CSC 446: Assignment #1

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Problem 1

$$E[XY] = \sum_{x=-\infty}^{+\infty} \sum_{y=-\infty}^{+\infty} xyp_{X,Y}(x,y)$$

bacause X and Y are independent,

$$p_{X,Y}(x,y) = p_X(x)p_Y(y)$$

therefore,

$$\begin{aligned}\sum_{x=-\infty}^{+\infty} \sum_{y=-\infty}^{+\infty} xyp_{X,Y}(x,y) &= \sum_{x=-\infty}^{+\infty} \sum_{y=-\infty}^{+\infty} xyp_X(x)p_Y(y) \\ &= \sum_{x=-\infty}^{+\infty} xp_X(x) \sum_{y=-\infty}^{+\infty} yp_Y(y) \\ &= E[X]E[Y]\end{aligned}$$

Problem 2

$$\begin{aligned}P(|X-\mu|^2 \geq k^2\sigma^2) &= P((X-\mu)^2 \geq k^2\sigma^2) \\ &\leq \frac{E[(X-\mu)^2]}{k^2\sigma^2} \\ &= \frac{\sigma^2}{k^2\sigma^2} \\ &= \frac{1}{k^2}\end{aligned}$$

Problem 3

(a)

$$E[R] = E[r_1 \cdot r_2 \cdot r_3 \cdot \ldots \cdot r_N]$$

because r_i is i.i.d,

$$E[R] = (E[r_1])^N \tag{1}$$

$$\begin{aligned}E[r_1] &= \sum_{x=-\infty}^{+\infty} xp_X(x) \\ &= \frac{1}{2} \times \frac{1}{2} + 2 \times \frac{1}{2} \\ &= \frac{5}{4}\end{aligned}$$

Therefore because of (1)

$$E[R] = \left(\frac{5}{4}\right)^N$$

Problem 3 continued on next page...

(b)

X could be seen as doing the same experiment for N times. Therefor the relationship between X and x_i should be similar to that between Binomial and Bernoulli.

$$\begin{aligned} E[x_1] &= \sum_{x=-\infty}^{+\infty} x p_{X_1}(x) \\ &= log\frac{1}{2} \times \frac{1}{2} + log2 \times \frac{1}{2} \\ &= 0 \end{aligned}$$

Hence,

$$E[X] = N \times E[x_1] = 0 \tag{2}$$

(c)

$$\begin{aligned} Var[X_i] &= Var[X_1] = E[X_1^2] - (E[X_1])^2 \\ E[X_1^2] &= \sum_{x=-\infty}^{+\infty} x_1^2 p_{X_1}(x) - (E[X_1])^2 \\ &= \left(log\frac{1}{2}\right)^2 \times \frac{1}{2} + (log2)^2 \times \frac{1}{2} \\ &= log^2 2 \end{aligned}$$

Because $E[X_1] = 0$, so

$$Var[X_i] = E[X_1^2] = log^2 2$$

(d)

Because x_i are i.i.d.
So,

$$Var[X] = NVar[X_1] = Nlog^2 2$$

Problem 3 continued on next page. . .

(e)

We want to know

$$P\left(X > \left(\frac{9}{8}\right)^N\right)$$

We already know from (2) that

$$\mu = E[X] = 0$$

$$P\left(X > \left(\frac{9}{8}\right)^N\right) = P\left(X - \mu > \left(\frac{9}{8}\right)^N\right)$$

Because $X > 0$, $\mu = 0$ so $X - \mu = |X - \mu|$ So

$$\begin{aligned}
P\left(X > \left(\frac{9}{8}\right)^N\right) &= P\left(|X - \mu| > \left(\frac{9}{8}\right)^N\right) \\
&\leq \frac{E((X-\mu)^2)}{\left(\frac{9}{8}\right)^{2N}} \\
&= \frac{Nlog^2 2}{\left(\frac{9}{8}\right)^{2N}} \\
&= \lim_{N \to +\infty} \frac{Nlog^2 2}{\left(\frac{9}{8}\right)^{2N}} \\
&= 0
\end{aligned}$$

Problem 4

(a)

$$r_i = \begin{cases} \frac{1}{4} & \text{with prob } \frac{1}{4} \\ \frac{5}{4} & \text{with prob } \frac{1}{2} \\ 2 & \text{with prob } \frac{1}{4} \end{cases}$$

(b)

$$\begin{aligned}
E[r_1] &= \sum_{x=-\infty}^{+\infty} x p_X(r) \\
&= \frac{1}{2} \times \frac{1}{4} + \frac{5}{4} \times \frac{1}{2} + 2 \times \frac{1}{4} \\
&= \frac{5}{4}
\end{aligned}$$

For similar reason as in problem #3,

$$E[R] = \left(\frac{5}{4}\right)^N$$

Problem 4 continued on next page...

(c)

$$
\begin{aligned}
E(x_1) &= log\left(\frac{1}{2}\right) \times \frac{1}{4} + log\left(\frac{5}{4}\right) \times \frac{1}{2} + log\,(2) \times \frac{1}{4} \\
&= \frac{1}{2} log\left(\frac{5}{4}\right)
\end{aligned}
$$

$$
\begin{aligned}
E(X) &= N \times \frac{1}{2} log\left(\frac{5}{4}\right) \\
&= \frac{N}{2} log\left(\frac{5}{4}\right)
\end{aligned}
$$

(d)

$$
Var[X_i] = Var[X_1] = E[X_1^2] - (E[X_1])^2
$$

$$
\begin{aligned}
E[X_1^2] &= \sum_{x_1=-\infty}^{+\infty} x_1^2 p_{X_1}(x_1) \\
&= log^2\left(\frac{1}{2}\right) \times \frac{1}{4} + log^2\left(\frac{5}{4}\right) \times \frac{1}{2} + log^2\,(2) \times \frac{1}{4} \\
&= \frac{1}{2} log^2\,(2) + \frac{1}{2} log^2\left(\frac{5}{4}\right)
\end{aligned}
$$

$$
\begin{aligned}
Var[X_i] &= \frac{1}{2} log^2\,(2) + \frac{1}{2} log^2\left(\frac{5}{4}\right) - \left(\frac{1}{2} log\left(\frac{5}{4}\right)\right)^2 \\
&= \frac{1}{2} log^2\,(2) + \frac{1}{4} log^2\left(\frac{5}{4}\right)
\end{aligned}
$$

(e)

$$
Var[X] = N\left(\frac{1}{2} log^2\,(2) + \frac{1}{4} log^2\left(\frac{5}{4}\right)\right)
$$

(f)

$$P\left(X<\left(\frac{9}{8}\right)^N\right)=1-P\left(X\geq\left(\frac{9}{8}\right)^N\right)$$

From the right hand side part,

$$\begin{aligned}
P\left(X\geq\left(\frac{9}{8}\right)^N\right) &= P\left(|X-\mu|^2\geq\left(\left(\frac{9}{8}\right)^N-\mu\right)^2\right)\\
&\leq \frac{E\left(|X-\mu|^2\right)=Var\left(X\right)}{\left(\left(\frac{9}{8}\right)-\mu\right)^2}\\
&= \frac{N\left(\frac{1}{2}log^2\left(2\right)+\frac{1}{4}log^2\left(\frac{5}{4}\right)\right)}{\left(\left(\frac{9}{8}\right)^N-\frac{N}{2}log\left(\frac{5}{4}\right)\right)^2}\\
&= \frac{\frac{1}{2}log^2\left(2\right)+\frac{1}{4}log^2\left(\frac{5}{4}\right)}{\frac{\left(\frac{9}{8}\right)^N}{N}+\frac{\frac{1}{4}log^2\left(\frac{5}{4}\right)}{\left(\frac{9}{8}\right)^N}-log^2\left(\frac{5}{4}\right)}
\end{aligned}$$

$$\lim_{N\to+\infty}\frac{\frac{1}{2}log^2\left(2\right)+\frac{1}{4}log^2\left(\frac{5}{4}\right)}{\frac{\left(\frac{9}{8}\right)^N}{N}+\frac{\frac{1}{4}log^2\left(\frac{5}{4}\right)}{\left(\frac{9}{8}\right)^N}-log^2\left(\frac{5}{4}\right)}=0$$

So when $N\to+\infty$

$$P\left(X\geq\left(\frac{9}{8}\right)^N\right)=P\left(|X-\mu|^2\geq\left(\left(\frac{9}{8}\right)^N-\mu\right)^2\right)=0$$

So

$$P\left(X<\left(\frac{9}{8}\right)^N\right)=1$$

(g)

For these two methods, their $E[R]$ are the same. Bue the variance of problem #4 is less than that of problem #3.

$$\begin{aligned}
Nlog^2 2-N\left(\frac{1}{2}log^2 2+\frac{1}{4}log^2(\frac{5}{4})\right) &= \frac{N}{2}log^2 2-\frac{N}{4}log^2(\frac{5}{4})\\
&\geq \frac{N}{4}log^2 2-\frac{N}{4}log^2(\frac{5}{4})\\
&> 0
\end{aligned}$$

This means if we choose the second method, we are more likely to be close to expectation. Therefore I will choose the latter one.