Examples of factor groupoids: Cantor functions and iterated function systems

Glasgow Analysis Seminar

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Groupoids

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Every element x of G has an inverse x^{-1}, but there are many "units". The set of units is denoted $G^{(0)}$ and they are of the form $x^{-1}x$. Define the maps $r(x) = xx^{-1}$ and $s(x) = x^{-1}x$.

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Example. Let X be a nonempty set and $R \subseteq X \times X$ an equivalence relation. Then R is a groupoid:

$$(x, y)(y', z) = (x, z) \qquad (x, y)^{-1} = (y, x)$$

only when $y = y'$.

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$C_c(G)$ = all continuous, compactly supported functions $f : G \to \mathbb{C}$.

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To get a complete norm, represent $C_c(G)$ on a Hilbert space and take the closure to get the *reduced* C*-algebra of G, called $C_r^*(G)$.

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If X is a locally compact Hausdorff space and $R = \{(x,x) \mid x \in X\}$, then $C_r^*(R) \cong C_0(X)$.

Let X be a compact Hausdorff space and Γ a discrete group acting on X by homeomorphisms. Then $X \times \Gamma$ is a locally compact Hausdorff étale groupoid:

$$(x, g)(y, h) = (x, gh) \qquad (x, g)^{-1} = (g \cdot x, g^{-1})$$

only when $g \cdot x = y$. Moreover, $C_r^*(X \times \Gamma)$ is isomorphic to the (reduced) crossed product $C(X) \rtimes \Gamma$.

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Notable references:

Li, X. "Every classifiable simple C^*-algebra has a Cartan subalgebra". Invent. math. 219, 653–699 (2020).

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Assume:

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Obtain an inclusion $C_r^*(G') \subseteq C_r^*(G)$ via $b \mapsto b \circ \pi$ (b in $C_c(G')$)

The Cantor function

Define the map $\varphi : \{0,1\}^{\mathbb{N}} \to S^1$ by

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S has a natural étale topology generated by the basic sets

$$\gamma(p,q) = \left\{ (\{x_n\},\{y_n\}) \mid \begin{array}{c} (x_1, x_2, \ldots, x_n) = (p_1, p_2, \ldots, p_n) \\ (y_1, y_2, \ldots, y_n) = (q_1, q_2, \ldots, q_n) \\ x_k = y_k \text{ for all } k \geq n+1 \end{array} \right\}$$

where $p, q \in \{0,1\}^n$, and $C_r^*(S) \cong M_{2^\infty}$.

The Cantor function

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$$T = \{(w, z) \in S^1 \times S^1 \mid w = e^{2\pi i\theta} z \text{ for some } \theta \in \mathbb{Z}[\tfrac{1}{2}]\}$$

with basic open sets $U_{W,\theta} = \{(z, e^{2\pi i\theta} z) \mid z \in W\}$ where $W \subseteq S^1$ is open and $\theta \in \mathbb{Z}[\frac{1}{2}]$.

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T is a second-countable locally compact Hausdorff étale groupoid in the quotient topology, and $C_r^*(T) \cong B$, where B is the Bunce-Deddens algebra of type 2^∞.

$$K_0(C_r^*(T)) \cong \mathbb{Z}[\tfrac{1}{2}] \qquad K_1(C_r^*(T)) \cong \mathbb{Z}$$

and $C_r^*(T)$ has a unique tracial state.

Bratteli diagrams

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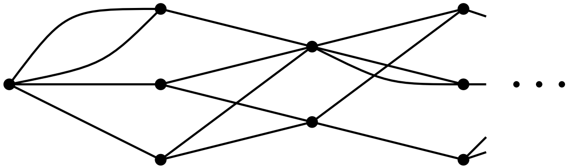

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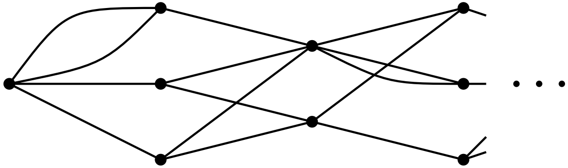

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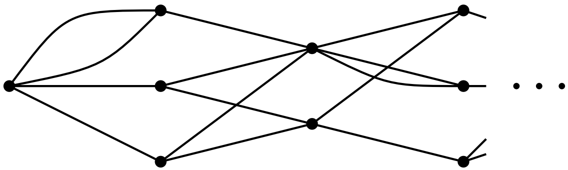

The *infinite path space* X_E of (V, E) is a totally disconnected compact metric space.

Tail-equivalence $R_E \subseteq X_E \times X_E$ has an étale topology in which $C_r^*(R_E)$ is an AF-algebra.

The space X_ξ

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Two *graph embeddings* $\xi^0, \xi^1 : (W, F) \to (V, E)$ with $\xi^0|_W = \xi^1|_W$ and $\xi^0(F) \cap \xi^1(F) = \varnothing$.

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Equivalence relation $\sim_\xi$ on X_E:

$$(x_1, x_2, \ldots, x_{n_0-1}, x_{n_0}, \xi^0(z_{n_0+1}), \xi^0(z_{n_0+2}), \ldots) \tag{1}$$

$$\sim_\xi (x_1, x_2, \ldots, x_{n_0-1}, x'_{n_0}, \xi^1(z_{n_0+1}), \xi^1(z_{n_0+2}), \ldots) \tag{2}$$

where $x_{n_0} = x'_{n_0}$ if $x_{n_0} \notin \xi^0(F) \cup \xi^1(F)$, and $x_{n_0} = \xi^1(f)$ if $x'_{n_0} = \xi^0(f)$.

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Denote $X_\xi := X_E / \sim_\xi$ and $\rho : X_E \to X_\xi$ the quotient map.

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Facts:

1. X_ξ is a second-countable compact metrizable space,
2. the covering dimension of X_ξ is 1,
3. each connected component is either a single point or homeomorphic to S^1.

Proposition. If $E = \xi^0(F) \cup \xi^1(F)$, then X_ξ is homeomorphic to $X_F \times S^1$.

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Example. Let (V, E) have one vertex and three edges at each level. Identify X_E with $\{0, 1, 2\}^{\mathbb{N}}$.

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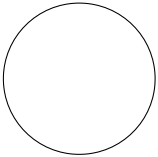
There is a nested sequence $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_E$ such that

$$X_E = \overline{\bigcup_{n=1}^{\infty} X_n}$$

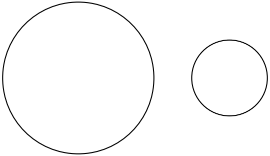
and each $\rho(X_n)$ is a disjoint union of finitely many circles.

Examples



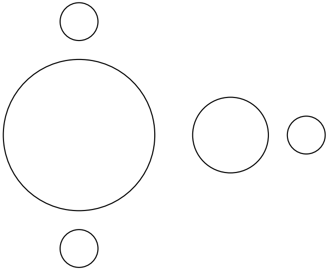
$\rho(X_1)$

Examples



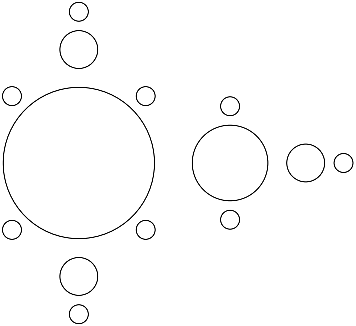
$\rho(X_2)$

Examples

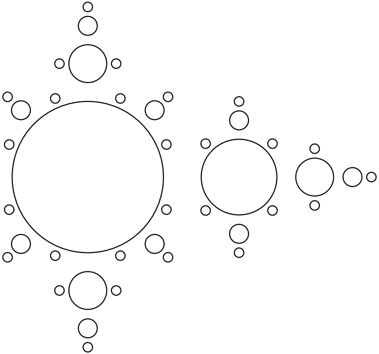


$\rho(X_3)$

Examples

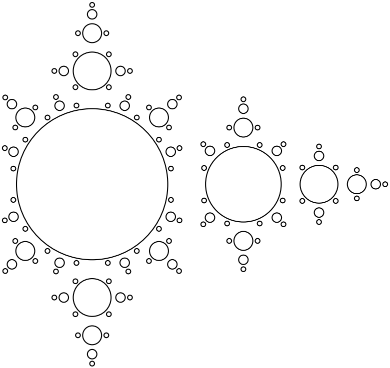


$\rho(X_4)$



$\rho(X_5)$

Examples



$\rho(X_6)$

The groupoid R_ξ

Let $R_\xi = \rho \times \rho(R_E)$.

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We want to analyze the K-theory of $C_r^*(R_\xi) \subseteq C_r^*(R_E)$.

K-theory

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K_1(A) \arrow[r] & K_0(A';A) \arrow[r] & K_0(A') \arrow[d, "\iota_*"] \\
K_1(A') \arrow[u, "\iota_*"] & K_1(A';A) \arrow[l] & K_0(A) \arrow[l]
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Excision: (Putnam, 2020)

$$K_*(C_r^*(G'); C_r^*(G)) \cong K_*(C_r^*(H'); C_r^*(H))$$

where $H \subseteq G$ and $H' \subseteq G'$ are where π is not one-to-one.

K-theory

Theorem (H.)

We have $K_0(C_r^*(R_\xi)) \cong K_0(C_r^*(R_E))$ and $K_1(C_r^*(R_\xi)) \cong K_0(C_r^*(R_F))$. If R_E is minimal, then $C_r^*(R_\xi)$ is classifiable.

K-theory

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Through the set-up $\xi^0, \xi^1 : (W, F) \to (V, E)$, we can prescribe $K_*(C_r^*(R_\xi))$.

Corollary

If G_0 is a simple acyclic dimension group and G_1 is a countable torsion free abelian group, we can find R_ξ such that $K_0(C_r^*(R_\xi)) \cong G_0$ (with order) and $K_1(C_r^*(R_\xi)) \cong G_1$.

Iterated function systems

An *iterated function system* $(X, \mathcal{F})$ (abbreviated IFS) is a complete metric space X with a finite set $\mathcal{F}$ of functions $f_j : X \to X$ for $j = 1, 2, \ldots, n$.

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If the IFS is hyberbolic (every f_j is a contraction) then there is a unique compact subset $C \subseteq X$ such that

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We will also assume every function f_j is injective.

Iterated function systems

Example. Take $X = \mathbb{R}$ and $f_1(x) = \frac{1}{2}x$ and $f_2(x) = \frac{1}{2}x + \frac{1}{2}$. The attractor is $C = [0, 1]$.

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Example. Take $X = \mathbb{R}^2$ and

$$f_1(x) = \frac{1}{2}x \qquad f_2(x) = \frac{1}{2}x + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \qquad f_3(x) = \frac{1}{2}x + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}$$

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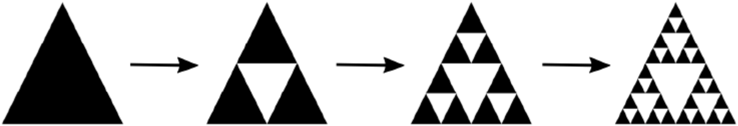
The attractor C is the Sierpiński triangle.



Nonhomogeneous extensions

Theorem (Deeley, Putnam, Strung. 2019)

Let $(C, \mathcal{F})$ be a compact hyperbolic IFS and (X, φ) a Cantor minimal system. There exists a minimal extension $(\tilde{X}, \tilde{\varphi})$ of (X, φ) with factor map $\tilde{\pi} : \tilde{X} \to X$ such that $\tilde{\pi}^{-1}(x)$ is either a single point or homeomorphic to C (both possibilities always occur).

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Idea: (X, φ) is topologically conjugate to a Bratteli-Vershik system on a Bratteli diagram (V, E). To each edge e in E, assign a function f_e in $\mathcal{F} \cup \{\mathrm{id}_C\}$.

$$\tilde{X}_n = \{(x, c) \in X_E \times C \mid c \in f_{x_1} \circ f_{x_2} \circ \cdots \circ f_{x_n}(C)\}$$

Set $\tilde{X} = \bigcap_{n=1}^{\infty} \tilde{X}_n$ and $\tilde{\pi}(x, c) = x$.

Nonhomogeneous extensions

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We get a surjective groupoid homomorphism $\tilde{\pi} \times \mathrm{id}_{\mathbb{Z}} : \tilde{X} \times \mathbb{Z} \to X_E \times \mathbb{Z}$. Using the excision theorem and the Pimsner-Voiculescu exact sequence, we get an exact sequence

$$0 \to K_0(C_r^*(R_E)) \to K_0(C_r^*(\tilde{X} \times \mathbb{Z})) \to K_0(C_r^*(R_E^C)) \otimes (K^0(C)/\mathbb{Z}) \to 0$$

and an exact sequence

$$0 \to \mathbb{Z} \to K_1(C_r^*(\tilde{X} \times \mathbb{Z})) \to K_0(C_r^*(R_E^C)) \otimes K^{-1}(C) \to 0$$

$R_E^C \subseteq R_E$ are the tail-equivalent paths that are "eventually id_C".

Nonhomogeneous extensions

If there is only one path in X_E with $f_{x_n} = \mathrm{id}_C$ for all n, then we get

$$K_0(C_r^*(\tilde{X} \times \mathbb{Z}))/K_0(C_r^*(R_E)) \cong K^0(C)/\mathbb{Z}$$

and

$$K_1(C_r^*(\tilde{X} \times \mathbb{Z}))/\mathbb{Z} \cong K^{-1}(C)$$

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1. Possibilities for $K^*(C)$?
2. When do the sequences split?

Thank you!

References

1. Deeley, R.J.; Putnam, I.F.; Strung, K.R. "Non-homogeneous extensions of Cantor minimal systems". to appear, Proc. A.M.S.
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