily map $\mathcal{S}$ into $\mathcal{S}$.

Lemma 11.18. *If $L\in\mathcal{S}'$, $u\in\mathcal{S}$, then the function $\langle\lambda(t)L, u\rangle$ is a smooth and slowly growing function; in particular, it is a tempered distribution: $L*u\in\mathcal{S}'$ whenever $L\in\mathcal{S}'$ and $u\in\mathcal{S}$.*

Proof. ☐

In particular, we can consider distributions in $\mathcal{S}'$ to be convolution *operators*, mapping $\mathcal{S}$ into the space of distributions $\mathcal{S}'$.

EXAMPLE 11.19. Let $L \in \mathcal{S}'$ as in Example 11.16. Then for $u \in \mathcal{S}$,

$$(L * u)(\xi) = \text{PV} \int_{-1}^{1} \frac{u(\xi - t)}{t} dt.$$

Since the distributional Fourier transform of the distribution defined (via principal value) by $\psi(t) = \frac{1}{t} \cdot \chi_{[-1,1]}(t)$ is the normalizing function φ of the Example (regarded as a distribution), convolution with L extends to a bounded (in fact contractive) operator $\lambda(L)\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$. In the notation (11.2), one would write

$$\lambda(L) = \int_{-1}^{1} \frac{U_t}{t} dt$$

for this operator; the same arguments can be used to show that the integral converges in the strong (or weak) operator topologies on $\mathbb{B}\big(L^2(\mathbb{R})\big)$. The element $\int_{-1}^{1} \frac{U_t}{t} dt$, in this notation, is not in $C_r^*(\mathbb{R})$, but is in the *multiplier algebra* $\mathcal{M}\big(C_r^*(\mathbb{R})\big)$.

Sobolev spaces

Recall that $L^2(\mathbb{R}) \subset \mathcal{S}'$, since L^2-functions are locally integrable. So any $f \in L^2(\mathbb{R})$ has distributional derivatives $\partial^k f$ of all orders. Of course they are distributions, in general, and not necessarily functions. We set

$$\begin{aligned} &H_k(\mathbb{R}) := \{f \in L^2(\mathbb{R}) \mid \partial^k f \in L^2(\mathbb{R})\ 0 \le k \le m\}, \\ &\qquad \langle f_1, f_2\rangle_m := \sum_{k=0}^{m} \langle \partial^k f_1, \partial^k f_2\rangle_{L^2(\mathbb{R})}, \quad \text{for } f_1, f_2 \in H_m(\mathbb{R}). \end{aligned} \tag{11.11}$$

The expression on the second line defines an inner product and makes $H_m(\mathbb{R})$ into a Hilbert space.

It is useful, however, to note that since Fourier transform is an unitary isomorphism on $L^2(\mathbb{R})$ conjugating (the distributions) ∂^k to multiplication by ξ^k, we see that

$$\langle f_1, f_2\rangle_m = \sum_{k=0}^{m} \int \partial^k f_1(x)\, \overline{\partial^k f_2(x)} dx = \sum_{k=0}^{m} \int \hat{f}_1(\xi) \overline{\hat{f}_2(\xi)} |\xi|^{2k} d\xi. \tag{11.12}$$

An easy exercise is to show that there are a pair of constants such that

$$C_1(1 + |\xi|^2)^m \le \sum_{k=0}^{m} |\xi|^{2k} \le C_2(1 + |\xi|^2)^m.$$

It follows that the definition of H_m gives above is equivalent to the following more flexible definition.

For $s \in \mathbb{R}$, the operator of multiplication by $(1 + |\xi|^2)^{\frac{s}{2}}$ (a slowly growing function) maps $\mathcal{S}$, and hence $\mathcal{S}'$, continuously to itself. If $f \in \mathcal{S}'$ we set

$$\Lambda_s(f) := \big((1 + |\xi|^2)^{\frac{s}{2}} \hat{f}\,\big)^{\vee}.$$

DEFINITION 11.20. The Sobolev spaces $H_s(\mathbb{R})$ ($s \in \mathbb{R}$) are defined by

$$H_s(\mathbb{R}) = \{f \in \mathcal{S}' \mid \Lambda_s f \in L^2(\mathbb{R})\},$$

with inner product

$$\langle f_1, f_2\rangle := \langle \Lambda_s(f_1), \Lambda_s(f_2)\rangle_{L^2(\mathbb{R})} := \int \Lambda_s(f_1)(\xi)\cdot \overline{\Lambda_s(f_2)(\xi)}\, d\xi \\ := \int f_1(\xi)\, \overline{\hat{f}_2(\xi)}\, (1+|\xi|^2)^{-\frac{s}{2}}\, d\xi. \tag{11.13}$$

The following facts about Sobolev spaces are easy to check and are left to the reader.

EXERCISE 11.21. Prove that

a) Fourier transform extends to a unitary isomorphism $F_{\mathbb{R}}\colon H_s(\mathbb{R}) \xrightarrow{\cong} L^2(\mathbb{R}, \mu_s)$ where $d\mu_s(\xi) = (1+|\xi|^2)^s d\xi$.
b) $C_c^\infty(\mathbb{R})$ and $\mathcal{S}$ are both dense in H_s for all $s \in \mathbb{R}$.
c) If $s < t$ then $H_t \subset H_s$ and $\|\cdot\|_s \leq \|\cdot\|_t$.
d) The map Λ_t determines a unitary isomorphism $H_s \to H_{s-t}$, for all s, t.
e) ∂^k extends continuously to a bounded linear operator $H_s \to H_{s-k}$.

REMARK 11.22. The elements of H_s are by definition, all distributions, and hence some are given by (locally integrable) functions, some are not. If $s > 0$, $H_s \subset L^2$ and hence all elements of H_s are given by functions. However, the Dirac distribution δ_0 is in H_s for all $s < -\frac{1}{2}$, since it's Fourier transform is the constant function 1, and the latter is in $L^2(\mathbb{R}, \mu_s)$ if (and only if) $s < -\frac{1}{2}$.

The result below shows in fact that if $s > \frac{1}{2}$, then the elements in $H_s(\mathbb{R})$ are actually given by *continuous* functions.

THEOREM 11.23. *(Sobolev embedding theorem.) Suppose $s > k + \frac{1}{2}$. Then*

a) *If $f \in H_s(\mathbb{R})$ then $\widehat{\partial^\alpha f} \in L^1(\mathbb{R})$ and $\|\widehat{\partial^k f}\|_{L^1(\mathbb{R})} \leq C\|f\|_{H_s(\mathbb{R})}$, for some fixed constant $C \geq 0$ and all $\alpha \leq k$.*
b) *$H_s(\mathbb{R}) \subset C_0^k(\mathbb{R})$ and $\|f\|_{C_0^k(\mathbb{R})} \leq C'\|f\|_{H_s(\mathbb{R})}$ for some fixed constant $C' \geq 0$ and all $f \in C_0^k(\mathbb{R})$.*

By definition, $\|f\|_{C_0^k(\mathbb{R})} := \max_{0\leq\alpha\leq k}\|\partial^\alpha f\|_{C_0(\mathbb{R})}$.

REMARK 11.24. In the Fourier-transformed picture, $H_s(\mathbb{R})$ becomes – denote it $H_s(\widehat{\mathbb{R}})$ – the collection of measureable functions $u = u(\xi)$ such that $\int |u(\xi)|^2(1+|\xi|^2)^s d\xi < \infty$.

The Fourier-transformed picture of Theorem 11.23 asserts then (in particular) that $H_s(\widehat{\mathbb{R}}) \subset C_r^*(\mathbb{R})$ continuously, for $s > \frac{1}{2}$.

PROOF. The first statement follows almost immediately from taking Fourier transforms and applying the Cauchy-Schwartz inequality:

$$\int |\hat{f}(\xi)||\xi|^\alpha\, d\xi \leq \int |\hat{f}(\xi)|(1+|\xi|^2)^{\frac{s}{2}}(1+|\xi|^2)^{\frac{k-s}{2}}\, d\xi \\ \leq \Big(\int |\hat{f}(\xi)|^2(1+|\xi|^2)^s\, d\xi\Big)^{\frac{1}{2}}\Big(\int (1+|\xi|^2)^{k-s}\, d\xi\Big)^{\frac{1}{2}} \leq C\|f\|_{H_s(\mathbb{R})} \tag{11.14}$$

for a constant C, since $2k - 2s < -1$ under the assumptions.

For the second statement, suppose $s > \frac{1}{2}$ and $f \in H_s(\mathbb{R})$. By part a), $\|\hat{f}\|_{L^1(\mathbb{R})} \leq C\,\|f\|_{H_s(\mathbb{R})}$. In particular, $\hat{f}$ is in $L^1(\mathbb{R})$, so is in $C_r^*(\mathbb{R})$, and hence f is in $C_0(\mathbb{R})$.

Furthermore, as $\|\hat{f}\|_{C_r^*(\mathbb{R})} = \|f\|_{C_0(\mathbb{R})}$ and $\|\hat{f}\|_{C_r^*(\mathbb{R})} \leq \|\hat{f}\|_{L^1(\mathbb{R})}$, putting everything together gives the norm estimate required. We leave the case where derivatives are present to the diligent reader; the arguments go the same way.

∎

REMARK 11.25. Elements of $H_s(\mathbb{R})$ for $s \geq 0$ are *a priori* L^2-functions, but it makes no sense to evaluate an L^2-function at a point. However, the Theorem says that point evaluation does make sense for $s > \frac{1}{2}$, and is even a continuous linear functional $H_s(\mathbb{R}) \to \mathbb{C}$.

(Note that this linear functional is pairing with the Dirac δ-function at that point, an element of $H_{-s}(\mathbb{R})$.)

We conclude this section with a key result about the Sobolev spaces.

The C*-algebra $C_0(\mathbb{R})$ acts by multiplication operators on all Sobolev spaces $H_s(\mathbb{R})$. Say an operator T between Sobolev spaces is *locally compact* if fT is a compact operator for all $f \in C_0(X)$ (equivalently, for all $f \in C_c(X)$.)

THEOREM 11.26. *(The Rellich Lemma.) If $s > t$ then the inclusion $H_s(\mathbb{R}) \to H_t(\mathbb{R})$ is a locally compact operator.*

The strategy of the proof is to use the Arzela-Ascoli Theorem: if (g_n) is a sequence of continuous functions on $\mathbb{R}$, which is pointwise bounded and equicontinuous, then (g_n) has a subsequence which converges uniformly on compact subsets of $\mathbb{R}$.

PROOF. We will prove that the inclusion $H_s(\mathbb{R}) \to L^2(\mathbb{R})$ is a locally compact operator; the general result follows easily and the details are left to the reader.

Let $\phi \in C_c(\mathbb{R})$, and $T\colon H_s(\mathbb{R}) \to L^2(\mathbb{R})$ the inclusion; we prove that the operator $T\phi\colon H_s(\mathbb{R}) \to L^2(R)$ is a compact operator. To do this, we need to show that the image of the unit ball in $H_s(\mathbb{R})$ is pre-compact.

So let (f_n) be a sequence in the unit ball of H_s. Then $(f_n\phi)$ is a sequence also in the unit ball, consisting of functions all supported within a fixed, compact subset $K \subset \mathbb{R}$. We need to show that a subsequence of the $(f_n\phi)$ converges in $L^2(\mathbb{R})$. We may as well simply denote $f_n\phi$ by f_n, keeping in mind these $H_s(\mathbb{R})$ vectors f_n are now all supported in K. Note that this implies that their Fourier transforms $\hat{f}_n$ are all smooth functions.

Let $\psi \in C_c^\infty(\mathbb{R})$ with $\psi = 1$ on K. Then $f_n = f_n\psi$, and hence

$$\hat{f}_n(\xi) = \hat{f}_n * \hat{\psi}\,(\xi) = \int \hat{f}_n(\eta)\hat{\psi}(\xi - \eta)\,d\eta. \tag{11.15}$$

Now we leave it to the reader to check the general inequality

$$(1 + |\xi|^2)^{\frac{s}{2}} \leq 2^s(1 + |\xi - \eta|^2)^s(1 + |\eta|^2)^s. \tag{11.16}$$

Multiplying (11.15) by $(1 + |\xi|^2)^{\frac{s}{2}}$ and using (11.16) now gives

$$\begin{aligned} (1 + |\xi|^2)^{\frac{s}{2}}\,|\hat{f}_n(\xi)| &\leq \int (1 + |\xi|^2)^{\frac{s}{2}}\,|\hat{f}_n(\eta)| \cdot |\hat{\psi}(\xi - \eta)|\,d\eta \\ &\leq 2^s \int |\hat{f}_n(\eta)|(1 + |\eta|^2)^s \cdot |\hat{\psi}(\xi - \eta)|(1 + |\xi - \eta|^2)^s\,d\eta. \end{aligned} \tag{11.17}$$

By the Cauchy-Schwartz inequality we get

$$(1 + |\xi|^2)^{\frac{s}{2}}\,|\hat{f}_n(\xi)| \leq 2^s \cdot \|\hat{f}_n\|_{H_s(\hat{\mathbb{R}})}\|\hat{\psi}\|_{H_s(\hat{\mathbb{R}})} \leq 2^s \cdot \|f_n\|_{H_s(\mathbb{R})}\|\psi\|_{H_s(\mathbb{R})} \leq C \tag{11.18}$$

for a fixed constant C, as the (f_n) were assumed unit vectors in $H_s(\mathbb{R})$.

This argument shows that the sequence $(\hat{f}_n)$ is a sequence of smooth, C_0-functions, which is equibounded in the $C_0(\mathbb{R})$-norm. To see that the family is also equicontinuous,

Now, following the same notation, $\partial(\hat{f}_n) = \partial(\hat{f}_n * \psi) = \hat{f}_n * \partial\hat{\psi}$. Replacing ψ by $\hat{\psi}$ in the previous argument therefore gives that the sequence of smooth functions $(\partial\hat{f}_n)$ is also equibounded in the $C_0(\mathbb{R})$-norm, and hence is also an equicontinuous family. Therefore, by the Arzela-Ascoli theorem, there exists a subsequence of the $(\hat{f}_n)$ which converges uniformly on compact subsets of $\mathbb{R}$. By labelling, we may assume the original sequence (f_n) has the property that $\hat{f}_n$ has this property.

We finish the proof by showing that the sequence (f_n) must be Cauchy in $L^2(\mathbb{R})$. For any $R > 0$ write

$$\|f_k - f_l\|^2_{L^2(\mathbb{R})} = \|\hat{f}_k - \hat{f}_l\|^2_{L^2(\hat{\mathbb{R}}} = \int_{|\xi|\leq R} |\hat{f}_k(\xi) - \hat{f}_l(\xi)|^2\, d\xi + \int_{|\xi|\geq R} |\hat{f}_k(\xi) - \hat{f}_l(\xi)|^2\, d\xi. \tag{11.19}$$

If $|\xi| \geq R$ then $1 \leq \frac{(1+|\xi|^2)^s}{(1+R^2)^2}$ and hence continuing with (11.19) we get

(11.20)

$$\|f_k - f_l\|^2_{L^2(\mathbb{R})} \leq \int_{|\xi|\leq R} |\hat{f}_k(\xi) - \hat{f}_l(\xi)|^2\, d\xi + (1+R^2)^{-s} \int_{|\xi|\geq R} |\hat{f}_k(\xi) - \hat{f}_l(\xi)|^2 (1+|\xi|^2)^s\, d\xi$$

$$\leq \int_{|\xi|\leq R} |\hat{f}_k(\xi) - \hat{f}_l(\xi)|^2\, d\xi + (1+R^2)^{-s} \leq \int_{|\xi|\leq R} |\hat{f}_k(\xi) - \hat{f}_l(\xi)|^2\, d\xi + (1+R^2)^{-s}\|f_k - f_l\|_{H_s(\mathbb{R})}$$

$$\leq \int_{|\xi|\leq R} |\hat{f}_k(\xi) - \hat{f}_l(\xi)|^2\, d\xi + C'(1+R^2)^{-s}.$$

for some constant C'. Since $(\hat{f}_n)$ converges uniformly on $\mathbb{R}$, the result follows.

□

12. The Haagerup-Sobolev property for discrete groups

One of the key results about Sobolev theory of the last section was the Sobolev Embedding Theorem, which in the form relevant to us now, asserts that

$$H_s(\mathbb{R}) \subset C_0(\mathbb{R}), \text{ and } \|f\|_{C_0(\mathbb{R})} \leq C\|f\|_s \text{ if } s > \frac{1}{2},$$

where $\|f\|_s$ is the Sobolev norm

$$\|f\|_s^2 = \int |\hat{f}(\xi)|^2 (1+|\xi|^2)^{\frac{s}{2}}.$$

More precisely, the inclusion

$$\mathcal{S}(\mathbb{R}) \to C_0(\mathbb{R})$$

extends continuously to $H_s(\mathbb{R})$ (as maps between normed vector spaces.)

The Sobolev spaces are defined in terms of Fourier transform and are even more naturally formulated in terms of the *group C*-algebra* of $\mathbb{R}$: the Fourier transformed Sobolev theorem asserts that there exists a constant $C > 0$ such that

$$\|\lambda(f)\| \leq C \cdot \|f\|_s, \quad f \in \mathcal{S}(\mathbb{R})),$$

where λ is the regular representation, and

$$\|f\|_s^2 := \int |f(\xi)|^2 (1+|\xi|^2)^{-\frac{s}{2}}.$$

Equivalently, the inclusion $\mathcal{S}(\mathbb{R}) \to C^*(\mathbb{R})$ extends continuously to an inclusion $H_s(\mathbb{R})$ if $s > \frac{1}{2}$, where $H_s(\mathbb{R})$ is defined by the Hilbert space norm $\|\cdot\|_s$ above.

We define Sobolev spaces for the integers $\mathbb{Z}$ in an analogous way. If $f \in C_c(\mathbb{Z})$, we let

$$\|f\|_s^2 := \sum_{n\in\mathbb{Z}} |f(n)|^2 (1+n^2)^{\frac{s}{2}}.$$

and $H_s(\mathbb{Z})$ the completion of $C_c(\mathbb{Z})$ with respect to this norm. We obtain, by essentially repeating the proof for $\mathbb{R}$ with integrals replaced by sums:

THEOREM 12.1. *With $H_s(\mathbb{Z})$ the Sobolev spaces for the group of integers $\mathbb{Z}$, there exists a constant $C > 0$ such that*

$$H_s(\mathbb{Z}) \subset C^*(\mathbb{Z}) \text{ if } s > \frac{1}{2}, \text{ and } \|f\|_{C^*(\mathbb{Z})} \leq C\|f\|_s, \ \forall f \in H_s(\mathbb{Z})..$$

The Sobolev theorem for the integers, therefore, gives a sufficient condition

$$\hat{f} \in H_s(\mathbb{Z}), \quad s > \frac{s}{2},$$

on the Fourier coefficients of f, for it to be continuous.

EXERCISE 12.2. Prove Theorem 12.1 by using the method of proof of Theorem 11.23.

EXERCISE 12.3. Prove that if $\alpha \in [-1, -\frac{3}{4})$, and

$$f(n) := n^{-\alpha},$$

then $\lambda(f) \in C_r^*(\mathbb{Z})$ but $f \notin l^1(\mathbb{Z})$.

In other words, the Fourier series

$$\sum_{n=0}^{\infty} n^{\alpha} z^n$$

represents a continuous function $\hat{f}$ on the circle $\mathbb{T}$ although its series of Fourier coefficients is not absolutely summable.

Let now G be any (countable) group. We will assume that G is finitely generated, so there is a finite subset $S \subset G$ such that every element of G can be written $g = s_1 \cdots s_n$. The minimal such n is the *word length* of g and denoted $|g|$.

DEFINITION 12.4. If G is a group generated by S finite, set, for $f \in C_c(G)$,

$$\|f\|_s^2 := \sum_{g \in G} |f(g)|^2 (1 + |g|^2)^{\frac{s}{2}}.$$

The completion of $C_c(G)$ with respect to $\|\cdot\|_s$ is denoted $H_s(G)$.

There is no need to specify the generating set S in the Sobolev spaces due to the following exercise.

EXERCISE 12.5. Prove that if S' is another generating set, then the Sobolev spaces $H_s(G)$ defined using S and using S' are exactly the same (the norms are different, but 'equivalent.')

We are mainly going to be studying the group $G = \mathbb{F}_2$ on two generators, say, a and b. Any element of G may thus be written as a product of a certain number of the elements $a^{\pm 1}$ and $b^{\pm 1}$. If the expression (as a 'word') contains no cancelling terms like aa^{-1} and so on, we call the word *reduced.* The word length $|g|$ is the minimum number of terms in a reduced expression of a group element.

In 1978, Ufe Haagerup proved the following result, amongst several others, about the group C*-algebra of $\mathbb{F}_2$.

THEOREM 12.6. *If $G = \mathbb{F}_2$, then the inclusion $\mathbb{C}[G] \to C^*(G)$ extends continuously to an inclusion*

$$H_s(G) \to C_r^*(G)$$

if $s > \frac{3}{2}$.

EXERCISE 12.7. Using the Theorem, if $G = \mathbb{F}_2$, find an element $f \in C_r^*(G)$ which is not in $l^1(G)$. Write down an explicit formula for f as a function on G.

We will say that a finitely generated group G has the *Haagerup-Sobolev property* for dimension $s > 0$ if if the inclusion $\mathbb{C}[G] \to C_r^*(G)$ extends continuously to a map

$$H_s(G) \to C_r^*(G).$$

Thus, we are going to prove that free groups have the Haagerup-Sobolev property.

For any finitely generated group G with word length $|\cdot|$, and for any $n \geq 0$ (an integer) let

$$S_n = \{g \in G \mid |g| = n\}.$$

LEMMA 12.8. *Suppose there is a polynomial p of degree d such that if n is a positive integer and $f \in C_c(G)$ is supported on group elements of length n then*

$$\|\lambda(f)\| \leq p(n) \cdot \|f\|_{l^2(G)}.$$

Then G has the Haagerup-Sobolev property for any $s > d + \frac{1}{2}$.

PROOF. Since $f = \sum_{n=0}^{\infty} f \cdot \chi_n$, with χ_n the characteristic function on S_n,

$$\|\lambda(f)\| \leq \sum_{n=0}^{\infty} \|\lambda(f \cdot \chi_n)\| \tag{12.1}$$

and by assumption $\|\lambda(f \cdot \chi_n)\| \leq p(n)\|f \cdot \chi_n\|_{l^2(G)}$ we obtain

$$\|\lambda(f)\| \leq \sum_{n=0}^{\infty} p(n) \cdot \|f \cdot \chi_n\|_{l^2(G)} \tag{12.2}$$

By Cauchy-Schwartz we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p(n) \cdot \|f \cdot \chi_n\|_{l^2(G)} &= \sum_{n=0}^{\infty} p(n) \cdot n^{-s} \cdot \|f \cdot \chi_n\|_{l^2(G)} \cdot n^s \\ &\leq \left(\sum_n p(n)^2 \cdot n^{-2s}\right)^{\frac{1}{2}} \cdot \left(\sum_n \|f \cdot \chi_n\|_{l^2(G)}^2 \cdot n^{2s}\right)^{\frac{1}{2}}.\end{aligned}$$

There exists a constant C, such that $p(n) \leq Cn^d$ for all n. Hence

$$\sum_n p(n)^2 \cdot n^{-2s} < \infty$$

converges if $s > d + \frac{1}{2}$, while

$$\sum_n \|f \cdot \chi_n\|_{l^2(G)}^2 \cdot n^{2s} = \|f\|_{2s} \leq c \cdot \|f\|_{2s}^2.$$

for a constant c, by definition of the Sobolev spaces. We obtain then a constant $C > 0$ such that

$$\|\lambda(f)\| \leq C \cdot \|f\|_{2s}, \quad \text{if} \ \ s > d + \frac{1}{2},$$

as claimed.

□

LEMMA 12.9. *Let $G = \mathbb{F}_2$ and suppose that f, g are functions on G supported in S_k and S_l respectively. Then*

$$\|(f * g) \cdot \chi_m\|_{l^2(G)} \leq \|f\|_{l^2(G)} \cdot \|g\|_{l^2(G)} \tag{12.3}$$

for any m such that

$$|k - l| \leq m \leq k + l.$$

and $(f * g) \cdot \chi_m = 0$ for m outside this range.

PROOF. We start with the following

EXERCISE 12.10. If $t, u \in G$ have lengths k, l respectively, then $m := |tu|$ has length one of the numbers

$$|k-l|, |k-l|+2, \dots k+l-2, k+l.$$

Now suppose $f, g,\ k, l$ are as in the Lemma. We have,

$$(f * g) \cdot \chi_m(s) = \sum_{tu=s, |t|=k, |u|=l} f(t)g(u), \quad s \in G,\ |s| = m, \tag{12.4}$$

with the value zero given to any s with $|s| \neq m$.

We know from Exercise 12.10 that m can only take one of the values

$$|k-l|, |k-l|+2, \dots k+l-2, k+l.$$

Suppose first that $m = k + l$. Then in (12.4) we are summing over pairs t, u of length k, l respectively, whose length is $k + l$, and there is a unique such pair, for each s, namely with $t = s_k$, the product of the first k letters of s, and $u = s_l$, the last l letters. Hence (12.4) gives, in this notation,

$$(f * g) \cdot \chi_m(s) = f(s_k) \cdot g(s_l) \tag{12.5}$$

from which the inequality

$$\begin{aligned} \|(f * g) \cdot \chi_m\|^2_{l^2(G)} &= \sum_{s \in G, |s|=m} (f * g)(s)^2 = \sum_{|s|=m} \left(\sum_{|t|=k, |u|=l, tu=s} f(t)g(u) \right)^2 \\ &= \sum_{|s|=m} f(s_k)^2 g(s_l)^2 \leq \sum_{|t|=k, |u|=l} f(t)^2 g(u)^2 = \|f\|^2_{l^2(G)} \cdot \|g\|^2_{l^2(G)}. \end{aligned} \tag{12.6}$$

we want to prove, follows.

Now suppose that $m = k + l - 2p$, for some p. If $|s| = m$, $|t| = k, |u| = l$, and $tu = s$, then the last p letters of t are cancelling with the first p letters in u, making the product tu have length $k + l - 2p$.

This in particular means that the first $k - p$ letters of t and the last $l - p$ letters of u are determined by s. Let t' be the product of the first $k - p$ letters and u' the product of the last $l - p$ letters of s. Any factorization $s = tu$ with $|t| = k$, $|s| = l$, has therefore the form $t = t'v$, $u = v^{-1}u'$. Moreover, $s = u'v'$ then gives a reduced expression of s, that is, with $|s| = |u'| + |v'|$.

Suppose now we define functions f' and g' by

$$f'(t) = \begin{cases} \left(\sum_{|v|=p} |f(tv)|^2 \right)^{\frac{1}{2}} & \text{if } |t| = k - p \\ 0 & \text{else} \end{cases} \tag{12.7}$$

$$g'(t) = \begin{cases} \left(\sum_{|v|=p} |g(v^{-1}u)|^2 \right)^{\frac{1}{2}} & \text{if } |t| = l - p \\ 0 & \text{else} \end{cases} \tag{12.8}$$

Then f' is supported in S_{k-p} and g' is supported on S_{l-p}. Continuing with the above notation, if $s = t'u'$ is a reduced expression of s into a word of length $k-p$ and a word of length $l - p$ then by the above remarks on the special case, we get that

$$(f' * g')(s) = f'(t') \cdot g'(u'), \tag{12.9}$$

and, as a consequence

$$\|(f' * g') \cdot \chi_m\|_{l^2(G)} \leq \|f'\|_{l^2(G)} \cdot \|g'\|_{l^2(G)},$$

which facts we employ as follows. We have

$$|(f * g)(s)| = |\sum_{tu=s,|t|=k,|u|=l} f(t)g(u)| \tag{12.10}$$
$$= |\sum_{|v|=p,|t'v|=k,|v^{-1}u'|=l} f(t'v)g(v^{-1}u')|$$
$$= |\sum_{|v|=p} f(t'v)g(v^{-1}u')|$$

where in the last step we used the fact that f is supported on words of length k, g on words of length l, so that some of the conditions in the sum could be dropped without changing it.

Now by the Cauchy-Schwartz inequality, the definitions, and (12.9) we get

$$|\sum_{|v|=p} f(t'v)g(v^{-1}u')| \leq \left(\sum_{|v|=p} |f(t'v)|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{|v|=p} |g(v^{-1}u')|^2\right)^{\frac{1}{2}} \tag{12.11}$$
$$= f'(t') \cdot g'(u') = (f' * g')(s)$$

We have proved that

$$|(f * g)(s)| \leq (f' * g')(s) \tag{12.12}$$

for every s of length $m = k + l - 2p$. Thus

$$\|(f * g) \cdot \chi_m\|_{l^2(G)} \leq \|(f' * g') \cdot \chi_m\|_{l^2(G)} \leq \|f'\|_{l^2(G)} \cdot \|g'\|_{l^2(G)},$$

Finally, by the uniqueness of expressions with no cancellations, we see that

$$\|f'\|_{l^2(G)} = \|f\|_{l^2(G)}, \quad \|g'\|_{l^2(G)} = \|g\|_{l^2(G)},$$

and the result follows.

□

Corollary 12.11. *If f is a function on G supported on words of length n then*

$$\|\lambda(f)\| \leq (n+1) \cdot \|f\|_{l^2(G)}.$$

Proof. Let $g \in l^2(G)$ and $g_k = g \cdot \chi_k$, with χ_k the characteristic function of S_k as usual. Then

$$\|(f * g) \cdot \chi_m\|_{l^2(G)} = \|\sum_{k=0}^{\infty} (f * g_k) \cdot \chi_m\|_{l^2(G)} \tag{12.13}$$

The kth term in the sum vanishes unless

$$|n - k| \leq m \leq n + k,$$

which is equivalent to saying that

$$|n - m| \leq k \leq n + m,$$

and, moreover, for any k, we have the inequality

$$\|(f * g_k) \cdot \chi_m\|_{l^2(G)} \leq \|f\|_{l^2(G)} \cdot \|g\|_{l^2(G)}$$

by the Lemma above. We obtain that

$$\|(f * g) \cdot \chi_m\|_{l^2(G)} \leq \|f\|_{l^2(G)} \cdot \sum_{k=|m-n|}^{m+n} \|g_k\|_{l^2(G)}. \tag{12.14}$$

The number of terms in the sum involving k is at most $n+1$. Hence, by the Cauchy-Schwartz inequality

$$\sum_{k=|m-n|}^{m+n} \|g_k\|_{l^2(G)} \leq \sqrt{n+1} \cdot \sum_{|m-n| \leq k \leq m+n,\ m+n-k \text{ even}} \|g_k\|^2_{l^2(G)}.$$

Putting things together, we have shown that

$$\|(f * g) \cdot \chi_m\|_{l^2(G)} \leq \sqrt{n+1} \cdot \|f\|_{l^2(G)} \cdot \sum_{|m-n| \leq k \leq m+n,\ m+n-k \text{ even}} \|g_k\|^2_{l^2(G)}. \tag{12.15}$$

For $|m-n| \leq k \leq k+n$ and $n+m-k$ even we can write

$$k = m+n-2l, \quad 0 \leq l \leq \min(m,n).$$

We then have

$$\begin{aligned}
\|f * g\|^2_{l^2(G)} = \sum_{m=0}^{\infty} \|(f * g) \cdot \chi_m\|^2_{l^2(G)} &\leq (n+1)\|f\|^2_{l^2(G)} \cdot \sum_{m=0}^{\infty} \sum_{l=0}^{\min(n,m)} \|g_{m+n-2l}\|^2_{l^2(G)} \\
&\leq (n+1) \cdot \|f\|^2_{l^2(G)} \cdot \sum_{l=0}^{n} \sum_{m=l}^{\infty} \|g_{n+m-2l}\|^2_{l^2(G)} \\
\leq (n+1) \cdot \|f\|^2_{l^2(G)} \cdot \sum_{l=0}^{n} \sum_{k=n-l}^{\infty} \|g_k\|^2_{l^2(G)} &\leq (n+1) \cdot \|f\|^2_{l^2(G)} \cdot \sum_{l=0}^{n} \|g\|^2_{l^2(G)} \\
&= (n+1)^2 \cdot \|f\|^2_{l^2(G)} \cdot \|g\|^2_{l^2(G)}
\end{aligned} \tag{12.16}$$

which gives the result.

□

Theorem 12.6 now follows from the combination of Lemmas 12.8 and Corollary 12.11.

EXERCISE 12.12. A finitely generated group G corollary:haagmain has *polynomial growth* if there exists $C \geq 0$ and a positive integer d such that

$$|\{g \in G \mid |g| \leq n\}| \leq Cn^d$$

for all n, where $||$ is the word length function, for any choice of finite generating set for G. Obviously $\mathbb{F}_2$ does not have polynomial growth, while $\mathbb{Z}$ does.

a) Prove that the group $\mathbb{Z}^d$ has polynomial growth.
b) Prove that if G has polynomial growth rate d with respect to one finite generating set, it does with respect to any other as well.
c) Prove that if G has polynomial growth d, then G has the Haagerup-Sobolev property.

EXERCISE 12.13. A (discrete) group G is *amenable* if the algebra homomorphism

$$\epsilon \colon \mathbb{C}[G] \to \mathbb{C}, \quad \epsilon(f) = \sum_{g\imath nG} f(g)$$

induced by the trivial representation of G, extends continuously to the reduced C*-algebra $C^*(G)$ of G. (Any group of polynomial growth is amenable, but to prove this would require considerable additional discussion about the definition of 'amenable.')

Prove the following partial converse: that if G is amenable and has the Haagerup-Sobolev property (for some s), then G has polynomial growth.

The semi-direct product group

$$\mathbb{Z}^2 \rtimes_A \mathbb{Z}$$

obtained by considering the automorphism of $\mathbb{Z}^2$ induced by the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

has exponential growth, but is amenable. Hence it does not have the Haagerup-Sobolev property.

CHAPTER 3

Spectral Theory and representations

In this chapter we return to the general theory of C*-algebras. It starts with a closer examination of the properties of *spectra*.

1. The spectral radius formula

An element a in a unital algebra A is invertible if there exists $b \in A$ such that $ab = ba = 1$.

DEFINITION 1.1. Let A be a unital Banach algebra and $a \in A$. The *spectrum* $\operatorname{Spec}_A(a)$ of a is the set $\lambda \in \mathbb{C}$ such that $\lambda - a$ is not invertible.

Here λ really means $\lambda \cdot 1$, where 1 is the unit, but we generally just write λ. Invertibility of course makes no sense unless the algebra is unital.

EXERCISE 1.2. If A is a Banach algebra and u is an invertible in A then $\operatorname{Spec}(a) = \operatorname{Spec}(uau^{-1})$.

If $\alpha\colon A \to B$ is a unital homomorphism of unital Banach algebras, continuous or not, then $\operatorname{Spec}\big(\alpha(a)\big) \subset \operatorname{Spec}(a)$ for every $a \in A$.

REMARK 1.3. Invertibility sometime depends on in which algebra one allows the inverse to be in, as in, for instance, if $A \subset B$ is a unital subalgebra of a unital Banach algebra B then by Exercise 1.2 $\operatorname{Spec}_B(a) \subset \operatorname{Spec}_A(a)$ but the containment may be strict.

For example, the function $f(z) = z$ in $C(\mathbb{T})$ is invertible in $C(\mathbb{T})$ but not in $\mathcal{A}(\mathbb{D})$, since its inverse would have to be $\frac{1}{z}$, which has a singularity at the origin. Hence $0 \in \operatorname{Spec}_{\mathcal{A}(\mathbb{D})}(z) \setminus \operatorname{Spec}_{C(\mathbb{T})}(z)$.

EXAMPLE 1.4. If $A = M_n(\mathbb{C})$ the spectrum reduces to the usual notion of eigenvalue of a matrix, since $T \in M_n(\mathbb{C})$ is invertible exactly when $\det(\lambda - T) \neq 0$.

More generally, the spectrum of $T \in \mathbb{B}(H)$ is the set of $\lambda \in \mathbb{C}$ such that $\lambda - T$ is not bijective (bijectivity of an operator is equivalent to its invertibility in $\mathbb{B}(H)$, by the Open Mapping Theorem).

EXERCISE 1.5. If X is a compact Hausdorff space and $f \in C(X)$, then f is invertible if and only if f does not vanish anywhere, so *the spectrum of f is the range of f*.

EXERCISE 1.6. Let A be a unital C*-algebra. Show that $a \in A$ is invertible if and only if a^* is invertible and in this case $(a^*)^{-1} = (a^{-1})^*$. Deduce that if A is a C*-algebra and $a \in A$ then $\operatorname{Spec}_A(a^*) = \{\bar{\lambda} \mid \lambda \in \operatorname{Spec}_A(a)\}$.

EXERCISE 1.7. Let T_z be the Toeplitz operator with symbol z.

a) If $0 \leq \lambda < 1$ prove that $\ker(\lambda - T_z^*) \neq 0$. Deduce (from this and Exercise 1.6) that the spectrum of T_z contains the entire closed interval $[0, 1]$.
b) By Exercise 7.8, $\lambda(w)T_z\lambda(w)^* = wT_z$ for any $w \in \mathbb{T}$, λ the regular representation of $\mathbb{T}$ on $L^2(\mathbb{T})$, restricted to the invariant subspace $\mathbf{H}^2$. Deduce that $\operatorname{Spec}(T_z)$ is a rotationally invariant closed subset of the unit disk whence that $\operatorname{Spec}(T_z) = \overline{\mathbb{D}}$.

EXERCISE 1.8. If T_z is the Toeplitz operator with symbol z, then $T_z^* T_z = 1$ is invertible, but $T_z T_z^*$ is not; however, it is true in general that for a, b in a unital Banach algebra, $\mathrm{Spec}(ab)\setminus\{0\} = \mathrm{Spec}(ba) \setminus \{0\}$. (see Exercise 1.3

The following lemma shows that the open disk in A centred at 1 consists entirely of invertibles.

LEMMA 1.9. *If A is unital, $a \in A$ and $\|a - 1\| < 1$, then a is invertible and the series $\sum_{n=0}^{\infty}(1 - a)^n$ converges in A to a^{-1}.*

PROOF. In a complete normed linear space, *e.g.* in a C*-algebra, or Hilbert space, if a series $\sum_n a_n$ converges absolutely, that is, if $\sum\|a_n\|$ converges, then the series converges. This is because of the triangle inequality implies that the sequence of partial sums of such a series is a Cauchy sequence.

Now since $\|a - 1\| < 1$, the series $\sum\|1 - a\|^n$ converges. Since $\|(1 - a)^n\| \leq \|1 - a\|^n$, the series $\sum_n\|(1 - a)^n\|$ converges, that is, $\sum_n(1 - a)^n$ is an absolutely convergent series in A. Hence it converges, say to b. By considering the partial sums of $(1 - a)\sum_{n=0}^{\infty}(1 - a)^n$ one sees easily that $(1 - a)b = b - 1$. Hence $b - ab = b - 1$, so $ab = 1$. Similarly, $ba = 1$.

□

COROLLARY 1.10. *Let A be a unital Banach algebra.*

a) *Let $a \in A$ be invertible. Then if $\|a - b\| < \frac{1}{\|a^{-1}\|}$, then b is invertible; moreover the absolutely convergent series $\sum_{n=0}^{\infty}(1 - ba^{-1})^n a$ converges to b^{-1}. In particular, the invertibles in A form an open subset of A in the norm topology.*
b) *If $|\lambda| > \|a\|$ then $\lambda - a$ is invertible, and the absolutely convergent series $\frac{1}{\lambda}\sum_{n=0}^{\infty}(\frac{a}{\lambda})^n$ converges to $(\lambda - a)^{-1}$. In particular, $\mathrm{Spec}(a) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$.*

COROLLARY 1.11. *The spectrum $\mathrm{Spec}(a)$ of any element of a unital Banach algebra A is a non-empty compact subset of the complex plane.*

PROOF. The first statement is immediate from Corollary 1.10, the first part of which implies that $\mathbb{C} \setminus \mathrm{Spec}(a)$ is open, and the second part that it is bounded.

For the last statement, we apply Liouville's theorem to the function

$$f\colon \mathbb{C} \setminus \mathrm{Spec}(a) \to A, \ \ f(\lambda) := (\lambda - a)^{-1},$$

which is easily checked to be holomorphic. Clearly $\lim_{|\lambda|\to\infty}\|f(\lambda)\| = 0$. In particular, f is bounded outside a compact set, and being continuous on the complement − on the compact set itself − it is bounded globally, so if its domain were $\mathbb{C}$, it would be a bounded, entire function $\mathbb{C} \to A$, whence constant by Liouville's Theorem, whence the zero function, since $\lim_{|\lambda|\to\infty}\|f(\lambda)\| = 0$. This contradiction implies that $\mathrm{Spec}(a) \neq \emptyset$. □

REMARK 1.12. In the case where $A = M_n(\mathbb{C})$, Corollary 1.11 amounts to the assertion that every matrix has a (complex) eigenvalue.

EXERCISE 1.13. Show that the map $a \mapsto a^{-1}$ is continuous on the (open) subset consisting of invertibles in A.

EXERCISE 1.14. Show that if $u \in A$ is an invertible in a unital Banach algebra and if $\lambda \in \mathrm{Spec}(u)$ then $\lambda^{-1} \in \mathrm{Spec}(u^{-1})$. Conclude that $\mathrm{Spec}(u^{-1}) = \{\frac{1}{\lambda} \mid \lambda \in \mathrm{Spec}(u)\}$.

Deduce that if u is a unitary, *i.e.* if $uu^* = u^*u = 1$ in a unital C*-algebra A, then u has norm 1 so $r(u) \leq 1$. Deduce from $u^* = u^{-1}$ that $\mathrm{Spec}(u) \subset \mathbb{T}$.

The *spectral radius* of an element $a \in A$ is defined

$$r(a) := \sup_{\lambda\in\mathrm{Spec}(a)} |\lambda|.$$

THEOREM 1.15. *(Spectral radius formula) If A is a unital Banach algebra and $a \in A$ then*
$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

PROOF. Consider the series $\frac{1}{\lambda}\sum_{n=0}^{\infty} a^n\lambda^{-n}$ of Corollary 1.10; it converges absolutely to $f(\lambda) := (\lambda - a)^{-1}$ for $|\lambda| > r(a)$. Set $g(\lambda) = f(\frac{1}{\lambda})$ if $\lambda \neq 0$, and set $g(0) = 0$. The power series expansion of g at 0 is obtained by substituting $\frac{1}{\lambda}$ into λ in the power series expansion of f, thus
$$(\lambda^{-1} - a)^{-1} = g(\lambda) = \lambda \cdot \sum_{n=0}^{\infty} a^n\lambda^n.$$
Now if $|\lambda| < \frac{1}{r(a)}$ then $\lambda^{-1} \notin \mathrm{Spec}(a)$ and so g will be holomorphic at λ. Thus g is holomorphic in the ball of radius $\frac{1}{r(a)}$ centred at zero. Hence, by the machinery of power series, the series converges absolutely on compact subsets of $\{\lambda \in \mathbb{C} \mid |\lambda| < \frac{1}{r(a)}\}$ and the radius of convergence of the series is $\frac{1}{r(a)}$. Since the nth coefficient of our series is now a^n, the result now follows from the radius of convergence formula of Theorem 9.3. □

THEOREM 1.16. *Let A be a unital C*-algebra and $a \in A$ a self-adjoint. Then $\|a\| = r(a)$.*

PROOF. By Theorem 1.15, $\lim_{n\to\infty}\|a^{2^n}\|^{\frac{1}{2^n}} = r(a)$ holds for any a even in a Banach algebra. . On the other hand, $\|a^2\| = \|a\|^2$ for self-adjoint elements in a C*-algebra, by the C*-identity. The result follows.

□

From Theorem 1.16, $r(a^*a) = \|a^*a\|$ for any $a \in A$, since a^*a is self-adjoint. So combining this with the C*-identity gives the following purely *algebraic* description of the norm on a C*-algebra:
$$\|a\|^2 = \sup\{|\lambda| \mid \lambda - a^*a \text{ is not invertible}\}.$$
In particular, C*-algebras are *rigid* in the following sense.

COROLLARY 1.17. *A unital *-homomorphism $\varphi\colon A \to B$ between unital C*-algebras is automatically continuous – in fact is a contraction $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$.*

PROOF. What is obvious is that if $\lambda - a$ is invertible, then so is $\varphi(\lambda - a) = \lambda - \varphi(a)$. Thus $\mathrm{Spec}\big(\varphi(a)\big) \subset \mathrm{Spec}(a)$, so $r\big(\varphi(a)\big) \leq r(a)$ holds for any $a \in A$, on general principals.

Now, since $\|a\| = r(a)$ for self-adjoint elements, and since $\varphi(a)$ is self-adjoint if a is, we see that $\|\varphi(a)\| = r\big(\varphi(a)\big) \leq r(a) = \|a\|$ for self-adjoints, using Theorem 1.16. Now in general, a^*a is self-adjoint, and using the C*-identity we get the result.

□

COROLLARY 1.18. *A C*-algebra isomorphism is automatically isometric: $\|\varphi(a)\| = \|a\|$ for all $a \in A$.*

REMARK 1.19. It will follow from the Spectral Permanence Theorem 3.1 that the hypothesis can be weakened from *isomorphism* to *injective*.

EXERCISE 1.20. . Show that if $f \in \mathcal{A}(\mathbb{D}) \subset C(\mathbb{T})$ and $\tilde{f}$ is the unique extension of f to a function $\tilde{f} \in C(\overline{\mathbb{D}})$ which is analytic in $\mathbb{D}$, then $\mathrm{Spec}_{\mathcal{A}(\mathbb{D})}(f) = \mathrm{ran}(\tilde{f})$. Since, by contrast, $\mathrm{Spec}_{C(\mathbb{T})}(f) = \mathrm{ran}(f)$, this gives many examples of elements whose spectrum in $\mathcal{A}(\mathbb{D})$ is different from their spectrum in $C(\mathbb{T})$, that is, $\mathrm{Spec}_{\mathcal{A}(\mathbb{D})}(f) \neq \mathrm{Spec}_{C(\mathbb{T})}(f)$ (like $\tilde{f}(z) = z$.)

Essential spectrum of a bounded operator

An interesting example of a spectrum is to take a bounded operator $T \in \mathbb{B}(H)$ and look at its image in the C*-algebra $\mathcal{Q}(H) := \mathbb{B}(H)/\mathcal{K}(H)$ (it was shown to be a C*-algebra under coset multiplication in the previous chapter.)

Definition 1.21. The *essential spectrum* $\mathrm{Spec}_{\mathrm{ess}}(T)$ of a bounded operator T, is the spectrum of T in $\mathcal{Q}(H)$, the set of $\lambda \in \mathbb{C}$ such that $\lambda - \pi(T)$ is not invertible in $\mathcal{Q}$.

By Exercise 1.2, $\mathrm{Spec}_{\mathrm{ess}}(T) \subset \mathrm{Spec}(T)$. The essential spectrum is the part of the spectrum which remains unchanged when T is replaced by a compact perturbation of T. By the definitions, T is *Fredholm* if and only if $0 \notin \mathrm{Spec}_{\mathrm{ess}}(T)$.

Exercise 1.22. Let $T \in \mathbb{B}\big(l^2(\mathbb{N})\big)$ be the multiplication operator M_f where $f \in l^\infty(\mathbb{N})$. Say $\|f\| \leq 1$ for simplicity. Prove that $\mathrm{Spec}(T) = \overline{\mathrm{ran(f)}}$, a closed subset of the unit disk, while

$$\mathrm{Spec}_{\mathrm{ess}}(T) = \mathrm{ran(f)}' \cup \{\lambda \in \mathbb{C} \mid f^{-1}(\lambda) \subset \mathbb{N} \text{ is infinite}\},$$

where $\mathrm{ran}(f)'$ means the set of *limit points.*

Exercise 1.23. Prove that if T is essentially unitary and $\mathrm{Spec}_{\mathrm{ess}}(T) \subset \mathbb{T}$ is a *proper* subset of the circle, then $\mathrm{index}(T) = 0$.

Exercise 1.24. Let T be any bounded operator. Show that $\mathrm{index}(\lambda - T)$ is constant on connected components of $\mathbb{C} \setminus \mathrm{Spec}_{\mathrm{ess}}(T)$ and vanishes on unbounded components. This gives a new proof that $\mathrm{index}(T) = 0$ if T is self-adjoint, and, proves as well that $\mathrm{index}(T) = 0$ if T is essentially unitary if $\mathrm{Spec}_{\mathrm{ess}}(T) \subset \mathbb{T}$ is a proper subset of the circle, *etc.*

A nice interpretation of the essential spectrum and its relation to the spectrum appears in Toeplitz operator theory.

Proposition 1.25. *If $f \in \mathcal{A}(\mathbb{D})$ and $\tilde{f}$ its analytic extension to the disk $\mathbb{D}$ the* $\mathrm{Spec}_{\mathrm{ess}}(T_f) = \mathrm{Spec}(f)(= \mathrm{ran}(f))$*, while* $\mathrm{Spec}(T_f) = \mathrm{ran}(\tilde{f})$*, and the diagram*

$$\begin{tikzcd} \mathrm{Spec}_{\mathrm{ess}}(T_f) \arrow[r] \arrow[d, "="'] & \mathrm{Spec}(T_f) \arrow[d, "="] \\ \mathrm{ran}(f) \arrow[r] & \mathrm{ran}(\tilde{f}) \end{tikzcd}$$

commutes, in which the horizontal maps are the natural inclusions.

Proof. The statement that $\mathrm{Spec}_{\mathrm{ess}}(T_f) = \mathrm{ran}(f)$ for $f \in \mathcal{A}(\mathbb{D})$ and even $f \in C(\mathbb{T})$, is the content of

Since $\mathcal{A}(\mathbb{D}) \to \mathcal{T}$, $f \mapsto T_f$, is a Banach algebra homomorphism, $\mathrm{Spec}(T_f) \subset \mathrm{Spec}_{\mathcal{A}(\mathbb{D})}(f)$. We need to show that equality holds. We need therefore show that for $f \in \mathcal{A}(\mathbb{D})$, if the Toeplitz operator T_f is invertible in $\mathcal{T}$, then f is invertible in $\mathcal{A}(\mathbb{D})$. This is quite awkward to try to prove directly.

One may argue alternatively as follows, using the *holomorphic functional calculus* for Banach algebras. If $f \in \mathcal{A}(\mathbb{D})$, $\tilde{f}$ its extension to an analytic function on the disk, then holomorphic functional calculus gives meaning to $\tilde{f}(T_z)$. Moreover, $\|\tilde{f}(T_z)\| \leq \|\tilde{f}\|_{\mathcal{A}(\mathbb{D})}$. On polynomials, this gives the same map as the Banach algebra map $\mathcal{A}(\mathbb{D}) \to \mathcal{T}$, and as both are continuous, the two maps agree on $\mathcal{A}(\mathbb{D})$. So we obtain that

$$\mathrm{Spec}_{\mathcal{T}}(T_f) = \mathrm{Spec}_{\mathcal{T}}\big(\tilde{f}(T_z)\big) = \tilde{f}\big(\mathrm{Spec}(T_z)\big) = \tilde{f}(\overline{\mathbb{D}}) = \mathrm{ran}(\tilde{f})$$

by the Spectral Mapping Theorem.

$\square$

Exercise 1.26. spectral radius of the Laplacian of a graph.

2. Characters and Gelfand's Theorem

Let A be a unital commutative Banach algebra. The set of algebraic, *i.e.* not necessarily closed, proper ideals in A is a poset to which Zorn's lemma can be applied. We deduce the existence of *maximal proper* ideals $\mathcal{M}$, *i.e.* proper ideals of A which are contained in no larger proper ideal.

Maximaity of a proper ideal $\mathcal{M}$ actually implies that it is closed. For otherwise, the closure of $\mathcal{M}$ would be a larger ideal. If this larger ideal where A itself, then $\mathcal{M}$ would be dense in A, and hence $\mathcal{M}$ itself would non-trivially intersect the open subset $\{a \in A \mid \|a-1\| < 1\}$, which consists entirely of invertibles. An ideal containing an invertible can only of course be A itself. This contradicts properness of $\mathcal{M}$.

EXERCISE 2.1. Let A be a Banach algebra and $J \subset A$ be any closed ideal. Show that A/J with the quotient norm

$$\|a+J\| := \inf_{x\in J} \|a+x\|$$

and quotient vector space and algebra structure, is a Banach algebra, and that the quotient map $\pi\colon A \to A/J$ is a contractive homomorphism of Banach algebras.

By the Exercise, $A/\mathcal{M}$ with the quotient norm is a Banach algebra, so for any $a \in A$ we can speak of the spectrum of the coset $a + \mathcal{M}$ in the Banach algebra $A/\mathcal{M}$.

LEMMA 2.2. *If $\mathcal{M}$ is any maximal ideal in a commutative, unital Banach algebra A, then the spectrum of any element $a+\mathcal{M}$ in the Banach algebra $A/\mathcal{M}$ consists of a single point in the spectrum of a in A. The mapping sending $a+\mathcal{M}$ to λ if $\mathrm{Spec}_{A/\mathcal{M}}(a+\mathcal{M}) = \{\lambda\}$ is an isometric isomorphism $A/\mathcal{M} \cong \mathbb{C}$ of Banach algebras.*

PROOF. If $a \notin \mathcal{M}$, *i.e.* if $a+\mathcal{M}$ is a nonzero element of $A/\mathcal{M}$, then $a+\mathcal{M}$ generates a nonzero principal ideal $\langle a+\mathcal{M}\rangle := \{ab+\mathcal{M} \mid b \in A\} \subset A/\mathcal{M}$ which, clearly, is proper in $A/\mathcal{M}$ if and only if $a+\mathcal{M}$ is not invertible in $A/\mathcal{M}$. The inverse image $\pi^{-1}(\langle a+\mathcal{M}\rangle) \subset A$ of this ideal is an ideal of A containing $\mathcal{M}$. Since $\mathcal{M}$ is maximal, this inverse image must be all of A. Thus the ideal $\langle a+\mathcal{M}\rangle$ we started with is actually $A/\mathcal{M}$. In particular, $a+\mathcal{M}$ must be invertible in $A/\mathcal{M}$. This shows that any nonzero element of $A/\mathcal{M}$ is invertible in $A/\mathcal{M}$.

From this, we deduce that the the spectrum

$$\mathrm{Spec}(a+\mathcal{M}) := \{\lambda \in \mathbb{C} \mid \lambda - a + \mathcal{M} \text{ invertible in } A/\mathcal{M}\}$$

of an element $a+\mathcal{M}$ of $A/\mathcal{M}$ is the same as $\{\lambda \in \mathbb{C} \mid a+\mathcal{M} = \lambda + \mathcal{M}\}$, or, equivalently, the same as $\{\lambda \in \mathbb{C} \mid \lambda - a \in \mathcal{M}\}$. Note that there can be at most one scalar λ such that $\lambda - a \in \mathcal{M}$, since if there were two, say λ_1 and λ_2, then we would get $\lambda_1 - \lambda_2 \in \mathcal{M}$, but $\mathcal{M}$, being proper, can contain no nonzero scalar. Furthermore, by Liouville's theorem, $\mathrm{Spec}(a+\mathcal{M})$ is non-empty.

This all shows that the Banach algebra $A/\mathcal{M}$ consists exactly of multiples of the unit $1+\mathcal{M}$, with $a+\mathcal{M}$ corresponding to the multiple λ if and only if $\{\lambda\} = \mathrm{Spec}_{A/\mathcal{M}}(a+\mathcal{M})$.

□

If A is a commutative unital Banach algebra we call a nonzero homomorphism $\chi\colon A \to \mathbb{C}$ of Banach algebras a *character* of A. If $\mathcal{M} := \ker(\chi)$ then χ determines an isomorphism $A/\mathcal{M} \cong \mathbb{C}$ and since $\mathbb{C}$ has no nonzero proper ideals, neither does $A/\mathcal{M}$, and hence $\mathcal{M}$ is a maximal ideal in A.

Hence the proof of Lemma 2.2 provides an isomorphism $A/\mathcal{M} \cong \mathbb{C}$ mapping $a+\mathcal{M}$ to the unique point $\lambda \in \mathrm{Spec}_A(a) \subset \mathbb{C}$ such that $\mathrm{Spec}_{A/\mathcal{M}}(a+\mathcal{M}) = \{\lambda\}$. Equivalently, λ is determined by the property that $\lambda - a \in \mathcal{M}$. Since $\chi(a)$ satisfies this condition, $\chi(a) = \lambda$.

In particular, $\chi(a) \in \mathrm{Spec}_A(a)$ for any character and any $a \in A$. Since $|\lambda| \leq r(a) \leq \|a\|$, we obtain the following facts about characters.

LEMMA 2.3. *If A is a commutative unital Banach algebra and $\chi\colon A \to \mathbb{C}$ is a character of A then $\chi(1) = 1$, $\chi(a) \in \mathrm{Spec}_A(a)$ for all $a \in A$, and (hence) $|\chi(a)| \le \|a\|$ for any a, that is, χ is automatically contractive.*

LEMMA 2.4. *If A is a commutative unital Banach algebra and $a \in A$, then for every $\lambda \in \mathrm{Spec}_A(a)$, there is a character $\chi\colon A \to \mathbb{C}$ such that $\chi(a) = \lambda$.*

In particular, for any commutative unital Banach algebra and any $a \in A$, the spectrum $\mathrm{Spec}_A(a)$ consists precisely of the values $\chi(a)$ of characters of A at a.

PROOF. Since $\lambda - a$ is not invertible, it generates a proper ideal $\langle \lambda - a\rangle$ in A. This is contained in a maximal ideal $\mathcal{M}$, by Zorn's lemma, which is the kernel of some character $\chi\colon A \to \mathbb{C}$. Moreover, $\chi(a) = \lambda$ if and only if $\lambda - a \in \mathcal{M}$, and since $\lambda := \chi(a)$ satisfies this condition, $\chi(a) = \lambda$.

□

EXERCISE 2.5. The following exercise demonstrates what the character space $\mathcal{A}(\mathbb{D})$ of the disk algebra is.

a) Show that if A is a unital commutative Banach algebra and $a \in A$ *generates* A then any character of A is completely determined by its value at a.
b) Let $\tilde{f}$ be an analytic function on $\mathbb{D}$, $\tilde{f}(z) = \sum_{n=0}^{\infty} a_n z^n$ its Taylor series expansion at the origin. Prove that if $\tilde{f}$ is known to extend continuously to $\overline{\mathbb{D}}$, then $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on $\overline{\mathbb{D}}$ to $\tilde{f}$.

   Deduce that $\mathcal{A}(\mathbb{D})$ is generated as a Banach algebra by $f(z) = z$.
c) Deduce that $\widehat{\mathcal{A}(\mathbb{D})} \cong \overline{\mathbb{D}}$ with points $z \in \partial\overline{\mathbb{D}}$ corresponding to point evaluations $\tilde{f} \mapsto \tilde{f}(z)$.

EXERCISE 2.6. If $\chi \in \widehat{\mathbb{Z}} \cong \mathbb{T}$ is a character of $\mathbb{Z}$ then the formula r

$$\int_G f(g)\overline{\chi(g)}d\mu(g)$$

defines a character the Banach algebra $l^1(\mathbb{Z})$.

Deduce that $\widehat{l^1(\mathbb{Z})} \cong \mathbb{T}$ as topological spaces, and write an explicit homeomorphism.

In the case of C*-algebras, we have the special result:

LEMMA 2.7. *If A is a commutative C*-algebra and $\chi\colon A \to \mathbb{C}$ is a character, then $\chi(a^*) = \overline{\chi(a)}$, that is, χ is automatically a *-homomorphism.*

This implies that $A/\mathcal{M}$ is a C*-algebra, since it is isometrically *-isomorphic to $\mathbb{C}$. It also implies that a maximal ideal in a C*-algebra is automatically both closed, and a *-ideal, *i.e.*, closed under adjoint.

PROOF. It clearly suffices to show that $\chi(a) \in \mathbb{R}$ for a self-adjoint. Write $\chi(a) = \alpha + i\beta$. Then $|\chi(a+it)|^2 = \alpha^2 + \beta^2 + 2\beta t + t^2$ for any $t \in \mathbb{R}$. On the other hand, χ is contractive, so $|\chi(a+it)|^2 \le \|a+it\|^2 = \|(a+it)^*(a+it)\| = \|a^2+t^2\| \le \|a\|^2 + t^2$. The resulting inequality

$$\alpha^2 + \beta^2 + 2\beta t + t^2 \le \|a\|^2 + t^2$$

valid for all $t \in \mathbb{R}$ implies immediately that $\beta = 0$.

□

COROLLARY 2.8. *If A is a C*-algebra and $a \in A$ is any self-adjoint element, then* $\mathrm{Spec}(a) \in \mathbb{R}$.

PROOF. Indeed, the elements of $\mathrm{Spec}_A(a)$ are precisely the values of characters $\chi\colon A \to \mathbb{C}$, and such a character maps self-adjoints to $\mathbb{R}$ by Lemma 2.7. □

REMARK 2.9. Here is an alternative proof that the spectrum of a self-adjoint is real that does not appeal to character theory. For an element a of a Banach algebra, the exponential e^a of A is defined by the usual absolutely convergent power series

$$e^a := \sum_{n=0}^{\infty} \frac{a^n}{n!}.$$

If a and b commute, then the standard method of multiplying power series gives that $e^a e^b$ can be expanded in a power series whose nth term is

$$\sum_{k=0}^{n} \frac{a^k}{k!} \frac{b^{n-k}}{(n-k)!}.$$

Since a and b commute, $(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}$, and it is immediate that the nth term of the power series expansion of $e^a e^b$ is the nth term of the expansion for e^{a+b} defined as above.

Hence $e^{a+b} = e^a e^b$ if a and b commute.

Clearly the exponential function also satisfies $(e^a)^* = e^{a^*}$. So if a is self-adjoint and t is any real number then $(e^{ita})^* = e^{-ita}$. Since $-ita$ commutes with ita, $e^{ita}e^{-ita} = 1 = e^{-ita}e^{ita}$. Hence e^{ita} is unitary for any self-adjoint and any $t \in \mathbb{R}$. Therefore $\mathrm{Spec}(e^{ita}) \subset \mathbb{T}$ by Exercise 1.14.

Notice also that the exponential function satisfies that for any $a \in A$, $e^a - 1$ is divisible by a in the sense that $e^a - 1 = ab$ for some $b \in A$ (factor out x from the power series defining $1 - e^a$.)

Now let λ be any complex number, then $e^{i\lambda} - e^{ia} = e^{i\lambda}(1 - e^{i(a-\lambda)})$. By the previous paragraph, $1 - e^{i(a-\lambda)})$ can be written in the form $(a-\lambda)b$ for some $b \in A$, and furthermore, b commutes with $a - \lambda$, by the way it is defined. This all shows that $e^{i\lambda} - e^{ia} = (a-\lambda)e^{i\lambda}b$ and hence if $e^{i\lambda} - e^{ia}$ were invertible, so would be $a - \lambda$.

We conclude that if $\lambda \in \mathrm{Spec}(a)$ then $e^{i\lambda} \in \mathrm{Spec}(e^{ia})$. But if a is self-adjoint, e^{ia} is unitary, and we've already noted that the spectrum of a unitary is contained in $\mathbb{T}$. Hence $e^{i\lambda} \in \mathbb{T}$ for any $\lambda \in \mathrm{Spec}(a)$, whence $\mathrm{Spec}(a) \subset \mathbb{R}$.

We complete this section with a proof of Gelfand's Theorem.

DEFINITION 2.10. If A is a commutative unital Banach algebra, $\widehat{A}$ denotes the space of characters $\chi \colon A \to \mathbb{C}$ endowed with the topology of pointwise convergence on A.

We remind the reader that in the case where A is a C*-algebra we do not need to require additionally that characters are *-homomorphisms, since this is automatic.

EXAMPLE 2.11. If $A = C(X)$ for a compact Hausdorff space X, then the characters of A are in natural 1-1 correspondence with the points of X, with a point $x \in X$ corresponding to the *-homomorphism $C(X) \to \mathbb{C}$ of evaluation $f \mapsto f(x)$ of functions at x. One easily checks that this map $X \to \hat{A}$ is actually a homeomorphism.

EXERCISE 2.12. Prove that the space $\widehat{\mathcal{A}(\mathbb{D})}$ of characters of the Banach algebra $\mathcal{A}(\mathbb{D})$, is naturally homeomorphic to the closed unit disk $\overline{\mathbb{D}}$.

LEMMA 2.13. *For any commutative unital Banach algebra A, $\widehat{A}$ is a compact Hausdorff space.*

PROOF. Since any characters is contractive, we may identity $\widehat{A}$ with a collection of functions $A_1 := \{a \in A \mid \|a\| \leq 1\}$ to $\mathbb{D}$, that is, as an element of the Cartesian product $X := \prod_{A_1} \mathbb{D}$. Endowing this product with the product topology results in a compact Hausdorff space since $\mathbb{D}$ is compact Hausdorff (and the Tychonoff theorem). The resulting map $\widehat{A} \to X$ is clearly injective since a character is determined by its values on A_1, and is continuous, by the definitions.

Moreover, its range is closed. Indeed, suppose that $\chi\colon A_1 \to \mathbb{C}$ is a map which is a limit point of the image of $\widehat{A}$. So there is a sequence of characters (χ_n) such that $\chi_n(a) \to \chi(a)$ for all $a \in A_1$. First we extend χ to A by setting $\tilde{\chi}(a) := \|a\|\chi(\frac{a}{\|a\|})$. If $a \in A$ is any nonzero element, $\chi_n(a) = \|a\|\chi_n(\frac{a}{\|a\|}) \to \|a\|\chi(\frac{a}{\|a\|}) = \tilde{\chi}(a)$. Hence $\chi_n \to \tilde{\chi}$ pointwise on all of A. Now the fact that $\tilde{\chi}$ is a character, and hence that χ is the restriction to A_1 of a character, follows immediately using limits. We leave the details to the reader.

□

If A is a commutative, unital Banach algebra, $C(\widehat{A})$ thus has the structure of a C*-algebra, and in particular a Banach algebra. If $a \in A$ is any element, we let $\hat{a}$ denote the function on $\widehat{A}$ defined by $\hat{a}(\chi) := \chi(a)$. Clearly $\hat{a}$ is continuous on $\widehat{A}$. The map $a \mapsto \hat{a}$ is the *Gelfand transform*

$$A \to C(\widehat{A}).$$

THEOREM 2.14. *(Gelfand) For any commutative unital Banach algebra A, the Gelfand transform $A \to C(\widehat{A})$ is a contractive, injective, Banach algebra homomorphism. If A is a C*-algebra, it is a C*-algebra isomorphism.*

PROOF. The Gelfand transform is easily checked to be an algebra homomorphism. Using Lemma 2.4, we get, for any $a \in A$, $\|\hat{a}\| = \sup_{\chi\in\widehat{A}}|\hat{a}(\chi)| = \sup_{\chi\in\widehat{A}}|\chi(a)| = \sup_{\lambda\in\mathrm{Spec}_A(a)}|\lambda| = r(a) \leq \|a\|$, so the Gelfand transform is a contractive homomorphism of Banach algebras. If A is a C*-algebra, $\chi(a^*) = \overline{\chi(a)}$ for any character, from Lemma 2.7. It follows that $\widehat{a^*} = \hat{a}^*$ if A is a C*-algebra, so in this case the Gelfand transform is a *-homomorphism. Moreover, if a is self-adjoint, then since $r(a) = \|a\|$ by Theorem 1.16, we get that $\|\hat{a}\| = \|a\|$ and the Gelfand transform is isometric, whence injective. Clearly $\hat{1}$ is the constant function 1 on $\widehat{A}$. Moreover, if $\chi_1 \neq \chi_2$ are different characters, then by definition, there is some $a \in A$ such that $\chi_1(a) \neq \chi_2(a)$, so the image of A in $C(\widehat{A})$ is a *-subalgebra of $C(\widehat{A})$ which separates points of $\widehat{A}$, contains the constant functions, and is closed under conjugation, so by the Stone-Weierstrass theorem it is dense in $C(\widehat{A})$. Since the Gelfand transform is isometric, however, and $C(\widehat{A})$ is complete, the range of the Gelfand transform is closed by a standard argument. Hence its image is $C(\widehat{A})$.

□

EXAMPLE 2.15. If $A = \mathcal{A}(\mathbb{D})$, then $\widehat{\mathcal{A}(\mathbb{D})} \cong C(\overline{\mathbb{D}})$. Therefore the Gelfand transform is the obvious inclusion $\mathcal{A}(\mathbb{D}) \to C(\widehat{\mathcal{A}(\mathbb{D})}) \cong C(\overline{\mathbb{D}})$. Clearly, the Gelfand transform is not surjective in this case.

EXAMPLE 2.16. If G is a discrete group, then the Fourier transform $F_G\colon l^1(G) \to C(\widehat{G})$ is a continuous homomorphism of Banach algebras, and under the identification $\widehat{G} \cong \widehat{l^1(\mathbb{Z})}$, F_G is just the Gelfand transform.

From the above discussion we obtain the following critically important theorem.

COROLLARY 2.17. *Every commutative unital C*-algebra is isomorphic to $C(X)$ where X is a compact Hausdorff space.*

This result is why general C*-algebras are sometimes considered philosophically as generalized spaces, or 'noncommutative' spaces.

REMARK 2.18. One can formulate Corollary 2.17 more precisely as follows. Let **Top** be the category with objects compact Hausdorff spaces and morphisms continuous maps and $\mathbf{C}^*_{\mathbf{Ab}}$ the category of commutative unital C*-algebras and C*-algebra homomorphisms. Define a (contravariant) functor $\mathbf{Top} \to \mathbf{C}^*_{\mathbf{Ab}}$ by sending an object X of **Top** to the object $C(X)$ of $\mathbf{C}^*_{\mathbf{Ab}}$ and a morphism $\phi\colon X \to Y$ to the induced *-homomorphism $C(Y) \to C(X)$, $f \mapsto f \circ \phi$. This functor is a (contravariant) equivalence of categories. Equivalently, it is an equivalence between **Top** and the opposite category ${\mathbf{C}^*_{\mathbf{Ab}}}^{op}$.

EXERCISE 2.19. Let X and Y be compact Hausdorff spaces.

a) As discussed in Remark 3.4, any *-homomorphism $\alpha\colon C(X) \to C(Y)$ has the form $\alpha(f) = f \circ \phi$ for a unique continuous map $\phi\colon Y \to X$. Show that if α is injective then ϕ is surjective, and if α is surjective, then ϕ is injective.
b) Deduce from a) that *any* injective *-homomorphism between C*-algebras – commutative or not – is isometric. (*Hint.* For the second part, note that to show that a *-homomorphism $\alpha\colon A \to B$ between C*-algebras is isometric it suffices to show that $\|\alpha(a)\| = \|a\|$ for all a self-adjoint (by the C*-identity.) Now if $a \in A$ is self-adjoint, then $C^*(a) \cong C\big(\mathrm{Spec}(a)\big)$ by the Spectral Theorem. Use the first part of the exercise.)

EXERCISE 2.20. A C*-algebra is *separable* if it is as a topological space, that is, if it contains a countable dense set. Show that if X is locally compact Hausdorff, then $C_0(X)$ is separable if and only if X is second countable. Thus, a commutative C*-algebra A is separable if and only if its spectrum $\widehat{A}$ is second countable.

EXERCISE 2.21. Viewing the circle $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$, we have a copy of $\mathbb{Q} \cap [0,1]$ inside of it (with 0 and 1 identified). Let A be the C*-algebra generated by $C(\mathbb{T})$ and the characteristic functions $\chi_{[p,q]}$ of the characteristic functions of closed intervals $[p,q]$, $p < q$, $p, q \in \mathbb{Q}$, with rational endpoints. Clearly A is commutative, and $C(\mathbb{T})$ is a subalgebra. Therefore $\widehat{A}$ is a compact space which maps to $\mathbb{T}$. Describe $\widehat{A}$ and the map.

EXERCISE 2.22. Let A be the C*-subalgebra of $L^\infty([0,1])$ consisting of all *left-continuous* functions f on the interval. (Show that this is a C*-algebra.) Since $C([0,1])$ is a C*-subalgebra of A, there is a canonical map $\widehat{A} \to [0,1]$. Describe $\widehat{A}$ and the map.

EXERCISE 2.23. Let X be any locally compact space. The *Stone-Cech compactification* βX of X is the maximal ideal space of the C*-algebra $C_b(X)$ of continuous, bounded functions on X.

EXERCISE 2.24. Let (X, Σ, μ) be a finite measure space. Then $L^\infty(X, \mu)$ is a C*-algebra (in fact a von Neumann algebra.) Therefore, by Gelfand's theorem, it is necessarily isomorphic to $C(Z)$ for some compact Hausdorff space Z. What is Z?

EXERCISE 2.25. It is clear that an inductive limit of commutative C*-algebras $A_n \cong C_0(X_n)$ results in a commutative C*-algebra $\varinjlim C_0(X_n)$ – what is its spectrum?

Let I be a directed system. An *inverse system of compact Hausdorff spaces* over I is a family $\{X_i \mid i \in I\}$ of compact Hausdorff spaces and a family $\{\phi_{ij}\colon X_j \to X_i \mid \text{ for } i \leq j\}$ of continuous maps such that $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ for all $i \leq j \leq k$. The corresponding *inverse limit* space is defined to be the set

$$\varprojlim X_j := \{(x_i)_{i \in I} \in \prod_i X_i \mid x_i = \phi_{ij}(x_j)\ \forall\ i \leq j\}.$$

topologized as a subspace of $\prod_i X_i$.

It is a closed subspace of a compact Hausdorff space, whence is itself compact Hausdorff.

The inverse limit satisfies the following universal property: if Y is a compact Hausdorff space and $\{\alpha_i\colon Y \to X_i;\ \mid i \in I\}$ is a family of maps such that if $i \geq j$ then $\phi_{ji} \circ \alpha_i = \alpha_j$, then there is a unique map $\alpha\colon Y \to \varprojlim X_i$ such that $\pi_i \circ \alpha = \alpha_i$ for all $i \in I$.

The maps ϕ_{ij} of an inverse system give rise to *-homomorphisms $\hat{\phi}_{ij}\colon C(X_j) \to C(X_i)$, for $i \leq j$, by $\hat{\phi}_{ij}(f) = f \circ \phi_{ji}$ and these make up a directed system of C*-algebras.

Prove that

$$C(\varprojlim X_i) \cong \varinjlim C(X_i).$$

The following exercise introduces some interesting compactifications of $\mathbb{R}$, and of more general metric spaces, which can be easily defined using the Gelfand transform.

EXERCISE 2.26. Let (X, d) be a non-compact metric space. Define

$$C(\eta X) := \{f \in C(X) \mid \lim_{x\to\infty} \sup_{d(x,y)\le R} |f(x) - f(y)| = 0,\ \forall R > 0\}.$$

Show that $C(\eta X)$ is a commutative C*-algebra. It's Gelfand dual ηX is, by definition, called the *Higson corona of* X, and plays a role in Index Theory.

a) With $\mathbb{R}$ given its standard metric, prove that smooth functions f on $\mathbb{R}$ with $f' \in C_0(\mathbb{R})$ are dense in $C(\eta\mathbb{R})$.
b) Prove that if $\phi\colon X \to Y$ is a proper Lipschitz map between metric spaces then

$$f \mapsto f \circ \phi$$

defines a *-homomorphism $C(\eta Y) \to C(\eta X)$, whence that a proper Lipschitz map $X \to Y$ determines a map $\eta Y \to \eta X$.
c) Prove that any group G of isometries of (X, d) acts naturally by homeomorphisms of the compact space ηX.

EXERCISE 2.27. Let (X, d) be a non-compact metric space. Recall that $f \in C(X)$ is *uniformly continuous* on X if for all $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Define

$$C_u(X) := \{f \in C(X)\ f \text{ is uniformly continuous on}\}.$$

Show that $C_u(X)$ is a unital commutative C*-algebra, determining, therefore, by Gelfand duality a compact space $\overline{X}^u$.

a) Prove that smooth functions f on $\mathbb{R}$ with $f' \in C_b(\mathbb{R})$ are dense in $C_u(\mathbb{R})$.
b) Prove that any group G of isometries of (X, d) acts naturally by homeomorphisms of the compact space $\overline{X}^u$.
c) Show that for $\mathbb{T}$ the circle, lifting functions to periodic functions determines an injection $C(\mathbb{T}) \to C_u(\mathbb{R})$, and an induced surjection $\overline{\mathbb{R}}^u \to \mathbb{T}$.

3. The Spectral Theorem

In this section, we apply our results on spectral theory for commutative C*-algebras to develop the functional calculus for 'normal' elements. The idea here is to assume we are given an element $a \in A$, where A is any C*-algebra. We may then consider the C*-algebra $C^*(a)$ generated by a inside A. This will be commutative exactly if a is *normal*, that is, if $aa^* = a^*a$. In this case we will be able to apply the results of the previous section.

We start with settling a technical point of some interest: if A is a unital C*-subalgebra of B, and if $a \in A$, is the spectrum of a the same in A and in B? Its affirmative answer for C*-algebras is sometimes called the *principal of spectral permanence*. It fails for more general Banach algebras (see Exercise 1.20.)

THEOREM 3.1. *(Spectral permanence). Let A be a unital C*-subalgebra of the unital C*-algebra B. Then* $\mathrm{Spec}_A(a) = \mathrm{Spec}_B(a)$ *for all* $a \in A$.

PROOF. Clearly $\mathrm{Spec}_B(a) \subset \mathrm{Spec}_A(a)$ for any $a \in A$. The issue then is to prove that if $a \in A$ is invertible in B, then it is also invertible in A. Notice first that it suffices to prove the statement for self-adjoints. For if a is invertible in B so is a^* (in B), and hence so are a^*a and aa^*. If we can show these are both invertible in A, invertibility of a in A will follow (if a^*a is invertible then a has a left inverse, *etc*).

If a is any-self-adjoint, it's spectrum is real, by Corollary 2.8. Since, of course, a is self-adjoint in both A and B, both $\mathrm{Spec}_A(a)$ and $\mathrm{Spec}_B(a)$ are subsets of $\mathbb{R}$. Let $\lambda \in \mathrm{Spec}_A(a)$. Assume, for purposes of finding a contradiction, that $\lambda \notin \mathrm{Spec}_B(a)$. Choose a sequence of complex numbers λ_n lying off the real axis and converging to λ. Since $\lambda_n \notin \mathbb{R}$ for all n, $\lambda_n - a$ is invertible in both A and B for all n. As inversion is continuous in any C*-algebra, $(\lambda_n - a)^{-1} \to (\lambda - a)^{-1}$ in B. But since the inclusion $A \to B$ is isometric, it follows that $(\lambda_n - a)^{-1}$ also converges in A (it is a Cauchy sequence and hence converges to something). A routine argument shows that it converges to the inverse of $\lambda - a$, *i.e.*, we obtain a contradiction to the assumption that $\lambda \in \mathrm{Spec}_A(a)$.

□

COROLLARY 3.2. *A unital *-homomorphism $A \to B$ is injective if and only if*

$$\mathrm{Spec}(a) = \mathrm{Spec}\big(\varphi(a)\big) \ \ \forall a \in A. \tag{3.1}$$

PROOF. The Principal of Spectral Permanence Theorem 3.1 asserts that injective unital *-homomorphisms are isospectral in the sense of (3.1).

For the converse, $\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\|\|\varphi(a^*a)\| = r\big(\varphi(a^*a)\big)$ by Theorem 1.16, for any $a \in A$, but since φ is assumed isospectral this equals $r(a^*a) = \|a\|^2$. □

Accordingly, from now on, when referring to the spectrum of an element a of a C*-algebra, we will just write $\mathrm{Spec}(a)$ rather than $\mathrm{Spec}_A(a)$, since the spectrum does not depend on the containing C*-algebra.

We now develop the *functional calculus*, for normal elements of a C*-algebra.

An element a in a C*-algebra A is *normal* if $a^*a = aa^*$. The set of all elements of A of the form $\sum_{n,m} \lambda_{n,m} a^n (a^*)^m$ is then a commutative *-subalgebra of A, whose closure is the C*-algebra $C^*(a)$ generated by a, that is, the smallest unital C*-subalgebra of A containing a. Clearly then $C^*(a)$ is commutative. From spectral permanence, the spectrum of a is the same if we regard a as an element of $C^*(a)$, or of the A we started with. So for brevity of notation, we replace A by $C^*(a)$ in the following. That is, we will assume that a generates A.

LEMMA 3.3. *The compact Hausdorff spaces $\widehat{A}$ and* $\mathrm{Spec}(a)$ *are homeomorphic by the map* $\widehat{A} \to \mathrm{Spec}(a)$ *of evaluation of characters at a.*

PROOF. Note that any character of $A = C^*(a)$ is determined by its value at a. This shows injectivity of the evaluation map. Surjectivity is immediate from the fact that $\mathrm{Spec}(a)$ consists exactly of values of characters at a (Lemma 2.4). □

By Gelfand's theorem, $A \cong C(\widehat{A})$. By the Lemma, $\widehat{A} \cong \mathrm{Spec}(a)$, whence $C(\widehat{A}) \cong C\big(\mathrm{Spec}(a)\big)$ as C*-algebras. Thus

$$A \cong C\big(\mathrm{Spec}(a)\big). \tag{3.2}$$

DEFINITION 3.4. (Functional calculus for normal elements). Let a be a normal element generating a unital C*-algebra A. Let $f\colon \mathrm{Spec}(a) \to \mathbb{C}$ be a continuous function on $\mathrm{Spec}(a)$. Then $f(a)$ denotes the element of A corresponding to f under the isomorphism (3.2).

We call the map $f \mapsto f(a)$ *functional calculus* for a.

If $f \in C\big(\mathrm{Spec}(a)\big)$, then the corresponding continuous function on $\widehat{A}$ given by Gelfand's map is $\chi \mapsto f\big(\chi(a)\big)$. For example, if f is a polynomial in z and $\bar{z}$, say $f(z) = \sum c_{n,m} z^n \bar{z}^m$, and if χ is a character, then $f\big(\chi(a)\big) = \sum c_{n,m}\chi(a)^n \overline{\chi(a)}^m$, because χ is a *-homomorphism. On the other hand, the element $\sum c_{n,m} a^n (a^*)^m \in C^*(a)$ under the Gelfand map goes to the continuous function on $\widehat{C^*(a)}$ sending χ to $\sum c_{n,m}\chi(a)^n\overline{\chi(a)}^m$. This shows that

$$\text{if } f(z) = \sum c_{n,m} z^n \bar{z}^m \text{ then } f(a) = \sum c_{n,m} a^n (a^*)^m,$$

so that, at least for polynomials, the functional calculus has the meaning one expects. The point of the Theorem is that if (p_n) is a sequence of polynomials in z and $\bar{z}$ which converges uniformly to f on $\mathrm{Spec}(a)$, then $p_n(a, a^*) \to f(a)$ in norm, in A. This of course follows from the fact that $f \mapsto f(a)$ is an isometric map $C\big(\mathrm{Spec}(a)\big) \to C^*(a)$.

Furthermore, since the inclusion function $z\colon \mathrm{Spec}(a) \to \mathbb{C}$ generates $C\big(\mathrm{Spec}(a)\big)$ as a C*-algebra, any *-homomorphism from $C\big(\mathrm{Spec}(a)\big)$ to another C*-algebra, is completely determined by the image of z ($\bar{z}$ is then sent to the adjoint of this element), and then polynomials in z and $\bar{z}$ are sent to the corresponding combinations in the C*-algebra. This leads to a compact formulation of the above discussion as a uniqueness result about functional calculus:

Lemma 3.5. *Let a be a normal element of a C*-algebra. Then the functional calculus is the unique unital *-homomorphism*

$$C\big(\mathrm{Spec}(a)\big) \to C^*(a)$$

which maps the inclusion function $z\colon \mathrm{Spec}(a) \to \mathbb{C}$, to a.

Lemma 3.5 implies the following result, the *Spectral mapping theorem.*

Proposition 3.6. *If a is a normal element of a unital C*-algebra and $f \in C\big(\mathrm{Spec}(a)\big)$ then $\mathrm{Spec}\big(f(a)\big) = f\big(\mathrm{Spec}(a)\big)$.*

Proof. The functional calculus is a C*-algebra isomorphism so $\mathrm{Spec}\big(f(a)\big) = \mathrm{Spec}(f)$, where $\mathrm{Spec}(f)$ means of course the spectrum of f as an element of the C*-algebra $C\big(\mathrm{Spec}(a)\big)$. Since the spectrum of a continuous function is its range, the result follows. ☐

Corollary 3.7. *Let a be normal, $g \in C\big(\mathrm{Spec}(a)\big)$ and f be a continuous function on $g\big(\mathrm{Spec}(a)\big)$. Then $(f \circ g)(a) = f\big(g(a)\big)$.*

In the statement of the Corollary, $(f\circ g)(a)$ refers to the functional calculus for a applied to $f \circ g$ and $f\big(g(a)\big)$ refers to applying first the functional calculus for a using g, then the functional calculus for $g(a)$ using f.

Proof. We apply the uniqueness result of functional calculus to $b := g(a)$. The map $f \mapsto f \circ g$ is a *-homomorphism $C\left[g\big(\mathrm{Spec}(a)\big)\right] \to C\big(\mathrm{Spec}(a)\big)$. By the Spectral Mapping Theorem $g\big(\mathrm{Spec}(a)\big) = \mathrm{Spec}\big(g(a)\big)$. Hence this is a *-homomorphism $C\big(\mathrm{Spec}(g(a)) \to C\big(\mathrm{Spec}(a)\big)$. Composing with the functional calculus for a then gives a *-homomorphism $C\big(\mathrm{Spec}(g(a)) \to C\big(\mathrm{Spec}(a)\big) \to C^*(a)$. This maps the inclusion function $z\colon \mathrm{Spec}(g(a)) \to \mathbb{C}$ to the function $g(a)$ by the definitions. By uniqueness, it must agree with functional calculus for $g(a)$. In other words, $f\big(g(a)\big) = (f \circ g)(a)$.

☐

Another result that is useful and follows immediately from density in $C\big(\mathrm{Spec}(a)\big)$ of polynomials in a, a^*, is the following.

Corollary 3.8. *If a is a normal element of a C*-algebra A and $b \in A$ commutes with a and a^* then b commutes with $f(a)$ for every $f \in C\big(\mathrm{Spec}(a)\big)$.*

Exponentials and logarithms

In Remark 2.9 we introduced the exponential e^a of an element a of a unital Banach algebra, as the sum $\sum_{n=)}^{\infty} \frac{a^n}{n!}$. Since this amounts to substituting a for z in the power series expansion $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ of e^z, which is continuous on the whole complex plane, in the case where A is a C*-algebra, the functional calculus gives the same meaning to e^a as does summing this series.

From complex analysis, one can define a branch of the complex logarithm function $\log z$ on a region of the form $W := \mathbb{C} \setminus \{re^{i\theta_0} \mid r \geq 0\}$, *i.e.* on the complement of a ray. And $\log z$ is then continuous on the spectrum of any normal element whose spectrum is contained in W. So functional calculus produces an element $\log a \in C^*(a)$. Since $\mathrm{e}^{\log z} = z$ for all $z \in W$, Corollary 3.7 then implies the corresponding identity $\mathrm{e}^{\log a} = a$.

COROLLARY 3.9. *If a is a normal element of a C*-algebra A whose spectrum is contained in the complement of a ray in $\mathbb{C}$, then a has a logarithm: an element* $\log a \in A$ *such that* $\mathrm{e}^{\log a} = a$.

A good application of Corollary 3.9 is to unitaries. The spectrum of a unitary u in a unital C*-algebra is contained in $\mathbb{T}$. If its spectrum is a *proper* subset of the circle, then the spectrum of u lies in the complement of a ray, and hence a branch of log is defined and is continuous on $\mathrm{Spec}(u)$, and $u = e^{\log u}$. Note that $\log u$ is skew-adjoint.

Hence $e^{t \log u}$ is unitary for all t. It is a 1-parameter unitary group in A containing u.

EXERCISE 3.10. The group $\mathbf{U}_n$ of n-by-n unitary matrices in $M_n(\mathbb{C})$ is path connected for any n.

REMARK 3.11. Not every unitary in a C*-algebra belongs to a 1-parameter group contained in the C*-algebra. For example, the inclusion $z\colon \mathbb{T} \to \mathbb{C}$ is a unitary in $C(\mathbb{T})$ which cannot be connected by a continuous path of unitaries in $C(\mathbb{T})$ to $1 \in C(\mathbb{T})$, for such a path would give a homotopy between the identity map $\mathbb{T} \to \mathbb{T}$ and the point map $\mathbb{T} \to \{1\} \subset \mathbb{T}$, which is impossible by elementary algebraic topology.

In particular, z doesn't belong to any 1-parameter group of unitaries in $C(\mathbb{T})$.

Another popular application of the functional calculus is to the construction of 'spectral projections' for (normal) elements whose spectrum is disconnected.

A *projection* in a C*-algebra is an element p such that $p^2 = p$ (that is, p is an an idempotent), and which are also self-adjoint. Trivial examples of projections are the unit 1 in a unital C*-algebra, and 0. But if X is a disconnected compact Hausdorff space and $A \subset X$ is any component of X, then the characteristic function χ_A of A is a nontrivial projection in $C(X)$. In the C*-algebra $\mathbb{B}(H)$ Hilbert space arguments imply that for any closed subspace $W \subset H$, the map which sends any $\xi \in H$ to the orthogonal projection of ξ to W is a projection $p_W \in \mathbb{B}(H)$. Thus $\mathbb{B}(H)$ has many nontrivial projections, while $C(X)$, X connected, has none, and in general, $C(X)$ has projections in 1-1 correspondence with the clopen subsets of $C(X)$.

By the functional calculus, this dictionary between connectedness and existence of nontrivial projections holds also for the C*-algebras generated by self-adjoint (or more generally normal) elements.

COROLLARY 3.12. *Let $a \in A$ be a normal element in a unital C*-algebra A. Assume that* $\mathrm{Spec}(a)$ *can be written as a union $I \cup J$ of two closed, disjoint, nonempty subsets of* $\mathrm{Spec}(a)$. *Then $\chi_I(a)$ and $\chi_J(a)$ are two nontrivial projections in $C^*(a)$ whose sum is the identity and whose product is zero. In particular, $C^*(a)$ has nontrivial projections if the spectrum of a is disconnected.*

The proof is trivial given the functional calculus, since χ_I and χ_J are each continuous on the spectrum of a, and in $C\big(\mathrm{Spec}(a)\big)$ they are projections whose product with each other is zero and which sum to the identity, so the same holds for $\chi_I(a)$ and $\chi_J(a)$.

A nice example illustrating the Spectral Theorem comes from considering the spectrum of a unitary $u \in \mathbb{B}(H)$. A choice of unitary on a Hilbert space is exactly the same thing as a unitary representation of the group $\mathbb{Z}$ of integers. This induces a *-representation

$$\pi\colon \mathbb{C}[\mathbb{Z}] \to \mathbb{B}(H), \;\; \pi(\sum a_n\delta_n) := \sum a_n u^n.$$

A basic and important question of representation theory is whether or not such a representation is weakly contained in the regular representation of the integers, that is, whether it extends continuously to a representation of the C*-algebra $C^*_r(\mathbb{Z})$ – equivalently, $\|\pi(f)\| \leq \|\lambda(f)\|$ for all $f \in C_c(\mathbb{Z})$. (The integers can be replaced by any discrete group, verbatim, in the above discussion.)

The Spectral theorem, however, provides an isomorphism $C(\big(\mathrm{Spec}(u)\big) \to C^*(u) \subset \mathbb{B}(H)$, and the spectrum of u is contained in $\mathbb{T}$ so by composition, and combining with the Fourier transform, we get a *-homomorphism $C^*_r(\mathbb{Z}) \to C(\mathbb{T}) \to C\big(\mathrm{Spec}(u)\big) \to \mathbb{B}(H)$ which can easily be checked to extend π.

Exercise 3.13. Use the Spectral Theorem to prove that every unitary representation $\pi\colon \mathbb{Z} \to \mathbf{U}(H)$ of the group of integers, extends to a *-representation $\overline{\pi}\colon C^*_r(\mathbb{Z}) \to \mathbb{B}(H)$. Prove that $\overline{\pi}$ is injective if and only if $\mathrm{Spec}\big(\pi(1)\big) = \mathbb{T}$, with $1 \in \mathbb{Z}$ the usual generator of the integers.

Exercise 3.14. A unitary representation $\pi\colon \mathbb{Z}^n \to \mathbf{U}(H)$ is the same as a choice of n commuting unitary operators $u_1, \ldots, u_n$ on H. Since $C^*(u_1, \ldots, u_n)$ is commutative, it has a Gelfand dual X, which we call the *joint spectrum* $\mathrm{Spec}(u_1, \ldots, u_n)$ of the unitaries. Prove that $\mathrm{Spec}(u_1, \ldots, u_n) \cong \mathrm{Spec}(u_1) \times \cdots \times \mathrm{Spec}(u_n) \subset \mathbb{T}^n$. Prove the analogue of Exercise 3.13 for $\mathbb{Z}^n$, $n \geq 1$.

The property that *all* unitary representations of $\mathbb{Z}^n$ extend to *-representations of $C^*_r(\mathbb{Z}^n)$ is actually quite special to $\mathbb{Z}^n$, and, more generally, to the class of *amenable* groups. The *trivial* representation of the free group $\mathbb{F}_2$ on two generators, extends, of course, to a *-algebra homomorphism $\overline{\pi}\colon \mathbb{C}[\mathbb{F}_2] \to \mathbb{C}$, but $\overline{\pi}$ does not extend continuously to $C^*_r(\mathbb{F}_2)$.

Another useful application of the Spectral Theorem is an analogue of the Hahn-Jordan decomposition theorem for measures.

Proposition 3.15. *if a is a self-adjoint in a unital C*-algebra A then there exist unique positive elements a_1 and a_2 in A such that $a = a_1 - a_2$ and $a_1a_2 = a_2a_1 = 0$.*

Proof. For existence set $f_1(t) = \max(t, 0)$ and $f_2(t) = -\min(t, 0)$, then $a_1 := f_1(a)$ and $a_2 := f_2(a)$ are the required elements. If A happens to (isomorphic to) $C(X)$, for X compact Hausdorff, the uniqueness part of the statement is easily checked by hand. We reduce the general case to this one by showing that if $a = b_1 - b_2$ has a second decomposition in A of the same kind, then the b_i's commute with the a_j's, whence $C^*(a_1, a_2, b_1, b_2)$ will be commutative, whence, by Gelfand's theorem, isomorphic to $C(X)$ for X compact Hausdorff, the special case just discussed. But given that $a = b_1 - b_2$ with $b_1b_2 = b_2b_1 = 0$, it follows immediately that $b_1a = b_1^2 = ab_1$, so b_1 commutes with a, and similarly b_2 commutes with a. Now by Corollary 3.8, each b_i then commutes with $f(a)$ for any continuous function f on $\mathrm{Spec}(a)$, so in particular each b_i commutes with both $a_1 = f_1(a)$ and $a_2 = f_2(a)$.

□

Exercise 3.16. Use the Spectral Theorem to prove that if S is the unilateral shift, and U is a unitary on a Hilbert space, and if

$$S' = \begin{bmatrix} S & 0 \\ 0 & U \end{bmatrix},$$

then S and S' generate isomorphic C*-algebras. (*Hint.* Every element of $C^*(S)$ is pseudo-Toeplitz, *i.e.* has the form $T_f + L$, where L is compact. Map $T_f + L$ in $C^*(S)$ to

$$\begin{bmatrix} T_f + L & 0 \\ 0 & f(U) \end{bmatrix}$$

in $C^*(S')$.

Spectral theory in the presence of a time evolution

Let A be a C*-algebra. Assume that A comes equipped with a *time evolution*: a 1-parameter group $(\sigma_t)_{t\in\mathbb{R}}$ of automorphisms of A. The continuity requirement is that $t \mapsto \sigma_t(a)$ is continuous for all $a \in A$.

An element $a \in A$ is *smooth* if the map $\mathbb{R} \to A$, $t \mapsto \sigma_t(a)$, is smooth.

For example if M is a compact manifold M admitting a smooth action of the group $\mathbb{R}$ on M), *i.e.* if M comes equipped with a smooth flow $(\tau_t)_{t\in\mathbb{R}}$ then $C(M)$ gets a time evolution $\sigma_t(f)(p) := f(\tau_{-t}p)$. The smooth elements are those functions such that $\mathcal{L}_X f(p)$ exists at all $p \in M$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to the tangent vector field of the flow

$$(\mathcal{L}_X f)(p) := \frac{d}{dt} f(\tau_t p), \quad p \in M.$$

Exercise 3.17. In the above notation:

a) Prove that A^∞ is a *-algebra, and that if $a \in A^\infty$ is invertible in A then it is invertible in A^∞. Moreover, $\delta(a^{-1}) = -a^{-1}\delta(a)a^{-1}$. (This exercise also shows that the inclusion $A^\infty \to A$ is *isospectral.*)

b) Prove that $\delta(a) := \frac{d}{dt}\sigma_t(a)$ is a *derivation* of A^∞:
$$\delta(ab) = \delta(a)b + a\delta(b).$$
The domain dom of δ is by definition the collection of all a such that $\sigma_t(a)$ is differentiable at $t = 0$.

c) Prove that $\frac{d}{dt}\sigma_t(a) = \sigma_t\big(\delta(a)\big)$, if $\sigma_t(a)$ is merely differentiable at $t = 0$. Deduce that if $\sigma_t(a)$ is differentiable at 0 it is differentiable everywhere, and prove further that that $A^\infty = \cap_{n=1}^\infty \mathrm{dom}(\delta^n)$.

The purpose of the following discussion is to show that A^∞ is stable under *smooth* functional calculus. Let $f \in C_c^\infty(\mathbb{R})$ be a smooth, compactly supported function. Since f is smooth, its Fourier transform

$$\hat{f}(\xi) = \int f(x)e^{-ix\xi}dx$$

is defined, and is a smooth function of rapid decay on $\mathbb{R}$: for every n there exists $C_n \geq 0$ such that $|\hat{f}(\xi)| \leq C_n|\xi|^n$. The Fourier inversion formula then expresses f as an integral

$$f(x) = \frac{1}{2\pi}\int \hat{f}(\xi)e^{ix\xi}d\xi.$$

Exercise 3.18. Let $a \in A$ be self-adjoint.

a) Prove that the integral $\int \hat{f}(\xi)e^{i\xi a}d\xi$ converges absolutely in A and that
$$f(a) = \frac{1}{2\pi}\int \hat{f}(\xi)e^{i\xi a}d\xi \tag{3.3}$$
holds in A.

b) By the usual technique of differentiating under the integral sign using (3.3), prove that $f(a)$ is smooth if a is smooth. Hence A^∞ is stable under smooth functional calculus.

c) Use (3.3) to deduce that

$$\frac{d}{dt}\sigma_t\big(f(a)\big) = \frac{d}{dt}\sigma_t(a)\, f'(a),$$

where of course $f(a)$ and $f'(a)$ refer to continuous functional calculus for a. This gives the particular case

$$\delta\big(f(a)\big) = \delta(a) f'(a) \tag{3.4}$$

(which could be checked more quickly by the Chain Rule.) If the reader prefers, the diagram

$$\begin{tikzcd} A^\infty \arrow[r, "\delta"] & A^\infty \\ C_0(\mathbb{R}) \arrow[u, "\text{funct. cal. for } a"] \arrow[r, "\frac{d}{dt}"] & C_0(\mathbb{R}) \arrow[u, "\text{funct. cal. for } a"'] \end{tikzcd}$$

commutes.

d) Check that if M is a smooth compact manifold, X a vector field on M, $A = C(M)$, then $\delta = \mathcal{L}_X$ is the Lie derivative with respect to X, and its domain is $C^\infty(M)$. Also, (3.4) follows from the chain rule, since functional calculus for $C(M)$ is composition of functions $f(a) = f \circ a$ for $a \in C(M), f \in C_0(\mathbb{R})$.

Exercise 3.19. The function $f(t) = \sqrt{1-t}$ is the uniform limit on any closed interval $|t| \leq 1 - \epsilon$, of its Taylor series expansion at $t = 0$: thus, for appropriate constants $c_0, c_1, \ldots$,

$$\sqrt{1-t} = \sum_{n=0}^{\infty} c_n t^n,$$

and the sum is norm convergent in the C*-algebra $C([0, 1-\epsilon])$. Use the fact that

$$a(a^*a)^n = (aa^*)^n a$$

in combination with the Spectral Theorem and the above remarks, to show that for any unital C*-algebra A and any $a \in A$ with $\|a\| \leq 1$,

$$a(1 - a^*a)^{\frac{1}{2}} = (1 - aa^*)^{\frac{1}{2}} a.$$

4. Positivity and states

Definition 4.1. Let A be a unital C*-algebra. An element $a \in A$ is *positive* if a is self-adjoint and $\mathrm{Spec}(a) \subseteq [0, \infty)$.

We write $a \geq 0$ for positive elements. More generally, if a and b are self-adjoints, we write $a \leq b$ if $b - a \geq 0$.

Remark 4.2. Clearly the square a^2 of any self-adjoint element a in a unital C*-algebra is positive, for by the Spectral Mapping Theorem (Proposition 3.6), $\mathrm{Spec}(a^2) = \{\lambda^2 \mid \lambda \in \mathrm{Spec}(a)\}$, and since $a = a^*$, $\mathrm{Spec}(a) \subset \mathbb{R}$, so $\mathrm{Spec}(a^2) \subset [0, \infty)$. Conversely, if $a \geq 0$ then $f(t) := \sqrt{t}$ is a continuous function on $\mathrm{Spec}(a)$, so $\sqrt{a}$ is defined and has square equal to a. Thus positivity for elements of a C*-algebra A is equivalent to being the square of a self-adjoint element of A.

It is clear that if $f \in C(X)$ for a compact Hausdorff space X then $f \geq 0$ in the C*-algebra sense if and only if f takes only non-negative values.

You can tell whether a self-adjoint operator on a Hilbert space is positive by the device suggested in the following.

Proposition 4.3. *If $T \in \mathbb{B}(H)$ is a self-adjoint then $T \geq 0$ if and only if $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in H$.*

PROOF. If $T \geq 0$ then $T = S^2$ for a self-adjoint S by Remark 4.2, and then $\langle T\xi, \xi\rangle = \langle S^2\xi, \xi\rangle = \langle S\xi, S\xi\rangle \geq 0$ holds evidently for all $\xi \in H$.

Conversely, suppose that $\langle T\xi, \xi\rangle \geq 0$ for all ξ. Then T is self-adjoint. By Proposition 3.15 we can write $T = T_1 - T_2$ for positive elements $T_i \in \mathbb{B}(H)$ such that $T_1 T_2 = 0$. Now

$$\|T_2^{\frac{3}{2}}\xi\|^2 = \langle T_2^{\frac{3}{2}}\xi, T_2^{\frac{3}{2}}\xi\rangle = \langle T_2^2\xi, T_2\xi\rangle.$$

But since $T_1T_2 = 0$, $TT_2 = -T_2^2$. So $\langle T_2^2\xi, T_2\xi\rangle = -\langle TT_2\xi, T_2\xi\rangle \leq 0$ follows. Therefore $\|T_2^{\frac{3}{2}}\xi\|^2 = 0$ so $T_2^{\frac{3}{2}}\xi = 0$. Since ξ was arbitrary, this shows that $T_2^{\frac{3}{2}} = 0$. Hence $T_2 = 0$. So $T = T_1$ and T is positive.

□

Note that Proposition 4.3 implies immediately that any operator of the form T^*T is positive. One of the trickier results in elementary C*-algebra theory is that a^*a is positive for any element a of any C*-algebra. To prove this is the main objective of this section. Note that it is clear for elements of abelian C*-algebras, since these are isomorphic to $C(X)$ for compact Hausdorff spaces X, and here the result is obvious, since a function of the form $f^*f = |f|^2$ obviously takes positive values only.

The following criterion for positivity will be useful for several arguments.

LEMMA 4.4. *If a is a self-adjoint in a C*-algebra with $\|a\| \leq 1$ then $a \geq 0$ if and only if $\|1 - a\| \leq 1$.*

PROOF. The statement is clearly true for complex numbers, then follows for $f \in C(X)$ by considering the values of f. Now the result for general a self-adjoint and contractive follows from the Spectral Theorem, giving an isomophism $C^*(a) \cong C\big(\mathrm{Spec}(a)\big)$. □

COROLLARY 4.5. *The sum of two positive elements is positive.*

PROOF. Let a and b be positive; assume first that $\|a\| \leq 1$ and $\|b\| \leq 1$. Then $\|\frac{a+b}{2}\| \leq 1$ as well, and

$$\|1 - (\frac{a+b}{2})\| = \frac{1}{2}\|(1-a) + (1-b)\| \leq \frac{1}{2}(\|1-a\| + \|1-b\|) \leq 1$$

since a and b are positive and have norm ≤ 1. This shows that $\frac{a+b}{2}$ is positive, and hence that $a + b$ is positive.

Now for the general case, replace a and b by ta and tb for suitable $t \geq 0$ making both ta and tb have norm ≤ 1. We deduce that $ta + tb = t(a + b)$ is positive and hence that $a + b$ is.

□

THEOREM 4.6. *If A is a C*-algebra and $a \in A$ then a^*a is positive.*

Hence an element of a C-algebra is positive if and only if it is of the form a^*a for some a.*

PROOF. Write $a^*a = u - v$ as a difference of positive elements as in Lemma **??** and let $e = av$. We are going to show $v = 0$.

First of all,

$$e^*e = va^*av = v(u - v)v = -v^3$$

since $uv = vu = 0$ and $v = v^*$. On the other hand, if we write $e = x + iy$ with x, y self-adjoint, then we see that $e^*e = x^2 + y^2 + i(xy - yx)$, while $ee^* = x^2 + y^2 - i(xy - yx)$. Hence $e^*e + ee^* = x^2 + y^2$. Since x^2 and y^2 are certainly positive, their sum is positive. Hence $e^*e + ee^* \geq 0$. On the other hand, we have just observed that $e^*e = -v^3$. Thus $ee^* = -e^*e + x^2 + y^2 = v^3 + x^2 + y^2$ which is another sum of positive elements, whence positive. Thus, ee^* is positive. On the other hand since $\mathrm{Spec}(ee^*) \setminus \{0\} = \mathrm{Spec}(e^*e) \setminus \{0\}$, by Exercise 1.8, and since ee^* has been shown to be positive, it must be that $e^*e \geq 0$ as well. But again,

$e^*e = -v^3 \leq 0$. So the spectrum of e^*e can consist only of 0. Since e^*e is self-adjoint, this implies that $e^*e = 0$ and hence that $-v^3 = 0$, whence, since v is self-adjoint, $v = 0$.

□

Remark 4.7. If the reader finds this proof dishearteningly ingenious may consider that it evaded Gelfand; indeed, to say that a^*a is positive for all $a \in A$, is equivalent to saying that $1 + a^*a$ is invertible for all $a \in A$ (see Exercise 4.8 below) and Gelfand included this as an axiom in his first definition of C*-algebra;.As we have seen, it actually follows from the other axioms.

Exercise 4.8. Prove that if A is any unital *-algebra, then $1 + a^*a$ is invertible for all $a \in A$, holds if and only if $\mathrm{Spec}(a^*a) \subset [0, \infty)$ for all $a \in A$. (*Hint.* For a fixed, and any $\lambda > 0$, the hypothesis implies that $0 \notin \mathrm{Spec}(1 + \lambda a^*a)$ and hence that $-\frac{1}{\lambda} \notin \mathrm{Spec}(a^*a)$.)

Since the converse is trivially true, we see that the property of C*-algebras that $a^*a \geq 0$ for all $a \in A$, is equivalent to the property that $1 + a^*a$ is invertible, for all $a \in A$.

We now prove that every C*-algebra is isomorphic to a closed C*-subalgebra of $\mathbb{B}(H)$ for some Hilbert space H (the Gelfand-Naimark Theorem).

Definition 4.9. Let A be a unital C*-algebra. A *state* on A is a linear functional $\varphi\colon A \to \mathbb{C}$ such that $\varphi(1) = 1$ and if $a \geq 0$ then $\varphi(a) \geq 0$.

A state, or, more generally, any linear map between normed linear spaces, is *contractive*, if $|\varphi(a)| \leq \|a\|$ for all $a \in A$.

Example 4.10. States of $C(X)$ are in 1-1 correspondence with Borel probability measures on X. (the Riesz Representation Theorem.)

Lemma 4.11. *Let $\varphi : A \to \mathbb{C}$ be a linear functional. Then φ is a state if and only if it unital $\varphi(1) = 1$, contractive, and maps self-adjoints to self-adjoints.*

In particular, states are automatically continuous.

Proof. Suppose φ is a state. If a is any self-adjoint, then $a = u - v$ by Lemma **??** where u, v are positive, whence $\varphi(a) = \varphi(u) - \varphi(v)$ is a difference of positive real numbers so $\varphi(a) \in \mathbb{R}$ for $a = a^*$.

Since $a \leq \|a\|$ holds for any $a \in A$ self-adjoint, $\varphi(a) \leq \|a\|\varphi(1) = \|a\|$.

Conversely, let $\varphi\colon A \to \mathbb{C}$ be linear and satisfy the stated conditions. If $a \in A$ with $\|a\| \leq 1$ then since a^*a is positive and $\|a^*a\| = \|a\|^2 \leq 1$, an application of Lemma 4.4 gives that $\|1 - a^*a\| \leq 1$. Since φ is a contraction, $|1 - \varphi(a^*a)| = |\varphi(1 - a^*a)| \leq 1$, and furthermore, $|\varphi(a^*a)| \leq 1$, and these two conditions together gives that $\varphi(a^*a) \geq 0$ for any a. Since positive elements are exactly those of the form a^*a, this shows that φ maps positive elements to non-negative real numbers as required. □

Theorem 4.12. *If A is any unital C*-algebra and $S(A)$ is the state space of A, then*

$$\|a\| = \sup_{\varphi \in S(A)} |\varphi(a)|$$

for any self-adjoint element $a \in A$.

Proof. If $a \in A$ is self-adjoint then $C^*(a) \cong C\big(\mathrm{Spec}(a)\big)$ with a corresponding to the inclusion $z : \mathrm{Spec}(a) \to \mathbb{C}$. Since $\mathrm{Spec}(a)$ is a compact subset of $\mathbb{R}$ the function $|z|$ obtains its maximum on $\mathrm{Spec}(a)$ and if λ is the maximum, then the map $\psi_0\colon C^*(a) \cong C\big(\mathrm{Spec}(a)\big) \to \mathbb{C}$ corresponding to evaluation $f \mapsto f(\lambda)$ of functions at λ is a character of $C^*(a)$ satisfying $|\psi_0(a)| = \|a\|$. Being a character, it is contractive on $C^*(a)$. By the Hahn-Banach Theorem it can be extended to a contractive linear functional $\psi\colon A \to \mathbb{C}$. Then $\varphi(a) := \frac{1}{2}(\psi(a) + \overline{\psi(a^*)})$ is still contractive, extends ψ_0, maps self-adjoints of A to real numbers, and has $\varphi(1) = 1$, so it is a state by Lemma 4.11 and by design satisfies $|\varphi(a)| = \|a\|$.

$\square$

EXERCISE 4.13. Using the idea of the proof of Theorem 4.12, prove that if A is unital, and $B \subset A$ is a unital subalgebra (a subalgebra containing the unit of A), and if $\varphi\colon B \to \mathbb{C}$ is a state of B, then φ extends to a state of A.

EXAMPLE 4.14. Let $P_+\colon C(\mathbb{T}) \to \mathcal{A}(\mathbb{D})$ the Szegö map, $(P_+f)(z) := \int_{\mathbb{T}} \frac{f(w)}{1-\bar{w}z} d\mu(w)$, for $|z| < 1$, but P_+f extends continuously to $\overline{\mathbb{D}}$.

The composition

$$\chi_{+,z}\colon C(\mathbb{T}) \xrightarrow{P_+} \mathcal{A}(\mathbb{D}) \xrightarrow{\mathrm{eval}_z} \mathbb{C}$$

is a continuous linear functional on $C(\mathbb{T})$, so it is given by integration against a measure on the circle, which is clearly the measure $d\mu_{+,z}(w) = \frac{d\mu(w)}{1-\bar{w}z}$, which is continuous with respect to Lebesgue measure.

As $z' \to z \in \mathbb{T}$ with $|z'| < 1$, $\chi_{+,z}(f)$ converges for every $f \in C(\mathbb{T})$, in fact as $z' \to z$ with $|z'| < 1$, $\chi_{+,z'}$ converges in the weak* topology of $C(\mathbb{T})^*$ to a continuous linear functional $\chi_{+,z}$. So $\chi_{+,z}$ is also integration with respect to some Borel measure on the circle, but it is not continuous with respect to Lebesgue measure because $\frac{1}{1-\bar{w}z}$ is not in $L^1(\mathbb{T})$ if $|z| = 1$. In fact, the target measure is point mass at z, as we discuss below.

Let P_- be the operator on $C(\mathbb{T})$ defined by convolution with $\frac{1}{1-\bar{z}}$ in place of the function $\frac{1}{1-z}$. Thus

$$(P_-f)(z) := \int_{\mathbb{T}} \frac{f(w)}{1 - w\bar{z}} d\mu(w).$$

Then it is easy to check that $P_-f(z) = \overline{P_+\bar{f}}$ and that P_- maps $C(\mathbb{T})$ into the space of continuous functions on $\overline{\mathbb{D}}$ which are *anti*-holomorphic on $\mathbb{D}$ (f is anti-holomorphic if $\overline{f(z)}$ is holomorphic, equivalently, if $f(\bar{z})$ is holomorphic.

So by the same argumentation as above we obtain continuous linear functionals

$$\chi_{-,z}\colon C(\mathbb{T}) \xrightarrow{P_-} C(\overline{\mathbb{D}}) \xrightarrow{\mathrm{eval}_z} \mathbb{C}.$$

Let $P = \frac{P_- + P_+}{2}$ and $\chi_z = \frac{\chi_{-,z} + \chi_{+,z}}{2}$ If $|z| < 1$ then

$$(Pf)(z) := \chi_z f = \int \frac{1 - \Re(\bar{w}z)}{|1-z|^2} f(w) d\mu(w). \tag{4.1}$$

so χ_z is integration against the measure $d\mu(w) = \frac{1-\Re(\bar{w}z)}{|1-z|^2} d\mu(w)$, the 'Poisson kernel.' The integral (4.1) is called the 'Poisson transform.' Clearly μ_z is absolutely continuous with respect to Lebesgue measure when $|z| < 1$, but as $z' \to z \in \mathbb{T}$ with $|z'| < 1$ then the measures $\mu_{z'}$ converge to *point masses*; more exactly $\mu_{z'} \to \delta_z$ as $z' \to z$, $|z'| < 1$. This is because the composition

$$C(\mathbb{T}) \xrightarrow{P_- + P_+} C(\overline{\mathbb{D}}) \xrightarrow{\mathrm{restr}} C(\mathbb{T})$$

is the identity map, because it is the identity map already on the dense *-subalgebra $\mathbb{C}[z, \bar{z}]$.

The measures $d\mu_z$, for $|z| \le 1$, are *states* on $C(\mathbb{T})$, in fact, they are a continuous family $\{\mu_z\}_{z \in \overline{\mathbb{D}}}$ of states, with $\mu_z = \delta_z$ on the boundary $z \in \partial\overline{\mathbb{D}}$.

EXERCISE 4.15. Let A be a unital C*-algebra. Answer the following questions about positivity.

a) $a \le \|a\| \cdot 1$ for all $a \in A$ self-adjoint.
b) If $x \le y$ are self-adjoints then $a^*xa \le a^*ya$ for all $a \in A$.

c) If $a, x \in A$ with $x \geq 0$ then $a^*xa \leq \|x\|a^*a$.
d) If $x \leq y$ and both are invertible, then $y^{-1} \leq x^{-1}$.
(*Hint* Observe first that $y^{-\frac{1}{2}}xy^{-\frac{1}{2}} \leq 1$. Deduce that $\|x^{\frac{1}{2}}y^{-\frac{1}{2}}\| \leq 1$, then that $x^{\frac{1}{2}}y^{-1}x^{\frac{1}{2}} \leq 1$.)

5. The GNS theorem

DEFINITION 5.1. A *representation* of a C*-algebra A is *-homomomorphism $\pi\colon A \to \mathbb{B}(H)$ for some Hilbert space H. The representation is *faithful* if π is an injective *-homomorphism, and is *non-degenerate* if $\pi(A)H = H$. It is *irreducible* if there is no proper nonzero closed subspace $H' \subset H$ of H such that $\phi(A)H' \subset H'$.

REMARK 5.2. For non-degeneracy, it can be proved (see Exercise below) that $\pi(A)H = \overline{\pi(A)H}$, so it is sufficient to check the apparently weaker condition that $\pi(A)H$ is dense in H.

LEMMA 5.3. *If $\varphi\colon M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is a unital *-homomorphism then $n|m$ and there is a unitary $u \in M_m(\mathbb{C})$ such that*

$$\varphi(A) = u \begin{bmatrix} A & 0 & \cdots \\ 0 & A & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & A \end{bmatrix} u^*$$

for all $A \in M_n(\mathbb{C})$).

Of course the number of diagonal summands in the matrix is $\frac{m}{n}$.

PROOF. Let us start by recalling that since $M_n(\mathbb{C})$ is always simple (Exercise 6.6), every nonzero *-homomorphism $M_n(\mathbb{C}) \to D$ to another C*-algebra is automatically injective, whence isometric.

The idea of the proof – and essentially the proof itself – is to think of the matrix units $e_{ij} \in M_n(\mathbb{C})$ geometrically in terms of the Hilbert space $\mathbb{C}^m$ on whcih $M_n(\mathbb{C})$ acts via the representation $\varphi\colon M_n(\mathbb{C}) \to M_m(\mathbb{C}) \cong \mathbb{B}(\mathbb{C}^m)$. Each e_{ii} acts as a projection ϵ_{ii} onto a subspace E_i of $\mathbb{C}^m$. The subspaces E_i are pairwise orthogonal and $\oplus_{i=1}^n E_i = \mathbb{C}^m$. (In particular $n|m$).

The matrix units e_{ij} for $i \neq j$ act as partial isometries ϵ_{ij} the range projection for ϵ_{ij} is $\epsilon_{ij}\epsilon_{ij}^* = \epsilon_{ij}\epsilon_{ji} = \epsilon_{ii}$ and the source projection is ϵ_{jj}. Thus ϵ_{ij} restricts to a unitary isomorphism $E_j\colon \to E_i$, and it is the zero operator on the orthogonal complement $E_j^\perp = \oplus_{k\neq j} E_k$.

Consider E_1. It has an orthonormal basis. Using $\epsilon_{j1}\colon E_1 \to E_j$, we construct the corresponding image orthonormal bases of $E_2, E_3, \ldots, E_n$. Putting all of these orthonormal bases together gives an orthonormal basis of $\mathbb{C}^m$. We leave it to the reader to check that with respect to this basis, $\varphi(A)$ has the form

$$\begin{bmatrix} A & 0 & \cdots \\ 0 & A & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & A \end{bmatrix}$$

as claimed. □

COROLLARY 5.4. *Up to unitary equivalence, the C*-algebra $M_n(\mathbb{C})$ has a unique irreducible representation – the standard one, given by the action of $M_n(\mathbb{C})$ on $\mathbb{C}^n$ by matrix multiplication.*

EXERCISE 5.5. Prove that if $H' \subset H$ is a closed subspace invariant under $\pi(A)$ then $H^\perp$ is also invariant under $\pi(A)$. With respect to this decomposition,

$$\pi(a) = \begin{bmatrix} \pi'(a) & 0 \\ 0 & \pi''(a) \end{bmatrix}$$

for a pair of representations of A on $H, H^{\perp}$.

EXERCISE 5.6. Prove that the representation $\pi\colon C(\mathbb{T}) \to \mathbb{B}\big(L^2(\mathbb{T})\big)$ of functions on $\mathbb{T}$ as multiplication operators on $L^2(\mathbb{T})$, contains no irreducible sub-representation of $C(\mathbb{T})$, but that it is a cyclic representation, and is non-degenerate.

EXERCISE 5.7. If X is a compact Hausdorff space, μ is probability measure on X, and $U \subset X$ is open, then the representation of $C_0(U)$ by multiplication operators on $L^2(X, \mu)$, extending such functions by zero outside of U as usual, is non-degenerate if and only if the support of μ is contained in U.

EXERCISE 5.8. If A is unital and $\pi\colon A \to \mathbb{B}(H)$ is a representation, then there exists a subspace H' of H so that with respect to the induced decoposition of H,

$$\pi(a) = \begin{bmatrix} \pi'(a) & 0 \\ 0 & 0 \end{bmatrix}$$

where $\pi'\colon A \to \mathbb{B}(H')$ is non-degenerate.

EXERCISE 5.9. Prove that any representation $\pi\colon A \to \mathbb{B}(H)$ is a direct sum of (countably many) cyclic representations. That is, there is a decomposition $H = H_1 \oplus H_2 \oplus \cdots$ and cyclic representations $\pi_n\colon A \to \mathbb{B}(H_n)$ such that $\pi(a) = \bigoplus_{n=1}^{\infty} \pi_n(a)$.

Normally, one is interested in studying representations up to unitary equivalence. Two representations $\pi\colon A \to \mathbb{B}(H)$ and $\pi'\colon A \to \mathbb{B}(H')$ are *unitarily equivalent* if there is a unitary isomorphism $u\colon H \to H'$ of Hilbert spaces such that $\pi'(a) = u\pi(a)u^*$.

Let $\pi\colon A \to \mathbb{B}(H)$ be a representation. If ξ is a unit vector in H, let $\varphi\colon A \to \mathbb{C}$ be the linear functional $\varphi(a) = \langle \pi(a)\xi, \xi\rangle$. Then clearly $\varphi(1) = 1$ and $\varphi(a^*a) = \|\pi(a)\|^2 \geq 0$ so φ is actually a state, usually called a *vector state*.

The GNS construction, explained below, reverses this, and produces a representation from a state. Let $\varphi\colon A \to \mathbb{C}$ be a state. Define a sesquilinear form on A by $\langle a, b\rangle := \varphi(b^*a)$. By the generalized Cauchy-Schwartz inequality (Theorem 4.16),

$$|\varphi(ab^*)|^2 \leq \varphi(a^*a)\varphi(b^*b) \tag{5.1}$$

for all $a, b \in A$, a fact used in the proof below.

PROPOSITION 5.10. *Let φ be a state of A. Then there is a cylic representation $\pi\colon A \to \mathbb{B}(H)$ of A such that $\varphi(a) = \langle \pi(a)\xi, \xi\rangle$, for some cyclic vector ξ.*

Furthermore, if $\rho\colon A \to \mathbb{B}(K)$ is another cyclic representation, and $\eta \in K$ is a cyclic vector giving the same state, i.e. *for which $\varphi(a) = \langle \rho(a)\eta, \eta\rangle$, then π and ρ are unitarily equivalent by a unitary isomorphism sending ξ to η.*

PROOF. In the notation above set $\langle a, b\rangle := \varphi(b^*a)$. Let $N \subset A$ be the subset of A consisting of a such that $\langle a, a\rangle = 0$. By (5.1) N is a linear subspace of A. Form the quotient vector space $H_0 := A/N$. Then $\langle\cdot,\cdot\rangle$ descends to an inner product on H_0. Let H be its completion, a Hilbert space. Write $\|a\|_H := \langle a, a\rangle := \varphi(a^*a)$ to distinguish from the C*-algebra norm.

If $a, b \in A$ then since $b^*a^*ab \leq \|a\|^2 b^*b$, and since φ is a state,

$$\|ab\|_H^2 = \langle ab, ab\rangle = \varphi(b^*a^*ab) \leq \|a\|^2\varphi(b^*b) = \|a\|_H^2\|b\|_H^2. \tag{5.2}$$

So if $b \in N$ then $ab \in N$. Hence N is actually a left ideal in A, and so there is a well-defined bilinear multiplication, $A \times H_0 \to H_0$, $(a, b+N) \mapsto ab+N$, where $b+N$ *etc* denotes the coset of b in $H_0 = A/N$. Equation (5.2) also shows that the linear map $H_0 \to H_0$ of left multiplication by $a \in A$ is bounded by $\|a\|$. Consequently, left multiplication by a extends to a linear map $H \to H$ which is a bounded operator with norm $\leq \|a\|$. Denote this bounded linear map by

$\pi(a)$. It is easily verified that $\pi\colon A \to \mathbb{B}(H)$ is a representation of A, and that if $\xi \in H$ is the coset of 1 in $H_0 \subset H$, then $\langle \pi(a)\xi, \xi\rangle = \varphi(a)$.

For the uniqueness statement, suppose $\rho\colon A \to \mathbb{B}(K)$ is another representation of A and $\eta \in H$ is a vector such that $\varphi(a) = \langle \rho(a)\eta, \eta\rangle$. Notice that from the definitions $\varphi(a^*a) = \|\pi(a)\|^2$ and for the same reason $= \|\rho(a)\|^2$. In particular if $a \in A$ then $\pi(a)\xi = 0$ if and only if $\rho(a)\eta = 0$, and the map $U\colon H \to K$, $U\big(\pi(a)\xi\big) := \pi(a)\eta$ is then well-defined. It is clearly linear and isometric on the dense subspace $\pi(A)\xi \subset H$ and hence extends uniquely to a unitary map $H \to K$. Since it was also assumed that $\rho(A)\eta$ is dense in K, it is a unitary isomorphism. From the definitions, if $\xi_1 := \pi(a)\xi \in H$ then $U\xi_1 = \rho(a)\eta \in K$ so

$$(U^*\rho(b)U)\xi_1 = U^*\rho(b)\rho(ba)\eta = U^*\rho(ba)\eta = \pi(ba)\xi = \pi(b)\pi(a)\xi = \pi(b)\xi_1.$$

This implies that π and ρ are unitarily equivalent by a unitary mapping ξ to η.

□

Example 5.11. Let G be a discrete group, and $\tau\colon C^*_r(G) \to \mathbb{C}$ the tracial state which on elements of $\mathbb{C}[G]$ is given by $\tau(\sum \lambda_g[g]) = \lambda_e$ (see Exercise Exercise 3.9.)

Then the cyclic representation associated to τ is the *left regular representation* $\lambda\colon C^*_r(G) \to \mathbb{B}\big(l^2(G)\big)$, with cyclic vector the point-mass e_1 at the identity element of G.

We recover the description

$$\tau(T) = \langle \lambda(T)e_1, e_1\rangle$$

of τ from Exercise 3.9.

Example 5.12. Let X be a compact Hausdorff space ad μ a Borel probability measure on X, φ_μ the corresponding state of $C(X)$. The associated GNS construction forms the Hilbert space obtained by completing $C(X)$ with the Hilbert space norm $\|f\|^2 = \int_X |f(x)|^2 d\mu(x)$, and represents $C(X)$ on $L^2(X,\mu)$ as multiplication operators. The cyclic vector is the constant function $1 \in L^2(X)$.

Corollary 5.13. *Every C*-algebra is isomorphic to a C*-subalgebra of $\mathbb{B}(H)$ for some Hilbert space H.*

Proof. From each unitary equivalence class of irreducible representation of A, pick a representative $\pi\colon A \to \mathbb{B}(H_\pi)$. Let X be the set of such representations. Let $H = \oplus_{\pi\in X} H_\pi$, and represent A on H diagonally by $\rho(a)(\xi_\pi)_{\pi\in X} := \big(\pi(a)\xi_\pi\big)_{\pi\in X}$. We need to show that ρ is injective. But if $\rho(a) = 0$ for some $a \in A$ then $\pi(a)\xi = 0$ for every $\pi \in X$ and every vector $\xi \in H_\pi$. Since every state of A can be represented as a vector state for some $\pi \in X$ and some $\xi \in H_\pi$, we get that $\varphi(a^*a) = 0$ for every state of A, which contradicts Theorem 4.12.

□

Exercise 5.14. Prove that the C*-algebra $M_2(\mathbb{C})$ has a unique irreducible representation, up to unitary equivalence, which is the standard representation (on $\mathbb{C}^2$).

Exercise 5.15. The following exercise addresses the question of whether or not a representation can be *extended* from a C*-subalgebra to a larger one.

Let A be a unital C*-algebra, $B \subset A$ a C*-subalgebra containing the unit of A, and $\pi\colon B \to \mathbb{B}(H)$ be a representation of B. Follow the steps below to prove the following:

Proposition 5.16. *In the above notation, there is a Hilbert space K and a representation $\rho\colon A \to K$ whose restriction $\rho|_B\colon B \to \mathbb{B}(K)$ splits into a direct sum of representations*

$$\rho(b) = \begin{bmatrix} \rho'(b) & 0 \\ 0 & \rho''(b) \end{bmatrix}$$

of B, where ρ' is unitarily equivalent to π.

a) Let ψ be a state of B, $\pi_\psi\colon B \to \mathbb{B}(H_\psi)$ the associated GNS representation. Let φ be a state of A extending ψ, and $\pi_\varphi\colon A \to \mathbb{B}(H_\psi)$ the corresponding GNS representation. Prove that the inclusion $B \to A$ induces a Hilbert space isometry $U\colon H_\psi \to H_\varphi$, that its image is invariant under $\pi_\varphi(B)$, and that $U\pi_\psi(b)U^*|_{U(H_\psi)} = \pi_\varphi(b)|_{U(H_\psi)}$ for all $b \in B$.

b) Prove that if $\pi\colon B \to \mathbb{B}(H)$ is a cyclic representation, then the conclusion of Proposition 5.16 holds.

c) Using Exercise 5.9 to deduce the Proposition for general representations π.

Exercise 5.17. Let A be a unital C*-algebra. Equip $M_2(A)$ with the obvious vector space structure, adjoint

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* := \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix},$$

and multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} := \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix}.$$

The result is a *-algebra.

Let $\pi\colon A \to \mathbb{B}(H)$ be a faithful representation of A on a Hilbert space H. Construct from this a faithful *-homomorphism $\bar{\pi}\colon M_2(A) \to \mathbb{B}(H \oplus H)$. We may define then

$$\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \| := \|\bar{\pi}(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \|)$$

where the norm on the right hand side is the norm in $\mathbb{B}(H \oplus H)$. Show that no nonzero element of $M_2(A)$ has zero norm, that $M_2(A)$ is already complete with respect to this norm, and that $M_2(A)$ (with the given norm) is a C*-algebra.

Remark 5.18. Since a C*-algebra can have only one C*-norm, the norm defined on $M_2(A)$ does not, in fact, depend on π. In fact, it may seem somewhat incredible that one needs the GNS theorem (which uses the Hahn-Banach Theorem) to prove that forming 2-by-2 matrices over a C*-algebra results in a C*-algebra, but it is in fact not obvious. An alternate proof involves interpreting $M_2(A)$ as the C*-algebra of bounded A-module operators on the (Hilbert module) $A \oplus A$, as in the next chapter, but this approach still requires the general (and somewhat difficult) fact that a^*a is positive, in a C*-algebra (but it doesn't require the Axiom of Choice..)

In the following Examples we construct injective representations of several C*-algebras already encountered in this book.

Example 5.19. The following construction produces an injective representation of the inductive limit C*-algebra $U(2^\infty)$ of Example 8.9. In other words, we show how to realize $U(2^\infty)$ as a C*-subalgebra of $\mathbb{B}(H)$, for some H, in a reasonably explicit way.

We will take $H = L^2([0,1])$.

A standard argument used to prove the Bolzano-Weierstrass theorem, proceeds as follows. Divide the interval $[0,1]$, which we denote just by I, into $I_1 := [0, \frac{1}{2}]$ and $I_2 := L^2([\frac{1}{2}, 1])$. We can further subdivide I_1 into equal-length subintervals I_{11}, I_{12} , and similarly subdivide I_2 into I_{21}, I_{22}. Continue in this way, for example dividing I_{21} (the interval $[\frac{1}{2}, \frac{3}{4}]$) into I_{211} (the interval $[\frac{1}{2}, \frac{5}{8}]$) and I_{212} (the interval $[\frac{5}{8}, \frac{3}{4}]$.)

We obtain a family of intervals I_μ, where μ ranges over all finite sequences of 1's and 2's.

Now let μ and ν be distinct sequences of equal length. Then the intervals I_μ and I_ν are disjoint, and of the same length, and hence same measure. There is then a canonical Hilbert space isometry (*c.f.* Exercise 7.4) $s_{\nu,\mu}\colon L^2(I_\nu) \to L^2(I_\mu)$ induced by translating the one interval onto the other. Viewing $L^2(I_\nu)$ and so on to be closed subspaces of $L^2([0,1])$, we can then extend $s_{\nu,\mu}$ to be zero on the orthogonal complement of the closed subspace $L^2(I_\nu)$ to get a partial isometry $s_{\mu,\nu}\colon L^2([0,1] \to L^2([0,1]$.

Now we may parameterize a basis for $\mathbb{C}^{2^n}$ by finite sequences of 1's and 2's of length n, since the number of such sequences is 2^n. If μ and ν are such sequences (of the same length), let $E_{\mu,\nu} \in M_{2^n}(\mathbb{C})$ be the corresponding matrix unit, and let $E_{\mu,\nu}$ act on $L^2([0,1])$ by $s_{\mu,\nu}$.

Exercise 5.20. Prove that this defines an injective representation of $U(2^\infty)$ as bounded operators on the Hilbert space $L^2([0,1])$.

In other words, $U(2^\infty)$ is isomorphic to the C*-subalgebra of $\mathbb{B}\big(L^2([0,1])\big)$ generated by the partial isometries $s_{\mu,\nu}\colon L^2([0,1]) \to L^2([0,1])$, where μ and ν range over all pairs of finite sequences of the same length.

The same idea produces a representation of $U(n^\infty)$ for any n.

Remark 5.21. The Cuntz algebra O_n was introduced (in Exercise 7.4 of Chapter 1) as the C*-algebra generated by a certain specific collection $s_1, \dots, s_n$ of isometries of $L^2([0,1])$.

Now if $\mu = (\mu_1, \dots, \mu_k)$ is a word (that is, a finite sequence) in the symbols $1, 2, \dots n$, let $s_\mu := s_{\mu_1} \cdots s_{\mu_k}$. Then it is easy to check that

$$s_{\mu,\nu} = s_\mu s_\nu^* \in U(n^\infty)$$

for all words μ, ν of equal length, where $s_{\mu,\nu}$ is the partial isometry of Example 5.19.

Thus, O_n contains $U(n^\infty)$ as a C*-subalgebra.

The following exercise produces another realization of $U(n^\infty)$ (and of O_n) on a different Hilbert space.

Exercise 5.22. Let $\Sigma = \prod_{k=1}^\infty \{1, 2, \dots, n\}$ with the product topology, a Cantor set.

If μ is a word in $1, 2, \dots, n$ of length k, let $U_\mu \subset \Sigma$ be the set of sequences which begin with μ. And let

$$\sigma\colon \Sigma \to \Sigma, \sigma(x_1, x_2, \dots) := (x_2, x_3, \dots)$$

be the left shift (it plays no role in the current exercise).

a) The collection $\{U_\mu \mid \mu \text{ a word }\}$ is a basis for the topology of Σ.
b) Assigning mass $m(U_\mu) := n^{-|\mu|}$, to U_μ generates a probability measure on Σ (called *Bernoulli measure*.)
c) Let α and β be words of the same length. Prove that substituting β for α defines a measure-preserving homeomorphism $f_{\alpha\beta}\colon U_\alpha \to U_\beta$.
d) Let $L^2(U_\mu)$ be the closed subspace of ${}^2(\Sigma, m)$ of L^2-functions supported on U_μ. For words α, β of equal length and $\xi \in L^2(U_\beta)$, let

$$s_{\alpha,\beta}(\xi)(y) := \xi\big(f_{\alpha,\beta}(y)\big).$$

Let $s_{\alpha,\beta}$ be defined to be zero on $L^2(U_\beta)^\perp$. Prove that $s_{\alpha,\beta}$ is a partial isometry with range $L^2(U_\alpha)$.
e) Prove that the linear span of the $s_{\alpha,\beta}$ with $|\alpha| = |\beta| = k$ generates a copy of $M_{n^k}(\mathbb{C})$ in bounded operators on $L^2(\Sigma, \mu)$.
f) Prove that in this way we can define an injective representation

$$U(n^\infty) \to \mathbb{B}\big(L^2(\Sigma, m)\big)$$

which is unitarily conjugate to the one discussed in Exercise 5.20. (*Hint*. The map $\phi\colon \Sigma \to [0,1]$, $\phi\big((x_k)\big) = \sum_{k=1}^\infty \frac{x_k - 1}{n^k}$ is a measure-preserving, finite-to-one continuous surjectiction; construct a Hilbert space unitary from it.)

g) Let $s_i\colon L^2(\Sigma, m) \to L^2(\Sigma, m)$ be the unique linear map such that $s_i(\chi_\mu) = \sqrt{n}\cdot\chi_{i\mu}$ for all words μ (this describes a dense subset of $L^2(\Sigma, m)$.) Prove that s_i is an isometry, and that the C*-algebra generated by $s_1, \ldots, s_n$ is isomorphic to the Cuntz algebra O_n (which contains $U(n^\infty)$, by Remark 5.21.) The universal property of O_n is useful here, *c.f.* Remark 7.5.

Exercise 5.23. Prove that the universal UHF algebra $\mathcal{N}$ of Example 8.10 can be realized as a C*-subalgebra of bounded operators on $L^2([0,1])$ in the following way. If n, k, l are natural numbers with $0 \le k, l < n$, let

$$s_{n;k,l}\colon L^2([0,1]) \to L^2([0,1])$$

be the partial isometry which is zero on the orthogonal complement of the closed subspace $L^2([\frac{l}{n}, \frac{l+1}{n}]$, and which maps $L^2([\frac{l}{n}, \frac{l+1}{n}]$ isomorphically onto $L^2([\frac{k}{n}, \frac{k+1}{n}]$ by the obvious map, induced by translating the one interval onto the other.

Prove that $\mathcal{N}$ is isomorphic to the C*-algebra generated by the $s_{n;k,l}$.

*Generalities about completions of *-algebras*

Many C*-algebras are defined by an algebraic construction followed by a 'completion' procedure. The algebraic construction makes a *-algebra. The completion procedure completes it to a C*-algebra. C*-algebras are themselves 'rigid' – a C*-algebra has a unique norm with respect to which it is a C*-algebra. But a given *-algebra may be endowed with many different submultiplicative norms satisfying the C*-norm axioms (except completeness, of course.) Each gives rise to a potentially different C*-algebra, by completion.

As in dealing with C*-algebras in practise, one is constantly having to think about things which are limits of things we understand, but are not themselves very easily described (like the Kazhdan projection), it is helpful to have a fairly good intuition for how completions work.

So the following discussion is intended to clarify a bit the general matter of completions.

Definition 5.24. A *pre-C*-algebra* will mean a *-algebra $\mathcal{A}$ which is equipped with a semi-norm $\|\cdot\|\colon \mathcal{A} \to [0,\infty)$ satisfying $\|ab\| \le \|a\|\|b\|$ for all $a, b \in A$, and $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$.

Exercise 5.25. Show that if $(\mathcal{A}, \|\cdot\|)$ is a pre-C*-algebra, then the elements of norm zero in $\mathcal{A}$ form a *-ideal $\mathrm{Null}(\|\cdot\|)$. Furthermore, if we define

$$\|a + \mathrm{Null}(\|\cdot\|)\| := \inf\{\|a+b\| \mid \|b\| = 0\},$$

then this defines a *norm* (not just a semi-norm) on the *-algebra $\mathcal{A}/\mathrm{Null}(\|\cdot\|)$. Show that the quotient map $\mathcal{A} \to \mathcal{A}/\mathrm{Null}(\|\cdot\|)$ is isometric.

Remark 5.26. A typical (and in fact the general) way of getting a pre-C*-algebra structure on a *-algebra $\mathcal{A}$ would be to find a *-representation $\pi\colon \mathcal{A} \to \mathbb{B}(H)$ of $\mathcal{A}$ as bounded operators on a Hilbert space. We could then define a semi-norm $\|\cdot\|_\pi$ satisfying the C*-identity by setting

$$\|a\|_\pi := \|\pi(a)\|$$

where the norm on the right hand side is the operator norm in $\mathbb{B}(H)$.

The Gelfand-Naimark-Segal Theorem combined with Proposition 5.30 below implies that every pre-C*-algebra structure on a given *-algebra arises in this way.

Definition 5.27. A *completion* of a pre-C*-algebra $(\mathcal{A}, \|\cdot\|)$ is a pair consisting of a C*-algebra A and an isometric *-homomorphism $\pi\colon \mathcal{A} \to A$ (with respect to the given semi-norm on $\mathcal{A}$ and the C*-algebra norm on A) which satisfies the following universal property. For any C*-algebra B and any contractive *-homomorphism $\varphi : \mathcal{A} \to B$, there is a unique *-homomorphism $\overline{\varphi}\colon A \to B$ such that $\overline{\varphi} \circ \pi = \varphi$.

Exercise 5.28. Assume the existence of a completion A of a pre-C*-algebra $(\mathcal{A}, \|\cdot\|)$. Suppose that B is a C*-algebra and $\varphi\colon \mathcal{A} \to B$ is an *isometric* *-homomorphism. Show that the induced map $\overline{\varphi}\colon A \to B$ is also isometric.

Exercise 5.29. Prove that if $\pi\colon \mathcal{A} \to A$ is a completion, then π has dense range.

Completions exist:

Proposition 5.30. *Every pre-C*-algebra has a completion; and if $\pi\colon \mathcal{A} \to A$ and $\pi'\colon \mathcal{A} \to A'$ are any two completions of $(\mathcal{A}, \|\cdot\|)$, then there is a unique C*-algebra isomorphism $\alpha\colon \mathcal{A} \to \mathcal{A}'$ such that $\pi' = \alpha \circ \pi$.*

Proof. Let $(\mathcal{A}, \|\cdot\|)$ be as in the statement. The set of elements of $\mathcal{A}$ of norm zero form a *-ideal in $\mathcal{A}$ as is easily checked. The quotient of $\mathcal{A}$ by this *-ideal then an algebra to which the seminorm descends, giving a a norm satisfying the C*-identity. To avoid introducing new notation, we just assume that $\|\cdot\|$ was a norm to begin with.

As any metric space has a completion, we may apply this to the current situation, obtaining a metric space A into which $\mathcal{A}$ is embedded isometrically and densely. Any Lipschitz map from $\mathcal{A}$ into a complete metric space extends uniquely to A. In particular, the norm on $\mathcal{A}$ extends to a norm on A. The algebra operations on $\mathcal{A}$ similarly extend to A: for example addition by a fixed $a \in \mathcal{A}$ regarded as a map $\mathcal{A} \to A$ is isometric, hence Lipschitz, and so extends continuously to a map $A \to A$. The adjoint $*$ also extends, by the same arguments. Thus A is actually a C*-algebra.

If $\varphi\colon \mathcal{A} \to B$ is a contractive *-homomorphism to a C*-algebra B, then φ maps null vectors in $\mathcal{A}$ to zero, and induces a (contractive) *-homomorphism $\mathcal{A}/\mathrm{Null}(\|\cdot\|) \to B$. Since a contractive map from a metric space into a complete metric space extends uniquely to the completion, φ extends uniquely to a contractive map $A \to B$. This extension is easily seen to be a *-homomorphism.

□

Exercise 5.31. Show, using the universal property of completions, that if $\pi\colon \mathcal{A} \to A$ is a C*-algebra completion of a pre-C*-algebra $(\mathcal{A}, \|\cdot\|)$ then $\ker(\pi) = \{a \in \mathcal{A} \mid \|a\| = 0\}$. *Hint.* Compare the completion of $(\mathcal{A}, \|\cdot\|)$ to the completion of the pre-C*-algebra $\mathcal{A}/\mathrm{Null}(\|\cdot\|)$.)

Exercise 5.32. Let $\mathcal{A}$ be any dense *-subalgebra of $C(X)$, where X is a compact Hausdorff space. Prove that if $\pi\colon \mathcal{A} \to A$ is a completion of $\mathcal{A}$ then $\|\pi(f)\| = \|f|_F\|$ for some closed subset $F \subset X$. In particular, A is C*-isomorphic to $C(F)$.

With the language of completions, we revisit group C*-algebras in the following example.

Example 5.33. As we have seen, a unitary representation $\pi\colon G \to \mathrm{U}(H)$ of a discrete group G on a Hilbert space H induces a *-algebra homomorphism $\pi\colon \mathbb{C}[G] \to \mathrm{U}(H)$,

$$\pi(\sum_{g\in G} \lambda_g [g]) := \sum_{g\in G} \lambda_g \pi(g) \in \mathbb{B}(H),$$

and hence every unitary representation of G gives a a pre-C*-algebra $(\mathbb{C}G, \|\cdot\|_\pi)$ where $\|f\|_\pi := \|\pi(f)\|$, for $f \in \mathbb{C}[G]$, and a corresponding C*-algebra completion $C^*_\pi G$.

In this notation, $C^*_\lambda(G) = C^*_r(G)$ where $\lambda\colon G \to \mathbf{U}\big(l^2(G)\big)$ is the left regular representation.

Exercise 5.34. Let $G = \mathbb{Z}$. Compute $C^*_\epsilon(\mathbb{Z})$ where $\epsilon\colon \mathbb{Z} \to \mathbb{T}$ is the trivial representation.

Exercise 5.35. Let $F \subset \mathbb{T}$ be a closed subset of the circle. Restriction of Laurent polynomials $f \in \mathbb{C}[z, z^{-1}]$ to F determines a completion $\mathbb{C}[z, z^{-1}] \to C(F)$ of the *-algebra of Laurent polynomials.

Using Fourier transform, we may regard this as giving a completion $\pi_F\colon \mathbb{C}[\mathbb{Z}] \to C(F)$ of the group algebra $\mathbb{C}[\mathbb{Z}]$ of the integers.

Prove that if F has an accumulation point, then π_F is injective on $\mathbb{C}[\mathbb{Z}]$.

The case of the group of integers has a special property (amenability), with regard to completions, not possessed by all discrete groups. If $F \subset F' \subset \mathbb{T}$ are two nested closed subsets of the circle, then $\|\pi(f)\|_F \leq \|f\|_{F'}$ for all $f \in \mathbb{C}[G]$. In particular, $\|f\|_F \leq \|f\|_{\mathbb{T}}$ for all $f \in \mathbb{C}[z, z^{-1}]$ (and it follows that the *-homomorphism $\pi_F\colon \mathbb{C}[z, z^{-1}] \to C(F)$ extends continuously to a C*-algebra homomorphism $\overline{\pi}_F\colon C(\mathbb{T}) \to C(F)$.)

It turns out that if $G = \mathbb{F}_n$ is a free group on $n \geq 2$ generators, then the *-algebra homomorphism $\epsilon\colon \mathbb{C}[G] \to \mathbb{C}$ induced by the trivial representation, does *not* extend continuously to $C^*_r(G)$ ($\mathbb{F}_n$ is not amenable, $n \geq 2$.)

Example 5.36. Another important example where a completion procedure is used is in connection with direct limits. We have already discussed direct limits, especially limits in which the structure maps of the system are injective. We now discuss the general case in the language of pre-C*-algebras.

Assume that $\{\varphi_{ij}\colon A_j \to A_i\}_{i\geq j}$ is a direct system of C*-algebras and *-homomorphisms. We start with forming the algebraic limit $\mathcal{A} := \sqcup A_i/\sim$ where $\sim$ is the equivalence relation generated by identifying $a \in A_i$ with $\varphi_{ji}(a_i) \in A_j$ for any $i \leq j$. Let $\varphi_i\colon A_i \to \mathcal{A}$ be the evident maps of A_i into $\mathcal{A}$. Note that if one of the structure maps $\varphi_{ij}\colon A_j \to A_i$ is not injective, and maps an element $a \in A_j$ to $0 \in A_i$, then $\varphi_j(a) = 0 \in \mathcal{A}$.

As discussed previously, $\mathcal{A}$ has a natural *-algebra structure. As before, if $a \in A_i$, we set $\|\varphi_i(a)\| := \lim_{j\to\infty}\|\varphi_{ji}(a)\|$. The limit exists because it is a decreasing net of positive real numbers, because *-homomorphisms are automatically contractive, by Lemma 8.6. We therefore have a pre-C*-algebra $\mathcal{A}, \|\cdot\|)$. The direct limit is defined to be the completion of this pre-C*-algebra.

Exercise 5.37. If $\varphi_i\colon A_i \to A := \varinjlim A_i$ is a directed system, prove that

$$\ker(\varphi_i) = \{a \in A_i \mid \lim_{j\to\infty}\|\varphi_{ji}(a)\| = 0\},$$

for all $i \in I$.

Exercise 5.38. Let $A_i := C([0,1])$ for $i = 1, 2, \ldots$, and let $\varphi_{ij}(f)(x) = f(2^{j-i}x)$, for $i \geq j$ and $f \in A_j = C([0,1])$. The *-homomorphisms in the system are not injective, since they are induced by non-surjective self-maps of the interval. Show that $\varinjlim A_i \cong \mathbb{C}$ by evaluation of functions at 0.

6. Ideals and quotients of C*-algebras

In this section we develop the extremely important theory of ideals and quotients in the category of C*-algebras. A technical issue has to be resolved first.

An *approximate* unit in a C*-algebra A is a net $(u_\lambda)_{\lambda\in\Lambda}$ in A such that $u_\lambda a \to a$ as $\lambda \to \infty$ for all $a \in A$. We are going to show that an approximate unit exists in any C*-algebra. In fact, one can put the index set for the net to be simply $\Lambda := \{a \in A_+ \mid \|a\| < 1\}$ and the net the 'identity net', whose value at $a \in \Lambda$ is a itself, but the important point is that the ordering on the index set Λ needed to make a net, is the usual ordering of positive elements $a \leq b$ if and only if $b - a \geq 0$.

It is clear that this relation is reflexive, transitivity is an extremely easy exercise, but the upper bound requirement, making this a directed set, needs to be verified. Clearly since the sum of two positive elements is positive (Corollary 4.5), if $a, b \in A_+$ then $a+b$ is an upper bound for a, b, so all of A_+ is a directed set, but if a, b are contractions, it is of course not true that $a+b$ need be a contraction. However, the map

$$\alpha\colon A_+ \to \Lambda, \ \ \alpha(a) := a(1+a)^{-1}$$

is an order isomorphism $A_+ \cong \Lambda$ with inverse $\beta\colon \Lambda \to A_+$, $\beta(a) := a(1-a)^{-1}$, and since the upper bound condition holds for the ordered set A_+, it does for Λ as well. To check these facts it helps to note that $a(1+a)^{-1} = 1-(1+a)^{-1}$.)

Exercise 6.1. If a and b are in Λ find an explicit formula for an upper bound of a, b in Λ.

Proposition 6.2. *Λ is an approximate unit for A. That is, if $a \in A$ and $\epsilon > 0$, then there exists $u_0 \in \Lambda$ such that if $u \in \Lambda$ and $u \geq u_0$ then $\|ua - a\| < \epsilon$.*

In the commutative case, the content of the proposition is as follows. Given $f \in C_0(X)$, and $\epsilon > 0$, there exists $\rho_0 \in C_0(X)$ such that $0 \leq \rho < 1$, and for all $\rho_0 \leq \rho < 1$, $\|\rho f - f\| < \epsilon$. It is clear, at any rate, if $f \in C_c(X)$, then one should take ρ_0 strictly less than, but very close to 1 on the support of f, which would certainly do the trick. An approximation arguments extends this to where $f \in C_0(X)$, in which case the support need not be compact necessarily. Alternatively, one can argue that one can simply choose, depending on $\epsilon > 0$, ρ_0 to be of the form $\mu f(1+\mu f)^{-1}$, where μ is large. The relevant estimates are done in the proof below.

Proof. If we can show that $ua \to a$ as $u \to \infty$ in Λ, for a positive and of norm < 1, then it follows that $ua \to a$ for any positive element. Since any self-adjoint can be written as a difference of two positive elements, $ua \to a$ for a self-adjoint. The result for arbitrary a follows from writing a as a linear combination of two self-adjoints.

So assume that $a \in A_+$, $\|a\| < 1$ (that is, that $a \in \Lambda$). Choose $\epsilon > 0$. Let $u_0 = \epsilon^{-2}a(1+\epsilon^{-2}a)^{-1}$. Let $u \geq u_0$. Since $1-u \leq 1$, $(1-u)^2 \leq 1-u$. Since $u \geq u_0$, $1-u \leq 1-u_0 = (1+\epsilon^{-2}a)^{-1}$. Putting these facts together gives

$$(1-u)^2 \leq (1+\epsilon^{-2}a)^{-1}.$$

So we deduce

$$a(1-u)^2 a \leq a(1+\epsilon^{-2}a)^{-1}a.$$

Since $a(1+\epsilon^{-2}a)^{-1}a \leq \epsilon^2 a$, we get $a(1-u)^2 a \leq \epsilon^2 a$. Hence $\|a(1-u)^2 a\| \leq \epsilon^2 \|a\| < \epsilon^2$. By the C*-identity, on the other hand, $\|a(1-u)^2 a\| = \|a(1-u)\|^2$. Thus $\|a - au\| < \epsilon$ as required. □

Exercise 6.3. If H is any Hilbert space it has an orthonormal basis $\mathcal{E}$ and the dimension of H is by definition the cardinality of $\mathcal{E}$. Make a directed set Λ consisting of all finite subsets of $\mathcal{E}$ under the subset relation, and let $(P_F)_{F\in\Lambda}$ be the net in which P_F is the projection to $\operatorname{span}(F)$ for any finite subset $F \subset \mathcal{E}$. Show that $(P_F)_{F\in\Lambda}$ is an approximate unit for $\mathcal{K}(H)$.

Exercise 6.4. Show that if A is separable, then A has an approximate unit which is an increasing sequence $u_1 \leq u_2 \leq \cdots$ of positive contractions in A. *Hint.* Let $a_1, a_2, \ldots$ be a countable dense set in A. Using Proposition 6.2, inductively construct (u_n) such that $0 \leq u_1 \leq u_2 \leq \cdots$ and $\|u_n a_i - a_i\| < \frac{1}{n}$ for $1 \leq i \leq n$

A *closed ideal* in a C*-algebra is an algebraic ideal $J \subset A$ which is also closed in the norm topology. This implies by the following Lemma that J is also closed under adjoints.

Lemma 6.5. *If J is a closed ideal in a C*-algebra A then J is automatically closed under adjoint (and hence is, in particular, a C*-subalgebra of A.)*

PROOF. If $a \in J$, then a^*a and hence the (non-unital) C*-algebra $C^*(a^*a)$ generated by a^*a is contained in J. indeed, $C^*(a^*a)$ is the completion of the *-algebra of polynomials $\sum_{k=0}^n \lambda_k (a^*a)^k$ in J.

In particular, $C^*(a^*a)$ has an approximate unit (u_λ) consisting of positive contractions. If we can show that $u_\lambda a \to a$, it will follow, since the adjoint is continuous, that $u_\lambda a^* \to a^*$, as well. Since $u_\lambda a^* \in J$ for all λ, it will then follow that $a^* \in J$ since J is closed.

But using the C*-identity and the facts that u_λ is self-adjoint and a contraction, $\|au_\lambda - a\|^2 = \|a(u_\lambda - 1)\|^2 = \|(u_\lambda - 1)a^*a(1-u_\lambda)\| \leq \|(u_\lambda - 1)a^*a\|\|u_\lambda - 1\| \leq \|(u_\lambda - 1)a^*a\| = \|u_\lambda a^*a - a^*a\|$. Since $u_\lambda a^*a \to a^*a$, we see that $u_\lambda a \to a$ as well, as required.

□

EXERCISE 6.6. A C*-algebra is *simple* if it has no nonzero, proper closed ideals. Prove that $M_n(\mathbb{C})$ is simple. (*Hint*. Suppose J was an ideal. If $a \in J$ is a nonzero element. If $E_{ij} \in M_n(\mathbb{C})$ are the usual matrix units of $M_n(\mathbb{C})$, then $a_{ij}E_{ij} = E_{ii}aE_{jj} \in J$ and hence if a_{ij} is a nonzero entry of a then $E_{ij} \in J$. Now multiplying on the left or right by permutation matrices σ, τ gives that $E_{\sigma(i),\tau(j)} \in J$ for all σ, τ and hence that J contains all the matrix units.)

EXERCISE 6.7. If X is a locally compact Hausdorff space and $F \subset X$ is a closed subset then show that $J_F := \{f \in C_0(X) \mid f_{|F} = 0\}$ is a closed ideal in $C_0(X)$, and that all closed ideals in $C_0(X)$ arise as J_F for some F, – the closed ideals in $C_0(X)$ are in 1-1 correspondence with the closed subsets of X.

EXAMPLE 6.8. The results of §6 show that the C*-subalgebra $\mathcal{K}(H)$ of $\mathbb{B}(H)$ is a closed ideal for any Hilbert space H.

EXERCISE 6.9. Prove that $\mathcal{K}(H)$ is the *unique* closed ideal in $\mathbb{B}(H)$.

Let $J \subset A$ be a closed ideal. The quotient vector space A/J has an evident structure of a *-algebra. We equip it with the quotient norm

$$\|a + J\| := \inf\{\|a - b\| \mid b \in J\}.$$

Since J is assumed closed, A/J is complete as a normed vector space, by an easy exercise. Thus, the cosets in A/J form a Banach *-algebra. We have already met one example: that of the Calkin algebra $\mathcal{Q}(H) := \mathbb{B}(H)/\mathcal{K}(H)$. We proved in Chapter 1 that the C*-identity holds for the quotient norm; we now prove this result in general.

LEMMA 6.10. *If $J \subset A$ is a closed ideal and (u_λ) is an approximate unit of positive contractions in J then $\|a + J\| = \lim_{\lambda\to\infty} \|au_\lambda - a\|$ for all $a \in A$.*

PROOF. If $b \in J$ then

$$\begin{aligned}\|a - au_\lambda\| = \|a + b - b + bu_\lambda - bu_\lambda - au_\lambda\| = \|a + b - (a+b)u_\lambda + bu_\lambda - b\| \leq \\ \|a + b - (a+b)u_\lambda\| + \|bu_\lambda - b\| = \|(a+b)(1-u_\lambda)\| + \|bu_\lambda - b\| \\ \leq \|a + b\| + \|bu_\lambda - b\|.\end{aligned}$$

Therefore $\limsup_{\lambda\to\infty} \|a - au_\lambda\| \leq \|a + b\| \leq \|a + J\|$. On the other hand $\|a + J\| \leq \|a - au_\lambda\|$ for all λ so $\|a + J\| \leq \liminf_{\lambda\to\infty} \|a - au_\lambda\|$. The result follows immediately.

□

COROLLARY 6.11. *The Banach *-algebra A/J with the quotient norm satisfies the C*-identity, and hence is a C*-algebra.*

Proof. If $a \in A$ then by the Lemma,

$$\|(a+J)^*(a+J)\| = \|a^*a+J\| = \lim_{\lambda\to\infty} \|a^*au_\lambda - a^*a\| = \lim_{\lambda\to\infty} \|a^*a(u_\lambda - 1)\|$$
$$\geq \lim_{\lambda\to\infty} \|(u_\lambda - 1)a^*a(u_\lambda - 1)\| = \lim_{\lambda\to\infty} \|a(u_\lambda - 1)\|^2 = \lim_{\lambda\to\infty} \|au_\lambda - a\|^2 = \|a+J\|^2.$$

The other inequality $\|(a+J)^*(a+J)\| \leq \|a^*+J\|\|a+J\| = \|a+J\|^2$ is just the statement made above (with proof left to the reader) that A/J is a Banach *-algebra. □

Example 6.12. If $F \subset X$, F closed, X locally compact Hausdorff, $J_F = \{f \in C_0(X) \mid f|_F = 0\}$ the closed ideal of $C_0(X)$ of Exercise 6.7, then the C*-algebra quotient $C_0(X)/J_F$ is naturally isomorphic to $C_0(F)$, by the homomorphism $f + J_F \mapsto f|_F$, that is, by the map of restriction of functions to F.

Corollary 6.13. *If $\alpha\colon A \to B$ is a *-homomorphism between C*-algebras, then the image of α is closed, and hence is a C*-subalgebra of B.*

Proof. The kernel $\ker(\alpha)$ is a closed ideal in A. By standard algebra, α induces an injective *-homomorphism $\dot\alpha\colon A/\ker(\alpha) \to B$. Since it is injective, it is isometric, by Exercise **??**, and hence its range is closed in B. □

Exercise 6.14. Show that if a unital C*-algebra A has no non-trivial *algebraic* ideals then it has no *closed* ideals either.

Exercise 6.15. Prove that $C_0(\mathbb{R})$ is a closed ideal in the C*-algebras $C_b(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$, $C(\eta\mathbb{R})$ the C*-algebra of functions of vanishing variation on $\mathbb{R}$, and $C_u(\mathbb{R})$, uniformly continuous functions on $\mathbb{R}$.

Deduce that $\mathbb{R}$ embeds as an open, dense subset of each of $\beta\mathbb{R}$, $\eta\mathbb{R}$ and $\overline{\mathbb{R}}^u$. The term ‘compactification is generally used for a compact space containing $\mathbb{R}$ as an open, dense subset.

7. Structure of crossed-products by proper actions of discrete groups

A condition on a group action $G \times X \to X$ which ensures that the space of orbits $G\backslash X$, with the quotient topology, is Hausdorff, is that the action is *proper* (see Proposition 7.4 below). The space of orbits can thus be studied purely topologically, since the quotient $G\backslash X$ is a reasonable topological space (unlike the space of orbits of irrational rotation on the circle.)

In fact, as we show in this section, in the case of a free and proper action of a discrete group, the crossed-product $C_0(X) \rtimes_r G$ is strongly Morita equivalent to $C(G\backslash X)$ so contains essentially the same information as the quotient space, and we will find some more general results as to the ideal structure of these crossed-products, in the possible presence of isotropy – which injects some ‘noncommutative’ phenomena into the structure.

Definition 7.1. An action $G \times X \to X$ of a discrete group on a locally compact space X is *proper* if for every pair K, K' of compact subsets of X, the subset $\{g \in G \mid g(K) \cap K' \neq \emptyset\}$ of G, is finite.

Example 7.2. Some examples of proper actions are:

a) Any action of a finite group is proper.
b) The action of $\mathbb{Z}$ on $\mathbb{R}$ by translation is proper, or of $\mathbb{Z}^n$ on $\mathbb{R}^n$.
c) The translation action of a discrete subgroup $G \subset G'$ of a locally compact group G', on G', is proper.

The action of the integers by irrational rotation is definitely not proper. Indeed, no infinite group can act properly on a compact space; proper actions of infinite groups are always on noncompact spaces.

EXERCISE 7.3. Prove that if G is a discrete group and H is a subgroup, the left multiplication action of G on G/H is proper if and only if H is finite.

The following is a standard result from basic topology and we omit the proof.

PROPOSITION 7.4. *If G is a discrete group acting properly on a locally compact Hausdorff space X, then $G\backslash X$ with the quotient topology, is locally compact and Hausdorff.*

This means that the quotient space $G\backslash X$, is a perfectly reasonable topological space, whose homology, cohomology, *etc* can be computed – in the case of a *proper* action. So there is in a sense no need for C*-algebras at this point, if one is interested in attaching invariants to group actions. However, the quotient space $G\backslash X$ is just a space, and contains as such no information about the isotropy groups of points of X. To build a kind of topological space, or analogue of one, containing all the isotropy information, requires a C*-algebra construction (which is noncommutative). Of course, the point remains, that the examples of group actions of most interest in dynamics, are not proper.

An important intuitive point about proper actions, is what happens when G acts on itself, or on a quotient space G/H, with H a finite subgroup of G. This is discussed in the exercise below.

Before going on, we describe some interesting geometric examples of finite group actions and proper actions.

EXAMPLE 7.5. Let D_4 be the dihedral group of symmetries of a square. We can realize D_4 as generated by a counter-clockwise rotation R of the plane through $\frac{\pi}{2}$ radians, and the reflection of the plane S across the x-axis. These two group elements, and the group of order 8 they generate, commute with the translation action of $\mathbb{Z}^2$ on the plane $\mathbb{R}^2$ and hence descend to homeomorphisms of the 2-torus $\mathbb{T}^2$. Thus D_4 acts on $\mathbb{T}^2$. The infinite group of maps of the plane generated by D_4 and $\mathbb{Z}^2$, which can be easily checked to be the semi-direct product $\mathbb{Z}^2 \rtimes D_4$ is an infinite group which acts properly on $\mathbb{R}^2$.

EXERCISE 7.6. Let $F \subset \mathbb{T}^2$ be the projection to the torus of the triangle $\{(s,t) \in \mathbb{R}^2 \mid 0 \leq s \leq t, \;\; 0 \leq t \leq \frac{1}{2}$ in the plane. Show that the restriction of the quotient map $\pi\colon \mathbb{T}^2 \to D_4\backslash\mathbb{T}^2$ to F is a homeomorphism. That is, $G\backslash\mathbb{T}^2 \cong F$. Compute the isotropy groups $\mathrm{Stab}_{D_4}(x)$ for all points $x \in F$.

EXAMPLE 7.7. The infinite dihedral group D_∞, is the group of homeomorphisms of $\mathbb{R}$ generated by the translation $T(s) = s + 1$ and reflection $S(x) = -x$. By construction, D_∞ acts on $\mathbb{R}$, and it is rather clear that the action is proper.

EXERCISE 7.8. The interval $[0, \frac{1}{2}] \subset \mathbb{R}$ is a fundamental domain for the D_∞ action on $\mathbb{R}$. Compute the isotropy groups at points of $[0, \frac{1}{2}]$.

EXAMPLE 7.9. The following example is an important one in topology, and is due to Connor and Floyd. Let p, q be two odd primes, $n = pq$ and ω a primitive nth root of unity in the circle $\mathbb{T} \subset \mathbb{C}$. Let $S =\subset \mathbb{CP}^2$ be the zero locus of the homogeneous polynomial $f(z_1, z_2, z_3) := z_1^n + z_2^n + z_3^n$. Then S is a smooth complex submanifold of $\mathbb{CP}^2$ of complex dimension 1, *i.e.* a curve. Let $\tau, T\colon S \to S$ be the maps

$$T([z_1, z_2, z_3]) := [z_1, \omega^p, z_2, \omega^q z_3], \quad \tau([z_1, z_2, z_3]) := [\omega z_1, z_2, z_3].$$

Then T and τ commute and generate the group $G := \mathbb{Z}/n \times \mathbb{Z}/n$, acting on S. We form the quotient space $X := \langle T\rangle\backslash S$ under the T-action; then τ acts on X, and X can be seen to be a closed Riemann surface of genus $\frac{(p-1)(q-1)}{2}$. The group $\mathbb{Z}/n$ generated by τ thus acts on the surface S with exactly one point of non-trivial isotropy.

EXERCISE 7.10. Find the point in X which has non-trivial isotropy.

Example 7.11. Let M be a finite complex and $G = \pi_1(M)$, the fundamental group of M. Let $X = \tilde{M}$, the universal cover of M. Then G acts on X, by 'deck-transformations,' that is, so that

$$\pi(gx) = \pi(x), \quad \forall x \in X, \ g \in G,$$

with $\pi\colon X \to M$ the covering map.

This G-action is always proper and free, and the quotient space $G\backslash X$ is naturally homeomorphic to M.

The results below will imply that $C_0(X) \rtimes G$ and $C(M)$ are Morita equivalent C*-algebras.

Important examples are when M is a compact manifold. Suppose for instance $M = M^g$ is a compact Riemann surface of genus g. The fundamental group Γ can be realized as a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ acting by Möbius transformations of the upper half-plane (see Example 7.13 below.)

Exercise 7.12. Let X be the geometric realization of the Cayley graph of the group $\mathbb{F}_2$. Prove that the left translation action of $\mathbb{F}_2$ on itself induces an action on X. Prove that this action is proper.

Example 7.13. Let G be the group $\mathrm{PSL}_2(\mathbb{Z})$. It acts by Möbius transformations on the upper half plane $X := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. A matrix $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ acs by the transformation

$$g(z) = \frac{az + b}{cz + d}.$$

Exercise 7.14. Prove that the action described above is proper. Is it free? Find all (conjugacy classes) of non-trivial isotropy.

More generally, any *Fuchsian group* acts properly on the hyperoblic plane; such groups may have torsion, and fixed-points.

Exercise 7.15. Find two examples of noncommutative discrete groups G which act properly by affine isometries of the plane $\mathbb{R}^2$.

An intuitive idea of a 'noncommutative space' is useful to have in thinking about C*-algebras associated to various kinds of groups, group actions, or dynamics. If $A = C(X)$ is a commutative C*-algebra, then A, then A encodes the compact space X up to canonical homeomorphism, as the space $\widehat{A}$ of characters $A \to \mathbb{C}$. A corresponds to the (ordinary) space X.

In general, it is sometimes useful to think about the set of equivalence classes of irreducible representations of a general C*-algebra A. This can be topologized in a natural way. We denote by $\widehat{A}$ this space. When A is commutative, every irreducible representation is 1-dimensional, *i.e.* isa character, and we recover Gelfand's dual. When A is noncommutative, the topology, though perhaps 'natural,' is non-Hausdorff in general, and thus is not very useful as a topological space. However, in certain cases, one can still get an intuitive idea of what a C*-algebra is doing, but thinking about its dual $\widehat{A}$. We will illustrate this in this section.

Remark 7.16. As we discuss in the next chapter, strongly Morita equivalent C*-algebras A and B have canonically isomorphic spaces $\widehat{A}$ and $\widehat{B}$ of irreducible representations. So thinking about duals $\widehat{A}$ is 'Morita invariant.'

For example, the C*-algebra of compact operators $\mathcal{K}$, has a unique irreducible representation, up to unitary equivalence. Hence the dual $\widehat{\mathcal{K}}$ of the compact operators is a single point.

The C*-algebras of interest in this section will be the crossed-products $C_0(X) \rtimes G$, with G discrete and acting properly on X.

We start with a basic example of a crossed-product, which turns out to very well illustrate the general situation.

Let the group $\mathbb{Z}/2$ act on $I := [-1, 1]$ by the homeomorphism $\sigma(x) = -x$. The crossed-product $A = C(I) \rtimes \mathbb{Z}/2$ is the same as the corresponding twisted group algebra, *i.e.* there is no completion involved. Elements of $A = C(I) \rtimes G$ can be written as sums $f + g[\sigma]$, here $[\sigma]$ denotes a unitary in A such that $[\sigma]f[\sigma]^* = f \circ \sigma$. The algebra multiplication in the crossed-product is determined by this rule and that ff' is the usual product of functions, and that $[\sigma] = [\sigma]^*$, *i.e* $[\sigma]^2 = 1$, the unit in A.

Choose any $x \in I$. We define a C*-algebra representation

$$\pi_x \colon C(I) \rtimes \mathbb{Z}/2 \to \mathbb{B}(\mathbb{C}^2) \cong M_2(\mathbb{C})$$

by the covariant pair

$$f \mapsto \begin{bmatrix} f(x) & 0 \\ 0 & f(-x) \end{bmatrix}, \quad [\sigma] \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The induced *-homomorphism $C(I) \rtimes \mathbb{Z}/2 \to M_2(\mathbb{C})$ is given by

$$\pi_x(f + g[\sigma]) = \begin{bmatrix} f(x) & g(x) \\ f(-x) & g(-x) \end{bmatrix}. \tag{7.1}$$

For example, since $x \neq -x$, we can find, for example, a function g such that $g(-x) = 0$ but $g(x) = 1$. For any such choice, we have $\pi_x(g[\sigma]) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Similarly, $\pi_x(g) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Arguing in this way, we see that the range of

$$\pi_x \colon C(I) \rtimes \mathbb{Z}/2 \to M_2(\mathbb{C})$$

contains all matrices. Thus π_x is a surjection for all $x \neq 0$. It is also clearly an irreducible representation, since its range contains all $M_2(\mathbb{C})$.

Remark 7.17. The unitary $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ gives a unitary conjugacy between the representations π_x and π_{-x}.

Now consider what happens when $x = 0$. The representation $\pi_0 \colon C(I) \rtimes \mathbb{Z}/2 \to M_2(\mathbb{C})$ is given by

$$\pi_0(f + g[\sigma]) = \begin{bmatrix} f(0) & g(0) \\ g(0) & f(0) \end{bmatrix}.$$

The collection of matrices of the form

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

with $a, b \in \mathbb{C}$, forms a C*-algebra naturally isomorphic to $C^*(\mathbb{Z}/2)$: it is the matrix picture of $C^*(\mathbb{Z}/2)$ acting by the regular representation on $l^2(\mathbb{Z}/2) \cong \mathbb{C}^2$. These matrices are simultaneously diagonalizable with unit eigenvectors $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, eigenvector of $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ with eigenvalue $a + b$, and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, an eigenvector of $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ with eigenvalue $a - b$.

The conclusion is that the representation π_0 splits as a direct sum of two 1-dimensional representations: namely the spans of the two given vectors. Thus

$$\pi_0 \cong \epsilon \oplus \chi,$$

with

$$\epsilon \colon C(I) \rtimes \mathbb{Z}/2 \to C^*(\mathbb{Z}/2) \to \mathbb{C}, \quad \epsilon(f + g[\sigma]) = f(0) + g(0),$$

and

$$\chi\colon C(I) \rtimes \mathbb{Z}/2 \to C^*(\mathbb{Z}/2) \to \mathbb{C}, \;\; \epsilon(f + g[\sigma]) = f(0) - g(0).$$

REMARK 7.18. In a suitable topology on the 'spectrum' $\widehat{A}$, with $A = C(I) \rtimes \mathbb{Z}/2$, the two characters ϵ and χ, form a 'double point.' Rather than explain what this topology on the spectrum is in general, we just describe what it boils down to in this example. Take the intervals $[-1, 0]$ and $[0, 1]$, form their disjoint union, and identify any *nonzero* x in $[-1, 0]$ with $-x$ in $[0, 1]$. The resulting identification space Z

$$[-1, 0] \sqcup [0, 1] \; / \; \sim$$

carries a quotient topology.

EXERCISE 7.19. The quotient topology on Z is not Hausdorff, but it is T_0.

The space just described parameterizes the spectrum of $C(I) \rtimes \mathbb{Z}/2$, by the map assigning to the equivalence class of nonzero x in the disjoint union, to the (class of the) irreducible representation $[\pi_x]$, and assigns to the two 0's in Z, the two characters ϵ and χ of $C(I) \rtimes \mathbb{Z}/2$ into which π_0 splits.

PROPOSITION 7.20. *The C*-algebra $C(I) \rtimes \mathbb{Z}/2$ is isomorphic to the C*-algebra*

$$C\big(I \times_{\mathbb{Z}/2} M_2(\mathbb{C})\big) := \{T\colon I \to M_2(\mathbb{C}) \mid T(-x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot T(x) \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \;\; \forall x \in I\}.$$

The isomorphism is given by considering the formula (7.1) as specifying a matrix-valued function, which is easily checked to transform as stated.

In this particular example we can go further. A matrix-valued function T on I such that

$$T(-x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot T(x) \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \;\; \forall x \in I,$$

is completely determined by its restriction to the 'fundamental domain' $[0, 1] \subset I$. Moreover, the condition implies that $T(0)$ *commutes* with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

COROLLARY 7.21. *The crossed-product $A := C(I) \rtimes \mathbb{Z}/2$ is isomorphic to the C*-algebra*

$$\{f\colon [0, 1] \to M_2(\mathbb{C}) \mid f(0) \text{ commutes with } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}.$$

It's space of irreducible representations is naturally parameterized by the closed interval $[0, 1]$ *with a double point at* 0 *added, by the parameterization described above.*

We now extend these ideas and describe the general case of a proper action of G discrete, acting on X. We start with a simple generic example of a proper action: if H is a finite subgroup of G, then G acts properly on G/H. We are going to analyze the structure of the C*-algebra $C_0(G/H) \rtimes G$.

The simplest case of all is where H is the trivial subgroup. In this case, we have already discussed the isomorphism

$$C_0(G) \rtimes G \cong \mathcal{K}\big(L^2(G)\big),$$

the isomorphism implemented by the covariant pair in which a function on G acts as a multiplication operator, and G acts on $L^2(G)$ by the left regular representation.

We will generally denote by λ this isomorphism

$$\lambda\colon C_0(G) \rtimes G \to \mathcal{K}\big(L^2(G)\big),$$

as it is closely related to the (left) regular representation of the group.

We are dealing only with discrete groups G here. The fact that λ defined as above maps to compact operators, and defines an isomorphism, is very easily checked in this case.

If $g \in G$, then $\lambda(g)\colon l^2(G) \to l^2(G)$ is the operator which on the standard basis is given by $\lambda(g)e_h = e_{gh}$, while if $f \in C_0(G)$ then $\lambda(f)(e_h) = f(h)e_h$ is clearly a compact (diagonal) operator. Thus, λ maps $\sum f_g[g] \in C_0(G)[G] \subset C_0(G) \rtimes G$ to the compact operator $\sum \lambda(f)\lambda(g)$.

Let $\rho(g)\colon l^2(G) \to l^2(G)$ be the unitary operator induced by *right* translation:

$$\rho(g)(e_h) = e_{hg}.$$

Then $\rho(g)$ clearly commutes with $\lambda(G)$, that is, with $\lambda(g)$ for every $g \in G$. The converse is described in the following exercise.

EXERCISE 7.22. If G is a finite group and $T\colon l^2(G) \to l^2(G)$ is an operator which commutes with $\rho(G)$, then T is in the C*-algebra $C^*(G)$ generated by $\lambda(G)$.

That is,

$$C^*(G) = \rho(G)',$$

with S' for a set of operators, denotes the commutant, the C*-algebra of operators on $l^2(G)$ which commute with S.

We can write the condition of commuting with $\rho(g)$ for $g \in G$ as

$$\rho(g)T\rho(g)^{-1} = T. \tag{7.2}$$

The above exercise shows that, for *finite* groups,

$$C^*_r(G) = \mathcal{K}(l^2G)^G, \tag{7.3}$$

where the superscript means those operators T such that (7.2) holds for all $g \in G$.

REMARK 7.23. The statement (7.3) when G is infinite, is not true; $\mathcal{K}(H)$ has to be replaced by $\mathbb{B}(H)$ for a start, since a left-translation operator $\lambda(g)$ on $l^2(G)$ is not compact if G is infinite.

Secondly, replacing $\mathcal{K}(l^2(G)^G$, by $\mathbb{B}\big(l^2(G)\big)^G$ results in a *larger* C*-algebra (namely the von Neumann algebra $C^*_r(G)''$ of G), rather than $C^*_r(G)$.

However, as we show below, crossed-products $C_0(X) \rtimes G$ for G acting *properly* on X, can be realized as fixed-point algebras in the spirit of (7.3).

DEFINITION 7.24. Let G be a discrete group acting properly on X. Let $\mathcal{K} := \mathcal{K}\big(l^2(G)\big)$, and denote by $C(X \times_G \mathcal{K})$ the C*-algebra of bounded, continuous functions

$$f\colon X \to \mathcal{K}$$

such that

$$f(gx) = \rho(g)f(x)\rho(g)^{-1}, \tag{7.4}$$

as operators on $l^2(G)$.

The following theorem and its proof generalize Proposition 7.20.

THEOREM 7.25. *The C*-algebras $C_0(X) \rtimes G$ and $C(X \times_G \mathcal{K})$ are canonically isomorphic.*

PROOF. We will construct an isomorphism $\pi\colon C_0(X) \rtimes G \to C(X \times_G \mathcal{K})$ by specifying a covariant pair, as follows.

Set

$$\pi(f)(x)(e_h) := f(hx),\ \ \pi(g)(x)(e_h) := e_{gh}, \quad f \in C_0(X),\ x \in X,\ g, h \in G,$$

where in each case we have given the action of the operator $\pi(\cdot)(x)$ on the standard basis $\{e_h\}_{h\in G}$ of $l^2(G)$. If $g \in G$, we are regarding $\pi(g) := \lambda(g)$, to be the constant, operator-valued function on X.

If $f \in C_0(X)$ and $g \in G$ then

$$\begin{aligned}(7.5)\quad \left[\pi(g)\pi(f)\pi(g)^{-1}\right](x)(e_h) &= \pi(g)\pi(f)(x)\pi(g)^{-1}(e_h)\\ &= \pi(g)\pi(f)(x)(e_{g^{-1}h}) = \pi(g)f(g^{-1}hx)\, e_{g^{-1}h} = f(g^{-1}hx)e_h\end{aligned}$$

whence

$$\pi(g)\pi(f)\pi(g)^{-1} = \pi(f \circ g^{-1}),$$

so that we have defined a covariant pair.

For fixed $x \in X$, and $f \in C_0(X)$, the operator $\pi(f)(x)$ on $l^2(G)$ is multiplication by the function on G with value $f(g^{-1}x)$ at $g \in G$. This function on G vanishes at infinity since the G-action is proper, since, as a *-homomorphism $C_0(X) \to C_b(G)$ it is Gelfand dual to the orbit map $G \to X$, $g \mapsto gx$, which is a proper map, since the G-action is assumed proper.

Hence $\pi(f)(x)$, and more generally, finite combinations $\pi(\sum_g f_g[g])(x) \in C_0(X)[G]$ are compact operators on $l^2(G)$, for any $x \in X$.

Next, if $g \in G$, and $f \in C_0(X)$, then

$$(7.6)\quad \left[\rho(g)\pi(f)(x)\rho(g)^{-1}\right](x)(e_h) = \rho(g)f(x)(e_{hg}) = \rho(g)f(hgx)e_{hg} = \pi(f)(x)(gx).$$

This shows that $\pi(f)$ satisfies (7.4), so is an element of $C(X \times_G \mathcal{K})$. If $g \in G$, then $\pi(g) = \lambda(g)$ is a constant, operator-valued function, and also satisfies (7.4), since $\lambda(g)$ commutes with $\rho(g)$.

This shows that our homomorphism maps $C_0(X) \rtimes G$ to $C(X \times_G \mathcal{K})$. The fact that it is an isomorphism is not difficult, and is left to the reader. $\square$

Remark 7.26. The C*-algebra $C(X \times_G \mathcal{K})$ may be considered a (section algebra of a) *bundle of C*-algebras over* $G\backslash X$, in a very natural way. This makes its ideal structure, and representation theory, rather transparent. On $X \times \mathcal{K}$, define an equivalence relation by

$$(x, T) \sim (gx, \rho(g)T\rho(g)^{-1}).$$

Then first coordinate projection of the quotient space $X \times_G \mathcal{K} := X \times \mathcal{K} / \sim$, gives $X \times_G \mathcal{K}$ the structure of a bundle of C*-algebras over $G\backslash X$. A section of this bundle is exactly equivalent to a map $X \to \mathcal{K}$ satisfying (7.4).

Fix any orbit $Gx \in X$. Then the fibre of $X \times_G \mathcal{K} \to G\backslash X$ over the orbit $Gx \in G$ identifies with $\mathcal{K}\big(l^2(G)\big)$, by choosing a representative point $x \in X$ in the orbit. We obtain a *-homomorphism

$$\pi_x\colon C(X \times_G \mathcal{K}) \to \mathcal{K}\big(l^2(G)\big).$$

The kernel $\ker(\pi_x)$ is an ideal, and π_x itself is a representation of $C(X \times_G \mathcal{K})$. The condition (7.4) implies the following:

Lemma 7.27. *The representation $\pi_x\colon C(X \times_G \mathcal{K}) \to \mathcal{K}\big(l^2(G)\big)$ of evaluation of a section at a point of Gx maps the C*-algebra $C(X \times_G \mathcal{K})$ isomorphically into the C*-algebra*

$$\mathcal{K}\big(l^2(G)\big)^{\mathrm{Stab}_G(x)},$$

where $\mathrm{Stab}_G(x) := \{h \in G \mid gx = x\}$ *is the stabilizer of x, and for $W \subset G$ a subgroup of G,*

$$\mathcal{K}\big(l^2(G)\big)^W := \{T \in \mathcal{K}\big(l^2(G)\big) \mid \rho(g)T\rho(g)^{-1} = T\ \forall g \in W\}.$$

Example 7.28. In the example of $\mathbb{Z}/2$ acting on I by reflection, the space $X \times_G \mathcal{K}\big(l^2(G)\big)$ and associated C*-algebra, amounts to the following. Here $X = I = [-1, 1]$, the generator of $\mathbb{Z}/2$ is $\sigma(x) = -x$, and, evidently, we can use the fundamental domain to identify $G\backslash X$ with $[0, 1]$.

We have already described (Corollary 7.21 and discussion) a family of *-homomorphisms

$$\pi_x\colon C(I) \rtimes G \to M_2(\mathbb{C}) = \mathcal{K}\big(l^2(\mathbb{Z}/2)\big),\quad x \in [0, 1],$$

and we noted several facts:

- The range of π_x for $x > 0$ is $M_2(\mathbb{C})$, and the range of π_0 consists of matrices which commute with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ – and thus is a copy of $C^*(\mathbb{Z}/2) = C^*\big(\mathrm{Stab}_{\mathbb{Z}/2}(0)\big)$.
- For $x = 0$, π_x splits into a direct sum of two 1-dimensional representations, *i.e.* characters,
$$\epsilon, \chi \colon C_0(X) \rtimes G \to \mathbb{C}.$$
These factor through the restriction map $C(I) \rtimes G \to C^*(G)$ induced by the evaluation of functions in $C(I)$ at 0, and the *-homomorphisms $C^*(\mathbb{Z}/2) \to \mathbb{C}$ induced by the two group characters ϵ and χ.

All of this data may be thought of as describing a 'bundle' of C*-algebras over $[0, 1]$. The fibre at any $x \in (0, 1]$ is $M_2(\mathbb{C})$. The fibre at $x = 0$ is $\mathbb{C} \oplus \mathbb{C}$.

Of course every point $x \in [0, 1]$ determines an ideal, $\ker(\pi_x)$. These are all primitive ideals for $x > 0$, since π_x is irreducible in this case. The origin $x = 0$ determines two ideals, the kernels of the two characters.

Exercise 7.29. Describe the two ideals $\ker(\epsilon)$ and $\ker(\chi)$ explicitly in $C(I) \rtimes \mathbb{Z}/2$.

We close this section with a discussion of what Theorem 7.25 says in the simple example of homogeneous spaces. Suppose G is a discrete group, and H is a finite subgroup. Then G acts properly on G/H.

By our results, $C_0(G/H) \rtimes H$ is isomorphic to fixed-point algebra of continuous $f \colon G/H \to \mathcal{K}(l^2G)$ which in particular satisfy the condition
$$f(gH) = \rho(g) f(eH) \rho(g)^{-1}.$$
Such functions are obviously determined by their value $T := f(eH)$ at the identity coset eH. Since H fixes the point eH, T has to satisfy
$$T = \rho(h) T \rho(h)^{-1} \;\; \forall h \in H.$$

Thus, we have a natural isomorphism
$$C_0(G/H) \rtimes G \cong \mathcal{K}(l^2G)^H. \tag{7.7}$$

We now describe the structure of the fixed-point algebra $\mathcal{K}(l^2G)^H$ in more useful terms. In the next chapter it will be shown to be strongly Morita equivalent to the group C*-algebra $C^*(H)$. But we give a more elementary approach to describing its structure here.

Proposition 7.30. *If G is a discrete group, H a finite subgroup, then there is an C*-algebra isomorphism*
$$C_0(G/H) \rtimes G \cong \varinjlim M_n\big(C^*(H)\big),$$
where $M_n\big(C^*(H)\big)$ is the C-algebra of n-by-n matrices with entries in H, and where the structure maps of the limit are the non-unital embeddings*
$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

With tensor products in place (next chapter) we will be able to reformulate this as an isomorphism
$$C_0(G/H) \rtimes G \cong C^*(H) \otimes \mathcal{K}\big(l^2(G/H)\big),$$
which is natural up to a choice of splitting
$$s \colon G/H \to G$$
of the quotient map.

Proof. The Hilbert space $l^2(G)$ decomposes into the subspaces $l^2(gH)$, as gH ranges over the points of G/H. Hence any bounded operator on $l^2(G)$ has a 'block description' in terms of a family, parameterized by cosets, of operators

$$T\colon l^2(g_1H) \to l^2(g_2H). \tag{7.8}$$

Now, for each coset, choose a representative of it in G. This determines an operator of left translation

$$U_{gH}\colon l^2(H) \to l^2(gH),$$

for each coset $gH \in G/H$.

It maps a standard basis vector $e_h \in l^2(H)$ to the standard basis vector (point mass) e_{gH}, if g has been selected as representative of the coset gH.

Now consider an operator T as in (7.9). Form the conjugated operator, denoted T', making the diagram

$$\begin{tikzcd} l^2(g_1H) \arrow[r, "T"] & l^2(g_2H) \arrow[d, "U_{g_2H}^{-1}"] \\ l^2(H) \arrow[u, "U_{g_1H}"] \arrow[r, "T'"] & l^2(H) \end{tikzcd} \tag{7.9}$$

Now observe that, by definition, the operators $U_{gH}\colon l^2(H) \to l^2(gH)$ commute with the right translation operatos $\rho(h)$, for $h \in H$. It follows that

$$T' \in \mathcal{K}(l^2H)^H,$$

with the superscript denoting fixed-point of the conjugation action of H on $\mathcal{K}(l^2H)$ induced by the right-regular representation of H on itself.

Since we have already noted that the operators on $l^2(H)$ commuting with $\rho(H)$ are precisely the elements of $C^*_r(H)$, we see that

$$T' \in C^*_r(H).$$

By applying the above unitary conjugacies to each block operator $T\colon l^2(g_1H) \to l^2(g_2H)$, we obtain a family of operators on $l^2(H)$, parameterized by the points of G/H, and each of them commuting with the right regular representation of H on $l^2(H)$, and hence each lying in the group C*-algebra $C^*_r(H)$. In this way, we can visualize operators T in $\mathcal{K}(l^2G)^H$ as G/H-by-G/H matrices of elements of $C^*_r(H)$. A dense set of such matrices are actually finitely supported, and can be regarded as *finite* matrices with entries in $C^*_r(H)$. This proves the result.

$\square$

Exercise 7.31. Let G act properly on X. Let $\mathcal{J}_X$ be the collection of $f \in C_c(X \times G) = C_c(X)[G] \subset C_0(X) \rtimes G$ satisfying

$$f(x, hg) = f(x, g), \quad \forall h \in \mathrm{Stab}_G(x), \;\; \forall x \in X.$$

Prove that $\mathcal{J}_X$ is an algebraic *-ideal in the twisted group algebra $C_0(X)[G]$. Deduce that its closure is a closed ideal in $C_0(X) \rtimes G$. This ideal will play an important role in the computation of the K-theory groups of such C*-algebras.

8. Tensor products of C*-algebras

If A is any C*-algebra, and n any positive integer, then the *-algebra $M_n(A)$ of n-by-n matrices with entries in A, is a C*-algebra. It is an example of a tensor product: in this case, it is the tensor product $A \otimes M_n(\mathbb{C})$. Another example, is the C*-algebra $C(X \times Y)$, with X, Y compact Hausdorff: this C*-algebra turns out to agree with the tensor product $C(X) \otimes C(Y)$ – tensor product is in some sense Gelfand dual to Cartesian products.

We start with tensor products of vector spaces.

Let V_1 and V_2 be vector spaces over $\mathbb{C}$. Their *tensor product* is a vector space denoted $V_1 \otimes V_2$ equipped with a bilinear map $V_1 \times V_2 \to V_1 \otimes V_2$, usually denoted simply $(v_1, v_2) \mapsto v_1 \otimes v_2$, and satisfying the following universal property: if $f \colon V_1 \times V_2 \to W$ is any bilinear map to a vector space W then there is a unique *linear* map $\bar{f} \colon V_1 \otimes V_2 \to W$ such that $\bar{f}(v_1 \otimes v_2) = f(v_1, v_2)$.

Assuming the existence of such an object, notice that the assumption that $(v_1, v_2) \mapsto v_1 \otimes v_2$ is bilinear, means that

$$(\lambda v_1) \otimes v_2 = v_1 \otimes (\lambda v_2) = \lambda\, (v_1 \otimes v_2). \tag{8.1}$$

for all scalars λ, vectors v_1, v_2, and that

$$(v_1 + v_1') \otimes v_2 = v_1 \otimes v_2 + v_1' \otimes v_2 \text{ and } v_1 \otimes (v_2 + v_2') = v_1 \otimes v_2 + v_1 \otimes v_2'. \tag{8.2}$$

Or, the diagram

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "f"] \arrow[d] & W \\ V_1 \otimes V_2 \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

can be made to commute, by an appropriate *linear* $\bar{f}$ and initial *bilinear* f.

To prove existence, one can construct $V_1 \otimes V_2$ in the following way. Consider initially the free vector space with basis the elements of $V_1 \times V_2$. This, by definition, is the vector space consisting of all formal, finite linear combinations of elements of $V_1 \times V_2$. Then let $V_1 \otimes V_2$ denote the quotient, in the category of vector spaces, of this vector space, by the subspace spanned by the vectors

$$(\lambda v_1, v_2) - \lambda(v_1, v_2),\ (v_1, \lambda v_2) - \lambda(v_1, v_2)$$

and the vectors

$$(v_1 + v_1', v_2) - (v_1, v_2) - (v_1', v_2),\ (v_1, v_2 + v_2') - (v_1, v_2) - (v_1, v_2').$$

If one denotes the equivalence class of the basis vector (v_1, v_2) by $v_1 \otimes v_2$, then the symbols $v_1 \otimes v_2$ clearly span $V_1 \otimes V_2$, satisfy the bilinearity relations (8.1) and (8.2), and it can be easily checked that $V_1 \otimes V_2$, together with the (bilinear) quotient map $(v_1, v_2) \mapsto v_1 \otimes v_2$ satisfy the required universal property.

We record a basic consequence of the universal property of a tensor product.

LEMMA 8.1. *if $T \colon V_1 \to W_1$ and $T_2 \colon V_2 \to W_2$ are two linear maps, then there is a unique linear map*

$$T_1 \otimes T_2 \colon V_1 \otimes V_2 \to W_1 \otimes W_2$$

such that $(T_1 \otimes T_2)(v_1 \otimes v_2) = T_1 v_1 \otimes T_2 v_2$.

EXERCISE 8.2. Show that if $w_1, \ldots, w_n$ is a basis for W then $V \otimes W \cong W \oplus \cdots \oplus W$ under the map

$$v \otimes w \mapsto (\lambda_1(w)v, \ldots, \lambda_n(w)v),$$

where $\lambda_1(w), \ldots, \lambda_n(w)$ are the coefficients of w with respect to the basis.

Show that in the notation of Lemma 8.1, $T \otimes 1_W \cong T \oplus T \cdots \oplus T$ with respect to this decomposition, for any $T \in \mathrm{End}(V)$, where here 1_W denotes the identity operator on W.

EXERCISE 8.3. Although by definition every element of a tensor product $V_1 \otimes V_2$ of vector spaces can be written in the form $\sum_i v_i \otimes w_i$ for some $v_1, \ldots, v_n \in V_1$ and $w_1, \ldots, w_n \in V_2$, it is less clear when two such expressions are equal in $V_1 \otimes V_2$. The following exercise helps with this.

a) If V_1, V_2 are vector spaces, show that any element of $V_1 \otimes V_2$ can be written in the form $\sum_i v_i \otimes w_i$ where the w_i are *linearly independent*. (*Hint.* Prove it by induction on n where $v = \sum_{i=1}^n v_i \otimes w_i$.)
b) Prove that if $w_1, \ldots, w_n$ are linearly independent vectors in V_1 and $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ are arbitrary vectors in V_2, then $\sum_i v_i \otimes w_i = \sum_i v'_i \otimes w_i$ implies $w_i = w'_i$ for all i. (*Hint.* Think about applying maps $f \otimes 1\colon V_1 \otimes V_2 \to V_2$, where $f \in V_1^*$ is appropriately chosen.)
c) Prove that if $e_1, \ldots, e_n$ is a basis for V_1 and $f_1, \ldots, f_m$ a basis for V_2 then $\{e_i \otimes f_j \mid i = 1, \ldots m, j = 1, \ldots, m\}$ is a basis for $V_1 \otimes V_2$. Deduce that $\mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{nm}$. Thus if V_1 and V_2 are finite-dimensional vector spaces then $\dim(V_1 \otimes V_2) = \dim(V_1)\dim(V_2)$.

Now let A and B be a pair of algebras; their tensor product $A \otimes B$ in the category of vector spaces has a natural structure of a algebra with

$$(\sum_i a_i \otimes b_i) \cdot (\sum_i c_i \otimes d_i) := \sum_{i,j} a_i c_i \otimes b_i d_j.$$

In particular, if $\mathrm{End}(V_i)$ denotes as usual the linear maps $V_i \to V_i$, $i = 1, 2$, then $\mathrm{End}(V_i)$ are algebras, and so $\mathrm{End}(V_1) \otimes \mathrm{End}(V_2)$ is an algebra.

Now, according to Lemma 8.1, for any $T_1 \in \mathrm{End}(V_1)$ and $T_2 \in \mathrm{End}(V_2)$ there is a unique linear map $T_1 \otimes T_2 \in \mathrm{End}(V_1 \otimes V_2)$ such that $(T_1 \otimes T_2)(v_1 \otimes v_2) = T_1 v_1 \otimes T_2 v_2$. This produces a map

$$\mathrm{End}(V_1) \times \mathrm{End}(V_2) \to \mathrm{End}(V_1 \otimes V_2)$$

mapping a pair (T_1, T_2) to $T_1 \otimes T_2$. It is easy to check that this map is bilinear. Hence by the universal property it determines a linear map

$$\mathrm{End}(V_1) \otimes \mathrm{End}(V_2) \to \mathrm{End}(V_1 \otimes V_2), \tag{8.3}$$

mapping the tensor $T \otimes S$ to the corresponding endomorphism. This map is easily checked to be an algebra homomorphism.

LEMMA 8.4. *The algebra homomorphism* (8.3) *is injective; it is surjective if and only if* V_1 *and* V_2 *are both finite-dimensional.*

PROOF. Let $\sum T_i \otimes S_i \in \mathrm{End}(V_1) \otimes \mathrm{End}(V_2)$, and suppose that its action on $V_1 \otimes V_2$ is the zero operator. By Exercise 8.3a), we may assume without loss of generality that the T_i are linearly independent in $\mathrm{End}(V_1)$. Now the hypothesis implies that

$$\sum_i \langle T_i v, v' \rangle \langle S_i w, w' \rangle = 0 \tag{8.4}$$

for all $v, v' \in V_1$, $w, w' \in V_2$, from which it is immediate that the operator $L := \sum \langle S_i w, w' \rangle T_i$ is the zero operator, since $\langle Lv, v' \rangle$ is equal to the expression (8.4) and so is zero, for all v, v'. The linear independence of the T_i now implies that each $\langle S_i w, w' \rangle = 0$. As w, w' were arbitrary vectors in V_2, each $S_i = 0$.

Surjectivity of (8.3) holds for dimension reasons when V_1 and V_2 are finite-dimensional, since

$$\begin{aligned} \dim \mathrm{End}(V_1 \otimes V_2) = \big(\dim(V_1)\dim(V_2)\big)^2 = \dim(V_1)^2 \dim(V_2)^2 &= \dim \mathrm{End}(V_1) \dim \mathrm{End}(V_2) \\ &= \dim\big(\mathrm{End}(V_1) \otimes \mathrm{End}(V_2)\big) \end{aligned}$$

□

Now let A and B be a pair of algebras; their tensor product $A \otimes B$ in the category of vector spaces has a natural structure of an *algebra* as well, with

$$(\sum_i a_i \otimes b_i) \cdot (\sum_i c_i \otimes d_i) := \sum_{i,j} a_i c_i \otimes b_i d_j.$$

If A, B are *-algebras, then so is $A \otimes B$, using

$$(\sum a_i \otimes b_i)^* := \sum a_i^* \otimes b_i^*.$$

We call $A \otimes B$ with this (*-)algebra structure the tensor product in the category of (*)-algebras.

Example 8.5. Let A be a C*-algebra. Then the tensor product $A \otimes M_n(\mathbb{C})$ in the category of *-algebras is isomorphic to the *-algebra $M_n(A)$ of n-by-n matrices with entries in A (c.f. Exercise 5.17). The map sends $a \otimes T \in A \otimes M_2(\mathbb{C})$ to the matrix $(aT)_{ij} := aT_{ij}$.

Exercise 8.6. Prove that the map just described is an isomorphism of *-algebras $A \otimes M_n(\mathbb{C}) \cong M_n(A)$.

Exercise 8.7. Prove that $M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \cong M_{nm}(\mathbb{C})$ for any positive integers n, m .

Exercise 8.8. Prove that if A, A', B, B' are algebras (or *-algebras), $\alpha \colon A \to A'$ an algebra (or *-algebra) homomorphism, $\beta \colon B \to B'$ another, then there is a unique (*-)algebra homomorphism $\alpha \otimes \beta \colon A \otimes B \to A' \otimes B'$ mapping $a \otimes b$ to $\alpha(a) \otimes \beta(b)$. Prove furthermore that $\alpha \otimes \beta$ is injective if α and β are each injective. (*Hint.* To prove injectivity, adapt the proof of Lemma 8.4, using more general linear functionals than the inner product functions $S \mapsto \langle Sv, v' \rangle$ used there.)

Exercise 8.9. Let A, B be two *-algebras, let $\pi \colon A \to \mathrm{End}(H)$, $\rho \colon B \to \mathrm{End}(K)$ be two representations of A, B as linear operators on vector spaces H, K. Prove that there is a unique representation

$$\pi \otimes \rho \colon A \otimes B \to \mathrm{End}(H \otimes K),$$

such that

$$(\pi \otimes \rho)(a \otimes b) = \pi(a) \otimes \rho(b) \in \mathrm{End}(H) \otimes \mathrm{End}(K) \subset \mathrm{End}(H \otimes K)$$

for all $a \in A, b \in B$. Prove that $\pi \otimes \rho$ is injective if π and ρ are each injective.

Tensor products in the category of Hilbert spaces

If V_1 and V_2 are Hilbert spaces, their tensor product $V_1 \otimes V_2$ can be made into a pre-Hilbert space (it satisfies all the conditions to be a Hilbert space except possibly completeness) by setting

$$\langle v_1 \otimes v_2, v_1' \otimes v_2' \rangle := \langle v_1, v_1' \rangle \langle v_2, v_2' \rangle. \tag{8.5}$$

By an easy exercise in the universal property of tensor products, it determines a corresponding sesquilinear form $V_1 \otimes V_2 \times V_1 \otimes V_2 \to \mathbb{C}$.

Lemma 8.10. *The sesquilinear form on* $V_1 \otimes V_2$ *determined by* (8.5) *is non-degenerate.*

Proof. Indeed, suppose otherwise, that there is an element x of $V_1 \otimes V_2$ such that $\langle x, y \rangle = 0$ for all y. We may write $x = \sum v_i \otimes w_i$ where the w_i's are linearly independent. The assumption then implies that

$$\langle v \otimes w, \sum_i v_i \otimes w_i \rangle = 0$$

for all vectors $v \in V_1, w \in V_2$. We may re-write this in the form

$$\sum \langle w, \langle v_i, v\rangle w_i\rangle = 0,\ \forall v \in V_1,\ w \in V_2.$$

In particular, for every $v \in V_1$, the linear functional

$$w \mapsto \langle w, \sum \langle v_i, v\rangle w_i\rangle$$

is the zero linear functional on V_2. Since it is inner product with the vector $\sum \langle v_i, v\rangle w_i$ we conclude that this latter vector is zero in V_2. Since the w_i's were assumed linearly independent, $\langle v_i, v\rangle = 0$ for all i. Finally, since v was arbitrary, we get that $v_i = 0$ for all i. ☐

Definition 8.11. The t*ensor product* $V_1 \otimes V_2$ of two Hilbert spaces V_1 and V_2 is defined to be the completion of the vector space tensor product of V_1 and V_2 with respect to the norm

$$\|\sum v_i \otimes w_i\|^2 := \sum_{i,j} \langle v_i, v_j\rangle\langle w_i, w_j\rangle.$$

In particular, $\|v \otimes w\| = \|v\|\|w\|$ for all vectors $v \in H, w \in K$.

The inner product $\langle \cdot\,, \cdot\rangle$ extends to the completion, so that the completion (still denoted $V_1 \otimes V_2$), is a Hilbert space.

Of course the tensor product *in the category of Hilbert spaces* involves a completion, and is *not* the same as the tensor product in the category of vector spaces. However, we will use the same notation, assuming the context makes it clear which we are using.

If H and K are Hilbert spaces, $H \otimes K$, unless otherwise specified, refers to the *Hilbert space* completion of H and K.

Exercise 8.12. Prove that if $\{e_i\}_{i\in I}$ is an orthonormal basis for H and if $\{e'_j\}_{j\in I}$ is an orthonormal basis for K then $\{e_i \otimes e'_j\}_{i\in I, j\in J}$ is an orthonormal basis for $H \otimes K$.

Lemma 8.13. *Let H and K be Hilbert spaces, $T \in \mathbb{B}(H)$, $S \in \mathbb{B}(K)$ bounded linear operators. Then there is a unique bounded linear operator $T \otimes S\colon H \otimes K \to H \otimes K$ such that*

$$(T \otimes S)(v \otimes w) = Tv \otimes Sw.$$

Furthermore, $\|T \otimes S\| = \|T\|\|S\|$, and $(T \otimes S)^* = T^* \otimes S^*$.

Proof. Start off with the tensor product of H and K in the category of *vector spaces.* By Lemma 8.1, there is a unique linear endomorphism $T \otimes S$ on this vector space, such that $(T \otimes S)(v \otimes w) = T(v) \otimes S(w)$. In order to show that $T \otimes S$ extends to a bounded linear operator on the completion, *i.e.* the Hilbert space tensor product, we need to show that

$$\|\sum_{i=1}^n Tv_i \otimes Sw_i\| \le \|T\| \cdot \|S\| \cdot \|\sum v_i \otimes w_i\|$$

for each finite collection of vectors $v_1, \ldots, v_n \in H, w_1, \ldots, w_n \in K$.

We show this for $S = \mathrm{id}_K$. The case $T = \mathrm{id}_H$ is similar, and together, these two partial results imply the result desired.

We can find a finite orthonormal basis $e_1, \ldots, e_m$ for the span of $w_1, \ldots, w_n$, and in this way re-write the vector $\sum v_i \otimes w_i$ in the form $\sum u_i \otimes e_i$ for some collection of vectors $u_1, \ldots, u_m$.

We compute

$$\|(T\otimes 1)(\sum u_i\otimes e_i)\|^2 = \|\sum Tu_i\otimes e_i\|^2 = \sum_{i,j}\langle Tu_i\otimes e_i, Tu_j\otimes e_j\rangle$$
$$= \sum_{i,j}\langle Tu_i, Tu_j\rangle\langle e_i, e_j\rangle = \sum_i \|Tu_i\|^2 \le \|T\|^2\cdot\sum\|u_i\|^2$$
$$= \|T\|^2\cdot\|\sum u_i\otimes e_i\|^2 = \|T\|^2\|\sum v_i\otimes w_i\|^2. \tag{8.6}$$

This proves the claim, and also proves, as sketched above, that $T\otimes S$ is bounded, and hence extends continuously to the Hilbert space tensor product $H\otimes K$ to itself, and, furthermore, that $\|T\otimes S\|\le\|T\|\|S\|$.

If $\epsilon>0$ then we can find a unit vector $\xi\in H$ and a unit vector $\eta\in K$ so that $\|T\xi\|\ge\|T\|-\epsilon$, $\|S\eta\|\ge\|S\|-\epsilon$. Then the unit vector $\xi\otimes\eta$ satisfies

$$\|(T\otimes S)(\xi\otimes\eta)\| = \|T\xi\otimes S\eta\| = \|T\xi\|\|S\eta\|\ge(\|T\|-\epsilon)(\|S\|-\epsilon),$$

whence $\|T\otimes S\|\ge\|T\|\|S\|$ follows by letting $\epsilon\to 0$.

The statement about the adjoints is left to the reader to prove.

□

EXERCISE 8.14. Prove directly using the definiitions, that if $T\ge 0$ is a positive operator on H such that

$$\langle Tv, v\rangle\ge\lambda\|v\|^2$$

for all nonzero vectors $v\in H$, then

$$\langle (T\otimes 1)x, x\rangle\ge\lambda\|x\|^2$$

for all nonzero vectors $x\in H\otimes K$. (*Hint.* Start by proving it for x in the algebraic tensor product, and write such an x in the form $x=\sum v_i\otimes w_i$, where the w_i's are orthonormal vectors.)

EXERCISE 8.15. Let A and B be self-adjoint operators on Hilbert spaces H, K. Prove that

$$\mathrm{Spec}(A\otimes 1 + 1\otimes B) = \mathrm{Spec}(A)+\mathrm{Spec}(B)$$

and that

$$\mathrm{Spec}(A\otimes B) = \mathrm{Spec}(A)\cdot\mathrm{Spec}(B).$$

The first set $\mathrm{Spec}(A)+\mathrm{Spec}(B)$ refers to all sums $\lambda+\mu$, with $\lambda\in\mathrm{Spec}(A)$, $\mu\in\mathrm{Spec}(B)$.

Tensor products in the category of C-algebras*

If A and B are C*-algebras, we may complete their algebraic tensor product, that is, the tensor product in the category of *-algebras, which we, for the moment, denote by $A\otimes_{\mathrm{alg}} B$, to a C*-algebra $A\otimes B$, using the following method.

Let $\pi\colon A\to\mathbb{B}(H)$ be a representation of A, $\rho\colon B\to\mathbb{B}(K)$ a representation of B. By Exercise 8.8 they combine to give a *-algebra homomorphism $\pi\otimes\rho\colon A\otimes_{\mathrm{alg}} B\to\mathbb{B}(H)\otimes_{\mathrm{alg}}\mathbb{B}(K)$. By Lemma 8.13, the algebraic tensor product $\mathbb{B}(H)\otimes_{\mathrm{alg}}\mathbb{B}(K)$ can be viewed as a *-subalgebra of $\mathbb{B}(H\otimes K)$. Therefore we obtain a representation $\pi\otimes\rho$ of $A\otimes_{\mathrm{alg}} B$ on $H\otimes K$ mapping $a\otimes b$ to the bounded operator $\pi(a)\otimes\rho(b)$.

DEFINITION 8.16. The *minimal (or spatial) tensor product* $A \otimes B$ of two C*-algebras A and B is the completion of their tensor product $A \otimes_{\mathrm{alg}} B$ in the category of *-algebras, with respect to the norm

$$\|\sum a_i \otimes b_i\| := \sup_{\pi,\rho} \|\sum a_i \otimes b_i\|_{\pi,\rho} := \|\sum \pi(a_i) \otimes \rho(b_i)\| \tag{8.7}$$

where the supremum is taken over all representations π, ρ of A, B.

The minimal tensor product, although it may feel somewhat inexplicit, has the advantage of satisfying the following important universal property.

PROPOSITION 8.17. *If A_1, A_2, B_1, B_2 are C*-algebras, and if $\alpha\colon A_1 \to B_1$ and $\alpha_2\colon A_2 \to B_2$ are *-homomorphisms, then there is a unique *-homomorphism*

$$\alpha_1 \otimes \alpha_2\colon A_1 \otimes A_2 \to B_1 \otimes B_2$$

such that $(\alpha_1 \otimes \alpha_2)(a_1 \otimes a_2) = \alpha_1(a_1) \otimes \alpha_2(a_2)$.

PROOF. The maps α_i combine to a *-homomorphism $\alpha_1 \otimes \alpha_2\colon A_1 \otimes_{\mathrm{alg}} A_2 \to B_1 \otimes_{\mathrm{alg}} B_2$, we need to show that that it extends continuously to the completions *i.e.*, that it is contractive with respect to the spatial tensor product norms, defined in (8.7). Let π_i be representations of B_i. Then $\pi_i \circ \alpha_i$ are representations of A_i. It is immediate then that if $x \in A_1 \otimes_{\mathrm{alg}} A_2$ then

$$\|(\pi_1 \otimes \pi_2)\big((\alpha_1 \otimes \alpha_2)(x)\big)\| = \|(\pi \circ \alpha_1 \otimes \pi_2 \circ \alpha_2)(x)\| \leq \|x\|.$$

Taking sups over all π_1, π_2 gives that

$$\|(\alpha_1 \otimes \alpha_2)(x)\| \leq \|x\|$$

as required.

□

EXERCISE 8.18. Suppose that $\pi \sim_u \pi'$ are unitarily equivalent representations of A, and that $\rho \sim_v \rho'$ are unitarily equivalent representations of B. Prove that the representations $\pi \otimes \rho$ and $\pi' \otimes \rho'$ of $A \otimes_{\mathrm{alg}} B$ are also unitarily equivalent. Deduce that $\|\cdot\|_{\pi,\rho} = \|\cdot\|_{\pi',\rho'}$.

EXERCISE 8.19. If A is a *-algebra and π, ρ are two representations of A as bounded operators on Hilbert spaces H, K, write $\pi \leq \rho$ if π is unitarily equivalent to a subrepresentation of ρ. That is, there is an orthogonal decomposition of the Hilbert space $K = K' \oplus K''$ such that with respect to this decomposition

$$\rho(a) = \begin{bmatrix} \rho'(a) & 0 \\ 0 & \rho''(a) \end{bmatrix}$$

for a pair of representations ρ', ρ'' of A on K', K'', such that ρ' unitarily equivalent to π. Start by checking the easy fact that if $\pi \leq \rho$ then $\|\pi(a)\| \leq \|\rho(a)\|$ for all $a \in A$.

a) If A and B are *-algebras, $\pi \leq \pi'$ are representations of A, ρ a representation of B, prove that $\pi \otimes \rho \leq \pi' \otimes \rho$, and hence that $\|\cdot\|_{\pi,\rho} = \|\cdot\|_{\pi',\rho'}$.
b) In the definition (8.7), prove that it suffices to take the sup over only *injective* representations π, ρ. (*Hint.* By the GNS theorem A has at least one injective representation ρ_0. Then $\rho := \pi \oplus \rho_0$ is still injective, and $\pi \leq \rho$.)
c) Suppose B has a representation ρ_0 with the property that any representation of B is unitarily equivalent to a subrepresentation of some number (possibly infinite) of copies of ρ_0. That is, suppose that for all ρ, there exists an index set Λ such that

$$\rho \leq \oplus_{\lambda \in \Lambda} \rho_0.$$

Prove that in the definition of the norm (8.7) on $A \otimes B$, it suffices to use the single representation π_0 for B.

d) Prove that $B := M_n(\mathbb{C})$ satisfies the property of c) for $\rho_0 \colon M_n(\mathbb{C}) \to \mathbb{B}(\mathbb{C}^n)$ the standard representation.

EXAMPLE 8.20. Let A be a C*-algebra. Then the tensor product of *-algebras $A \otimes M_n(\mathbb{C})$ (in the category of *-algebras) is isomorphic to $M_n(A)$, as discussed in Example 8.5. Under this identification, if $\pi \colon A \to \mathbb{B}(H)$ is an injective representation of A, and if $\rho_0 \colon M_n(\mathbb{C}) \to \mathbb{B}(\mathbb{C}^n)$ is the standard representation of $M_n(\mathbb{C})$, then the tensor product representation $\pi \otimes \rho_0$ can be identified with the representation

$$\bar{\pi} \colon M_n(A) \to \mathbb{B}(H \oplus \cdots \oplus H), \quad \bar{\pi}(T) = \begin{bmatrix} \pi(T_{11}) & \cdots & \pi(T_{1n}) \\ \cdots & \cdots & \cdots \\ \pi(T_{n1}) & \cdots & \pi(T_{nn}) \end{bmatrix}$$

of $M_n(A)$ on $H^n = H \oplus \cdots \oplus H$. Note that it is obviously injective. Furthermore,

$$\sup_{i,j} \|\pi(T_{ij})\| \leq \|\bar{\pi}(T)\| \leq n \sup_{i,j} \|\pi(T_{ij})\|,$$

holds for any T, so that $M_n(A)$, or, equivalently, the algebraic tensor product $A \otimes M_n(\mathbb{C})$, is already complete with respect to $\|\cdot\|_{\bar{\pi}}$. Thus, it is a C*-algebra.

Note that a C*-algebra can have only one norm, so norm produced by π in the previous paragraph is actually independent of π amongst injective representations of A. Actually, the same applies to variation of ρ. This is proved more generally below.

EXAMPLE 8.21. Another extremely important example of a tensor product is the tensor product $A \otimes \mathcal{K}$ of any C*-algebra, with the compact operators (say, on a separable Hilbert space.) The norm on the tensor product in this case is rather easy to understand. The compact operators $\mathcal{K}(H)$ has a unique irreducible representation, the obvious one, on H. Choosing any injective representation of A on a Hilbert space L gives a tensor product representation

$$A \otimes \mathcal{K} \to \mathbb{B}(L \otimes H).$$

Another way of describing this is as follows. Fix an orthonormal basis $e_1, e_2, \ldots$ for H. The tensor product Hilbert space $L \otimes H$ can then be identified with $L \oplus L \oplus \cdots$ and operators on $L \otimes H$ can be represented as $\mathbb{N}$-by-$\mathbb{N}$ matrices A with entries A_{ij} in $\mathbb{B}(L)$. Elements of the algebraic tensor product of A and finitely supported elements of $\mathcal{K}(H)$ correspond to matrices with only finitely many nonzero entries. The entries can be considered elements of A, or as operators on L by the given representation.

The tensor product $A \otimes \mathcal{K}$ is thus the closure of this algebra $M_\infty(A)$ of infinite matrices (with only finitely nonzero entries) with entries in A.

C*-algebras A and B such that $A \otimes \mathcal{K}$ is isomorphic to $B \otimes \mathcal{K}$ are said to be *Morita equivalent* (Morita equivalence is discussed in the next section.) Since $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$, A is Morita equivalent to $A \otimes \mathcal{K}$, for any A.

EXERCISE 8.22. Prove that if $M_n(A) \cong M_n(B)$ for some n then A and B are Morita equivalent.

EXERCISE 8.23. Prove that any finite dimensional C*-algebra A is Morita equivalent to $C(X)$, for a finite set of points X.

We now return to the general theory. Let A and B be C*-algebras.

PROPOSITION 8.24. *For any pair of injective representations $\pi \colon A \to \mathbb{B}(H)$ and $\rho \colon B \to \mathbb{B}(K)$, the norms $\|\cdot\|_{\pi,\rho}$ are equal,.*

That is, in computing the norm on $A \otimes B$, we can do so with any, *fixed* pair of injective representations. This is obviously quite helpful in thinking about specific examples, where typically, there is an 'obvious' such pair.

Proof. Suppose that π and ρ are injective representations on H, K. Let ρ' be another injective representation. Let $x = \sum a_i \otimes b_i \in A \otimes_{\mathrm{alg}} B$. Let (P_n) be a sequence of finite-rank projections in $\mathbb{B}(H)$ with P_n of rank n, and $P_n\xi \to \xi$ for all $\xi \in H$. By Exercise 2.7,

$$\|T\| = \sup_n \|(P_n \otimes \mathrm{id}_K)T(P_n \otimes \mathrm{id}_K)\|, \ \ \forall T \in \mathbb{B}(H \otimes K).$$

In particular, this applies to the operator $T = \sum \pi(a_i) \otimes \rho(b_i)$. and implies that

$$\|\sum a_i \otimes b_i\|\pi, \rho = \sup_n \|(P_n\pi(a_i)P_n \otimes \rho(b_i)\|.$$

Therefore, to prove that $\|_{\|}\pi, \rho = \|\pi, \rho'\|$ it suffices to fix n and prove that

$$\|\sum_i P_n\pi(a_i)P_n \otimes \rho(b_i)\| = \|\sum_i P_n\pi(a_i)P_n \otimes \rho'(b_i)\|. \tag{8.8}$$

By the definitions, the left hand side of this equation is the operator norm on $\mathbb{B}(P_nH \otimes K)$, and the right-hand side is the operator norm on $\mathbb{B}(P_nH \otimes K')$. On the other hand, $M_n(\mathbb{C}) \otimes B$ (equivalently, $M_n(\mathbb{C}) \otimes_{\mathrm{alg}} B$), is a C*-algebra, and has a unique norm. Now let $i\colon M_n(\mathbb{C}) \to \mathbb{B}(P_nH)$ be the inclusion, then the representation $i \otimes \rho\colon M_n(\mathbb{C}) \otimes B \to \mathbb{B}(P_nH \otimes K)$ is injective and so results in the same norm on $M_n(\mathbb{C}) \otimes B$ as does the one $i \otimes \rho'$ using ρ' instead (as ρ' is also assumed injective) because $M_n(\mathbb{C}) \otimes B$ is already a C*-algebra. This proves the equality (8.8). Now, reversing the roles of π and ρ in the obvious way gives that

$$\|\cdot\|_{\pi,\rho} = \|\cdot\|_{\pi',\rho'}$$

as required.

$\square$

The above proof is due to N. Brown and N. Ozawa.

Exercise 8.25. Prove that if X is any locally compact Hausdorff space and A is any C*-algebra, then $C_0(X) \otimes A \cong C_0(X, A)$, where $C_0(X, A)$ has the C*-norm explained in Exercise 1.8.

We conclude this chapter with two basic results about the spatial tensor product.

Proposition 8.26. *Let $\{A_i, \phi_{ij} \mid i < j\}$ be an inductive system of C*-algebras with injective structure maps. Then $\{A_i \otimes B, \phi_{ij} \otimes \mathrm{id}_B \mid i < j\}$ is another inductive system (with injective structure maps), and*

$$(\varinjlim_i A_i) \otimes B \cong \varinjlim_i A_i \otimes B$$

for any C-algebra B.*

Proof. Choose an injective representation $\pi\colon \varinjlim_i A_i$. If $\phi_i\colon A_i \to \varinjlim_i A_i$ are the canonical inclusions, then the representations $\pi \circ \phi_i\colon A_i \to \mathbb{B}(H)$ are also injective. Fixing an injective representation of B, we can compute the norm on $A_i \otimes B$ using $\|\cdot\|_{\pi\circ\phi_i,\rho}$. The norm of an element of the algebraic inductive limit $\varinjlim_i A_i \otimes B$ is its norm in $A_i \otimes B$, for i sufficiently large. and hence its $\|\cdot\|_{\pi\circ\phi_i,\rho}$ norm. On the other hand, the norm on $(\varinjlim_i A_i) \otimes B$ is the $\|\cdot\|_{\pi,\rho}$ norm. Since this reduces to the $\|\cdot\|_{\pi\circ\phi_i,\rho}$ norm on elements of $\phi_i(A_i) \subset \varinjlim_i A_i$, for sufficiently large i, the two ways of completing are the same, proving the result. $\square$

Finally, we prove the C*-algebraic version of the injectivity part of Exercise 8.8.

PROPOSITION 8.27. *If A_1, A_2, B_1, B_2 are C*-algebras, and if $\alpha\colon A_1 \to B_1$ and $\alpha_2\colon A_2 \to B_2$ are injective *-homomorphisms, then*

$$\alpha_1 \otimes \alpha_2\colon A_1 \otimes A_2 \to B_1 \otimes B_2$$

is also injective.

This follows from the following

LEMMA 8.28. *Suppose that A, B are unital C*-algebras and $i\colon B \to A$ an injective *-homomorphism. Then $i \otimes \mathrm{id}_D\colon B \otimes D \to A \otimes D$ is an injective *-homomorphism, for any C*-algebra D.*

PROOF. From Exercise, 8.8 the map $i \otimes \mathrm{id}_D\colon B \otimes_{\mathrm{alg}} D \to A \otimes_{\mathrm{alg}} D$ is injective on the algebraic tensor product, and extends to a C*-algebra homomorphism $i \otimes \mathrm{id}_D\colon B \otimes D \to A \otimes D$, which we need to show is isometric. Thus, it suffices to show that if $x \in B \otimes_{\mathrm{alg}} D \subset B \otimes D$, then $\|(i \otimes \mathrm{id}_D)(x)\| \geq \|x\|$. But by Proposition 5.16, given any representation π of B, there is a representation π' of A such that $\pi \leq \pi'|_B$. It follows that $\pi \otimes \rho \leq \pi'|_B \otimes \rho$ for any representation ρ of D, and hence that

$$\|\cdot\|_{\pi,\rho} \leq \|\cdot\|_{\pi'|_B,\rho}.$$

Taking sups over all π and ρ gives $\|x\| \leq \|i(x)\|$. That is, $\|i(x)\| \geq \|x\|$ for all $x \in B \otimes_{\mathrm{alg}} D$, as required.

□

EXERCISE 8.29. This exercise does Example 8.9 again, using tensor products. Let $A_n := M_2(\mathbb{C}) \otimes \cdots M_2(\mathbb{C})$ (n-times). Let $\varphi_n\colon A_n \to A_{n+1}$ be the map $\varphi_n(T) := T \otimes 1_{M_2(\mathbb{C})}$.

a) Prove that
$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \to \cdots$$
is an inductive system of C*-algebras and that
$$\varinjlim_n A_n \cong U(2^\infty),$$
with $U(2^\infty)$ the UHF algebra of Type 2^∞.

b) Let $v = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and let $\alpha_n\colon A_n \to A_n$ be the C*-algebra automorphism of conjugation by the unitary element $v \otimes \cdots \otimes v \in M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C}) = A_n$. Prove that the family $\{\alpha_n\colon A_n \to A_n\}_{n=1}^\infty$ assembles to give an automorphism of $U(2^\infty)$ of order 2.

EXERCISE 8.30. Let $\pi\colon A \to B$ be a surjective *-homomorphism of unital C*-algebras.

a) Prove that π has the 'path-lifting property': if $\gamma\colon [0,1] \to B$ is a continuous path in B, and if $a \in A$ with $\pi(a) = \gamma(0)$, then there exists a continous path $\tilde{\gamma}\colon [0,1] \to A$ such that $\tilde{\gamma}(0) = a$ and $\pi \circ \tilde{\gamma} = \gamma$. (*Hint.* Paths in A correspond to elements of the C*-algebra $C([0,1], A)$, and π determines a surjective *-homomorphism $C([0,1], A) \to C([0,1], B)$.)
b) Show that the restriction of π to a map $A_{sa} \to B_{sa}$ to the space of self-adjoints in A, to the space of self-adjoints in B, has the path lifting property.
c) Show that the restriction of π to a map $\mathbf{U}(A) \to \mathbf{U}(B)$ also has the path-lifting property. (*Hint.* Start by showing that if $(u_t)_{t\in[0,1]}$ is a path of unitaries with $u_0 = 1 \in A$, then for some $\epsilon > 0$, then the portion $(u_t)_{t\in[0,\epsilon]}$ of the given path with $0 \leq t \leq \epsilon$, lifts to a path $(\tilde{u}_t)_{t\in[0,\epsilon]}$ by using a logarithm to write $u_t = e^{ix_t}$ for a path $(x_t)_{t\in[0,\epsilon]}$ of self-adjoints.)

The following exercise returns to the notion of 'Morita equivalence', in connection with group actions. It is an extremely important example.

EXERCISE 8.31. Let H be a finite subgroup of a discrete group G. Let G act on G/H by left group multiplication. Show that the crossed-product $C_0(G/H) \rtimes G$ is isomorphic to $C^*(H) \otimes \mathcal{K}\big(l^2(G/H)\big)$.

In particular, $C_0(G/H) \rtimes G$ and $C^*(H)$ are Morita equivalent.

We have already shown that $C_0(G/H) \rtimes G$ is isomorphic to $\mathcal{K}(l^2G)^H$, the *-algebra of compact operators on $l^2(G)$ which commute with the right translation action $\rho\colon H \to \mathbf{U}(l^2G)$ of H.

a) Let $s\colon G/H \to G$ be any section of the quotient map $\pi\colon G \to G/H$. Prove that $\hat{s}\colon G/H \times H \to G$,
$$\hat{s}(gH, h) := s(gH)h$$
is a bijection that is equivariant with respect to the right translation action of H on the second variable of $G/H \times H$, and the right translation action of H on G.

b) Let
$$U\colon l^2(G/H) \otimes l^2(H) \to l^2(G)$$
be the unitary map sending the standard basis vector $e_{gH} \otimes e_h$ to the standard basis vector $e_{\hat{s}(gH,h)} = e_{s(gH)h}$ for $l^2(G)$.

c) Prove that U conjugates the representation $\rho\colon H \to \mathbf{U}(l^2)$ of H on $l^2(G)$ by right translation, to the tensor product representation $1 \otimes \rho$ of the right regular representation of H on $l^2(H)$, and the identity on $l^2(G/H)$. That is,
$$U^*\rho(h)U = 1 \otimes \rho(h) \in \mathbf{U}\big(l^2(G/H) \otimes l^2(H)\big).$$

d) Deduce that the C*-algebra of compact operators on $l^2(G)$ which commute with $\rho(H)$ is isomorphic, by a unitary conjugacy, to the C*-algebra of compact operators on $l^2(G/H) \otimes l^2(H)$ which commute with $(1 \otimes \rho)(H)$.

e) Prove that the C*-algebra of compact operators on $l^2(G/H) \otimes l^2(H)$ which commute with $(1 \otimes \rho)(H)$ is isomorphic to
$$\mathcal{K}\big(l^2(G/H)\big) \otimes \mathcal{K}(l^2H)^H.$$
Since $\mathcal{K}(l^2H)^H$, operators on $L^2(H)$ which commute with the right regular representation of H, is equal to $C^*_r(H)$, the result is proved.

9. Appendix: Calculus for Banach algebra-valued functions; the holomorphic functional calculus

Much of calculus – the theory of the derivative, the Riemann integral, power series, for functions with real or complex values – goes through for functions valued in a C*-algebra; that is, where $\mathbb{C}$ is replaced by an arbitrary C*-algebra A. In this section we give a brief summary. This will be necessary to develop the *functional calculus* for self-adjoint (or more general, normal) elements in a C*-algebra, but is used in many other constructions as well (see...)

Fix a C*-algebra A. A function $f\colon (a,b) \to A$ from an open interval in $\mathbb{R}$ to a C*-algebra A, is differentiable at $t_0 \in (a,b)$ if $\lim_{t\to t_0} \frac{f(t)-f(t_0)}{t-t_0}$ exists, in this case we denote the limit $f'(t_0)$. The standard properties of the derivative, like its linearity, the Leibniz rule $(f_1f_2)'(t) = f_1'(t)f_2(t) + f_1(t)f_2'(t)$ go through for A-valued functions, we may speak of $C^1, C^2, \ldots, C^k$ or C^∞-functions in the evident way, and so on.

Similarly, the Riemann integral is defined for continuous functions $f\colon [a,b] \to A$ is defined using nets. If $\mathcal{P}$ is the set of all partitions of $[a,b]$, and $P \in \mathcal{P}$ is one of them with intervals' endpoints $a = x_0 < x_1 < \cdots < x_n = b$, we associate to it the element
$$\langle f, P\rangle = \sum_{i=1}^n f(t_i)(t_i - t_{i-1}) \in A. \tag{9.1}$$

The net $(\langle f, P\rangle)_{P\in\mathcal{P}}$ is Cauchy and hence converges. We define

$$\int_a^b f(t)dt := \lim_{P\in\mathcal{P}} \langle f, P\rangle.$$

The linearity and other expected basic properties of the integral are easily checked. In fact, the space $C([a,b],A)$ is itself a C*-algebra, and integration defines a linear functional $C([a,b],A) \to A$ which is continuous, since

$$\|\int_a^b f(t)dt\| \leq \sup_{t\in[a,b]} \|f(t)\|(b-a) = \|f\|(b-a)$$

is easily checked from the definition.

Let $f\colon [a,b] \to A$ be a C^1-function on some open neighbourhood of $[a,b]$. The derivative of f being continuous implies that the function

$$\tilde{f}\colon [a,b]\times[a,b] \to A, \;\; \tilde{f}(s,t) := \frac{f(s)-f(t)}{s-t} \;\text{ if } t = s, \;\text{ else }\; \tilde{f}(t,t) := f'(t)$$

is continuous. Since the square is compact, it is uniformly continuous, and it follows that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|s-t| < \delta$ then $\|\frac{f(s)-f(t)}{s-t} - f'(s)\| < \frac{\epsilon}{b-a}$. For a sufficiently fine partition P with points $a = x_0 < x_1 < \cdots < x_n = b$, we have thus

$$\frac{f(t_{i+1}) - f(t_i)}{t_{i+1}-t_i} = f'(t_i) + a_i$$

where $a_i \in A$ has norm $< \frac{\epsilon}{b-a}$. Now pairing f' with P yields

$$\langle f', P\rangle = \sum_{i=1}^n f'(t_i)(t_i - t_{i-1}) = \sum_{i=1}^n \big(f(t_{i+1}) - f(t_i)\big) - a_i(t_i - t_{i-1})$$

and $\sum_{i=1}^n \big(f(t_{i+1}) - f(t_i)\big) - a_i(t_i - t_{i-1})$ is within ϵ of $f(b) - f(a)$.

Using approximation of domains in the plane by rectangles, one similarly defines the integral $\int\int_D f$ of a continous function $f\colon D \to A$, on a suitable class of regions $D \subset \mathbb{C}$ of the plane. Fubini's theorem holds, so such integrals can be computed by the method of iterated integrals. The class of regions for which all this can be checked includes those enclosed by piecewise smooth, simple closed curves in $\mathbb{C}$.

We now discuss line integrals. Let $\gamma\colon [0,1] \to W \subset \mathbb{C}$ be a smooth curve with $\gamma(t) = x(t) + iy(t)$, W the domain of a continuous function $f\colon W \to A$. Set

$$\int_\gamma f dx := \int_0^1 f\big(\gamma(t)\big)x'(t)dt, \;\; \int_\gamma f dy := \int_0^1 f\big(\gamma(t)\big)y'(t)dt, \tag{9.2}$$

these 'line integrals' and any complex linear combination of them define continuous linear functionals $C_b(D,A) \to A$. An important such linear combination is the contour integral

$$\int_\gamma f dz := \int_\gamma f dx + i\int_\gamma f dy. \tag{9.3}$$

If γ is merely a piecewise smooth curve, it is the union of finitely many smooth segments, and by adding up the relevant integrals, one extends the all the definitions (9.2) above to work for piecewise smooth curves as well.

Suppose $D = [a,b]\times[c,d]$ is a rectangle in the complex plane. A direct calculation using Fubini's theorem yields Green's Theorem for the rectangle:

$$\int_{\partial D} f dx + g dy = \int\int_D \big(-\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\big)\, dxdy$$

where the boundary is oriented positively in the usual way. Green's theorem can then be extended to all D which are interiors of piecewise smooth simple closed curves.

For purposes of spectral theory, we are most interested in holomorphic functions. A function $f\colon W \to A$ is *holomorphic* at a point $z_0 \in W$ if

$$\frac{\partial f}{\partial z}(z_0) := \lim_{z\in W z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \quad \text{exists.}$$

f is holomorphic in W if it is holomorphic at every point of W.

EXAMPLE 9.1. It is clear that any polynomial $f(\lambda) = \sum_{i=)}^n a_i\lambda^i$ with coefficients in A is holomorphic on $\mathbb{C}$.

EXAMPLE 9.2. Let a be an element of a unital Banach algebra. Then

$$f(z) := (z-a)^{-1}$$

is an (important) example of an analytic A-valued function defined on the open subset $U := \mathbb{C} \setminus \mathrm{Spec}_a(a)$.

If f is holomorphic at $z_0 = x_0 + iy_0$, then in particular the limits

$$\lim_{x\to x_0} \frac{f(x+iy_0)-f(z_0)}{x-x_0}, \quad \lim_{x\to x_0} \frac{f(x_0+iy)-f(z_0)}{iy-iy_0}$$

exist in A, *i.e.* $\frac{\partial f}{\partial x}$ and $-i\frac{\partial f}{\partial y}$ exist at z_0, where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are the standard vector fields on $\mathbb{C}$, and are each equal to $\frac{\partial f}{\partial z}$ whence to each other; we get the Cauchy-Riemann equations

$$f_x = -if_y \in A.$$

Now let D be a region whose boundary is a piecewise smooth curve γ. Let f be an A-valued function which is holomorphic on a neighbourhood of D. With $dz := dx + idy$ as in (9.3) above, we obtain the analogue of the Cauchy-Goursat Theorem as an immediate consequence of Green's Theorem

$$\int_\gamma f dz = 0.$$

The rest of the standard machinery of complex analysis then proceeds in the usual way. The Cauchy Integral formula

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz$$

holds for all $w \in D$ as a consequence of the Cauchy-Goursat Theorem applied to the (holomorphic) function $\tilde{f}(z) = \frac{f(z)}{z-w}$ if $z \neq w$ and else $= f'(z)$. By differentiation under the integral sign we get the generalized Cauchy Integral formula

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^{n+1}}dz$$

THEOREM 9.3. *Let $f\colon W \to A$ be a holomorphic function defined on an open set W. Then at any point $z_0 \in W$, f has a power series expansion*

$$f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n$$

which converges absolutely and uniformly on compact subsets of the open disk $|z-z_0| < R$ to f, where R its the distance from z_0 to $\mathbb{C}\setminus W$. Moreover,

$$\frac{1}{R} = \lim_{n\to\infty} \|a_n\|^{\frac{1}{n}}$$

gives the radius of convergence in terms of the coefficients.

Theorem 9.4. *(Liouville's Theorem) If $f\colon \mathbb{C} \to A$ is bounded and holomorphic everywhere, then f is constant.*

Indeed, we show that the complex derivative f' vanishes everywhere. For if $z_0 \in \mathbb{C}$ and if $|f(z)| \le C$ for all z then the Cauchy integral formula applied to the circle of radius n around z_0, gives

$$\|f'(z_0)\| = \|\int_{\gamma_n} \frac{f(z)}{(z-z_0)^2} dz\| \le \frac{2\pi C}{n},$$

which implies the result by letting $n \to \infty$.

The holomorphic functional calculus

We apply the above discussion in the following way to define a kind of functional calculus which is limited to analytic (or holomorphic) functions, but is valid in general Banach algebras.

Let A be a commutative, unital Banach algebra and $a \in A$, $X := \mathrm{Spec}_A(a)$ its spectrum, a compact subset of $\mathbb{C}$.

Exercise 9.5.

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$$

is a power series with radius of convergence $> \|a\|$, then

$$f(a) := \sum_{n=0}^{\infty} \alpha_n\, a^n$$

is a norm convergent series in A.

The reasoning of the Exercise above leads to definitions of for example e^a, for example, where $f(z) = e^z$, as we did above. To show that there is a reasonable definition of, for example, $\log a$, where a has spectrum contained, for example, in the right half plane $\mathrm{Re}(z) > 0$, cannot be done so directly. The method described below uses the Cauchy Integral formula as the main tool, and is called *holomorphic functional calculus.*

Let U be an open neighbourhood of X and f an analytic (holomorphic) function on U. Let γ be a simple closed, rectifiable curve in U. Any such curve splits the plane into two connected components, one bounded (the 'inside' of γ) and the other unbounded (the 'outside.'). We propose to define an element

$$f(a) := \frac{1}{2\pi i} \oint_{\gamma} f(w)(w-a)^{-1} dw. \tag{9.4}$$

where the integral is the contour, or line integral, for Banach algebra-valued functions discussed in the Appendix.

The first observation is that the formula does not depend on γ. For if γ' were another rectifiable closed curve with $\mathrm{Spec}_A(a)$ inside both of them, then connecting the two curves by an arc, and then going around γ and then γ' and along the inverse of the arc, describes a closed curve on whose inside the function $f(z)(z-a)^{-1}$ is analytic, giving

$$\oint_{\gamma_1} f(z)(z-a)^{-1} dz = \oint_{\gamma_2} f(z)(z-a)^{-1} dz$$

The second example is that the construction meshes with a case where the problem of defining $f(a)$ is rather obviously solved.

EXAMPLE 9.6. Let A be the Banach algebra (the C*-algebra) $C(X)$, with $X \subset \mathbb{C}$ any compact subset. Let $a \in A$ be the function z. Then $\operatorname{Spec}_A(a) = X$. Let U be an open neighbourhood of X. If f is analytic on U then by the usual Cauchy Integral formula

$$f(z) = \frac{1}{2\pi i} \oint_\gamma f(w)(w-z)^{-1} dw$$

holds for all $z \in X$. The right hand side is the value of the integral in (9.4), evaluated at z. In other words, the map $f \mapsto f(a)$ defined in (9.4), agrees with the natural inclusion of Banach algebras

$$\operatorname{Hol}(G) \to C(X), \quad f \mapsto f|_X.$$

PROPOSITION 9.7. *In the above notation, suppose that $f, g \in \operatorname{Hol}(G)$, with $X \subset G$, X the spectrum of $a \in A$, a unital Banach algebra. Then*

$$f(a) \cdot g(a) = (f \cdot g)(a).$$

That is, the map $f \mapsto f(a)$ is an algebra homomorphism

$$\operatorname{Hol}(A) \to A.$$

PROOF. Let α and β be two curves with $\operatorname{Spec}_A(a)$ on the inside of both of them and α inside β. Using α to define $f(a)$ and β to define $g(a)$, we write

$$\begin{aligned} f(a) \cdot g(a) &= \int_\alpha \int_\beta f(z) g(w)(z-a)^{-1}(w-a)^{-1} dw dz \\ &= \oint_\alpha \int_\beta f(z) g(w) \cdot \left(\frac{(z-a)^{-1} - (w-a)^{-1})}{w-z} \right) dw dz \\ &= \oint_\alpha f(z) \cdot \int_\beta \frac{g(w)}{w-z} dw \cdot (z-a)^{-1} dz + \oint_\beta g(w) \cdot \int_\alpha \frac{f(w)}{w-z} \cdot (w-a)^{-1} dw \end{aligned} \tag{9.5}$$

but in the second term we see the integral

$$\oint_\alpha \frac{f(w)}{w-z} dw$$

where w is a point on β, and hence is outside the loop α, so the integral is zero. Thus, the second term vanishes, while in the first integral, we see the integral

$$\oint_\beta \frac{g(w)}{w-z} dw dz$$

where z is on α, and hence is inside β. From the Cauchy Integral Formula

$$\oint_\beta \frac{g(w)}{w-z} dw dz = 2\pi i \cdot g(a).$$

Going back to (9.5) we see that we have showed that

$$f(a) \cdot g(a) = 2\pi i \oint_\alpha f(z) g(z)(z-a)^{-1} dz,$$

which is $(f \cdot g)(a)$, defined using the curve α.

□

EXERCISE 9.8. if $f(z) = \sum_{n=0}^\infty \alpha_n z^n$ is a power series with radius of convergence $> \|a\|$, then f is holomorphic on a neighbourhood U of a: prove that the integral definitions and series definitions of $f(a)$ agree.

EXERCISE 9.9. If (f_n) is a sequence of analytic functions in U and $f_n \to f$ uniformly on compact subsets of U (so that f is therefore analytic on U as well), then

$$f_n(a) \to f(a)$$

in the Banach algebra A.

CHAPTER 4

Vector bundles, Hilbert modules and applications

1. Vector bundles

A *section* of a surjective map $\pi\colon E \to X$ between topological spaces, is a continuous map $s\colon X \to E$ such that $\pi \circ s = \mathrm{id}_X$. If $Z \subset X$, a *section of* E *on* Z is a map $s\colon Z \to E$ such that $\pi \circ s = \mathrm{id}_Z$.

Definition 1.1. Let X be a locally compact Hausdorff space. A real, or complex *vector bundle* over X is a locally compact Hausdorff topological space E together with a continuous surjective map $\pi\colon E \to X$, satisfying the following additional properties.

a) The fibres $E_x := p^{-1}(x)$, $x \in X$ are all (real or) complex vector spaces.
b) The vector space operations are fibrewise continuous.
c) For each $p \in X$ there exists a neighbourhood U of p and continuous sections $s_1, \ldots, s_n$ of $\pi\colon E \to X$ on U such that the vectors $s_1(x), \ldots, s_n(x)$ are linearly independent in E_x for all $x \in U$.

A *vector bundle map* $T\colon E \to E'$ between vector bundles over X is a continuous map such that T restricts to a (real or complex depending on whether the bundle is real or complex) linear map $E_x \to E'_x$ for all $x \in X$. Equivalently, $\pi' \circ T = \pi$.

The identity map $\mathrm{id}_E\colon E \to E$ is a vector bundle map. We say that a vector bundle map $T\colon E \to E'$ is an *isomorphism* if there is a vector bundle map $T'\colon E' \to E$ such that $T \circ T' = \mathrm{id}_{E'}$ and $T' \circ T = \mathrm{id}_E$.

The space of *sections* of a vector bundle E is the $C(X)$-module of continuous maps $s\colon X \to E$ such that $\pi \circ s = \mathrm{id}_X$. Thus, for each $x \in X$, $s(x)$ is a vector in E_x, in a continuous manner. A section can be multiplied by a continuous function $f \in C(X)$ in an obvious way, using the fibrewise scalar multiplication on E. . The $C(X)$-module of sections of E is denoted $\Gamma(E)$.

Example 1.2. The second projection map $\mathrm{pr}_2\colon X \times \mathbb{C}^n$ endows the project space $X \times \mathbb{C}^n$ (with the product topology) with the structure of a complex vector bundle. Condition c) of Definition 1.1 is met since we may take $U = X$ and $s_i(x) := (x, e_i)$ where $e_1, \ldots, e_n \in \mathbb{C}^n$ is the standard basis of $\mathbb{C}^n$. Similarly, $X \times \mathbb{R}^n$ is a real vector bundle.

Such bundles are topologically uninteresting. More generally, we call any vector bundle $\pi\colon E \to X$ *trivial* if it is isomorphic, as a vector bundle, to a product bundle $X \times \mathbb{C}^n$ (or $X \times \mathbb{R}^n$).

Generally, we denote by 1_n the trivial bundle over X of rank n (real or complex, depending on the context.)

Note that the space $\Gamma(E)$ of sections of a trivial bundle $E = X \times \mathbb{C}^n$ is $C(X, \mathbb{C}^n) \cong C(X)^n$, a free $C(X)$-module.

If $\pi\colon E \to X$ is a vector bundle and $Z \subset X$ is a subspace, the restriction $E_{|Z}$ of E to Z is the topological space $\pi^{-1}(Z)$ with projection map $E_Z \to Z$ the restriction of π. It is an easy exercise to check that $E_{|Z}$ is a vector bundle over Z.

Triviality, or local triviality, can also be described in terms of sections.

Proposition 1.3. *A vector bundle $\pi\colon E \to X$ over X is trivial if and only if there is a finite collection $s_1, \ldots, s_n\colon X \to E$ of sections of $\pi\colon E \to X$ such that the vectors $s_1(x), \ldots, s_n(x)$ are linearly independent for all $x \in X$.*

Proof. Given n linearly independent sections $s_1, \ldots, s_n\colon X \to E$, define a vector bundle isomorphism $\varphi\colon X \times \mathbb{C}^n \to E$ by $\varphi\big((x, (t_1, \ldots, t_n)\big) := t_1 s_1(x) + \cdots + t_n s_n(x)$. Then φ is fibrewise an isomorphism, and is clearly continuous and a bundle map, so is an isomorphism of vector bundles.

Conversely, if $\varphi\colon X \times \mathbb{R}^n \to E$ is a vector bundle isomorphism, define $s_i(x) := \varphi(x, e_i)$, where e_i is the ith standard basis vector of $\mathbb{C}^n$. Then $s_1, \ldots, s_n$ are fibrewise everywhere linearly independent sections as required.

□

In particular, every vector bundle $\pi\colon E \to X$ is *locally trivial* in the sense that every point of X has a neighbourhood U such that $E|_U$ is trivial. Typically, the corresponding isomorphisms $\varphi\colon E|_U \to U \times \mathbb{R}^n$ are called *local trivializations* of E.

The Lemma suggests that the failure of a vector bundle $\pi\colon E \to X$ to be trivial can only depend on the global topology of X, since it is automatically locally trivial.

The following Example gives a bit of intuition for how a vector bundle can twist around a topologically interesting space (like the circle) in such a way as not to be trivial.

Example 1.4. (The Möbius bundle). Let $E = [0,1] \times \mathbb{R}/\sim$ where $\sim$ identifies the points $(0,t)$ and $(1,-t)$, for all $t \in \mathbb{R}$ Thus, E is obtained by taking a vertically bi-infinite strip, and identifying the sides with a twist. Projecting to the first coordinate determines a map from E to the unit interval with endpoints identified – that is, to the circle S^1.

To show that E is a vector bundle, let $U \subset S^1$ be the image of the open interval $(0,1) \subset [0,1]$. We denote points of S^1 by their equivalence classes $[x]$. This notation reflects, of course, an implicit choice of representative x. There is a unique choice on U, however, and we can just define $s([x]) = [(x,1)] \in E_{[x]}$. This is a non-vanishing section on U.

Now let U' be the image in S^1 of $[0,1] \setminus \{\frac{1}{2}\}$. We define a section s' on U' by setting $s'([x]) := [(x,1)]$ for $x < \frac{1}{2}$ and $s'([x]) := [(x,-1)]$ for $x > \frac{1}{2}$. This procedure makes s' well-defined at the endoints, and yields a continuous map $s'\colon U' \subset S^1 \to E$ which clearly does not vanish anywhere.

Exercise 1.5. Prove that there is a canonical bijective correspondence between the space of sections $\Gamma(E)$ of the Möbius bundle, and continuous maps $f\colon [0,1] \to \mathbb{R}$ such that $f(0) = -f(1)$. Deduce, using the Intermediate value Theorem, that E is not trivial.

Example 1.6. (The tangent bundle to the n-sphere). Consider the n-sphere S^n, the space of unit vectors in $\mathbb{R}^{n+1}$ with respect to the usual Euclidean metric. The *tangent bundle* TS^n is the vector bundle over S^n given by

$$TS^n := \{(x,v) \in S^n \times \mathbb{R}^{n+1} \mid x \perp v\}.$$

The first coordinate projection $S^n \times \mathbb{R}^{n+1} \to S^n$ restricts to a continuous surjection $\pi\colon TS^n \to S^n$. It is clear that with the usual vector space operations of $\mathbb{R}^n$, each fibre $\pi^{-1}(x)$ is a vector space; it is the orthogonal complement of x and so is a linear subspace of $\mathbb{R}^{n+1}$.

To prove that it's a real vector bundle, if $x \in S^n$, let $p_{x^\perp}\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the orthogonal projection to the linear subspace $x^\perp \subset \mathbb{R}^{n+1}$ followed by any, fixed, identification $x^\perp \cong \mathbb{R}^n$. Let $p_{x^\perp}(y)$ denote the restriction of $p_{x^\perp}$ to T_yS^n, for any y. Then it is easily checked that as long as y lies on the same side in S^n of the hyperplane $x^\perp$ as x, the linear map $p_{x^\perp}(y)\colon T_yS^n \to \mathbb{R}^n$ is a vector space isomorphism. Since these isomorphisms obviously vary continously, they trivialize TS^n in a neighbourhood of x.

EXAMPLE 1.7. (The Hopf bundle). The following procedure defines a nontrivial complex vector bundle over n-dimensional complex projective space $\mathbb{CP}^n$, the space of 1-dimensional complex subspaces of $\mathbb{C}^{n+1}$. To describe the topology on $\mathbb{CP}^n$, we can identify the set of 1-dimensional subspaces of $\mathbb{C}^{n+1}$ with the quotient of the space $\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$ of nonzero vectors in $\mathbb{C}^{n+1}$ by the equivalence relation which identifies two nonzero vectors if they are scalar multiplies of each other. With this identification, we can give $\mathbb{CP}^n$ the corresponding quotient topology.

There is a completely canonical (continuous) family of 1-dimensional vector spaces parameterized by the points L of $\mathbb{CP}^n$: set

$$H := \{(L, v) \mid v \in L\} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$$

with the subspace topology of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$. The first projection map $\mathrm{pr}_1 \colon \mathbb{CP}^n \times \mathbb{C}^{n+1} \to \mathbb{CP}^n$ restricts to a surjection $\pi \colon H \to \mathbb{CP}^n$.

I claim that $\pi \colon H \to \mathbb{CP}^n$ is a vector bundle. First, let us describe points of $\mathbb{CP}^n$ by their homogeneous coordinates: if L is a line in $\mathbb{C}^{n+1}$, and $(z_0, \ldots, z_n)$ is a point on the line, denote by $[z_0, \ldots, z_n]$ the equivalence class of the nonzero vector $(z_0, \ldots, z_n)$.

Let

$$U_i := \{[(z_0, \ldots, z_n)] \mid z_i \neq 0\} \subset \mathbb{CP}^n,$$

for $i = 0, 1, \ldots, n$. Then each U_i is open and $\cup_{i=0}^n U_i = \mathbb{CP}^n$. Since we are dealing with a one-dimensional vector bundle, to verify Condition c) of the definition of vector bundle, it is sufficient to produce a non-vanishing section $s_i \colon U_i \to H$ on each U_i. Since on U_i, the coordinate z_i does not vanish, we can set

$$s_i([z_0, \ldots, z_n]) := (\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}).$$

This is well-defined, continuous, and non-vanishing on U_i, since the ith coordinate is 1.

EXAMPLE 1.8. (Induced bundles). Let Γ be a discrete group, acting properly on X. Then there is a natural way of associating a vector bundle over $\Gamma \backslash X$ to any finite dimensonal representation $\pi \colon \Gamma \to \mathrm{GL}(V)$, V a complex vector space. As a space, let $X \times_{\Gamma, \pi} V$ be the quotient of $X \times V$ by the equivalence relation $(gx, \pi(g)v) = (x, v)$, that is, the quotient of $X \times V$ by the given group action. The first coordinate projection $X \times V$ to X induces a well-defined map $\pi \colon X \times_{\Gamma, \pi} V \to \Gamma \backslash X$.

The fibres of π are clearly copies of V.

LEMMA 1.9. *$\pi \colon X \times_\Gamma V \to \Gamma \backslash X$ is a complex vector bundle.*

PROOF. Let $x \in X$. There is a neighbourhood U of x such that $g(U) \cap U = \emptyset$ for all non-identity elements $g \in \Gamma \setminus \{e\}$.

□

The following is an excellent and important exercise.

EXERCISE 1.10. Let $T = \mathbb{R}/\mathbb{Z}$, $T' = \widehat{\mathbb{Z}}$. Both T and T' are circles, clearly, but they have different roles in the following construction. Define an equivalence relation on $\mathbb{R} \times T' \times \mathbb{C}$ by $(x, \chi, z) \sim (x + n, \chi, \chi(n)z)$. Let L be the quotient space, where we understand χ to mean a character $\chi \colon \mathbb{Z} \to \mathbb{T}$ of $\mathbb{Z}$.

The coordinate projections define a map $\pi \colon L \to T \times T'$. Show that L is a rank-one complex vector bundle over $T \times T'$, whose restriction to each slice $T \times \{\chi\}$ is the induced bundle $\mathbb{R} \times_{\mathbb{Z}, \chi} \mathbb{C}$ from the 1-dimensional representation χ.

EXERCISE 1.11. Let $\pi \colon E \to X$ be a vector bundle (either real or complex). Prove that the function $x \mapsto \dim(E_x)$ is a locally constant function on X. Deduce that the fibres of a vector bundle over a connected space all have the same dimensions as vector spaces. This is called the *rank* of the vector bundle.

EXERCISE 1.12. Prove that if $\pi\colon E \to X$ is a real or complex vector bundle over a locally compact space, then π is an open map.

We close this section with an important definition.

DEFINITION 1.13. Let $\varphi\colon X \to Y$ be a map and $\pi\colon V \to Y$ be a vector bundle over Y. Then, the pulled-back bundle $\varphi^*(V)$ is the vector bundle over X defined as follows. As a space,

$$\varphi^*(V) := \{(x, v) \in X \times V \ \ \varphi(x) = \pi(v)\},$$

topologized as a subspace of $X \times V$. The restriction $\mathrm{pr}_1|_{\varphi^*(V)} \to X$ of the first coordinate map $\mathrm{pr}_1\colon X \times V \to X$ supplies the vector bundle projection; note that the fibre of $\mathrm{pr}_1|_{\varphi^*(V)}$ over $x \in X$ is $V_{\varphi(x)}$, so the fibres have natural vector space structures.

It takes only a small amount of thought to check that $\varphi^*(V)$ really is locally trivial. Indeed, suppose that $V \subset Y$ is the domain of a chart with local sections $s_1, \ldots, s_n$. Let $U := \varphi^{-1}(V) \subset X$. Then $s_1 \circ \varphi, \ldots, s_n \circ \varphi$ are continuously defined on U and by the definitions are linearly independent sections of $\varphi^*(V)$.

DEFINITION 1.14. If $\varphi\colon X \to Y$ is a map and $\pi\colon V \to Y$ is a vector bundle over Y, $\varphi^*(V)$ denotes the vector bundle over X described above.

For a simple example, the pull-back of any vector space (in other words, vector bundle over a point) to any compact X under the map from X to a point, is a trivial bundle over X.

EXERCISE 1.15. Prove that if $Z \subset X$ is a subspace and $i\colon Z \to X$ is the inclusion then $i^*(V) = V_{|Z}$ for any vector bundle V over X.

EXERCISE 1.16. if E is the Möbius bundle over the circle $\mathbb{T}$ and $f\colon \mathbb{T} \to \mathbb{T}$ is the map $f(z) = z^2$, prove that $f^*(E)$ is trivial.

Direct sums and tensor products of vector bundles

Let V and W be vector bundles over X. Their *direct sum* $V \oplus W$, is defined as a space to be the (closed) subspace of $V_1 \times V_2$ (with the product topology) consisting of all (v_1, v_2) such that $\pi_1(v_1) = \pi_2(v_2)$. By the definitions, there is an obvious (continuous) map $\pi\colon V_1 \oplus V_2 \to X$. And each fibre $\pi^{-1}(x)$ is just $V_x \times W_x$ which can be endowed with the usual product vector space structure, making it the direct sum vector space $V_x \oplus W_x$.

EXERCISE 1.17. The direct sum $V_1 \oplus V_2$ with projection map defined above, is a vector bundle.

REMARK 1.18. It is easy to check that there are two natural inclusions $i\colon V \to V \oplus W$ and $j\colon W \to V \oplus W$, that these are vector bundle maps, and that the direct sum construction is a categorical co-product: if V_1 and V_2 are vector bundles, W a third vector bundle, and i_1 and i_2 the inclusions $V_i \to V_1 \oplus V_2$, then for any pair of bundle maps $\varphi_1\colon V_1 \to W$ and $\varphi_2\colon V_2 \to W$, there is a unique vector bundle map $\varphi\colon V_1 \oplus V_2 \to W$ such that $\varphi \circ i_1 = \varphi_1$ and $\varphi \circ i_2 = \varphi_2$.

The *tensor product* $V \otimes W$ of two vector bundles V, W over X, is defined in roughly the same way. It will be the vector bundle whose fibre at x is $V_x \otimes W_x$. Thus, as a set, $V_1 \otimes V_2$ is by definition, $\bigsqcup_{x \in X} V_x \otimes W_x$. There is of course a natural projection from this set to X.

In order to topologize the tensor product, let us cover X by open sets U on which *both* V and W are trivial. Fix such U. Suppose then that $V|_U \cong U \times \mathbb{R}^k$ and $W|_U \cong U \times \mathbb{R}^m$ by a certain pair of isomorphisms. It follows, by taking the tensor product of these isomorphisms, that we get a canonical set bijection $\bigsqcup_{x\in U} V_x \otimes W_x$ and $U \times \mathbb{R}^k \otimes \mathbb{R}^m$, which, furthermore, maps each $V_x \otimes W_x$ linearly and isomorphically to $\mathbb{R}^k \otimes \mathbb{R}^m$.

We can now specify a collection of subsets of $\bigsqcup_{x\in U} V_x \otimes U_x$ by taking images, under this isomorphism, of open subsets of $U \times \mathbb{R}^k \otimes \mathbb{R}^m$.

As U varies, the collection of all open subsets so obtained, forms a basis for a topology on $\bigsqcup_{x\in X} V_x \otimes W_x$, as the reader will easily check, and makes $V \otimes W$ into a vector bundle over X.

Exercise 1.19. Let V, W be vector bundles over X. Let $\mathrm{HOM}(V, W)$ be defined as a set to be $\bigsqcup_{x\in X} \mathrm{Hom}(V_x, W_x)$. Topologize this in such a way as to make a vector bundle, and prove that the fibrewise evaluation maps $V_x \otimes \mathrm{Hom}(V_x, W_x) \to W_x$ piece together to give a natural vector bundle map $V \otimes \mathrm{HOM}(V, W) \to W$. Furthermore, the vector space $\mathrm{Hom}(V, W)$ of vector bundle maps from V to W, is precisely the space of sections of $\mathrm{HOM}(V, W)$, by the definitions.

Exercise 1.20. If V is a vector bundle, its dual is the vector bundle $V^* = \mathrm{HOM}(V, X \times 1)$, where 1 denotes the trivial line bundle over X (real if one is working with real bundles, complex else.) It is the bundle whose fibre at $x \in X$ is the dual V_x^* of the vector space V_x.

Prove that if V and W are vector bundles over X then $\mathrm{HOM}(V, W) \cong V^* \otimes W$ as vector bundles over X.

Exercise 1.21. Prove that $V \otimes V^*$ is trivial for any complex line bundle V. (*Hint.* Identify it with $\mathrm{HOM}(V, V)$ and deduce the existence of a non-vanishing section.)

2. Projective modules and vector bundles: and Swan's Theorem

Lemma 2.1. *Let X be a locally compact Hausdorff space and $p\colon X \to M_n(\mathbb{R})$ (respectively $M_n(\mathbb{C})$) be a continuous idempotent-valued map. Let $E := \{(x, v) \in X \times \mathbb{R}^n \mid p(x)v = v\}$ (respectively $\{(x, v) \in X \times \mathbb{C}^n \mid p(x)v = v\}$), equipped with the subspace topology; let $\pi\colon E \to X$, be the restriction of the first coordinate projection to E.*

Then E is a real (respectively complex) vector bundle over X.

Thus, the fibre E_x of E at $x \in X$ is the range of $p(x)$, a subspace of $\mathbb{R}^n$.

Before going to the proof, we introduce one of the standard tools of vector bundle theory. The property of locally compact Hausdorff spaces stated in the Lemma is called *paracompactness.*

Lemma 2.2. *If X is a locally compact Hausdorff space, and if $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ is any open cover of X then there exist*

- *An open cover $\mathcal{V} = \{V_i\}_{i\in I}$ of X, such that every V_i is contained in some U_α, and such that if $F \subset I$ then $\cap_{i\in F} V_i \neq \emptyset$ only if F is finite.*
- *A family $\{\rho_i\}_{i\in I}$ of continuous functions $\rho_i \in C_c(X)$ of compact support, such that $0 \le \rho_i \le 1$ for all $i \in I$, $\mathrm{supp}(\rho_i) \subset V_i$, and such that $\sum_{i\in I} \rho_i(x) = 1$ for all $x \in X$.*

We refer to the data consisting of the locally finite refinement $\mathcal{V} = \{V_i\}_{i\in I}$ of $\mathcal{U}$, in the above Lemma, and a collection of functions $\{\rho_i\}_{i\in I}$, subordinate to $\mathcal{V}$, as a *partition of unity subordinate to the cover $\mathcal{U}$.*

Partitions of unity are useful for proving the following facts about vector bundles. A *Euclidean structure* on a real vector bundle $\pi\colon E \to X$ is a family $\{\langle\cdot,\cdot\rangle_x \mid x \in X\}$ of inner products on the fibres of E such that for any two continuous sections s_1, s_2 of E, the function $x \mapsto \langle s_1(x), s_2(x)\rangle$ on X is continuous. A *Hermitian structure* on a complex vector bundle $\pi\colon E \to X$ is a family of Hermitian inner products on the fibres of E, which is continuous in the same sense.

Proposition 2.3. *Any real vector bundle over a locally compact space has a Euclidean structure, and any complex bundle has a Hermitian structure.*

PROOF. If $\{U_i, \varphi_i\}$ is an atlas for the real vector bundle E, and $\{\rho_i \mid i \in I\}$ is a subordinate partition of unity, then we can set, for $e, e' \in E_x$,

$$\langle e, e' \rangle_x = \sum_i \sqrt{\rho_i(x)}\, \langle \varphi_i(e), \varphi_i(e') \rangle$$

where the right-hand-side refers to the usual inner product on $\mathbb{R}^n$.

This defines a Euclidean structure on E. The complex case is similar.

□

With partitions of unity in hand, we now prove Lemma 2.1.

PROOF. (Of Lemma 2.1). We just do the real case; the complex case works exactly the same.

Let $x \in X$. Then $p(x)$ is an idempotent matrix in $\mathbf{GL}(n, \mathbb{R})$, with range the subspace $E_x \subset \mathbb{R}^n$. Let $v_1, \ldots, v_k$ be a basis for this subspace, and extend it to a basis $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ for $\mathbb{R}^n$. Let $f\colon X \to M_n(\mathbb{R})$ by setting $f(y)$ equal to the n-by-n matrix with columns

$$P(y)v_1, P(y)v_2, \ldots, P(y)v_k, v_{k+1}, \ldots v_n.$$

Then f takes an invertible value at x, and hence takes invertible values in a neighbourhood U of x. In particular, the vectors $P(y)v_1, \ldots, P(y)v_k$ must be linearly independent $\forall y \in U$, and so they form n linearly independent sections of $\pi\colon E \to X$ on U. This results in a local trivialization of E on U as required.

□

We will denote the vector bundle $\pi\colon E \to X$ defined by an idempotent-valued function $p\colon X \to M_n(\mathbb{R})$ by $\mathrm{Im}(p)$. A section of $\mathrm{Im}(p)$ is by definition a continuous map $s\colon X \to \mathbb{C}^n$ such that $s(x) \in \mathrm{Im}\big(p(x)\big)$, or, equivalently, such that $p(x)s(x) = s(x)$ for all $x \in X$.

Recall that a finitely generated right A-module Γ, where A is a unital ring, is *projective* if it is a direct summand of the trivial right A-module A^n. Equivalently, there is an idempotent $p \in M_n(A)$ such that $E \cong pA^n$ as right A-modules.

Now set $A = C(X)$. If $p\colon X \to M_n(\mathbb{C})$ is a continuous, idempotent-valued function, as in Lemma 2.1, then the $C(X)$-module $\Gamma\big(\mathrm{Im}(p)\big)$ of sections of $\mathrm{Im}(p)$ is exactly equal to $p\,C(X)^n$. Therefore, it is a finitely generated projective $C(X)$-module.

We prove below that every vector bundle has the form $\mathrm{Im}(p)$ for some p, and hence that $\Gamma(E)$ is finitely generated projective for *every* vector bundle E over a compact space:

LEMMA 2.4. *Let $\pi\colon E \to X$ be a real vector bundle over a compact Hausdorff space X. Then E is isomorphic to a sub-bundle of a trivial bundle $X \times \mathbb{R}^n$ for some n. Furthermore, if $p\colon X \to M_n(\mathbb{R})$ is defined by setting $p(x)$ equal to the orthogonal projection onto $E_x \subset \mathbb{R}^n$, then p is a continuous, projection-valued map and $\mathrm{Im}(p) = E$.*

In particular, the $C(X)$-module $\Gamma(E)$ is a finitely generated projective $C(X)$-module for any vector bundle E over X.

PROOF. Suppose $U_1, \ldots, U_m$ is a finite cover of X such that $E|_{U_i}$ is trivial for all i. Let $\varphi_i\colon E|_{U_i} \to \mathbb{R}^n$ the corresponding trivializations. Let ρ_i be a partition of unity subordinate to this cover.

Define

$$\Phi\colon E \to \mathbb{R}^m,\ \Phi(e) := \oplus_{i=1}^m \big(\pi(e), \sqrt{(\rho_i \circ \pi)(e)}\varphi_i(e)\big) \in \{\pi(e)\} \times \mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n \cong \{\pi(e)\} \times \mathbb{R}^{nm},$$

which is well-defined since φ_i is zero outside of U_i. If $\pi(e) = x$, $\Phi(e) = \oplus(x, \rho_i(x)\varphi_i(e))$. If one takes the squared norm of this vector in $\mathbb{R}^{nm}$ one gets $\sum_i \rho_i(x)\|\varphi_i(e)\|^2$ which is a convex combination of non-negative numbers at least one of which is nonzero. Hence Φ is injective on each fibre, and so defines fibrewise injective vector bundle map $E \to X \times \mathbb{R}^m$ as required.

□

For example, if $X \times \mathbb{C}^n$ is a trivial bundle, or is isomorphic to one, then its space of sections is $C(X, \mathbb{C}^n) \cong C(X) \oplus \cdots \oplus C(X)$, a free $C(X)$-module. The existence of non-trivial vector bundles over X, in general, is equivalent to the existence of finitely generated projective $C(X)$-modules, which are not free.

EXAMPLE 2.5. (Following Example 1.7). Let $X = \mathbb{CP}^1$ and $\pi\colon H \to \mathbb{CP}^1$ the Hopf bundle. Then by the very definition, $H = \mathrm{Im}(P)$ where $p\colon \mathbb{CP}^1 \to M_2(\mathbb{C})$ is the following projection valued function. A point of $\mathbb{CP}^1$ is a line L in $\mathbb{C}^2$, by definition. So we let $P(L)$ be orthogonal projection onto this line.

To find an explicit formula for P is not difficult, using basic linear algebra. In terms of homogeneous coordinates on $\mathbb{CP}^1$,

$$P([z,w]) = \frac{1}{|z|^2 + |w|^2} \begin{bmatrix} |z|^2 & \bar{w}z \\ \bar{z}w & |w|^2 \end{bmatrix}.$$

Note also that if we restrict it to the natural copy of $\mathbb{C} \subset \mathbb{CP}^1$ by $z \mapsto [z,1]$ we get the projection-valued map

$$p\colon \mathbb{C} \to M_2(\mathbb{C}), \quad p(z) = \frac{1}{|z|^2 + 1} \begin{bmatrix} |z|^2 & z \\ \bar{z} & 1 \end{bmatrix}.$$

on $\mathbb{C}$. It has the property that

$$\lim_{z \to \infty} p(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

In particular, it extends continuously to the one-point compactification $\mathbb{C}^+$ of $\mathbb{C}$, which, of course, is the same as $\mathbb{CP}^1$.

EXERCISE 2.6. Find an explicit formula for a projection-valued function $p\colon S^2 \to M_3(\mathbb{R})$ whose image $\mathrm{Im}(p)$ is the tangent bundle TS^2.

LEMMA 2.7. *If E and E' are vector bundles over X compact, then $E \cong E'$ if and only if the $C(X)$-modules $\Gamma(E)$ and $\Gamma(E')$ are isomorphic.*

PROOF. One direction is clear; we prove that if $\tau\colon \Gamma(E) \to \Gamma(E')$ is a module isomorphism, then $E \cong E'$.

First observe that if s is a section of E then $\mathrm{supp}(s) \subset U$ if and only if $\rho s = 0$ for all $\rho \in C_c(X \setminus U)$ (this is essentially obvious from the definition of support). Since $\rho s = 0$ if and only if $\tau(\rho s) = \rho\tau(s) = 0$, we see that $\mathrm{supp}(\tau s) = \mathrm{supp}(s)$ for all sections s of E.

From this it follows that if s_1 and s_2 are two sections of E which agree at a single point, then $\tau(s_1)$ and $\tau(s_2)$ also agree at that same point.

Now let $v \in E$ over $x \in X$. Choose any section $s_v\colon X \to E$ such that $s_v(x) = v$. Set

$$Tv := \tau(s_v)(x) \in E'_x.$$

By the observations above, Tv does not depend on the choice of section s_v, and it is a routine matter to check that $T\colon E \to E'$ is a vector bundle isomorphism.

□

Putting all this together produces the following theorem, called *Swan's Theorem*.

THEOREM 2.8. *Let X be a compact Hausdorff space. Then the assignment $E \to \Gamma(E)$ determines a 1-1 correspondence between isomorphism classes of vector bundles over X and isomorphism classes of finitely generated projective (f.g.p.) modules over $C(X)$. In this correspondence, trivial vector bundles correspond to finitely generated free $C(X)$-modules.*

EXERCISE 2.9. Let M be the linear space of continuous functions f on $\mathbb{R}^2$ with the property that

$$f(s+n,t) = e^{-2\pi int} f(s,t)$$

for all $(s,t) \in \mathbb{R}^2, , n \in \mathbb{Z}$ Give M the structure of a (right) $C(\mathbb{T}\times\mathbb{T})$-module by interpreting continuous functions on $\mathbb{T}$ with $\mathbb{Z}$-periodic functions on $\mathbb{R}$, and setting

$$(f \cdot \varphi)(s,t) := f(s,t)\varphi(s,t).$$

a) Check that the module structure maps M to itself.
b) Find a $C(\mathbb{T}\times\mathbb{T})$-valued inner product making M into a right Hilbert $C(\mathbb{T}\times\mathbb{T})$-module. Prove that it is finitely generated and projective.
c) Prove that M is the section module of a rank-one complex vector bundle L over $\mathbb{T}\times\mathbb{T}$; describe the bundle concretely.
   See Exercise 1.10 for a more conceptual way of looking at the bundle L, and Exercise 4.11 its generalization to discrete groups other than $\mathbb{Z}$.
d) If we describe the bundle L in terms of a bundle over $\mathbb{T}$ and $\hat{\mathbb{Z}}$ there is a much more conceptual description of it. Find this description. (It has already been given, in Exercise 1.10.)

Remarks on projective modules vs idempotents vs projections

We finish this section with a brief discussion of the exact relationship between finitely generated projective modules, and the idempotents which go along with them.

If pA^n and qA^m are isomorphic projective modules over A, with $\alpha'\colon pA^n \to qA^m$ the isomorphism, $\beta'\colon qA^m \to pA^n$ its inverse, then we can extend α' to an A-module map $A^n \to qA^m$ which is zero on $(1-p)A^n$, which we denote by α. Note that α is given by the left multiplication action of an m-by-n matrix with entries in A; similarly, $\beta \in M_{nm}(A)$. Multiplying these matrices one way gives $\alpha\beta = p$, and multiplying them the other way gives $\beta\alpha = q$.

Conversely, if $\alpha \in M_{mn}(A)$ and $\beta \in M_{nm}(A)$ with $\alpha\beta = p$ an idempotent in $M_n(A)$ and $\beta\alpha = q$ an idempotent in $M_m(A)$, then the projective A-modules pA^n and qA^m are isomorphic; the isomorphism is multiplication by the matrix α and its inverse is multiplication by the matrix β.

PROPOSITION 2.10. *Two finitely generated projective modules pA^m and qA^n are isomorphic if and only if there exists $\alpha \in M_{mn}(A), \beta \in M_{nm}(A)$ such that $\alpha\beta = p, \quad \beta\alpha = q$.*

We call this *algebraic equivaelnce* of idempotents. A special case is *similarity*, as in the following easy

EXERCISE 2.11. Prove that if $p, q \in M_n(A)$ are *similar*, *i.e.* if there is an invertible $u \in M_n(A)$ such that $upu^{-1} = q$, then p and q are algebraically equivalent.

PROPOSITION 2.12. *If A is a unital C^*-algebra and p and q are similar, then they are unitarily equivalent.*

PROOF. Suppose that $apa^{-1} = q$. Then $ap = qa$, and taking adjoints, $pa^* = a^*q$. It follows that $pa^*a = a^*qa = a^*ap$ so that p commutes with a^*a. Hence it commutes with the unitary $u := |a|^{-1} = (a^*a)^{-\frac{1}{2}}$ of the polar decomposition $a = u|a|$ of a. We get

$$up = a|a|^{-1}p = ap|a|^{-1} = qa|a|^{-1} = qu$$

and hence $upu^* = q$.

$\square$

The following Lemma refines this result:

LEMMA 2.13. *If A is a unital C*-algebra then every idempotent in A is similar to a projection.*

Furthermore, if p and q are algebraically equivalent projections, with $\alpha\beta = p$ and $\beta\alpha = q$, then there is a partial isometry $u \in M_{nm}(A)$ such that $uu^* = \alpha$ and $u^*u = \beta$.

PROOF. The idea of the proof is to think of $A \subset \mathbb{B}(H)$ for a HIlbert space H. In this case, there is an orthogonal decomposition of H into two subspaces with respect to which e has the block matrix representation

$$e = \begin{bmatrix} 1 & R \\ 0 & 0 \end{bmatrix}$$

for some operator R. Let p be the operator with matrix representation $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and s the operator with matrix representation $s = \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix}$. It is clear that $ses^{-1} = p$. It remains to show that p and s are actually in the C*-algebra A. To see this, compute with the matrices that

$$p\big(1 + ee^* + e^*e - (e + e^*)\big) = ee^* \tag{2.1}$$

Since $1 + ee^* + e^*e - (e + e^*) = 1 + (e - e^*)(e - e^*)^*$, it is a strictly positive and in particular invertible element of A. Hence by (2.1) we get

$$p = ee^*\big(1 + (e - e^*)(e - e^*)^*\big)^{-1} \tag{2.2}$$

and so $p \in A$. Since $e - p = \begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix}$ we get $1 + e - p = s$, so that $s \in A$ as well.

For the second statement, suppose that $p = \alpha'\beta'$ and $q = \beta'\alpha'$ for some $\alpha', \beta' \in A$. Let $\alpha = p\alpha' q$ and $\beta = q\beta p$. Then the equations $\alpha\beta = p$ and $\beta\alpha = q$ still hold, but now α and β satisfy $p\alpha q = \alpha$ and $q\beta p = \beta$.

Now

$$p = pp^* = \alpha\beta\beta^*\alpha^* \leq \|\beta\|^2\alpha\alpha^*,$$

which shows that $\alpha\alpha^*$ is invertible in the unital C*-algebra pAp. Set $u = \beta|\beta|$, the partial isometry in the polar decomposition $\beta \in pAp$. Then $u^*u = p$ is immediate. Also, uu^* is a projection, since u is a partial isometry. And

$$quu^* = q\beta|\beta|^2\beta^* = \beta|\beta|^2\beta^* = uu^*$$

since $q\beta = \beta$. This shows that uu^* is a subprojection of q. On the other hand,

$$q = qq^* = \beta\alpha\alpha^*\beta^* \leq \|\alpha\|^2\beta\beta^*$$

and since $\beta\beta^* = u\beta^*\beta u^*$, we get $q \leq \|\alpha\|^2 u\beta^*\beta u^* \leq \|\alpha\|^2\|\beta\|^2 uu^*$. Putting things together gives

$$uu^* \leq q \leq \|\alpha\|^2\|\beta\|^2 uu^*$$

which implies that $uu^* = q$.

□

DEFINITION 2.14. Let A be any C*-algebra. Two projections $p \in M_n(A)$ and $q \in M_m(A)$ are said to be *Murray-von-Neumann equivalent* if there is a partial isometry $u \in M_{mn}(A)$ such that $uu^* = p$ and $u^*u = q$.

This discussion shows that as far as classifying projective modules over a unital C*-algebra, the problem is that of classifying Murray-von-Neumann equivalence classes of projections in $M_\infty(A)$.

The following exercise shows that Murray-von-Neumann equivalence is not very far from unitary equivalence.

EXERCISE 2.15. Let p and q be projections in a unital C*-algebra A which are Murray-von-Neumann equivalent. Let v is the partial isometry implementing the equivalence, with $v^*v = p, vv^* = q$,

a)
$$u := \begin{bmatrix} v & 1 - vv^* \\ v^*v - 1 & v^* \end{bmatrix}$$
is a unitary satisfying
$$u \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} u^* = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}.$$
Thus, $\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}$ are unitarily equivalent.

b) The unitary u is connected by a continuous path of unitaries in $M_2(A)$ to the identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (Consider the path
$$u_t := \begin{bmatrix} \cos t\, v & 1 - (1 - \sin t)vv^*) \\ (1 - \sin t)v^*v - 1 & \cos t v^* \end{bmatrix},$$
which connects u to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, for $t \in [0, \frac{\pi}{2}]$.

The latter matrix can then be connected with $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by the same trick.)

EXERCISE 2.16. Let X be compact Hausdorff. Prove that if $p\colon X \to M_n(\mathbb{C})$ and $q\colon X \to M_n(\mathbb{C})$ are continuous, projection-valued functions, and if $\|p - q\| < 1$ as elements of the C*-algebra $C\big(X, M_n(\mathbb{C})\big)$, then $\mathrm{Im}(p) \cong \mathrm{Im}(q)$ as vector bundles.

REMARK 2.17. This implies a certain 'discreteness' of the space of isomorphism classes of vector bundles over a compact, second countable topological space: prove that this set is countable, using the fact that the C*-algebra $C(X) \otimes M_n(\mathbb{C})$ is separable, for all n.

3. Multiplier algebras

Let A be a C*-algebra. We are going to develop the general theory of Hilbert modules over A. This is a critical step in the evolution of the subject.

DEFINITION 3.1. A *multiplier* of A is a linear map $L\colon A \to A$ such that

- $L(ab) = L(a)b$ for all $a, b \in A$,
- L is *adjointable* in the sense that there exists a linear map $L^*\colon A \to A$ such that $L(a)^*b = a^*L^*(b)$ for all $a, b \in A$.

EXAMPLE 3.2. Suppose that A is an ideal in a larger C*-algebra B and $x \in B$. Then left multiplication by x defines a multiplier $L_x\colon A \to A$ (with adjoint $L_x^* = L_{x^*}$). This contains the following two examples:

a) Any bounded, continuous function f on X locally compact Hausdorff, defines by point-wise multiplication, a multiplier of $C_0(X)$.
b) Any bounded operator $T \in \mathbb{B}(H)$ defines a multiplier of $\mathcal{K}(H)$, by multiplication.

For the first example, $A = C_0(X)$ is an ideal in the C*-algebra $B = C_b(X)$, and in the second, $A = \mathcal{K}(H)$ is an ideal in $B = \mathbb{B}(H)$.

Exercise 3.3. The Szego projection P_+ of group convolution with $\chi(z) = \frac{1}{1-z}$, taken in a distributional sense, is a self-adjoint multiplier of $C^*_r(\mathbb{T})$.

Exercise 3.4. If A is a unital C*-algebra then $A \cong \mathcal{M}(A)$ by mapping $x \in A$ to the left multiplication operator $L_x \colon A \to A$ (whose adjoint is L_{x^*}).

Exercise 3.5. The linear map $L^* \colon A \to A$ specified by Definition 3.1, provided that it exists, is both right A-linear: $L^*(ab) = L^*(a)b$ for all $a, b \in A$, and unique. We call it the *adjoint* of L.

Lemma 3.6. *Multipliers are bounded: there exists $C \geq 0$ such that $\|La\| \leq C\|a\|$ for all $a \in A$.*

Proof. This is a standard exercise in the Closed Graph Theorem. Note that A is in particular a Banach space, so the Closed Graph Theorem applies to a multiplier $L \colon A \to A$; to show that it is bounded it suffices to show that the graph $\{(x, y) \in A \oplus A \mid y = L(x)\}$ is closed.

So let $(a_\lambda) \subset A$, $a_\lambda \to a$, and suppose that $L(a_\lambda) \to b$. We need show that $L(a) = b$. But if $c \in A$ then

$$(b - L(a))^* c = \lim_{\lambda\to\infty} (L(a_\lambda) - L(a))^* c = \lim_{\lambda\to\infty} L(a_\lambda - a)^* b = \lim_{\lambda\to\infty} (a_\lambda - a) L^*(c) = 0$$

so the result follows from Exercise 1.7.

□

Exercise 3.7. Prove that if $L \colon A \to A$ is a multiplier, and if $a \in A$, then La and aL are both multipliers, where $(La)(b) = L\big(ab\big)$, $(aL)(b) := aL(b)$.

What are these multipliers?

Exercise 3.8. Prove that the adjoint operation on $\mathcal{M}(A)$ is conjugate A-linear in the sense that $(Ta)^* = a^* T^*$. Also, prove that $L^{**} = L$ for any multiplier L.

It is easy to check that a linear combination or product of multipliers is again a multiplier, whilst the set $\mathcal{M}(A)$ of multipliers has an obvious adjoint operation as well, so $\mathcal{M}(A)$ is a *-algebra containing A. Exercise 3.7 shows that A is an ideal in $\mathcal{M}(A)$.

Set

$$\|L\| = \sup_{\|a\|\leq 1} \|L(a)\|,$$

then $\|L\|$ defines a norm on $\mathcal{M}(A)$, which restricts to the given norm on A.

Exercise 3.9. , Show that if L_1 and L_2 are multipliers then $\|L_1 L_2\| \leq \|L_1\|\|L_2\|$, and if $a \in A$ and L_a the corresponding multiplier then $\|L_a\| = \|a\|$.

Lemma 3.10. *If L is a multiplier, L^* its adjoint, then $\|L\| = \|L^*\|$.*

Proof. We argue as follows: if $a \in A$ then

$$\begin{aligned} \|L^*(a)\| = \sup_{\|b^*\|\leq 1} \|b^* L^*(a)\| = \sup_{\|b^*\|\leq 1} \|L(b)^* a\| \leq \sup_{\|b^*\|\leq 1} \|L(b)^*\| \\ = \sup_{\|b\|\leq 1} \{\|L(b)\| = \|L\|, \end{aligned}$$

The first step is by Exercise 1.7 and the last step is because $\|x\| = \|x^*\|$ for x in a C*-algebra. Therefore $\|L^*(a)\| \leq \|L\|$ and so $\|L^*\| \leq \|L\|$. Replacing L by L^* completes the argument.

□

Lemma 3.11. *$\|L^*L\| = \|L\|^2$ for all multipliers L.*

Proof. Since $\|L^*L\| \leq \|L^*\|\|L\|$ by Exercise 3.9, and $\|L^*\| = \|L\|$ by Lemma 3.10, $\|L^*L\| \leq \|L\|^2$. On the other hand

$$\|L^*L\| = \sup_{\|a\|\leq 1}\|L^*L(a)\| = \sup_{\|a\|,\|b\|\leq 1}\|b^*L^*L(a)\| = \sup_{\|a\|,\|b\|\leq 1}\|L(b)^*L(a)\|$$
$$\geq \sup_{\|a\|\leq 1}\|L(a)^*L(a)\| = \|L\|^2,$$

□

Lemma 3.12. *In the multiplier norm, $\mathcal{M}(A)$ is complete.*

Proof. Let (L_λ) be a Cauchy net of mutipliers, then $L_\lambda \to L$ where L is some bounded linear operator on the Banach space A, since the space of all bounded linear operators on a Banach space is complete in the operator norm. So it suffices to show that L is a multiplier. It follows immediately from the continuity of multiplication in A that L is right A-linear. So we are reduced to showing that L is adjointable. But since $L_\lambda \to L$ in norm, it follows from Lemma 3.10 that L_λ^* converges, and it is easy to check that it converges to the adjoint of L.

□

Definition 3.13. We say that a net (L_i) in $\mathcal{M}(A)$ converges in the *strict* topology to a multiplier L if $L_i(a) \to L(a)$ and $L_i^*(a) \to L^*(a)$ for all $a \in A$.

For example, by the definitions, if (u_i) is an approximate unit in A, then the net (L_{u_i}) converges strictly in $\mathcal{M}(A)$ to 1, the identity of $\mathcal{M}(A)$.

Similarly, if L is any multiplier of A, then $L(x) = \lim_{i\to\infty} L(u_i x) = \lim_{i\to\infty} L(u_i)x$. The conclusion is that the net $(L(u_i))$ (or, if one prefes, the net (u_i) itself), converges, in the strict topology, to L.

A non-degenerate homomorphism $\alpha\colon A \to B$ is one for which $\alpha(A)B = B$. This is equivalent to $\alpha(A)B$ being dense in B, the equivalence being a result called *Cohen's factorization theorem.* Another equivalent condition is that α maps an approximate unit for A to an approximate unit for B.

Corollary 3.14. *If A is any C*-algebra, then $\mathcal{M}(A)$ is a unital C*-algebra containing A as a closed and strictly dense ideal.*

*In particular, if $\varphi\colon A \to B$ is a strictly continuous $*$-homomorphism, then φ extends to a *-homomorphism $\mathcal{M}(A) \to \mathcal{M}(B)$.*

*Non-degenerate *-homomorphisms are strictly continuous, and hence extend.*

Proof. □

Exercise 3.15. Show that if $x_i \to x$ strictly, (x_i) a net in A, then $L(x_i) \to L(x)$ strictly, for any $L \in \mathcal{M}(A)$.

Example 3.16. Let X be a locally compact Hausdorff space. The Stone-Cech compactification βX of X is, by definition, the maximal ideal space of the commutative C*-algebra $C_b(X)$ of bounded, continuous functions on X, with the uniform norm.

Since bounded functions clearly act by multipliers of $C_0(X)$, here is a natural *-homomorphism $C_b(X) \to \mathcal{M}\big(C_0(X)\big)$. Since one can always realize, up to ϵ, the norm of a bounded function, within a fixed compact set, it follows from an easy argument that $\|f\| = \sup_{g\in C_0(X),\|g\|\leq 1}\|fg\|$, for any $f \in C_b(X)$, so the inclusion $C_b(X) \to \mathcal{M}\big(C_0(X)\big)$ is isometric.

Let (u_i) be an approximate unit for $C_0(X)$, and L a multiplier of $C_0(X)$, then since $u_i \to 1$ strictly, the net of functions $L(u_i)$ converges strictly as well, as multipliers of $C_0(X)$. It follows easily that the functions $(L(u_i)$ converge uniformly on compact subsets of X. Let f be the target function. It is clearly bounded, as $\|L(u_i)\| \le \|L\|$ for all i, so that the functions $L(u_i)$ are all uniformly bounded by a fixed constant, giving that f is as well. It is now easy to verify that $L(h) = fh$ for any $h \in C_0(X)$. So $L = L_f$.

Hence $C_b(X) = \mathcal{M}\big(C_0(X)\big)$.

EXERCISE 3.17. Let H be a Hilbert space. Prove that $\mathcal{M}\big(\mathcal{K}(H)\big) \cong \mathbb{B}(H)$, and prove that the strict topology on $\mathbb{B}(H)$, regarded as the multiplier algebra of $\mathcal{K}(H)$, corresponds to the strong* operator topology on $\mathbb{B}(H)$, in which a net (T_i) of bounded operators converges in the strong* topology to T if and only if $\lim_{i\to\infty} T_i\xi = T\xi$ and $\lim_{i\to\infty} T_i^*\xi = T^*\xi$ for all $\xi \in H$.

EXERCISE 3.18. Let A be faithfully and non-degenerately represented as bounded operators on a Hilbert space H. Prove that

$$\mathcal{M}(A) \cong \{T \in \mathbb{B}(H) \mid T\pi(a) \text{ and } \pi(a)T \in \pi(A)\ \forall a \in A\}$$

as C*-algebras. Describe the strict topology on $\mathcal{M}(A)$ in terms of the Hilbert space.

EXERCISE 3.19. Let X be a locally compact Hausdorff space and A be any C*-algebra. Prove that $\mathcal{M}(C_0(X) \otimes A)$ is the C*-algebra of bounded, strictly continuous maps $X \to \mathcal{M}(A)$.

REMARK 3.20. An important example is where $A = \mathcal{K}$. The multiplier algebra of $C_0(X)\otimes\mathcal{K}$ is thus the C*-algebra of bounded, strictly continuous maps $X \to \mathbb{B}(H)$.

An important theorem of Kuiper asserts that the unitary group of this C*-algebra is contractible in the norm topology. This result has repercussions in K-theory.

EXERCISE 3.21. For $t \in \mathbb{R}$, let u_t be the unitary $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ of translation by t: $(u_t\xi)(x) = \xi(x-t)$. Let $T := \textsc{pv}\int_{-1}^{1} \frac{u_t}{t}$ be the 'convolution operator' defined by taking the principal value

$$(Tf)(x) := \textsc{pv}\int_{-1}^{1} \frac{f(x-t)}{t} dt := \lim_{\epsilon\to 0}\int_{I_\epsilon} \frac{f(x-t)}{t} dt,$$

where $I_\epsilon := [-1,-\epsilon] \cup [\epsilon, 1]$, $f \in \mathcal{S}(\mathbb{R})$. Prove that T is a self-adjoint contractive multiplier of $C^*(\mathbb{R})$ and that $T^2 - 1 \in C^*(\mathbb{R})$. It's Fourier transform is a multiplier of $C_0(\mathbb{R})$, and hence is multiplication by a bounded function. What is the function? (See Example 11.16).

4. Hilbert modules

DEFINITION 4.1. A *Hermitian right A-module* is a complex vector space which is also a right A-module, with linear A-multiplication, with the following piece of additional structure. We require a Hermitian A-valued form on E: a map

$$\langle \cdot\,,\ \cdot\rangle_A \colon E \times E \to A$$

linear in the second variable, conjugate linear in the first, and such that

- $\langle x, ya\rangle = \langle x, y\rangle a$ for all $x, y \in E$, $a \in A$,
- $\langle x, y\rangle = \langle y, x\rangle^*$ and $\langle x, x\rangle \ge 0$ for all $x \in E$,
- $\langle x, x\rangle = 0$ only $x = 0$.

The *support* of E is the closed $\mathbb{C}$-linear span of the set $\langle x, y\rangle$ of inner products of elements of E, and we say E is *full* if $\mathrm{supp}(E) = A$.

If E is complete with respect to the norm $\|x\| := \|\langle x, x\rangle\|$, then we say E is a right *HIlbert A-module*. In this case, we also refer to $\langle\cdot,\cdot\rangle$ as an (A-valued) *inner-product*.

The fact that $\|x\| := \|\langle x, x\rangle\|$ is a norm is proved below.

Sometimes we refer to a right *semi-Hermitian* A-module, if the form $\langle\cdot,\cdot\rangle$ is possibly degenerate, *i.e.*, if there exist nonzero $x \in E$ such that $\langle x, x\rangle = 0$.

Occasionally, when the need arises, we will denote an A-valued inner project in the form $\langle \cdot, \rangle_A$. We will mostly only do this when there is more than one inner product under consideration at the same time.

EXAMPLE 4.2. A is a right Hilbert A-module over itself, for any A, with right module structure right algebra multiplication, and inner product $\langle a, b\rangle := a^*b$. It is full.

More generally, if J is a closed *right* ideal in a C*-algebra A then J has the structure of a right Hilbert A-module with the evident right multiplication, and inner product $\langle a, b\rangle := a^*b \in A$. It's support is J.

A good and important geometric example is when $A = C_0(X)$, $J = C_0(U)$, where $U \subset X$ is an open subset.

EXERCISE 4.3. Suppose X is locally compact Hausdorff and $\mathcal{E}$ is a right Hilbert $C_0(X)$-module. Since the support of $\mathcal{E}$ is an ideal of $C_0(X)$, it has the form $C_0(U)$ for some $U \subset X$ open. In this notation, prove that

$$\overline{U} = \cap_{f\in \mathrm{ann}(\mathcal{E})} f^{-1}(0),$$

where $\mathrm{ann}(\mathcal{E}) := \{f \in C_0(X) \mid \xi f = 0\ \forall \xi \in \mathcal{E}\}$.

For another simple example, let $A = M_n(\mathbb{C})$ and let E be the linear space $M_{k,n}(\mathbb{C})$ of k-by-n-matrices with complex entries. If $x, y \in M_{k,n}(\mathbb{C})$ then $x^*y \in M_n(\mathbb{C})$, and if $a \in M_n(\mathbb{C})$ then $xa \in M_{k,n}(\mathbb{C})$ so that $M_{k,n}(\mathbb{C})$ is a right Hilbert A-module.

In fact this is of the general kind discussed in the first paragraph, with J the (closed) right ideal of $M_n(\mathbb{C})$ of matrices A with $A_{ij} = 0$ for $i > k$.

EXERCISE 4.4. Suppose E is a full right Hilbert B-module, B unital. Prove that there exist $x_1, \ldots, x_n$ such that $\sum_{i=1}^n \langle x_i, x_i\rangle = 1_B$. (*Hint.* There exist x_i, y_i such that $\sum \langle x_i, y_i\rangle = 1$, now expand $\sum\langle x_i + y_i, x_i + y_i\rangle$ and do a little algebra to get an expansion of the required kind.)

EXERCISE 4.5. Prove that if $x_0 \in X$ is any point, and if we view $\mathbb{C}$ as a (right) $C(X)$-module by evaluation of functions at x_0, then there is *no* $C(X)$-valued inner product making $\mathbb{C}$ (with this $C(X)$-module structure) into a Hilbert $C(X)$-module, unless x_0 is an isolated point of X. Try to draw a more general conclusion (about what algebraic modules over $C(X)$ can actually be Hilbert modules.)

Further interesting examples of Hilbert modules over $C^*(G)$, for compact groups G, are discussed at the end of the section.

DEFINITION 4.6. The *standard Hilbert A-module of rank n* is $A^n := A \oplus \cdots \oplus A$ with right A-module structure $(x_1, \ldots, x_n)a := (x_1a, \ldots, x_na)$ and inner product

$$\langle x, y\rangle := \sum_{i=1}^n x_i^* x_i.$$

The *standard Hilbert A-module of rank $\mathbb{N}$*, which we sometimes denote by H_A, other times by $A^{\mathbb{N}}$, is the *completion* of the collection of finitely supported sequences $(x_n)_{n=1}^\infty$ of elements of A, with right module structure $(x_n)a := (x_na)$ and inner product

$$\langle x, y\rangle = \sum_{n=1}^\infty x_n^* y_n,$$

the completion taken with respect to norm

$$\|x\| := \sqrt{\|\sum_{n=1}^\infty x_n^* x_n\|}.$$

More generally, if H is a Hilbert space, A is a C*-algebra, then the algebraic tensor product $A \otimes_{\mathbb{C}} H$ (of vector spaces) has a natural A-valued Hermitian form $\langle a \otimes \xi, b\rangle \otimes \eta\rangle := a^* b\langle \xi, \eta\rangle$. The completion of the algebraic tensor product is then a right Hilbert A-module $A \otimes_{\mathbb{C}} H$. (We develop tensor products of Hiilbert modules later – the notation is consistent.)

Fixing an isomorphism $H \cong l^2(\mathbb{N})$ determines an obvious unitary isomorphism of right Hilbert A-modules $A \otimes_{\mathbb{C}} H \cong A^{\mathbb{N}}$. We leave it to the reader to verify all of this.

We present a few more examples of a geometric origin below in the form of exercises.

EXAMPLE 4.7. Let G be a finite group. Let $\mathcal{E}_G$ be the following right Hilbert $C^*(G) \otimes C^*(G)$-module. The right module structure is given by

$$[a] \cdot ([h] \otimes [k]) := [h^{-1}ak], \quad a, h, k \in G, [a], \ etc \in \mathbb{C}G = C^*(G),$$

and inner product

$$\langle [a], [b]\rangle := \sum_{c \in G} [a^{-1}cb] \otimes [a^{-1}c].$$

EXERCISE 4.8. Prove that this all defines a right Hilbert $C^*(G) \otimes C^*(G)$ module.

Notice that $\langle 1, 1\rangle_{C^*(G) \otimes C^*(G)} = \sum_{g \in G} [g] \otimes [g]$. In fact the map $\Delta \colon C^*(G) \to C^*(G) \otimes C^*(G)$, $\Delta(f) := \langle f, f\rangle$ defines a co-multiplication on $C^*(G)$.

EXAMPLE 4.9. For $\xi, \eta \in C_c(\mathbb{R})$ let $\langle \cdot, \cdot\rangle$ be the following $C^*(\mathbb{Z})$-valued inner product:

$$\langle \xi, \eta\rangle := \sum_{n \in \mathbb{Z}} \Big(\int_{\mathbb{R}} \overline{\xi(x+n)} \eta(x) dx \Big) \cdot [n]. \tag{4.1}$$

Note that (4.1) is in the group algebra $\mathbb{C}[\mathbb{Z}]$, because f_i are compactly supported.

Give $C_c(\mathbb{R})$ the right $C^*(\mathbb{Z})$-module structure

$$(\xi \cdot [n])(x) := \xi(x+n). \tag{4.2}$$

Then $C_c(\mathbb{R})$ completes under this inner product to a right Hilbert $C^*(\mathbb{Z})$-module $\mathcal{E}_{\mathbb{Z},\mathbb{R}}$.

EXERCISE 4.10. Exhibit an explicit isomorphism $\mathcal{E}_{\mathbb{Z},\mathbb{R}} \cong L^2(\mathbb{R}) \otimes C^*(\mathbb{Z})$.

EXAMPLE 4.11. Let G be a discrete group acting *properly and co-compactly* on a locally compact Hausdorff space X.

Consider the linear space of continuous functions $f \colon X \to C^*_r(G)$, where $C^*_r(G)$ has the C*-algebra norm, such that $f(gx) = [g]f(x)$ for all $x \in X$. Here we are using group algebra notation, with $[g] \in \mathbb{C}G \subset C^*(G)$ the generator corresponding to $g \in G$.

Define for two such functions f_1, f_2, an element

$$\langle f_1, f_2\rangle \in C(G\backslash X) \otimes C^*_r(G) \cong C\big(G\backslash X, C^*_r(G)\big),$$

by

$$\langle f_1, f_2\rangle(\dot{x}) := f_1(x)^* f_2(x),$$

where $\dot{x}$ is the orbit of x. The expression on the right is well-defined, *i.e.* does not depend on the choice of x, since by the equivariance condition on the functions,

$$f_1(gx)^* f_2(gx) = \big([g]f_1(x)\big)^* \big([g]f_2(x)\big) = f_1(x)^*[g]^*[g]f_2(x) = f_1(x)^* f_2(x).$$

We define a right $C(G\backslash X) \otimes C^*(G)$-module structure by

$$(f \cdot g)(x) := f(x)[g], \quad f \cdot \varphi(x) := f(x)\varphi(\dot{x}).$$

EXERCISE 4.12. Check that the above determines a Hermitian right $C(G\backslash X) \otimes C^*_r(G)$-module.

Denote the associated right Hilbert $C(G\backslash X) \otimes C^*_r(G)$-module $\mathcal{E}_{G,X}$, sometimes called the *Mishenko* module.

Exercise 4.13. Prove that $\mathcal{E}_{G,X}$ is f.g.p. as a right $C(G\backslash X) \otimes C_r^*(G)$-module.

We now return to the general theory of Hilbert modules.

Lemma 4.14. *Let A be a semi-Hermitian right A-module with semi-Hermitian form $\langle \cdot, \cdot \rangle \colon E \times E \to A$. Then*

$$L(y,x)L(x,y) \leq \|L(x,x)\| \cdot L(y,y) \tag{4.3}$$

holds for all $x, y \in E$.

Remark 4.15. This fundamental Lemma includes the case of $A := \mathbb{C}$-sesquilinear forms, such as, for example, $E := A$, a C*-algebra, and $L(a,b) := \varphi(a^*b)$, where $\varphi \colon A \to \mathbb{C}$ is a state, in which case it asserts that

$$\varphi(a^*b) \leq \varphi(a^*a)\varphi(b^*b).$$

It contains the usual Hilbert space situation as well, of course, with the Hilbert space inner product.

Proof. Fix $x, y \in E$, $a \in X$ and $t \in \mathbb{R}$. Then the positivity condition on L gives

$$\begin{aligned} 0 \leq L(xa - ty, xa - ty) &= L(xa, xa) - L(ty, xa) - L(xa, ty) + L(ty, ty) \\ &= a^* L(x,x)a - t\big(L(y,x)a - a^* L(x,y)\big) + t^2 L(y,y). \end{aligned} \tag{4.4}$$

Now set $a = L(x,y)$, then we derive that $2ta^*a \leq a^* L(x,x)a + t^2 L(y,y)$, that is,

$$2tL(y,x)L(x,y) \leq L(y,x)L(x,x)L(x,y) + t^2 L(y,y) \tag{4.5}$$

In the case $L(x,x) = 0$, we get $2tL(y,x)L(x,y) \leq t^2 L(y,y)$ for all $t \in \mathbb{R}$, whence it is immediate that $L(y,x)L(x,y) = 0$ as well. Otherwise, suppose $L(x,x) \neq 0$. Since $a^*ba \leq \|b\| a^*a$ holds for any a, b in a C*-algebra, $L(y,x)L(x,x)L(x,y) \leq \|L(x,x)\| L(y,x)L(x,y)$ holds, and hence from (4.5) we get

$$2tL(y,x)L(x,y) \leq \|L(x,x)\| \cdot L(y,x)L(x,y) + t^2 L(y,y) \tag{4.6}$$

Now set $t = \|L(x,x)\|$, divide both sides by it, and we get

$$L(y,x)L(x,y) \leq \|L(x,x)\|\, L(y,y)$$

as required.

□

Theorem 4.16. *(The Cauchy-Schwartz inequality for semi Hermitian A-modules). If E, with A-valued inner product $\langle \cdot, \cdot \rangle$ is a semi Hermitian right A-module then*

$$\|\langle x, y \rangle\|^2 \leq \|\langle x, x \rangle\| \|\langle y, y \rangle\| \tag{4.7}$$

for all $x, y \in E$.

Exercise 4.17. If $E, \langle \cdot, \cdot \rangle$ is a Hermitian right A-module with, prove that $\|x\| := \|\langle x, x \rangle\|$ satisfies the triangle inequality. Hence it is a normed linear space, and can be completed, if necessary, to a Banach space.

In the semi-Hermitian case, $\|\cdot\|$ is a semi-norm, and we may mod out by zero-length vectors to get a Banach space.

Exercise 4.18. Let E be a semi-Hermitian right Hilbert A-module, $\|x\| := \|\langle x, x \rangle\|$ the induced seminorm on E.

a) Prove that for all $x \in E$, $a \in A$, $\|xa\| \leq \|x\| \|a\|$. Show by an example that the inequality may be strict even if $a \geq 0$ (but nonzero).

b) Prove that
$$\|x\| = \sup_{\|y\|\le 1} \|\langle x, y\rangle\|$$
for any $x \in E$.

Exercise 4.19. Let E be a right Hilbert A-module with support $J \subset A$.

a) Prove that J is a closed ideal in A.
b) Prove that if $a \in J$ then $xa = 0$ for all $x \in E$ implies $a = 0$. (*Hint.* Show that $a^*\langle x, y\rangle = 0$ for all $x, y \in E$ and deduce that $a^*J = 0$.)
c) Prove that if A is unital then $x \cdot 1 = x$ for all $x \in E$, where $1 \in A$ is the unit.
d) Prove that if (u_λ) is an approximate unit for the support ideal J then $\lim_{\lambda\to\infty} xu_\lambda = x$ for all $x \in E$.

Exercise 4.20. Let $\mathcal{E}$ be a right Hilbert A-module.

Prove that the right multiplication action of A on $\mathcal{E}$ extends to an action of the multiplier algebra $\mathcal{M}(A)$ on $\mathcal{E}$. (*Hint.* Show that if $(a_n) \subset A$ with $a_n \to a$ strictly, then$(\xi \cdot a_n)$ is a Cauchy sequence of vectors in $\mathcal{E}$.)

Exercise 4.21. Let p be a projection in a C*-algebra A. Prove that pA is a closed right ideal in A and hence is a right Hilbert A-module, with inner product $\langle pa, pb\rangle := a^*pb$. Check that its support is the closed (2-sided) ideal ApA generated by p.

Remark 4.22. The focus on (semi-) Hermiitian *right* modules in the discussion above is by convention; a (semi-) Hermitian *left* A-module is defined analogously to right modules; the (semi-)Hermitian form is then required to be conjugate linear in the second variable, linear in the first, and satisfy $a\langle x, y\rangle = \langle ax, y\rangle$, for all $a \in A, x, y \in E$, but othewise all axioms and the corresponding results (especially the Cauchy-Schwartz inequality) remain the same.

Exercise 4.23. Suppose $\pi\colon Y \to X$ is a smooth submersion between smooth manifolds. Let $V = \ker(D\pi) \subset TY$ be the 'vertical tangent bundle.' Then the inclusion $V \to TY$ of vector bundles restricts on each fibre $\pi^{-1}(x)$ to an isomorphism
$$V|_{\pi^{-1}(x)} \cong T\big(\pi^{-1}(x)\big).$$
Assume that V admits an orientation. Put an inner product on it. Then each fibre $\pi^{-1}(x)$ has in this way been endowed with the structure of a smooth, oriented manifold, and in particular has a volume form ω_x. Define a Hermitian form
$$\langle \cdot, \cdot\rangle\colon C_c^\infty(Y) \to C^\infty(X),\ \ \langle \xi, \eta\rangle(x) := \int_{\pi^{-1}(x)} \overline{\xi(y)}\eta(y)\omega_x.$$
Show that $C_c^\infty(Y)$ completes in this way to a right Hilbert $C_0(X)$-module $\mathcal{E}_\pi$.

Adjointable operators, compact operators

It turns out that for Hilbert module maps, bounded does *not* imply the existence of an adjoint, so this has to be assumed to obtain a C*-algebraic structure on the Hilbert module maps.

Definition 4.24. Let E_1 and E_2 be Hilbert A-modules. An A-module map $T\colon E_1 \to E_2$ is *adjointable* if there exists an A-module map $T^*\colon E_2 \to E_1$ such that
$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad \forall x \in E_1, y \in E_2. \tag{4.8}$$

The collection of adjointable operators $E_1 \to E_2$ is denoted $\mathbb{B}(E_1, E_2)$. When $E_1 = E_2$ we write $\mathbb{B}(E)$, which we will show shortly is a C*-algebra.

EXAMPLE 4.25. A multiplier $L\colon A \to A$ of a C*-algebra A is an adjointable A module map, with inverse L^*.

EXAMPLE 4.26. If $E = A^n$ the standard rank n Hilbert A-module, then any A-module map $T\colon A^n \to A^n$ has a matrix representation, and if T^* is the endomorphism corresponding to its conjugate transpose $(T^*)_{ij} := T^*_{ji}$, then it is easily checked that T^* is the adjoint of T.

Thus any A-module endomorphism of A^n is adjointable.

EXERCISE 4.27. Let A be a C*-algebra and T be an n-by-n matrix of multipliers T_{ij} of A. By using the 'obvious' adjoint, prove that matrix multiplication by T is an adjointable operator on A^n.

Returning to the general situation, the uniqueness of the adjoint of $T\colon E_1 \to E_2$, if it exists, follows from a standard argument from Hilbert space theory:

$$\langle x, (T_1^* - T_2^*)y\rangle = \langle x, T_1^* y\rangle - \langle x, T_2^* y\rangle = \langle Tx, y\rangle - \langle Tx, y\rangle = 0.$$

for two A-linear maps satisfying (4.8). In Lemma 3.6 we proved that multipliers are bounded; we leave it as an exercise to adapt the proof cosmetically to work for general adjointable operators.

EXERCISE 4.28. Using the Closed Graph Theorem along the lines of the proof of Lemma 3.6, prove that an adjointable operator $T\colon E_1 \to E_2$ is bounded in the respective Hilbert module norms: there exists $C \geq 0$ such that $\|Tx\| \leq C\|x\|$, $\forall x \in E_1$.

The operator norm $\|T\|$ for $T \in \mathbb{B}(E_1, E_2)$ an adjointable operator, is defined in the usual way by

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\|.$$

LEMMA 4.29. *Let A be any C*-algebra and E a right Hilbert A-module. Then $\mathbb{B}(E)$, with the operator norm, is a C*-algebra.*

PROOF. The proof works exactly the same as it does when $A = \mathbb{C}$, where we are talking about bounded operators on a Hilbert space. If T is adjointable, and $x \in E$ is a unit vector, then by the Cauch-Schwartz inequality

$$\|Tx\|^2 = \|\langle Tx, Tx\rangle\| = \|\langle T^*Tx, x\rangle\| \leq \|T^*Tx\| \leq \|T^*T\| \leq \|T^*\|\|T\|. \tag{4.9}$$

Thus $\|Tx\|^2 \leq \|T^*\|\|T\|$ for unit vectors x, and hence taking sups, we get

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\|. \tag{4.10}$$

Interchanging the roles of T and T^* gives that $\|T\| = \|T^*\|$ and making the corresponding adjustment to (4.10) we deduce without further ado that $\|T\|^2 = \|T^*T\|$.

$\square$

If E_1, E_2 are right Hillbert A-modules, and $x \in E_1, y \in E_2$, let

$$\theta_{x,y}\colon E_1 \to E_2,\ \ \theta_{x,y}(z) := y\langle x, z\rangle. \tag{4.11}$$

From the right A-linearity of the inner product, $\theta_{x,y}$ is right A-linear, *i.e.* is a module map, and clearly has range yA the rank one submodule of E_2 generated by y.

EXERCISE 4.30. Prove that if E is merely assumed a semi-Hermitian right A-module, with form $\langle\cdot,\cdot\rangle$, then the collection $\mathbb{B}(E)$ of adjointable A-linear operators on E is a pre-C*-algebra with semi-norm $\|T\| := \sup_{\|x\|\leq 1}\|Tx\|$, where $\|x\| := \|\langle x, x\rangle\|$ as usual. Hence $\mathbb{B}(E)$ can be completed in this case to a C*-algebra, by modding out by zero length vectors in the usual manner.

Deduce that $\|xb\| \leq \|x\|\|b\|$ for any $x \in E, b \in B$.

Exercise 4.31. Let E_1, E_2 be right Hilbert A-modules.

a) If $x \in E_2, y \in E_1$ then $\theta_{x,y}$ is adjointable and $\theta_{x,y}^* = \theta_{y,x}$.
b) If E_3 is a third right Hilbert A-module, $x \in E_3, y, x' \in E_2, y', z \in E_1$, then $\theta_{x,y} \circ \theta_{x',y'} = \theta_{x\langle y,x'\rangle,y'}$.
c) If E_3 is a third right Hilbert A-module, $T\colon E_2 \to E_3$ an adjointable operator, $x \in E_2, y \in E_1$, then $T \circ \theta_{x,y} = \theta_{Tx,y}$. Similarly, if $T\colon E_1 \to E_2$, $x \in E_3, y \in E_2$, then $\theta_{x,y} \circ T = \theta_{x,T^*y}$.
d) Prove that $\|\theta_{x,y}\| \leq \|x\|\|y\|$.

Exercise 4.32. If E is a right Hilbert B-module and $x \in E$, prove that the operator

$$T\colon E \to B, \;\; T(z) := \langle x, z\rangle$$

is an adjointable operator $E \to B$ between Hilbert B-modules, and that $T^*(b) = xb$. Check that $T^*T = \theta_{x,x}$ and that TT^* is multiplication by $\langle x, x\rangle$, and deduce that $\theta_{x,x} \geq 0$.

Definition 4.33. If E_1 and E_2 are Hilbert modules, a linear combination of operators $E_1 \to E_2$ of the form (4.11) is a *finite rank operator* $E_1 \to E_2$. A norm limit of finite rank operators $E_1 \to E_2$ is a *compact operator* $E_1 \to E_2$. The collection of compact operators $E_1 \to E_2$ is denoted $\mathcal{K}(E_1, E_2)$.

As with bounded operators we just write $\mathcal{K}(E)$ for the compact operators $E \to E$.

Exercise 4.34. Let $E_1 = E_2 = E$. Prove that the the collection of finite-rank operators $\sum \lambda_j \theta_{x_j,y_j}\colon E \to E$ (finite sum, $\lambda_j \in \mathbb{C}$, $x_j, y_j \in E$), is a *-subalgebra of $\mathbb{B}(E)$, an algebraic ideal in $\mathbb{B}(E)$. Deduce that $\mathcal{K}(E)$ is a closed ideal of $\mathbb{B}(E)$.

The previous exercise shows that $\mathcal{K}(E)$ is a closed ideal in $\mathbb{B}(E)$ and hence is a C*-algebra in its own right.

Proposition 4.35. *If A is a C*-algebra, regarded as a right Hilbert A-module, then $\mathcal{K}(A) = A$, and $\mathbb{B}(A) = \mathcal{M}(A)$.*

Proof. Firstly, adjointable operators on the right Hilbert A-module A are precisely multipliers of A by definition of 'multiplier'. So $\mathbb{B}(A) \cong \mathcal{M}(A)$. If $x, y \in A$, the rank-one operator $\theta_{x,y}\colon A \to A$ is the map $\theta_{x,y}(a) = xy^*a$, thus is multiplication by $xy^* \in A$, so it is in the image of (the isometric *-homomorphism) $A \to \mathcal{M}(A) = \mathbb{B}(A)$. Hence the closed span of the $\theta_{x,y}$ is in the image, so $\mathcal{K}(A)$ is contained in the image. Conversely, let $x \in A$, then we claim that the multiplier $L_x(a) := xa$ is compact. Let (u_λ) be an approximate unit for A, then we have

$$L_x(a) = \lim_{\lambda \to \infty} xu_\lambda a = \lim_{\lambda to \infty} \theta_{x,u_\lambda^*}(a)$$

and since $xu_\lambda \to x$ in norm, $\theta_{x,u_\lambda^*} = L_{xu_\lambda}$ converges in the norm topology to L_x. Hence every multiplier L_x with $x \in A$ is compact. This completes the proof.

□

Exercise 4.36. Let A be a C*-algebra.

a) Generalize the Proposition 4.35 and prove that $\mathcal{K}(A^n) \cong M_n(A)$ for all $n = 1, 2, \cdots$.
b) Prove that $\mathbb{B}(A^n) \cong M_n\big(\mathcal{M}(A)\big)$ (*c.f.* Exercise 4.27) for any $n = 1, 2, \ldots$. Hence if A is unital then
$$\mathbb{B}(A^n) \cong \mathcal{K}(A^n) \cong M_n(A).$$
c) Prove that $\mathcal{K}(A^{\mathbb{N}}) \cong \varinjlim_n M_n(A)$.
d) Prove that $\mathcal{K}(A^{\mathbb{N}}) \cong A \otimes \mathcal{K}$.
It is *not* true that $\mathbb{B}(A^{\mathbb{N}}) \cong \mathbb{B}(\mathbb{C}^{\mathbb{N}}) \otimes A$. (Why?)

The folowing result is very useful. It's proof is easy.

LEMMA 4.37. *Let A and D be C*-algebras and let $\rho\colon A \to \mathcal{M}(D)$ be a non-degenerate *-homomorphism. Then ρ induces a canonical *-homomorphism*

$$\mathbb{B}(A^{\mathbb{N}}) \to \mathbb{B}(D^{\mathbb{N}}), \tag{4.12}$$

which agrees with $\rho \otimes \mathrm{id}_{\mathcal{K}}$ on the closed ideal $A \otimes \mathcal{K} \cong \mathcal{K}(A^{\mathbb{N}})$. Moreover, (4.12) *is injective if ρ is injective.*

EXAMPLE 4.38. Let $\mathcal{E}_{\mathbb{Z},\mathbb{R}}$ be the right Hilbert $C^*(\mathbb{Z})$-module of Example 4.9: the completion of $C_c(\mathbb{R})$ with respect to the $C^*(\mathbb{Z})$-valued Hermitian form

$$\langle \xi, \eta \rangle := \sum_{n\in\mathbb{Z}} \langle \xi \cdot n, \eta\rangle [n] \in \mathbb{C}\mathbb{Z} \subset C^*(\mathbb{Z})$$

(using group-algebra notation). Here $\xi \cdot n$ denotes the function $(\xi\cdot n)(x) = \xi(x+n)$; this formula for $\xi \cdot n$ also determines the right $C^*(\mathbb{Z})$-module structure.

If $f \in C_c(\mathbb{R})$, let $\lambda(f)\colon \mathcal{E}_{\mathbb{Z},\mathbb{R}} \to \mathcal{E}_{\mathbb{Z},\mathbb{R}}$ be the map $\lambda(f)$ of convolution by f:

$$\lambda(f)\xi(x) := \int_{\mathbb{R}} f(y)\xi(x-y)dy.$$

EXERCISE 4.39. Prove the following.

a) Prove that if $f \in C_c(\mathbb{R}) \subset C^*(\mathbb{R})$, then $\lambda(f)$ is an adjointable right Hilbert $C^*(\mathbb{Z})$-module map, and that λ extends to a C*-algebra homomorphism $C^*(\mathbb{R}) \to \mathbb{B}(\mathcal{E}_{\mathbb{Z},\mathbb{R}})$.
b) Prove that, in fact, $\lambda(f)$ acts by a *compact* Hilbert $C^*(\mathbb{Z})$-module operator on $\mathcal{E}_{\mathbb{Z},\mathbb{R}}$, for all $f \in C^*(\mathbb{R})$.

   (*Note.* We will show that $\mathcal{E}_{\mathbb{Z},\mathbb{R}}$ can actually be given the structure of a $C(\mathbb{T}) \rtimes \mathbb{R}$ - $C^*(\mathbb{Z})$ strong Morita equivalence bimdodule, in Section 26. The left action by $C^*(\mathbb{R}) \subset C(\mathbb{T}) \rtimes \mathbb{R}$ in this bimodule context, is the same as that presented here. Since in the strong Morita equivalence context, the left algebra always acts by compact operators, the assertion of the Exercise would follow from this.)
c) Let $D = i\frac{d}{dx}$, acting on compactly supported functions in $C_c^\infty(\mathbb{R})$. Prove that $(1 + D^2)^{-1}$ is a compact Hilbert $C^*(\mathbb{Z})$-module map (it follows from part b).

The example is important in Index Theory, since it gives an example of where a natural differential operator (differentiation on the line in this case) on a *noncompact* space is Fredholm (in the sense that $(1+D^2)^{-1}$ is compact) when regarded not as acting on its natural Hilbert space, but rather when acting as a Hilbert *module* map, over an appropriate group C*-algebra, with a co-compact action on the space.

A remark on f.g.p. Hilbert modules

Finitely generated projective modules (f.g.p.) have already been studied in the case of commutative C*-algebras $A = C(X)$; they correspond to complex vector bundles over X, by Swan's theorem. Recall that a finitely generated right A-module E is *finitely generated projective* (or f.g.p.) if it is a direct summand, purely in the algebraic sense, of A^n, for some n, so that there exists an idempotent $e \in M_n(A)$ such that E is isomorphic to eA^n. We have already proved that every idempotent is similar to a projection, and so the module is isomorphic as a module to pA^n, for some projection $p \in M_n(A)$, which has the structure of a right Hilbert A-module as a subset of A^n, and which is also orthogonally complemented in A^n with orthogonal complement $(1-p)A^n$.

LEMMA 4.40. *Let E be a right Hilbert A-module. Then if the identity operator* $\mathrm{id}_E\colon E \to E$ *is compact, then E is f.g.p.*

PROOF. If the identity is compact, there exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in E$ such that $\|\mathrm{id}_E - \sum_i \theta_{x_i,y_i}\| < 1$. This makes the finite-rank operator $\sum_i \theta_{x_i,y_i}$ invertible. If $S \in \mathbb{B}(E)$ is its inverse, then $\mathrm{id}_E = S\sum_i \theta_{x_i,y_i} = \sum \theta_{Sx_i,y_i}$.

So we may as assume assume, after replacing x_i with Sx_i for each i, that $\sum_i \theta_{x_i,y_i} = \mathrm{id}_E$ to begin with. Therefore, we have the identity

$$x = \sum_i x_i\langle y_i, x\rangle \quad \forall x \in E. \tag{4.13}$$

Applying (4.13) to $x = x_j$ and taking the inner product with y_k give sthe identity

$$\langle y_k, x_j\rangle = \sum_i \langle y_k, x_i\rangle\langle y_i, x_j\rangle. \tag{4.14}$$

Let $p \in M_n(A)$ the matrix $p_{ij} := \langle y_i, x_j\rangle$. The above identity can be written

$$p_{kj} = \sum_i p_{ki}p_{ij}. \tag{4.15}$$

Hence p is an idempotent. We claim that the f.g.p. module pA^n is isomorphic to E. Let $T\colon E \to A^n$ by the map $T(x) = (\langle y_1, x\rangle, \ldots, \langle y_n, x\rangle)$. Let $S\colon A^n \to E$ be the map $s(a_1, \ldots, a_n) := \sum x_i a_i$. Then $TSa = pa$, as is easily checked, with p the projection matrix above, and any $a \in A$, and $STx = x$, for all $x \in E$, from (4.13). So T defines an isomorphism from E onto the direct summand pA^n of A^n.

□

EXERCISE 4.41. Give an example of an f.g.p. module over $C(S^2)$ which is not a free module.

Tensor products of Hilbert modules

Let E be a right Hilbert A-module, E' a right Hilbert B-module, and $\pi\colon A \to \mathbb{B}(E')$ a *-homomorphism.

The algebraic tensor product of modules in this situation is the quotient of the $\mathbb{C}$ vector space tensor product $E \otimes_{\mathbb{C}} E'$ (a right B-module), by the right B-submodule generated by the elements

$$xa \otimes y - x \otimes \pi(a)y.$$

In the algebraic setting, the tensor product just defined, is usually denoted $E \otimes_A E'$. But we will reserve this notation for it's completed version:

DEFINITION 4.42. In the above notation, with E a right Hilbert A-module, E' a right Hilbert B-module, and $\pi\colon A \to \mathbb{B}(E')$ a representation, we will let $E \otimes_A E'$ denote the completion of the algebraic tensor product of modules described above, with respect to the semi-Hermitian B-valued form

$$\langle x \otimes_A x', y \otimes_A y'\rangle_B := \langle \pi(\langle x, y\rangle_A)x', y'\rangle_B \tag{4.16}$$

EXERCISE 4.43. Check that (4.16) really is a semi-Hermitian B-valued form.

The completion $E \otimes_A E'$ is a right Hilbert A-module. We leave it to the reader to check the following:

Exercise 4.44. In the above notation, if $T \in \mathbb{B}(E)$, then there is a unique adjointable operator $T \otimes 1$ on $E \otimes_A E'$ such that

$$(T \otimes 1)(x \otimes_A y) = T(x) \otimes_A y, \ \ \forall x \in E, y \in E'.$$

The adjoint of $T \otimes 1$ is $T^* \otimes 1$.

Remark 4.45. It is clear that $T \mapsto T \otimes 1$ in fact defines a *-homomorphism $A \to \mathbb{B}(E \otimes_A E')$. In particular, it is contractive, and hence automatic that $\|T \otimes 1\| \leq \|T\|$, where the norms are the respective operator norms.

Exercise 4.46. Let $\mathcal{E}$ be a f.g.p. Hilbert A-module, *i.e.* an orthogonally complemented Hilbert submodule of A^n for some n. Let $\alpha \colon A \to \mathbb{B}(\mathcal{E}_B)$ be a representative of A as bounded adjointable operators on a right Hilbert B-module $\mathcal{E}_B$. Show that the algebraic tensor product $\mathcal{E}_A \otimes_A \mathcal{E}_B$ is already complete with respect to the tensor product Hermitian form defined above, so that the tensor product in the category of Hilbert modules is the same as the algebraic tensor product.

Example 4.47. A particular case of the tensor product construction shows that if A and B are C*-algebras and $\varphi \colon A \to \mathcal{M}(B)$ a *-homomorphism, then, regarding φ as a representation of A in $\mathbb{B}(B)$, (by multipliers) we can form, for any right Hilbert A-module $\mathcal{E}$, the tensor product $\mathcal{E} \otimes_A B$. This results in a Hilbert B-module: the 'pushforward' of $\mathcal{E}$ under φ.

Exercise 4.48. Suppose $\mathcal{E}_A$ is a right Hilbert A-module, that $\pi \colon A \to \mathbb{B}(\mathcal{E}'_B)$ is a non-degenerate representation as bounded adjointable operators on $\mathcal{E}'_B$, a right Hilbert B-module. The non-degeneracy implies that the representation extends to the multiplier algebra $\mathcal{M}(A)$.

By Exercise 4.20, the right multiplication action of A on $\mathcal{E}_A$ also extends to an action of the multiplier algebra $\mathcal{M}(A)$.

Prove that in the tensor product of Hilbert modules

$$\mathcal{E}_A \otimes_A \mathcal{E}'_B,$$

that the vectors

$$\xi \cdot a \otimes_A \eta - \xi \otimes_A a \cdot \eta$$

are zero even if $a \in \mathcal{M}(A)$. (*Hint.* Compute, for $a_n \to a$ strictly, $a_n \in A, a \in \mathcal{M}(A)$,

$$\|\xi \cdot a_n \otimes_A \eta - \xi \cdot a \otimes_A \eta\|^2 \to 0$$

directly, for vectors $\xi, \eta \in \mathcal{E}_A, \mathcal{E}'_B$), and $n \to \infty$.)

Exercise 4.49. Consider the right $C^*(\mathbb{Z})$-module $\mathcal{E}_{\mathbb{Z},\mathbb{R}}$ of Example 4.9. Every point $\omega \in \mathbb{T}$ determines a *-homomorphism $C^*(\mathbb{Z}) \to \mathbb{C}$. Tensoring over this *-homomorphism produces a right Hilbert $\mathbb{C}$-module, denoted $H_\omega := \mathcal{E}_{\mathbb{Z},\mathbb{R}} \otimes_\omega \mathbb{C}$, that is, a Hilbert space.

Prove that this 'bundle' of Hilbert spaces $\{H_\omega \mid \omega \in \mathbb{T}\}$ can described as follows. For each ω, H_ω is the Hilbert space completion of the space of continuous functions φ on $\mathbb{R}$ such that $\varphi(x+n) = \omega^n \varphi(x)$ for all integers n, and inner product

$$\langle \varphi, \psi \rangle := \int_0^1 \overline{\varphi(t)} \psi(t) dt.$$

Although all the Hilbert spaces H_ω are isomorphic to each other as Hilbert spaces, one cannot find a *continuous* (in ω) family of such isomorphisms.

Exercise 4.50. This exercise follows from Exercise 4.23 and give a geometric interpretation of certain tensor products of Hilbert modules occurring in index theory.

Assume X, Y, Y' are smooth manifolds. Let $\pi \colon Y \to X$ be a submersion with orientable fibres. Let $\mathcal{E}_\pi$ the Hilbert $C_0(X)$-module constructed in the cited exercise. Let $\pi' \colon Y' \to Z$ be another such submersion, $E_{\pi'}$ the corresponding right Hilbert $C_0(Z)$-module. And suppose that $b\colon Y' \to X$ is a smooth map which is *transverse* to $\pi\colon Y \to X$.

a) $Y \times_X Y' := \{(y, y') \mid \pi(y) = \pi'(y')\}$ is a smooth manifold of dimension $\dim Y + \dim Y' - \dim X$.
b) Let $\pi''\colon Y \times_X Y' \to Z$ the second projection map, restricted to $Y \times_X Y'$, followed by $\pi'\colon Y' \to Z$. Show that π'' is a smooth submersion with oriented fibres.
c) The map b determines a *-homomorphism $C_0(X) \to C_b(Y')$. By construction, there is a natural representation of $C_0(Y')$ in $\mathcal{E}_{\pi'}$. Hence we can take the product of Hilbert modules $\mathcal{E}_\pi \otimes_{C_0(X)} \mathcal{E}_{\pi'}$. Show that
$$\mathcal{E}_\pi \otimes_{C_0(X)} \mathcal{E}_{\pi'} \cong \mathcal{E}_{\pi''}$$
as right Hilbert $C_0(Z)$-modules.

EXERCISE 4.51. The following construction amounts to a type of GNS construction for Hilbert modules.

Let A be a unital C*-algebra and $\varphi\colon A \to \mathbb{C}$ a state. Suppose that $\mathcal{E}$ is a right Hilbert A-module.

a) Prove that the sesquilinear form $(\xi, \eta) := \varphi(\langle \xi, \eta\rangle)$ makes $\mathcal{E}$ into a semi-Hermitian right Hilbert $\mathbb{C}$-module. So it's completion is a Hilbert space $L^2(\mathcal{E}, \varphi)$.
b) Prove that any adjointable operator T on $\mathcal{E}$ determines a bounded operator on $L^2(\mathcal{E}, \varphi)$, so that we obtain a representation
$$\pi\colon \mathbb{B}(\mathcal{E}) \to \mathbb{B}\big(L^2(\mathcal{E}, \varphi)\big)$$
of the C*-algebra $\mathbb{B}(\mathcal{E})$ on the Hilbert space $L^2(\mathcal{E}, \varphi)$.
c) Prove that if φ is a *trace* then 'scalar' multiplication by elements of A determines a representation $\rho\colon A \to \mathbb{B}\big(L^2(\mathcal{E}, \varphi),\big)$ of the C*-algebra A (on the right; that is, it is a representation of the opposite algebra A^{op} of A.)
d) Check that with the assumption that φ is a trace, that the two actions (representations) of A and of $\mathbb{B}(\mathcal{E})$ commute. That is, $\mathbb{B}(\mathcal{E}) \subset A'$.

5. The definition of crossed-products via Hilbert modules

Hilbert modules give an excellent way of approaching completions of crossed-products, and we discuss this in this section.

Let G be a discrete group and A a G-C*-algebra. Form the twisted group *-algebra $A[G]$. In order to make $A[G]$ into a C*-algebra, we will produce a natural representation of A as bounded (adjointable) operators on the right Hilbert A-module $l^2(G, A) := \{(a_g)_{g\in G} \mid \|\sum_{g\in G} a_g^* a_g\| < \infty\}$, with right A-valued inner product $\langle a, b\rangle = \sum_{g\in G} a_g^* b_g$. Define a *-homomorphism
$$\lambda_{A,G}\colon A[G] \to \mathbb{B}(l^2(G, A)\big)$$
from the *-algebra $A[G]$ to the C*-algebra of adjointable operators on $l^2(G, A)$ by the covariant pair

$$\lambda_{A,G}(g)(a_h)_{h\in G} = (a_{g^{-1}h})_{h\in G}, \quad \lambda_{A,G}(a)(a_h)_{h\in G} := \big(h^{-1}(a)a_h)_{h\in G}. \tag{5.1}$$

It is left to the reader to check that
$$\lambda_{G,A}\big(h(a)\big) = \lambda_{A,G}(h)\lambda_{A,G}(a)\lambda_{A,G}(h)^*,$$
so that this is a covariant pair, and determines a *-homomorphism as required (see Exercise 9.5 e)).

We may now give a much better definition of the crossed-product:

DEFINITION 5.1. The *reduced crossed-product* $A \rtimes_r G$ of a discrete group G acting by automorphisms of a C*-algebra A is the completion of the pre-C*-algebra $(A[G], \|\cdot\|_{\lambda_{A,G}})$.

The covariant pairs discussed in Exercise 9.5 produce from such a pair, say with $\alpha\colon A \to B$ a *-homomorphism, $\pi\colon G \to \mathbf{U}(B)$ a group homomorphism, such that $\alpha\big(g(a)\big) = \pi(g)\alpha(a)\pi(g)^*$ for all $g \in G$, $a \in A$, give rise to 'integrated' *-homomorphisms, for which we temporarily use the awkward notation

$$\alpha \rtimes \pi\colon A[G] \to B$$

and defined in the obvious way by

$$(\alpha \rtimes \pi)(\sum a_g[g]) := \sum \alpha(a_g)\pi(g) \in B.$$

All *-homomorphisms $A[G] \to B$ have this form. And hence all *-homomorphisms $A \rtimes G \to B$ arise from extending such *-homomorphisms $A[G] \to B$ continuously to $A \rtimes G$.

Thus, all *-homomorphisms $A \rtimes G \to B$ arise from covariant pairs, but it is far from obvious which covariant pairs actually extend continuously from $A[G]$ to $A \rtimes G$. This issue is closely connected with amenability, and will be discussed elsewhere. Corollary 5.7 gives a nontrivial construction of covariant pairs which *do* extend as C*-algebra homomorphisms from $A[G]$ to $A \rtimes G$.

We note that obviously this 'extension problem' does not arise for finite groups. It doesn't hold for abelian groups as well, but this is harder to prove.

REMARK 5.2. As a general matter regarding cross-products, the *-homomorphism

$$\lambda_{A,G}\colon A[G] \to \mathbb{B}(l^2(G, A)\big)$$

maps the group algebra $A[G]$ into Hilbert A-module maps on $l^2(G, A)$. So we can represent the operator corresponding to a given element of $A[G]$ as a G-by-G matrix of elements of A, by trivializing the Hilbert A-module $l^2(G, A)$ in the standard way.

For example, if $a \in A$, its matrix is diagonal with $g^{-1}(a)$ in the (g, g)th coordinate. If $g \in G$ is a group element, and A is unital, so that we can view the elements of G as (unitary) elements of $A[G]$, then $\lambda_{A,G}(g)$ has constant g-diagonal (collection of entries of the form (h, hg)) with 1's along it, so has the form

$$S = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

Thus, the matrix representation of the general element of $A[G]$ (not of $A \rtimes G$, of course), has only finitely many diagonals.

EXERCISE 5.3. Write down a formula for the matrix representation of $\sum_{g \in F} a_g[g]$, with $F \subset G$ a finite subset.

We now return to the general theory.

PROPOSITION 5.4. *If $\alpha\colon A \to B$ is a G-equivariant *-homomorphism, then α extends uniquely to a *-homomorphism $\lambda(\alpha)\colon A \rtimes_r G \to B \rtimes_r G$.*

Moreover, $\lambda(\alpha)$ is injective if α is injective.

PROOF. The *-homomorphism

$$\alpha \otimes \mathrm{id}_{\mathcal{K}}\colon A \otimes \mathcal{K} \to B \otimes \mathcal{K}$$

is non-degenerate and extends to the multiplier algebras, so that we have a *-homomorphism, which we denote by $\overline{\alpha \otimes \mathrm{id}_{\mathcal{K}}}$ from $\mathbb{B}\big(l^2(G, A)\big) \to \mathbb{B}\big(l^2(G, B)\big)$.

Let $\lambda(\alpha)\colon A[G] \to B[G]$ be the *-homomorphism determined by α.

We claim that

$$\lambda_{G,B} \circ \lambda(\alpha) = (\overline{\alpha \otimes \mathrm{id}_{\mathcal{K}}}) \circ \lambda_{G,A}. \tag{5.2}$$

To check this, note that with respect to the obvious decomposition $l^2(G, B) = \oplus_{h \in G} B$, $\lambda_{G,A}(a)$ is a diagonal operator, and acts in the hth coordinate by left multiplication $A \to A$ by $h^{-1}(a)$. So $(\alpha \otimes \mathrm{id})\big(\lambda_{G,A} a\big)$ is the diagonal operator with hth entry $\alpha\big(h(a)\big)$. Since α is assumed equivariant, this $= h\big(\alpha(a)\big)$, so the operator we have described is $\lambda_{G,B}\big(\lambda(\alpha)(a)\big)$. This verifies the equation (5.2) for elements of A, and the remainder of the check (on the unitaries $[g] \in A[G]$, assuming A is unital, and the elements $a[g]$ otherwise) is left to the reader.

The equation (5.2) implies that

$$\|\lambda(\alpha)x\|_{B \rtimes_r G} := \|\big(\lambda_{G,B} \circ \lambda(\alpha)\big)(x)\| = \|(\overline{\alpha \otimes \mathrm{id}_{\mathcal{K}}})\big(\lambda_{G,A}(x)\big)\| \leq \|\lambda_{G,A}(x)\| \tag{5.3}$$

for all $x \in A[G]$, using the fact that any *-homomorphism is contractive. This proves the result.

For the injectivity statement, injectivity of α implies that of $\alpha \otimes \mathrm{id}_{\mathcal{K}}$ and then that of it's extension to a *-homomorphism $\mathbb{B}\big(l^2(G, A)\big) \to \mathbb{B}\big(l^2(G, B)\big)$. The result follows from (5.3).

□

EXERCISE 5.5. Prove that if A is unital then the (unique) unital *-algebra inclusion $\mathbb{C}[G] \to A[G]$ extends to a C*-algebra injection $C^*_r(G) \to A \rtimes_r G$.

The following result often helps to make crossed-products more concrete by realizing them as C*-subalgebras of bounded operators on a Hilbert space.

Let A be a G-C*-algebra.

Let $\rho\colon A \to B$ be a *-homomorphism, not necessarily equivariant, indeed, we do not assume that B has a G-C*-algebra structure at all. We produce a *-homomorphism

$$\mathrm{Ind}(\rho)\colon A[G] \to \mathbb{B}\big(l^2(G, B)\big)$$

by defining

$$\mathrm{Ind}(\rho)(a)(b \otimes e_h) := \rho\big(h^{-1}(a)\big)b \otimes e_h, \quad \mathrm{Ind}(g)(b \otimes e_h) := b \otimes e_{gh}.$$

It is easy to check that this defines a covariant pair into the C*-algebra $\mathbb{B}\big(l^2(G, B)\big)$, with G mapping into unitaries in this C*-algebra. We obtain a *-algebra homomorphism

$$\mathrm{Ind}(\rho)\colon A[G] \to \mathbb{B}\big(l^2(G, B)\big).$$

LEMMA 5.6. *In the above notation,* $\mathrm{Ind}(\rho) = (\rho \otimes \mathrm{id}_{\mathcal{K}}) \circ \lambda_{G,A}\colon A[G] \to \mathbb{B}\big(l^2(G, B)\big)$.

Hence $\mathrm{Ind}(\rho)$ *extends to a C*-algebra homomorphism* $A \rtimes_r G \to \mathbb{B}\big(l^2(G, B)\big)$*. It is injective if* ρ *is.*

PROOF. The identity $\mathrm{Ind}(\rho) = (\overline{\rho \otimes \mathrm{id}_{\mathcal{K}}}) \circ \lambda_{G,A}$ is easily checked on generators $a[g] \in A[G]$ and is left to the reader. This identity implies that $\|\mathrm{Ind}(\rho)\|_{\mathbb{B}\left(l^2(G,B)\right)} \leq \|x\|_{A \rtimes_r G}$ and hence that $\mathrm{Ind}(\rho)$ extends continuously to $A \rtimes_r G$ as claimed.

Injectivity in the case when ρ is injective follows from injectivity of $\rho \otimes \mathrm{id}_{\mathcal{K}}$.

□

COROLLARY 5.7. *If* $\rho\colon A \to \mathbb{B}(H)$ *is a representation of* A *on a Hilbert space, then the map* $\mathrm{Ind}(\rho)\colon A \rtimes_r G \to \mathbb{B}\big(l^2(G, H)\big)$ *is a representation of* $A \rtimes_r G$ *on* $l^2(G, H)$*. The latter is injective if* ρ *is injective.*

The proof consists merely of re-interpreting the target $\mathbb{B}\big(l^2(G,B)\big)$ of $\mathrm{Ind}(\rho)$ as $\mathbb{B}\big(l^2(G,H)\big)$, when $B := \mathbb{B}(H)$.

As a consequence,

COROLLARY 5.8. *Definitions 5.1 and 9.8 agree, that is, produce the same completion $A \rtimes G$ of $A[G]$, for any G, and any choice of injective representation involved in Definition 9.8.*

In many examples, there is an evident, or natural, representation of A on a Hilbert space H; the Ind construction hows to realize the crossed-product in a somewhat more concrete way as a C*-algebra of bounded operators on $l^2(G,H)$.

EXAMPLE 5.9. Let $A = C_0(X)$, where G acts on X, locally compact. To each orbit $Gx_0 \subset X$ of the action, we associate a representation

$$\pi_{x_0} \colon C_0(X) \rtimes G \to \mathbb{B}(l^2G),$$

of the crossed-product $C_0(X) \rtimes G)$ as follows. Let $\mathrm{ev}_{x_0} \colon C_0(X) \to \mathbb{C}$ be the *-homomorphism of evaluation of functions at x_0: a representation of $C_0(X0$ on a one-dimensional Hilbert space.

Applying the construction of $\mathrm{Ind}(\mathrm{ev}_{x_0}) \colon C_0(X) \rtimes G \to \mathbb{B}(l^2G)$ above we see that

$$\mathrm{Ind}(\mathrm{ev}_{x_0})(f)(e_h) := \mathrm{ev}_{x_0}\big(h^{-1}(f)\big) := \mathrm{ev}_{x_0}(f \circ h) = f\big(h(x_0)\big), \quad g(e_h) := e_{gh}.$$

Hence $\mathrm{Ind}(\mathrm{ev}_{x_0})$ is induced by the covariant pair

$$C_0(X) \to C_b(G) \subset \mathbb{B}(l^2G), \quad \text{Gelfand dual to the orbit map } \ G \to X, g \mapsto gx_0,$$
$$\text{and } G \to \mathbf{U}(l^2G), \text{the left regular representation.}$$

EXERCISE 5.10. Prove that if x_0 and x_1 are in the same G-orbit, then the representations π_{x_0} and π_{x_1} are unitarily equivalent representations of $C_0(X) \rtimes G$. (In many examples, in fact, the representations π_{x_0} and π_{x_1} are clearly not equivalent in general: they may not even have the same dimension, if the orbits are finite and of different size.)

EXERCISE 5.11. Prove that the representation π_{x_0} associated to an orbit of G acting on X is injective if and only if the orbit Gx_0 of x_0 is dense in X.

6. Morita equivalence

Let A and B be C*-algebras.

DEFINITION 6.1. A *Morita equivalence A-B-bimodule* is a linear space E which is both a Hermitian right B-module, with form $\langle \cdot, \cdot \rangle_B \colon E \times E \to B$, and a Hermitian *left* A-module, with form ${}_A\langle \cdot, \cdot \rangle \colon E \times E \to A$, such that the two module structures commute

$$(ax)b = a(xb), \ \ \forall a \in A, b \in B, x \in E,$$

the identities

$$(6.1) \qquad {}_A\langle x, y\rangle z = x\langle y, z\rangle_B, \quad \text{and} \ \ {}_A\langle x, yb\rangle = {}_A\langle xb^*, y\rangle, \text{ and } \langle ax, y\rangle_B = \langle x, a^*y\rangle_B,$$

hold for all $x, y, z \in E, a \in A, b \in B$, and such that the linear spans of ${}_A\langle E, E\rangle$ and of $\langle E, E\rangle_B$ are dense in A, B, respectively.

We emphasize that the inner products must also be Hermitian in the module sense: in the case of ${}_A\langle \cdot, \cdot \rangle$, it must be A-linear in the first variable, A-conjugate linear in the second. The analogous conditions must hold for $\langle \cdot, \cdot \rangle_B$ as well.

EXAMPLE 6.2. Let A be a C*-algebra. Then $M_n(A)$ is strongly Morita equivalent to $M_m(A)$ for any n, m.

Indeed, let $E = M_{n\times m}(A)$. If $S, T \in E$ then $ST^* \in M_n(A)$ and $S^*T \in M_m(A)$. This provides two inner products

$$_{M_n(A)}\langle \cdot, \cdot \rangle \colon E \times E \to M_n(A), \quad \text{and} \quad \langle \cdot, \cdot \rangle_{M_m(A)} \colon E \times E \to M_m(A)$$

and the conditions are easily checked to be met for this to give a strong Morita equivalence bimodule.

PROPOSITION 6.3. *Let E be any right Hilbert B-module. Then $\mathcal{K}(E)$ is strongly Morita equivalent to B.*

PROOF. For a strong Morita equivalence bimodule, we use E as a right Hermitian B-module, and set

$$\langle x, y \rangle_{\mathcal{K}(E)} := \theta_{x,y}.$$

If $T \in \mathcal{K}(E)$ (or more generally if $T \in \mathbb{B}(E)$) then $T\theta_{x,y} = \theta_{Tx,y}$ so that $\langle \cdot, \cdot \rangle_{\mathcal{K}(E)}$ is $\mathcal{K}(E)$-linear in the first coordinate. Since $\theta_{x,y} = \theta_{y,x}^*$, we get $\langle x, y \rangle_{\mathcal{K}(E)} = \langle y, x \rangle_{\mathcal{K}(E)}^*$. Exercise 4.32 implies that $\langle x, x \rangle_{\mathcal{K}(E)} \geq 0$ for all $x \in E$. Also, $\theta_{x,yb} = \theta_{xb^*,y}$ by an easy computation and hence $\langle x, yb \rangle_{\mathcal{K}(E)} = \langle xb^*, y \rangle_{\mathcal{K}(E)}$.

Finally, $\langle Tx, y \rangle_B = \langle x, T^*y \rangle_B$ since compact operators are adjointable, and the condition (6.1) follows from the definition of $\theta_{x,y}$. □

Returning to the general situation of a strong Morita equivalence A-B-bimodule, note that the proof of the Cauchy-Schwartz inequality for semi-Hermitian *right* A-modules (Lemma 4.14), works equally well for semi-Hermitian *left* A-modules, so that in the setting of Definition 6.1 it holds that

$$\|{}_A\langle x, y \rangle\| \leq \|{}_A\langle x, x \rangle\| \|{}_A\langle y, y \rangle\|,$$

as well as the corresponding statement

$$\|\langle x, y \rangle_B\| \leq \|\langle x, x \rangle_B\| \|\langle y, y \rangle_B\|.$$

for the B-valued Hermitian form.

Secondly, right scalar multiplication by $b \in B$ defines an adjointable operator $E \to E$ as a left Hermitian A-module, and it follows that

$$\|xb\|_A \leq \|x\|_A \|b\|.$$

as well as

$$\|ax\|_B \leq \|a\| \|x\|_B.$$

Coupling these two statements gives the following.

LEMMA 6.4. *If E is a strong Morita equivalence A-B-bimodule then*

$$\|{}_A\langle x, x \rangle\| = \|\langle x, x \rangle_B\|$$

for all $x \in E$.

PROOF. If $x \in E$ then

$$\begin{aligned} \|x\|_A^4 = \|\langle x, x \rangle_A\|^2 = \|\langle x, x \rangle_A \langle x, x \rangle_A\| &= \|\langle \langle x, x \rangle_A x, x \rangle_A\| \\ &= \|\langle x\langle x, x \rangle_B, x \rangle_A\| \leq \|x\|_A \cdot \|x\langle x, x \rangle_B\|_A \leq \|x\|_A^2 \|x\|_B^2, \end{aligned}$$

where we used the Cauchy-Schwartz inequality for $\langle \cdot, \cdot \rangle_A$, and the fact that $\|zb\|_A \leq \|z\| \|b\|$ for all $z \in E, b \in B$. Hence $\|x\|_A^2 \leq \|x\|_B^2$ and the result follows by switching the roles of A and B. □

In particular, if E is any strong Morita equivalence A-B-bimodule, then E can be completed to a Banach space using either $\|\cdot\|_A$ or $\|\cdot\|_B$, the result is the same, and is a right Hilbert B-module (as well as a left Hilbert A-module.)

PROPOSITION 6.5. *Let E be a strong Morita equivalence A-B-bimodule. Then $A \cong \mathcal{K}(E)$. by the left multiplication action of A on E, where here $\mathcal{K}(E)$ refers to the compact operators on E as a right Hilbert B-module.*

PROOF. Left multiplication by $a = \langle x, y\rangle_A \in A$ maps $z \in E$ to $\langle x, y\rangle_A z = x\langle y, z\rangle_B = \theta_{x,y}(z)$. So left multiplication by $a \in A$ of this form is a compact operator. Since the span of the $\langle x, y\rangle_A$ is dense in A, the result follows. $\square$

PROPOSITION 6.6. *Let E be a strong Morita equivalence A-B-bimodule with A unital. Then the completion of E to a right Hilbert B module is f.g.p., as a right B-module.*

PROOF. Since E is full as a right B-module, there exist (see Exercise below) $x_1, \dots x_n$ such that $\sum_{i=1}^n \langle x_i, x_i\rangle_B = 1_B$. Let

$$\Phi\colon A \to M_n(B), \;\; \Phi(a)_{ij} := \langle ax_i, x_j\rangle_B.$$

We leave it to the reader to verify that Φ is a *-isomorphism from A onto $pM_n(B)p$, where $p := \Phi(1)$. $\square$

The following is sometimes useful to explicitly compute the K-theory maps induced by a strong Morita equivalence.

COROLLARY 6.7. *Let A and B be strongly Morita equivalent by an A-B-equivalence bimodule E. Assume A is unital. Let $p \in M_n(B)$ a projection such that $E = \mathrm{im}(p)$ as a right Hilbert B-module. (A construction of such p is done in the proof of Proposition 6.6 above).*

Then A is isomorphic to the corner $pM_n(B)p$ of $M_n(B)$.

PROOF. As a right Hilbert B-module $E \oplus E' \cong B^n$ for some n and some right Hilbert B-module E', and the projection $p \in M_n(B)$ appearing in the statement corresponds under this isomorphism to projection to E. By Proposition 6.5, the left multiplication action of A on E gives an isomorphism $A \cong \mathcal{K}(E)$, while clearly $\mathcal{K}(E) = p\mathcal{K}(B^n)p = pM_n(B)p$. $\square$

EXERCISE 6.8. Let A be a C*-algebra. A *full corner of A* is a C*-subalgebra of the form pAp, where p is a projection in A for which the ideal generated by p in A is all of A.

a) Prove that the ideal generated by p is the closure of ApA.
b) Prove that if B is a full corner of A, $B = pAp$, then Ap with inner products

$$_A\langle ap, bp\rangle := apb^*p, \quad \langle ap, bp\rangle_B := pa^*b$$

and evident left A-module structure, and right B-module structure, is a strong Morita equivalence between A and B.
c) Prove that A can be realized as a full corner of $\mathcal{K}(A^{\mathbb{N}}) \cong A \otimes \mathcal{K}$ using any rank 1 projection $p \in \mathcal{K}$. Compare the present proof to the one given in Proposition 6.5, with $\mathcal{E} = A^{\mathbb{N}}$.

We close this section with a discussion of compositions of strong Morita equivalences. We first show that the relation of being strong Morita equivalent is symmetric.

Let E be a strong Morita equivalence A-B-bimodule.

LEMMA 6.9. *Let E be a strong Morita equivalence A-B-bimodule with A-valued inner product ${}_A\langle\cdot,\cdot\rangle$ and B-valued inner product $\langle\cdot,\cdot\rangle_B$.*

Let E^* be E as an additive group, but with the conjugate $\mathbb{C}$-multiplication $\lambda x := \bar{\lambda}x$.

Then E^* together with the B-A-bimodule structure

$$bxa := a^*xb^*,$$

and inner products

$${}_B\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_B, \quad \langle\cdot,\cdot\rangle_A :={}_A \langle\cdot,\cdot\rangle,$$

is a strong Morita equivalence B-A-bimodule.

LEMMA 6.10. *Let E be a strong Morita equivalence A-B-bimodule, and E' be a strong Morita equivalence B-C-bimodule.*

Let $E \otimes_B E'$ be the tensor product as defined in Definition 4.42, with $\pi\colon B \to \mathbb{B}(E')$ the left multiplication action of B on E'. Then the left A action on $E \otimes_B E'$, $a(x \otimes x') := ax \otimes x'$, and Hermitian A-valued form

$$ {}_A\langle x \otimes x', y \otimes y'\rangle := {}_A\langle x, y \cdot {}_B\langle x', y'\rangle\rangle \tag{6.2}$$

makes $E \otimes_B E'$ into a strong Morita equivalence A-C-bimodule.

This category turns out to be the one we are more interested in than the category of C*-algebras and *-homomorphisms, because of the following Theorem.

THEOREM 6.11. *Strongly Morita equivalent C*-algebras have naturally homeomorphic primitive ideal spaces, spaces of irreducible representations, and tracial state spaces.*

We will later show that strongly Morita equivalences have naturally isomorphic K-theory groups as well.

In other words, strong Morita equivalent C*-algebras have all of the important C*-algebraic structural data exactly the same. They should be considered, for most purposes of extracting invariants, equivalent objects.

7. Hilbert modules from proper actions of discrete groups

Strong Morita equivalence and proper actions

THEOREM 7.1. *If G is a discrete group acting properly and freely on X, then $C_0(X) \rtimes G$ is strongly Morita equivalent to $C_0(G\backslash X)$.*

To prove the theorem, we construct an appropriate strong Morita equivalence bimodule. For much of the construction, we will not need that the action is *free*, only that it is proper.

Let then G act *properly* on X.

Start with the linear space $C_c(X)$. We denote elements by ξ, η.. and so on. Denote by $\dot{x}$ the image of $x \in X$ in the quotient space $G\backslash X$.

We give $C_c(X)$ a left module structure over $C_0(G\backslash X)$ by

$$(f\xi)(x) := f(\dot{x})\xi(x), \tag{7.1}$$

and left $C_0(G\backslash X)$-valued Hermitian form

$$ {}_{C_0(G\backslash X)}\langle\xi,\eta\rangle(\dot{x}) := \sum_{g\in G} \xi\big(g^{-1}x\big)\overline{\eta}\big(g^{-1}x\big). \tag{7.2}$$

By the properness assumption, there are only finitely many $g \in G$ such that $g^{-1}x \in \operatorname{supp}(\xi)$, because the support of ξ is compact, so the sum is finite.

Note also that the positivity condition ${}_{C_0(G\backslash X)}\langle \xi, \xi\rangle \geq 0$ for any ξ, is met, and $\xi \neq 0$ implies the inequality is strict.

Exercise 7.2. Verify that

$$\langle f \cdot \xi, \eta\rangle_{C_0(G\backslash X)} = f\langle \xi, \eta\rangle_{C_0(G\backslash X)}.$$

for all $\xi, \eta \in C_c(X)$, $f \in C_c(G\backslash X)$, that

$$\langle \xi, \eta\rangle_{C_0(G\backslash X)} = \langle \eta, \xi\rangle^*_{C_0(G\backslash X)}$$

and that $\langle \xi, \xi\rangle_{C_0(G\backslash X)} \geq 0$ for all ξ (and is > 0 for $\xi \neq 0$.)

Next, we define a right $C_0(X) \rtimes_r G$-module structure and inner product by

$$\xi \cdot g := g^{-1}(\xi) := \xi \circ g,\ (\xi f) := \xi \cdot f, \quad \langle \xi, \eta\rangle_{C_0(X)\rtimes_r G} := \sum_{h\in G} \bar{\xi} \cdot h(\eta)\,[h], \tag{7.3}$$

where $\xi, \eta \in C_c(X)$, $g \in G, f \in C_c(X)$. The last sum is finite by properness, since there are only finitely many $h \in G$ such that $\mathrm{supp}\xi \cap h^{-1}\big(\mathrm{supp}((\xi)\big) \neq \emptyset$.

Exercise 7.3. Verify that if $\xi, \eta, f \in C_c(X)$, and $g \in G$, then

$$\langle \xi, \eta \cdot g\rangle_{C_0(X)\rtimes_r G} = \langle \xi, \eta\rangle_{C_0(X)\rtimes_r G} \cdot [g] \in C_0(X) \rtimes_r G$$

and that

$$\langle \xi, \eta \cdot f\rangle_{C_0(X)\rtimes_r G} = \langle \xi, \eta\rangle_{C_0(X)\rtimes_r G} f \in C_0(X) \rtimes_r G,$$

if $f \in C_c(X)$. Hence $\langle \xi, \eta b\rangle_{C_0(X)\rtimes_r G} = \langle \xi, \eta\rangle_{C_0(X)\rtimes G} b$ for all $b \in C_c(X) \rtimes_r G$.

The *positivity* of the $C_0(X) \rtimes_r G$-valued inner product is more subtle. We need to show that if $\varphi \in C_c(X)$ then

$$C_0(X) \rtimes_r G \ni \langle \varphi, \varphi\rangle_{C_0(X)\rtimes_r G} \geq 0$$

Now

$$\langle \varphi, \varphi\rangle_{C_0(X)\rtimes_r G} = \sum_{g\in G} \bar{\varphi} g(\varphi)\,[g]. \tag{7.4}$$

To see why (7.4) is a positive element of $C_0(X) \rtimes G$, let us start with a *finite* group action. This is convenient because we can represent elements of $C_0(X) \rtimes_r G = C_0(X)[G]$ as a C*-subalgebra of adjointable operators on $l^2(G, C_0(X))\big)$ by G-by-G-matrices of multipliers of $C_0(X)$, by the representation (5.1).

In this notation, we compute the entries of the matrix T corresponding to $\langle \varphi, \psi\rangle_{C_0(X)\rtimes_r G}$. These entries are multipliers of $C_0(X)$ (elements of $C_0(X)$ in this case), and since the operator $\langle \varphi, \psi\rangle_{C_0(X)\rtimes_r G}$ maps the vector $fe_g \in l^2\big(G, C_0(X)\big)$ (for $f \in C_0(X)$) to the vector

$$\sum_k (kg)^{-1}\big(\overline{\psi} k(\psi)\big) f e_{kg},$$

taking the Hilbert module inner product with the vector e_h therefore describes the matrix: it has h, gth coordinate

$$T_{h,g} = h^{-1}(\overline{\psi}) g^{-1}(\psi).$$

Note that $T^*_{g,h} = T_{hg}$.

Now let E be the G-by-G matrix with all 1's, determining an adjointable operator on $l^2\big(G, C_0(X)\big)$, let $M_G(\varphi)$ the diagonal matrix with gth entry $g^{-1}(\varphi)$, and $M_G(\psi)$ similarly defined. Then it is a matter of elementary matrix multiplication to verify that T can be factored as

$$T = M_G(\overline{\varphi})^* \cdot E \cdot M_G(\psi).$$

Now E is a multiple of a projection: $P := \frac{1}{n}E$. We derive the formula

$$\langle \varphi, \psi\rangle_{C_0(X)\rtimes_r G} = nM_G(\varphi)^* PM_G(\psi) = n\big(PM_G(\varphi)\big)^*\big(PM_G(\psi)\big),$$

whence positivity is immediate if $\varphi = \psi$.

For the general case, we introduce the following useful notion.

DEFINITION 7.4. If $G \times X \to X$ is a proper action of a discrete group, a *cutoff function* for the action will be a continuous function $\rho \in C_b(X)$ such that $0 \leq \rho \leq 1$ everywhere, and

$$\sum_{g\in G} g(\rho)^2 = 1. \tag{7.5}$$

Note that if $G\backslash X$ is compact, and we take a cutoff function of compact support, then as an element of the bimodule we are building, its defining condition (7.5) can be written

$$_{C(G\backslash X)}\langle \rho, \rho\rangle = 1_{C(G\backslash X)}.$$

EXERCISE 7.5. Prove that the collection of cutoff-functions is a convex space, and that a path between any two cutoff functions gives rise to a path of projections in $C_0(X) \rtimes G$ between the projections associated by (7.6) to them.

LEMMA 7.6. *If $K \subset X$ is any compact subset, then there exists a cutoff function ρ which equals 1 on K. If $G\backslash X$ is compact, such ρ may also be taken to have compact support.*

If ρ is any cutoff function, then

$$P_\rho := \sum_{g\in G} \rho g(\rho)\,[g] \tag{7.6}$$

is a projection in the multiplier algebra of $C_0(X) \rtimes_r G$. If ρ has compact support, then P_ρ is a projection in $C_0(X) \rtimes G$.

REMARK 7.7. If G is finite and X is compact, then the constant function $\rho := \frac{1}{|G|}$ is a cutoff-function. The associated projection (7.6) is the projection $\frac{1}{|G|}\sum_{g\in G}[g] \in \mathbb{C}[G] \subset C(X) \rtimes G$.

LEMMA 7.8. *Let $\varphi, \psi \in C_c(X)$. Let $K \subset X$ be a compact subset containing* $\mathrm{supp}(\varphi) \cup \mathrm{supp}(\psi)$*, and ρ be a cutoff function which is 1 on K.*

Then the equality

$$\varphi^* P_\rho \psi = \langle \varphi, \psi\rangle_{C_0(X)\rtimes G},$$

holds in $C_0(X) \rtimes_r G$, where, on the left-hand-side, we are considering φ, ψ as elements of $C_0(X) \rtimes G$, by the inclusion $C_c(X) \subset C_0(X) \rtimes G$.

PROOF. Since $\rho = 1$ on $\mathrm{supp}(\varphi) \cup \mathrm{supp}(\psi)$,

$$\overline{\varphi} P_\rho \psi = \sum_{g\in G} \overline{\varphi}\rho g(\rho) g(\psi)[g] = \sum_{g\in G} \overline{\varphi} g(\rho\psi)[g] = \sum_{g\in G} \overline{\varphi} g(\psi)[g] = \langle \varphi, \psi\rangle_{C_0(X)\rtimes G}. \tag{7.7}$$

$\square$

The Lemma immediately implies the positivity result we are looking for, since now for $\varphi \in C_c(X)$ and ρ a cutoff function which is 1 on the support of φ, we have

$$\langle \varphi, \varphi\rangle_{C_0(X)\rtimes_r G} = \overline{\varphi} P_\rho \varphi = (P_\rho\varphi)^*(P_\rho\varphi) \geq 0 \in C_0(X) \rtimes_r G.$$

We leave it to the reader to check that the two inner products just defined satisfy

$$\varphi_1\langle \varphi_2, \varphi_3\rangle_{C_0(X)\rtimes_r G} = _{C_0(G\backslash X)}\langle \varphi_1, \varphi_2\rangle \cdot \varphi_3.$$

Furthermore, the $C_0(G\backslash X)$-valued inner product, has dense span, as one sees as follows. Let ρ be a cutoff function, with or without compact support. Then if $f, f' \in C_c(G\backslash X)$, then

$$f\overline{f'} = _{C_0(G\backslash X)}\langle f\rho, f'\rho\rangle$$

follows from the defining condition for ρ. This proves density as required.

Corollary 7.9. *Suppose that the discrete group G acts properly on X. Let $\mathcal{E}$ be equipped with its left $C_0(G\backslash X)$-module structure and $C_0(G\backslash X)$-valued inner product, it's right $C_0(X)\rtimes G$-module structure and $C_0(X)\rtimes G$-valued inner product, defined in* (7.1), (7.2), *and* (7.3). *Then:*

a) *If G acts freely on X, then the range of the $C_0(X)\rtimes G$-valued inner product is dense, and $\mathcal{E}$ is a $C_0(G\backslash X)$-$C_0(X)\rtimes G$ strong Morita equivalence bimodule.*
b) *If the action is not free, the range of the $C_0(X)\rtimes G$-valued inner product is an ideal in $C_0(X)\rtimes G$, and $\mathcal{E}$ gives a Morita equivalence between this ideal, and $C_0(G\backslash X)$.*

Corollary 7.10. *If G is discrete and acts properly and co-compactly on X, then the right $C_0(X)\rtimes G$-Hilbert module $\mathcal{E}$ of Corollary 7.9 is finitely generated projective.*

As a f.g.p. module over $C_0(X)\rtimes G$, it isomorphic to $p_\rho\cdot C_0(X)\rtimes_r G$ for any cutoff function ρ with compact support, where p_ρ is the projection (7.6).

In order to describe more explicitly the ideal given by the range of the inner product

$$\langle\cdot,\cdot\rangle_{C_0(X)\rtimes G}\colon \mathcal{E}\to C_0(X)\rtimes G,$$

we start by considering the relatively simple case of a homogeneous G space G/H. For these purposes, we recall the structure of the group C*-algebra $C^*(H)$, where H is a finite group.

Let $\widehat{H}$ denote the dual of H. Then we have shown that for each (equivalence class of) irreducible representation $[\sigma]\in\widehat{H}$, $\sigma\colon H\to \mathbf{U}(V_\sigma)$, there is a direct summand $L^2(H)_\sigma$ of the left regular representation $\lambda\colon H\to\mathbf{U}(l^2H)$, isomorphic to $n_\sigma := \dim(V_\sigma)$ copies of σ. Since each such summand is $\lambda(H)$-invariant, the C*-algebra $C^*(H)$ splits into a direct sum

$$C^*(H)\cong \oplus_{[\sigma]\in\widehat{H}} M_{n_\sigma}(\mathbb{C}) = \oplus_{[\sigma]\in\widehat{H}}\mathcal{K}(V_\sigma) \tag{7.8}$$

of matrix algebras,.

More exactly, $L^2(H)_\sigma\cong V_\sigma\otimes V_\sigma^*$, identifying the action of $\lambda(h)$ on $L^2(H)_\sigma$, with $\sigma(h)\otimes 1_{V_\sigma^*}$ acting on $V_\sigma\otimes V_\sigma^*$.

The projections to the various summands in this decomposition $C^*(H)$ correspond to the representations themselves.

For example, let

$$\epsilon\colon H\to\mathbb{C}$$

be the trivial representation. Since it is 1-dimensional, it contributes a 1-dimensional summand of the regular representation, given by the subspace of constant functions on $L^2(H)$. If

$$e := \frac{1}{|H|}\sum_{h\in H}[h]\in C^*(H)$$

then e projects to this summand. It is a central projection in $C^*(H)$. It generates a C*-subalgebra $\mathbb{C}\cong(\mathbb{C}e\subset C^*(H)$, which is a 1-dimensional summand

$$e\,C^*(H)\,e\cong\mathbb{C}e$$

of $C^*(H)$.

Let I_ϵ be the ideal in $C^*(H)$ corresponding, in the direct sum decomposition (7.8) to $\widehat{H}$-tuples supported in the ϵ-coordinate. By the definitions,

$$C^*(H)/I_\epsilon\cong\oplus_{[\sigma]\neq\epsilon}\mathcal{K}(V_\sigma)$$

Now let H be finite in G, and consider the crossed-product $C_0(G/H)\rtimes G$. We have shown that a section $s\colon G/H\to G$ of the quotient map determines a unitary,

$$U\colon l^2(G/H)\otimes l^2(H)\to l^2(G),$$

defined in terms of the standard bases as

$$U(e_{gH} \otimes e_h) := e_{gh}.$$

The unitary conjugates the right translation action of H on $l^2(G)$ to the action given by the representation

$$1 \otimes \rho \colon H \to \mathbf{U}\big(l^2(G/H) \otimes l^2(H)\big).$$

where ρ is the right regular representation of H on $l^2(H)$.

From this it follows that conjugation by U gives an isomorphism

$$\mathcal{K}(l^2G)^H \cong \mathcal{K}\big(l^2(G/H)\big) \otimes \mathcal{K}(l^2H)^H = \mathcal{K}\big(l^2(G/H)\big) \otimes C^*(H). \tag{7.9}$$

And $\mathcal{K}(l^2G)^H \cong C_0(G/H) \rtimes G$. Thus, we have an isomorphism

$$C_0(G/H) \rtimes G \cong \mathcal{K}\big(l^2(G/H)\big) \otimes C^*(H), \tag{7.10}$$

and hence the direct sum decomposition

$$C^*(H) \cong \oplus_{[\sigma]\in\widehat{H}} \mathcal{K}\big(l^2(V_\sigma)\big)$$

leads to a direct sum decomposition

$$C_0(G/H) \rtimes G \cong \oplus_{[\sigma]\in\widehat{H}} \mathcal{K}\big(l^2(G/H)\big) \otimes \mathcal{K}(V_\sigma) \cong \oplus_{[\sigma]\in\widehat{H}} \mathcal{K}\big(l^2(G/H) \otimes V_\sigma\big). \tag{7.11}$$

The summand corresponding to the trivial representation ϵ determines an ideal $\tilde{I}_\epsilon \subset C_0(G/H) \rtimes G$ consisting of $\widehat{H}$-tuples of compact operators supported in the ϵ coordinate, and we have a natural isomorphism

$$C_0(G/H) \rtimes G \;/\; \tilde{I}_\epsilon \cong \oplus_{[\sigma]\in\widehat{H}, [\sigma]\neq\epsilon} \mathcal{K}\big(l^2(G/H) \otimes V_\sigma\big). \tag{7.12}$$

Now let G act properly on X. For each $x \in X$, restriction defines a *-homomorphism

$$\lambda_x \colon C_0(X) \rtimes G \to C_0(Gx) \rtimes G \cong C_0\big(G/\mathrm{Stab}_G(x)\big) \rtimes G.$$

Since the action is proper, $\mathrm{Stab}_G(x)$ is finite. Applying the above discussion yields an ideal

$$\tilde{I}_\epsilon(x) \subset C_0(Gx) \rtimes G.$$

Definition 7.11. If G acts properly on X, then we denote by J_X the ideal in $C_0(X) \rtimes G$ given by

$$J_X := \{a \in C_0(X) \rtimes G \mid \lambda_x(a) \in \tilde{I}_\epsilon(x),\ \forall x \in X\}.$$

Remark 7.12. The ideal J_X has already appeared in Exercise 7.31. If we think of elements of the crossed-product as functions on $X \times G$, then the ideal J_X is the closure in $C_0(X) \rtimes G$ of the algebraic *-ideal of functions $f \in C_c(X \times G)$ satisfying

$$f(x, hg) = f(x, g) \ \ \forall h \in \mathrm{Stab}_G(x),\ \forall g \in G,\ \forall x \in X.$$

Exercise 7.13. Let ρ be a cut-off function, and P_ρ the corresponding projection in $C_0(X) \rtimes G$.

a) Prove that $P_\rho \in J_X$.
b) Prove that J_X is generated as an ideal by P_ρ.

We summarize what we have proved for proper actions with possible isotropy.

Corollary 7.14. *If G acts properly on X, then $C_0(G/X)$ is Morita equivalent to the ideal J_X in $C_0(X) \rtimes G$ generated by the projection P_ρ associated to any cutoff function on X.*

Equivariant vector bundles and Hilbert modules over crossed-products

Some of the constructions used in above extend rather naturally in a way we will explain, to where one has an 'equivariant vector bundle' over a proper G-space. These ideas will come up again in an important way when we discuss K-theory of crossed-products (by proper actions.)

Let G be any locally compact group. Recall that a *G-equivariant vector bundle* $\pi\colon E \to X$ over a G-space X is a vector bundle $\pi\colon E \to X$ which is also equipped with a group action for which

a) $\pi\colon E \to X$ is a G-equivariant map.
b) The action of any $g \in G$ maps the fibre E_x linearly to the fibre E_{gx}, for any $x \in X$.

.

Exercise 7.15. If G acts properly on X and E is a G-equivariant vector bundle on X, then there is a Hermitian metric on E making the G-action fibrewise unitary.

The following is a very important example of a G-equivariant vector bundle over $\mathbb{CP}^1$, where G is a *compact* group.

Example 7.16. Let G be the group $\mathbf{SU}_2(\mathbb{C})$ of 2-by-2 unitary matrices with determinant 1. Clearly G acts (lineatly) on $\mathbb{C}^2$, so there is an induced action of G on the space $\mathbb{CP}^1$ of lines in $\mathbb{C}^2$. If $L \subset \mathbb{C}^2$ is a line, spanned, say, by a vector $(z, w) \in \mathbb{C}^2$, then $g(L)$ is the line spanned by the vector $g \cdot (z, w)$. Moreover, g maps the line L to the line $g(L)$ linearly, in the obvious way, sending $\lambda \cdot (z, w) \in L$ to $\lambda \cdot g(z, w) \in g(L)$. So the Hopf bundle over $\mathbb{CP}^1$, carries a natural structure of a G-equivariant vector bundle over the G-space $\mathbb{CP}^1$.

Exercise 7.17. The above G-action may be understood in the following way. Let G act on the extended complex plane $\mathbb{C} \cup \{\infty\}$ by letting $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ act by

$$g(z) := \frac{az + b}{cz + d},$$

if $z \in \mathbb{C}$, and if $cz + d \neq 0$. Show that this formula extends to give a well-defined map $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. Show that the standard identification of $\mathbb{C} \cup \{\infty\}$ with $\mathbb{CP}^1$ (sending $z \in \mathbb{C}$ to the line in $\mathbb{C}^2$ spanned by $(z, 1)$, and sending ∞ to the line spanned by $(1, 0)$), identifies this action with the one described in terms of projectivisation above.

We now construct a right Hilbert $C_0(X) \rtimes G$-module $\mathcal{E}_E$ from any G-equivariant vector bundle on a *proper* G-space X. We set $\Gamma(E)$ to be the linear space of compactly supported sections of E. Give it the right $C_0(X) \rtimes G$-module structure by letting

$$(\xi \cdot g)(x) := g^{-1} \cdot \xi(gx), \quad (\xi \cdot f)(x) := \xi(x) f(x).$$

It is easily checked these formulas satisfy a covariance condition and extend to a right module multiplication by $C_0(X) \rtimes G$. We define an inner product valued in $C_0(X) \rtimes G$ by

$$\langle \xi, \eta \rangle_{C_0(X) \rtimes G}(x, g) := \langle \xi(x), g \cdot \eta(g^{-1}x) \rangle,$$

where the inner product on the right hand side is the Hermitian inner product on E.

Definition 7.18. Let G be a locally compact group acting properly on X. Then the completion of the pre-Hilbert $C_0(X) \rtimes G$-module $\Gamma(E)$ according to the module structure and inner product above, to a right Hilbert $C_0(X) \rtimes G$-module, is denoted $\mathcal{E}_E$.

PROPOSITION 7.19. *If G is a locally compact group acting properly on X, with $G\backslash X$ is compact, then $\mathcal{E}_E$ is f.g.p. over $C_0(X) \rtimes G$. That is, the identity operator $\mathcal{E}_E \to \mathcal{E}_E$ is a compact Hilbert $C_0(X) \rtimes G$-module operator.*

Hence $\mathcal{E}_E$ is a f.g.p. module over $C_0(X) \rtimes G$.

We outline a method of proof.

Given a G-equivariant vector bundle over X, endowed with a G-invariant inner product, we construct $\mathcal{E}_E$ as above.

Now consider the C*-algebra of bounded sections of the bundle $\mathrm{End}(E)$ over X, whose fibre at x is the C*-algebra $\mathbb{B}(E_x)$, which are G-equivariant, *i.e.*

$$a(gx) = g \cdot s(x) \cdot g^{-1} \ \ \forall x \in X.$$

Let A_E denote this (unital) C*-algebra.

If ξ is a compactly supported section of E, and $a \in A_E$, set

$$(a \cdot \xi)(x) := a(x)\xi(x).$$

Since $a(gx) = ga(x)g^{-1}$, if ξ is a compactly supported section of E then

$$\begin{aligned}
\text{(7.13)} \quad [(a \cdot \xi) \cdot h]\,(x) = h^{-1} \cdot [(a \cdot \xi)(hx)] &= (h^{-1} \cdot a(hx) \cdot h)h^{-1} \cdot \xi)(hx) \\
&= a(x)h^{-1} \cdot \xi(hx) = [a \cdot (\xi \cdot h)]\,(x).
\end{aligned}$$

and hence

$$(a \cdot \xi) \cdot h = a \cdot (\xi \cdot h).$$

Similarly

$$(a \cdot \xi) \cdot f = a \cdot (\xi \cdot f)$$

for any $f \in C_0(X) \rtimes G$. Hence the left multiplication action by $a \in A_E$ is a right $C_0(X) \rtimes G$-module map.

Next, we define a right A_E-valued inner product by

$${}_{A_E}\langle \xi, \eta \rangle(x) := \sum_{g \in G} \left[g \cdot \xi(g^{-1}x)) \right]^* \otimes (g \cdot \eta)(g^{-1}x),$$

where for vectors $v, w \in E_x$, we let $v^* \otimes w$ be the corresponding rank-one operator $E_x \to E_x$.

Now one checks that the inner product condition for a SME bimodule is met. Find a finite collection $\xi_1, \ldots, \xi_n$ of compactly supported sections of E such that

$$\sum {}_{A_E}\langle \xi_i, \xi_i \rangle = 1_{A_E},$$

from which it follows that id_E is a compact operator on $\mathcal{E}_E$.

It also shows that the projection p_E with matrix the entries

$$\langle \xi_i, \xi_j \rangle_{C_0(X) \rtimes G}$$

satisfies

$$\mathcal{E}_E \cong p_E \cdot (C_0(X) \rtimes G)^n$$

as right Hilbert $C_0(X) \rtimes G$-modules.

EXERCISE 7.20. In the proof of Theorem 7.1 we constructed a certain $C(G\backslash X)$-$C_0(X) \rtimes G$ Morita equivalence bimodule. Let $\mathcal{E}^*$ be its conjugate bimodule, a $C_0(X) \rtimes G$-$C(G\backslash X)$ Morita equivalence bimodule (defined in the same way, but with the order of multiplications and inner products reversed.)

If X is compact, G finite and acts freely on X, then $C_0(X) \rtimes G = C(X) \rtimes G$ is unital and hence $\mathcal{E}^*$ is finitely generated projective over $C(G\backslash X)$. (Verify this statement.) By Swan's theorem it is isomorphic to the module of sections of a vector bundle over $G\backslash X$. Prove that this vector bundle is the induced bundle

$$X \times_G l^2(G),$$

with the product over the left regular representation $\lambda\colon G \to \mathbf{U}\big(l^2(G)\big)$.

(*Hint.* The section module of an induced bundle $X \times_G V_\rho$ for any representation ρ of G on V_ρ, is the linear space $C(X,V)^G$ of continuous maps $f\colon X \to V$ such that $f(gx) = \rho(g)f(x)$ for all $x \in X$ – the $C(G\backslash X)$-module structure is by lifting $f \in C(G\backslash X)$ to a G-invariant function on X, it then multiplies against f as above in the obvious way.)

8. Some geometric applications of Hilbert modules related to groups

In this section, we present some relationships between the representation theory of discrete groups, and Hilbert modules over their C*-algebras $C^*_r(G)$ (or von Neumann algebras $L(G)$).

The 'higher index maps' suggested by these constructions make up the Baum-Connes assembly map (see [**1**]), which the Baum-Connes Conjecture conjectures is an isomorphism – a statement with striking geometric and algebraic corollaries.

We start with the connection between the representation theory of finite (and more generally compact) groups, and the Hilbert module theory of their C*-algebras.

The Green-Julg correspondence

Hilbert modules (over a C*-algebra A, say) are a little special amongst general (algebraic) modules, since they are required to come equipped with an A-valued inner product. We have already discussed the idea of the support of a Hilbert module: a Hilbert module over $C(X)$ for a compact Hausdorff space X, cannot be supported at a single point of X, unless the point is isolated.

But such a point gives a *-homomophism $C(X) \to \mathbb{C}$ and hence gives $\mathbb{C}$ the structure of a $C(X)$-module. But it cannot be given the structure of a *Hilbert* $C(X)$-module.

One naturally wonders about a similar question in connection with groups.

If G is, say, for simplicity, a discrete group, and if $\pi\colon G \to \mathbf{U}(V)$ is a unitary representation of G on a Hilbert space H, in such a way that the representation integrates to a representation $C^*_r(G) \to \mathbb{B}(H)$, then we may, in a purely algebraic sense, interpret H as being a (left) $C^*_r(G)$-module. Similarly, if the representation extends to the von Neumann algebra $L(G)$, then H becomes a left $L(G)$-module.

If right modules are preferred, note that group C*-algebras are canonically isomorphic to their opposite algebras: if $f \in C_c(G)$, set

$$\tilde{f}(g) := f(g^{-1}).$$

This extends to an anti-isomorphism $C^*_r(G) \to C^*_r(G)$, and to obtain a right module from a left module we simply define

$$\xi f := \tilde{f}\xi.$$

The same remarks hold if one works with the von Neumann algebra $L(G)$.

For *compact* groups, whose duals $\widehat{G}$ are *discrete*, it is possible to put a Hilbert $C^*_r(G)$-module structure on the algebraic right module associated to a finite-dimensional representation, roughly because, since points of $\widehat{G}$ are all isolated, it is possible to have a Hilbert module supported at a single point of that space. More precisely, given a representation, *.i.e.* a strongly continuous unitary action of G on a Hilbert space H, with inner-product conjugate-linear in the first variable, set

$$\langle \xi, \eta \rangle_{C^*(G)}(g) := \langle \pi(g^{-1})\xi, \eta \rangle. \tag{8.1}$$

Since the representation is assumed strongly continuous, $\langle \cdot, \cdot \rangle$ so defined is continuous on G, and hence defines an element of $C^*(G)$.

In group-algebra notation , if the group is discrete, then

$$\langle \xi, \eta \rangle_{C^*(G)} = \sum_{g \in G} \langle \xi, \pi(g)^* \eta \rangle \cdot [g] \in \mathbb{C}[G] = C^*(G). \tag{8.2}$$

In this case, if $h \in G$, $h] \in \mathbb{C}[G] = C^*(G)$ the corresponding algebra element, then

$$\begin{aligned} \langle \xi, \eta \rangle_{C^*(G)} \cdot [h] &= \sum_{g \in G} \langle \pi(g^{-1})\xi, \eta \rangle \cdot [gh] = \sum_{G \in G} \langle \pi(h)\pi(g^{-1})\xi, \eta \rangle \cdot [g] \\ &= \sum_{G \in G} \langle \pi(g^{-1})\xi, \pi(h^{-1})\eta \rangle \cdot [g] = \langle \xi, \pi(h^{-1})\eta \rangle_{C^*(G)} = \langle \xi, \eta \cdot [h] \rangle_{C^*(G)} \end{aligned} \tag{8.3}$$

which shows right $C^*(G)$-linearity of the inner product. The other conditions and the general case of compact groups is left to the reader.

Exercise 8.1. Verify in general for compact groups G that (8.1) satisfies the conditions to be a Hermitian $C^*(G)$-valued form on the right $C^*(G)$-module H.

Exercise 8.2. Let

$$f^{\pi}_{\eta,\xi} \colon G \to \mathbb{C}, \quad f^{\pi}_{\eta,\xi}(g) := \langle \pi(g)\eta, \xi \rangle$$

be the matrix coefficient, as in Proposition 5.5, of a representation $\pi \colon G \to \mathbb{B}(H_\pi)$, and vectors $\xi, \eta \in H_\pi$.

Show that

$$\langle \xi, \eta \rangle_{C^*(G)} = \overline{f^{\pi}_{\eta,\xi}}$$

where $f^{\pi}_{\eta,\xi}$ is the matrix coefficient and $\langle \cdot, \cdot \rangle_{C^*(G)}$ is the right $C^*(G)$-valued inner product defined above.

Definition 8.3. We denote by $\mathcal{E}_{\pi,H}$ the Hilbert module H_π equipped with the right $C^*(G)$-valued inner product $\langle \cdot, \cdot \rangle_{C^*(G)}$ defined above.

Exercise 8.4. Prove that there is a 1-1 correspondence between G-equivariant (bounded) linear operators on the Hilbert space H, and (adjointable) $C^*_r(G)$-module operators on $\mathcal{E}_{\pi,H}$. More generally, prove that if $\pi \colon G \to \mathbf{U}(H)$ and $\pi' \colon G \to \mathbf{U}(H')$ are two isomorphic (unitarily equivalent) representations, then $\mathcal{E}_{\pi,V} \cong \mathcal{E}_{\pi',H'}$ as right Hilbert $C^*_r(G)$-modules.

Remark 8.5. If G is not compact, for example, if $G = \mathbb{Z}$, then the matrix coefficient $\langle \pi(g)v, w \rangle$ may not be in $l^2(G)$ and hence certainly is not an element of $C^*_r(G)$.

For example, fix any point $\omega \in \mathbb{T}$ and take the representation $\pi \colon \mathbb{Z} \to \mathbb{T} = \mathbf{U}(\mathbb{C})$, $\pi(n) := \omega^n$. Then $\mathbb{C}$ becomes a right module over $C^*_r(\mathbb{Z})$, using the above ideas, and setting $\lambda \cdot u := \bar{\omega}\lambda$, for $\lambda \in \mathbb{C}$, and where u is the unitary generator of $C^*(\mathbb{Z})$. But the inner product formula (8.1) would dictate for example, that $\langle 1, 1 \rangle$ would be the function $\langle 1, 1 \rangle_{C^*(\mathbb{Z})}(n) = \omega^n$ and this function on the integers is obviously not in $C^*_r(\mathbb{Z})$.

(In fact it corresponds under Fourier transform to the Dirac delta distribution δ_ω on the circle $\mathbb{T} = \widehat{\mathbb{Z}}$, which, though a distribution, is of course not in $C(\mathbb{T})$.)

So this situation corresponds to the geometric idea that we are trying to make a Hilbert $C^*(\mathbb{Z}) \cong C(\mathbb{T})$-module, supported at the single point $\omega \in \mathbb{T}$, which is impossible, because G is not compact (*i.e.* $\widehat{G}$ is not discrete).

PROPOSITION 8.6. *Let G be a finite group and $\pi\colon G \to \mathbb{B}(H_\pi)$ be a finite-dimensional representation. Then $\mathcal{E}_{G,\pi}$ is f.g.p. over $C^*(G)$. Moreover,*

$$\mathcal{E}_{G,\pi} \cong e_\pi \cdot C^*(G)$$

as right f.g.p. Hilbert $C^*(G)$-modules, where e_π is the idempotent

$$e_\pi := \frac{\dim H_\pi}{|G|} \cdot \chi_\pi^* \in C^*(G),$$

of Proposition 5.8.

The general statement for compact groups is called the Green-Julg theorem.

THEOREM 8.7. *Let G be a compact group. The association to a finite-dimensional representation $\pi\colon G \to \mathbf{U}(H)$, the right Hilbert f.g.p. $C_r^*(G)$-module $\mathcal{E}_{\pi,H}$, determines an isomorphism between the semi-group of unitary equivalence classes of finite-dimensional unitary representations of G, and the semigroup of unitary isomorphism classes of f.g.p. right Hilbert $C_r^*(G)$-modules.*

In the case of a *finite* group the following exercise shows that we can realize the Green-Julg correspondence

$$\{\text{f.d. repns of } G\} \longrightarrow \{\text{f.g.p. right Hilbert } C_r^*(G)\text{-modules}\}.$$

as a kind of tensoring map with a *fixed* module $\mathcal{E}$.

EXERCISE 8.8. Let $\pi\colon G \to \mathbf{U}(V)$ be a finite-dimensional representation of a finite group G, and $\mathcal{E}_G$ the right Hilbert $C^*(G) \otimes C^*(G)$-module of Exercise 4.7.

Show that $\mathcal{E} \otimes_{C^*(G)\otimes C^*(G)} (C^*(G) \otimes V) \cong \mathcal{E}_{\pi,V}$ as right Hilbert $C^*(G)$-modules (they are unitarily isomorphic), where the tensor product is over the representation

$$\mathrm{id}_{C^*(G)} \otimes \pi\colon C^*(G) \otimes C^*(G) \to C^*(G) \otimes \mathbb{B}(V) \cong \mathbb{B}\big(C^*(G) \otimes V\big)$$

of $C^*(G)$ as bounded $C^*(G)$-module operators on the right Hilbert $C^*(G)$-module $C^*(G) \otimes V$.

L^2-dimension and L^2-Betti numbers

We now discuss an analogue of this construction for *discrete*, infinite groups, coming from considering proper actions of such groups. For the most interesting applications, we will assume that we are given a complete Riemannian manifold X, on which a discrete group G acts properly, isometrically, and co-compactly.

Spectral theory and index theory of elliptic operators on compact manifolds entails as one of the basic starting points that if D is an elliptic operator, then $(1 + D^*D)^{-1}$ is a compact operator. This in itself implies that $f(D)$ is compact for all $f \in C_0(\mathbb{R})$, that the eigenspaces of D in L^2 are finite-dimensional. Of course $\ker(D)$ is the solution set to a system of differential equations, so it's finite-dimensionality is likely to be important in applications. (We discuss some of these things a bit more in Chapter 7 and ask the reader to take them on faith provisionally.)

For non-compact manifolds, elliptic operators D are 'locally Fredholm' in the sense that $f(1+D^*D)^{-1}$ is compact if $f \in C_0(X)$. The relevant solution spaces $\ker(D)$, are not necessarily finite-dimensional anymore, and so the operator does not have a traditional Fredholm index.

Atiyah observed that if X is acted on isometrically, properly and freely by a discrete group G, then one can build a unitary representation of G on $L^2(X)$, and D commutes with it. One can then consider D not as a Hilbert *space* operator, but a Hilbert $\mathbb{C}[G]$-*module* operator, leading to a new, 'module version' of the Fredholm property (and index), and new invariants of D (and G).

The easiest example is to set $G = \mathbb{Z}$ acting on $X = \mathbb{R}$ by translation. In Example 4.38 we considered the Hilbert space $L^2(\mathbb{R})$, and the densely defined differential operator $D = -i\frac{d}{dx}$ on $L^2(\mathbb{R})$. It is essentially self-adjoint, and generates the regular representation $C^*(\mathbb{R}) \to \mathbb{B}\big(L^2(\mathbb{R})\big)$ of $C^*(\mathbb{R})$.

But the operator $\lambda(f)$, for $f \in C^*(\mathbb{R})$, is not, however, compact. This corresponds to the fact that the unbounded operator D is not 'Fredholm' in this representation: that is, $(1+D^2)^{-1}$ is not a compact operator on $L^2(\mathbb{R})$. This in turn, is a consequence of the fact that the space $X = \mathbb{R}$ being acted on (by $\mathbb{R}$) is not compact.

REMARK 8.9. For *compact* spaces with $\mathbb{R}$ actions, one *does* get compactness, as we note in the following exercise.

EXERCISE 8.10. Consider the group representation of $\mathbb{R}$, and then of $C^*(\mathbb{R})$, on $L^2(\mathbb{T})$ obtained by the group translation action of $\mathbb{R}$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Denote it $\lambda\colon C^*(\mathbb{R}) \to \mathbb{B}\big(L^2(\mathbb{T})\big)$. Prove that $\lambda(f)$ is a compact operator for all $f \in C^*(\mathbb{R})$. and that the unbounded generator D for the action is the usual differentiation operator $-i\frac{d}{d\theta}$ on the circle.

Returning to the example of $\mathbb{R}$ acting on $\mathbb{R}$ by translation, with $D = -i\frac{d}{dx}$ the unbounded generator of the action, although D is not Fredholm, *i.e.* $(1 + D^2)^{-1}$ is not compact, it *is* compact as a right Hilbert $C^*(\mathbb{Z})$-module operator on a closely related space. For Example 4.38 shows that if we consider D as not acting on the Hilbert *space* $L^2(\mathbb{R})$, we consider it acting on the Hilbert *module* $\mathcal{E}_{\mathbb{Z},\mathbb{R}}$, an appropriate completion of $C_c(\mathbb{R})$ to a right Hilbert $C^*(\mathbb{Z})$-module, then $(1 + D^2)^{-1} \in \mathcal{K}(\mathcal{E}_{\mathbb{Z},\mathbb{R}})$. That is, D is 'Fredholm' in the sense of operators on Hilbert $C^*(\mathbb{Z})$-modules.

We now show how to build, for any discrete group G, and any proper action of G, with G-invariant measure μ, a a Hilbert $C^*_r(G)$-module, giving an alternative space to $L^2(X,\mu)$ on which various natural geometric operators can act. Observe first that we have the unitary representation

$$(g \cdot \xi)(x) := \xi(g^{-1}x).$$

of G on $L^2(X,\mu)$.

This means that we can regard $L^2(X,\mu)$ as a (left) module over $\mathbb{C}G$. Alternatively, we can regard it as a right module, by letting $\xi \cdot [g] := g^{-1}(\xi) := \xi \circ g$.

We prove below that this right module action extends to an action of $C^*_r(G)$. (Otherwise put, the left action extends to a representation of $C^*_r(G)$.) So we can view $L^2(X,\mu)$ as a right $C^*_r(G)$-module. In order to make a right Hilbert $C^*_r(G)$-module, we attempt to define a Hermitian $\mathbb{C}[G] \subset C^*_r(G)$-valued form by the formula

$$\langle \xi, \eta\rangle = \sum_{g\in G} \langle \xi \cdot [g], \eta\rangle_{L^2((X,\mu)} \cdot [g] := \sum_{g\in G} \langle g^{-1}(\xi), \eta\rangle_{L^2(X,\mu)} \cdot [g] \tag{8.4}$$

The difficulty is that for a pair of L^2-functions, the sum may not be convergent. Instead, we define the form (8.4) initially for $\xi, \eta \in C_c(X)$. We view the result (which is then in $\mathbb{C}G$) as an element of $C^*_r(G)$. The formula defines a Hermitian $\mathbb{C}G \subset C^*_r(G)$-valued form, and we leave it to the reader to check that it is $C^*_r(G)$-sesquilinear.

Let $\mathcal{E}_\mu$ denote the completion of $C_c(X)$ with respect to the Hermitian $C^*_r(G)$-valued form (4.38). It is a right Hilbert $C^*_r(G)$-module.

EXERCISE 8.11. Suppose that in the above discussion, $X = G$ (with counting measure), with G acting by left translation on $X = G$. Show that on the standard basis vectors $e_x, x \in G$, the right module action of $\mathbb{C}G$ is given by $e_x \cdot [g] = e_{g^{-1}x}$, and that the $C^*_r(G)$-valued inner product is determined by setting $\langle e_x, e_y\rangle := [xy^{-1}] \in \mathbb{C}G \subset C^*_r(G)$.

Prove that inversion on the group determines a unitary isomorphism $\mathcal{E}_\mu \cong C^*_r(G)$ of right Hilbert $C^*_r(G)$-modules with the standard, rank-one Hilbert $C^*_r(G)$-module.

EXERCISE 8.12. Show that, in general, $\mathcal{E}_\mu$ is a (non-closed) $C^*_r(G)$-submodule of $L^2(X,\mu)$.

Now suppose that $T \in \mathbb{B}\big(L^2(X)\big)$ is a bounded, G-equivariant operator, *i.e.* T commutes with the unitary action of G. If $\xi \in C_c(X) \subset L^2(X)$ and $g \in G$, then

$$(T\xi)[g] = g^{-1} \cdot T(\xi) = T(g^{-1} \cdot \xi) = T\big(\xi \cdot [g]\big),$$

that is, T defines a $\mathbb{C}[G]$-module map on $C_c(X) \subset \mathcal{E}_\mu$. It is easy to see that $\|\langle T\xi, T\xi\rangle_{C^*_r(G)}\| \leq \|T\|^2$, where $\|T\|^2$ denotes the norm of T as a bounded operator on a Hilbert space. It follows that T extends continuously to an operator on $\mathcal{E}_\mu$, which, necessarily, is a $C^*_r(G)$-module map. Finally, it is obvious that $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$, with T^* the Hilbert space adjoint of T, so T canonically defines an adjointable right Hilbert $C^*_r(G)$-module map $\mathcal{E}_\mu \to \mathcal{E}_\mu$.

Higher index theory and the Baum-Connes assembly map

The above construction can be generalized in an extremely important way. We briefly outline it. Let M be a compact manifold with fundamental group G and $\pi\colon \tilde{M} \to M$ the universal cover. The group G acts properly, freely, and isometrically on X with respect to any Riemannian metric lifted from M. We set $X = \tilde{M}$. If we let, for example, μ be the Riemannian volume form on X, it is G-invariant, and we may proceed as above to construct from the Hilbert space $L^2(X,\mu)$, with it's unitary G-action, the right Hilbert $C^*_r(G)$-module $\mathcal{E}_\mu$.

More generally, suppose that S is a Hermitian vector bundle over M. Let $\tilde{S}$ be it's lift to a G-equivariant vector bundle over X. Since S has by assumption a Hermitian metric on it's fibres, we can make it into a Hilbert space by setting

$$\langle \xi, \eta\rangle := \int_X \langle \xi(x), \eta(x)\rangle \, d\mu(x).$$

Moreover, the space $\Gamma_c(\tilde{S})$ of compactly supported sections of $\tilde{S}$, can be endowed with the Hermitian $C^*_r(G)$-valued form

$$\langle \xi, \eta\rangle := \sum_{g\in G} \langle g^{-1}(\xi), \eta\rangle \cdot [g] \in \mathbb{C}G \subset C^*_r(G).$$

So we can, as in the case of the trivial rank-one bundle above, complete $\Gamma_c(\tilde{S})$ to a right Hilbert $C^*_r(G)$-module $\mathcal{E}_{S,\mu}$.

The setting of 'higher index theory' is that one now assumes in addition, that one is given an elliptic differential operator D on sections of S. It can be viewed as a densely defined Hilbert space operator on $L^2(S,\mu)$, with initial domain the smooth sections.

The operator D is 'local', *i.e.* the value of a section Ds at $x \in M$ only depends on the value of s in a neighbourhood. This implies that D 'lifts' in a natural way to an elliptic differential operator $\tilde{D}$ on sections of $\tilde{S}$: to define $\tilde{D}s\,(x)$ one notes that $s(x) \in S_{\pi(x)}$, and we can choose a diffeomorphism of a neighbourhood of $\pi(x)$ with a neighbourhood of x. Using the diffeomorphism, we can consider s as, locally, near x, equivalent for all purposes to a section of S defined near $\pi(x)$. We may apply D to this local section, and re-interpret the result as the value of the section Ds of $\tilde{S}$ near x.

Different local diffeomorphisms differ by a constant map. Hence $\tilde{D}s(x)$ does not depend on the choice of local diffeomorphism (or 'sheet').

By the obvious analogue of our discussion above, we may not interpret $\tilde{D}$ as a G-equivariant, unbounded, operator on the Hilbert space $L^2(\tilde{S})$ – or, alternatively, as a densely defined right Hilbert $C^*_r(G)$-module map on $\mathcal{E}_{S,\mu}$.

THEOREM 8.13. *The densely defined right Hilbert $C_r^*(G)$-module operator $\tilde{D}\colon \Gamma_c^\infty(\tilde{S}) \subset \mathcal{E}_{S,\mu} \to \mathcal{E}_{S,\mu}$, is Fredholm in the sense that $(1+\tilde{D}^2)^{-1}$ is a compact Hilbert $C_r^*(G)$-module map on $\mathcal{E}_{S,\mu}$.*

This implies, by a rather general procedure, that D (or rather, $\tilde{D}$), defines in a natural way, a class $\mathrm{index}_G(D) \in \mathrm{K}_0(C_r^*G)$ in the K-theory group of $C_r^*(G)$.

9. C*-algebras of étale groupoids

A *groupoid* is a category whose object space is a set, and for which every morphism is an isomorphism. A groupoid is a more general object than a group action, and groupoids are sometimes useful even if one is interested mainly in group actions, and cover a number of other examples, like holonomy groupoids of foliations, or certain tail-equivalence C*-algebras connected with dynamics, which are not quite of the group action type.

A groupoid is usually denoted G and the morphisms in the category – elements γ of the groupoid – have targets $r(\gamma)$ and sources $s(\gamma)$, lying in the object space G^0 of the groupoid. We write $\gamma_1\gamma_2$ rather than $\gamma_1 \circ \gamma_2$. In order for the composition to make sense, $r(\gamma_2)$ must equal $s(\gamma_1)$.

Any category is forced to have identity morphisms. If $x \in G^0$ is a point, we let $e_x \in G$ be the identity morphism from x to itself. Thus, $r(e_x) = s(e_x) = x$, and $e_x\gamma = \gamma$ for any γ with range x, and $\gamma e_x = \gamma$ for any γ with source x. A standard argument from group theory proves that identity morphisms are unique.

The extra condition making G a groupoid asserts that every γ is an isomorphism: that is, there is a morphism γ^{-1} with range and source of γ switched, so that $\gamma^{-1}\gamma = e_x$ and $\gamma\gamma^{-1} = e_y$, where $x = s(\gamma)$, $y = r(\gamma)$.

The usual associativity conditions

$$(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$$

are required to hold, for composable morphisms, or groupoid elements, $\gamma_1, \gamma_2, \gamma_3$.

If the object space G^0 is a point, this is a rather abstract way of defining a group (a category with one object, in which every morphism is an isomorphism.) So we can reasonably view groupoids as generalizing groups.

EXAMPLE 9.1. Any group is a groupoid, with exactly one object. The morphisms are the elements of the group.

EXERCISE 9.2. Prove that if G is a groupoid, then

$$\mathrm{Stab}_G(x) := \{\gamma \in G \mid s(\gamma) = r(\gamma) = x\}$$

is a group under the groupoid composition rule.

EXAMPLE 9.3. (Transformation groupoids).

If $G \times X \to X$ is an action of a group on a space, then the *transformation groupoid* $G \rtimes X$, has unique, or object set X, and morphisms the elements of $G \times X$, with $s(g,x) = g^{-1}x$ and $r(g,x) = x$. The composition rule is

$$(x,g) \cdot (g^{-1}(x), h) := (x, gh).$$

We can think of an element (g,x) as an arrow, which I draw going from right to left, from the point $g^{-1}x$ to x.

EXERCISE 9.4. Let G be any groupoid and $F \subset G^0$ is any subset of its unit space.

a) Show that the restriction

$$G|_F := \{g \in G \mid r(g), s(g) \in F\}$$

of G to F, with composition rules and range and source maps $r, s\colon G|_F \to F$, is a groupoid.

b) If G was a transformation groupoid $\Gamma \ltimes X$ to begin with, show that $G|_F$ is just $G \ltimes F$ if F is Γ-invariant, but otherwise is not in general a transformation groupoid.

EXAMPLE 9.5. If X is any set and $\sim$ is an equivalence relation on X, then its graph $G := \{(x, y) \in X \times X \mid x \sim y\}$ can be considered a groupoid with object space X, range and source maps $r(x, y) = x$, $s(x, y) = y$, and composition $(x, y)(y, z) := (x, z)$.

In particular, setting $\sim$ equal to the trivial equivalence relation, we obtain the set X, understood as a groupoid with range and source maps the identity maps $X \to X$.

In order to make a C*-algebra from a groupoid, one needs additional hypotheses: especially a topology on the groupoid. In this section, we restrict attention to the analogues for groupoids, of actions of *discrete* groups: these are called *étale* groupioids ('étale' means 'spread out.')

A *topological groupoid* is a groupoid equipped with a locally compact Hausdorff topology in which the composition and inversion of morphisms are all continuous maps.

DEFINITION 9.6. A second countable topological groupoid G is *étale* if G^0 is open in G and the range and source maps $r, s\colon G \to G^0$ are both local homeomorphisms.

EXAMPLE 9.7. If $G \ltimes X$ is a transformation groupoid with G a discrete group, and if we equip $G \ltimes X$ with the product topology, then it becomes an étale groupoid. $G \ltimes X$ as a groupoid is étale.

We now show how to associate a C*-algebra to an étale groupoid G.

If $f_1, f_2 \in C_c(G)$, their *convolution* is defined by

$$(f_1 * f_2)(g) := \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2).$$

The sum can be re-written

$$= \sum_{h \in G,\, r(h) = r(g)} f_1(h) f_2(h^{-1} g). \tag{9.1}$$

Now since $r\colon G \to X$ is a local homeomorphism, it's fibres $r^{-1}(x)$ are discrete. Since f_i are compactly supported, the sum (9.1) is a finite sum.

An adjoint is defined by setting

$$f^*(g) := \overline{f(g^{-1})} \tag{9.2}$$

EXERCISE 9.8. If G is a discrete group (for simplicity) acting on X, then the convolution formula (9.1) for the groupoid $G \ltimes X$, agrees with the twisted convolution operation we introduced on $C_c(G \times X)$ in the definition of C*-algebra crossed-products (9.1).

Finally, in a manner similar to the one we used to form crossed-products above, we complete the *-algebra $C_c(G)$ to a C*-algebra to a C*-algebra by representing it as adjointable operators on a certain right Hilbert $C(\Sigma)$-module, which we now construct as a completion of the linear space $C_c(G)$, with the right $C(\Sigma)$-module structure given by

$$(f \cdot \phi)(x, y) := f(x, y)\phi(y)$$

and inner product valued in $C(\Sigma)$ by

$$\langle f_1, f_2\rangle(y) := \sum_{x\in[y]} \overline{f_1(x,y)} f_2(x,y).$$

Clearly $\langle f_1, f_2\rangle = \langle f_2, f_1\rangle^* \in C(\Sigma)$, $\langle f, f\rangle \geq 0$ for all f, and $\langle\cdot,\cdot\rangle$ is right $C(\Sigma)$-linear.

Let $\mathcal{E}$ be the completion of $C_c(G)$ with respect to this inner product. And if $f \in C_c(G)$, we let $\pi(f)$ be the $C(\Sigma)$-linear operator on $\mathcal{E}$ given by

$$\pi(f)\xi\ (x,y) := \sum_{z\in[x]} f(x,z)\xi(z,y).$$

Then $\pi(f)$ is adjointable with adjoint $\pi(f^*)$, with f^* defined as in (9.2). Therefore, we obtain a *-representation

$$\pi\colon C_c(G) \to \mathbb{B}(\mathcal{E}).$$

Definition 9.9. The (reduced) C*-algebra $C^*_r(G)$ of the étale groupoid G is the completion of the pre-C*-algebra $\big(C_c(G), \|\cdot\|_\pi\big)$.

The C-algebra of tail equivalence on the full n-shift*

Let

$$\Sigma = \prod_{n=1}^{\infty}\{1,2,\ldots,n\}.$$

the Cantor space of sequences (x_i) with $x_i \in \{1,2,\ldots,n\}$, with the product topology. Of interest here, is the equivalence relation of *tail equivalence*

$$x \sim y \ \Leftrightarrow\ x_i = y_i\ \forall i \geq N, \ \text{ some } N.$$

Thus, two sequences are equivalent if they agree after a certain point. Put another way, let $\sigma\colon \Sigma \to \Sigma$ be the left shift:

$$\sigma(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Then $x \sim y$ if and only if $\sigma^k(x) = \sigma^k(y)$ for some k. Of course this implies the same statement for all larger k as well.

For example

$$(123112211221122\cdots) \sim (56382211221122\cdots).$$

for after removing the initial strings 1231 and 5638 from the first and second sequence respectively, we obtain exactly the same sequence, namely $(2211221122\cdots)$.

Let $G \subset \Sigma\times\Sigma$ be the graph of the equivalence relation of tail equivalence. Let

$$r, s\colon G \to \Sigma, \ \ r(x,y) := x, \ s(x,y) := y.$$

Exercise 9.10. Show that every equivalence class $[x]$ is dense in Σ.

We are going to show how to associate a C*-algebra $C^*_r(G)$ to this equivalence relation, which will turn out to be isomorphic to the C*-algebra $U(n^\infty) := \varinjlim_k M_{n^k}(\mathbb{C})$ of UHF type, discussed earlier. The construction is more general, *i.e.* applies to more general equivalence relations, but we will focus on the example for the moment.

We start by topologizing G, not using the subspace topology as a subset of $\Sigma\times\Sigma$, but in a topology in which the maps $r, s\colon G \to \Sigma$ are *local homeomorphism*.

If α is a finite word in the alphabet and $x \in \Sigma$, we write $o(x) = \alpha$ if the sequence x begins with α. If β is another word of the same length, say both have length k, we set

$$Z_{\alpha,\beta} := \{(x, y) \in G \mid o(x) = \alpha,\ o(y) = \beta, \text{ and } \sigma^k(x) = \sigma^k(y).\}$$

A pair (x, y) is in $Z_{\alpha,\beta}$ if and only if x starts with α, y starts with β, and they agree immediately after α, β (by definition α and β have the same length.)

Lemma 9.11. *The sets $Z_{\alpha,\beta}$ form a basis for a topology on G. Moreover, $Z_{\alpha,\beta} \cap Z_{\alpha',\beta'}$ is empty unless either*

a) *There exists a word ϵ such that $\alpha' = \alpha\epsilon$ and $\beta' = \beta\epsilon$, or*
b) *There exists a word ϵ such that $\alpha = \alpha'\epsilon$ and $\beta = \beta'\epsilon$.*

In case a), $Z_{\alpha,\beta} \cap Z_{\alpha',\beta'} = Z_{\alpha',\beta'}$ and in case b), $Z_{\alpha,\beta} \cap Z_{\alpha',\beta'} = Z_{\alpha,\beta}$.

Proof. If $(x, y) \in G$, then $\sigma^k(x) = \sigma^k(y)$ for some k, and if α is the word consisting of the first k letters $x_1, \ldots, x_k$ of x and β is the first k letters of y then $(x, y) \in Z_{\alpha,\beta}$. Hence the union of the $Z_{\alpha,\beta}$ is all of G.

It remains to verify the statement regarding intersections $Z_{\alpha,\beta} \cap Z_{\alpha',\beta'}$, where $|\alpha| = |\beta|$ and $|\alpha'| = |\beta'|$.

Suppose that the intersection is non-empty. Assume that $|\alpha| < |\alpha'|$. This will imply scenario a); the other case implies scenario b). Let $(x, y) \in Z_{\alpha,\beta} \cap Z_{\alpha',\beta'}$. Then $x = (\alpha_1, \ldots, \alpha_k, \cdots\cdots)$ and at the same time $x = (\alpha'_1, \ldots, \alpha'_l, \cdots, \cdots)$. Since α' is longer than α, it must be that $\alpha'_1 = \alpha_1, \ldots, \alpha'_k = \alpha_k$. Hence for some string ϵ, $\alpha' = \alpha\epsilon$.

Now by assumption x and y agree after α and β respectively. Given that x also starts with α' and y starts with β', it must be that β' also ends in ϵ. This proves the result.

□

From now on, we give G the topology generated by the basis $\{Z_{\alpha,\beta} \mid |\alpha| = |\beta|\}$.

Exercise 9.12. If α and β are words of the same length k then the map $f_{\beta,\alpha} \colon U_\alpha \to U_\beta$ defined $f_{\beta,\alpha}(\alpha_1\alpha_2\cdots\alpha_k x_{k+1}x_{k+2}\cdots) := (\beta_1\beta_1\ldots\beta_k x_{k+1}, x_{k+2}, \ldots)$ is a local homeomorphism whose graph is the basic open subset $Z_{\alpha,\beta}$.

Lemma 9.13. *Then G is Hausdorff, second countable, locally compact, and the maps r, s are each local homeomorphisms. Furthermore, the basic open subsets $Z_{\alpha,\beta}$ are clopen, so G is totally disconnected.*

Of course G is not compact, since the fibres $r^{-1}(x)$ are all infinite and discrete in G, for $x \in \Sigma$.

Proof. If $(x, y) \in G$, let $Z_{\alpha,\beta}$ be a basic open neighbourhood containing (x, y). By the definitions, then, x begins with α and y begins with β, and x and y agree after α, β respectively. So if $(x, y') \in Z_{\alpha,\beta}$, for some y', then y' begins with β, and agrees with x, and hence y, after β. As both y and y' begin with β and agree with x and hence each other afterwards, we conclude that $y = y'$. Hence the restriction of r to $Z_{\alpha,\beta}$ is injective, and surjects to the clopen subset $U_\alpha \subset \Sigma$ of all sequences beginning with α. This shows that r is a local homeomorphism. Similarly, s is a local homeomorphism.

For the proof that $Z_{\alpha,\beta}$ is clopen, we show that its complement is open by showing that if $(x, y) \notin Z_{\alpha,\beta}$ then there exist α', β' words of the same length as each other such that $(x, y) \in Z_{\alpha',\beta'}$ and $Z_{\alpha',\beta'}$ is disjoint from $Z_{\alpha,\beta}$.

If either x does not begin with α or y does not begin with β, then this is easy. Supposing that x and y agree after k steps (*i.e.* that $\sigma^k(x) = \sigma^k(y)$), and assuming as well without loss of generality that k is larger than $|\alpha| = |\beta|$, set $\alpha' = x_1x_2\cdots x_k, \beta' = y_1y_2\cdots y_k$. The required conditions are easily checked.

If then x begins with α, y with β, and if they agree with each other after l steps and yet $(x, y) \notin Z_{\alpha,\beta}$, then x and y must fail to agree at some point between k and l steps, so that if we put $\alpha' = x_1 x_2 \cdots x_l$ and $\beta' = y_1 y_2 \cdots y_l$ then it is easy to check that α', β' have the right properties.

□

EXERCISE 9.14. The local homeomorphism $r\colon G \to \Sigma$ (and similarly for s) is not a *covering map*. Why not?

Let $C_c(G)$ be compactly supported continuous functions on G. If $f \in C_c(G)$, then due to the fact that r and s are local homeomorphisms, if $x \in \Sigma$, then $f(x, y) \neq 0$ *for only finitely many* $y \in [x]$. This shows that if f_1 and f_2 are compactly supported functions, then their 'convolution'

$$(f_1 \cdot f_2)(x, y) := \sum_{z \in [x]} f_1(x, z) f_2(z, y) \tag{9.3}$$

is defined, for each $(x, y) \in G$, by a finite sum.

The compactly supported, continuous functions to focus on in this example are the characteristic functions

$$s_{\alpha,\beta} := \chi_{Z_{\alpha,\beta}} \in C_c(G), \tag{9.4}$$

of the basic open sets $Z_{\alpha,\beta}$, where $|\alpha| = |\beta|$.

The convolution of two such functions $s_{\alpha,\beta}$ and $s_{\alpha'\beta'}$ is given by

$$s_{\alpha,\beta} \cdot s_{\alpha',\beta'}(x, y) = \sum_{z \in [x]} s_{\alpha,\beta}(x, z) s_{\alpha',\beta'}(z, y). \tag{9.5}$$

Assume that $|\alpha| = |\beta| = |\alpha'| = |\beta'| = k$. If some term in (9.5), corresponding to z, is nonzero, then x must begin with α, z with β, as well as with α', while y begins with β'. This forces $\alpha' = \beta$, and furthermore, if this holds, there is in any case exactly one z for which all this happens: z begins with β and matches the sequence x afterwards.

The conclusion is thus that here is exactly one nonzero term in the sum (9.5), and we have that

$$s_{\alpha,\beta} \cdot s_{\alpha',\beta'} = \delta_{\beta,\alpha'}\, s_{\alpha,\beta'}, \tag{9.6}$$

under the assumption that $\alpha, \beta, \alpha', \beta'$ all have the same length.

EXERCISE 9.15. Prove that

$$s_{\alpha,\beta} = \sum_{i=1}^{n} s_{\alpha i,\beta i} \in C_r^*(G)$$

for any pair α and β of the same length.

Then $C_c(G)$ with these operators becomes a *-algebra.

In the present situation specifically, there is an alternative method to completing $C_c(G)$ to a C*-algebra, due to the following:

PROPOSITION 9.16. *The linear span W_k of the $s_{\alpha,\beta}$ where $|\alpha| = |\beta| = k$ is a C*-subalgebra of $C_r^*(G)$ isomorphic to $M_{n^k}(\mathbb{C})$. Furthermore, $W_k \subset W_{k+1}$ for all k and $\cup_{n=1}^{\infty} W_k$ is dense in $C_r^*(G)$.*

Hence $C_r^*(G) \cong U(n^\infty)$, where $U(n^\infty)$ is the UHF algebra of type n^∞.

REMARK 9.17. To see $W_k \subset W_{k+1}$ note that $s_{\mu,\nu} = \sum_{i=1}^{n} s_{\mu i,\nu i}$ puts $s_{\mu,\nu} \in W_k$, also in W_{k+1}. This is the same structure map $M_{n^k}(\mathbb{C}) \to M_{n^{k+1}}(\mathbb{C})$ used to define the UHF algebra.

We close this section by showing how to build a natural representation of $C_r^*(G)$ as bounded operators on a Hilbert space, by choosing a probability measure on the base $G^0 = \Sigma$. The construction applies much more broadly to groupoid situations. The representation turns out to agree with one we have already discussed.

Let m be a probability measure on Σ. It gives rise to a representation $\pi\colon C(\Sigma) \to \mathbb{B}\big(L^2(\Sigma, m)\big)$ by multiplication operators.

The right Hilbert $C(\Sigma)$-module $\mathcal{E}$ constructed above can now be tensored with the representation, making Hilbert *space*

$$H_\mu := \mathcal{E} \otimes_{C(\Sigma)} L^2(\Sigma, m).$$

Since $C_r^*(G)$ acts on $\mathcal{E}$, we can tensor the action with the identity on $L^2(\Sigma)$ to get an action of $C_r^*(G)$ on H_μ, *i.e.*, a representation

$$\pi_\mu\colon C_r^*(G) \cong U(n^\infty) \to \mathbb{B}(H_\mu),$$

with $\pi_\mu(f) := \pi(f) \otimes 1$.

Now $\mathcal{E}$ is itself a certain completion of $C_c(G)$ with respect to the $C(\Sigma)$-valued inner product developed above. In particular, the characteristic functions $\chi_{Z_{\alpha,\beta}}$ of the sets $Z_{\alpha,\beta}$ may be regarded as vectors in $\mathcal{E}$ and hence, tensoring them with the constant function $1 \in L^2(\Sigma, m)$, yields vectors $\xi_{\alpha,\beta} := \chi_{Z_{\alpha,\beta}} \otimes 1 \in \mathcal{E} \otimes_{C(\Sigma)} L^2(\Sigma, m)$.

Let us compute the geometry of these vectors.

LEMMA 9.18. *In the above notation, with α, β words of equal length, and α', β' words of equal length, $\xi_{\alpha,\beta}$ and $\xi_{\alpha',\beta'}$ are orthogonal vectors in H_μ unless $\alpha = \alpha'$ and $\beta = \beta'$. And*

$$\langle \xi_{\alpha,\beta}, \xi_{\alpha,\beta} \rangle = n^{-|\beta|}.$$

EXERCISE 9.19. The following exercise makes use of the isomorphism $C_r^*(G) \cong U(n^\infty)$ for the tail-equivalence groupoid G discussed above. Let $\tau\colon U(n^\infty) \to \mathbb{C}$ be the state whose restriction to $M_{n^k}(\mathbb{C})$ is the normalized trace $\tau_k := n^{-k}\mathrm{Trace}\colon M_{n^k}(\mathbb{C}) \to \mathbb{C}$ on matrices, for each k (see Exercise **??**).

Prove that the GNS representation of $U(n^\infty)$ associated to τ is isomorphic to the representation π_μ of $C_r^*(G) \cong U(n^\infty)$ described above.

a) Write down a formula for τ directly as a functional on $C_c(G)$, without appealing to the inductive limit structure.
b) By showing that the two representations π_τ and π_μ are both cyclic with cyclic vectors yield exactly the same state on $C_r^*(G) \cong U(n^\infty)$, and Proposition 5.10.
c) By writing down a direct unitary equivalence between π_τ and π_μ.
d) Prove that the representation π_μ contains the representation of $U(n^\infty)$ developed in Exercise 5.22 f) as a closed subrepresentation (in particular, π_μ is not irreducible.)

10. Strong Morita equivalence and transversals

THEOREM 10.1. *Let $G \times X \to X$ be a smooth action of a locally compact group G on a compact manifold X, and let $L \subset X$ be a closed submanifold transverse to every G-orbit in X.*

Let G_L be the groupoid obtained by restricting the transformation groupoid $G \ltimes X$ to L.

Then G_L is étale, and the C^*-algebras $C^*(G_L)$ and $C(X) \rtimes G$ are strongly Morita equivalent.

We will apply this result in the next section to produce a family of f.g.p. modules over the irrational rotation algebra A_θ of extreme relevance to the K-theory of this C*-algebra.

EXAMPLE 10.2. Let $\mathbb{Z}$ act by a diffeomorphism $\alpha\colon X \to X$ of a compact manifold X. Let the integers act on $\mathbb{R} \times X$ by $n \cdot (x, t) := (\alpha^n(x), t + n)$. This action is proper (and free). Form the quotient space by this action; the result is usually denoted $\mathbb{R} \times_{\mathbb{Z}} X$. This is called the *mapping cylinder* of the action.

We can visualize $\mathbb{R} \times_{\mathbb{Z}} X$ as obtained by taking $X \times [0,1]$ and identifying the 'sides' by glueing $(0,x)$ to $(1,\alpha(x))$, for all $x \in X$.

Note that the second projection map $\mathrm{pr}_1 \colon \mathbb{R} \times X \to \mathbb{R}$ induces a well-defined surjection $\pi \colon \mathbb{R} \times_{\mathbb{Z}} X \to \mathbb{R}/\mathbb{Z} = \mathbb{T}$. So the mapping cylinder $\mathbb{R} \times_{\mathbb{Z}} X$ is a bundle of spaces over the circle $\mathbb{T}$.

Another important point, is that $\mathbb{R} \times_{\mathbb{Z}} X$ carries a natural action of $\mathbb{R}$, with $\alpha_t([(s,x]) := [(s+t,x)]$. A natural transversal to the action is a copy of the original space X embedded in the mapping cylinder as $\{0\} \times X$. Restricting the flow on the cylinder to the transversal gives back the original $\mathbb{Z}$-action on X.

As a corollary of the theorem, the C*-algebras $C(X) \rtimes \mathbb{Z}$ and $C(X \times_{\mathbb{Z}} \mathbb{R}) \rtimes \mathbb{R}$ are therefore, strongly Morita equivalent.

We are going to construct a strong Morita equivalance $C^*(G_L)$-$C_0(X) \rtimes G$ bimodule $\mathcal{E}_L$. We break the task into steps. We start with the space $P := L \times G$, with the product topology. The bimodule we construct will be a completion of $C_c(P)$.

Before starting, let us point out a useful *ansatz* for the set of formulas given below. Consider $G \ltimes X$ as an abstract groupoid G'.

The space P may then be labelled as $\{\gamma \in G' \mid r(\gamma) \in L\}$. Now the idea behind making $C_c(P_L)$ into a left $C^*(G_L)$-module, is to let

$$(f \cdot \xi)(\gamma) = \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)\xi(\gamma_2), \tag{10.1}$$

where the exact meaning of the integral sign is to be determined.

And the idea for the $C^*(G_L)$-valued inner product is going to be

$${}_{C^*(G_L)}\langle \xi_1, \xi_2\rangle(\gamma) := \int_{\gamma_1\gamma_2=\gamma} \xi_1(\gamma_1)\xi_2^*(\gamma_2), \tag{10.2}$$

where $\xi^*(\gamma) := \overline{\xi(\gamma^{-1}})$.

Similarly, the right $C^*(G')$-module structure will be given by, for $f \in C_c(G')$,

$$(\xi \cdot f)(\gamma) = \int_{\gamma_1\gamma_2=\gamma} \xi_1(\gamma_1) f(\gamma_2) \tag{10.3}$$

and inner product by

$$\langle \xi_1, \xi_2\rangle_{C^*(G')}(\gamma) := \int_{\gamma_1\gamma_2=\gamma} \xi_1^*(\gamma_1)\xi_2(\gamma_2). \tag{10.4}$$

All the compatibility conditions for the Morita equivalence can now be checked in an algebraic, formal manner, and it is just a matter of assessing when these formulas make sense, and what the integral signs mean. The fact that these algebraic methods actually work topologically isa consequence of the transversality hypothesis.

To see this, examine (10.1). The element γ lies in P so $y := r(\gamma) \in L$. In order for $\gamma_1\gamma_2 = \gamma$ to make sense, $r(\gamma_1) = y$ as well, so $\gamma \in r^{-1}(y)$, a topologically discrete set. So the integral can be given meaning as a sum. Also, in order for $f(\gamma_1)$ to make sense, $x = s(\gamma_1) \in L$. Finally, γ_2 is determined by γ and γ_1, since $\gamma_2 = \gamma_1^{-1}\gamma$, so we define

$$(f \cdot \xi)(\gamma) = \sum_{\gamma_1 \in G_L,\, r(\gamma_1)=r(\gamma)} f(\gamma_1)\xi(\gamma_1^{-1}\gamma), \tag{10.5}$$

for the left action of $C_c(G_L)$ on $C_c(P)$.

For the $C^*(G_L)$-valued inner product, looking at (10.2), we note that if $\gamma = (x, g) \in G_L$, so $x, g^{-1}(x) \in L$, the only factorizations $\gamma = \gamma_1\gamma_2$ possible, with $\gamma_1, \gamma_2^{-1} \in P$, are $\gamma_1 = (x, h)$, $\gamma_2 = (h^{-1}x, h^{-1}g))$, for any $h \in G$. So the integral should be an integral with respect to Haar measure on G, and we define

$$ {}_{C^*(G_N)}\langle \xi_1, \xi_2\rangle(x, g) := \int_G \xi_1(x, h)\overline{\xi_2(g^{-1}x, g^{-1}h))}\, dh. \tag{10.6} $$

Next, we make sense of (10.10). In the formula, we may set $\gamma = (x, g) \in P = L \times G$. We change variables formally, to 'integrate' with respect to γ_1, and write $\gamma_2 = \gamma_1^{-1}\gamma$, where $\gamma_1\gamma_2 = \gamma$. Note that $r(\gamma_1) = r(\gamma) = x$. We obtain, formally,

$$ (\xi \cdot f)(\gamma) = \int_{r(\gamma_1)=x} \xi(\gamma_1) f(\gamma_1^{-1}\gamma). \tag{10.7} $$

Now if $\gamma_1 \in P$ and $r(\gamma_1) = x$ then $\gamma_1 = (x, h)$ for any $h \in G$, and $\gamma_1^{-1}\gamma = (h^{-1}x, h^{-1}g)$. So we set

$$ (\xi \cdot f)(x, g) = \int_G = \xi(x, h) f(h^{-1}x, h^{-1}g)\, dh, \quad x \in L, g \in G. \tag{10.8} $$

where $f \in C_c(G') = C_c(X \times G)$.

Similarly, we define a $C^*(G') = C(X) \rtimes G$-valued inner product by examining (10.11), which suggests the formula

$$ \langle \xi_1, \xi_2\rangle_{C_0(X)\rtimes G)}(x, g) := \sum_{h\in G, h^{-1}x\in L} \overline{\xi_1(h^{-1}x,\ h^{-1})}\xi_2(h^{-1}x, h^{-1}g), \quad x \in X, g \in G. \tag{10.9} $$

Theorem 10.3. *Let G, X, L be as in Theorem 10.1.*
Set $P := L \times G$ with the product topology. Let $C_c(G_L) \subset C^*(G_L)$ act on $C_c(P)$ by

$$ (f \cdot \xi)(x, g) = \sum_{h\in G,\ h^{-1}x\in L} f(x, h)\xi(h^{-1}x, h^{-1}g), \quad x \in L, g \in G. $$

Define a Hermitian $C^*(G_L)$-valued inner product by

$$ {}_{C^*(G_N)}\langle \xi_1, \xi_2\rangle(x, g) := \int_G \xi_1(x, h)\overline{\xi_2(g^{-1}x, g^{-1}h)}\, dh, \quad x \in L,\ g^{-1}x \in L. $$

Define a right $C_0(X) \rtimes G$-module structure by

$$ (\xi \cdot f)(x, g) = \int_G \xi(x, h) f(h^{-1}x, h^{-1}g)\, dh, \quad x \in L, g \in G, $$

and right $C_0(X) \rtimes G$-valued inner product by

$$ \langle \xi_1, \xi_2\rangle_{C_0(X)\rtimes G)}(x, g) := \sum_{h\in G, h^{-1}x\in L} \overline{\xi_1(h^{-1}x,\ h^{-1})}\xi_2(h^{-1}x, h^{-1}g), \quad x \in X, g \in G. $$

Then these structures complete $C_c(P_L)$ to a strong Morita equivalence $C^*(G_L)$-$C_0(X) \rtimes G$-bimodule $\mathcal{E}_L$.

Theorem 10.3 has an interesting application to constructing strong Morita equivalence bimodules between $C^*(G_L)$ and $C^*(G_{L'})$ for *different* transversals L, L' to the G-action on X. In this setting we can take the Hilbert module product of the strong Morita equivalence bimodules $\mathcal{E}_L$ and $\overline{\mathcal{E}}_{L'}$, where $\overline{\mathcal{E}}_{L'}$ is the conjugate, $C^*(G')$-$C^*(G_{L'})$-bimodule to $\mathcal{E}_{L'}$.

The strong Morita equivalence bimodules $\mathcal{E}_L, \mathcal{E}_{L'}$ are then defined as certain completions of $C_c(P), C_c(P')$ respectively, where P is the space of arrows in $G \ltimes X$ ending in a point of L, and

P' is the set of arrows ending in a point of L'. Let $\bar{P}'$ be the space of arrows *beginning* at a point of L'. In terms of the group G and space X,

$$P = \{(x,g) \in X \times G \mid x \in L\}, \quad \bar{P}' = \{(x,g) \in X \times G \mid g^{-1}(x) \in L'\}.$$

On $C_c(\bar{P}')$ we put a right $C^*(G_{L'})$-module structure as follows. We use groupoid language, as it seems clearer. If $f \in C_c(G_{L'})$ and $\xi \in C_c(\bar{P}')$, let

$$(\xi \cdot f)(\gamma) := \sum_{\gamma_1 \in G',\, s(\gamma_1) \in L'} \xi(\gamma_1) f(\gamma_1^{-1}\gamma),$$

This is basically the same formula as (10.5). Similarly, for an inner product valued in $C^*(G_{L'})$ we set

$$\langle \xi_1, \xi_2 \rangle_{C^*(G_{L'})}(\gamma) := \int_{r(\gamma_1)=r(\gamma)} \overline{\xi_1(\gamma_1^{-1})} \xi_2(\gamma_1^{-1}\gamma)\, d\gamma_1$$

where the integral is with respect to Haar measure on G under the identification of $r^{-1}(y) = \{y\} \times G \cong G$, for any y (*e.g.* $y = r(\gamma)$.) This is the analogue of the formula (10.6), written in more groupoid-theoretic language.

So $C_c(\bar{P}')$ has a right $C^*(G_{L'})$-module structure, and inner product. It also has a left $C^*(G')$-module structure, and inner product, again, as in the case of $C_c(P)$. The left $C^*(G')$-action is given by

$$(f \cdot \xi)(\gamma) = \int_{\gamma_1\gamma_2 = \gamma} f(\gamma_1)\xi(\gamma_1^{-1}\gamma) \tag{10.10}$$

for $f \in C_c(G')$, where $s(\gamma) \in L'$, and $d\gamma_1$ again is defined by Haar measure on G. The inner product is

$$_{C^*(G')}\langle \xi_1, \xi_2 \rangle(\gamma) \sum_{\gamma_1,\, s(\gamma_1) \in L',\, r(\gamma_1) = r(\gamma)} \xi_1(\gamma_1)\overline{\xi_2(\gamma^{-1}\gamma_1)}. \tag{10.11}$$

LEMMA 10.4. *The conjugate $C^*(G')$-$C^*(G_{L'})$-bimodule $\overline{\mathcal{E}}_{L'}$ is the completion of $C_c(\bar{P}')$ with the bimodule structures described above.*

The proof is a routine consequence of the definitions.

So far, we have $C_c(P)$ completing to a $C^*(G_L)$-$C^*(G')$-bimodule $\mathcal{E}_L$ and $C_c(\bar{P}')$ completing to a $C^*(G')$-$C^*(G_{L'})$-bimodule $\bar{\mathcal{E}}_{L'}$.

We aim to compute the composition, or tensor product

$$\mathcal{E}_L \otimes_{C^*(G')} \bar{\mathcal{E}}_{L'},$$

which will be a strong Morita equivalence $C^*(G_L)$-$C^*(G_{L'})$-bimodule.

DEFINITION 10.5. For a pair L, L' of transversals to G (a Lie group) acting smoothly on X, let $G' := G \ltimes X$ the corresponding transformation groupoid.

Let

$$P_{L,L'} = \{\gamma \in G' \mid r(\gamma) \in L, s(\gamma) \in L'\} \subset G'$$

Endow $C_c(P_{L,L'})$ with the structure of a strong Morita equivalence $C^*(G_L)$-$C^*(G_{L'})$ module using the actions and inner products as follows: (a picture, with L and L', with arrows, representing groupoid elements, between them, is quite helpful here).

$$\langle \xi, \eta \rangle_{C^*(G_{L'})}(\gamma) = \sum_{r(\gamma_1)=r(\gamma),\, s(\gamma_1) \in L} \overline{\xi(\gamma_1^{-1})}\eta(\gamma_1^{-1}\gamma), \quad \gamma \in G',\ r(\gamma) \in L', s(\gamma) \in L'. \tag{10.12}$$

where $\xi, \eta \in C_c(P_{L,L})$, $f \in C_c(G_{L'})$.

For the right $C^*(G_{L'})$-module structure we set

$$(\xi \cdot f)(\gamma) = \sum_{r(\gamma_1)=r(\gamma),\, s(\gamma_1)\in L'} \xi(\gamma_1) f(\gamma_1^{-1}\gamma). \quad \gamma \in G',\ r(\gamma) \in L', s(\gamma) \in L'. \tag{10.13}$$

Similarly for the left $C^*(G_L)$-module structures we define

$$_{C^*(G_L)}\langle \xi, \eta \rangle(\gamma) = \sum_{r(\gamma_1)=r(\gamma),\, s(\gamma_1)\in L'} \xi(\gamma_1)\overline{\eta(\gamma^{-1}\gamma_1)}, \quad \gamma \in G',\ r(\gamma) \in L, s(\gamma) \in L. \tag{10.14}$$

and

$$(f \cdot \xi)(\gamma) = \sum_{r(\gamma_1)=r(\gamma),\, s(\gamma_1)\in L} f(\gamma_1)\xi(\gamma_1^{-1}\gamma). \quad \gamma \in G',\ r(\gamma) \in L, s(\gamma) \in L'. \tag{10.15}$$

We let $\mathcal{E}_{L,L'}$ be the corresponding completion.

Theorem 10.6. *$\mathcal{E}_{L,L'}$ is naturally isomorphic, as a strong Morita equivalence $C^*(G_L)$-$C^*(G_{L'})$-bimodule, to $\mathcal{E}_L \otimes_{C^*(G')} \bar{\mathcal{E}}_{L'}$.*

The module tensor product $\mathcal{E}_L \otimes_{C^*(G')} \bar{\mathcal{E}}_{L'}$ starts by taking the quotient of the (algebraic) *external* tensor product $\mathcal{E}_L \otimes_{\mathbb{C}} \bar{\mathcal{E}}_{L'}$, *i.e.* the tensor product over $\mathbb{C}$, by certain relations, those spanned by the elements

$$\xi \cdot a \otimes \eta - \xi \otimes a \cdot \eta,\ \ \xi \in \mathcal{E}_L, \eta \in \bar{\mathcal{E}}_{L'},\ a \in C^*(G'). \tag{10.16}$$

In the following, we will work largely with $\xi \in C_c(P)$, $\eta \in C_c(\bar{P}')$, so that $\xi \otimes \eta$ can be considered, to begin with, either as an element of the tensor product over $\mathbb{C}$ of $\mathcal{E}_L$ and $\bar{\mathcal{E}}_{L'}$, or, subsequently, as an element of the module tensor product $\mathcal{E}_L \otimes_{C^*(G')} \bar{\mathcal{E}}_{L'}$. An element of the tensor product over $\mathbb{C}$ can be considered a function on $P \times \bar{P}'$.

The tensor product is over $C^*(G')$, and $G' = G \ltimes X$ is a transformation groupoid. We extricate its right multiplication on $C_c(P)$ and its left multiplication on $C_c(\bar{P}')$ in terms of X and G.

Let $\xi \in C_c(P)$ and $\eta \in C_c(\bar{P}')$. Both the right action of $C^*(G') = C(X) \rtimes G$ on $C_c(P)$ and the left action of $C^*(G')$ on $C_c(\bar{P}')$ are given by covariant pairs. In terms of the group action, P is points (x, g) where $x \in L$, and $\bar{P}'$ is pairs (x, g) where $g^{-1}x \in L'$.

Let $f \in C(X)$. Then f acts on the left and right respectively by

$$(\xi \cdot f)(x, h) = \xi(x, h) f(h^{-1}x), \quad (f \cdot \eta)(x, h) = f(x)\eta(x, h)(g^{-1}x, g^{-1}h), \tag{10.17}$$

where $x \in L, h^{-1}x \in L'$.

The group G acts, on the right, on $C_c(P)$, and on the left, on $C_c(\bar{P}')$, by

$$(\xi \cdot g)(x, h) = \xi(x, hg^{-1}), \quad (g \cdot \eta)(x, h) = (g^{-1}x, g^{-1}h). \tag{10.18}$$

By exercise 4.49, the vectors

$$\xi \cdot a \otimes \eta - \xi \otimes a \cdot \eta \in \mathcal{E}_L \otimes_{C(X)\rtimes G} \bar{\mathcal{E}}_{L'}$$

are zero even for a in the multiplier algebra of $C^*(G') = C(X) \rtimes G$. In particular, it applies to delta functions δ_g at points $g \in G$. That is,

$$\xi \cdot g \otimes \eta = \xi \otimes g \cdot \eta$$

in the tensor product $\mathcal{E}_L \otimes_{C(X)\rtimes G} \bar{\mathcal{E}}_{L'}$.

More abstractly, let $\pi \colon P \to X$ be $\pi(x, h) = h^{-1}x$ and $\pi' \colon \bar{P}' \to X$ be $\pi'(x, h) = x$. Define a linear map

$$\Phi' \colon C_c(P) \otimes_{\mathbb{C}} C_c(\bar{P}') \to C_c(P \times_X \bar{P}')$$

by

$$\Phi'(\xi \otimes \eta)(p, p') := \xi(p)\eta(p'). \tag{10.19}$$

This map passes to the quotient of $C_c(P) \otimes_{\mathbb{C}} C_c(\bar{P}')$ by the subspace generated by elements $\xi \cdot f \otimes \eta - \xi \otimes f \cdot \eta$ since

$$\Phi'\left[(\xi \cdot f) \otimes \eta - \xi \otimes f \cdot \eta\right](p, p') = \xi(p) f\big(\pi(p)\big)\eta(p) - \xi(p) f\big(\pi'(p')\big)\eta(p') = 0 \tag{10.20}$$

since $\pi(p) = \pi'(p')$.

It follows from this discussion that the tensor product $\mathcal{E}_L \otimes_{C^*(G')} \bar{\mathcal{E}}_{L'}$ is (a completion of) the quotient of the linear space $C_c(P \times_X P)$, equipped with certain left and right inner products, by the span of the elements

$$\xi \cdot g \otimes \eta - \xi \otimes g \cdot \eta, \tag{10.21}$$

with $g \in G$.

An element of $P \times_X \bar{P}'$ is a pair $\big((x, g), (y, h)\big)$ where $x \in L$, $g^{-1}x = y$, and $h^{-1}y \in L'$.

We are going to define a Hilbert bimodule isomorphism

$$\Phi \colon \mathcal{E}_L \otimes_{C^*(G')} \bar{\mathcal{E}}_{L'} \to \mathcal{E}_{L,L'},$$

by ending the map $\Phi \colon C_c(P \times_X \bar{P}') \to C_c(P_{L,L'})$ given by

$$\Phi(\xi \otimes \eta)(x, g) := \int_G \xi(x, s)\eta(s^{-1}x, s^{-1}g)\, ds, \tag{10.22}$$

where $\xi \in C_c(P)$, $\eta \in C_c(\bar{P}')$, $x \in L$, $g \in G$ and $g^{-1}x \in L'$.

We have, for $f \in C(X)$,

$$\Phi(\xi \cdot f \otimes \eta)(x, g) := \int_G \xi(x, s) f(s^{-1}x)\eta(s^{-1}x, s^{-1}g)\, ds = \Phi(\xi \otimes f \cdot \eta) \tag{10.23}$$

by the definitions.

Using the definitions (10.18) of the left and right group actions, we compute next that

$$\begin{aligned} \Phi(\xi \cdot h \otimes h^{-1}\eta)(x, g) &= \int_G \xi(x, sh^{-1})\eta(hs^{-1}x, hs^{-1}g)\, ds \\ &= \int_G \xi(x, s)\eta(s^{-1}x, s^{-1}g)\, ds = \Phi(\xi \otimes \eta)(x, g), \end{aligned} \tag{10.24}$$

by making the change of variables s to sh, so that Φ is invariant under the diagonal action $h \cdot (\xi \otimes \eta) := \xi \cdot h^{-1} \otimes h \cdot \eta$.

Therefore, we have constructed a map

$$\Phi \colon C_c(P) \otimes_{C^*(G')} C_c(\bar{P}') \to C_c(P_{L,L'}). \tag{10.25}$$

THEOREM 10.7. *The map Φ of* (10.25) *extends to an isomorphism $\mathcal{E}_L \otimes_{C^*(G')} \bar{\mathcal{E}}_{L'} \to \mathcal{E}_{L,L'}$ of strong Morita equivalence $C^*(G_L)$-$C^*(G_{L'})$-bimodules.*

PROOF. The map Φ is given by

$$\Phi(\xi \otimes \eta)(x, g) := \int_G \xi(x, s)\eta(s^{-1}x, s^{-1}g)\, ds, \tag{10.26}$$

where $\xi \in C_c(P)$, $\eta \in C_c(\bar{P}')$, $x \in L$, $g \in G$ and $g^{-1}x \in L'$.

Let $\varphi \in C_c(G)$ with $\int_G \varphi(s)\, ds = 1$. We define a map of bimodules

$$\Psi \colon \mathcal{E}_{L,L'} \to \mathcal{E}_L \otimes_{C^*(G')} \bar{\mathcal{E}}_{L'}$$

by

$$(\Psi\xi)(x,h,h^{-1}x,g) := \xi(x,hg)\varphi(h) \tag{10.27}$$

Then

$$\begin{aligned}(\Phi\circ\Psi)\xi(x,t) &= \int_G (\Psi\xi)(x,s,s^{-1}x,s^{-1}t)\,ds \\ &= \int_G \xi(x,t)\varphi(s)\,ds = \xi(x,t), \quad x\in L,\ t^{-1}x\in L'.\end{aligned} \tag{10.28}$$

so that $\Phi\circ\Psi$ is the identity $C_c(P_{L,L'}) \to C_c(P\times_X \bar{P}') \to C_c(P_{L,L'})$. We leave it to the reader to check that $(\Psi\circ\Phi)\xi - \xi$ is a null vector in $\mathcal{E}_L \otimes_{C^*(G')} \bar{\mathcal{E}}_{L'}$.

$\square$

Kronecker flow on the 2*-torus, the irrational rotation algebra*

Let $X = \mathbb{T}$, understood as $\mathbb{R}/\mathbb{Z}$, and the action of $\mathbb{Z}$ be by irrational rotation: $\alpha(x) = x+\theta \bmod \mathbb{Z}$. Then $\mathbb{R}\times_{\mathbb{Z}}\mathbb{T}$ is obtained by taking the square $[0,1]\times[0,1]$, identifying the top and bottom edges by the identity map, so $(0,t)$ is identified with $(1,t)$, for all $t\in[01,]$, and identifying the sides by glueing $(0,s)$ to $(1, s+\theta \bmod \mathbb{Z})$.

There is a smooth $\mathbb{R}$ action on $\mathbb{R}\times_{\mathbb{Z}}\mathbb{T}$ by $\alpha_t([(x,y)]) := [(x,y+t)]$.

We can visualize the construction as follows. The glueing of the top and bottom sides of the square results in a cylinder. One then attaches the ends together after twisting one of them by $2\pi\theta$ radians. The space obtained is clearly homeomorphic to a torus. In fact

$$h\colon \mathbb{R}\times\mathbb{T} \to \mathbb{R}\times_{\mathbb{Z}}\mathbb{T}, \quad h(x,y) := (x, y-x\theta)$$

satisfies

$$h(x+1,y) = (x+1, y-(x+1)\theta) = (x+1, y-x\theta-\theta)$$

and in $\mathbb{R}\times_{\mathbb{Z}}\mathbb{T}$ this point is equivalent to $(x,y-x\theta) = h(x,y)$. This shows that h descends to a map $\mathbb{T}\times\mathbb{T}\to\mathbb{T}\times_{\mathbb{Z}}\mathbb{R}$.

The map h conjugates the flow on $\mathbb{R}\times_{\mathbb{Z}}\mathbb{T}$ to an $\mathbb{R}$-action on $\mathbb{T}^2$ whose orbits are the projections to $\mathbb{T}^2$ of lines in the plane $\mathbb{R}^2$ of slope θ. These lines make a 1-dimensional foliation of $\mathbb{T}^2$ called the *Kronecker foliation*. In usual $\mathbb{T}^2$-coordinates we have, thus

$$\alpha_t(x,y) = (x+t, y+t\theta) \bmod \mathbb{Z}^2.$$

The space X we began with, still sits after these identifications, as the slice $\{0\}\times\mathbb{T}\in\mathbb{T}^2$.

Let p and q be two relatively prime integers, say positive. Let $L_{p,q}$ be the line in the plane of slope $\frac{p}{q}$ through the origin. On its way from the origin in the plane to the lattice point (q,p), it meets no other lattice points, so it describes a closed, simple curve in the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, (and in particular it describes a a homeomorphic copy of the circle $\mathbb{T}$.) It follows that if one starts at any point of $\mathbb{T}^2$ and moves along an irrational flow line, one will eventually hit $L_{p,q}$. The closed subset $L_{p,q}\subset\mathbb{T}^2$ is a transversal for the $\mathbb{R}$-action, and the general results above apply to this situation

Let $G_{p,q}$ denote the restriction of the groupoid $\mathbb{R}\ltimes\mathbb{T}^2$ to the transversal $L_{p,q}$.

Lemma 10.8. *The restriction $G_{1,0}$ of the Kronecker flow to the transversal $L_{1,0}$ is the integer action of translation by θ on $\mathbb{T}$: that is, $G_{1,0}\cong\mathbb{Z}\ltimes_\theta\mathbb{T}$ as (topological) groupoids.*

PROOF. Let $y \in \mathbb{R}/\mathbb{Z}$, represent it as a real number on the y-axis of $\mathbb{R}^2$. We follow the line $l(t) = (t, y + t\theta)$ of irrational slope θ through $(0, y)$, until it hits an integer translate of the y-axis. This obviously occurs first when the irrational line hits the line $x = 1$. The points on the line are given by $(t, y + t\theta)$ and so $t = 1$ at the crossing point and y has been shifted to $y + t\theta$.

□

By Theorem 10.3, we have:

PROPOSITION 10.9. *The irrational rotation algebra $A_\theta := C(\mathbb{T}) \rtimes \mathbb{Z}$ with $\mathbb{Z}$ acting by irrational rotation θ on $\mathbb{R}/\mathbb{Z}$, is strongly Morita equivalent to $C(\mathbb{T}^2) \rtimes \mathbb{R}$ with $\mathbb{R}$ acting by the Kronecker flow on $\mathbb{T}^2$ along lines of slope θ.*

Let $\mathcal{E}_{p,q}$ the $C^*(G_{p,q})$ - $C(\mathbb{T}^2) \rtimes \mathbb{R}$ bimodule constructed in Theorem 10.3. We describe it explicitly for this particular example.

A point of $G_{p,q}$ is a morphism in $\mathbb{R} \ltimes \mathbb{T}^2$ which begins and ends in $L_{p,q}$. So it is labelled by a pair (x, t) where $x, \alpha_{-t}(x) \in L_{p,q}$.

THEOREM 10.10. *Endow $C_c(L_{p,q} \times \mathbb{R})$ with the left action of $C^*(G_{p,q})$ induced by setting, for $f \in C_c(G_{p,q})$ and $\xi \in C_c(L_{p,q} \times \mathbb{R})$:*

$$(f \cdot \xi)(x,t) = \sum_{s \in \mathbb{R},\ \alpha_{-s}(x) \in L_{p,q}} f(x,s)\xi(\alpha_{-s}(x), -s + t), \qquad x \in L_{p,q}, t \in \mathbb{R}, \alpha_{-t}(x) \in L_{p,q}, \tag{10.29}$$

For the $C^*(G_{p,q})$-valued inner product, we set

$$_{C^*(G_{p,q})}\langle \xi_1, \xi_2 \rangle(x,t) := \int_{\mathbb{R}} \xi_1(x, t-s)\overline{\xi_2(x,-s)}\, ds, \qquad x \in L_{p,q}, \alpha_{-t}(x) \in L_{p,q}. \tag{10.30}$$

Endow the same space with the right action of $C(\mathbb{T}^2) \rtimes \mathbb{R}$ given on $f \in C_c(\mathbb{T}^2 \times \mathbb{R})$ by

$$(\xi \cdot f)(x,t) = \int_{\mathbb{R}} \xi(x,s) f(\alpha_{-s}(x), -s + t)\, ds, \quad x \in L_{p,q}, t \in \mathbb{R}. \tag{10.31}$$

and $C(\mathbb{T}^2) \rtimes \mathbb{R}$-valued inner product

$$\langle \xi_1, \xi_2 \rangle_{C(\mathbb{T}^2)\rtimes\mathbb{R}}(x,t) := \sum_{s \in \mathbb{R}, \alpha_{-s}(x) \in L_{P,q}} \overline{\xi_1(\alpha_{-s}(x), -s)}\xi_2(\alpha_{-s}(x), t-s), \quad x \in \mathbb{T}^2, t \in \mathbb{R}. \tag{10.32}$$

Then $C_c(L_{p,q} \times \mathbb{R})$ completes under these inner products to a strong Morita $C^*(G_{p,q})$-$C_0(\mathbb{T}^2) \rtimes \mathbb{R}$-equivalence bimodule $\mathcal{E}_{p,q}$.

REMARK 10.11. The right $C(\mathbb{T}^2) \rtimes \mathbb{R}$-module structure is given by a covariant pair for the opposite (negated) action of $\mathbb{R}$ on $C(\mathbb{T}^2)$. Namely, consider the 1-parameter family of maps $\mathcal{E}_{p,q} \to \mathcal{E}_{p,q}$ given by $(\xi \cdot U_s)(x,t) := \xi(x, t+s)$, and the right action of $C(\mathbb{T}^2)$ given by $(\xi \cdot f)(x,t) := \xi(x,t) f\big(\alpha_{-t}(x)\big)$. The map on functions is the *-homomorphism $C_c(\mathbb{T}^2 \times \mathbb{R}) \to C_b(L_{p,q} \times \mathbb{R})$ Gelfand dual to the map $L_{p,q} \times \mathbb{R} \to \mathbb{T}^2,\ (x,t) \mapsto \alpha_{-t}(x)$, and it is suitably intertwined by the family $\{U_s\}_{s \in \mathbb{R}}$ to give a covariant pair, for the opposite action, because we are dealing with a right module.

In particular:

COROLLARY 10.12. *The right Hilbert $C(\mathbb{T}^2) \rtimes \mathbb{R}$-modules $\mathcal{E}_{p,q}$ described in Theorem 10.10 are f.g.p.modules over $C(\mathbb{T}^2) \rtimes \mathbb{R}$.*

Next, by Theorem 10.6, the product, or composition, of strong Morita equivalence bimodules

$$\mathcal{L}_{p,q} := \mathcal{E}_{p,q} \otimes_{C(\mathbb{T}^2)\rtimes\mathbb{R})} \bar{\mathcal{E}}_{1,0} \tag{10.33}$$

is isomorphic, as a $C^*(G_{p,q})$-$C^*(G_{1,0})$ bimodule, to the bimodule $\mathcal{E}_{L_{p,q},L_{1,0}}$, associated to a *pair* of transversals. We will just denote it by by $\mathcal{L}_{p,q}$.

By Lemma 10.8, $C^*(G_{1,0}) \cong A_\theta$ so the $\mathcal{L}_{p,q}$ are strong Morita equivalence $C^*(G_{p,q})$-A_θ bimodules. We aim to describe them as concretely as possible. To do this, a group-theoretic interpretation of the Kronecker flow, the transversals, and so on, is quite helpful.

Let $H = L_{p,q}$ and $K = L_{1,0}$, which are two closed subgroups of the compact group $\mathbb{T}^2$. Let $U \subset \mathbb{T}^2$ the *dense* 1-parameter subgroup generated by $(1,\theta) \in \mathbb{T}^2$. Let

$$e\colon \mathbb{R} \to \mathbb{T}^2$$

the group isomorphism $e(t) = t \cdot (1,\theta)$, with image U.

We set

$$P := \{(x,y) \in H \times K \mid x - y \in U\}.$$

We topologize P not as a subspace of $H \times K$ but as a subspace of $H \times \mathbb{R}$ under the embedding

$$P \to H \times \mathbb{R}, \quad i(x,y) := \big(x, e^{-1}(x-y)\big)$$

With this topology, the projections

$$\pi_1\colon P \to H \quad \pi_2\colon P \to K$$

are each local homeomorphisms, thus with topologically discrete fibres. Observe that if $y \in K$ then

$$\pi_2^{-1}(y) = \{(x,y) \in H \times K \mid x - y \in U\} \cong \{x \in H \mid x \in y + U\}$$

is in natural bijective correspondence with the intersection of the group coset $y + U$ with K . Similarly, the fibre $\pi_1^{-1}(x)$ over $x \in H$ is $\cong H \cap (x+U)$. Note that the cosets $x + U$ are the orbits of the $\mathbb{R}$-action through x. Intersecting these cosets with H or K yields countable sets.

From the maps π_1, π_2 we obtain a left module structure of $C_c(P)$ over $C(H)$ and a right module structure of $C_c(P)$ over $C(K)$:

$$f \cdot \xi \cdot g := (f \circ \pi_1) \cdot \xi \cdot (g \circ \pi_2) \in C_c(P). \tag{10.34}$$

Let

$$\Gamma_1 := U \cap H, \quad \Gamma_2 := U \cap K.$$

Each of Γ_1 and Γ_2 are countable groups, and under e they correspond to a pair of countable subgroups of $\mathbb{R}$. We define a left action of Γ_1 on P and a right action of Γ_2 on P by

$$\gamma_1 \cdot (x,y) \cdot \gamma_2 := (x + \gamma_1, y + \gamma_2).$$

These actions induce actions on $C_c(P)$ as usual by

$$(\gamma_1 \cdot \xi)(x,y) := \xi(x - \gamma_1, y), \quad (\xi \cdot \gamma_2)(x,y) := \xi(x, y - \gamma_2) \tag{10.35}$$

and it is routine to check the covariance conditions are met to give $C_c(P)$ the structure of a $C(H) \rtimes \Gamma_1$-$C(K) \rtimes \Gamma_2$ bimodule structure.

For a $C(K) \rtimes \Gamma_2$-valued inner product we set

$$\langle \xi, \eta \rangle_{C(K)\rtimes\Gamma_2}(y,\gamma) := \sum_{x \in\ H \cap (y+U)} \overline{\xi(x,y)}\eta(x, y+\gamma). \tag{10.36}$$

and for a $C(H) \rtimes \Gamma_1$-valued inner product we set

$$_{C(H)\rtimes\Gamma_1}\langle \xi, \eta \rangle(x,\gamma) := \sum_{y \in\ K \cap (x+U)} \xi(x+\gamma, y)\overline{\eta(x,y)}. \tag{10.37}$$

EXERCISE 10.13. Show that the conditions are met for a strong Morita equivalence bimodule.

LEMMA 10.14. *In the above notation:*

a) *The subgroup* $\Gamma_2 \subset \mathbb{R}$ *is the group* $\mathbb{Z}$ *of integers. Moreover, under the identification of* $K = L_{1,0}$ *with* $\mathbb{T}$*, the translation action of* Γ_1 *on* K *corresponds to the action of* $\mathbb{Z}$ *on* $\mathbb{T}$ *by irrational rotation. Hence* $C(K) \rtimes \Gamma_2$ *is naturally isomorphic to* A_θ*.*
b) *The groupoid* $G_{p,q}$ *is naturally isomorphic to the transformation groupoid* $\Gamma_1 \ltimes L_{p,q}$*.*

Note that we have already computed in Lemma 10.8 that the groupoid $G_{1,0}$ is also the irrational rotation action, so part a) says that $G_{1,0} \cong \Gamma_2 \ltimes K$.

PROOF. In the present notation $K = L_{1,0} = \{0\} \times \mathbb{T}$ and therefore the condition that $e(t) \in K$ is that $e(t) := (t, t\theta)$ has first coordinate zero modulo $\mathbb{Z}$, and hence says that $t \in \mathbb{Z}$. Hence $e^{-1}(\Gamma_2) = \mathbb{Z}$. By the definitions, Γ_2 acts on $K := L_{1,0}$ by group translation. We have to compute this action and show it is the action of translation by θ, after identifying $L_{1,0}$ with $\mathbb{T}$.

The calculation just done shows that the integer n, considered as an element of the group $e^{-1}(\Gamma_2) \subset \mathbb{R}$, acts on $\mathbb{T}^2$ by translation by the group element $(n, n\theta)$. Hence it maps $(0, x)$ to $(n, x+n\theta) = (0, x+n\theta)$ as elements of $\mathbb{T}^2$. This verifies that Γ_2 acting on K by group translation is exactly the action of $\mathbb{Z}$ on $\mathbb{T}$ by translation by θ, as claimed.

For b), an element of $C^*(G_{p,q})$ is a pair $\gamma = (x, t) \in L_{p,q} \times \mathbb{R}$ where $x \in L_{p,q}$ and $x - e(t) \in L_{p,q}$. Hence $e(t) \in \Gamma_1$, and so we may regard (x, t) as an element of the transformation groupoid $\Gamma_1 \ltimes L_{p,q}$. The reader can easily check that the composition rules are the same.

□

THEOREM 10.15. *The completion of* $C_c(P)$ *using the inner product* (10.36) *is a strong Morita equivalence* $C(H) \ltimes \Gamma_1$*-*A_θ*-bimodule, isomorphic to* $\mathcal{L}_{p,q}$*; that is, to the composition of Morita equivalences* $\mathcal{E}_{p,q} \otimes_{C(\mathbb{T}^2) \rtimes \mathbb{R})} \bar{\mathcal{E}}_{1,0}$*.*

PROOF. Going back to Definition 10.5, recall that an element of $P_{L,L'}$ is a groupoid element $\gamma \in G' := \mathbb{R} \ltimes \mathbb{T}^2$ beginning in L' and ending in L. It is thus given by a pair $(x, t) \in \mathbb{T}^2 \times \mathbb{R}$ such that $x \in L$ and $\alpha_{-t}(x) \in L'$. In the present context, $L = H$ and $L' = K$, and $\alpha_{-t}(x) = x - e(t)$. So $P_{L,L'}$ can be parameterized by pairs $x \in H$ and $t \in \mathbb{R}$ such that $x - e(t) \in K$. This is equivalent to

$$e(t) \in x + K, \quad \text{or} \ \ t \in e^{-1}(x + K).$$

(and in the above notation, $e(t) \in U$).

The map $P_{L,L'} \to P$, $(x, t) \mapsto \big(x, x - e(t)\big)$ therefore defines a canonical homeomorphism. We leave it to the reader that the rest of the definitions of Definition 10.5 translate into those given above.

□

We next analyze the C*-algebras $C^*(G_{p,q})$, and show that in fact they are other irrational rotation algebras $A_{\theta'}$. We have already checked that $C^*(G_{p,q})$ is the transformation groupoid of the group $\Gamma_1 := U \cap L_{p,q}$ acting by group translation on $L_{p,q}$.

LEMMA 10.16. $e^{-1}(\Gamma_1)$ *is the infinite cyclic subgroup of* $\mathbb{R}$ *generated by* $\frac{1}{q\theta - p}$*, and* Γ_1 *is the infinite cyclic subgroup of* $L_{p,q}$ *generated by the element* $(\frac{1}{q\theta-p}, \frac{\theta}{q\theta-p})$ *of the group* $\mathbb{T}^2$*.*

PROOF. An element of Γ_1 is of the form $(t, t\theta)$ which is in $L_{p,q}$. The subgroup $L_{p,q}$ is the kernel of $\gamma_{p,q} \colon \mathbb{T}^2 \to \mathbb{T}$, $\gamma_{p,q}(x, y) := -px + qy$. We need to solve for $t \in \mathbb{R}$ for which

$$-pt + qt\theta = 0 \bmod \mathbb{Z}$$

and we obtain

$$t = \frac{n}{q\theta - p}$$

for some integer n. Clearly the collection of solutions is the cyclic group generated by $\frac{1}{q\theta-p}$, as claimed.

□

LEMMA 10.17. *Fix integers m, n such that*

$$np + mq = 1. \tag{10.38}$$

Then the action of Γ_1 by group translation on $L_{p,q}$ is naturally conjugate to the action of the integers $\mathbb{Z}$ on $\mathbb{T}$ by translation by $= \frac{n\theta+m}{-q\theta+p} \in \mathbb{R}/\mathbb{Z}$.

PROOF. The group $L_{p,q}$ is naturally isomorphic to $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by the map $\phi\colon \mathbb{T} \to L_{p,q}$, $\phi(x) = x \cdot (q,p) \bmod \mathbb{Z}^2$. The inverse map is $\phi^{-1}(x,y) = mx + ny$. We have shown above that the group Γ_1 is isomorphic to the integers. We prefer to fix the isomorphism $\Gamma_1 \cong \mathbb{Z}$ which maps $(\frac{1}{-q\theta+p}, \frac{\theta}{-q\theta+p})$ (the negative of the generator discussed above) to $1 \in \mathbb{Z}$.

Then conjugating the action of the generator acting on $L_{p,q}$, by the homeomorphism ϕ gives, since the homeomorphism is also a group isomorphism, the action of the group element

$$\phi^{-1}(\frac{1}{-q\theta+p}, \frac{\theta}{-q\theta+p}) = \frac{m}{-q\theta+p} + \frac{n\theta}{-q\theta+p} = \frac{n\theta+m}{-q\theta+p}$$

of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by group translation. □

The element

$$\theta' = \frac{n\theta+m}{-q\theta+p}.$$

is the result of applying the Möbius transformation

$$z \mapsto \frac{nz+m}{-qz+p}$$

to θ. The matrix $\begin{bmatrix} n & m \\ -q & p \end{bmatrix}$ is in $\mathrm{SL}_2(\mathbb{Z})$ due to (10.38). Since the action of a matrix by the corresponding Möbius transformation only depends on the class of the matrix in $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$, we prefer to regard $\begin{bmatrix} n & m \\ -q & p \end{bmatrix}$ as an element of $\mathrm{PSL}_2(\mathbb{Z})$. As p, q run over all relatively prime integers, it is now easy to see that the matrix $\begin{bmatrix} n & m \\ -q & p \end{bmatrix}$ runs over all elements of $\mathrm{PSL}_2(\mathbb{Z})$. We have therefore in particular proved the following result:

COROLLARY 10.18. *Suppose $\theta, \theta' \in \mathbb{R}$ are in the same orbit of $\mathrm{PSL}_2(\mathbb{Z})$ acting on $\mathbb{R}$ by Möbius transformations. Then the irrational rotation algebras A_θ and $A_{\theta'}$ are strongly Morita equivalent.*

We finish this section by constructing projections $e_{p,q} \in A_\theta$ which correspond to the finitely generated projective modules $\mathcal{L}_{p,q}$. We use Proposition 6.6. If we can find ξ in the strong Morita equivalence bimodule $\mathcal{L}_{p,q}$ such that

$$_{C^*(G_{p,q})}\langle \xi, \xi \rangle = 1$$

then

$$e := \langle \xi, \xi \rangle_{A_\theta}$$

will be a projection in A_θ such that $\mathcal{L}_{p,q} \cong eA$ as right Hilbert A-modules.

The space P is actually a smooth manifold. In fact it is naturally parameterized by $\mathbb{R} \times \mathbb{Z}/q$. Indeed, an element of P is an element (x,t) of $G' = \mathbb{R} \ltimes \mathbb{T}^2$ with source in $L_{1,0}$ and range in

$L_{p,q}$, and they differ by an element of the dense subgroup $U = \{(1, t\theta) \mid t \in \mathbb{R}\} \subset \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Thus,

$$x - (t, t\theta) \in L_{1,0}, \quad \text{where } x \in L_{p,q}.$$

If $y \in \mathbb{R}$ is such that $(0, y) = x - (t, t\theta) \bmod \mathbb{Z}^2$ then equivalently,

$$(0, y) + (t, t\theta) \in L_{p,q}.$$

Hence, by the equation defining the line $L_{p,q}$,

$$(t, y + t\theta) \in L_{p,q}, \quad \text{or } - pt + q(y + t\theta) = 0 \bmod \mathbb{Z}$$

from which

$$qy = t(p - q\theta) \mod \mathbb{Z} \tag{10.39}$$

follows.

The equation (10.39) does not have a unique solution $y \bmod \mathbb{Z}$, since if y is a solution, so is $y' = y + \frac{k}{q}$, for $0 \leq k < q$. However, these are all the solutions. There is in particular a unique solution in the interval $[0, \frac{1}{q})$. Label this solution y_0 and let $y = y_0 + \frac{k}{q}$. Then we assign our groupoid element the coordinates (t, k).

A geometric picture is as follows. To produce elements of P with real coordinate t, we shift the line $L_{p,q}$ through the origin in the plane, by the vector $-(t, t\theta)$. An intersection point of this shifted line with the $x = 0$ axis in the plane, defines an element $y \in \mathbb{T}$ and the k-coordinate of the corresponding groupoid element is $k = 0$.

The same shifted line intersects the vertical lines $x = 1, x = 2, \ldots, x = q-1$. An intersection point belonging to the $x = k$ line defines a groupoid element with coordinates t, k.

Since the source of an arrow with coordinates (t, k) is given by

$$y := t \cdot (\frac{p}{q} - \theta) + \frac{k}{q} \ \in \mathbb{R}/\mathbb{Z},$$

the right $C(\mathbb{T})$-module structure is determined by the map $\mathbb{R} \times \mathbb{Z}/q \to \mathbb{T}$ sending $(t, k) \to y_k$ (the source of (t, k) if one thinks of (t, k) as a groupoid element.) Thus, $f \in C(\mathbb{T})$ acts on the kth copy of $C_c(\mathbb{R})$ by

$$(\xi \cdot f)(t, k) = \xi(t, k) \cdot f\left[(\frac{p}{q} - \theta)\, t + \frac{k}{q}\right]. \tag{10.40}$$

If y is replaced by $y' = y + \theta$ in the equation (10.39) then t is replaced by

$$t' = t + \frac{q\theta}{p - q\theta}. \tag{10.41}$$

Therefore the $\mathbb{Z}$ action on $C_c(\mathbb{R} \times \mathbb{Z}/q)$ has the generator $1 \in \mathbb{Z}$ acting the map

$$(\xi \cdot U)(t, k) = \xi(t + \frac{q\theta}{p - q\theta}, k).$$

EXERCISE 10.19. Verify that

$$\xi \cdot UfU^{-1} = \xi \cdot \alpha(f)$$

where α is the automorphism of $C(\mathbb{T})$ defining A_θ. This is the covariant pair condition for the right bimodule structure defined above.

The right A_θ-valued inner product is given by

$$\langle \xi, \eta \rangle_{A_\theta}(y, n) = \sum_{k=0}^{q-1} \sum_{m=-\infty}^{\infty} \overline{\xi(\frac{qy + m}{p - q\theta}, k)}\, \eta(\frac{qy + m}{p - q\theta} + \frac{nq\theta}{p - q\theta}, k). \tag{10.42}$$

As discussed above, $G_{p,q} \cong \mathbb{Z} \ltimes L_{p,q}$, with the integer action on $L_{p,q}$ being group translation by $\frac{1}{p-q\theta}$.

A similar exercise in the definitions produces the following formula for the left $C^*(G_{p,q})$-valued inner product, in which we identify $C^*(G_{p,q})$ with a crossed-product $C(L_{p,q}) \rtimes \mathbb{Z}$ by the action explained above.

$$ {}_{C^*(G_{p,q})}\langle \xi, \eta \rangle(x,n) = \sum_{k=0}^{q-1} \sum_{m=-\infty}^{\infty} \xi(x+m),k) \cdot \overline{\eta\big(x+m+\frac{n}{p-q\theta}\big)}, \tag{10.43} $$

where in the formula, x is the first coordinate of a point of $L_{p,q}$.

In particular, if ξ is a compactly supported real-valued function on $\mathbb{R}$, extended to $\mathbb{R} \times \mathbb{Z}/q$ by zero outside $\mathbb{R} \times \{0\}$, then

$$ {}_{C^*(G_{p,q})}\langle \xi, \xi \rangle(x,n) = \sum_{m=-\infty}^{\infty} \xi(x+m)\xi(x+m+\frac{n}{p-q\theta}), \tag{10.44} $$

Now suppose that $\frac{1}{|p-q\theta|} > 1$, or, equivalently, that

$$ |\frac{p}{q} - \theta| < \frac{1}{q}. $$

Let $\frac{1}{|p-q\theta|} = 1 + \epsilon$. We can then easily construct a function ξ supported in the interval

$$ [0, 1 + \frac{\epsilon}{2}] $$

such that

$$ \sum_{m \in \mathbb{Z}} \xi(x+m) = 1 $$

for all $x \in \mathbb{R}$, since $[0,1]$ is a fundamental domain of the $\mathbb{Z}$-action. Furthermore, by construction

$$ \xi(x)\xi(x+\frac{n}{p-q\theta}) = 0 \quad \forall x \in \mathbb{R},\ \forall n \in \mathbb{Z},\ n \neq 0, $$

because ξ and its translates by $\pm\frac{1}{p-q\theta}$ have disjoint supports. We get

$$ {}_{C^*(G_{p,q})}\langle \xi, \xi \rangle = 1 \in C^*(G_{p,q}) $$

and hence

$$ \langle \xi, \xi \rangle_{A_\theta} =: e_\theta $$

is a projection such that

$$ e_\theta A_\theta \cong \mathcal{E}_{p,q}. $$

We have

$$ e_\theta(y,n) = \langle \xi, \xi \rangle_{A_\theta}(y,n) = \sum_{m=-\infty}^{\infty} \xi(y+m)\, \xi(y+m+\frac{nq\theta}{p-q\theta}). \tag{10.45} $$

CHAPTER 5

Topological K-theory

In this chapter, we introduce K-theory for C*-algebras. K-theory came into prominence in the 1950's in part because it gave the correct language for formulating and proving the Index Theorem of Atiyah and Singer. The Index Theorem had to do with manifolds, and the K-theory used in that context was for commutative C*-algebras, that is, locally compact Hausdorff spaces. In that case, K-theory can be described rather geometrically in terms of certain objects called 'vector bundles,' and we start by defining and analyzing K-theory for commutative C*-algebras in this language.

Given our ability to construct a C*-algebra from a groupoid, however, the observation that actually K-theory can be extended in a highly natural way from spaces – commutative C*-algebras – to general, possibly noncommutative C*-algebras, presents the interesting possibility of defining various kinds of homology theories associated to various classes of groupoids, such as group actions, or various kinds of equivalence relations, or groups. This idea is at the heart of Noncommutative Geometry. The link comes from Swan's Theorem, (Theorem 2.8 of Chapter 4), which describes vector bundles as special kinds of modules.

1. The definition of K-theory, the K-theory of the circle

From the last section, the collection of vector bundles over a fixed space X has a natural additive structure: given two vector bundles V, W, their direct sum $V \oplus W$ is another vector bundle.

It is rather easy to see that $V \oplus W$ only depends on the isomorphism classes of V and W. Therefore, the direct sum operation descends to an addition operation on the collection $\mathrm{Vect}(X)$ of isomorphism classes of complex vector bundles over X, and similarly, the collection of isomorphism classes $\mathrm{Vect}_{\mathbb{R}}(X)$ of real vector bundles over X, has an addition operation. Thus, each of $\mathrm{Vect}(X)$ and $\mathrm{Vect}_{\mathbb{R}}(X)$ has a natural structure of *abelian semigroup with identity* (the zero vector bundle is the identity.)

The *Grothendieck completion* of an abelian semigroup A (think of $A = \mathbb{N}$ the natural numbers (including zero) under addition, or $A = \mathbb{N}^*$ the nonzero natural numbers, under multiplication) is the group defined in the following manner.

Let $G(A) := A \times A \,/\sim$, modulo the equivalence relation $(a, b) \sim (c, d)$ if $a+d+\epsilon = b+c+\epsilon$ for some $\epsilon \in A$.

Denote the equivalence class of a pair (a, b) in $G(A)$ by $a - b$.

It is easy to check that the operation $(a - b) + (c - d) := (a + c) - (b + d)$ is well-defined. There is a natural pair of semigroup homomorphisms $A \to G(A)$, in product notation, mapping $a \in A$ to the equivalence class of $(a, 0)$, and respectively, to the equivalence class of $(0, a)$. We write simply a for $a - 0$, and $-a$ for $0 - a$.

Then it is easy to verify that $-a$ is the additive inverse of a, and, more generally, the additive inverse of $a - b$ is $b - a$, in this notation.

Thus $G(A)$ is a group.

REMARK 1.1. The condition, for $a, b \in A$, that $a = b$ as elements of $G(A)$, says that $a + \epsilon = b + \epsilon$ for some $\epsilon \in A$, which is weaker of course than to say $a = b$ as elements of A. If the semigroup has the property that this implies that $a = b$, we say it has the *cancellation property*. Many semigroups of interest for us do not have this property; for them, $G(A)$ does not contain A injectively, but rather only a homomorphic image of it. This is the case for $A = \mathrm{Vect}(X)$, for example, which fails cancellation in general. The real vector bundle TS^2 satisfies $TS^2 \oplus 1 \cong 1_3 = 1_2 \oplus 1$ but TS^2 is not isomorphic to 1_2 (by the Hopf Theorem, for example, since TS^2 has no non-varnishing section.)

EXERCISE 1.2. Prove that $G(A)$ has the following universal property. Let $f\colon A \to H$ be a semigroup homomorphism to an abelian group H mapping the zero element of A to the identity of H. Then f extends uniquely to a group homomorphism $\bar{f}\colon G(A) \to H$ such that $\bar{f} \circ i = f$, where $i\colon A \to G(A)$ is the canonical map discussed in part b).

EXERCISE 1.3. Prove that the Grothendieck completion of the natural numbers (including zero) $\mathbb{N}$ under addition, is the integers, and that the Grothendieck completion of the nonzero natural numbers $\mathbb{N}^*$ under multiplication, is the nonzero rational numbers $\mathbb{Q}^*$ under multiplication.

DEFINITION 1.4. If X is a compact space, then $\mathrm{K}^0(X)$ is the Grothendieck completion of the semigroup $\mathrm{Vect}(X)$.

$\mathrm{KO}^0(X)$ is the Grothendieck completion of $\mathrm{Vect}_{\mathbb{R}}(X)$.

We generally denote by $[V]$ the class in $\mathrm{K}^0(X)$ of a vector bundle V over X.

By Remark 2.17, the K-theory of X is a countable group for any compact second countable Hausdorff space X.

EXERCISE 1.5. Prove that if x is any point of a compact Hausdorff space X then the map $V \mapsto \dim(V_x)$ determines a group homomorphism $\mathrm{K}^0(X) \to \mathbb{Z}$.

Deduce from this that if V is any nonzero real or complex vector bundle over a compact space X, then $[V] \neq 0 \in \mathrm{K}^0(X)$. Similarly for real vector bundles.

In particular, $\mathrm{K}^0(X)$ is not the zero group, for any compact X, because the subgroup generated by the 1-dimensional trivial bundle $[1]$ over X generates an infinite cyclic subgroup (and similarly $\mathrm{KO}^0(X)$ is never zero.)

The simplest example of a space is the 1-point space pt; it is completely obvious from the definitions that $\mathrm{Vect}(\mathrm{pt}) \cong \mathbb{N}$, by the map associating a vector bundle V over the point, which is exactly the same as a finite-dimensional vector space, with its rank, or the dimension of the vector space.

Hence $\mathrm{K}^0(\mathrm{pt}) \cong \mathbb{Z}$. Similarly $\mathrm{KO}^0(\mathrm{pt}) \cong \mathbb{Z}$. The following easy exercise implies that for any finite space X, $\mathrm{K}^0(X)$ is the free abelian group on the points of X.

EXERCISE 1.6. If a compact Hausdorff space X is the disjoint union of two clopen (both closed and open) subsets U and V, then $\mathrm{K}^0(X) \cong \mathrm{K}^0(U) \oplus \mathrm{K}^0(V)$, the direct sum in the category of abelian groups.

We next compute $\mathrm{K}^0([0,1])$ and $\mathrm{KO}^0([0,1]$.

THEOREM 1.7. *Any real or complex vector bundle over* $[0,1]$ *is trivial. In particular,* $\mathrm{Vect}([0,1]) \cong \mathbb{N}$ *by the map* $V \mapsto \mathrm{rank}(V)$*, and* $\mathrm{K}^0([0,1]) \cong \mathbb{Z}$*,* $\mathrm{KO}^0([0,1]) \cong \mathbb{Z}$*.*

Of course there is an analogous statement for real K-theory.

PROOF. By connectedness of $[0,1]$ and Exercise 1.11, E has constant fibre dimension n, for some n.

The interval is covered by open subintervals on which E is trivializable, by definition of vector bundle. By compactness, there exists a finite subcover of $[0, 1]$ by such intervals. Thus, we can find open intervals $I_1, \dots, I_m$, moving from left to right, overlapping, and such that E_{I_k} is trivializable for $k = 1, 2, \dots m$.

Let $s_i^{(k)}$ be sections of E on I_k, everywhere linearly independent, $i = 1, 2, \dots, n$. Moving from left to right along the interval we build n globally defined sections s_i which are everywhere linearly independent, as follows. Fix a point $t_0 \in I_1 \cap I_2$. We have two bases $s_i^{(1)}(t_0)$ and $s_i^{(2)}(t_0)$, $i = 1, \dots, n$ for the fibre E_{t_0} of E at t_0. Let A be the matrix defined by

$$s_i^{(2)}(t_0) = \sum_j A_{ij} s_j^{(1)}(t_0).$$

Then A is invertible. And for each i the section $x \mapsto \sum_j (A^{-1})_{ij} s_j^{(2)}(x)$ on I_2 agrees with s_i at $t_0 \in I_1 \cap I_2$ and can thus be used to extend $s_i^{(1)}$ on $[0, t] \subset I_1$ to $I_1 \cup I_2$. We then choose a point $t_2 \in I_2 \cap I_3$, and continue this process until we have constructed n linearly independent global sections of E, showing that it is trivial.

$\square$

EXERCISE 1.8. Let $v_1 \dots, v_n$ and $w_1, \dots, w_n$ be two bases for $\mathbb{C}^n$. Let p and q be two points of the interval $[0, 1]$. Prove that there are n everywhere linearly independent sections $s_1, \dots, s_n$ of the trivial bundle $[p, q] \times \mathbb{C}^n$ such that $s_i(p) = v_i$, $s_i(q) = w_i$, $i = 1, \dots, n$. (Hint. This is equivalent to showing that $\mathrm{GL}_n(\mathbb{C})$ is path connected.)

EXERCISE 1.9. Do Exercise 1.8 in the case of two bases for $\mathbb{R}^n$, in the case when the two bases are positively related in the sense that the change-of-basis linear transformation of $\mathbb{R}^n$ has positive determinant. What happens when the two bases are negatively related? Can it still be done?

Extending the argument of the Lemma a little for *complex* vector bundles produces the following result – but it definitely doesn't work for real vector bundles, since the Möbius bundle is not trivial.

PROPOSITION 1.10. *Any complex vector bundle over S^1 is trivial. Hence* $\mathrm{Vect}(S^1) \cong \mathbb{N}$ *and* $\mathrm{K}^0(S^1) \cong \mathbb{Z}$.

PROOF. Cover the circle with a finite family $I_1, I_2, \dots, I_m$ of open intervals (in the angular sense) such that $E_{|_{I_k}}$ is trivial for $k = 1, \dots, m$. By the technique of proof of Lemma **??**, we can take n linearly independent sections, call them $s_1, \dots, s_n$ of $E_{|_{I_1}}$ and extend them one interval at a time to $I_1 \cup I_2$, $I_1 \cup I_2 \cup I_3$ and so on, until they are defined on $I_1 \cup \cdots \cup I_{m-1}$. Proceeding to the next step produces two choices for our sections on $I_1 \cap I_m$, for by extending the constructed sections on I_{m-1} to sections on $I_{m-1} \cup I_m$ produces n sections $s'_1, \dots, s'_n$ which may not agree with the initially defined sections $s_1, \dots, s_n$ defined on I_1, on the intersection $I_1 \cap I_m$.

To remedy this, choose two points $z, w \in S^1$ in $I_1 \cap I_m$, with w past z in the counter-clockwise direction. The bundle E is trivial over $I_1 \cap I_m$. Let $\varphi \colon E_{|_{I_1 \cap I_m}} \to I_1 \cap I_m \times \mathbb{C}^n$ be a trivialization. Consider the bases $\varphi\big(s_1(z)\big), \varphi\big(s_2(z)\big), \dots, \varphi\big(s_n(z)\big)$, and $\varphi\big(s'_1(w)\big), \varphi\big(s'_2(w)\big), \dots, \varphi\big(s'_n(w)\big)$ for $\mathbb{C}^n$. By Exercise 1.8, there is a family $t_1, \dots, t_n$ of everywhere linearly independent sections of the trivial bundle $[z, w] \times \mathbb{C}^n$ such that $t_i(z) = \varphi\big(s'_i(z)\big)$ and $t_i(w) = \varphi\big(s_i(w)\big)$, $i = 1, 2, \dots, n$. We can then glue the sections $\varphi^{-1}(t_i)$ to s'_i at z and to s_i at w. This produces the required family of n linearly independent sections of E on S^1. $\square$

EXERCISE 1.11. What goes wrong if one tries to run the same argument through for the Möbius bundle?

The determinant map $\det\colon \mathbf{GL}_n(\mathbb{R}) \to \mathbb{R}^*$ is continuous for every n and so $\mathbf{GL}_n(\mathbb{R})$ has two components, since $\mathbb{R}^*$ does.

Use this to show that if V is any orientable real vector bundle of rank n over the circle, then $V \cong 1_n$. Deduce that $[M] - [1] \in \mathrm{K}^0_{\mathbb{R}}(S^1)$ is 2-torsion, where M is the Möbius bundle. Can you take a guess at the group $\mathrm{KO}^0_{\mathbb{R}}(S^1)$ (we are still not in a position to prove it.)

More generally, we outline how torsion classes arise in K-theory. Suppose G is a finite group acting freely on $\tilde{X}$ compact, let $X := G\backslash X$. And suppose that

$$\alpha\colon G \to \mathbf{U}_n$$

is a finite-dimensional, unitary representation of G. Let $\tilde{X} \times_G \mathbb{C}^n := X \times \mathbb{C}^n \ / \ \sim$, where $\sim$ is the equivalence relation $(x, v) \sim (gx, \alpha(g)v)$. The first coordinate projection descends to a well-defined map

$$E_\alpha := \tilde{X} \times_G \mathbb{C}^n \to G\backslash\tilde{X} = X. \tag{1.1}$$

And we leave it as an exercise to check that this gives E_α the structure of a vector bundle over X.

Exercise 1.12. Prove that $E_\alpha \to X$ defined above, is a vector bundle over X.

Proposition 1.13. *If $X = G\backslash X$ for a free action of a finite group on a compact space, and $\alpha\colon G \to \mathbf{U}_n$ is a representation of G on $\mathbb{C}^n$, E_α the bundle over X defined above, then*

$$|G| \cdot ([E_\alpha] - [1_n]) = 0 \in \mathrm{K}^0(X). \tag{1.2}$$

In particular, $[E_\alpha] - [1_n] \in \mathrm{K}^0(X)$ is always a torsion class, of order a divisor of $|G|$.

The proof uses a simple device that is slightly more general, so we give this slightly more general statement.

Let $\pi\colon \tilde{X} \to X$ be a finite covering map. We define a *push-forward* operation on vector bundles as follows. If E is a vector bundle over $\tilde{X}$, define a vector bundle $\pi_\sharp(E)$ over X by setting the fibre at $x \in X$ to be

$$\pi_\sharp(E)_x := \sum_{y \in \pi^{-1}x} E_y. \tag{1.3}$$

Exercise 1.14. Prove that $\pi_\sharp(E)$ defined above is a vector bundle over X. If E and E' are isomorphic vector bundles over X, then $\pi_\sharp(E)$ and $\pi_\sharp(E')$ are isomorphic.

The push-forward construction therefore gives rise to a *group homomorphism*

$$\pi_*\colon \mathrm{K}^0(\tilde{X}) \to \mathrm{K}^0(X). \tag{1.4}$$

Lemma 1.15. *In the above notation, let $\pi^*\colon \mathrm{K}^0(X) \to \mathrm{K}^0(\tilde{X})$ be the map induced by the finite covering map $\pi\colon \tilde{X} \to X$. Then*

$$\pi^* \circ \pi_* = |G| \cdot \mathrm{id}_{\mathrm{K}^0(\tilde{X})}, \ \ \pi_* \circ \pi^* = |G| \cdot \mathrm{id}_{\mathrm{K}^0(X)}. \tag{1.5}$$

hold.

In particular, the push-foward rationally *inverts the pull-back map.*

Exercise 1.16. Verify that (1.5) holds.

We now prove Proposition 1.13.

Proof. Let $\pi\colon \tilde{X} \to X$ the quotient map – a covering map. It is obvious that $\pi^*(E_\alpha) \cong \tilde{X} \times \mathbb{C}^n$, that is, the pull-back of E_α to $\tilde{X}$ is trivial. Hence

$$\pi^*([E_\alpha]) = [1_n].$$

Applying the push-forward map π_* gives

$$\pi_*\big(\pi^*([E_\alpha] - [1_n])\big) = 0 \in \mathrm{K}^0(X).$$

By (1.5)

$$|G| \cdot ([E_\alpha] - [1_n]) = 0 \in \mathrm{K}^0(X),$$

as required. □

Clutching constructions, a homotopy description of vector bundles over spheres

We finish this section with a discussion of 'clutching.' We restrict ourselves to complex vector bundles for simplicity; the analogous discussion goes through for real bundles.

Let $X = U \cup U'$ be the union of two open sets, let E be a complex vector bundle over U and E' a complex vector bundle over U', and let $\varphi\colon E|_{U\cap U'} \to E'|_{U\cap U'}$ be a bundle isomorphism.

Then the *clutching* of E and E' over φ is denoted $E \cup_\varphi E'$, is defined as follows. As a space, $E \cup_\varphi E'$ is the quotient of $E \sqcup E'$ by the equivalence relation which identifies $(x, v) \in E$ with $\varphi(x)v$ in E'. The projection maps $E \to U$ and $E' \to U'$ splice together to make a projection map $\pi\colon E \cup_\varphi E' \to U \cup U' = X$.

Each fibre of π has the structure of a vector space, since the glueing map $\varphi(x)\colon E_x \to E'_x$ is linear for all $x \in U \cap U'$, this is well-defined on $E \cup_\varphi E'$, and the addition and scalar multiplication operators on $E \cup_\varphi E'$ are easily checked to be continuous, fibrewise, with respect to the quotient topology.

EXERCISE 1.17. In the above notation, prove that $\pi\colon E \cup_\varphi E' \to X$ is locally trivial and that the isomorphism class of $E \cup_\varphi E'$ only depends on the homotopy class of the vector bundle isomorphism $\varphi\colon E_{|_{U\cap U'}} \to E'_{|_{U\cap U'}}$. (*Hint.* A homotopy $(\varphi_t)_{t\in[0,1]}$ of bundle isomorphisms $E_{|_{U\cap U'}} \to E'_{|_{U\cap U'}}$ is equivalent to a single bundle isomorphism $\pi^*(E)_{|_{U\cap U'\times[0,1]}} \to \pi^*(E')_{|_{U\cap U'\times[0,1]}}$, where $\pi\colon X \times [0,1] \to X$ is the projection. Now prove that $E \cup_{\varphi_t} E' \cong f_t^*\big(\pi^* E \cup_\Phi \pi^* E'\big)$, with $f_t\colon X \to X \times [0,1]$ the map $f_t(x) = (x, t)$.)

EXERCISE 1.18. Prove that if U and U' are open in X, E and E' are vector bundles over U, U', and if V is a vector bundle over X whose restriction to U is isomorphic to E, and whose restriction to U' is isomorphic to E', then V is isomorphic to the clutching $E \cup_\varphi E'$, using the clutching function manufactured on $U \cap U'$ by using first the isomorphism $E' \cong V$ (on U) followed by the inverse of the isomorphism $V \cong E$ (on U').

Deduce from this that if a vector bundle is trivial over an open set $U \subset X$, then it is isomorphic to a vector bundle which is actually *equal* over U to a product bundle.

The following exercise generalizes the clutching idea over two open sets, to an arbitrary collection of them.

EXERCISE 1.19. (Clutching using a cocycle).

Suppose that $\{U_i\}_{i\in I}$ is a cover of X by open sets. And suppose we are given a family $\{\varphi_{ij}\colon U_i \cap U_j \to \mathbf{GL}(n, \mathbb{C}) \mid i, j \in I\}$ of maps satisfying the cocycle conditions

- $\varphi_{ii}(x) = \mathrm{id}$ for all i,
- $\varphi_{ij}(x)\varphi_{jk}(x) = \varphi_{ik}(x)$, $\forall i, j, k$.

Then the relation $\sim$ on $\bigsqcup_{i\in I} U_i \times \mathbb{R}^n$ defined $(x, v) \sim (x, \varphi_{ij}(x)v)$ for $x \in U_i \cap U_j$, is an equivalence relation, and the quotient space has a canonical structure of an n-dimensional vector bundle over X.

EXERCISE 1.20. Let $\pi\colon E \to X$ be an n-dimensional real or complex vector bundle. Suppose that $\{U_i, \varphi_i\}_{i\in I}$ is an atlas for E, *i.e.* $\varphi_i\colon E|_{U_i} \to U_i \times \mathbb{C}^n$ is a local trivialization for all $i \in I$. Let $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ the transition functions for the atlas, understood as maps $\varphi_{ij}\colon U_i \cap U_j \to \mathbf{GL}(n,\mathbb{C})$. Check that they satisfy the cocycle condition and that the vector bundle $\bigsqcup_{i\in I} U_i \times \mathbb{C}^n \;/\; \sim$ as in Exercise 1.19, is isomorphic to E.

EXERCISE 1.21. Let $\{\varphi_{ij}\colon U_i \cap U_j \to \mathbf{GL}(n,\mathbb{C})\}$ be a cocycle as in Exercise 1.19 which is a *coboundary* in the sense that there are maps $\psi_i\colon U_i \to \mathbf{GL}(n,\mathbb{C})$ for which $\varphi_{ij}(x) = \psi_j(x)\psi_i(x)^{-1}$ for $x \in U_i \cap U_j$. Prove that the 'clutched' bundle $\bigsqcup_{i\in I} U_i \times \mathbb{C}^n \;/\; \sim$ described in the Exercise 1.19 , is trivial.

EXERCISE 1.22. Let G be a finite group acting freely on $\tilde{X}$ compact. Show that the vector bundle E_α over X associated to a finite-dimensional representation $\alpha\colon G \to \mathbf{U}_n$, may be considered as being obtained by clutching in the following way. Cover X by the open images of sets $U_i \subset \tilde{X}$ for which $g(U_i) \cap U_i = \emptyset$ for $g \neq e$. The projection map $\pi\colon \tilde{X} \to X$ restricts to a homeomorphism on each U_i, and the composition $(\pi|_{U_j})^{-1} \circ \pi|_{U_i}$ is a homeomorphism on $U_i \cap U_j$ onto an open subset of $\tilde{X}$. Show that this homeomorphism is the restriction of a group element $g_{ij} \in G$, and if we set $\varphi_{ij} := \alpha(g_{ij})$ then the system $\varphi_{ij}\colon U_i \cap U_j \to \mathbf{U}_n$ defines a cocyle, whose functions are locally constant.

EXERCISE 1.23. Prove that a pair of vector bundles can be 'cluched' over two *closed* sets, as well as over two open sets. More precisely, let $A_1, A_2 \subset X$ be two closed subsets of X, let E_i be vector bundles over A_i, and let $\varphi\colon E_1|_{A_1\cap A_2} \xrightarrow{\cong} E_2|_{A_1\cap A_2}$ be a vector bundle isomorphism. Forming the quotient space of $E_1 \sqcup E_2$ by the equivalence relation which identifies $v \in E_1$ with $\varphi(v) \in E_2$, results in a vector bundle over $X = A_1 \cup A_2$, which is isomorphic to E.

We close this section with a homotopy-theoretic description of vector bundles over a sphere.

Let $\pi\colon V \to S^n$ be a vector bundle over the n-sphere. Let S^n_+ be the (closed) upper hemisphere, S^n_- the lower hemisphere, so that $S^n_+ \cap S^n_- \cong S^{n-1}$.

Let E be a k-dimensional complex vector bundle over S^n. Since $S^n_\pm$ are each contractible compact spaces, $E_{|_{S^n_\pm}}$ is trivial. Fix trivializations

$$\alpha_\pm\colon E_{|_{S^n_\pm}} \xrightarrow{\cong} S^n_\pm \times \mathbb{C}^k.$$

The restriction of $\alpha_- \circ \alpha_+^{-1}$ to $S^n_+ \cap S^n_- \cong S^{n-1}$ is a bundle map $S^{n-1} \times \mathbb{C}^n \to S^{n-1} \times \mathbb{C}^k$, which is equivalent to a map $\alpha\colon S^{n-1} \to \mathbf{GL}(k,\mathbb{C})$. Let $[\alpha] \in [S^{n-1},\ \mathbf{GL}(k,\mathbb{C})]$ be the corresponding homotopy class of map.

EXERCISE 1.24. In the above notation, answer the following.

a) Prove that the homotopy class $[\alpha] \in [S^{n-1},\ \mathbf{GL}(n,\mathbb{C})]$ does not depend on the choice of trivializations $\alpha_\pm$. (*Hint.* Due to contractibility of $S^n_\pm$, even the homotopy classes of the bundle maps $\alpha_\pm$ are uniquely defined.)
b) Using clutching to produce a map inverse to the construction above, prove that

$$\mathrm{Vect}_k(S^n) \cong [S^{n-1},\ \mathbf{GL}(k,\mathbb{C})],$$

where $\mathrm{Vect}_k(S^n)$ is the set of isomorphism classes of k-dimensional complex vector bundles over the sphere.

EXERCISE 1.25. Let $S^2_\pm$ be the upper and lower closed hemispheres of the 2-sphere. Prove that the Hopf bundle is obtained by clutching two trivial bundles over $S^2_\pm$ using the $\mathbf{GL}(1,\mathbb{C}) \cong \mathbb{C}^*$-valued function

$$\varphi\colon S^2_+ \cap S^2_- \cong S^1 \to \mathbb{C}^*, \;\; \varphi(z) = \bar{z}.$$

EXERCISE 1.26. Let $\pi\colon V \to X$ be an n-dimensional vector bundle over a compact space. Let $\mathcal{F}(V)$ be the bundle of frames of V: a point of $\mathcal{F}$ is a pair $(x, \mathbf{v})$ where $\mathbf{v}$ is an n-tuple $(v_1, \dots, v_n)$ of linearly independent vectors in V_x. Topologize $\mathcal{F}(V)$ to be a compact space, and prove that the projection $p\colon \mathcal{F}(V) \to X$ pulls V back to a trivial bundle over $\mathcal{F}(V)$.

EXERCISE 1.27. (*Flat bundles*). The following is an important geometric construction of vector bundles. Let X be a locally compact space, and G a discrete group acting *freely and properly* on X. For essentially notational purposes, we view it as a right action $(x, g) \mapsto xg$, with $xg := g^{-1}(x)$.

Let $\pi\colon G \to \mathbf{GL}(V)$ be a finite-dimensional representation of G on a real or complex vector space V. Set $X \times_G V$ to be the quotient of the space $X \times V$ by the equivalence relation $(x, v) \sim (xg, \pi(g)v)$ for all $x \in X, v \in V, g \in G$. Let

$$\pi\colon X \times_G V \to X/G, \quad \pi([(x, v)] := [x].$$

a) Prove that $\pi\colon X \times_G V \to X/G$ defines a complex $\dim(V)$-dimensional vector bundle over X/G by showing that it is locally trivial.
b) Show that the transition functions $\varphi\colon W \to \mathbf{GL}(V)$ associated to the local vector bundle trivializations you probably found in a), are locally constant (and in fact are given by the action of elements of G on V.)
c) Prove that the Möbius vector bundle of Example 1.4 is the flat bundle over $\mathbb{R}/\mathbb{Z} \cong S^1$ associated to the one-dimensional representation $\chi(n) = (-1)^n$ of the integers $\mathbb{Z}$.
d) Since $\mathbb{RP}^n$ is the quotient of S^n by an action of the group $\mathbb{Z}/2$, exhibit a corresponding flat, one-dimensional real vector bundle L_n over $\mathbb{RP}^n$ for all n. For a challenge, prove that $L_n \oplus L_n$ is a trivial bundle.

Such bundles are often called *flat*. The reason is that the transition functions of the obvious atlas of such a bundle, are locally constant. This allows construction, for example, of a *flat connection* on such bundles – *i.e.* a connection with zero curvature.

Orientations on vector bundles

Let $\pi\colon V \to X$ be any real vector bundle. Due to being a real vector bundle, V has local frames. Thus, for any point of X, there is a neighbourhood U of the point, and a frame $\mathbf{e}$ – *i.e.* a collection of sections $e_1, \dots, e_n$ of V defined on U, such that $e_1(x), \dots, e_n(x)$ is a basis for V_x for all $x \in U$.

If U' is another, intersecting open set, with another frame $\mathbf{e}'$ on it, then we say the frames are *compatibly oriented* on $U \cap U'$ if $\mathbf{e}(x)$ and $\mathbf{e}'(x)$ are compatibly oriented frames of V_x for all $x \in U$, equivalently, $e_1(x) \wedge \cdots e_n(x)$ is a *positive* multiple of $e'_1(x) \wedge \cdots \wedge e'_n(x)$ in $\Lambda^n(V_x)$ for all $x \in U \cap U'$.

A vector bundle is *orientable* if there is a cover of X by open sets U_i, and a frame $\mathbf{e}_i$ on U_i such that if $U_i \cap U_j \neq \emptyset$ then $\mathbf{e}_i$ and $\mathbf{e}'_i$ are compatibly oriented frames. We call any such data an *orientation* on V.

EXERCISE 1.28. The following are equivalent for a real vector bundle $\pi\colon V \to X$.

a) V is orientable.
b) There exists an atlas $\{U_i, \varphi_i\}_{i \in I}$ for V, for which the transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}\colon U_i \cap U_j \to \mathbf{GL}(n, \mathbb{R})$ take values in the subgroup $\mathbf{GL}^+(n, \mathbb{R})$ of matrices of positive determinant.
c) There exists an atlas $\{U_i, \varphi_i\}_{i \in I}$ for V, for which the transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}\colon U_i \cap U_j \to \mathbf{GL}(n, \mathbb{R})$ take values in $\mathbf{SO}(n, \mathbb{R})$.

2. Vector bundles on manifolds

In this chapter we review some of the standard (real) vector bundles that come up in smooth manifold theory.

An *n-dimensional locally Euclidean space* M is a Hausdorff, second countable topological space with the property that every point $p \in M$ has a neighbourhood U homeomorphic to an open subset of $\mathbb{R}^n$. A *smooth atlas* on an n-dimensional locally Euclidean space is a collection of pairs $\{U_i, \varphi_i\}$ with U_i an open subset of M and $\varphi_i \colon U_i \to \mathbb{R}^n$ a homeomorphism onto an open subset, such that $\cup U_i = M$ and $\varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ a *smooth* map, for all $i.j$. Each such pair is called a (smooth) *local coordinate chart*.

If $r_1, \ldots, r_n \colon \mathbb{R}^n \to \mathbb{R}$ are the usual coordinate projections, we can write $\varphi = (x_1, \ldots, x_n)$ where $x_i := r_i \circ \varphi$ and we can label points in U by their corresponding coordinate vectors $(x_1, \ldots, x_n)$.

A maximal smooth atlas is a *differentiable structure* on M. A continuous function $f \colon M \to \mathbb{R}$ is *smooth* if $f \circ \varphi^{-1} \colon \varphi(U) \to \mathbb{R}$ is smooth for every local coordinate chart on M. $C^\infty(M)$ denotes the real algebra of smooth functions on M.

If $p \in M$ is a point, a *point derivation* $X_p \colon C^\infty(M) \to \mathbb{R}$ is a linear map satisfying the Leibnitz rule

$$X_p(fg) = f(p)X_p(g) + g(p)X_p(f),$$

for f and g smooth functions on M.

The *tangent bundle* of M is the disjoint union $TM = \sqcup_{p \in M} T_p(M)$ where $T_p(M)$ is the real vector space of point derivations of $C^\infty(M)$ at p. There is an evident projection $\pi \colon TM \to M$; we want to show that TM can be given the structure of a vector bundle over M.

Exercise 2.1. If $X_p \in T_p(M)$ is a point derivation at p, and $f \in C^\infty(M)$ is a smooth function which vanishes in a neighbourhood of p, then $X_p f = 0$. Deduce that $X_p f$, for $f \in C^\infty(M)$, really only depends on the *germ* of f at p (germs are discussed below.)

Exercise 2.2. Suppose that $p \in M$ and $\gamma \colon (-\epsilon, \epsilon) \to M$ is a smooth curve such that $\gamma(0) = 0$. Show that γ determines a point derivation $\gamma'(0)$ at p by

$$\gamma'(0)f := (f \circ \gamma)'(0).$$

It is a standard result from basic manifold theory that all point derivations at p arise in this way.

If p is in the domain of a coordinate chart $\varphi \colon U \to \mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$, let $\frac{\partial}{\partial x_i}|_p$ denote the point derivation $f \mapsto \frac{\partial f}{\partial x_i}(p) := \frac{\partial (f \circ \varphi^{-1})}{\partial r_i}(p)$ at p.

Taylor's lemma asserts that if f is a smooth function in a neighbourhood of a point $p \in \mathbb{R}^n$, then

$$f(r) = f(p) + \sum_{i=1}^n g_i(r)(r_i - p_i)$$

where $g_1, \ldots, g_n$ are smooth functions in a neighbourhood of p satisfying $g_i(p) = \frac{\partial f}{\partial r_i}$.

Taylor's Lemma extends more or less verbatim to points $p \in M$ in a smooth manifold M, and to smooth functions on M. If (U, φ), $\varphi = (x^1, \ldots, x^n)$ is a local coordinate chart, then any $f \in C^\infty(M)$ may be written

$$f(x) = f(p) + \sum_{i=1}^n g_i(x)(x^i - p^i)$$

for a collection $g_1, \ldots, g_n$ of functions smooth in a neighbourhood of p.

Now, if X_p is a point derivation at p, then by the Leibnitz rule, and the fact that all the $x^i - p^i$ vanish at p,

$$X_p(f) = \sum_{i=1}^n g_i(p) X_p(x_i - p_i) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}(p)$$

with a_i the constants $a_i = X_p(x_i - r_i)$ obtained by applying X_p to the functions $x^i - p^i$, which vanish at p. That is, any X_p, for $p \in U$, can be expanded uniquely in the form

$$X_p = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}|_p, \text{ where } a_i = X_p(x^i - p^i).$$

It is also easy to check that the $\frac{\partial}{\partial x^i}|_p$ are linearly independent at each $p \in U$. Hence they form linearly independent and spanning sections of $TM|_U := \pi^{-1}(U) \subset TM$. This supplies local trivializations of TM, and a basis for a topology, and TM therefore becomes a real vector bundle over M of dimension $n = \dim(M)$.

The *dual* T^*M of the tangent bundle also has a nice geometric description. Fix a point $p \in M$. Let A_p be the algebra over $\mathbb{R}$ of germs (f, U) of smooth functions at p, which vanish at p. Thus, $f \in C_c^\infty(U)$, $p \in U$, $f(p) = 0$, and two such pairs (f, U) and (g, V) for which $f = g$ on $U \cap V$ are considered equivalent.

Now form $T_p^*(M) := A_p/A_p^2$. This is the algebra of germs which vanish to first order at p modulo the germs which vanish to second order at p.

Now if f is a smooth function defined in a neighbourhood of p, we let $df(p)$ be the class in A_p/A_p^2 of the smooth $f - f(p)$, which vanishes at p.

By Taylor's Lemma, we can find smooth functions $g_1, \ldots, g_n$ in a neighbourhood of p such that $g_i(p) = \frac{\partial f}{\partial x^i}(p)$. Applying Taylor's Lemma to each such g_i then yields smooth functions hij in a neighbourhood of p such that

$$g_i(x) = \frac{\partial f}{\partial x^i}(p) + \sum_{j=1}^n h_{ij}(x)(x^j - p^j).$$

Substituting into the formula for $f - f(p)$ yields

$$f - f(p) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p)(x^i - p^i) + \sum_{i.j} h_{ij}(x)(x^i - p^i)(x^j - p^j).$$

This immediately implies that

$$df(p) = \sum_{i=1}^n a_i dx^i(p), \quad \text{where } a_i = \frac{\partial f}{\partial x^i}(p)$$

since the germ $\sum_{i.j} h_{ij}(x)(x^i - p^i)(x^j - p^j)$ vanishes to order 2 at p.

It follows that $\sqcup_{p \in M} A_p/A_p^2$ can be given the structure of a real, n-dimensional vector bundle over M, with local sections given on the domains of coordinate charts by the cosets $dx^1, \ldots, dx^n$. In fact, this vector bundle can be naturally identified with the dual bundle T^*M of the tangent bundle. To prove this, define a map $A_p/A_p^2 \to T_p(M)^*$ by letting $df(p) \in A_p/A_p^2$ act on point derivations at p by

$$\langle df(p), X_p \rangle := X_p(f).$$

This formula is well-defined, because any point derivation must vanish on functions which vanish to order 2 at p, so that $X_p(f)$ only depends on the coset of f modulo A_p^2.

The details are left as an exercise.

A section of the tangent bundle is called a *vector field* on M. A section of T^*M is called a *differential* 1*-form* on M. A section of the exterior algebra bundle $\Lambda^k T^*M$ is called a *differential*

k-form on M. Any differential k-form ω on M can be locally expanded into a linear combination of the standard differential k-forms $dx^I := dx^{i_1} \wedge d^{i_2} \wedge \cdots \wedge dx^{i_k}$, where $I = (i_1, \ldots, i_k)$ is a multi-index: thus, on the domain of a local coordinate system, we can write

$$\omega = \sum_I a_I dx^I$$

for some collection of smooth functions a_I on the domain of the chart.

Exercise 2.3. A smooth manifold whose tangent bundle is trivial is called *parallelizable*. Prove that the n-torus $\mathbb{T}^n$ is parallelizable.

Exercise 2.4. Let M be any smooth manifold. Prove that the tangent bundle $T(TM)$, as a vector bundle over the space TM, is isomorphic to $\pi^*(TM) \oplus \pi^*(TM)$, where $\pi \colon TM \to M$ is the projection map, and $\pi^*(TM)$ is the pull-back of the vector bundle TM over M, to a vector bundle over TM.

Deduce that the tangent bundle to TM has a complex structure.

Smooth structures on vector bundles

Definition 2.5. A vector bundle $\pi \colon E \to M$ over a manifold, is *smooth* if E is a smooth manifold, and if there exists an atlas for E consisting of smooth maps.

By an easy exercise, if E is smooth, then $\pi \colon E \to M$ is a smooth map, and M embeds by the zero section of E as a regular submanifold of E.

In this section, we show the important basic result that every vector bundle over a smooth manifold may as well be taken to be a smooth vector bundle. This idea is an important one in Noncommutative Geometry: it means that one may for purposes of K-theory computations, assume that all the K-theory data is smooth. The proof we give here is fairly C*-algebraic in nature.

Theorem 2.6. *Every real vector bundle $\pi \colon E \to M$ over a smooth compact manifold is isomorphic to a smooth vector bundle.*

Remark 2.7. The theorem can be phrased a bit more concretely as follows: any vector bundle E over smooth M be given a differentiable structure, and, moreover, one can find a system of local trivializations of E which are smooth.

Lemma 2.8. *Let a be a self-adjoint element of a C*-algebra of norm ≤ 1. Then if $\|a - a^2\| < \frac{1}{4}$ then $\frac{1}{2} \notin \mathrm{Spec}(a)$.*

Proof. If a is self-adjoint then the functional calculus produces a *-isomorphism $C^*(a) \cong C\big(\mathrm{Spec}(a)\big)$ mapping a to $f(t) = t$, so $\|a - a^2\| < \frac{1}{4}$ implies that $|t - t^2| < \frac{1}{4}$ for all $t \in \mathrm{Spec}(a)$, and hence that $\frac{1}{2} \notin \mathrm{Spec}(a)$ since $t - t^2$ assumes the value $-\frac{1}{4}$ there.

□

Lemma 2.9. *Let M be a smooth compact manifold and let $H \in C^\infty\big(M, M_n(\mathbb{C})\big)$ be a smooth element of the C*-algebra $C\big(M, M_n(\mathbb{C})\big)$. Let ψ be a continuous function on $\mathrm{Spec}(H)$. Then if ψ extends to a holomorphic function on a neighbourhood in $\mathbb{C}$ of $\mathrm{Spec}(H)$, then $\psi(H)$ is also smooth.*

Proof. Let $\tilde{\psi}$ be an extension of ψ to a holomorphic function in a neighbourhood U of $\mathrm{Spec}(H)$, and let γ be a simple closed, positively oriented contour in U with $\mathrm{Spec}(H)$ contained in its interior. By the holomorphic functional calculus

$$\psi(H) = \frac{1}{2\pi i}\oint_\gamma \tilde{\psi}(w)(w-H)^{-1}dw.$$

As a function on X, thus,

$$\psi(H)(x) = \frac{1}{2\pi i}\oint_\gamma \tilde{\psi}(w)(w-H(x))^{-1}dw.$$

The usual technique of differentiating under the integral sign implies that this function of x is smooth, because H is assumed smooth.

□

Lemma 2.10. *If M is a smooth manifold and $p\colon M \to M_n(\mathbb{C})$ is a smooth projection-valued function, then the vector bundle* $\mathrm{Im}(p)$ *is a smooth vector bundle over M.*

Proof. By definition, $\mathrm{Im}(p)$ is a *subset* of the smooth manifold $M \times \mathbb{R}^m$. So to show it has the structure of a smooth manifold, it suffices to show that it is a regular submanifold of $M \times \mathbb{R}^m$. Choose any point $a \in M$. We have already shown that if $v_1, \dots, v_m$ is a basis for $\mathbb{R}^m$ with the first k vectors a basis for $\mathrm{Im}(p(a)$, then the sections $s_1(b) := p(b)v_1, \dots, s_k(b) := p(b)v_k, s_{k+1}(b) := v_{k+1}, \dots, s_m(q) := v_m$ form a basis for $\mathbb{R}^m$ for all q in a neighbourhood U of a. After possibly shrinking U we may also assume it is the domain of a coordinate chart $(U, x^1, \dots, x^n)$ for M.

Now if $v \in \mathbb{R}^m$ and $q \in U$ then we can find unique scalars $t^1(v,q), \dots, t^m(v,q)$ such that $v = \sum_{i=1}^m t^i(v,q)v_i$. The functions t^i are smooth, and are linear in v for fixed q. We now make a local coordinate system for $M \times \mathbb{R}^m$ around $(a,0)$ by $(q,v) \mapsto (x^1(q), \dots, x^n(q), t^1(v,q), \dots, t^m(v,q))$. This forms a coordinate system, and the last $m-k$ coordinates of (q,v) vanish if and only if v has the form $v = \sum_{i=1}^k t^i(v,q)P(q)v_i$ which lies in $\pi^{-1}(U) \subset \mathrm{Im}(p)$, so that locally $\mathrm{Im}(p)$ may be represented as

$$\{(x^1, \dots, x^n, t^1, \dots, t^m) \mid t_{k+1} = \dots = t_m = 0\}$$

which is the condition for being a regular submanifold.

Note that with this differential structure, the bundle trivializations are smooth, indeed, in our local coordinates the bundle trivializations are, in the above notation,

$$\varphi(q,v) = (q, t^1(q,v), \dots, t^k(q,v)) \in M \times \mathbb{R}^k, \text{ for } (q,v) \in \mathrm{Im}(p).$$

□

Proof. (Of Theorem 2.6). Let $p\colon M \to M_n(\mathbb{C})$ be a continuous projection-valued function such that $\mathrm{Im}(p) \cong E$. Since smooth functions $M \to \mathbb{C}$ are dense in continuous functions, by the Stone-Weierstrass Theorem, it follows that smooth, matrix-valued functions $M \to M_n(\mathbb{C})$ are also dense in continuous matrix-valued functions. So there exists a sequence (H_n) of smooth functions $H_n\colon M \to M_n(\mathbb{C})$ with $H_n \to P$ in the C*-algebra $C(M, M_n(\mathbb{C}))$. Since $\frac{H_n + H_n^*}{2} \to P$ as well, we may as well assume that the H_n are also self-adjoint.

In particular, there exists a smooth self-adjoint element $H \in C^\infty\big(M, M_n(\mathbb{C})\big)$ such that $\|H - P\| < \frac{1}{4}$. By Lemma 2.8, $\frac{1}{2} \notin \mathrm{Spec}(H)$. The spectrum of H is compact and does not contain $\frac{1}{2}$ and hence there exists a pair of open sets $U, V \subset \mathbb{C}$ such that $U \cap V = \emptyset$, V contains $\mathrm{Spec}(H) \cap (\frac{1}{2}, +\infty)$, U contains $\mathrm{Spec}(H) \cap (-\infty, \frac{1}{2})$. Let $\tilde{f}$ be the function assuming value 1 on V and 0 on U, then $\tilde{f}$ is clearly holomorphic on $U \cup V$, and if γ is a simple closed contour in V encircling $(\frac{1}{2}, +\infty) \cap \mathrm{Spec}(H)$, then $f(H)$ is then smooth by Lemma 2.9, and is a projection, call it Q, since it is the image of a characteristic function on $\mathrm{Spec}(H)$ under holomorphic (and hence continuous) functional calculus $C\big(\mathrm{Spec}(H)\big) \to C^*(H)$.

Since $\|Q - P\| < 1$, $\mathrm{Im}(Q) \cong \mathrm{Im}(P) \cong E$ as vector bundles over M. Finally, an application of Lemma 2.10 gives that $\mathrm{Im}(Q)$ is a smooth vector bundle, and we conclude that E is isomorphic to a smooth vector bundle as initially claimed.

□

REMARK 2.11. The reader who wishes to avoid holomorphic functional calculus may instead use the *smooth* functional calculus developed in Exercise 3.18 and in the environmental discussion.

Indeed, in the notation of the above proof, f is any smooth, compactly supported function on $\mathbb{R}$ such that $0 \leq f \leq 1$, and $f(t) = 0$ if $t \in \mathrm{Spec}(H), t < \frac{1}{2}$, and $f(t) = 1$ if $t \in \mathrm{Spec}(H)$ with $t > \frac{1}{2}$, then $f(H) = Q$ is a projection having the integral formula

$$f(H) = \int \hat{f}(\xi) e^{i\xi H} d\xi.$$

As a matrix-valued function on M we therefore have the integral formula at each point $x \in M$:

$$f(H)(x) = \int \hat{f}(\xi) e^{i\xi H(x)} d\xi,$$

and the usual trick of differentiating under the integral sign implies this is a smooth function of $x \in M$, since H is smooth, and the Fourier transform of f is also smooth.

EXERCISE 2.12. Suppose that $A \subset X$ is a closed subspace of a locally compact space X and E a vector bundle over A. Prove that E can be extended to a vector bundle over a neighbourhood of A. That is, prove that there exists an open neighbourhood U containing A and a vector bundle $\tilde{E}$ over U whose restriction to A is E. (*Hint.* Find a projection valued map $p\colon X \to M_n(\mathbb{C})$ such that $\mathrm{Im}(p) \cong E$. Extend p to a map $\tilde{p}\colon X \to M_n(\mathbb{C})$, argue that for some neighbourhood U of A, $\frac{1}{2} \notin \mathrm{Spec}\big(\tilde{p}(x)\big)$ for all $x \in U$, and use the functional calculus methods of the proof of Theorem 2.6 perturb $\tilde{p}$ so that it is projection-valued in a neighbourhood of A.)

3. Functoriality and homotopy-invariance

Let $\varphi\colon X \to Y$ be a continuous map of compact spaces. The pull-back operation $V \mapsto \varphi^*(V)$ from vector bundles on Y to vector bundles on X, defined in Definition 1.13, can be easily checked to take isomorphic vector bundles to isomorphic vector bundles, and respects direct sums. Hence it induces a homomorphism $\varphi^*\colon \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ of abelian semi-groups. This results in a pair of abelian group homomorphisms $\varphi^*\colon \mathrm{K}^0(Y) \to \mathrm{K}^0(X)$ and $\mathrm{KO}^*(Y) \to \mathrm{KO}^*(X)$.

It is routine to check that as maps on K-theory, or KO-theory, $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$, for $\varphi\colon Y \to Z$ and $\psi\colon X \to Y$, so that *the assignment $X \mapsto \mathrm{K}^0(X)$, $\varphi \mapsto \varphi^*$ defines a contravariant functor from the category of compact Hausdorff spaces and continuous maps, to the category of abelian groups and group homomorphisms* (and similarly for KO^0-theory.)

The main result of this section is the homotopy-invariance of these K-theory functors: that is, that homotopic maps induce the same map on K-theory.

LEMMA 3.1. *Let V be a vector bundle over a locally compact Hausdorff space X and $Y \subset X$ be a closed subspace. Then any section $s\colon Y \to V$ can be extended to a section $\bar{s}\colon X \to V$ of V on all of X.*

PROOF. For product bundles $X \times \mathbb{R}^n$, the result follows immediately from the Tietze Extension Theorem. For trivial bundles, it follows as well, since if $\varphi\colon V \xrightarrow{\cong} X \times \mathbb{R}^n$ is a bundle isomorphism, s a section of V on a closed subset, $Y \subset X$, then $\varphi \circ s$ is a section of a product bundle on Y, and if t is an extension of it to a section on X, then $\varphi^{-1} \circ t$ is a section of V which extends s on Y, as required.

Now let $\pi\colon V \to X$ be any vector bundle. Then X is covered by open sets on which V is trivial. Let $\{U_i\}_{i\in I}$ and $\{\rho_i\}_{i\in I}$ a partition of unity subordinate to this cover. Let $\varphi_i\colon V|_{U_i} \xrightarrow{\cong} U_i \times \mathbb{R}^n$ trivializations of V on each U_i.

Now using the partition of unity, it will be enough to extend $s|_{U_i\cap Y}$ to a section $s_i\colon U_i \to V$ on U_i. For then we may define $s(x) = \sum_{i\in I} \rho_i(x)s_i(x)$. The sum will be finite, for any fixed $x \in X$, because of local finiteness. And if $x \in Y$, it equals $s(x)$, because $s_i(x) = s(x)$ for every i, since we are assuming s_i extends s on $U_i \cap Y$, and since $\sum \rho_i(x) = 1$, for all $x \in X$.

So we are reduced to showing that any section of $V|_{U_i}$ can be extended from $U_i \cap Y$ to U_i. But by construction $V|_{U_i}$ is trivial, and hence, the extension property follows from our initial remarks.

□

COROLLARY 3.2. *If V_1 and V_2 are vector bundles over X, and if $Y \subset X$ a closed subset, then any isomorphism $V_1|_Y \cong V_2|_Y$ can be extended to an isomorphism $V_1|_U \cong V_2|_U$ on a neighbourhood of Y.*

PROOF. Bundle maps $V_1 \to V_2$ are exactly continuous sections of the bundle $\mathrm{HOM}(V_1, V_2)$ (see Exercise 1.19). So an isomorphism $V_1|_Y \to V_2|_Y$, since it is a section of $\mathrm{HOM}(V_1, V_2)$ on $Y \subset X$, extends, by Lemma 3.1 to a section on X, in other words, to a bundle map $T\colon V_1 \to V_2$ defined on all of X, and such that $T(y)$ is an isomorphism for all $y \in Y$. The result will then follow from the following

Claim. If $T\colon V_1 \to V_2$ is any vector bundle map, then the set $\{x \in X \mid T(x) \text{ an isomorphism }\}$ is open in X.

To see this, let $x_0 \in X$ for which $T(x_0)$ is an isomorphism. We may find trivializations V_1 and V_2 on a neighbourhood V of x_0, and hence we can find a frame for V_1 and a frame for V_2, defined on V, and write T in terms of these frames, as a matrix. Let $\tilde{T}$ be the corresponding map $U \to M_n(\mathbb{R})$ – it is continuous, and takes an invertible value at x_0. Since the inveribles $\mathbf{GL}(n, \mathbb{R}) \subset M_n(\mathbb{R})$ are open in $M_n(\mathbb{R})$, it follows that $\tilde{T}$ takes invertible values in a neighbourhood of x_0, so there exists U a neighbourhood of x_0 on which $\tilde{T}$ takes invertible values. It follows immediately that T is an isomorphism on U. This completes the claim.

□

LEMMA 3.3. *Let X be any compact Hausdorff space and let $i_0, i_1\colon X \to X \times [0,1]$ be the maps $i_0(x) := (x, 0)$, $i_1(x) := (x, 1)$. Then $i_0^* = i_1^*\colon \mathrm{KO}^0(X \times [0,1]) \to \mathrm{KO}^0(X)$, and similarly $i_0^* = i_1^*$ as maps $\mathrm{K}^*(X \times [0,1]) \to \mathrm{K}^*(X)$.*

PROOF. Let $\pi\colon V \to X \times [0,1]$ be a vector bundle. For each t, let $i_t\colon X \to X \times [0,1]$, $i_t(x) := (x,t)$ be the inclusion of X as the slice at t. Let $V_t := i_t^*(V)$, a vector bundle over X. We show the following

Claim. In the above notation, there exists $\epsilon > 0$ such that $V_s \cong V_t$ if $|s - t| < \epsilon$.

To prove the claim, choose any t and consider the bundle $\mathrm{pr}_1^*(V_t)$ on $X \times [0,1]$, with $\mathrm{pr}_1\colon X \times [0,1] \to X$ the first projection map.

Obviously, by the definintions, $\mathrm{pr}_1^*(V_t)$ agrees on the nose with V on the slice $X \times \{t\}$, which is a closed subset of $X \times [0,1]$. In particular, there is a bundle isomorphism $\mathrm{pr}_1^*(V_t) \to V$ defined on the slice. By Lemma this extends to a bundle isomorphism in a neighbourhood of the slice. A routine compactness argument implies that any such neighbourhood contains one of the form $X \times (t - \epsilon, t + \epsilon)$. In particular, if $|t - s| < \epsilon$, V_t is isomorphic to V_s, as claimed.

The result we are trying to prove – that V_0 is isomorphic to V_1 – now follows from a routine compactness argument, producing a list of points $0 < t_1 < \cdots < \epsilon_n < 1$ of the interval close enough to each other that $V_{t_i} \cong V_{t_{i+1}}$, $i = 0, 1, \dots n$.

□

COROLLARY 3.4. *Let φ_0 and φ_1 be homotopic maps $X \to Y$, where X and Y are compact. Then the induced group homomorphisms φ_1^* and $\varphi_2^* \colon \mathrm{KO}^*(Y) \to \mathrm{KO}^*(X)$ are equal. Similarly, $\varphi_1^* = \varphi_2^* \colon \mathrm{K}^*(Y) \to \mathrm{K}^*(X)$.*

PROOF. By definition of homotopy, there exists a map $F \colon X \times [0,1] \to Y$ such that $F \circ i_0 = \varphi_0$ and $F \circ i_1 = \varphi_1$. By functoriality and Lemma 3.3, we get

$$\varphi_0^* = (F \circ i_0)^* = i_0^* \circ F^* = i_1^* \circ F^* = (F \circ i_1)^* = \varphi_1^*,$$

which completes the proof.

□

Ring and module structures on K^0

We close this section with a discussion of the very important *ring* structure on the K^0-group of a compact space.

If V_1 and V_2 are vector bundles over X, then their tensor product $V_1 \otimes V_2$ is a vector bundle over X. If $V_1 \cong V_1'$ and $V_2 \cong V_2'$ then $V_1 \otimes V_2 \cong V_1' \otimes V_2'$, so the tensor product operation descends to an operation on the semigroup of isomorphism classes $\mathrm{Vect}(X)$ (or on $\mathrm{Vect}_{\mathbb{R}}(X)$, if one is working with real bundles.) By the universal property of the Grothendieck completion, tensor products on real and respectively complex bundles descends to a pair of multiplication operations

$$\mathrm{K}^0(X) \times \mathrm{K}^0(X) \to \mathrm{K}^0(X), \;\; \mathrm{KO}^0(X) \times \mathrm{KO}^0(X) \to \mathrm{KO}^0(X).$$

PROPOSITION 3.5. *Under direct sum and tensor product,* $\mathrm{K}^0(X)$ *(respectively* $\mathrm{KO}^0(X)$*) is a commutative ring with identity.*

EXERCISE 3.6. If X is compact and $a = [E^1] - [E^2] \in \mathrm{K}^0(X)$, $b = [F^1] - [F^2] \in \mathrm{K}^0(X)$, then the ring product $a \cdot b \in \mathrm{K}^0(X)$ equals the difference $[(E^1 \otimes F^1) \,\oplus\, (E^2 \otimes F^2)] - [(E^2 \otimes F^1) \,\oplus\, (E^1 \otimes F^2)]$.

The multiplicative identity of $\mathrm{K}^0(X)$ is the class of the trivial line bundle over X (and similarly in KO^0-theory.)

EXERCISE 3.7. If $\varphi \colon X \to Y$ is a map of compact spaces, the induced map abelian group homomorphism $\varphi^* \colon \mathrm{KO}^0(Y) \to \mathrm{KO}^0(X)$ is also a ring homomorphism. (Similarly for complex K-theory.)

EXERCISE 3.8. Let A be a closed, contractible subspace of a compact space X. Prove that $\mathrm{K}^0(X) \cong \mathrm{K}^0(X/A)$, where X/A is the quotient space obtained by crushing A to a point.

4. K-theory for noncompact spaces, higher K-groups

Everything we say in this chapter is equally valid for K-theory and KO-theory. We mainly just focus on K-theory.

Let X be locally compact Hausdorff, X^+ its one-point compactification. Neighbourhoods of the point ∞ at infinity are complements of compact subsets of X, with the point at ∞ added. See Exercise 1.14. The space X^+ is compact Hausdorff. Let $\epsilon_X \colon \mathrm{pt} \to X^+$ the inclusion of the one-point space as the point at infinity. It induces a map $\epsilon_X^* \colon \mathrm{K}^0(X^+) \to \mathrm{K}^0(\mathrm{pt}) \cong \mathbb{Z}$, and similarly induces a map $\mathrm{KO}^0(X^+) \to \mathrm{KO}^0(\mathrm{pt}) \cong \mathbb{Z}$.

Definition 4.1. If X is a locally compact Hausdorff space, we define $\mathrm{K}^0(X)$ to be the kernel of the map $\epsilon_X^*\colon \mathrm{K}^0(X^+) \to \mathbb{Z}$. Similarly, we define $\mathrm{KO}^0(X)$.

Exercise 4.2. Prove that $\mathrm{K}^0\big([0,1)\big)$ is the zero group.

Remark 4.3. Elements in $\mathrm{K}^0(X^+)$ are differences $[V_1]-[V_2]$ of stable isomorphism classes $[V_1]$ and $[V_2]$ of vector bundles over X^+. At the level of vector bundles, the map ϵ_X^* just maps a vector bundle V over X^+ to its restriction V_∞ to the point at ∞; this results in a vector space, and the corresponding integer is its dimension. Thus, $\epsilon_X^*([V]-[W]) = \dim(V_\infty) - \dim(W_\infty)$.

In particular, $\mathrm{K}^0(X)$ is always an *ideal* in the ring $\mathrm{K}^0(X^+)$. In particular, it is of course a subring, and hence a (non-unital) ring in its own right.

Similarly $\mathrm{KO}(X)$-theory is a ring, with the ring structure inherited from $\mathrm{KO}^0(X^+)$.

Example 4.4. $\mathrm{K}^0(\mathbb{R}) = 0$. Indeed, by definition, $\mathrm{K}^0(\mathbb{R})$ is the kernel of the augmentation map $\epsilon_{\mathbb{R}}^*\colon \mathrm{K}^0(\mathbb{R}^+) \to \mathbb{Z}$, while $\mathbb{R}^+ \cong S^1$ is the circle, whose K^0 has already been computed (Proposition 1.10) to be infinite cyclic with generator the class $[1] \in \mathrm{K}^0(S^1)$ of the trivial line bundle on S^1. If $n[1] \in \mathrm{K}^0(S^1)$ is any element, then $\epsilon_{\mathbb{R}}^*(n[1]) = n$ so $\epsilon_{\mathbb{R}}^*$ is injective and $\mathrm{K}^0(\mathbb{R}) := \ker(\epsilon_{\mathbb{R}}^*)$ is the zero group.

Exercise 4.5. Let E be a vector bundle over X. Prove that if E is trivial outside a compact subset of X, then X is (isomorphic to) the restriction of a vector bundle over X^+ to X. (That is, if E is trivial outside a compact set, then E 'extends' to a vector bundle over X^+.)

Exercise 4.6. From the previous exercise, check in detail that if X is noncompact, then $\mathrm{K}^0(X)$ can be described as formal differences $[E^1]-[E^2]$ where E^i are each vector bundles over X, each trivial outside a compact subset of X, and, such that each have the same dimension outside some compact subset. Write down when two such formal differences correspond to the same element of $\mathrm{K}^0(X)$.

Exercise 4.7. Let X be compact and $Y \subset X$ a finite set of points. Let $p\colon X \to (X\setminus Y)^+$ the canonical map, which is the identity on $X\setminus Y$ and maps Y to the point at infinity of $X\setminus Y$. Prove that if E is a vector bundle over $(X\setminus Y)^+$ describe its pull-back p^*E to a vector bundle over X, and prove that p^*E has the same fibre dimension at all the points of Y.

Functoriality of K-theory for non-compact spaces involves a nuance. It is *not* functorial under arbitrary maps $f\colon X \to Y$, but only *proper* maps, for these are precisely the maps which extend continuously to maps $f_+\colon X^+ \to Y^+$ mapping the point at infinity to the point at infinity. Due to this property, $f_+ \circ \epsilon_X = \epsilon_Y$, and hence by functoriality of KO^0 or K^0, $\epsilon_X^* \circ f_+^* = \epsilon_Y^*$ and hence f_+^* maps $\ker(\epsilon_Y^*)$ into $\ker(\epsilon_X^*)$.

Thus, a proper map $f\colon X \to Y$ induces maps $f^*\colon \mathrm{K}^*(Y) \to \mathrm{K}^*(X)$ and $f^*\colon \mathrm{KO}^0(Y) \to \mathrm{KO}^0(X)$.

Suppose that X is already compact. Then ∞ is isolated in X^+ (is an open set) and hence $\mathrm{K}^0(X^+) \cong \mathrm{K}^0(X) \oplus \mathrm{K}^0(\{\infty\}) \cong \mathrm{K}^0(X) \oplus \mathrm{K}^0(\mathrm{pt}) = \mathrm{K}^0(X) \oplus \mathbb{Z}$, with ϵ_X^* corresponding to the second projection map (by Exercise 1.6). It is immediate that $\ker(\epsilon_X^*) = \mathrm{K}^0(X)$, so we recover our old definition of K-theory for compact spaces. The same remarks go through verbatim for KO^0-theory.

Exercise 4.8. Prove that a proper map is a closed map.

Exercise 4.9. Prove that if $\varphi\colon X \to Y$ is a proper map and $f \in C_c(X)$ is a continuous, complex-valued function with compact support, then $f \circ \varphi$ has compact support.

Two proper maps $\varphi_0, \varphi_1\colon X \to Y$ are *properly homotopic* if there is a proper map $F\colon X \times [0,1] \to Y$ such that $F \circ i_0 = \varphi_0$ and $F \circ i_1 = \varphi_1$, with i_0, i_1 the inclusions of X at the endpoints, as usual.

PROPOSITION 4.10. *If X and Y are locally compact Hausdorff and $\varphi_0, \varphi_1 \colon X \to Y$ are properly homotopic proper maps, then $\varphi_0^* = \varphi_1^*$ as maps $\mathrm{K}^0(Y) \to \mathrm{K}^0(X)$. Similarly, $\varphi_0^* = \varphi_1^* \colon \mathrm{KO}^0(Y) \to \mathrm{KO}^0(X)$*

PROOF. For the proof, we restrict ourselves to complex K-theory. The same proof works for the real version.

As before, it is enough to prove that the maps $i_0^*, i_1^* \colon \mathrm{K}^0(X \times [0,1]) \to \mathrm{K}^0(X)$, are equal. (Note that they are each proper.) By the definitions, it is sufficient to show that i_0^+ and i_1^+ induce the same map $\mathrm{K}^0\big((X \times [0,1])^+\big) \to \mathrm{K}^0(X^+)$.

Let $H \colon X^+ \times [0,1] \to (X \times [0,1])^+$ map any $(x,t) \in X \times [0,1]$ to the image of (x,t) in $(X \times [0,1])^+$, and let $H(\infty, t) = \infty$ for every $t \in [0,1]$. The reader can easily verify that H is continuous. It gives a homotopy between i_0^+ and i_1^+, as maps between two compact spaces. Hence $(i_0^+)^* = (i_1^+)^*$ from Theorem 3.4. This proves the result. □

We can now define the higher K-theory groups of a space.

DEFINITION 4.11. For X locally compact Hausdorff, $\mathrm{K}^{-n}(X)$ is defined to be $\mathrm{K}^0(X \times \mathbb{R}^n)$, and likewise $\mathrm{KO}^{-n}(X) := \mathrm{KO}^0(X \times \mathbb{R}^n)$.

EXAMPLE 4.12. Since $\mathrm{K}^0(\mathbb{R}) = 0$, (Example 4.4) we have so far determined that $\mathrm{K}^0(\mathrm{pt}) \cong \mathbb{Z}$ and $\mathrm{K}^{-1}(\mathrm{pt}) = 0$. The computation of $\mathrm{K}^{-2}(\mathrm{pt})$ and the higher groups $\mathrm{K}^{-3}(\mathrm{pt}), \mathrm{K}^{-4}(\mathrm{pt}), \dots$, which turn out to be 2-periodic, is much harder.

REMARK 4.13. $\mathbb{R}^n$ is obviously a contractible space, but it is not *properly* contractible. Hence there is no *a priori* reason to suppose the K-theory or KO-theory groups of $\mathbb{R}^n$, equivalently, the higher K-theory groups $\mathrm{K}^{-n}(\mathrm{pt})$, of a point, are uninteresting. (And similarly for KO-theory.)

Since $(\mathbb{R}^n)^+ \cong S^n$, $\mathrm{K}^{-n}(\mathrm{pt}) := \mathrm{K}^0(\mathbb{R}^n)$ is the subgroup of $\mathrm{K}^0(S^n)$ consisting of differences $[V_1] - [V_2]$ of vector bundles over the sphere, of the same dimension. The difference $[H] - [1]$ is an example of such a difference, where H is the Hopf bundle.

The Hopf bundle is non-trivial, so there is no immediate reason to conclude that this difference is zero in $\mathrm{K}^0(S^n)$ (in fact it is not); the difference, in fact, measures exactly the non-triviality of the Hopf bundle.

In fact, it is the computation of the K^0 and KO^0-groups of $\mathbb{R}^n$ that is the key one in the whole subject of K-theory – they exhibit an interesting periodicity in n due to Bott, and called Bott Periodicity. It is discussed in the next chapter.

K-*theory classes from triples;* K-*theory 'germs'*

One of the key points in the construction of K-theory or KO-theory classes from geometric considerations is that they can be constructed on various interesting *non*-compact spaces, by considering pairs of bundles, isomorphic to each other off a compact set. Since many spaces (like manifolds) have interesting open subsets, one can often splice a K-theory class for the (non-compact) open 'subset, into a K-theory class for X, with interesting results.

We start with some basic observations about the K-theory of open subsets of a space.

EXERCISE 4.14. Let $U \subset X$ be an open subset. Show that mapping the complement of U in X^+ to the point at infinity of U^+ results in a continuous map $i^+ \colon X^+ \to U^+$ mapping the points at infinity to each other.

The following constructions work in either K-theory or KO-theory; for brevity we restrict ourselves to K-theory.

a) If $i_U \colon U \to X$ is the inclusion of an open set in X, and $i^+ \colon X^+ \to U^+$ the map described above, show that $(i^+)^* \colon \mathrm{K}^0(U^+) \to \mathrm{K}^0(X^+)$ maps $\ker(\epsilon_U^*)$ to $\ker(\epsilon_X^*)$. Let $i_U! \colon \mathrm{K}^0(U) \to \mathrm{K}^0(X)$ be the corresponding map.

b) Prove that if $U \xrightarrow{i_U} V$ and $V \xrightarrow{j_V} W$ are two open inclusions then $(j_V \circ i_U)! = j_V! \circ i_U! \colon \mathrm{K}^0(U) \to \mathrm{K}^0(W)$.

c) Prove that the groups $\mathrm{K}^0(U)$, as U runs over the open subsets of X, directed by inclusion, and the group homomorphisms $i! \colon \mathrm{K}^0(U) \to \mathrm{K}^0(V)$, for $i \colon U \to V$ an inclusion, make up a directed system of groups, and prove that

$$\mathrm{K}^0(X) \cong \varinjlim_{U} \mathrm{K}^0(U).$$

d) Prove that (for X locally compact Hausdorff as usual), the result of c) holds if we restrict the directed system just to the collection of *pre-compact* open subsets of X.

As a consequence of the result in part d) of the Exercise, is that every K^0-class for X has the form $i_U!(a)$ for some K-theory class $a \in \mathrm{K}^0(U)$, for an open and *pre-compact* subset $U \subset X$.

Definition 4.15. Let X be a locally compact space. A K-triple E for X (respectively a KO-triple) consists of a pair E^0 and E^1 of complex (respectively real) vector bundles over X, and a bundle map $\varphi \colon E^0 \to E^1$, which is an isomorphism on the complement of a compact subset of X.

Two triples $E = (E^0, E^1, \varphi)$ and $F = (F^0, F^1, \psi)$ are *isomorphic* if there are vector bundle isomorphisms $\alpha \colon E^0 \to F^0$ and $\beta \colon E^1 \to F^1$ such that the diagram

$$\begin{tikzcd} E^0 \arrow[r, "\varphi"] \arrow[d, "\alpha"] & E^1 \arrow[d, "\beta"] \\ F^0 \arrow[r, "\varphi'"] & F^1 \end{tikzcd}$$

commutes.

A *homotopy* of triples is a triple (E^0, E^1, φ) over $X \times [0,1]$; the inclusions $i_0 \colon X \to X \times [0,1]$ and $i_1 \colon X \to X \times [0,1]$ at the endpoints of the interval pull such a triple to a pair of triples for X, which we call *homotopic triples.*

The *support* of a triple (V, W, φ) is the set of points $x \in X$ for which $\varphi(x)$ is *not* an isomorphism. The support is compact, by the definitions.

A *degenerate triple* is one for which φ is an isomorphism everywhere.

On the collection of isomorphism classes of triples, we put the equivalence relation generated by homotopy and addition of degenerate triples. Let $L(X)$ denote the correspond semigroup, with addition operation direct sum of triples. (It turns out to be a group, but we may not prove this. ON the other hand, we may..)

We are going to describe the map $L(X) \to \mathrm{K}^0(X)$ – it can be shown to be an isomorphism, but we will not need this fact. The *map* is of most importance – it is particularly relevant because it is as complexes that natural K-theory classes in topology come presented.

Suppose $\tau = (E^0, E^1, \varphi)$ is a triple. Let $U \subset X$ be any neighbourhood of its support, a compact subset of X (U could be X, for example). We define a K^0-class $[\tau_U] \in \mathrm{K}^0(U)$ in the following way. Let W and V be open subsets of X with $\mathrm{supp}(E) \subset W \subset \overline{W} \subset V \subset \overline{V} \subset U$, and $\overline{V}$ compact. Since $\overline{V}$ is compact, there is a vector bundle F over $\overline{V}$ such that $E^1 \oplus F$ is trivial on $\overline{V}$. Adding the degenerate triple (F, F, id) to E results in a triple for $\overline{V}$ in which the second vector bundle is trivial.

Instead of introducing new notation for this, we just denote by $\tau = (E^0, E^1, \varphi)$ the triple we have constructed, for $\overline{V}$, in which now the bundle E^1 is a product bundle.

We now proceed as in the examples. Let $A = \overline{W}$ and $B \subset V^+$ be the complement of W in V, together with the point of infinity of V^+. Then A and B are closed in V^+.

Take the bundle E^0 on A, and clutch it to the trivial bundle $E^1 = B \times \mathbb{C}^n$ on B using the clutching function $E^0|_{A\cap B} \xrightarrow{\varphi} E^1_{A\cap B} = A \cap B \times \mathbb{C}^n = (B \times \mathbb{C}^n)|_{A\cap B}$. The clutching results in a vector bundle $\tilde{E}$ on V^+. The difference $[\tilde{E}] - [1_n]$ is in $\mathrm{K}^0(V)$, where $n = \dim(E^0)$. We now set

Definition 4.16. $\tau_U := i_{U,V}!([\tilde{E}] - [1_n])) \in \mathrm{K}^0(U)$ where $i_{U,V} \colon V \to U$ is the inclusion.

Exercise 4.17. In the above notation, if $j \colon U \to U'$ is an inclusion of open sets, then $[\tau_{U'}] = j!([\tau_U])$.

In particular, any triple τ over X determines a class $\tau_U \in \mathrm{K}^0(U)$ for any neighbourhood $U \subset X$ of its support, and in particular, determines a class $\tau_X \in \mathrm{K}^0(X)$.

Example 4.18. (The Bott element for $\mathbb{R}^2$). For $(x, y) \in \mathbb{R}^2$ let $c(x, y) \colon \mathbb{C} \to \mathbb{C}$ be multiplication by the complex number $x + iy$. We may interpret c as a vector bundle map from the trivial bundle $\mathbf{1}_2 := \mathbb{R}^2 \times \mathbb{C}$ over $\mathbb{R}^2$, to itself.

Now, c is a bundle isomorphism away from 0. Let U be any neighbourhood of the origin (it could be all of $\mathbb{R}^2$), let $A \subset U$ be a small closed Euclidean ball contained in U and centred at the origin. Let B be the closure in U^+ of $U^+ \setminus A$. Thus, B consists of the closure of the complement of A in U, together with the point at infinity of U.

On A we put the product bundle $A \times \mathbb{C}$, on B we put the product bundle $B \times \mathbb{C}$, and we clutch them (see Exercise 1.23) using the function c on $A \cap B$. This results in a complex vector bundle H_U on U^+, and a class $\beta_U := [H_U] - [1] \in \mathrm{K}^0(U)$, where $[1] \in \mathrm{K}^0(U^+)$ is the class of the trivial line bundle on U^+, because $\epsilon_U^*(\beta_U) = 0$.

Exercise 4.19. In the above notation, prove that the complex vector bundle $H_{\mathbb{R}^2}$ over $(\mathbb{R}^2)^+ = S^2$ is isomorphic to the Hopf bundle.

Hence $\beta_{\mathbb{R}^2} \in \mathrm{K}^0(\mathbb{R}^2) \subset \mathrm{K}^0(S^2)$ is equal to the difference $[H] - [1]$, where 1 is the trivial complex line bundle over S^2 and H is the Hopf bundle over S^2.

Exercise 4.20. Verify that if $i \colon U \to V$ is an inclusion of neighbourhoods of the origin in $\mathbb{R}^2$ then $i_U!(\beta_U) = \beta_V$.

It will be a consequence of Bott Periodicity that $\mathrm{K}^0(\mathbb{R}^2)$ is an infinite cyclic group generated by $\beta_{\mathbb{R}^2}$.

Example 4.21. The system of 'Bott elements' $\beta_U \in \mathrm{K}^0(U)$, one attached to each neighbourhood of the origin $\mathbb{R}^2$, suggests might be thought of as the specification of a kind of a 'germ' of a K-theory class, around the origin.

One can also get (1-dimensional, now) 'K-theory 'germs' in this (informal) sense around smooth curves in the plane, as we now show. In order to make things topologically nontrivial, remove a finite set of points, let $X = \mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$ from the plane.

We consider a smooth closed curve C in the plane looping around some of these points. By the Jordan Curve Theorem one can select a (smooth) field of unit vectors $\mathbf{n}(x)$ as $x \in C$, such that $\mathbf{n}(x)$ is perpendicular to the tangent of the curve at x. We can thus label points in a neighbourhood of the curve by pairs (x, t) where $x \in C$ and $t \in \mathbb{R}$, but making this pair correspond to the point $x + t\mathbf{n}(x)$. Our labelling determines a natural diffeomorphism and system of coordinates on the neighbourhood U of the curve consisting of points (x, t) where $|t| < \epsilon$. Let $V = U \times \mathbb{R}$. On U let $c(x, t, s) := t + is$ where (x, t) are the coordinates as explained above, of a point of U.

One finds suitable closed sets A and B to argue that the bundle map c determines, by clutching, a canonical class in $\mathrm{K}^0(V^+)$ and then, by subtracting the class of a trivial line bundle over $A \cap B$, a class in $\mathrm{K}^0(V) = \mathrm{K}^0(U \times \mathbb{R}) = \mathrm{K}^{-1}(U)$, which can then be pushed forward to a class $\beta_C \in \mathrm{K}^{-1}(\mathbb{R}^2 \setminus \{p_1, \dots, p_n\})$.

It can be shown that this 'germ' is a non-trivial K-theory class for X f the curve loops around at least some of the points.

Graded ring structure on higher K*-theory*

A graded ring is *graded commutative* if $ab = (-1)^{\partial a \partial b} ba$ for any homogeneous elements a, b of degrees ∂a and ∂b. It turns out that $\mathrm{K}^*(X) := \oplus_{n=0}^{\infty} \mathrm{K}^{-n}(X)$ has a graded commutative ring structure extending, in an appropriate sense, the ring structure on $\mathrm{K}^0(X)$ by tensor product of vector bundles.

Before proceeding, let X and Y be locally compact spaces and $Z = X \times Y$, $\pi_X \colon Z \to X$ and $\pi_Y \colon Z \to Y$ the projection maps. These will not be proper maps, if the spaces are not compact. So if $a \in \mathrm{K}^0(X)$, it does not quite make sense to write $\pi_X^*(a) \in \mathrm{K}^0(Z)$, as π being not proper, does not give a map on K-theory. However, $\pi_X^*(a) \cdot \pi_Y^*(b)$ does in fact make sense as an element of $\mathrm{K}^0(X \times Y)$, it's 'support' is roughly speaking, the product of the support of a and the support of b, which will be compact.

Suppose $a = [E^1] - [E^2]$ for two vector bundles E^i on X, trivial and of the same dimension outside a compact subset $K \subset X$. Write $b = [F^1] - [F^2]$, F^i trivial and of the same dimension off $L \subset Y$.

Let $\tilde{E}^i := \pi_X^*(E^i)$, $\tilde{F}^i := \pi_Y^*(F^i)$. Then $\tilde{E}^i$ are trivial and isomorphic to each other outside $K \times Y$, and the $\tilde{F}^i$ are trivial and isomorphic to each other outside $X \times L$.

Consider the vector bundles

$$V^1 := (\tilde{E}^1 \otimes \tilde{F}^1) \ \oplus \ (\tilde{E}^2 \otimes \tilde{F}^2),$$

and

$$V^2 := (\tilde{E}^2 \otimes \tilde{F}^1) \ \oplus \ (\tilde{E}^1 \oplus \tilde{F}^2).$$

Now $\tilde{E}^1 \cong \tilde{E}^2$ outside $K \times Y$, so the first summand $\tilde{E}^1 \otimes \tilde{F}^1$ of V^1 is isomorphic to the first summand $\tilde{E}^2 \otimes \tilde{F}^1$ of V^2 outside $K \times Y$. By the same reasoning, the second summand of V^1 is isomorphic to the second summand of V^2 outside $K \times Y$. Therefore, V^1 is isomorphic to V^2 outside $K \times L$.

On the other hand, outside $X \times L$, the second summand of V^1 is isomorphic to the first summand of V^1, and, likewise, the first summand of V^1 is isomorphic to the second summand of V^2, so that in this case also, we see that V^1 is isomorphic to V^2.

We conclude therefore that V^1 is isomorphic to V^2 outside $K \times L$, a compact subset of $X \times Y$, and fixing the isomorphism, we obtain a triple (V^1, V^2, φ) representing an element of $\mathrm{K}^0(X \times Y)$, which we denote by $\pi^*(a) \cdot \pi^*(b)$.

REMARK 4.22. The idea is that the product $\pi^*(a) \cdot \pi^*(b)$ should be represented by the product, formally speaking,

$$[\tilde{E}^1] - [\tilde{E}^2]) \cdot ([\tilde{F}^1] - [\tilde{F}^2]), \tag{4.1}$$

the problem of course being that neither of the terms actually define K-theory classes for $X \times Y$.

However, the first term 'vanishes' outside $K \times Y$, and the second term vanishes outside $X \times L$, so the idea is that the product should vanish outside $K \times L$, which of course is compact, making the product define a K-theory class.

In fact, if one multiplies, somewhat formally, the equality (4.1) out, one obtains the formal difference $[(\tilde{E}^1 \otimes F^1) \ \oplus \ (\tilde{E}^2 \otimes F^2)] - [(\tilde{E}^2 \otimes F^1) \ \oplus \ (\tilde{E}^1 \oplus F^2)]$, that is, one obtains $[V_1] - [V_2]$ with V_i defined as above.

EXERCISE 4.23. Construct an explicit formula for the isomorphism between V_1 and V_2 based on the assumed isomorphisms $E^1 \cong E^2$ and $F^1 \cong F^2$ (outside suitable compact sets.)

By similar arguments (see Exercise 4.25) one can argue that there is a multiplication operation between K-theory classes $a \in \mathrm{K}^0(X)$ and K-theory classes $c \in \mathrm{K}^0((X \times Y)$, with values in $\mathrm{K}^0(X \times Y)$, which we denote by $\pi^*(a) \cdot c$.

Thus, $\mathrm{K}^0(X \times Y)$ has the structure of a *module* over the ring $\mathrm{K}^0(X)$. Similarly, $\mathrm{K}^0(X \times Y)$ is a module over $\mathrm{K}^0(Y)$.

PROPOSITION 4.24. *The pairings and module structures defined above, are all well-defined, $\mathbb{Z}$-bilinear, and associative in the sense that*

$$\pi_X^*(a) \cdot \big(\pi_X^*(a') \cdot \pi_Y^*(b)\big) = \big(\pi_X^*(a \cdot a')\big) \cdot \pi_Y^*(b),$$

and

$$\big(\pi_X^*(a) \cdot \pi_Y^*(b)\big) \cdot \pi_Y^*(b') = \pi_X^*(a) \cdot \big(\pi_Y^*(b \cdot b')\big),$$

for $a, a' \in \mathrm{K}^0(X)$, $b, b' \in \mathrm{K}^0(Y)$.

EXERCISE 4.25. Let X and Y be locally compact spaces and let $f\colon Y \to X$ be any map (not necessarily proper). Then there is a well-defined, $\mathbb{Z}$-bilinear multiplication operation $\mathrm{K}^0(Y) \times \mathrm{K}^0(X) \to \mathrm{K}^0(Y)$ mapping a pair $a \in \mathrm{K}^0(X)$ and $c \in \mathrm{K}^0(Y)$ to an element $f^*(a) \cdot c \in \mathrm{K}^0(Y)$, which makes $\mathrm{K}^0(Y)$ into a module over the ring $\mathrm{K}^0(X)$.

EXERCISE 4.26. Generalize the $\mathbb{Z}$-bilinear multiplication operation $\mathrm{K}^0(X) \times \mathrm{K}^0(Y) \to \mathrm{K}^0(X \times Y)$ developed above to a multiplication operation $\mathrm{K}^0(X) \times \mathrm{K}^0(Y) \to \mathrm{K}^0(Z)$, producing an element $\rho_1^*(a) \cdot \rho_2^*(b)$ from $a \in \mathrm{K}^0(X)$, $b \in \mathrm{K}^0(Y)$, whenever $\rho_1\colon Z \to X$ and $\rho_2\colon Z \to Y$ are two maps with the property that *for any pair of compact subsets $K \subset X$ and $L \subset Y$, $\rho_1^{-1}(K) \cap \rho_2^{-1}(L)$ is compact in Z.*

We may now define a *graded ring structure* on $\mathrm{K}^*(X) = \oplus_{i=0}^{\infty} \mathrm{K}^{-i}(X)$.

Choose $r, s \geq 0$ and let $\pi_1\colon \mathbb{R}^{r+s} \to \mathbb{R}^r$, $\pi_2\colon \mathbb{R}^{r+s} \to \mathbb{R}^s$ be the projection maps. Then for any locally compact space X, consider the maps $\rho_1 := \mathrm{id}_X \times \pi_1\colon X \times \mathbb{R}^{r+s} \to X \times \mathbb{R}^r$, and $\rho_2\colon X \times \mathbb{R}^{r+s} \to X \times \mathbb{R}^s$. It is easily checked that $\rho_1^{-1}(K) \cap \rho_2^{-1}(L)$ is compact in $X \times \mathbb{R}^{r+s}$, for any compact $K \subset X \times \mathbb{R}^r$ and any compact $L \subset X \times \mathbb{R}^s$. By Exercise 4.26, there is a well-defined product class $\rho_1^*(a) \cdot \rho_2^*(b) \in \mathrm{K}^0(X \times \mathbb{R}^{k+s})$ for any $a \in \mathrm{K}^0(X \times \mathbb{R}^r)$ and $b \in \mathrm{K}^0(X \times \mathbb{R}^s)$.

DEFINITION 4.27. If $a \in \mathrm{K}^{-r}(X)$, $b \in \mathrm{K}^{-s}(X)$, we let

$$a \wedge b \in \mathrm{K}^{-(r+s)}(X)$$

denote the class $\rho_1^*(a) \cdot \rho_2^*(b)$ described above.

The wedge product notation is requried to distinguish our graded multiplication from ordinary multiplication, when the situation is ambiguous. If, for example, $a, b \in \mathrm{K}^{-2}(\mathrm{pt}) := \mathrm{K}^0(\mathbb{R}^2)$, then since $\mathrm{K}^0(X)$ is always a ring, for any X, and in particular for $X = \mathbb{R}^2$, we can form $a \cdot b \in \mathrm{K}^0(\mathbb{R}^2)$, whereas, $a \wedge b \in \mathrm{K}^{-4}(\mathrm{pt}) = \mathrm{K}^0(\mathbb{R}^2 \times \mathbb{R}^2)$, lies, of course, in a different group. The two products are related by the following

EXERCISE 4.28. In the above notation, if $\delta\colon \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ is the diagonal map, then $a \cdot b = \delta^*(a \wedge b)$.

The multiplication $a \wedge b$ is easily checked to be associative, and by the definitions, extends the usual ring structure on the summand $\mathrm{K}^0(X)$. Recall that the latter ring structure is *commutative*. The more general multiplication turns out to be *graded commutative*.

PROPOSITION 4.29. *If $a \in \mathrm{K}^{-r}(X)$ and $b \in \mathrm{K}^{-s}(X)$, then $a \wedge b = (-1)^{rs} b \wedge a$. That is, $\mathrm{K}^*(X)$ is a* graded commutative *ring.*

5. The long exact sequence of a pair

In ordinary cohomology, defined for spaces using chain complexes, the Snake Lemma implies that associated to a closed subspace $A \subset X$ is a long exact cohomology sequence.

In this section we show that such a pair generates a similar such long exact sequence of K-theory groups. This is based on some fairly simple abstract tricks.

THEOREM 5.1. *Let $A \subset X$ be a closed subspace of a locally compact space. Then there exist natural maps $\delta\colon \mathrm{K}^{-i}(A) \to \mathrm{K}^{-i+1}(X \setminus A)$ for which the sequence, infinite to the left,*

$$\begin{aligned}(5.1)\qquad \cdots \to \mathrm{K}^{-i-1}(A) \xrightarrow{\delta} \mathrm{K}^{-i}(X \setminus A) \xrightarrow{i!} \mathrm{K}^{-i}(X) \xrightarrow{j^*} \mathrm{K}^{-i}(A) \longrightarrow \cdots \\ \cdots \xrightarrow{\delta} \mathrm{K}^0(X \setminus A) \to \mathrm{K}^0(X) \to \mathrm{K}^0(A)\end{aligned}$$

is exact, where $i\colon X \setminus A \to X$ is the (open) inclusion, $j\colon A \to X$ the (closed) inclusion.

This long exact sequence is natural with respect to maps $(X, A) \to (X', A')$ of pairs of locally compact spaces.

We will know nothing about the range of the last map until Bott Periodicity. This makes the long exact sequence not very helpful for computations.

We give a sketch of the proof below. Use of *reduced* K-theory makes the argument go much cleaner, because the reduced K-theory of a point becomes zero.

DEFINITION 5.2. The *reduced* K-theory $\tilde{\mathrm{K}}^0(X)$ of X compact, is the quotient of $\mathrm{K}^0(X)$ by the subgroup generated by the trivial bundles on X.

It is clear that $\tilde{\mathrm{K}}$ is still a functor, since if $\varphi\colon X \to Y$ is a map, then the induced homomorphism $\varphi^*\colon \mathrm{K}^0(Y) \to \mathrm{K}^0(X)$ maps classes of trivial bundles to classes of trivial bundles.

EXERCISE 5.3. A map $X \to Y$ is *nullhomotopic* if it is homotopic to a map $X \to Y$ which factors through the 1-point space. Prove that if $\varphi\colon X \to Y$ is null-homotopic, then $\varphi^*\colon \tilde{\mathrm{K}}^0(Y) \to \tilde{\mathrm{K}}^0(X)$ is the zero homomorphism.

EXERCISE 5.4. Recall that two vector bundles V and W over X are *stably isomorphic* if $V \oplus 1_k \cong W \oplus 1_l$, as vector bundles, for a pair of trivial bundles 1_k and 1_l. Prove that under direct sum, the collection of stable isomorphism classes of vector bundles over X forms an abelian *group* isomorphic to $\tilde{\mathrm{K}}^0(X)$.

If X and Y are compact spaces containing a common subspace A we let $X \cup_A Y$ be their usual topological sum along A: it is the quotient of $X \sqcup Y$ by the equivalence relation that identifies the copies of A in X and Y.

LEMMA 5.5. *Let X and Y be compact spaces containing a common closed subspace A. Let $j\colon A \to X$ and $i\colon X \to X \cup_A Y$ be the obvious inclusions. Then the kernel of $j^*\colon \tilde{\mathrm{K}}^0(X) \to \tilde{\mathrm{K}}^0(A)$ is contained in the range of $i^*\colon \tilde{\mathrm{K}}^0(X \cup_A Y) \to \tilde{\mathrm{K}}^0(X)$.*

PROOF. We can represent any element of $\tilde{\mathrm{K}}^0(X)$ by the stable isomorphism class of a bundle E over X. Then by the definitions, $i^*([E]) = 0$ if and only if $E_{|A}$ is stably isomorphism to a trivial bundle on A. Thus, $E_{|A} \oplus 1_k \cong 1_l$ for some k, l. Now clutch the bundle $E \oplus 1_k$ (over X, and the trivial bundle 1_l of rank l, over Y, over the subspace A, using the isomorphism $E_{|A} \oplus 1_k \cong 1_l$. This results in a bundle $\tilde{E}$ over X whose restriction to X is $E \oplus 1_k$, which is stably isomorphic to E.

□

If Z is any space, we let $CZ := Z \times [0,1]/\, Z \times \{0\}$ (the 'cone' on Z.) It contains a copy of Z as the image of $Z \times \{1\}$. Note that this embedding $Z \to CZ$ is nullhomotopic. The *suspension* of X is defined $SX := CX \cup_X CX$. It is obtained from $X \times [0,1]$ by identifying $X \times \{0\}$ and $X \times \{1\}$ to (separate) points.

LEMMA 5.6. *Let X be any compact space and $A \subset X$ a closed subspace. Then the sequence of groups*

$$\tilde{\mathrm{K}}^0(X \cup_A CA) \xrightarrow{i^*} \tilde{\mathrm{K}}^0(X) \xrightarrow{j^*} \tilde{\mathrm{K}}^0(A) \tag{5.2}$$

is exact, where the maps are those induced by the inclusion $i\colon X \to X \cup_A CA$ and the inclusion $j\colon A \to X$, respectively.

PROOF. The composition $A \xrightarrow{j} X \xrightarrow{i} X \cup_A CA$ agrees with the composition $A \to CA \to X \cup_A CA$ and the latter is nullhomotopic. Hence it induces the zero map on $\tilde{\mathrm{K}}^0$. This shows that $(i \circ j)^* = j^* \circ i^*$ is the zero map, *I.e.* that $\mathrm{ran}(i^*) \subset \ker(j^*)$. The converse follows from Lemma 5.5.

□

Now the above procedure of attaching a cone to X along A can be iterated with now $X \cup_A CA$ the subspace, along which we attach a cone to the larger space $CX \cup_A CA$. This produces the exact sequence

$$\tilde{\mathrm{K}}^0\big(CX \cup_A CA \cup_{X \cup_A CA} C(X \cup_A CA)\big) \xrightarrow{i^*} \tilde{\mathrm{K}}^0(CX \cup_A CA) \xrightarrow{j^*} \tilde{\mathrm{K}}^0(X \cup_A CA), \tag{5.3}$$

which can be then spliced to the sequence (5.2)

By the definitions, $C(X \cup_A CA) \cong CX \cup_{CA} C(C(A))$. Hence

$$CX \cup_A CA \cup_{X \cup_A CA} C(X \cup CA) \cong CX \cup_A CA \cup_{X \cup_A CA} \big(CX \cup_{CA} C(C(A))\big),$$

which, bringing one of the CA's across the union, is homeomorphic to

$$CX \cup_X CX \cup_{CA} C(C(A)) \cong SX \cup_{CA} C(C(A)). \tag{5.4}$$

On the other hand, it is not hard to see that $C(C(A))$ is homotopy-equivalent to CA by a homotopy leaving CA fixed. Therefore, (5.4) is homotopy-equivalent to SX. We leave it to the reader to check that $CX \cup_A CA$ is homotopy-equivalent to $CA \cup_A CA = SA$.

Finally, observe that since CA is a closed, contractible subspace of $X \cup_A CA$, and since clearly $X/A \cong X \cup_A CA \,/\, CA$, we get an isomorphism $\mathrm{K}^0(X \cup_A CA) \cong \mathrm{K}^0(X \cup_A CA/\, CA) \cong \mathrm{K}^0(X/A)$, and similarly with reduced K-theory.

Making all these adjustments to the sliced-together sequences produces the exact sequence of groups

$$\tilde{\mathrm{K}}^0\big(SX\big) \xrightarrow{i^*} \tilde{\mathrm{K}}^0(SA) \xrightarrow{\delta} \tilde{\mathrm{K}}^0(X/A) \xrightarrow{i^*} \tilde{\mathrm{K}}^0(X) \xrightarrow{j^*} \tilde{\mathrm{K}}^0(A). \tag{5.5}$$

Since $\tilde{K}^0(SZ) \cong \tilde{\mathrm{K}}^{-1}(Z)$, we may re-write this

$$\tilde{\mathrm{K}}^{-1}(X) \xrightarrow{i^*} \tilde{\mathrm{K}}^{-1}(A) \xrightarrow{\delta} \tilde{\mathrm{K}}^0(X/A) \xrightarrow{i^*} \tilde{\mathrm{K}}^0(X) \xrightarrow{j^*} \tilde{\mathrm{K}}^0(A). \tag{5.6}$$

The last step of the proof is to replace reduced K-theory with ordinary K-theory. If X is compact, this follows from the fact that $\tilde{\mathrm{K}}^0(X^+) \cong \mathrm{K}^0(X)$, with X^+ the one-point compactification of X.

EXERCISE 5.7. Let X be a compact space and $Y \subset X$ a finite subset. Let X/Y be the *quotient* space obtained by identifying all the points of Y with each other. Let $\pi\colon X \to X/Y$ be the quotient map.

a) Prove that $\pi^*\colon \mathrm{K}^0(X/Y) \to \mathrm{K}^0(X)$ is always injective.

b) Prove that π^* is an isomorphism if all points of Y lie in the same connected component of X.

(*Hint.* X/Y can be identified with $(X \setminus Y)^+$. We get a long exact sequence from the pair $Y \subset X$. On the other hand, X/Y comes with a natural basepoint and this generates another long exact sequence. The second sequence maps naturally to the first; examine the corresponding commutative diagram, and use the fact that $\mathrm{K}^{-1}(Y)$ is zero for any finite Y.)

Note that π^* does *not* induce an isomorphism on K^{-i} for $i > 0$; for example identifying the endpoints of $[0,1]$ results in S^1, and $\mathrm{K}^{-1}(S^1) \cong \mathbb{Z}$ will follow from Bott Periodicity. But $\mathrm{K}^{-1}([0,1]) \cong \mathrm{K}^{-1}(\mathrm{pt}) = 0$. Nor is π^* an isomorphism even on K^0, if the connectedness assumption is dropped (consider the 2-point space X.)

REMARK 5.8. The last exercise makes it a bit easier to visualize what space one is dealing with in computing $\mathrm{K}^{-1}(X)$, for X compact (say). By the definitions, $\mathrm{K}^{-1}(X) = \mathrm{K}^0(X \times \mathbb{R}) := \ker \epsilon_X^* \colon \mathrm{K}^0\big((X \times \mathbb{R})^+\big) \to \mathbb{Z}$. The space $(X \times \mathbb{R})^+$ is thus what is of interest here. We can consider it alternately as the quotient space $X \times [0,1] \;/\; \sim$ where the equivalence relation collapses $X \times \{0\} \cup X \times \{1\}$ to a single point. The exercise above shows that this results in the same K^0 group as for the quotient space obtained by each of $X \times \{0\}$ and $X \times \{1\}$ to (different) points.

EXERCISE 5.9. Deduce from Exercise 5.7 (and Remark 5.8) that $\mathrm{K}^0(S^2) \cong \mathrm{K}^0((S^1 \times \mathbb{R})^+)$, by a natural isomorphism, and that as a consequence,

$$\mathrm{K}^0(\mathbb{R}^2) \cong \mathrm{K}^{-1}(S^1).$$

(*C.f.* Exercise 1.24.) The same reasoning proves the more general result that $\mathrm{K}^0(\mathbb{R}^n) \cong \mathrm{K}^{-1}(S^{n-1})$ for all n (the case $n = 0$ has in effect already been proved.)

EXERCISE 5.10. Give another proof that $\mathrm{K}^{-1}(S^1) \cong \mathrm{K}^0(\mathbb{R}^2)$ (even though at this stage, we are still not in a position to say what either of these groups are), using the following method. The closed subset $S^1 \subset \overline{\mathbb{D}}$ generates a long exact sequence

$$\cdots \to \mathrm{K}^{-1}(\overline{\mathbb{D}}) \to \mathrm{K}^{-1}(S^1) \xrightarrow{\delta} \mathrm{K}^0(\mathbb{R}^2) \to \mathrm{K}^0(\overline{\mathbb{D}}) \to \mathrm{K}^0(S^1). \tag{5.7}$$

Argue that the last map is an isomorphism and deduce that δ is an isomorphism.

EXERCISE 5.11. Let A be a contractible subspace (that is, contractible as a topological space in its own right) of a compact space X. Let X/A be the space obtained from X by identifying A to a point. Prove that the quotient map $\pi \colon X \to X/A$ induces an isomorphism $\pi^* \colon \mathrm{K}^0(X/A) \to \mathrm{K}^0(X)$. (*Hint.* Define an inverse map $\mathrm{K}^0(X) \to \mathrm{K}^0(X/A)$ as follows. If E is a vector bundle over X, find a trivialization $E \cong 1_n$ of E restricted to A; one exists, since A is contractible. If $u \colon E_{|A} \to 1_n = A \times \mathbb{C}^n$ is a trivialization, extend it to a bundle map $\bar{u}$ from E to 1_n in a neighbourhood of A. Now clutch the bundle 1_n over a suitable (slightly smaller) neighbourhood of A with E on the complement. This bundle now has a single fibre over A and can be considered a bundle over X/A.)

An alternative description of $\mathrm{K}^{-1}(X)$ *and the boundary map for the long exact sequence*

We first discuss an interesting topological group, whose connected components are relevant to K-theory.

Let A be a unital C*-algebra. Let $\mathbf{U}_\infty(A)$ be the group of all $\mathbb{N}$-by-N-matrices with entries in A, which have a block diagonal form $\begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$ with u a (square) unitary matrix in $M_n(A)$, and 1 denoting the identity operator. There is an evident group structure on $\mathbf{U}_\infty(A)$ by multiplication, and we can regard, in the obvious way, all of the groups $\mathbf{U}\big(M_n(A)\big)$ as subgroups of $\mathbf{U}_\infty(A)$.

We give $\mathbf{U}_\infty(A)$ the inductive limit topology): a subset $U \subset \mathbf{U}_\infty(A)$ is open if and only if $U \cap \mathbf{U}\big(M_n(A)\big)$ is open for all n.

We are particularly interested in the path components $\pi_0\big(\mathbf{U}_\infty(A)\big)$. If $u, v \in \mathbf{U}_\infty(A)$ let $u \sim v$ mean that u and v are in the same path component of $\mathbf{U}_\infty(A)$.

Assume that u and v are invertible matrices of a fixed size n, understood as elements of $\mathbf{U}_\infty(A)$. Form the matrix $\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \in \mathbf{U}_\infty(A)$. With respect to the same block decomposition put $R_t := \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$. Then $R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $R_{\frac{\pi}{2}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

We have:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}. \tag{5.8}$$

We obtain a path of unitaries $R_t^{-1} \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} R_t$ between $\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$ and $\begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}$. That is,

$$\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \sim \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}. \tag{5.9}$$

Now multiply both sides of this identity by $\begin{bmatrix} v^* & 0 \\ 0 & 1 \end{bmatrix}$. We obtain

$$\begin{bmatrix} v^*u & 0 \\ 0 & v \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}. \tag{5.10}$$

Taking $u = 1$ for example gives then that

$$\begin{bmatrix} v^* & 0 \\ 0 & v \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.11}$$

Multiplying (5.9) on both sides by the matrix $\begin{bmatrix} v^* & 0 \\ 0 & v \end{bmatrix}$, therefore, gives the identity

$$\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & uv \end{bmatrix}. \tag{5.12}$$

Of course, this is $\sim \begin{bmatrix} uv & 0 \\ 0 & 1 \end{bmatrix}$, the group product of u and v in $\mathbf{U}_\infty(A)$.

Proposition 5.12. *Let A be a unital C^*-algebra. Then the group $\pi_0\big(\mathbf{U}_\infty(A)\big)$ of path components of $\mathbf{U}_\infty(A)$ is abelian. Moreover, if $[u], [v] \in \pi_0\big(\mathbf{U}_\infty(A)\big)$ are two elements of this group, with u, v unitary-valued matrices of the same size, then*

$$[u] \cdot [v] := [uv] = [\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}] \in \pi_0\big(\mathbf{U}_\infty(A)\big)$$

Now suppose that X is compact Hausdorff and that $A = C(X)$. If $u\colon X \to \mathbf{U}_n = \mathbf{U}\big(M_n(\mathbb{C})\big)$ is a continuous map, then we may consider it alternatively as a unitary matrix in $M_n\big(C(X)\big)$, and then as an element of $\mathbf{U}_\infty\big(C(X)\big)$. We see that the group $\mathbf{U}_\infty\big(C(X)\big)$ is the same as the group $[X, \mathbf{U}_\infty]$ of continuous maps $X \to \mathbf{U}_\infty := \mathbf{U}_\infty(\mathbb{C})$, where such maps are multiplied pointwise in the obvious way.

Moreover, to say that two elements of $\mathbf{U}_\infty\big(C(X)\big)$ are in the same path component of the group, is equivalent to saying that the corresponding maps $X \to \mathbf{U}_\infty$ are homotopic.

EXAMPLE 5.13. The $\mathbb{T} = \mathbf{U}_1$-valued map on the circle $S^1 \subset \mathbb{C}$ defined by the inclusion, determines a class $[z] \in \mathrm{K}^{-1}(S^1)$. It will be a consequence of Bott Periodicity that $\mathrm{K}^{-1}(S^1)$ is infinite cyclic, and $[z]$ generates it.

THEOREM 5.14. *There is a canonical, natural isomorphism of abelian groups* $\mathrm{K}^{-1}(X) \cong [X, \mathbf{U}_\infty(\mathbb{C})]$.

Furthermore, under this identification, suppose that $A \subset X$ is a closed subspace of X. Then the boundary map $\delta\colon \mathrm{K}^{-1}(A) \to \mathrm{K}^0(X \setminus A)$ in the long exact sequence, sends the K*-theory class corresponding to a homotopy class $u\colon X \to \mathbf{U}_n$, to the class of the* K*-theory triple $(1_n, 1_n, \bar{u})$, where $\bar{u}$ is any extension of u to a matrix-valued function $\bar{u}\colon X \to M_n(\mathbb{C})$.*

The above description of the boundary map is very helpful in doing computations.

EXAMPLE 5.15. Consider the setting of Example 5.10, where we considered the pair $(\overline{\mathbb{D}}, S^1)$ and the associated long exact sequence. It was argued there (or rather left to the reader to argue) that $\delta\colon \mathrm{K}^{-1}(S^1) \to \mathrm{K}^0(\mathbb{D})$ is an isomorphism. In example 5.13 we pointed out the tautological class $[z] \in \mathrm{K}^{-1}(S^1)$. According to Theorem 5.14, the boundary map

$$\delta\colon \mathrm{K}^{-1}(S^1) \to \mathrm{K}^0(\mathbb{D})$$

maps $[z]$ to the class of the triple $(\mathbf{1}, \mathbf{1}, z)$, since z can be extended in the obvious way from a map S^1 to $\mathbb{C}^*$ to a map $\overline{\mathbb{D}}$ into $\mathbb{C}$. But this also describes the Bott class $\beta_{\mathbb{D}}$ described in Example 4.18 .

In other words, the boundary map $\delta\colon \mathrm{K}^{-1}(S^1) \to \mathrm{K}^0(\mathbb{D})$ maps the class $[z]$ to the Bott class $\beta_{\mathbb{R}^2}$ for the open disk.

6. Bott Periodicity, the 6-term exact sequence

Let X be any locally compact space. In this section we describe Bott's celebrate Periodicity Theorem. The key character is the Bott class $\beta_{\mathbb{R}^2} \in \mathrm{K}^{-2}(\mathrm{pt}) = \mathrm{K}^0(\mathbb{R}^2)$ described in Example 4.18.

The graded ring $\mathrm{K}^*(X) := \oplus_{i=0}^{\infty} \mathrm{K}^{-i}(X)$ is a (graded) module over the graded ring $\mathrm{K}^*(\mathrm{pt})$. Therefore, multiplication by the Bott element defines a map

$$\beta_X : \mathrm{K}^*(X) \to \mathrm{K}^{*-2}(X),$$

shifting degrees by -2.

The Bott Periodicity theorem in complex K-theory is the following statement.

THEOREM 6.1. *For every locally compact space X, ring multiplication $\beta_X : \mathrm{K}^*(X) \to \mathrm{K}^{*-2}(X)$ by the Bott element, is an isomorphism.*

Periodicity says in particular that as an abstract group, $\mathrm{K}^{-2}(\mathrm{pt}) := \mathrm{K}^0(\mathbb{R}^2)$ is isomorphic to the group $\mathbb{Z}$ of integers with generator the Bott element $\beta_{\mathbb{R}^2} \in \mathrm{K}^0(\mathbb{R}^2)$. Furthermore, $\beta_{\mathbb{R}^2}^2$ generates $\mathrm{K}^{-4}(\mathrm{pt})$, $\beta_{\mathbb{R}^2}^3$ generates $\mathrm{K}^{-6}(\mathrm{pt})$ and so on.

On the other hand we have already established (it was relatively easy, based on a computation of $\mathrm{K}^0(S^1)$) that $\mathrm{K}^{-1}(\mathrm{pt}) = 0$. Combining this observation with Bott Periodicity gives that all of the groups $\mathrm{K}^{-3}(\mathrm{pt}), \mathrm{K}^{-5}(\mathrm{pt}), \ldots$ and so on, are zero.

Since we have already proved that $\mathrm{K}^{-1}(S^1) \cong \mathrm{K}^0(\mathbb{R}^2)$, we also get:

COROLLARY 6.2. $\mathrm{K}^{-1}(S^1) \cong \mathbb{Z}$ *with generator the $\mathbf{GL}_1(\mathbb{C})$-valued function $u(z) = z$. Moreover, the boundary map $\delta\colon \mathrm{K}^{-1}(S^1) \to \mathrm{K}^0(\mathbb{D}) \cong \mathrm{K}^0(\mathbb{R}^2)$ for the long exact sequence associated to $S^1 \subset \overline{\mathbb{D}}$ maps $[z] \in \mathrm{K}^{-1}(S^1)$ to the Bott element $\beta_{\mathbb{R}^2} \in \mathrm{K}^0(\mathbb{R}^2)$.*

Exercise 6.3. Verify that the system of maps β_X satisfies the following two naturality conditions: firstly, it is *natural* in X in the sense that if $f\colon X \to Y$ is a continuous, proper map, then the diagram

$$\begin{tikzcd} \mathrm{K}^0(Y) \arrow[r, "f^*"] \arrow[d, "\beta_Y"'] & \mathrm{K}^0(X) \arrow[d, "\beta_X"] \\ \mathrm{K}^{-2}(Y) \arrow[r, "f^*"] & \mathrm{K}^{-2}(X) \end{tikzcd} ,$$

commutes. This says that β defines a *natural transformation* from the functor K^0 (from locally compact Hausdorff spaces, to abelian groups), to the functor K^{-2}.

Secondly, β_X, is compatible with the ring structure on K-theory in the sense that

$$\beta_X(ab) = \beta_X(a)b, \quad a, b \in \mathrm{K}^*(X). \tag{6.1}$$

Exercise 6.4. Use the graded multiplicativity of K-theory and (6.1) to deduce that $\beta_X(ab) = a\beta_X(b)$ for all $a, b \in \mathrm{K}^*(X)$. (*Hint.* Prove it first for homogeneous elements $a \in \mathrm{K}^{-i}(X)$ and $b \in \mathrm{K}^{-j}(X)$. That is, β_X is a bimodule homomorphism of bimodules.)

Bott Periodicity also plays a role analogous to the essential Excision Theorem of cohomology in the sense that coupling it with the Long Exact sequence results in a periodic exact sequence of length 6, which makes it possible in principal to compute the K-groups of spaces which are inductively made up of simpler pieces (simplicial complexes.)

Let $A \subset X$ be a closed subset of X locally compact, and consider the associated long exact sequence

$$\cdots \to \mathrm{K}^{-1}(A) \xrightarrow{\delta} \mathrm{K}^{-1}(X \setminus A) \xrightarrow{i!} \mathrm{K}^{-1}(X) \xrightarrow{j^*} \mathrm{K}^{-1}(A) \longrightarrow \cdots$$
$$\cdots \xrightarrow{\delta} \mathrm{K}^0(X \setminus A) \to \mathrm{K}^0(X) \to \mathrm{K}^0(A) \tag{6.2}$$

of Theorem 5.1. By Bott Periodicity, $\mathrm{K}^0(A) \cong \mathrm{K}^{-2}(A)$ by the Bott map β_A. Composing β_A with the boundary map $\delta\colon \mathrm{K}^{-2}(A) \to \mathrm{K}^{-1}(X \setminus A)$ thus produces a map

$$\delta' := \delta \circ \beta_A \colon \mathrm{K}^0(A) \to \mathrm{K}^{-1}(X \setminus A). \tag{6.3}$$

To describe this map, let E be a vector bundle over A, assuming that A is compact, and $p\colon A \to M_n(\mathbb{C})$ a projection-valued map such that $\mathrm{Im}(p) \cong E$. Extend p to a continuous map $\bar{p}\colon X \to M_n(\mathbb{C})$ taking self-adjoint values. As a self-adjoint of the C*-algebra $C(X, M_n(\mathbb{C}))$, we have available functional calculus, and in particular the $\mathbb{T}$-valued function $e(x) = e^{2\pi i x}$. Applying this function, which is obviously continuous on the spectrum of $\bar{p}$, to $\bar{p}$ defines a unitary $e(\bar{p}) \in C(X, M_n(\mathbb{C}))$, that is, a unitary matrix-valued map on X.

Since the spectrum of $p(x)$ consists of 0 and 1 alone, for $x \in A$, the function $e(\bar{p})$ takes the constant value 1 (meaning the identity operator in $M_n(\mathbb{C})$) on A.

Hence it can be considered as a unitary matrix-valued function on $(X \setminus A)^+ \cong X/A$.

Corollary 6.5. *Let $A \subset X$ be a closed subspace of a locally compact space. Then there is a natural (with respect to maps of pairs) 6-term exact sequence*

$$\begin{tikzcd} \mathrm{K}^0(X \setminus A) \arrow[r] & \mathrm{K}^0(X) \arrow[r] & \mathrm{K}^0(A) \arrow[d, "\delta"] \\ \mathrm{K}^{-1}(A) \arrow[u, "\delta"] & \mathrm{K}^{-1}(X) \arrow[l] & \mathrm{K}^{-1}(X \setminus A) \arrow[l] \end{tikzcd}$$

Moreover:

a) *The boundary map* $\delta\colon \mathrm{K}^{-1}(A) \to \mathrm{K}^0(X \setminus A)$ *maps the homotopy class of a map* $u\colon A \to \mathbf{GL}(n, \mathbb{C})$ *to the class of the* K*-theory triple* $(1_n, 1_n, \bar{u})$*, where* u *is any extension of* u *to a map* $X \to M_n$*.*
b) *The map* $\delta\colon \mathrm{K}^0(A) \to \mathrm{K}^{-1}(X \setminus A)$ *maps the class of a bundle* $E = \mathrm{Im}(p)$*, for some projection-valued map* $p\colon A \to M_n(\mathbb{C})$*, to the (homotopy class of the)* $\mathbf{U}_n$*-valued map* $e(\bar{p})\colon X/A \to \mathbf{U}_n$*, described above, for any extension* $\bar{p}\colon X \to M_n(\mathbb{C})$ *of* p *to* X*.*

Exercise 6.6. Compute the K-theory groups of a (closed) annulus $a \leq |z| \leq b$, and of an open annulus $a < |z| < b$, respectively.

K-*theory of spheres*

There are two methods of deducing the K-theory of spheres from Bott Periodicity and the 6-term exact sequence. The first might be described as viewing S^n as the one-point compactification $(\mathbb{R}^n)^+$ so that, the pair consisting of S^n together with the closed subspace $\{\infty\}$ consisting of the single point 'at infinity,' gives a 6-term exact sequence

(6.4)
$$\begin{tikzcd} \mathrm{K}^0(\mathbb{R}^n) \arrow[r] & \mathrm{K}^0(S^n) \arrow[r] & \mathrm{K}^0(\mathrm{pt}) \arrow[d, "\delta"] \\ \mathrm{K}^{-1}(\mathrm{pt}) \arrow[u, "\delta"] & \mathrm{K}^{-1}(S^n) \arrow[l] & \mathrm{K}^{-1}(\mathbb{R}^n) \arrow[l] \end{tikzcd}\ .$$

Let n be even. Plugging in what we know (the K-theory of $\mathbb{R}^n$, and of points), this boils down to the sequence

(6.5)
$$\begin{tikzcd} \mathbb{Z} \arrow[r] & \mathrm{K}^0(S^n) \arrow[r] & \mathbb{Z} \arrow[d, "\delta"] \\ 0 \arrow[u, "\delta"] & \mathrm{K}^{-1}(S^n) \arrow[l] & 0 \arrow[l] \end{tikzcd}\ .$$

from which it is immediate that $\mathrm{K}^{-1}(S^n) = 0$, and that there is an exact sequence

(6.6)
$$0 \to \mathbb{Z} \to \mathrm{K}^0(S^n) \to \mathbb{Z} \to 0,$$

which can be described as follows. The first map corresponds to the using the open embedding of $\mathbb{R}^n$ as an open subset of S^n to map the Bott class $\beta_{\mathbb{R}^n} \in \mathrm{K}^0(\mathbb{R}^n)$ to $\mathrm{K}^0(S^n)$. If $i\colon \mathbb{R}^n \to S^n$ denotes this embedding, then, therefore, the first map $\mathbb{Z} \to \mathrm{K}^0(S^n)$ maps the generator $1 \in \mathbb{Z}$ to $i!(\beta_{\mathbb{R}^n})$, which we will call b.

On the other hand, the quotient map sends the class $[\mathbf{1}]$ of the trivial line bundle over S^n to the generator $1 \in \mathbb{Z}$. It follows that the map $\mathbb{Z} \oplus \mathbb{Z} \to \mathrm{K}^0(S^n)$, sending (n, m) to $nb + m[\mathbf{1}]$, is a group isomorphism.

Now suppose that n is odd. Then plugging what we know into (6.4) gives the sequence

(6.7)
$$\begin{tikzcd} 0 \arrow[r] & \mathrm{K}^0(S^n) \arrow[r] & \mathbb{Z} \arrow[d, "\delta"] \\ 0 \arrow[u, "\delta"] & \mathrm{K}^{-1}(S^n) \arrow[l] & \mathbb{Z} \arrow[l] \end{tikzcd}\ .$$

The generator of the integers (the K^0 of a point) in the upper right corner, obviously is in the image of the map from $\mathrm{K}^0(S^n)$, since it is the image of the class $[\mathbf{1}]$ of the trivial line bundle on S^n. So the quotient map is surjective, and so both connecting maps δ vanish.

It follows that the map $\mathrm{K}^{-1}(\mathbb{R}^n) \to \mathrm{K}^{-1}(S^n)$, induced by the open embedding of $\mathbb{R}^n$ in S^n, is an isomorphism, and that $\mathrm{K}^0(S^n) \cong \mathbb{Z}$ with generator $[\mathbf{1}]$.

Remark 6.7. In view of the fact that, rather than suspending a space X to define it's K^{-1}-group, we can instead use Theorem 5.14 and look for unitary-valued maps on the space, it would seem reasonable to look for such a description of $\mathrm{K}^{-1}(S^n) \cong \mathbb{Z}$, when n is odd. We will do this once we have a bit of Clifford algebra theory in hand.

The second method of computing the K-theory of spheres is to view S^n as the boundary of the $n-1$-disk $\mathbb{D}^{n-1} \cong \mathbb{R}^{n-1}$. We obtain the 6-term exact sequence..

K-*theory of real projective spaces*

Real projective spaces $\mathbb{RP}^n$ is the quotient space obtained by identifying antipodal points x and $-x$ of the sphere S^n. The case $n = 1$ is slightly special; in this case the map

$$\mathbb{RP}^1 \to S^1, \ \ [z] \mapsto z^2 \tag{6.8}$$

is a homeomorphism of $\mathbb{RP}^1$ with S^1 (but the other projective spaces are not homeomorphic to spheres.)

Proposition 6.8. *$\mathrm{K}^1(\mathbb{RP}^n) = 0$ and $\mathrm{K}^0(\mathbb{RP}^n) \cong \mathbb{Z}/2 \oplus \mathbb{Z}$ with generators a certain 'Bott element' $b \in \mathrm{K}^0(\mathbb{RP}^n)$, of order 2, and described in the proof, and free generator $[\mathbf{1}]$, the class of the trivial line bundle.*

Proof. We can think of $\mathbb{RP}^2$ as obtained from the closed disk $\overline{\mathbb{D}}$ by the equivalence relation that identifies antipodal boundary points of the disk. Since no points of the interior $\mathbb{D} \cong \mathbb{R}^2$ of the disk are identified, $\mathbb{D}$ may be considered an open subset of $\mathbb{RP}^2$ with complement $\mathbb{RP}^1$.

This generates a six-term exact sequence

$$\begin{tikzcd} \mathrm{K}^0(\mathbb{D}) \arrow[r] & \mathrm{K}^0(\mathbb{RP}^2) \arrow[r] & \mathrm{K}^0(\mathbb{RP}^1) \arrow[d, "\delta"] \\ \mathrm{K}^1(\mathbb{RP}^1) \arrow[u, "\delta"] & \mathrm{K}^1(\mathbb{RP}^2) \arrow[l] & \mathrm{K}^1(\mathbb{D}) \arrow[l] \end{tikzcd} \tag{6.9}$$

By Bott Periodicity, $\mathrm{K}^0(\mathbb{D}) = \mathbb{Z}$ with generator the Bott element $\beta_{\mathbb{D}}$, and $\mathrm{K}^1(\mathbb{D}) = 0$. By the remarks above, $\mathrm{K}^0(\mathbb{RP}^1)$ and $\mathrm{K}^1(\mathbb{RP}^2)$ are both infinite cyclic, with generators $[1] \in \mathrm{K}^0(\mathbb{RP}^1)$ the class of the trivial line bundle, and the map (6.8), a unitary-valued map, the generator of $\mathrm{K}^1(\mathbb{RP}^1)$. We denote it's class $[z^2]$.

Inserting all this information into (6.9) we get the diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r] & \mathrm{K}^0(\mathbb{RP}^2) \arrow[r] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[u, "\delta"] & \mathrm{K}^1(\mathbb{RP}^2) \arrow[l] & 0 \arrow[l] \end{tikzcd} \tag{6.10}$$

To compute the vertical map δ on the left, we take the known generator, the class of z^2, of $\mathrm{K}^1(\mathbb{RP}^1)$, and we extend it from $\mathbb{RP}^1 \subset \mathbb{RP}^2$ to a map $\mathbb{RP}^2 \to \mathbb{C}$. One can clearly do this in a number of ways, for example, to simply take the extension to be defined by the function z^2, now defined on the whole closed disk.

Then we get a K-theory triple $(\mathbf{1}, \mathbf{1}, z^2)$ consisting of the trivial bundles $\mathbf{1}$ over $\mathbb{D}$ with z^2 the bundle map between them, and we have

$$\delta([z^2]) = [(\mathbf{1}, \mathbf{1}, z^2)].$$

As a triple, this is homotopic to the triple $(\mathbf{1}_2, \mathbf{1}_2, \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix})$. consisting of the direct sum of $(\mathbf{1}, \mathbf{1}, z)$ with itself. Since the latter represents the Bott element $\beta_{\mathbb{D}}$ for the open disk of radius 1 around 0, we get that

$$\delta([z^2]) = 2\beta_{\mathbb{D}}. \tag{6.11}$$

Hence the vertical left map $\mathbb{Z} \to \mathbb{Z}$ induced by δ is multiplication by 2.

Since the kernel of this map is trivial, and since $\mathrm{K}^1(\mathbb{RP}^2)$ injects to the kernel of this map, by the diagram, and since multiplication by 2 is injective, $\mathrm{K}^1(\mathbb{RP}^2) = 0$.

The diagram now shows that the map $\mathbb{Z} \to \mathrm{K}^0(\mathbb{RP}^2)$ induced by the open disk $\mathbb{D}$ sitting in $\mathbb{RP}^2$, and the corresponding map $\mathrm{K}^0(\mathbb{D}) \to \mathrm{K}^0(\mathbb{RP}^2)$, vanishes on the even integers. We obtain an injection $\mathbb{Z}/2\mathbb{Z} \to \mathrm{K}^0(\mathbb{RP}^2)$. The corresponding element b of order 2 in $\mathrm{K}^0(\mathbb{RP}^2)$ is simply obtained by mapping the open disk $\mathbb{D}$ into $\mathbb{RP}^2$, and in this way allowing us to view the Bott element $\beta_{\mathbb{D}} \in \mathrm{K}^0(\mathbb{D})$ as a K^0-class, which we are calling b, for $\mathbb{RP}^2$.

Our calculations show thus that

$$2b = 0 \in \mathrm{K}^0(\mathbb{RP}^2).$$

Finally, the class $[\mathbf{1}] \in \mathrm{K}^0(\mathbb{RP}^2)$ of the trivial line bundle over $\mathbb{RP}^2$ is the other generator for $\mathrm{K}^0(\mathbb{RP}^2)$; the two thus generate a copy of the group $\mathbb{Z}/2 \oplus \mathbb{Z}$. To check this, observe that we have produced a group extension

$$0 \to \mathbb{Z}/2 \to \mathrm{K}^0(\mathbb{RP}^2) \to \mathbb{Z} \to 0,$$

and any such extension is split, since $\mathbb{Z}$ is free abelian. This makes $\mathrm{K}^0(\mathbb{RP}^2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}$ as claimed, and it is left to the reader to check that $[\mathbf{1}]$ can be taken to generate the copy of $\mathbb{Z}$.

$\square$

EXERCISE 6.9. Here is an alternative proof that the boundary map in (6.9) satisfies $\delta([z^2]) = 2\beta_{\mathbb{D}}$.

Consider $\mathbb{RP}^2$ as the quotient of the closed disk $\overline{\mathbb{D}}$ by identifying antipodal points. Let $\pi\colon \overline{\mathbb{D}} \to \mathbb{RP}^2$ be the quotient map. It restricts to a map $S^1 = \partial\overline{\mathbb{D}} \to \mathbb{RP}^1 \subset \mathbb{RP}^2$ and so can be considered as a map of pairs. By naturality of the connecting map, owe obtain a commutative diagram

$$\begin{tikzcd} \mathrm{K}^0(\mathbb{D}) \arrow[d, "\pi^*"'] & \mathrm{K}^1(\mathbb{RP}^1) \arrow[l, "\delta"'] \arrow[d, "\pi^*"] \\ \mathrm{K}^0(\mathbb{D}) & \mathrm{K}^1(S^1) \arrow[l, "\delta"] \end{tikzcd} \tag{6.12}$$

Now check that π^* maps the generator $[z^2]$ for $\mathrm{K}^1(\mathbb{RP}^1)$ to $[z^2] \in \mathrm{K}^1(S^1)$, *i.e.* to $2[z]$. The lower map δ thus has $\delta \circ \pi^*([z^2]) = 2\beta_{\mathbb{R}^2}$, since we've already computed that $\delta([z]) = \beta_{\mathbb{R}^2}$. We conclude that $\pi^*(\delta([z^2]) = 2\beta_{\mathbb{R}^2}$. On the other hand, the restriction of π to $\mathbb{D}$ is the identity map. Hence $\delta([z^2]) = 2b$ follows immediately, with b the Bott element, considered as an element of $\mathrm{K}^0(\mathbb{RP}^2)$.

EXERCISE 6.10. Recall the complex line bundle L over the 2-torus $\mathbb{T}^2$ defined in Exercise 1.10 (see also Exercise 2.9).

a) Prove that $\mathrm{K}^0(\mathbb{T}^2)$ is generated by the classes [1] of the trivial line bundle, and the class $[L]$.
b) In Example 4.18 we discussed the K-theory 'germ' around the origin $0 \in \mathbb{R}^2$, in the sense that in any neighbourhood U of 0, there is an associated 'Bott class' $\beta_U \in \mathrm{K}^0(U)$). Since a small enough neighbourhood of any point of $\mathbb{R}^2$ can be fit into $\mathbb{T}^2$, we obtain corresponding K-theory germs, which we still denote by β_U, at points of $\mathbb{T}^2$. Show that
$$\beta_U = [L] - [1] \in \mathrm{K}^0(\mathbb{T}^2),$$
that is, $[L] - [1]$ is the K-theory germ of a point of the torus.

7. Spin geometry and Clifford algebras

Further progress in K-theory requires an understanding of some points of classical algebra: namely, the theory of *Clifford algebras*, and their representations. With the aid of some understanding of Clifford algebras, we will obtain clear geometric descriptions of some important K-theory classes, like for example, the higher-dimensional Bott classes $\beta^k_{\mathbb{R}^2} \in \mathrm{K}^{-2k}(\mathrm{pt})$ discussed in the previous section, the generators of K^{-1} of odd-dimensional spheres, projective spaces, and so on.

Clifford algebras also play a role in geometric problems related to K-theory, like the construction of vector bundles on spheres, and most importantly, to index theory of Dirac operators.

Clifford algebras and the Spin double-covering.

DEFINITION 7.1. The *Clifford algebra* $\mathrm{Cliff}_{\mathbb{R}}(V)$ of a Euclidean vector space V is the unital algebra $T(V)/IV$, where

- $T(V) := \oplus_{n=0}^{\infty} V^{\otimes n}$ is the tensor algebra of V.
- I is the ideal in $T(V)$ generated by the elements $v \otimes w + w \otimes v - 2\langle v, w\rangle 1$, for $v, w \in V$.

The *complex* Clifford algebra of V is $\mathrm{Cliff}_{\mathbb{R}}(V) \otimes_{\mathbb{R}} \mathbb{C}$ and will be simply denoted $\mathrm{Cliff}(V)$; it is the one we will be primarily working with.

By the definitions, $\mathrm{Cliff}(V)$ is generated as a unital algebra by the elements of V, and has the relation

$$v \cdot w + w \cdot v = -2\langle v, w\rangle. \tag{7.1}$$

The *grading* on a Clifford algebra is quite important. A $\mathbb{Z}/2$-*graded algebra* A is an algebra, either real or complex, which decomposes into a direct sum $A = A^0 \oplus A^1$, in such a way that $A_i A_j \subset A_{i+j \bmod 2}$. One usually refers to the elements of A_0 as 'even', those of A_1 'odd'. One way of ensuring a $\mathbb{Z}/2$-grading on an algebra is to specify an automorphism $\epsilon\colon A \to A$ such that $\epsilon \circ \epsilon = \mathrm{id}$. Then A decomposes into a direct sum with even part $A^0 := \{a \in A \mid \epsilon(a) = a\}$ and odd part $A^1 := \{a \in A \mid \epsilon(a) = -a\}$. We call ϵ the *grading operator*.

PROPOSITION 7.2. $\mathrm{Cliff}_{\mathbb{R}}(V)$ *and* $\mathrm{Cliff}(V)$ *are* 2^n*-dimensional,* $\mathbb{Z}/2$*-graded algebras over* $\mathbb{R}$ *and* $\mathbb{C}$ *respectively, with in each case the grading operator the automorphism of* $\mathrm{Cliff}_{\mathbb{R}}(V)$ *(or* $\mathrm{Cliff}(V)$*) induced by the map* $\epsilon(v) = -v$.

PROOF. We consider V as a subset of $\mathrm{Cliff}_{\mathbb{R}}(V)$ for the proof (and more generally), and write $v_1 \cdots v_k$ for the coset of $v_1 \otimes \cdots \otimes v_k$ in $\mathrm{Cliff}_{\mathbb{R}}(V) := T(V)/I$. The key relation is $v \cdot w + w \cdot v = 2\langle v, w\rangle 1$. Fix an orthonormal basis $e_1, \ldots, e_n$ for V. Then the algebra in $\mathrm{Cliff}_{\mathbb{R}}(V)$ generated by $e_1, \ldots, e_n$ already clearly contains V, and V, by the definitions, certainly generates $\mathrm{Cliff}_{\mathbb{R}}(V)$ as an algebra, so $\mathrm{Cliff}_{\mathbb{R}}(V)$ is generated by $e_1, \ldots, e_n$. Since they are orthonormal, $e_i \cdot e_j = -e_j \cdot e_i$ and $e_i^2 = -1$. It follows immediately that any product $e_{i_1} \cdots e_{i_k}$ in which $k > n$ contains at least two occurrences of some e_i, and moving them adjacent to each other in the monomial, and cancelling them, results in a product of smaller length. So it follows that every element of $\mathrm{Cliff}_{\mathbb{R}}(V)$ can be written as a linear combination of monomials $e_{i_1} \cdots e_{i_k}$, with $k \leq n$, and $i_1, \ldots, i_k$ a set of k *distinct* indices. We leave it as an exercise to show that if one takes multi-indices $i_1 < i_2 < \cdots < i_k$ in increasing order, then the corresponding monomials $e_{i_1} \cdots e_{i_k}$ are linearly independent, and so form a basis for $\mathrm{Cliff}_{\mathbb{R}}(V)$. The basis is by definition in 1-1 correspondence with the collection of subsets of $\{1, 2, \ldots, n\}$ so the dimension of $\mathrm{Cliff}_{\mathbb{R}}(V)$ is 2^n. □

EXERCISE 7.3. The *transpose* map on $\mathrm{Cliff}_{\mathbb{R}}(V)$ is induced by letting $v^t := -v$ and requiring that it's extension to $\mathrm{Cliff}_{\mathbb{R}}(V)$ is linear and satisfies $(x \cdot y)^t = y^t x^t$ for $x, y \in \mathrm{Cliff}_{\mathbb{R}}(V)$.

These statements can be adjusted appropriately to get a *conjugate* linear involution on $\mathrm{Cliff}(V)$, for which we use the usual C*-algebra notation $x \mapsto x^*$. For the same reasons as in the real case, $\Gamma^* = \Gamma$ (it is self-adjoint) if n is odd, and $\Gamma^* = -\Gamma$ (skew-adjoint) if n is even.

DEFINITION 7.4. Let V be a Euclidean vector space and $\mathbf{e} = \{e_1, \ldots, e_n\}$ an orthonormal basis for V The corresponding *volume element* is the element $\Gamma(\mathbf{e}) := e_1 \cdots e_n$.

EXERCISE 7.5. In the above notation, prove that $\Gamma(\mathbf{e})$ only depends on the *orientation class* of the orthonormal basis. That is, prove that $\Gamma(\mathbf{e}) = \pm\Gamma(\mathbf{e}')$ for any $\mathbf{e}, \mathbf{e}'$, and the sign is $+1$ if and only if $\mathbf{e}$ and $\mathbf{e}'$ determine the same *orientation* on V.

In accordance with the above exercise, we usually just write Γ for the volume element, with the understanding that an orientation on V has been fixed.

The proof of the following is routine and is left to the reader.

LEMMA 7.6. *Let V be Euclidean and oriented, let $\Gamma \in \mathrm{Cliff}_{\mathbb{R}}(V)$ the volume element $\Gamma := e_1 \cdots e_n$, for a positively oriented orthonormal basis $e_1, \ldots e_n$.*

a) *If n is even then $\Gamma^2 = (-1)^{\frac{n}{2}}$, $\Gamma^t = (-1)^{\frac{n}{2}}\Gamma$, and Γ graded commutes with the vectors e_i, $i = 1, \ldots n$.*
b) *If n is odd then $\Gamma^2 = (-1)^{\frac{n+1}{2}}$, $\Gamma^t = (-1)^{\frac{n+1}{2}}\Gamma$ and Γ commutes with the e_i's, $i = 1, \ldots, n$.*

EXERCISE 7.7. Prove the following about Clifford algebras.

a) Prove that a bijective linear isometry $L \colon V \to V'$ of Euclidean vector spaces, extends uniquely to an algebra isomorphism $\mathrm{Cliff}_{\mathbb{R}}(V) \to \mathrm{Cliff}_{\mathbb{R}}(V')$.
b) Prove that if V is any Euclidean vector space, then the orthogonal group $\mathbf{O}(V)$ acts on $\mathrm{Cliff}_{\mathbb{R}}(V)$ by $\mathbb{Z}/2$-grading preserving C*-algebra automomorphisms, extending the action of $\mathbf{O}(V)$ on $V \subset \mathrm{Cliff}_{\mathbb{R}}(V)$.

EXAMPLE 7.8. Endowing $\mathbb{R}^n$ with it's standard inner product. If $n = 1$, and $e \in \mathbb{R}^1$ is a unit vector, then $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^1)$ is spanned over $\mathbb{R}$ by 1 and e and the map $a + be \mapsto a + bi$ defines an isomorphism of $\mathbb{Z}/2$-graded real algebras $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^1) \cong \mathbb{C}$, where $\mathbb{C}$ (understood as a an algebra over $\mathbb{R}$) is graded by complex conjugation, $\epsilon(z) := \bar{z}$.

For $n = 2$ let $e_1 = (1, 0)$, $e_2 = (0, 1)$, say. Then $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^2)$ is linearly spanned over $\mathbb{R}$ by $1, e_1, e_2$ and $e_1 e_2$, and is isomorphic to the algebra of quaternions $\mathbb{H}$, by the formula $\varphi(a \cdot 1 + be_1 + ce_2 + de_1 e_2) := a + bi + cj + dk$.

EXERCISE 7.9. Prove that the *even* part $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^n)^0$ of the $\mathbb{Z}/2$-graded algebra $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^n)$, is isomorphic to $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^{n-1})$. (*Hint.* $\mathrm{Cliff}(R^{n-1})$ is generated by $e_1, \ldots, e_{n-1}$ with relations $e_i e_j + e_j e_i = -2\delta_{ij}$. Map generators to $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^n)$ by $e_i \mapsto e_n e_i$, and check that the relations are preserved.)

We define two subgroups of the group of invertibles in the algebra $\mathrm{Cliff}_{\mathbb{R}}(V)$ as follows.

$$\mathbf{Pin}(V) := \{v_1 \cdots v_k \mid v_1, \ldots, v_k \text{ unit vectors in } V\}$$

$$\mathbf{Spin}(V) := \{v_1 \cdots v_k \mid v_1, \ldots, v_k \text{ unit vectors in } V,\ k \text{ even.}\}$$

Clearly both are subgroups of the group of invertibles of $\mathrm{Cliff}_{\mathbb{R}}(V)$. The more important for our purposes is $\mathbf{Spin}(V)$.

EXERCISE 7.10. Prove that $\mathbf{Spin}(V)$ is a compact, connected group. (*Hint.* To prove it's path connected, move a vector v along a path of unit vectors in V from v_n to $-v_{n-1}$, we obtain a path $v_1 \cdots v_{n-1} \cdot v$ in $\mathbf{Spin}(V)$ from $v_1 \cdots v_n$ to $v_1 \cdots v_{n-2}$. Now continue this process.)

EXERCISE 7.11. Prove that $\mathbf{Spin}_3$ is diffeomorphic to the 3-sphere S^3. (*Hint.* The elements of $\mathbf{Spin}_3$ may be uniquely written $a + be_2e_1 + ce_3e_2 + de_3e_1$ where $a^2 + b^2 + c^2 + d^2 = 1$.)

We now describe the *spin covering* $\rho\colon \mathbf{Spin}(V) \to \mathbf{SO}(V)$ of the special orthogonal group, V Euclidean. Basic covering space theory and the fact that $\pi_1\big(\mathbf{SO}(V)\big) \cong \mathbb{Z}/2$ for $\dim(V) \geq 3$, implies that the space $\mathbf{SO}(V)$ *has* such a cover, which is also it's universal cover.

A short calculation shows that for V Euclidean, and $v \in V$ a *unit* vector, and if $w \in V$, then

$$v \cdot w \cdot v = v(-2\langle v, w\rangle - v \cdot w) = w - 2\langle v, w\rangle v = \mathrm{refl}_{v^\perp}(w), \tag{7.2}$$

where $\mathrm{refl}_{v^\perp}(w)$ is the orthogonal reflection of w through the hyperplane $v^\perp$ determined by v – thus $\mathrm{refl}_{v^\perp}$ is an orthogonal transformation of V. Motivated by this, we want to define a group homomorphism

$$\rho\colon \mathbf{Pin}(V) \to \mathbf{O}(V),\ \rho(\pm 1) = 1,\ \rho(v_1 \cdots v_k) := \mathrm{refl}_{v_1^\perp} \cdots \mathrm{refl}_{v_n^\perp} \in \mathbf{O}(V). \tag{7.3}$$

It turns out that ρ is well-defined. Of more interest to us is the restriction of ρ to a homomorphism $\mathbf{Spin}(V) \to \mathbf{SO}(V)$.

REMARK 7.12. If $\theta \in \mathbb{R}$ and v and w are orthogonal unit vectors in V, then $\cos\theta v + \sin\theta w$ is also a unit vector in V, and hence $(\cos\theta v + \sin\theta w) \cdot (-w) \in \mathbf{Spin}(V)$. A short calculation shows that this equals $\cos\theta + \sin\theta\, v \cdot w$, and that $\rho(\cos\theta + \sin\theta\, v \cdot w)$ is rotation through an angle of 2θ in the plane spanned by v and w (in the direction from v towards w.)

The main difficulty is that we need to prove that (7.3) is well-defined, since although every element of $\mathbf{Pin}(V)$ is a product $v_1 \cdots v_k$ in some way, there may be, in principal, at least, different representations of it as such.

LEMMA 7.13. *The expression* (7.3) *is well-defined, and defines a surjective,* 2*-to-*1 *group homomorphism* $\mathbf{Pin}(V) \to \mathbf{O}(V)$*, and, its restriction to* $\mathbf{Spin}(V)$ *is a* 2*-to-*1 *group homomorphism onto* $\mathbf{SO}(V)$*.*

Since $\mathbf{Spin}(V)$ *is connected,* ρ *is a nontrivial double cover of* $\mathbf{SO}(V)$*.*

PROOF. If $x \in \mathrm{Cliff}_{\mathbb{R}}(V)$ is an invertible in the Clifford algebra, then $\mathrm{Ad}_\epsilon(x)y := xy\epsilon(x)^{-1}$ defines an algebra homomorphism $\mathrm{Cliff}_{\mathbb{R}}(V) \to \mathrm{Cliff}_{\mathbb{R}}(V)$. A further easy check shows that $\mathrm{Ad}_\epsilon(xy) = \mathrm{Ad}_\epsilon(x)\mathrm{Ad}_\epsilon(y)$. Furthermore, if $x = v \in V \subset \mathrm{Cliff}_{\mathbb{R}}(V)$ then since $\epsilon(v) = -v$, and by the calculation (7.2) above, we get that $\mathrm{Ad}_\epsilon(v) = \mathrm{refl}_{v^\perp}$. The result follows: we have shown that (7.3) actually represents the value $\mathrm{Ad}_\epsilon(v_1 \cdots v_k)$, and hence only depends on $v_1 \cdots v_k$ as an element of the Clifford algebra, and not on its representation.

Surjectivity of $\rho\colon \mathbf{Pin}(V) \to \mathbf{O}(V)$ now follows from the classical result of Cartan and Dieudonné that the orthogonal group is generated by reflections.

$\square$

EXERCISE 7.14. Compute the action of the linear map $\mathrm{Ad}_\epsilon(v)$ (notation as in the proof above) on the subspace $V \subset \mathrm{Cliff}_{\mathbb{R}}(V)$, when v is not necessarily a unit vector.

EXERCISE 7.15. Prove that $\mathbf{Spin_2} \cong \mathbb{T}$, and that the spin covering $\rho\colon \mathbf{Spin_2} \to \mathbf{SO}(2,\mathbb{R}) \cong \mathbb{T}$ corresponds to the map $z \mapsto z^2$ on the circle.

EXERCISE 7.16. Prove that the action of $\mathbf{O}(V)$ on $\mathrm{Cliff}_{\mathbb{R}}(V)$ developed in Exercise 7.7 b), is actually an action by *inner automorphisms*.

EXAMPLE 7.17. This example examines a bit further the geometry of, especially, $\mathbf{Spin}_3$, and the double covering $\mathbf{Spin}_3 \to \mathbf{SO}(3,\mathbb{R})$.

Firstly, $\mathbf{Spin}_3$ is generated as a group by the elements $v \cdot w$, where $v, w \in \mathbb{R}^3$ are unit vectors. Let e_1, e_2, e_3 be the standard orthonormal basis for $\mathbb{R}^3$.

If $v, v' \in \mathbb{R}^3$ are unit vectors, write $v = ae_1 + be_2 + ce_3$ where $a^2 + b^2 + c^2 = 1$, $v' = a'e_1 + b'e_2 + c'e_3$ where $(a')^2 + (b')^2 + (c')^2 = 1$.

EXERCISE 7.18. Multiply out $v \cdot v'$ and expand in the basis $1, e_1e_2, e_1e_3, e_2e_3$. Check that the first coordinate is the dot product $\langle v, v'\rangle$, or inner product of v and v', and that the next 3 coordinates are those of the crossed-product $v \times v'$.

It follows from elementary properties of the dot and crossed-products, that the map $\phi\colon \mathbf{Spin}_3 \to \mathbb{R}^4$, $\phi(v \cdot w) := (-\langle v, w\rangle, v \times w)$ takes values in unit vectors in $\mathbb{R}^4$, and, in fact, can be easily checked to define a diffeomorphism $\mathbf{Spin}_3 \cong S^3$.

Let us see what the double covering $\rho\colon \mathbf{Spin}_3 \to \mathbf{SO}(3,\mathbb{R})$ looks like in this picture.

EXERCISE 7.19. 3-dimensional real projective space $\mathbb{RP}^3$ is by definition the space $S^3/\sim$ obtained by identifying antipodal points of the 3-sphere. Since every equivalence class in $\mathbb{RP}^3$ is represented by a unit vector $v \in \mathbb{R}^4$ with z-coordinate ≥ 0, we may also consider $\mathbb{RP}^3$ as obtained by taking the closed upper hemisphere in S^3 and identifying antipodal points of its boundary. The map $(x, y, z) \mapsto (x, y, z, \sqrt{1 - x^2 - y^2 - z^2})$ identifies the upper hemisphere in S^3 with the closed 3-ball B^3. Thus, we can consider $\mathbb{RP}^3$ as the space obtained from the closed 3-ball B^3 by identified antipodal points on the boundary of the ball.

If $v \in B^3$ is a nonzero vector, with $t = \|v\|$, let $\alpha(v) \in \mathbf{SO}(3,\mathbb{R})$ be rotation in the axis determined by v, by an angle of πt, in the sense determined by the right-hand-rule. Since $\lim_{v\to 0} \alpha(v) = \mathrm{id}_{\mathbf{SO}(3,\mathbb{R})}$, as is easy to check, α extends to a continuous map $B^3 \to \mathbf{SO}(3,\mathbb{R})$. Prove this map is a homeomorphism. Deduce that $\mathbf{SO}(3,\mathbb{R}) \cong \mathbb{RP}^3$.

By the preceding exercise, $\mathbf{SO}(3,\mathbb{R})$ may be identified with $S^3/\sim$, where $\sim$ identifies antipodal points of the sphere. It is easily checked that the diagram

$$\begin{tikzcd}
\mathbf{Spin}_3 \arrow[r, "\rho"] \arrow[d, "\cong"] & \mathbf{SO}(3,\mathbb{R}) \arrow[d, "\cong"] \\
S^3 \arrow[r, "\pi"] & \mathbb{RP}^3
\end{tikzcd}$$

commutes where π is the quotient map .

REMARK 7.20. Another interesting fact about the double cover $\rho\colon \mathbf{Spin}_3 \to \mathbf{SO}(3,\mathbb{R})$, is that if it is followed by an orbit map $\mathbf{SO}(3,\mathbb{R}) \to S^2$, $A \mapsto Ae$, e a chosen fixed unit vector in $S^2 \subset \mathbb{R}^3$, then the resulting map $\rho_e\colon \mathbf{Spin}_3 \cong S^3 \to S^2$ gives a fibration,

$$\begin{tikzcd}
S^1 \arrow[r] & S^3 \arrow[d, "\rho_e"] \\
 & S^2
\end{tikzcd}$$

called the *Hopf fibration*. It turns out to represent a non-trivial element of $\pi_3(S^2)$.

REMARK 7.21. It can be checked that $\mathbf{Spin}(V)$ is a (compact, connected) *Lie* group in a natural way, and it's Lie algebra $\mathfrak{spin}(V)$ can be identified with the Lie subalgebra of $\mathrm{Cliff}_{\mathbb{R}}(V)$ spanned by the $e_i \cdot e_j$, with $i \neq j$; the $e_i \cdot e_j$ with $i < j$ form a basis, giving a $\frac{n(n+1)}{2}$-dimensional space (and it is closed under commutators, as the reader can easily check.)

The exponential map $\exp\colon \mathfrak{spin}(V) \to \mathbf{Spin}(V)$ in this case has the form of the convergent series $\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$, where X is supposed an element of $\mathfrak{spin}(V) \subset \mathrm{Cliff}_{\mathbb{R}}(V)$ and X^n has it's obvious meaning of X multiplied by itself n times. The series converges in the topology of $\mathrm{Cliff}_{\mathbb{R}}(V)$ (which as a linear space is just a finite-dimensional real vector space) to an element of $\mathrm{Cliff}^*_{\mathbb{R}}(V)$, and to an element of $\mathbf{Spin}(V)$ if $X \in \mathfrak{spin}(V)$.

It is an easy exercise (plug in $te_i \cdot e_j$ into the power series for exp, use the basic properties of the Clifford multiplication) to see that $\exp(te_i \cdot e_j) = \cos t + \sin te_i \cdot e_j$. These elements form a closed subgroup of $\mathbf{Spin}(V)$ isomorphic to the circle $\mathbb{T}$. Thus, exp maps the line through $e_i \cdot e_j$ to this subgroup (in a $\mathbb{Z}$-to-1 fashion; it is a covering map like the usual complex exponential function $\mathbb{R} \to \mathbb{T}$.)

To see the connection to the orthogonal group and the spin covering $\rho\colon \mathbf{Spin}_n \to \mathbf{SO}(n, \mathbb{R})$, let E_{ij} be the n-by-n matrix with 1 in the i, jth coordinate, -1 in the j, ith coordinate, and zeros elsewhere. Then E_{ij} is an element of the Lie algebra $\mathfrak{so}(n, \mathbb{R})$ of $\mathbf{SO}(n, \mathbb{R})$ and for any $t \in \mathbb{R}$, $\exp(tE_{ij}) = \cos t + \sin tE_{ij}$ is rotation by t in the plane spanned by e_i and e_j, in the direction from e_i to e_j.

In particular, the identification of Lie algebras $\mathfrak{spin}(\mathbb{R}^n)$ and $\mathfrak{so}(n, \mathbb{R})$, maps the elements $e_i \cdot e_j \in \mathfrak{spin}(\mathbb{R}^n) \subset \mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^n)$ to the matrices E_{ij} of $\mathfrak{so}(n, \mathbb{R})$.

We close this section with the remark that complex Clifford algebras have the structure of C*-algebras in a natural way.

LEMMA 7.22. *If V is any Euclidean vector space, then* $\mathrm{Cliff}(V)$ *has the structure of a ($\mathbb{Z}/2$-graded) C*-algebra with adjoint determined by conjugate linearity and the rules $(vw)^* = w^*v^*$ and $v^* = -v$, for $v \in V \subset \mathrm{Cliff}(V)$.*

PROOF. Let $e_1, \ldots, e_n$ be an orthonormal basis for V. The monomials $e_{i_1} \cdots e_{i_k}$ in $\mathrm{Cliff}(V)$, with $i_1 < i_< \cdots < i_k$ span $\mathrm{Cliff}(V)$ as a vector space, and they are linearly independent. We may therefore define an inner product $\langle \cdot, \cdot \rangle$ on $\mathrm{Cliff}(V)$ making it a ($\mathbb{Z}/2$-graded) Hilbert space for which the monomials $e_{i_1} \cdots e_{i_k}$ consitute an orthonormal basis, and on which $\mathrm{Cliff}(V)$ acts by left multiplication.

We leave it to the reader that this is a *-representation. It is injective for dimension reasons. $\square$

8. Representation theory of Clifford algebras

Let V be a Euclidean vector space, as in the previous section. We will usually refer to an C*-algebra representation $\pi\colon \mathrm{Cliff}(V) \to \mathrm{End}(W)$ on a Hilbert space W as a $\mathrm{Cliff}(V)$-*module*. Such representations we will always assume to be non-degenerate (*i.e.* unital, *i.e.* $1 \in \mathrm{Cliff}(V)$ acts by the identity map on W.)

THEOREM 8.1. *For any even-dimensional Euclidean vector space V, there is, up to isomorphism, exactly one irreducible* $\mathrm{Cliff}(V)$ *module S of dimension $2^{\frac{\dim V}{2}}$. Furthermore, an orientation on V determines a $\mathbb{Z}/2$-grading on S, for which Clifford multiplication by vectors are odd operators.*

Furthermore, any $\mathrm{Cliff}(V)$*-module of dimension $2^{\frac{\dim V}{2}}$ is irreducible.*

If $\dim(V)$ *is odd, there are, up to isomorphism, exactly two irreducible* $\mathrm{Cliff}(V)$*-modules of dimension $2^{\frac{\dim V - 1}{2}}$, and any* $\mathrm{Cliff}(V)$*-module of dimension $2^{\frac{\dim V-1}{2}}$, is necessarily irreducible.*

The isomorphism in the statement refers to the obvious notion of isomorphism of $\mathrm{Cliff}(V)$-modules.

In the case V is even-dimensional, if $c\colon \mathrm{Cliff}(V) \to \mathrm{End}(S)$ is an irreducible $\mathrm{Cliff}(V)$-module, and if $e_1, \ldots, e_n$ is an orthonormal basis of V, then the operator

$$\epsilon_S := i \cdot c(e_1) \cdots c(e_n)$$

satisfies $\epsilon_S^* = \epsilon_S, \epsilon^2 = 1$, and $\epsilon_S c(v) = -c(v)\epsilon_S$ for all $v \in V$, and so determines a $\mathbb{Z}/2$-grading on S which is a $\mathbb{Z}/2$-graded Clifford module.

Note that, conversely, if we assume from the beginning that S carries a $\mathbb{Z}/2$-graded Clifford module structure, with grading operator ϵ, then since $\epsilon_S \epsilon$ commutes with $\mathrm{Cliff}(V)$, it is a multiple of the identity, and in fact must be ± 1.

Remark 8.2. One can always realize a given $\mathrm{Cliff}(V)$-module, up to isomorphism, by one in which the Clifford multiplication action by vectors $v \in V$ is skew-adjoint. We sometimes call these *Hermitian* $\mathrm{Cliff}(V)$-modules, if the context absolutely demands it.

Indeed, fix any Hermitian metric on V, and then average it over the compact group $\mathbf{Pin} \subset \mathrm{Cliff}(V)$. This produces a metric with which $\mathbf{Pin}(V)$ acts by unitary maps. Since also $v^2 = -1$ as an endomorphism of W, we get $v^2 = -1 = -vv^*$ (since v is a unitary) and, cancelling the v's, that $v^* = -v$.

Exercise 8.3. Prove that the averaging of the previous Remark could have been simply done over the *finite* subgroup of the invertibles in $\mathrm{Cliff}_{\mathbb{R}}(V)$ generated by a fixed orthonormal basis $e_1, \ldots, e_n$ for V. (Check that these vectors really do generate a finite subgroup.)

Example 8.4. (*The unique $\mathbb{Z}/2$-graded irreducible* $\mathrm{Cliff}(\mathbb{R}^{2n})$*-module.*

If V is a Hermitian vector space, $\Lambda^*(V)$ its exterior algebra, then $\Lambda^*(V)$ inherits an inner product from V by the formula

$$\langle v_1 \wedge \cdots \wedge v_k, v_1' \wedge \cdots \wedge v_k \rangle := \det(\langle v_i v_j' \rangle).$$

In the obvious way, we equip $\Lambda^*(V)$ with a $\mathbb{Z}/2$-grading.

Choose $v \in V$. The operator of exterior multiplication $\lambda_v \colon \Lambda^*(V) \to \Lambda^*(V)$ is odd with respect to the grading. We leave it as an exercise to check that it's adjoint is given by *interior* multiplication, defined on elementary products by

$$i_v(w_1 \wedge \cdots \wedge w_k) = \sum_{i=1}^{k} (-1)^{i+1} \langle w_i, v \rangle w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_k. \tag{8.1}$$

Now for $v \in V$ let

$$c(v) := \lambda_v + i_v.$$

Clearly $c(v) = c(v)^*$, and moreover, if $w_1 \ldots w_k \in V$, then

(8.2)

$$\begin{aligned} i_v \lambda_v\, (w_1 \wedge \cdots \wedge w_k) = \langle v, v \rangle\, w_1 \wedge \cdots \wedge w_k - \sum_{i=1}^{k} (-1)^{i+1} \langle w_i, v \rangle\, v \wedge w_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge w_k \\ = \|v\|^2 - \lambda_v i_v\, (w_1 \wedge \cdots \wedge w_k) \end{aligned}$$

giving that

$$(\lambda_v + i_v)^2 = \|v\|^2.$$

Let

$$c_V \colon V \to \mathrm{End}\big(\Lambda^* V\big), \quad c_V(v) := i\,(\lambda_v + i_v),$$

then $c_V(v)^2 = -\|v\|^2$ and $c_V(v)^* = -c_V(v)$.

In particular, the above discussion applies to $V = \mathbb{C}^n$ with it's standard Hermitian structure. Define now

$$c\colon \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \to \operatorname{End}\big(\Lambda^* V\big), \ \ c(u,v) := c_{\mathbb{C}^n}(u+iv).$$

Then from the above discussion

$$c(u,v)^2 = -\|u+iv\|^2 = -\|u\|^2 - \|v\|^2 = -\|(u,v)\|^2.$$

Since $\dim \Lambda^*(\mathbb{C}^n) = 2^n$, this construction produces the unique irreducible representation of $\mathrm{Cliff}(\mathbb{R}^{2n})$.

Exercise 8.5. Show that if $n = 2$, then up to isomorphism, the $\mathrm{Cliff}(\mathbb{R}^2)$-module constructed above the $\mathbb{Z}/2$-graded vector space $\mathbb{C}^2$, with grading operator $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and Clifford representation in which a vector $(x,y) \in \mathbb{R}^2$ acts by

$$c(x,y) := i \begin{bmatrix} 0 & x+iy \\ x-iy & 0 \end{bmatrix}.$$

Example 8.6. How do we construct the irreducible representations of $\mathrm{Cliff}(V)$ when V is odd-dimensional? Then the orthogonal sum $V \oplus \mathbb{R}$ is even-dimensional. Let S be an irreducible $\mathrm{Cliff}(V \oplus \mathbb{R})$-module, with grading operator ϵ_S. Suppose $\dim V = n$. Thus S is $2^{\frac{n+1}{2}}$-dimensional. To be concrete, let $e_1, \dots e_n$ be an orthonormal basis for V, extend it by adding $e_{n+1} = (0,1) \in V \oplus \mathbb{R}$.

Now, since n is odd, the vectors $e_1, \dots, e_n$ commute with $\Gamma := e_1 \cdots e_n \in \mathrm{Cliff}(V) \subset \mathrm{Cliff}(V \oplus \mathbb{R})$ Therefore, the element $\Gamma' := i^{\frac{n+1}{2}} \cdot \Gamma$ satisfies $(\Gamma')^* = \Gamma'$, $(\Gamma')^2 = 1$, and Γ commutes with $\mathrm{Cliff}(V)$. Hence $\mathrm{Cliff}(V)$ leaves each of it's ± 1-eigenspaces invariant. These subspaces $S_\pm$ realize the two irreducible $\mathrm{Cliff}(V)$-modules.

For example, take the $\mathrm{Cliff}(\mathbb{R}^2)$-module $\mathbb{C}^2$ constructed in Exercise 8.5 with

$$c(x,y) = i \begin{bmatrix} 0 & x+iy \\ x-iy & 0 \end{bmatrix}.$$

The operator $\Gamma' := i\, c(1,0)$ is multiplication by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ whose ± 1 eigenspaces are the subspaces $x = y$, and $x = -y$ in $\mathbb{C}^2$, and on these two 1-dimensional subspaces, Clifford multiplication by $x \in \mathbb{R}$ is multiplication by the (real) scalar $x \in \mathbb{C}$, and respectively, the (real) scalar $-x$.

So we get the two irreducible, one-dimensional complex representations of $\mathrm{Cliff}(\mathbb{R})$.

An equivalent way of phrasing this construction is as follows. Let $e = (0,1) \in V \oplus \mathbb{R}$. It graded commutes with the subalgebra $\mathrm{Cliff}(V)$ of $\mathrm{Cliff}(V \oplus \mathbb{R})$. If we combine it with the grading operator ϵ_S we obtain $\Gamma := \epsilon_S c(e) \in \operatorname{End}(S)$. If $v \in V$, then $c(v)\Gamma = -\epsilon_S c(v) c(e) = \epsilon_S c(e) c(v) = \Gamma c(v)$ so Γ commutes with $\mathrm{Cliff}(V)$. Furthermore, since $c(e)$ graded commutes with ϵ_S,

$$\Gamma^* = \big(\epsilon_S c(e)\big)^* = c(e)^* \epsilon_S^* = -c(e)\epsilon_S = \epsilon_S c(e) = \Gamma$$

so Γ is a self-adjoint operator on S. Similar calculations show that $\Gamma^2 = 1$. Hence S deomposes into the ± 1-eigenspaces of Γ, each is left invariant by $\mathrm{Cliff}(V)$, and realize the two irreducible representations of $\mathrm{Cliff}(V)$.

Exercise 8.7. Show that the two descriptions of the irreducible representations of $\mathrm{Cliff}(V)$, with V odd-dimensional, given above, are equivalent. (*Hint.* The element $e_1 \cdots e_{n+1}$ in the first description acts as the grading operator on S, so $\epsilon_S c(e) = \Gamma$ in the second description agrees with $-\Gamma'$ from the first description.)

The following argument shows that when $\dim(V)$ is even-dimensional, then an orientation on V supplies any irreducible irreducible $\mathrm{Cliff}(V)$-module with a natural $\mathbb{Z}/2$-grading.

PROPOSITION 8.8. *Let V be even-dimensional and oriented. Let $\Gamma \in \mathrm{Cliff}(V)$ be the volume element associated to the orientation. Then $\Gamma' := i^{\frac{n}{2}} \cdot \Gamma$ is self-adjoint, $(\Gamma')^2 = 1$, and if S is an irreducible $\mathrm{Cliff}(V)$-module, then the ± 1-eigenspaces of Γ' induce a $\mathbb{Z}/2$-grading on S with respect to which Clifford multiplication by a vector in V is an odd operator.*

If $n = \dim(V)$ is odd, V oriented, Γ the volume element, then Γ acts by ± 1 in any irreducible $\mathrm{Cliff}(V)$-module, and the sign depending on which of the two irreducible representations it is.

All of these statements follow from Lemma 7.6. Note that since v graded-commutes with Γ, and hence Γ', for any vector $v \in V$, it follows that $c(v)$ interchanges the ± 1 eigenspaces of Γ'.

, because it commutes with $\mathrm{Cliff}(V)$, and Schur's Lemma. The two distinct possibilities, of course, correspond to the two distinct isomorphism classes of irreducible representations.

EXERCISE 8.9. This gives a slightly different way of looking at irreducible modules in the odd-dimensional case.

Suppose n is odd and that S is an irreducible $\mathrm{Cliff}(\mathbb{R}^{n+1})$-module, with the $\mathbb{Z}/2$-grading coming from the standard orientation on $\mathbb{R}^{n+1}$ (with associated volume $\Gamma := e_1 \cdots e_n e_{n+1}$). Let $S_\pm$ be the even and odd parts of S.

The even part of $\mathrm{Cliff}(\mathbb{R}^{n+1})$ leaves each of $S_\pm$ invariant. Show that composing the the isomorphism $\mathrm{Cliff}(\mathbb{R}^n) \cong \mathrm{Cliff}(\mathbb{R}^{n+1})^0$ of Exercise 7.9 with the two representations of $\mathrm{Cliff}(\mathbb{R}^{n+1})^0$ on $S_\pm$ we obtain the two irreducible representations of $\mathrm{Cliff}(\mathbb{R}^n)$. (*Hint.* Check how the volume element for $\mathrm{Cliff}(\mathbb{R}^n)$ acts in each of these representations.)

EXERCISE 8.10. Let V be odd-dimensional and suppose that $c\colon \mathrm{Cliff}(V) \to \mathrm{End}(S)$ is an irreducible $\mathrm{Cliff}(V)$ module. Recall that there are two such modules, up to isomorphism. Show that letting $c'(v) := c(-v)$ we obtain the other one.

EXERCISE 8.11. Let A and B be $\mathbb{Z}/2$-graded *-algebras (over $\mathbb{C}$.) Their *graded tensor product* $A\hat{\otimes}B$ is, as a complex vector space, the same as their ordinary tensor product $A \otimes B$, but with the multiplication on homogeneous (*i.e.* either even or odd) elements

$$(a\hat{\otimes}b) \cdot (c\hat{\otimes}d) := (-1)^{\partial b \partial c}\, ac\hat{\otimes}bd, \quad (a\hat{\otimes}b)^* := (-1)^{\partial a \partial b}\, a^*\hat{\otimes}b^*,$$

and grading having even part $(A\hat{\otimes}B)^0 := A^0\hat{\otimes}B^0 \oplus A^1\hat{\otimes}B^1$, and odd part $A^0\hat{\otimes}B^1 \oplus A^1\hat{\otimes}B^0$.

Prove that in this notation, there is a natural isomorphism

$$\mathrm{Cliff}(V)\hat{\otimes}\mathrm{Cliff}(W) \cong \mathrm{Cliff}(V \oplus W)$$

if V and W are Euclidean vector space (and $V \otimes W$ is the orthogonal direct sum.)

EXERCISE 8.12. Prove that any $\mathbb{Z}/2$-graded C*-algebra (*i.e.* a C*-algebra, which is also $\mathbb{Z}/2$-graded as a *-algebra), is isomorphic to a closed C*-subalgebra of $\mathbb{B}(H)$ for some *graded* Hilbert space $H = H^0 \oplus H^1$, in such a way that A^0 acts by grading-preserving (*i.e.* 'even') operators on H, and A^1 acts by grading-reversing ('odd') operators. (*Hint.* Extend a faithful state for A^0 to A and apply the GNS representation to get a faithful $\mathbb{Z}/2$-graded representation.)

EXERCISE 8.13. Adapt the definition of the spatial tensor product of C*-algebras (Definition 8.16) to work for $\mathbb{Z}/2$-graded C*-algebras, using Exercise 8.12.

EXERCISE 8.14. This exercise introduces a bit more material on $\mathbb{Z}/2$-graded spaces and algebras.

A $\mathbb{Z}/2$-graded Hilbert space is of course a Hilbert space with a $\mathbb{Z}/2$-grading. Suppose that H_1 and H_2 are two $\mathbb{Z}/2$-graded Hilbert spaces. Their graded tensor product $H_1\hat{\otimes}H_2$ is the same as $H_1 \otimes H_2$ as a Hilbert space, but $\mathbb{Z}/2$-graded as in the case of tensor products of $\mathbb{Z}/2$-graded algebras as discussed above.

Suppose that $T \in \mathbb{B}(H_1)$ and $S \in \mathbb{B}(H_2)$. Assume that T, S, v_1 and v_2 are all homogeneous elements for the gradings. Define

$$(T\hat{\otimes}S)\,(v_1\hat{\otimes}v_2) := (-1)^{\partial S\,\partial v_1}\; Tv_1\hat{\otimes}Sv_2.$$

Show that the above definition extends to a $\mathbb{Z}/2$-graded *-representation of the $\mathbb{Z}/2$-graded algebra $\mathbb{B}(H_1)\hat{\otimes}\mathbb{B}(H_2)$ on $H_1\hat{\otimes}H_2$.

LEMMA 8.15. *Assume that V and W are even-dimensional Euclidean spaces, $c_V\colon \mathrm{Cliff}(V) \to \mathrm{End}(S_V)$ and $c_W\colon \mathrm{Cliff}(W) \to \mathrm{End}(S_W)$ are two $\mathbb{Z}/2$-graded irreducible Clifford modules. Set $S_{V\oplus W} := S_V\hat{\otimes}S_W$, the* graded *tensor product of graded vector spaces. Let $c_{V\oplus W}\colon V\oplus W \to \mathrm{End}(S_{V\oplus W})$,*

$$c_{V\oplus W}(v,w) := c(v)\hat{\otimes}1 + 1\hat{\otimes}c(w).$$

Then $c_{V\oplus W}$ $c_V \oplus W\colon \mathrm{Cliff}(V\oplus W) \to \mathrm{End}(S_{V\oplus W})$ is an irreducible $\mathbb{Z}/2$-graded representation of $\mathrm{Cliff}(V\oplus W)$.

PROOF. It is easily checked that $c_V(v)\hat{\otimes}1 + 1\hat{\otimes}c_W(w)$ is an odd operator on the $\mathbb{Z}/2$-graded $S_V\hat{\otimes}S_W$. On the other hand, by definition of the multiplication in $\mathbb{B}(S_V)\hat{\otimes}\mathbb{B}(S_W)$, we have

$$\begin{aligned}(8.3)\quad \big(c_V(v)\hat{\otimes}11 + 1\hat{\otimes}c_W(w)\big)^2 &= c_V(v)^2\hat{\otimes}1 - c_V(v)\hat{\otimes}c_W(w) + c_V(v)\hat{\otimes}c_W(w) + 1\hat{\otimes}c_W(w)^2\\ &= -\|v\|^2 - \|w\|^2 = -\|(v,w)\|^2\end{aligned}$$

as required.

Since $\dim S_V\hat{\otimes}S_W = \dim S_V\,\dim S_W = 2^{\frac{\dim V + \dim W}{2}} = 2^{\frac{\dim(V\oplus W)}{2}}$, this is an irreducible, $\mathbb{Z}/2$-graded representation, as required.

□

EXERCISE 8.16. In Example 8.4 we described the irreducible $\mathrm{Cliff}(\mathbb{R}^n)$-module whenever n is even. On the other hand, Lemma 8.15 allows, in principal, us to describe this representation for all n (even), up to isomorphism, as soon as we know what it is for $n = 2$. Verify.

When V is even-dimensional and W is odd-dimensional, we proceed as follows.

LEMMA 8.17. *Suppose that V is even-dimensional, W is odd-dimensional, $c_V\colon \mathrm{Cliff}(V) \to \mathrm{End}(S_V)$ a $\mathbb{Z}/2$-graded, irreducible representation of $\mathrm{Cliff}(V)$, and $c_W\colon \mathrm{Cliff}(W) \to \mathrm{End}(S_W)$ an irreducible representation.*

With respect to the decomposition $S_V\otimes S_W = S_V^0\otimes S_W \,\oplus\, S_V^1\otimes S_W$, let

$$c_{V\oplus W}(v,w) := \begin{bmatrix} 0 & c_V(v)\otimes 1 + 1\otimes ic_W(w)\\ c_V(v)\otimes 1 - 1\otimes ic_W(w) & 0\end{bmatrix}$$

Then $c_{V\oplus W}$ $c_V\oplus W\colon \mathrm{Cliff}(V\oplus W) \to \mathrm{End}(S_{V\oplus W})$ is an irreducible representation of $\mathrm{Cliff}(V\oplus W)$.

Finally, we deal with the case when V and W are both odd-dimensional:

LEMMA 8.18. *Suppose that V and W are odd-dimensional Euclidean vector spaces, and that $c_V\colon \mathrm{Cliff}(V) \to \mathrm{End}(S_V)$ is an irreducible representation of $\mathrm{Cliff}(V)$, and $c_W\colon \mathrm{Cliff}(W) \to \mathrm{End}(S_W)$ is an irreducible representation of $\mathrm{Cliff}(W)$.*

Set $S_{V\oplus W} := S_V\otimes S_W \,\oplus\, S_V\otimes S_W$, endow $S_{V\oplus W}$ with the $\mathbb{Z}/2$-grading with even part the first factor, odd part the second factor. Let

$$c_{V\oplus W}(v,w) := \begin{bmatrix} 0 & c_V(v)\otimes 1 + 1\otimes ic_W(w)\\ c_V(v)\otimes 1 - 1\otimes ic_W(w) & 0\end{bmatrix}$$

Then $c_{V\oplus W}$ extends to an irreducible, $\mathbb{Z}/2$-graded representation $c_{V\oplus W}\colon \mathrm{Cliff}(V\oplus W) \to \mathrm{End}(S_V\otimes S_W)$.

The two-out-of-three Lemma

Above we proved that a representation of $\mathrm{Cliff}(V)$ and one of $\mathrm{Cliff}(W)$ induced one of $\mathrm{Cliff}(V \oplus W)$. Here we show a partial converse. This is highly relevant to K-theory, as explained in the next section.

LEMMA 8.19. *Assume that V and W are even dimensional Euclidean vector spaces, $V \oplus W$ their orthogonal direct sum.*

Suppose that $c_V : \mathrm{Cliff}(V) \to \mathrm{End}(S_V)$ is a $\mathbb{Z}/2$-graded irreducible representation of $\mathrm{Cliff}(V)$, and $c_{V\oplus W} : \mathrm{Cliff}(V \oplus W) \to \mathrm{End}(S_{V\oplus W})$ is a $\mathbb{Z}/2$-graded irreducible representation of $\mathrm{Cliff}(V \oplus W)$. Let

$$S_W := \mathrm{Hom}_{\mathrm{Cliff}(V)}(S_V, S_{V\oplus W}) := \{T : S_V \to S_{V\oplus W} \mid c_V(v)T(s) = T\big(c_V(v)s\big)\ \forall s \in S_V\},$$

graded into even and odd operators. Define, for $w \in W$, and $T \in S_{V\oplus W}$ homogeneous,

$$c_W(w)T := (-1)^{\partial T}\, c_{V\oplus W}(w) \circ T \circ \epsilon_V \tag{8.4}$$

Then this extends to a $\mathbb{Z}/2$-graded, irreducible representation of $\mathrm{Cliff}(W)$.

PROOF. We leave it to the reader to verify that (8.8) really is well-defined (that the indicated action of $c_W(w)$ maps the space of invariant operators to itself), and extends to a $\mathbb{Z}/2$-graded Clifford representation.

Since there a unique irreducible representation of $\mathrm{Cliff}(V)$ up to equivalence, the evaluation map

$$\mathrm{Hom}_{\mathrm{Cliff}(V)}(S_V, S_{V\oplus W}) \otimes_{\mathbb{C}} S_V \to S_{V\oplus W} \tag{8.5}$$

is an isomorphism. As a result, In particular,

$$\dim \mathrm{Hom}_{\mathrm{Cliff}(V)}(S_V, S_{V\oplus W}) \cdot 2^{\frac{\dim V}{2}} = 2^{\frac{\dim S_{V\oplus W}}{2}}$$

from which $\dim S_W = 2^{\frac{\dim W}{2}}$ follows, and hence irreducibility.

□

LEMMA 8.20. *Assume that V and W are Euclidean vector spaces, one even-dimensional, one odd-dimensional.*

a) *If W is odd-dimensional, V even-dimensional, $c_V : \mathrm{Cliff}(V) \to \mathrm{End}(S_V)$ is a $\mathbb{Z}/2$-graded irreducible representation of $\mathrm{Cliff}(V)$, and $c_{V\oplus W} : \mathrm{Cliff}(V \oplus W) \to \mathrm{End}(S_{V\oplus W})$ is an irreducible representation of $\mathrm{Cliff}(V \oplus W)$, let*

$$S_W := \mathrm{Hom}_{\mathrm{Cliff}(V)}(S_V, S_{V\oplus W}) := \{T : S_V \to S_{V\oplus W} \mid c_V(v)T(s) = T\big(c_V(v)s\big)\ \forall s \in S_V\}, \tag{8.6}$$

(with no grading). Define, for $w \in W$, and $T \in S_{V\oplus W}$ homogeneous,

$$c_W(w)T := \ c_{V\oplus W}(w) \circ T \circ \epsilon_V \tag{8.7}$$

where ϵ_V is the grading operator on V Then $c_W : \mathrm{Cliff}(W) \to \mathrm{End}(S_W)$ is an irreducible representation of $\mathrm{Cliff}(W)$.

b) *If V is odd-dimensional, W even-dimensional, $c_V : \mathrm{Cliff}(V) \to \mathrm{End}(S_V)$ is an irreducible representation of $\mathrm{Cliff}(V)$, and $c_{V\oplus W} : \mathrm{Cliff}(V \oplus W) \to \mathrm{End}(S_{V\oplus W})$ is an irreducible representation of $\mathrm{Cliff}(V \oplus W)$, S_W still as in (8.6), we let ϵ_W be the $\mathbb{Z}/2$-grading operator on $S_{V\oplus W}$ induced by the orientation on W, and*

$$c_W(w)T := \epsilon_W \circ c_{V\oplus W}(w) \circ T, \quad w \in W, T \in \mathrm{Hom}_{\mathrm{Cliff}(V)}(S_V, S_{V\oplus W})..$$

Then $c_W : \mathrm{Cliff}(W) \to \mathrm{End}(S_W)$ is an irreducible, $\mathbb{Z}/2$-graded representation of $\mathrm{Cliff}(W)$.

The verification of the above statements is a good exercise and is left to the reader.

Finally, we finish with the last case:

LEMMA 8.21. *Assume that V and W are odd-dimensional Euclidean vector spaces, $V \oplus W$ their orthogonal direct sum.*

Suppose that $c_V\colon \mathrm{Cliff}(V) \to \mathrm{End}(S_V)$ is an irreducible representation of $\mathrm{Cliff}(V)$, and $c_{V\oplus W}\colon \mathrm{Cliff}(V\oplus W) \to \mathrm{End}(S_{V\oplus W})$ is a $\mathbb{Z}/2$-graded irreducible representation of $\mathrm{Cliff}(V \oplus W)$, with grading operator ϵ. Let

$$S_W := \mathrm{Hom}_{\mathrm{Cliff}(V)}(S_V, S_{V\oplus W}) := \{T\colon S_V \to S_{V\oplus W} \mid c_{V\oplus W}(v)T(s) = T\big(c_V(v)s\big)\ \forall s \in S_V\}.$$

Define, for $w \in W$, and $T \in S_{V\oplus W}$ homogeneous,

$$c_W(w)T := i\, c_{V\oplus W}(w) \circ \epsilon \circ T \tag{8.8}$$

Then this extends to a $\mathbb{Z}/2$-graded, irreducible representation of $\mathrm{Cliff}(W)$.

EXERCISE 8.22. In the notation of the Lemma, show that if one sets

$$S_W^- := \{T\colon S_V \to S_{V\oplus W} \mid c_V(v)T(s) = -T\big(c_V(v)s\big)\ \forall s \in S_V\}$$

then one also obtains another irreducible $\mathrm{Cliff}(W)$-module (recall that when W is odd-dimensional, there are, up to isomorphism, two irreducible $\mathrm{Cliff}(W)$-modules).

Representation theory of real Clifford algebras

We will be focusing on complex K-theory in the following, and correspondingly will not discuss much the representation theory of the *real* Clifford algebras $\mathrm{Cliff}_{\mathbb{R}}(V)$ of a Euclidean vector space. However, we give some examples of real Clifford modules below, and an application to the problem of vector fields on spheres. Real Clifford algebra theory is important in Riemannian geometry.

EXAMPLE 8.23. $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^1) \cong \mathbb{C}$, so a $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^1)$-module W, disregarding the grading for the moment, corresponds to a real vector space, together with a linear operator $c(e)$, where $e = (1, 0)$, with $c(e)^2 = -\mathrm{id}_W$, which we can view as a complex structure on W. With this point of view, Clifford multiplication by e is multiplication by the complex scalar $i \in \mathbb{C}$.

In particular, W has real dimension a multiple of 2. Of course, 2 can be achieved (by $W := \mathbb{C}$, viewed as a module over $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^1) = \mathbb{C}$ by scalar multiplication.

There is a natural $\mathbb{Z}/2$-grading with even part the real axis $\mathbb{R}$ in $\mathbb{C}$, and odd part the imaginary axis $\mathbb{R}i$, so that $c(e)$ (multiplication by i) acts as an odd operator.

EXAMPLE 8.24. We have already seen that $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^2) \cong \mathbb{H}$, the real algebra of quaternions. A $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^2)$-module is therefore the endowment of a real vector space with a 'quaternionic' structure,' with i, j, k acting respectively as $c(e_1), c(e_2)$ and $c(e_1e_2)$.

Note that this forces W to have real dimension a multiple of 4. Using the standard embedding of the quaternions in $M_2(\mathbb{C})$ (sending the unit quaternions to $\mathbf{SU}_2$) results in a real representation of real dimension 4 of $\mathrm{Cliff}(\mathbb{R}^2)$.

Some applications to vector fields over spheres

Some of the ideas above can be used to prove significant theorems about vector fields on spheres. We give a rather simple such result here.

Let $c\colon \mathbb{R}^n \to \mathrm{End}_{\mathbb{R}}(W)$ be a $\mathrm{Cliff}_{\mathbb{R}}(\mathbb{R}^n)$-module, with $\dim_{\mathbb{R}}(W) = m$. Endow W with an inner product with respect to which Clifford multiplication by vectors is orthogonal, whence skew-symmetric. Now, if $\xi \in \mathbb{R}^n$, let $V_\xi \colon S^{m-1} \to \mathbb{R}^m$ be $V_\xi(x) := c(\xi)x$. Since $c(\xi)$ is skew-adjoint, $\langle c(\xi)x, x\rangle = 0$. Hence $c(\xi)x \in T_x S^{m-1}$. The map $\xi \to V_\xi$ defines a linear injective map from $\mathbb{R}^n$ into the space of vector fields on S^{m-1}.

For example, the construction in Example 8.4 has a real counterpart, producing a real representation
$$\mathrm{Cliff}(\mathbb{R}^n) \to \mathrm{End}_{\mathbb{R}}\big(\Lambda^*_{\mathbb{R}}(\mathbb{R}^n)\big),$$
with $c(v) := \lambda_v + i_v$ (exterior and interior multiplication defined as in the complex case.) The space on which the representation occurs is thus of real dimension 2^n. So we get n linearly independent vector fields on S^{2^n-1}, *e.g.* there are 2 on S^3 by this method, 3 on S^7 and so on. One can do much better. We refer the reader to Michaelsohn and Lawson's book, or to the papers of Adams, for 'more information.

9. The Atiyah-Bott-Shapiro Theorem

Suppose that $c\colon \mathrm{Cliff}(\mathbb{R}^n) \to \mathrm{End}(S)$ is a $\mathbb{Z}/2$-graded Clifford module for $\mathbb{R}^n$. Let
$$\sigma_c \colon \mathbb{R}^n \to \mathrm{Hom}_{\mathbb{C}}(S_+, S_-), \;\; \sigma_S(\xi) := c(\xi).$$
Then σ_c determines a vector bundle map $\mathbb{R}^n \times S_+ \to \mathbb{R}^n \times S_-$ between the trivial bundles $\mathbb{R}^n \times S_\pm$, which is invertible away from $\{0\}$. We get a K-theory triple $(\mathbb{R}^n \times S_+, \mathbb{R}^n \times S_-, \sigma_c)$ and a corresponding class $[\sigma_c] \in \mathrm{K}^0(\mathbb{R}^n)$.

Lemma 9.1. *View* $\mathrm{Cliff}(\mathbb{R}^n) \subset \mathrm{Cliff}(\mathbb{R}^{n+1})$ *in the usual way.*

Then if a $\mathbb{Z}/2$-graded Clifford module $c\colon \mathbb{R}^n \to \mathrm{End}(S)$ is the restriction to $\mathrm{Cliff}(\mathbb{R}^n) \subset \mathrm{Cliff}(\mathbb{R}^{n+1}$ *of a representation* $c'\colon \mathrm{Cliff}(\mathbb{R}^{n+1}) \to \mathrm{End}(S)$ *over* $\mathbb{R}^{n+1}$*, then the corresponding class* $[\sigma_c] \in \mathrm{K}^0(\mathbb{R}^n)$ *is zero.*

Proof. Let e_{n+1} be the $n+1$th standard basis vector of $\mathbb{R}^{n+1}$. We show that there is a vector bundle map $\sigma'\colon \mathbb{R}^n \times S_+ \to \mathbb{R}^n \times S_-$ which agrees with σ_c off a compact set, and is invertible everywhere. The result will follow. To accomplish this, if $v \in \mathbb{R}^n$, $\|v\| \leq 1$, set
$$\sigma'(v) := c'(v + \sqrt{1 - \|v\|^2}\, e_{n+1}).$$
and if $\|v\| \geq 1$ let $\sigma'(v) = \sigma_c(v)$.

□

This gives a rather direct argument that at least *some* candidates for classes in $\mathrm{K}^0(\mathbb{R}^n)$, when n is *odd*, are zero:

Proposition 9.2. *Suppose n is odd. Then any $\mathbb{Z}/2$-graded Clifford module* $c\colon \mathrm{Cliff}(\mathbb{R}^n) \to \mathrm{End}(S)$ *is the restriction of a $\mathbb{Z}/2$-graded Clifford module for $\mathbb{R}^{n+1}$, and hence $[\sigma_c] \in \mathrm{K}^0(\mathbb{R}^n)$ is the zero class.*

Proof. Let $\epsilon\colon S \to S$ be the grading operator. Let $e_{n+1} \in \mathbb{R}^{n+1}$ act on S by $i\epsilon$. Since by hypothesis, ϵ graded commutes with Clifford multiplication by vectors in $\mathbb{R}^n$, we do indeed obtain a representation $\mathrm{Cliff}(\mathbb{R}^{n+1}) \to \mathrm{End}(S)$.

□

Now let $n = 2m$ be even. Let $c\colon \mathrm{Cliff}(\mathbb{R}^n) \to \mathrm{End}(\Lambda^*\mathbb{C}^m)$ be the $\mathbb{Z}/2$-graded Clifford module of Example 8.4. Let $\beta_n := \beta^m_{\mathbb{R}^2} \in \mathrm{K}^0(\mathbb{R}^n)$ be the Bott generator of $\mathrm{K}^0(\mathbb{R}^n) = \mathrm{K}^{-n}(\mathrm{pt})$.

Proposition 9.3. *With c as above,*
$$[\sigma_c] = \beta_{\mathbb{R}^n} \in \mathrm{K}^0(\mathbb{R}^n) = \mathrm{K}^{-n}(\mathrm{pt}),$$
where $\beta_{\mathbb{R}^n}$ is the Bott generator for $\mathrm{K}^0(\mathbb{R}^n)$.

The following exercise gives a nice application to describing the K-theory of odd spheres.

Exercise 9.4. Let n be odd. Let $: \mathbb{R}^{n+1} \to \mathrm{End}(S)$ be a $\mathbb{Z}/2$-graded Clifford module over $\mathrm{Cliff}(\mathbb{R}^{n+1})$. Fix $v_0 \in S^n \subset \mathbb{R}^{n+1}$ and let

$$u_c \colon S^n \to \mathrm{End}(S_+)$$

be the map $u(v) := c(v)c(v_0)|_{S_+}$. Then u is a unitary-valued function on S^n and if c is irreducible, the

$$[u_c] \in [S^n, \mathbf{U}_\infty] \cong \mathrm{K}^{-1}(S^n)$$

generates $\mathrm{K}^{-1}(S^n)$.

For each n, the collection of isomorphism classes of $\mathbb{Z}/2$-graded $\mathrm{Cliff}(\mathbb{R}^n)$-modules forms a semi-group under addition of modules. By taking the Grothendiek completion of this semigroup, we obtain a $\mathbb{Z}/2$-graded abelian group, which we denote by $\widehat{\mathcal{M}}_n$ and let $\widehat{\mathcal{M}}_* := \oplus_{n=0}^\infty \widehat{\mathcal{M}}_n$.

Note that $\widehat{\mathcal{M}}_*$ has a natural ring structure, for if $c_i \colon \mathrm{Cliff}(\mathbb{R}^n) \to \mathrm{End}(S_i)$ is a pair of $\mathbb{Z}/2$-graded Clifford modules over $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, then their $\mathbb{Z}/2$-graded tensor product $S_1 \hat{\otimes} S_2$ is a $\mathbb{Z}/2$-graded $\mathrm{Cliff}(\mathbb{R}^{n+m})$-module with $c(v_1, v_2) := c_1(v_1)\hat{\otimes}1 + 1\hat{\otimes}c_2(v_2)$. This multiplication descends to $\widehat{\mathcal{M}}_*$ to give a $\mathbb{Z}/2$-graded ring structure to $\widehat{\mathcal{M}}_*$. To check all this is left as an easy exercise for the reader.

Now, for each n let $i \colon \widehat{\mathcal{M}}_{n+1} \to \widehat{\mathcal{M}}_n$ be the group homomorphism induced by the inclusion $\mathrm{Cliff}(\mathbb{R}^n) \to \mathrm{Cliff}(\mathbb{R}^{n+1})$. We consider the quotient $\widehat{\mathcal{M}}_n / i^*(\widehat{\mathcal{M}}_{n+1})$. It maps, by Lemma **??**, to $\mathrm{K}^{-n}(\mathrm{pt})$.

Hence, by taking the direct sum of all these homomorphisms we obtain a homomorphism

$$\widehat{Q}_* := \widehat{\mathcal{M}}_*/i^*(\widehat{\mathcal{M}}_{*+1}) := \oplus_{n=0}^\infty \widehat{\mathcal{M}}_n/i^*(\widehat{\mathcal{M}}_{n+1}) \to \oplus_{n=0}^\infty \mathrm{K}^{-n}(\mathrm{pt}) := \mathrm{K}^*(\mathrm{pt}). \tag{9.1}$$

The Bott Periodicity Theorem now be stated in the following way.

Theorem 9.5. *(Atiyah-Bott-Shapiro) The map*

$$\widehat{Q}_* \to \mathrm{K}^*(\mathrm{pt})$$

of (9.1) *is an isomorphism of* $\mathbb{Z}/2$*-graded rings.*

The irreducible $\mathrm{Cliff}(\mathbb{R}^2)$-module of Example 8.5, in which Clifford multiplication by $(x, y) \in \mathbb{R}^2$ acts by the matrix $\begin{bmatrix} 0 & x+iy \\ x-iy & \end{bmatrix}$, has thus an isomorphism class Λ in $\widehat{Q}_*$. Then in this ring, Λ^n is the class of Example 8.4 (by a routine exercise.) The ring $\widehat{Q}_*$ is isomorphic as a ring, to the polynomial ring $\mathbb{Z}[x]$ of polynomials in one variable, by the map sending a polynomial $f(x) = \sum_{k=0}^n a_k x^k$ to $= \sum_{k=0}^n a_k \Lambda^k$.

On the K-theory side, such a polynomial identifies with the $\mathrm{K}^*(\mathrm{pt})$ element $\sum_{k=0}^n a_k \beta_{\mathbb{R}^2}^k$, with $\beta_{\mathbb{R}^2} \in \mathrm{K}^{-2}(\mathrm{pt})$ the Bott generator.

in the real case, the Atiyah-Bott-Shapiro Theorem is much more interesting. We omit the statement. The ring in question is, over the reals, more complicated. It is no longer 2-periodic but 8-periodic.

10. K-orientations and the Thom isomorphism theorem

The Thom isomorphism is a central result in K-theory (as is its analogue in ordinary cohomology) because it allows the construction in topological K-theory of so-called *wrong-way maps*. These appeared in the work of Aityah and Singer in connection with the Index Theorem, which, indeed, was eventually stated in terms of them.

Let $\pi \colon V \to X$ be a Euclidean vector bundle over a locally compact space X.

Since each fibre $V_x := \pi^{-1}(x)$ of V is a Euclidean vector space, we can form its Clifford algebra $\mathrm{Cliff}(V_x)$. The bundle of algebras $\{\mathrm{Cliff}(V_x)\}_{x\in X}$ is easily checked to be locally trivial as a bundle of algebras, and hence is, in particular, a complex vector bundle over X. We denote it $\mathrm{Cliff}(V)$.

There is an obvious bundle-version of the idea of a Clifford module. By a $\mathbb{Z}/2$-*graded vector bundle* we mean a vector bundle equipped with a fibrewise $\mathbb{Z}/2$-grading such that the even and odd parts $V^{\pm}$ of V are themselves (sub-) vector bundles (of V).

Definition 10.1. Let V be a Euclidean vector bundle over X.

By a $\mathrm{Cliff}(V)$-*module*, we shall mean a complex, Hermitian vector bundle S over X, together with a vector bundle map $c\colon \mathrm{Cliff}(V) \to \mathrm{End}(S)$, whose restriction $c_x\colon \mathrm{Cliff}(V_x) \to \mathrm{End}(S_x)$, to each fibre of $\mathrm{Cliff}(V)$, is a $\mathrm{Cliff}(V_x)$-module.

We sometimes require the bundle S to be $\mathbb{Z}/2$-graded as a vector bundle, and for the Cllifford action to be by (fibrewise) odd operators. In this case we refer to a $\mathbb{Z}/2$-graded $\mathrm{Cliff}(V)$-module.

The case where the Clifford bundle is, fibrewise, an irreducible module, is particularly important, and we will generally call the S of the previous definition a *spinor bundle* in this case.

There is a fairly obvious notation of *isomorphism* of $\mathrm{Cliff}(V)$-modules: one requires a fibrewise unitary bundle map $S \to S'$ intertwining the two representations. In the case when the modules are graded, we generally require the isomorphism to be grading-preserving.

Exercise 10.2. Suppose n is even. Let $c\colon \mathrm{Cliff}(\mathbb{R}^n) \to \mathrm{End}(S)$ be a $\mathbb{Z}/2$-graded irreducible $\mathrm{Cliff}(\mathbb{R}^n)$-module. Let A be an n-by-n orthogonal matrix. Show that the map $c'(v) := c\big(Av\big)$ extends to a $\mathbb{Z}/2$-graded irreducible $\mathrm{Cliff}(\mathbb{R}^n)$-module which is isomorphic, as a $\mathbb{Z}/2$-graded module, to c, if $\det(A) = 1$, and is isomorphic to c^{op} (the same Clifford module, but with grading reversed), if $\det(A) = -1$.

Definition 10.3. Suppose $\pi\colon V \to X$ is a a real vector bundle over X locally compact.

A K-*orientation* on V is a pair, consisting of

i) An inner product q on V.
ii) A fibrewise irreducible $\mathrm{Cliff}(V)$-module, with V equipped with the Euclidean structure from the metric in i). We require the module to be $\mathbb{Z}/2$-graded if V is even-dimensional.

Example 10.4. The zero vector bundle $\mathbf{0}$, over any space, can be K-oriented in two different ways.

Indeed, the Clifford algebra of the zero vector space is $\mathbb{C}$, graded as an algebra with $\mathbb{C}^+ := \mathbb{C}$, $\mathbb{C}^- := \{0\}$. Of course the C*-algebra $\mathbb{C}$ has a unique irreducible representation, on the one-dimensional Hilbert space $\mathbb{C}$. Either choice of $\mathbb{Z}/2$-grading on this Hilbert space, setting $\mathbb{C}^+ := \mathbb{C}$, $\mathbb{C}^- := \{0\}$, or setting $\mathbb{C}^+ := \{0\}$ and $\mathbb{C}^+ := \mathbb{C}$ yield $\mathbb{Z}/2$-graded Clifford modules, and they are clearly non-isomorphic as $\mathbb{Z}/2$-graded modules.

If one is speaking of the zero vector bundle over X, it's Clifford algebra is the algebra bundle with fibres $\mathbb{C}$-everywhere (*i.e.* the trivial rank-one bundle), so a K-orientation amounts to endowing the trivial line bundle over X with a $\mathbb{Z}/2$-grading. Note that such a grading might vary from component to component (of X).

Definition 10.5. A $\mathrm{spin}^{\mathrm{c}}$-*structure* on a smooth manifold X is a pair, consisting of a Riemannian metric on X, and a fibrewise irreducible $\mathrm{Cliff}(TX)$-module, graded if $\dim X$ is even. That is, X is $\mathrm{spin}^{\mathrm{c}}$ if the real vector bundle TX is K-oriented.

It is completely obvious that a trivial Euclidean vector bundle $X \times \mathbb{R}^n$ bundle (with the standard Euclidean metric fixed on the fibres) is K-orientable.

Furthermore, a K-orientation on a vector bundle determines an orientation on it, as the following exercise shows.

Exercise 10.6. Suppose that V is an even-dimensional Euclidean vector bundle over X. and $c\colon \mathrm{Cliff}(V) \to \mathrm{End}(S)$ a $\mathbb{Z}/2$-graded irreducible $\mathrm{Cliff}(V)$-module. If $e_1, \ldots, e_n$ is a local orthonormal frame for V, defined, say on a connected open set U, then we declare this local frame to be positively oriented if Clifford multiplication by $c(x, e_1(x) \cdots e_n(x)) \in \mathrm{End}(S_x)$ equal to the grading operator ϵ on S_x, and call it negatively oriented if it equals $-\epsilon$ (one of these possibilities must occur, for all $x \in U$).

a) Check that the above prescription orients V.
b) Prove the analogous result (K-oriented implies oriented) for odd-dimensional vector bundles.

In particular, the Möbius bundle over the circle is not K-orientable.

Example 10.7. If $\pi\colon V \to X$ has a complex structure (*i.e.* if it is isomorphic as a real vector bundle to a complex vector bundle E), and equipped with a Hermitian metric, and induced Euclidean metric, then V is K-oriented by letting

$$S := \Lambda^*_{\mathbb{C}}(E), \quad c(v) := \lambda_v + i_v\colon \Lambda^*_x E \to \Lambda^*_x E, \quad v \in E_x, \tag{10.1}$$

where λ_v is external product with v and i_v intterior product.

This all follows immediately from the definitions and the discussion in Example 8.4.

Exercise 10.8. Deduce from Example 10.7 that $V \oplus V$ is (canonically) K-oriented, for *any* real vector bundle V.

Definition 10.9. Suppose that V is a vector bundle over X, and $c\colon \mathrm{Cliff}(V) \to \mathrm{End}(S)$ is a $\mathbb{Z}/2$-graded $\mathrm{Cliff}(V)$-module, and that L is a $\mathbb{Z}/2$-graded Hermitian line bundle over X. Let $S' := S \hat{\otimes} L$ be the $\mathbb{Z}/2$-graded tensor product of these two vector bundles, endowed with the tensor product Hermitian structure, and the $\mathrm{Cliff}(V)$-module structure

$$c_L(v) := c(v) \otimes \mathrm{id} \in \mathrm{End}(S_x \otimes L_x) \quad x \in X,\ v \in V_x, s \in S_x, l \in L_x. \tag{10.2}$$

We call S' the $\mathrm{Cliff}(V)$-module obtained from S by *twisting by L*.

If S is an ungraded $\mathrm{Cliff}(V)$-module, we twist in exactly the same way, dropping all mentions of the gradings. The outcome is another non-graded $\mathrm{Cliff}(V)$-module.

Allowing a grading on a line bundle L essentially just means allowing 'negative' line bundle: if L is a $\mathbb{Z}/2$-graded line bundle, then either $L = L_+$ or $L = L_-$.

Proposition 10.10. *Let V be a Euclidean vector bundle over X. Then if V admits one* K-*orientation S, then any other* K-*orientation S' on V is obtained by twisting S by some $\mathbb{Z}/2$-graded Hermitian line bundle L over X, as in* (10.2)

In the case V is odd-dimensional, the same statement holds, with L ungraded.

Proof. Assume first that V is even-dimensional, and $c\colon \mathrm{Cliff}(V) \to \mathrm{End}(S)$ is a $\mathbb{Z}/2$-graded irreducible representation of $\mathrm{Cliff}(V)$, and $c'\colon \mathrm{Cliff}(V) \to \mathrm{End}(S')$ another. Let L be the Hermitian vector bundle with fibres

$$L_x := \mathrm{Hom}_{\mathrm{Cliff}(V_x)}(S_x, S'_x) := \{T \in \mathrm{Hom}_{\mathbb{C}}(S_x, S'_x) \mid c(v)T = Tc(v)\ \forall v \in V_x\},$$

graded into even and odd operators.

Then L is a complex line bundle over X and $S' \cong S \otimes L$ as $\mathbb{Z}/2$-graded Clifford modules.

In the odd-dimensional case, we simply drop the gradings and use the same argument.

□

Lemma 10.11. *If $f\colon X \to Y$ is a map and V is a vector bundle over Y, then a* K-*orientation on V pulls back to a* K-*orientation on $f^*(V)$.*

Proof. Suppose $c\colon V \to \mathrm{End}(S)$ is a spinor bundle for V, equipped with some Euclidean metric. The pulled-back bundle f^*V has fibre at x the Euclidean vector space $V_{f(x)}$, so f^*V inherits a natural pulled-back Euclidean structure. And the associated bundle of Clifford algebras has fibres $\mathrm{Cliff}(V_{f(x)})$, which map to $\mathrm{End}(S_{f(x)})$. This provides the pull-back f^*S of the spinor bundle for V with an action of $\mathrm{Cliff}(f^*V)$ as required. $\square$

Lemma 10.12. (The 2-out-of-3 Lemma). *Suppose that*

$$0 \to W \xrightarrow{i} V \xrightarrow{\pi} Q \to 0$$

is an exact sequence of real vector bundles over X. Then a K*-orientation on any two of them, determines a canonical* K*-orientation on the third.*

Proof. Choosing a splitting $s\colon Q \to V$ determines an isomorphism of real vector bundles $V \cong Q \oplus W$. So need to prove that if V and Q is K-oriented, then so is Q, and if Q and W are K-oriented, then so is V. This all follows from the obvious bundle versions of Lemmas 8.15, 8.17, 8.21 and 8.20.

$\square$

Exercise 10.13. On the tangent bundle TS^2 to the 2-sphere, there are two natural K-orientations that spring to mind. The first is based on the complex structure on S^2, seen as $\mathbb{CP}^1$. The tangent bundle TS^2 thus has a complex structure, and one builds an irreducible Clifford module bundle accordingly (as in Example 10.7).

On the other hand, S^2 is the boundary of the closed ball $\bar{\mathbb{D}}^3$, with trivial normal bundle, so that one has an exact sequence of vector bundles over S^2 given by

$$0 \to TS^2 \to T\bar{\mathbb{D}}^3 \to S^2 \times \mathbb{C},$$

and $T\bar{\mathbb{D}}^3$ is also a trivial bundle and so canonically K-orientable.

So we get a canonical K-orientation on TS^2, by the 2-out-of-3 result, the *boundary* K-*orientation.*

Find the complex line bundle L which interpolates between these two K-orientations.

Definition 10.14. Let $\pi\colon V \to X$ be a K-oriented, even-dimensional Euclidean vector bundle over X, with associated $\mathbb{Z}/2$-graded irreducible representation $c\colon \mathrm{Cliff}(V) \to \mathrm{End}(S)$.

On the total space V of V, define a K-theory triple by $(\pi^*S^+, \pi^*S^-, \sigma_V)$, where $\sigma_V(x, v) := c(x, v)\colon S_x^+ \to S_x^-$.

The associated class $\xi_V \in \mathrm{K}^0(V)$ is the *Thom class of V*.

If V is odd-dimensional, identify $V \times \mathbb{R}$ with the total space of the vector bundle $V \oplus \mathbf{1}$, where $\mathbf{1}$ is the trivial line bundle over X.

Since V and $\mathbf{1}$ are K-oriented, so is their sum, which is even-dimensional. We let in this case the Thom class ξ_V be the class in $\mathrm{K}^{-1}(V) \cong \mathrm{K}^0(V \oplus \mathbf{1})$ associated to the even-dimensional K-oriented bundle $V \oplus \mathbf{1}$.

Unifying the two definitions above, we typically understand the Thom class ξ_V of V as lying in the group $\mathrm{K}^{-\dim V}(V)$, with the superscript $-\dim V$ to be interpreted mod 2.

For the purposes of the following central theorem, recall that if $\pi\colon V \to X$ is a vector bundle, then using pull-back by π and the ring multiplication on bundles on V gives the $\mathbb{Z}/2$-graded abelian group $\mathrm{K}^*(V)$ the structure of a (graded) module over the $\mathbb{Z}/2$-graded abelian group $\mathrm{K}^*(X)$.

Theorem 10.15. (The Thom isomorphism theorem). *Let V be a* K*-oriented vector bundle over X locally compact. Then as a $\mathrm{K}^*(X)$-module, $\mathrm{K}^*(V)$ is free and rank-one, and is generated freely as a rank-one $\mathrm{K}^*(X)$-module by the Thom class $\xi_V \in \mathrm{K}^{-\dim V}(V)$. That is, the map*

$$\tau_V\colon \mathrm{K}^*(X) \to \mathrm{K}^{*-\dim V}(V), \quad \tau_V(a) := \pi^*(a)\xi_V, \tag{10.3}$$

is a $\mathrm{K}^*(X)$*-module isomorphism.*

Furthermore, if $f\colon X \to Y$ *is a map, then* $\xi_{f^*V} = f^*(\xi_V)$*, so that the Thom isomorphism is natural with respect to maps, and pull-backs of* K*-oriented vector bundles.*

The last (naturality) statement means the following. Suppose that if $f\colon X \to Y$ is a map, V is a real vector bundle over Y, and f^*V is given the pull-back K-orientation, using f. There is an obvious extension of f to a map $\bar{f}\colon f^*V \to V$.

Then naturality of the Thom isomorphism is the statement that the diagram

$$\begin{tikzcd} \mathrm{K}^*(f^*V) \arrow[d, "\tau_{f^*V)}"] & \mathrm{K}^*(V) \arrow[l, "\bar{f}^*"'] \arrow[d, "\tau_V"] \\ \mathrm{K}^*(Y) & \mathrm{K}^*(X) \arrow[l, "f^*"'] \end{tikzcd} \tag{10.4}$$

commutes.

The Thom isomorphism is a 'bundle' version of Bott Periodicity. If V is a K-oriented, n-dimensional vector bundle over X, with, initially, let us say, n even, then it's restriction to a point $p \in X$ can be identified linearly with $\mathbb{R}^n$, and it's K-orientation thus restricts to a K-orientation on the real vector space $\mathbb{R}^n$.

Since there are only two possible K-orientations of $\mathbb{R}^n$, for n even, up to equivalence, the standard one, and it's opposite, it follows that

$$\xi_V\,|_{\mathrm{pt}} = \pm\beta_{\mathbb{R}^n} \in \mathrm{K}^0(\mathbb{R}^n) = \mathrm{K}^{-n}(\mathrm{pt}).$$

The sign is positive if and only if the orientation determined on $\mathbb{R}^n$ by identifying it with V, and then taking the orientation determined by the K-orientation on V, agrees with the standard K-orientation on $\mathbb{R}^n$. One can tell the difference as follows. Take a positively oriented linear basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$, so that as vectors in V, they form an orthonormal set in V_p, and let $\mathrm{sign}_p(V) = 1$ if $c(e_1)\cdots c(e_n) = \epsilon$, and $\mathrm{sign}_p(V) := -1$ if $c(e_1)\cdots c(e_n) = -\epsilon_p \in \mathrm{End}(S_p)$, where S_p is the spinor bundle at p, ϵ_p the grading operator at p.

Proposition 10.16. *If* V *is a* K*-oriented* n*-dimensional vector bundle over* X*, then the restriction of the Thom class for* V *to any point* $p \in X$ *is given by*

$$(\xi_V)|_{\mathrm{pt}} = \mathrm{sign}(V)\cdot\beta_{\mathbb{R}^n} \in \mathrm{K}^{-n}(\mathrm{pt}),$$

where $\beta_{\mathbb{R}^n}$ *is the Bott generator of* $\mathrm{K}^{-n}(\mathbb{R}^n)$*.*

Thus, intuitively, the Thom isomorphism restricts to Bott periodicity on the fibres of any K-oriented vector bundle – but one must take care with the K-orientations.

11. Wrong-way maps, spin numbers and Baum-Douglas cycles

Let $f\colon X \to Y$ be a smooth map between smooth manifolds. Set

$$T_f := TX \oplus f^*(TY), \tag{11.1}$$

a real vector bundle over X.

Definition 11.1. Let X and Y be smooth manifolds. A K-*orientation* on a smooth map $f\colon X \to Y$ is a K-orientation on the real vector bundle T_f of (11.1).

Of course not all maps are 'K-orientable'. A K-orientation on the map $X \to \mathrm{pt}$ would correspond to a K-orientation on TX, *i.e.* a spin$^{\mathrm{c}}$-structure on X.

Example 11.2. For any smooth manifold X, $TX \oplus TX$ has a complex structure, so carries a canonical K-orientation. Hence the identity map $\mathrm{id}_X\colon X \to X$ is K-oriented for any smooth manifold X.

LEMMA 11.3. *. if $f\colon X \to Y$ and $g\colon Y \to Z$ are* K*-oriented maps between manifolds, then $g \circ f$ inherits a natural* K*-orientation those on on f and g.*

PROOF. We have

$$T_{g\circ f} = TX \oplus (g \circ f)^*(TZ) = TX \oplus f^*\big(g^*(TZ)\big) \tag{11.2}$$

from which it follows, adding $f^*(TY)$ to each side, that

$$T_{g\circ f} \oplus f^*(TY) \cong TX \oplus f^*(T_g). \tag{11.3}$$

Adding TX to each side now gives

$$T_{g\circ f} \oplus T_f \cong TX \oplus TX \oplus f^*(T_g). \tag{11.4}$$

To K-orient $T_{g\circ f}$, we therefore pull back a K-orientation on T_g to get one on $f^*(T_g)$. This K-orients the right hand side of (11.4) and hence the left-hand side. The assumed K-orientation on T_f and the 2-out-of-3 lemma then produces a canonical K-orientation on $T_{g\circ f}$.

∎

EXERCISE 11.4. Any diffeomorphism $f\colon X \to U \subset Y$ from a manifold X onto an open subset of a manifold Y carries a canonical K-orientation.

THEOREM 11.5. *Let X, Y be smooth manifolds and $f\colon X \to Y$ a smooth* K*-oriented map. Then f induces a group homomorphism*

$$f!\colon \mathrm{K}^*(X) \to \mathrm{K}^*(Y)$$

shifting degrees by $\dim X - \dim Y$.

Moreover:

a) $\mathrm{id}_X! = \mathrm{id}_{\mathrm{K}^*(X)}$ *for any X.*
b) *If f is as above and $g\colon Y \to Z$ is another* K*-oriented map then $(g \circ f)! = g! \circ f!\colon \mathrm{K}^*(X) \to \mathrm{K}^*(Z)$, where $g \circ f$ is* K*-oriented as in Lemma 11.3.*

PROOF. First suppose that $f\colon X \to Y$ is a smooth embedding. Let ν be its normal bundle, U the corresponding tubular neighbourhood of $f(X)$ in Y.

Observe first that a K-orientation on T_f determines one on the normal bundle ν and conversely. Indeed, $\nu \oplus TX \cong f^*(TY)$ and hence

$$T_f = TX \oplus f^*(TY) \cong \nu \oplus TX \oplus TX.$$

Since $TX \oplus TX$ is canonically K-oriented, by the 2-out-of-3 Lemma, K-orientations on ν are in 1-to-1 correspondence with K-orientations on $\nu \oplus TX \oplus TX$ *i.e.* with T_f, as claimed.

Note that $\dim \nu = \dim Y - \dim X$. The Thom isomorphism for the K-oriented bundle ν has the form

$$\tau_\nu\colon \mathrm{K}^*(X) \to \mathrm{K}^{*+\dim X - \dim Y}(\nu).$$

The map $f!$ is defined to be the composition

$$\mathrm{K}^*(X) \xrightarrow{\tau_\nu} \mathrm{K}^{*+\dim X - \dim Y}(\nu) \to \mathrm{K}^{*+\dim X - \dim Y}(Y), \tag{11.5}$$

where the last map is induced by the open embedding $\nu \cong U \subset Y$.

In the general case, let $i\colon X \to \mathbb{R}^n$ be a smooth embedding, $\tilde{f}\colon X \to Y \times \mathbb{R}^n$ the map

$$\tilde{f}(x) = (f(x), i(x)).$$

Then $\tilde{f}$ is an embedding, with normal bundle of dimension $\dim Y - \dim X + n$, with a K-orientation based on the given one on T_f, and the above discussion yields

$$\tilde{f}!\colon \mathrm{K}^*(X) \to \mathrm{K}^{*+\dim X - \dim Y - n}(Y \times \mathbb{R}^n) \to \mathrm{K}^{*+\dim X - \dim Y}(Y),$$

where the last map is the inverse of the Bott Periodicity map

$$\beta_Y : \mathrm{K}^*(Y) \to \mathrm{K}^{*-n}(Y \times \mathbb{R}^n), \tag{11.6}$$

a map shifting degrees by $+n$, since the Bott Periodicity map shifts degrees by $-n$.

□

EXAMPLE 11.6. Let M be a smooth Riemannian manifold, let $p \in M$ and denote also by p the corresponding map $p\colon \mathrm{pt} \to M$ from the 1-point space to M. To K-orient the map p means to K-orient the vector bundle, *i.e.* vector space, T_pM, over the 1-point space. Choosing a frame $\mathbf{e}_p$ for T_pM gives an isometric isomorphism $T_pM \xrightarrow{\cong} \mathbb{R}^n$, and using the standard K-orientation on the vector space $\mathbb{R}^n$ we get a corresponding K-orientation on the map $p\colon \mathrm{pt} \to M$, which depends on the frame. Let

$$(p, \mathbf{e}_p)! \in \mathrm{K}^{-n}(M)$$

be the class of the corresponding wrong-way map $\mathrm{pt} \to M$. We will use this construction in the context of $\mathbb{RP}^2$ to give a geometric proof of the 2-torsion part of its K^0-group, below.

EXERCISE 11.7. The following two exercises explore two important types of shriek maps. They refer to the proof of Theorem 11.5 for the definition of $f!$, for a K-oriented map f.

Let X be a smooth manifold.

a) Suppose $\pi\colon V \to X$ is a K-oriented n-dimensional vector bundle. Prove that $\zeta_V : X \to V$ is a K-oriented map, and that $\zeta_V! = \tau_V : \mathrm{K}^*(X) \to \mathrm{K}^{*-n}(V)$.
b) With the same hypotheses as a), prove that $\pi! = \tau_V^{-1} : \mathrm{K}^*(V) \to \mathrm{K}^{*+n}(V)$.

Given the results of the Exercise, one sees that the construction of $f!$ may be summarized by the diagram

$$\begin{tikzcd} N \arrow[r, "\hat{\varphi}"] & Y \times \mathbb{R}^n \arrow[d, "\mathrm{pr}_Y"] \\ X \arrow[u, "\zeta_N"] \arrow[r, "f"] & Y \end{tikzcd} . \tag{11.7}$$

The vertical map ζ_N is the zero section of the normal bundle N to X sitting inside $Y \times \mathbb{R}^n$, in our proof. The map $\hat{\varphi}$ is the corresponding open embedding. The diagram commutes. And each edge, excluding the one corresponding to f itself, induces a map on K-theory.

The map ζ_N induces the map $\zeta_N!\colon \mathrm{K}^*(X) \to \mathrm{K}^*(N)$ (by the Exercise above), the open embedding induces a natural map $\mathrm{K}^*(N) \to \mathrm{K}^*(Y \times \mathbb{R}^n)$, and the projection pr_Y induces $\mathrm{pr}_Y!\colon \mathrm{K}^*(Y \times \mathbb{R}^n) \to \mathrm{K}^*(Y)$.

Each of these maps except the open embedding shifts degrees, but the net degree shift turns out to be by $\dim X - \dim Y$, as we have seen.

Spin^c-numbers

A particular important case of the 'shriek' construction above, is where $p_X : X \to \mathrm{pt}$ is the map from a smooth, K-oriented (or 'spin^c') manifold, to the one-point manifold. From the discussion above, we have shown:

PROPOSITION 11.8. *Let X be an n-dimensional manifold. Then a* spin^c*-structure on X induces a group homomorphism*

$$p_X!\colon \mathrm{K}^{-n}(X) \to \mathbb{Z}. \tag{11.8}$$

This prompts introduction of some terminology.

DEFINITION 11.9. Let X be a smooth n-dimensional, compact manifold equipped with a spin^c-structure.

a) If n is even and $E \to X$ is a complex vector bundle over X, the spin^c-*number of* E is the integer
$$\mathrm{spin}^c(E) := p_X!([E]) \in \mathbb{Z},$$
and
b) if n is odd, and $u\colon X \to \mathbf{U}_n$ is a map to the unitary group $\mathbf{U}_n$, then the spin^c-*number of* u is
$$\mathrm{spin}^c(u) := p_X!([u]),$$
where $p_X!\colon \mathrm{K}^{-1}(X) \to \mathbb{Z}$ is the wrong-way map (11.8).

In the even-dimensional case, we call the spin^c-*number of* X the spin^c-number of the trivial line bundle $\mathbf{1}$ over X, and denote it $\mathrm{spin}^c(X)$.

It is obvious from its definition that the spin^c number is defined not just for vector bundles, but for K^0-classes, and similarly for unitaries and K^1-classes, and will correspondingly sometimes refer to the spin^c-number of a K-theory class.

The following Exercise shows that the spin^c-number reduces to the winding number, in the case $X = \mathbb{T}$.

EXERCISE 11.10. The spin^c-number for the circle $\mathbb{T}$ agrees with the winding number.

More precisely, let $u\colon \mathbb{T} \to \mathbf{U}_n$ be a map. Then
$$\mathrm{spin}^c(u) = \mathrm{wind}(\det u),$$
where $\det(u) = \det \circ u$, the determinant of u, and wind is the winding number around the origin $0 \in \mathbb{C}$.

EXERCISE 11.11. Give $\mathbb{CP}^1$ the spin^c-structure associated to the complex structure on its tangent bundle. Compute the spin^c-number $\mathrm{index}_{\mathrm{spin}^c}(\mathbb{CP}^1)$ of the corresponding spin^c-manifold (the answer is $+1$.)

More generally, compute the spin^c-numbers of powers H^n of the Hopf bundle: *i.e.* compute
$$\mathrm{index}_{\mathrm{spin}^c}(H^n)$$
for any integer n (the answer is zero unless $n = 0$.)

EXERCISE 11.12. Let $\mathbb{T}^2$ be the torus with its standard spin^c-structure (coming from its complex structure).

Compute the spin^c-numbers of the line bundles L^n over $\mathbb{T}^2$, where L is the complex line bundle defined in Exercise 1.10, and the torus $\mathbb{T}^2$ is given its standard, product spin^c-structure (equivalently, from its complex structure.)

REMARK 11.13. Let X^g be a compact Riemann surface of genus g. The complex structure on X^g gives a spin^c-structure, and one way of stating the Riemann-Roch Theorem is that
$$\mathrm{index}_{\mathrm{spin}^c}(X^g) = 1 - g.$$
The Euler characteristic of such a surface is $2 - 2g$, so for these relatively simple examples, the spin^c-index turns out to be $\frac{1}{2}$ of the Euler characteristic of X (and equals the Euler characteristic of the Dolbeault complex of X.)

The construction of spin^c-numbers is an important first step towards devising a *homology* theory for (locally compact Hausdorff) spaces which is *dual* to K-theory. This geometric model of K-*homology*, denoted $\mathrm{K}_*(X)$, is due to Baum and Douglas. A second, purely analytic model of the same theory, is due to Atiyah and Kasparov; the equality of the two theories is one way of formulating the Index Theorem.

We give a brief description of this 'Baum-Douglas theory,' before discussing the more general framework of K-theory correspondences, in the next section.

DEFINITION 11.14. A *Baum-Douglas cycle for X* is a triple (M, ξ, b) where M is a smooth spin^c-manifold, $\xi \in \mathrm{K}^*(M)$ is a K-theory class, and $b\colon M \to X$ is a map.
The cycle is *proper* if b is a proper map.

Say that a Baum-Douglas cycle has *dimension d* if $\deg(\xi) + \dim M = d \bmod 2$, where $\deg(\xi)$ is the degree of the K-theory class (mod 2.)

A d-dimensional Baum-Douglas cycle for X determines a group homomorphism

$$\mathrm{K}^{-d}(X) \to \mathbb{Z}, \quad \eta \mapsto \mathrm{index}_{\mathrm{spin}^c}(f^*\eta \cdot \xi). \tag{11.9}$$

by combining

- Ring structure on topological K-theory,
- spin^c-numbers, and
- ordinary functoriality for K-theory

This homomorphism turns out to remain unchanged if a Baum-Douglas cycle is replaced by either a *bordant* one, or if it is *Thom modified* by a K-oriented vector bundle, as we now explain.

LEMMA 11.15. *Let X be a* spin^c *manifold-with-boundary ∂X. Then ∂X has a canonical* spin^c*-structure.*

PROOF. Let ν be the normal bundle to the boundary. We have the canonical vector bundle isomorphism

$$T(\partial X) \oplus \nu \cong TX|_{\partial X}$$

of vector bundles over ∂X. The normal bundle ν is trivial, and hence K-orientable, whence by the 2-out-of-3 Lemma, K-orientability of $T(\partial X)$ follows from that of $TX|_{\partial X}$.

$\square$

We call the spin^c-structure on ∂X specified in the proof of the Lemma, the *boundary* spin^c-*structure*.

DEFINITION 11.16. A *bordism* of Baum-Douglas cycles for X consists of

a) A smooth manifold-with-boundary Z, equipped with a spin^c-structure,
b) a K-theory class $\xi \in \mathrm{K}^*(Z)$,
c) a map $b\colon Z \to X$.

The *boundary* of such a Baum-Douglas bordism is the Baum-Douglas cycle consisting of ∂Z, equipped with its boundary spin^c-structure, and the restriction $i^*(\xi) \in \mathrm{K}^*(\partial Z)$ of the class ξ to the boundary ($i\colon \partial Z \to Z$ denoting the inclusion.)

Suppose that (M, ξ, b) is a Baum-Douglas cycle for X. By reversing the spin^c-structure on M, we obtain another spin^c-manifold $-M$. This new spin^c-manifold is the same, as a manifold, as the old one, and so carries the map b and class ξ.

EXERCISE 11.17. Prove that $(M \sqcup -M, \xi \sqcup \xi, b \sqcup b)$ is the boundary of the Baum-Douglas bordism $(M \times [0,1], \mathrm{pr}_M^*(\xi), b \circ \mathrm{pr}_M)$, where $M \times [0,1]$ is given the product spin^c-structure.

DEFINITION 11.18. Two Baum-Douglas cycles (M_i, ξ_i, b_i) for X are *bordant* if their difference $(M_1 \sqcup -M_2, \xi_1 \sqcup \xi_2, b_1 \sqcup b_2)$ is a boundary of a Baum-Douglas bordism.

DEFINITION 11.19. The *Thom modification* of a Baum-Douglas cycle (M, ξ, b) for X, using a K-oriented vector bundle $\pi\colon V \to M$ over M, is the Baum-Douglas cycle

$$(V, \pi^*(\xi) \cdot \tau_V, b \circ \pi)$$

for X, where V is given the spin^c-structure discussed above, and $\tau_V \in \mathrm{K}^*(V)$ is the Thom class of V.

Note that Thom modification results in an improper cycle.

THEOREM 11.20. *The homomorphism* (11.9) *defined by a Baum-Douglas cycle* (M, ξ, b) *for* X *does not change if the cycle is replaced by a bordant cycle, or if the cycle is Thom modified by a* K*-oriented vector bundle over* M*.*

Based on these results, Baum and Douglas define a K-*homology* functor on compact spaces X by defining $\mathrm{K}_j(X)$ to be the abelian group with a generator for every Baum-Douglas cycle, and relations given by

$$(M, \xi, b) + (M', \xi', b') = (M \sqcup M', \xi \sqcup \xi', b \sqcup b'). \tag{11.10}$$

$$\partial(W, \xi, b) = \emptyset, \tag{11.11}$$

$$(M, \xi, b) = (V, \pi^*(\xi) \cdot \tau_V, b \circ \pi), \tag{11.12}$$

and

$$(M, \xi, b) + (M, \xi', b) = (M, \xi + \xi', 0), \tag{11.13}$$

where $\pi\colon V \to X$ is a K-oriented vector bundle over M, and (W, ξ, b) is a Baum-Douglas cycle with boundary.

EXERCISE 11.21. Prove that $-(M, \xi, b) = (-M, \xi, b)$ in the group $\mathrm{K}_*(X)$, for any Baum-Douglas cycle for X (see Exercise 11.17).

This geometric K-homology group determines a pairing

$$\mathrm{K}_*(X) \times \mathrm{K}^*(X) \to \mathbb{Z},$$

by the formula (11.9).

Kasparov, based on an idea of Atiyah, proposed a different, and entirely *analytic* description of $\mathrm{K}_*(X)$, which adapts (unlike the Baum-Douglas model) to the category of (possibly noncommutative) C*-algebras. We will discuss Kasparov's model briefly at the end of the book.

12. Correspondences and KK-theory for manifolds

We now combine all of the different types of functorialities we have been developing for K-theory into a single extremely convenient definition – the notion of a *correspondence*. The concept is due to Connes and Skandalis. It is ufficiently general to contain both the examples of the 'K-theory germs' around submanifolds we have discussed earlier, and the Baum-Douglas cycles of the previous chapter.

Furthermore:

a) Equivalence classes of correspondences from X to Y form a $\mathbb{Z}/2$-graded abelian group
$$\mathrm{KK}_*(X, Y).$$
b) There is a bilinear composition operation, for any smooth manifolds X, Y, Z,
$$\mathrm{KK}_*(X, Y) \times \mathrm{KK}_*(Y, Z) \to \mathrm{KK}_*(X, Z)$$
behaving naturally with respect to the gradings.

c) The composition in b) determines a ($\mathbb{Z}/2$-graded) additive category, with objects smooth manifolds, and morphisms $X \to Y$ the elements of $\mathrm{KK}_*(X, Y)$,
d) $\mathrm{KK}_*(\mathrm{pt}, X) \cong \mathrm{K}^*(X)$ and $\mathrm{KK}_*(X, \mathrm{pt}) \cong \mathrm{K}_*(X)$, for any smooth manifold X, where $\mathrm{K}_*(X)$ is the Baum-Douglas K-homology group defined in the previous section.

As a consequence of d), a correspondence – call it α – from X to Y (where X and Y are smooth manifolds) determines a group homomorphism

$$\mathrm{K}^*(X) \to \mathrm{K}^*(Y),$$

by

$$\mathrm{K}^*(X) \cong \mathrm{KK}_*(\mathrm{pt}, X) \xrightarrow{\alpha} \mathrm{KK}_*(\mathrm{pt}, Y) \cong \mathrm{K}^*(Y).$$

where the map in the middle is the operation of composition of morphisms with α. We now define these 'correspondences.'

Definition 12.1. Let X and Y be smooth manifolds. A K-*theory correspondence from* X *to* Y consists of the following 4 pieces of data.

a) A smooth manifold M.
b) A smooth map $b\colon M \to X$.
c) A K-theory class $\xi \in \mathrm{K}^*(M)$, for some j.
d) A smooth, K-oriented map $f\colon M \to Y$.

We often designate a correspondence diagrammatically in the form

$$X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y. \tag{12.1}$$

The *degree* of the correspondence (12.1) is $\dim(M) - \dim(X) + j \bmod 2$, if this is well-defined.

The *sum* of two correspondences is their disjoint union:

$$(M_1, b_1, f_1, \xi_1) + (M_2, b_2, f_2, \xi_2) := (M_1 \sqcup M_2, b_1 \sqcup b_2, f_1 \sqcup f_2, \xi_1 \sqcup \xi_2).$$

This is is well-defined, associative, and commutative up to isomorphism. The empty correspondence with $M = \emptyset$ acts as zero. Any correspondence decomposes uniquely as a sum of correspondences of degrees $j \in \mathbb{Z}$; this is intuitively clear, by decomposing M into the (finite set of) disjoint subsets of different dimensions.

Example 12.2. If $\xi \in \mathrm{K}^*(X)$ we can use it to build a a correspondence from a point to X by

$$\mathrm{pt} \leftarrow (X, \xi) \to X$$

of degree $-j$, or to define a correspnodence from X to X by

$$X \xleftarrow{\mathrm{id}} (X, \xi) \xrightarrow{\mathrm{id}} X,$$

also of degree $-j$.

Example 12.3. (K-theory germs). Suppose X is a smooth manifold of dimension n and $N \subset X$ is a closed submanifold of dimension d whose normal bundle ν is K-oriented. Consider the inclusion

$$f\colon N \to X.$$

We have

$$T_f := TN \oplus f^*(TX) \cong TN \oplus TN \oplus \nu,$$

so that by the 2-out-of-3 result, a K-orientation on ν induces one on T_f. So T_f is a K-oriented map, and for any $\xi \in \mathrm{K}^*(N)$, one gets a correspondence

$$\mathrm{pt} \leftarrow (N, \xi) \xrightarrow{f} X \tag{12.2}$$

is a K-theory correspondence from a point to X of degree $d - n - j$ of N in X.

EXAMPLE 12.4. (Baum-Douglas cycles) A Baum-Douglas cycle (M,ξ,b) for X, with $\xi \in \mathrm{K}^{-j}(M)$, defines a correspondence from X to a point by

$$X \xleftarrow{b} (M,\xi) \to \mathrm{pt}$$

of degree $\dim M - j$,

A *correspondence-with-boundary* is defined similarly to a correspondence, given by a diagram

$$X \xleftarrow{b} (M,\xi) \xrightarrow{f} Y, \tag{12.3}$$

where now M to be a manifold with boundary, with tangent bundle TM, and, as before, the requirement on $f\colon M \to Y$ is the same: that the vector bundle

$$T_f := TM \oplus f^*(TY)$$

over M carries a K-orientation.

The boundary of such a correspondence – call it γ – as in (12.3) is defined similarly to the way we defined the boundary of a Baum-Douglas cycle. The maps b, f restrict to ∂M and the splitting

$$TM \cong T(\partial M) \oplus \nu$$

with ν the normal bundle to the boundary, which is trivial, and the 2-out-of-3 Lemma, determines a K-orientation on the map $f|_{\partial M}\colon \partial M \to Y$.

If ∂M decomposes into disjoint subsets $\partial_0 M$ and $\partial_1 M$, then by restricting the maps b and f and the K-theory class we obtain correspondences $\partial_0\gamma$ and $\partial_1\gamma$ given by

$$X \xleftarrow{b|_{\partial_i M}} (\partial_i M, \xi|_{\partial_i M}) \xrightarrow{f|_{\partial_i M}} Y.$$

We say in this case that the correspondences $\partial_i\gamma$ are *bordant*.

REMARK 12.5. If γ is a correspondence-with-boundary as above, then one can take one of the subsets $\partial_1 M$ (say) to be the empty set., giving that, by definition, the boundary of any correspondence-with-boundary is bordant to the empty correspondence.

The *Thom modification* of a correspondence

$$X \xleftarrow{b} (M,\xi) \xrightarrow{f} Y$$

by a K-oriented vector bundle $\pi\colon V \to M$, is the correspondence

$$X \xleftarrow{b\circ\pi} (M, \pi^*(\xi)\cdot\tau_V) \xrightarrow{f\circ\pi} Y,$$

where $\tau_V \in \mathrm{K}^{-\dim V}(V)$ is the Thom class of V.

Equivalence of correspondences is the equivalence relation on the set of correspondences from X to Y generated by

a) bordism, and
b) Thom modification.

The degree of a correspondence modulo 2, does not change under any of the above steps. So the set of equivalence classes of correspondences from X to Y is naturally $\mathbb{Z}/2$-graded, by degree. Finally, if

$$X \xleftarrow{b} (M,\xi) \xrightarrow{f} Y$$

is a correspondence, call it γ, then by reversing the K-orientation on f we obtain another correspondence, call it $-\gamma$. By the same argument as in Exercise 11.17, $\gamma \sqcup \gamma$ is bordant to the empty correspondence. Therefore, the collection of equivalence classes of correspondences from X to Y has a natural (abelian) group structure.

Definition 12.6. $\mathrm{KK}_*(X,Y)$ denotes the set of equivalence classes of correspondences from X to Y. It is a $\mathbb{Z}/2$-graded abelian group under disjoint union.

Fix a correspondence, γ,

$$X \xleftarrow{b} (M,\xi) \xrightarrow{f} Y.$$

By varying the K-theory datum $\xi \in \mathrm{K}^*(M)$ we get a map

$$\mathrm{K}^*(X) \to \mathrm{KK}_*(\mathrm{pt}, X). \tag{12.4}$$

We first show that (12.4) is additive.

Lemma 12.7. *The sum (*i.e. *disjoint union) of the correspondences*

$$X \xleftarrow{b} (M,\xi_i) \xrightarrow{b} Y$$

for $i=1,2$, is bordant to the Thom modification of

$$X \xleftarrow{b} (M,\xi_1+\xi_2) \xrightarrow{f} Y$$

induced by the trivial (real) line bundle.

In particular, the map (12.4) *is a group homomorphism.*

Proof. We prove the Lemma only for the case M is a point. The general case follows by taking products of everything with M, and is left to the reader – the maps b and f are irrelevant in the argument.

Take the manifold-with-boundary $[0,1]\times\mathbb{R}$, and remove the origin. Let W be the result, an open subset of a manifold-with-boundary, and so a manifold-with-boundary, with boundary

$$\{0\}\times\mathbb{R}\setminus\{0\}\sqcup\{1\}\times\mathbb{R}.$$

We divide the boundary accordingly into two pieces, so, identifying $\mathbb{R}\setminus\{0\}$ with $\mathbb{R}\sqcup\mathbb{R}$ gives

$$\partial_0 W := \mathbb{R}\sqcup\mathbb{R},\;\; \partial_1 W := \mathbb{R}.$$

Now, the Thom modification of

$$X \xleftarrow{b} (\mathrm{pt},\xi_1+\xi_2) \xrightarrow{b} Y$$

along the trivial line bundle is

$$X \xleftarrow{b} (\mathbb{R},(\xi_1+\xi_2)\cdot\beta_{\mathbb{R}}) \xrightarrow{b} Y$$

where $\beta_{\mathbb{R}} \in \mathrm{K}^{-1}(\mathbb{R})$ is the Bott class (here $\xi_i\in\mathrm{K}^*(\mathbb{C})$, so correspond to integers, and the product is the corresponding product of the sum of these two integers and the Bott class.) We are building a bordism using the manifold-with-boundary W above, with boundary the disjoint union of the the Thom modifications of the two correspondences we wish to show equivalent. To do this, we need a class $\xi\in\mathrm{K}^{*+1}(W)$ which restricts to $(\xi_1+\xi_2)\cdot\beta_{\mathbb{R}}$ on $\partial_1 W\cong\mathbb{R}$, and to $\xi_1\sqcup\xi_2$ on $\partial_0 W=\mathbb{R}\sqcup\mathbb{R}$.

Let $i_0\colon \partial_0 W\to W$ and $i_1\colon\partial_1 W\to W$ be the inclusions. If we inspect the 6-term exact sequence associated with the closed subset $\partial_0 W\subset W$ we see, that since $W\setminus\partial_0 W$ has zero K-theory,

$$i_0^*\colon \mathrm{K}^*(W)\to\mathrm{K}^*(\partial_0 W)$$

is an isomorphism. In particular, there is a class $\xi\in\mathrm{K}^*(W)$ which restricts to $\xi_1\sqcup\xi_2\in\mathrm{K}^*(\partial_0 W)$.

On the other hand, the map

$$i_0\colon \mathbb{R}\sqcup = \partial_0 W\to W$$

induces a map

$$i_0^*\colon \mathrm{K}^*(\mathbb{R})\oplus\mathrm{K}^*(\mathbb{R})$$

it is rather easy to see that

$$i_0^*(\xi_1 \sqcup \xi_2) = i_1^*(\xi_1) + \xi_1^*(\xi_2),$$

because all three of inclusions of $\mathbb{R}$ into the pieces of ∂W under discussion here, are properly homotopic.

□

We conclude this section with two very important results, which we do not prove here, although they are not difficult.

THEOREM 12.8. *Let X be any smooth manifold. Then the map*

$$\alpha\colon \mathrm{K}^*(X) \to \mathrm{KK}_*(\mathrm{pt}, X), \quad \alpha(\xi) := \Big[\mathrm{pt} \leftarrow (X, \xi) \xrightarrow{\mathrm{id}_X} X\Big] \tag{12.5}$$

is a group isomorphism with inverse

$$\beta\colon \mathrm{KK}_*(\mathrm{pt}, X) \to \mathrm{K}^*(X), \quad \beta(\Big[\mathrm{pt} \leftarrow (M, \xi) \xrightarrow{f} X\Big]) := f!(\xi). \tag{12.6}$$

where the square brackets denote equivalence class.

More generally, let X and Y be smooth manifolds. A correspondence

$$\gamma\colon \quad X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y$$

induces a map

$$\gamma_*\colon \mathrm{K}^*(X) \to \mathrm{K}^*(Y), \ \ \gamma_*(\eta) := f!\big(b^*(\eta)\cdot \xi\big),$$

shifting degrees by the degree of γ. This map only depends on the equivalence class of γ in $\mathrm{KK}_*(X, Y)$*.*

By the theorem, we get an abelian group homomorphism

$$\mathrm{KK}_*(X, Y) \to \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{K}^*(X), \mathrm{K}^*(Y)\big). \tag{12.7}$$

We aim to sketch a proof that the KK_* has the structure of a *category*, with objects smooth manifolds, and morphisms $X \to Y$ equivalence classes of correspondences from X to Y, and for which (12.7) is a functor (to the category of abelian groups, and abelian group homomorphisms.)

Composition of correspondences

Two smooth maps

$$f\colon X \to W, \quad g\colon Y \to W$$

between smooth manifolds, are *transverse* if at every point $(x, y) \in X \times Y$ for which $f(x) = g(y)$,

$$D_x f(T_x X) + D_y g(T_y Y) = T_{f(x)} W.$$

DEFINITION 12.9. Two correspondences

$$X \xleftarrow{b'} (M, \xi) \xrightarrow{f} Y \tag{12.8}$$

and

$$Y \xleftarrow{b} (N, \eta) \xrightarrow{f'} Z \tag{12.9}$$

are *transverse* if f is transverse to b.

Proposition 12.10. *Suppose that α is a correspondence from $X \to Y$ as in* (12.8) *and β is a correspondence from Y to Z, as in* (12.9)*. If α and β are transverse, then the space*

$$M \times_Y N := \{(m,n) \in M \times N \mid f(m) = b(n)\}$$

has a smooth manifold structure for which the projection maps

$$\mathrm{pr}_N \colon M \times_Y N \to N, \quad \mathrm{pr}_M \colon M \times_Y N \to M$$

are smooth maps. Moreover, the sequence of vector bundles over $M \times_Y N$

$$0 \to T(M \times_Y N) \to \mathrm{pr}_M^*(TM) \oplus \mathrm{pr}_N^*(TN) \xrightarrow{Df - Dg} (f \circ \mathrm{pr}_M)^*(TY) \to 0 \tag{12.10}$$

is exact, where the first map is induced by the derivatives of the projections pr_N *and* pr_N*, and determines a canonical* K*-orientation on* pr_N *depending only on the* K*-orientation on f.*

Proof. The statements regarding smooth manifold structures and the exactness of the sequence (12.10) are standard results from differential topology and we omit the proofs. The exact sequence implies that

$$T(M \times_Y N) \oplus (f \circ \mathrm{pr}_M)^*(TY) \cong \mathrm{pr}_M^*(TM) \oplus \mathrm{pr}_N^*(TN). \tag{12.11}$$

Adding $\mathrm{pr}_M^*(TM) \oplus \mathrm{pr}_N^*(TN)$ to each side and identifying

$$TM \oplus TM \cong TM \otimes \mathbb{C}, \;\; TN \oplus TN \cong TN \otimes \mathbb{C}$$

gives

$$T_{\mathrm{pr}_N} \oplus \mathrm{pr}_M^*(T_f) \cong \mathrm{pr}_M^*(TM \otimes \mathbb{C}) \oplus \mathrm{pr}_N^*(TN \otimes \mathbb{C}) \tag{12.12}$$

and the 2-out-of-3 result now determines a K-orientation on T_{pr_N}, as claimed.

□

Let α and β be transverse correspondences as in Proposition 12.10. With pr_N K-oriented as in the Proposition, and K-orienting $g \circ \mathrm{pr}_N \colon M \times_Y N \to Z$ as a composition of K-oriented maps, we obtain a correspondence from X to Z:

$$X \xleftarrow{b' \circ \mathrm{pr}_M} (M \times_Y N, \mathrm{pr}_M^*(\xi) \cdot \mathrm{pr}_N^*(\eta)) \xrightarrow{g \circ \mathrm{pr}_N} Z. \tag{12.13}$$

Definition 12.11. The *intersection product* of the correspondences (12.8) and (12.9), provided that they are transverse, is the correspondence (12.13).

Lemma 12.12. *Any correspondence from X to Y is equivalent to a correspondence*

$$X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y$$

from X to Y for which the map f is a submersion.

The lemma shows that any two correspondences are equivalent to a pair which are transverse, and for which therefore their intersection product is defined. As a consequence, we obtain the following

Theorem 12.13. *For any smooth manifolds X, Y, Z, there is a well-defined $\mathbb{Z}$-bilinear map*

$$\mathrm{KK}_*(X,Y) \times \mathrm{KK}_*(Y,Z) \to \mathrm{KK}_*(X,Z), \qquad (\alpha, \beta) \mapsto \alpha \otimes_Y \beta \;, \tag{12.14}$$

which maps a pair of classes represented by a pair of transverse correspondences, to (the class of) their intersection product.

a) *If $\alpha \in \mathrm{KK}_i(X,Y), \beta \in \mathrm{KK}_j(Y,Z)$ then $\alpha \otimes_Y \beta \in \mathrm{KK}_{i+j}(X,Z)$.*

b) *The operation* $\otimes$ (12.14) *gives* KK_* *the structure of a* $\mathbb{Z}/2$*-graded, additive category, with objects smooth manifolds, and morphisms* $X \to Y$ *the elements of* $\mathrm{KK}_*(X, Y)$*, and composition of morphisms given by the intersection product* (12.14) *(that is,* $g \circ f := f \otimes_B g$ *if* $f \in \mathrm{KK}(A, B)$ *and* $g \in \mathrm{KK}(B, C)$*.)*

c) *For any* X*, the correspondence*

$$X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$$

acts as the identity morphism $X \to X$ *in* $\mathrm{KK}_0(X, X) \subset \mathrm{KK}_*(X, X)$*.*

d) *The map* (12.7) *is a functor from the category* KK *to the category of abelian groups and abelian group homomorphisms, mapping an object* X *to the group* $\mathrm{K}^*(X)$*, and a morphism* $\alpha \in \mathrm{KK}_*(X, Y)$ *to the group homomorphism* $\alpha_* \colon \mathrm{K}^*(X) \to \mathrm{K}^*(Y)$ *defined by* (12.7)*.*

Exercise 12.14. Suppose

$$X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y \tag{12.15}$$

is a correspondence from X to Y, that $U \subset M$ is an open subset, $\xi' \in \mathrm{K}^*(U)$ a K-theory class such that

$$i_U!(\xi') = \xi,$$

where $i_U! \colon \mathrm{K}^*(U) \to \mathrm{K}^*(M)$ is the homomorphism induced by the open inclusion $i_U \colon U \to M$. Then the correspondences (12.15) and

$$X \xleftarrow{b|_U} (U, \xi') \xrightarrow{f|_U} Y$$

are bordant, where $f|_U$ is K-oriented as the restriction of f.

(*Hint.* $M \times [0, 1]$ is a manifold-with-boundary, and removing the closed set $M \setminus U \times \{0\}$ from it results in an open subset W of a manifold-with-boundary, and hence has itself the structure of a manifold-with-boundary. (The boundary of W is $U \sqcup M$.) The projection map $\mathrm{pr}_M \colon M \times [0, 1] \to M$ restricts to a proper map $W \to M$ and pulls back ξ to a class $\mathrm{pr}^*(\xi) \in \mathrm{K}^*(W)$ whose restrictions to the two boundary components $U \times \{0\}$ and $M \times \{1\}$ of W are the classes ξ' and ξ respectively.)

For example, if $\beta_U \in \mathrm{K}^0(U)$ is a model of the Bott element with support in a neighbourhood U of the origin (as in Example 4.21), so that $i_U!(\beta_U) = \beta_{\mathbb{R}^2}$, then the correspondences

$$\mathrm{pt} \leftarrow (\mathbb{R}^2, \beta_{\mathbb{R}^2}) \xrightarrow{\mathrm{id}} \mathbb{R}^2$$

and

$$\mathrm{pt} \leftarrow (U, \beta_U) \xrightarrow{i_U} \mathbb{R}^2$$

are bordant.

Point correspondences and torsion in $\mathrm{K}^*(\mathbb{RP}^2)$

We continue from Example 11.6, and show how some very simple manipulations with correspondences can give a simple geometric explanation of torsion appearing in certain simple, non-orientable manifolds.

Let M be a smooth Riemannian manifold, $p \colon \mathrm{pt} \to M$ the inclusion of the one-point space in M, $\mathbf{e}_p$ a frame at $p \in M$, and

$$(p, \mathbf{e}_p)! \in \mathrm{K}^{-n}(M)$$

be the class of the corresponding wrong-way map pt $\to M$, defined in Example 11.6. As a correspondence, it is

$$\mathrm{pt} \leftarrow (\mathrm{pt}, [1]) \xrightarrow{p} M,$$

where the map $p\colon \mathrm{pt} \to M$ is K-oriented by the frame.

EXERCISE 12.15. Suppose $\mathbf{e}'_p$ is another orthonormal frame for T_pM. If $\mathbf{e}_p$ and $\mathbf{e}'_p$ are in the same orientation class for $\mathbb{R}^n$, then

$$(p, \mathbf{e}_p)! = (p, \mathbf{e}'_p)! = \in \mathrm{K}^{-n}(M).$$

If they are in the opposite orientation class, then

$$(p, \mathbf{e}_p)! = -(p, \mathbf{e}'_p)! = \in \mathrm{K}^{-n}(M).$$

Now suppose that $\gamma\colon [0,1] \to M$ is a smooth path, and that $\{\mathbf{e}_t\}_{t\in[0,1]}$ is a moving frame along the path (it could be for example associated with a parallel transport along the path, using the Levi-Civita connection.)

We then obtain a K-oriented map-with-boundary

$$\gamma\colon [0,1] \to M \tag{12.16}$$

by K-orienting the vector bundle $\gamma^*(TM)$ over $[0,1]$, with fibre $T_{\gamma(t)}(M)$ at $t \in [01,]$, using the isometric isomorphism $\mathbb{R}^n \xrightarrow{\cong} T_{\gamma(t)}M$ induced from the frame $\mathbf{e}_t$ and the standard K-orientation on $\mathbb{R}^n$.

EXERCISE 12.16. In the above notation:

a) Check that the K-orientation on the boundary

$$\mathrm{pt} \sqcup \mathrm{pt} \to M \tag{12.17}$$

of the wrong-way-map-with-boundary (12.16) is obtained by K-orienting the map

$$\mathrm{pt} \sqcup \mathrm{pt} \to M$$

of the 2-point space to M, by using the standard K-orientation on $\mathbb{R}^n$, combined with, firstly, the obvious isomorphism $(\mathrm{pt} \sqcup \mathrm{pt})^*(TM) = T_pM \oplus T_qM$, and then the isomorphisms $T_pM \cong \mathbb{R}^n$ and $T_qM \cong \mathbb{R}^n$ using the two frames $\mathbf{e}_0$ and $\mathbf{e}_1$ respectively,.

b) Deduce that

$$(p, \mathbf{e}_0)! = (p, \mathbf{e}_1)! \in \mathrm{K}^{-n}(M). \tag{12.18}$$

c) Suppose that M is a smooth Riemannian manifold and that $\alpha, \beta\colon [0,1] \to M$ are two smooth paths between a pair of points p, q. Assume that the parallel transport maps $\tau_\alpha, \tau_\beta\colon T_pM \to T_qM$ satisfy:

$$\tau_\beta \circ \tau_\alpha^{-1}\colon T_pM \to T_pM$$

is orientation-reversing.

Prove that

$$2(p, \mathbf{e})! = 0 \in \mathrm{K}^{-n}(M).$$

(*Hint* Fix a frame $\mathbf{e}_p$ for T_pM. Let $\mathbf{e}'_q$ and $\mathbf{e}''_q$ be the frames for T_qM obtaining by parallel transporting the frame $\mathbf{e}_p$ along α and β respectively. Observe, from the above discussion, that the three equations

$$(p, \mathbf{e}_p)! = (q, \mathbf{e}'_q)!, \quad (p, \mathbf{e}_p)! = (q, \mathbf{e}''_q)!, \quad \text{and} \quad (q, \mathbf{e}'_q)! = -(q, \mathbf{e}''_q)!$$

hold in $\mathrm{K}^{-n}(M)$.)

d) In particular, c) applies if parallel transport $\tau_\gamma\colon T_pM \to T_pM$ around some smooth loop at p is orientation-reversing, then $(p,\mathbf{e})! \in \mathrm{K}^{-n}(M)$ is 2-torsion, for any frame $\mathbf{e}$ at p.

   Show that the hypotheses of c) are met in the case of the open Möbius band. In other words, the 'K-theory germ' of a point in the open Möbius band, is 2-torsion.

e) Prove the analogue of d) for $M := \mathbb{RP}^2$.

CHAPTER 6

K-theory for C*-algebras

1. K-theory for C*-algebras and basic properties of the K-theory functor

We have already noted (Swan's Theorem 2.8) that for any second-countable compact Hausdorff space X, there is a bijective correspondence

$$\{\text{iso. classes of vector bundles over } X\} \cong \{\text{iso. classes of f.g.p. modules over } C(X)\}.$$

obtained by mapping a vector bundle E to the right $C(X)$-module $\Gamma(E)$ of its sections.

This immediately suggests how to define K-theory of a possibly noncommutative C*-algebra, in such a way as to generalize topological K-theory.

If A is a a unital ring, and L_1 and L_2 are f.g.p. modules over A, their sum $L_1 \oplus L_2$, as a right A-module, is also f.g.p. Obviously $L_1 \oplus L_2 \cong L_2 \oplus L_1$ as A-modules, and this addition operation is well-defined on isomorphism classes of f.g.p. right A-modules. We write $[L]$ for the isomorphism class of L.

The collection of isomorphism classes of finitely generated projective modules over A defines a commutative semigroup $\mathcal{M}(A)$. If $A = C(X)$, this semigroup is in 1-1 correspondence with the semigroup $\mathrm{Vect}(X)$ of isomorphism classes of vector bundles over X, as we have already proved.

We may think of f.g.p. modules as kinds of 'noncommutative vector bundles,' and we correspondingly define

Definition 1.1. If A is a unital ring, and in particular, if A is a unital C*-algebra, $\mathrm{K}_0(A)$ is the Grothendiek completion of the semi-group $\mathcal{M}(A)$ of finitely generated projective (f.g.p.) right A-modules.

An element of $\mathrm{K}_0(A)$ is a formal difference $[L] - [M]$, of isomorphism classes of right f.g.p. A-modules L, M.

Two such formal differences $[L_1] - [L_2]$ and $[L_1'] - [L_2']$ are equal in $\mathrm{K}_0(A)$ if there exists an f.g.p. module L such that $L_1 \oplus L_2' \oplus L \cong L_1' \oplus L_2 \oplus L$.

Example 1.2. $\mathcal{M}(\mathbb{C}) \cong \mathbb{N}$ and $\mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}$. Indeed, an f.g.p. module $\mathcal{E}$ over $\mathbb{C}$ is exactly the same as a finite-dimensional complex vector space, and isomorphism of f.g.p. modules over $\mathbb{C}$ corresponds to linear isomorphism of $\mathbb{C}$-vector spaces. Since a vector space over $\mathbb{C}$, or any field, is determined up to isomorphism by its dimension, $\mathcal{M}(\mathbb{C}) \cong \mathbb{N}$ by the map $\mathcal{E} \mapsto \dim_{\mathbb{C}}(\mathcal{E})$. Taking Grothendiek completions we get $\mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}$ by the map $[\mathcal{E}_1] - [\mathcal{E}_2] \mapsto \dim_{\mathbb{C}}(\mathcal{E}_1) - \dim_{\mathbb{C}}(\mathcal{E}_2)$.

Example 1.3. Let $\pi\colon H \to \mathbb{CP}^1$ be the Hopf bundle. Then by the very definition, $H = \mathrm{Im}(P)$ where $p\colon \mathbb{CP}^1 \to M_2(\mathbb{C})$ is the projection-valued function $p\colon \mathbb{CP}^1 \to M_2(\mathbb{C})$ mapping a line $L \subset \mathbb{C}^2$ to orthogonal projection pr_L onto that line. In terms of homogeneous coordinates on $\mathbb{CP}^1$,

$$P([z,w]) = \frac{1}{|z|^2 + |w|^2} \begin{bmatrix} |z|^2 & \bar{w}z \\ \bar{z}w & |w|^2 \end{bmatrix}.$$

We can restrict p to $\mathbb{C} \subset \mathbb{CP}^1$, *i.e.* set

$$p\colon \mathbb{C} \to M_2(\mathbb{C}), \quad p(z) = p([z,1]) = \frac{1}{|z|^2+1}\begin{bmatrix} |z|^2 & z \\ \bar{z} & 1 \end{bmatrix}.$$

on $\mathbb{C}$. Note that

$$\lim_{z\to\infty} p(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

that is, p on $\mathbb{CP}^1$ takes the value $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ at 'infinity' – the point with homogeneous coordinates $[1,0]$.

The class

$$\beta_{\mathbb{R}^2} := [p] - [1] \in \mathrm{K}_0(\mathbb{R}^2) \tag{1.1}$$

is the *Bott element* for $\mathbb{R}^2$, already discussed in the context of topological K-theory. Bott Periodicity implies that $\mathrm{K}^0(\mathbb{R}^2)$ is infinite cyclic with generator $\beta_{\mathbb{R}^2}$

If A is a unital ring, we let

$$M_\infty(A) := \bigcup_{n=1}^{\infty} M_n(A),$$

the *-algebra of infinite $\mathbb{N}$-by-$\mathbb{N}$ matrices with entries in A, which have only finitely many nonzero terms. Recall that two idempotents $p, q \in M_n(A)$ are *algebraically equivalent* if there exists $u, v \in M_n(A)$ elements such that $uv = p$, $vu = q$. We have already seen that if A is a unital C*-algebra, then any idempotent is algebraically equivalent to a projection, and two projections are algebraically equivalent if and only if they are *Murray-von-Neumann* equivalent: *i.e.* if and only if there is a partial isometry $u \in M_n(A)$ such that $uu^* = p, u^*u = q$.

If A is a unital ring, we set $\mathcal{P}(A)$ to be the collection of algebraic equivalence classes of idempotents in $M_\infty(A)$. If A is a unital C*-algebra, this is equivalent to looking at Murray-von-Neumann equivalence classes of projections, and we normally take this latter picture as defining $\mathcal{P}(A)$, when A is a C*-algebra.

Exercise 1.4. Let $p, q \in M_n(A)$ be projections, where A is a unital C*-algebra. Show that

a) The partial isometry
$$u := \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}$$
gives a Murray-von-Neumann equivalence between the projections $\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ and $\begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix}$.

b) If $pq = qp = 0 \in M_n(A)$, so that $p + q$ is a projection, then the partial isometry
$$u := \begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix}$$
gives a Murray-von-Neumann equivalence between $\begin{bmatrix} p+q & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$.

c) Prove that if $p \sim p' \in M_n(A)$ and $q \sim q' \in M_n(A)$, then
$$\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \sim \begin{bmatrix} p' & 0 \\ 0 & q' \end{bmatrix} \in M_{2n}(A),$$
where $\sim$ means Murray-von-Neumann equivalence.

This shows that $\sim$-equivalence classes of projections in $\mathcal{P}(A)$ can be added.

The results of the Exercise, with appropriate adjustments to the statements, still hold for unital rings, and algebraic equivalence classes of idempotents. We leave this generalization to the reader to check.

From the exercise, the collection $\mathcal{P}(A)$ has the structure of a commutative semigroup under the addition operation

$$[p] + [q] := \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix},$$

corresponding to the direct sum operation on f.g.p. modules.

PROPOSITION 1.5. *If A is a unital C*-algebra, then the map sending the Murray-von-Neumann equivalence class $[p]$ of a projection $p \in M_n(A)$ to the isomorphism class of the f.g.p. A-module pA^n in $\mathrm{K}_0(A)$ defines an isomorphism*

$$\mathcal{P}(A) \cong \mathcal{M}(A).$$

Hence $\mathrm{K}_0(A)$ is naturally isomorphic to the Grothendiek completion of the abelian semigroup $\mathcal{P}(A)$.

EXERCISE 1.6. Let p and q be projections in $M_\infty(A)$ such that $[p] = [q] \in \mathrm{K}_0(A)$. Then for some k, $p \oplus 1_k$ is Murray-von-Neumann equivalent to $q \oplus 1_k$, as projections in $M_\infty(A)$, where 1_k is the k-by-k identity matrix.

(*Hint.* By the definitions, $p \oplus e \cong q \oplus e$ for some projection e. Now $e \oplus 1 - e$ is Murray-von-Neumann equivalent to 1_k for suitable k, by Exercise 1.4 b), and the result follows.)

Due to Swan's Theorem, K_0 for unital C*-algebras as described above generalizes topological K-theory for compact spaces:

PROPOSITION 1.7. $\mathrm{K}^0(X) \cong \mathrm{K}_0\big(C(X)\big)$ *for any second-countable compact Hausdorff space X.*

PROOF. The semigroup $\mathrm{Vect}(X)$ of isomorphism classes of vector bundles is isomorphic to the semigroup of isomorphism classes of f.g.p. modules by mapping the bundle $E \to X$ to the f.g.p. module $\Gamma(E)$ of sections of E. Lemma 2.7 proved that $E \cong E'$ if and only if $\Gamma(E) \cong \Gamma(E')$. □

We next compute $\mathrm{K}_0\big(\mathbb{B}(H)\big)$, for any Hilbert space H.

LEMMA 1.8. *If H is a Hilbert space, and p, q are projections in $\mathbb{B}(H)$, then p and q are Murray-von-Neumann equivalent in $\mathbb{B}(H)$ if and only if they have the same rank as operators on H.*

PROOF. if p and q have the same rank, their ranges are isomorphic as Hilbert spaces. Let $u\colon pH \to qH$ be such a unitary isomorphism, and extend it to $H = pH \oplus (pH)^\perp$ by setting it equal to the zero map on $(pH)^\perp$. Then $uu^* = q$, $u^*u = p$. □

COROLLARY 1.9. *If H is finite-dimensional, then $\mathrm{K}_0\big(\mathbb{B}(H)\big) \cong \mathbb{Z}$. If H is infinite-dimensional, $\mathrm{K}_0\big(\mathbb{B}(H)\big) = \{0\}$.*

PROOF. First let H be infinite-dimensional. Let $p \in \mathbb{B}(H)$. If p has infinite rank, then apply the lemma to the projections $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$ and $\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$ to get that they are Murray-von-Neumann equivalent in $\mathbb{B}(H \oplus H) \cong M_2\big(\mathbb{B}(H)\big)$. Hence $[p] + [p] = [p] \in \mathrm{K}_0$. Hence $[p] = 0$.

If H is finite-dimensional, $\mathbb{B}(H) \cong M_n(\mathbb{C})$ for some n, and the result follows from our computation of $\mathrm{K}_0(\mathbb{C})$ and Lemma **??**. □

To be more concrete, for each n, $\mathrm{K}_0\big(M_n(\mathbb{C})\big) \cong \mathbb{Z}$ by an isomorphism sending the generator $1 \in \mathbb{Z}$ to the class $[p] \in \mathrm{K}_0(M_n(\mathbb{C}))$ of any projection $p \in M_n(\mathbb{C})$ of rank one.

We now continue with the general theory.

If $\alpha\colon A \to B$ is a *-homomorphism, then it induces, by applying α pointwise to the entries of a matrix, a C*-algebra homomorphism $M_n(A) \to M_n(B)$ for all n and a *-algebra homomorphism $M_\infty(A) \to M_\infty(B)$. Such a *-homomorphism maps projections to projections, and maps partial isometries to partial isometries, and hence induces a canonical semigroup homomorphism

$$\alpha_*\colon \mathcal{P}(A) \to \mathcal{P}(B).$$

which then determines a group homomorphism

$$\alpha_*\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B).$$

It is obvious from the definitions that if $\beta\colon B \to C$ is another *-homomorphism, then

$$(\beta \circ \alpha)_* = \beta_* \circ \alpha_*\colon \mathrm{K}_0(A) \to \mathrm{K}_0(C).$$

Hence the assignment

$$A \to \mathrm{K}_0(A), \quad \alpha \mapsto \alpha_*$$

defines a *functor* from the category of unital C*-algebras and *-homomorphisms, to the category of abelian groups, and abelian group homomorphisms.

EXERCISE 1.10. Let A be a unital C*-algebra and $p \in A$ be a projection. Then p determines a *-homomorphism $\alpha_p\colon \mathbb{C} \to A$. Such a *-homomorphism determines a group homomorphism $(\alpha_p)_*\colon \mathrm{K}_0(\mathbb{C}) \to \mathrm{K}_0(A)$. Check that $(\alpha_p)_*([1]) = [p]$. Formulate and prove a more general statement, where p is allowed to be in a matrix algebra $M_n(A)$ over A.

EXERCISE 1.11. . If A is any unital C*-algebra, then the inclusion

$$A \to M_n(A), \quad i(a) := \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

of A as a corner of $M_n(A)$, determines an isomorphism

$$i_*\colon \mathrm{K}_0(A) \cong \mathrm{K}_0\big(M_n(A)\big).$$

EXERCISE 1.12. This exercise describes functoriality in terms of f.g.p. modules. Let A and B be C*-algebras and $\alpha\colon A \to B$ be a *-homomorphism.

a) If $\mathcal{E}_A$ is a f.g.p. module over A, then the algebraic tensor product

$$\alpha_*(\mathcal{E}_A) := \mathcal{E}_A \otimes_A B$$

of the right A-module $\mathcal{E}_A$ over the homomorphism $\alpha\colon A \to B$, with the left B-module B, gives a f.g.p. module over B.

b) If $\mathcal{E}_A$ and $\mathcal{E}'_A$ are isomorphic f.g.p. A-modules, then $\alpha_*(\mathcal{E}_A)$ and $\alpha_*(\mathcal{E}'_A)$ are isomorphic f.g.p. B-modules.

c) If $\mathcal{E}_A \cong pA^n$ for a projection $p \in M_n(A)$, then the pushed-forward module $\alpha_*(\mathcal{E}_A)$ is isomorphic to $\alpha(p)B^n$.

We are going to study the properties of the K_0-functor. Firstly, it is clearly *additive* in the following sense. Suppose A, B are unital C*-algebras. Firstly,

$$M_n(A \oplus B) \cong M_n(A) \oplus M_n(B), \quad \text{and} \quad M_\infty(A \oplus B) \cong M_\infty(A) \oplus M_\infty(B),$$

as *-algebras. Since *-algebra isomorphisms send projections to projections, and partial isometries to partial isometries, it follows that

$$\mathcal{P}(A \oplus B) \cong \mathcal{P}(A) \oplus \mathcal{P}(B)$$

as semigroups, and hence

$$\mathrm{K}_0(A \oplus B) \cong \mathrm{K}_0(A) \oplus \mathrm{K}_0(B).$$

More exactly:

LEMMA 1.13. *Let A, B be unital C*-algebras, $i_A\colon A \to A \oplus B$ and $i_B\colon B \to A \oplus B$, the inclusions and $\mathrm{pr}_A\colon A \oplus B \to A$ and $\mathrm{pr}_B\colon A \oplus B \to B$ the projections. Then*

$$(i_A)_* + (i_B)_*\colon \mathrm{K}_0(A) \oplus \mathrm{K}_0(B) \to \mathrm{K}_0(A \oplus B)$$

is an isomorphism of groups, with inverse

$$(\mathrm{pr}_A)_* \oplus (\mathrm{pr}_B)_*\colon \mathrm{K}_0(A \oplus B) \to \mathrm{K}_0(A) \oplus \mathrm{K}_0(B).$$

We now extend the K_0-functor to (possibly) non-unital C*-algebras. Let A be any C*-algebra, unital or not, and A^+ its unitization. As a vector space $A^+ = A \oplus \mathbb{C}$, but with algebra multiplication and adjoint

$$(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu), \quad (a, \lambda)^* = (a^*, \bar{\lambda})$$

and the supremum norm makes A^+ a C*-algebra.

If A is already unital, then the map

$$\gamma\colon A^+ \to A \oplus \mathbb{C}, \quad \gamma(a, \lambda) = (a + \lambda \cdot 1, \lambda) \tag{1.2}$$

is an isomorphism with the direct sum C*-algebra $A \oplus \mathbb{C}$.

Let

$$\epsilon\colon A^+ \to \mathbb{C}, \quad \epsilon(a, \lambda) = \lambda.$$

Then $\ker(\epsilon) = A$, embedded in the first copy of A^+. By functoriality of K_0 for unital C*-algebras, we obtain a group homomorphism

$$\epsilon_*\colon \mathrm{K}_0(A^+) \to \mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}.$$

DEFINITION 1.14. If A is any C*-algebra, unital or not, we define $\mathrm{K}_0(A)$ to be the kernel of the homomorphism $\epsilon_*\colon \mathrm{K}_0(A^+) \to \mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}$.

LEMMA 1.15. *Definitions 1.1 and 1.14 agree for unital C*-algebras.*

PROOF. The isomorphism (1.2) identifies $\epsilon\colon A^+ \to \mathbb{C}$ with the projection $\mathrm{pr}_{\mathbb{C}}\colon A \oplus \mathbb{C} \to \mathbb{C}$. Since K_0 is additive (for unital C*-algebras), $\mathrm{K}_0(A \oplus \mathbb{C}) \cong \mathrm{K}_0(A) \oplus \mathrm{K}_0(\mathbb{C})$ and under this isomorphism, $(\mathrm{pr}_{\mathbb{C}})_*$ becomes the second projection map of groups (see Lemma 1.13). The kernel of the second projection map $\mathrm{K}_0(A) \oplus \mathrm{K}_0(\mathbb{C}) \to \mathrm{K}_0(\mathbb{C})$ is obviously $\mathrm{K}_0(A)$.

□

REMARK 1.16. Suppose A is non-unital, let A^+ be its unitization, $\epsilon\colon A \to \mathbb{C}$ the augmentation.

Suppose that $\mathcal{E}$ is a f.g.p. module over A^+, with class $[\mathcal{E}] \in \mathrm{K}_0(A^+)$. Let $[1] \in \mathrm{K}_0(A^+)$ be the class of the trivial right A-module given by A itself. And suppose that

$$[\mathcal{E}] - m[1] \in \mathrm{K}_0(A^+)$$

is in the kernel of

$$\epsilon_*\colon \mathrm{K}_0(A^+) \to \mathrm{K}_0(\mathbb{C}) = \mathbb{Z}.$$

Since

$$B^+ \otimes_{B^+} \mathbb{C} \cong B^+$$

as right B^+-modules, by the obvious map $b \otimes \lambda \mapsto \epsilon(b)\lambda$, application of ϵ_* to the given difference gives the formal difference in $\mathrm{K}_0(\mathbb{C})$ of the isomorphism classes of $\mathcal{E} \otimes_{B^+} \mathbb{C}$ and $\mathbb{C}$. The isomorphism $\mathrm{K}_0(\mathbb{C}) \to \mathbb{Z}$ is of course by taking the complex dimension of a complex vector space, and the difference is thus mapped to

$$\dim_{\mathbb{C}}(\mathcal{E} \otimes_{B^+} \mathbb{C}) - m.$$

For this to be zero means therefore that

$$m = \dim_{\mathbb{C}}(\mathcal{E} \otimes_{B^+} \mathbb{C}) - m.$$

So, rather generally, we can parameterize $\mathrm{K}_0(A)$, for A non-unital, by isomorphism classes $[\mathcal{E}]$ of f.g.p. modules over A^+, the corresponding classes being

$$[\mathcal{E}] - \dim_{\mathbb{C}}(\mathcal{E} \otimes_{A^+} \mathbb{C}) \cdot [1] \in \mathrm{K}_0(A) \subset \mathrm{K}_0(A^+).$$

Before proceeding, we prove a useful lemma, which implies in particular that the K_0-group of a separable C*-algebra is always a *countable* abelian group.

Lemma 1.17. *If A is a unital C*-algebra and $p, q \in A$ are projections with $\|p - q\| < 1$, then p and q are unitarily equivalent.*

Proof. Let $a = pq + (1-p)(1-q)$. A direct calculation shows that

$$1 - a = (2p - 1)(p - q).$$

Since $\|pq\| < 1$ and $\|2p - 1\| \leq 1$, we see that $\|1 - a\| < 1$ and so a is invertible. Since $pa = pq = aq$, $a^{-1}pa = q$, and p, q are similar. From Proposition 2.12 they are unitarily equivalent (using the unitary $u = a(a^*a)^{-\frac{1}{2}}$ in the polar decomposition of a.)

□

Corollary 1.18. *If A is a separable C*-algebra then* $\mathrm{K}_0(A)$ *is countable.*

The proof is left as an exercise.

Two projections p, q in a C*-algebra A are *homotopic* if there is a continuous, projection-valued map $p \colon [0,1] \to A$ with values p, q at the endpoints.

Corollary 1.19. *If A is a unital C*-algebra and $p, q \in A$ are homotopic projections, then p and q are unitarily equivalent, and in particular, are Murray-von-Neumann equivalent.*

Proof. By compactness of the interval we can find $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\|p(t_k) - p(t_{k+1})\| < 1$ for all k. The result follows from Lemma 1.17.

□

The converse is almost true as well:

Proposition 1.20. *Let p, q be Murray-von-Neumann equivalent projections in a unital C*-algebra A. Then $\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}$ are homotopic.*

Proof. Assume v is a partial isometry in A such that $v^*v = p, vv^* = q$. By Exercise 2.15 of Chapter 4, the matrix

$$u := \begin{bmatrix} v & 1 - vv^* \\ v^*v - 1 & v^* \end{bmatrix}$$

is unitary, is connected by a path of unitaries to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and

$$upu^* = q.$$

The result follows.

□

EXERCISE 1.21. Suppose p is a projection in A (unital). Conjugate $\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$ by the rotation matrix $R_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$. Check that this gives a path of projections between $\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & p \end{bmatrix}$.

EXERCISE 1.22. Suppose p, q are projections in A and $pq = qp = 0$. Show that

$$\begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} + R_t \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} R_t^*$$

gives a path of projections between $\begin{bmatrix} p+q & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$, where R_t is the rotation matrix $\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$.

EXERCISE 1.23. Let $p \in M_2\big(C(\mathbb{T})\big)$ be the projection

$$p(z) := \frac{1}{2} \begin{bmatrix} 1 & z \\ \bar{z} & 1 \end{bmatrix}.$$

Find a loop $u(z)$ of unitaries in $M_2(\mathbb{C})$ such that

$$upu^* = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in M_2\big(C(\mathbb{T})\big).$$

Note that p is the restriction to $\mathbb{T} \subset \mathbb{C}$ of the Bott projection of Example 1.3.

DEFINITION 1.24. Two *-homomorphisms $\alpha \colon A \to B$ and $\beta \colon A \to B$ between unital C*-algebras are *homotopic* if there is a *-homomorphism

$$\gamma \colon A \to C([0,1], B)$$

such that

$$\gamma(a)(0) = \alpha(a), \ \ \gamma(a)(1) = \beta(a), \quad \forall a \in A.$$

Such a homotopy determines a 1-parameter family $(\alpha_t)_{t\in[0,1]}$ of *-homomorphisms $A \to B$ by $\alpha_t(a) := \gamma(a)(t)$, with $\alpha_0 = \alpha$, and $\alpha_1 = \beta$, which is continuous in the sense that for each $a \in A$, the map $[0,1] \to A$, $t \mapsto \alpha_t(a)$, is continuous.

COROLLARY 1.25. *If $\alpha \colon A \to B$ and $\beta \colon A \to B$ are homotopic *-homomorphisms, then* $\alpha_* = \beta_* \colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$.

PROOF. Assume A and B are unital; the general case is dealt with by taking unitizations. For any n, γ induces a *-homomorphism $M_n(A) \to C\big([0,1], M_n(B)\big)$, and a family of *-homomorphisms which we still denote by $\alpha_t \colon M_n(A) \to M_n(B)$. If $p \in M_n(A)$, then $q(t) := \alpha_t(p)$ is a path of projections in $M_n(B)$ between $\alpha(p)$ and $\beta(p)$ Hence these two projections determine the same class in $\mathrm{K}_0(B)$.

$\square$

EXERCISE 1.26. A C*-algebra A is *contractible* if the identity homomorphism and zero homomorphisms $A \to A$ are homotopic.

a) Prove that if A is contractible (unital or not) then $\mathrm{K}_0(A) = 0$.
b) Prove that if A is any C*-algebra, then the C*-algebra $C_0\big([0,1), A\big)$ is contractible.

Exercise 1.27. Let $p\colon [0,1] \to A$ be a path of projections in a unital C*-algebra A. Use the proof of Lemma 1.17 to show that there is a continuous path of unitaries $(u_t)_{t\in[0,1]}$ such that

$$p_t = u_t p_1 u_t^*.$$

(*Hint.* The proof of Lemma 1.17 shows that if $t_0 \in [0,1]$ is any point, and

$$a_s := p_t p_s + (1-p_t)(1-p_s)$$

then a_s is invertible for $|t-s|$ sufficiently small, and

$$u_s := a_s(a_s^* a_s)^{-\frac{1}{2}}$$

is a unitary such that

$$u_s p_t u_s^* = p_s$$

holds.)

Exercise 1.28. Let $\pi\colon A \to B$ be a surjective C*-algebra homomorphism between unital C*-algebras. Prove that if $(p_t)_{t\in[0,1]}$ is a path of projections in B, $\tilde{p}_1 \in A$ is a projection such that $\pi(\tilde{p}_1) = p_1$, then there is a path $(\tilde{p}_t)_{t\in[0,1]}$ of projections in A, ending in $\tilde{p}_1$, and such that $\pi(\tilde{p}_t) = p_t$ for all t.

That is, show that the map $\mathcal{P}(A) \to \mathcal{P}(B)$ induced by a surjective *-homomorphism, has the path lifting property. (*Hint.* Use Exercise 1.27 and the fact proved in Exercise 8.30 that the restriction of π to the unitary group $\mathbf{U}(A)$ has the path-lifting property.)

We next discuss a technique with broad applications in Noncommutative Geometry, in preparation for proving continuity of K_0 under inductive limits.

Definition 1.29. Let $\mathcal{A}$ be a *-subalgebra of a C*-algebra A. We say that $\mathcal{A}$ is *spectral* in A if $\mathcal{A}$ is dense in A, and whenever $a \in M_n(\mathcal{A})$ and f is a holomorphic function on $\mathrm{Spec}_A(a)$, then $f(a) \in M_n(\mathcal{A})$.

Notice that if $\mathcal{A}$ is spectral in A and $a \in \mathcal{A}$ is invertible in A, then $a^{-1} \in \mathcal{A}$. It follows that the spectrum of $a \in \mathcal{A}$, as an element of $\mathcal{A}$, is the same as its spectrum as an element of A.

Exercise 1.30. If M is a compact manifold, then $C^k(M)$ is spectral in $C(M)$ for every $k \in \mathbb{N} \cup \{\infty\}$.

Exercise 1.31. Suppose $A_1 \subset A_2 \subset \cdots \subset A$ is an increasing union of unital C*-subalgebras of a fixed unital C*-algebra A. Show that $\mathcal{A} := \bigcup_{n=1}^\infty A_n$ is a spectral *-subalgebra of A; in fact, show that it is even closed under *continuous* functional calculus.

We have already remarked that K_0, as defined in terms of idempotents in $\mathrm{M}_\infty(\mathcal{A})$, is defined for any algebra (in fact any ring), and in particular for any *-algebra. Moreover, if $\mathcal{A} \subset A$ happens to be spectral, then since it is closed under holomorphic functional calculus in A, it follows easily that the idempotent picture of K_0 agrees with the projection picture, with algebraic equivalence corresponding to Murray von-Neumann equivalence, because any spectral $\mathcal{A}$ will be closed under the operation of taking square roots of positive elements and hence polar decompositions of elements of $\mathcal{A}$ which are invertible in A, end up having their constituents in $\mathcal{A}$. (A spectral subalgebra is very nearly a C*-algebra).

Thus in the following, we will understand that K_0 as defined in terms of projections and Murray von-Neumann equivalence, extends directly to spectral *-subalgebras. With this convention:

Theorem 1.32. *Suppose A is a unital C*-algebra and $\mathcal{A} \subset A$ is a spectral *-subalgebra. Then the homomorphisms*

$$i_*\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(A), \quad i_*\colon \mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(A)$$

induced from the inclusion $i\colon \mathcal{A} \to A$, are isomorphisms.

LEMMA 1.33. *Let A be a unital C^*-algebra and p a projection in A. If $a \in A$ is a normal element with $\|a - p\| < \delta$, then*

$$\mathrm{Spec}(a) \subset B_\delta(0) \cup B_\delta(1) \subset \mathbb{C},$$

with $B_\delta(\cdot)$ the disk of radius δ.

PROOF. Suppose $\lambda \notin B_\delta(0) \cup B_\delta(1)$, *i.e.* that $\min\{|\lambda|, |1-\lambda|\} > \delta$. Then $\lambda - p$ is invertible in A. And

$$\|(\lambda - p)^{-1}\| = \max\{|\lambda|^{-1}, |1-\lambda|^{-1}\} < \frac{1}{\delta}.$$

Consequently

$$\begin{aligned} \|(\lambda - p)^{-1}(\lambda - a) - 1\| &= \|(\lambda - p)^{-1}(\lambda - a) - (\lambda - p)^{-1}(\lambda - p)\| \\ &\leq \|(\lambda - p)^{-1}\| \cdot \|a - p\| < 1. \end{aligned} \tag{1.3}$$

Hence $(\lambda - p)^{-1}(\lambda - a)$ is invertible, and hence so is $\lambda - a$. □

PROOF. (Of Theorem 1.32). Let $p \in M_n(A)$ be a projection. By density of $M_n(\mathcal{A})$ in $M_n(A)$ there exists $h \in M_n(\mathcal{A})$ such that $\|h - p\| < \frac{1}{2}$. By replacing h with $\frac{h+h^*}{2}$, we may assume that h is self-adjoint. By Lemma 1.33, $\frac{1}{2} \notin \mathrm{Spec}(h)$, and hence the characteristic function $\chi := \chi_{[\frac{1}{2},1]}$ is holomorphic on $\mathrm{Spec}(h)$. Since $M_n(\mathcal{A})$ is spectral in $M_n(A)$, the projection $q := \chi(h)$ is in $M_n(\mathcal{A})$. Also, $\|\chi(h) - h\| < \frac{1}{2}$ since $\frac{1}{2} \notin \mathrm{Spec}(h)$, and hence

$$\|q - p\| \leq \|q - h\| + \|h - p\| < \frac{1}{2} + \frac{1}{2} = 1$$

so that p and q are unitarily equivalent and hence Murray-von-Neumann equivalent in A by Lemma 1.17, and hence $i_*([q]) = [p] \in \mathrm{K}_0(A)$.

We have shown that $i_*\colon \mathcal{P}(\mathcal{A}) \to \mathcal{P}(A)$ is surjective.

Injectivity is similar. Suppose p, q are projections in $M_n(\mathcal{A})$ which become Murray-von-Neumann equivalent in $M_n(A)$, by a partial isometry u such that

$$u^*u = p, \;\; uu^* = q.$$

It follows from density of $M_n(\mathcal{A})$ in $M_n(A)$, that we can find a contraction $x \in M_n(\mathcal{A})$ such that $\|x - u\| < \frac{1}{4}$. As

$$\|x^*x - p\| = \|x^*x - x^*u + x^*u - u^*u\| \leq (\|x\| + \|u\|)\|x - u\| \leq 2\|x - u\|,$$

we have that $\|x^*x - p\| < \frac{1}{2}$. Similarly $\|xx^* - q\| < \frac{1}{2}$.

By Lemma 1.33, $\mathrm{Spec}(x^*x) \cup \mathrm{Spec}(xx^*) \subset B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(1)$. So $\chi = \chi_{[0,\frac{1}{2}]}$ as in the proof of the same Lemma, is continuous on the spectra both of x^*x and xx^*, and $\|\chi(x^*x) - x^*x\| < \frac{1}{2}$, whence $\|\chi(x^*x) - p\| < 1$ so p is unitarily equivalent to $p' := \chi(x^*x)$ in A by Lemma 1.17. Similarly, q is unitarily equivalent to $q' := \chi(xx^*)$. We show that p' and q' are Murray von-Neumann equivalent in $\mathcal{A}$. Let $f(t) = \sqrt{\frac{\chi(t)}{t}}$, a continuous function on $\mathrm{Spec}(x^*x) \cup \mathrm{Spec}(xx^*)$ satisfying

$$tf(t)^2 = \chi(t) \;\; \forall t \in \mathrm{Spec}(x^*x) \cup \mathrm{Spec}(xx^*).$$

Set

$$v := xf(x^*x).$$

Since $\mathcal{A}$ is spectral and $x \in \mathcal{A}$, $v \in \mathcal{A}$ as well.

Since x^*x commutes with $f(x^*x)$,

$$v^*v = f(x^*x)x^*xf(x^*x) = f(x^*x)^2x^*x = \chi(x^*x) = p'.$$

Now notice that
$$x(x^*x)^k x = (xx^*)^k xx^*$$
for any positive integer k. It follows that
$$xh(x^*x)x^* = h(xx^*)xx^*$$
for any polynomial h and hence for any continuous function. In particular, taking $h = f^2$, we see
$$vv^* = xf(x^*x)^2x^* = xg(x^*x)^2x^* = g(xx^*)^2xx^* = \chi(xx^*) = q',$$
so that $p' \sim q'$ in $\mathcal{A}$ as claimed.

□

In the course of the proof, we proved the following, which will be used again:

LEMMA 1.34. *If A is a unital C*-algebra, $x \in A$, p, q are projections in A, then if there exists $x \in A$ such that $\|x^*x - p\| < \frac{1}{2}$ and $\|xx^* - q\| < \frac{1}{2}$, then p and q are Murray-von-Neumann equivalent in A.*

We next verify the continuity of K_0 under inductive limits. Suppose that $\{A_i,\ \varphi_{ij}\}_{i\in I}$ is an inductive system of C*-algebras, and $\varphi_i\colon A_i \to \varinjlim A_i$ the associated C*-algebra homomorphisms. The homomorphisms φ_i and the homomorphisms φ_{ij} all determine maps $(\varphi_i)_*$, *etc* between the appropriate K-theory groups. By functoriality of K_0,
$$(\varphi_{ij})_* \circ (\varphi_{jk})_* = (\varphi_{ik})_*$$
for all relevant indices $i.j, k$ so we obtain an inductive system $\{\mathrm{K}_0(A_i),\ (\varphi_{ij})_*\}$ of abelian groups.

Similarly, $\varphi_i \circ \varphi_{ij} = \varphi_j$ implies $(\varphi_i)_* \circ (\varphi_{ij})_* = (\varphi_j)_*$. Hence, by the universal property of inductive limits of groups, we obtain a unique group homomorphism
$$\Phi\colon \varinjlim \mathrm{K}_0(A) \cong \mathrm{K}_0\big(\varinjlim A_i\big).$$

THEOREM 1.35. *If $\{A_i,\ \varphi_{ij}\}_{i\in I}$ is an inductive system of C*-algebras, and $\varphi_i\colon A_i \to \varinjlim A_i$ the associated C*-algebra homomorphisms, then $\{\mathrm{K}_0(A_i),\ (\varphi_{ij})_*\}$ is an inductive system of abelian groups, and*
$$\varinjlim \mathrm{K}_0(A) \cong \mathrm{K}_0\big(\varinjlim A_i\big).$$
The isomorphism is induced from the coherent family of group homomorphisms $(\varphi_i)_*\colon \mathrm{K}_0(A_i) \to \mathrm{K}_0(\varinjlim A_i)$ induced by the φ_i.

PROOF. Observe first that $\mathcal{A} := \cup_{i\in I}\varphi_i(A_i)$ is spectral in A, and the map Φ factors through $\mathrm{K}_0(\mathcal{A})$. So we are reduced to showing that
$$\Phi\colon \varinjlim \mathrm{K}_0(A_i) \to \mathrm{K}_0(\mathcal{A})$$
is an isomorphism.

If p is a projection in $M_n(\mathcal{A})$, then $p \in M_n\big(\varphi_i(A_i)\big)$ for some i. Let $h \in M_n(A_i)$ self-adjoint with $\varphi_i(h) = p$. Then $h^2 - h \in \ker(\varphi_i) = \{a \in A_i \mid \lim_{j\to\infty}\|\varphi_{ji}(a)\| = 0\}$. so that for j large enough $\|\varphi_{ji}(h)^2 - \varphi_{ji}(h)\| < \frac{1}{4}$. Since $\varphi_j \circ \varphi_{ji}(h) = \varphi_i(h) = p$, due to the definitions, by replacing $h \in A_j$ by $\varphi_{ji}(h) \in A_j$, we may as well have assumed from the beginning that
$$h \in A_j,\ h = h^*,\ \varphi_j(h) = p,\ \|h^2 - h\| < \frac{1}{4}.$$
Then $\frac{1}{2} \notin \mathrm{Spec}(h)$, and $q := \chi(h)$ is a projection in $M_n(A_j)$ such that $\|q - h\| < \frac{1}{2}$. Hence $\|\varphi_j(q) - p\| = \|\varphi_j q) - \varphi_j(h)\| \le \|q - h\| < \frac{1}{2}$ so that $\varphi_j(q)$ is unitarily equivalent to p in $M_n(A)$ and
$$\Phi([q]) = (\varphi_j)_*([q]) = [\varphi_j(q)] = [p].$$
This shows that Φ is surjective.

Injectivity: suppoes that p and q are in $M_n(A_i)$ whose φ_i-images are Murray-von-Neumann equivalent by a partial isometry u in $M_n\big(\varphi_i(A_i)\big) \subset \mathcal{A}$, with, say, $u^*u = p$ and $uu^* = q$. We can lift u to an $x \in A_i$. As previously, we have

$$\lim_{j\to\infty} \|\varphi_{ji}(x)^*\varphi_{ji}(x) - \varphi_{ji}(p)\| = 0.$$

and similarly

$$\lim_{j\to\infty} \|\varphi_{ji}(x)\varphi_{ji}(x)^* - \varphi_{ji}(q)\| = 0.$$

Moreover, the image in $\varinjlim \mathrm{K}_0(A_i)$ of the class of $\varphi_{ji}(p)$ in $\mathrm{K}_0(A_i)$ is the same as the image of $[p]$ in this inductive limit of groups, and similarly for q. So we may as well have assumed from the beginning that

$$x \in M_n(A_j),\ \|x^*x - p\| < \frac{1}{2},\ \|xx^* - q\| < \frac{1}{2}.$$

By Lemma 1.34, p and q are Murray-von-Neumann equivalent in A_i. Hence $[p] = [q] \in \varinjlim \mathrm{K}_0(A_i)$, as required.

□

Remark 1.36. The proof Theorem 1.35 is really just an amplification of that of Theorem 1.32. In fact, if an inductive system has injective structure maps, then $\mathrm{K}_0(\varinjlim A_i) \cong \varinjlim \mathrm{K}_0(A_i)$ is a direct consequence of Theorem 1.32, because then the union $\mathcal{A} := \cup_{i\in I}\varphi_i(A_i)$ is spectral in the C*-algebra limit A, and the following basic exercise in the definitions:

Exercise 1.37. Using only the basic definitions of K_0, show that if $\mathcal{A} = \bigcup_{i\in I} A_i$ is an ascending union of unital C*-algebras, then $\mathrm{K}_0(\mathcal{A}) \cong \varinjlim \mathrm{K}_0(A_i)$.

Corollary 1.38. *If $\mathcal{K}$ is the C*-algebra of compact operators on a Hilbert space H, then $\mathrm{K}_0(\mathcal{K}) \cong \mathbb{Z}$, under an isomorphism sending $1 \in \mathbb{Z}$ to the class $[p] \in \mathrm{K}_0\big(\mathcal{K}(H)\big)$ of any rank-one projection.*

Proof. Indeed, $\mathcal{K}(H) \cong \varinjlim \mathbb{B}(V)$, the inductive limit of the C*-algebras $\mathbb{B}(V)$, as V ranges over the directed set of finite-dimensional subspaces of H. The structure maps $\mathbb{B}(V) \to \mathbb{B}(V')$, for $V \subset V'$, are by setting $T \in \mathbb{B}(V)$ to be zero on the orthogonal complement $V^\perp$ of V in V'.

For each finite-dimensional Hilbert space V, $\mathrm{K}_0\big(\mathbb{B}(V)\big) \cong \mathbb{Z}$. A generator for the K_0-group can be taken to be the class of any rank-one projection p_V. With V fixed, any two such projections are unitarily equivalent in $\mathbb{B}(V)$ and so determine the same K_0-class for $\mathbb{B}(V)$.

Now if if $V \subset V'$, then the image of p_V in $\mathbb{B}(V')$ under the structure map is the projection $\begin{bmatrix} p_V & 0 \\ 0 & 0\end{bmatrix}$, (with respect to the decomposition $V' = V \oplus V^\perp$, and this image is clearly also a minimal projection in $\mathbb{B}(V')$. Hence the induced maps $\mathrm{K}_0\big(\mathbb{B}(V)\big) \to \mathrm{K}_0\big(\mathbb{B}(V')\big)$ are the identity maps $\mathbb{Z} \to \mathbb{Z}$, for every inclusion of subspaces. The result follows.

□

Exercise 1.39. Prove the *stability* of the K_0-functor: that the C*-algebra inclusion $A \to A \otimes \mathcal{K}$ mapping $a \in A$ to $a \otimes p$, where $p \in \mathcal{K}$ is any rank-one projection, induces an isomorphism

$$\mathrm{K}_0(A) \cong \mathrm{K}_0(A \otimes \mathcal{K}).$$

(*Hint.* Write $A \otimes \mathcal{K}$ as an inductive limit.)

Corollary 1.40. *For the UHF algebra $U(d^\infty)$,*

$$\mathrm{K}_0\big(U(d^\infty)\big) \cong \mathbb{Z}[\frac{1}{d}],$$

the additive subgroup underlying the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{d}$.

If $\mathcal{N}$ is the universal UHF algebra of Example 8.10, then

$$\mathrm{K}_0(\mathcal{N}) \cong \mathbb{Q},$$

with $\mathbb{Q}$ as a group under addition.

PROOF. We just do the case $d = 2$, since it is typographically simpler. The UHF is the inductive limit of the system

$$\mathbb{C} \subset M_2(\mathbb{C}) \to M_{2^2}(\mathbb{C}) \to M_{2^3}(\mathbb{C}) \to \cdots$$

where the structure maps between adjacent C*-algebras are of the form

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}.$$

A rank-one projection in $M_{2^n}(\mathbb{C})$ is thus identified in the inductive limit with a rank-two projection in $M_{2^{n+1}}(\mathbb{C})$. The K_0-group of each $M_{2^k}(\mathbb{C})$ is $\mathbb{Z}$ with $1 \in \mathbb{Z}$ corresponding to the class of a rank-one projection. It follows that the maps $\mathbb{Z} \cong \mathrm{K}_0\big(M_{2^n}(\mathbb{C})\big) \to \mathrm{K}_0\big(M_{2^{n+1}}(\mathbb{C})\big) \cong \mathbb{Z}$ induced by the corresponding structure map, is multiplication by 2.

Thus

$$\mathrm{K}_0\big(U(2^\infty)\big) \cong \varinjlim \mathbb{Z},$$

where in the inductive limit on the right hand side, the structure maps are given between adjacent groups by multiplication by 2. The result follows from Example 8.3.

The second assertion follows from similar arguments and Exercise 8.4. □

EXERCISE 1.41. Model the Cantor set X as $X := \prod_{n=1}^\infty \{0,1\}$. For an element $\mu := (i_1, \ldots, i_k) \in \{0,1\} \times \cdots \times \{0,1\} = \{0,1\}^n$, let $U_\mu \subset X$ be the clopen set of all sequences (x_n) starting with μ. Let χ_μ be the characteristic function of U_μ. Then X is the inverse limit of the spaces $\{0,1\}^n \leftarrow \{0,1\}^{n+1}$, with $S^n = S \times \cdots \times S$ where the maps drop the last coordinate, and $C(X) \cong \varinjlim C(\{0,1\}^n)$, in such a way that a delta function at a point $\mu \in \{0,1\}^n$, defining an element of $C(\{0,1\}^n)$, corresponds under the map into the inductive limit, to the the characteristic function χ_μ.

Prove, using compactness of X, that if $f \in C(X, \mathbb{Z})$ is any continuous, integer-valued function on X, then f is a finite, $\mathbb{Z}$-linear combination of the χ_μ's. (*Hint.* Start with f a characteristic function of a clopen set.)

Deduce that the natural map

$$C(X, \mathbb{Z}) \to \mathrm{K}^0(X)$$

is an isomorphism, with $C(X, \mathbb{Z})$ the group, under addition, of integer-valued, continuous functions on X.

EXERCISE 1.42. This exercise explores the K_0-group of the C*-algebra $C^*(G)$ of a finite group.

For any such group, the *representation ring* $\mathrm{Rep}(G)$ of G, as a group, is the Grothendieck completion of the semigroup of unitary isomorphism classes of finite-dimensional unitary representations of G.

Elements of the representation ring can be designated $[\pi_1] - [\pi_2]$ where π_i are finite-dimensional unitary representations of G.

Prove that the Green-Julg correspondence determines an isomorphism

$$\mathrm{K}_0\big(C^*(G)\big) \cong \mathrm{Rep}(G).$$

And if π is irreducible, then this isomorphism maps the class $[e_\pi] \in \mathrm{K}_0(C^*(G))$ of the projection $e_\pi := \frac{\dim H_\pi}{|G|} \cdot \chi_\pi^* \in C^*(G)$, with χ_π^* the (conjugate) character of π, to the class $[\pi] \in \mathrm{Rep}(G)$ of the representation π.

In particular, $\mathrm{K}_0\big(C^*(G)\big)$ is a finitely generated free abelian group with free generators the classes of the projections e_π, with π an irreducible representation of G.

See Theorem 8.7.

Exercise 1.43. Let G be the group $\mathbb{Z}/2$ acting on the interval $I := [-1,1]$ by $\sigma(x) = -x$. Compute the K-theory of the crossed-product $C(I) \rtimes \mathbb{Z}/2$. (*Hint.* The interval is $\mathbb{Z}/2$-equivariantly contractible. Use the homotopy-invariance of K-theory.)

Morita invariance of K*-theory*

Let G be a locally compact group acting properly on X. We have seen that the crossed-product $C_0(X) \rtimes G$ can be identified with the fixed-point algebra

$$C_0(X, \mathcal{K})^G$$
$$:= \{f\colon X \to \mathcal{K}\big(l^2G\big) \mid f \text{ is continuous, and } f(gx) = \rho(g)f(x)\rho(g)^{-1}\ \forall g \in G,\ x \in X.\}$$

Exercise 1.44. Suppose that p is a projection in $M_n\big(C_0(X) \rtimes G\big)$. Thinking of p as a map $X \to \mathcal{K}((l^2G)^n$ which is G-equivariant in the above sense, the image $p(x) \subset l^2(G)$ for any $x \in X$, is a finite-dimensional, linear subspace $E_x \subset l^2(G)^n := l^2(G) \oplus \cdots l^2(G)$..

a) Show that the family of subspaces $\{E_x\}_{x\in X}$ are the fibres of a vector bundle E over X.
b) Show that if $g \in G$, then the right translation operator $\rho(g)\colon l^2(G) \to l^2(G)$ induces a linear isomorphism $E_x \to E_{gx}$ for any $x \in X$. Show that E has in this way the structure of a G-equivariant vector bundle over X, and whose associated section module $\Gamma(E)$ over $C_0(X) \rtimes G$, is isomorphic to $p \cdot \big(C_0(X) \rtimes G\big)^n$.

For G acting properly on X, the *G-equivariant* K*-theory of X* is defined similarly to the ordinary K-theory, but with G-equivariant vector bundles instead of ordinary vector bundles. One has an obvious notion of *isomorphism* of G-equivariant vector bundles, and one can add them, so one forms a corresponding semigroup and Grotheniek completion, denoted

$$\mathrm{K}^0_G(X).$$

Exercise 1.45. Suppose that G is discrete and acts properly, freely and co-compactly on X. Suppose that $E \to X$ is a G-equivariant vector bundle over X. Let $G\backslash E$ be the quotient of the total space of E, by the group action.

a) Prove that $G\backslash E$, has a natural structure of vector bundle over $G\backslash X$.
b) Let $\mathcal{E}$ be the Morita equivalence $C_0(X)\rtimes G$-$C(G\backslash X)$ bimodule constructed in the proof of Theorem 7.1 (see below). Let E be a G-equivariant vector bundle over X, let $\Gamma(E)$ be its f.g.p. right $C_0(X) \rtimes G$-module of sections. Show that
satisfies
$$\Gamma(E) \otimes_{C_0(X)\rtimes G} \mathcal{E} \cong \Gamma(G\backslash E).$$
c) Let $\mathcal{E}^*$ be the conjugate bimodule, a Morita equivalence $C(G\backslash X)$-$C_0(X)\rtimes G$ bimodule. Show that
$$\Gamma(G\backslash E) \otimes_{C(G\backslash X)} \mathcal{E}^* \cong \Gamma(E)$$
as right $C_0(X) \rtimes G$-bimodules, for any G-equivariant vector bundle over X.

The exercise above supplies the proof that

$$\mathrm{K}^0_G(X) \cong \mathrm{K}_0\big(C_0(X) \rtimes G\big).$$

The Morita equivalence bimodule $\mathcal{E}$ figuring in the above Exercise is quite important, and we recall its definition, or, more precisely, the dual $\mathcal{E}$. To construct $\mathcal{E}$, we take $C_c(X)$. We regard it as a left module over $C(G\backslash X)$ with module structure and inner product

$$(f \cdot \xi)(x) := f(\dot{x})\xi(x), \quad \xi \in C_c(X),\ f \in C(G\backslash X),\ g \in G, \tag{1.4}$$

and right $C_0(X) \rtimes G$-module structure and inner product given by the (anti-)covariant pair

$$(\xi \cdot f)(x) := f(x)\xi(x),\ \ (\xi \cdot g)(x) = \xi(gx),\ \ \xi \in C_c(X),\ f \in C_0(X),\ g \in G. \tag{1.5}$$

EXERCISE 1.46. Let G be a finite group acting on X compact. Let $i\colon C^*(G) \to C(X) \rtimes G$ be the inclusion. Let

$$p_\epsilon = \frac{1}{|G|}\sum_{g\in G}[g] \in \mathbb{C}[G] = C^*(G),$$

a projection, which we regard as an element of the crossed-product $C(X) \rtimes G$. Show that if G acts freely on X, then the isomorphism

$$\mathcal{E}_*\colon \mathrm{K}^0(G\backslash X) \to \mathrm{K}_0(C(X) \rtimes G)$$

induced by the Morita equivalence between $C(G\backslash X)$ and $C(X) \rtimes G$ maps the class $[1] \in \mathrm{K}^0(G\backslash X)$ of the trivial line bundle on $G\backslash X$, to the class $i_*([p_\epsilon]) \in \mathrm{K}_0\big(C^*(G)\big)$.

EXERCISE 1.47. Let G be a finite group acting freely on X. Let $\mathcal{E}$ be the strong Morita $C(G\backslash X)$-$C(X) \rtimes G$-bimodule discussed above. Let

$$\mathcal{E}_*\colon \mathrm{K}^0(G\backslash X) \to \mathrm{K}_0\big(C(X) \rtimes G\big)$$

be the induced group isomorphism, of tensoring with $\mathcal{E}$.

a) Let $i\colon C^*(G) \to C(X) \rtimes G$ be the inclusion. Let

$$p_\epsilon = \frac{1}{|G|}\sum_{g\in G}[g] \in \mathbb{C}[G] = C^*(G),$$

a projection, defining a class $[p_\epsilon] \in \mathrm{K}_0(C^*G)$. Show that

$$\mathcal{E}_*([1_{G\backslash X}]) = i_*([p_\epsilon]) \in \mathrm{K}_0(C_0(X) \rtimes G).$$

b) Prove that the diagram

$$\begin{tikzcd} \mathrm{K}_0(C^*G) \arrow[r] & \mathrm{K}_0(C(X) \rtimes G) \\ \mathrm{Rep}(G) \arrow[u, "\cong"] \arrow[r] & \mathrm{K}^0_G(X) \arrow[u, "\cong"'] \end{tikzcd} \tag{1.6}$$

commutes, where
- the vertical map on the left is the Green-Julg isomorphism,
- the vertical map on the right is the map sending the class $[E] \in \mathrm{Vect}_G(X)$ to the class $[\Gamma(E)]$ of its right $C(X) \rtimes G$-module of sections
- The top horizontal map is induced from the inclusion $i\colon C^*(G) \to C(X) \rtimes G$.
- The bottom horizontal map is: if $\rho\colon G \to \mathbf{U}(V_\rho)$ is a finite-dimensional representation of G, then $X \times V_\rho$ carries a (diagonal) G-action, making it into a G-equivariant vector bundle over X. This gives an obvious group homomorphism $\mathrm{Rep}(G) \to \mathrm{K}^0_G(X)$.

c) Consider the group homomorphism

$$i_* \colon \mathrm{Rep}(G) \to \mathrm{K}^0(G\backslash X)$$

sending the class of a representation $\rho\colon G \to \mathbf{U}(V_\rho)$ to the class of the 'flat' vector bundle $X \times_G V_\rho$ over $G\backslash X$. Define another map

$$j_* \colon \mathrm{Rep}(G) \to \mathrm{K}_0(C_0(X) \rtimes G)$$

by mapping the class of a representation $\rho\colon G \to \mathbf{U}(V_\rho)$ to the class of the right $C(X) \rtimes G$-module $C(X, V_\rho)^G$ of continuous functions $f\colon X \to V_\rho$ such that $f(gx) = \rho(g)f(x)$.

– Prove that $C(X, V_\rho)^G$ has the following structure of a f.g.p. right $C(X) \rtimes G$-module:

$$(\xi \cdot f)(x) = \xi(x)f(x), \quad (\xi \cdot g)(x) := \rho(g)^{-1}\xi(gx).$$

– Let

$$k_* \colon \mathrm{Rep}(G) \to \mathrm{K}_0(C(X) \rtimes G)$$

be the map sending $[V_\rho]$ to the class $[\Gamma(X, V_\rho)^G]$ of the f.g.p. module corresponding to it by the construction in the previous item. Show that it fits into a commutative diagram

$$\begin{tikzcd} \mathrm{Rep}(G) \arrow[r, "k_*"] \arrow[dr, "i_*"'] & \mathrm{K}^0_G(X) \arrow[d] \\ & \mathrm{K}^0(G\backslash X) \end{tikzcd} \quad . \tag{1.7}$$

d) Let $i\colon \mathrm{Rep}(G) \to \mathrm{K}^0(G\backslash X)$ be the induction map $i([V_\rho]) := [X \times_G V_\rho] \in \mathrm{K}^0(G\backslash X)$ as above. Show that

$$i_*[\lambda] = [1_{G\backslash X}] \in \mathrm{K}^0(G\backslash X),$$

where λ is the regular representation.

e) If ρ is any finite-dimensional representation of G then the tensor product representation $\rho \otimes \lambda$ is unitarily equivalent to $\dim(\rho)$ copies of λ.

f) From the constructions above, pre-prove that if ρ is any finite-dimensional representation of G, the the class $[X \times_G V_\rho] \in \mathrm{K}^0(G\backslash X)$ has the property that

$$m \cdot [X \times_G V_\rho] = k \cdot [1_{G\backslash X}] \in \mathrm{K}^0(G\backslash X),$$

for some k, where $[1_{G\backslash X}]$ is the class of the trivial line bundle on $G\backslash X$. (*Hint.* Use the results of the previous exercises.)

Exercise 1.48. Let G be a locally compact group and $H \subset G$ a compact subgroup. Let $\pi\colon H \to \mathbf{U}(V_\pi)$ be any finite-dimensional representation of H. Let

$$E_\pi := G \times_H V_\pi,$$

the quotient of $G \times V_\pi$ by the equivalence relation

$$(g, v) \sim (gh^{-1}, \pi(h)v).$$

Show that E_π is a G-equivariant vector bundle over G/H.

An excellent example of the 'induction' procedure above, comes from considering the group $G = \mathbf{SU}(2, \mathbb{C})$ and the subgroup $H \subset G$ of diagonal matrices in G. Then H is nothing but the circle group $\mathbb{T}$, and any character

$$\chi_n \colon \mathbb{T} \to \mathbb{T}$$

of the circle gives rise to a G-equivariant vector bundle on G/H. In this case, the space G/H identifies with $\mathbb{CP}^1$, and the G-equivariant vector bundle induced from the character χ_n is a corresponding tensor power of the Hopf bundle.

2. Higher K-theory, loops and unitaries

Higher K-theory groups are described as in topological K-theory by suspension.

DEFINITION 2.1. If A is a C*-algebra, $i = 0, 1, 2, \ldots$, then we define $\mathrm{K}_i(A) := \mathrm{K}_0(S^n A)$, where $S^n(A) = C_0(\mathbb{R}^n) \otimes A$.

It is clear that K_i is functorial with respect to *-homomorphisms, and stable (that is, Morita invariant), for all i. Bott Periodicity will tell us that the groups $\mathrm{K}_i(A)$ are actually automatically 2-periodic, and hence there are in effect only two of them.

In topological K-theory, we noted that a vector bundle over $(X \times \mathbb{R})^+$ can be trivialized over the closure in $(X \times \mathbb{R})^+$ of $X \times (-\infty, 0]$, and similarly can be trivalized over the closure of $X \times [0, \infty)$. The difference of the two trivializations on the intersection $\cong X$ of these two closed subsets (neglecting the point at infinity) gives a unitary map $u\colon X \to \mathbf{U}_n$. The argument shows that homotopy classes of such u's give a group isomorphic to $\mathrm{K}^{-1}(X)$. (See Proposition 5.12). We now extend the construction to noncommutative C*-algebras. .

Let A be a unital C*-algebra. Recall (see Proposition 5.12 and the that $\mathbf{U}_\infty(A)$ denotes the group of all $\mathbb{N}$-by-$\mathbb{N}$-matrices with entries in A, which have a block diagonal form $\begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$ with u a (square) unitary matrix in $M_n(A)$, and 1 denoting the identity operator. There is an evident group structure on $\mathbf{U}_\infty(A)$ by multiplication, and we can regard, in the obvious way, all of the groups $\mathbf{U}\big(M_n(A)\big)$ as subgroups of $\mathbf{U}_\infty(A)$.

Clearly $\mathbf{U}_\infty(A)$ is the inductive limit of the groups

$$\mathbf{U}_n(A) := \{u \in M_n(A) \mid u \text{ is unitary}\}.$$

We give it the corresponding inductive limit topology): a subset $U \subset \mathbf{U}_\infty(A)$ is open if and only if $U \cap \mathbf{U}\big(M_n(A)\big)$ is open for all n.

We write $u, v \in \mathbf{U}_\infty(A)$ let $u \sim v$ if u and v are in the same path component of $\mathbf{U}_\infty(A)$. The quotient $\pi_0\big(\mathbf{U}_\infty(A)\big)$ is an abelian group, by Proposition 5.12.

PROPOSITION 2.2. *If A is any unital C*-algebra, then $\mathrm{K}_1(A)$ is naturally isomorphic to the abelian group $\pi_0\big(\mathbf{U}_\infty(A)\big)$.*

PROOF. By definition , $\mathrm{K}_1(A) := \mathrm{K}_0\big(S(A)\big)$ is the kernel of the augmentation homomorphism $\epsilon_*\colon \mathrm{K}_0\big((S(A)^+\big) \to \mathbb{Z}$, and in particular is a subgroup of $\mathrm{K}_0\big((S(A)^+\big)$. The latter consists of equivalence classes of projections in $S(A)^+$, and a projection in $S(A)^+$ is a loop

$$p\colon [0,1] \to M_n(A)$$

of projections in A, such that $p(0) = p(1) \ \in M_n(\mathbb{C}) \subset M_n(A)$. Any projection in $M_n(\mathbb{C})$ is unitarily equivalent to $\begin{bmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{bmatrix}$ for some k, and conjugating the loop by this unitary gives an equivalent loop with

$$p(0) = p(1) = \begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix}.$$

so we replace the original loop with this one without change in notation.

By Exercise 1.27, there is a path of unitaries $u\colon [0,1] \to M_n(A)$ with $u(1) = 1_n$ and

$$p(t) = u(t) p(1) u(t)^*, \quad t \in [0,1].$$

We have

$$p(0) = u(0)p(1)u(0)^* = u(0)p(0)u(0)^*$$

and hence $u(0)$ commutes with $p(0) = p(1) = \begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix}$. Therefore it has a block-diagonal form

$$u(0) = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix}$$

for some pair of unitaries $v \in M_k(A)$ and $w \in M_{n-k}(A)$.

We map

$$\mathrm{K}_1(A) \to \pi_0\big(\mathbf{U}_\infty(A)\big)$$

by sending $[p]$ to $[v]$. We leave it to the reader to check that this assignment is well-defined on K-theory classes $[p]$, and that it is a group homomorphism. To construct an inverse, let $v \in \mathbf{U}_m(A) \subset M_m(A)$ be a unitary. Then

$$\begin{bmatrix} v & 0 \\ 0 & v^* \end{bmatrix}$$

is a unitary which is in the same path component of $\mathbf{U}_\infty(A)$ as the identity $1_{2m} \in M_{2m}(\mathbb{C}) \subset M_n(A)$, by (5.12). Let, therefore,

$$u\colon [0,1] \to \mathbf{U}_{2m}(A)$$

be a unitary-valued function such that

$$u(0) = \begin{bmatrix} v & 0 \\ 0 & v^* \end{bmatrix}, \quad u(1) = \begin{bmatrix} 1_m & 0 \\ 0 & 1_m \end{bmatrix}.$$

Now let

$$p\colon [0,1] \to M_{2m}(A), \quad p(t) := u(t) \begin{bmatrix} 1_m & 0 \\ 0 & 0_m \end{bmatrix} u(t)^*.$$

Note that

$$p(0) = \begin{bmatrix} 1_m & 0 \\ 0 & 0_m \end{bmatrix} = p(1),$$

so that p is an element of $S(A)^+$. If now $q\colon [0,1] \to M_{2m}(A)$ is the constant loop $q(t) = \begin{bmatrix} 1_m & 0 \\ 0 & 0_m \end{bmatrix}$, then

$$[p] - [q] \in \mathrm{K}_0\big(S(A)\big)$$

and maps to $[u]$ under our construction above.

□

To define K_1 for non-unital C*-algebras in terms of unitaries, let A be possibly non-unital. Then the augmentation homomorphism

$$\epsilon\colon A^+ \to \mathbb{C}$$

induces a group homomorphism $\mathbf{U}_\infty(A^+) \to \mathbf{U}_\infty(\mathbb{C})$ and then an induced group homomorphism

$$\pi_0\big(\mathbf{U}_\infty(A^+)\big) \to \pi_0\big(\mathbf{U}_\infty(\mathbb{C})\big).$$

PROPOSITION 2.3. *For any C*algebra A,*

$$\mathrm{K}_1(A) \cong \ker(\epsilon_*\colon \pi_0\big(\mathbf{U}_\infty(A^+)\big) \to \pi_0\big(\mathbf{U}_\infty(\mathbb{C})\big).$$

We leave the proof to the reader, using the ideas and constructions from the unital case.

In fact, since $\mathrm{K}_1(\mathbb{C}) = \mathrm{K}^{-1}(\mathrm{pt}) = 0$, the augmentation homomorphism ϵ_* is actually the zero map, so actually

$$\mathrm{K}_1(A) \cong \mathrm{K}_1(A^+) \cong \pi_0\big(\mathbf{U}_\infty(A^+)\big)$$

holds for any A.

One should take a bit of care with this statement, however. It is not true in KO-theory (since it is no longer true that $\mathrm{KO}^{-1}(\mathrm{pt}) = 0$.)

Remark 2.4. The discussion above shows that we now have two different ways of describing $\mathrm{K}_2(\mathbb{C}) = \mathrm{K}^{-2}(\mathrm{pt})$. The first is the definition

$$\mathrm{K}_2(\mathbb{C}) := \mathrm{K}_0\big(C_0(\mathbb{R}^2)\big),$$

defined as a certain subgroup of $\mathrm{K}^0((\mathbb{R}^2)^+) = \mathrm{K}^0(S^2)$ (the kernel of the augmentation homomorphism.)

On the other hand, $\mathrm{K}^0(\mathbb{R}^2) = \mathrm{K}^{-1}(\mathbb{R}) = \mathrm{K}_1\big(C_0(\mathbb{R})\big)$ and according to the discussion above,

$$\mathrm{K}_1\big(C_0(\mathbb{R})\big) \cong \pi_0\big(\mathbf{U}_\infty(C(\mathbb{T})\big)$$

since $C_0(\mathbb{R})^+ = C(\mathbb{T})$.

Therefore, the 'unitary description' in this case produces a group isomorphism

$$\ker(\epsilon_*) \subset \mathrm{K}^0(S^2) \to \pi_0\big(\mathbf{U}_\infty(C(\mathbb{T})\big) \cong [\mathbb{T}, \mathbf{U}_\infty], \tag{2.1}$$

the last group being homotopy classes of maps $\mathbb{T} \to \mathbf{U}_\infty$, a group under pointise multiplication of homotopy classes (another and briefer way of describing $\pi_0\big(\mathbf{U}_\infty(C(\mathbb{T})\big)$.)

It is not difficult to check that this isomorphism is the clutching construction of Theorem 5.14. If $E \to S^2$ is any complex vector bundle, it can be trivialized over the top and bottom $S^2_\pm$ of the sphere. On the equator $\mathbb{T} = S^2_- \cap S^2_+$, one obtains, by following the inverse of the one trivialization, followed by the other, a map

$$\mathbb{T} \times \mathbb{C}^n \to \mathbf{GL}_n(\mathbb{C}),$$

and such a map determines a unique homotopy class $u(E) \in [\mathbb{T}, \mathbf{GL}_n(\mathbb{C})] \cong [\mathbb{T}, \mathbf{U}_\infty]$.

Notice that $u([1])$ is the zero element of the group $[\mathbb{T}, \mathbf{U}]$. However, $[1] \in \mathrm{K}^0(S^2)$ is not zero. Thus, clutching directly describes a group homomorphism

$$\mathrm{K}^0(S^2) \to [\mathbb{T}, \mathbf{U}_\infty]$$

which annihilates the class $[1]$ of the trivial bundle, and maps the class $[H] \in \mathrm{K}^0(S^2)$ of the Hopf bundle $H \to S^2$ to the class of the unitary $z\colon \mathbb{T} \to \mathbb{C}$. In particular, it maps

$$\beta := [H^*] - [1]$$

to the class $[\bar{z}]$ of the unitary $\bar{z} \in C(\mathbb{T})$.

Exercise 2.5. Let $u \in \mathbf{U}(A)$ be a unitary in a unital C*-algebra A. It determines, by functional calculus, a *-homomorphism

$$C(\mathbb{T}) \to C\big(\mathrm{Spec}(u)\big) \to A,$$

where the first map is restrictions of functions on the circle $\mathbb{T}$ to the spectrum of u, and second is functional calculus for u.

The Cayley transform $T\colon \mathbb{R} \to \mathbb{T}$ is $T(x) = \frac{x-i}{x+i}$. It maps the point at infinity of $\mathbb{R}$ to $1 \in \mathbb{T}$, and gives a natural identification of $\mathbb{R}$ with the open subset $\mathbb{T} - \{1\}$ of the circle.

Since there is a *-algebra inclusion of the ideal $C_0(\mathbb{T} - \{1\}) \subset C(\mathbb{T})$ we obtain a *-homomorphism

$$\alpha_u\colon C_0(\mathbb{R}) \subset C(\mathbb{T}) \to C\big(\mathrm{Spec}(u)\big) \to A,$$

with the last map functional calculus.

By functoriality of K-theory α_u determines a group homomorphism

$$(\alpha_u)_*\colon \mathrm{K}_1\big(C_0(\mathbb{R})\big) \to \mathrm{K}_1(A).$$

Prove that

$$(\alpha_u)_*(\beta) = [u],$$

where $\beta \in \mathrm{K}_1\big(C_0(\mathbb{R})\big) = \mathrm{K}^0(\mathbb{R}^2)$ is the Bott element.

Deduce that if $\mathrm{Spec}(u) \subset \mathbb{T}$ is a *proper* subset of the circle, then $[u] = 0 \in \mathrm{K}_1(A)$. (*Hint.* The unitary $z|_{\mathrm{Spec}(u)} \in C\big(\mathrm{Spec}(u)\big)$ is connected by a path of unitaries in $C\big(\mathrm{Spec}(u)\big)$ to 1.)

3. The long exact sequence

We now develop the *long exact sequence* in K-theory associated to an ideal $J \subset A$ of a C*-algebra. For this it will be convenient to describe a 'relative' version of K_0, similar to the 'K-theory triples' discussed previously.

Let A be a unital C*-algebra and $J \subset A$ an ideal. Let $\pi\colon A \to A/J$ be the quotient map

A *relative triple* for this situation is a triple (p, q, x) where p, q are projections in $M_n(A)$, for some n, $x \in M_n(A)$, and

$$\pi(x)^*\pi(x) = p, \;\; \pi(x)\pi(x)^* = q.$$

A triple is *degenerate* if $x^*x = p$ and $xx^* = q$ in A.

A homotopy of triples is a triple of continuous paths p_t, q_t and x_t, in $M_n(A)$, $t \in [0,1]$, such that (p_t, q_t, x_t) is a triple for all t. We say the endpoints (p_0, q_0, x_0) and (p_1, q_1, x_1) are *homotopic* triples.

Exercise 3.1. Show that (p, q, x) is a relative triple, and $x' \in A$ with $x - x' \in J$, then (p, q, x') is a relative triple which is homotopic to (p, q, x). (*Hint.* Straight line homotopy.)

The reader might recognize the similarity to the K-theory triples (E^0, E^1, φ) we introduced in connection with K-theory of noncompact spaces X. Let $U \subset X$ be an open subset and $\varphi\colon E^0 \to E^1$ be a vector bundle map which is an isomorphism on $X \setminus U$.

Then the triple defines a relative triple for the ideal $C_0(U)$ in $C_0(X)$, since $\varphi|_{X\setminus U}$ is an isomorphism between the two bundles, and hence determines a Murray-von-Neumann equivalence between projections p_0, p_1 such that $E^i \cong \mathrm{Im}(p_i)$.

For example, let $X = \overline{\mathbb{D}}$ the closed unit disk in $\mathbb{C}$, and $z \in C(\overline{\mathbb{D}})$ the usual complex variable, here considered as a bundle map from the trivial line bundle $\mathbf{1}$ on $\overline{\mathbb{D}}$, to itself.

Then its restriction to $\partial\mathbb{D}$ is unitary, and hence $(\mathbf{1}, \mathbf{1}, z)$ defines a relative triple for the ideal $C_0(\mathbb{D})$ of $C(\overline{\mathbb{D}})$.

Definition 3.2. The relative group $\mathrm{K}_0(A, A/J)$ is defined to be the free abelian group with one generator for each *homotopy class* of relative triple (p, q, x), subject to the relations

a) $(p_0, q_0, x_0) + (p_1, q_1, x_1) = (p_0 \oplus p_1, q_0 \oplus q_1, x_0 \oplus x_1)$, for any pair of triples (p_0, q_0, x_0) and (p_1, q_1, x_1);
b) Degenerate triples are zero.

Remark 3.3. Let A be any C*-algebra, A^+ its unitization. Then A is an ideal in A^+, with quotient map $\epsilon\colon A^+ \to A^+/A \cong \mathbb{C}$. By definition $\mathrm{K}_0(A) = \ker(\epsilon_*)$ with $\epsilon_*\colon \mathrm{K}_0(A^+) \to \mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}$ the induced group homomorphism.

Define a map

$$\mathrm{K}_0(A^+, A^+/A) \to \mathrm{K}_0(A) \tag{3.1}$$

by sending a triple (p, q, x) of elements of $M_n(A^+)$ to $[p] - [q] \in \mathrm{K}_0(A^+)$. The element $\epsilon(x) =: v \in M_n(\mathbb{C})$ is a partial isometry with $v^*v = \epsilon(p), vv^* = \epsilon(q)$, by the definitions, whence $[p]-[q] \in \ker(\epsilon_*)$, so our map has range in $\mathrm{K}_0(A)$.

Conversely, if $[p]-[q] \in \ker(\epsilon_*) = \mathrm{K}_0(A)$ with $p, q \in M_n(A^+)$, then the projections $\epsilon(p)$ and $\epsilon(q)$ in $M_n(\mathbb{C})$ determine the same K-theory class for $\mathbb{C}$ and hence have the same rank. There is then a partial isometry $z \in M_n(\mathbb{C})$ such that $z^*z = \epsilon(p)$ and $zz^* = \epsilon(q)$, so if $x \in M_n(A^+)$ is any lift of x under ϵ then (p, q, x) is a relative triple for the ideal A in A^+, which maps to $[p]-[q]$ under (3.1). Hence the the map is surjective. Injectivity is left to the reader as an exercise; the conclusion is that

$$\mathrm{K}_0(A) \cong \mathrm{K}_0(A^+, A^+/A). \tag{3.2}$$

So ordinary K-theory is a special case of the relative theory.

EXERCISE 3.4. Let A, A' be unital C*-algebras, $J \subset A$ and $J' \subset A'$ ideals. Then a *-homomorphism

$$\alpha\colon A \to A'$$

which maps J into J', induces a group homomorphism

$$\mathrm{K}_0(A, A/J) \to \mathrm{K}_0(A', A'/J').$$

Let $J \subset A$, A unital. Then J^+ can be identified with a C*-subalgebra of A. The inclusion maps the ideal J into A. So we have an induced map

$$\mathrm{K}_0(J) \cong \mathrm{K}_0(J^+, J^+/J) \to \mathrm{K}_0(A, A/J) \tag{3.3}$$

which we call the *excision map*. (The first isomorphism is a case of (3.2)).

We are going to show that the excision map is an isomorphism. We prove it in several steps.

As in our discussion of K-theory triples for noncompact spaces, we start by showing that the first projection in a relative triple can be taken to be 'trivial,' without changing the class of the relative triple.

LEMMA 3.5. *Any relative triple for the ideal J of A is equivalent to a relative triple in which*

$$p = \begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix} \in M_n(\mathbb{C}) \subset M_n(A),$$

for some k and n.

PROOF. Suppose that (p, q, x) be a relative triple with $p, q, x \in A$ for simplicity. Note that $(1-p, 1-p, 1-p)$ is a degenerate triple. Adding it to the original triple gives the equivalent triple $(p \oplus 1-p, q', x')$ for some q', x'.

Using rotation matrices $R_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ we construct the homotopy of triples

$$\Big(\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} + R_t \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix} R_t^*, R_t q' R_t^*, R_t x R_t^*\Big).$$

As $R_{\frac{\pi}{2}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ conjugates $\begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}$ to $\begin{bmatrix} 1-p & 0 \\ 0 & 0 \end{bmatrix}$, at the $t = \frac{\pi}{2}$ end of the path, we get a triple of the required kind, and at the $t = 0$ end of the path we get the cycle $(p \oplus 1-p, q', x')$. The result follows.

□

LEMMA 3.6. *Any relative triple is equivalent to one of the form (p, q, x), where $x = up$ for some unitary $u \in M_n(A)$ connected to the identity in $M_n(A)$ by a path of unitaries, and satisfying $upu^* = q$ mod J.*

PROOF. As in Exercise 2.15, let (p, q, x) be a triple. Let

$$w = \begin{bmatrix} \pi(x), & 1 - \pi(x)\pi(x)^* \\ \pi(x)^*\pi(x) - 1 & \pi(x)^* \end{bmatrix}.$$

Then w is unitary in $M_2(A/J)$ and is connected to the identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by a path of unitaries in $M_2(A/J)$. Since a path of unitaries starting at the identity can be lifted under $\pi \colon A \to A/J$ to a path of unitaries starting at the identity in $M_2(A)$, we get a unitary $u \in A$, connected to the identity, and such that $\pi(u) = w$. Then, working mod J, we compute

$$u \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & 1 - xx^* \\ x^*x - 1 & x^* \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$$

since $xp = x \bmod J$ and $(x^*x - 1)p = 0 \bmod J$, due to $x^*x = p \bmod J$. The triple

$$(\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix})$$

is a degenerate perturbation of the cycle we started with, and $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} - u \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$ is in J by the computation just done. So the triple we started with is equivalent to

$$(\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}, u \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}).$$

The fact that $upu^* = q \bmod J$ follows from the construction of u, which equals $w \bmod J$.

This proves the Lemma. □

LEMMA 3.7. *Any triple (p, q, pu) where u is a unitary in $M_n(A)$ connected to the identity by a path of unitaries, and such that $upu^* = q$, is equivalent to one of the form (p, q, p).*

Note that if one has a triple of the form (p, q, p) then by the definitions, it must be that $p - q \in J$.

PROOF. Let $(u_t)_{t\in[0,1]}$ be a path of unitaries with $u_1 = u$ and $u_0 = 1$. Then

$$(p, u_t^* uqu^* u_t, pu_t)$$

is a path of triples. When $t = 0$ we get (p, uqu^*, p). When $t = 1$ we get (p, q, pu), the original triple. The result is proved.

□

THEOREM 3.8. *For any ideal J in a unital C^*-algebra A, the excision map* (3.3) *is an isomorphism.*

PROOF. We prove surjectivity and leave injectivity as an exercise.

Suppose that (p, q, x) is a relative triple. By Lemma 3.5, it is equivalent to a triple in which $p = \begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix}$, and in particular, p is in the range of the inclusion $J^+ \to A$. Next, by Lemma 3.6, the triple can be replaced by one in which $x = up$, where u is a unitary in A, connected to the identity by a path of unitaries, and by Lemma 3.7 the unitary u can be removed by a homotopy, to get a triple now of the form (p, q, p). The projection p remains of the form $\begin{bmatrix} 1_m & 0 \\ 0 & 0 \end{bmatrix}$ for some m, since p has not been changed through any of the previous steps, except having zeros added to it. In particular, since $p - q \in J$. Now since $p = \begin{bmatrix} 1_m & 0 \\ 0 & 0 \end{bmatrix}$, both p and q are in the range of the inclusion $J^+ \to A$. The excision map sends $[p] - [q]$ to the class of the triple (p, q, x) as claimed, and so excision is surjective.

∎

As a consequence of the excision isomorphism, we deduce the existence of the long exact sequence in C*-algebra K-theory, as follows.

Let A be unital, $J \subset A$ an ideal. Let $\alpha\colon \mathrm{K}_0(A/A/J) \to \mathrm{K}_0(A)$ be the group homomorphism

$$\alpha([(p,q,x)]) := [p] - [q] \in \mathrm{K}_0(A).$$

LEMMA 3.9. *The sequence of groups*

$$\mathrm{K}_0(A, A/J) \xrightarrow{\alpha} \mathrm{K}_0(A) \xrightarrow{\pi_*} \mathrm{K}_0(A/J)$$

is exact in the middle.

PROOF. Clearly $\pi_* \circ \alpha$ is the zero homomorphism, so $\mathrm{ran}(\alpha) \subset \ker(\pi^*)$. To show the other equality, let $[p] - [q] \in \mathrm{K}_0(A)$, where p, q are projections in $M_n(A)$ for some n, such that $\pi_*([p]-[q]) = 0$. Then by Exercise 1.6, $\pi(p) \oplus 1_k$ is Murray-von-Neumann equivalent to $\pi(q) \oplus 1_k$, for some k, so there exists $z \in M_{n+k}(A/J)$ such that $z^*z = \pi(p) \oplus 1_k$, $zz^* = \pi(q) \oplus 1_k$. Let $x \in M_{n+k}(A)$ be any lift of z. Then (p,q,x) is a triple such that $\alpha([(p,q,x)]) = [p] - [q]$.

∎

COROLLARY 3.10. *Let J be an ideal in a unital C*-algebra A, Then the sequence of groups*

$$\mathrm{K}_0(J) \to \mathrm{K}_0(A) \to \mathrm{K}_0(A/J)$$

is exact in the middle.

The proof is easy, by the Excision theorem.

Let F be any functor from the category of C*-algebras and C*-algebra homomorphisms, to the category of abelian groups. Since F is a functor, the inclusion $i\colon J \to A$ of an ideal in a C*-algebra determine a sequence of group homomorphisms

$$F(J) \xrightarrow{F(i)} F(A) \xrightarrow{F(\pi)} F(A/J).$$

The functor is called *half-exact* if this sequence of group homomorphisms is exact in the middle. The functor is *homotopy-invariant* if $F(\alpha) = F(\beta)$ for any pair of homotopic *-homomorphisms $\alpha, \beta\colon A \to B$.

We have proved so far that the functor K_0 is both homotopy invariant and half-exact.

We are going to show that *any* half-exact, homotopy-invariant functor determines a long exact sequence of the form

$$\begin{aligned}(3.4)\quad F(J) \xrightarrow{F(i)} F(A) \xrightarrow{F(\pi)} F(A/J)&\\ \xrightarrow{\delta} F(SJ) \xrightarrow{F(Si)} F(SA) \xrightarrow{F(S\pi)} F\big(S(A/J)\big)&\\ \xrightarrow{\delta} F(S^2J) \xrightarrow{F(S^2i)} F(S^2A) \xrightarrow{F(S^2\pi)} F\big(S^2(A/J)\big) \xrightarrow{\delta} \cdots&\end{aligned}$$

This will in particular hold for the functor K_0.

Let $\pi\colon A \to A/J$ be the quotient map. We define two auxilliary C*-algebras.

Let

$$\begin{aligned}(3.5)\quad C_\pi := \{(a,f) \in A \oplus C([0,1], A/J) \mid f(0) = 0,\ f(1) = \pi(a)\},&\\ Q := \{f\colon [0,1] \to A \mid f(0) \in J\}.&\end{aligned}$$

called the *mapping cone* of π.

There is an obvious inclusion $i\colon J \to Q$, as constant functions. There is also a map $\rho\colon Q \to J$, defined $\rho(f) := f(0)$. Clearly $\rho\colon i = \mathrm{id}_J$. On the other hand,

$$\iota_t(f)(s) := f(ts)$$

gives a homotopy between $i \circ \rho$ and the identity homomorphism $Q \to Q$.

That is, i and ρ are homotopy-inverses of each other.

Let $k\colon J \to C_\pi$ be the inclusion $j(a) := (a, 0)$.

LEMMA 3.11. *k induces an isomorphism $k_*\colon \mathrm{K}_*(J) \to \mathrm{K}_*\big(C_\pi\big)$.*

PROOF. Consider the map

$$\alpha\colon Q \to C_\pi, \quad \alpha(f) = (f(1), \pi \circ f). \tag{3.6}$$

Note that $(\pi \circ f)(0) = 0$ since $f(0) \in J$. The kernel of α is the ideal $\{f \in Q \mid f(1) = 0,\ \pi \circ f = 0\}$ and such f map into J. It follows that

$$\ker(\alpha) \cong C_0\big((0,1], J\big),$$

which is a contractible C*-algebra (Exercise 1.26). The sequence of groups

$$\mathrm{K}_0\big(\ker(\alpha)\big) \to \mathrm{K}_0(Q) \to \mathrm{K}_0(C_\pi) \tag{3.7}$$

is exact in the middle, and $\mathrm{K}_0\big(\ker(\alpha)\big)$ is the zero group by the above discussion. Hence

$$\alpha_*\colon \mathrm{K}_0(Q) \to \mathrm{K}_0(C_\pi)$$

is injective.

Now the composition

$$J \xrightarrow{i} Q \xrightarrow{\alpha} C_\pi$$

equals k. Hence $k_* = \alpha_* \circ i_*$ and i_* is an isomorphism and α_* is injective.

Hence k_* is injective.

For surjectivity of k_* observe that k actually embeds J as an *ideal* in C_π. It is the kernel of the map

$$\beta\colon C_\pi \to C\big((0,1], A/J\big), \;\; \beta(a, f) := f.$$

We obtain a sequence of groups

$$\mathrm{K}_0(J) \xrightarrow{k_*} \mathrm{K}_0(C_\pi) \to \mathrm{K}_0\left[C\big((0,1], A/J\big)\right] \tag{3.8}$$

and $\mathrm{K}_0\left[C_0\big((0,1], A/J\big)\right]$ is the zero group, since $C_0\big((0,1], A/J\big)$ is contractible.

This shows that k_* is surjective.

□

Note that $S(A/J) := \{f\colon [0,1] \to A/J \mid f(0) = f(1) = 0\}$ is an ideal in C_π. We let $s\colon S(A/J) \to C_\pi$ be the inclusion.

DEFINITION 3.12. The *connecting homomorphism*, or boundary homomorphism,

$$\partial\colon \mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$$

is defined to be the composition

$$\mathrm{K}_1(A/J) := \mathrm{K}_0\big(S(A/J)\big) \xrightarrow{s_*} \mathrm{K}_0(C_\pi) \xrightarrow{k_*^{-1}} \mathrm{K}_0(J), \tag{3.9}$$

where the first map is induced from the inclusion $s\colon S(A/J) \to C_\pi$ and the second map is the inverse of the group isomorphism $k_*\colon \mathrm{K}_0(J) \to \mathrm{K}_0(C_\pi)$ induced by the inclusion $k\colon J \to C_\pi$.

We now take our extension $0 \to J \to A \to A/J \to 0$. It generates the sequence of groups and group homomorphisms

$$\mathrm{K}_0(J) \xrightarrow{j_*} \mathrm{K}_0(A) \xrightarrow{\pi_*} \mathrm{K}_0(A/J), \tag{3.10}$$

which is exact in the middle.

The sequence $0 \to C_0(\mathbb{R}) \otimes J \to C_0(\mathbb{R}) \otimes A \to C_0(\mathbb{R}) \otimes A/J \to 0$ is still exact, and writing $C_0(\mathbb{R}) \otimes J = SJ$ and so on, we get a sequence of groups

$$\mathrm{K}_0(SJ) \to \mathrm{K}_0(SA) \to \mathrm{K}_0\big(S(A(A/J)\big) \tag{3.11}$$

which by the definitions can be written

$$\mathrm{K}_1(J) \xrightarrow{j_*} \mathrm{K}_1(A) \xrightarrow{\pi_*} \mathrm{K}_1(A/J). \tag{3.12}$$

It is exact in the middle.

We then connect the end of (3.12) with (3.10) using the boundary map (3.9). to get the spliced-together sequence

$$\mathrm{K}_1(J) \xrightarrow{j_*} \mathrm{K}_1(A) \xrightarrow{\pi_*} \mathrm{K}_1(A/J) \xrightarrow{\partial} \mathrm{K}_0(J) \xrightarrow{j_*} \mathrm{K}_0(A) \xrightarrow{\pi_*} \mathrm{K}_0(A/J). \tag{3.13}$$

LEMMA 3.13. *In reference to* (3.13), *we have* $\ker(\partial) = \operatorname{ran}(\pi_*)$ *and* $\operatorname{ran}(\partial) = \ker(j_*)$.

PROOF. Consider the diagram

$$\begin{tikzcd} \mathrm{K}_1(A/J) \arrow[r, "s_*"] \arrow[dr, "\partial"'] & \mathrm{K}_0(C_\pi) \arrow[r, "p_*"] & \mathrm{K}_0(A) \\ & \mathrm{K}_0(J) \arrow[u, "k_*"] \arrow[ur, "j_*"'] & \end{tikzcd}$$

Here s is the map induced by the inclusion $s\colon S(A/J) \to C_\pi$ of $S(A/J)$ as an ideal in C_π, p_* is induced by the map $C_\pi \to A$, whose kernel is the image of s, and $k\colon J \to C_\pi$ is the inclusion as constant functions.

The diagram commutes by the definitions. The top row is exact in the middle because it comes from a short exact sequence of C*-algebras. It is now apparent that $\operatorname{ran}(\partial) = \ker(j_*)$, as claimed.

Now consider the exact sequence

$$0 \to S(A/J) \xrightarrow{s} C_\pi \xrightarrow{q} A \to 0. \tag{3.14}$$

where $q(a, f) = a$. It generates a connecting map ∂' fitting into a sequence

$$\mathrm{K}_1\big(S(A/J)\big) \to \mathrm{K}_1(C_\pi) \to \mathrm{K}_1(A) \xrightarrow{\partial'} \mathrm{K}_0\big(S(A/J)\big) \xrightarrow{s_*} \mathrm{K}_0(C_\pi) \to \mathrm{K}_0(A). \tag{3.15}$$

and $\operatorname{ran}(\partial') = \ker(s_*)$ by what has already been proved, while $\ker(s_*) = \ker(\partial)$ by the definitions, so we get that

$$\ker(\partial) = \operatorname{ran}(\partial').$$

But the map ∂' is simply equal to π_* as a map $\mathrm{K}_1(A) \to \mathrm{K}_1(A/J)$. The result follows.

$\square$

THEOREM 3.14. *Let J be an ideal in a C*-algebra A. Then there exist connecting homomorphisms $\partial\colon \mathrm{K}_{i+1}(A/J) \to \mathrm{K}_i(A)$, for each $i = 0, 1, \ldots$, making the sequence of groups and group homomorphisms*

$$\begin{aligned} \cdots \to \mathrm{K}_2(J) \xrightarrow{i_*} \mathrm{K}_2(A) \xrightarrow{\pi_*} \mathrm{K}_0(A/J) & \\ \xrightarrow{\partial} \mathrm{K}_1(J) \xrightarrow{i_*} \mathrm{K}_1(A) \xrightarrow{\pi_*} \mathrm{K}_1(A/J) & \\ \xrightarrow{\partial} \mathrm{K}_0(J) \xrightarrow{i_*} \mathrm{K}_0(A) \xrightarrow{\pi_*} \mathrm{K}_0(A/J)\big) & \end{aligned} \tag{3.16}$$

exact (with nothing known about the right end-point), and with the following naturality property.

*If $\varphi\colon A_1 \to A_2$ is a *-homomorphism, mapping an ideal $J_1 \subset A_1$ to an ideal J_2 in A_2, then the diagram*

(3.17)

$$\begin{tikzcd} \cdots & \mathrm{K}_{i+1}(J_1) \arrow[r] \arrow[d] & \mathrm{K}_{i+1}(A_1) \arrow[r] \arrow[d] & \mathrm{K}_{i+1}(A_1/J_1) \arrow[r] & \mathrm{K}_{i-1}(J_1) \arrow[r] \arrow[d] & \mathrm{K}_{i-1}(A) \arrow[r] & \cdots \\ \cdots & \mathrm{K}_{i+1}(J_2) \arrow[r] & \mathrm{K}_{i+1}(A_2) \arrow[r] & \mathrm{K}_{i+1}(A_2/J_2) \arrow[r] & \mathrm{K}_{i-1}(J_2) \arrow[r] & \mathrm{K}_{i-1}(A_2) \arrow[r] & \cdots \end{tikzcd}$$

commutes, with the top and bottom being the long exact sequences associated to the ideals $J_1 \subset A_1$ and $J_2 \subset A_2$.

An explicit description of the connecting homomorphism

We finish this section with a fairly specific description of the connecting homomorphism

$$\delta\colon \mathrm{K}_1(A/J) := \mathrm{K}_0\big(S(A/J)\big) \to \mathrm{K}_0(J).$$

of Definition 3.12, associated to an ideal $J \subset A$.

Assume A is unital, so $\mathbb{C} \subset A$ naturally, by multiplying against the unit of A. The quotient mapping is unital. So we also have a copy of $\mathbb{C}$ in A/J.

We consider $S(A/J)$ to be the C*-algebra of continuous $f\colon [0,1] \to A/J$ with $f(0) = f(1) = 0$. Then its unitization $S(A/J)^+$ is the C*-algebra of continuous $f\colon [0,1] \to A/J$ such that $f(0) = f(1) \in \mathbb{C} \subset A/J$.

On the other hand, the mapping cone C_π is the C*-algebra of pairs (a, f) in the direct sum $A \oplus C([0,1], A/J)$ such that $f(0) = 0$ and $f(1) = \pi(a)$. Its unitization C_π^+ is pairs (a, f) where $f\colon [0,1] \to A/J$ with $f(0) \in \mathbb{C} \subset A/J$ and $f(1) = \pi(a)$. As above, we have a natural injective *-homomorphism $s\colon S(A/J) \to C_\pi$ and it extends canonically to a *-homomorphism

$$s\colon S(A/J)^+ \to C_\pi^+, \quad s(f) = \big(f(0), f\big). \tag{3.18}$$

Note that $f(0) = f(1) \in \mathbb{C}$ in this formula, so $s(f)$ lies in C_π^+. The other ingredient in the boundary map is the inclusion $k\colon J \to C_\pi$. It extends to a *-homomorphism

$$k\colon J^+ \to C_\pi^+, \quad k(a) := (a, \pi(a)), \tag{3.19}$$

for $a \in J^+$ understood as a C*-subalgebra of A. Note that the restriction of the quotient map π to J^+, has kernel exactly equal to J.

Now take a projection $p \in S(A/J)^+$. It is a loop of projections: a continuous, projection-valued map $p\colon [0,1] \to A/J$ with $p(0) = p(1) = \lambda \in \mathbb{C}$. By the path-lifting property of projections, p lifts under the quotient map $\pi\colon A \to A/J$ to a path

$$\tilde{p}\colon [0,1] \to A$$

of projections in A such that $\tilde{p}(1) = \lambda$. We know that $\pi\big(\tilde{p}(0)\big) = p(0) = \lambda$, so that $\tilde{p}(0) - \lambda$ maps to zero under the quotient map $\pi\colon J^+ \to A$ and hence $\tilde{p}(0) - \lambda \in J$, whence $\tilde{p}(0) \in J^+$.

We set

$$\mathrm{Twist}(p) := [\tilde{p}(0)] \in \mathrm{K}_0(J^+). \tag{3.20}$$

Now consider the image of $[p]$ under $s_*\colon \mathrm{K}_0\big(S(A/J)^+\big) \to \mathrm{K}_0(C_\pi^+)$. By the definitions, see (3.12), the class $s_*([p])$ is the class of the projection in C_π^+ given by the element

$$\big(p(0), p\big) \in C_\pi^+.$$

On the other hand, consider the map $k\colon J^+ \to C_\pi^+$, given by (3.19). By its definition, $k_*([\tilde{p}(0)]) \in \mathrm{K}_0(C_\pi^+)$ is the class of the projection $\big(\tilde{p}(0), p(0)\big) \in C_\pi^+$.

LEMMA 3.15. *In the above notation, the projections $\big(\tilde{p}(0), p(0)\big)$ and $\big(p(0), p\big)$ in C_π^+ are homotopic.*

In particular,

$$k_*([\tilde{p}(0)]) = s_*([p]) \in \mathrm{K}_0(C_\pi^+).$$

PROOF. Let $q_s \in C_\pi^+$ be the projection $q_s := \big(\tilde{p}(s), p_s\big)$, where $p_s\colon [0,1] \to A/J$ is the projection-valued map $p_s(t) := p(ts)$. For each $s \in [0,1]$, $\pi\big(\tilde{p}(s)\big) = p(s) = p_s(1)$, and, moreover, $p_s(0) = p(0) \in \mathbb{C}$, so q_s really is in C_π^+. It is clearly a projection. When $s = 0$, since p_0 is the constant function $p(0)$ we obtain the projection

$$\big(\tilde{p}(0), p(0))\big)$$

and when $s = 1$ we get, since $p_1 = p$,

$$\big(\tilde{p}(1), p\big).$$

Moreover, $\tilde{p}(1) = p(1) = p(0) \in \mathbb{C}$, by construction, and so the two endpoints of our path are the two given projections, as claimed.

$\square$

Now let $p, q \in S(A/J)^+$ be projections such that $[p] - [q] \in \mathrm{K}_0\big(S(A/J)\big) = \ker(\epsilon_*)$, with $\epsilon\colon S(A/J)^+ \to \mathbb{C}$ the usual augmentation.

I claim that

$$\delta([p] - [q]) = \mathrm{Twist}(p) - \mathrm{Twist}(q) \in \mathrm{K}_0(J).$$

Indeed, from Lemma 3.15,

$$s_*([p] - [q]) = s_*([p]) - s_*([q]) = k_*([\tilde{p}(0)]) - k_*([\tilde{q}(0)]) = k_*\big(\mathrm{Twist}(p) - \mathrm{Twist}(q)\big)$$

and hence

$$\delta([p] - [q]) := (k_*^{-1} \circ s_*)([p] - [q]) = \mathrm{Twist}(p) - \mathrm{Twist}(q)$$

follows.

The extension of this argument to where the projections are matrix-valued, is routine and is left to the reader. In fact it essentially follows from simply replacing A by $M_n(A)$, J by $M_n(J)$, *etc*, in the given argument.

THEOREM 3.16. *Let $J \subset A$ be an ideal., A unital. Let $p\colon [0,1] \to M_n(A/J)$ be a continuous, projection-valued map with $p(0) = p(1) \in M_n(\mathbb{C}) \subset M_n(A/J)$. Let $\tilde{p}(1)$ be lift of $p(1) \in M_n(\mathbb{C})$ to $M_n(A)$, and let $\tilde{p}\colon [0,1] \to M_n(A)$ be a lifting of the path p with $\tilde{p}(1)$ prescribed as in the previous sentence. Then $\tilde{p}(0) \in J^+ \subset A$, and if*

$$\mathrm{Twist}(p) := [\tilde{p}(0)] \in \mathrm{K}_0(J^+),$$

then the connecting map

$$\delta\colon \mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$$

satisfies

$$\delta([p] - [q]) = \mathrm{Twist}(p) - \mathrm{Twist}(q),$$

for any group element $[p] - [q] \in \mathrm{K}_1(A/J)$.

EXERCISE 3.17. The Mischenko element β defines a canonical loop of vector bundles over the circle $\mathbb{T}$, which is the boundary of the closed disk $\overline{\mathbb{D}}$. Since $\mathbb{D}^+ \cong S^2$, the 2-sphere,

$$\mathrm{Twist}(\beta) \in \mathrm{K}_0(S^2).$$

Show that $\mathrm{Twist}(\beta) = [H]$ is the class of the Hopf bundle.

Finally, we complete this section by describing the connecting map

$$\delta\colon \mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$$

in terms of the description of $\mathrm{K}_1(A/J)$ as equivalence classes of unitaries in (matrix algebras over) A/J.

LEMMA 3.18. *If A is a unital C^*-algebra and $a \in A$ with $\|a\| \leq 1$, then*

$$w := \begin{bmatrix} a & -(1-aa^*)^{\frac{1}{2}} \\ (1-a^*a)^{\frac{1}{2}} & a^* \end{bmatrix} \tag{3.21}$$

is a unitary in $M_2(A)$.

PROOF. This follows from a direct calculation using the fact that

$$a(1-a^*a)^{\frac{1}{2}} = (1-aa^*)^{\frac{1}{2}}a$$

(see Exercise 3.19.) □

Now let $w \in M_n(A/J)$ be a unitary. We need to describe the corresponding cycle for $\mathrm{K}_0(SA)$ and describe its twist, which will be a cycle for $\mathrm{K}_0(J)$, so there will be two steps in the calculation.

Firstly, looking back at the proof of Proposition 2.2, we need to find a path of unitaries $w\colon [0,1] \to M_{2n}(A/J)$ such that

$$w(0) = \begin{bmatrix} w & 0 \\ 0 & w^* \end{bmatrix}, \quad w(1) = \begin{bmatrix} 1_n & 0 \\ 0 & 1_n \end{bmatrix}. \tag{3.22}$$

From this is obtained a loop of projections in $M_{2n}(A)$ by setting

$$p_t := w(t) \begin{bmatrix} 1_n & 0 \\ 0 & 0_n \end{bmatrix} w(t)^*.$$

In the second step, the twist $\mathrm{Twist}(p)$ is by definition obtained by lifting this path of projections in A/J to a path of projections in A, starting at $\begin{bmatrix} 1_n & 0 \\ 0 & 0_n \end{bmatrix} \in M_n(J^+) \subset M_n(A)$.

We can carry out both steps at once quite efficiently, however. Let $a \in M_n(A)$ be any lift of $u \in M_n(A/J)$ to an element of A with $\|a\| \leq 1$. Set

$$a_t := t \cdot 1_n + (1-t) \cdot a \in M_n(A).$$

Then $\|a_t\| \leq 1$, and hence

$$u(t) := \begin{bmatrix} a_t & -(1-a_ta_t^*)^{\frac{1}{2}} \\ (1-a_t^*a_t)^{\frac{1}{2}} & a_t^* \end{bmatrix} \tag{3.23}$$

is unitary in $M_{2n}(A)$. Note that $\pi(a_0) = \pi(a) = u$ and that $u_1 = \begin{bmatrix} 1_n & 0 \\ 0 & 1_n \end{bmatrix} = 1_{2n}$. Therefore the path $w := \pi \circ u\colon [0,1] \to M_{2n}(A/J)$ is as required by (3.22). We obtain the loop of projections in $M_{2n}(A/J)$,

$$p_t := w(t) \begin{bmatrix} 1_n & 0 \\ 0 & 0_n \end{bmatrix} w(t)^*.$$

This loop, however, has a ready-make lift to a path of projections in $\tilde{p}_t \in M_{2n}(A)$ starting at $\begin{bmatrix} 1_n & 0 \\ 0 & 0_n \end{bmatrix}$ since we may set

$$\tilde{p}(t) := u(t) \begin{bmatrix} 1_n & 0 \\ 0 & 0_n \end{bmatrix} u(t) * .$$

with $u(t)$ the path of unitaries in (3.23) .

This lifted path has endpoints

$$\tilde{p}(1) = \begin{bmatrix} 1_n & 0 \\ 0 & 0_n \end{bmatrix}$$

and

$$\tilde{p}(0) = \begin{bmatrix} a & -(1-aa^*)^{\frac{1}{2}} \\ (1-a^*a)^{\frac{1}{2}} & a^* \end{bmatrix} \begin{bmatrix} 1_n & 0 \\ 0 & 0_n \end{bmatrix} \begin{bmatrix} a^* & (1-a^*a)^{\frac{1}{2}} \\ -(1-aa^*)^{\frac{1}{2}} & a \end{bmatrix} = \begin{bmatrix} aa^* & a(1-a^*a)^{\frac{1}{2}} \\ (1-a^*a)^{\frac{1}{2}} & 1-aa^* \end{bmatrix}, \tag{3.24}$$

which is a projection in $M_{2n}(A)$. By the definitions,

$$\operatorname{Twist}(p) = [\tilde{p}(0)] - [1_n] \in \mathrm{K}_0(J^+).$$

We have proved the following.

THEOREM 3.19. *Let $u \in M_n(A/J)$ be a unitary representing a class $[u] \in \mathrm{K}_1(A/J)$. Let $a \in M_n(A)$ such that $\|a\| \leq 1$ and $\pi(a) = u$.*
Set

$$q := \begin{bmatrix} aa^* & a(1-a^*a)^{\frac{1}{2}} \\ (1-a^*a)^{\frac{1}{2}}a^* & 1-a^*a \end{bmatrix} \in M_{2n}(A). \tag{3.25}$$

Then q is a projection in $M_{2n}(J^+)$ such that $\pi(q) = \begin{bmatrix} 1_n & 0 \\ 0 & 0_n \end{bmatrix} \in M_{2n}(A/J)$, and

$$\delta([u]) = [q] - [1_n] \in \mathrm{K}_0(J),$$

holds, where $\delta\colon \mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ is the connecting map in the long exact sequence.

4. Examples of the connecting homomorphism

We start by considering an extremely important instance of an exact sequence of C*-algebras: namely the sequence

$$0 \to \mathcal{K}(H) \to \mathbb{B}(H) \to \mathcal{Q}(H) \to 0,$$

where $\mathcal{Q}(H) = \mathbb{B}(H)/\mathcal{K}(H)$ is the Calkin algebra. This exact sequence generates a long exact sequence of K-theory groups, and noting that $\mathrm{K}_i(\mathcal{K}) \cong \mathrm{K}_i(\mathbb{C})$, it has the form

$$\begin{aligned} \cdots \to \mathrm{K}_2(\mathbb{C}) \to \mathrm{K}_2(\mathbb{B}) \to \mathrm{K}_2(\mathcal{Q}) &\xrightarrow{\delta} \\ \mathrm{K}_1(\mathbb{C}) \to \mathrm{K}_1(\mathbb{B}) \to \mathrm{K}_1(\mathcal{Q}) &\xrightarrow{\delta} \\ \mathrm{K}_0(\mathbb{C}) \to \mathrm{K}_0(\mathbb{B}) \to \mathrm{K}_0(\mathcal{Q})& \end{aligned} \tag{4.1}$$

At this stage, we are interesting in computing the connecting maps δ. Using the isomorphism $\mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}$, the first of these connecting maps boils down to a group homomorphism

$$\delta\colon \mathrm{K}_1(\mathcal{Q}) \to \mathbb{Z}.$$

In order to compute it explicitly, we start with a general lemma.

LEMMA 4.1. *Suppose that A is unital and $J \subset A$ is an ideal.*
Suppose that $u \in M_n(A/J)$ and that u lifts to a partial isometry *$v \in M_n(A)$. Then*

$$\delta([u]) = [1 - v^*v] - [1 - vv^*] \in \mathrm{K}_0(J),$$

where $\delta\colon \mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ is the connecting map of the previous section.

PROOF. From Theorem 3.19,

$$\delta([u]) = [q] - [1_n]$$

where $q = \begin{bmatrix} vv^* & v(1-v^*v)^{\frac{1}{2}} \\ (1-v^*v)^{\frac{1}{2}}v^* & 1-v^*v \end{bmatrix} \in M_{2n}(A)$. Since v is a partial isometry, $(1-v^*v)^{\frac{1}{2}} = 1-v^*v$ is projection to the kernel of v and so the off-diagonal entries in the matrix q are zero. We get

$$[q] - [1_n] = [vv^*] + [1-v^*v] - [1_n] = [1-v^*v] - [1-vv^*]$$

as claimed.

$\square$

LEMMA 4.2. *If $u \in \mathcal{Q}$ is a unitary, then u has a lift to $\mathbb{B}(H)$ which is a partial isometry.*

PROOF. Let $a \in \mathbb{B}$ be any lift of u. Then it has a polar decomposition $a = v|a|$ where v is a partial isometry from $\ker(a)^\perp$ to $\overline{\operatorname{ran}(a)}$. If $\pi \colon \mathbb{B} \to \mathcal{Q}$ is the quotient map, then

$$u = \pi(a) = \pi(v)\pi(|a|).$$

Now it follows from the definitions that the projections vv^* and v^*v satisfy

$$a \cdot v^*v = a, \quad vv^* \cdot a = a.$$

Hence, projecting these equations to the Calkin algebra we get

$$u \cdot \pi(v)^*\pi(v) = u, \ \ \pi(v)\pi(v)^* \cdot u = u \in \mathcal{Q}.$$

Since u is unitary, we get

$$\pi(v)^*\pi(v) = 1, \quad \pi(v)\pi(v)^* = 1.$$

Hence $\pi(v)$ is unitary in the Calkin algebra, and hence $\pi(|a|)$ is also unitary, and is also positive. Any positive unitary in a C*-algebra must be equal to 1. Hence

$$u = \pi(v).$$

So v provides the required lift of u to a partial isometry in $\mathbb{B}$.

$\square$

COROLLARY 4.3. *Let H be a separable Hilbert space, $\mathcal{Q} = \mathbb{B}(H)/\mathcal{K}(H)$ the Calkin algebra of H. Let u be a unitary in $M_n(\mathcal{Q})$ for some n, representing a class $[u] \in \mathrm{K}_1(\mathcal{Q})$. Lift u to a bounded operator*

$$T \colon H \oplus \cdots H \to H \oplus \cdots \oplus H.$$

Then T is Fredholm, and

$$\delta([u]) = \operatorname{index}(T) \in \mathrm{K}_0(\mathcal{K}) \cong \mathbb{Z},$$

where

$$\delta \colon \mathrm{K}_1(\mathcal{Q}) \to \mathrm{K}_0(\mathcal{K}) \cong \mathbb{Z}$$

is the connecting homomorphism of the exact sequence

$$0 \to \mathcal{K} \to \mathbb{B} \to \mathcal{Q} \to 0.$$

PROOF. Since $M_n\big(\mathcal{Q}(H)\big) \cong \mathcal{Q}(H \oplus \cdots H)$ we assume for simplicity that $u \in \mathcal{Q}$ from the start. If T is a lift of u to $\mathbb{B}(H)$ then T is essentially unitary and hence Fredholm. Its Fredholm index $\operatorname{index}(T) \in \mathbb{Z}$ is therefore well-defined, and independent of the lift.

By Lemma 4.2, we can find a lift T which is a partial isometry. By Lemma 4.1,

$$\delta([u]) = [1 - T^*T] - [1 - TT^*] = [\mathrm{pr}_{\ker(T)}] - [\mathrm{pr}_{\ker(T^*)}] \in \mathrm{K}_0(\mathcal{K}),$$

the isomorphism $\mathrm{K}_0(\mathcal{K}) \cong \mathbb{Z}$ maps $[\mathrm{pr}_{\ker(T)}] - [\mathrm{pr}_{\ker(T^*)}]$ to the difference of integers

$$\dim \ker(T) - \dim \ker(T^*) = \operatorname{index}(T)$$

as required.

□

Corollary 4.4. *Let*

$$\delta\colon \mathrm{K}^{-1}(\mathbb{T}) \to \mathrm{K}_0(\mathcal{K}) \cong \mathbb{Z}$$

be the connecting homomorphism of the Toeplitz extension

$$0 \to \mathcal{K} \to \mathcal{T} \to C(\mathbb{T}) \to 0.$$

Then if $u \in M_n(C(\mathbb{T}))$ is a unitary, lift u to a matrix of Toeplitz operators on $l^2(\mathbb{N})$ and let

$$T_u\colon l^2(\mathbb{N}) \oplus \cdots \oplus l^2(\mathbb{N})$$

be the associated generalized Toeplitz operator. Then T is Fredholm and

$$\delta([u]) = \mathrm{index}(T_u),$$

where $\mathrm{index}(T_u)$ *is the Fredholm index of T_u.*

Proof. We have a commutative diagram of C*-algebras and homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{T} \arrow[r] \arrow[d, "i"] & C(\mathbb{T}) \arrow[r] \arrow[d, "\tau"] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r] & \mathbb{B} \arrow[r] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd} \tag{4.2}$$

The map $i\colon \mathcal{T} \to \mathbb{B}$ is the natural inclusion; the map $\tau\colon C(\mathbb{T}) \to \mathcal{Q}$ maps $f \in C(\mathbb{T})$ to the image of

$$T_f \in \mathbb{B}\big(l^2(\mathbb{N})\big)/\mathcal{K}\big(l^2(\mathbb{N})\big) \cong \mathcal{Q}\big(l^2(\mathbb{N})\big).$$

By naturaity of connecting maps with respect to *-homomorphisms, the diagram

$$\begin{tikzcd} \mathrm{K}^{-1}(\mathbb{T}) \arrow[r, "\delta"] \arrow[d, "\tau_*"] & \mathrm{K}_0(\mathcal{K}) = \mathbb{Z} \arrow[d, "\mathrm{id}"] \\ \mathrm{K}_1(\mathcal{Q}) \arrow[r, "\delta"] & \mathrm{K}_0(\mathcal{K}) = \mathbb{Z} \end{tikzcd} \tag{4.3}$$

commutes, where the δ's on the top and bottom are associated to the two exact sequences. The result follows from Corollary 4.4.

□

The connecting homomorphism for the boundary extension of the disk.

We next consider a more purely topological instance of the connecting homomorphism, *i.e.* one which involves only topological K-theory of spaces.

Consider the exact sequence of C*-algebras

$$0 \to C_0(\mathbb{D}) \to C(\overline{\mathbb{D}}) \to C(\partial\mathbb{D}) = C(\mathbb{T}) \to 0 \tag{4.4}$$

of $C(\mathbb{T})$ by $C_0(\mathbb{D})$, with $\mathbb{D}$ the open disk. Since $\overline{\mathbb{D}}$ is compact and contractible, $\mathrm{K}^{-1}(\overline{\mathbb{D}}) = \mathrm{K}^{-1}(\mathrm{pt}) = 0$ and $\mathrm{K}^0(\mathbb{D}) = \mathrm{K}^0(\mathrm{pt}) = \mathbb{Z}$. Since $\mathrm{K}^0(\mathbb{T}) = \mathbb{Z}$ with generator the class of the trivial line bundle over $\mathbb{T}$, the restriction map $C(\overline{\mathbb{D}}) \to C(\mathbb{T})$ induces a surjection on K^0. Hence we get an exact sequence of groups

$$0 \to \mathrm{K}^{-1}(\mathbb{T}) \xrightarrow{\delta_t} \mathrm{K}^0(\mathbb{D}) \to \mathbb{Z} \to 0, \tag{4.5}$$

where δ_t denotes the connecting homomorphism; we subscript it by t (standing for 'topological') to distinguish it from the Toeplitz connecting map.

The map $\mathrm{K}^0(\mathbb{D}) \to \mathbb{Z}$ is induced by the C*-algebra homomorphism $C_0(\mathbb{D}) \to C(\overline{\mathbb{D}})$ and the isomorphism $\mathrm{K}^0(\overline{\mathbb{D}}) \cong \mathbb{Z}$ due to contractibility of the closed disk.

The following exercise is a good one, and does not require Bott Periodicity to solve it.

EXERCISE 4.5. Prove that if $U \subset X$ is an open subset of a compact, contractible space X (so that $\mathrm{K}_0\big(C(X)\big) = \mathrm{K}^0(X) \cong \mathbb{Z}$), then the C*-algebra inclusion $C_0(U) \to C(X)$ induces the zero homomorphism

$$\mathrm{K}^0(U) \to \mathrm{K}^0(X).$$

(*Hint.* Let $i\colon \mathrm{pt} \to X$ be an inclusion of the one-point space in X mapping the point to $x_0 \in X$; contractibility of X implies that $i^*\colon \mathrm{K}^0(X) \to \mathbb{Z}$ is an isomorphism, for any choice of x_0. But the point x_0 can be moved to be disjoint from the support of any K-theory class for U, since any such class has a compact support inside U.)

From the exercise,

$$\delta_t\colon \mathrm{K}^{-1}(\mathbb{T}) \to \mathrm{K}^0(\mathbb{D})$$

is an isomorphism of groups.

Now let $z\colon \mathbb{T} \to \mathbb{C}$ the inclusion, so $z \in C(\mathbb{T})$ is unitary and defines a class $[z] \in \mathrm{K}^{-1}(\mathbb{T})$. It lifts to the inclusion $z\colon \overline{\mathbb{D}} \to \mathbb{C}$, an element of $C(\overline{\mathbb{D}})$ of norm ≤ 1.

By Theorem 3.19,

$$q := \begin{bmatrix} |z|^2 & z\sqrt{1-|z|^2} \\ \bar{z}\sqrt{1-|z|^2} & 1-|z|^2 \end{bmatrix} \in M_2\big(C(\overline{\mathbb{D}})\big). \tag{4.6}$$

is a projection in $M_2(C_0(\mathbb{D})^+)$ such that and

$$\delta_t([z]) = [q] - [1] \in \mathrm{K}^0(\mathbb{D}). \tag{4.7}$$

EXERCISE 4.6. Let $\varphi\colon \mathbb{R}^2 = \mathbb{C} \to \mathbb{D}$ be the diffeomorphism

$$\varphi(z) := \frac{z}{\sqrt{1+|z|^2}}.$$

Show that

$$q\big(\varphi(z)\big) = p(z) := \frac{1}{1+|z|^2}\begin{bmatrix} |z|^2 & z \\ \bar{z} & 1 \end{bmatrix},$$

the projection-valued map $p\colon \mathbb{C} \to M_2(\mathbb{C})$ defined in Example 1.3; the class $[p]-[1]$ is a representation of the *Bott element* $\beta \in \mathrm{K}^0(\mathbb{R}^2)$. This shows that the isomorphism

$$\varphi^*\colon \mathrm{K}^0(\mathbb{D}) \to \mathrm{K}^0(\mathbb{R}^2)$$

satisfies

$$\varphi^*\big(\delta_t([z])\big) = \beta_{\mathbb{R}^2} \in \mathrm{K}^0(\mathbb{R}^2).$$

Thus, up to the canonical isomorphism $\mathrm{K}^0(\mathbb{D}) \cong \mathrm{K}^0(\mathbb{R}^2)$, the class $\delta([z])$ is the Bott element.

Since it is going to play a significant role in what follows, we review the construction of the 'Bott element.' It's initial source was the Hopf bundle H over $\mathbb{CP}^1$. The Hopf bundle has a rather convenient representation in terms of a projection: the map $p\colon \mathbb{CP}^1 \to M_2(\mathbb{C})$ is the projection-valued function $p\colon \mathbb{CP}^1 \to M_2(\mathbb{C})$ mapping a line $L \subset \mathbb{C}^2$ to orthogonal projection pr_L onto that line. In terms of homogeneous coordinates on $\mathbb{CP}^1$,

$$P([z,w]) = \frac{1}{|z|^2+|w|^2}\begin{bmatrix} |z|^2 & \bar{w}z \\ \bar{z}w & |w|^2 \end{bmatrix}.$$

Under the standard embedding of $\mathbb{C}$ as an open subset of $\mathbb{CP}^1$, $z \mapsto [z, 1]$ we obtain the restriction p of P to $\mathbb{C}$, given by the formula

$$p(z) = \frac{1}{|z|^2 + 1} \begin{bmatrix} |z|^2 & z \\ \bar{z} & 1 \end{bmatrix}.$$

Thinking of $\mathbb{CP}^1$ as $(\mathbb{R}^2)^+$, note that p takes the value

$$p(\infty) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and at the origin $0 \in \mathbb{C}$, takes the value

$$p(0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The Bott element is by definition

$$\beta := [p] - [1] \in \mathrm{K}^0(\mathbb{R}^2).$$

As noted above, the particular choice of homeomorphism $\mathbb{R}^2 \cong \mathbb{D}$ maps p (and β) into a corresponding projection, and K-theory class, for $C_0(\mathbb{D})$, with $\mathbb{D}$ the unit disk; the formula of this new, projection-valued function on the disk, is conveniently given by

$$q(z) = \begin{bmatrix} |z|^2 & z\sqrt{1-|z|^2} \\ \bar{z}\sqrt{1-|z|^2} & 1-|z|^2 \end{bmatrix} \in M_2\big(C(\overline{\mathbb{D}})\big). \tag{4.8}$$

– conveniently because this is precisely the formula for the projection involved in the formula produced by us for

$$\partial([z]) \in \mathrm{K}_0\big(C_0(\mathbb{R}^2)\big),$$

with δ the connecting homomorphism for the disk and the closed disk.

Note that on the boundary of the disk $q(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. In particular, since it is constant on the boundary, we can extend it (by the same constant matrix value to a function $\tilde{q}\colon \mathbb{C} \to M_2(\mathbb{C})$, and even further to $\mathbb{C}^+ = (\mathbb{R}^2)^{\to} M_2(\mathbb{C})$, with value $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ at ∞.

EXERCISE 4.7. The projections $\tilde{q}$ and p are homotopic as projections in $C_0(\mathbb{R}^2)^+$. (*Hint.* Argue that the composition of the map $\varphi\colon \mathbb{R}^2 \to \mathbb{D}$ of Exercise 4.6 and the open embedding $\mathbb{D} \to \mathbb{R}^2$ is homotopic to the identity map $\mathbb{R}^2 \to \mathbb{R}^2$, through a homotopy of open embeddings fixing the point at infinity.

From the Exercise above, $[p] - [1]$ and $[q] - [1]$ define the same element of $\mathrm{K}^0(\mathbb{R}^2)$.

Note that q (or $\tilde{q}$) can be re-scaled into an arbitrarily small disk around $0 \in \mathbb{C}$, or, of course, moved into a disk centred at another point. One thus obtains varieties of formulas for projection-valued functions on $(\mathbb{R}^2)^+$, all taking the constant value $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ outside of a small open disk in the plane. The classes $[q]-[1]$ are equal to the Bott element β in the group $\mathrm{K}^0(\mathbb{R}^2)$. We have already discussed classes defined in this manner; we call them K-theory 'germs,' and the one under discussion was referred to as the 'K-theory germ of a point in $\mathbb{R}^2$.' The general outcome of that discussion was that one can produce a K-theory class for $\mathbb{R}^2$ in the following way.

Take a point $p \in \mathbb{R}^2 = \mathbb{C}$ and let $\varphi(z) = z - p$, which is non-vanishing away from p. Now let B be any closed ball around p and $B' = \mathbb{C} \setminus \mathrm{int}(B) \cup \{\infty\}$. Each are closed, contractible subsets of $(\mathbb{R}^2)^+$ and we can clutch the trivial bundles $B \times \mathbb{C}$ and $B' \times \mathbb{C}$ over $B \cap B' = \partial B$ using the clutching function $z - p$. This results in a complex line E bundle over S^2, and the difference $[E] - [1] \in \mathrm{K}^0(\mathbb{R}^2)$ equals the Bott element β.

One can obviously use more general clutching functions than $z - p$. Any complex-valued continuous function with p as as isolated zero determines enough data to use it to clutch two trivial bundles, one defined over a neighbourhood of p, one defined over its complement in S^2, to produce a vector bundle over S^2.

Let us fix $p = 0$ and clutch using the closed disk $D = \overline{\mathbb{D}}$ and D', its complement in $(\mathbb{R}^2)^+$. The intersection $D \cap T' = \mathbb{T}$ is the circle. If $u\colon \mathbb{T} \to \mathbf{U}_n$ is any unitary valued function, let E_φ be the vector bundle over S^2 defined by clutching $D \times \mathbb{C}^n$ and $D' \times \mathbb{C}^n$ using φ. Let

$$b(\varphi) := [E_\varphi] - [1_n] \in \mathrm{K}^0(\mathbb{R}^2)$$

be the corresponding 'Bott-type' element. It is an element of $\mathrm{K}^0(S^2)$ which is in the kernel of the augmentation homomorphism

$$\epsilon^*\colon \mathrm{K}^0(S^2) = \mathrm{K}^0((\mathbb{R}^2)^+) \to \mathrm{K}^0(\mathrm{pt}) = \mathbb{Z}$$

and hence defines a class in $\mathrm{K}^0(\mathbb{R}^2)$.

It is an easy exercise to check that if u and u' are homotopic $\mathbf{U}_n$-valued maps, amongst such maps, then the vector bundles E_u and $E_{u'}$ are homotopic vector bundles on S^2, and hence are isomorphic. In particular, since the clutching function

$$\begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix}$$

is homotopic to the constant clutching function $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we obtain that

$$E_{z \oplus \bar{z}} \sim S^2 \times \mathbb{C}^2,$$

isomorphic as vector bundles, and thus in $\mathrm{K}^0(S^2)$ it follows that

$$[E_z] + [E_{\bar{z}}] = [1_2] \in \mathrm{K}^0(S^2)$$

holds, so that

$$[E_z] - [1] + [E_{\bar{z}}] - [1] = 0 \in \mathrm{K}^0(S^2).$$

Hence

$$b(z) = -b(\bar{z}) \in \mathrm{K}^0(S^2).$$

Exercise 4.8. Let E_u be the bundle over S^2 obtained from $u\colon \mathbb{T} \to \mathbf{U}_n$, as in the above discussion. Let $\alpha\colon S^2 \to S^2$ be the extension of a linear, isometric map (an element of $\mathrm{O}(2, \mathbb{R})$.) Such a map restricts to a map $\alpha\colon \mathbb{T} \to \mathbb{T}$, and $u \circ \alpha$ is another clutching function determining a bundle $E_{u\circ\alpha}$ over S^2.

Show that

$$E_{u\circ\alpha} \cong \alpha^*(E_u)$$

as vector bundles over S^2.

Combining the discussion above with the exercise we obtain the following simple result.

Proposition 4.9. *If* $\alpha\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is an orthogonal map,*

$$\alpha^*\colon \mathrm{K}^0(\mathbb{R}^2) \to \mathrm{K}^0(\mathbb{R}^2)$$

the induced map, then

$$\alpha^*(\beta) = \det(\alpha) \cdot \beta,$$

with $\det(\alpha)$ *the determinant of the matrix* α*, and* β *is the Bott element.*

PROOF. The Bott element in the notation of the discuss above, is given by

$$\beta = b(z),$$

where $z\colon \mathbb{T} \to \mathbb{C}$ is the usual coordinate. Since α is homotopic to either the identity map $\mathbb{R}^2 \to \mathbb{R}^2$ or to the complex conjugation map, through elements of $\mathrm{O}(2, \mathbb{R})$, the result follows from the above discussion.

$\square$

A K*-theoretic perspective on the Toeplitz index theorem.*

A pseudo-Toeplitz operator $T = T_u + S$, for $u\colon \mathbb{T} \to \mathbb{C}^*$ smooth, say, and S a smoothing operator on the circle, is a special case of a singular integral operator. As we have seen, such an operator is Fredholm, and has a Fredholm index

$$\mathrm{index}(T) := \dim \ker(T) - \dim \ker(T^*).$$

We have shown that this 'analytic index' admits a K-theoretic interpretation involving the Toeplitz algebra and the Toeplitz extension:

$$\mathrm{index}(T) = \delta([u]),$$

where

$$\delta\colon \mathrm{K}^{-1}(\mathbb{T}) \to \mathrm{K}_0(\mathcal{K}) = \mathbb{Z}$$

is the connecting map for the Toeplitz extension.

The essential idea leading to the Atiyah-Singer index theorem, is that this index also has a purely *topological* K-theoretic interpretation. That is, it can be described purely in terms of a certain *topological* K-theoretic invariant of the symbol, which involves no noncommutative C*-algebras.

We describe this more general statement now, assuming Bott Periodicity, which implies that $\mathrm{K}^0(\mathbb{D}) = \mathbb{Z} \cdot \beta_{\mathbb{D}}$, where $\beta_{\mathbb{D}}$ is the Bott element of the open disk. Consider the connecting homomorphism

$$\delta_t\colon \mathrm{K}^{-1}(\mathbb{T}) \to \mathrm{K}^0(\mathbb{D})$$

associated to the exact sequence

$$0 \to C_0(\mathbb{D}) \to C(\overline{\mathbb{D}}) \to C(\mathbb{T}) \to 0.$$

Now for u the symbol of T as above, set

$$\mathrm{index}_t(T) := n \iff \delta_t([u]) = n \cdot \beta_{\mathbb{D}} \in \mathrm{K}^0(\mathbb{D}). \tag{4.9}$$

Then

THEOREM 4.10. *If T is a pseudo-Toeplitz operator on $\mathbb{T}$ with symbol u, then*

$$\mathrm{index}(T) = \mathrm{index}_t([u]),$$

where $[u] \in \mathrm{K}^{-1}(\mathbb{T})$ is the K*-theory class determined by u.*

This statement of the Toeplitz index theorem generalizes significantly. Let n be an odd positive integer, S^n the n-dimensional sphere. Such a sphere has a natural self-adjoint elliptic operator D (the Dirac operator) acting on sections of a certain spinor bundle over S^n. The operator D is self-adjoint, and diagonalizable. Let P_+ be the projection onto the direct sum of the non-negative eigenspaces of D. Then

$$[P_+, M_f] \in \mathcal{K}$$

for all continuous $f \in C(S^n)$, and the operators $P_+ M_f P_+$ are Fredholm operators. On the other hand, the boundary extension

$$0 \to C_0(\mathbb{D}^{n+1}) \to C(\overline{\mathbb{D}}^{n+1}) \to C(S^n) \to 0$$

determines a topologically defined connecting homomorphism

$$\delta_t \colon \mathrm{K}^{-1}(S^n) \to \mathrm{K}^0(\mathbb{D}^{n+1})$$

and following this homomorphism by the inverse of Bott Periodicity $\mathrm{K}^0(\mathbb{D}^{n+1}) \cong \mathrm{K}^0(\mathbb{R}^{n+1}) \cong \mathbb{Z}$ (as $n+1$ is even) gives a map depending only on considerations of topological K-theory

$$\mathrm{index}_t \colon \mathrm{K}^{-1}(S^n) \to \mathbb{Z}.$$

The generalized Toeplitz index theorem asserts that

$$\mathrm{index}(T) = \mathrm{index}_t([u]),$$

where T is pseudo-Toeplitz in the above sense, with symbol u, the left-hand side of this equation is the Fredholm index, the right-hand side the topological index.

Exercise 4.11. Let $i \colon \mathbb{D} \to \mathbb{R}^2$ be the inclusion of $\mathbb{D}$ as an open subset of $\mathbb{R}^2$. It determines a map

$$i! \colon \mathrm{K}^0(\mathbb{D}) \to \mathrm{K}^0(\mathbb{R}^2).$$

Let $\beta_{\mathbb{D}} \in \mathrm{K}^0(\mathbb{D})$ be the Bott element of the disk (4.7). Let $\varphi \colon \mathbb{R}^2 \to \mathbb{D}$ the diffeomorphism of Example 4.6, $\beta_{\mathbb{R}^2}$ the Bott element for $\mathbb{R}^2$. Prove that

$$i!(\beta_{\mathbb{D}}) = \varphi^*(\beta_{\mathbb{D}}).$$

The proof only involves some simple homotopies.

Exercise 4.12. If u is a non-vanishing function on $\mathbb{T}$, define $\mathrm{index}_t([u])$ by (4.9). Let E_u be the vector bundle over S^2 obtained by clutching the trivial vector bundles $S^2_\pm \times \mathbb{C}$ over the top and bottom hemispheres, using $u \colon S^2_+ \cap S^2_- \to \mathbb{C}^*$. Prove that

$$[E_u] - [1] = \mathrm{index}_t([u]) \cdot [H^*] \in \mathrm{K}^0(S^2),$$

where H^* is the dual of the Hop bundle.

This gives another way of looking at the topological index.

5. The external product operation on K-theory

While K-theory group of a commutative C*-algebras has a natural ring structure (induced by the tensor product of vector bundles, fibrewise), the K-theory of a noncommutative C*-algebra has in general no natural ring structure.

Formally, the fibrewise tensor product of two vector bundles $E \to X$ and $E' \to X$ may be interpreted in the following way. First, one forms the *external tensor product* of the two bundles, forming the bundle over $X \times X$ whose fibre at (x, y) is $E_x \otimes E'_y$, which can easily be checked to be a vector bundle over $X \times X$.

Then one *restricts* this vector bundle over $X \times X$ to the diagonal, a copy of X inside $X \times X$.

This results, obviously, in precisely the (fibrewise) tensor product bundle $E \otimes E'$.

The first step makes sense for noncommutative C*-algebras. The second step involves the diagonal map $\delta\colon X \to X \times X$, whose Gelfand dual is the multiplication map $C(X) \otimes C(X) \to C(X)$. The multiplication map makes sense for general C*-algebras but is not a *-homomorphism, unless they are commutative, and so it does not induce a product at the level of K-theory.

In this section we describe the extension of the first step to general C*-algebras. We will show that tensor product (of f.g.p. modules) gives rise to a natural bilinear map

$$\mathrm{K}_i(A) \times \mathrm{K}_j(B) \to \mathrm{K}_{i+j}(A \otimes B), \quad (x, y) \mapsto x \hat{\otimes} y$$

for any A, B, which we will call the *external product.*

Let E_A be a finitely generated right A-module. As we have shown, E_A is isomorphic to a direct summand pA^n for some projection $p \in \mathrm{End}_A(A^n) \cong M_n(A)$, and, therefore, E_A is equipped with a Hermitian form $\langle p\xi, p\eta\rangle := p\xi^*\eta p \in A$, with $\xi, \eta \in A^n$ and $\xi^*\eta := \sum_{i=1}^n \xi_i^*\eta_i$, and making E_A a Hilbert module.

Definition 5.1. Let E_A and E_B be Hilbert modules over A, B, respectively. Their *external product* $E_A \otimes_{\mathbb{C}} E_B$ of the two modules, is the completion of the algebraic tensor product of E_A and E_B over $\mathbb{C}$, with respect to the right $A \otimes B$-valued inner product

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle := \langle \xi_1, \xi_2\rangle \otimes \langle \eta_1, \eta_2\rangle.$$

Observe that the external product of a finitely generated *free* right A-module E_A, and a finitely generated free right B-module E_B, in this sense, is a free right $A \otimes B$-module. Indeed, choosing an isomorphism $E_A \cong A^n$ and an isomorphism $E_B \cong B^m$, we obtain an isomorphism on the algebraic tensor product $E_A \otimes_{\mathbb{C}} E_B$ with $A^n \otimes B^m \cong (A \otimes B)^{nm}$, where the tensor product is algebraic, that is, to the direct sum of nm copies of the algebraic tensor product $A \otimes_{\mathbb{C}} B$ of A and B.

Now completing this direct sum with respect to the Hermitian form above, results in the direct sum nm copies of the C*-algebraic tensor product $A \otimes B$ of A and B.

If E_A and E'_A are isomorphic, then they have Hermitian forms and an isometric isomorphism between the two corresponding Hilbert modules. It follows that $E_A \otimes_{\mathbb{C}} E_B \cong E'_A \otimes_{\mathbb{C}} E_B$, if E_B is a right B-module. By similar such simple arguments, one verifies that one obtains a well-defined 'external product' operation on K-theory, as summarized by the following

Proposition 5.2. *Let A and B be unital C*-algebras. If E_A and E_B be finitely generated projective A, B-modules, respectively, $E_A \otimes E_B$ their tensor product as in Definition 5.1, then $E_A \otimes_{\mathbb{C}} E_B$ is finitely generated projective over $A \otimes B$.*

1. Defining

$$[E_A]\hat{\otimes}[E_B] := [E \otimes_{\mathbb{C}} E_B],$$

gives a well-defined, $\mathbb{Z}$-bilinear map

$$\hat{\otimes}\colon \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(A \otimes B), \tag{5.1}$$

on the semigroups of isomorphism classes of f.g.p. modules, and consequent $\mathbb{Z}$-bilinear map

$$\mathrm{K}_0(A) \times \mathrm{K}_0(B) \to \mathrm{K}_0(A \otimes B).$$

2. The pairing (5.8) *is natural in the sense that if $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ are *-homomorphisms between unital C*-algebras, $\alpha \otimes \beta\colon A \otimes B \to A' \otimes B'$ their tensor product, then*

$$\alpha_*([E_A])\hat{\otimes}\beta_*([E_B]) = (\alpha \otimes \beta)_*([E_A]\hat{\otimes}[E_B]) \in \mathrm{K}_0(A' \otimes B') \tag{5.2}$$

holds, for all f.g.p. modules E_A over A, and E_B over B.

3. If $x \in \mathrm{K}_0(A)$ and $[1] \in \mathrm{K}_0(\mathbb{C})$ denotes the positive generator of $\mathrm{K}_0(\mathbb{C})$, then under the identifications $A \otimes \mathbb{C} \cong A$, $\mathbb{C} \otimes A \cong A$,

$$x \hat{\otimes} [1] = x = [1] \hat{\otimes} x$$

holds for all $x \in \mathrm{K}_0(A)$.

REMARK 5.3. If one is thinking of projections, rather than f.g.p. modules, let $p \in M_n(A)$ and $q \in M_m(B)$ be two projections. Then $p \otimes q \in M_n(A) \otimes M_m(B)$. Choosing any bijection between $\{1, \ldots, n\} \times \{1, 2, \ldots, m\} \cong \{1, 2, \ldots, nm\}$ gives an isomorphism $M_n(A) \otimes M_m(B) \cong M_{nm}(A \otimes B)$. The projection $p \otimes q$ so defined, orthogonally projections $(A \otimes B)^{nm}$ to an isomorphic copy of the external product of modules $pA^n \otimes_{\mathbb{C}} qB^m$ of Definition 5.1.

EXERCISE 5.4. Suppose $E \to X$ and $E' \to X'$ are vector bundles over X, X' (compact). Prove that the section module of the external product of the two bundles, a bundle over $X \times X'$, defined at the beginning of this section, is isomorphic to the external product $\Gamma(E) \otimes_{\mathbb{C}} \Gamma(E')$ of Definition 5.1.

EXERCISE 5.5. If $1_n = \mathbb{C}^n$ is the trivial rank n free $\mathbb{C}$-module, then

$$[1_n] \hat{\otimes} x = nx$$

for any A unital, any $x \in \mathrm{K}_0(A)$.

EXERCISE 5.6. Recall that $\mathcal{P}(\mathbb{C}) \cong \mathbb{N}$ and $\mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}$. Show directly that the external product

$$\mathcal{P}(\mathbb{C}) \times \mathcal{P}(\mathbb{C}) \to \mathcal{P}(\mathbb{C} \otimes \mathbb{C}) \cong \mathcal{P}(\mathbb{C})$$

defined above corresponds to multiplication of natural numbers.

Analogously, show that

$$\mathrm{K}_0(\mathbb{C}) \times \mathrm{K}_0(\mathbb{C}) \to \mathrm{K}_0(\mathbb{C} \otimes \mathbb{C}) \cong \mathrm{K}_0(\mathbb{C})$$

corresponds to multiplication of integers.

You can use Exercise 5.5 to show, more generally, that the external product map

$$\mathrm{K}_0(\mathbb{C}) \times \mathrm{K}_0(A) \to \mathrm{K}_0(\mathbb{C} \otimes A) = \mathrm{K}_0(A)$$

identifies with the obvious $\mathbb{Z}$-multiplication map

$$\mathbb{Z} \times \mathrm{K}_0(A) \to \mathrm{K}_0(A).$$

The external product on the K_0-groups of a pair of possibly non-unital algebras is slightly more complicated to define. Suppose A_1 and A_2 are two, possibly non-unital algebras. Let $\epsilon_1 \colon A_1^+ \to \mathbb{C}$ and $\epsilon_2 \colon A_2^+ \to \mathbb{C}$ the usual augmentation *-homomorphisms. They induce *-homomorphisms

$$\epsilon_1 \otimes 1_{A_2^+} \colon A_1^+ \otimes A_2^+ \to \mathbb{C} \otimes A_2^+ \cong A_2^+, \quad \text{and} \;\; 1_{A_1^+} \otimes \epsilon_2 \colon A_1^+ \otimes A_2^+ \to A_1^+.$$

Let

$$\pi \colon A_1^+ \otimes A_2^+ \to A_1^+ \oplus A_2^+, \qquad \pi(a_1 \otimes a_2) := \big((1_{A_1^+} \otimes \epsilon_2)(a_1 \otimes a_2), (\epsilon_1 \otimes 1_{A_2^+})(a_1 \otimes a_2)\big). \tag{5.3}$$

be the direct sum of the *-homomorphisms $1_{A_1^+} \otimes \epsilon_2$ and $\epsilon_1 \otimes 1_{A_2^+}$.

Note that $(A_1 \otimes A_2)^+$ embeds in $A_1^+ \otimes A_2^+$ by extending the obvious embedding $A_1 \otimes A_2 \to A_1^+ \otimes A_2^+$ and then extending it to the unitization by mapping the unit to the unit $1 \otimes 1$ of $A_1^+ \otimes A_2^+$.

LEMMA 5.7. *For π as in (5.3), the map* $\mathrm{K}_0((A_1 \otimes A_2)^+) \to \mathrm{K}_0(A_1^+ \otimes A_2^+)$ *induced by the inclusion* $(A_1 \otimes A_2)^+ \to A_1^+ \otimes A_2^+$*, maps* $\mathrm{K}_0(A_1 \otimes A_2)$ *to the subgroup* $\ker(\pi_*) \subset \mathrm{K}_0(A_1^+ \otimes A_2^+)$ *of* $\mathrm{K}_0(A_1^+ \otimes A_2^+)$.

Note that (omitting subscripts)

$$\ker(\pi) = \ker(\epsilon_1 \otimes 1)_* \cap \ker(1 \otimes \epsilon_2)_*.$$

PROOF. The restriction $\epsilon_1 \otimes 1_{A_2}$ of $\epsilon_1 \otimes 1_{A_2^+}$ to $A_1^+ \otimes A_2$ has kernel $A_1 \otimes A_2$. Hence the sequence

$$0 \to A_1 \otimes A_2 \to A_1^+ \otimes A_2 \to A_2 \to 0$$

is exact. It is actually split exact, using the splitting $A_2 \to A_1^+ \otimes A_2$, $a_2 \mapsto 1 \otimes a_2$. Hence we obtain an exact sequence of K_0-groups

$$0 \to \mathrm{K}_0(A_1 \otimes A_2) \to \mathrm{K}_0(A_1^+ \otimes A_2) \to \mathrm{K}_0(A_2) \to 0.$$

The quotient map is the map induced on K_0 from $\epsilon_1 \otimes 1_{A_2}$. Therefore, $\mathrm{K}_0(A_1 \otimes A_2)$ embeds in $\mathrm{K}_0(A_1^+ \otimes A_2)$ as the kernel of $(\epsilon_1 \otimes 1_{A_2})_*$.

On the other hand, $\mathrm{K}_0(A_1^+ \otimes A_2)$ injects in $\mathrm{K}_0(A_1^+ \otimes A_2^+)$ as the kernel of $(1_{A_1^+} \otimes \epsilon_2)_*$. by arguing similarly with the exact sequence

$$0 \to A_1^+ \otimes A_2 \to A_1^+ \otimes A_2^+ \to A_1^+ \to 0,$$

which is also easily checked to be split exact.

Putting these two observations together, we conclude that $\mathrm{K}_0(A_1 \otimes A_2)$ injects naturally into $\mathrm{K}_0(A_1^+ \otimes A_2^+)$ with kernel

$$\ker\big((\epsilon_1 \otimes 1_{A_2})_*\big) \cap \ker\big((1_{A_1^+} \otimes \epsilon_2)_*\big). \tag{5.4}$$

Note that the (injective) map on K_0-theory induced by the inclusion $A_2 \to A_2^+$ identifies the kernels of $(\epsilon_1 \otimes 1_{A_2})_*$ and $(\epsilon_1 \otimes 1_{A_2^+})_*$. Hence (5.4) is the same as the subgroup $\ker(\pi_*)$, by the additivity property of K_0.

□

From the Lemma, we obtain the following recipe for taking external products in the non-unital case.

Let A_1, A_2 be two, possibly non-unital algebras, A_i^+ their unitizations, the map π defined as above. Suppose $x \in \mathrm{K}_0(A_1)$, $y \in \mathrm{K}_0(A_2)$. So $x \in K_0(A_1^+)$ is in the kernel of $(\epsilon_1)_* \colon \mathrm{K}_0(A_1^+) \to \mathbb{Z}$, and $y \in \mathrm{K}_0(A_2^+)$ is in the kernel of $(\epsilon_2)_* \colon \mathrm{K}_0(A_2^+) \to \mathbb{Z}$, and

Since $(\epsilon_2)_*(y) = 0$,

$$(1_{A_1^+} \otimes \epsilon_2)_*(x \hat{\otimes} y) = x \hat{\otimes} (\epsilon_2)_*(y) = 0,$$

where we have used Proposition 5.2 2). Similarly,

$$(\epsilon_1 \otimes 1_{A_2^+})_*(x \hat{\otimes} y) = 0.$$

Hence

$$\pi_*(x \hat{\otimes} y) = 0$$

and therefore $x \hat{\otimes} y \in \mathrm{K}_0(A_1^+ \otimes A_2^+)$ is in the kernel of $\pi_* \colon \mathrm{K}_0(A_1^+ \otimes A_2^+) \to \mathrm{K}_0(A_1^+) \oplus \mathrm{K}_0(A_2^+)$. Applying the identification of this subgroup with $\mathrm{K}_0(A_1 \otimes A_2)$, we obtain therefore a map

$$\mathrm{K}_0(A_1) \times \mathrm{K}_0(A_2) \to \mathrm{K}_0(A_1 \otimes A_2) \tag{5.5}$$

for arbitrary C*-algebras A, B. Furthermore, for any n if we replace in (5.5) the C*-algebra A_1 by $S^n(A_1) = C_0(\mathbb{R}^n) \otimes A_1$, then we obtain a bilinear pairing

$$\begin{aligned}\mathrm{K}_n(A_1) \times \mathrm{K}_0(A_2) := \mathrm{K}_0\big(S^n(A_1)\big) \times \mathrm{K}_0(A_2) \to \mathrm{K}_0(S^n(A_1) \otimes A_2) \\ \cong \mathrm{K}_0\big(C_0(\mathbb{R}^n) \otimes A_1 \otimes A_2\big) \cong \mathrm{K}_n(A_1 \otimes A_2),\end{aligned} \tag{5.6}$$

which plays a role in part 3) of the Theorem below.

THEOREM 5.8. *Let A and B be C*-algebras.*
1. There is a $\mathbb{Z}$-bilinear pairing

$$\mathrm{K}_0(A) \times \mathrm{K}_0(B) \to \mathrm{K}_0(A \otimes B) \tag{5.7}$$

mapping $(x, y) \in \mathrm{K}_0(A) \times \mathrm{K}_0(B)$ to their external product $x \hat{\otimes} y$.

*2. The external product (5.7) is natural in the sense that if $\alpha \colon A \to A'$ and $\beta \colon B \to B'$ are *-homomorphisms $\alpha \otimes \beta \colon A \otimes B \to A' \otimes B'$ their tensor product *-homomorphism, then*

$$\alpha_*(x) \hat{\otimes} \beta_*(y) = (\alpha \otimes \beta)_*(x \hat{\otimes} y) \in \mathrm{K}_0(A' \otimes B') \tag{5.8}$$

holds, for all $x \in \mathrm{K}_0(A)$ and $y \in \mathrm{K}_0(B)$.

3. If

$$0 \to J \to A \to A/J \to 0 \tag{5.9}$$

is a c.p. split exact sequence of C^*-algebras and B is any C-algebra, so that*

$$0 \to J \otimes B \to A \otimes B \to A/J \otimes B \to 0 \tag{5.10}$$

is also short exact, then

$$\partial(x \hat{\otimes} y) = \partial(x) \hat{\otimes} y \tag{5.11}$$

for all $x \in \mathrm{K}_1(A/J) = \mathrm{K}_0\big(S(A/J)\big)$ and $y \in \mathrm{K}_0(B)$. The boundary map on the left-hand-side is the K*-theory connecting map for the exact sequence* (5.17)*, and the boundary map on the right-hand-side is the* K*-theory map associated to the exact sequence* (5.9)*.*

PROOF. The statement 2 follows routinely from Proposition 5.2. For 3, we recall that definition of the boundary map

$$\partial \colon \mathrm{K}_1(A/J) \to \mathrm{K}_0(J). \tag{5.12}$$

By definition, $\mathrm{K}_1(A/J) = \mathrm{K}_0\big(S(A/J)\big)$, and $S(A/J)$ is naturally isomorphic to the ideal $\{f \in C_\pi \mid f(1) = 0\}$, where C_π is the mapping cone of the quotient map. The inclusion $s \colon S(A/J) \to C_\pi$ induces a map

$$s_* \colon \mathrm{K}_0\big(S(A/J)\big) \to \mathrm{K}_0(C_\pi). \tag{5.13}$$

On the other hand the inclusion

$$k \colon J \to C_\pi, \quad k(a) := (a, 0), \tag{5.14}$$

induces an isomorphism

$$k_* \colon \mathrm{K}_0(J) \to \mathrm{K}_0(C_\pi). \tag{5.15}$$

The boundary map ∂ is defined

$$\partial := k_*^{-1} \circ s_* \colon \mathrm{K}_1(A/J) \to \mathrm{K}_0(J). \tag{5.16}$$

On the other hand, the sequence

$$0 \to J \otimes B \to A \otimes B \to A/J \otimes B \to 0. \tag{5.17}$$

This implies by an easy exercise that

$$A/J \otimes B \cong A \otimes B \,/\, J \otimes B.$$

Associated therefore to the ideal $J \otimes B$ in $A \otimes B$, we have the quotient map, which we denote

$$\pi_B \colon A \otimes B \to A \otimes B / \, J \otimes B,$$

and its mapping cone, which we denote by C_{π_B}, the inclusions $k_B \colon J \otimes B \to C_{\pi_B}$ and $s_B \colon S(A/J) \otimes B) \to C_{\pi_B}$.

On the other hand, in a natural way

$$C_{\pi_B} \cong C_\pi \otimes B,$$

and under this identification, the inclusion

$$s_B \colon S(A \otimes B/J \otimes B) \to C_{\pi_B}$$

identifies with $s \otimes 1_B \colon S(A/J) \otimes B \to C_\pi \otimes B$. In particular, by part 2) of the Theorem, if $x \in \mathrm{K}_0\big(S(A/J)\big)$ and $y \in \mathrm{K}_0(B)$, the

$$(s_B)_*(x \hat{\otimes} y) = (s \otimes 1_B)_*(x \hat{\otimes} y) = s_*(x) \hat{\otimes} y. \tag{5.18}$$

Similarly, under the identification $C_{\pi_B} \cong C_\pi \otimes B$, the inclusion $k_B \colon J \otimes B \to C_{\pi_B}$ for the ideal $J \otimes B \subset A \otimes B$ identifies with $k \otimes 1_B$. Hence

$$(k_B)_* \cong (k \otimes 1)_* \colon \mathrm{K}_0(J \otimes B) \to \mathrm{K}_0(C_\pi \otimes B)$$

and thus

$$(k_B)_*^{-1} \cong (k \otimes 1)_*^{-1} \colon \mathrm{K}_0(C_\pi \otimes B) \to \mathrm{K}_0(J \otimes B) \tag{5.19}$$

Now functoriality of $\hat{\otimes}$ with respect to *-homomorphisms gives

$$(k \otimes 1)_*\big(k_* s_*(x) \hat{\otimes} y\big) = k_* k_*^{-1} s_*(x) \hat{\otimes} y = s_*(x) \hat{\otimes} y. \tag{5.20}$$

Since $(k \otimes 1)_*$ is an isomorphism, we get

$$(k \otimes 1)_*^{-1}\big(s_*(x) \hat{\otimes} y\big) = k_* s_*(x) \hat{\otimes} y \tag{5.21}$$

and under our standard identifications this says that

$$(k_B)_*^{-1}\big((s_B)_*(x \hat{\otimes} y)\big) = k_* s_*(x) \hat{\otimes} y$$

giving

$$\partial(x \hat{\otimes} y) = \partial(x) \hat{\otimes} y$$

as required. ☐

We conclude this section with an extension of the external product

$$\mathrm{K}_0(A) \times \mathrm{K}_0(B) \to \mathrm{K}_0(A \otimes B), \quad (x, y) \in \mathrm{K}_0(A) \times \mathrm{K}_0(B) \mapsto x \hat{\otimes} y \tag{5.22}$$

to a bilinear pairing

$$\mathrm{K}_i(A) \times \mathrm{K}_j(B) \to \mathrm{K}_{i+j}(A \otimes B).$$

This is very easily done. If

$$x \in \mathrm{K}_i(A) := \mathrm{K}_0(C_0(\mathbb{R}^i) \otimes A), \quad y \in \mathrm{K}_j(B) := \mathrm{K}_0(C_0(\mathbb{R}^j) \otimes B),$$

then the product already defined gives an element

$$x \hat{\otimes} y \in \mathrm{K}_0(C_0(\mathbb{R}^i) \otimes A \otimes C_0(\mathbb{R}^j) \otimes B).$$

Re-arranging factors gives a canonical isomorphism

$$C_0(\mathbb{R}^i) \otimes A \otimes C_0(\mathbb{R}^j) \otimes B \cong C_0(\mathbb{R}^i \times \mathbb{R}^j) \otimes A \otimes B.$$

Futhermore, we can identify $C_0(\mathbb{R}^i \times \mathbb{R}^j)$ with $C_0(\mathbb{R}^{i+j})$ by identifying a pair $x \in \mathbb{R}^i$ and $y \in \mathbb{R}^j$ with the element $(x, y) = (x_1, \ldots, x_i, y_1, \ldots, y_j)$ of $\mathbb{R}^{i+j}$. This gives a further isomorphism with $C_0(\mathbb{R}^{i+j}) \otimes A \otimes B$, so that we may interpret the product $x \hat{\otimes} y$ already defined for K_0-classes as lying in $\mathrm{K}_{i+j}(A \otimes B)$.

If one identifies a pair $(x, y) \in \mathbb{R}^i \times \mathbb{R}^j$ with the element $(y, x) = (y_1, \ldots, y_j, x_1, \ldots, x_i)$ instead, the two differ by a permutation of the coordinates of sign $(-1)^{ij}$. This accounts for an important *graded* commutativity of the external product, acting on higher K-theory:

THEOREM 5.9. *The external product of Theorem 5.8 extends to a more general bilinear, natural pairing*

$$\mathrm{K}_i(A) \times \mathrm{K}_j(B) \to \mathrm{K}_{i+j}(A \otimes B)$$

mapping x, y to $x \hat{\otimes} y$. The generalized external product is graded commutative *in the sense that if $x \in \mathrm{K}_i(A)$ and $y \in \mathrm{K}_j(B)$, then*

$$x \hat{\otimes} y = (-1)^{ij}\ \sigma_*(y \hat{\otimes} x),$$

where $\sigma \colon B \otimes A \to A \otimes B$ is the flip isomorphism.

The sign is of course material only when both classes x and y involved in the product, are odd-dimensional classes.

EXERCISE 5.10. Let A, B be unital C*-algebras and $u \in A$ be a unitary. Let $p \in B$ be a projection. Show that the external product

$$[u] \hat{\otimes} [p] \in \mathrm{K}_1(A \otimes B)$$

is represented by the class of the unitary

$$u \otimes p + 1 \otimes (1 - p) \in A \otimes B.$$

6. The Bott Periodicity theorem

We now give the proof of Bott Periodicity.

Let $\beta \in \mathrm{K}^0(\mathbb{R}^2) = \mathrm{K}_0\big(C_0(\mathbb{R}^2)\big)$ be the Bott element.

External product with β, discussed in the previous section, defines a map

$$\beta_A \colon \mathrm{K}_0(A) \to \mathrm{K}_2(A) := \mathrm{K}_0(C_0(\mathbb{R}^2) \otimes A), \quad \beta_A(x) := \beta \hat{\otimes} x, \;\; x \in \mathrm{K}_0(A), \tag{6.1}$$

and for any C*-algebra A. We aim to show that it is an isomorphism.

We start with some general remarks about β.

The first point is that the map

$$\beta_A \colon \mathrm{K}_0(A) \to \mathrm{K}_2(A)$$

is natural in A in the sense that if $\alpha \colon A \to B$ is a C*-algebra homomorphism, then the diagram of groups and group homomorphisms

$$\begin{tikzcd} \mathrm{K}_0(A) \arrow[r, "\beta_A"] \arrow[d, "\alpha_*"'] & \mathrm{K}_2(A) \arrow[d, "\alpha_*"] \\ \mathrm{K}_0(B) \arrow[r, "\beta_B"] & \mathrm{K}_2(B) \end{tikzcd} \tag{6.2}$$

commutes.

In the language of functors, β is a *natural transformation* between the functors K_0 and K_2 (each is a functor from the category of C*-algebras and C*-algebra homomorphisms, to the category of abelian groups, and group homomorphisms.)

The second important point is that β *commutes with external products* in the sense that

$$\beta_{A\otimes B}(x\hat{\otimes}y) = \beta_A(x)\hat{\otimes}y.$$

This statement is merely the associativity of the external product, since the left-hand side is $\beta\hat{\otimes}(x\hat{\otimes}y)$ and the right hand side is $(\beta\hat{\otimes}x)\hat{\otimes}y$.

In order to invert Bott Periodicity, we will define a similar, natural transformation of functors: a group homomorphism, for each A,

$$\alpha_A\colon \mathrm{K}_2(A) \to \mathrm{K}_0(A),$$

using the Toeplitz extension.

Let A be a unital C*-algebra. The first step in defining α_A is the rather trivial one of identifying

$$\mathrm{K}_2(A) := \mathrm{K}_0(C_0(\mathbb{R}^2)\otimes A) = \mathrm{K}_1(C_0(\mathbb{R})\otimes A). \tag{6.3}$$

Now let $\mathcal{T}$ be the Toeplitz algebra. Form the exact sequence of C*-algebras

$$0 \to \mathcal{K}\otimes A \to \mathcal{T}\otimes A \to C(\mathbb{T})\otimes A \to 0.$$

There is an associated connecting homomorphism

$$\delta_A\colon \mathrm{K}_1(C(\mathbb{T})\otimes A) \to \mathrm{K}_0(\mathcal{K}\otimes A) \cong \mathrm{K}_0(A).$$

Now, identify $\mathbb{R}$ with the open subset $\mathbb{T}\setminus\{1\}$ of the circle, using (say) the Cayley transform. This gives an embedding

$$i\colon C_0(\mathbb{R})\otimes A \subset C(\mathbb{T})\otimes A.$$

Putting things together we obtain the map

$$\alpha_A\colon \mathrm{K}_2(A) = \mathrm{K}_1\big(C_0(\mathbb{R})\otimes A\big) \xrightarrow{i_*} \mathrm{K}_1(C(\mathbb{T})\otimes A) \xrightarrow{\delta_A} \mathrm{K}_0(A). \tag{6.4}$$

LEMMA 6.1. *α is a natural transformation $\mathrm{K}_2 \to \mathrm{K}_0$, which commutes with external products in the sense that*

$$\alpha_{A\otimes B}(x\hat{\otimes}y) = \alpha_A(x)\hat{\otimes}y \tag{6.5}$$

for any $x \in \mathrm{K}_2(A)$ and $y \in \mathrm{K}_0(A)$.

Furthermore,

$$\alpha_{\mathbb{C}}(\beta) = [1] \in \mathrm{K}_0(\mathbb{C}),$$

where $\beta \in \mathrm{K}_2(\mathbb{C}) = \mathrm{K}_0\big(C_0(\mathbb{R}^2)\big)$ is the Bott element, and $[1]$ is the class of the unit $1 \in \mathbb{C}$, the generator of $\mathrm{K}_0(\mathbb{C}) \cong \mathbb{Z}$.

PROOF. The first two statements are obvious. For the last one, recall that the unitary in $C_0(\mathbb{R})^+ \cong C(\mathbb{T})$ corresponding to the projection $p \in C_0(\mathbb{R}^2)^+$ defining the Bott element is the inclusion $\bar{z}\colon \mathbb{T} \to \mathbb{C}$. We have already proved that

$$\delta_{\mathbb{C}}([\bar{z}]) = \mathrm{index}(T_{\bar{z}}) = -\mathrm{index}(T_z) = 1,$$

where $T_{\bar{z}}$ is the Toeplitz operator with symbol $\bar{z}$, and index is the Fredholm index. This proves the Lemma.

□

COROLLARY 6.2. *The transformations α and β satisfy*

$$\alpha_A \circ \beta_A = \mathrm{id}_{\mathrm{K}_0(A)},$$

for any C-algebra A.*

PROOF. Both transformations are natural with respect to external products, and we may write any $x \in \mathrm{K}_0(A)$ as the external product

$$x = [1]\hat{\otimes}x$$

where $[1] \in \mathrm{K}_0(\mathbb{C})$ is the generator. We get

$$(\alpha_A \circ \beta_A)(x) = \alpha_A\big(\beta_{\mathbb{C}}([1])\hat{\otimes}x\big)$$

since β commutes with external products. By the same reasoning with α

$$= \alpha_{\mathbb{C}}\big(\beta_{\mathbb{C}}([1])\big)\hat{\otimes}x = \alpha_{\mathbb{C}}(\beta)\hat{\otimes}x$$

and by the Lemma

$$= [1]\hat{\otimes}x = x.$$

□

THEOREM 6.3. *(*Bott Periodicity*). The Toeplitz transformation α and the Bott transformation β are inverse to each other. That is,*

$$\alpha_A \circ \beta_A = \mathrm{id}_{K_0(A)}, \quad \beta_A \circ \alpha_A = \mathrm{id}_{\mathrm{K}_2(A)},$$

for any C^*-algebra A.

Therefore, K*-theory is Bott periodic:* $\mathrm{K}_i(A) \cong \mathrm{K}_{i+2}(A)$ *for any C^*-algebra A, any nonnegative integer i.*

The proof boils down to a 'rotation trick' initially devised by Atiyah, to reduce left-invertibility of the transformation β, to right invertibility, which has already been proved in Corollary 6.2.

PROOF. Let $x \in \mathrm{K}_2(A) := \mathrm{K}_0(C_0(\mathbb{R}^2) \otimes A)$. We want to show that

$$(\beta_A \circ \alpha_A)(x) = x \in \mathrm{K}_2(A) = \mathrm{K}_0(C_0(\mathbb{R}^2) \otimes A).$$

By definition, $\alpha_A(x) \in \mathrm{K}_0(A)$ and

$$(\beta_A \circ \alpha_A)(x) = \beta\hat{\otimes}\alpha_A(x).$$

By commutativity of the external product this equals

$$\sigma_*\big(\alpha_A(x)\hat{\otimes}\beta\big),$$

where

$$\sigma\colon A \otimes C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2) \otimes A$$

is the flip homomorphism. Moreover, since α commutes with external products, our product can be re-written as

$$\sigma_*\big(\alpha_{A\otimes C_0(\mathbb{R}^2)}(x\hat{\otimes}\beta)\big). \tag{6.6}$$

Now, the naturality property of α with respect to *-homomorphisms is that if $\nu\colon B \to B'$ is a *-homomorphism, then

$$\alpha_{B'}\big((\mathrm{id}_{C_0(\mathbb{R}^2)} \otimes \nu)_*(x)\big) = \nu_*\big(\alpha_B(x)\big).$$

Applying naturality to (6.6) gives that it equals

$$\alpha_{A\otimes C_0(\mathbb{R}^2)}\big((\mathrm{id}_{C_0(\mathbb{R}^2)} \otimes \sigma)_*(\beta\hat{\otimes}x)\big). \tag{6.7}$$

By commutativity of $\hat{\otimes}$, this can be re-written

$$\alpha_{A\otimes C_0(\mathbb{R}^2)}\big((\mathrm{id}_{C_0(\mathbb{R}^2)} \otimes \sigma)_* \circ \sigma'_*(x\hat{\otimes}\beta\big) \tag{6.8}$$

where

$$\sigma'\colon C_0(\mathbb{R}^2) \otimes A \otimes C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2) \otimes A$$

is the flip homomorphism given permuting the factors cyclically:

$$\sigma'(f \otimes a \otimes f') := f' \otimes f \otimes a.$$

By functoriality of K-theory we can write (6.8) as

$$\alpha_{A\otimes C_0(\mathbb{R}^2)} \left[\left((\mathrm{id}_{C_0(\mathbb{R}^2)} \otimes \sigma) \circ \sigma' \right)_* (\beta \hat{\otimes} x) \right] = \alpha_{A\otimes C_0(\mathbb{R}^2)} \big((\sigma'')_* (\beta \hat{\otimes} x) \big) \tag{6.9}$$

with σ'' the composition

$$C_0(\mathbb{R}^2) \otimes A \otimes C_0(\mathbb{R}^2) \xrightarrow{\sigma'} C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2) \otimes A \xrightarrow{\mathrm{id}_{C_0(\mathbb{R}^2)} \otimes \sigma} C_0(\mathbb{R}^2) \otimes A \otimes C_0(\mathbb{R}^2)$$

– which just flips the first and third factors (of $C_0(\mathbb{R}^2)$. It is therefore homotopic to the identity *-homomorphism because the flip homomorphism

$$C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2)$$

is already homotopic to the identity homomorphism, since $C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}^2) \cong C_0(\mathbb{R}^4)$ and our homomorphism is induced by the corresponding map $\mathbb{R}^4 \to \mathbb{R}^4$, which is matrix multiplication by $\begin{bmatrix} 0 & 1_2 \\ 1_2 & 0 \end{bmatrix}$, an orthogonal matrix with determinant $+1$.

Since σ'' is homotopic to the identity and since, by definition, $\beta \hat{\otimes} x = \beta_{C_0(\mathbb{R}^2)\otimes A}(x)$ we can write (6.9) as

$$\alpha_{A\otimes C_0(\mathbb{R}^2)}(\beta \hat{\otimes} x) = (\alpha_{A\otimes C_0(\mathbb{R}^2)} \circ \beta_{C_0(\mathbb{R}^2)\otimes A})(x) = x, \tag{6.10}$$

the last step by Corollary 6.2.

□

This concludes the proof of Bott Periodicity.

The 6*-term exact sequence*

Suppose now that $J \subset A$ is an ideal. The associated long exact sequence

$$\begin{aligned} \cdots \to \mathrm{K}_2(J) \xrightarrow{i_*} \mathrm{K}_2(A) \xrightarrow{\pi_*} \mathrm{K}_2(A/J) & \\ \xrightarrow{\partial} \mathrm{K}_1(J) \xrightarrow{i_*} \mathrm{K}_1(A) \xrightarrow{\pi_*} \mathrm{K}_1(A/J) & \\ \xrightarrow{\partial} \mathrm{K}_0(J) \xrightarrow{i_*} \mathrm{K}_0(A) \xrightarrow{\pi_*} \mathrm{K}_0(A/J)\big) & \end{aligned} \tag{6.11}$$

give no information about the final map $\pi_* \colon \mathrm{K}_0(A) \to \mathrm{K}_0(A/J)$. However, by naturality of the Bott Periodicity isomorphism β, the diagram

$$\begin{tikzcd} \mathrm{K}_0(A) \arrow[r, "\pi_*"] \arrow[d, "\beta_A", "\cong"'] & \mathrm{K}_0(A/J) \arrow[d, "\beta_{A/J}", "\cong"'] \\ \mathrm{K}_2(A) \arrow[r, "\pi_*"] & \mathrm{K}_2(A/J) \end{tikzcd} \tag{6.12}$$

and the range of $\pi_* \colon \mathrm{K}_0(A) \to \mathrm{K}_0(A/J)$ identifies under Bott Periodicity with the range of

$$\pi_* \colon \mathrm{K}_2(A) \to \mathrm{K}_2(A/J).$$

which equals the kernel of the connecting homorphism

$$\delta \colon \mathrm{K}_2(J) \to \mathrm{K}_1(J),$$

by exactness of the long exact sequence. Let

$$\delta'\colon \mathrm{K}_0(A/J) \xrightarrow{\beta_{A/J}} \mathrm{K}_2(A/J) \xrightarrow{\delta} \mathrm{K}_1(J)$$

be the indicated composition.

Then the *periodic* sequence

$$\begin{tikzcd} \mathrm{K}_0(J) \arrow[r, "i_*"] & \mathrm{K}_0(A) \arrow[r, "\pi_*"] & \mathrm{K}_0(A/J) \arrow[d, "\delta'"] \\ \mathrm{K}_1(A/J) \arrow[u, "\delta"] & \mathrm{K}_1(A) \arrow[l, "\pi_*"] & \mathrm{K}_1(J) \arrow[l, "i_*"] \end{tikzcd} \tag{6.13}$$

is exact.

THEOREM 6.4. *For any ideal J in a C*-algebra A, the sequence* (6.13) *is exact. Furthermore, the sequence is natural with respect to *-homomorphisms, and the boundary maps commute with external products.*

The map $\delta'\colon \mathrm{K}_0(A/J) \to \mathrm{K}_1(J)$ has a particularly simple description. Assume that A is unital. Let $p \in M_n(A/J)$ be a projection. Lift p to a self-adjoint $H \in M_n(A)$. Note that $H^2 - H \in J$. Using functional calculus for self-adjoints, we form the unitary

$$e^{2\pi i H} \in M_n(A).$$

Identifying J^+ with the C*-subalgebra of A generated by J and the unit $1 \in A$, I claim that $e^{2\pi i H} \in J^+$. Indeed, for $\pi\colon M_n(A) \to M_n(A/J)$ the quotient map, since $p = \pi(H)$ is a projection, its spectrum consists of 0 and 1, and hence $e^{2\pi i p} = 1$. Hence

$$e^{2\pi i H} = 1 \bmod J,$$

proving the claim.

We call the map associating to a projection p the corresponding exponentiated unitary $e^{2\pi i H}$, H a lift of p to a self-adjoint in A, the *exponential map*

PROPOSITION 6.5. *In terms of the description of* $\mathrm{K}_1(J)$ *as* $\pi_0\big(\mathbf{U}_\infty(J^+)\big)$*, the homomorphism*

$$\delta'\colon \mathrm{K}_0(A/J) \to \mathrm{K}_1(J)$$

is the exponential map.

We will omit the proof.

EXAMPLE 6.6. Let $A = C_0\big([0,1)\big)$, a contractible C*-algebra containing $C_0\big((0,1)\big)$ as an ideal with quotient $\mathbb{C}$, under the map evaluating a function at 0. The 6-term exact sequence looks like

$$\begin{tikzcd} \mathrm{K}^0\big((0,1)\big) \arrow[r, "i_*"] & \mathrm{K}^0\big([0,1)\big) \arrow[r] & \mathrm{K}^0(\mathrm{pt}) \arrow[d, "\delta'"] \\ \mathrm{K}^{-1}(\mathrm{pt}) \arrow[u, "\delta"] & \mathrm{K}^{-1}\big([0,1)\big) \arrow[l] & \mathrm{K}^{-1}\big((0,1)\big) \arrow[l] \end{tikzcd} \tag{6.14}$$

Substituting into the sequence the identities

$$\mathrm{K}^0\big((0,1)\big) = \mathrm{K}^0\big([0,1)\big) = \mathrm{K}^1\big([0,1)\big) = \mathrm{K}^{-1}(\mathrm{pt}) = 0, \quad \mathrm{K}^0(\mathrm{pt}) = \mathbb{Z}$$

gives the exact sequence

$$0 \to \mathrm{K}^0(\mathrm{pt}) = \mathbb{Z} \xrightarrow{\delta'} \mathrm{K}^{-1}\big((0,1)\big) \to 0, \tag{6.15}$$

so that δ' is an isomorphism.

The generator of $\mathrm{K}^0(\mathrm{pt})$ is the projection $1 \in \mathbb{C}$, and lifting it to $C_0\big([0,1)\big)$ in this case amounts to extending the function 1 at 0 to a continuous, real-valued function $f(t)$ on $[0,1)$. There is an obvious explicit such extension:

$$f(t) := 1 - t.$$

Applying the exponential map to f we get the unitary function

$$u(t) = e^{2\pi i(1-t)} = e^{-2\pi i t}$$

on the interval $[0,1]$; notice that u assumes the same value 1 at each endpoint, so it is a unitary in $C_0\big((0,1)\big)^+$. Of course, under the standard identification

$$C_0\big((0,1)\big)^+ \cong C(\mathbb{T}),$$

this unitary is nothing but the usual complex (conjugate) coordinate $\bar{z}\colon \mathbb{T} \to \mathbb{C}$.

Exercise 6.7. Consider the extension of C*-algebras

$$0 \to C_0(\mathbb{T} - \{1\}) \to C(\mathbb{T}) \to \mathbb{C} \to 0 \tag{6.16}$$

obtained by removing a point (say, the point $1 \in \mathbb{T}$).

By direct computation using the 'exponential map' description, as in the discussion above, show that the connecting map

$$\delta'\colon \mathrm{K}_0(\mathbb{C}) \to \mathrm{K}_1\big(C_0(\mathbb{T} - \{1\})\big) = \mathrm{K}^{-1}(\mathbb{R})$$

is the zero map.

This is also clear from the 6-term exact sequence. Why?

7. K-theory of crossed-products by proper actions of discrete groups

In this section we will draw from the material developed in Section 7 about the basic structure of crossed-products $C_0(X) \rtimes G$ of discrete groups, acting *properly* on spaces X, and the strong Morita equivalence results of Section, together with the Morita invariance of K-theory, to compute the K-theory of some of these examples. This includes some examples of finite group actions.

If G acts properly on X then the C*-algebra $C_0(X) \rtimes G)$ is isomorphic to the fixed-point algebra $C(X \times_G \mathcal{K})$, where $\mathcal{K} := \mathcal{K}\big(l^2(G)\big)$, and the 'fixed-point algebra' $C(X \times_G \mathcal{K})$ is by definition the C*-algebra of all bounded $f\colon X \to \mathcal{K}$ such that $f(gx) = \rho(g)f(x)\rho(g)^{-1}$ for all $x \in X$, where ρ is the right-regular representation of G.

A good way to think of these functions is as sections of a bundle of C*-algebras over $G\backslash X$. The fibre of this bundle at an orbit Gx is $\mathcal{K}(l^2G)^H$, where $H = \mathrm{Stab}_G(x)$, that is, compact operators on $l^2(G)$ which commute with the right translation action of H on $l^2(G)$.

As we have shown, actually

$$\mathcal{K}(l^2G)^H \cong C^*(H) \otimes \mathcal{K}\big(l^2(G/H)\big).$$

Given our results on finite groups, $C^*(H)$ decomposes into a direct sum

$$C^*(H) \cong \oplus_{[\sigma]\in\widehat{H}} \mathcal{K}(V_\sigma)$$

of matrix algebras, with the summands parameterized by the points $\widehat{H}$, the irreducible representations of H. This induces a direct sum decomposition of $L^2(G)$ respected by the action of $\mathcal{K}(l^2G)^H$. Putting everything together we obtain

$$\mathcal{K}(l^2G)^H \cong \oplus_{[\sigma]\in\widehat{H}} \mathcal{K}\big(l^2(G/H) \otimes V_\sigma\big). \tag{7.1}$$

We can summarize all of this in terms of the idea of a bundle of C*-algebras. if $A = C_0(X) \rtimes G$ and A_{Gx} is the 'fibre' at Gx, then

$$A_{Gx} \cong \mathcal{K}(l^2G)^H \cong \oplus_{[\sigma]\in\widehat{H}}\mathcal{K}\big(l^2(G/H)\otimes V_\sigma\big), \quad \text{where } H = \mathrm{Stab}_G(x).$$

In particular, in each fibre, one can single out the ideal corresponding to the ϵ-coordinate in the direct sum. This gives an ideal J_x in A_{Gx}, and the quotient is given by

$$A_x/J_x \cong \mathcal{K}(l^2G)^H \cong \oplus_{\widehat{H}\ni[\sigma]\neq\epsilon}\mathcal{K}\big(l^2(G/H)\otimes V_\sigma\big). \tag{7.2}$$

Of course $A_x/J_x = 0$ if x has no non-trivial isotropy.

The bundle of ideals $\{J_x\}$ corresponds to the ideal J_X discussed in the section on Morita equivalence, by the definitions: it is the ideal corresponding to the range of a certain inner product involved in a Morita equivalence between J_X and $C(G\backslash X)$.

Proposition 7.1. *Suppose that G acts properly on X, with only a finite set of orbits with non-trivial isotropy. Let J_X the ideal of $C_0(X) \rtimes G$ discussed above. Then $C_0(X) \rtimes G \,/\, J_X$ is isomorphic to a direct sum of compact operators. More exactly, if $F \subset G\backslash X$ denotes the set of points with non-trivial isotropy, then*

$$C_0(X) \rtimes G \,/\, J_X \cong \oplus_{Gx\in F} \oplus_{\widehat{\mathrm{Stab}_G(x)}\ni[\sigma]\neq\epsilon} \mathcal{K}\big(l^2(Gx)\otimes V_\sigma\big), \tag{7.3}$$

where $\widehat{\mathrm{Stab}_G(x)}$ denotes, as usual, the collection of irreducible representations of the finite group $\mathrm{Stab}_G(x)$.

In particular, the K-theory groups of the quotient $C_0(X) \rtimes G \,/\, J_X$ are very easy: the K_0-group is

$$\oplus_{Gx\in F} \oplus_{\widehat{\mathrm{Stab}_G(x)}\ni[\sigma]\neq\epsilon} \mathbb{Z}$$

and K_1-group of the quotient is the zero group.

The 6-term exact sequence associated with the exact sequence

$$0 \to J_X \to C_0(X) \rtimes G \to C_0(X) \rtimes G \,/\, J_X \to 0$$

has has therefore the form

$$\begin{aligned} 0 \to \mathrm{K}^0(G/X) \to \mathrm{K}_0(C_0(X) \rtimes G) \to \oplus_{Gx\in F} \oplus_{\widehat{\mathrm{Stab}_G(x)}\ni[\sigma]\neq\epsilon} \mathbb{Z} & \\ \xrightarrow{\delta} \mathrm{K}^{-1}(G\backslash X) \to \mathrm{K}_1(C_0(X) \rtimes G) \to 0.& \end{aligned} \tag{7.4}$$

Before stating a theorem, recall that the Green-Julg Theorem identifies, for any finite group H, the group $\mathrm{K}_0(C^*H)$ with the additive group $\mathrm{Rep}(H)$ underlying the representation ring of H. The Green-Julg isomorphism is straightforward: if $\pi\colon H \to \mathbf{U}(V)$ is a unitary representation of H on V, finite-dimensional, then we can simply consider V as a (right) module over $\mathbb{C}[H] = C^*(H)$, using the given action of H on V, and by specifying an appropriate $C^*(H)$-valued inner product on V. The resulting (right) $C^*(H)$-module is f.g.p.

Conversely, if $\mathcal{E}$ is an f.g.p. module over $C^*(H)$, with $p \in M_n(C^*H)$ a projection with $\mathcal{E} \cong p \cdot C^*(H)^n$, then, we may consider p as projection onto a certain subspace of $l^2(H)^n$. Since p commutes with the right regular representation $\rho\colon H \to \mathbb{B}(l^2H)$, the subspace $\mathrm{ran}(p)$ is invariant under $\rho(H)$, so we obtain a representation of H on it, using ρ.

In Section 2, we also noted that for a proper action of G locally compact on X, one can define a group $\mathrm{K}^0_G(X)$, the *G-equivariant* K-*theory of X* , made by taking the Grothendiek completion of the semi-group of isomorphism classes of G-equivariant vector bundles on X. We described there an isomorphism

$$\mathrm{K}_G^0(X) \cong \mathrm{K}_0\big(C_0(X) \rtimes G\big),$$

if $G\backslash X$ is compact, so that we can also understand, a bit more geometrically, the K_0-group of the C*-algebra $C_0(X) \rtimes G$, in terms of equivariant vector bundles.

Now if $E \to X$ is a G-equivariant vector bundle over X, and $x \in X$, then the fibre E_x carries, by the assumptions, a representation of the compact group $\mathrm{Stab}_G(x)$. This results in a canonical group homomorphism

$$\mathrm{K}_G^0(X) \to \mathrm{Rep}\left[(C^*(\mathrm{Stab}_G(x))\right].$$

On the other hand, in our discussion above of the structure of $C_0(X) \rtimes G$, there is, for any $x \in X$, a natural *-homomorphism

$$\mathrm{K}_0\big(C_0(X) \rtimes G\big) \to \mathrm{K}_0\big(C^*(\mathrm{Stab}_G(x))\big)).$$

by restriction to the orbit of x. These two maps fit into a diagram

(7.5)
$$\begin{tikzcd}
\mathrm{K}_G^0(X) \arrow[r] \arrow[d] & \mathrm{Rep}\big(\mathrm{Stab}_G(x)\big) \arrow[d] \\
\mathrm{K}_0\big(C_0(X) \rtimes G\big) \arrow[r] & \mathrm{K}_0\big(C^*(\mathrm{Stab}_G(x))\big)
\end{tikzcd}$$

where the vertical map on the left was discussed above, the vertical map on the right is the Green-Julg isomorphism.

Interpreting $\mathrm{K}_0\big(C_0(X) \rtimes G\big)$ in this way as $\mathrm{K}_G^0(X)$ allows us to describe the exact sequence 7.4 as follows.

THEOREM 7.2. *Let G be a locally compact group acting properly on X with only finitely many points in $G\backslash X$ having non-trivial isotropy. Then if $F \subset X$ is a set of representatives of these points, then there is an exact sequence*

$$(7.6)\quad 0 \to \mathrm{K}^0(G/X) \to \mathrm{K}_0\big(C_0(X) \rtimes G\big) \cong \mathrm{K}_G^0(X) \xrightarrow{r_*} \oplus_{x\in F}\mathrm{Rep}^*\big(\mathrm{Stab}_G(x)\big)$$
$$\xrightarrow{\delta} \mathrm{K}^{-1}(G\backslash X) \to \mathrm{K}_1(C_0(X) \rtimes G) \to 0.$$

where, for any finite group H, $\mathrm{Rep}^*(H)$ denotes the free abelian group with one generator for each non-trivial *irreducible representation of H.*

What can we say about the map δ in the above sequence? In fact, it is rather subtle. It turns out that the question has to do with *torsion* in the K-theory of $G\backslash X$, and reflects a somewhat more general result, to the effect that it is much easier to compute *rationalized* K-theory of crossed-products of the kind we are discussing, than it is to compute ordinary, integral K-theory.

In this section, we are going to prove the following

THEOREM 7.3. *The connecting homomorphism δ vanishes rationally. In particular, if the group $\mathrm{K}^{-1}(G\backslash X)$ has no torsion, then δ is the zero map.*

We will actually show, more precisely, that

$$m \cdot \delta(x) = 0, \ \ \forall x \in \mathrm{K}_0(C_0(X) \rtimes G),$$

where m is the least common multiple of the cardinalities of the subgroups $\mathrm{Stab}_G(x)$.

PROOF. This is equivalent to showing that *mx lifts* to an element of $\mathrm{K}_0(C_0(X) \rtimes G)$ under the map r_* of (7.6),

$$m \cdot x = r_*(y), \quad y \in \mathrm{K}_0(C_0(X) \rtimes G) = \mathrm{K}_0^G(X). \tag{7.7}$$

In order to do this, fix a point x with non-trivial isotropy. Denote

$$H := \mathrm{Stab}_G(x).$$

Let U be an H-slice at x: thus for some neighbourhood V of x, the natural map

$$G \times_H V \to U$$

is a homeomorphism. We have already discussed that there is a natural 'induction' map

$$\mathrm{Vect}_H(V) \to \mathrm{Vect}_G(G \times_H V) = \mathrm{Vect}(U),$$

applying in this situation. To induce an H-equivariant vector bundle on V to a G-equivariant vector bundle on U, we form

$$\tilde{E}_V := G \times_H E,$$

defined similarly as with $G \times_H V$. Let 1_H denote in this argument the trivial H-equivariant vector bundle, over whatever space, say, W, is under discussion. Thus, $1_H = W \times \mathbb{C}$ with the trivial action of H on the factor $\mathbb{C}$. Similarly for 1_G, the trivial G-equivariant vector bundle. Induction clearly maps $1_H \in \mathrm{Vect}_H(V)$ to $1_G \in \mathrm{Vect}_G(U)$.

Let $\rho\colon H \to \mathbf{U}(V_\rho)$ be a unitary representation of H. We can view V_ρ as a H-equivariant vector bundle over the 1-point H-space $\{x\}$. Inducing it results in a G-equivariant vector bundle $\tilde{V}_\rho$ over the orbit Gx. Now consider the restriction of $\tilde{V}_\rho$ to $U - G \cdot x$. I claim that for some positive integer m, $m \cdot \tilde{V}_\rho = \tilde{V}_\rho \oplus \cdots \oplus \tilde{V}_\rho$, is isomorphic, as a G-equivariant vector bundle over $U - G \cdot x$, to a multiple $k \cdot 1_G$ of the trivial G-equivariant vector bundle over $U - Gx$.

Note that if we can prove this, the extension problem has been solved for $x := [V_\rho] \in \mathrm{Rep}(H)$. Indeed, take the G-equivariant vector bundle $\tilde{V}_\rho$ over the G-invariant open set $W_1 := U = G \times_H V$ obtained by inducing the H-equivariant vector bundle $W_1 \times V_\rho$ (with diagonal H-action.) On the G-invariant open set $W_2 := X \setminus Gx$ take the trivial G-vector bundle $k \cdot 1_G = W_2 \times \mathbb{C}^k$ with $k = \dim(V_\rho)m$. Now glue these two G-equivariant vector bundles together to form a G-equivariant vector bundle over $W_1 \cup W_2 = X$. The its class $y \in \mathrm{K}_G^0(X)$ is the required lift of x.

In order to prove the claim, we only need to observe that H acts freely on $V^* := V \setminus \{y\}$, and due to this,

$$\mathrm{K}_H^0(V) \cong \mathrm{K}^0(H \backslash V^*)$$

by a map sending the class of the H-equivariant vector bundle $V \times V_\rho$ to the class of the flat bundle $[V \times_H V_\rho] \in \mathrm{K}^0(G \backslash V^*)$. We have already proved that for in this situation, exists m so that $m \cdot [V \times_H V_\rho] = k \cdot [1_H]$, with $[1] \in \mathrm{K}^0(H \backslash V)$ the class of the trivial line bundle. It follows that the bundles $V^* \times V_\rho$ and $V^* \times \mathbb{C}^k$, with H acting trivially on $\mathbb{C}^k$, are H-equivariantly isomorphic, for some k, over $V^* = V \setminus \{y\}$. Now inducing this result to $\mathrm{K}_G^0(G \times_H V \setminus G \cdot x) = \mathrm{K}_G^0(U \setminus G \cdot x)$ gives the required statement.

□

CHAPTER 7

The Atiyah-Singer Index Theorem and analytic KK-theory

On a compact, spin^c-manifold X, the Dirac operator D is a densely defined unbounded self-adjoint operator acting on an appropriate $\mathbb{Z}/2$-graded Hilbert space (L^2-sections of a spinor bundle). This Hilbert space also carries a natural representation of $C(X)$ (by multiplication operators). The Index Theorem computes the Fredholm index of D, and, more generally, the Fredholm index of twisted versions D_E by complex vector bundles over X, by a topological formula. Although this topological formula is usually given in the form of an integral of appropriate characteristic classes, the right-had-side of

$$\operatorname{index}(D_E) = \int_X \mathcal{A}(X) \cdot \operatorname{ch}(E),$$

the Index Theorem, perhaps more usefully sometimes, may be stated rather more abstractly. Atiyah and Singer define a *topological index* of D_E by embedding X in Euclidean space $\mathbb{R}^n$, and then, by a sequence of simple manipulations using normal bundles obtain a map

$$\mathrm{K}^*(X) \to \mathrm{K}^*(\mathrm{pt}) = \mathbb{Z}$$

by repeated applications of the Thom isomorphism (for the normal bundle), wrong-way functoriality of K-theory for open embeddings, and Bott Periodicity itself, which identifies $\mathrm{K}^0(\mathbb{R}^n)$ with the K-theory of the 1-point space, *i.e.* the integers $\mathbb{Z}$. The functoriality of the Thom Isomorphism implies that this sequence of steps is independent of the choices, and it can be translated into the language of characteristic classes without too much difficulty.

To say it another way, the topological index of D_E as Atiyah and Singer describe it, is precisely the integer corresponding to the correspondence

$$\mathrm{pt} \leftarrow (X, E) \to \mathrm{pt},$$

in $\mathrm{KK}_0(\mathrm{pt}, \mathrm{pt}) = \mathbb{Z}$, and the 3 steps used by Atiyah and Singer to define their topological index, correspond to applying the two basic kinds of equivalence which the geometric KK-category possesses by definition: vector bundle, or 'Thom' modification (i.e. Bott Periodicity), and bordism, to modify the correspondence above to the trivial correspondence

$$\mathrm{pt} \leftarrow (\mathrm{pt}, [n]) \to \mathrm{pt},$$

for an integer n giving the bundle data over a point, computing the index.

Atiyah observed that a rather abstract way of stating the *analytic index* of D_E, used the following idea. The spectrum of D in this situation is a discrete subset of $\mathbb{R}$, and application of functional calculus to D produces a a bounded operator $F = \chi(D)$ (for χ, a small perturbation of the sign function $\chi(x) = \frac{x}{|x|}$) on the same Hilbert space. It has the properties that

$$F^2 - 1, \quad \text{and} \quad [f, F]$$

are all *compact* operators. The first condition is a Fredholm condition on D, while the second is a link between the C*-algebra $C(X)$ and the operator.

Now F is a grading-reversing self-adjoint operator on a $\mathbb{Z}/2$-graded Hilbert space $H = H^+ \oplus H^-$ such that $F^2 = 1$ mod compacts, so it corresponds to an essentially unitary operator

$$F^+ \colon H^+ \to H^-$$

between two Hilbert spaces. Now by Swan's Theorem, the vector bundle E determines a projection p in the C*-algebra $C\big(X, M_n(\mathbb{C})\big)$, and if, for simplicity, $n = 1$, then the operator

$$F_E^+ := pUp + (1-p) \colon H^+ \to H^-$$

is also Fredholm. The Fredholm index $F_E^+)$ is the same as the analytic index of D twisted by E defined in the context of the Index Theorem.

Actually, this whole procedure, seen at this level of abstraction, clearly has the potential to define analytic index-type pairings in a variety of situations: all one is using is a Fredholm operator on a Hilbert space, and a representation of a C*-algebra on the same Hilbert space, together with an essential commutation condition. Atiyah's idea was that perhaps such pairs (or triples) might define a theory (K-homology) dual to K-theory, and the idea was developed by Kasparov into a mechanical device for studying K-theory for C*-algebras which has already had breathtakingly diverse and impressive applications, not the least, the Novikov Conjecture in topology, for classes of groups, but also is an absolutely essential tool in Noncommutative Geometry, so much so, that in the author's view, it almost defines what Noncommutative Geometry is.

1. Differential operators on Euclidean space

A *differential operator* (of order $\leq m$) on $\mathbb{R}^n$ is a linear operator on the complex vector space $C_c^\infty(\mathbb{R}^n)$, of the form

$$(Df)(x) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha f\,(x) \tag{1.1}$$

Our notation is that

$$D^\alpha := (-i)^{|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}}, \qquad |\alpha| := \sum_{i=1}^n \alpha_i.$$

The Fourier transform allows us to re-write this as follows. If $f \in C_c(\mathbb{R}^n)$, or, more generally, if $f \in L^1(\mathbb{R}^n)$, then its Fourier transform $\hat{f}$ is the function on the dual group $\widehat{\mathbb{R}^n}$ defined

$$\hat{f}(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx.$$

Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of infinitely differentiable functions on $\mathbb{R}^n$ such that for all α, β there exists a constant $C_{\alpha\beta}$ such that $|D^\alpha f\,(x)| \leq C_{\alpha\beta}(1 + |x|)^\beta$ for all $x \in \mathbb{R}^n$. It is routine to check that $f \in \mathcal{S}(\mathbb{R}^n)$ implies $\hat{f}$ is in $\mathcal{S}(\widehat{\mathbb{R}^n})$. So Fourier transform defines a linear map

$$F \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\widehat{\mathbb{R}^n}).$$

The Fourier inversion formula says that $f \in \mathcal{S}(\mathbb{R}^n)$ then

$$f(x) = \int_{\widehat{\mathbb{R}^n}} \hat{f}(\xi) e^{ix\cdot\xi} d\xi. \tag{1.2}$$

Differentiating (1.2) under the integral sign and using that $D_x^\alpha(e^{ix\cdot\xi}) = \xi^\alpha e^{ix\cdot\xi}$ we see that

$$Df\,(x) = \int_{\widehat{\mathbb{R}^n}} \sigma(x,\xi) e^{ix\cdot\xi} \hat{f}(\xi) d\xi$$

where $\sigma(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$, an infinitely differentiable function of (x,ξ) which is polynomial in ξ of order $\leq m$. The top order term of the symbol is called the *principal symbol* given by

$$\sigma_p(x,\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha.$$

Directly in terms of σ, using a limit:

$$\sigma_p(x,\xi) = \lim_{t\to\infty} \frac{\sigma(x,t\xi)}{t^m}.$$

The symbol is *elliptic* if its principal symbol satisfies $\sigma_p(x,\xi) \neq 0$ for $\xi \neq 0$. The operator D is an *elliptic differential operator of order m* if its principal symbol is an elliptic symbol of order m.

EXERCISE 1.1. Let $\mathrm{Diff}(\mathbb{R}^n)$ denote the collection of operators of the form (1.1).

a) Show that if S and T are differential operators of orders m, m', then ST is a differential operator of order $m + m'$.
b) Check that the commutator of differential operators $[\frac{\partial}{\partial x_k}, a]$ has order zero, for any coordinate x_k, and any smooth function a.
c) Extend the result of b) to show that if S and T are as in a), then the commutator $[S,T] = ST - TS$ is a differential operator of order $m + m' - 1$.
d) Deduce from c) that if S and T are as above, then the principal symbol of the differential operator ST of order $m + m'$, is the pointwise product of the principal symbols of S and T:
$$\sigma^p_{ST}(x,\xi) = \sigma^p_T(x,\xi)\cdot\sigma^p_S(x,\xi).$$

An important special case of part c) of the Exercise above is that if $f \in C^\infty(\mathbb{R}^n)$ is a smooth function, acting by multiplication on $C_c^\infty(\mathbb{R}^n)$, and if D is a differential operator of order m on $\mathbb{R}^n$, then the commutator

$$[f, D]$$

is a differential operator of order $m-1$. In the important special case $m = 1$, this means $[D, f]$ is order zero, and thus is a *bounded* operator.

DEFINITION 1.2. A *pseudodifferential operator on* $\mathbb{R}^n$ is an operator on $\mathcal{S}(\mathbb{R}^n)$ of the form

$$T_\sigma f\,(x) := \int_{\widehat{\mathbb{R}^n}} \sigma(x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)d\xi,$$

where σ is a symbol of order $\leq m$.

Let $\Psi^m(\mathbb{R}^n)$ denote the pseudodifferential operators on $\mathbb{R}^n$ of order $\leq m$.

Every differential operator of order $\leq m$ is pseudodifferential of order $\leq m$. But clearly there are more pseudodifferential operators. They will come up briefly at the end, when we prove Bott Periodicity by KK-theory, but we will set them aside for the moment.

In the applications of elliptic operator theory one usually works with not scalar-valued functions a_α, but *matrix*-valued functions. Fix a positive integer m. For $f\colon \mathbb{R}^n \to \mathbb{C}^m$ a smooth, vector valued function, with entries $f(x) = (f_1(x), \ldots, f_m(x))$, and α a multi-index, set $D^\alpha f := (D^\alpha f_1, \ldots D^\alpha f_m)$. Now, as above, if we are given a family of smooth functions

$$a_\alpha\colon \mathbb{R}^n \to M_m(\mathbb{C}),$$

for various multi-indices α, with $|\alpha| \leq m$ as before, we can define an operator

$$D\colon C_c^\infty(\mathbb{R}^n, \mathbb{C}^m) \to C_c^\infty(\mathbb{R}^n, \mathbb{C}^m)$$

by setting,

$$(Df)(x) = \sum_{|\alpha|\leq m} a_\alpha(x)(D^\alpha f)(x), \quad f \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^m).$$

The *symbol* of such an operator, and the principal symbol, are defined just as in the scalar case, and now are matrix-valued functions

$$\sigma\colon \mathbb{R}^n \times \widehat{\mathbb{R}^n} \to M_n(\mathbb{C})$$

(and similarly for σ_p.) The symbol is *elliptic* if $\sigma_p(x,\xi)$ is *invertible* in $M_n(\mathbb{C})$ for all non-zero ξ, and all $x \in \mathbb{R}^n$.

Slightly more abstractly, if V is any finite-dimensional Hilbert space, then the partial differentiation operators $\frac{\partial}{\partial x_j}$ act on $C_c^\infty(\mathbb{R}^n, V)$; to see this one can define the action directly, by a limit, in the usual way, or one can fix a basis for V, and identify $C_c^\infty(\mathbb{R}^n, V)$ with functions valued in $\mathbb{C}^m$ for some m, by expanding vectors into their coefficients. Functions valued in $\mathbb{C}^m$ can then be differentiated component-wise, and it checked that the result is independent of the choice of basis for V. A differential operator D on $C_c^\infty(\mathbb{R}^n, V)$ is then one of the form

$$(Df)(x) = \sum_{|\alpha|\leq m} a_\alpha(x)(D^\alpha f)(x), \quad f \in C_c^\infty(\mathbb{R}^n, V). \tag{1.3}$$

The coefficients a_α are smooth functions $\mathbb{R}^n \to \mathrm{End}(V)$, the symbol $\sigma\colon \mathbb{R}^n \times \widehat{\mathbb{R}^n} \to \mathrm{End}(V)$ is defined as in the scalar case by

$$\sigma(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha,$$

and the principal symbol is the top-order part as before. Ellipticity means that the principal symbol takes values in invertible operators on V.

Exercise 1.3. Show that the results of Exercise 1.1 still hold, for the algebra of differential operators on $C_c^\infty(\mathbb{R}^n, V)$, with V a finite-dimensional Hilbert space.

Example 1.4. Let $m = 2$, $n = 2$ and define a matrix-valued function

$$\sigma\colon \mathbb{R}^2 \times \widehat{\mathbb{R}^2} \to M_2(\mathbb{C}), \quad \sigma(x,\xi) = \begin{bmatrix} 0 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & 0 \end{bmatrix}. \tag{1.4}$$

Then σ is elliptic, for $\det \sigma(x,\xi) = \|\xi\|^2 \neq 0$ if $\xi \neq 0$.

The associated operator on $C_c^\infty(\mathbb{R}^2, \mathbb{C}^2)$ is

$$D = \begin{bmatrix} 0 & D_1 - iD_2 \\ D_1 + iD_2 & 0 \end{bmatrix}.$$

We will show below that $D_{\bar\partial} = \overline{\partial} + \overline{\partial}^*$, where $\overline{\partial}$ is the Dolbeault operator, acting as a map between complex differential forms of type $(0,0)$ and of type $(0,1)$, on $\mathbb{R}^2$, with its usual complex structure.

The previous example might remind the reader of Clifford algebras. In fact the map $\sigma\colon \mathbb{R}^2 \to \mathbb{B}(\mathbb{C}^2)$ in (1.4) satisfies $\sigma(\xi) = \sigma(\xi)^*$, and $\sigma(\xi)^2 = \|\xi\|^2$, for ξ a vector in $\mathbb{R}^2$. This is no accident. Let

$$c\colon \mathrm{Cliff}(\mathbb{R}^n) \to \mathrm{End}(S)$$

be a $\mathrm{Cliff}(\mathbb{R}^n)$-module; we may as well assume that it is one of the irreducible representations of Theorem 8.1 – the unique one, if n is even. We will build an associated *Dirac operator* D. The operator will act on the space $C_c^\infty(\mathbb{R}^n, S)$ of smooth, compactly supported functions $s\colon \mathbb{R}^n \to S$.

The Clifford module structure gives, for each unit vector $\xi \in \mathbb{R}^n$, a linear operator $c(\xi)$ on the Hilbert space S, which is $\mathbb{Z}/2$-graded, if n is even, and with the properties that

$$c(\xi)^2 = -1, \;\; c(\xi)^* = -c(\xi),$$

and $c(\xi)$ is odd with respect to the grading, in the case n is even.

For ξ ranging over, let us say, the standard orthonormal basis $\xi_1, \ldots, \xi_n$ for $\mathbb{R}^n$, we form the composition of the partial differentiation operator $\frac{\partial}{\partial x_i}$ and the Clifford multiplication operator $c(\xi_i)$. Adding them up gives a differential operator with constant coefficients

$$D := \sum_{i=1}^n c(\xi_i)\, \frac{\partial}{\partial x_i} = \sum_{i=1}^n -ic(\xi_i) D_i. \tag{1.5}$$

on $C_c^\infty(\mathbb{R}^n, S)$.

Note that D is elliptic. Indeed, it's symbol is given by the self-adjoint operator

$$\sigma(x,\xi) = -i \sum_{i=1}^n c(\xi_i) \cdot \xi_i \in \mathrm{End}(S).$$

Hence

$$\sigma(x,\xi)^2 = -\sum_{i,j} \xi_i \xi_j c(\xi_i) c(\xi_j).$$

Since

$$c(\xi_i)c(\xi_j) = -c(\xi_j)c(\xi_i$$

for all $i \neq j$, and $c(\xi_i)^2 = -1$ this equals

$$\sum_{i=1}^n \xi_i^2 = \|\xi\|^2,$$

a nonzero scalar multiple of the identity operator on S, provided that $\xi \neq 0$, whence $\sigma(x,\xi)$ is also invertible.

Note also that in a slightly formal sense, D is also self-adjoint. Indeed, $c(\xi_i)$ obviously commutes with $\frac{\partial}{\partial_{x_j}}$ as linear operators on $C_c^\infty(\mathbb{R}^n, S)$. The spin representation space S is, by assumption, a Hilbert space, with an inner product. We endow $C_c^\infty(\mathbb{R}^n, S)$ with the inner product

$$\langle s, s' \rangle := \int_{\mathbb{R}^n} \langle s(x), s'(x) \rangle, \quad s, s' \in C_c^\infty(\mathbb{R}^n, S).$$

In this notation, integration-by-parts gives that

$$\langle \frac{\partial s}{\partial x_j}, s' \rangle = -\langle s, \frac{\partial s'}{\partial x_j} \rangle,$$

so that partial differentiation is a skew-adjoint operator. Since the operator $c(\xi_i)$ on $C_c^\infty(\mathbb{R}^n, S)$ is also skew-adjoint, for any vector $\xi \in \mathbb{R}^n$, we get that D, a linear combination of compositions of two skew-adjoint operators, is self-adjoint, *i.e.*

$$\langle Ds, s' \rangle = \langle s, Ds' \rangle, \quad s, s' \in C^\infty(\mathbb{R}^n, S).$$

Finally, we note that if we give the linear space $C_c^\infty(\mathbb{R}^n, S)$ the $\mathbb{Z}/2$-grading induced by the $\mathbb{Z}/2$-grading on S, then the Dirac operator D an *odd* operator $C_c^\infty(\mathbb{R}^n, S) \to C_c^\infty(\mathbb{R}^n, S)$, *i.e.* interchanges the even and odd parts of $C_c^\infty(\mathbb{R}^n, S)$.

EXAMPLE 1.5. The simplest example of all of a Dirac operator is $D = -i\frac{d}{dx}$ acting on $C_c^\infty(\mathbb{R})$. This is associated to the Clifford module

$$c\colon \mathrm{Cliff}(\mathbb{R}) \to \mathbb{C}$$

of its positive irreducible representation on the one-dimensional spinor space $S = \mathbb{C}$, which maps the unit vector $1 \in \mathbb{R} \subset \mathrm{Cliff}(\mathbb{R})$ to the scalar $+1$.

We conclude this section with some general remarks regarding changes of coordinates.

Differential operators on $\mathbb{R}^n$ are *local*, in the sense that if D is such an operator, thus, of the form (1.3), and if $\rho \in C_c^\infty(\mathbb{R}^n)$ is zero on an open set $W \subset \mathbb{R}^n$, then $D\rho$ is also zero on W.

It follows that any such operator D restricts to an operator

$$D|_U\colon C_c^\infty(U) \to C_c^\infty(U)$$

for any open set $U \subset \mathbb{R}^n$. We let $\mathrm{Diff}(U)$ be the algebra of differential operators on U: it is generated by the partial differentiation operators $\frac{\partial}{\partial x_j}$ and the multiplication operators by smooth functions $f \in C^\infty(U)$.

Now, suppose that $\phi\colon U \to V$ is a diffeomorphism between two open subsets of $\mathbb{R}^n$. Let $T_\phi\colon C_c^\infty(V) \to C_c^\infty(U)$ be the linear map of composition with ϕ,

$$(T_\phi f)(x) = f\big(\phi(x)\big).$$

Let $\phi_1, \ldots, \phi_n$ be the coordinate functions of ϕ. Then by the Chain Rule

$$\frac{\partial}{\partial x_j}(f \circ \phi)(x) = \sum_i \frac{\partial f}{\partial x_i}\big(\phi(x)\big) \cdot \frac{\partial \phi_i}{\partial x_j}(x),$$

from which it follows that

$$T_\phi^{-1}\,\frac{\partial}{\partial x_j}\,T_\phi = \sum_i \Big(\frac{\partial \phi_i}{\partial x_j} \circ \phi^{-1}\Big) \cdot \frac{\partial}{\partial x_i}$$

as operators on $C_c^\infty(V)$.

Since

$$T_\phi^{-1} \circ f \circ T_\phi = f \circ \phi$$

as (multiplication) operators on $C_c^\infty(V)$, for any $f \in C^\infty(U)$, and since such functions, and the partial differentiation operators, generate $\mathrm{Diff}(U)$ as an algebra, it follows that conjugation $L \mapsto T_\phi^{-1} L T_\phi$ by T_ϕ maps $\mathrm{Diff}(U)$ to $\mathrm{Diff}(V)$, in a canonical manner.

EXERCISE 1.6. If $L \in \mathrm{Diff}(U)$, $\phi\colon U \to V$ a diffeomorphism, T_ϕ the operator of composition with ϕ as above, and if $\sigma_L^p \in C^\infty(U)$ is the principal symbol of L, then the principal symbol of $T_\phi^{-1} L T_\phi \in \mathrm{Diff}(V)$ is given by

$$\sigma^p_{T_\phi^{-1} L T_\phi}(x, \xi) = \sigma_T^p(\phi^{-1}x, {}^t D_{\phi^{-1}x}\phi \cdot \xi), \quad x \in V, \;\; \xi \in \widehat{\mathbb{R}^n}. \tag{1.6}$$

where ${}^t D_{\phi^{-1}x}\phi \cdot \xi$ is shorthand for

$$\sum_i \frac{\partial \phi_i}{\partial x_j}(\phi^{-1}x) \cdot \xi_i.$$

2. The Atiyah-Singer Index Theorem

DEFINITION 2.1. Let M be a smooth manifold, and $L\colon C_c^\infty(M) \to C_c^\infty(M)$ be a linear, *local* operator: that is, L leaves the subspaces $C_c^\infty(U)$ invariant, for every $U \subset M$ open.

We say that L is a *differential operator of order* m on M if for every $p \in M$, there exists a coordinate chart

$$\phi\colon U \to \mathbb{R}^n,$$

such that the operator

$$T_\phi^{-1} L T_\phi\colon C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$$

is a differential operator of order m on $\mathbb{R}^n$.

The discussion preceding the Definition shows that L is differential of order m on M if and only if *for every* manifold chart $\phi\colon U \to \mathbb{R}^n$ $T_\phi^{-1} L T_\phi$ is differential of order m.

EXAMPLE 2.2. One of the most famous differential operators is the *Laplacian* operator on a Riemannian manifold. Let M be a manifold, equipped with a Riemannian metric. Let g be the coefficient matrix in a chart with coordinates $x_1, \ldots, x_n$, then locally

$$\Delta f = \frac{-1}{\sqrt{\det(g)}} \sum_{i,j} \frac{\partial}{\partial x_i}\big(\sqrt{\det g} \cdot (g^{-1})_{ij}\, \frac{\partial f}{\partial x_j}\big). \tag{2.1}$$

It has order 2.

Suppose L is a differential operator of order m on M, and that in the domain $U \subset M$ of a coordinate chart, with coordinates $x_1, \ldots, x_n$, L can be represented in the form

$$L|_U = \sum_\mu a_\mu D^\mu, \tag{2.2}$$

with a_μ smooth functions on U, $D_i := -i\frac{\partial}{\partial x_i}$, differentiation in the coordinate direction x_i, and D_μ the corresponding product of such operators, according to the multi-index μ. We are assuming that the top-order part of this operator is in degree m.

On U, the tangent and co-tangent bundles $TU = TM|_U$ and $T^*U = T^*M|_U$ are trivial. A trivialization of TU is given by assigning coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ in $\mathbb{R}^n \times \widehat{\mathbb{R}^n}$ to the tangent vector $\sum_i \xi_i \frac{\partial}{\partial x_i}$ at the point of U with coordinates $(x_1, \ldots, x_n)$. For current purposes, $\widehat{\mathbb{R}^n}$ just denotes another copy of n-dimensional Euclidean space, whose coordinate projections we are now calling $\xi_1, \ldots, \xi_n$.

Similarly, T^*U can be given coordinates using coordinates on U and the frame $dx_1, \ldots, dx_n$ on U, for the co-tangent bundle.

With this notation, we define the symbol of D as follows. At a point $x \in U$, the coordinate functions $\xi_1, \ldots, \xi_n$ are, by definition, linear functions $T_x^*M \to \mathbb{R}$. If μ is a multi-index, the corresponding product ξ^μ is, therefore, a polynomial function $T_x^*M \to \mathbb{R}$, of degree m, if the length $|\mu|$ of the multi-index is m. To define the symbol of D we simply replace the expression (2.2) by the function

$$\sum_\mu a_\mu(x)\, \xi^\mu, \tag{2.3}$$

on T^*U.

To see that this is actually independent of the coordinate system, let us describe the symbol in a slightly more abstract way.

Recall that a smooth function f in a neighbourhood of a point $p \in \mathbb{R}^n$ *vanishes to order* 1 at p if $f(p) = 0$. It vanishes to order 2 at p if it is a product of two functions each vanishing to order 1 at p, and so on.

Let J_p^k be the algebra of germs of smooth functions at p which vanish to order k at p. By definition, $J_p^1 \supset J_p^2 \supset \cdots$.

Then the cotangent bundle T^*M, has fibre J_p^1/J_p^2, by definition. The differential 1-form *df* of a germ of a smooth function at p, is by definition the class modulo J_p^2 of $f - f(p)$. The duality with the tangent bundle is by virtue of the fact that the pairing between a tangent vector v at p and an element *df* of T_p^*M is by taking the derivative by v of f at p, and this process annihilates constant functions, and functions which vanish to order 2 at p.

In fact, vanishing to order 2 at p might be rephrased in terms of differential operators by observing that $f \in J_p^2$ is *equivalent* to saying that $(Df)(p) = 0$ for all differential operators D of order 1, defined in a neighbourhood fo p.

LEMMA 2.3. *Let D be a differential operator of order m on $\mathbb{R}^n$, and $f \in C_c^\infty(\mathbb{R}^n)$ be a smooth function which vanishes to order $m+1$ at p. Then $(Df)(p) = 0$.*

PROOF. Suppose first that $m = 1$. If f vanishes to order 2 at p, then $f = f_1 f_2$ with f_i vanishing to order 1 at p. Hence for any i,

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial f_1}{\partial x_i}(p) \cdot f_2(p) + f_1(p) \cdot \frac{\partial f_2}{\partial x_i}(p) = 0,$$

since each of f_1, f_2 vanish at p.

This implies the result for $m = 1$, and the general result follows from induction.

□

LEMMA 2.4. *If f_1 and f_2 each vanish to order 1 at p and $f_1 - f_2$ vanishes to order 2 at p, then $f_1^m - f_2^m$ vanishes to order $m+1$ at p.*

PROOF. We use the identity

$$f_1^m - f_2^m = (f_1 - f_2) \cdot f_1^{m-1} + f_2 \cdot (f_1^{m-1} - f_2^{m-1}). \tag{2.4}$$

If $f_i \in J_p^1$ and $f_1 - f_2 \in J_p^2$ then it follows that the first term is in J_p^{m+1}. By induction, $f_1^{m-1} - f_2^{m-1} \in J_p^m$. Hence the second term is also in J_p^{m+1}.

□

As a corollary:

LEMMA 2.5. *If D is a differential operator of order m in a neighbourhood of p, and if $f \in J_p^1$ is a germ of smooth function at p, then the value of $D(f^m)$ at p only depends on the coset of f in J_p^1/J_p^2 – that is, depends only on $df \in J_p/J_p^2 = T_p^*M$.*

As a consequence, the symbol can be defined in the following natural way.

DEFINITION 2.6. If D is a differential operator of order m, its principal symbol σ is the function on T^*M whose value at a point $df \in T_p^*M$, is the value

$$\frac{i^m}{m!} \cdot D(f^m)(p).$$

The constant is needed to be consistent with our earlier definitions.

EXAMPLE 2.7. Suppose D is Lie derivative with respect to a vector field V on M, or an open subset. Then for $p \in M$, $df \in T_p^*M$, for f vanishing to order 1 at p, $(Df)(p) := V(f)(p)$. This is a linear function of $df \in T_p^*M$. If in a coordinate system near p, centred at 0 for convenience, the vector field is $V = \sum_i a_i \cdot \frac{\partial}{\partial x_i}$, and if $f = x_j$, so $df = dx_j$ then

$$(Vf)(p) = \sum a_i \cdot \frac{\partial x_j}{\partial x_i}(p) = a_j$$

so that in cotangent coordinates $\xi_1, \dots \xi_n$, the symbol is given by

$$\sigma_D(x, \xi) = i \cdot \sum a_i \xi_i,$$

which is (except for the multiplication by i) precisely the pairing between T_p^*M and T_pM, applied to the value of the vector field $V(p)$ and the cotangent vector df.

EXERCISE 2.8. Show that if $D = \frac{\partial^2}{\partial x_i \partial x_j}$ then t

$$\sigma_D(x, \xi) = -\xi_i \xi_j.$$

Deduce that if $\Delta := -\sum \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on $\mathbb{R}^n$, then

$$\sigma_\Delta(x, \xi) = \|\xi^2\| := \sum \xi_i^2.$$

EXERCISE 2.9. Show that the symbol of the Laplacian is given in local coordinates by

$$\sigma_\Delta(x, \xi) = \sum_{i,j} (g^{-1})_{ij}(x) \cdot \xi_i \xi_j \quad \xi \in T_x^*M.$$

That is, the symbol of the Laplacian on M is the Riemannian metric

$$\sigma_\Delta(x, \xi) = \|\xi\|^2.$$

on the co-tangent bundle.

Our discussion of differential operators on $\mathbb{R}^n$ contained variants involving an auxillary (finite-dimensional) Hilbert space. In the context of manifolds, the interesting examples of elliptic differential operators related to geometry are operators not on $C_c^\infty(M)$ but on the spaces of smooth sections of a smooth vector bundle $\pi \colon S \to M$ over M: we use the notation $C_c^\infty(M, S)$ for this linear space of smooth, compactly supported sections. More generally, we consider pairs of vector bundles, and maps between their spaces of smooth sections.

DEFINITION 2.10. A *differential operator of order m*

$$D \colon C_c^\infty(M, S^+) \to C_c^\infty(M, S^-)$$

between sections of a pair of bundles S^+, S^- over M, is a linear operator which is, firstly, *local*, in the sense discussed above: D does not change supports of smooth sections, and, secondly, such that every point of M has a neighbourhood U such that the restriction $D \colon C_c^\infty(U, S^+) \to C_c^\infty(U, S^-)$ can be written in local coordinates on M in the form

$$(Ds)(x) = \sum_{|\mu| \leq m} a_\mu(x) \cdot (A^{-1} D^\mu)(As)\,(x), \quad s \in C_c^\infty(U, S^+|_U) \tag{2.5}$$

for some (any) smooth trivialization $A \colon S^+|_U \to U \times \mathbb{C}^n$ of the bundle S^+ over U, and a family $a_\mu \in C_c^\infty\big(U, \mathrm{Hom}(S^+, S^-)\big)$ of bundle maps $S^+ \to S^-$.

Of course any section $T \in C^\infty(M, \mathrm{End} S)$ of the endomorphism bundle of a single bundle S, that is, any bundle map $S \to S$, defines a differential operator of order zero. Of course one can consider (smooth) bundle maps between different bundles as well.

EXAMPLE 2.11. Suppose that $\pi \colon S \to M$ is a vector bundle over M and that ∇ is a connection on S. Then ∇ restricts to a connection on $S|_U$ for any open subset. Pick U with $S|_U$ is trivial, with $A \colon S|_U \to U \times \mathbb{C}^n$ a trivialization. On the trivial bundle $U \times \mathbb{C}^n$ we always have the trivial connection

$$\nabla_X^{\mathrm{triv}}(s_1, \dots, s_n) := (X(s_1), \dots X(s_n)),$$

where $(s_1, \dots, s_n)$ is a section of $U \times \mathbb{C}^n$.

Hence $A^{-1} \cdot \nabla^{\mathrm{triv}} \cdot A$ is another connection on $S|_U$.

Now any two connections ∇^1 and ∇^2 on $S|_U$ differ by an $\mathrm{End}(S)$-valued 1-form: that is, a bundle map $T^M \to \mathrm{End}(S)$, given by the pairing

$$\langle X, s\rangle_p := \nabla^1_X s - \nabla^2_X s,$$

where s is a section of $S|_U$ and X is a tangent vector. This expression is $C^\infty(M)$-linear in the variable s since

$$\nabla^1_X(fs) - \nabla^2_X(fs) = X(f)s + f\nabla^1_X s - X(f)s + f\nabla^2_X s,$$

by the connection property. Hence $\langle X, s\rangle(p)$, for any $p \in U$, only depends on the value of X at p, and the value of s at p.

In particular, any covariant derivative ∇_V, for V a vector field on M, is locally the sum of a section of $\mathrm{End}(S)$, and a conjugate, as above, of Lie differentiation by V, acting on sections of a trivial bundle. In particular, it locally has the form specified in (2.5). Therefore, ∇_V is a differential operator of order 1 on $C_c^\infty(M, S)$.

Exercise 2.12. Prove that if D is an order 1 differential operator $D\colon C_c^\infty(M, S^+) \to C_c^\infty(M, S^-)$ between sections of a pair of bundles over M, and if $f \in C^\infty(M)$ acts on sections of each of these bundles by multiplication, then the commutator $[f, D]$ is an operator of order zero, and in particular, is a bounded operator.

Definition 2.13. Let $D\colon C_c^\infty(M, S^+) \to C_c^\infty(M, S^-)$ be a differential operator of order m between section spaces of a pair of bundles $S^\pm$ over a smooth manifold M.

The *symbol* of D is the smooth bundle map

$$\sigma_D\colon \pi^*(S^+) \to \pi^*(S^-)$$

mapping a covector $\xi := df \in T_p^*M$, where f is smooth and vanishes to order 1 at x, and an element $w \in S_x^+$, to

$$\sigma_D(x, df) \cdot w := D(f^m s)(x) \in S_x^-, \tag{2.6}$$

where s is any smooth extension of s to a smooth section defined near p.

D is *elliptic* if $\sigma_D(x, \xi)\colon S_x^+ \to S_x^-$ is invertible for every $\xi \neq 0$ in T_x^*M.

The formula (2.6) is well-defined, since if s and s' are sections that agree at x, then $s - s'$ vanishes at x, and hence the section $f^m(s - s')$ vanishes to order $m + 1$ at x, and by a slight generalization of Lemma 2.3, we deduce that $D\big(f^m(s - s')\big)$ vanishes at x.

We will discuss some specific examples related to geometry in the next section.

If D is elliptic, mapping $C_c^\infty(M, S^+)$ to $C_c^\infty(M, S^-)$, then the pair of bundles $\pi^*(S^\pm)$ over T^*M and the bundle map $\sigma\colon \pi^*(S^+) \to \pi^*(S^-)$ between them provided by the symbol, constitutes a K-theory triple for the noncompact space T^*M. The support of the section is M, embedded as the zero section, and this is compact, so the triple has compact support as required.

Definition 2.14. Let M be a compact manifold, D an elliptic differential operator on M with symbol σ_D^p. Then the *symbol class* of D is the class $[\sigma_D]$ in $\mathrm{K}^0(T^*M)$ of the K-theory triple $\big(\pi^*(S^+), \pi^*(S^-), \sigma_D^p\big)$ for T^*M.

Recall that in Definition **??** we introduced a certain 'spin$^\mathrm{c}$-index' map $\mathrm{index}_{\mathrm{spin}^\mathrm{c}}\colon \mathrm{K}^{-d}(M) \to \mathbb{Z}$, where M is any d-dimensional spin$^\mathrm{c}$-manifold, *i.e.* for which the map $p_M\colon M \to \mathrm{pt}$, is K-oriented.

Lemma 2.15. *The tangent bundle of the cotangent bundle T^*M (or tangent bundle TM) of any smooth manifold, admits a complex structure. In particular, it admits a canonical* K-*orientation. Thus, T^*M has a canonical* spin$^\mathrm{c}$-*structure, for any smooth manifold M.*

PROOF. $TM \cong \pi^*(TM) \oplus \pi^*(TM)$ as vector bundles over TM, where $\pi\colon TM \to M$ is the projection map. $\square$

THEOREM 2.16. *(The Atiyah-Singer Index Theorem.) Let M be a compact smooth manifold. Let*

$$D\colon C_c^\infty(M, S^+) \to C_c^\infty(M, S^-)$$

be an elliptic operator, for any pair of bundles $S^\pm$ over M. Let $[\sigma_D] \in \mathrm{K}^0(T^*M)$ be its symbol class, and

$$\mathrm{index}_{\mathrm{spin^c}}\colon \mathrm{K}^0(T^*M) \to \mathbb{Z}$$

be the spin$^\mathrm{c}$*-index map for the* spin$^\mathrm{c}$*-manifold T^*M, from its almost-complex structure.*

Then

a) *The kernel and cokernels of D as a linear operator $C_c^\infty(M,S) \to C_c^\infty(M,S)$, are each finite-dimensional.*
b) *Set* $\mathrm{index}_a(D) := \dim\ker D - \dim\operatorname{coker} D$, *the* analytic index *of D. Then*

$$\mathrm{index}_a(D) = \mathrm{index}_{\mathrm{spin^c}}([\sigma_D]) \in \mathbb{Z} \tag{2.7}$$

holds, with $[\sigma_D] \in \mathrm{K}^0(T^*M)$ the symbol class.

3. Dirac operators

In this section, we describe an important class of elliptic operators, called *Dirac operators*. Such operators are associated to spin$^\mathrm{c}$-structures on smooth manifolds, in a reasonably canonical manner, and an application of the Atiyah-Singer Index Theorem gives an analytic interpretation to spin$^\mathrm{c}$-index maps for spin$^\mathrm{c}$-manifolds. This is in a sense one of the key underlyig ideas of Noncommutative Geometry, since the *analytic* description of spin$^\mathrm{c}$-index maps, to be explained here, adapts easily to the C*-algebraic category.

Let X be a Riemannian manifold. The Fundamental Theorem of Riemannian geometry asserts that there is a unique connection ∇^{LC} on the tangent bundle TX of X – the Levi-Cevita connection – which is both torsion-free, and compatible with the metric. Compatibility with the metric asserts that

$$\langle \nabla^{\mathrm{LC}}_X Y, Z\rangle + \langle Y, \nabla^{\mathrm{LC}}_X Z\rangle = X(\langle Y, Z\rangle),$$

for vector fields X, Y, Z. Our immediate goal, is to refine the Levi-Cevita connection so as to extend, in a sense, to a connection on any spinor bundle on X, which, of course, depends, by definition, on the Clifford algebra bundle of X, which depends oVn the metric. We will then describe an elliptic operator (the Dirac operator) to which the Atiyah-Singer IndeV Theorem applies.

Let $e_1, \ldots, e_n$ be a (pointwise) orthogonal, oriented frame for the tangent bundle TX, defined on an open subset $U \subset X$. For each vector field V on U there is a family of smooth real-valued functions $\omega_{ij}(V)$ such that

$$\nabla_V e_i = \sum_{i,j} \omega_{ij}(V) e_j. \tag{3.1}$$

This expression is $C^\infty(U)$-linear in the variable V. Note that $\omega_{ij}(V) = \langle \nabla^{\mathrm{LC}}_V e_i, e_j\rangle$. Compatibility of ∇^{LC} with the metric gives

$$\langle \nabla^{\mathrm{LC}}_V e_i, e_j\rangle + \langle e_i, \nabla^{\mathrm{LC}}_V e_j\rangle = V(\langle e_i, e_j\rangle) = 0$$

for all i, j. Hence

$$\omega_{ij}(V) + \omega_{ji}(V) = 0,$$

so that the matrix $\omega(V)$ defined by its coordinates $\omega_{Ij}(V)$ lies in the Lie algebra $\mathfrak{so}(n,\mathbb{R})$ of $\mathbf{SO}(n,\mathbb{R})$. The *connection* 1-*form* of ∇^{LC} is the map

$$\omega\colon TU \to \mathfrak{so}(n,\mathbb{R}), \tag{3.2}$$

defined on U, and depending on the initial choice of frame. If E_{ij} denotes the matrix in $\mathfrak{so}(n,\mathbb{R})$ with $+1$ in entry (i,j), -1 in entry (j,i), then our initial choice of frame gives the formula

$$\omega(V) = \sum_{i<j} \omega_{ij}(V)\cdot E_{ij} \in \mathfrak{so}(n,\mathbb{R})$$

for this map.

Note that one can recover the connection ∇^{LC} the 1-form ω in (3.2). The frame $e_1,\ldots,e_n$ for TU gives an identification $C^\infty(U,TU)$ with $C^\infty(U,\mathbb{R}^n)$, and in terms of this identification, we have

$$\nabla_V(\sigma) := \big(V(\sigma_1),\ldots,V(\sigma_n)\big) + \omega(V)\cdot \begin{bmatrix}\sigma_1\\ \cdots\\ \cdots\\ \sigma_n\end{bmatrix}, \tag{3.3}$$

for $\sigma = (\sigma_1,\ldots,\sigma_n)$ a smooth $\mathbb{R}^n$-valued function on U.

Exercise 3.1. Suppose the frame $e_1,\ldots,e_n$ is transformed into a frame $e_1',\ldots,e_n'$ by the action of an orthogonal matrix $g \in \mathbf{SO}(n,\mathbb{R})$, where

$$e_i' = \sum_j g_{ij} e_j. \tag{3.4}$$

Show that the 1-form in terms of the new basis is given by

$$\omega'(v) = \mathrm{Ad}_g\big(\omega(v)\big),$$

where

$$\mathrm{Ad}\,:\mathbf{SO}(n,\mathbb{R}) \to \mathrm{End}\big(\mathfrak{so}(n,\mathbb{R})\big)$$

is the adjoint representation.

Now suppose the

$$\rho\colon \mathrm{Cliff}(|\mathbb{R}^n) \to \mathrm{End}(\Delta)$$

is a spin representation for $\mathrm{Cliff}(\mathbb{R}^n)$. It restricts to a representation of $\mathbf{Spin}_n$ and differentiating gives a representation

$$\rho_*\colon \mathfrak{spin}_n \to \mathrm{End}(\Delta)$$

of the Lie algebra $\mathfrak{spin}_n$.

Now the Lie algebras $\mathfrak{spin}_n$ and $\mathfrak{so}(n,\mathbb{R})$ of $\mathbf{Spin}_n$ and of $\mathbf{SO}(n,\mathbb{R})$ are naturally isomorphic, by differentiating the standard double covering $\mathbf{Spin}_n \to \mathbf{SO}(n,\mathbb{R})$. On the other hand, there is a natural embedding of $\mathfrak{spin}_n$ in $\mathrm{Cliff}(\mathbb{R}^n)$. Putting things together, one checks that under the resulting embedding of $\mathfrak{so}(n,\mathbb{R})$ in $\mathrm{Cliff}(\mathbb{R}^n)$, the matrix E_{ij} with $+1$ in entry (i,j), -1 in entry (j,i), and zeros elsewhere, corresponds to the Clifford algebra element

$$e_ie_j \in \mathrm{Cliff}(\mathbb{R}^n).$$

With these preliminary remarks aside, we define a connection 1-form

$$\omega_\Delta\colon TU \to \mathrm{End}(\Delta)$$

depending on our initial choice of frame, by setting

$$\omega_\Delta(V) := \frac{1}{2}\sum_{i<j} \omega_{ij}(V)\cdot c(e_ie_j) \in \mathrm{End}(\Delta), \tag{3.5}$$

for a vector field V on U.

DEFINITION 3.2. Let $U \subset X$ be an open subset of a Riemannian manifold on which an orthonormal framing $e_1, \dots, e_n$ of TX is defined. If $\rho\colon \mathrm{Cliff}(\mathbb{R}^n) \to \mathrm{End}(\Delta)$ is the spin representation, then the composition

$$\mathrm{Cliff}(TU) \cong U \times \mathrm{Cliff}(\mathbb{R}^n) \to \mathrm{End}(U \times \Delta),$$

where the first map is induced by the frame, defines a $\mathrm{Cliff}(TU)$-module. The *spin* connection on the product spinor bundle $S := U \times \Delta$, whose sections we understand as smooth maps $\sigma\colon U \to \Delta$, is given by

$$\nabla^S_V(\sigma) = V(\sigma) + \omega_\Delta(V) \cdot \sigma, \tag{3.6}$$

where ω_Δ is the 1-form valued in $\mathrm{End}(\Delta)$ give by (3.5), and $V(\sigma)$ is the usual Lie derivative of a vector-valued function.

The crucial property of the spin connection constructed locally above is the following.

LEMMA 3.3. *If ∇^S is the spin connection on sections of $U \times \Delta$ as above, $w\colon U \to \mathbb{R}^n \subset$ $\mathrm{Cliff}(\mathbb{R}^n)$ is a smooth map, and $\sigma\colon U \to \Delta$ is a smooth section of $U \times \Delta$, then*

$$\nabla^S_X\bigl(c(w) \cdot \sigma\bigr) = c(w) \cdot \nabla^S_X(\sigma) + c\bigl(\nabla^{\mathrm{LC}}_X w\bigr) \cdot \sigma \tag{3.7}$$

where ∇^{LC} is the Levi-Civita connection.

The proof is left as an exercise.

DEFINITION 3.4. Let S be a $\mathrm{Cliff}(TX)$-module, a Hermitian vector bundle, with Clifford multiplication $c\colon \mathrm{Cliff}(TX) \to \mathrm{End}(S)$. We say that a connection ∇^S on S is *compatible* with the Clifford module structure if

$$\nabla^S_X\bigl(c(w) \cdot \sigma\bigr) = c(w) \cdot \nabla^S_X(\sigma) + c\bigl(\nabla^{\mathrm{LC}}_X w\bigr) \cdot \sigma \tag{3.8}$$

holds for all smooth vector fields w on X, and smooth sections s of S. The connection ∇^S will be called a *Dirac connection* if it is compatible with the Clifford multiplication, and compatible with the Hermitian metric on S.

We aim to prove the following.

PROPOSITION 3.5. *Every fibrewise irreducible* $\mathrm{Cliff}(TX)$*-module has a Dirac connection.*

LEMMA 3.6. *Suppose $c\colon \mathrm{Cliff}(TX) \to \mathrm{End}(S)$ and $c'\colon \mathrm{Cliff}(TX) \to \mathrm{End}(S')$ are two Clifford modules over $\mathrm{Cliff}(TX)$, and that $U\colon S \to S'$ is a unitary bundle isomorphism intertwining the two Clifford multiplications.*

Then if ∇ is a Dirac connection on S, then the conjugate $\nabla' := U\nabla U^*$ connection is a Dirac connection on S'.

PROOF. The conjugate connection is defined

$$\nabla'_X(s) := U\nabla_X(U^*s) \tag{3.9}$$

for a vector field X and smooth section s of S'. It is easily checked that ∇' is a connection. If s_1, s_2 are smooth sections of S', X a vector field, then

$$\begin{aligned}\langle \nabla'_X(s_1), s_2\rangle + \langle s_1, \nabla'_X(s_2)\rangle &= \langle U\nabla'_X(U^*s_1), s_2\rangle + \langle s_1, U\nabla'_X(U^*s_2)\rangle \\ &= \langle \nabla'_X(U^*s_1), U^*s_2\rangle + \langle U^*s_1, \nabla'_X(U^*s_2)\rangle = X(\langle U^*s_1, U^*s_2\rangle) = X(\langle s_1, s_2\rangle)\end{aligned} \tag{3.10}$$

shows that it is compatible with the metric. Finally, the assumption $Uc(w)^* = c'(w)$ for a tangent vector field w, and the assumed Clifford compatibility of ∇, gives

$$U\nabla_X U^*\bigl(c(w)s\bigr) = c\bigl(\nabla^{\mathrm{LC}}_X(w)\bigr)s + c(w)U\nabla_X(U^*s)$$

so the conjugate connection is Clifford multiplication compatible as well. $\square$

LEMMA 3.7. *if S is a fibrewise irreducible* $\mathrm{Cliff}(TX)$*-module and L is a complex Hermitian line bundle over X, then $S \otimes L$ is a fibrewise irreducible* $\mathrm{Cliff}(TX)$*-module with module structure*

$$c(w)(s \otimes g) := c(w)s \otimes w,$$

for $w \in T_xX$, $s \in S_x$ and $w \in L_x$, $x \in X$.

Finally, we recall the following result observed earlier.

LEMMA 3.8. *If S and S' are two fibrewise irreducible* $\mathrm{Cliff}(TX)$*-modules, then there is a Hermitian line bundle L and a unitary isomorphism of Clifford modules $S \cong S' \otimes L$.*

PROOF. Set $L := \mathrm{Hom}_{\mathrm{Cliff}(TX)}(S, S')$, the bundle of bundle maps $S \to S'$ which commute with the Clifford module structures. Then L is a complex line bundle, with a natural Hermitian structure, and the obvious map $S \otimes L \to S'$ sending $s \otimes T$ to $T(s)$ is a bundle isomorphism intertwining the Clifford multiplications.

□

PROOF. (Of Proposition 3.5). Let S be a fibrewise irreducible $\mathrm{Cliff}(TX)$-module. If $U \subset X$, with the restricted Riemannian metric, then $S|_U$ is also a $\mathrm{Cliff}(TU)$-module, which is fibrewise irreducible. Suppose that U has a globally defined orthonormal frame. The frame gives a unitary bundle isomorphism $TU \cong U \times \mathbb{R}^n$ and an induced isomorphism $\mathrm{Cliff}(TU) \cong U \times \mathrm{Cliff}(\mathbb{R}^n)$. Composing this isomorphism with the product $\mathrm{Cliff}(U \times \mathbb{R}^n)$-module $U \times \Delta$, gives a new, fibrewise irreducible $\mathrm{Cliff}(TU)$-module. Therefore, there is a complex Hermitian line bundle L over U such that $S|_U \otimes L \cong \mathbf{U} \times \Delta$. If U is also contractible, every line bundle is trivial, and hence we get a unitary isomorphism

$$S|_U \cong U \times \Delta$$

of $\mathrm{Cliff}(TU)$-modules.

On the other hand, we have already shown in Lemma 3.3 that $U \times \Delta$ has a $\mathrm{Cliff}(U \times \mathbb{R}^n)$-compatible connection, and hence a $\mathrm{Cliff}(TU)$-compatible connection. Hence $S|_U$ has a compatible connection as well.

This shows that compatible connections exist locally. We may then piece them together using a partition of unity $\{\rho_i\}$, setting

$$\nabla := \sum_i \rho_i \nabla_i,$$

where $\{U_i\}_{i \in I}$ is a locally finite open cover of X by contractible open sets U_i each of which has a globally defined orthonormal frame, and ∇_i is a Clifford-compatible connection on $S|_{U_i}$.

□

We are finally in a position to define the Dirac operator associated to a spin$^{\mathrm{c}}$-manifold.

DEFINITION 3.9. Let X be a spin$^{\mathrm{c}}$ Riemannian manifold. Let $c\colon \mathrm{Cliff}(TX) \to \mathrm{End}(S)$ be an irreducible Clifford module, and ∇^S be a Dirac connection on S. The *Dirac operator* is the differential operator on sections $C_c^\infty(M, S)$ of the spin bundle, given locally in terms of a local orthonormal frame $e_1, \ldots, e_n$ of TX by the formula

$$D = \sum_i c(e_i) \cdot \nabla_{e_i} \colon C_c^\infty(X, S) \to C_c^\infty(X, S), \tag{3.11}$$

where ∇_{e_i} is covariant differentiation by e_i.

EXERCISE 3.10. The expression (3.11) is independent of the frame.

PROPOSITION 3.11. *The Dirac operator is an order* 1 *elliptic, differential operator on sections* $C_c^\infty(X,S)$ *of the spinor bundle. The symbol of* D *is the composition of the bundle isomorphism* $T^*X \cong TX$ *given by the Riemannian metric, and the Clifford multiplication* $c\colon TX \to \mathrm{End}(S)$.

In the case X *is even-dimensional,* D *is* $\mathbb{Z}/2$*-grading-reversing, and so maps* $C_c^\infty(X,S^+)$ *to* $C_c^\infty(X,S^-)$.

In the case when X is even-dimensional, the fact that D is grading-reversing means that, with respect to the decomposition $C_c^\infty(X,S) = C_c^\infty(X,S^+)\oplus C_c^\infty(X,S^-)$, D has a 2-by-2 matrix decomposition

$$D = \begin{bmatrix} 0 & D_+ \\ D_- & 0 \end{bmatrix},$$

and the (formal) self-adjointedness of D implies that $D_+^* = D_-$. The *analytic index* of D is then by definition

$$\mathrm{index}_{\mathrm{an}}(D_+) := \dim\ker(D_+) - \dim\ker(D_-) \in \mathbb{Z}. \tag{3.12}$$

It agrees with the Fredholm index described earlier, since $D_- = D_+^*$.

In the present setting, X spin$^{\mathrm{c}}$, and in the language of wrong-way maps and correspondences, the map $p_X\colon X \to \mathrm{pt}$ from X to a point is K-oriented. Thus, the triple $(X, p_X, [1])$ is a Baum-Douglas cycle, *i.e.* a correspondence from X to a point. We have previously observed that such a correspondence induces a map

$$p_X!\colon \mathrm{K}^0(X) \to \mathrm{K}^0(\mathrm{pt}),$$

by the following recipe. Choose an embedding

$$i\colon X \to \mathbb{R}^n$$

for some n even. Let ν be the normal bundle to the embedding. Since $\nu\oplus TX \cong T(\mathbb{R}^n)|_X \cong [1_n]$, with $[1_n]$ the class of the trivial bundle over X of dimension n, by the 2-out-of-3 Lemma, K-orientations on TX and on ν are in 1-1-correspondence. So in particular the fixed K-orientation on TX induces one on ν, so we obtain a resulting Thom Isomorphism map

$$\zeta_\nu!\colon \mathrm{K}^*(X) \to \mathrm{K}^*(\nu).$$

On the other hand, if

$$\varphi\colon \nu \to \mathbb{R}^n$$

is the tubular neighbourhood embedding associated to the normal bundle, with image an open subset of $\mathbb{R}^n$, we obtain another wrong-way map

$$\varphi!\colon \mathrm{K}^*(\nu) \to \mathrm{K}^*(\mathbb{R}^n).$$

Finally, Bott Periodicity gives an isomorphism

$$\beta_{\mathbb{R}^n}\colon \mathbb{Z} \cong \mathrm{K}^*(\mathrm{pt}) \to \mathrm{K}^*(\mathbb{R}^n)$$

and so combining these three ingredients gives the ‘topological index map, which is only interesting in even-dimension and is defined

$$p_M! := \beta_{\mathbb{R}^n}^{-1} \circ \varphi! \circ \zeta_\nu!\colon \mathrm{K}^0(X) \to \mathbb{Z}. \tag{3.13}$$

If $E \to X$ is a complex vector bundle, we called $p_M!([E]) \in \mathbb{Z}$ the spin$^{\mathrm{c}}$-*number* of E and denoted it $\mathrm{index}_{\mathrm{spin}^{\mathrm{c}}}(E)$; the spin$^{\mathrm{c}}$-number of X is the spin$^{\mathrm{c}}$-number of $[1] \in \mathrm{K}^0(X)$.

The following is the Atiyah-Singer Index Theorem for the Dirac operator on an even-dimensional manifold.

THEOREM 3.12. *Let X be an even-dimensional Riemannian* $\mathrm{spin}^{\mathrm{c}}$*-manifold. Then the analytic index* (3.12) *of the Dirac operator on X is equal to the* $\mathrm{spin}^{\mathrm{c}}$*-number of X:*

$$\mathrm{index}_{\mathrm{an}}(D) = \mathrm{index}_{\mathrm{spin}^{\mathrm{c}}}([E]).$$

More generally, the analytic index of D twisted by a complex vector bundle E over X, is the $\mathrm{spin}^{\mathrm{c}}$*-number* (3.13) *of E:*

$$\mathrm{index}_{\mathrm{an}}(D_E) = \mathrm{index}_{\mathrm{spin}^{\mathrm{c}}}(E).$$

We now describe the Atiyah-Singer formula in the odd-dimensional case – the generalization of the Toeplitz index theorem promised in Chapter 2.

Suppose X is odd-dimensional, Riemannian, $\mathrm{spin}^{\mathrm{c}}$, then we are assuming given a spinor bundle S over X, with a fibrewise irreducible representation of the Clifford algebra bundle $\mathrm{Cliff}(TX)$.

The bundle S is no longer, however, graded. According to the results above, the Dirac operator $D\colon C_c^\infty(X,S) \to C_c^\infty(X,S)$ is (formally) self-adjoint. Using Hilbert space theory, one can show that D admits an extension to a self-adjoint operator on a suitable Hilbert space closure $L^2(X,S)$ of $C_c^\infty(M,S)$, which means that one can do Hilbert space spectral theory with it. Since the operator is self-adjoint, its spectrum is contained in $\mathbb{R}$, it can be shown to be discrete, with finite-multiplicities, and there is a direct sum decomposition of $L^2(X,S)$ into eigenspaces of D.

DEFINITION 3.13. Let X be odd-dimensional Riemannian $\mathrm{spin}^{\mathrm{c}}$, D the (self-adjoint) Hilbert space Dirac operator on $L^2(X,S)$ discussed above. The *Dirac-Szegö projection p_D* is defined to be the projection $p_D\colon L^2(X,S) \to L^2(X,S)$ onto the span H_D^+ of the non-negative eigenspaces of D. If $u\colon X \to \mathrm{End}(S)$ is a unitary bundle endomorphism of S, the *generalized Toeplitz operator* T_u is defined to be

$$T_u := p_D T_u p_D \colon H_D^+ \to H_D^+.$$

LEMMA 3.14. *If p_D is the Dirac-Szegö projection, and $a \in C^\infty(X)$, acting as a multiplication operator on $L^2(X,S)$, then:*

- *The commutator $[a, p_D]$ is a compact operator on $L^2(X,S)$.*
- *If a is non-vanishing on X, then $T_a := p_D a p_D$ is a Fredholm operator on H_D^+.*

These statements will follow from techniques explained in the next chapter.

In the following, we will want to consider a 'matrix-valued' version of the Dirac-Szegö projection, *etc*, so we note that if $a\colon X \to M_n(\mathbb{C})$ is a matrix-valued continuous map, it acts as a multiplication operator on $L^2(X,S^n)$, with $S^n := S \oplus \cdots S$. The projection p_D acts on $L^2(X,S^n)$ by $\tilde{p}_D := p_D \oplus \cdots p_D$, and the statements of Lemma 3.14 extend to this situation (they follow from the scalar version, by an easy exercise), and in particular the operator $T_u := \tilde{p}_D u \tilde{P}_D$ is Fredholm on $H_D^+ \oplus \cdots \oplus H_D$.

The following is the Atiyah-Singer Index Theorem for the Dirac operator on an odd-dimensional manifold – the generalized Toeplitz Index Theorem promised in Chapter 1.

THEOREM 3.15. *Let X be an odd-dimensional Riemannian* $\mathrm{spin}^{\mathrm{c}}$*-manifold, D the Dirac operator, $p_D\colon L^2(X,S) \to H_D^+$ the Dirac-Szegö projection, $u\colon X \to \mathbf{U}_n$ a unitary-valued map, and $T_u = \tilde{p}_D u \tilde{p}_D$ the corresponding Toeplitz operator.*

Then the Fredholm index of T_u is equal to the $\mathrm{spin}^{\mathrm{c}}$*-number of u:*

$$\mathrm{index}(T_u) = \mathrm{index}_{\mathrm{spin}^{\mathrm{c}}}(u) \in \mathbb{Z}.$$

Thus, the $\mathrm{spin}^{\mathrm{c}}$-number plays the role of the winding number around the origin, of a unitary map on the circle, that was the invariant we had in the context of the Toeplitz Index Theorem.

4. Dirac operators and analytic K-homology

Deeper insight into K-theory, as we have seen, comes from the construction of interesting *maps* on K-theory groups: maps to the integers, as in the construction of spin^c-numbers of vector bundles, or unitaries, arising from Baum-Douglas cycles, or maps *between* K-theory groups, as arise from the correspondences discussed in Chapter 5. Such maps enable us, for example, to detect when K-theory classes are nonzero.

More formally, in Chapter 5 we described a *dual* theory to K-theory, called K-*homology*, and more generally, a bivariant theory KK-theory, defined by equivalence classes of correspondences.

This "KK-theory" we have described only makes sense for spaces, or, if one likes to think of C*-algebras, it only is defined for commutative C*-algebras. It was the contribution of Kasparov, motivated by the Index Theorem, and suggestions of Atiyah, to extend it to noncommutative C*-algebras, using the concept of a *Fredholm module*, which we start by explaining.

Fredholm modules

Let X be a smooth, compact manifold and D an elliptic differential operator on $C_c^\infty(M, S)$, for a Hermitian vector bundle S over X. In order to form a Hilbert space, fix a Borel measure on X which is smoothly equivalent, in the domains of manifold charts, to Lebesgue measure on $\mathbb{R}^n$. Let $L^2(X, S)$ be the Hilbert space completion of $C_c^\infty(M, S)$ under the inner product

$$\langle s_1, s_2\rangle := \int_X \langle s_1(x), s_2(x)\rangle d\mu(x).$$

Sobolev theory implies that the original elliptic operator extends in a unique way to a densely defined, self-adjoint operator on $L^2(X, S)$. As such, it has a spectrum, the spectrum of an unbounded operator on a Hilbert space being defined to be all $\lambda \in \mathbb{C}$ for which $(\lambda - D)^{-1}$ is defined, and extends continuously to a bounded operator on $L^2(X, S)$. The spectrum of D can be shown to be both infinite and topologically *discrete* in $\mathbb{R}$, each point of it is an eigenvalue, and the eigenspaces are finite-dimensional and consist entirely of smooth sections.

In particular, we obtain an orthogonal decomposition

$$L^2(X, S) = \bigoplus_{\lambda \in \mathrm{Spec}(D)} H_\lambda, \tag{4.1}$$

where H_λ is the λ-eigenspace of D.

We define the *sign of* D to be the operator

$$F_D = \oplus_{\lambda \in \mathrm{Spec}(D)} \mathrm{sign}(\lambda) \cdot 1_{H_\lambda},$$

where the sign is defined as usual for real numbers, and the sign of 0 being defined in this case to be 0, ad 1_{H_λ} the identity operator on H_λ. The operator F_D satisfies

a) If $f \in C^\infty(X)$ is a smooth function, acting as a multiplication operator on $L^2(X, S)$, then the operator commutator $[f, F_D]$ is a *compact operator* on $L^2(X, S)$.
b) $\ker(F) = \ker(D)$.

The second assertion is clear from the definition, and the first is discussed below.

Now suppose that S is a $\mathbb{Z}/2$-graded bundle, such as for example happens if S is the spinor bundle for an even-dimensional spin^c-manifold, and D is the Dirac operator. Let $\epsilon \colon L^2(X, S) \to L^2(X, S)$ be the grading operator, and assume that D anti-commutes with ϵ: $D\epsilon = -\epsilon D$. If $s \in H_\lambda$, then

$$D(\epsilon s) = -\epsilon D s = -\lambda \cdot \epsilon s.$$

Hence $\epsilon s \in H_{-\lambda}$ if $s \in H_\lambda$. In particular, ϵ maps $\ker(D)$ to itself, and F_D also anti-commutes with ϵ. That is, F_D is an odd operator with respect to the grading. If we define the analytic index of such an odd operator by

$$\mathrm{index}_{\mathrm{an}}(F_D) := \dim\ker(F_D^+) - \dim\ker(F_D^-),$$

where $F_D^+ \colon L^2(X,S)^+ \to L^2(X,S)^-$ is the restriction of F_D to the the $+1$-eigenspace of ϵ, then observe that

$$\mathrm{index}_{\mathrm{an}}(F_D) = \mathrm{index}_{\mathrm{an}}(D),$$

with the index of D being defined as in (3.12) for an odd, elliptic operator on a $\mathbb{Z}/2$-graded bundle. In principal, this method reduces the problem of studying the analytic Fredholm index of an unbounded operator, to that of a bounded one.

LEMMA 4.1. *Let M be a compact manifold, S a complex vector bundle over M, and D be an elliptic operator on $C^\infty(M,S)$, understood as an unbounded, self-adjoint operator on the Hilbert space $L^2(M,S)$.*

Then if $\psi \in C_0(\mathbb{R})$, then the (bounded) operator $\psi(D)$ obtained by functional calculus for D, is a compact operator on $L^2(M,S)$.

PROOF. The operator $\psi(D)$ is diagonal with respect to the decomposition (4.1), with λ-th entry $\psi(\lambda)$. Since $L^2(M,S)$ is infinite-dimensional, and $\mathrm{Spec}(D)$ is topologically discrete, ψ as a function on $\mathrm{Spec}(D)$ is a C_0-function, and hence, since every eigenspace is also finite-dimensional, $\psi(D)$ is compact. $\square$

REMARK 4.2. Suppose that $\chi\colon \mathbb{R} \to [-1,1]$ is any continuous, odd function, such that $\lim_{t\to\pm\infty}\chi(t) = \pm 1$. Such a function is called a *normalizing function.* An example is the sign function used above to define F_D from D.

Then if D is a self-adjoint operator such that $\psi(D)$ is compact for all $\psi \in C_0(\mathbb{R})$, and if χ and χ' are any two normalizing functions, observe that

$$\chi(D) - \chi'(D) = (\chi - \chi')(D)$$

and $\chi - \chi' \in C_0(\mathbb{R})$. So an application of the lemma shows that $\chi(D) - \chi'(D) \in \mathcal{K}(H)$.

REMARK 4.3. For a self-adjoint unbounded operator D, and a bounded operator S, the condition

$$(1 + D^2)^{-1} \in \mathcal{K}(H).$$

is equivalent to the condition

$$\psi(D) \in \mathcal{K}(H)$$

for all $\psi \in C_0(\mathbb{R})$.

The reason is that

$$(1 + D^2)^{-1} = (i + D)^{-1}(-i + D)^{-1} = TT^*,$$

where $T = (i + D)^{-1}$, and compactness of TT^* implies that of T as well, by an easy exercise. This shows that $\psi_\pm(D)$ is compact for the particular functions $\psi_\pm(t) := (\pm i + t)^{-1}$, and these generate $C_0(\mathbb{R})$ as a C*-algebra, by the Stone-Weierstrass theorem, and so compactness of $\psi(D)$ for all $\psi \in C_0(\mathbb{R})$ follows.

The exercise below give a slight variation of these remarks useful for some examples.

LEMMA 4.4. *Let D be a densely defined self-adjoint operator on a Hilbert space H such that $(1 + D^2)^{-1} \in \mathcal{K}(H)$.*

Let S be a bounded operator on H, leaving the domain of D invariant, and such that the commutator $[S,D]$ is bounded, Then

Then, for any normalizing function χ, the commutator

$$[S, \chi(D)]$$

is compact.

PROOF. The first statement follows from the discussion before the Lemma and the assumption a). As a consequence, it suffices to prove the second assertion for the particular normalizing function

$$\chi(t) := t(1+t^2)^{-\frac{1}{2}}.$$

This particular function has an integral formula

$$(1+t^2)^{-\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda+t^2)^{-1}d\lambda$$

– the integral converges absolutely because the integrand is bounded by

$$\frac{1}{2}\lambda^{-\frac{1}{2}}(1+\lambda)^{-1} \sim \frac{1}{2}\lambda^{-\frac{3}{2}}.$$

It follows that

$$(1+D^2)^{-\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda+D^2)^{-1}d\lambda,$$

where the integral converges absolutely in $\mathcal{K}(H)$.

Now, by the derivation property of taking commutators,

$$[S, \chi(D)] = [S, D(1+D^2)^{-\frac{1}{2}}] = [S, D]\cdot(1+D^2)^{-\frac{1}{2}} + D\cdot[S, (1+D^2)^{-\frac{1}{2}}].$$

The first term is compact, since $[S, D]$ is bounded, and $(1+D^2)^{-\frac{1}{2}}$ is compact, by the assumptions. Therefore it suffices to prove that the second term

$$D\cdot[S, (1+D^2)^{-\frac{1}{2}}].$$

is compact. Using the integral formula we get

$$[S, (1+D^2)^{-\frac{1}{2}}] = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\cdot[S, (1+\lambda+D^2)^{-1}]d\lambda,$$

Multiplying by D gives

$$D\cdot[S, (1+D^2)^{-\frac{1}{2}}] = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}D\cdot[S, (1+\lambda+D^2)^{-1}]d\lambda = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}D\cdot[S, T_\lambda^{-1}]d\lambda \tag{4.2}$$

where $T_\lambda := 1+\lambda+D^2$. Now

$$[S, T_\lambda^{-1}] = T_\lambda^{-1}[S, T_\lambda]T_\lambda^{-1} = T_\lambda^{-1}[S, D^2]T_\lambda^{-1} = T_\lambda^{-1}[S, D]DT_\lambda^{-1} + T_\lambda^{-1}D[S, D]T_\lambda^{-1}. \tag{4.3}$$

Multiplying by D gives

$$D\cdot[S, T_\lambda^{-1}] = DT_\lambda^{-1}[S, D]DT_\lambda^{-1} + DT_\lambda^{-1}D[S, D]T_\lambda^{-1}. \tag{4.4}$$

The operator DT_λ^{-1} is obtained from D by application of functional calculus to the function

$$\phi_\lambda(t) = t\cdot(1+\lambda+t^2)^{-1},$$

which is in $C_0(\mathbb{R})$. Our assumption that $(1+D^2)^{-1}$ is compact implies that $\phi(D)$ is compact for any $\psi \in C_0(\mathbb{R})$, because the hypothesis implies that $(D \pm i)^{-1}$ are compact operators, and these generate $C_0(\mathbb{R})$ as a C*-algebra. Applying this observation to the given ϕ_λ we get that

DT_λ^{-1} is compact. An exercise in calculus shows that the maximum value(s) obtained by ϕ_λ are at $\pm\sqrt{1+\lambda}$ and their common value is $\frac{1}{2}(1+\lambda)^{-\frac{1}{2}}$. Hence

$$\|\phi_\lambda\|_{C_0(\mathbb{R})} = \frac{1}{2}(1+\lambda)^{-\frac{1}{2}}$$

and therefore, since $\phi \mapsto \phi(D)$ is a *-homomorphism on $C_0(\mathbb{R}$ and so contractive,

$$\|DT_\lambda^{-1}\| = \frac{1}{2}(1+\lambda)^{-\frac{1}{2}}.$$

The operator $[S, D]$ is bounded. It follows that

$$DT_\lambda^{-1}[S,D]DT_\lambda^{-1}$$

is a compact operator with operator norm bounded by $\frac{1}{4}\|[S,D]\|(1+\lambda)^{-1}$. The same analysis applies to the second term of (4.4), and it follows that the integral (4.2) is an absolutely convergent integral of compact operators in the operator norm, and hence that

$$D \cdot [S, (1+D^2)^{-\frac{1}{2}}]$$

is compact, as claimed.

□

Remark 4.5. The point of the integral trick used in the proof above, is that one is interested in computing a commutator of an operator S with $(1+D^2)^{-\frac{1}{2}}$, which is a square root (of $(1+D^2)^{-1}$) and square roots are tricky to deal with; the integral removes the square root from the computation.

Note also that the proof gives the following norm estimate on a commutator $[S, D]$:

$$\|S, D(1+D^2)^{-\frac{1}{2}}]\| \leq \text{const.}\|[S,D]\| \cdot \int_0^\infty \lambda^{-\frac{1}{2}} \cdot \|D(1+\lambda+D^2)^{-1}\|^2 d\lambda \tag{4.5}$$

Definition 4.6. Let A be a C*-algebra. An *even Fredholm module over A* is a triple consisting of a $\mathbb{Z}/2$-graded Hilbert space H, a representation

$$\pi\colon A \to \mathbb{B}(H),$$

as even operators, and a self-adjoint, odd operator F on H such that

$$\pi(a) \cdot (F^2 - 1) \in \mathcal{K}(H)$$

for all $a \in A$.

If A is unital, and $\pi\colon A \to \mathbb{B}(H)$ is non-degenerate, then the *index* of the even Fredholm module (H, π, F) is the Fredholm index index(F_D) of the operator

$$F|_{H^+}\colon H^+ \to H^-.$$

An *odd* Fredholm module is defined the same way, except we drop the assumption of a $\mathbb{Z}/2$-grading.

We define equivalence of Fredholm modules in the following way.

Definition 4.7. A Fredholm module (H, π, F) is *degenerate* if $\pi(a)(F^2-1), [\pi(a), F]$ are both zero for all $a \in A$.

Two Fredholm modules (H_i, π_i, F_i) are *operator homotopic* if, up to a unitary isomorphism, they differ only by the operators F_i, and the F_i are homotopic in $\mathbb{B}(H)$ through operators F_t for which (H, π, F_t) are Fredolm modules.

Exercise 4.8. Prove that if (H, π, F) is a Fredholm module, and (H, π, F') is another, such that $F - F' \in \mathcal{K}(H)$, then the two Fredholm modules are operator homotopic.

They are called *compact perturbations* of each other.

Remark 4.9. The index of an even Fredholm module doesn't involve the representation of the C*-algebra A, but the map induced on $\mathrm{K}_0(A)$, defined below, does use it. In the odd case, an odd Fredholm module has no index. But it will induce map on $\mathrm{K}_1(A)$.

We also note that an arbitrary Fredholm module over unital A can be replaced by an equivalent one, in which the representation is non-degenerate (that is, unital). If $\pi(1) = p \in \mathbb{B}(H)$, then we compress the Fredholm module by the projection p

$$(pH, \pi, pFp).$$

This results in an equivalent Fredholm module, because the original Fredholm module differs from this one by a compact perturbation of a degenerate Fredholm module, namely

$$((1-p)H, 0, (1-p)F(1-p)),$$

where 0 denotes the zero representation.

Example 4.10. From the preceding discussion that if X is a spin$^{\mathrm{c}}$-manifold, D the Dirac operator on $C_c^\infty(M, S)$, F_D the sign of D, and $\pi\colon C(X) \to \mathbb{B}\big(L^2(X, S)\big)$ the representation of $C(X)$ by multiplication operators, constitutes a Fredholm module over $C(X)$. It is of even dimension $j = 0$ if X is even-dimensional (since in this case the Hilbert space comes from a $\mathbb{Z}/2$-grading), and of odd dimension $j = 1$ otherwise.

Since the process of building the Fredholm module $(L^2(X, S), \pi, F_D)$ only used that D was elliptic, and that S was $\mathbb{Z}/2$-graded, in the even-dimensional case, exactly the same procedure produces a Fredholm module from the 'Dirac-type' operators D_E, where E is a vector bundle over X (because they are elliptic, still formally self-adjoint, and are odd with respect to the induced grading.)

In particular, the Dirac operator $D = -i\frac{d}{d\theta}$ on the circle $\mathbb{T}$ gives rise to the following Fredholm module. The Hilbert space is $L^2(\mathbb{T})$, and the operator is the Fourier transform of the operator on $l^2(\mathbb{Z})$ which is diagonal, equal to 0 at 0, and otherwise is $= \operatorname{sign}(n)$. The representation $\pi\colon C(\mathbb{T}) \to \mathbb{B}\big(L^2(\mathbb{T})\big)$ is by multiplication operators.

Notice that we have essentially assembled all the ingredients for the Toeplitz Index Theorem. If P_+ is the Toeplitz, or Szegö projection, the Fourier transform of the Hilbert space projection to the Hardy space $\mathbf{H}^2$ inside $l^2(\mathbb{Z})$, then

$$F_D = -(1 - P_+) + P_+$$

(up to finite rank.)

This Fredholm module over $C(\mathbb{T})$ is *odd*.

Fredholm modules can be added (by direct sum), and equivalence classes of them form an abelian $\mathbb{Z}/2$-graded group

$$\mathrm{KK}_*(A, \mathbb{C}) := \mathrm{KK}_0(A, \mathbb{C}) \oplus \mathrm{KK}_1(A, \mathbb{C})$$

which we call the *Atiyah-Kasparov* K-*homology* of A. It pairs with K-theory in the following way.

Pairing of even Fredholm modules with even K-*theory classes*

Suppose that (H, π, F) is a Fredholm module over A

Consider the operator F. We know that $F = F^*$, and that $F^2 - 1$ is compact. Furthermore, F is odd with respect to the grading. So with respect to the orthogonal decompositon $H = H_+ \oplus H_-$, the operator F and representation have the forms

$$F = \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix}, \quad \pi(a) = \begin{bmatrix} \pi^+(a) & 0 \\ 0 & \pi^-(a) \end{bmatrix},$$

where $U\colon H^+ \to H^-$ is an essentially unitary operator, with the property that

$$\pi^-(a)U - U\pi^+(a) \in \mathcal{K}(H^+, H^-)$$

is a compact operator for all $a \in A$.

Now suppose that p is a projection in A. Since $\pi(p)$ commutes mod compact operators with F, the operator

$$\pi^-(p)U\pi^+(p)\colon \pi^+(p)H^+ \to \pi^-(p)H^- \tag{4.6}$$

is a Fredholm operator. Then we make the definition

$$\langle (H, \pi, F), p\rangle := \operatorname{index}\big(\pi^-(p)U\pi^+(p)\big). \tag{4.7}$$

Exercise 4.11. Show that this 'index pairing' is invariant under changing the Fredholm module to either a homotopic one, or perturbing it by a degenerate.

To define the pairing of (H, π, F) with a projection not necessarily in A, but in $M_n(A)$, for some n, observe that $(H \oplus \cdots H, \pi \oplus \cdots \oplus \pi, F \oplus \cdots F)$ is also a Fredholm module over A, where the $\mathbb{Z}/2$-grading is induced by $\epsilon \oplus \cdots \oplus \epsilon$. Hence, if $p \in M_n(A)$ is a projection, then we can pair p with the original Fredholm module by first forming a direct sum of n copies of (H, π, F), compressing the operator with the projections $\pi^{\pm}(p)$ as in (4.8), and taking its Fredholm index, that is, the index of:

$$\pi^-(p)(U \oplus \cdots \oplus U)\pi^+(p)\colon \pi^+(p)H^+ \oplus \cdots \oplus \pi^+(p)H^+ \to \pi^-(p)H^- \oplus \cdots \oplus \pi^-(p)H^- \tag{4.8}$$

We obtain a bilinear pairing

$$\mathrm{K}_0(A) \times \mathrm{KK}_0(A, \mathbb{C}) \to \mathbb{Z}.$$

Note that if M be a compact, even-dimensional, spin^c-manifold, and D the Dirac operator on M, F_D its sign, then the pairing between the even Fredholm module $(L^2(M, S), \pi, F_D)$ and the projection $1 \in C(M)$, is the Fredholm index of F_D, and hence equals the analytic index of D.

Pairing of odd Fredholm modules with odd K*-theory classes*

To define a pairing

$$\mathrm{K}_1(A) \times \mathrm{KK}_1(A, \mathbb{C}) \to \mathbb{Z}.$$

between $\mathrm{K}_1(A)$-classes, and *odd* Fredholm modules, we use ideas familiar from the Toeplitz Index Theorem. Assume that (H, π, F) is a non-generate odd Fredholm module, Then $P := \frac{F+1}{2} \in \mathbb{B}(H)$ satisfies

$$P^* = P, \quad P^2 - P \in \mathcal{K}(H).$$

It follows that if $u \in A$ is a unitary, then

$$PuP + (1 - P) \in \mathbb{B}(H)$$

is an essentially unitary operator and we can set

$$\langle (H, \pi, F), u\rangle := \operatorname{index}(P\pi(u)P + 1 - P). \tag{4.9}$$

Exercise 4.12. In the above notation, show that

a) $P\pi(u)P + (1 - P)$ is an essentially unitary operator.
b) Show that the pairing (4.9) does not change if the Fredholm module is replaced by an operator homotopic one, or under addiing a degenerate (odd) Fredholm module to it.
c) Show that if u is replaced by another unitary, connected by a path of unitaries to u, then the pairing (4.9) does not change.

The pairing above extends to unitaries in $M_n(A)$ by the same procedure as in the even case.

EXERCISE 4.13. Follow the steps indicated to show that if D_1 and D_2 are densely defined, self-adjoint operators on a Hilbert space H sharing a common domain, and such that $D_1 - D_2$ is bounded and $(1 + D_i^2)^{-1}$ is compact, then the operators $\chi(D_1)$ and $\chi(D_2)$ are compact perturbations of each other, for any normalizing function φ.

a) Let X and Y be densely defined self-adjoint operators on a Hilbert space, sharing a common dense domain. Then

$$e^{isX} - e^{isY} = i \int_0^s e^{itX}(X - Y)e^{i(s-t)Y} u dt. \tag{4.10}$$

(*Hint.* Equation (4.10) is equivalent to

$$e^{isX} \cdot e^{-isY} - 1 = i \int_0^s e^{itX}(X - Y)e^{-itY} u dt. \tag{4.11}$$

Differentiating the path

$$e^{itX} \cdot e^{-itY}$$

using the product rule gives

$$\frac{d}{dt}\left(e^{itD} \cdot e^{-itD_2}\right) = ie^{itX}(X - Y)e^{-itY},$$

and integrating this identity from 0 to s gives, by the Fundamental Theorem of Calculus,

$$i \int_0^s e^{itX}(X - Y)e^{-itY} dt = e^{isX} \cdot e^{-isY} - 1 \tag{4.12}$$

as required.)

b) Let X and Y be densely defined self-adjoint operators on a Hilbert space such that $(1 + X^2)^{-1}$ and $(1 + Y^2)^{-1}$ are compact and $X - Y$ is bounded. Show that

$$\chi(X) - \chi(Y)$$

is a compact operator, for any normalizing function χ. (*Hint.* For the proof we select χ be a normalizing function with compactly supported Fourier transform and such that $s \cdot \hat{\chi}(s)$ is a smooth function. Suppose $\hat{\chi}$ is supported in the interval $[-a, a]$. Notice that $\chi_\epsilon(t) := \chi(\epsilon t)$ is also a normalizing function.

We have

$$\chi_\epsilon(X) - \chi_\epsilon(Y) = \chi(\epsilon X) - \chi(\epsilon Y) = \int \hat{\chi}(s)(e^{is\epsilon X} - e^{is\epsilon Y}) ds \tag{4.13}$$

By Lemma 4.13,

$$e^{isX} - e^{isY} = i \int_0^s e^{itX}(X - Y)e^{i(s-t)Y} dt,$$

and substituting this expression into (4.13) gives

$$\chi_\epsilon(X) - \chi_\epsilon(Y) = \epsilon \cdot \int \hat{\chi}(s) \cdot \int_0^s e^{it\epsilon X}(X - Y)e^{i(s-t)\epsilon Y} dt\, ds \tag{4.14}$$

so

$$\chi_\epsilon(X) - \chi_\epsilon(Y) = \epsilon \cdot \int \hat{\chi}(s) \cdot \int_0^s e^{it\epsilon X} \cdot (X - Y) e^{i(s-t)\epsilon Y} dt\, ds \tag{4.15}$$

Since $s \in [-a, a]$ and t is bounded by $|s|$ and hence by a, and as $X - Y$ is bounded, we see that $e^{it\epsilon X} \cdot (X - Y)e^{i(s-t)\epsilon Y}$ is a bounded operator and

$$\|e^{it\epsilon X} \cdot (X - Y)e^{i(s-t)\epsilon Y}\| \leq C$$

where C is a constant independent of s, t or ϵ. Finally, we obtain the norm estimate of bounded operators

$$\|\chi_\epsilon(X) - \chi_\epsilon(Y)\| \leq C'\epsilon \cdot \int |\hat{\chi}(s)| \cdot |s| ds. \tag{4.16}$$

This proves that

$$\lim_{\epsilon \to 0} \|\chi_\epsilon(X) - \chi_\epsilon(Y)\| = 0. \tag{4.17}$$

$$\lim_{\epsilon \to 0} \chi_\epsilon(X) - \chi_\epsilon(Y) = 0 \in \mathbb{B}(H). \tag{4.18}$$

But, modulo compact operators $\chi_\epsilon(X) = \chi(X)$, $i = 1, 2$, and all $\epsilon > 0$, and similarly for Y, so, projecting the identity (4.18) to the Calkin algebra gives that $\chi(X) = \chi(Y)$ mod compact operators, as claimed.)

c) Let D_1 and D_2 be order 1 elliptic operators on a compact manifold M, with the same principal symbol. Prove that the operators $\chi(D_1)$ and $\chi(D_2)$ are compact perturbations of each other. (*Hint.* $D_1 - D_2$ is bounded.)

5. KK-theory

Kasparov defines his bivariant groups $\mathrm{KK}_*(A, B)$, where A and B are C*-algebras, in a very similar way to the way in which we have defined analytic K-homology above. The definition goes essentially unchanged, except that H is replaced by a right Hilbert B-*module*.

Definition 5.1. Let A and B be C*-algebras. A *Fredholm A-B-bimodule* is a triple $(\mathcal{E}, \pi, F)$, where

a) $\mathcal{E}$ is a right Hilbert B-module.
c) $\pi\colon A \to \mathbb{B}(\mathcal{E})$ is a *-homomorphism, *i.e.* a representation of A by adjointable Hilbert B-module operators on $\mathcal{E}$.
d) $F \in \mathbb{B}(\mathcal{E})$ is a self-adjoint Hilbert B-module operator, satisfying

$$\pi(a)(F^2 - 1),\ \ [\pi(a), F]\ \in \mathcal{K}(\mathcal{E}) \tag{5.1}$$

for all $a \in A$.

The bimodule is *even* if it carries the additional data of a $\mathbb{Z}/2$ grading

$$\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-,$$

on $\mathcal{E}$, into orthogonal B-submodules, with respect to which elements of A act as even (grading-preserving) operators, and the operator F acts as an odd (grading-reversing) operator.

Otherwise, the bimodule is *odd*.

Clearly, a Fredholm $\mathcal{A}$-$\mathbb{C}$-bimodule is the same as a Fredholm module, as in the previous section.

For any A, B, the triple $(0,0,0)$, consisting of the zero Hilbert B-module, understood as a $\mathbb{Z}/2$-graded bimodule, in the only possible way, the zero representation of A, and the zero operator, is a Fredholm A-B-bimodule, and so is the triple $(A, \mathrm{id}_A, 0)$, thus, where A acts on the right Hilbert module A by left multiplication. The grading is $A = A^+$ (in other words, $A^- = \{0\}$.

There is an obvious notation of unitary isomorphism of such bimodules, and one can clearly take the direct sum of two of them. Let $\mathcal{E}_i(A, B)$ denote the corresponding semigroup of unitary isomorphism classes of even (if $i = 0$) and odd ($i = 1$) Fredholm A-B bimodules. It clearly has a certain natural functoriality. If $\beta \colon B \to B'$ is a *-homomorphism, then form the right Hilbert B'-module

$$\mathcal{E} \otimes_B B',$$

using tensor product of Hilbert modules: that of the right Hilbert B-module $\mathcal{E}$, and the right Hilbert B'-module B', over the representation $B \to B' \subset \mathcal{M}(B) = \mathbb{B}(B)$. If $a \in A$ let it act on $\mathcal{E} \otimes_B B'$ by $\pi(a) \otimes 1$, and form the operator $F \otimes 1$. The corresponding triple is a Fredholm A-B'-bimodule $\beta_*(\mathcal{E}, \pi, F)$. It is even or odd according as the original one was.

Thus β determines a semigroup homomorphism

$$\beta_* \colon \mathcal{E}(A, B) \to \mathcal{E}(A, B').$$

It is even more straightforward that if $\alpha \colon A' \to A$ is a *-homomorphism, then it induces a semigroup homomorphism

$$\alpha^* \colon \mathcal{E}(A, B) \to \mathcal{E}(A', B),$$

simply by replacing the representation π in an A-B-bimodule by $\pi \circ \alpha$.

If a Fredholm A-B-bimodule $(\mathcal{E}, \pi, F)$ has the property that all the terms in (5.1) are zero, not just compact, then we say it is *degenerate*.

As a particular case of the (forward) functoriality of the $\mathcal{E}(A, B)$ semigroups, note that the point evaluations at $t = 0$ and $t = 1$ give two *-homomorphisms

$$\epsilon_0, \epsilon_1 \colon C([0,1]) \to \mathbb{C},$$

and then semi-group homomorphisms

$$(\epsilon_0)_*,\ (\epsilon_1)_* \colon \mathcal{E}(A,\ C([0,1]) \otimes B) \to \mathcal{E}(A, B). \tag{5.2}$$

Definition 5.2. Two Fredhom A-B-bimodules are *homotopic* if they are unitarily isomorphic to the endpoints $(\epsilon_i)_*(\mathcal{E}, \pi, F)$ of a Fredholm A-$C([0,1]) \otimes B$-bimodule $(\mathcal{E}, \pi, F)$.

Lemma 5.3. *If* $\mathcal{E}, \pi, F)$ *is a degenerate, even, Fredholm* A-B*-bimodule,then* $(\mathcal{E}, \pi, F)$ *is homotopic to the zero bimodule* $(0,0,0)$.

Proof. The homotopy uses the $\mathbb{Z}/2$-graded right Hilbert $B \otimes C(I)$-module $C_0([0,1), \mathcal{E})$. The representation of A is by

$$\big(\tilde{\pi}(a)\xi\big)(t) := \pi(a)\xi(t),$$

and the operator

$$(\tilde{F}\xi)(t) := F\big(\xi(t)\big).$$

The triple $\big(C_0([0,1), \mathcal{E}), \tilde{\pi}, \tilde{F}\big)$ is a Fredholm bimodule because the operators

$$\tilde{\pi}(a) \cdot (\tilde{F}^2 - 1),\ \ [\tilde{\pi}(a), \tilde{F}]$$

are actually zero, and hence compact (which would not be the case if we merely assumed that $\pi(a) \cdot (F^2 - 1)$ and $[\pi(a), F]$ were merely *compact*.)

The endpoints of our homotopy clearly are respectively the zero bimodule, and our degenerate one, proving the result.

□

EXERCISE 5.4. Show that if $\beta_1, \beta_2 \colon B \to B'$ are homotopic *-homomorphisms, then

$$(\beta_1)_*(\mathcal{E}, \pi, F)$$

is homotopic to

$$(\beta_1)_*(\mathcal{E}, \pi, F)$$

, for any Fredholm A-B-bimodule $(\mathcal{E}, \pi, F)$.

DEFINITION 5.5. Let A, B be C*-algebras.

Then $\mathrm{KK}_0(A, B)$ is the quotient of the semigroup $\mathcal{E}_0(A, B)$ of $\mathbb{Z}/2$-graded (that is, even) Fredholm A-B-bimodules, by the equivalence relation of homotopy.

$\mathrm{KK}_1(A, B)$ is defined in exactly the same way, using odd bimodules.

A *Kasparov morphism* $A \to B$ is an element of $\mathrm{KK}_*(A, B) := \mathrm{KK}_0(A, B) \oplus \mathrm{KK}_1(A, B)$.

REMARK 5.6. In order to immediately correct an apparent conflict of notation, we point out the following. It is possible to define the equivalence relation(s) on cycles determining KK in a different way than the way we have done, and in the same way we did with K-homology: taking the equivalence relation on cycles generated by addition of degenerates, and operator homotopy (a homotopy $\{F_t\}_{t \in [0,1]}$ of operators in the norm topology, but where none of the other data varies.)

In the end it turns out that these two approaches agree. Thus, two cycles are homotopic if and only if, and addition of degenerates, they becomes operator homotopic. This somewhat remarkable result is due to G. Skandalis. We use this fact below in our proof that KK-theory recovers K-theory when the first variable is put equal to $\mathbb{C}$. The book [**2**] contains a proof of these statements.

LEMMA 5.7. *With the direct sum operation,* $\mathrm{KK}_0(A, B)$ *is a group.*

PROOF. Let $(\mathcal{E}, \pi, F)$ be a Fredholm A-B-bimodule. Consider the triple $(-\mathcal{E}, \pi, -F)$, where $-\mathcal{E}$ denotes $\mathcal{E}$ but with the *opposite* grading. The sum of $(\mathcal{E}, \pi, F)$ and $(-\mathcal{E}, \pi, -F)$ is

$$(\mathcal{E} \oplus -\mathcal{E}, \pi \oplus \pi, F \oplus -F).$$

Now let

$$\tilde{F}_t := \begin{bmatrix} \cos t \cdot F & \sin t \\ \sin t & -\cos t \cdot F \end{bmatrix} \in \mathbb{B}(\mathcal{E} \oplus -\mathcal{E}).$$

Using the operators $\tilde{F}_t$ we get a homotopy between $(\mathcal{E} \oplus -\mathcal{E}, \pi \oplus \pi, F \oplus -F)$. and the degenerate bimodule

$$(\mathcal{E} \oplus -\mathcal{E}, \pi \oplus \pi, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}).$$

□

A *-homomorphism $\alpha \colon A \to B$ determines immediately an element of $\mathcal{E}_0(A, B)$ by setting $\mathcal{E}^+ = B, \mathcal{E}^- = 0$, $\pi :=\colon A \to B = \mathcal{K}(\mathcal{E})$ and $F := 0$. The corresponding degree-zero Kasparov morphism is denoted

$$[\alpha] \in \mathrm{KK}_0(A, B).$$

More generally, if $\mathcal{E}$ is a Hilbert B-module and $\pi \colon A \to \mathcal{K}(\mathcal{E})$ is a representation of A as *compact operators* on B, then we can assign the grading $\mathcal{E}^+ := \mathcal{E}, \mathcal{E}^- := \{0\}$, set $F := 0$, then we obtain a cycle $(\mathcal{E}, \pi, 0) \in \mathcal{E}_0(A, B)$, because the terms in (5.1) are all compact.

This situation applies in particular if $\mathcal{E}$ is the underlying right B-module of a *strong Morita A-B-bimodule*, in which π is the left action.

Thus, *a strong Morita A-B-equivalence bimodule determines a Kasparov morphism*

$$[\mathcal{E}] \in \mathrm{KK}_0(A, B).$$

Kasparov morphisms also appear naturally in connection with K-theory of non-compact spaces. Let (E^+, E^-, u) be a K-theory triple for X. Thus, $E^\pm$ are complex vector bundles over X and u is a bundle map $E^+ \to E^-$ which is invertible off a compact subset of X.

We can put Hermitian metrics on $E^\pm$, making the spaces of C_0-sections $C_0(X, E^\pm)$ into right Hilbert $C_0(X)$-modules $\mathcal{E}^\pm$. Form the $\mathbb{Z}/2$-graded Hilbert $C_0(X)$-module $\mathcal{E} := \mathcal{E}^+ \oplus \mathcal{E}^-$. We can also assume without loss of generality (by a simple homotopy) that u is unitary off a compact set. Hence the $C_0(X)$-module operator

$$F := \begin{bmatrix} 0 & u* \\ u & 0 \end{bmatrix},$$

acting on $\mathcal{E}$, is self-adjoint, and $F^2 - 1$ is compact, because (because $F^2 - 1$ it vanishes off a compact set.)

The triple $(\mathcal{E}, 1, F)$ is a Fredholm $\mathbb{C}$-$C_0(X)$-bimodule and gives and element of $\mathrm{KK}_0\big(\mathbb{C}, C_0(X)\big)$.

A particular case of a triple is when X is actually compact, and then the bundle map $E^+ \to E^-$ can be taken to be the zero map. The procedure above produces a corresponding Kasparov morphism. These observations show that one has a natural map

$$\mathrm{K}^0(X) \to \mathrm{KK}_0(\mathbb{C}, C_0(X))$$

for any locally compact space X. It can be shown to be an isomorphism.

This is a special case of the following construction.

Let A be a C*-algebra (perhaps not unital) and (p, q, u) be a 'relative triple' for (A^+, A), where A^+ is its unitization. So $p, q, u \in M_n(A^+)$, and $uu^* - 1$, $u^*u - 1$ and $upu^* - q$ all lie in the ideal $M_n(A)$ of $M_n(A^+)$, for some n. Such triples are by definition cycles for $\mathrm{K}_0(A)$.

Now any $a \in M_n(A^+)$ is in the multiplier algebra of $M_n(A)$, and hence defines a right Hilbert A^n-module map, which we continue to just call

$$a \colon A^n \to A^n.$$

Moreover, if $a \in M_n(A)$, then a acts as a compact operator on A^n.

Proposition 5.8. *If (p, q, u) is a relative triple for A^+ over A, and*

$$w := qup \colon pA^n \to qA^n,$$

then the triple

$$\left((pA^n \oplus qA^n, 1, F := \begin{bmatrix} 0 & w^* \\ w & 0 \end{bmatrix} \right)$$

is a cycle for $\mathrm{KK}_0(\mathbb{C}, A)$.

Proof. The assumption implies that

$$u^*qu = p \bmod A.$$

Write

$$u^*qu = p + s, \qquad s \in A.$$

Let

$$w \colon pA^n \to qA^n, \ \ w := qup.$$

We have

$$w^*w = (qup)^*qup = pu^*qup = p(p+s)p = p + psp,$$

which is a perturbation of p by an element of $pM_n(A)p$, and hence by a compact operator on pA^n.

Therefore $w^*w - 1 \in \mathcal{K}(pA^n)$, and similarly $ww^* - 1 \in \mathcal{K}(pA^n)$. The result follows. ∎

EXERCISE 5.9. Show that unitarily equivalent relative triples map under the above construction to unitarily equivalent KK-cycles, and that degenerate triples map to degenerate KK-cycles.

THEOREM 5.10. *If A is a σ-unital C^*-algebra, then the map on cycles defined by Proposition 5.8 descends to a group isomorphism*

$$\mathrm{K}_0(A) \to \mathrm{KK}_0(\mathbb{C}, A).$$

We will sketch the proof of the theorem modulo two important technical theorem.

1. The Stabilization theorem .

The *Stabilization Theorem* for Hilbert modules asserts that if A is a σ-unital C*-algebra, then any right Hilbert A-module $\mathcal{E}$ is a direct summand (in the sense that it is orthogonally complemented) in the standard Hilbert A-module $A \otimes l^2$. To be explicit, $\mathcal{E} \oplus \mathcal{F} \cong A \otimes l^2$ for some Hilbert A-module $\mathcal{F}$, and where, of course, the direct sum is in the category of Hilbert modules, and the isomorphism is a unitary isomorphism of Hilbert modules.

2. Kuiper theorem.

Let A be any C*-algebra and $\mathcal{M}^s(A) := \mathcal{M}(A \otimes \mathcal{K})$. Then

$$\mathrm{K}_i(\mathcal{M}^s(A)) = 0, \quad i = 1, 2.$$

This is a consequence of what is sometimes called *Kuiper's theorem*, the contractibility of the unitary group in $\mathcal{M}^s(A)$, for any A.

PROOF. Let

$$\Phi\colon \mathrm{K}_0(A) \to \mathrm{KK}_0(\mathbb{C}, A)$$

be the map constructed using relative triples, above. We construct a map

$$\Psi\colon \mathrm{KK}_0(\mathbb{C}, A) \to \mathrm{K}_0(A)$$

inverting Φ by rephrasing $\mathrm{KK}_0(\mathbb{C}, A)$ in terms of the exact sequence of C*-algebras

$$0 \to A \otimes \mathcal{K} \to \mathcal{M}^s(A) \to Q^s(A) \to 0. \tag{5.3}$$

where $\mathcal{M}^s(A) := \mathcal{M}(A \otimes \mathcal{K})$ and $Q^s(A) = \mathcal{M}^s(A)/A \otimes \mathcal{K}$. The associated long exact sequence in K-theory of Theorem 3.14 gives an exact sequence

$$\begin{aligned} \cdots \mathrm{K}_2(Q^s(A)) \xrightarrow{\delta} \mathrm{K}_1(A) \to \mathrm{K}_1(\mathcal{M}^2(A)) \\ \to \mathrm{K}_1(Q^s(A)) \xrightarrow{\delta} \mathrm{K}_0(A) \to \mathrm{K}_0(\mathcal{M}^s(A)) \to \mathrm{K}_0(Q^s(A)). \end{aligned} \tag{5.4}$$

and by the Theorem referred to above, $\mathrm{K}_i(\mathcal{M}^s(A)) = 0, \ \ i = 0, 1$, so

$$\delta\colon \mathrm{K}_q(Q^s(A)) \to \mathrm{K}_0(A)$$

is an isomorphism.

Finally, we observe that we can rather easily map

$$\mathrm{KK}_0(\mathbb{C}, A) \to \mathrm{K}_1(Q^s(A))$$

using the Stabilization Theorem for Hilbert modules. Suppose that $(\mathcal{E}, \pi, F)$ is a cycle for $\mathrm{KK}_0(\mathbb{C}, A)$. We may assume that $\pi\colon \mathbb{C} \to \mathbb{B}(\mathcal{E})$ is unital, otherwise replace the cycle by the one obtained by compressing everything by the projection $\pi(1)$. The original cycle is then the direct sum of this one and a degenerate cycle. So we may assume that the representation involved in our cycle is unital, so the cycle has the form $(\mathcal{E}, 1, F)$.

Next, using the Stabilization Theorem, there exists Hilbert A-modules $\mathcal{F}^\pm$ such that

$$\mathcal{E}^\pm \oplus \mathcal{F}^\pm,$$

and we may assume in addition, without loss of generality, that $\mathcal{F}^+ \cong \mathcal{F}^-$. Let $\mathcal{F} = \mathcal{F}^+ \oplus \mathcal{F}^-$.

We now modify our cycle $(\mathcal{E}, 1, F)$ by adding to it the degenerate cycle

$$(\mathcal{F}, 1, \begin{bmatrix} 0 & w^* \\ w & 0 \end{bmatrix}).$$

This results in a cycle for $\mathrm{KK}_0(\mathbb{C}, A)$ in which the only remaining variable is the operator F, which has the form

$$F = \begin{bmatrix} 0 & u \\ u & 0 \end{bmatrix}$$

where

$$u \in \mathbb{B}(A \otimes l^2) \cong \mathcal{M}^s(A),$$

is an *essential unitary.* In particular, u defines a class

$$[u] \in \mathrm{K}_1(Q^s(A)).$$

We let Ψ be defined on the cycle we started with by

$$\delta([u]) \in \mathrm{K}_0(A),$$

where δ is the connecting map in the exact sequence (5.4).

Notice that if the cycle, reduced to one of the form

$$(A \otimes l^2 \oplus A \otimes l^2, 1, \begin{bmatrix} 0 & u^* \\ u & 0 \end{bmatrix})$$

is degenerate, then u is actually unitary in $\mathcal{M}^s(A)$, and hence by the exact sequence (5.4),

$$\delta([u]) = 0.$$

Finally, if two Kasparov cycles are operator homotopic, then they determine homotopic unitaries in $Q^s(A)$, as we leave it to the reader to check. Since the equivalence relation defining KK can be taken to be operator homotopy and addition of degenerates, we obtain a well-defined map

$$\mathrm{KK}_0(\mathbb{C}, A) \to \mathrm{K}_0(A).$$

The fact that it inverts Φ is left to the assiduous reader.

□

Elliptic operators on compact (or non-compact) manifolds also determine classes in KK, as we have already discussed in the previous Section.

Thus, if X is a compact spin^c-manifold, D the Dirac operator on X, extended to a self-adjoint unbounded operator on $L^2(X, S^\pm)$, and if $F_D := \mathrm{sign}(D)$ is the sign operator (or $\chi(D)$, for a normalizing function χ), then $(L^2(X, S^\pm), \pi, F_D)$ is a cycle for $\mathrm{KK}_*(C(X), \mathbb{C})$. The same is true for non-compact manifolds, although in general some additional discussion is needed about completeness of the Riemannian metric. We refer to Higson and Roe for the details.

We usually denote the class of the Dirac operator by

$$[D] \in \mathrm{KK}_{\dim X}(C_0(X), \mathbb{C}).$$

The most important result about KK-theory is that it has a category structure: Kasparov morphisms can be composed. This composition generalizes, one might say, the pairing between K-theory ($\mathrm{KK}_*(\mathbb{C}, A)$ and K-homology $(\mathrm{KK}_*(A, \mathbb{C}))$. Alternatively, it shows that this is simply a special case of composition of morphisms in the KK-category.

THEOREM 5.11. *For any A, B there is a bilinear pairing*

$$\mathrm{KK}_*(A,B) \times \mathrm{KK}_*(B,C) \to \mathrm{KK}_*(A,C) \tag{5.5}$$

mapping a pair of morphisms $f \in \mathrm{KK}_i(A,B)$ and $g \in \mathrm{KK}_j(B,C)$ to a morphism

$$f \hat{\otimes}_B g \in \mathrm{KK}_{i+j}(A,C),$$

and called the intersection product*, which is natural with respect to *-homomorphisms, and gives KK_* the structure of a $\mathbb{Z}/2$-graded, additive category with objects C^*-algebras, and morphisms $A \to B$ the elements of $\mathrm{KK}_*(A,B)$.*

*Moreover, mapping a *-homorphism $\alpha\colon A \to B$ to its class $[\alpha] \in \mathrm{KK}_0(A,B)$ defines a functor from the category of C^*-algebras and *-homomorphisms, to the category KK_*, with objects C^*-algebras, and morphisms given by the intersection product.*

The *existence* of the intersection product is a hard technical theorem due to Kasparov. His existence proof, moreover, is not constructive, so it is not helpful to know it in order to compute a *specific* intersection product. Instead, for this, one uses an axiomatic approach to the product, due to Connes and Skandalis.

THEOREM 5.12. *Let $(\mathcal{E}_1, \pi_1, F_1)$ be a cycle for $\mathrm{KK}_1(A,B)$, defining a class x, and $(\mathcal{E}_2, \pi_2, F_2)$ a cycle for $\mathrm{KK}_1(B,C)$, with class y. Let*

$$\mathcal{E} := \mathcal{E}_1 \otimes_B \mathcal{E}_2,$$

*the Hilbert module tensor product over the *-homomorphism $\pi\colon B \to \mathbb{B}(\mathcal{E}_2)$, and a right Hilbert C-module.*

Let

$$\pi\colon A \to \mathbb{B}(\mathcal{E}), \;\; \pi(a) = \pi_1(a) \otimes 1_{\mathcal{E}_2}.$$

In addition, assume that $u \in \mathbb{B}(\mathcal{E})$ is a bounded operator satisfying the following conditions:

a) $\pi(a) \cdot (u^*u - 1)$, $\pi(a) \cdot (uu^* - 1)$, $[\pi(a), u] \in \mathcal{K}(\mathcal{E})$ *for all $a \in A$.*

b) *For all $\xi \in \mathcal{E}_1$, the operators*

$$T_\xi \circ iF_2 - u \circ T_\xi \in \mathbb{B}(\mathcal{E}_2, \mathcal{E}), \quad -iF_2 \circ T_\xi^* - T_\xi^* \circ u \in \mathbb{B}(\mathcal{E}, \mathcal{E}_2) \tag{5.6}$$

are compact operators, where $T_\xi\colon \mathcal{E}_2 \to \mathcal{E}$ is the operator

$$T_\xi(\eta) := \xi \otimes \eta.$$

c) *For any $a \in A$, the operators*

$$\pi(a) \cdot (F_1 u + u^* F_1) \cdot \pi(a^*), \qquad \pi(a) \cdot (F_1 u^* + u F_1) \cdot \pi(a^*) \tag{5.7}$$

are positive *in the C^*-algebra $\mathbb{B}(\mathcal{E})/\mathcal{K}(\mathcal{E})$.*

Then the intersection product

$$x \hat{\otimes}_B y. \in \mathrm{KK}_0(A,C).$$

is represented by the triple $(\mathcal{E} \oplus \mathcal{E}, \pi \oplus \pi, F := \begin{bmatrix} 0 & u^* \\ u & 0 \end{bmatrix})$.

The axioms present certain relationships between the operators F_1, F_2 and the operator F. These relationships guarantee that any two F's satisfying the axioms, are actually operator homotopic, as we show below.

The axioms for the intersection product may be phrased slightly more transparently with more circumspect use of the gradings. Firstly, the conditions on u when phrased in terms of $F \in \mathbb{B}(\mathcal{E} \oplus \mathcal{E})$ assert that

$$\pi(a) \cdot (F^2 - 1), \;\; [\pi(a), F] \;\; \in \mathcal{K}(\mathcal{E} \oplus \mathcal{E}), \tag{5.8}$$

where here π denotes the direct sum of two copies of the original representation. Let

$$\tilde{F}_2 = \begin{bmatrix} 0 & -iF_2 \\ iF_2 & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{E}_2 \oplus \mathcal{E}_2).$$

For $\xi \in \mathcal{E}_1$, let $\tilde{T}_\xi \colon \mathcal{E}_2 \oplus \mathcal{E}_2 \to \mathcal{E} \oplus \mathcal{E}$, be the direct sum of two copies of T_ξ. Then the connection condition b) asserts that

$$\tilde{T}_\xi \cdot \tilde{F}_2 - F \cdot \tilde{F}_\xi \tag{5.9}$$

is a compact operator. Finally, the alignment condition can be written as follows: let

$$\tilde{F}_1 = \begin{bmatrix} 0 & F_1 \\ F_1 & 0 \end{bmatrix}.$$

The alignment condition c) says in this notation

$$\pi(a) \cdot [\tilde{F}_1, F]_s \cdot \pi(a)^* \geq 0 \mod \mathcal{K}(\mathcal{E} \oplus \mathcal{E}). \tag{5.10}$$

where $[\cdot, \cdot]_s$ denotes the *graded commutator*

$$[A, B]_s := AB + BA$$

of two (odd) operators on a graded space.

Proposition 5.13. *Let F and F' be two self-adjoint, odd operators on $\mathcal{E} \oplus \mathcal{E}$ which satisfy conditions* (5.8)*,* (5.9) *and* (5.10)*. Then there is a path of self-adjoint odd operators F_t between then, also satisfying the axioms.*

Proof. Suppose first that the self-adjoint operator $[F, F']_s := FF' + F'F$ is actually positive, and, in fact, we only need to assume that

$$[F, F']_s > -2$$

for the argument. Also, for simplicity, assume that A, and the representation π, is unital, to simplify. Then $F^2 = 1$ and $(F')^2 = 1$ mod compacts. It follows that each of F and F' commute mod compacts with $[F, F']_s$.

Consider the path

$$F_t := (\cos t \cdot F + \sin t \cdot F') \cdot (1 + \sin t \cos t \cdot [F, F']_s)^{-1}, \;\; t \in [0, \frac{\pi}{2}]. \tag{5.11}$$

As $\sin t \cos t \geq \frac{1}{2}$, it follows that $1 + \sin t \cos t \cdot [F, F']_s$ will be invertible if $[F, F']_s \geq \lambda$, where $\lambda > -2$. So the inverse in the second term is well-defined. The two terms also commute mod compacts, by the observations above. As

$$(\cos t \cdot F + \sin t \cdot F')^2 = 1 + 2 \cos t \sin t \cdot [F, F']_s, \quad \mod \mathcal{K}$$

we get $F_t^2 = 1$ mod compacts. It is routine to check that the F_t's satisfy (5.8), (5.9) and (5.10).

In the general case, write

$$[F, F']_s = P - Q$$

where $P, Q \geq 0,\ PQ = QP = 0$. If $a \in A$ we have

$$\pi(a) \cdot [F, F']_s \cdot \pi(a) = \pi(a) P \pi(a)^* - \pi(a) Q \pi(a)^*$$

is a positive operator mod compacts. Moreover, $\pi(a)$ commutes with P and with Q mod compacts, and it follows that

$$\pi(a) Q \pi(a)^* = 0 \mod \mathcal{K}$$

and hence that $\pi(a)Q = 0$ mod compacts. We set

$$F_t := (\cos t \cdot F + \sin t \cdot F') \cdot (1 + \sin t \cos t \cdot P)^{-1}, \;\; t \in [0, \frac{\pi}{2}]. \tag{5.12}$$

We leave it to the reader to check that he conditions are met.

□

We will show how the axioms work in practise in the next section, where we re-prove the Bott Periodicity theorem, and an important generalization of it, using KK-theory.

EXERCISE 5.14. Prove that $C_0\big([0,+\infty)\big)$ is KK-equivalent to the zero C*-algebra by showing the the identity

$$1_{C_0\left([0,+\infty)\right)} = 0 \in \mathrm{KK}_0(C_0\big([0,+\infty)\big), C_0\big([0,+\infty)\big)).$$

(*Hint.* Produce an appropriate homotopy.)

6. The Dirac-Schrödinger proof of Bott Periodicity

In this section we compute the single most important example of an intersection product – one which gives rise to an important 'KK-proof' of Bott Periodicity, and even of 'equivariant Bott Periodicity' discussed in the next section. The beauty of this Dirac-Schrödinger proof, aside from bringing our exposition elegantly back to the quantum-mechanical sources of the subject, is that it is built in a way reflecting the geometry of $\mathbb{R}$ and, for example, of the integers $\mathbb{Z}$ sitting in it as a subgroup. These features lead to strengthening of the Periodicity Theorem to make it equivariant, in a sense explained in the next section.

Let $\chi\colon \mathbb{R} \to [-1,1]$ be a normalizing function.

Set $\mathcal{E}_1 := C_0(\mathbb{R})$, a right Hilbert $C_0(\mathbb{R})$-module. Let

$$F_1 \in \mathbb{B}(\mathcal{E}_1), \ \ (F_1\xi)(t) = \chi(t)\cdot\xi(t),$$

that is, F_1 is multiplication by the bounded continuous function χ. Clearly $F_1^2 - 1$ is compact, since it is multiplication by the C_0-function $\chi(t)^2 - 1$. We obtain a cycle $(\mathcal{E}_1, 1, F_1)$ for $\mathrm{KK}_1(\mathbb{C}, C_0(\mathbb{R}))$.

Let

$$x \in \mathrm{KK}_1(\mathbb{C}, C_0(\mathbb{R}))$$

be the class of the odd Fredholm $\mathbb{C}$-$C_0(\mathbb{R})$-bimodule $(C_0(\mathbb{R}), 1, F_1)$. We will call it the *Bott morphism*.

Next, let $\mathcal{E}_2 := L^2(\mathbb{R})$, D_2 the self-adjoint extension of the densely defined unbounded operator $-i\frac{d}{dx}$, on $L^2(\mathbb{R})$. Let $F_2 \in \mathbb{B}(L^2\mathbb{R})$ be $\chi(D_2)$, (note, it is the Fourier conjugate of F_1), so that

$$F_2 = \chi(D_2)$$

in the sense of functional calculus for unbounded, self-adjoint operators.

If $f \in C_c^\infty(\mathbb{R})$, then

$$[f, D_2] = -if',$$

and in particular, the commutator is bounded for smooth and compactly supported functions. Furthermore, $\rho\cdot(1+D_2^2)^{-1}$ is a compact operator for any $\rho \in C_c^\infty(\mathbb{R})$, whence

$$[f, D_2]\cdot(1+D_2)^{-1} \in \mathcal{K}(L^2\mathbb{R})$$

for any $f \in C_c^\infty(\mathbb{R})$, and hence by Exercise **??**,

$$[f, \chi(D_2)] \in \mathcal{K}(L^2\mathbb{R}),$$

for all $f \in C_c^\infty(\mathbb{R})$, and hence for all $f \in C_0(\mathbb{R})$ as well.

Therefore $(L^2\mathbb{R}, \pi, D_2)$ is an odd, Fredholm $C_0(\mathbb{R})$-$\mathbb{C}$-bimodule.

Let

$$y \in \mathrm{KK}_1(C_0(\mathbb{R}), \mathbb{C})$$

the class of the cycle $(L^2(\mathbb{R}), \pi, F_2)$, where $\pi\colon C_0(\mathbb{R}) \to \mathbb{B}\big(L^2(\mathbb{R})\big)$ is the representation by multiplication operators. We will call it the *Dirac morphism*.

We wish to compute the intersection product

$$x\hat{\otimes}_{C_0(\mathbb{R})}y \in \mathrm{KK}_0(\mathbb{C},\mathbb{C}).$$

We are going to show that just as x and y were represented, in a sense, by unbounded operators, namely x and $-i\frac{d}{dx}$, respectively, to which we applied normalizing functions, the intersection product $x\hat{\otimes}_{C_0(\mathbb{R})}y$ will involve an unbounded operator: the *creation operator*

$$S := x + \frac{d}{dx}.$$

Its adjoint, in a formal sense, is $x - \frac{d}{dx}$.

Set

$$D := \begin{bmatrix} 0 & -\frac{d}{dx} + x \\ x + \frac{d}{dx} & 0 \end{bmatrix} = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}, \tag{6.1}$$

which we consider as acting initially on the Schwartz space $\mathcal{S}(\mathbb{R})$. D is self-adjoint, and has the form

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$$

where $A = x + \frac{d}{dx}$.

If

$$H := -\frac{d^2}{dx^2} + x^2$$

is the 'harmonic oscillator', then the formal adjoint $A^* = x - \frac{d}{dx}$ of A satisfies

$$AA^* = H + 1, \quad A^*A = H - 1, \quad [A, A^*] = 2, \quad [H, A] = -2A, \quad [H, A^*] = 2A^*. \tag{6.2}$$

Now set

$$\xi_0 := \sqrt{\pi} \cdot e^{-\frac{x^2}{2}} \in L^2(\mathbb{R}).$$

In quantum mechanics, ξ_0 is called the *ground state*, and the states inductively defined by

$$\xi_k := (2k)^{-\frac{1}{2}} \cdot A^*\xi_{k-1}$$

the 'excited states'. Observe that due to

$$HA^* = A^*H + 2A^*,$$

from (6.2), we see that ξ_k is an eigenvector of H with eigenvalue $2k-1$:

$$\begin{aligned} H\xi_k = HA^*\xi_{k-1}(A^*H + 2A^*)\xi_{k-1} &= A^*H\psi_{k-1} + 2\xi_k \\ &= (2(k-1)+1) \cdot A^*\xi_{k-1} + 2\xi_k = (2k+1) \cdot \xi_k. \end{aligned} \tag{6.3}$$

It follows from $[H, A] = -2A$ that A shifts the eigenspace $\mathbb{C}\psi_k$ of H to the eigenspace $\mathbb{C}\psi_{k-1}$, annihilating ψ_0. it also multiplies by an increasing sequence of scalars; we leave it to the reader to compute them in the following

EXERCISE 6.1. By repeatedly applying the relation $HA^* = A^*H + 2A^*$, show that if $\psi_0, \psi_1, \psi_2, \ldots$ are the normalized eigenvectors of H discussed above, then

$$A\psi_k = \sqrt{2k} \cdot \psi_{k-1}, \quad A^*\psi_k = \sqrt{2k+2} \cdot \psi_{k+1}.$$

LEMMA 6.2. *The eigenvectors of H have the form*

$$\xi_k = h_k(x)e^{-\frac{x^2}{2}}$$

where h_k is a polynomial of degree k.

PROOF. By induction, using the recurrence

$$h_k(x) = (2k)^{-\frac{1}{2}} \cdot \big(2xh_{k-1}(x) - h'_{k-1}(x)\big).$$

□

The polynomials h_k are the classical *Hermite polynomials*. The reader can easily check for example that

$$h_1(x) = \frac{\sqrt{\pi}}{2} \cdot x.$$

It follows from the Stone-Weierstrass theorem that the span of the ξ_k is dense in $L^2(\mathbb{R})$. In particular, we have proved:

PROPOSITION 6.3. *The vectors $\{\xi_k\}$ form an orthonormal basis for $L^2(\mathbb{R})$. Each ξ_k is in the Schwartz class $\mathcal{S}(\mathbb{R})$, and is an eigenvector of H with eigenvalue $2k+1$. With respect to this basis, H is diagonal with eigenvalues the odd integers $1, 2, 3, \ldots$:*

$$H = \begin{bmatrix} 1 & 0 & \cdots & \cdots \\ 0 & 3 & 0 & \cdots \\ 0 & 0 & 5 & \cdots \\ 0 & \cdots & \cdots & \cdots \end{bmatrix}.$$

In particular, H has a canonical extension to a self-adjoint operator on $L^2(\mathbb{R})$, and $f(H)$ is a compact operator for all $f \in C_0(\mathbb{R})$, and a bounded operator for all $f \in C_b(\mathbb{R})$.

The following lemma connects the 'spectral geometric' picture of the real line suggested by the spectrum of the harmonic oscillator, and the ordinary geometry of $\mathbb{R}$.

If $f \in L^2(\mathbb{R})$, let $(\hat{f}(n))$ denote the sequence of its Fourier coefficients with respect to the eigenbasis for $L^2(\mathbb{R})$ for H discussed above.

LEMMA 6.4. *if $f \in L^2(\mathbb{R})$, then $f \in \mathcal{S}$ if and only if $(\hat{f}(n))$ is a rapidly decreasing sequence of integers:*

$$|\hat{f}(n)| = O(n^{-k})$$

for any k.

PROOF. If $f \in \mathcal{S}$ then Hf is in $\mathcal{S}$, as is clear from the definition of H. Similarly, $H^k f \in \mathcal{S}$ for all k. Since

$$\widehat{H^k f}(n) = (2n+1)^k \cdot \hat{f}(n),$$

and since this is an L^2-sequence, and hence bounded, we get, for each k a constant C such that

$$(2n+1)^k \cdot |\hat{f}(n)| \le C_k$$

and so

$$|\hat{f}(n)| = O(n^{-k})$$

follows.

Conversely, suppose that $f \in L^2(\mathbb{R})$ and that $(\hat{f}(n))$ is a rapidly decreasing sequence. Then f is, by definition, in the domains of A, A^*, H, and all positive powers of these operators. Since $A + A^* = 2x$ and $A - A^* = \frac{d}{dx}$, it follows that $x^k f$ and $\frac{d^k f}{dx^k}$ are in $L^2(\mathbb{R})$ for all k. Hence $f \in \mathcal{S}$.

□

The algebra of Dirac-Schrödinger operators

Consider the *-algebra $\mathcal{D}$ generated by the unbounded operator A. This *-algebra contains H of course. The spectrum of H grows linearly, its eigenvalues are the odd positive integers, with the nth eigenvalue $\lambda_n = 2n+1$ attached to the n-th eigenvector ψ_n discussed above. The operator A acts by a weighted shift

$$A\psi_n = \sqrt{2n} \cdot \psi_{n-1} \quad A^*\psi_n = \sqrt{2n+2} \cdot \psi_{n+1}.$$

Let s be a complex number.

We say that an operator $T \in \mathcal{D}$ *has order* $\leq s$ if

$$T \cdot H^{-s}$$

extends continuously to a bounded operator on $L^2(\mathbb{R})$.

We write

$$\mathrm{ord}(T) \leq s.$$

Thus, by definition, a bounded operator in $\mathcal{D}$ (of which there are none, except multiples of of the identity) has order ≤ 0, the operator H has order ≤ 1, while A has order $\leq \frac{1}{2}$, since $AH^{-\frac{1}{2}}$ is bounded, indeed, is essentially unitary (c.f. (6.15)).

The correct algebra to consider in the arguments that follows is given below.

Definition 6.5. Let $\mathcal{D}'$ be the *-algebra of linear operators $T\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ for which, for some $q \in \mathbb{R}$, the operators

$$T,\ T^{(1)} := [T, H],\ T^{(2)} := [H, [H, T]], \dots$$

all have order $\leq q$.

If T satisfies a) and b) we say T has *filtration order* $\leq a$.)

Example 6.6. The annihilation operator $A = x + \frac{d}{dx}$ is in $\mathcal{D}$. Indeed, $AH^{-\frac{1}{2}}$ is bounded, so A has order $\frac{1}{2}$. Furthermore,

$$[H, A] = -2A.$$

Hence the order of $[H, A]$ is $\frac{1}{2}$ as well, and repeating this gives that A has filtration order $\leq \frac{1}{2}$.

Similarly for A^*. Furthermore, $\mathcal{D}$ contains all complex powers H^s of H.

Example 6.7. Let f be any smooth, bounded function, for example a function in $\mathcal{S}$. Then multiplication by f defines an operator on $\mathcal{S}$, which is bounded, so an operator of order zero. However,

$$[f, H] = 2f'' + f' \cdot \frac{d}{dx}.$$

which is clearly not bounded unless f is constant. Hence f does not have filtration order ≤ 0.

It is clear, however, that any bounded smooth function all of whose derivatives are bounded, has filtration order $\leq \frac{1}{2}$, as is clear.

It is not particularly obvious that $\mathcal{D}'$ even is as *-algebra. This follows from the following lemmas.

Lemma 6.8. *Let $T \in \mathcal{D}'$, of filtration order $\leq q$. Then there is a 'Taylor series' expansion, for any positive integer k,*

$$[H^s, T] = \frac{1}{2\pi i}\binom{s}{1} T^{(1)} \cdot H^{s-1} + \frac{1}{2\pi i}\binom{s}{2} T^{(2)} \cdot H^{s-2} + \dots + \frac{1}{2\pi i}\binom{s}{k} T^{(k)} \cdot H^{s-k} + \frac{1}{2\pi i}\int_C \lambda^s \cdot (\lambda - H)^{-1} T^{(k)} (\lambda - H)^{-k} d\lambda. \tag{6.4}$$

Moreover, the remainder (integral) term has order

$$\leq q + \mathrm{Re}(s) - k - 1.$$

In particular, if $T \in \mathcal{D}'$ has filtration order $\leq q$ and s is a complex number with $\mathrm{Re}(s) < 0$, *then*

$$\mathrm{ord}([H^s, T]) \leq \mathrm{Re}(s) + q - 1.$$

For the proof, we will need a suitable version of Cauchy's theorem, for unbounded operators.

Let C be the contour in the complex plane given by the straight line, $\mathrm{Re}(\lambda) = \epsilon$, for any $0 < \epsilon < 1$, oriented straight downwards. It misses the spectrum of H (the odd integers). Cauchy's theorem gives that

$$H^s = \frac{1}{2\pi i} \int_C \lambda^s (\lambda - H)^{-1} d\lambda. \tag{6.5}$$

The integrand is a function valued in $\mathcal{K}(L^2\mathbb{R})$. Since

$$\|(\lambda - H)^{-1}\| = O(|\lambda|), \quad |\lambda| \to \infty,$$

we get that $\|\lambda^s(\lambda - H)^{-1}\| = O(|\lambda|^{-s-1})$ and hence the integral converges absolutely if $\mathrm{Re}(s) < 0$.

More generally, if $\binom{n}{p}$ is the usual binomial coefficient, $p = 0, 1, 2, \ldots$, then Cauchy's formula gives

$$\binom{s}{p} H^{s-p} = \frac{1}{2\pi i} \int_C \lambda^s (\lambda - H)^{-p-1} d\lambda. \tag{6.6}$$

Proof. Proof. By (6.5),

$$[H^s, T] = \frac{1}{2\pi i} \int_C \lambda^s \cdot [T, (\lambda - H)^{-1}] d\lambda. \tag{6.7}$$

Plugging

$$[T, (\lambda - H)^{-1}] = (\lambda - H)^{-1} \cdot [T, H] \cdot (\lambda - H)^{-1}.$$

into (6.7) gives

$$\begin{aligned} [H^s, T] &= \frac{1}{2\pi i} \int_C \lambda^s (\lambda - H)^{-1} \cdot [T, H] \cdot (\lambda - H)^{-1} d\lambda \\ &= \frac{1}{2\pi i} [T, H] \cdot \int_C \lambda^s (\lambda - H)^{-2} d\lambda + R = \binom{s}{1} [T, H] \cdot H^{s-1} + R, \end{aligned} \tag{6.8}$$

where the remainder term R is given by

$$\begin{aligned} R = \frac{1}{2\pi i} \int_C \lambda^s (\lambda - H)^{-1} \, [[T, H], H] \cdot (\lambda - H)^{-2} d\lambda. \\ = \frac{1}{2\pi i} \int_C \lambda^s (\lambda - H)^{-1} T^{(2)} \cdot (\lambda - H)^{-2} d. \end{aligned} \tag{6.9}$$

We have thus shown that

$$[H^s, T] = \frac{1}{2\pi i} \binom{-s}{1} T^{(1)} \cdot H^{s-1} + R. \tag{6.10}$$

Now similarly,

$$R = \frac{1}{2\pi i} \int_C \lambda^s (\lambda - H)^{-1} \, [[T, H], H] \cdot (\lambda - H)^{-2} d\lambda = \frac{1}{2\pi i} \binom{s}{2} T^{(2)} \cdot H^{s-2} + R'',$$

for a remainder term R'', and continuing in this fashion we obtain the result.

□

Applying the Taylor series gives

$$[H^s, T] \cdot H^{-s-q+1} = \frac{1}{2\pi i}\binom{s}{1} \cdot T^{(1)} H^{-q} + \text{lower order...}$$

and this is a bounded operator since $T^{(1)}$ has order q by assumption, so the first term $T^{(1)} \cdot H^{-q}$ has order ≤ 0, and the further terms of even smaller order so are in particular bounded.

This shows that

$$\operatorname{ord}([H^s, T]) \leq \operatorname{Re}(s) + q - 1.$$

□

Corollary 6.9. *If T, S in $\mathcal{D}'$ have filtration orders $\leq t$ and $\leq s$ respectively, then $ST \in \mathcal{D}'$ and has filtration order $\leq s + t$. Moreover, $T^* \in \mathcal{D}'$ of order $\leq t$. So $\mathcal{D}'$ is a filtered *-algebra.*

Proof. We first prove that if $T \in \mathcal{D}'$ of filtration order $\leq q$, then $H^{-q} \cdot T \in \mathcal{D}'$ is bounded, that is, is of order ≤ 0.

By Lemma 6.8, the commutator $[H^{-q}, T]$ has order ≤ -1. We have

$$H^{-q}T = TH^{-q} + [H^{-q}, T]$$

which is therefore a ≤ -1-order perturbation of a bounded operator, and hence is bounded.

Next, it is easy to check that if $T \in \mathcal{D}'$ of filtration order $\leq q$, then $TH^s \in \mathcal{D}'$ of filtration order $\leq q + s$ for any complex number s. In particular, we can apply the above remark to TH^s, giving that

$$H^{-s-q}TH^s$$

is bounded. Otherwise phrased, this shows that if $\alpha, \beta \in \mathbb{C}$ with $\alpha + \beta = q$, then we can write

$$T = H^{\alpha} \cdot V \cdot H^{\beta}$$

for some bounded operator V.

Now let T, S have orders $\leq t, s$ respectively. By the above remarks we can write $S = VH^s$ with V bounded, and $T = H^{-s}WH^{t+s}$, with W bounded. Then

$$ST \cdot H^{-s-t} = VW$$

by a trivial computation, which is a composition of bounded operators and hence is bounded.

To summarize: we have proved that *if T and S are in $\mathcal{D}'$ of filtration orders $\leq t, s$ then the order of TS is $\leq s + t$.* We can improve this replacing 'order' with 'filtration order' quite easily, as follows. Consider a commutator

$$[H, TS] = [H, T] \cdot S + T \cdot [H, S].$$

Clearly if $T \in \mathcal{D}'$ then $[H, T] \in \mathcal{D}'$ as well, by the definitions, of the same filtration order. Sine $S \in \mathcal{D}'$ by assumption, by what was just proved,

$$\operatorname{ord}([H, T] \cdot S) \leq t + s.$$

The same remarks hold for the second term, giving that

$$\operatorname{ord}([H, TS]) \leq t + s,$$

and now, iterating this argument shows that $TS \in \mathcal{D}'$, of filtration order $\leq t + s$.

□

Since $\mathcal{D}'$ has now been shown to be a *-algebra, and it contains the creation and annihilation operators A, A^*, we obtain

Corollary 6.10. *$\mathcal{D}'$ contains the algebra $\mathcal{D}$, as well as all complex powers H^s of H.*

Since $A, A^* \in \mathcal{D}'$ of filtration order $\leq \frac{1}{2}$, it follows that $2x = A + A^* \in \mathcal{D}'$ of the same filtration order $\frac{1}{2}$. Since we are in a filtered algebra, we get:

COROLLARY 6.11. *$\mathcal{D}'$ contains all polynomials, and if $f(x)$ is a polynomial of degree n, it has filtration degree $\leq \frac{n}{2}$.*

Simlarly, if $D = \sum_{k=0}^{n} a_i \frac{d^k}{dx^k}$ is a linear differential operator of order n then $D \in \mathcal{D}'$ of filtration order $\leq \frac{n}{2}$.

We are going to need the following result for our computation of Kasparov products.

LEMMA 6.12. *Let $f(x) = x(1+x^2)^{-\frac{1}{4}}$. Then the operator*

$$f \cdot [H^{-\frac{1}{2}}, f]$$

is compact.

PROOF. Note that $f^2 - x$ vanishes at infinity, and hence $H^{-\frac{1}{4}}(f^2 - x)H^{-\frac{1}{4}}$ is compact, and in particular is bounded. Since $H^{-\frac{1}{4}} x H^{-\frac{1}{4}}$ is bounded, we deduce that

$$H^{-\frac{1}{4}} f^2 H^{-\frac{1}{4}} \text{ is bounded.} \tag{6.11}$$

Now if A is an operator on the Schwartz space such that A^*A is bounded, it follows that A is bounded. Applying this to the operator $fH^{-\frac{1}{4}}$ and our observations above, gives that

$$fH^{-\frac{1}{4}} \text{ is bounded.}$$

So $f = VH^{\frac{1}{4}}$ for V bounded.

Next, we apply the asymptotic expansion of $[f, H^{-\frac{1}{2}}]$. We write

$$H^{-\frac{1}{2}} = \frac{1}{2\pi i} \int_C \lambda^{-\frac{1}{2}} (\lambda - H)^{-1} d\lambda.$$

The integrand is a compact-operator valued function and

$$\|\lambda^{-\frac{1}{2}} (\lambda - H)^{-1}\| = O(|\lambda|^{-\frac{3}{2}}),$$

so it is an absolutely convergent integral of compact operators.

As before, taking commutators with $H^{-\frac{1}{2}}$ gives

$$[f, H^{-\frac{1}{2}}] = \frac{1}{2\pi i} \int_C \lambda^{-\frac{1}{2}} (\lambda - H)^{-1} \cdot [f, H] \cdot (\lambda - H)^{-1} d\lambda.$$

Multiplying by f gives

$$f \cdot [f, H^{-\frac{1}{2}}] = \frac{1}{2\pi i} V \cdot \int_C \lambda^{-\frac{1}{2}} H^{\frac{1}{4}} (\lambda - H)^{-1} \cdot [f, H] \cdot (\lambda - H)^{-1} d\lambda.$$

Now $[f, H]$ has order $\leq \frac{1}{2}$. Write $[f, H] = WH^{\frac{1}{2}}$, W bounded. We get

$$f \cdot [f, H^{-\frac{1}{2}}] = \frac{1}{2\pi i} V \cdot \int_C \lambda^{-\frac{1}{2}} H^{\frac{1}{4}} (\lambda - H)^{-1} \cdot W \cdot H^{\frac{1}{2}} (\lambda - H)^{-1} d\lambda.$$

Now for all λ, $\lambda^{-\frac{1}{2}} H^{\frac{1}{4}} (\lambda - H)^{-1} \cdot W \cdot H^{\frac{1}{2}} (\lambda - H)^{-1}$ is a compact operator and

$$\|\lambda^{-\frac{1}{2}} H^{\frac{1}{4}} (\lambda - H)^{-1} \cdot W \cdot H^{\frac{1}{2}} (\lambda - H)^{-1}\| = O(|\lambda|^{-\frac{5}{2}})$$

so we are done, the integral converges absolutely to a compact operator.

$\square$

We now return to KK-theory.

From the definition of $D = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$, we have

$$D^2 = \begin{bmatrix} A^*A & 0 \\ 0 & AA^* \end{bmatrix}.$$

Since $A^*A = H - 1$ and $AA^* = H + 1$, we get

$$1 + D^2 = \begin{bmatrix} H & 0 \\ & \\ 0 & H+2 \end{bmatrix}. \tag{6.12}$$

Hence if $f \in C_0(\mathbb{R})$,

$$f(1+D^2) = \begin{bmatrix} f(H) & 0 \\ & \\ 0 & f(H+2) \end{bmatrix}.$$

In particular, taking $f(x) = (1+x^2)^{-1}$ gives that $(1+D^2)^{-1}$ is compact, and hence that

$$F := \chi(D) = D(1+D^2)^{-\frac{1}{2}} = \begin{bmatrix} 0 & A^*(H+2)^{-\frac{1}{2}} \\ & \\ AH^{-\frac{1}{2}} & 0 \end{bmatrix}. \tag{6.13}$$

satisfies $F^2 = 1$ mod compact operators, where χ is any normalizing function.

Putting everything together, we obtain a Fredholm $\mathbb{C}$-$\mathbb{C}$-bimodule

$$(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), F := \chi(D)),$$

for $\mathrm{KK}_0(\mathbb{C}, \mathbb{C})$, where

$$D := \begin{bmatrix} 0 & -\frac{d}{dx} + x \\ x + \frac{d}{dx} & 0 \end{bmatrix} = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix},$$

and χ is any normalizing function.

THEOREM 6.13. *Let x and y be the Bott and Dirac morphisms. Then their intersection product $x \hat{\otimes}_{C_0(\mathbb{R})} y \in \mathrm{KK}_0(\mathbb{C}, \mathbb{C})$ is represented by the cycle*

$$(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), F := \chi(D)),$$

where the Hilbert space is graded with first summand even, second odd, χ is any normalizing function, D is the Schröndinger operator (6.1)*, and $\chi(D)$ is obtained from D by functional calculus.*

PROOF. The tensor product of Hilbert modules is

$$\mathcal{E} = \mathcal{E}_1 \otimes_{C_0(\mathbb{R})} \mathcal{E}_2 = C_0(\mathbb{R}) \otimes_{C_0(\mathbb{R})} L^2(\mathbb{R}) \cong L^2(\mathbb{R}).$$

So the intersection product will be represented by a cycle of the form

$$(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), F)$$

where F is a suitable odd, self-adjoint operator. We need therefore to check that our proposal $\chi(D)$ for F, with D defined above, satisfies the axioms for the Kasparov product.

The Fredholm condition has of course already been verified.

We discuss the connection condition b).

We can write

$$F = \begin{bmatrix} 0 & w^* \\ w & 0 \end{bmatrix}$$

where

$$w = AH^{-\frac{1}{2}}.$$

An inspection of the connection condition shows that the operators T_ξ correspond to multiplication operators on $L^2(\mathbb{R})$ under the identification $C_0(\mathbb{R}) \otimes_{C_0(\mathbb{R})} L^2(\mathbb{R})$, and that the connection condition boils down to showing that

$$iF_2\rho - \rho w \tag{6.14}$$

is a compact operator, for any $\rho \in C_c^\infty(\mathbb{R})$, where $F_2 = \chi(-i\frac{d}{dx})$ as before. Since ρ commutes mod $\mathcal{K}(L^2\mathbb{R})$ with F_2 we are reduced therefore to showing that

$$\rho \cdot (iF_2 - w) \tag{6.15}$$

is a compact operator.

Let

$$D_2' := \begin{bmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{bmatrix}.$$

Then D_2' is self-adjoint and

$$(D_2')^2 = \begin{bmatrix} -\frac{d^2}{dx^2} & 0 \\ 0 & -\frac{d^2}{dx^2} \end{bmatrix}.$$

Therefore

$$\chi(D_2') = \begin{bmatrix} 0 & -\frac{d}{dx}\,(1-\frac{d^2}{dx^2})^{-\frac{1}{2}} \\ \frac{d}{dx}\,(1-\frac{d^2}{dx^2})^{-\frac{1}{2}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -iF_2 \\ iF_2 & 0 \end{bmatrix}.$$

Now notice that

$$\rho \cdot (D_2' - D) = \begin{bmatrix} 0 & -x\rho \\ x\rho & 0 \end{bmatrix}$$

is bounded. Furthermore, $\rho(1+(D_2')^2)^{-1}$ and $\rho \cdot (1+D^2)^{-1}$ are both compact. By Exercise 4.13,

$$\rho \cdot \big(\chi(D_2') - \chi(D)\big) \in \mathcal{K}(L^2\mathbb{R}). \tag{6.16}$$

By our computations above, this says explicitly that

$$\begin{bmatrix} 0 & \rho \cdot (-iF_2 - w^*) \\ \rho \cdot (iF_2 - w) & 0 \end{bmatrix} \tag{6.17}$$

is a matrix of compact operators, as required.

We now address the positivity, or ‘alignment’ condition. As in the previous argument, this reduces to showing that

$$F_1 w + w^* F_1 \geq 0$$

in the Calkin algebra $\mathbb{B}(L^2\mathbb{R})/\mathcal{K}(L^2\mathbb{R})$, where F_1 is multiplication by $\chi(x) = x(1+x^2)^{-\frac{1}{2}}$. This function has vanishing derivative at ∞. Hence the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_1 \end{bmatrix} \tag{6.18}$$

commutes modulo bounded operators with the matrix of operators D. By Lemma 4.4, $\begin{bmatrix} F_1 & 0 \\ 0 & F_1 \end{bmatrix}$ commutes mod compact operators with $\chi(D) = F$. It follows immediately that F_1 commutes mod compact operators with w and w^*. So the positivity condition reduces to showing that

$$F_1 \cdot (w + w^*) \geq 0 \bmod \mathcal{K}(L^2\mathbb{R}).$$

We have

$$w + w^* = AH^{-\frac{1}{2}} + H^{-\frac{1}{2}}A^*.$$

Now it is easy to check by directly inspecting the actions of these operators on the orthonormal basis diagonalizing H, that

$$AH^{-\frac{1}{2}} = (H+2)^{-\frac{1}{2}}A.$$

Taking adjoints gives

$$H^{-\frac{1}{2}}A^* = A^*(H+2)^{-\frac{1}{2}}. \tag{6.19}$$

Hence

$$w + w^* = AH^{-\frac{1}{2}} + A^*(H+2)^{-\frac{1}{2}}. \tag{6.20}$$

By Lemma **??** $A^*(H+2)^{-\frac{1}{2}}. = A^*H^{-\frac{1}{2}}$ mod compact operators. Therefore

$$w + w^* = AH^{-\frac{1}{2}} + A^*H^{-\frac{1}{2}} = (A + A^*) \cdot H^{-\frac{1}{2}} \tag{6.21}$$

mod compact operators. The operator $A + A^*$ is multiplication by $2x$. The operator F_1 is multiplication by $\chi(x)$, with χ a normalizing function. Putting everything together gives that

$$F_1 \cdot (w + w^*) = 2x^2(1+x^2)^{-\frac{1}{2}}H^{-\frac{1}{2}} = 2f^2H^{-\frac{1}{2}} \quad \text{mod compact operators.} \tag{6.22}$$

with $f(x) = x(1+x^2)^{-\frac{1}{4}}$. And

$$f^2H^{-\frac{1}{2}} = fH^{-\frac{1}{2}}f + f \cdot [H^{-\frac{1}{2}}, f]$$

by algebra, and the first term S is clearly positive in the sense that

$$\langle Su, u \rangle \geq 0$$

for all $u \in \mathcal{S}$, and also bounded, whence is a bounded, positive operator, while the second term is compact by Lemma 6.12.

COROLLARY 6.14. *The class in* $\mathrm{KK}_0(\mathbb{C}, \mathbb{C})$ *of the cycle* $(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), F := \chi(D))$, *described above, is the same as the class*

$$1_{\mathbb{C}} \in \mathrm{KK}_0(\mathbb{C}, \mathbb{C}),$$

with $1_{\mathbb{C}}$ *the unit.*

PROOF. The operator $A = x + \frac{d}{dx}$ acts by a weighted left-shift in the basis described above, and it follows that

$$\ker(A) = \mathbb{C}\psi_0,$$

while

$$\ker(A^*) = \{0\}.$$

In particular:

$$\mathrm{index}(D) = \ker(A) - \ker(A^*) = 1.$$

□

□

An alternative and more geometric proof of the Corollary is given at the end of the chapter.

As with Atiyah's arguments in topological K-theory, a simple rotation trick proves that x and y are two-sided inverses of each other in KK:

LEMMA 6.15. *With* x, y *the Bott and Dirac morphisms,*

$$y \hat{\otimes}_{\mathbb{C}} x = 1_{C_0(\mathbb{R})} \in \mathrm{KK}_0(C_0(\mathbb{R}), C_0(\mathbb{R})).$$

PROOF. The external product $\hat{\otimes}_{\mathbb{C}}$ over $\mathbb{C}$ is graded commutative. Hence

$$y \hat{\otimes}_{\mathbb{C}} x = -x \hat{\otimes}_{\mathbb{C}} y.$$

By definition,

$$x \hat{\otimes}_{\mathbb{C}} y = (x \hat{\otimes} 1_{C_0(\mathbb{R})}) \otimes_{C_0(\mathbb{R}^2)} (1_{C_0(\mathbb{R})} \hat{\otimes} y) \tag{6.23}$$

We have already noted that the *-homomorphism

$$S \colon C_0(\mathbb{R}) \to C_0(\mathbb{R}), \quad S(f)(x) = f(-x),$$

satisfies

$$[S] = -1_{C_0(\mathbb{R})} \in \mathrm{KK}_0(C_0(\mathbb{R}), C_0(\mathbb{R})).$$

This implies that the class of the flip homomorphism

$$\sigma\colon C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2), \quad \sigma(f)(x,y) = f(y,x)$$

satisfies

$$[\sigma] = -1_{C_0(\mathbb{R}^2)} \in \mathrm{KK}_0(C_0(\mathbb{R}^2), C_0(\mathbb{R}^2))$$

because σ is homotopic to $S \otimes 1$ (by a homotopy between

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$y\hat{\otimes}_{\mathbb{C}}x = -\sigma_*(x\hat{\otimes}1_{C_0(\mathbb{R})}) \otimes_{C_0(\mathbb{R}^2)} (1_{C_0(\mathbb{R})}\hat{\otimes}y) \tag{6.24}$$

and

$$-\sigma_*(x\hat{\otimes}1_{C_0(\mathbb{R})}) = 1_{C_0(\mathbb{R})}\hat{\otimes}x.$$

Therefore (6.24) equals

$$(-1_{C_0(\mathbb{R})}\hat{\otimes}x)\hat{\otimes}_{C_0(\mathbb{R}^2)}(1_{C_0(\mathbb{R})}\hat{\otimes}y) = -1_{C_0(\mathbb{R})}\hat{\otimes}(x\hat{\otimes}_{C_0(\mathbb{R})}y) = -1_{C_0(\mathbb{R})}$$

by what has already been proved, and the general mechanics of KK. This proves the result. □

We have established the following.

THEOREM 6.16. *The Bott and Dirac morphisms x and y are KK_1-equivalences. In particular, $A \otimes C_0(\mathbb{R})$ is KK_1-equivalent to A, for any separable C^*-algebra A.*

PROOF. We have already shown that

$$y \otimes_{\mathbb{C}} x = 1_{C_0(\mathbb{R})}, \quad x \otimes_{C_0(\mathbb{R})} y = 1_{\mathbb{C}}.$$

It follows that $x\hat{\otimes}1_A \in \mathrm{KK}_1(A, C_0(\mathbb{R}) \otimes A)$ is a KK_1-equivalence for any A, with inverse $y\hat{\otimes}1_A$. □

7. More on the spectra of Dirac-Schrödinger operators and geometry

An interesting thing to do is to take the annihilation or creation operators $A = x + \frac{d}{dx}$, $A^* = x - \frac{d}{dx}$ and scale them in one variable, thus considering the family of operators

$$A_\lambda = \lambda x + \frac{d}{dx},$$

with $\lambda \geq 0$ (and similarly A_λ^*.) This idea will lead to a very geometric proof that the Dirac-Schrödinger cycle in $\mathrm{KK}_0(\mathbb{C}, \mathbb{C})$, is equivalent to $1_{\mathbb{C}}$, and generally illuminates the nature of these interesting operators.

Let

$$D_\lambda = \begin{bmatrix} 0 & \lambda x - \frac{d}{dx} \\ \lambda x + \frac{d}{dx} & 0 \end{bmatrix}.$$

Let

$$H_\lambda = \lambda^2 \cdot x^2 - \frac{d^2}{dx^2}.$$

Proceeding as in the case of the unscaled operators, one verifies the relations

$$A_\lambda A_\lambda^* = H_\lambda + \lambda, \quad A_\lambda^* A_\lambda = H_\lambda - \lambda, \qquad [A_\lambda, A_\lambda^*] = 2\lambda, \quad [H_\lambda, A_\lambda] = -2\lambda A_\lambda, \qquad [H_\lambda, A_\lambda^*] = 2\lambda A_\lambda^*. \tag{7.1}$$

By the same computations we we did before, we have:

Lemma 7.1. *Let $\lambda > 0$, and*

$$\psi_{0,\lambda}(x) = \left(\frac{\lambda}{\sqrt{\pi}}\right)^{\frac{1}{2}} \cdot e^{-\frac{\lambda x^2}{2}}.$$

Then $\psi_{0,\lambda} \in \ker(A_\lambda)$, is a unit vector in $L^2(\mathbb{R})$. If

$$\psi_{k,\lambda} := \frac{1}{\sqrt{2k\lambda}} \cdot A_\lambda^*(\psi_{k-1}),$$

then $\psi_{k,\lambda}$ is an eigenvalue of H_λ with eigenvalue $(2k+1)\cdot\lambda$, and $\psi_{0,\lambda}, \psi_{1,\lambda}, \dots$ form a complete orthonormal basis of $L^2(\mathbb{R})$.

For each λ we obtain therefore an orthonormal basis of $L^2(\mathbb{R})$, but these bases are changing with λ. Furthermore, as $\lambda \to \infty$, one should notice that the spectrum of H_λ scales linearly with λ, so that the first nonzero eigenvalue of, for instance, $A_\lambda^* A_\lambda$ (namely, 2λ), is increasingly large as λ increases.

The kernel of $A_\lambda^* A_\lambda$, meanwhile, is spanned by $e^{-\frac{\lambda x^2}{2}}$. The kernel of $A_\lambda A_\lambda^*$ is trivial and its spectrum is that of A^*A, minus zero. A_λ thus has Fredholm index $+1$, as we have already verified, and so does the self-adjoint unbounded operator

$$D_\lambda = \begin{bmatrix} 0 & \lambda x - \frac{d}{dx} \\ \lambda x + \frac{d}{dx} & 0 \end{bmatrix}.,$$

whose spectrum is $2\lambda \cdot \mathbb{Z}$.

For each λ, let pr_λ be projection to the kernel of D_λ, that is, the span of $e^{-\frac{\lambda x^2}{2}}$. It is a subspace of the 'first' summand $L^2(\mathbb{R})$ of the Hilbert space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Note that p_λ is the integral operator on $L^2(\mathbb{R})$ with kernel

$$k_\lambda(x,y) := \left(\frac{\lambda}{\sqrt{\pi}}\right) \cdot e^{-\lambda\left(\frac{x^2+y^2}{2}\right)}, \tag{7.2}$$

since, writing $\psi_{0,\lambda}(x) = c_\lambda e^{-\frac{\lambda x^2}{2}}$, $c_\lambda := \left(\frac{\lambda}{\sqrt{\pi}}\right)^{\frac{1}{2}}$, for brevity, we have by the definitions

$$\begin{aligned} \mathrm{pr}_\lambda(\xi) &= \langle \psi_{0,\lambda}, \xi\rangle \psi_{0,\lambda} = c_\lambda^2 \langle e^{-\frac{\lambda x^2}{2}}, \xi\rangle e^{-\frac{\lambda x^2}{2}} \\ &= c_\lambda^2 \left(\int \xi(y) e^{-\frac{\lambda y^2}{2}} dy\right) e^{-\frac{\lambda x^2}{2}} = c_\lambda^2 \int \xi(y) e^{-\lambda\left(\frac{x^2+y^2}{2}\right)} dxdy = \int \xi(y) k_\lambda(x,y) dxdy \end{aligned} \tag{7.3}$$

as claimed.

We are going to construct a right Hilbert $C([1,+\infty])$-module by specifying a continuous family of Hilbert spaces, parameterized by $[1,+\infty]$.

Definition 7.2. Define a family $\{H_\lambda^\pm\}_{\lambda\in[1,+\infty]}$ of Hilbert spaces by setting $H_\lambda^- := L^2(\mathbb{R})$ for all $\lambda \in [1,+\infty]$, and

$$H_\lambda^+ = \begin{cases} L^2(\mathbb{R}) & \text{if } 1 \le \lambda < \infty \\ L^2(\mathbb{R}) \oplus \mathbb{C} & \text{if } \lambda = +\infty \end{cases} \tag{7.4}$$

I will denote by δ_0 the unit vector corresponding to $(0,1) \in H^+_\infty = L^2(\mathbb{R}) \oplus \mathbb{C}$, and will think of it intuitively, as a Dirac-delta distribution at 0, which we have added to $L^2(\mathbb{R})$, as a unit vector, orthogonal to $L^2(\mathbb{R})$.

We endow this field with a structure of continuous field as follows. We only need be concerned about the point $+\infty$.

A section ξ of the field $\{H^+_\lambda\}_{\lambda \in [1,+\infty]}$ with value $\eta + z\delta_0$ at $\lambda = +\infty$ is *continuous at* $+\infty$ if

$$\|\xi(\lambda) - (\eta + z\psi_{0,\lambda})\|_{L^2(\mathbb{R})} \to 0 \quad \text{as } \lambda \to +\infty,$$

where $\psi_{0,\lambda} \in L^2(\mathbb{R})$ is the normalized 0-eigenvector of D_λ.

Let $\mathcal{E}^+$ denote the right Hilbert $C([1,+\infty])$-module of sections of this continuous field, and let $\mathcal{E}^-$ sections of the constant field $\{H^-_\lambda := L^2(\mathbb{R})\}_{\lambda \in [1,+\infty]}$.

Elements of $\mathcal{E}^+$ might be described like this. They are, firstly, continuous families ξ_λ of vectors in $L^2(\mathbb{R})$ to points $\lambda \in [1,+\infty)$. The family

$$\xi_\lambda\,(x) := \psi_{0,\lambda}(x) = \left(\frac{\lambda}{\sqrt{\pi}}\right)^{\frac{1}{2}} \cdot e^{-\frac{\lambda x^2}{2}}.$$

is an example, and it is deemed to be continuous at $+\infty$. We also admit families which are asymptotic to this one, and in addition, we allow 'ordinary' families which converge as $\lambda \to +\infty$ to a vector in $L^2(\mathbb{R})$. Finally, we allow linear combinations of these two types of families.

We now describe a continuous family of self-adjoint, grading-reversing operators

$$F_\lambda \colon H_\lambda \to H_\lambda,$$

for $\lambda \in [1,+\infty]$. For finite λ,

$$F_\lambda := \begin{bmatrix} 0 & A_\lambda^*(1 + A_\lambda A_\lambda^*)^{-\frac{1}{2}} \\ A_\lambda(1 + A_\lambda^* A_\lambda)^{-\frac{1}{2}} & 0 \end{bmatrix} = \chi(D_\lambda), \text{ where } D_\lambda := \begin{bmatrix} 0 & \lambda x - \frac{d}{dx} \\ \lambda x + \frac{d}{dx} & 0 \end{bmatrix}.$$

We can use the shorter notation

$$F_\lambda = \begin{bmatrix} 0 & G_\lambda^* \\ G_\lambda & 0 \end{bmatrix}.$$

Set then

$$G_\infty \colon L^2(\mathbb{R}) \oplus \mathbb{C} \to L^2(\mathbb{R})$$

be multiplication by the Borel function $\frac{x}{|x|}$ on the summand $L^2(\mathbb{R})$, and zero on the $\mathbb{C}$-summand. As $H_\infty := L^2(\mathbb{R}) \oplus \mathbb{C} \ \oplus L^2(\mathbb{R})$ with the first summand $L^2(\mathbb{R}) \oplus \mathbb{C}$ graded even, the second summand $L^2(\mathbb{R})$ graded odd. The operator F_∞ will then be the odd, self-adjoint operator on H_∞ given by the matrix

$$F_\infty := \begin{bmatrix} 0 & G_\infty^* \\ G_\infty & 0 \end{bmatrix}.$$

To be clear, the operator $G_\infty^* \colon L^2(\mathbb{R}) \to L^2(\mathbb{R}) \oplus \mathbb{C}$ is multiplication by the (Borel) function $\frac{x}{|x|}$ on $L^2(\mathbb{R})$, followed by the inclusion into $L^2(\mathbb{R}) \oplus \mathbb{C}$ by zero in the second summand.

Notice what happens regarding the Fredholm index. When λ is finite, F_λ has the 1-dimensional kernel spanned by $\psi_{0,\lambda}$. The self-adjoint operator F_λ has spectrum

$$0, \pm 2\lambda, \pm 3\lambda, \ldots$$

and so has a spectral gap between 0 and its first nonzero eigenvalue(s), of size 4λ, which is growing as $\lambda \to +\infty$. The index of F_λ is $+1$. This is by the definitions, the difference of dimensions of the kernel of G_λ and its cokernel (the kernel of G_λ^*).

The operator F_∞ is built from $G_\infty \colon L^2(\mathbb{R}) \oplus \mathbb{C} \to L^2(\mathbb{R})$ as explained above. Now the operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ of multiplication by $\frac{x}{|x|}$ has no kernel. Since, however, G_∞ kills the second summand $\mathbb{C}$ of $L^2(\mathbb{R}) \oplus \mathbb{C}$, the operator G_∞ has a 1-dimensional kernel. The cokernel of G_∞ is clearly trivial. So G_∞ also has index $+1$.

The family of operators $\{F_\lambda\}_{\lambda \in [1,+\infty]}$ induces a self-adjoint grading-reversing operator F on sections of the field $\{H_\lambda\}_{\lambda \in [1,+\infty]}$; that is, we have constructed a $\mathbb{Z}/2$-graded Hilbert $C([1,+\infty])$-module $\mathcal{E}$ and an odd, self-adjoint operator F on $\mathcal{E}$ such that $F^2 - 1$ is compact.

We are going to discuss the spectral geometry involved in all of this, but first state the result we aim to prove.

Theorem 7.3. *The triple $(\mathcal{E}, 1, F)$ defines a cycle for* $\mathrm{KK}_0(\mathbb{C}, C([1,+\infty]))$, *whose value at infinity is the sum of the triple $(\mathbb{C} \oplus 0, 1, 0)$, where $\mathbb{C} \oplus 0$ is $\mathbb{Z}/2$-graded with first summand even and second odd, and the degenerate cycle*

$$\left(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), 1, \begin{bmatrix} 0 & \frac{x}{|x|} \\ \frac{x}{|x|} & 0 \end{bmatrix} \right),$$

where $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is $\mathbb{Z}/2$-graded with first summand even, second odd.

In particular, if $1 \leq \lambda < \infty$, then the cycle $\big(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), 1, F_\lambda\big)$ represents the class $1_{\mathbb{C}} \in \mathrm{KK}_0(\mathbb{C}, \mathbb{C})$.

Consider the family $\{D_\lambda\}_{[1,+\infty]}$,

$$D_\lambda := \begin{bmatrix} 0 & \lambda x - \frac{d}{dx} \\ \lambda x + \frac{d}{dx} & 0 \end{bmatrix}.$$

The operator D_λ has is orthogonally diagonalizable with spectrum $2\lambda\mathbb{Z}$, as we have discussed above. It has a one-dimensional kernel spanned by the unit 0-eigenvector $\psi_{0,\lambda}$.

As $\lambda \to \infty$, the 0 remains in the spectrum, but the rest of the eigenvalues fly out to $\pm\infty$. To remove the distracting zero, let

$$D'_\lambda = D_\lambda|_{\ker(D_\lambda)^\perp} \colon \ker(D_\lambda)^\perp \to \ker(D_\lambda)^\perp.$$

The spectrum of D'_λ is then $2n\lambda\mathbb{Z} \setminus \{0\}$.

Now let f be a C_0-function on $\mathbb{R}$. Then $f(D'_\lambda)$ is a bounded operator for all λ (even a compact operator). By the Spectral Theorem

$$\|f(D'_\lambda)\| = \sup_{x \in \mathrm{Spec}(D'_\lambda)} |f(x)|.$$

But since the spectrum of D'_λ leaves any compact subset of $\mathbb{R}$ for λ large enough, it is clear that

$$\lim_{\lambda \to \infty} \|f(D'_\lambda)\| = 0.$$

Now returning to D_λ, as an operator on $L^2(\mathbb{R})$, it is a direct sum

$$D_\lambda = 0 \cdot \mathrm{pr}_\lambda \oplus D'_\lambda$$

and if f is a C_0-function on $\mathbb{R}$, or even a bounded Borel function on $\mathbb{R}$, then

$$f(D_\lambda) = f(0) \cdot \mathrm{pr}_\lambda \oplus f(D'_\lambda). \tag{7.5}$$

and the second term converges to 0 in operator norm if f is C_0. We have proved the following.

Lemma 7.4. *If $f \in C_0(\mathbb{R})$ then*

$$\lim_{\lambda \to \infty} \|f(D_\lambda) - f(0) \cdot \mathrm{pr}_\lambda\| = 0,$$

the norm being the operator norm in $\mathbb{B}\big(L^2(\mathbb{R})\big)$.

More generally, if f_1 and f_2 are two bounded Borel functions on $\mathbb{R}$ such that $\lim_{x\to\pm\infty}|f_1(x)-f_2(x)|=0$, *then*

$$\lim_{\lambda\to\infty}\|f_1(D_\lambda)-f_2(D_\lambda)\|=0.$$

We next show the following.

LEMMA 7.5. *Let f be a bounded Borel function on $\mathbb{R}$, and assume that f extends to a Borel function on $[-\infty,+\infty]$ which is continuous at the points $\pm\infty$. Let $\hat{f}$ be the matrix-valued Borel function*

$$\hat{f}:=\begin{bmatrix}\frac{f(-\infty)+f(+\infty)}{2} & \left(\frac{f(+\infty)-f(-\infty)}{2}\right)\cdot\frac{x}{|x|}\\ \left(\frac{f(+\infty)-f(-\infty)}{2}\right)\cdot\frac{x}{|x|} & \frac{f(-\infty)+f(+\infty)}{2}\end{bmatrix}.$$

Then

$$\lim_{\lambda\to\infty}\|f(D_\lambda)-f(0)\cdot\mathrm{pr}_\lambda-\hat{f}\|=0,$$

the limit in the operator norm.

Note that when $f=\chi$ is a normalizing function, then $\hat{\chi}=\frac{x}{|x|}$, so we obtain the operator

$$\left(L^2(\mathbb{R})\oplus L^2(\mathbb{R}),1,\begin{bmatrix}0 & \frac{x}{|x|}\\ \frac{x}{|x|} & 0\end{bmatrix}\right),$$

as the 'limiting value' of the operators $F_\lambda:=\chi(D_\lambda)$.

PROOF. Let

$$Q_{\lambda,+}:=\chi_{(0,\infty)}(D_\lambda),$$

the positive spectral projection of D_λ, using the Borel function $\chi_{(0,\infty)}$ (which is, of course, continuous on $\mathrm{Spec}(D_\lambda)$ but not on $\mathbb{R}$.) Similarly, let

$$Q_{\lambda,-}:=\chi_{(-\infty,0)}(D_\lambda).$$

Then with pr_λ projection to $\ker(D_\lambda)$,

$$Q_{\lambda,+}+\mathrm{pr}_\lambda+Q_{\lambda,-}=1$$

as (orthogonal) projections, for all $\lambda>0$.

Now the operators $\frac{1}{\lambda}D_\lambda$ are given by

$$\frac{1}{\lambda}D_\lambda=\begin{bmatrix}0 & x-\frac{1}{\lambda}\frac{d}{dx}\\ x+\frac{1}{\lambda}\frac{d}{dx} & 0\end{bmatrix}.$$

which, pointwise acting on Schwartz functions, converges as $\lambda\to\infty$ to the operator

$$X:=\begin{bmatrix}0 & x\\ x & 0\end{bmatrix}.$$

It follows that

$$\chi_{(0,+\infty)}(\frac{1}{\lambda}D_\lambda)\to\chi_{(0,+\infty)}(X),$$

for example by an exercise in the continuity of the functional calculus left to the reader. The convergences discussed here, are, we emphasize, in the strong operator topology on Schwartz functions.

On the other hand, $\frac{1}{\lambda}D_\lambda$ and D_λ have exactly the same eigenspaces; the operators differ only by a scaling by a factor of the positive scalar λ. In particular, the sum of the positive eigenspaces is the same, for all λ, and we deduce that

$$\chi_{(-\infty,0)}(\frac{1}{\lambda}D_\lambda)=Q_{\lambda,-},\qquad\chi_{(0,+\infty)}(\frac{1}{\lambda}D_\lambda)=Q_{\lambda,+}.$$

We deduce therefore that

$$Q_{\lambda,-} \to \chi_{(-\infty,0)}(X) =: Q_-, \qquad Q_{\lambda,+} \to \chi_{(0,+\infty)}(X) =: Q_+, \quad \lambda \to \infty.$$

Now let f be a bounded Borel function on $\mathbb{R}$ with limits $f(\pm\infty)$ at $\pm\infty$. Let δ_0 be the Dirac delta function at 0. The Borel function

$$\tilde{f} := f - \big(f(-\infty) \cdot \chi_{(-\infty,0)} + f(0) \cdot \delta_0 + f(+\infty) \cdot \chi_{(0,\infty)}$$

vanishes at $\pm\infty$ and at 0. By Lemma 7.4,

$$f(D_\lambda) \sim f(-\infty) \cdot \chi_{(-\infty,0)}(D_\lambda) + f(0) \cdot \mathrm{pr}_\lambda + f(+\infty) \cdot \chi_{(0,\infty)}(D_\lambda)$$

where $\sim$ means that the difference converges to zero in the strong operator topology.

In the notation above, $\chi_{(-\infty,0)}(D_\lambda) = Q_{\lambda,-}$, and so on, so we may reformulate the conclusion as

$$f(D_\lambda) \sim f(-\infty) \cdot Q_{\lambda,-} + f(0) \cdot \mathrm{pr}_\lambda + f(+\infty) \cdot Q_{\lambda,+}. \tag{7.6}$$

And by the preceding discussion, $Q_{\lambda,\pm} \sim Q_\pm$, so we obtain

$$f(D_\lambda) \sim f(-\infty) \cdot Q_- + f(0) \cdot \mathrm{pr}_\lambda + f(+\infty) \cdot Q_+. \tag{7.7}$$

so that

$$f(D_\lambda) - f(0) \cdot \mathrm{pr}_\lambda \sim f(-\infty) \cdot Q_- + f(+\infty) \cdot Q_+. \tag{7.8}$$

By definition $Q_+ = \chi_{(0,\infty)}(X)$, *etc*, so we need to compute $h(X)$ for various Borel functions h with

$$X = \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}.$$

Proceeding by diagonalizing a matrix $\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ gives the conjugation

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -x & 0 \\ 0 & x \end{bmatrix}.$$

It follows that if h is a Borel function and $\hat{h}(x) = h(-x)$ then, since, obviously, $h(x)$ is the multiplication operator by h,

$$h(X) = \frac{1}{2}\begin{bmatrix} \frac{h+\hat{h}}{2} & \frac{h-\hat{h}}{2} \\ \frac{h-\hat{h}}{2} & \frac{h+\hat{h}}{2} \end{bmatrix}$$

Applying this discussion to *e.g.* $h = \chi_{(0,\infty)}$, where $\hat{h} = \chi_{(-\infty,0)}$, we get

$$Q_+ := \chi_{(0,\infty)}(X) = \frac{1}{2}\begin{bmatrix} 1 & S \\ S & 1 \end{bmatrix}$$

where S is the sign function

$$S(x) = \frac{x}{|x|}.$$

Applying it to Q_- gives

$$Q_- := \chi_{(-\infty,0)}(X) = \frac{1}{2}\begin{bmatrix} 1 & -S \\ -S & 1 \end{bmatrix}$$

Therefore for any $a, b \in \mathbb{C}$

$$(a\chi_{(-\infty,0)} + b\chi_{(0,\infty)})(X) = \begin{bmatrix} \frac{a+b}{2} & \frac{b-a}{2} \\ \frac{b-a}{2} & \frac{a+b}{2} \end{bmatrix}.$$

Setting $a = f(-\infty)$ and $b = f(+\infty)$ and using (7.9) gives the required

$$f(D_\lambda) - f(0)\cdot \mathrm{pr}_\lambda \sim \begin{bmatrix} \frac{f(-\infty)+f(+\infty)}{2} & \left(\frac{f(+\infty)-f(-\infty)}{2}\right) S \\ \left(\frac{f(+\infty)-f(-\infty)}{2}\right) S & \frac{f(-\infty)+f(+\infty)}{2} \end{bmatrix}. \tag{7.9}$$

□

We end this chapter on some applications of the techniques we have been discussing, to the construction of interesting K-homology classes over certain compact spaces defined by Gelfand duality (see Exercise 2.27.)

Lemma 7.6. *Let*

$$D_\lambda := \begin{bmatrix} 0 & \lambda x - \frac{d}{dx} \\ \lambda x + \frac{d}{dx} & 0 \end{bmatrix}.$$

and $\{F_\lambda\}_{\lambda\in[1,+\infty]}$ *the family of operators obtained from the* D_λ *by applying the normalizing function* $\chi(x) = x(1+x^2)^{-\frac{1}{2}}$.

Then if $\phi \in C_b^\infty(\mathbb{R})$ *is a smooth function with* ϕ' *bounded, acting as a multiplication operator on* $L^2(\mathbb{R})$ *(and on* $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$*), then*

$$\lim_{\lambda\to+\infty} \|[F_\lambda, \phi]\| = 0.$$

We will leave the proof as a guided exercise.

Exercise 7.7. The operators D_λ of self-adjoint operators all have the same domain, all with discrete spectrum (and finite multiplicities) but the spectra are becoming more ‘spread-out’ as $\lambda \to \infty$

$$\mathrm{Spec}(D_\lambda) = \{0, \pm\lambda, 2\pm\lambda, 3\pm\lambda, \ldots\}.$$

Now observe that multiplication by a smooth function with bounded derivative defines an operator S whose commutators $[S, D_\lambda]$ are the same.

a) Using the description of $\mathrm{Spec}(D_\lambda)$ above, prove that

$$\|D_\lambda(1+t+D_\lambda^2)^{-1}\| \le \text{const.}\cdot\left(\frac{\lambda}{1+t+\lambda^2}\right), \quad \lambda\to\infty.$$

b) By inspecting the integral trick prove that

$$\|[S, D_\lambda(1+D_\lambda^2)^{-\frac{1}{2}}]\| \le \text{const.}\cdot \|[S, D_1]\|\cdot\lambda^{-2}, \ \ \lambda > 0.$$

Deduce that in particular, if $F_\lambda := D_\lambda(1+D_\lambda^2)^{-\frac{1}{2}}$, we have

$$\|[S, F_\lambda]\| \to 0, \ \ \lambda\to\infty.$$

Now consider the family $\{H_\lambda^\pm\}_{\lambda\in[1,+\infty]}$ of Hilbert spaces discussed above, with $H_\lambda^- := L^2(\mathbb{R})$ for all $\lambda \in [1, +\infty]$, and

$$H_\lambda^+ = \begin{cases} L^2(\mathbb{R}) & \text{if } 1 \le \lambda < \infty \\ & \\ L^2(\mathbb{R}) \oplus \mathbb{C} & \text{if } \lambda = +\infty \end{cases} \tag{7.10}$$

Let $C_u(\mathbb{R})$ be the C*-algebra of uniformly continuous functions on $\mathbb{R}$. For $f \in C_u(\mathbb{R})$ we let f act on all appearances of $\mathrm{L}^2(\mathbb{R})$ by the corresponding multiplication operator. On the copy $\mathbb{C}$, we let f act by the complex number $f(0)$. Denote by

$$\mu\colon C_u(\mathbb{R}) \to \mathbb{B}(\mathcal{E})$$

this representation. Since the commutators $[f, D_\lambda]$ are (uniformly) bounded by $\|f'\|$, the commutators $[f, F_\lambda]$ are all compact, and furthermore,

$$\|[f, F_\lambda]\| \to 0$$

as $\lambda \to \infty$. We obtain the following.

Now let $\mathcal{E}$ be the right Hilbert $C([1,+\infty])$ module constructed above, then endowing it with the representation of $C_u(\mathbb{R})$ defined above, gives a cycle $(\mathcal{E}, \mu, F)$ for $\mathrm{KK}_0(C_u(\mathbb{R}), C([1,+\infty]))$, whose value at infinity is the sum of the triple $(\mathbb{C} \oplus 0, \mu_1, 0)$, where $\mathbb{C} \oplus 0$ is $\mathbb{Z}/2$-graded with first summand even and second odd, and carries the representation of $C_u(\mathbb{R})$ by

$$f \mapsto f(0),$$

and a degenerate cycle

$$\left(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), 1, \begin{bmatrix} 0 & \frac{x}{|x|} \\ \frac{x}{|x|} & 0 \end{bmatrix} \right),$$

where $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is $\mathbb{Z}/2$-graded with first summand even, second odd.

We have proved the following result:

THEOREM 7.8. *If $1 \leq \lambda < \infty$, then the cycle for $\mathrm{KK}_0(C_u(\mathbb{R}), \mathbb{C})$, $\big(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), 1, F_\lambda\big)$, where the Hilbert space carries the representation of $C_u(\mathbb{R})$ by multiplication operators, represents the class in $\mathrm{KK}_0(C_u(\mathbb{R}), \mathbb{C})$ of the *-homomorphism*

$$C_u(\mathbb{R}) \to \mathbb{C}, \quad f \mapsto f(0).$$

To pursue the reasoning further, note that each of the C*-algebras $C_0(\mathbb{R}), C(\eta\mathbb{R})$ also embed in $C_u(\mathbb{R})$, or, from the Gelfand dual point of view, there are natural maps

$$\overline{\mathbb{R}}^u \to \mathbb{R}, \quad \overline{\mathbb{R}}^u \to \eta\mathbb{R}.$$

Using these maps, we can pull our cycle $(\mathcal{E}, \mu, F)$ for $\mathrm{KK}_0(C_u(\mathbb{R}), C([1,+\infty]))$ back to a cycle for $\mathrm{KK}_0(C(\eta\mathbb{R}), C([1,+\infty]))$, or for $\mathrm{KK}_0(C_0(\mathbb{R}), C([1,+\infty]))$, respectively. The endpoints of the class determine a corresponding identity of KK-classes,

The case of $C_0(\mathbb{R})$ has some special features. Let us call the *de Rham cycle* for $\mathbb{R}$ the unbounded $\mathrm{KK}_0(C_0(\mathbb{R}), \mathbb{C})$ cycle

$$(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \mu_1, D_0 := \begin{bmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{bmatrix}).$$

where μ_1 denotes the usual representation of $C_0(\mathbb{R})$ by multiplication operators on $L^2(\mathbb{R})$. Applying a normalizing function we obtain the bounded cycle

$$(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \mu_1, D_0 := \begin{bmatrix} 0 & -F_0 \\ F_0 & 0 \end{bmatrix}).$$

where F_0 is singular convolution with $\hat{\chi}$, with χ the standard normalizing function.

The de Rham cycle can be defined appropriately for any Riemannian manifold, and defines a class in KK-theory.

COROLLARY 7.9. *The de Rham cycle for $\mathbb{R}$ defines the same class in $\mathrm{KK}_0(C_0(\mathbb{R}), \mathbb{C})$ as the class of the *-homomorphism $C_0(\mathbb{R}) \to \mathbb{C}$, $f \mapsto f(0)$.*

Somewhat unfortunately for our highly specialized example, the de Rham class vanishes for simple dimension reasons for odd-dimensional manifolds, for example $\mathbb{R}$, because of some basic considerations of Clifford algebras. The class of a point evaluation must also therefore vanish in $\mathrm{KK}_0(C_0(\mathbb{R}), \mathbb{C})$, according to Corollary 7.9.

EXERCISE 7.10. Show that the class of the evaluation *-homomorphism $C_0(\mathbb{R}) \to \mathbb{C}$, $f \mapsto f(0)$, defines the zero element of $\mathrm{KK}_0(C_0(\mathbb{R}), \mathbb{C})$, or, even more generally, if X is any non-compact, connected space, x_0 a point of X and $[x_0] \in \mathrm{KK}_0(C_0(X), \mathbb{C})$ the corresponding evaluation morphism, then $[x_0] = 0$.

(*Hint.* Since X is connected and compact, argue that it contains a ray, *i.e.* a map $r\colon [0,+\infty) \to X$, whose value, say, at 0 is the point $x_0 \in X$. Deduce that the point evaluation morphism $[x_0]$ factors through $C_0\big([0,\infty)\big)$, which is KK-equivalent to the zero C*-algebra.)

As the exercise shows, the vanishing of a point K-homology class in $\mathrm{KK}_0(C_0(\mathbb{R}),\mathbb{C})$ is a consequence of a rather elementary aspect of the global topology of $\mathbb{R}$: it is non-compact. On the other hand, the vanishing of the de Rham class has nothing to do with global topology, but only a symmetry due to odd-dimensionality of $\mathbb{R}$. The reasons for vanishing in each case are thus rather different.

In fact the constructions behind Corollary 7.9 can be adapted to produce a theorem of more interesting topological content. To close this section, we give a sketch of of a special case of a result of Lück and Rosenberg along these lines.

The operator $\frac{d}{dx}$ acting on $L^2(\mathbb{R})$ is $\mathbb{Z}$-equivariant, with respect to the unitary action of the integers on $L^2(\mathbb{R})$ by translation. It follows that $\frac{d}{dx}$ descends to an operator on $L^2(\mathbb{R}/\mathbb{Z}) = L^2(\mathbb{T})$, which is of course the angular derivative on the circle $\mathbb{T}$ – the Dirac operator on the circle.

Form the Hilbert space $L^2(\mathbb{T})$ and take two copies of it to form $L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$, graded with first copy even and second odd.

We next let $f \in C^\infty(\mathbb{T})$ be a smooth, *real*-valued function, considered as a periodic function on $\mathbb{R}$. We will assume that 0 is a regular value, so that $Z_f := f^{-1}(0)$ is a finite subset of $\mathbb{T}$ and $f'(x) \neq 0$ at any of these points. As in our discussion with Dirac-Schrödinger operators, we form the 'annihilation and creation operators'

$$A_{f,\lambda} := \lambda f + \frac{d}{dx}, A^*_{f,\lambda} := \lambda f - \frac{d}{dx},\ \ D_{f,\lambda} := \begin{bmatrix} 0 & \lambda f - \frac{d}{dx} \\ \lambda f + \frac{d}{dx} & 0 \end{bmatrix}.$$

Now the idea is that, as in the case of operators on $\mathbb{R}$, as $\lambda \to +\infty$, the operators $A_{f,\lambda}$ are, up to addition of degenerates, more or more concentrated around the zeros of f.

We build a family of $\mathbb{Z}/2$-graded Hilbert spaces H_λ, $\lambda \in [1,+\infty]$, as follows. As before, if $\lambda < +\infty$ let $H_\lambda = L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$. If $\lambda = +\infty$, we form a $\mathbb{Z}/2$-graded Hilbert space $l^2(Z_f)^\pm$ by the splitting

$$l^2(Z_f) = l^2(Z_f^+) \oplus l^2(Z_f^-),$$

where $x \in Z_f^+$ if $f'(x) > 0$ and $x \in Z_f^-$ if $f'(x) < 0$.

The *-homomorphism

$$\rho_f\colon C(\mathbb{T}) \to \mathcal{K}\big((l^2(Z_f^\pm)\big)$$

defines an element

$$[\rho_f] \in \mathrm{KK}_0(C(\mathbb{T}),\mathbb{C}), \quad [\rho_f] := [(l^2(Z_f^\pm), 0).$$

where the graded Hilbert space $l^2(Z_f^\pm)$ carries the action of $C(\mathbb{T})$ just described, using the representation ρ_f.

THEOREM 7.11. *In the above notation, letting $\lambda \to +\infty$ determines a homotopy between the class in* $\mathrm{KK}_0(\mathbb{C}(\mathbb{T}),\mathbb{C})$ *of the de Rham cycle*

$$(L^2(\mathbb{T}) \oplus L^2(\mathbb{T}), \begin{bmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{bmatrix}), \tag{7.11}$$

for $\mathbb{T}$, and the class $[\rho_f]$.

Since the class of the cycle (7.11) in $\mathrm{KK}_0(C(\mathbb{T}),\mathbb{C})$ vanishes, we obtain that, for any smooth function f on the circle with isolated zeros and non-vanishing derivative at each of them, we obtain the Poincaré-Hopf Theorem

$$\sum_{x\in f^{-1}(0)} \operatorname{sign} f'(x) \ = 0$$

for the circle.

Obviously, this example is not meant to impress. It is meant to indicate a technique which might profitably be applied to far more general geometric situations.

Exercise 7.12. Use the Intermediate Value Theorem to prove that if f is a smooth function on $\mathbb{T}$ with $f'(x) \neq 0$ for every $x \in f^{-1}(0)$, then $\sum_{x\in f^{-1}(0)} \operatorname{sign} f'(x) \ = 0$.

8. Equivariant Bott periodicity and the K-theory of crossed products

In this book we have given (modulo the Kasparov Technical theorem) two proofs of Bott Periodicity – the Toeplitz proof, and the KK-proof. There are several other well-known ones. Atiyah has proved Bott Periodicity using elementary linear algebra applied to matrix-valued Laurent polynomials on the circle. Joachim Cuntz has given an extremely general argument that Bott Periodicity is forced by basic properties of K-theory functor (like half-exactness, and stabliity under tensoring by $\mathcal{K}$.)

The merit, however, of the 'Schrödinger proof' we have given in the previous section, is that it is quite precisely built on the geometry and analysis of $\mathbb{R}$, and is, in a certain sense, *essentially translation-invariant* (a feature not possessed by the other proofs), as we now explain, after first stating the result.

If A is any $\mathbb{Z}$-C*-algebra, that is, a C*-algebra with an action of the integers by automorphisms, then $C_0(\mathbb{R}) \otimes A \cong C_0(\mathbb{R}, A)$ is a $\mathbb{Z}$-C*-algebra, using the diagonal action of $\mathbb{Z}$. One has the crossed-products $A \rtimes \mathbb{Z}$, and $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$.

The $\mathbb{Z}$-equivariant Bott Periodicity Theorem is the following statement.

Theorem 8.1. *(*$\mathbb{Z}$-equivariant Bott Periodicity*). If A is any $\mathbb{Z}$-C*-algebra, then $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$ and $A \rtimes \mathbb{Z}$ are* KK_1*-equivalent.*

The power of this theorem is quite simple to see. Suppose that $A = C_0(X)$, for a locally compact $\mathbb{Z}$-space X. Then the diagonal action of $\mathbb{Z}$ on $\mathbb{R} \times X$ is a proper action. In fact it is a free and proper action, and therefore

$$C_0(\mathbb{R} \times X) \rtimes \mathbb{Z} \sim C_0(\mathbb{R} \times_{\mathbb{Z}} X),$$

where $\mathbb{R} \times_{\mathbb{Z}} X$ is the quotient of $\mathbb{R} \times X$ by the diagonal group action, and $\sim$ is strong Morita equivalence. We deduce

Corollary 8.2. *For any integer action on a locally compact space, $C_0(X) \rtimes \mathbb{Z}$ is* KK_1*-equivalent to the mapping cylinder $C_0(\mathbb{R} \times_{\mathbb{Z}} X)$.*

An integer action on X is determined by where the generator 1 goes. Say that two $\mathbb{Z}$ actions are *isotopic* if the corresponding pair of homeomorphisms can be connected to each other by a continuous path of homeomorphisms.

It is not at all obvious that the K-theory $\mathrm{K}_*(C_0(X) \rtimes \mathbb{Z})$ of the crossed-product, only depends on the isotopy class of the action, although one would imagine this must be true. It follows, however, from equivariant Bott Periodicity.

Corollary 8.3. *If two $\mathbb{Z}$-actions are isotopic, then the* K*-theory groups of the corresponding crossed-products $\mathrm{K}_*(C_0(X) \rtimes \mathbb{Z})$ are isomorphic.*

Proof. It is obvious that isotopic $\mathbb{Z}$-actions lead to homeomorphic mapping cylinders $\mathbb{R} \times_{\mathbb{Z}} X$. The result follows from ordinary homotopy-invariance of K-theory (and equivariant Bott Periodicity.)

□

For the case of the irrational rotation algebra $A_\theta := C(\mathbb{T}) \rtimes \mathbb{Z}$, defined by letting the homeomorphism be rotation by an irrational angle $\theta \in \mathbb{R}/\mathbb{Z}$, we obtain

Corollary 8.4. $\mathrm{K}_0(A_\theta) \cong \mathbb{Z} \oplus \mathbb{Z}$, *with generators the class* $[1_{A_\theta}]$ *of the unit in* A_θ, *and the class* $[\mathcal{E}_{0,1}]$ *of the f.g.p. module constructed from a transversal as in* (10.33). $\mathrm{K}_1(A_\theta) \cong \mathbb{Z} \oplus \mathbb{Z}$ *with generators the class* $[z]$ *of the unitary complex coordinate on* $\mathbb{T}$, *and the class* $[u]$ *of the generator* $u \in C(\mathbb{T}) \rtimes \mathbb{Z}$ *of* $\mathbb{Z}$ *in the crossed-product.*

Proof. By equivariant Bott Perioicity, the crossed-product $C(\mathbb{T}) \rtimes \mathbb{Z}$ is KK_1-equivalent to the mapping cylinder $\mathbb{R} \times_{\mathbb{Z}} \mathbb{T}$, which, as we have noted, is naturally homeomorphic to the ordinary 2-torus $\mathbb{T}^2$ (note that the homeomorphism uses, precisely, the obvious isotopy between rotation by θ, and the identity.) Since $\mathrm{K}_0(\mathbb{T}^2) \cong \mathbb{Z}^2$ and $\mathrm{K}^1(\mathbb{T}^2) \cong \mathbb{Z}^2$, $\mathrm{K}_1(A_\theta) \cong \mathbb{Z}^2$ and $\mathrm{K}_0(A_\theta) \cong \mathbb{Z}^2$ follow, as abstract groups. To check the assertions about the generators is an excellent exercise in Kasparov products, and is left to the reader.

□

Corollary 8.5. *Suppose* $\phi\colon X \to X$ *is a minimal homeomorphism of the Cantor set* X. *Then* $\mathrm{K}_0(C(X) \rtimes \mathbb{Z})$ *is naturally isomorphic to the cokernel of the abelian group homomorphism*

$$\mathrm{id} - \phi^*\colon C(X, \mathbb{Z}) \to C(X, \mathbb{Z}),$$

that is

$$\mathrm{K}_0(C(X) \rtimes \mathbb{Z}) \cong C(X, Z)/\mathrm{ran}(\mathrm{id} - \phi^*),$$

with $\phi^*(f) := f \circ \phi$, *and* $C(X, \mathbb{Z})$ *the group of integer-valued, continuous functions on* X.

And $\mathrm{K}_1(C(X) \rtimes \mathbb{Z}) \cong \mathbb{Z}$, *with generator the class* $[u]$ *of the unitary* $u \in C(X) \rtimes \mathbb{Z}$ *generating the action.*

This is an immediate corollary of the more general sequence, called the *Pimsner-Voiculescu sequence.*

Corollary 8.6. *For any* $\mathbb{Z}$*-action on a* C^**-algebra* A, *there is a cyclic* 6*-term exact sequence of the form*

(8.1)
$$\begin{tikzcd} \mathrm{K}_0(A) \arrow[r, "\mathrm{id}-\alpha_*"] & \mathrm{K}_0(A) \arrow[r, "i_*"] & \mathrm{K}_0(A \rtimes \mathbb{Z}) \arrow[d, "\delta"] \\ \mathrm{K}_1(A \rtimes \mathbb{Z}) \arrow[u, "\delta"] & \mathrm{K}_1(A) \arrow[l, "i_*"'] & \mathrm{K}_1(A) \arrow[l, "\mathrm{id}-\alpha_*"'] \end{tikzcd}$$

where $\alpha\colon A \to A$ *is the automorphism generating the action,* $i\colon A \to A \rtimes \mathbb{Z}$ *is the inclusion.*

The maps δ are not easy to describe or compute in general, and we do not go into this.

Proof. This is an application of Corollary 8.6 in the case $A = C(X)$. In this case, $\mathrm{K}_1(A)$ is the zero group, and hence the Pimsner-Voiculescu sequence reduces to a sequence of the form

(8.2)
$$0 \to \mathrm{K}_1(C(X) \rtimes \mathbb{Z}) \xrightarrow{\delta} \mathrm{K}^0(X) \xrightarrow{\mathrm{id}-\alpha_*} \mathrm{K}^0(X) \xrightarrow{i_*} \mathrm{K}_0(C(X) \rtimes \mathbb{Z}) \to 0.$$

Since $\mathrm{K}^0(X) \cong C(X, \mathbb{Z})$ as abelian groups, it follows that

$$\mathrm{K}_0(C(X) \rtimes \mathbb{Z}) \cong C(X, \mathbb{Z})/(\mathrm{id} - \alpha_*)C(X, \mathbb{Z}),$$

that is $\mathrm{K}_0(C(X) \rtimes \mathbb{Z}$ is isomorphic to the cokernel of $\mathrm{id} - \alpha_*$ acting on $C(X, \mathbb{Z})$. Furthermore, if the $\mathbb{Z}$-action is minimal, then no continuous, complex-valued function $f\colon X \to \mathbb{C}$ can satisfy

$$f \circ \alpha = f,$$

unless it is constant. Hence $\ker(\mathrm{id}-\alpha_*)$ consists of the subgroup of constant functions in $C(X,\mathbb{Z})$ and thus is infinite cyclic. Hence $\mathrm{K}_1(C(X)\rtimes\mathbb{Z})$ is also infinite cyclic, and we leave it as an exercise to verify that the class $[u]$ described in the statement, is a generator. □

We now proceed with the proof of the equivariant Periodicity theorem.

The descent construction

The proof of the $\mathbb{Z}$-equivariant Periodicity theorem starts with the observation that both the cycle involved in the Bott morphism $(C_0(\mathbb{R}),1,\varphi)$ and the cycle involved in the Dirac morphism $(L^2(\mathbb{R}),\pi,F:=\varphi(\frac{id}{dx}))$, have a certain natural essential equivariance under the translation action of the integers on $\mathbb{R}$.

Let

$$\pi\colon\mathbb{Z}\to\mathbb{B}(L^2\mathbb{R}),\quad (\pi(n)\xi(x):=\xi(x-n). \tag{8.3}$$

Then π is a unitary representation. To abbreviate notation later, let

$$\xi^n(x):=\xi(x-n).$$

Now any translation operator $\pi(n)$ obviously commutes with $D=-i\frac{d}{dx}$ in the strongest sense (leaving its domain invariant, *etc*) and hence $\pi(n)$ also commutes with the bounded Fredholm operators $F:=\chi(D)$, with χ a normalizing function.

Now let A be any $\mathbb{Z}$-C*-algebra. The tensor product

$$L^2(\mathbb{R})\otimes A$$

is a right Hilbert A-module, carrying a diagonal action of the integers $\mathbb{Z}$. Let

$$\pi\otimes 1_A\colon C_0(\mathbb{R},A)\to\mathbb{B}\big(L^2(\mathbb{R})\otimes A\big).$$

be the representation obtained by tensoring π with the identity on A. This results in a $\mathbb{Z}$-*equivariant right Hilbert A-module*, a notion which will be useful in the following discussion, so we formalize it in a

Definition 8.7. Let A be a $\mathbb{Z}$-C*-algebra. Then a $\mathbb{Z}$-*equivariant right Hilbert A-module* is a right Hilbert A-module $\mathcal{E}$ with a linear action of $\mathbb{Z}$, satisfying

$$n(\xi\cdot a)=n(\xi)\cdot n(a),\quad \langle n(\xi),n(\eta)\rangle=n(\langle\xi,\eta\rangle).$$

Note that the group $\mathbb{Z}$ does not act by Hilbert module maps.

The following procedure can be applied to any $\mathbb{Z}$-equivariant right Hilbert A-module to produce a right Hilbert $A\rtimes\mathbb{Z}$-module.

Definition 8.8. Let A be a $\mathbb{Z}$-C*-algebra and $\mathcal{E}$ be a $\mathbb{Z}$-equivariant right Hilbert A-module. The module $\mathcal{E}_{\mathbb{Z}}$ obtained by applying *descent* to $\mathcal{E}$ is the right Hilbert $A\rtimes\mathbb{Z}$-module which is the completion of the algebraic tensor product $\mathcal{E}\otimes\mathbb{C}[\mathbb{Z}]$ with respect to the inner product

$$\langle\xi\otimes[n],\eta\otimes[m]\rangle:=\langle(-n)(\xi),(-n)(\eta)\rangle\cdot[m-n]\in A[\mathbb{Z}],$$

and right $A\rtimes\mathbb{Z}$-module structure with

$$(\xi\otimes[n])\cdot[k]:=\xi\otimes[n+k],\quad (\xi\otimes[n])\cdot a:=\xi\cdot n(a)\otimes[n].$$

Example 8.9. If A is any $\mathbb{Z}$-C*-algebra, then A with its canonical right Hilbert A-module structure, is a $\mathbb{Z}$-equivariant Hilbert A-module. Application of descent to it produces

$$A_{\mathbb{Z}}\cong A\rtimes\mathbb{Z},$$

as right $A\rtimes\mathbb{Z}$-modules.

EXAMPLE 8.10. Suppose H is a Hilbert space equipped with a unitary action of $\mathbb{Z}$. Then H is a $\mathbb{Z}$-equivariant right Hilbert $\mathbb{C}$-module, and applying descent to it results in simply the tensor product

$$H_{\mathbb{Z}} \cong H \otimes C^*(\mathbb{Z}),$$

as $C^*(\mathbb{Z})$-modules.

More generally, with the same hypotheses on H, if A is any $\mathbb{Z}$-C*-algebra, then $H \otimes A$ is a $\mathbb{Z}$-equivariant right Hilbert A-module, and application of descent to it yields

$$(H \otimes A)_{\mathbb{Z}} \cong H \otimes A \rtimes \mathbb{Z}$$

as right Hilbert $A \rtimes \mathbb{Z}$-modules.

'Descent' is functorial, in the following sense (the proof is easy, and is left to the reader).

LEMMA 8.11. *Let A, B be $\mathbb{Z}$-C*-algebras.*

Suppose that $\mathcal{E}_1$ is a $\mathbb{Z}$-equivariant Hilbert A-module, and that $\pi\colon A \to \mathbb{B}(\mathcal{E}_2)$ is a $\mathbb{Z}$-equivariant representation of A on a $\mathbb{Z}$-equivariant Hilbert B-module $\mathcal{E}_2$.

Then

$$(\mathcal{E} \otimes_A \mathcal{E}')_{\mathbb{Z}} \cong \mathcal{E}_{\mathbb{Z}} \otimes_{A \rtimes \mathbb{Z}} \mathcal{E}'$$

as right Hilbert $B \rtimes \mathbb{Z}$-modules.

LEMMA 8.12. *Let A, B be $\mathbb{Z}$-C*-algebras and $\mathcal{E}$ is a $\mathbb{Z}$-equivariant Hilbert B-module. Suppose*

$$\pi\colon A \to \mathbb{B}(\mathcal{E})$$

is a $\mathbb{Z}$-equivariant representation of A on $\mathcal{E}$.

Then the covariant pair, defined initially on elementary tensors in $\mathcal{E}_{\mathbb{Z}}$ by

$$\pi_{\mathbb{Z}}(a)(\xi \otimes [n]) := \pi(a)\xi \otimes [n], \quad k \cdot (\xi \otimes [n]) := k(\xi) \otimes [k+n],$$

extends to a representation

$$\pi_{\mathbb{Z}}\colon A \rtimes \mathbb{Z} \to \mathbb{B}(\mathcal{E}_{\mathbb{Z}}),$$

of $A \rtimes \mathbb{Z}$ as right Hilbert $A \rtimes \mathbb{Z}$-module operators.

Finally, we note the following easy

LEMMA 8.13. *If $\mathcal{E}$ is a $\mathbb{Z}$-equivariant right Hilbert A-module, $\mathcal{E}_{\mathbb{Z}}$ the right Hilbert $A \rtimes \mathbb{Z}$-module obtained by descent, then for any $T \in \mathbb{B}(\mathcal{E})$, the operator on the algebraic tensor product $\mathcal{E} \otimes \mathbb{C}[\mathbb{Z}]$,*

$$\xi \otimes [n] \mapsto T(\xi) \otimes [n]$$

extends continuously to an adjointable operator $T_{\mathbb{Z}} \in \mathbb{B}(\mathcal{E}_{\mathbb{Z}})$.

The resulting map

$$\mathbb{B}(\mathcal{E}) \to \mathbb{B}(\mathcal{E}_{\mathbb{Z}})$$

*is a *-homomorphism mapping $\mathcal{K}(\mathcal{E})$ to $\mathcal{K}(\mathcal{E}_{\mathbb{Z}})$.*

In the case $\mathcal{E} = A$, the content of the lemma is that there is a natural inclusion

$$\mathcal{M}(A) \subset \mathcal{M}(A \rtimes \mathbb{Z}),$$

which, clearly, maps A into $A \rtimes \mathbb{Z}$ (the 'compact operators' on $A \rtimes \mathbb{Z}$.)

With these generalities discussed, we return to the discussion prior to Definition 8.7.

The $\mathbb{Z}$-equivariant Dirac morphism

Let A be any $\mathbb{Z}$-C*-algebra. We construct a certain KK-cycle, using the ideas of the previous section.

a) Firstly, $C_0(\mathbb{R}) \otimes A \cong C_0(\mathbb{R}, A)$ is also a $\mathbb{Z}$-C*-algebra, with the diagonal action.
b) $L^2(\mathbb{R}) \otimes A$ is a $\mathbb{Z}$-equivariant Hilbert A-module, to which, therefore, we can apply the descent construction, yielding the right Hilbert $A \rtimes \mathbb{Z}$-module

$$\big(L^2(\mathbb{R}) \otimes A\big)_{\mathbb{Z}} \cong L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z}.$$

c) The representation $M\colon C_0(\mathbb{R}) \to \mathbb{B}(L^2\mathbb{R})$ induces a representation $M \otimes 1_A\colon C_0(\mathbb{R}, A) \to \mathbb{B}(L^2\mathbb{R} \otimes A)$, which is also clearly $\mathbb{Z}$-equivariant. By Lemma 8.12 we obtain a representation

$$(M \otimes 1_A)_{\mathbb{Z}}\colon C_0(\mathbb{R}, A) \rtimes \mathbb{Z} \to \mathbb{B}(L^2\mathbb{R} \otimes A \rtimes \mathbb{Z}).$$

d) If χ is any normalizing function, and $F := \chi(D) \in \mathbb{B}(L^2\mathbb{R})$ is the sign of the Dirac operator on $\mathbb{R}$, then $F \otimes 1_A \in \mathbb{B}(L^2\mathbb{R} \otimes A)$ and then descends to a Hilbert $A \rtimes \mathbb{Z}$-module operator $(F \otimes 1_A)_{\mathbb{Z}}$ on $L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z}$, which, furthermore, is self-adjoint and, last but not least, *commutes* with the $\mathbb{Z}$-action on on $L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z}$ which is part of the representation $(M \otimes 1_A)_{\mathbb{Z}}$ described in part c).
e) If $f \in C_0(\mathbb{R})$ then $[M_f, F]$ is compact on $L^2(\mathbb{R})$. It follows that

$$[(M \otimes 1_A)_{\mathbb{Z}}(f \otimes a), (F \otimes 1_A)_{\mathbb{Z}}] \in \mathbb{B}(L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z})$$

is a compact operator on $L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z}$, for any $f \in C_0(\mathbb{R}), a \in A$, in fact, it is of the form $T_{\mathbb{Z}}$ in the sense of Lemma 8.13, where $T = [f, F] \otimes a \in \mathbb{B}(L^2\mathbb{R} \otimes A)$, a compact operator on $L^2(\mathbb{R}) \otimes A$ since F is compact, and $a \in A$. For the same reasons,

$$(M \otimes 1_A)_{\mathbb{Z}}(f \otimes a) \cdot (F \otimes 1_A)^2 - 1)$$

is a compact operator on $L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z}$, since $f \cdot (F^2 - 1)$ is a compact operator on $L^2(\mathbb{R})$.

As a consequence of these observations, we have the following.

LEMMA 8.14. *For any $\mathbb{Z}$-C*-algebra A, the triple* $(L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z}, (M \otimes 1_A)_{\mathbb{Z}}, (F \otimes 1_A)_{\mathbb{Z}})$ *defines an odd Fredholm $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$-$A \rtimes \mathbb{Z}$-bimodule.*

We let

$$\alpha_A \in \mathrm{KK}_1(C_0(\mathbb{R}, A) \rtimes \mathbb{Z}, A \rtimes \mathbb{Z})$$

be the class of the triple of Lemma 8.14.

REMARK 8.15. Let $\mathbb{Z}$-**C*-alg** denote the category with objects $\mathbb{Z}$-C*-algebras, and morphisms $\mathbb{Z}$-equivariant *-homomorphisms.

Now the assignment of the $\mathbb{Z}/2$-graded abelian group $\mathrm{K}_*(A \rtimes \mathbb{Z})$ to an object A in $\mathbb{Z}$-**C*-alg**, and to a $\mathbb{Z}$-equivariant *-homomorphism $A \to B$, the abelian group homomorphism (of degree zero)

$$\alpha_{\mathbb{Z}}\colon \mathrm{K}_*(A \rtimes \mathbb{Z}) \to \mathrm{K}_*(B \rtimes \mathbb{Z})$$

induced by the *-homomorphism $\alpha_{\mathbb{Z}}$, is a functor from the category $\mathbb{Z}$-**C*-alg** to the category **Ab** of $\mathbb{Z}/2$-graded abelian groups.

We can also produce a functor between these two categories by sending an object A to the group $\mathrm{K}_*(C_0(\mathbb{R}, A) \rtimes \mathbb{Z})$, and to a $\mathbb{Z}$-equivariant *-homomorphism $\varphi\colon A \to B$, the group homomorphism

$$(\varphi_{\mathbb{Z}})_*\colon \mathrm{K}_*(C_0(\mathbb{R}, A) \rtimes \mathbb{Z}) \to \mathrm{K}_*(C_0(\mathbb{R}, B) \rtimes \mathbb{Z}).$$

Then the assignment

$$A \mapsto (\alpha_A)_*\colon \mathrm{K}_*(C_0(\mathbb{R}, A) \rtimes \mathbb{Z}) \to \mathrm{K}_*(A \rtimes \mathbb{Z})$$

where $(\alpha_A)_*$ is the map on K-theory induced by the Kasparov morphism $\alpha_A \in \mathrm{KK}_1(C_0(\mathbb{R}, A) \rtimes \mathbb{Z}, A \rtimes \mathbb{Z})$, defines a *natural transformation* between these two functors.

We will show below that α_A is a KK-equivalence for any $\mathbb{Z}$-C*-algebra A. Its inverse, as we will show, is an enrichment in a similar sense, taking into account $\mathbb{Z}$-equivariance, of the Bott morphism.

Before proceeding, we first describe the Dirac morphism in geometric terms.

Since $\mathbb{R}$ acts freely and properly on $\mathbb{Z}$ by translation,

$$C_0(\mathbb{R}) \rtimes \mathbb{Z} \sim C(\mathbb{R}/\mathbb{Z}),$$

where $\sim$ is strong Morita equivalence. Kasparov product with the Morita equivalence $C(\mathbb{T})$-$C_0(\mathbb{R}) \rtimes \mathbb{Z}$-bimdodule $\mathcal{E}_{\mathbb{Z},\mathbb{R}}$ discussed earlier thus induces an isomorphism

$$\mathcal{E}_{\mathbb{Z},\mathbb{R}}\hat{\otimes}\colon \mathrm{KK}_1\big(C_0(\mathbb{R}) \rtimes \mathbb{Z}, C^*(\mathbb{Z})\big) \cong \mathrm{KK}_1\big(C(\mathbb{T}), C^*(\mathbb{Z})\big)$$

induced by Kasparov product with the class $[\mathcal{E}_{\mathbb{Z},\mathbb{R}}] \in \mathrm{KK}_0(C(\mathbb{T}), C_0(\mathbb{R}) \rtimes \mathbb{Z})$ of the bimodule.

On the other hand, under Fourier transform, $C^*(\mathbb{Z}) \cong C(\widehat{\mathbb{Z}})$, which is continuous functions on another circle, which we denote by $\widehat{\mathbb{T}}$. If $F\colon C^*(\mathbb{Z}) \to C(\widehat{\mathbb{T}})$ is the Fourier isomorphism, it determines a KK-morphism and we obtain a class

$$\alpha'_{\mathbb{C}} \in \mathrm{KK}_1\big(C(\mathbb{T}), C(\widehat{\mathbb{T}})\big), \quad [\mathcal{E}_{\mathbb{Z},\mathbb{R}}]\hat{\otimes}_{C_0(\mathbb{R})\rtimes\mathbb{Z}}\alpha_{\mathbb{C}}\hat{\otimes}_{C^*(\mathbb{Z})}[F].$$

PROPOSITION 8.16. *The class*

$$\alpha'_{\mathbb{C}} \in \mathrm{KK}_1\big(C(\mathbb{T}), C(\widehat{\mathbb{T}})\big), \tag{8.4}$$

obtained by applying Morita invariance and Fourier transform to the $\mathbb{Z}$-equivariant Dirac element $\alpha_{\mathbb{C}}$, is the image in $\mathrm{KK}_1(C(\mathbb{T}), C(\widehat{\mathbb{T}}))$ of the Fourier-Mukai correspondence

$$\mathbb{T} \xleftarrow{\mathrm{pr}_{\mathbb{T}}} (\mathbb{T} \times \widehat{\mathbb{T}}, \beta) \xrightarrow{\mathrm{pr}_{\widehat{\mathbb{T}}}} \widehat{\mathbb{T}}, \tag{8.5}$$

under the canonical map

$$\mathrm{KK}_1(\mathbb{T}, \widehat{\mathbb{T}}) \to \mathrm{KK}_1(C(\mathbb{T}), C(\widehat{\mathbb{T}})).$$

That is, up to some standard identifications, the Dirac morphism is exactly the Fourier-Mukai correspondence.

We give a sketch of the proof, through a guided sequence of exercises.

PROOF. The relevant Kasparov product involves firstly computing the product of Hilbert modules

$$\mathcal{E}_{\mathbb{Z},\mathbb{R}} \otimes_{C_0(\mathbb{R})\rtimes\mathbb{Z}} L^2(\mathbb{R})_{\mathbb{Z}}.$$

We discuss this via the following

EXERCISE 8.17. Show that the right $C^*(\mathbb{Z})$-module

$$\mathcal{E}_{\mathbb{Z},\mathbb{R}} \otimes_{C_0(\mathbb{R})\rtimes\mathbb{Z}} L^2(\mathbb{R})_{\mathbb{Z}}$$

(recall that $L^2(\mathbb{R})_{\mathbb{Z}} = L^2(\mathbb{R}) \otimes C^*(\mathbb{Z})$) can be described more concretely as the space of measurable functions $f\colon \mathbb{R} \to C^*(\mathbb{Z})$ which are $\mathbb{Z}$-equivariant in the sense that $f(x+n) = f(x)\cdot[n]$ for all $x \in \mathbb{R}$, with inner product

$$\langle f_1, f_2\rangle := \int_0^1 f_1(t)^* \cdot f_2(t)\, dt$$

and the right $C^*(\mathbb{Z})$-module structure

$$(\xi \cdot [n])(t) := \xi(t)\cdot[n] \in C^*(\mathbb{Z}).$$

Note that this right $C^*(\mathbb{Z})$-module carries a natural representation of $C(\mathbb{T})$, by lifting them to periodic functions on $\mathbb{R}$ and then act by pointwise scalar multiplication on $C^*(\mathbb{Z})$.

Let $\widehat{\mathbb{T}}$ denote the Pontryagin dual of the integers $\widehat{\mathbb{Z}}$. Define a complex line bundle over $\widehat{\mathbb{T}}$ over $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ by inducing the character χ:

$$L_\chi := \mathbb{R} \times_{\mathbb{Z}} \mathbb{C}.$$

The space of continuous sections $C(\mathbb{T}, L_\chi)$ of this line bundle over $\mathbb{T}$ can be completed to a Hilbert space

$$L^2(\mathbb{T}, L_\chi)$$

since the bundle is Hermitian, and using Lebesegue measure over the circle.

Note that there is a natural representation of $C(\mathbb{T})$ on this Hilbert space, by multiplication operators.

Now using Fourier transform $F\colon C^*(\mathbb{Z}) \to C(\widehat{\mathbb{T}})$, to push forward $C^*(\mathbb{Z})$-modules to $C(\widehat{\mathbb{T}})$-modules, we form the right $C(\widehat{\mathbb{T}})$-module

$$\mathcal{E}_{\mathbb{Z},\mathbb{R}} \otimes_{C_0(\mathbb{R})\rtimes\mathbb{Z}} L^2(\mathbb{R})_{\mathbb{Z}} \otimes_{C^*(\mathbb{Z})} C(\widehat{\mathbb{T}}).$$

It is isomorphic to the right Hilbert $C(\widehat{\mathbb{T}})$-module

$$C\big(\widehat{\mathbb{T}}, \{L^2(L_\chi)\}_{\chi\in\widehat{\mathbb{Z}}}\big)$$

of sections of the the field of Hilbert spaces

$$\{L^2(L_\chi) \mid \chi \in \widehat{\mathbb{T}}\}.$$

over $\widehat{\mathbb{T}}$. The C*-algebra $C(\mathbb{T})$ acts by adjointable operators, by letting $f \in C(\mathbb{T})$ act as a multiplication operator on each Hilbert space $L^2(L_\chi)$.

Finally, we discuss the operator. We focus on the unbounded source $D = -i\frac{d}{dx}$ of the operator $F := \chi(D)$ figuring in the Dirac morphism. Since it is $\mathbb{Z}$-equivariant, that is, commutes with the unitary translation action of $\mathbb{Z}$ on $L^2(\mathbb{R})$, D descends to an elliptic operator on $\mathbb{T}$. More generally, D determines an elliptic, first-order differential operator on sections $C^\infty(\mathbb{T}, L_\chi)$, for any character χ of $\mathbb{Z}$, and then a densely defined Hilbert space operator on $L^2(L_\chi)$. Indeed, sections of L_χ are equivalent to maps

$$f\colon \mathbb{R} \to \mathbb{C}, \text{ such that } f(x+n) = \chi(n)f(x) \ \ \forall x \in \mathbb{R}.$$

The derivative f' of such f, provided it is smooth, satisfies the same invariance condition. Therefore, we obtain an operator

$$D_\chi := -i\frac{d}{dx}\colon C^\infty(\mathbb{T}, L_\chi) \to C^\infty(\mathbb{T}, L_\chi)$$

on sections of the (flat) bundle L_χ over $\mathbb{T}$.

The totality of the D_χ make up a *bundle* of elliptic operators, over the dual circle $\widehat{\mathbb{T}}$, and together make a densely defined right Hilbert $C(\widehat{\mathbb{T}})$-module operator

$$D\colon C\big(\widehat{\mathbb{T}}, \{L^2(L_\chi)\}_{\chi\in\widehat{\mathbb{Z}}}\big) \to C\big(\widehat{\mathbb{T}}, \{L^2(L_\chi)\}_{\chi\in\widehat{\mathbb{Z}}}\big),$$

with an evident dense domain. Functional calculus applied in each fibre now produces a bounded version F, thus completing the description of the Kasparov product. The fact that this describes the image in KK of the Fourier-Mukai correspondence we leave to the reader to check.

$\square$

The $\mathbb{Z}$-equivariant Bott morphism

At this stage, the Bott morphism is much easier to define. Let A be a $\mathbb{Z}$-C*-algebra. Then $C_0(\mathbb{R}, A)$ is a $\mathbb{Z}$-C*-algebra, and so can be viewed as a $\mathbb{Z}$-equivariant right Hilbert $C_0(\mathbb{R}, A)$-module over itself. It carries an evident left multiplication action of A, which is $\mathbb{Z}$-equivariant, in the evident sense.

Applying the descent apparatus yields the (trivial, rank 1) right Hilbert $C_0(\mathbb{R}, A)\rtimes\mathbb{Z}$-module $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$, and a representation

$$i_A \colon A \rtimes \mathbb{Z} \to \mathbb{B}(C_0(\mathbb{R}, A) \rtimes \mathbb{Z}), \tag{8.6}$$

which is, in this case, essentially the identity map.

Let φ be a normalizing function. It defines a multiplier of $C_0(\mathbb{R})$ such that $\varphi^2 - 1 \in \mathcal{K}(C_0(\mathbb{R})) = C_0(\mathbb{R})$. Similarly, it defines a muliplier of $C_0(\mathbb{R}, A)$ for any A. From this discussion, we obtain, in a very obvious way, a cycle for

$$\mathrm{KK}_1(A, C_0(\mathbb{R}, A) \rtimes \mathbb{Z}).$$

It is in replacing A by $A \rtimes \mathbb{Z}$ in the above group, that requires the *essential equivariance* of φ:

$$\lim_{t\to\pm\infty} \varphi(t+n) - \varphi(t) = 0,$$

for any integer n, which, of course, follows from the assumption that

$$\lim_{\to\pm\infty} \varphi(t) = \pm 1.$$

We have thus shown

LEMMA 8.18. *With respect to the $\mathbb{Z}$-action on the $\mathbb{Z}$-equivariant right $C_0(\mathbb{R}, A)$-module $C_0(\mathbb{R}, A)$, the operator φ of multiplication by φ, satisfies*

$$n \cdot \varphi - \varphi \in \mathcal{K}\big(C_0(\mathbb{R}, A)\big) = C_0(\mathbb{R}, A).$$

Applying descent, we obtain therefore the following result.

LEMMA 8.19. *Consider $C_0(\mathbb{R}, A)\rtimes\mathbb{Z}$ module as a right module over itself. Represent $A\rtimes\mathbb{Z}$ as operators on this module by* (8.6)*. And let F'_A be the operator, that is, multiplier, of $C_0(\mathbb{R}, A)\rtimes\mathbb{Z}$, induced as in the above discussion by the multiplier φ of $C_0(\mathbb{R})$.*

Then the triple

$$(C_0(\mathbb{R}, A) \rtimes \mathbb{Z}, i_A, F'_A)$$

defines a cycle for $\mathrm{KK}_1(A \rtimes \mathbb{Z}, C_0(\mathbb{R}, A) \rtimes \mathbb{Z})$.

We let

$$\beta_A \in \mathrm{KK}_1(A \rtimes \mathbb{Z}, C_0(\mathbb{R}, A) \rtimes \mathbb{Z})$$

be the class of the triple of Lemma 8.19.

The content of the $\mathbb{Z}$-equivariant Bott Periodicity Theorem is then the following.

THEOREM 8.20. *Let A be any $\mathbb{Z}$-C^*-algebra. Then*

$$\beta_A \hat{\otimes}_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}} \alpha_A = 1_{A\rtimes\mathbb{Z}},$$

and

$$\alpha \hat{\otimes}_{A\rtimes\mathbb{Z}} \beta = 1_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}}.$$

In particular, α and β determine KK*-equivalences*

$$A \rtimes \mathbb{Z} \cong_{\mathrm{KK}} C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$$

for any $\mathbb{Z}$-C^*-algebra A.

PROOF. We want to compute the Kasparov product

$$\beta_A \hat{\otimes}_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}} \alpha_A \in \mathrm{KK}_0(A \rtimes \mathbb{Z}, A \rtimes \mathbb{Z}),$$

and argue that it is equal to

$$1_{A\rtimes\mathbb{Z}} \in \mathrm{KK}_0(A \rtimes \mathbb{Z}, A \rtimes \mathbb{Z}).$$

The proof is almost exactly the same as our Schrödinger proof of Bott Periodicity, with only one extra ingredient.

Step 1.

We first describe the composition of *modules* in the product $\beta_A \hat{\otimes}_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}} \alpha_A$. The right Hilbert $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$-module, involved in β_A, in the language of descent, is simply $C_0(\mathbb{R}, A)_{\mathbb{Z}}$ (obtained by applying descent to the trivial rank-one $\mathbb{Z}$-equivariant $C_0(\mathbb{R}, A)$-module.)

The module for α_A is $(L^2(\mathbb{R}) \otimes A)_{\mathbb{Z}}$, which is naturally isomorphic to $L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z}$, as an $A \rtimes \mathbb{Z}$-module. It carries an action of $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$ as Hilbert module operators, induced by the action of $C_0(\mathbb{R})$ as multiplication operators on $L^2(\mathbb{R})$.

We thus see, using functoriality of descent, or simply by direct inspection, that the module involved in the Kasparov product is simply

$$\begin{aligned} C_0(\mathbb{R}, A)_{\mathbb{Z}} \otimes_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}} (L^2(\mathbb{R}) \otimes A)_{\mathbb{Z}} &= (C_0(\mathbb{R}, A) \otimes_{C_0(\mathbb{R},A)} (L^2(\mathbb{R}) \otimes A)_{\mathbb{Z}} \\ &\cong L^2(\mathbb{R}) \otimes A)_{\mathbb{Z}} \cong L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z}. \end{aligned} \tag{8.7}$$

Step 2.

Let $A = \mathbb{C}$ for the moment.

Let

$$F'' := \chi(D''), \quad D'' = \begin{bmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{bmatrix},$$

an unbounded operator on $L^2(\mathbb{R})$.

The integers $\mathbb{Z}$ act, as we have discussed, as translation on $L^2(\mathbb{R})$, this is a unitary representation, which we denote by

$$\rho\colon \mathbb{Z} \to \mathbf{U}(L^2\mathbb{R}).$$

Furthermore, D'' satisfies

$$\rho(n) \cdot D'' \cdot \rho(-n) - D'' = \begin{bmatrix} 0 & n \\ n & 0 \end{bmatrix},$$

which is in particular bounded. It follows that $\rho(n) \cdot F'' \cdot \rho(-n) - F''$ is a compact operator on $L^2(\mathbb{R})$.

Now application of descent to $L^2(\mathbb{R})$ yields

$$L^2(\mathbb{R})_{\mathbb{Z}} \cong L^2(\mathbb{R}) \otimes C^*(\mathbb{Z}).$$

The operator D'' (or F'') descends to a operator $D''_{\mathbb{Z}}$, or ($F''_{\mathbb{Z}}$, respectively) on $L^2(\mathbb{R})_{\mathbb{Z}} \oplus L^2(\mathbb{R})_{\mathbb{Z}}$, and since $\rho(n) \cdot F'' \cdot \rho(-n) - F''$ is compact on $L^2(\mathbb{R})$, and it follows immediately that

$$\rho(n) \cdot F''_{\mathbb{Z}} \cdot \rho(-n) - F''_{\mathbb{Z}} \in \mathbb{B}(L^2(\mathbb{R})_{\mathbb{Z}})$$

is a compact operator on the Hilbert $C^*(\mathbb{Z})$-module $L^2(\mathbb{R})_{\mathbb{Z}}$.

Furthermore, since $(F'')^2 - 1 \in \mathcal{K}(L^2\mathbb{R} \oplus L^2\mathbb{R})$, it follows that

$$(F''_{\mathbb{Z}})^2 - 1 \in \mathcal{K}(L^2(\mathbb{R})_{\mathbb{Z}}) \oplus L^2(\mathbb{R})_{\mathbb{Z}}.$$

as well. The conclusion is that the triple

$$(L^2(\mathbb{R})_{\mathbb{Z}} \oplus L^2(\mathbb{R})_{\mathbb{Z}}, \rho_{\mathbb{Z}}, F_{\mathbb{Z}})$$

defines a cycle for $\mathrm{KK}_0(C^*(\mathbb{Z}), C^*(\mathbb{Z}))$.

More generally, one can easily 'twist' the above cycle, by a $\mathbb{Z}$-C*-algebra A, just as we did with the Dirac and Bott morphisms. We construct a cycle for

$$\mathrm{KK}_0(A \rtimes \mathbb{Z}, A \rtimes \mathbb{Z})$$

by constructing a $\mathbb{Z}$-equivariant Fredholm A-A-bimodule, using the $\mathbb{Z}$-equivariant right Hilbert A-module $L^2(\mathbb{R}) \otimes A$, with the diagonal $\mathbb{Z}$-action, standard right A-module structure, and representation of A by Hilbert A-module maps given by

$$a \cdot (\xi \otimes b) := \xi \otimes ab.$$

As operator we use $D'' \otimes 1_A$, or $F'' \otimes 1_A$, in the unbounded or bounded pictures respectively. The commutators

$$[D'', a], \quad [F'', a]$$

are obviously zero for elements $a \in A$, as operators on $L^2(\mathbb{R}) \otimes A$. If $n \in \mathbb{Z}$, and let, say $U_n \colon L^2(\mathbb{R}) \otimes A \to L^2(\mathbb{R}) \otimes A$ be the action of n,

$$U_n(\xi \otimes a) = \rho(n)\xi \otimes n(a),$$

then

$$U_n \cdot (F'' \otimes 1_A) \cdot U_{-n} - F \otimes 1_A = (\rho(n)F''\rho(-n) - F'') \otimes 1_A.$$

This has the form

$$T \otimes 1_A$$

where T is a compact operator on $L^2(\mathbb{R})$. Hence, if $a \in A$, then

$$a \cdot [U_n, F'' \otimes 1_A]$$

is a compact operator on $L^2(\mathbb{R}) \otimes A$.

All of the conditions to be a $\mathbb{Z}$-equivariant Fredholm A-A-bimodule are therefore met, and applying descent to it yields a morphism

$$\gamma_A \in \mathrm{KK}_0(A \rtimes \mathbb{Z}, A \rtimes \mathbb{Z}).$$

LEMMA 8.21. *If A is any $\mathbb{Z}$-C^*-algebra, then*

$$\beta_A \hat{\otimes}_{A \rtimes \mathbb{Z}} \alpha_A = \gamma_A \in \mathrm{KK}_0(A \rtimes \mathbb{Z}, A \rtimes \mathbb{Z}).$$

PROOF. Recall firstly from Lemma 8.19 that $\beta_A \in \mathrm{KK}_1(A \rtimes \mathbb{Z}, C_0(\mathbb{R}, A) \rtimes \mathbb{Z})$ is represented by the following Fredholm bimodule. For the module, we consider $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$ as a right module over itself. Then $A \rtimes \mathbb{Z}$ is represented in an obvious way as operators on this module, indeed, as multipliers of $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$. Let

$$i_A \colon A \rtimes \mathbb{Z} \to \mathcal{M}\big(C_0(\mathbb{R}, A) \rtimes \mathbb{Z}\big)$$

denote this representation.

Let F'_A be the particular multiplier of $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$, induced as in the above discussion by the multiplier χ of $C_0(\mathbb{R})$, where χ is a normalizing function. Then the Bott morphism β_A is represented by the triple

$$(C_0(\mathbb{R}, A) \rtimes \mathbb{Z}, i_A, F'_A).$$

The Fredholm bimodule for $\alpha_A \in \mathrm{KK}_1(C_0(\mathbb{R}, A) \rtimes \mathbb{Z}, A \rtimes \mathbb{Z})$ is

$$\big(L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z}, \rho_A, F \otimes 1_{A \rtimes \mathbb{Z}}\big),$$

where $F = \chi(D)$, $D = -\frac{d}{dx}$, acting on $L^2(\mathbb{R})$, and

$$\rho_A \colon C_0(\mathbb{R}, A) \rtimes \mathbb{Z} \to \mathbb{B}\big(L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z}\big)$$

is induced by the covariant pair which lets an integer $n \in \mathbb{Z}$ act on the module by

$$n \cdot (\xi \otimes a[m]) := n(\xi) \otimes n(a)[n+m].$$

and lets $C_0(\mathbb{R}, A)$ act by

$$(f \otimes a)(\xi \otimes b[n]) := f\xi \otimes ab[n].$$

Now we have already observed (see (8.7)) in Step 1, that the composition of the *modules*

$$C_0(\mathbb{R}, A) \rtimes \mathbb{Z} \otimes_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}} L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z},$$

which is going to result in a right Hilbert $A \rtimes \mathbb{Z}$-module, with a representation of $A \rtimes \mathbb{Z}$ on it, is simply

$$L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z}.$$

The homomorphism ρ_A described above represents $A \rtimes \mathbb{Z}$ on it.

Now the operator $F'' \in \mathbb{B}\big(L^2(\mathbb{R}) \otimes A \rtimes \mathbb{Z}\big)$ is now easily checked to satisfy the axioms for an intersection product of the operator F' involved in β_A, and F, involved in α_A. Indeed, the computations showing this in the context of *non-equivariant* Bott Periodicity go through in precisely the same way, by tensoring everything by A. We leave the somewhat tedious details of this to the reader.

□

Step 3. $\gamma_A = 1_{A\rtimes\mathbb{Z}}$.

At this stage, the reader will have noticed the repeated use of the following technique. Let $(H, 1, F)$ be an even, $\mathbb{Z}$-equivariant, Fredholm $\mathbb{C}$-$\mathbb{C}$-bimodule, with 1 denoting the usual unital representation of $\mathbb{C}$ on H. The assumption is that we are given a unitary representation of $\mathbb{Z}$ on H, and that

$$n \cdot F \cdot (-n) - F$$

is compact, for all $n \in \mathbb{Z}$.

From this data, we can construct, for any $\mathbb{Z}$-C*-algebra A, a Fredholm

$$\mathrm{KK}(A \rtimes \mathbb{Z}, A \rtimes \mathbb{Z})$$

module, by first replacing the original bimodule by the 'twisted' bimodule $(H \otimes A, 1_A, F \otimes 1_A)$, then applying the descent procedure, which results in

$$(H \otimes A \rtimes \mathbb{Z}, 1_{A\rtimes\mathbb{Z}}, (F \otimes 1_A)_{\mathbb{Z}}).$$

As a simple example, the identity morphism $1_{\mathbb{C}} \in \mathrm{KK}_0(\mathbb{C}, \mathbb{C})$ is represented by the Fredholm bimodule $(\mathbb{C}, 1, 0)$, where $\mathbb{C}$ is $\mathbb{Z}/2$-graded with $\mathbb{C}^+ = \mathbb{C}$, $\mathbb{C}^- = 0$. Let us endow this bimodule with the trivial $\mathbb{Z}$-action. Then it is a very simple exercise to show that application of the 'twisting' procedure above produces the Fredholm $A \rtimes \mathbb{Z}$-$A \rtimes \mathbb{Z}$-bimodule

$$(A \rtimes \mathbb{Z}, 1_{A\rtimes\mathbb{Z}}, 0),$$

which represents

$$1_{A\rtimes\mathbb{Z}} \in \mathrm{KK}_0(A \rtimes \mathbb{Z}, A \rtimes \mathbb{Z}).$$

The case of interest here is $(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), 1, F'')$ where the operator F'' is $\chi(D'')$ where

$$D'' := \begin{bmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{bmatrix}.$$

The integers acts by translation on $L^2(\mathbb{R})$, and commutes modulo *bounded* operators with D'', and hence modulo compact operators, with F''. It is somewhat better to consider F'' as an operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, rather than a matrix, for this discussion. Its kernel as such is 1-dimensional, as we have already seen in our discussion of the harmonic oscillator, spanned by the ‘ground state $\psi_0(x) = e^{-x^2}$. The cokernel of F'' is zero, that is, F'' is surjective. The index of F'' is thus $+1$. Now we have a direct sum decomposition of cycles

$$(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), 1, F'') \cong (\mathbb{C}\psi_0, 1, 0) \oplus ((\mathbb{C}\psi_0)^\perp \oplus (\mathbb{C}\psi_0)^\perp, 1, T)$$

where T is the restriction of F'' to $(\mathbb{C}\psi_0)^\perp$. Since T is invertible, by the above discussion, the second cycle is degenerate, and this yields directly using the definitions of KK that the Fredholm $\mathbb{C}$-$\mathbb{C}$-bimodule $(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), 1, F'')$ is equivalent to the Fredholm $\mathbb{C}$-$\mathbb{C}$-bimodule $(\mathbb{C}, 1, 0)$ representing the identity $1_\mathbb{C} \in \mathrm{KK}_0(\mathbb{C}, \mathbb{C})$.

With this review of our proof of ordinary Bott Periodicity, one might then hope for a similar direct-sum decomposition of the cycle obtained by twisting our cycle $(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), 1, F'')$, but the difficulty is that the direct sum is not left invariant by the unitary $\mathbb{Z}$-action on $L^2(\mathbb{R})$. The operator F'' does not commute with translation operators on $L^2(\mathbb{R}$, but only does so *up to compact operators*. In particular, the ground state ψ_0 is not – obviously – translation-invariant.

This problem can be solved by an extremely naïve argument, which demonstrates the power of Kasparov theory, and, specifically, the relation of ‘homotopy.’

Let $\rho\colon \mathbb{Z} \to \mathbf{U}(L^2\mathbb{R})$ be the unitary representation by translation operators, as above. For all $t \in [0, 1]$ let

$$\rho_t\colon \mathbb{Z} \to \mathbf{U}(L^2\mathbb{R}), \ \ (\rho_t(n)f)(s) = f(s - tn).$$

This is a path of unitary representations of $\mathbb{Z}$. When $t = 1$ we get

$$(\rho_1(n)f)(s) = (\rho(n)f)(s) = f(s - n)$$

and when $t = 0$ we obtain a trivial representation

$$(\rho_0(n)f)(s) = f(s).$$

Now, let us return to our $\mathbb{Z}$-equivariant $\mathbb{C}$-$\mathbb{C}$-bimodule $(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), 1, F'')$ where the operator F'' is $\chi(D'')$ where

$$D'' := \begin{bmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{bmatrix}.$$

The $\mathbb{Z}$-action is by the unitary representation ρ. With the $\mathbb{Z}$-action and the extra condition that

$$[F'', \rho(n)] \in \mathcal{K}(L^2\mathbb{R}),$$

we obtained by descent a Fredholm $C^*(\mathbb{Z})$-$C^*(\mathbb{Z})$-bimodule, which we want to show is equivalent to $1_{C^*(\mathbb{Z})}$.

But by varying the representation ρ_t, for $t \in I := [0, 1]$, and changing none of the other data, we obtain a 1-parameter family of $\mathbb{Z}$-equivariant Fredholm bimodules. The commutators $[\rho_t(n), F'']$ are all compact, and even uniformly bounded in norm, as the reader can easily check. This means that we have a *homotopy of $\mathbb{Z}$-equivariant $\mathbb{C}$-$C(I)$-Fredholm bimodules* (where $\mathbb{Z}$ acts trivially on $C(I)$): namely the $\mathbb{Z}$-equivariant Fredholm $\mathbb{C}$-$C(I)$-bimodule

$$(L^2(\mathbb{R}) \otimes C(I) \ \oplus \ L^2(\mathbb{R}) \otimes C(I), 1, F'' \otimes 1_{C(I)}),$$

where the $\mathbb{Z}$-action on the Hilbert module $L^2(\mathbb{R}) \otimes C(I)$ is

$$(\tilde{\rho}(n)(f)(s, t) = f(s - tn, t), \quad (s, t) \in \mathbb{R} \times I.$$

The endpoints of this homotopy of $\mathbb{Z}$-equivariant bimodules all have the same underlying $\mathbb{C}$-$\mathbb{C}$-bimodule, but the representations of $\mathbb{Z}$ are different. When $t = 0$, the $\mathbb{Z}$-action on $L^2(\mathbb{R})$ is trivial. When $t = 1$ we obtain the original $\mathbb{Z}$-action.

Finally, using the $\mathbb{Z}$-action, we apply our descent procedure to get a homotopy of cycles for $\mathrm{KK}_0(C^*(\mathbb{Z}), C^*(\mathbb{Z}))$, that is, a cycle for

$$\mathrm{KK}_0(C^*(\mathbb{Z}), C^*(\mathbb{Z}) \otimes C(I)).$$

The module is obtained by descent so is given by simply

$$(L^2(\mathbb{R}) \otimes C(I))_{\mathbb{Z}} \cong L^2(\mathbb{R}) \otimes C(I) \otimes C^*(\mathbb{Z}),$$

a right $C(I) \otimes C^*(\mathbb{Z})$-module in the usual way. The operator is $F'' \otimes 1_{C(I)} \otimes 1_{C^*(\mathbb{Z})}$. The most important part of the data is the representation

$$\tilde{\rho}_{\mathbb{Z}} \colon C^*(\mathbb{Z}) \to \mathbb{B}\big((L^2(\mathbb{R}) \otimes C(I))_{\mathbb{Z}}\big),$$

which is induced by the unitary representation

$$(\tilde{\rho}_{\mathbb{Z}})(n)(f)(s,t,m) = f(s - tn, t, m),$$

where we are thinking in this formula of elements of $L^2(\mathbb{R}) \otimes C(I) \otimes C^*(\mathbb{Z})$ as functions on $\mathbb{R} \times I \times \mathbb{Z}$.

Now what are the two endpoints of the homotopy? Clearly when $t = 1$ we obtain our Fredholm module $(L^2(\mathbb{R})_{\mathbb{Z}} \oplus L^2(\mathbb{R})_{\mathbb{Z}}, \rho_{\mathbb{Z}}, F''_{\mathbb{Z}})$ representing $\gamma_{\mathbb{C}} \in \mathrm{KK}_0(\mathbb{C}, \mathbb{C})$.

When $t = 0$, we get the cycle obtained by descent applying to the *trivial* $\mathbb{Z}$-equivariant Hilbert space $L^2(\mathbb{R})$: namely $(L^2(\mathbb{R})_{\mathbb{Z}} \oplus L^2(\mathbb{R})_{\mathbb{Z}}, \epsilon_{\mathbb{Z}}, F''_{\mathbb{Z}})$ where $\epsilon \colon \mathbb{Z} \to \mathbf{U}(L^2\mathbb{R})$ is the trivial action.

Now, by our previous discussion, since in this bimodule, $\mathbb{Z}$ now acts trivially, we can now split off the kernel of F'' as in the above discussion, even $\mathbb{Z}$-equivariantly. This implies a corresponding splitting of the module obtained by descent, and shows that the latter is a degenerate perturbation of $(C^*(\mathbb{Z}), \mathrm{id}_{C^*(\mathbb{Z})}, 0)$, representing $1_{C^*(\mathbb{Z})} \in \mathrm{KK}_0(C^*(\mathbb{Z}), C^*(\mathbb{Z})))$.

Finally, the desired result

$$\gamma_{\mathbb{C}} = 1_{\mathbb{C}} \in \mathrm{KK}_0(C^*(\mathbb{Z}), C^*(\mathbb{Z}))$$

follows from the homotopy-invariance of KK-theory.

Now it is a trivial matter to 'twist' the above argument by a $\mathbb{Z}$-C*-algebra A and obtain

$$\gamma_A = 1_{A \rtimes \mathbb{Z}} \in \mathrm{KK}_0(A \rtimes \mathbb{Z}, A \rtimes \mathbb{Z}),$$

and we leave the details to the reader. In fact, aside from a standard construction (descent), applied to various $\mathbb{Z}$-equivariant bimodules, the only non-trivial step in the above argument was to prove that the Fredholm $\mathbb{C}$-$\mathbb{C}$-bimodule $(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), 1, F'')$ is not only equivalent to $(\mathbb{C}, 1, 0)$, but is $\mathbb{Z}$*-equivariantly equivalent to it*, by an appropriate homotopy. This is still true if one 'twists' by a $\mathbb{Z}$-C*-algebra A, and all of this descends to give a homotopy between the the given representative of γ_A, and the cycle $(A \rtimes \mathbb{Z}, \mathrm{id}_{A \rtimes \mathbb{Z}} 0)$ representing $1_{A \rtimes \mathbb{Z}} \in \mathrm{KK}_0(A \rtimes \mathbb{Z}, A \rtimes \mathbb{Z})$. □

We end by noting that the pattern of proof used to prove that

$$A \rtimes \mathbb{Z} \cong_{\mathrm{KK}_1} C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$$

is called the *Dirac-dual-Dirac method*, and was invented by Kasparov, and others. It is the only known general method of computing crossed-products by infinite groups.

The following exercises are slightly challenging, but are do-able! They illustrates that our argument above is not particularly special so much to the integers $\mathbb{Z}$, but is built around the geometry of the real line $\mathbb{R}$, and the various groups which act on it properly.

EXERCISE 8.22. Show that our proof of $\mathbb{Z}$-equivariant Bott Periodicity goes through for *any* closed subgroup G of $\mathbb{R}$ (the only possibilities are therefore the trivial group, $\mathbb{R}$ itself, and the cyclic groups generated by elements of $\mathbb{R}$.) Mainly, the meat of the exercise is to check that descent works equally well for the locally compact group $\mathbb{R}$ as it does for (the discrete group) $\mathbb{Z}$.

EXERCISE 8.23. Let $G := D_\infty$ be the infinite dihedral group: the group of motions of $\mathbb{R}$ generated by $u(x) = -x$ and $v(x) = x + 1$.

a) Where does our proof of equivariant Bott Periodicity for $\mathbb{Z}$ fail, if we try to make it D_∞-equivariant, rather just just $\mathbb{Z}$-equivariant? (*Hint.* The issue is that the Dirac operator $D = -i\frac{d}{dx}$ no longer commutes with the group action.)

b) If our proof *did* generalize in an obvious way, it would yield the statement

$$A \rtimes D_\infty \cong_{\mathrm{KK}_1} C_0(\mathbb{R}, A) \rtimes D_\infty$$

for any D_∞-C*-algebra A. Show that this statement is false. (Actually, it suffices to take $A = \mathbb{C}$ and check that $C^*(D_\infty)$ is not KK_1-equivalent to $C_0(\mathbb{R}) \rtimes D_\infty$. But actually, they are strong Morita equivalent, and hence KK_0-equivalent, but clearly, by inspection, not KK_1-equivalent.)

b) If $g \in G$, or more generally, if g is any smooth map $\mathbb{R} \to \mathbb{R}$, let g act on $\mathbb{R}^2$ by the differentiated action

$$g(s,t) := (s, g'(s)t).$$

Show that

$$A \rtimes D_\infty \cong_{\mathrm{KK}_0} C_0(\mathbb{R}^2, A) \rtimes D_\infty,$$

where in $C_0(\mathbb{R}^2, A) \rtimes D_\infty$, we are using the differentiated action of D_∞ on $\mathbb{R}^2$.

c) Show that the Dolbeault operator $\frac{\partial}{\partial \bar{z}} := \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ acting on smooth compactly supported functions $C_c^\infty(\mathbb{R}^2)$ in the plane, commutes with the action of any $g \in D_\infty$ (in fact even more generally with holomorphic maps $g\colon \mathbb{R}^2 \to \mathbb{R}^2$.) Build an even, G-equivariant Fredholm $C_0(\mathbb{R}^2)$-$\mathbb{C}$-bimodule using the $\mathbb{Z}/2$-graded Hilbert space

$$L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2),$$

operator

$$D := \begin{bmatrix} 0 & \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & \end{bmatrix},$$

(or its normalized version $F := \chi(D)$), and the representation of $C_0(\mathbb{R}^2)$ by multiplication operators.

d) Build a G-equivariant 'Bott morphism' constructed as usual by normalizing the unbounded multiplier

$$c\colon \mathbb{R}^2 \to M_2(\mathbb{C}), \qquad c(x,y) := \begin{bmatrix} 0 & x+iy \\ x-iy & 0 \end{bmatrix}.$$

e) Prove that the G-equivariant Dirac morphisms constructed in parts c) and d) compose to the identity, and deduce G-equivariant Bott Periodicity.

Methods somewhat similar to the ones described above apply to various classes of groups, for example, to surface groups $G = \pi_1(M^g)$, M^g a compact Riemann surface of genus g, in which context one can construct Bott and Dirac morphisms on the hyperbolic plane, with its complex structure, which are invariant under the group of Möbius transformations. The method can be much further extended than that: we refer the reader to the extensive literature. The general statement that this procedure can in principal always be carried out (for any group, or even any locally compact group) is the content of the Baum-Connes conjecture (see [**1**]).

9. Appendix: Further remarks on the KK proof of Bott Periodicity

In this section, we re-compute the intersection product

$$x \hat{\otimes}_{\mathbb{C}} y \in \mathrm{KK}_0(\mathbb{C}, \mathbb{C})$$

where x and y are the Bott and Dirac morphisms for $\mathbb{R}$. This proof is a little different, and essentially only uses bounded operator theory.

We start by summarizing the main important properties of the operators x and $\frac{d}{dx}$ and their normalizations.

LEMMA 9.1. *Let $D_1 = x$ and $D_2 = -i\frac{d}{dx}$, self-adjoint unbounded operators on $L^2(\mathbb{R})$, and $F_i = \chi(D_i)$, with χ a normalizing function. Then*

$$(1 - F_1^2) \cdot (1 - F_2^2), \quad \text{and} \quad [F_1, F_2]$$

are compact operators.

PROOF. In Lemma 4.4 set $S := F_1$, with $F_1 = \chi(D_2)$, where the normalizing function χ is chosen so that $\chi' \in C - c^\infty(\mathbb{R})$. Then

$$[F_1, D_2] = -i\chi' \in C_c^\infty(\mathbb{R})$$

is bounded and

$$[F_1, D_2] \cdot (1 + D_2^2)^{-1}$$

is compact, so it follows from the Lemma that $[F_1, F_2] \in \mathcal{K}(L^2\mathbb{R})$.

The other statement follows from $(1 - F_1^2) \in C_0(\mathbb{R})$, and $F_2^2 - 1$ is compact after multiplying it by a C_0-function.

□

THEOREM 9.2. *Let x and y be the Bott and Dirac morphisms. Then their intersection product $x\hat{\otimes}_{C_0(\mathbb{R})}y \in \mathrm{KK}_0(\mathbb{C}, \mathbb{C})$ is represented by the even Fredholm $\mathbb{C}$-$\mathbb{C}$-bimodule*

$$(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), F := \begin{bmatrix} 0 & -iF_2 + (1 - F_2^2)^{\frac{1}{4}} F_1 (1 - F_2^2)^{\frac{1}{4}} \\ iF_2 + (1 - F_2^2)^{\frac{1}{4}} F_1 (1 - F_2^2)^{\frac{1}{4}} & 0 \end{bmatrix}),$$

where the Hilbert space is graded with first summand even, second odd,

PROOF. Firstly,

$$\mathcal{E} = \mathcal{E}_1 \otimes_{C_0(\mathbb{R})} \mathcal{E}_2 = C_0(\mathbb{R}) \otimes_{C_0(\mathbb{R})} L^2(\mathbb{R}) \cong L^2(\mathbb{R}).$$

So the intersection product will be represented by a cycle of the form

$$(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), F)$$

where F is a suitable odd, self-adjoint operator. We need therefore to check that our proposal $\chi(D)$ for F, with D defined above, satisfies the axioms for the Kasparov product.

The operator F is self-adjoint. We can write

$$F = \begin{bmatrix} 0 & u^* \\ u & 0 \end{bmatrix},$$

where $u = iF_2 + (1 - F_2^2)^{\frac{1}{4}} F_1 (1 - F_2^2)^{\frac{1}{4}}$, which is a compact perturbation of

$$w = iF_2 + (1 - F_2)^{\frac{1}{2}} F_1,$$

and we generally work with w as its formula is shorter. We have

$$ww^* = (iF_2 + (1 - F_2)^{\frac{1}{2}} F_1)(-iF_2 + F_1(1 - F_2)^{\frac{1}{2}}) = F_2^2 + (1 - F_2^2)F_1^2 \mod \mathcal{K}$$

since, by Lemma 9.1, F_1 and F_2 commute mod compacts. If $\psi(x) = -\frac{1}{1+x^2} \in C_0(\mathbb{R})$, then again by Lemma 9.1,

$$\psi(D_1)(F_2^2 - 1) \in \mathcal{K}$$

from which we get that $(1 - F_2^2)F_1^2$ is a compact perturbation of $1 - F_2^2$. We get

$$ww^* = 1 \mod \mathcal{K}.$$

An exactly similar calculation shows that $w^*w - 1 \in \mathcal{K}$. Therefore

$$F^2 - 1 \in \mathcal{K}(L^2\mathbb{R}).$$

Next, we discuss the connection condition b). An inspection of the condition shows that the operators T_ξ correspond to multipication operators on $L^2(\mathbb{R})$ under the identification $C_0(\mathbb{R})\otimes_{C_0(\mathbb{R})} L^2(\mathbb{R})$, and so the connection condition boils down to showing that

$$\rho \cdot iF_2 - w \cdot \rho$$

is a compact operator for any $\rho \in C_c^\infty(\mathbb{R})$, where, as above,

$$w = iF_2 + (1 - F_2)^{\frac{1}{2}} F_1.$$

As ρ commutes mod $\mathcal{K}$ with F_2, this boils down to showing that

$$\rho \cdot (iF_2 - w)$$

is compact. But

$$iF_2 - w = -(1 - F_2^2)^{\frac{1}{2}} F_1,$$

and already $\rho(1 - F_2^2)^{\frac{1}{2}}$ is a compact operator, so compactness of $\rho(iF_2 - w)$ follows.

Finally, we check the positivity condition. It asserts that

$$F_1 w + w^* F_1 \geq 0 \quad \bmod \mathcal{K},$$

that is, that

$$F_1 w + w^* F_1$$

is a compact perturbation of a positive operator. But, since F_1 and F_2 commute mod compact operators,

$$F_1 w + w^* F_1 = F_1 \cdot (w + w^*) = 2F_1 \cdot (1 - F_2^2)^{\frac{1}{2}} F_1 \mod \mathcal{K}.$$

and this is clearly a positive operator.

$\square$

Bibliography

[1] P. Baum, A. Connes, N. Higson: *Classifying space for proper actions and K-theory of group C^*-algebras*, from *C^*-Algebras: 1943–1993*, San Antonio, TX, Contemp. Math.167 (1994), Amer. Math. Soc., Providence, RI, 240–291.

[2] B. Blackadar: *K-theory for Operator Algebras*, 2nd edition, MSRI Publications (1998)

[3] A. Connes: *Cyclic cohomology and the transverse fundamental class of a foliation*, Geometric methods in operator algebras (Kyoto, 1983), Pitman Res. Notes Math. Ser. 123 (1986), pp. 52–144

[4] A. Connes: *Compact metric spaces, Fredholm modules, and hyperfiniteness*, Ergodic Theory Dynam. Systems 9 (1989), no. 2, 207–220

[5] A. Connes: *Noncommutative Geometry*, Academic Press 1994

[6] A. Connes, G. Skandalis: *The longitudinal index theorem for foliations*, Publ. Res. Inst. Math. Sci. 20 (1984), no. 6, 1139–1183

[7] H. Emerson, R. Meyer: *Equivariant representable K-theory*, J. Topology. (2009), no. 2 123–156.

[8] H. Emerson, R. Meyer: *Equivariant embedding theorems and topological index maps*, Adv. Math. 225 (2010), 2840–2882

[9] H. Emerson, R. Meyer: *Bivariant K-theory via correspondences*, Adv. Math. 225 (2010), 2883–2919

[10] T. Fack, G. Skandalis, *L'analogue de l'isomorphisme de Thom pour les groups de Kasparov,* C.R. Acad. Sci. Paris (A-B) 291 no. 10 (1980), 579–581.

[11] M. Gromov: *Hyperbolic groups*, in *Essays in group theory* (Publ. MSRI 8, Springer 1987), 75–263

[12] : N. Higson, G. G. Kasparov: *E-theory and KK-theory for groups which act properly and isometrically on Hilbert space*, Invent. Math. 144 (2001), no. 1, 23–74

[13] : G. Kasparov: *Equivariant KK-theory and the Novikov Conjecture*, Invent. Math. 91 (1988), 147–201.

[14] N. Higson, J. Roe: *Analytic K-homology*, Oxford Science Publications (2000).

[15] G. Skandalis, *Some Remarks on Kasparov Theory*, Journal of Functional Analysis 56 (1984), 337-447.

[16] J.-L. Tu: *La conjecture de Baum-Connes pour les feuilletages moyennables*, K-Theory 17 (1999), no. 3, 215–264