CSC 446: Assignment #2

Due on Wed, Jan. 29, 2014

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Problem 1

(a)

$$\begin{aligned}
H(Y|X) &= -\sum_{x,y} p(x,y) log P(y|x) \\
&= -\sum_{x,y} p(x,y) log \frac{P(x,y)}{P(x)} \\
&= -\sum_{x,y} p(x,y)(log P(x,y) - log P(x)) \\
&= -\sum_{x,y} p(x,y) log P(x,y) - -\sum_{x,y} p(x,y) log P(x) \\
&= -\sum_{x,y} p(x,y) log P(x,y) - -\sum_{x} p(x) log P(x) \\
&= H(X,Y) - H(X)
\end{aligned}$$

Therefore:

$$H(X,Y) = H(X) + H(Y|X)$$

(b)

$$\begin{aligned}
H(X) - H(X|Y) &= -\sum_{x} p(x) log P(x) + \sum_{x,y} p(x,y) log \frac{P(x,y)}{P(y)} \\
&= \sum_{x} p(x) log \frac{1}{P(x)} + \sum_{x,y} p(x,y) log \frac{P(x,y)}{P(y)} \\
&= \sum_{x,y} p(x,y) log \frac{1}{P(x)} + \sum_{x,y} p(x,y) log \frac{P(x,y)}{P(y)} \\
&= \sum_{x,y} p(x,y) log \frac{P(x,y)}{P(x)P(y)} \\
&= I(X;Y)
\end{aligned}$$

Therefore:

$$I(X;Y) = H(X) - H(X|Y)$$

(c)

From (a) we know that

$$H(X,Y) = H(X) + H(Y|X) \tag{1}$$

$$H(Y,X) = H(Y) + H(X|Y) \tag{2}$$

(1) - (2), we have

$$0 = H(X) - H(Y) + H(Y|Z) - H(X|Y)$$

with a a little bit of rearrangement:

$$H(X) - H(X|Y) = H(Y) - H(Y|X)$$

Problem 1 [(c)] continued on next page...

(d)

From (a) we know

$$H(X,Y) = H(X) + H(Y|X)$$

$$H(X) + H(Y) \geq H(X) + H(Y|X)$$

$$H(Y) \geq H(Y|X)$$

$$I(Y;X) \geq 0 \text{ which is always true}$$

Therefore we have

$$H(X) + H(Y) \geq H(X,Y)$$

Problem 2

(a)

$$L(\vec{x},\lambda) = \|\vec{x}\|^2 + \lambda(\sum_i \vec{x} - 1)$$

which is equivalent to

$$\begin{cases} \frac{\partial L}{\partial \vec{x}} = 0 \\ \sum_i x_i - 1 = 0 \end{cases} \tag{3}$$

which is equivalent to

$$\begin{cases} \frac{\partial L}{\partial x_i} = 0 \\ \sum_i x_i - 1 = 0 \end{cases} \tag{4}$$

which is equivalent to

$$\begin{cases} x_i = \frac{-\lambda}{2} \\ \sum_i x_i - 1 = 0 \end{cases} \tag{5}$$

Therefore

$$\begin{cases} \lambda = \frac{-2}{n} \\ x_i = \frac{1}{n} \end{cases} \tag{6}$$

(b)

$$L(\vec{x},\lambda) = \sum_i x_i + \lambda \left(\sum_i x_i^2 - 1 \right)$$

which is equivalent to

$$\begin{cases} \frac{\partial L}{\partial \vec{x}} = 0 \\ \sum_i x_i^2 - 1 = 0 \end{cases} \tag{7}$$

which is equivalent to

$$\begin{cases} 1 + 2\lambda x_i = 0 \\ \sum_i x_i^2 - 1 = 0 \end{cases} \tag{8}$$

solve (8), we get:

$$\begin{cases} \lambda = \frac{\sqrt{n}}{2} \\ x_i = \frac{-1}{\sqrt{n}} \end{cases} \tag{9}$$

(c)

$$L(Q,\lambda) = \sum_i \ln \left(\frac{P(i)}{Q(i)} \right) P(i) - \lambda \left(\sum_i Q(i) - 1 \right)$$

which is equivalent to

$$\begin{cases} \frac{\partial L}{\partial Q(i)} = 0 \\ \sum_i Q(i) - 1 = 0 \end{cases} \tag{10}$$

which is equivalent to

$$\begin{cases} Q(i) = \frac{-P(i)}{\lambda} \\ \sum_i Q(i) - 1 = 0 \end{cases} \tag{11}$$

solve (11), we get:

$$\begin{cases} \lambda = -1 \\ Q(i) = P(i) \end{cases} \tag{12}$$

(d)

$$L(Q,\lambda) = -\sum_k Q(k)\ln Q(k) + \lambda\left(\sum_k Q(k) - 1\right)$$

which is equivalent to

$$\begin{cases} \frac{\partial L}{\partial Q(k)} = 0 \\ \sum_i Q(k) - 1 = 0 \end{cases} \tag{13}$$

which is equivalent to

$$\begin{cases} Q(k) = e^{\lambda - 1} \\ \sum_k Q(k) - 1 = 0 \end{cases} \tag{14}$$

solve (14), we get:

$$\begin{cases} \lambda = 1 - \ln K \\ Q(k) = \frac{1}{K} \end{cases} \tag{15}$$