AMAT 503: CWT notes

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1 The continuous wavelet transform

The continuous wavelet transform can be motivated by a formula in multi-resolution analysis, where we take a function (or signal) $f(t)$ on the real line and compute various discrete wavelet coefficients from the inner products

$$c_{j,k} = \langle f, \psi_{j,k} \rangle = \int_{\mathbb{R}} f(t)\overline{\psi_{j,k}(t)}\, dt, \tag{1}$$

where the functions $\psi_{j,k}$ are scaled, translated versions of a given wavelet $\psi(t)$.

Writing this out in full, we have the formula

$$c_{j,k} = \langle f, \psi_{j,k} \rangle = \frac{1}{2^{j/2}} \int_{\mathbb{R}} f(t)\overline{\psi(\frac{t}{2^j} - k)}\, dt, \tag{2}$$

where the indices j, k are discrete integers.

The continuous wavelet transform is obtained by replacing the indices j, k with continuous parameters, traditionally labeled $a \sim 2^j$ (for scale) and $b \sim 2^j k$ for translation. This suggests defining a new transform with parameters a, b to obtain coefficients

$$W(a, b) = \frac{1}{|a|^{1/2}} \int_{\mathbb{R}} f(t)\overline{\psi(\frac{t-b}{a})}\, dt, \tag{3}$$

where the parameters a, b can be any real numbers. This function $W(a, b)$ is called the *continuous wavelet transform* of the function f with respect to the wavelet ψ. You should think of W as a function that represents signal f in a new, transformed domain. The wavelet ψ is usually fixed for a certain application, while we examine a wide variety of different signals f.

Before proceeding, we should ask: what does this transform look like? Figure 1 shows some examples – CWT transforms of a constant tone, a sequence of three tones, and a linear sweep. For comparison, Figure 2 shows the spectrogram (Gabor transform) of the same three examples. Notice in the CWT, the scale parameter runs from 1 to 64, where 64 corresponds to large scale (low frequency) while 1 corresponds to short scale (high frequency). Meanwhile, the Gabor transform shows an exact frequency measure on the vertical scale, indicated in Hertz. We can read directly in the 3rd image that the linear sweep starts at zero Hertz and increases to 880 Herts. So, while the CWT is essentially a time-frequency display, with time along the horizontal axis, and inverse frequency on the vertical axis, the units of "scale" can be a bit hard to interpret.

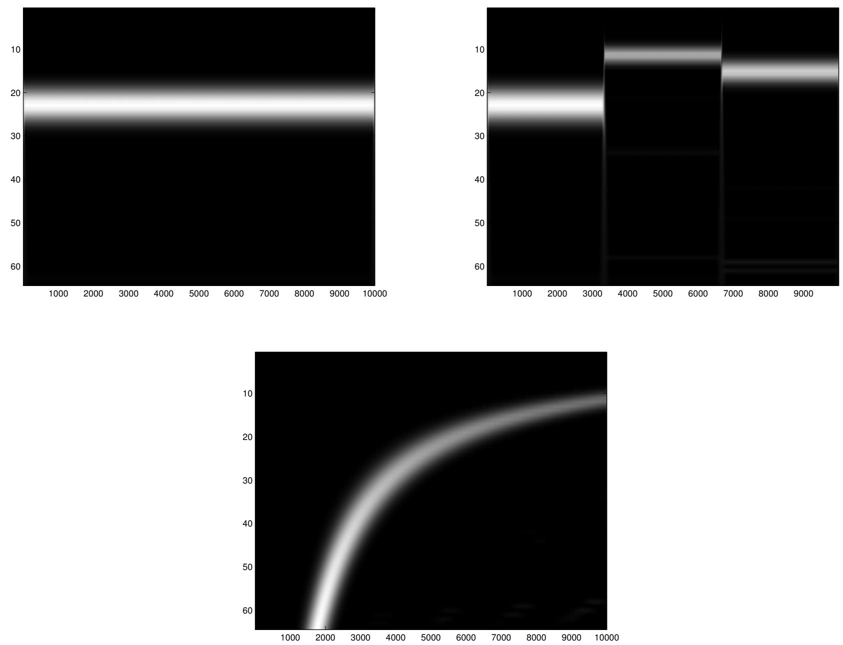

Figure 1: Continuous wavelet transforms: tone, three tones, linear sweep.

1.1 Admissibility condition

To define the continuous wavelet transform, at each a, b we need to know that we can take the dot product of the functions f and ψ. This is easily assured if we work with square-integrable functions, thus we usually assume that both the signal f and

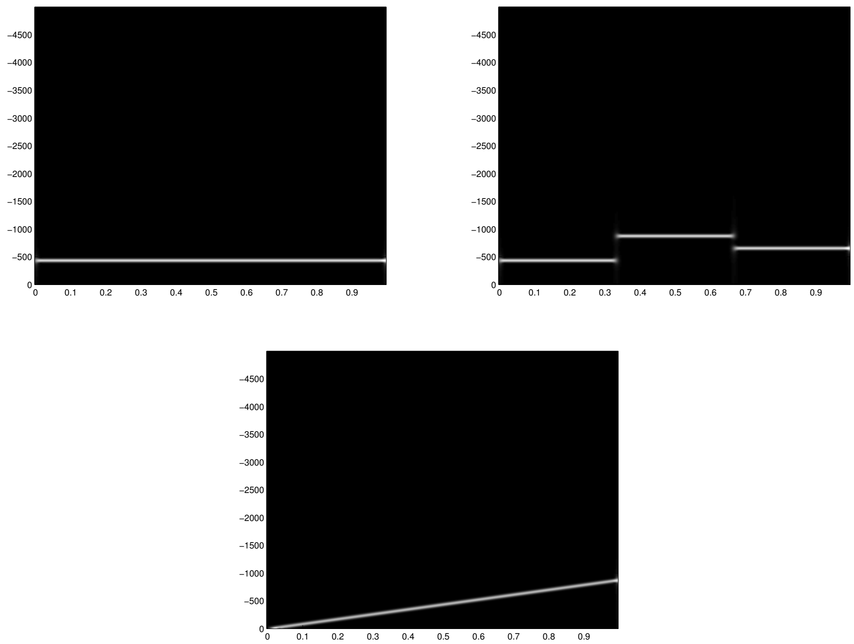

Figure 2: Gabor transforms: tone, three tones, linear sweep

the wavelet ψ have finite energy. That is, these two integrals

$$\int_R |f(t)|^2\,dt, \int_R |\psi(t)|^2\,dt \tag{4}$$

must be finite.

However, we need a bit more. To ensure the transform is invertible, we require that the constant

$$C_\psi = \int_{\mathbb{R}} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} d\omega \tag{5}$$

is finite and non-zero, where $\widehat{\psi}$ is the usual Fourier transform of wavelet ψ. In particular, the $1/\omega$ singularity must be cancelled by a zero of the Fourier transform of $\widehat{\psi}$ at $\omega = 0$. This implies

$$\widehat{\psi}(0) = \int_R \psi(t)\,dt = 0, \tag{6}$$

from which we conclude the wavelet ψ has both negative and positive parts, which cancel out on average. Informally, we see that ψ must "wiggle" above and below zero, so it really does have a wave-like behaviour.

1.2 Preserving norm

Once we have the admissibility condition, we can conclude that the L^2 norm of the function f is preserved under the wavelet transform, subject to a weighting condition:

$$\|f\|^2 = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|W(a,b)|^2}{a^2} \, da \, db. \tag{7}$$

This simply says the integral of $|f|^2$ is proportional to the weighted integral of $|W|^2$, where the weight is just $1/a^2$.

1.3 Inverse formula

We can recover the function f from its wavelet transform using the following integral

$$f = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} W(a,b) \psi_{a,b} \, \frac{da \, db}{a^2}, \tag{8}$$

which says that f is a superposition of the dilated, translated wavelets $\psi_{a,b}$ defined as

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi(\frac{t-b}{a}). \tag{9}$$

More fully, we expand the functions $\psi_{a,b}$ in the integral and write

$$f(t) = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} W(a,b) \frac{1}{\sqrt{|a|}} \psi(\frac{t-b}{a}) \, \frac{da \, db}{a^2}, \tag{10}$$

where the convergence of the integral is understood in the weak sense (if you know what that means).

1.4 Proving the inverse formula - Resolution of the identity

It is surprisingly easy to prove the norm preservation and inverse formula for the continuous wavelet transform, by making good use of our Fourier theory.

The outline of the proof of norm preservation is as follows:

- write the wavelet transform as a convolution in time;
- take a Fourier transform w.r.t. parameter b, to convert the convolution in the time domain to a product in the frequency domain;
- the Fourier transform of a dilated wavelet, is a scaled, dilated version of the transform of the original wavelet;
- then we just compute norms.

Let first compute the Fourier transform of the dilated wavelet $\psi_a(t) = \psi(\frac{t}{a})$:

$$\widehat{\psi_a}(\omega) = \int_{\mathbb{R}} \psi(\frac{t}{a}) e^{-2\pi i t\omega}\, dt = a \int_{\mathbb{R}} \psi(t') e^{-2\pi i t' a\omega}\, dt' = |a| \widehat{\psi}(a\omega), \tag{11}$$

using the change of variables $t'a = t$ in the first integral. (If you're not sure why the absolute value appears, try a few examples. For instance, take $\psi(t)$ to be a Gaussian. Its Fourier transform is also a Gaussian – dilating it by $\pm a$ will not change the fact that the Gaussian and its FT are both positive functions.)

Fix a and write $W_a(b) = W(a, b)$, so we can write the wavelet transform from Equation 3 in the form

$$W_a(b) = \frac{1}{|a|^{1/2}} \int_{\mathbb{R}} f(t) \overline{\psi_{-a}(b - t)}\, dt \tag{12}$$

which we recognize as a convolution of f with ψ_{-a}. Taking a Fourier transform converts the convolution to a product in the frequency domain, so we obtain

$$\widehat{W_a}(\omega) = \frac{1}{|a|^{1/2}} \widehat{f}(\omega) \widehat{\overline{\phi_{-a}}}(\omega) = |a|^{1/2} \widehat{f}(\omega) \overline{\widehat{\phi}(a\omega)}, \tag{13}$$

where we carefully swapped the FT and complex conjugate on ψ.

Square the functions and integrate with the weight da/a^2 give the identity

$$\int_{\mathbb{R}} |\widehat{W_a}(\omega)|^2 \frac{da}{a^2} = |\widehat{f}(\omega)|^2 \int_{\mathbb{R}} |\widehat{\phi}(a\omega)|^2 \frac{da}{|a|} = |\widehat{f}(\omega)|^2 \int_{\mathbb{R}} |\widehat{\phi}(\omega')|^2 \frac{d\omega'}{|\omega'|} = |\widehat{f}(\omega)|^2 C_\psi, \tag{14}$$

where we obtain the last integral from a change of variables $\omega' = aw$, $dw'/w' = da/a$. In other words, we have

$$\int_{\mathbb{R}} |\widehat{W_a}(\omega)|^2 \frac{da}{a^2} = C_\psi |\widehat{f}(\omega)|^2. \tag{15}$$

Now the rest follows from the fact the Fourier transform preserves energy, so

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} |W(a,b)|^2 \frac{da\,db}{a^2} &= \int_{\mathbb{R}}\int_{\mathbb{R}} |W_a(b)|^2 \frac{da\,db}{a^2} = \int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{W_a}(\omega)|^2 \frac{da\,d\omega}{a^2} \\ &= C_\psi \int_{\mathbb{R}} |\widehat{f}(\omega)|^2\,d\omega = C_\psi \int_{\mathbb{R}} |f(t)|^2\,dt,\end{aligned}$$

where again the last equality follows from the FT preserving the norm of $f(t)$. In other words, we have an equality of L^2 norms, with

$$\|W\|^2 = C_\psi \|f\|^2, \tag{16}$$

where the squared norm on the function W is a weighted squared norm.

This completes the proof of the energy-preserving property of the CWT.

To derive the inverse formula, we define a function $g(t)$ as a linear superposition of the wavelets $\psi_{a,b}$, with

$$g(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} W(a,b)\phi_{a,b}(t)\frac{da\,db}{a^2}. \tag{17}$$

Again writing $W_a(b) = W(a,b)$ and $\psi_a(t) = \psi(\frac{t}{a})$, we have

$$g(t) = \int_{\mathbb{R}}\int_{R} W_a(b)\psi_a(t-b)\frac{da\,db}{|a|^{5/2}} = \int_{\mathbb{R}} (W_a * \psi_a)(t)\frac{da}{|a|^{5/2}}, \tag{18}$$

where we note the integral over b gives a convolution. Apply the Fourier transform over t changes the convolution to a product, so we have

$$\widehat{g}(\omega) = \int_{\mathbb{R}} \widehat{W_a}(\omega)\widehat{\psi_a}(\omega)\frac{da}{|a|^{5/2}} = \int_{\mathbb{R}} \widehat{f}(\omega)\overline{\widehat{\psi_a}(\omega)}\widehat{\psi_a}(\omega)\frac{da}{|a|^3}, \tag{19}$$

where we have used Equation 13 to expand $\widehat{W_a}(\omega)$. Now the $\widehat{f}(\omega)$ comes out of the integral, and the $\widehat{\psi_a}(\omega)$ simplifies to $|a|\widehat{\psi}(a\omega)$ so we get

$$\widehat{g}(\omega) = \widehat{f}(\omega)\int_{\mathbb{R}} |\widehat{\psi}(a\omega)|^2\frac{da}{|a|} = \widehat{f}(\omega)C_\psi, \tag{20}$$

since the last integral there was our admissibility constant. Inverting the Fourier transform shows that $g(t) = f(t)C_\psi$, and so from Equation 18 we conclude that

$$f(t) = \frac{1}{C_\psi}\int_{\mathbb{R}}\int_{\mathbb{R}} W(a,b)\phi_{a,b}(t)\frac{da\,db}{a^2}. \tag{21}$$

This concludes the proof of the inversion formula.

2 A time-frequency representation

There is a particular wavelet where the CWT gives a precise time-frequency representation of a signal, with parameter a indicating inverse frequency, and b indicating location in time. We fix the wavelet $\psi(t)$ to be a modulated Gaussian, with

$$\psi(t) = e^{2\pi i t} e^{-t^2/2}. \tag{22}$$

The real and imaginary parts of the wavelet look like tapered cosine and sine waves, respectively, as shown in Figure 3. This wavelet is called the complex Morlet wavelet, and is available in the MATLAB wavelet toolbox. Incidentally, Jean Morlet was a French geophysicist who introduced the wavelet transform to the geophysics community in the 1970s.

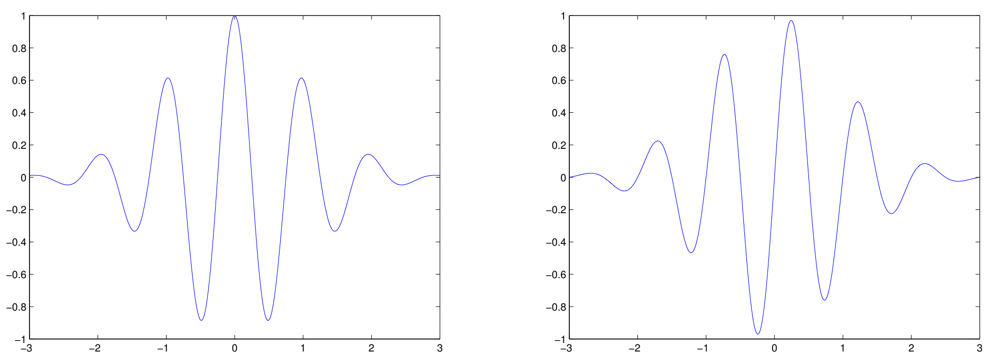

Figure 3: Real and imaginary parts of the complex Morlet wavelet; a modulated Gaussian.

Given a signal $f(t)$, its wavelet coefficients with respect to the Morlet wavelet are given as

$$\begin{aligned} W(a,b) &= \int_{\mathbb{R}} f(t)\overline{\psi_{a,b}(t)}\, dt \qquad (23)\\ &= \frac{1}{|a|^{1/2}} \int_{\mathbb{R}} f(t)\overline{\psi(\frac{t-b}{a})}\, dt \qquad (24)\\ &= \frac{1}{|a|^{1/2}} \int_{\mathbb{R}} f(t) e^{-\frac{(t-b)^2}{2a^2}} e^{-2\pi i (t-b)/a}\, dt \qquad (25) \end{aligned}$$

and making a change of variable $a = 1/\omega$ we have

$$W(\omega^{-1}, b) = |w|^{1/2} e^{2\pi i b\omega} \int_{\mathbb{R}} f(t) e^{-\frac{(t-b)^2\omega^2}{2}} e^{-2\pi i t\omega}\, dt. \tag{26}$$

How do we interpret this? The variable ω is exactly frequency, in Hertz (if t is in seconds), while the variable b is position (also in seconds). The integral

$$\int_{\mathbb{R}} f(t) e^{-\frac{(t-b)^2}{2a^2}} e^{-2\pi i(t-b)/a}\, dt \tag{27}$$

can be thought of as as Fourier transform (because of the exponential factor $e^{-2\pi i t\omega}$) of the function $f(t)e^{-\frac{(t-b)^2\omega^2}{2}}$, which is just the signal f multiplied by a Gaussian window centred at b, and width adjusted by parameter ω. So basically, this transform shows us the frequency content of the signal (parameterized by ω), near the time $t = b$. Notice that the width of the Gaussian decreases as the frequency goes up, which is a useful attribute for getting finer time resolution for higher frequency events.

In applications, we typically make a 2D plot of the two parameter function $TF(t,\omega) = |W(\omega^{-1}, t)|$, with parameter $t = b$ as the time, and $\omega = a^{-1}$ as the frequency. The factors in front of the integral in Equation 26, $|w|^{1/2}e^{2\pi i t\omega}$ are simply phase and amplitude adjustments which make the transform invertible in the nice way.

You might like to check that the Morlet wavelet does not quite satisfy the admissibility condition, so technically it is not invertible. However, it is easy enough to fix – there are a variety of methods to tweak the wavelet to make it satisfy admissibility.

There is a closely related transform, the S-transform, created by R. Stockwell in the 1990s, which is defined as the integral

$$S(\omega, b) = |w| \int_{\mathbb{R}} f(t) e^{-\frac{(t-b)^2\omega^2}{2}} e^{-2\pi i t\omega}\, dt. \tag{28}$$

It is essentially the continuous wavelet transform with a Morlet wavelet, as in Equation 26, but dropping the phase factor and using a different amplitude adjustment. It can be inverted simply by integrating over the b variable, to obtain the usual Fourier transform of f, which may be useful in some applications.

Another related, and well-studied transform is the Gabor transform, created by Dennis Gabor in the 1940s, which fixes the width of the Gaussians, so we write

$$G(\omega, b) = \int_{\mathbb{R}} f(t) e^{-\frac{(t-b)^2}{\sigma^2}} e^{-2\pi i t\omega}\, dt. \tag{29}$$

The fixed window width, as determined by constant σ, has its advantages as well.

2.1 Non-stationary filtering

The above transforms give us a time-frequency representation of a signal. We can denote these representations as some function $TF(t, \omega)$ where t is time and ω is frequency.

The idea of non stationary filtering is to transform the signal in the t-f domain, modify it in this domain, then invert the transform. This should give us a new signal that has been modified in some way, hopefully in a useful way.

A simple way is by multipliers. Fix some function $M(t, \omega)$ which varies in the time-frequency domain. For instance, it might be equal to zero in certain domains where you want to kill off particular frequencies at particular times, and equal to one in other regions where you want no change.

Take a signal $f(t)$. Find its time-frequency representation $FT(t, \omega)$, and modify it by multiplying with the function $M(t, \omega)$, to get a new function, the product $FT(t, \omega)M(t, \omega)$. Now invert this, using whatever inversion formula you have, to get a new function $g(t) = invert[FT(t, \omega)M(t, \omega)]$. This is the modified signal.

A very interesting example involves pseudo-differential operators, where a differential operator

$$Df(t) = A(t)f''(t) + B(t)f'(t) + c(t)f(t) \tag{30}$$

can be approximately represented by a multiplier of the form

$$M(t, \omega) = A(t)(2\pi i\omega)^2 + B(t)(2\pi i\omega) + C(t). \tag{31}$$

In principle, the time-frequency multiplier can give good approximations to these operators. There is a question, though, of how quickly they can be implemented numerically.

In our seismic work, we have had good success with the Gabor transform and multipliers. So far, not much success with the CWT and S-transform. But that is not to say this would never work...

Appendix = code

Some code I used to create the figures in this chapter.

```
%% Testing the continuous wavelet transform

t = linspace(0,1,10000);
```

```
t3 = linspace(0,.3333,3333);
x = sin(2*pi*440*t);
y = sin(2*pi*440*t.^2);
z = [sin(2*pi*440*t3),sin(2*pi*880*t3),sin(2*pi*660*t3)];
a = cwt(x,1:64,'cmor4-1');
b = cwt(y,1:64,'cmor4-1');
c = cwt(z,1:64,'cmor4-1');
figure(1)
imagesc(abs(a)); colormap('gray')
figure(2)
imagesc(abs(b)); colormap('gray')
figure(3)
imagesc(abs(c)); colormap('gray')

%% Testing the Gabor transform

[tvs1,tout,fout] = fgabor(x,t,.01,.005,1);
[tvs2,tout,fout] = fgabor(y,t,.01,.005,1);
[tvs3,tout,fout] = fgabor(z,t,.01,.005,1);
figure(1)
imagesc(tout,-fout,abs(tvs1')); colormap('gray')
figure(2)
imagesc(tout,-fout,abs(tvs2')); colormap('gray')
figure(3)
imagesc(tout,-fout,abs(tvs3')); colormap('gray')

%% Draw the Morlet wavelet

t = linspace(-3,3,10000);
x = real(exp(2*pi*i*t).*exp(-(t.^2)/2));
y = imag(exp(2*pi*i*t).*exp(-(t.^2)/2));
figure(1)
plot(t,x)
figure(2)
plot(t,y)
```