$$X = \begin{array}{c} \\ 1 \\ 2 \\ \cdot \\ k \end{array} \begin{array}{c} 1\ 2\ \ldots \qquad w \\ \left(\begin{array}{c} AACCAGT \\ \\ \\ \\ \end{array} \right) \end{array} \begin{array}{l} \\ \leftarrow z_1 \\ z_2 \\ \cdot \\ z_n \end{array}$$

$z_i \in \{0, 1\}$

$P(z_i = 1) = \lambda$

$$\Theta = 4 \begin{array}{c} 1 \ldots \qquad w \\ \left(\square\ \square\ \square \qquad \right) \end{array} \qquad z_1$$

$$\Theta^b = \square \qquad w\ w\ \ldots\ \underset{w}{w}$$

Algorithm 1 The General Expectation-Minimization (EM) Algorithm.

Input: Observations $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n), \mathbf{x}_i \in \mathbb{R}^d$.

$z_1 \to x_1 \quad z_2 \to x_2 \quad \ldots \quad z_n \to x_n$

1: $t = 0$. Initialize (randomly) $\boldsymbol{\theta}^{(t)}$

2: **E(xpectation) step**. Compute $Q_i^{(t)}(j) = P(Z_i = j | \mathbf{X}; \boldsymbol{\theta}^{(t)})$ $= x_i$

3: **M(aximization) step**.

4: M-Step 1: Compute

$$\mathcal{Q}(\boldsymbol{\theta}, \boldsymbol{\theta}^{(t)}) = \sum_{i=1}^{n} \sum_{j=1}^{M} Q_i^{(t)}(j) \log \underbrace{P(\mathbf{x}_i, Z_i = j; \boldsymbol{\theta})}_{\cancel{Q_i^{(t)}(j)}}$$

5: M-Step 2: Find

$$\boldsymbol{\theta}^{(t+1)} = \underbrace{\arg\max_{\boldsymbol{\theta}} \mathcal{Q}(\boldsymbol{\theta}, \boldsymbol{\theta}^{(t)})}.$$

6: **If** converged **then** stop; **else** Goto 2.

$\Theta^t = (\Theta^t, \Theta^{b,t})$

$$Q_i^{(t)}(0) = P(Z_i = 0 \mid x_i, \Theta^t, \Theta^{b,t}) = \frac{P(Z_i = 0, x_1, \Theta^t)}{C}$$

$$= \frac{P(x_i \mid Z_i = 0, \Theta^t)\, P(Z_i = 0)}{C} = \frac{1}{C}(1-\lambda)\prod_{j=1}^{W} \Theta^{b,t}_{x_{ij}}$$

$$Q_i^t(1) = \frac{P(x_i \mid Z_i = 1, \Theta^t)\, P(Z_i = 1)}{C} = \frac{1}{C}\lambda \prod_{j=1}^{W} \Theta^t_{x_{ij},j}$$

$$\Theta = (\Theta, \Theta^b)$$

$Z_i = 1 \to$ MOTIF

$$Q(\Theta, \Theta^t) = \sum_{i=1}^{k} \sum_{j=0}^{1} Q_i^{(t)}(j) \cdot \log P(x_i, Z_i = j, \Theta)$$

$$= \sum_{i=1}^{k} \Big[\underbrace{Q_i^{(t)}(0) \log \underbrace{P(x_i, Z_i = 0, \Theta)}_{\prod_{j=1}^{W} \Theta^b_{x_{ij}} \cdot (1-\lambda)}}_{Q_1} + \underbrace{Q_i^t(1) \log \underbrace{P(x_i, Z_i = 1, \Theta)}_{\prod_{j=1}^{W} \Theta_{x_{ij},j} \cdot \lambda}}_{Q_2} \Big]$$

$\cancel{\partial\Theta}$

$$\Theta^{b,t+1} = \arg\max_{\Theta^b} Q(\Theta, \Theta^t) \quad \text{s.t.} \quad \Theta^b_1 + \Theta^b_2 + \ldots \Theta^b_4 = 1$$

$$L(\Theta^b) = Q_1(\Theta^b) - \lambda\, g(\Theta^b) \qquad g(\Theta^b) = \underbrace{\Theta_1^b + \ldots \Theta_4^b - 1}$$

$$\lambda(\ldots\ldots 4 - 1)$$

$$\frac{\partial L(\Theta^b)}{\partial \Theta^b_1} = \sum_{l=1}^{k} Q_l^{(t)}(c) \frac{\partial}{\partial \Theta^b_1} \left[\sum_{j=1}^{W} \log \Theta^b_{x_{lj}} \right] - \lambda$$

$$= \sum_{l=1}^{k} Q_l^{(t)}(c) \sum_{j=1}^{W} \cdot \frac{1}{\Theta^b_{x_{lj}}} \mathbb{1}(x_{lj}=1) - \lambda = 0$$

$$\Theta^{t+1,b}_{x_1} = \frac{\sum_{l=1}^{k} Q_l^{(t)}(c) \cdot \#\{j : x_{lj}=1\}}{\lambda}$$

$$\vdots$$

$$\Theta^{t+1,b}_{x_3} = \frac{\sum_l \dots \quad \#\{j : x_{lj}=3\}}{\lambda}$$

$$\lambda = \sum_{r=1}^{4} W \cdot \sum_{l=1}^{k} Q_l^{(t)}(c)$$

$$\Theta^{(t)} = (\Theta_1^t, \Theta^{b,t}, \lambda^t)$$

.